

Vibrations and Waves

Benjamin Crowell

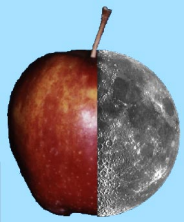


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Vibrations and Waves

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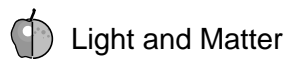
- 1 Newtonian Physics
- 2 Conservation Laws
- 3 Vibrations and Waves
- 4 Electricity and Magnetism
- 5 Optics
- 6 The Modern Revolution in Physics



Vibrations and Waves

Benjamin Crowell

www.lightandmatter.com



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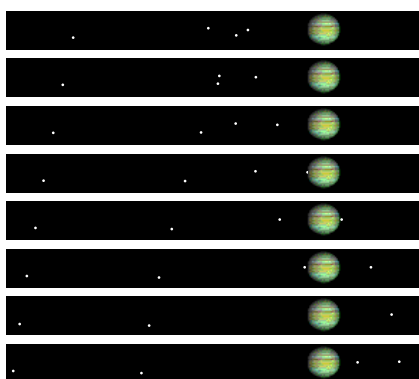
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Contents



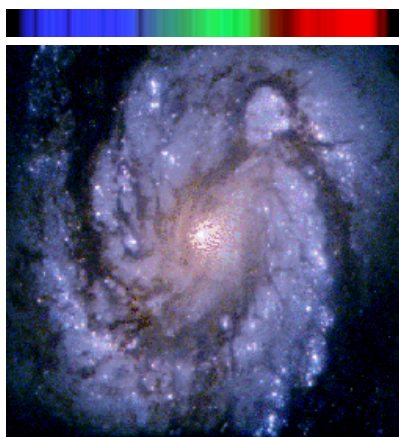
1 Vibrations 11

1.1	Period, Frequency, and Amplitude	12
1.2	Simple Harmonic Motion	14
1.3*	Proofs	16
	Summary	18
	Homework Problems	19



2 Resonance 21

2.1	Energy in Vibrations	22
2.2	Energy Lost From Vibrations	24
2.3	Putting Energy Into Vibrations	26
2.4*	Proofs	33
	Summary	36
	Homework Problems	37



3 Free Waves 39

3.1	Wave Motion	40
3.2	Waves on a String	44
3.3	Sound and Light Waves	48
3.4	Periodic Waves	50
3.5	The Doppler Effect	54
	Summary	58
	Homework Problems	59



4 Bounded Waves 61

4.1	Reflection, Transmission, and Absorption	62
4.2*	Quantitative Treatment of Reflection ..	67
4.3	Interference Effects	70
4.4	Waves Bounded on Both Sides	72
	Summary	77
	Homework Problems	78

Exercises 79

Glossary 81

Index 83

Photo Credits 85

Useful Data 92

Note: See Simple Nature (www.lightandmatter.com/area1sn.html) for coverage of complex number techniques.



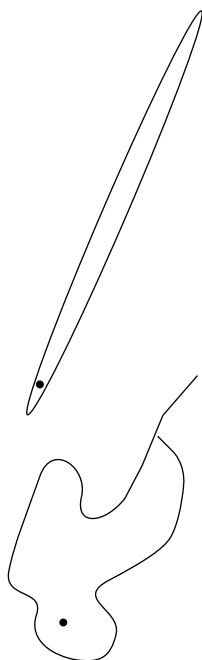
The vibrations of this electric bass string are converted to electrical vibrations, then to sound vibrations, and finally to vibrations of our eardrums.

1 Vibrations

Dandelion. Cello. Read those two words, and your brain instantly conjures a stream of associations, the most prominent of which have to do with vibrations. Our mental category of “dandelion-ness” is strongly linked to the color of light waves that vibrate about half a million billion times a second: yellow. The velvety throb of a cello has as its most obvious characteristic a relatively low musical pitch — the note you are spontaneously imagining right now might be one whose sound vibrations repeat at a rate of a hundred times a second.

Evolution has designed our two most important senses around the assumption that not only will our environment be drenched with information-bearing vibrations, but in addition those vibrations will often be repetitive, so that we can judge colors and pitches by the rate of repetition. Granting that we do sometimes encounter nonrepeating waves such as the consonant “sh,” which has no recognizable pitch, why was Nature’s assumption of repetition nevertheless so right in general?

Repeating phenomena occur throughout nature, from the orbits of electrons in atoms to the reappearance of Halley’s Comet every 75 years. Ancient cultures tended to attribute repetitious phenomena like the seasons to the cyclical nature of time itself, but we now have a less mystical explanation. Suppose that instead of Halley’s Comet’s true, repeating elliptical orbit that closes seamlessly upon itself with each revolution, we decide to take a pen and draw a whimsical alternative path that never repeats. We will not be able to draw for very long without having the path cross itself. But at

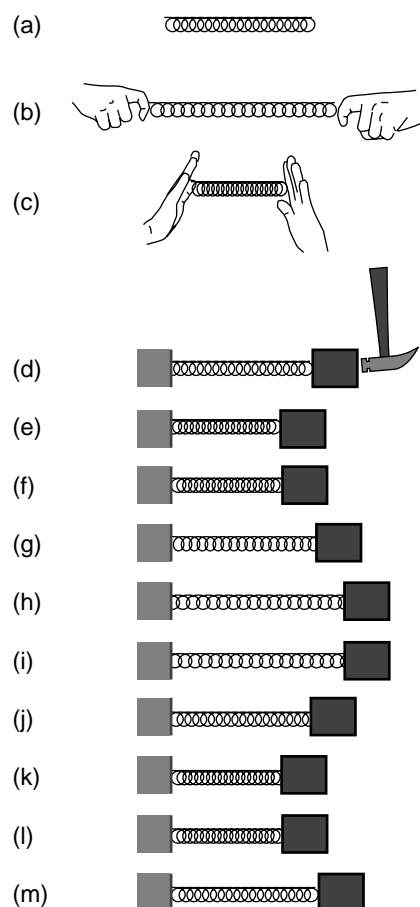


If we try to draw a non-repeating orbit for Halley's Comet, it will inevitably end up crossing itself.

such a crossing point, the comet has returned to a place it visited once before, and since its potential energy is the same as it was on the last visit, conservation of energy proves that it must again have the same kinetic energy and therefore the same speed. Not only that, but the comet's direction of motion cannot be randomly chosen, because angular momentum must be conserved as well. Although this falls short of being an ironclad proof that the comet's orbit must repeat, it no longer seems surprising that it does.

Conservation laws, then, provide us with a good reason why repetitive motion is so prevalent in the universe. But it goes deeper than that. Up to this point in your study of physics, I have been indoctrinating you with a mechanistic vision of the universe as a giant piece of clockwork. Breaking the clockwork down into smaller and smaller bits, we end up at the atomic level, where the electrons circling the nucleus resemble — well, little clocks! From this point of view, particles of matter are the fundamental building blocks of everything, and vibrations and waves are just a couple of the tricks that groups of particles can do. But at the beginning of the 20th century, the tables were turned. A chain of discoveries initiated by Albert Einstein led to the realization that the so-called subatomic “particles” were in fact waves. In this new world-view, it is vibrations and waves that are fundamental, and the formation of matter is just one of the tricks that waves can do.

1.1 Period, Frequency, and Amplitude



The figure shows our most basic example of a vibration. With no forces on it, the spring assumes its equilibrium length, (a). It can be stretched, (b), or compressed, (c). We attach the spring to a wall on the left and to a mass on the right. If we now hit the mass with a hammer, (d), it oscillates as shown in the series of snapshots, (d)-(m). If we assume that the mass slides back and forth without friction and that the motion is one-dimensional, then conservation of energy proves that the motion must be repetitive. When the block comes back to its initial position again, (g), its potential energy is the same again, so it must have the same kinetic energy again. The motion is in the opposite direction, however. Finally, at (j), it returns to its initial position with the same kinetic energy and the same direction of motion. The motion has gone through one complete cycle, and will now repeat forever in the absence of friction.

The usual physics terminology for motion that repeats itself over and over is **periodic motion**, and the time required for one repetition is called the **period**, T . (The symbol P is not used because of the possible confusion with momentum.) One complete repetition of the motion is called a **cycle**.

We are used to referring to short-period sound vibrations as “high” in pitch, and it sounds odd to have to say that high pitches have low periods. It is therefore more common to discuss the rapidity of a vibration in terms of the number of vibrations per second, a quantity called the **frequency**, f . Since the period is the number of seconds per cycle and the frequency is the number of cycles per second, they are reciprocals of each other,

$$f = 1/T \text{ .}$$



Example: a carnival game

In the carnival game shown in the figure, the rube is supposed to push the bowling ball on the track just hard enough so that it goes over the hump and into the valley, but does not come back out again. If the only types of energy involved are kinetic and potential, this is impossible. Suppose you expect the ball to come back to a point such as the one shown with the dashed outline, then stop and turn around. It would already have passed through this point once before, going to the left on its way into the valley. It was moving then, so conservation of energy tells us that it cannot be at rest when it comes back to the same point. The motion that the customer hopes for is physically impossible. There is a physically possible periodic motion in which the ball rolls back and forth, staying confined within the valley, but there is no way to get the ball into that motion beginning from the place where we start. There is a way to beat the game, though. If you put enough spin on the ball, you can create enough kinetic friction so that a significant amount of heat is generated. Conservation of energy then allows the ball to be at rest when it comes back to a point like the outlined one, because kinetic energy has been converted into heat.

Example: Period and frequency of a fly's wing-beats

A Victorian parlor trick was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and announce how many times the fly's wings had flapped in one second. If the fly's wings flap, say, 200 times in one second, then the frequency of their motion is $f = 200/1 \text{ s} = 200 \text{ s}^{-1}$. The period is one 200th of a second, $T = 1/f = (1/200) \text{ s} = 0.005 \text{ s}$.

Units of inverse second, s^{-1} , are awkward in speech, so an abbreviation has been created. One Hertz, named in honor of a pioneer of radio technology, is one cycle per second. In abbreviated form, $1 \text{ Hz} = 1 \text{ s}^{-1}$. This is the familiar unit used for the frequencies on the radio dial.

Example: frequency of a radio station

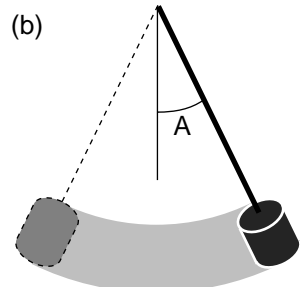
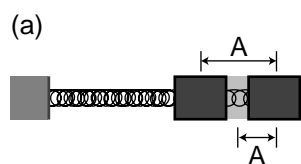
Question: KLON's frequency is 88.1 MHz. What does this mean, and what period does this correspond to?

Solution: The metric prefix M- is mega-, i.e. millions. The radio waves emitted by KLON's transmitting antenna vibrate 88.1 million times per second. This corresponds to a period of

$$T = 1/f = 1.14 \times 10^{-8} \text{ s}.$$

This example shows a second reason why we normally speak in terms of frequency rather than period: it would be painful to have to refer to such small time intervals routinely. I could abbreviate by telling people that KLON's period was 11.4 nanoseconds, but most people are more familiar with the big metric prefixes than with the small ones.

Units of frequency are also commonly used to specify the speeds of computers. The idea is that all the little circuits on a computer chip are synchronized by the very fast ticks of an electronic clock, so that the circuits can all cooperate on a task without getting ahead or behind. Adding two numbers might require, say, 30 clock cycles. Microcomputers these days operate at clock frequencies of about a gigahertz.



The amplitude of the vibrations of the mass on a spring could be defined in two different ways, (a). It would have units of distance. The amplitude of a swinging pendulum would more naturally be defined as an angle, (b).

We have discussed how to measure how fast something vibrates, but not how big the vibrations are. The general term for this is **amplitude**, A . The definition of amplitude depends on the system being discussed, and two people discussing the same system may not even use the same definition. In the example of the block on the end of the spring, the amplitude will be measured in distance units such as cm. One could work in terms of the distance traveled by the block from the extreme left to the extreme right, but it would be somewhat more common in physics to use the distance from the center to one extreme. The former is usually referred to as the **peak-to-peak amplitude**, since the extremes of the motion look like mountain peaks or upside-down mountain peaks on a graph of position versus time.

In other situations we would not even use the same units for amplitude. The amplitude of a child on a swing would most conveniently be measured as an angle, not a distance, since her feet will move a greater distance than her head. The electrical vibrations in a radio receiver would be measured in electrical units such as volts or amperes.

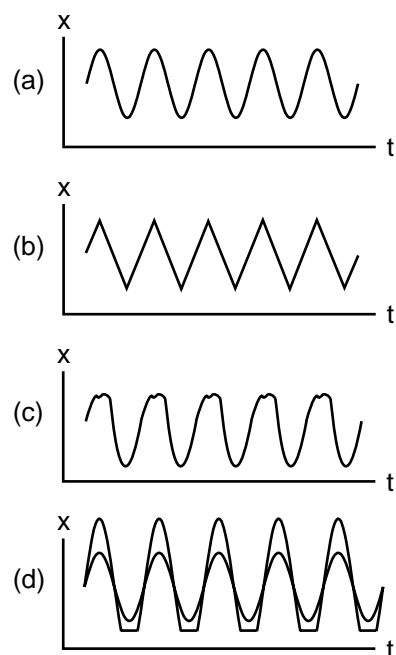
1.2 Simple Harmonic Motion

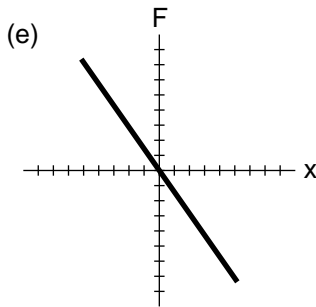
Why are sine-wave vibrations so common?

If we actually construct the mass-on-a-spring system discussed in the previous section and measure its motion accurately, we will find that its x - t graph is nearly a perfect sine-wave shape, as shown in figure (a). (We call it a “sine wave” or “sinusoidal” even if it is a cosine, or a sine or cosine shifted by some arbitrary horizontal amount.) It may not be surprising that it is a wiggle of this general sort, but why is it a specific mathematically perfect shape? Why is it not a sawtooth shape like (b) or some other shape like (c)? The mystery deepens as we find that a vast number of apparently unrelated vibrating systems show the same mathematical feature. A tuning fork, a sapling pulled to one side and released, a car bouncing on its shock absorbers, all these systems will exhibit sine-wave motion under one condition: the amplitude of the motion must be small.

It is not hard to see intuitively why extremes of amplitude would act differently. For example, a car that is bouncing lightly on its shock absorbers may behave smoothly, but if we try to double the amplitude of the vibrations the bottom of the car may begin hitting the ground, (d). (Although we are assuming for simplicity in this chapter that energy is never dissipated, this is clearly not a very realistic assumption in this example. Each time the car hits the ground it will convert quite a bit of its potential and kinetic energy into heat and sound, so the vibrations would actually die out quite quickly, rather than repeating for many cycles as shown in the figure.)

The key to understanding how an object vibrates is to know how the force on the object depends on the object's position. If an object is vibrating to the right and left, then it must have a leftward force on it when it is on the right side, and a rightward force when it is on the left side. In one dimension, we can represent the direction of the force using a positive or





negative sign, and since the force changes from positive to negative there must be a point in the middle where the force is zero. This is the equilibrium point, where the object would stay at rest if it was released at rest. For convenience of notation throughout this chapter, we will define the origin of our coordinate system so that x equals zero at equilibrium.

The simplest example is the mass on a spring, for which force on the mass is given by Hooke's law,

$$F = -kx .$$

We can visualize the behavior of this force using a graph of F versus x , fig. (e). The graph is a line, and the spring constant, k , is equal to minus its slope. A stiffer spring has a larger value of k and a steeper slope. Hooke's law is only an approximation, but it works very well for most springs in real life, as long as the spring isn't compressed or stretched so much that it is permanently bent or damaged.

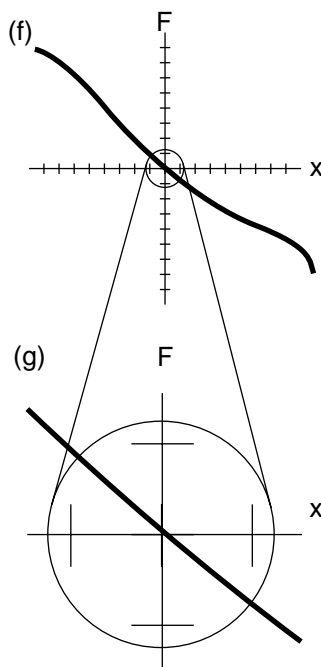
The following important theorem, whose proof is given in optional section 1.3, relates the motion graph to the force graph.

Theorem: A linear force graph makes a sinusoidal motion graph.

If the total force on a vibrating object depends only on the object's position, and is related to the object's displacement from equilibrium by an equation of the form $F = -kx$, then the object's motion displays a

sinusoidal graph with period $T = 2\pi\sqrt{m/k}$.

Even if you do not read the proof, it is not too hard to understand why the equation for the period makes sense. A greater mass causes a greater period, since the force will not be able to whip a massive object back and forth very rapidly. A larger value of k causes a shorter period, because a stronger force can whip the object back and forth more rapidly.



This may seem like only an obscure theorem about the mass-on-a-spring system, but figures (f) and (g) show it to be far more general than that. Figure (f) depicts a force curve that is not a straight line. A system with this F - x curve would have large-amplitude vibrations that were complex and not sinusoidal. But the same system would exhibit sinusoidal small-amplitude vibrations. This is because any curve looks linear from very close up. If we magnify the F - x graph as shown in (g), it becomes very difficult to tell that the graph is not a straight line. If the vibrations were confined to the region shown in (g), they would be very nearly sinusoidal. This is the reason why sinusoidal vibrations are a universal feature of all vibrating systems, if we restrict ourselves to small amplitudes. The theorem is therefore of great general significance. It applies throughout the universe, to objects ranging from vibrating stars to vibrating nuclei. A sinusoidal vibration is known as **simple harmonic motion**.

Period is independent of amplitude.

Until now we have not even mentioned the most counterintuitive aspect of the equation $T = 2\pi\sqrt{m/k}$: it does not depend on amplitude at all. Intuitively, most people would expect the mass-on-a-spring system to take longer to complete a cycle if the amplitude was larger. (We are comparing amplitudes that are different from each other, but both small enough that the theorem applies.) In fact the larger-amplitude vibrations take the

same amount of time as the small-amplitude ones

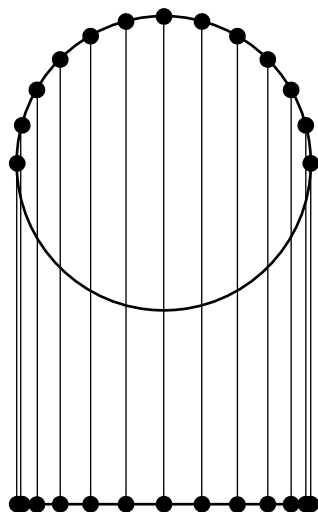
Legend has it that this fact was first noticed by Galileo during what was apparently a less than enthralling church service. A gust of wind would now and then start one of the chandeliers in the cathedral swaying back and forth, and he noticed that regardless of the amplitude of the vibrations, the period of oscillation seemed to be the same. Up until that time, he had been carrying out his physics experiments with such crude time-measuring techniques as feeling his own pulse or singing a tune to keep a musical beat. But after going home and testing a pendulum, he convinced himself that he had found a superior method of measuring time. Even without a fancy system of pulleys to keep the pendulum's vibrations from dying down, he could get very accurate time measurements, because the gradual decrease in amplitude due to friction would have no effect on the pendulum's period. (Galileo never produced a modern-style pendulum clock with pulleys, a minute hand, and a second hand, but within a generation the device had taken on the form that persisted for hundreds of years after.)

Example: the pendulum

Question: Compare the periods of pendula having bobs with different masses.

Solution: From the equation $T = 2\pi\sqrt{m/k}$, we might expect that a larger mass would lead to a longer period. However, increasing the mass also increases the forces that act on the pendulum: gravity and the tension in the string. This increases k as well as m , so the period of a pendulum is independent of m .

1.3* Proofs



The object moves along the circle at constant speed, but even though its overall speed is constant, the x and y components of its velocity are continuously changing, as shown by the unequal spacing of the points when projected onto the line below. Projected onto the line, its motion is the same as that of an object experiencing a force $F = -kx$.

In this section we prove (1) that a linear F - x graph gives sinusoidal motion, (2) that the period of the motion is $T = 2\pi\sqrt{m/k}$, and (3) that the period is independent of the amplitude. You may omit this section without losing the continuity of the chapter.

The basic idea of the proof can be understood by imagining that you are watching a child on a merry-go-round from far away. Because you are in the same horizontal plane as her motion, she appears to be moving from side to side along a line. Circular motion viewed edge-on doesn't just look like any kind of back-and-forth motion, it looks like motion with a sinusoidal x - t graph, because the sine and cosine functions can be defined as the x and y coordinates of a point at angle θ on the unit circle. The idea of the proof, then, is to show that an object acted on by a force that varies as $F = -kx$ has motion that is identical to circular motion projected down to one dimension. The equation $T = 2\pi\sqrt{m/k}$ will also fall out nicely at the end.

For an object performing uniform circular motion, we have

$$|a| = v^2/r.$$

The x component of the acceleration is therefore

$$a_x = -\frac{v^2}{r} \cos \theta,$$

where θ is the angle measured counterclockwise from the x axis. Applying Newton's second law,

$$F_x / m = -\frac{v^2}{r} \cos \theta, \text{ so}$$

$$F_x = -m \frac{v^2}{r} \cos \theta.$$

Since our goal is an equation involving the period, it is natural to eliminate the variable $v = \text{circumference}/T = 2\pi r/T$, giving

$$F_x = -\frac{4\pi^2 m r}{T^2} \cos \theta.$$

The quantity $r \cos \theta$ is the same as x , so we have

$$F_x = -\frac{4\pi^2 m}{T^2} x.$$

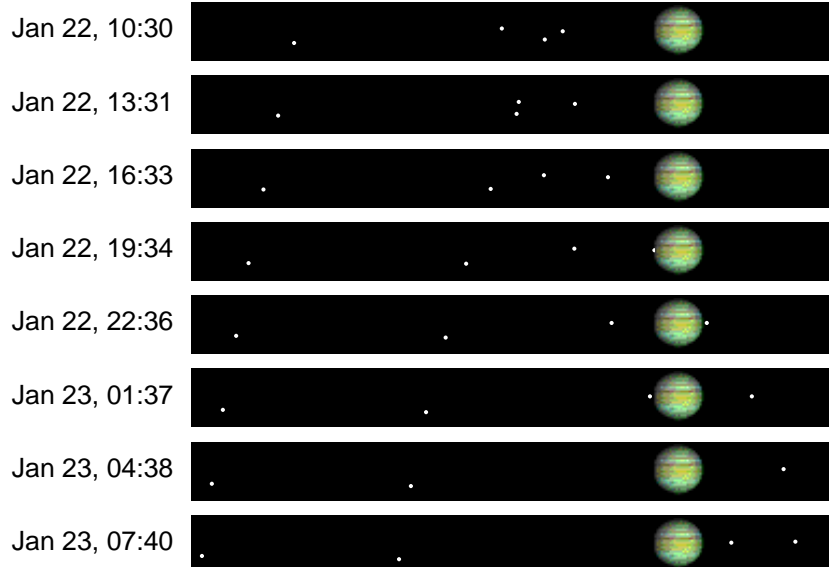
Since everything is constant in this equation except for x , we have proven that motion with force proportional to x is the same as circular motion projected onto a line, and therefore that a force proportional to x gives sinusoidal motion. Finally, we identify the constant factor of $4\pi^2 m / T^2$ with k , and solving for T gives the desired equation for the period,

$$T = 2\pi \sqrt{\frac{m}{k}}.$$

Since this equation is independent of r , T is independent of the amplitude.

