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DIAGRAMMAR

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## 1. INTRODUCTION

With the advent of gauge theories it became necessary to reconsider many well-established ideas in quantum field theories. The canonical formalism, formerly regarded as the most conventional and rigorous approach, has now been abandoned by many authors. The path-integral concept cannot replace the canonical formalism in defining a theory, since path integrals in four dimensions are meaningless without additional and rather *ad hoc* renormalization prescriptions.

Whatever approach is used, the result is always that the S-matrix is expressed in terms of a certain set of Feynman diagrams. Few physicists object nowadays to the idea that diagrams contain more truth than the underlying formalism, and it seems only rational to take the final step and abandon operator formalism and path integrals as instruments of analysis.

Yet it would be very shortsighted to turn away completely from these methods. Many useful relations have been derived, and many more may be in the future. What must be done is to put them on a solid footing. The situation must be reversed: diagrams form the basis from which everything must be derived. They define the operational rules, and tell us when to worry about Schwinger terms, subtractions, and whatever other mythological objects need to be introduced.

The development of gauge theories owes much to path integrals and it is tempting to attach more than a heuristic value to path integral derivations. Although we do not rely on path integrals in this paper, one may think of expanding the exponent of the interaction Lagrangian in a Taylor series, so that the algebra of the Gaussian integrals becomes exactly identical to the scheme of manipulations with Feynman diagrams. That would leave us with the problems of giving the correct  $\epsilon$  prescription in the propagators, and to find a decent renormalization scheme.

There is another aspect that needs emphasis. From the outset the canonical operator formalism is not a perturbation theory, while diagrams certainly are perturbative objects. Using diagrams as a starting point seems therefore to be a capitulation in the struggle to go beyond perturbation theory. It is unthinkable to accept as a final goal a perturbation theory, and it is not our purpose to forward such a notion. On the contrary, it becomes more and more clear that perturbation theory is a very useful device to discover equations and properties that may hold true even if the perturbation expansion fails. There are already several examples of this mechanism: on the simplest level there is for instance the treatment of unstable particles, while if it comes to unfathomed depths the Callan-Symanzik equation may be quoted. All such treatments have in common that global properties are established for diagrams and then extrapolated beyond perturbation theory. Global properties are those that hold in arbitrary order of perturbation theory for the grand total of all diagrams entering at any given order. It is here that very naturally the concept of the global diagram enters: it is for a given order of perturbation theory, for a given number of external lines the sum of all contributing diagrams. This object, very often presented as a blob, an empty circle, in the following pages, is supposed to have a significance beyond perturbation theory. Practically all equations of the canonical formalism can be rewritten in terms of such global diagrams, thereby opening up the arsenal of the canonical formalism for this approach.

A further deficiency is related to the divergencies of the perturbation series. Traditionally it was possible to make the theory finite within the context of the canonical formalism. For instance quantum electrodynamics can be made finite by means of Pauli-Villars regulator fields, representing heavy particles with wrong metric or wrong statistics. Judicial choice of masses and coupling constants makes everything finite and gauge invariant and turns the canonical formalism into a reasonably well-behaving machine, free of objects such as  $\delta(0)$ , to name one. Unfortunately this is not the situation in the case of gauge theories. There the most suitable regulator method, the dimensional regularization scheme, is defined exclusively for diagrams, and up to now nobody has seen a way to introduce a dimensional canonical formalism or path integral. The very concept of a field, and the notion of a Hilbert space are too rigid to allow such generalizations.

The treatment outlined in the following pages is not supposed to be complete, but rather meant as a first, more or less pedagogical attempt to implement the above point of view featuring global diagrams as primary objects. The most important properties of the canonical as well as path integral formalisms are rederived: unitarity, causality, Faddeev-Popov determinants, etc. The starting point is always a set of Feynman rules succinctly given by means of a Lagrangian. No derivation of these rules is given: corresponding to any Lagrangian (with very few limitations concerning its form) the rules are simply defined.

Subsequently, Green's functions, a Hilbert space and an S-matrix are defined in terms of diagrams. Next we examine properties like unitarity and causality of the resulting theory. The basic tool for that are the cutting equations derived in the text. The use of these equations relates very closely to the classical work of Bogoliubov, and Bogoliubov's definition of causality is seen to hold. The equations remain true within the framework of the continuous dimension method: renormalization can therefore be treated *à la* Bogoliubov.

Of course, the cutting equations will tell us in general that the theory is not unitary, unless the Lagrangian from which we started satisfies certain relations. In a gauge theory moreover, the S-matrix is only unitary in a "physical" Hilbert space, which is a subspace of the original Hilbert space (the one that was suggested by the form of the Lagrangian).

To illustrate in detail the complications of gauge theory we have turned to good old quantum electrodynamics. Even if this theory lacks some of the complications that may arise in the general case it turns out to be sufficiently structured to show how everything works.

The metric used throughout the paper is

$$x_\mu = (\vec{x}, ict), \quad k^2 = \vec{k}^2 + k_4^2, \quad k_4 = ik_0 = \frac{iE}{c}.$$

The factors  $i$  in the fourth components are only there for ease of notation, and should not be reversed when taking the complex conjugate of a four-vector

$$j_\mu = (\vec{j}, ij_0), \\ j_\mu^* = (\vec{j}^*, ij_0^*) = (\vec{j}^*, -j_4^*).$$

In our Feynman rules we have explicitly denoted the relevant factors  $(2\pi)^4 i$ , but often omitted the  $\delta$  functions for energy-momentum conservation.

## 2. DEFINITIONS

### 2.1 Definition of the Feynman rules