Example: The moons of Jupiter.

The idea behind this proof is aptly illustrated by the moons of Jupiter. Their discovery by Galileo was an epochal event in astronomy, because it proved that not everything in the universe had to revolve around the earth as had been believed. Galileo's telescope was of poor quality by modern standards, but the figure below shows a simulation of how Jupiter and its moons might appear at intervals of three hours through a large present-day instrument. Because we see the moons' circular orbits edge-on, they appear to perform sinusoidal vibrations. Over this time period, the innermost moon, Io, completes half a cycle.



Summary

Selected Vocabulary

periodic motion	motion that repeats itself over and over
period	the time required for one cycle of a periodic motion
frequency	the number of cycles per second, the inverse of the period
amplitude	the amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration
simple harmonic motion ...	motion whose x - t graph is a sine wave

Notation

T	period
f	frequency
A	amplitude
k	the slope of the graph of F versus x , where F is the total force acting on an object and x is the object's position; For a spring, this is known as the spring constant.

Notation Used in Other Books

ν	The Greek letter ν , nu, is used in many books for frequency.
ω	The Greek letter ω , omega, is often used as an abbreviation for $2\pi f$.

Summary

Periodic motion is common in the world around us because of conservation laws. An important example is one-dimensional motion in which the only two forms of energy involved are potential and kinetic; in such a situation, conservation of energy requires that an object repeat its motion, because otherwise when it came back to the same point, it would have to have a different kinetic energy and therefore a different total energy.

Not only are periodic vibrations very common, but small-amplitude vibrations are always sinusoidal as well. That is, the x - t graph is a sine wave. This is because the graph of force versus position will always look like a straight line on a sufficiently small scale. This type of vibration is called simple harmonic motion. In simple harmonic motion, the period is independent of the amplitude, and is given by

$$T = 2\pi\sqrt{\frac{m}{k}} .$$

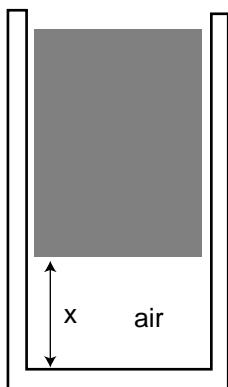
Homework Problems

1. Find an equation for the frequency of simple harmonic motion in terms of k and m .

2. Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail's vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond?

3. (a) Pendulum 2 has a string twice as long as pendulum 1. If we define x as the distance traveled by the bob along a circle away from the bottom, how does the k of pendulum 2 compare with the k of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of x .]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.



Problems 4 and 5.

4 ✓. A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by $F_{\text{air}} = ax^{-1.4}$, where a is a constant with funny units of $\text{N}\cdot\text{m}^{1.4}$. For simplicity, assume the air only supports the weight, F_{w} , of the piston itself, although in practice this device is used to support some other object. The equilibrium position, x_0 , is where F_{w} equals $-F_{\text{air}}$. (Note that in the main text I have assumed the equilibrium position to be at $x=0$, but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Let $a=1 \text{ N}\cdot\text{m}^{1.4}$, $x_0=1.00 \text{ m}$, and $F_{\text{w}}=-1.00 \text{ N}$. The piston is released from $x=1.01 \text{ m}$. Draw a neat, accurate graph of the total force, F , as a function of x , on graph paper, covering the range from $x=0.98 \text{ m}$ to 1.02 m . Over this small range, you will find that the force is very nearly proportional to $x-x_0$. Approximate the curve with a straight line, find its slope, and derive the approximate period of oscillation.

5. Consider the same pneumatic piston described in the previous problem, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of x vs t , showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the curvature of the x - t graph, so if the force is greater, the graph should curve around more quickly.]

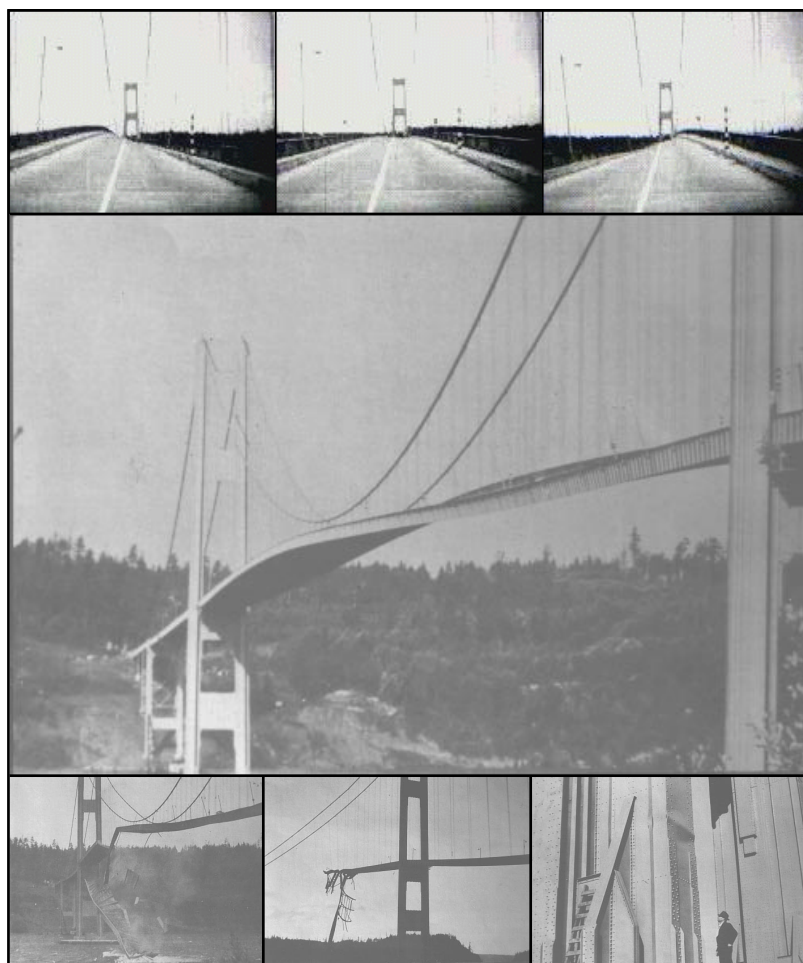
S A solution is given in the back of the book.

✓ A computerized answer check is available.

★ A difficult problem.

∫ A problem that requires calculus.

6. Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new "chunk" of water. (a) Show that a cube of mass m with edges of length b floating upright (not tilted) in a fluid of density ρ will have a draft (depth to which it sinks below the waterline) h given at equilibrium by $h_o = m / b^2 \rho$. (b) Find the total force on the cube when its draft is h , and verify that plugging in $h = h_o$ gives a total force of zero. (c) Find the cube's period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of h_o and g only.



Top: A series of images from a silent movie of the bridge vibrating on the day it was to collapse, taken by an unknown amateur photographer.

Middle: The bridge immediately before the collapse, with the sides vibrating 8.5 meters (28 feet) up and down. Note that the bridge is over a mile long.

Bottom: During and after the final collapse. The right-hand picture gives a sense of the massive scale of the construction.

2 Resonance

Soon after the mile-long Tacoma Narrows Bridge opened in July 1940, motorists began to notice its tendency to vibrate frighteningly in even a moderate wind. Nicknamed “Galloping Gertie,” the bridge collapsed in a steady 42-mile-per-hour wind on November 7 of the same year. The following is an eyewitness report from a newspaper editor who found himself on the bridge as the vibrations approached the breaking point.

“Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car... I jammed on the brakes and got out, only to be thrown onto my face against the curb.

“Around me I could hear concrete cracking. I started to get my dog Tubby, but was thrown again before I could reach the car. The car itself began to slide from side to side of the roadway.

“On hands and knees most of the time, I crawled 500 yards or more to the towers... My breath was coming in gasps; my knees were raw and bleeding, my hands bruised and swollen from gripping the concrete curb... Toward the last, I risked rising to my feet and running a few yards at a time... Safely back at the toll plaza, I saw the bridge in its final collapse and

saw my car plunge into the Narrows.”

The ruins of the bridge formed an artificial reef, one of the world's largest. It was not replaced for ten years. The reason for its collapse was not substandard materials or construction, nor was the bridge underdesigned: the piers were hundred-foot blocks of concrete, the girders massive and made of carbon steel. The bridge was destroyed because of the physical phenomenon of resonance, the same effect that allows an opera singer to break a wine glass with her voice and that lets you tune in the radio station you want. The replacement bridge, which has lasted half a century so far, was built smarter, not stronger. The engineers learned their lesson and simply included some slight modifications to avoid the resonance phenomenon that spelled the doom of the first one.

2.1 Energy in Vibrations

One way of describing the collapse of the bridge is that the bridge kept taking energy from the steadily blowing wind and building up more and more energetic vibrations. In this section, we discuss the energy contained in a vibration, and in the subsequent sections we will move on to the loss of energy and the adding of energy to a vibrating system, all with the goal of understanding the important phenomenon of resonance.

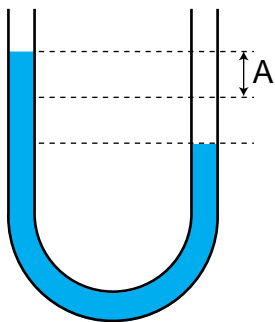
Going back to our standard example of a mass on a spring, we find that there are two forms of energy involved: the potential energy stored in the spring and the kinetic energy of the moving mass. We may start the system in motion either by hitting the mass to put in kinetic energy by pulling it to one side to put in potential energy. Either way, the subsequent behavior of the system is identical. It trades energy back and forth between kinetic and potential energy. (We are still assuming there is no friction, so that no energy is converted to heat, and the system never runs down.)

The most important thing to understand about the energy content of vibrations is that the total energy is proportional to the square of the amplitude. Although the total energy is constant, it is instructive to consider two specific moments in the motion of the mass on a spring as examples. When the mass is all the way to one side, at rest and ready to reverse directions, all its energy is potential. We have already seen that the potential energy stored in a spring equals $\frac{1}{2}kx^2$, so the energy is proportional to the square of the amplitude. Now consider the moment when the mass is passing through the equilibrium point at $x=0$. At this point it has no potential energy, but it does have kinetic energy. The velocity is proportional to the amplitude of the motion, and the kinetic energy, $\frac{1}{2}mv^2$, is proportional to the square of the velocity, so again we find that the energy is proportional to the square of the amplitude. The reason for singling out these two points is merely instructive; proving that energy is proportional to A^2 at any point would suffice to prove that energy is proportional to A^2 in general, since the energy is constant.

Are these conclusions restricted to the mass-on-a-spring example? No. We have already seen that $F=-kx$ is a valid approximation for any vibrating object, as long as the amplitude is small. We are thus left with a very general

conclusion: the energy of any vibration is approximately proportional to the square of the amplitude, provided that the amplitude is small.

Example: water in a U-tube



If water is poured into a U-shaped tube as shown in the figure, it can undergo vibrations about equilibrium. The energy of such a vibration is most easily calculated by considering the “turnaround point” when the water has stopped and is about to reverse directions. At this point, it has only potential energy and no kinetic energy, so by calculating its potential energy we can find the energy of the vibration. This potential energy is the same as the work that would have to be done to take the water out of the right-hand side down to a depth A below the equilibrium level, raise it through a height A , and place it in the left-hand side. The weight of this chunk of water is proportional to A , and so is the height through which it must be lifted, so the energy is proportional to A^2 .

Example: the range of energies of sound waves

Question: The amplitude of vibration of your eardrum at the threshold of pain is about 10^6 times greater than the amplitude with which it vibrates in response to the softest sound you can hear. How many times greater is the energy with which your ear has to cope for the painfully loud sound, compared to the soft sound?

Solution: The amplitude is 10^6 times greater, and energy is proportional to the square of the amplitude, so the energy is greater by a factor of 10^{12} . This is a phenomenally large factor!

We are only studying vibrations right now, not waves, so we are not yet concerned with how a sound wave works, or how the energy gets to us through the air. Note that because of the huge range of energies that our ear can sense, it would not be reasonable to have a sense of loudness that was additive. Consider, for instance, the following three levels of sound:

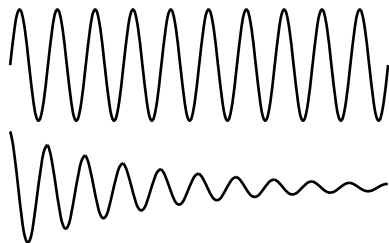
barely audible, gentle wind

quiet conversation 10^5 times more energy than the wind

heavy metal concert 10^{12} times more energy than the wind

In terms of addition and subtraction, the difference between the wind and the quiet conversation is nothing compared to the difference between the quiet conversation and the heavy metal concert. Evolution wanted our sense of hearing to be able to encompass all these sounds without collapsing the bottom of the scale so that anything softer than the crack of doom would sound the same. So rather than making our sense of loudness additive, mother nature made it multiplicative. We sense the difference between the wind and the quiet conversation as spanning a range of about 5/12 as much as the whole range from the wind to the heavy metal concert. Although a detailed discussion of the decibel scale is not relevant here, the basic point to note about the decibel scale is that it is logarithmic. The zero of the decibel scale is close to the lower limit of human hearing, and adding 1 unit to the decibel measurement corresponds to *multiplying* the energy level (or actually the power per unit area) by a certain factor.

2.2 Energy Lost From Vibrations



Friction has the effect of pinching the x - t graph of a vibrating object.

Until now, we have been making the relatively unrealistic assumption that a vibration would never die out. For a realistic mass on a spring, there will be friction, and the kinetic and potential energy of the vibrations will therefore be gradually converted into heat. Similarly, a guitar string will slowly convert its kinetic and potential energy into sound. In all cases, the effect is to “pinch” the sinusoidal x - t graph more and more with passing time. Friction is not necessarily bad in this context — a musical instrument that never got rid of any of its energy would be completely silent! The dissipation of the energy in a vibration is known as damping.

Self-Check



Most people who try to draw graphs like those shown on the left will tend to shrink their wiggles horizontally as well as vertically. Why is this wrong?

In the graphs on the left, I have not shown any point at which the damped vibration finally stops completely. Is this realistic? Yes and no. If energy is being lost due to friction between two solid surfaces, then we expect the force of friction to be nearly independent of velocity. This constant friction force puts an upper limit on the total distance that the vibrating object can ever travel without replenishing its energy, since work equals force times distance, and the object must stop doing work when its energy is all converted into heat. (The friction force does reverse directions when the object turns around, but reversing the direction of the motion at the same time that we reverse the direction of the force makes it certain that the object is always doing positive work, not negative work.)

Damping due to a constant friction force is not the only possibility however, or even the most common one. A pendulum may be damped mainly by air friction, which is approximately proportional to v^2 , while other systems may exhibit friction forces that are proportional to v . It turns out that friction proportional to v is the simplest case to analyze mathematically, and anyhow all the important physical insights can be gained by studying this case.

If the friction force is proportional to v , then as the vibrations die down, the frictional forces get weaker due to the lower speeds. The less energy is left in the system, the more miserly the system becomes with giving away any more energy. Under these conditions, the vibrations theoretically never die out completely, and mathematically, the loss of energy from the system is exponential: the system loses a fixed percentage of its energy per cycle. This is referred to as exponential decay.

A nonrigorous proof is as follows. The force of friction is proportional to v , and v is proportional to how far the objects travels in one cycle, so the frictional force is proportional to amplitude. The amount of work done by friction is proportional to the force and to the distance traveled, so the work done in one cycle is proportional to the square of the amplitude. Since both the work and the energy are proportional to A^2 , the amount of energy taken away by friction in one cycle is a fixed percentage of the amount of energy the system has.



The horizontal axis is a time axis, and the period of the vibrations is independent of amplitude. Shrinking the amplitude does not make the cycles any faster.

Self-Check



The figure shows an x - t graph for a strongly damped vibration, which loses half of its amplitude with every cycle. What fraction of the energy is lost in each cycle?



It is customary to describe the amount of damping with a quantity called the **quality factor**, Q , defined as the number of cycles required for the energy to fall off by a factor of 535. (The origin of this obscure numerical factor is $e^{2\pi}$, where $e=2.71828\dots$ is the base of natural logarithms.) The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

Example: exponential decay in a trumpet

Question: The vibrations of the air column inside a trumpet have a Q of about 10. This means that even after the trumpet player stops blowing, the note will keep sounding for a short time. If the player suddenly stops blowing, how will the sound intensity 20 cycles later compare with the sound intensity while she was still blowing?

Solution: The trumpet's Q is 10, so after 10 cycles the energy will have fallen off by a factor of 535. After another 10 cycles we lose another factor of 535, so the sound intensity is reduced by a factor of $535 \times 535 = 2.9 \times 10^5$.

The decay of a musical sound is part of what gives it its character, and a good musical instrument should have the right Q , but the Q that is considered desirable is different for different instruments. A guitar is meant to keep on sounding for a long time after a string has been plucked, and might have a Q of 1000 or 10000. One of the reasons why a cheap synthesizer sounds so bad is that the sound suddenly cuts off after a key is released.

Example: Q of a stereo speaker

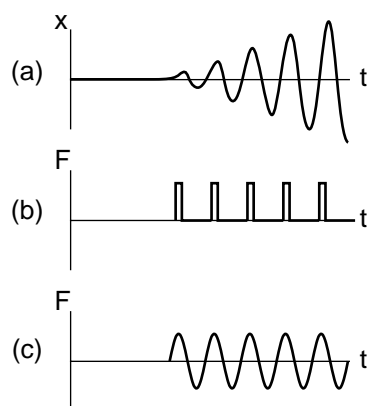
Stereo speakers are not supposed to reverberate or “ring” after an electrical signal that stops suddenly. After all, the recorded music was made by musicians who knew how to shape the decays of their notes correctly. Adding a longer “tail” on every note would make it sound wrong. We therefore expect that stereo speaker will have a very low Q , and indeed, most speakers are designed with a Q of about 1. (Low-quality speakers with larger Q values are referred to as “boomy.”)

We will see later in the chapter that there are other reasons why a speaker should not have a high Q .

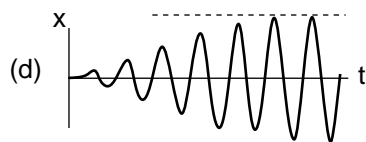


Energy is proportional to the square of amplitude, so its energy is four times smaller after every cycle. It loses three quarters of its energy with each cycle.

2.3 Putting Energy Into Vibrations



When pushing a child on a swing, you cannot just apply a constant force. A constant force will move the swing out to a certain angle, but will not allow the swing to start swinging. Nor can you give short pushes at randomly chosen times. That type of random pushing would increase the child's kinetic energy whenever you happened to be pushing in the same direction as her motion, but it would reduce her energy when your pushing happened to be in the opposite direction compared to her motion. To make her build up her energy, you need to make your pushes rhythmic, pushing at the same point in each cycle. In other words, your force needs to form a repeating pattern with the same frequency as the normal frequency of vibration of the swing. Graph (a) shows what the child's x - t graph would look like as you gradually put more and more energy into her vibrations. A graph of your *force* versus time would probably look something like graph (b). It turns out, however, that it is much simpler mathematically to consider a vibration with energy being pumped into it by a driving force that is itself a sine-wave, (c). A good example of this is your eardrum being driven by the force of a sound wave.



Now we know realistically that the child on the swing will not keep increasing her energy forever, nor does your eardrum end up exploding because a continuing sound wave keeps pumping more and more energy into it. In any realistic system, there is energy going out as well as in. As the vibrations increase in amplitude, there is an increase in the amount of energy taken away by damping with each cycle. This occurs for two reasons. Work equals force times distance (or, more accurately, the area under the force-distance curve). As the amplitude of the vibrations increases, the damping force is being applied over a longer distance. Furthermore, the damping force usually increases with velocity (we usually assume for simplicity that it is proportional to velocity), and this also serves to increase the rate at which damping forces remove energy as the amplitude increases. Eventually (and small children and our eardrums are thankful for this!), the amplitude approaches a maximum value (d) at which energy is removed by the damping force just as quickly as it is being put in by the driving force.

This process of approaching a maximum amplitude happens extremely quickly in many cases, e.g. the ear or a radio receiver, and we don't even notice that it took a millisecond or a microsecond for the vibrations to "build up steam." We are therefore mainly interested in predicting the behavior of the system once it has had enough time to reach essentially its maximum amplitude. This is known as the steady-state behavior of a vibrating system.

Now comes the interesting part: what happens if the frequency of the driving force is mismatched to the frequency at which the system would naturally vibrate on its own? We all know that a radio station doesn't have to be tuned in exactly, although there is only a small range over which a given station can be received. The designers of the radio had to make the

range fairly small to make it possible eliminate unwanted stations that happened to be nearby in frequency, but it couldn't be too small or you wouldn't be able to adjust the knob accurately enough. (Even a digital radio can be tuned to 88.0 MHz and still bring in a station at 88.1 MHz.) The ear also has some natural frequency of vibration, but in this case the range of frequencies to which it can respond is quite broad. Evolution has made the ear's frequency response as broad as possible because it was to our ancestors' advantage to be able to hear everything from a low roars to a high-pitched shriek.

The remainder of this section develops four important facts about the response of a system to a driving force whose frequency is not necessarily the same is the system's natural frequency of vibration. The style is approximate and intuitive, but proofs are given in the subsequent optional section.

First, although we know the ear has a frequency — about 4000 Hz — at which it would vibrate naturally, it does not vibrate at 4000 Hz in response to a low-pitched 200 Hz tone. It always responds at the frequency at which it is driven. Otherwise all pitches would sound like 4000 Hz to us. This is a general fact about driven vibrations:

- (1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

Now let's think about the amplitude of the steady-state response. Imagine that a child on a swing has a natural frequency of vibration of 1 Hz, but we are going to try to make her swing back and forth at 3 Hz. We intuitively realize that quite a large force would be needed to achieve an amplitude of even 30 cm, i.e. the amplitude is less in proportion to the force. When we push at the natural frequency of 1 Hz, we are essentially just pumping energy back into the system to compensate for the loss of energy due to the damping (friction) force. At 3 Hz, however, we are not just counteracting friction. We are also providing an extra force to make the child's momentum reverse itself more rapidly than it would if gravity and the tension in the chain were the only forces acting. It is as if we are artificially increasing the k of the swing, but this is wasted effort because we spend just as much time decelerating the child (taking energy out of the system) as accelerating her (putting energy in).

Now imagine the case in which we drive the child at a very low frequency, say 0.02 Hz or about one vibration per minute. We are essentially just holding the child in position while very slowly walking back and forth. Again we intuitively recognize that the amplitude will be very small in proportion to our driving force. Imagine how hard it would be to hold the child at our own head-level when she is at the end of her swing! As in the too-fast 3 Hz case, we are spending most of our effort in artificially changing the k of the swing, but now rather than reinforcing the gravity and tension forces we are working against them, effectively reducing k . Only a very small part of our force goes into counteracting friction, and the rest is used in repetitively putting potential energy in on the upswing and taking it back out on the downswing, without any long-term gain.

We can now generalize to make the following statement, which is true for all driven vibrations:

(2) A vibrating system **resonates** at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

Example: an opera singer breaking a wineglass

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back.

Example: collapse of the Nimitz Freeway in an earthquake

I led off the chapter with the dramatic collapse of the Tacoma Narrows Bridge, mainly because it was well documented by a local physics professor, and an unknown person made a movie of the collapse. The collapse a section of the Nimitz Freeway in Oakland, CA, during a 1989 earthquake is however a simpler example to analyze.



The collapsed section of the Nimitz Freeway.

An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

Now all the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer's resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

Example: Collapse of the Tacoma Narrows Bridge

Let's now examine the more conceptually difficult case of the Tacoma Narrows Bridge. The surprise here is that the wind was steady. If the wind was blowing at constant velocity, why did it shake the bridge back and forth? The answer is that once the bridge started twisting a little, it began acting like a kite or an airplane wing. It established swirling patterns of air flow around itself, of the kind that you can see in a moving cloud of smoke. Thus even though the wind velocity measured far away from any structures was constant, the bridge's vibration at its own natural frequency of 0.2 Hz set up an alternating pattern of wind gusts in the air immediately around it, which then increased the amplitude of the bridge's vibrations. This vicious cycle fed upon itself, increasing the amplitude of the vibrations until the bridge finally collapsed.

As long as we're on the subject of collapsing bridges, it is worth bringing up the reports of bridges falling down when soldiers marching over them happened to step in rhythm with the bridge's natural frequency of oscillation. This is supposed to have happened in 1831 in Manchester, England, and again in 1849 in Anjou, France. Many modern engineers and scientists, however, are suspicious of the analysis of these reports. It is possible that the collapses had more to do with poor construction and overloading than with resonance. The Nimitz Freeway and Tacoma Narrows Bridge are far better documented, and occurred in an era when engineers' abilities to analyze the vibrations of a complex structure were much more advanced.

Example: emission and absorption of light waves by atoms

In a very thin gas, the atoms are sufficiently far apart that they can act as individual vibrating systems. Although the vibrations are of a very strange and abstract type described by the theory of quantum mechanics, they nevertheless obey the same basic rules as ordinary mechanical vibrations. When a thin gas made of a certain element is heated, it emits light waves with certain specific frequencies, which are like a fingerprint of that element. As with all other vibrations, these atomic vibrations respond most strongly to a driving force that matches their own natural frequency. Thus if we have a relatively cold gas with light waves of various frequencies passing through it, the gas will absorb light at precisely those frequencies at which it would emit light if heated.

- (3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q .

This is fairly intuitive. The steady-state behavior is an equilibrium between energy input from the driving force and energy loss due to damping. A low- Q oscillator, i.e. one with strong damping, dumps its energy faster, resulting in lower-amplitude steady-state motion.

Self-Check



If an opera singer is shopping for a wine glass that she can impress her friends by breaking, what should she look for?

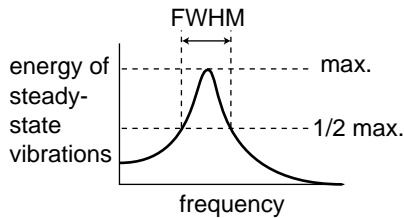
Example: Piano strings ringing in sympathy with a sung note

Question: A sufficiently loud musical note sung near a piano with the lid raised can cause the corresponding strings in the piano to vibrate. (A piano has a set of three strings for each note, all struck by the same hammer.) Why would this trick be unlikely to work with a violin?

Solution: If you have heard the sound of a violin being plucked (the pizzicato effect), you know that the note dies away very quickly. In other words, a violin's Q is much lower than a piano's. This means that its resonances are much weaker in amplitude.



She should tap the wineglasses she finds in the store and look for one with a high Q , i.e. one whose vibrations die out very slowly. The one with the highest Q will have the highest-amplitude response to her driving force, making it more likely to break.



Our fourth and final fact about resonance is perhaps the most surprising. It gives us a way to determine numerically how wide a range of driving frequencies will produce a strong response. As shown in the graph, resonances do not suddenly fall off to zero outside a certain frequency range. It is usual to describe the width of a resonance by its full width at half-maximum (FWHM) as illustrated on the graph.

(4) The FWHM of a resonance is related to its Q and its resonant frequency f_{res} by the equation

$$\text{FWHM} = \frac{f_{\text{res}}}{Q}.$$

(This equation is only a good approximation when Q is large.)

Why? It is not immediately obvious that there should be any logical relationship between Q and the FWHM. Here's the idea. As we have seen already, the reason why the response of an oscillator is smaller away from resonance is that much of the driving force is being used to make the system act as if it had a different k . Roughly speaking, the half-maximum points on the graph correspond to the places where the amount of the driving force being wasted in this way is the same as the amount of driving force being used productively to replace the energy being dumped out by the damping force. If the damping force is strong, then a large amount of force is needed to counteract it, and we can waste quite a bit of driving force on changing k before it becomes comparable to the k to it. If, on the other hand, the damping force is weak, then even a small amount of force being wasted on changing k will become significant in proportion, and we cannot get very far from the resonant frequency before the two are comparable.

Example: Changing the pitch of a wind instrument

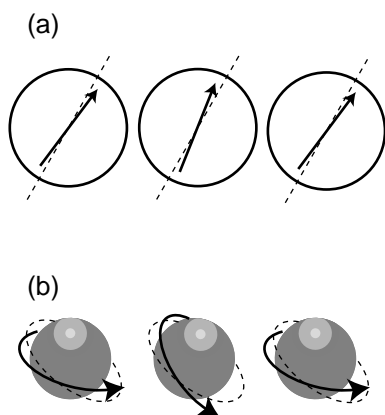
Question: A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the Q of a saxophone?

Solution: Five percent is the width on one side of the resonance, so the full width is about 10%, $\text{FWHM} / f_{\text{res}} = 0.1$. This implies a Q of about 10, i.e. once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that has a low Q , so that they can produce the bluesy pitch-slides typical of their style. "Legit," i.e. classically oriented players, use a higher- Q setup because their style only calls for enough pitch variation to produce a vibrato.)

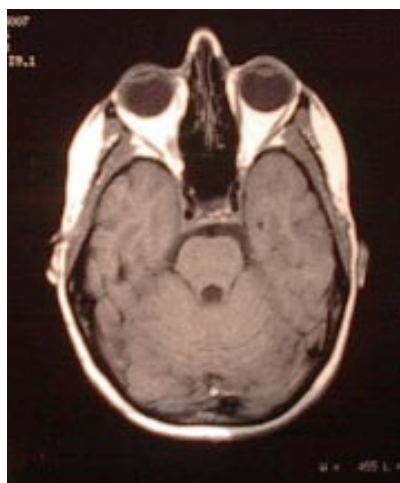
Example: decay of a saxophone tone

Question: If a typical saxophone setup has a Q of about 10, how long will it take for a 100-Hz tone played on a baritone saxophone to die down by a factor of 535 in energy, after the player suddenly stops blowing?

Solution: A Q of 10 means that it takes 10 cycles for the vibrations to die down in energy by a factor of 535. Ten cycles at a



(a) A compass needle vibrates about its equilibrium position under the influence of the earth's magnetic forces.
 (b) The orientation of a proton's spin vibrates about its equilibrium direction under the influence of the magnetic forces coming from the surrounding electrons and nuclei.



A member of the author's family, who turned out to be healthy.



A three-dimensional computer reconstruction of the shape of a human brain, based on magnetic resonance data. R. Malladi, LBNL.

frequency of 100 Hz would correspond to a time of 0.1 seconds, which is not very long. This is why a saxophone note doesn't "ring" like a note played on a piano or an electric guitar.

Example: Q of a radio receiver

Question: A radio receiver used in the FM band needs to be tuned in to within about 0.1 MHz for signals at about 100 MHz. What is its Q ?

Solution: $Q = f_{\text{res}} / \text{FWHM} = 1000$. This is an extremely high Q compared to most mechanical systems.

Example: Q of a stereo speaker

We have already given one reason why a stereo speaker should have a low Q : otherwise it would continue ringing after the end of the musical note on the recording. The second reason is that we want it to be able to respond to a large range of frequencies.

Example: Nuclear magnetic resonance

If you have ever played with a magnetic compass, you have undoubtedly noticed that if you shake it, it takes some time to settle down. As it settles down, it acts like a damped oscillator of the type we have been discussing. The compass needle is simply a small magnet, and the planet earth is a big magnet. The magnetic forces between them tend to bring the needle to an equilibrium position in which it lines up with the planet-earth-magnet.

Essentially the same physics lies behind the technique called Nuclear Magnetic Resonance (NMR). NMR is a technique used to deduce the molecular structure of unknown chemical substances, and it is also used for making medical images of the inside of people's bodies. If you ever have an NMR scan, they will actually tell you you are undergoing "magnetic resonance imaging" or "MRI," because people are scared of the word "nuclear." In fact, the nuclei being referred to are simply the nonradioactive nuclei of atoms found naturally in your body.

Here's how NMR works. Your body contains large numbers of hydrogen atoms, each consisting of a small, lightweight electron orbiting around a large, heavy proton. That is, the nucleus of a hydrogen atom is just one proton. A proton is always spinning on its own axis, and the combination of its spin and its electrical charge cause it to behave like a tiny magnet. The principle identical to that of an electromagnet, which consists of a coil of wire through which electrical charges pass; the circling motion of the charges in the coil of wire makes it magnetic, and in the same way, the circling motion of the proton's charge makes it magnetic.

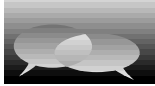
Now a proton in one of your body's hydrogen atoms finds itself surrounded by many other whirling, electrically charged particles: its own electron, plus the electrons and nuclei of the other nearby atoms. These neighbors act like magnets, and exert magnetic forces on the proton. The k of the vibrating proton is simply a measure of the total strength of these magnetic forces. Depending on the structure of the molecule in which the hydrogen atom finds itself, there will be a particular set of magnetic forces acting on the proton and a particular value of k . The NMR apparatus bombards the sample with radio waves, and if the frequency of the radio waves matches the resonant frequency of the proton, the proton will absorb radio-wave energy strongly and

oscillate wildly. Its vibrations are damped not by friction, because there is no friction inside an atom, but by the reemission of radio waves.

By working backward through this chain of reasoning, one can determine the geometric arrangement of the hydrogen atom's neighboring atoms. It is also possible to locate atoms in space, allowing medical images to be made.

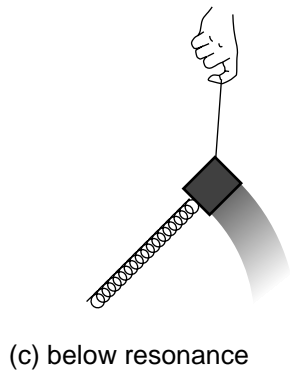
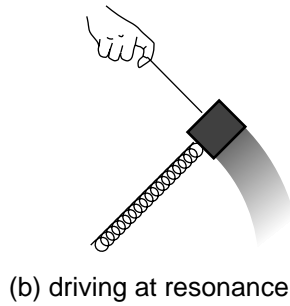
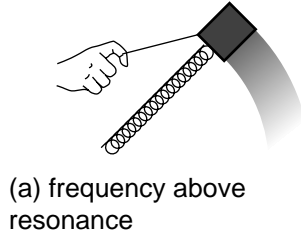
Finally, it should be noted that the behavior of the proton cannot be described entirely correctly by Newtonian physics. Its vibrations are of the strange and spooky kind described by the laws of quantum mechanics. It is impressive, however, that the few simple ideas we have learned about resonance can still be applied successfully to describe many aspects of this exotic system.

Discussion Question



Nikola Tesla, one of the inventors of radio and an archetypical mad scientist, told a credulous reporter the following story about an application of resonance. He built an electric vibrator that fit in his pocket, and attached it to one of the steel beams of a building that was under construction in New York. Although the article in which he was quoted didn't say so, he presumably claimed to have tuned it to the resonant frequency of the building. "In a few minutes, I could feel the beam trembling. Gradually the trembling increased in intensity and extended throughout the whole great mass of steel. Finally, the structure began to creak and weave, and the steelworkers came to the ground panick-stricken, believing that there had been an earthquake. ... [If] I had kept on ten minutes more, I could have laid that building flat in the street." Is this physically plausible?

2.4* Proofs



Our first goal is to predict the amplitude of the steady-state vibrations as a function of the frequency of the driving force and the amplitude of the driving force. With that equation in hand, we will then prove statements 2, 3, and 4 from the previous section. We assume without proof statement 1, that the steady-state motion occurs at the same frequency as the driving force.

As with the proof in the previous chapter, we make use of the fact that a sinusoidal vibration is the same as the projection of circular motion onto a line. We visualize the system shown in the figure, in which the mass swings in a circle on the end of a spring. The spring does not actually change its length at all, but it appears to from the flattened perspective of a person viewing the system edge-on. The radius of the circle is the amplitude, A , of the vibrations as seen edge-on. The damping force can be imagined as a backward drag force supplied by some fluid through which the mass is moving. As usual, we assume that the damping is proportional to velocity, and we use the symbol b for the proportionality constant, $|F_d| = bv$. The driving force, represented by a hand towing the mass with a string, has a tangential component $|F_t|$ which counteracts the damping force, $|F_t| = |F_d|$, and a radial component F_r which works either with or against the spring's force, depending on whether we are driving the system above or below its resonant frequency.

The speed of the rotating mass is the circumference of the circle divided by the period, $v = 2\pi A/T$, its acceleration (which is directly inward) is $a = v^2/r$, and Newton's second law gives $a = F/m = (kA + F_r)/m$. We write f_{res} for $(1/2\pi)\sqrt{k/m}$. Straightforward algebra yields

$$\frac{F_r}{F_t} = \frac{2\pi m}{bf} (f^2 - f_{\text{res}}^2) \quad (1)$$

This is the ratio of the wasted force to the useful force, and we see that it becomes zero when the system is driven at resonance.

The amplitude of the vibrations can be found by attacking the equation $|F_t| = bv = 2\pi bAf$, which gives

$$A = \frac{|F_t|}{2\pi bf} \quad (2)$$

However, we wish to know the amplitude in terms of $|F|$, not $|F_t|$. From now on, let's drop the cumbersome magnitude symbols. With the Pythagorean theorem, it is easily proven that

$$F_t = \frac{F}{\sqrt{1 + \left(\frac{F_r}{F_t}\right)^2}} \quad (3)$$

and equations 1-3 are readily combined to give the final result

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2 (f^2 - f_{res}^2)^2 + b^2 f^2}} \quad (4)$$

Statement 2: maximum amplitude at resonance

Equation 4 shows directly that the amplitude is maximized when the system is driven at its resonant frequency. At resonance, the first term inside the square root vanishes, and this makes the denominator as small as possible, causing the amplitude to be as big as possible. (Actually this is only approximately true, because it is possible to make A a little bigger by decreasing f a little below f_{res} , which makes the second term smaller. This technical issue is addressed in the homework problems.)

Statement 3: amplitude at resonance proportional to Q

Equation 4 shows that the amplitude at resonance is proportional to $1/b$, and the Q of the system is inversely proportional to b , so the amplitude at resonance is proportional to Q .

Statement 4: FWHM related to Q

We will satisfy ourselves by proving only the proportionality $FWHM \propto f_{res}/Q$, not the actual equation $FWHM = f_{res}/Q$. The energy is proportional to A^2 , i.e. to the inverse of the quantity inside the square root in equation 4. At resonance, the first term inside the square root vanishes, and the half-maximum points occur at frequencies for which the whole quantity inside the square root is double its value at resonance, i.e. when the two terms are equal. At the half-maximum points, we have

$$\begin{aligned} f^2 - f_{res}^2 &= \left(f_{res} \pm \frac{FWHM}{2} \right)^2 - f_{res}^2 \\ &= \pm f_{res} FWHM + \frac{1}{4} FWHM^2 \quad (5) \end{aligned}$$

If we assume that the width of the resonance is small compared to the resonant frequency, then the $FWHM^2$ term in equation 5 is negligible compared to the $f_{res} FWHM$ term, and setting the terms in equation 4 equal to each other gives

$$4\pi^2 m^2 (f_{res} FWHM)^2 = b^2 f^2 \quad .$$

We are assuming that the width of the resonance is small compared to the resonant frequency, so f and f_{res} can be taken as synonyms. Thus,

$$FWHM = \frac{b}{2\pi m} \quad .$$

We wish to connect this to Q , which can be interpreted as the energy of the free (undriven) vibrations divided by the work done by damping in one cycle. The former equals $kA^2/2$, and the latter is proportional to the force, $bv \propto bAf_{\text{res}}$, multiplied by the distance traveled, A . (This is only a proportionality, not an equation, since the force is not constant.) We therefore find that Q is proportional to k/bf_{res} . The equation for the FWHM can then be restated as a proportionality $FWHM \propto k/Q_{\text{res}} m \propto f_{\text{res}}/Q$.

Summary

Selected Vocabulary

damping	the dissipation of a vibration's energy into heat energy, or the frictional force that causes the loss of energy
quality factor	the number of oscillations required for a system's energy to fall off by a factor of 535 due to damping
driving force	an external force that pumps energy into a vibrating system
resonance	the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration
steady state	the behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force

Notation

Q	the quality factor
f_{res}	the natural (resonant) frequency of a vibrating system, i.e. the frequency at which it would vibrate if it was simply kicked and left alone
f	the frequency at which the system actually vibrates, which in the case of a driven system is equal to the frequency of the driving force, not the natural frequency

Summary

The energy of a vibration is always proportional to the square of the amplitude, assuming the amplitude is small. Energy is lost from a vibrating system for various reasons such as the conversion to heat via friction or the emission of sound. This effect, called damping, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss. A driving force that pumps energy into the system may drive the system at its own natural frequency or at some other frequency. When a vibrating system is driven by an external force, we are usually interested in its *steady-state behavior*, i.e. its behavior after it has had time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

The following are four important facts about a vibrating system being driven by an external force:

- (1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.
- (2) A vibrating system *resonates* at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.
- (3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q .
- (4) The FWHM of a resonance is related to its Q and its resonant frequency f_{res} by the equation

$$FWHM = \frac{F_{\text{res}}}{Q} .$$

(This equation is only a good approximation when Q is large.)

Homework Problems

1. If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

2. Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish's buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder's Q is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

3. As noted in section 2.4, it is only approximately true that the amplitude has its maximum at $f = 2\pi\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_0 = 2\pi\sqrt{k/m}$ and f_{res} for the slightly different frequency at which the amplitude is a maximum, i.e. the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2 (f^2 - f_0^2)^2 + b^2 f^2}}.$$

Show that the maximum occurs not at f_0 but rather at the frequency

$$f_{\text{res}} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2} \text{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.

S A solution is given in the back of the book.

✓ A computerized answer check is available.

★ A difficult problem.

∫ A problem that requires calculus.

4. (a) Let W be the amount of work done by friction per cycle of oscillation, i.e. the amount of energy lost to heat. Find the fraction of energy *remaining* in the oscillations after n cycles of motion.

(b) From this prove the equation $(1 - W/E)^Q = e^{-2\pi}$.

(c) Use this to prove the approximation $1/Q \approx (1/2\pi) W/E$. [Hint: Use the approximation $\ln(1+x) \approx x$, which is valid for small values of x .]

5 ★ J. The goal of this problem is to refine the proportionality $\text{FWHM} \propto f_{\text{res}}/Q$ into the equation $\text{FWHM} = f_{\text{res}}/Q$, i.e. to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force $F = -bv$ over one cycle of steady-state motion equals $W_{\text{damp}} = -2\pi^2 b f A^2$. Hint: It is less confusing to calculate the work done over half a cycle, from $x = -A$ to $x = +A$, and then double it.

(b) Show that the fraction of the undriven oscillator's energy lost to damping over one cycle is $|W_{\text{damp}}|/E = 4\pi^2 b f/k$.

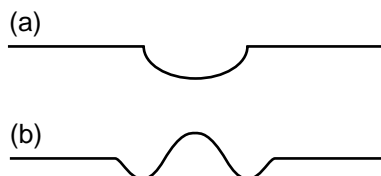
(c) Use the previous result, combined with the result of problem 4, to prove that Q equals $k/2\pi b f$.

(d) Combine the preceding result for Q with the equation $\text{FWHM} = b/2\pi m$ from section 2.4 to prove the equation $\text{FWHM} = f_{\text{res}}/Q$.



3 Free Waves

Your vocal cords or a saxophone reed can vibrate, but being able to vibrate wouldn't be of much use unless the vibrations could be transmitted to the listener's ear by sound waves. What are waves and why do they exist? Put your fingertip in the middle of a cup of water and then remove it suddenly. You will have noticed two results that are surprising to most people. First, the flat surface of the water does not simply sink uniformly to fill in the volume vacated by your finger. Instead, ripples spread out, and the process of flattening out occurs over a long period of time, during which the water at the center vibrates above and below the normal water level. This type of wave motion is the topic of the present chapter. Second, you have found that the ripples bounce off of the walls of the cup, in much the same way that a ball would bounce off of a wall. In the next chapter we discuss what happens to waves that have a boundary around them. Until then, we confine ourselves to wave phenomena that can be analyzed as if the medium (e.g. the water) was infinite and the same everywhere.



It isn't hard to understand why removing your fingertip creates ripples rather than simply allowing the water to sink back down uniformly. The initial crater, (a), left behind by your finger has sloping sides, and the water next to the crater flows downhill to fill in the hole. The water far away, on the other hand, initially has no way of knowing what has happened, because there is no slope for it to flow down. As the hole fills up, the rising water at the center gains upward momentum, and overshoots, creating a little hill where there had been a hole originally. The area just outside of this region has been robbed of some of its water in order to build the hill, so a depressed "moat" is formed, (b). This effect cascades outward, producing ripples.



The two circular patterns of ripples pass through each other. Unlike material objects, wave patterns can overlap in space, and when this happens they combine by addition.

3.1 Wave Motion

There are three main ways in which wave motion differs from the motion of objects made of matter.

1. Superposition

The first, and most profound, difference between wave motion and the motion of objects is that waves do not display any repulsion of each other analogous to the normal forces between objects that come in contact. Two wave patterns can therefore overlap in the same region of space, as shown in the figure at the top of the page. Where the two waves coincide, they add together. For instance, suppose that at a certain location in at a certain moment in time, each wave would have had a crest 3 cm above the normal water level. The waves combine at this point to make a 6-cm crest. We use negative numbers to represent depressions in the water. If both waves would have had a troughs measuring -3 cm, then they combine to make an extra-deep -6 cm trough. A $+3$ cm crest and a -3 cm trough result in a height of zero, i.e. the waves momentarily cancel each other out at that point. This additive rule is referred to as the *principle of superposition*, “superposition” being merely a fancy word for “adding.”

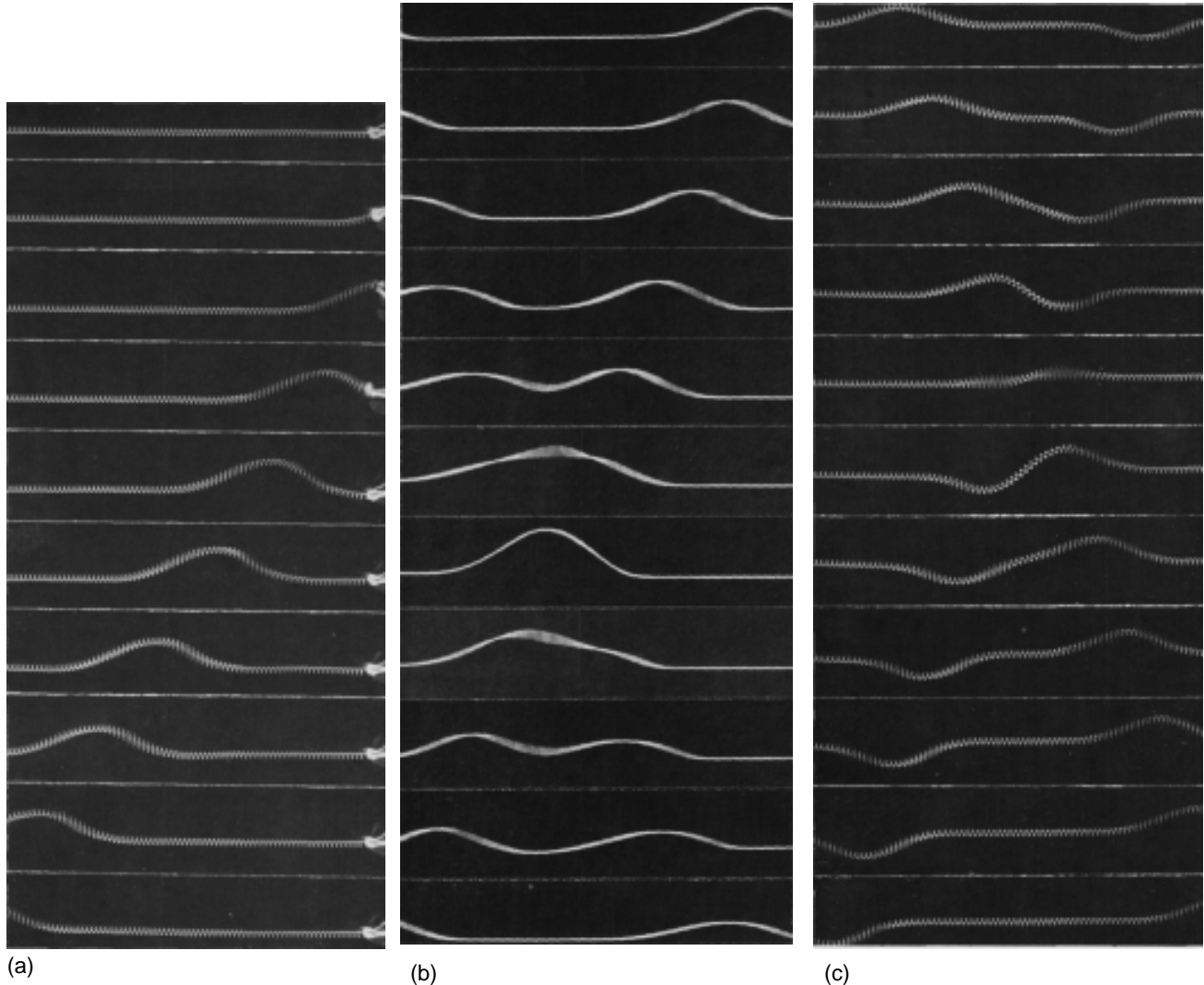
Superposition can occur not just with sinusoidal waves like the ones in the figure above but with waves of any shape. The figures on the following page show superposition of wave *pulses*. A pulse is simply a wave of very short duration. These pulses consist only of a single hump or trough. If you hit a clothesline sharply, you will observe pulses heading off in both directions. This is analogous to the way ripples spread out in all directions when you make a disturbance at one point on water. The same occurs when the hammer on a piano comes up and hits a string.

Experiments to date have not shown any deviation from the principle of superposition in the case of light waves. For other types of waves, it is typically a very good approximation for low-energy waves.

Discussion Question



A. In figure (c) below, the fifth frame shows the spring just about perfectly flat. If the two pulses have essentially canceled each other out perfectly, then why does the motion pick up again? Why doesn't the spring just stay flat?



(a)

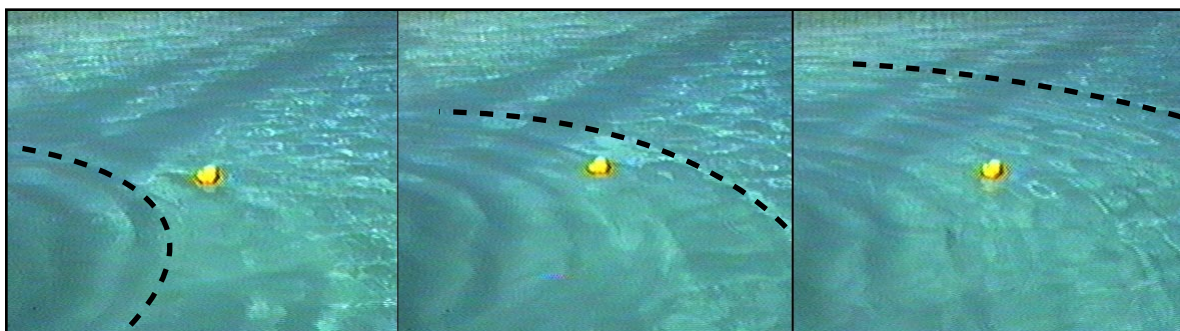
(b)

(c)

These pictures show the motion of wave pulses along a spring. To make a pulse, one end of the spring was shaken by hand. Movies were filmed, and a series of frames chosen to show the motion.

(a) A pulse travels to the left. (b) Superposition of two colliding positive pulses. (c) Superposition of two colliding pulses, one positive and one negative.

Uncopyrighted photographs from PSSC Physics.



As the wave pattern passes the rubber duck, the duck stays put. The water isn't moving with the wave.

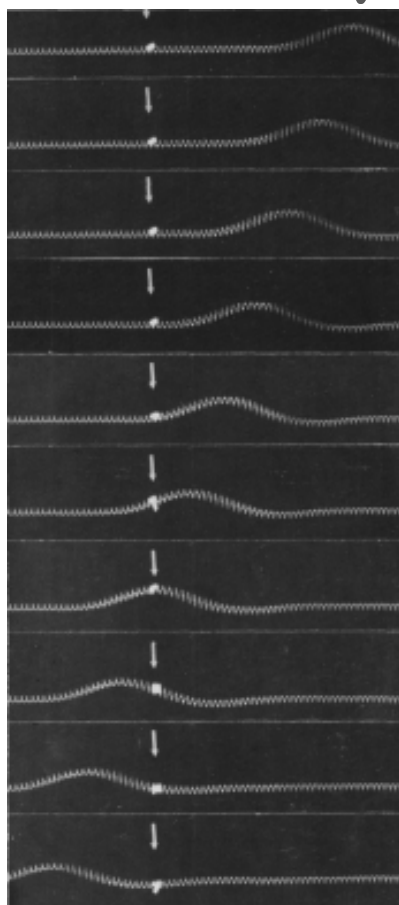
2. The medium is not transported with the wave.

The sequence of three photos above shows a series of water waves before it has reached a rubber duck (left), having just passed the duck (middle) and having progressed about a meter beyond the duck (right). The duck bobs around its initial position, but is not carried along with the wave. This shows that the water itself does not flow outward with the wave. If it did, we could empty one end of a swimming pool simply by kicking up waves! We must distinguish between the motion of the medium (water in this case) and the motion of the wave pattern through the medium. The medium vibrates; the wave progresses through space.

Self-Check



In the photos on the left, you can detect the side-to-side motion of the spring because the spring appears blurry. At a certain instant, represented by a single photo, how would you describe the motion of the different parts of the spring? Other than the flat parts, do any parts of the spring have zero velocity?



The incorrect belief that the medium moves with the wave is often reinforced by garbled secondhand knowledge of surfing. Anyone who has actually surfed knows that the front of the board pushes the water to the sides, creating a wake. If the water was moving along with the wave and the surfer, this wouldn't happen. The surfer is carried forward because forward is downhill, not because of any forward flow of the water. If the water was flowing forward, then a person floating in the water up to her neck would be carried along just as quickly as someone on a surfboard. In fact, it is even possible to surf down the back side of a wave, although the ride wouldn't last very long because the surfer and the wave would quickly part company.

As the wave pulse goes by, the ribbon tied to the spring is not carried along. The motion of the wave pattern is to the right, but the medium (spring) is moving from side to side, not to the right. Uncopyrighted photos from PSSC Physics.



The leading edge is moving up, the trailing edge is moving down, and the top of the hump is motionless for one instant.

3. A wave's velocity depends on the medium.

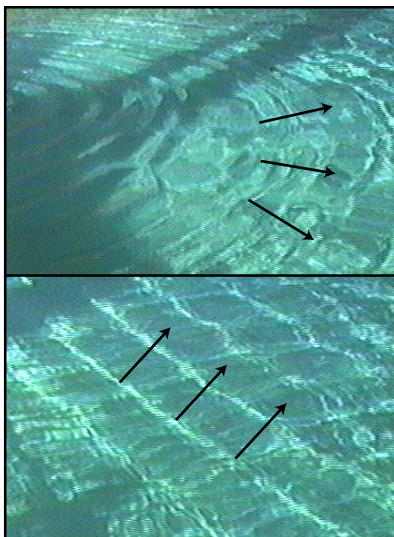
A material object can move with any velocity, and can be sped up or slowed down by a force that increases or decreases its kinetic energy. Not so with waves. The magnitude of a wave's velocity depends on the properties of the medium (and perhaps also on the shape of the wave, for certain types of waves). Sound waves travel at about 340 m/s in air, 1000 m/s in helium. If you kick up water waves in a pool, you will find that kicking harder makes waves that are taller (and therefore carry more energy), not faster. The sound waves from an exploding stick of dynamite carry a lot of energy, but are no faster than any other waves. In the following section we will give an example of the physical relationship between the wave speed and the properties of the medium.

Once a wave is created, the only reason its speed will change is if it enters a different medium or if the properties of the medium change. It is not so surprising that a change in medium can slow down a wave, but the reverse can also happen. A sound wave traveling through a helium balloon will slow down when it emerges into the air, but if it enters another balloon it will speed back up again! Similarly, water waves travel more quickly over deeper water, so a wave will slow down as it passes over an underwater ridge, but speed up again as it emerges into deeper water.

Example: Hull speed

The speeds of most boats (and of some surface-swimming animals) are limited by the fact that they make a wave due to their motion through the water. A fast motor-powered boat can go faster and faster, until it is going at the same speed as the waves it creates. It may then be unable to go any faster, because it cannot climb over the wave crest that builds up in front of it. Increasing the power to the propeller may not help at all. Putting more energy into the waves doesn't make them go any faster, it just makes them taller and more energetic, and that much more difficult to climb over.

A water wave, unlike many other types of wave, has a speed that depends on its shape: a broader wave moves faster. The shape of the wave made by a boat tends to mold itself to the shape of the boat's hull, so a boat with a longer hull makes a broader wave that moves faster. The maximum speed of a boat whose speed is limited by this effect is therefore closely related to the length of its hull, and the maximum speed is called the hull speed. Small racing boats ("cigarette boats") are not just long and skinny to make them more streamlined — they are also long so that their hull speeds will be high.



Circular and linear wave patterns, with velocity vectors shown at selected points.

Wave patterns

If the magnitude of a wave's velocity vector is preordained, what about its direction? Waves spread out in all directions from every point on the disturbance that created them. If the disturbance is small, we may consider it as a single point, and in the case of water waves the resulting wave pattern is the familiar circular ripple. If, on the other hand, we lay a pole on the surface of the water and wiggle it up and down, we create a linear wave pattern. For a three-dimensional wave such as a sound wave, the analogous patterns would be spherical waves (visualize concentric spheres) and plane waves (visualize a series of pieces of paper, each separated from the next by the same gap).

Infinitely many patterns are possible, but linear or plane waves are often the simplest to analyze, because the velocity vector is in the same direction no matter what part of the wave we look at. Since all the velocity vectors are parallel to one another, the problem is effectively one-dimensional. Throughout this chapter and the next, we will restrict ourselves mainly to wave motion in one dimension, while not hesitating to broaden our horizons when it can be done without too much complication.

Discussion Questions



- A. [see above]
- B. Sketch two positive wave pulses on a string that are overlapping but not right on top of each other, and draw their superposition. Do the same for a positive pulse running into a negative pulse.
- C. A traveling wave pulse is moving to the right on a string. Sketch the velocity vectors of the various parts of the string. Now do the same for a pulse moving to the left.
- D. In a spherical sound wave spreading out from a point, how would the energy of the wave fall off with distance?

3.2 Waves on a String



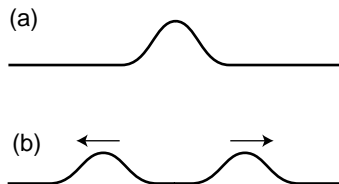
Hitting a key on a piano causes a hammer to come up from underneath and hit a string (actually a set of three). The result is a pair of pulses moving away from the point of impact.

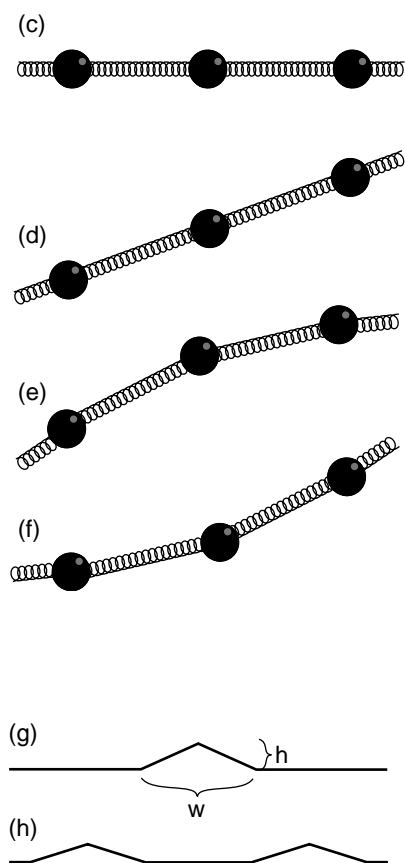
So far you have learned some counterintuitive things about the behavior of waves, but intuition can be trained. The first half of this section aims to build your intuition by investigating a simple, one-dimensional type of wave: a wave on a string. If you have ever stretched a string between the bottoms of two open-mouthed cans to talk to a friend, you were putting this type of wave to work. Stringed instruments are another good example. Although we usually think of a piano wire simply as vibrating, the hammer actually strikes it quickly and makes a dent in it, which then ripples out in both directions. Since this chapter is about free waves, not bounded ones, we pretend that our string is infinitely long.

After the qualitative discussion, we will use simple approximations to investigate the speed of a wave pulse on a string. This quick and dirty treatment is then followed by a rigorous attack using the methods of calculus, which may be skipped by the student who has not studied calculus. How far you penetrate in this section is up to you, and depends on your mathematical self-confidence. If you skip the later parts and proceed to the next section, you should nevertheless be aware of the important result that the speed at which a pulse moves does not depend on the size or shape of the pulse. This is a fact that is true for many other types of waves.

Intuitive ideas

Consider a string that has been struck, (a), resulting in the creation of two wave pulses, (b), one traveling to the left and one to the right. This is analogous to the way ripples spread out in all directions from a splash in water, but on a one-dimensional string, “all directions” becomes “both directions.”





We can gain insight by modeling the string as a series of masses connected by springs. (In the actual string the mass and the springiness are both contributed by the molecules themselves.) If we look at various microscopic portions of the string, there will be some areas that are flat, (c), some that are sloping but not curved, (d), and some that are curved, (e) and (f). In example (c) it is clear that both the forces on the central mass cancel out, so it will not accelerate. The same is true of (d), however. Only in curved regions such as (e) and (f) is an acceleration produced. In these examples, the vector sum of the two forces acting on the central mass is not zero. The important concept is that *curvature makes force*: the curved areas of a wave tend to experience forces resulting in an acceleration toward the mouth of the curve. Note, however, that an uncurved portion of the string need not remain motionless. It may move at constant velocity to either side.

Approximate treatment

We now carry out an approximate treatment of the speed at which two pulses will spread out from an initial indentation on a string. For simplicity, we imagine a hammer blow that creates a triangular dent, (g). We will estimate the amount of time, t , required until each of the pulses has traveled a distance equal to the width of the pulse itself. The velocity of the pulses is then $\pm w/t$.

As always, the velocity of a wave depends on the properties of the medium, in this case the string. The properties of the string can be summarized by two variables: the tension, T , and the mass per unit length, μ (Greek letter mu).

If we consider the part of the string encompassed by the initial dent as a single object, then this object has a mass of approximately μw (mass/length \times length = mass). (Here, and throughout the derivation, we assume that h is much less than w , so that we can ignore the fact that this segment of the string has a length slightly greater than w .) Although the downward acceleration of this segment of the string will be neither constant over time nor uniform across the string, we will pretend that it is constant for the sake of our simple estimate. Roughly speaking, the time interval between (g) and (h) is the amount of time required for the initial dent to accelerate from rest and reach its normal, flattened position. Of course the tip of the triangle has a longer distance to travel than the edges, but again we ignore the complications and simply assume that the segment as a whole must travel a distance h . Indeed, it might seem surprising that the triangle would so neatly spring back to a perfectly flat shape. It is an experimental fact that it does, but our analysis is too crude to address such details.

The string is kinked, i.e. tightly curved, at the edges of the triangle, so it is here that there will be large forces that do not cancel out to zero. There are two forces acting on the triangular hump, one of magnitude T acting down and to the right, and one of the same magnitude acting down and to the left. If the angle of the sloping sides is θ , then the total force on the segment equals $2T \sin \theta$. Dividing the triangle into two right triangles, we see that $\sin \theta$ equals h divided by the length of one of the sloping sides. Since h is much less than w , the length of the sloping side is essentially the same as $w/2$, so we have $\sin \theta = 2h/w$, and $F = 4Th/w$. The acceleration of the segment (actually the acceleration of its center of mass) is

$$\begin{aligned}
 a &= F/m \\
 &= 4Th/\mu w^2 .
 \end{aligned}$$

The time required to move a distance h under constant acceleration a is found by solving $h = \frac{1}{2}at^2$ to yield

$$\begin{aligned}
 t &= \sqrt{2h/a} \\
 &= w\sqrt{\frac{\mu}{2T}} .
 \end{aligned}$$

Our final result for the velocity of the pulses is

$$\begin{aligned}
 |v| &= w/t \\
 &= \sqrt{\frac{2T}{\mu}} .
 \end{aligned}$$

The velocity of a wave on a string does not depend on the shape of the wave. The same is true for many other types of waves.

The remarkable feature of this result is that the velocity of the pulses does not depend at all on w or h , i.e. any triangular pulse has the same speed. It is an experimental fact (and we will also prove rigorously in the following subsection) that any pulse of any kind, triangular or otherwise, travels along the string at the same speed. Of course, after so many approximations we cannot expect to have gotten all the numerical factors right. The correct result for the velocity of the pulses is

$$v = \sqrt{\frac{T}{\mu}} .$$

The importance of the above derivation lies in the insight it brings —that all pulses move with the same speed — rather than in the details of the numerical result. The reason for our too-high value for the velocity is not hard to guess. It comes from the assumption that the acceleration was constant, when actually the total force on the segment would diminish as it flattened out.

Rigorous derivation using calculus (optional)

After expending considerable effort for an approximate solution, we now display the power of calculus with a rigorous and completely general treatment that is nevertheless much shorter and easier. Let the flat position of the string define the x axis, so that y measures how far a point on the string is from equilibrium. The motion of the string is characterized by $y(x,t)$, a function of two variables. Knowing that the force on any small segment of string depends on the curvature of the string in that area, and that the second derivative is a measure of curvature, it is not surprising to find that the infinitesimal force dF acting on an infinitesimal segment dx is given by

$$dF = T \frac{d^2y}{dx^2} dx .$$

proof of the principle of superposition, in the case of waves on a string

(This can be proven by vector addition of the two infinitesimal forces acting on either side.) The acceleration is then $a = dF/dm$, or, substituting $dm = \mu dx$,

$$\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2} .$$

The second derivative with respect to time is related to the second derivative with respect to position. This is no more than a fancy mathematical statement of the intuitive fact developed above, that the string accelerates so as to flatten out its curves.

proof that all wave shapes travel at the same speed, in the case of waves on a string

Before even bothering to look for solutions to this equation, we note that it already proves the principle of superposition, because the derivative of a sum is the sum of the derivatives. Therefore the sum of any two solutions will also be a solution.

Based on experiment, we expect that this equation will be satisfied by any function $y(x,t)$ that describes a pulse or wave pattern moving to the left or right at the correct speed v . In general, such a function will be of the form $y=f(x-vt)$ or $y=f(x+vt)$, where f is any function of one variable. Because of the chain rule, each derivative with respect to time brings out a factor of $\pm v$. Evaluating the second derivatives on both sides of the equation gives

$$(\pm v)^2 f'' = \frac{T}{\mu} f'' .$$

Squaring gets rid of the sign, and we find that we have a valid solution for any function f , provided that v is given by

$$v = \sqrt{\frac{T}{\mu}} .$$

3.3 Sound and Light Waves

Sound waves

The phenomenon of sound is easily found to have all the characteristics we expect from a wave phenomenon:

- Sound waves obey superposition. Sounds do not knock other sounds out of the way when they collide, and we can hear more than one sound at once if they both reach our ear simultaneously.
- The medium does not move with the sound. Even standing in front of a titanic speaker playing earsplitting music, we do not feel the slightest breeze.
- The velocity of sound depends on the medium. Sound travels faster in helium than in air, and faster in water than in helium. Putting more energy into the wave makes it more intense, not faster. For example, you can easily detect an echo when you clap your hands a short distance from a large, flat wall, and the delay of the echo is no shorter for a louder clap.

Although not all waves have a speed that is independent of the shape of the wave, and this property therefore is irrelevant to our collection of evidence that sound is a wave phenomenon, sound does nevertheless have this property. For instance, the music in a large concert hall or stadium may take on the order of a second to reach someone seated in the nosebleed section, but we do not notice or care, because the delay is the same for every sound. Bass, drums, and vocals all head outward from the stage at 340 m/s, regardless of their differing wave shapes.

If sound has all the properties we expect from a wave, then what type of wave is it? It must be a vibration of a physical medium such as air, since the speed of sound is different in different media, such as helium or water. Further evidence is that we don't receive sound signals that have come to our planet through outer space. The roars and whooshes of Hollywood's space ships are fun, but scientifically wrong.*

We can also tell that sound waves consist of compressions and expansions, rather than sideways vibrations like the shimmying of a snake. Only compressional vibrations would be able to cause your eardrums to vibrate in and out. Even for a very loud sound, the compression is extremely weak; the increase or decrease compared to normal atmospheric pressure is no more than a part per million. Our ears are apparently very sensitive receivers!

*Outer space is not a perfect vacuum, so it is possible for sound waves to travel through it. However, if we want to create a sound wave, we typically do it by creating vibrations of a physical object, such as the sounding board of a guitar, the reed of a saxophone, or a speaker cone. The lower the density of the surrounding medium, the less efficiently the energy can be converted into sound and carried away. An isolated tuning fork, left to vibrate in interstellar space, would dissipate the energy of its vibration into internal heat at a rate billions of times greater than the rate of sound emission into the nearly perfect vacuum around it.

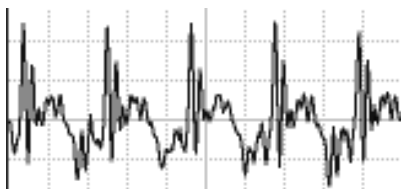
Light waves

Entirely similar observations lead us to believe that light is a wave, although the concept of light as a wave had a long and tortuous history. It is interesting to note that Isaac Newton very influentially advocated a contrary idea about light. The belief that matter was made of atoms was stylish at the time among radical thinkers (although there was no experimental evidence for their existence), and it seemed logical to Newton that light as well should be made of tiny particles, which he called corpuscles (Latin for “small objects”). Newton’s triumphs in the science of mechanics, i.e. the study of matter, brought him such great prestige that nobody bothered to question his incorrect theory of light for 150 years. One persuasive proof that light is a wave is that according to Newton’s theory, two intersecting beams of light should experience at least some disruption because of collisions between their corpuscles. Even if the corpuscles were extremely small, and collisions therefore very infrequent, at least some dimming should have been measurable. In fact, very delicate experiments have shown that there is no dimming.

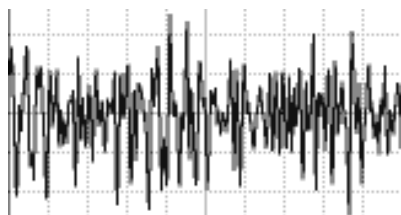
The wave theory of light was entirely successful up until the 20th century, when it was discovered that not all the phenomena of light could be explained with a pure wave theory. It is now believed that both light and matter are made out of tiny chunks which have *both* wave and particle properties. For now, we will content ourselves with the wave theory of light, which is capable of explaining a great many things, from cameras to rainbows.

If light is a wave, what is waving? What is the medium that wiggles when a light wave goes by? It isn’t air. A vacuum is impenetrable to sound, but light from the stars travels happily through zillions of miles of empty space. Light bulbs have no air inside them, but that doesn’t prevent the light waves from leaving the filament. For a long time, physicists assumed that there must be a mysterious medium for light waves, and they called it the aether (not to be confused with the chemical). Supposedly the aether existed everywhere in space, and was immune to vacuum pumps. The details of the story are more fittingly reserved for later in this course, but the end result was that a long series of experiments failed to detect any evidence for the aether, and it is no longer believed to exist. Instead, light can be explained as a wave pattern made up of electrical and magnetic fields.

3.4 Periodic Waves



(a) A graph of pressure versus time for a periodic sound wave, the vowel "ah."



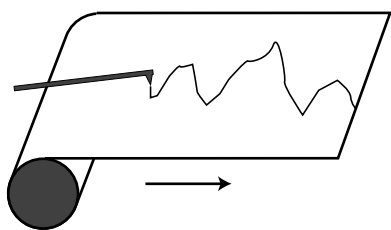
(b) A similar graph for a nonperiodic wave, "sh."

Period and frequency of a periodic wave

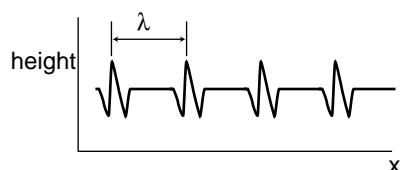
You choose a radio station by selecting a certain frequency. We have already defined period and frequency for vibrations, but what do they signify in the case of a wave? We can recycle our previous definition simply by stating it in terms of the vibrations that the wave causes as it passes a receiving instrument at a certain point in space. For a sound wave, this receiver could be an eardrum or a microphone. If the vibrations of the eardrum repeat themselves over and over, i.e. are periodic, then we describe the sound wave that caused them as periodic. Likewise we can define the period and frequency of a wave in terms of the period and frequency of the vibrations it causes. As another example, a periodic water wave would be one that caused a rubber duck to bob in a periodic manner as they passed by it.

The period of a sound wave correlates with our sensory impression of musical pitch. A high frequency (short period) is a high note. The sounds that really define the musical notes of a song are only the ones that are periodic. It is not possible to sing a nonperiodic sound like "sh" with a definite pitch.

The frequency of a light wave corresponds to color. Violet is the high-frequency end of the rainbow, red the low-frequency end. A color like brown that does not occur in a rainbow is not a periodic light wave. Many phenomena that we do not normally think of as light are actually just forms of light that are invisible because they fall outside the range of frequencies our eyes can detect. Beyond the red end of the visible rainbow, there are infrared and radio waves. Past the violet end, we have ultraviolet, x-rays, and gamma rays.



A strip chart recorder.



A water wave profile created by a series of repeating pulses.

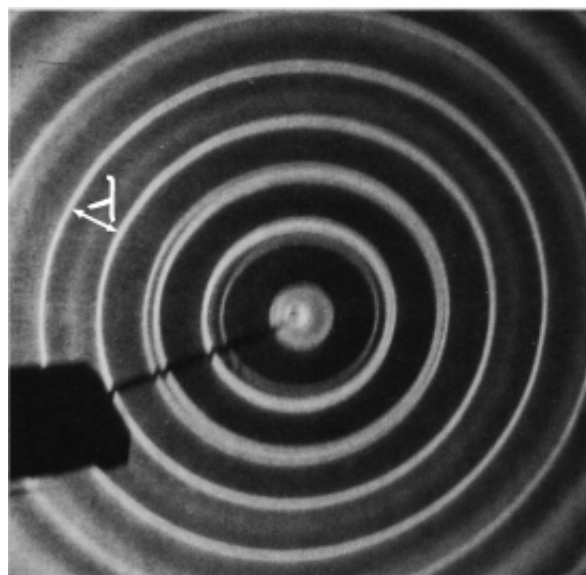
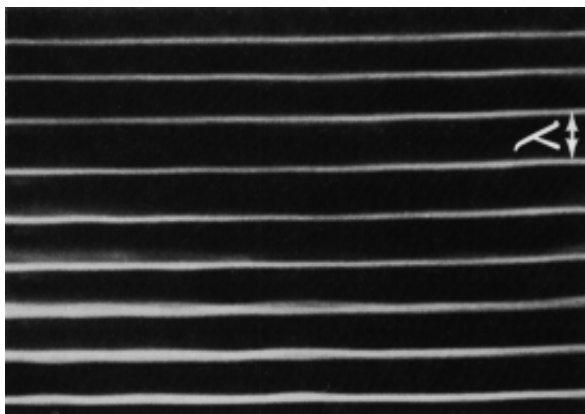
Graphs of waves as a function of position

Some waves, light sound waves, are easy to study by placing a detector at a certain location in space and studying the motion as a function of time. The result is a graph whose horizontal axis is time. With a water wave, on the other hand, it is simpler just to look at the wave directly. This visual snapshot amounts to a graph of the height of the water wave as a function of *position*. Any wave can be represented in either way.

An easy way to visualize this is in terms of a strip chart recorder, an obsolescing device consisting of a pen that wiggles back and forth as a roll of paper is fed under it. It can be used to record a person's electrocardiogram, or seismic waves too small to be felt as a noticeable earthquake but detectable by a seismometer. Taking the seismometer as an example, the chart is essentially a record of the ground's wave motion as a function of time, but if the paper was set to feed at the same velocity as the motion of an earthquake wave, it would also be a full-scale representation of the profile of the actual wave pattern itself. Assuming, as is usually the case, that the wave velocity is a constant number regardless of the wave's shape, knowing the wave motion as a function of time is equivalent to knowing it as a function of position.

Wavelength

Any wave that is periodic will also display a repeating pattern when graphed as a function of position. The distance spanned by one repetition is referred to as one *wavelength*. The usual notation for wavelength is λ , the Greek letter lambda. Wavelength is to space as period is to time.



Wavelengths of linear and circular waves. (PSSC Physics)

Wave velocity related to frequency and wavelength

Suppose that we create a repetitive disturbance by kicking the surface of a swimming pool. We are essentially making a series of wave pulses. The wavelength is simply the distance a pulse is able to travel before we make the next pulse. The distance between pulses is λ , and the time between pulses is the period, T , so the speed of the wave is the distance divided by the time,

$$v = \lambda/T .$$

This important and useful relationship is more commonly written in terms of the frequency,

$$v = f\lambda .$$

Example: Wavelength of radio waves

Question: The speed of light is 3.0×10^8 m/s. What is the wavelength of the radio waves emitted by KLON, a station whose frequency is 88.1 MHz?

Solution: Solving for wavelength, we have

$$\begin{aligned} \lambda &= v/f \\ &= (3.0 \times 10^8 \text{ m/s}) / (88.1 \times 10^6 \text{ s}^{-1}) \\ &= 3.4 \text{ m} \end{aligned}$$

The size of a radio antenna is closely related to the wavelength of the waves it is intended to receive. The match need not be exact (since after all one antenna can receive more than one wavelength!), but the ordinary “whip” antenna such as a car’s is $1/4$ of a wavelength. An antenna optimized to receive KLON’s signal (which is the only one my car radio is ever tuned to) would have a length of $3.4 \text{ m}/4 = 0.85 \text{ m}$.

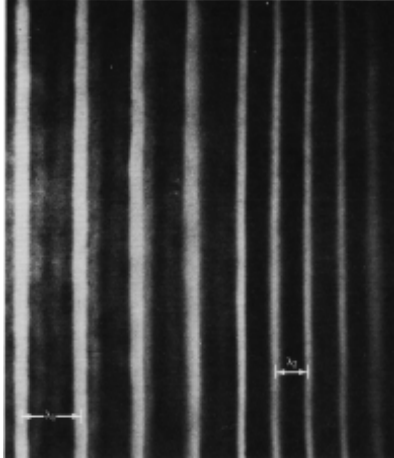
A note on dispersive waves

The discussion of wave velocity given here is actually a little bit of an oversimplification for a wave whose velocity depends on its frequency and wavelength. Such a wave is called a dispersive wave. Nearly all the waves we deal with in this course are nondispersive, but the issue becomes important in book 6 of this series, where it is discussed in detail in optional section 5.2.



Ultrasound, i.e. sound with frequencies higher than the range of human hearing, was used to make this image of a fetus. The resolution of the image is related to the wavelength, since details smaller than about one wavelength cannot be resolved. High resolution therefore requires a short wavelength, corresponding to a high frequency.

The equation $v=f\lambda$ defines a fixed relationship between any two of the variables if the other is held fixed. The speed of radio waves in air is almost exactly the same for all wavelengths and frequencies (it is exactly the same if they are in a vacuum), so there is a fixed relationship between their frequency and wavelength. Thus we can say either “Are we on the same wavelength?” or “Are we on the same frequency?”



A water wave traveling into a region with different depth will change its wavelength.

A different example is the behavior of a wave that travels from a region where the medium has one set of properties to an area where the medium behaves differently. The frequency is now fixed, because otherwise the two portions of the wave would otherwise get out of step, causing a kink or discontinuity at the boundary, which would be unphysical. (A more careful argument is that a kink or discontinuity would have infinite curvature, and waves tend to flatten out their curvature. An infinite curvature would flatten out infinitely fast, i.e. it could never occur in the first place.) Since the frequency must stay the same, any change in the velocity that results from the new medium must cause a change in wavelength.

The velocity of water waves depends on the depth of the water, so based on $\lambda=v/f$, we see that water waves that move into a region of different depth must change their wavelength, as shown in the figure on the left. This effect can be observed when ocean waves come up to the shore. If the deceleration of the wave pattern is sudden enough, the tip of the wave can curl over, resulting in a breaking wave.

Sinusoidal waves

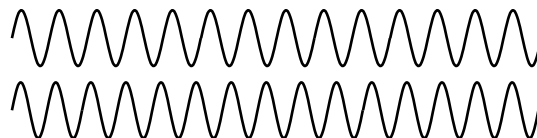
Sinusoidal waves are the most important special case of periodic waves. In fact, many scientists and engineers would be uncomfortable with defining a waveform like the “ah” vowel sound as having a definite frequency and wavelength, because they consider only sine waves to be pure examples of a certain frequency and wavelengths. Their bias is not unreasonable, since the French mathematician Fourier showed that any periodic wave with frequency f can be constructed as a superposition of sine waves with frequencies $f, 2f, 3f, \dots$. In this sense, sine waves are the basic, pure building blocks of all waves. (Fourier’s result so surprised the mathematical community of France that he was ridiculed the first time he publicly presented his theorem.)

However, what definition to use is a matter of utility. Our sense of hearing perceives any two sounds having the same period as possessing the same pitch, regardless of whether they are sine waves or not. This is undoubtedly because our ear-brain system evolved to be able to interpret human speech and animal noises, which are periodic but not sinusoidal. Our eyes, on the other hand, judge a color as pure (belonging to the rainbow set of colors) only if it is a sine wave.

Discussion Question



Suppose we superimpose two sine waves with equal amplitudes but slightly different frequencies, as shown in the figure. What will the superposition look like? What would this sound like if they were sound waves?



3.5 The Doppler Effect



The pattern of waves made by a point source moving to the right across the water. Note the shorter wavelength of the forward-emitted waves and the longer wavelength of the backward-going ones.

The figure shows the wave pattern made by the tip of a vibrating rod which is moving across the water. If the rod had been vibrating in one place, we would have seen the familiar pattern of concentric circles, all centered on the same point. But since the source of the waves is moving, the wavelength is shortened on one side and lengthened on the other. This is known as the *Doppler effect*.

Note that the velocity of the waves is a fixed property of the medium, so for example the forward-going waves do not get an extra boost in speed as would a material object like a bullet being shot forward from an airplane.

We can also infer a change in frequency. Since the velocity is constant, the equation $v=f\lambda$ tells us that the change in wavelength must be matched by an opposite change in frequency: higher frequency for the waves emitted forward, and lower for the ones emitted backward. The frequency Doppler effect is the reason for the familiar dropping-pitch sound of a race car going by. As the car approaches us, we hear a higher pitch, but after it passes us we hear a frequency that is lower than normal.

The Doppler effect will also occur if the observer is moving but the source is stationary. For instance, an observer moving toward a stationary source will perceive one crest of the wave, and will then be surrounded by the next crest sooner than she otherwise would have, because she has moved toward it and hastened her encounter with it. Roughly speaking, the Doppler effect depends only the relative motion of the source and the observer, not on their absolute state of motion (which is not a well-defined notion in physics) or on their velocity relative to the medium.

Restricting ourselves to the case of a moving source, and to waves emitted either directly along or directly against the direction of motion, we can easily calculate the wavelength, or equivalently the frequency, of the Doppler-shifted waves. Let v be the velocity of the waves, and v_s the velocity of the source. The wavelength of the forward-emitted waves is shortened by an amount $v_s T$ equal to the distance traveled by the source over the course of one period. Using the definition $f=1/T$ and the equation $v=f\lambda$, we find for the wavelength λ' of the Doppler-shifted wave the equation

$$\lambda' = \left(1 - \frac{v_s}{v}\right)\lambda .$$

A similar equation can be used for the backward-emitted waves, but with a plus sign rather than a minus sign.

Example: Doppler-shifted sound from a race car

Question: If a race car moves at a velocity of 50 m/s, and the velocity of sound is 340 m/s, by what percentage are the wavelength and frequency of its sound waves shifted for an observer lying along its line of motion?

Solution: For an observer whom the car is approaching, we find

$$1 - \frac{v_s}{v} = 0.85 ,$$

so the shift in wavelength is 15%. Since the frequency is inversely proportional to the wavelength for a fixed value of the speed of sound, the frequency is shifted upward by

$$1/0.85 = 1.18 ,$$

i.e. a change of 18%. (For velocities that are small compared to the wave velocities, the Doppler shifts of the wavelength and frequency are about the same.)

Example: Doppler shift of the light emitted by a race car

Question: What is the percent shift in the wavelength of the light waves emitted by a race car's headlights?

Solution: Looking up the speed of light in the front of the book, $c = 3.0 \times 10^8$ m/s, we find

$$1 - \frac{v_s}{v} = 0.99999983 ,$$

i.e. the percentage shift is only 0.000017%.

The second example shows that under ordinary earthbound circumstances, Doppler shifts of light are negligible because ordinary things go so much slower than the speed of light. It's a different story, however, when it comes to stars and galaxies, and this leads us to a story that has profound implications for our understanding of the origin of the universe.

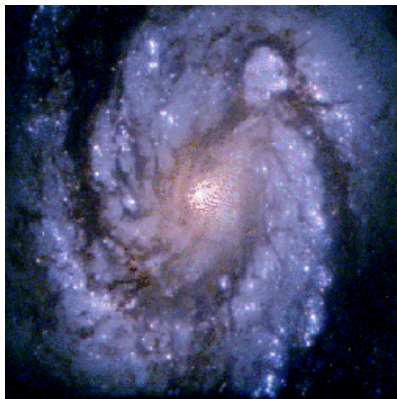
Optional Topic: A Note on Doppler Shifts of Light

If Doppler shifts depend only on the relative motion of the source and receiver, then there is no way for a person moving with the source and another person moving with the receiver to determine who is moving and who isn't. Either can blame the Doppler shift entirely on the other's motion and claim to be at rest herself. This is entirely in agreement with the principle stated originally by Galileo that all motion is relative.

On the other hand, a careful analysis of the Doppler shifts of water or sound waves shows that it is only approximately true, at low speeds, that the shifts just depend on the relative motion of the source and observer. For instance, it is possible for a jet plane to keep up with its own sound waves, so that the sound waves appear to stand still to the pilot of the plane. The pilot then knows she is moving at exactly the speed of sound. The reason this doesn't disprove the relativity of mo-

tion is that the pilot is not really determining her absolute motion but rather her motion *relative to the air*, which is the medium of the sound waves.

Einstein realized that this solved the problem for sound or water waves, but would not salvage the principle of relative motion in the case of light waves, since light is not a vibration of any physical medium such as water or air. Beginning by imagining what a beam of light would look like to a person riding a motorcycle alongside it, Einstein eventually came up with a radical new way of describing the universe, in which space and time are distorted as measured by observers in different states of motion. As a consequence of this Theory of Relativity, he showed that light waves would have Doppler shifts that would exactly, not just approximately, depend only on the relative motion of the source and receiver.



The galaxy M100 in the constellation Coma Berenices. Under higher magnification, the milky clouds reveal themselves to be composed of trillions of stars.

The Big Bang

As soon as astronomers began looking at the sky through telescopes, they began noticing certain objects that looked like clouds in deep space. The fact that they looked the same night after night meant that they were beyond the earth's atmosphere. Not knowing what they really were, but wanting to sound official, they called them "nebulae," a Latin word meaning "clouds" but sounding more impressive. In the early 20th century, astronomers realized that although some really were clouds of gas (e.g. the middle "star" of Orion's sword, which is visibly fuzzy even to the naked eye when conditions are good), others were what we now call galaxies: virtual island universes consisting of trillions of stars (for example the Andromeda Galaxy, which is visible as a fuzzy patch through binoculars). Three hundred years after Galileo had resolved the Milky Way into individual stars through his telescope, astronomers realized that the universe is made of galaxies of stars, and the Milky Way is simply the visible part of the flat disk of our own galaxy, seen from inside.

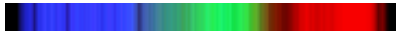
This opened up the scientific study of cosmology, the structure and history of the universe as a whole, a field that had not been seriously attacked since the days of Newton. Newton had realized that if gravity was always attractive, never repulsive, the universe would have a tendency to collapse. His solution to the problem was to posit a universe that was infinite and uniformly populated with matter, so that it would have no geometrical center. The gravitational forces in such a universe would always tend to cancel out by symmetry, so there would be no collapse. By the 20th century, the belief in an unchanging and infinite universe had become conventional wisdom in science, partly as a reaction against the time that had been wasted trying to find explanations of ancient geological phenomena based on catastrophes suggested by biblical events like Noah's flood.

In the 1920's astronomer Edwin Hubble began studying the Doppler shifts of the light emitted by galaxies. A former college football player with a serious nicotine addiction, Hubble did not set out to change our image of the beginning of the universe. His autobiography seldom even mentions the cosmological discovery for which he is now remembered. When astronomers began to study the Doppler shifts of galaxies, they expected that each galaxy's direction and velocity of motion would be essentially random. Some would be approaching us, and their light would therefore be Doppler-shifted to the blue end of the spectrum, while an equal number would be expected to have red shifts. What Hubble discovered instead was that except for a few very nearby ones, all the galaxies had red shifts, indicating that they were receding from us at a hefty fraction of the speed of light. Not only that, but the ones farther away were receding more quickly. The speeds were directly proportional to their distance from us.



Edwin Hubble

Did this mean that the earth (or at least our galaxy) was the center of the universe? No, because Doppler shifts of light only depend on the relative motion of the source and the observer. If we see a distant galaxy moving away from us at 10% of the speed of light, we can be assured that the astronomers who live in that galaxy will see ours receding from them at the same speed in the opposite direction. The whole universe can be envisioned as a rising loaf of raisin bread. As the bread expands, there is more and more space between the raisins. The farther apart two raisins are,



How do astronomers know what mixture of wavelengths a star emitted originally, so that they can tell how much the Doppler shift was? This image (obtained by the author with equipment costing about \$5, and no telescope) shows the mixture of colors emitted by the star Sirius. (If you have the book in black and white, blue is on the left and red on the right.) The star appears white or bluish-white to the eye, but any light looks white if it contains roughly an equal mixture of the rainbow colors, i.e. of all the pure sinusoidal waves with wavelengths lying in the visible range. Note the black “gap teeth.” These are the fingerprint of hydrogen in the outer atmosphere of Sirius. These wavelengths are selectively absorbed by hydrogen. Sirius is in our own galaxy, but similar stars in other galaxies would have the whole pattern shifted toward the red end, indicating they are moving away from us.

the greater the speed with which they move apart.

Extrapolating backward in time using the known laws of physics, the universe must have been denser and denser at earlier and earlier times. At some point, it must have been extremely dense and hot, and we can even detect the radiation from this early fireball, in the form of microwave radiation that permeates space. The phrase Big Bang was originally coined by the doubters of the theory to make it sound ridiculous, but it stuck, and today essentially all astronomers accept the Big Bang theory based on the very direct evidence of the red shifts and the cosmic microwave background radiation.

What the Big Bang is not

Finally it should be noted what the Big Bang theory is not. It is not an explanation of *why* the universe exists. Such questions belong to the realm of religion, not science. Science can find ever simpler and ever more fundamental explanations for a variety of phenomena, but ultimately science takes the universe as it is according to observations.

Furthermore, there is an unfortunate tendency, even among many scientists, to speak of the Big Bang theory as a description of the very first event in the universe, which caused everything after it. Although it is true that time may have had a beginning (Einstein's theory of general relativity admits such a possibility), the methods of science can only work within a certain range of conditions such as temperature and density. Beyond a temperature of about 10^9 degrees C, the random thermal motion of subatomic particles becomes so rapid that its velocity is comparable to the speed of light. Early enough in the history of the universe, when these temperatures existed, Newtonian physics becomes less accurate, and we must describe nature using the more general description given by Einstein's theory of relativity, which encompasses Newtonian physics as a special case. At even higher temperatures, beyond about 10^{33} degrees, physicists know that Einstein's theory as well begins to fall apart, but we don't know how to construct the even more general theory of nature that would work at those temperatures. No matter how far physics progresses, we will never be able to describe nature at infinitely high temperatures, since there is a limit to the temperatures we can explore by experiment and observation in order to guide us to the right theory. We are confident that we understand the basic physics involved in the evolution of the universe starting a few minutes after the Big Bang, and we may be able to push back to milliseconds or microseconds after it, but we cannot use the methods of science to deal with the beginning of time itself.

Discussion Questions



- A. If an airplane travels at exactly the speed of sound, what would be the wavelength of the forward-emitted part of the sound waves it emitted? How should this be interpreted, and what would actually happen?
- B. If bullets go slower than the speed of sound, why can a supersonic fighter plane catch up to its own sound, but not to its own bullets?
- C. If someone inside a plane is talking to you, should their speech be Doppler shifted?

Summary

Selected Vocabulary

superposition	the adding together of waves that overlap with each other
medium	a physical substance whose vibrations constitute a wave
wavelength	the distance in space between repetitions of a periodic wave
Doppler effect	the change in a wave's frequency and wavelength due to the motion of the source or the observer or both

Notation

λ	wavelength (Greek letter lambda)
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Summary

Wave motion differs in three important ways from the motion of material objects:

- (1) Waves obey the principle of superposition. When two waves collide, they simply add together.
- (2) The medium is not transported along with the wave. The motion of any given point in the medium is a vibration about its equilibrium location, not a steady forward motion.
- (3) The velocity of a wave depends on the medium, not on the amount of energy in the wave. (For some types of waves, notably water waves, the velocity may also depend on the shape of the wave.)

Sound waves consist of increases and decreases (typically very small ones) in the density of the air. Light is a wave, but it is a vibration of electric and magnetic fields, not of any physical medium. Light can travel through a vacuum.

A periodic wave is one that creates a periodic motion in a receiver as it passes it. Such a wave has a well-defined period and frequency, and it will also have a wavelength, which is the distance in space between repetitions of the wave pattern. The velocity, frequency, and wavelength of a periodic wave are related by the equation

$$v = f\lambda .$$

A wave emitted by a moving source will be shifted in wavelength and frequency. The shifted wavelength is given by the equation

$$\lambda' = \left(1 - \frac{v_s}{v}\right) \lambda ,$$

where v is the velocity of the waves and v_s is the velocity of the source, taken to be positive or negative so as to produce a Doppler-lengthened wavelength if the source is receding and a Doppler-shortened one if it approaches. A similar shift occurs if the observer is moving, and in general the Doppler shift depends approximately only on the relative motion of the source and observer if their velocities are both small compared to the waves' velocity. (This is not just approximately but exactly true for light waves, and this fact forms the basis of Einstein's Theory of Relativity.)

Homework Problems

1. The following is a graph of the height of a water wave as a function of *position*, at a certain moment in time.



Trace this graph onto another piece of paper, and then sketch below it the corresponding graphs that would be obtained if

- (a) the amplitude and frequency were doubled while the velocity remained the same;
- (b) the frequency and velocity were both doubled while the amplitude remained unchanged;
- (c) the wavelength and amplitude were reduced by a factor of three while the velocity was doubled.

[Problem by Arnold Arons.]



Problem 2.

- 2. (a) The graph shows the height of a water wave pulse as a function of position. Draw a graph of height as a function of time for a specific point on the water. Assume the pulse is traveling to the right.
- (b) Repeat part a, but assume the pulse is traveling to the left.
- (c) Now assume the original graph was of height as a function of time, and draw a graph of height as a function of position, assuming the pulse is traveling to the right.
- (d) Repeat part c, but assume the pulse is traveling to the left.

[Problem by Arnold Arons.]



Problem 3.

3. The figure shows one wavelength of a steady sinusoidal wave traveling to the right along a string. Define a coordinate system in which the positive x axis points to the right and the positive y axis up, such that the flattened string would have $y=0$. Copy the figure, and label with " $y=0$ " all the appropriate parts of the string. Similarly, label with " $v=0$ " all parts of the string whose velocities are zero, and with " $a=0$ " all parts whose accelerations are zero. There is more than one point whose velocity is of the greatest magnitude. Pick one of these, and indicate the direction of its velocity vector. Do the same for a point having the maximum magnitude of acceleration.

[Problem by Arnold Arons.]

- 4. Find an equation for the relationship between the Doppler-shifted frequency of a wave and the frequency of the original wave, for the case of a stationary observer and a source moving directly toward or away from the observer.
- 5. Suggest a quantitative experiment to look for any deviation from the principle of superposition for surface waves in water. Make it simple and practical.

S A solution is given in the back of the book.

✓ A computerized answer check is available.

★ A difficult problem.

∫ A problem that requires calculus.

6✓. The musical note middle C has a frequency of 262 Hz. What are its period and wavelength?

7✓. Singing that is off-pitch by more than about 1% sounds bad. How fast would a singer have to be moving relative to the rest of a band to make this much of a change in pitch due to the Doppler effect?



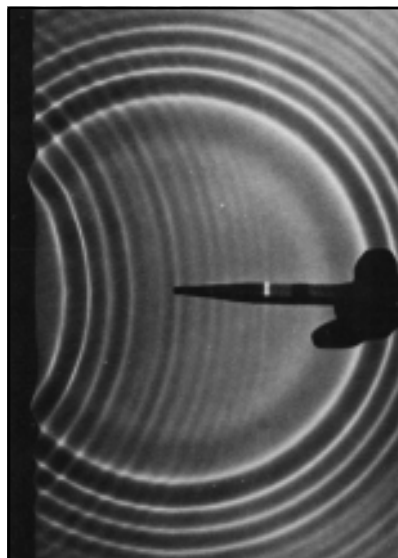
A cross-sectional view of a human body, showing the vocal tract.

4 Bounded Waves

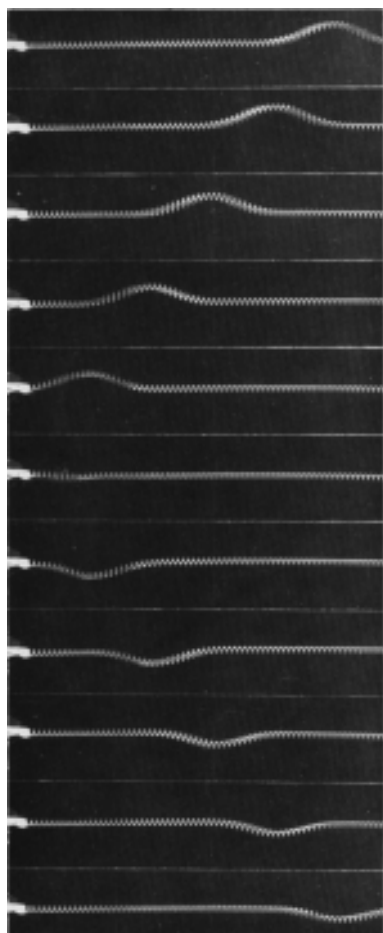
Speech is what separates humans most decisively from animals. No other species can master syntax, and even though chimpanzees can learn a vocabulary of hand signs, there is an unmistakable difference between a human infant and a baby chimp: starting from birth, the human experiments with the production of complex speech sounds.

Since speech sounds are instinctive for us, we seldom think about them consciously. How do we do control sound waves so skillfully? Mostly we do it by changing the shape of a connected set of hollow cavities in our chest, throat, and head. Somehow by moving the boundaries of this space in and out, we can produce all the vowel sounds. Up until now, we have been studying only those properties of waves that can be understood as if they existed in an infinite, open space with no boundaries. In this chapter we address what happens when a wave is confined within a certain space, or when a wave pattern encounters the boundary between two different media, such as when a light wave moving through air encounters a glass window-pane.

4.1 Reflection, Transmission, and Absorption



(a) Circular water waves are reflected from a boundary on the left. Uncopyrighted photo from PSSC Physics.



(b) A wave on a coil spring, initially traveling to the left, is reflected from the fixed end. Uncopyrighted photo from PSSC Physics.

Reflection and transmission

Sound waves can echo back from a cliff, and light waves are reflected from the surface of a pond. We use the word *reflection*, normally applied only to light waves in ordinary speech, to describe any such case of a wave rebounding from a barrier. Figure (a) shows a circular water wave being reflected from a straight wall. In this chapter, we will concentrate mainly on reflection of waves that move in one dimension, as in figure (b).

Wave reflection does not surprise us. After all, a material object such as a rubber ball would bounce back in the same way. But waves are not objects, and there are some surprises in store.

First, only part of the wave is usually reflected. Looking out through a window, we see light waves that passed through it, but a person standing outside would also be able to see her reflection in the glass. A light wave that strikes the glass is partly *reflected* and partly *transmitted* (passed) by the glass. The energy of the original wave is split between the two. This is different from the behavior of the rubber ball, which must go one way or the other, not both.

Second, consider what you see if you are swimming underwater and you look up at the surface. You see your own reflection. This is utterly counterintuitive, since we would expect the light waves to burst forth to freedom in the wide-open air. A material projectile shot up toward the surface would never rebound from the water-air boundary!

What is it about the difference between two media that causes waves to be partly reflected at the boundary between them? Is it their density? Their chemical composition? Ultimately all that matters is the speed of the wave in the two media. *A wave is partially reflected and partially transmitted at the boundary between media in which it has different speeds.* For example, the speed of light waves in window glass is about 30% less than in air, which explains why windows always make reflections. Figures (c) and (d) show examples of wave pulses being reflected at the boundary between two coil springs of different weights, in which the wave speed is different.

Reflections such as (a) and (b), where a wave encounters a massive fixed object, can usually be understood on the same basis as cases like (c) and (d) later in this section, where two media meet. Example (b), for instance, is like a more extreme version of example (c). If the heavy coil spring in (c) was made heavier and heavier, it would end up acting like the fixed wall to which the light spring in (b) has been attached.

Self-Check



A. In figure (b), the reflected pulse is upside-down, but its depth is just as big as the original pulse's height. How does the energy of the reflected pulse compare with that of the original?

Example: Fish have internal ears.

Why don't fish have ear-holes? The speed of sound waves in a fish's body is not much different from their speed in water, so sound waves are not strongly reflected from a fish's skin. They pass right through its body, so fish can have internal ears.

Example: Whale songs traveling long distances

Sound waves travel at drastically different speeds through rock, water, and air. Whale songs are thus strongly reflected at both the bottom and the surface. The sound waves can travel hundreds of miles, bouncing repeatedly between the bottom and the surface, and still be detectable. Sadly, noise pollution from ships has nearly shut down this cetacean version of the internet.

Example: Long-distance radio communication.

Radio communication can occur between stations on opposite sides of the planet. The mechanism is similar to the one explained in the previous example, but the three media involved are the earth, the atmosphere, and the ionosphere.

Self-Check



B. Sonar is a method for ships and submarines to detect each other by producing sound waves and listening for echoes. What properties would an underwater object have to have in order to be invisible to sonar?

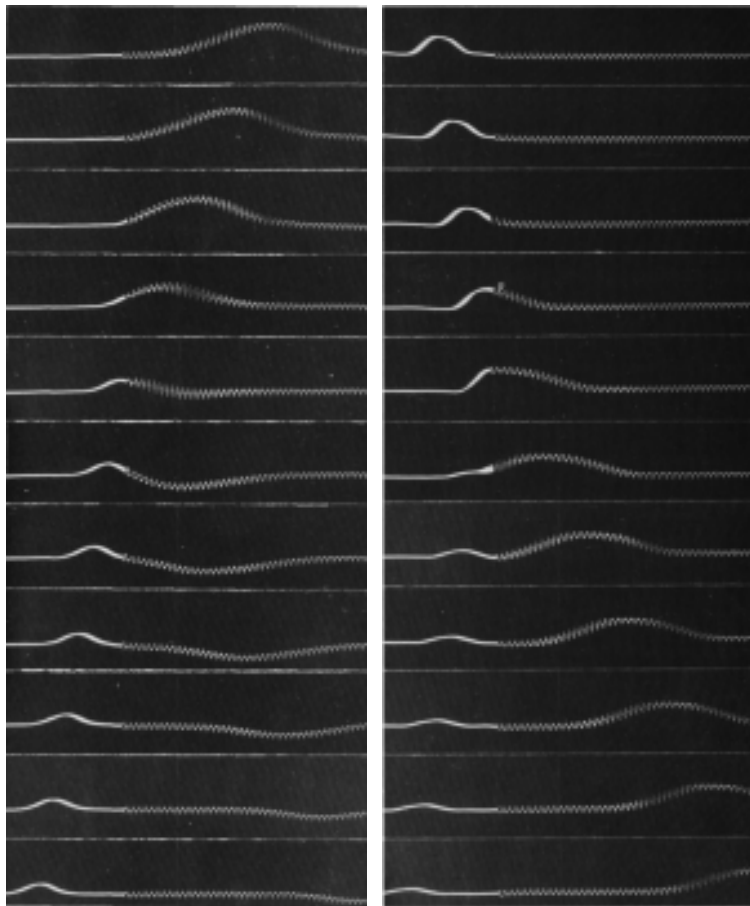


A. The energy of a wave is usually proportional to the square of the amplitude. Squaring a negative number gives a positive result, so the energy is the same. **B.** A substance is invisible to sonar if the speed of sound waves in it is the same as in water. Reflections occur only at boundaries between media in which the wave speed is different.

The use of the word “reflection” naturally brings to mind the creation of an image by a mirror, but this might be confusing, because we do not normally refer to “reflection” when we look at surfaces that are not shiny. Nevertheless, reflection is how we see the surfaces of all objects, not just polished ones. When we look at a sidewalk, for example, we are actually seeing the reflecting of the sun from the concrete. The reason we don’t see an image of the sun at our feet is simply that the rough surface blurs the image so drastically.

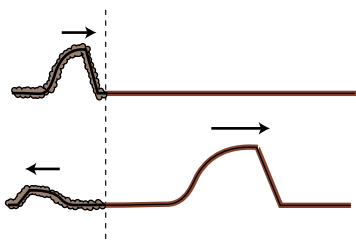
Inverted and uninverted reflections

Notice how the pulse reflected back to the right in example (c) comes back upside-down, whereas the one reflected back to the left in (d) returns in its original upright form. This is true for other waves as well. In general, there are two possible types of reflections, a reflection back into a faster medium and a reflection back into a slower medium. One type will always be an inverting reflection and one noninverting.

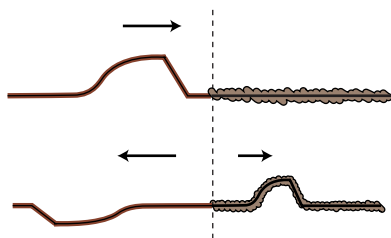


(c) A wave in the lighter spring, where the wave speed is greater, travels to the left and is then partly reflected and partly transmitted at the boundary with the heavier coil spring, which has a lower wave speed. The reflection is inverted. Uncopyrighted figure from PSSC Physics.

(d) A wave moving to the right in the heavier spring is partly reflected at the boundary with the lighter spring. The reflection is uninverted. Uncopyrighted figure from PSSC Physics.



(e) An uninverted reflection. The reflected pulse is reversed front to back, but is not upside-down.



(f) An inverted reflection. The reflected pulse is reversed both front to back and top to bottom.

It's important to realize that when we discuss inverted and uninverted reflections on a string, we are talking about whether the wave is flipped across the direction of motion (i.e. upside-down in these drawings). The reflected pulse will always be reversed front to back, as shown in figures (e) and (f). This is because it is traveling in the other direction. The leading edge of the pulse is what gets reflected first, so it is still ahead when it starts back to the left — it's just that "ahead" is now in the opposite direction.

Absorption

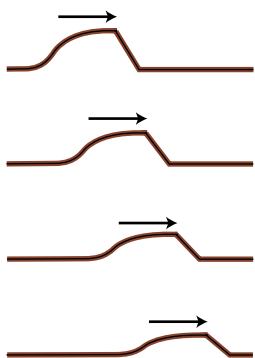
So far we have tacitly assumed that wave energy remains as wave energy, and is not converted to any other form. If this was true, then the world would become more and more full of sound waves, which could never escape into the vacuum of outer space. In reality, any mechanical wave consists of a traveling pattern of vibrations of some physical medium, and vibrations of matter always produce heat, as when you bend a coathanger back and forth and it becomes hot. We can thus expect that in mechanical waves such as water waves, sound waves, or waves on a string, the wave energy will gradually be converted into heat. This is referred to as *absorption*.

The wave suffers a decrease in amplitude, as shown in figure (g). The decrease in amplitude amounts to the same fractional change for each unit of distance covered. For example, if a wave decreases from amplitude 2 to amplitude 1 over a distance of 1 meter, then after traveling another meter it will have an amplitude of $1/2$. That is, the reduction in amplitude is exponential. This can be proven as follows. By the principle of superposition, we know that a wave of amplitude 2 must behave like the superposition of two identical waves of amplitude 1. If a single amplitude-1 wave would die down to amplitude $1/2$ over a certain distance, then two amplitude-1 waves superposed on top of one another to make amplitude $1+1=2$ must die down to amplitude $1/2+1/2=1$ over the same distance.

Self-Check



As a wave undergoes absorption, it loses energy. Does this mean that it slows down?



(g) A pulse traveling through a highly absorptive medium.

In many cases, this frictional heating effect is quite weak. Sound waves in air, for instance, dissipate into heat extremely slowly, and the sound of church music in a cathedral may reverberate for as much as 3 or 4 seconds before it becomes inaudible. During this time it has traveled over a kilometer! Even this very gradual dissipation of energy occurs mostly as heating of the church's walls and by the leaking of sound to the outside (where it will eventually end up as heat). Under the right conditions (humid air and low frequency), a sound wave in a straight pipe could theoretically travel hundreds of kilometers before being noticeable attenuated.

In general, the absorption of mechanical waves depends a great deal on the chemical composition and microscopic structure of the medium. Ripples on the surface of antifreeze, for instance, die out extremely rapidly compared to ripples on water. For sound waves and surface waves in liquids and gases, what matters is the viscosity of the substance, i.e. whether it flows



No. A material object that loses kinetic energy slows down, but a wave is not a material object. The velocity of a wave ordinarily only depends on the medium, not on the amplitude. The speed of soft sound, for example, is the same as the speed of loud sound.

easily like water or mercury or more sluggishly like molasses or antifreeze. This explains why our intuitive expectation of strong absorption of sound in water is incorrect. Water is a very weak absorber of sound (viz. whale songs and sonar), and our incorrect intuition arises from focusing on the wrong property of the substance: water's high density, which is irrelevant, rather than its low viscosity, which is what matters.

Light is an interesting case, since although it can travel through matter, it is not itself a vibration of any material substance. Thus we can look at the star Sirius, 10^{14} km away from us, and be assured that none of its light was absorbed in the vacuum of outer space during its 9-year journey to us. The Hubble Space Telescope routinely observes light that has been on its way to us since the early history of the universe, billions of years ago. Of course the energy of light can be dissipated if it does pass through matter (and the light from distant galaxies is often absorbed if there happen to be clouds of gas or dust in between).

Example: soundproofing

Typical amateur musicians setting out to soundproof their garages tend to think that they should simply cover the walls with the densest possible substance. In fact, sound is not absorbed very strongly even by passing through several inches of wood. A better strategy for soundproofing is to create a sandwich of alternating layers of materials in which the speed of sound is very different, to encourage reflection.

The classic design is alternating layers of fiberglass and plywood. The speed of sound in plywood is very high, due to its stiffness, while its speed in fiberglass is essentially the same as its speed in air. Both materials are fairly good sound absorbers, but sound waves passing through a few inches of them are still not going to be absorbed sufficiently. The point of combining them is that a sound wave that tries to get out will be strongly reflected at each of the fiberglass-plywood boundaries, and will bounce back and forth many times like a ping pong ball. Due to all the back-and-forth motion, the sound may end up traveling a total distance equal to ten times the actual thickness of the soundproofing before it escapes. This is the equivalent of having ten times the thickness of sound-absorbing material.

Example: radio transmission

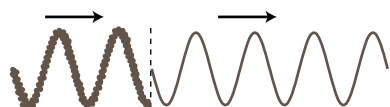
A radio transmitting station, such as a commercial station or an amateur "ham" radio station, must have a length of wire or cable connecting the amplifier to the antenna. The cable and the antenna act as two different media for radio waves, and there will therefore be partial reflection of the waves as they come from the cable to the antenna. If the waves bounce back and forth many times between the amplifier and the antenna, a great deal of their energy will be absorbed. There are two ways to attack the problem. One possibility is to design the antenna so that the speed of the waves in it is the same as the speed of the waves in the cable. There is then no reflection. The other method is to connect the amplifier to the antenna using a type of wire or cable that does not strongly absorb the waves. Partial reflection then becomes irrelevant, since all the wave energy will eventually exit through the antenna.



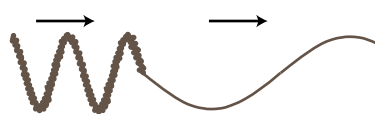
Discussion Question

A sound wave that underwent a pressure-inverting reflection would have its compressions converted to expansions and vice versa. How would its energy and frequency compare with those of the original sound? Would it sound any different? What happens if you swap the two wires where they connect to a stereo speaker, resulting in waves that vibrate in the opposite way?

4.2* Quantitative Treatment of Reflection



(a) A change in frequency without a change in wavelength would produce a discontinuity in the wave.



(b) A simple change in wavelength without a reflection would result in a sharp kink in the wave.

In this optional section we analyze the reasons why reflections occur at a speed-changing boundary, predict quantitatively the intensities of reflection and transmission, and discuss how to predict for any type of wave which reflections are inverting and which are uninverting. The gory details are likely to be of interest mainly to students with concentrations in the physical sciences, but all readers are encouraged at least to skim the first two subsections for physical insight.

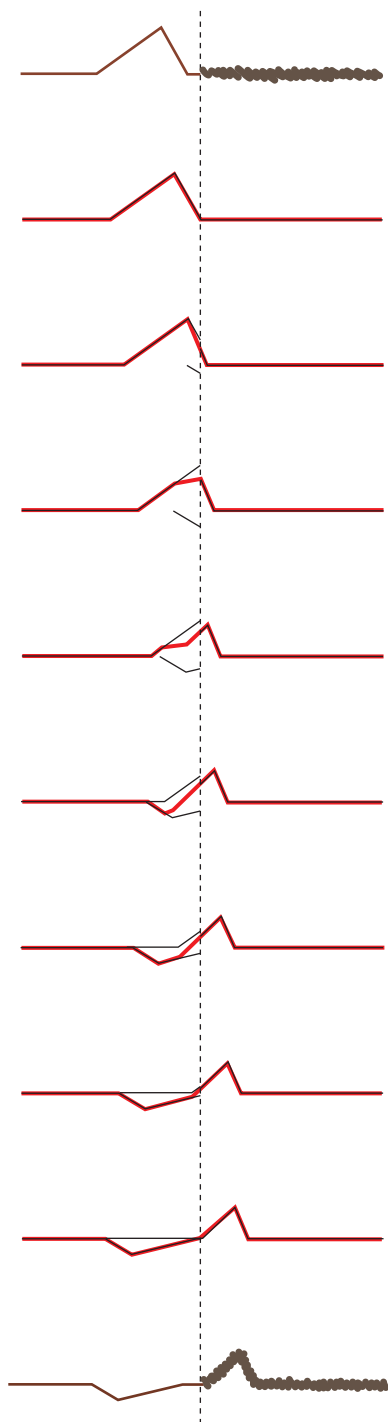
Why reflection occurs

To understand the fundamental reasons for what does occur at the boundary between media, let's first discuss what doesn't happen. For the sake of concreteness, consider a sinusoidal wave on a string. If the wave progresses from a heavier portion of the string, in which its velocity is low, to a lighter-weight part, in which it is high, then the equation $v=f\lambda$ tells us that it must change its frequency, or its wavelength, or both. If only the frequency changed, then the parts of the wave in the two different portions of the string would quickly get out of step with each other, producing a discontinuity in the wave, (a). This is unphysical, so we know that the wavelength must change while the frequency remains constant, (b).

But there is still something unphysical about figure (b). The sudden change in the shape of the wave has resulted in a sharp kink at the boundary. This can't really happen, because the medium tends to accelerate in such a way as to eliminate curvature. A sharp kink corresponds to an infinite curvature at one point, which would produce an infinite acceleration, which would not be consistent with the smooth pattern of wave motion envisioned in fig. (b). Waves can have kinks, but not stationary kinks.

We conclude that without positing partial reflection of the wave, we cannot simultaneously satisfy the requirements of (1) continuity of the wave, and (2) no sudden changes in the slope of the wave. (The student who has studied calculus will recognize this as amounting to an assumption that both the wave and its derivative are continuous functions.)

Does this amount to a proof that reflection occurs? Not quite. We have only proven that certain types of wave motion are not valid solutions. In the following subsection, we prove that a valid solution can always be found in which a reflection occurs. Now in physics, we normally assume (but seldom prove formally) that the equations of motion have a unique solution, since otherwise a given set of initial conditions could lead to different behavior later on, but the Newtonian universe is supposed to be deterministic. Since the solution must be unique, and we derive below a valid solution involving a reflected pulse, we will have ended up with what amounts to a proof of reflection.



A pulse being partially reflected and partially transmitted at the boundary between two strings in which the wave speed is different. The top drawing shows the pulse heading to the right, toward the heavier string. For clarity, all but the first and last drawings are schematic. Once the reflected pulse begins to emerge from the boundary, it adds together with the trailing parts of the incident pulse. Their sum, shown as a wider line, is what is actually observed.

Intensity of reflection

We will now show, in the case of waves on a string, that it is possible to satisfy the physical requirements given above by constructing a reflected wave, and as a bonus this will produce an equation for the proportions of reflection and transmission and a prediction as to which conditions will lead to inverted and which to uninverted reflection. We assume only that the principle of superposition holds, which is a good approximation for waves on a string of sufficiently small amplitude.

Let the unknown amplitudes of the reflected and transmitted waves be R and T , respectively. An inverted reflection would be represented by a negative value of R . We can without loss of generality take the incident (original) wave to have unit amplitude. Superposition tells us that if, for instance, the incident wave had double this amplitude, we could immediately find a corresponding solution simply by doubling R and T .

Just to the left of the boundary, the height of the wave is given by the height 1 of the incident wave, plus the height R of the part of the reflected wave that has just been created and begun heading back, for a total height of $1+R$. On the right side immediately next to the boundary, the transmitted wave has a height T . To avoid a discontinuity, we must have

$$1+R = T \quad .$$

Next we turn to the requirement of equal slopes on both sides of the boundary. Let the slope of the incoming wave be s immediately to the left of the junction. If the wave was 100% reflected, and without inversion, then the slope of the reflected wave would be $-s$, since the wave has been reversed in direction. In general, the slope of the reflected wave equals $-sR$, and the slopes of the superposed waves on the left side add up to $s-sR$. On the right, the slope depends on the amplitude, T , but is also changed by the stretching or compression of the wave due to the change in speed. If, for example, the wave speed is twice as great on the right side, then the slope is cut in half by this effect. The slope on the right is therefore $s(v_1/v_2)T$, where v_1 is the velocity in the original medium and v_2 the velocity in the new medium. Equality of slopes gives $s-sR = s(v_1/v_2)T$, or

$$1 - R = \frac{v_1}{v_2} T \quad .$$

Solving the two equations for the unknowns R and T gives

$$R = \frac{v_2 - v_1}{v_2 + v_1} \quad \text{and} \quad T = \frac{2v_2}{v_2 + v_1} \quad .$$

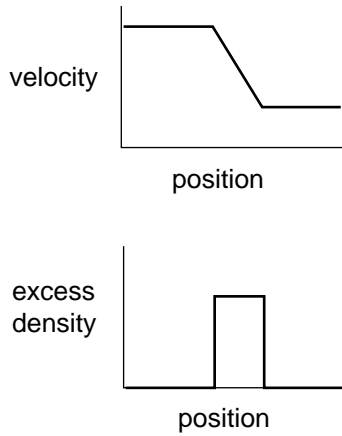
The first equation shows that there is no reflection unless the two wave speeds are different, and that the reflection is inverted in reflection back into a fast medium.

The energies of the transmitted and reflected waves always add up to the same as the energy of the original wave. There is never any abrupt loss (or gain) in energy when a wave crosses a boundary. (Conversion of wave energy to heat occurs for many types of waves, but it occurs throughout the medium.) The equation for T , surprisingly, allows the amplitude of the transmitted wave to be greater than 1, i.e. greater than that of the incident wave. This does not violate conservation of energy, because this occurs

when the second string is less massive, reducing its kinetic energy, and the transmitted pulse is broader and less strongly curved, which lessens its potential energy.

Inverted and uninverted reflections in general

For waves on a string, reflections back into a faster medium are inverted, while those back into a slower medium are uninverted. Is this true for all types of waves? The rather subtle answer is that it depends on what property of the wave you are discussing.



Let's start by considering wave disturbances of freeway traffic. Anyone who has driven frequently on crowded freeways has observed the phenomenon in which one driver taps the brakes, starting a chain reaction that travels backward down the freeway as each person in turn exercises caution in order to avoid rear-ending anyone. The reason why this type of wave is relevant is that it gives a simple, easily visualized example of our description of a wave depends on which aspect of the wave we have in mind. In steadily flowing freeway traffic, both the density of cars and their velocity are constant all along the road. Since there is no disturbance in this pattern of constant velocity and density, we say that there is no wave. Now if a wave is touched off by a person tapping the brakes, we can either describe it as a region of high density or as a region of decreasing velocity.

The freeway traffic wave is in fact a good model of a sound wave, and a sound wave can likewise be described either by the density (or pressure) of the air or by its speed. Likewise many other types of waves can be described by either of two functions, one of which is often the derivative of the other with respect to position.

Now let's consider reflections. If we observe the freeway wave in a mirror, the high-density area will still appear high in density, but velocity in the opposite direction will now be described by a negative number. A person observing the mirror image will draw the same density graph, but the velocity graph will be flipped across the x axis, and its original region of negative slope will now have positive slope. Although I don't know any physical situation that would correspond to the reflection of a traffic wave, we can immediately apply the same reasoning to sound waves, which often do get reflected, and determine that a reflection can either be density-inverting and velocity-uninverting or density-uninverting and velocity-inverting.

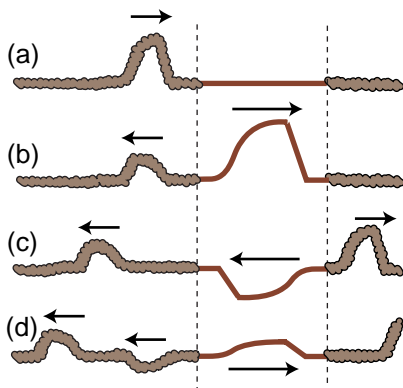
This same type of situation will occur over and over as one encounters new types of waves, and to apply the analogy we need only determine which quantities, like velocity, become negated in a mirror image and which, like density, stay the same.

A light wave, for instance consists of a traveling pattern of electric and magnetic fields. All you need to know in order to analyze the reflection of light waves is how electric and magnetic fields behave under reflection; you don't need to know any of the detailed physics of electricity and magnetism. An electric field can be detected, for example, by the way one's hair stands on end. The direction of the hair indicates the direction of the electric field. In a mirror image, the hair points the other way, so the electric field is apparently reversed in a mirror image. The behavior of magnetic fields, however, is a little tricky. The magnetic properties of a bar magnet, for

instance, are caused by the aligned rotation of the outermost orbiting electrons of the atoms. In a mirror image, the direction of rotation is reversed, say from clockwise to counterclockwise, and so the magnetic field is reversed twice: once simply because the whole picture is flipped and once because of the reversed rotation of the electrons. In other words, magnetic fields do not reverse themselves in a mirror image. We can thus predict that there will be two possible types of reflection of light waves. In one, the electric field is inverted and the magnetic field uninverted. In the other, the electric field is uninverted and the magnetic field inverted.

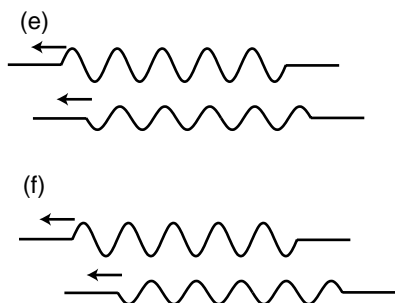
4.3 Interference Effects

If you look at the front of a pair of high-quality binoculars, you will notice a greenish-blue coating on the lenses. This is advertised as a coating to prevent reflection. Now reflection is clearly undesirable — we want the light to go *in* the binoculars — but so far I've described reflection as an unalterable fact of nature, depending only on the properties of the two wave media. The coating can't change the speed of light in air or in glass, so how can it work? The key is that the coating itself is a wave medium. In other words, we have a three-layer sandwich of materials: air, coating, and glass. We will analyze the way the coating works, not because optical coatings are an important part of your education but because it provides a good example of the general phenomenon of wave interference effects.



There are two different interfaces between media: an air-coating boundary and a coating-glass boundary. Partial reflection and partial transmission will occur at each boundary. For ease of visualization let's start by considering an equivalent system consisting of three dissimilar pieces of string tied together, and a wave pattern consisting initially of a single pulse. Figure (a) shows the incident pulse moving through the heavy rope, in which its velocity is low. When it encounters the lighter-weight rope in the middle, a faster medium, it is partially reflected and partially transmitted. (The transmitted pulse is bigger, but nevertheless has only part of the original energy.) The pulse transmitted by the first interface is then partially reflected and partially transmitted by the second boundary, (c). In figure (d), two pulses are on the way back out to the left, and a single pulse is heading off to the right. (There is still a weak pulse caught between the two boundaries, and this will rattle back and forth, rapidly getting too weak to detect as it leaks energy to the outside with each partial reflection.)

Note how, of the two reflected pulses in (d), one is inverted and one uninverted. One underwent reflection at the first boundary (a reflection back into a slower medium is uninverted), but the other was reflected at the second boundary (reflection back into a faster medium is inverted).



Now let's imagine what would have happened if the incoming wave pattern had been a long sinusoidal wave train instead of a single pulse. The first two waves to reemerge on the left could be in phase, (e), or out of phase, (f), or anywhere in between. The amount of lag between them depends entirely on the width of the middle segment of string. If we choose the width of the middle string segment correctly, then we can arrange for destructive interference to occur, (f), with cancellation resulting in a very weak reflected wave.

This whole analysis applies directly to our original case of optical coatings. Visible light from most sources does consist of a stream of short sinusoidal wave-trains such as the ones drawn above. The only real difference between the waves-on-a-rope example and the case of an optical coating is that the first and third media are air and glass, in which light does not have the same speed. However, the general result is the same as long as the air and the glass have light-wave speeds that either both greater than the coating's or both less than the coating's.



(g) A soap bubble.

The business of optical coatings turns out to be a very arcane one, with a plethora of trade secrets and “black magic” techniques handed down from master to apprentice. Nevertheless, the ideas you have learned about waves in general are sufficient to allow you to come to some definite conclusions without any further technical knowledge. The self-check and discussion questions will direct you along these lines of thought.

The example of an optical coating was typical of a wide variety of wave interference effects. With a little guidance, you are now ready to figure out for yourself other examples such as the rainbow pattern made by a compact disc, a layer of oil on a puddle, or a soap bubble.

Self-Check



1. Color corresponds to wavelength of light waves. Is it possible to choose a thickness for an optical coating that will produce destructive interference for all colors of light?
2. How can you explain the rainbow colors on the soap bubble in figure g?

Discussion Questions



- A. Is it possible to get *complete* destructive interference in an optical coating, at least for light of one specific wavelength?
- B. Sunlight consists of sinusoidal wave-trains containing on the order of a hundred cycles back-to-back, for a length of something like a tenth of a millimeter. What happens if you try to make an optical coating thicker than this?
- C. Suppose you take two microscope slides and lay one on top of the other so that one of its edges is resting on the corresponding edge of the bottom one. If you insert a sliver of paper or a hair at the opposite end, a wedge-shaped layer of air will exist in the middle, with a thickness that changes gradually from one end to the other. What would you expect to see if the slides were illuminated from above by light of a single color? How would this change if you gradually lifted the lower edge of the top slide until the two slides were finally parallel?
- D. An observation like the one described in the previous discussion question was used by Newton as evidence *against* the wave theory of light! If Newton didn't know about inverting and noninverting reflections, what would have seemed inexplicable to him about the region where the air layer had zero or nearly zero thickness?



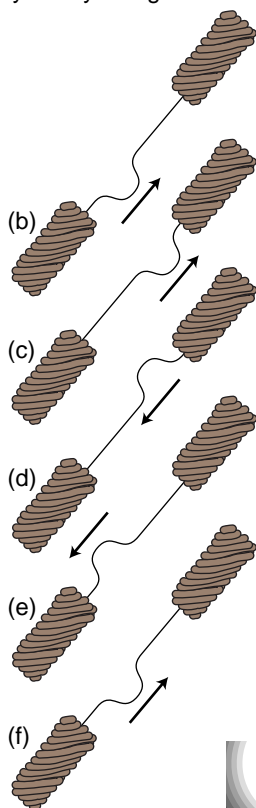
1. No. To get the best possible interference, the thickness of the coating must be such that the second reflected wave train lags behind the first by an integer number of wavelengths. Optimal performance can therefore only be produced for one specific color of light. The typical greenish color of the coatings shows that it does the worst job for green light. 2. The white sunlight is a mixture of all the colors of the rainbow. A particular color will be reflected more or less depending on the thickness of the soap film in relation to its wavelength. (We have also been assuming that the wave came in perpendicular to the film, but that is not necessarily true here, and that will also have an effect. If the light comes in at an oblique angle, it has to traverse a greater amount of the film.)

4.4 Waves Bounded on Both Sides



(a)

We model a guitar string attached to the guitar's body at both ends as a light-weight string attached to extremely heavy strings at its ends.



Self-Check

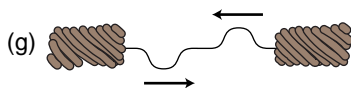


Notice that from (b) to (f), the pulse has passed by every point on the string exactly twice. This means that the total distance it has traveled equals $2L$, where L is the length of the string. Given this fact, what are the period and frequency of the sound it produces, expressed in terms of L and v , the velocity of the wave? [answer on next page]

In the example of the previous section, it was theoretically true that a pulse would be trapped permanently in the middle medium, but that pulse was not central to our discussion, and in any case it was weakening severely with each partial reflection. Now consider a guitar string. At its ends it is tied to the body of the instrument itself, and since the body is very massive, the behavior of the waves when they reach the end of the string can be understood in the same way as if the actual guitar string was attached on the ends to strings that were extremely massive. Reflections are most intense when the two media are very dissimilar. Because the wave speed in the body is so radically different from the speed in the string, we should expect nearly 100% reflection.

Although this may seem like a rather bizarre physical model of the actual guitar string, it already tells us something interesting about the behavior of a guitar that we would not otherwise have understood. The body, far from being a passive frame for attaching the strings to, is actually the exit path for the wave energy in the strings. With every reflection, the wave pattern on the string loses a tiny fraction of its energy, which is then conducted through the body and out into the air. (The string has too little cross-section to make sound waves efficiently by itself.) By changing the properties of the body, moreover, we should expect to have an effect on the manner in which sound escapes from the instrument. This is clearly demonstrated by the electric guitar, which has an extremely massive, solid wooden body. Here the dissimilarity between the two wave media is even more pronounced, with the result that wave energy leaks out of the string even more slowly. This is why an electric guitar with no electric pickup can hardly be heard at all, and it is also the reason why notes on an electric guitar can be sustained for longer than notes on an acoustic guitar.

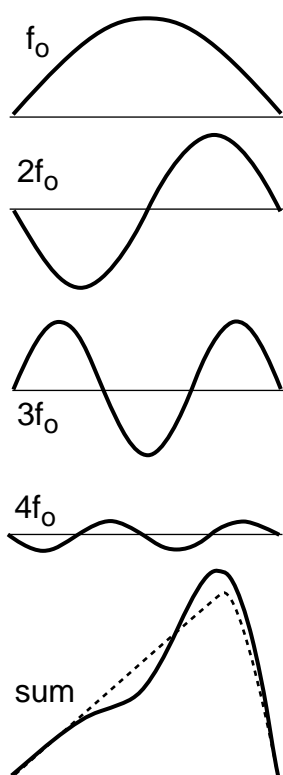
If we initially create a disturbance on a guitar string, how will the reflections behave? In reality, the finger or pick will give the string a triangular shape before letting it go, and we may think of this triangular shape as a very broad “dent” in the string which will spread out in both directions. For simplicity, however, let's just imagine a wave pattern that initially consists of a single, narrow pulse traveling up the neck, (b). After reflection from the top end, it is inverted, (d). Now something interesting happens: figure (f) is identical to figure (b). After two reflections, the pulse has been inverted twice and has changed direction twice. It is now back where it started. The motion is periodic. This is why a guitar produces sounds that have a definite sensation of pitch.



Note that if the waves on the string obey the principle of superposition, then the velocity must be independent of amplitude, and the guitar will produce the same pitch regardless of whether it is played loudly or softly. In reality, waves on a string obey the principle of superposition approximately, but not exactly. The guitar, like just about any acoustic instrument, is a little out of tune when played loudly. (The effect is more pronounced for wind instruments than for strings, but wind players are able to compensate for it.)

Now there is only one hole in our reasoning. Suppose we somehow arrange to have an initial setup consisting of two identical pulses heading toward each other, as in figure (g). They will pass through each other, undergo a single inverting reflection, and come back to a configuration in which their positions have been exactly interchanged. This means that the period of vibration is half as long. The frequency is twice as high.

This might seem like a purely academic possibility, since nobody actually plays the guitar with two picks at once! But in fact it is an example of a very general fact about waves that are bounded on both sides. A mathematical theorem called Fourier's theorem states that any wave can be created by superposing sine waves. The figure on the left shows how even by using only four sine waves with appropriately chosen amplitudes, we can arrive at a sum which is a decent approximation to the realistic triangular shape of a guitar string being plucked. The one-hump wave, in which half a wavelength fits on the string, will behave like the single pulse we originally discussed. We call its frequency f_0 . The two-hump wave, with one whole wavelength, is very much like the two-pulse example. For the reasons discussed above, its frequency is $2f_0$. Similarly, the three-hump and four-hump waves have frequencies of $3f_0$ and $4f_0$.

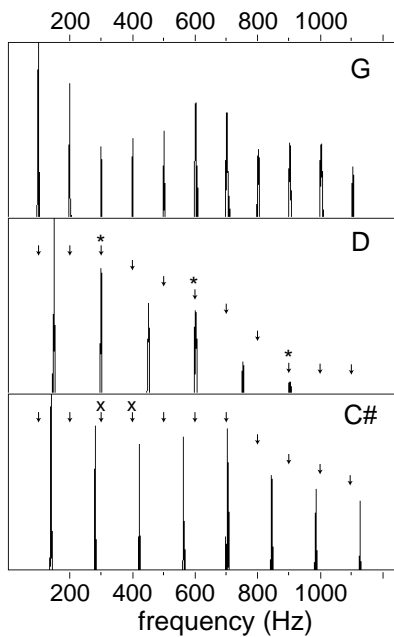


Theoretically we would need to add together infinitely many such wave patterns to describe the initial triangular shape of the string exactly, although the amplitudes required for the very high frequency parts would be very small, and an excellent approximation could be achieved with as few as ten waves.

We thus arrive at the following very general conclusion. Whenever a wave pattern exists in a medium bounded on both sides by media in which the wave speed is very different, the motion can be broken down into the motion of a (theoretically infinite) series of sine waves, with frequencies f_0 , $2f_0$, $3f_0$, ... Except for some technical details, to be discussed below, this analysis applies to a vast range of sound-producing systems, including the air column within the human vocal tract. Because sounds composed of this kind of pattern of frequencies are so common, our ear-brain system has evolved so as to perceive them as a single, fused sensation of tone.



[answer to self-check on previous page] The period is the time required to travel a distance $2L$ at speed v , $T=2L/v$. The frequency is $f=1/T=v/2L$.



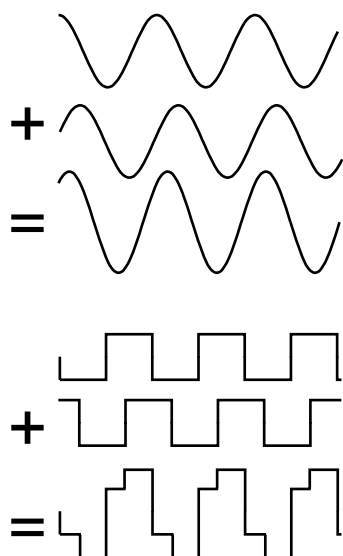
Graphs of loudness versus frequency for the vowel “ah,” sung as three different musical notes. G is consonant with D, since every overtone of G that is close to an overtone of D (*) is at exactly the same frequency. G and C# are dissonant together, since some of the overtones of G (x) are close to, but not right on top of, those of C#.

Musical applications

Many musicians claim to be able to identify individually several of the frequencies above the first one, called *overtones* or *harmonics*, but they are kidding themselves. In reality, the overtone series has two important roles in music, neither of which depends on this fictitious ability to “hear out” the individual overtones.

First, the relative strengths of the overtones is an important part of the personality of a sound, called its *timbre* (rhymes with “amber”). The characteristic tone of the brass instruments, for example, is a sound that starts out with a very strong harmonic series extending up to very high frequencies, but whose higher harmonics die down drastically as the attack changes to the sustained portion of the note.

Second, although the ear cannot separate the individual harmonics of a single musical tone, it is very sensitive to clashes between the overtones of notes played simultaneously, i.e. in harmony. We tend to perceive a combination of notes as being dissonant if they have overtones that are close but not the same. Roughly speaking, strong overtones whose frequencies differ by more than 1% and less than 10% cause the notes to sound dissonant. It is important to realize that the term “dissonance” is not a negative one in music. No matter how long you search the radio dial, you will never hear more than three seconds of music without at least one dissonant combination of notes. Dissonance is a necessary ingredient in the creation of a musical cycle of tension and release. Musically knowledgeable people do not usually use the word “dissonant” as a criticism of music, and if they do, what they are really saying is that the dissonance has been used in a clumsy way, or without providing any contrast between dissonance and consonance.



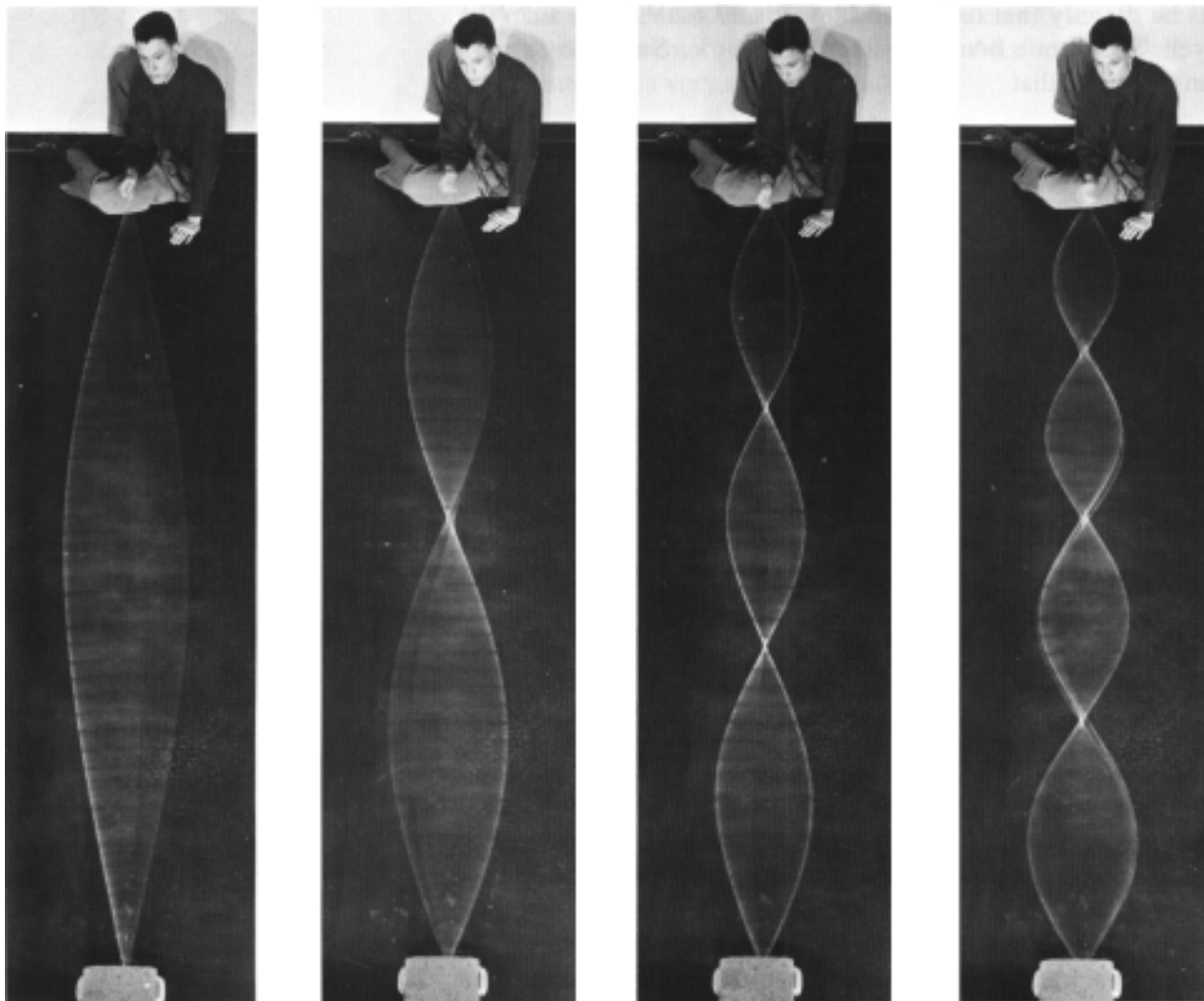
If you take a sine wave and make a copy of it shifted over, their sum is still a sine wave. The same is not true for a square wave.

Standing waves

The photos below show sinusoidal wave patterns made by shaking a rope. I used to enjoy doing this at the bank with the pens on chains, back in the days when people actually went to the bank. You might think that I and the person in the photos had to practice for a long time in order to get such nice sine waves. In fact, a sine wave is the only shape that can create this kind of wave pattern, called a *standing wave*, which simply vibrates back and forth in one place without moving. The sine wave just creates itself automatically when you find the right frequency, because no other shape is possible.

If you think about it, it's not even obvious that sine waves should be able to do this trick. After all, waves are supposed to travel at a set speed, aren't they? The speed isn't supposed to be zero! Well, we can actually think of a standing wave as a superposition of a moving sine wave with its own reflection, which is moving the opposite way. Sine waves have the unique mathematical property that the sum of sine waves of equal wavelength is simply a new sine wave with the same wavelength. As the two sine waves go back and forth, they always cancel perfectly at the ends, and their sum appears to stand still.

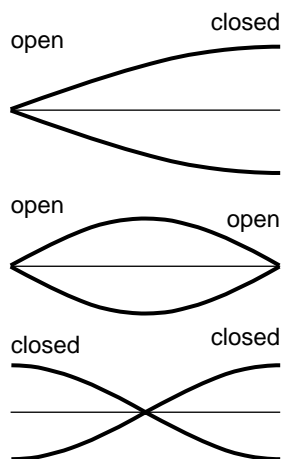
Standing wave patterns are rather important, since atoms are really standing-wave patterns of electron waves. You are a standing wave!



(PSSC Physics)



Surprisingly, sound waves undergo partial reflection at the open ends of tubes as well as closed ones. The reason has to do with the readjustment of the wave pattern from a plane wave to a spherical wave. If the readjustment was as sudden as that shown in the figure, then there would be kinks in the wave. Waves don't like to develop kinks. In section 4.2 we deduced the strength of the reflected wave at a change in medium from the requirement that the wave would not have discontinuities or kinks. Here there is no change in medium, but a reflected wave is still required in order to avoid kinks.



Graphs of excess density versus position for the lowest-frequency standing waves of three types of air columns. Points on the axis have normal air density.

Standing-wave patterns of air columns

The air column inside a wind instrument or the human vocal tract behaves very much like the wave-on-a-string example we've been concentrating on so far, the main difference being that we may have either inverting or noninverting reflections at the ends.

Inverting reflection at one end and uninverting at the other

If you blow over the mouth of a beer bottle to produce a tone, the bottle outlines an air column that is closed at the bottom and open at the top. Sound waves will be reflected at the bottom because of the difference in the speed of sound in air and glass. The speed of sound is greater in glass (because its stiffness more than compensates for its higher density compared to air). Using the type of reasoning outlined in optional section 4.2, we find that this reflection will be density-uninverting: a compression comes back as a compression, and a rarefaction as a rarefaction. There will be strong reflection and very weak transmission, since the difference in speeds is so great. But why do we get a reflection at the *mouth* of the bottle? There is no change in medium there, and the air inside the bottle is connected to the air in the room. This is a type of reflection that has to do with the three-dimensional shape of the sound waves, and cannot be treated the same way as the other types of reflection we have encountered. Since this chapter is supposed to be confined mainly to wave motion in one dimension, and it would take us too far afield here to explain it in detail, but a general justification is given in the caption of the figure.

The important point is that whereas the reflection at the bottom was density-uninverting, the one at the top is density-inverting. This means that at the top of the bottle, a compression superimposes with its own reflection, which is a rarefaction. The two nearly cancel, and so the wave has almost zero amplitude at the mouth of the bottle. The opposite is true at the bottom — here we have a peak in the standing-wave pattern, not a stationary point. The standing wave with the lowest frequency, i.e. the longest wave length, is therefore one in which $1/4$ of a wavelength fits along the length of the tube.

Both ends the same

If both ends are open (as in the flute) or both ends closed (as in some organ pipes), then the standing wave pattern must be symmetric. The lowest-frequency wave fits half a wavelength in the tube.

Self-Check

Draw a graph of pressure versus position for the first overtone of the air column in a tube open at one end and closed at the other. This will be the next-to-longest possible wavelength that allows for a point of maximum vibration at one end and a point of no vibration at the other. How many times shorter will its wavelength be compared to the frequency of the lowest-frequency standing wave, shown in the figure? Based on this, how many times greater will its frequency be? [Answer on next page.]

Summary

Selected Vocabulary

reflection the bouncing back of part of a wave from a boundary
transmission the continuation of part of a wave through a boundary
absorption the gradual conversion of wave energy into heating of the medium
standing wave a wave pattern that stays in one place

Notation


λ wavelength (Greek letter lambda)

Summary

Whenever a wave encounters the boundary between two media in which its speeds are different, part of the wave is reflected and part is transmitted. The reflection is always reversed front-to-back, but may also be inverted in amplitude. Whether the reflection is inverted depends on how the wave speeds in the two media compare, e.g. a wave on a string is uninverted when it is reflected back into a segment of string where its speed is lower. The greater the difference in wave speed between the two media, the greater the fraction of the wave energy that is reflected. Surprisingly, a wave in a dense material like wood will be strongly reflected back into the wood at a wood-air boundary.

A one-dimensional wave confined by highly reflective boundaries on two sides will display motion which is periodic. For example, if both reflections are inverting, the wave will have a period equal to twice the time required to traverse the region, or to that time divided by an integer. An important special case is a sinusoidal wave; in this case, the wave forms a stationary pattern composed of a superposition of sine waves moving in opposite direction.



[Answer to self-check on previous page.] The wave pattern will look like this: . Three quarters of a wavelength fit in the tube, so the wavelength is three times shorter than that of the lowest-frequency mode, in which one quarter of a wave fits. Since the wavelength is smaller by a factor of three, the frequency is three times higher. Instead of $f_0, 2f_0, 3f_0, 4f_0, \dots$, the pattern of wave frequencies of this air column goes $f_0, 3f_0, 5f_0, 7f_0, \dots$

Homework Problems

1. Light travels faster in warmer air. Use this fact to explain the formation of a mirage appearing like the shiny surface of a pool of water when there is a layer of hot air above a road.

2. (a) Using the equations from optional section 4.2, compute the amplitude of light that is reflected back into air at an air-water interface, relative to the amplitude of the incident wave. The speeds of light in air and water are 3.0×10^8 and 2.2×10^8 m/s, respectively.

(b✓) Find the energy of the reflected wave as a fraction of the incident energy. [Hint: The answers to the two parts are not the same.]

3. A B-flat clarinet (the most common kind) produces its lowest note, at about 230 Hz, when half of a wavelength fits inside its tube. Compute the length of the clarinet. [Check: The actual length of a clarinet is about 67 cm from the tip of the mouthpiece to the end of the bell. Because the behavior of the clarinet and its coupling to air outside it is a little more complex than that of a simple tube enclosing a cylindrical air column, your answer will be close to this value, but not exactly equal to it.]

4. (a) A good tenor saxophone player can play all of the following notes without changing her fingering, simply by altering the tightness of her lips: Eb (150 Hz), Eb (300 Hz), Bb (450 Hz), and Eb (600 Hz). How is this possible? (b) Some saxophone players are known for their ability to use this technique to play “freak notes,” i.e. notes above the normal range of the instrument. Why isn’t it possible to play notes below the normal range using this technique?

5. The table gives the frequencies of the notes that make up the key of F major, starting from middle C and going up through all seven notes. (a) Calculate the first five or six harmonics of C and G, and determine whether these two notes will be consonant or dissonant. (b) Do the same for C and B flat. [Hint: Remember that harmonics that differ by about 1-10% cause dissonance.]

C	261.6 Hz
D	293.7
E	329.6
F	349.2
G	392.0
A	440.0
B flat	466.2

Problem 5.

S A solution is given in the back of the book.

✓ A computerized answer check is available.

★ A difficult problem.

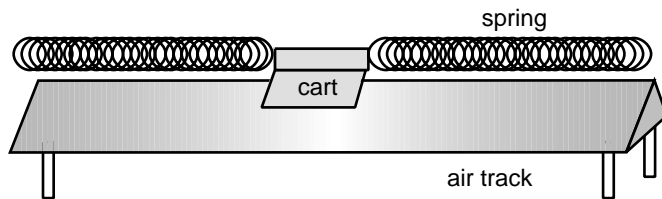
∫ A problem that requires calculus.

Exercises

Exercise 1A: Vibration

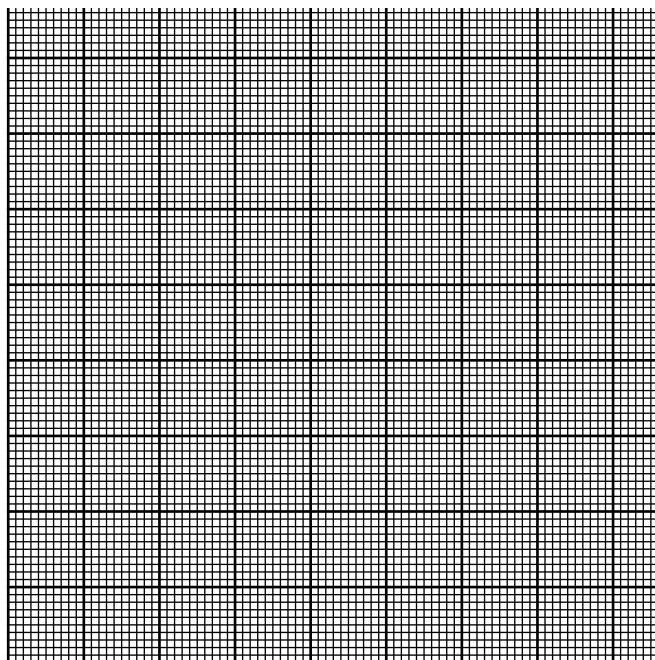
Equipment:

air track and carts of two different masses
springs
alligator clips
meter sticks
spring scales
stopwatches



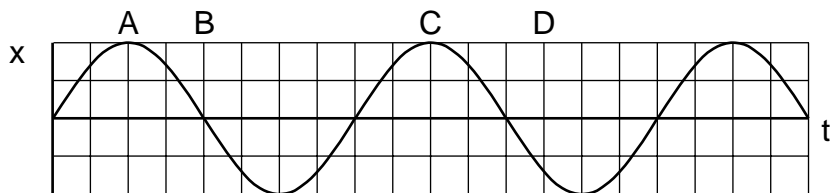
Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least two moderate amplitudes, for which the springs do not go slack, and at least one amplitude that is large enough so that they do go slack.
2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $T=2\pi\sqrt{m/k}$?
3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?
4. Test the equation $T=2\pi\sqrt{m/k}$ numerically.

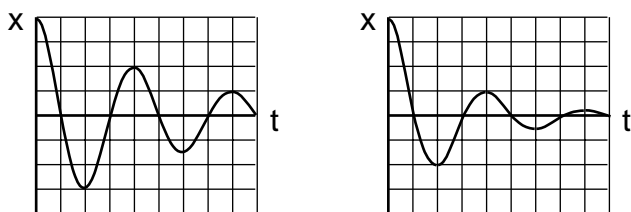


Exercise 2A: Worksheet on Resonance

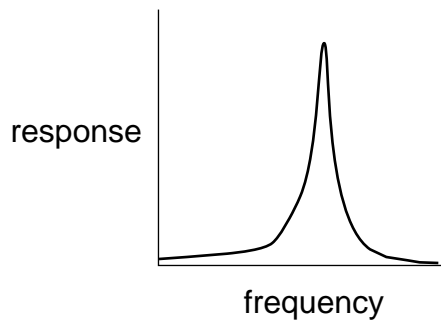
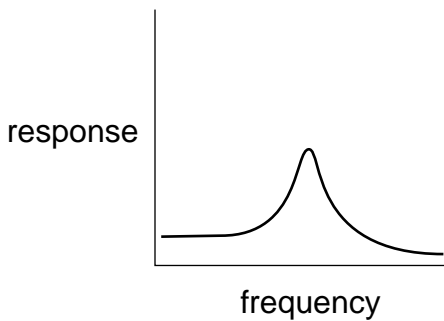
1. Compare the oscillator's energies at A, B, C, and D.



2. Compare the Q values of the two oscillators.



3. Match the $x-t$ graphs in #2 with the amplitude-frequency graphs below.



Glossary

Amplitude. The amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration.

Damping. the dissipation of a vibration's energy into heat energy, or the frictional force that causes the loss of energy

Driving force. An external force that pumps energy into a vibrating system.

Frequency. The number of cycles per second, the inverse of the period (q.v.).

Period. The time required for one cycle of a periodic motion (q.v.).

Periodic motion. Motion that repeats itself over and over.

Resonance. The tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration.

Simple harmonic motion. Motion whose x - t graph is a sine wave.

Steady state. The behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

Quality factor. The number of oscillations required for a system's energy to fall off by a factor of 535 due to damping.

Index

A

absorption of waves 65
amplitude
 defined 14
 peak-to-peak 14

C

comet 11

D

damping
 defined 24
decibel scale 23
Doppler effect 54
driving force 26

E

eardrum 26
Einstein, Albert 12
exponential decay
 defined 24

F

Fourier's theorem 73
frequency
 defined 12

G

Galileo 16

H

Halley's Comet 11
harmonics 74
Hooke's law 15

I

interference effects 70

M

motion
 periodic 12

O

overtones 74

P

period
 defined 12
 of simple harmonic motion. *See* simple harmonic motion: period of
periodic motion. *See* motion: periodic
pitch 11
principle of superposition 40
pulse
 defined 40

Q

quality factor
 defined 25

R

reflection
 of waves
 defined 62
reflection of waves 62
resonance
 defined 28

S

simple harmonic motion
 defined 15
 period of 15
standing wave 75
steady-state behavior 26
swing 26

T

timbre 74
tuning fork 14

W

work
 done by a varying force 12, 14, 16

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Chapter 2

Tacoma Narrows Bridge: Still photos and a movie of the bridge's collapse were taken by an unknown photographer.

Nimitz Freeway: Unknown photographer, courtesy of the UC Berkeley Earth Sciences and Map Library.

Brain: R. Malladi, LBNL.

Chapter 3

Painting of ocean waves: Hokusai

M100: Hubble Space Telescope.

Chapter 4

Human Cross-Section: Courtesy of the Visible Human Project, National Library of Medicine, US NIH.

Useful Data

Metric Prefixes

M-	mega-	10^6
k-	kilo-	10^3
m-	milli-	10^{-3}
μ - (Greek mu)	micro-	10^{-6}
n-	nano-	10^{-9}

(Centi-, 10^{-2} , is used only in the centimeter.)

Notation and Units

quantity	unit	symbol
distance	meter, m	$x, \Delta x$
time	second, s	$t, \Delta t$
mass	kilogram, kg	m
density	kg/m^3	ρ
force	newton, 1 N=1 kg m/s^2	F
velocity	m/s	v
acceleration	m/s^2	a
energy	joule, J	E
momentum	kg m/s	p
angular momentum	$\text{kg m}^2/\text{s}$	L
period	s	T
wavelength	m	λ
frequency	s^{-1} or Hz	f

symbol	meaning
\propto	is proportional to
\approx	is approximately equal to
\sim	on the order of

The Greek Alphabet

α	A	alpha	ν	N	nu
β	B	beta	ξ	Ξ	xi
γ	Γ	gamma	\omicron	O	omicron
δ	Δ	delta	π	Π	pi
ϵ	E	epsilon	ρ	P	rho
ζ	Z	zeta	σ	Σ	sigma
η	H	eta	τ	T	tau
θ	Θ	theta	υ	Y	upsilon
ι	I	iota	ϕ	Φ	phi
κ	K	kappa	χ	X	chi
λ	Λ	lambda	ψ	Ψ	psi
μ	M	mu	ω	Ω	omega

Conversions

Conversions between SI and other units:

1 inch	=	2.54 cm (exactly)
1 mile	=	1.61 km
1 pound	=	4.45 N
(1 kg):g	=	2.2 lb
1 gallon	=	$3.78 \times 10^3 \text{ cm}^3$
1 horsepower	=	746 W
1 kcal*	=	$4.18 \times 10^3 \text{ J}$

*When speaking of food energy, the word "Calorie" is used to mean 1 kcal, i.e. 1000 calories. In writing, the capital C may be used to indicate

1 Calorie=1000 calories.

Conversions between U.S. units:

1 foot	=	12 inches
1 yard	=	3 feet
1 mile	=	5280 ft

Earth, Moon, and Sun

body	mass (kg)	radius (km)	radius of orbit (km)
earth	5.97×10^{24}	6.4×10^3	1.49×10^8
moon	7.35×10^{22}	1.7×10^3	3.84×10^5
sun	1.99×10^{30}	7.0×10^5	

The radii and radii of orbits are average values. The moon orbits the earth and the earth orbits the sun.

Subatomic Particles

particle	mass (kg)	radius (m)
electron	9.109×10^{-31}	? – less than about 10^{-17}
proton	1.673×10^{-27}	about 1.1×10^{-15}
neutron	1.675×10^{-27}	about 1.1×10^{-15}

The radii of protons and neutrons can only be given approximately, since they have fuzzy surfaces. For comparison, a typical atom is about 10^{-9} m in radius.

Speeds of Light and Sound

speed of light	$c \approx 3.00 \times 10^8 \text{ m/s}$
speed of sound	340 m/s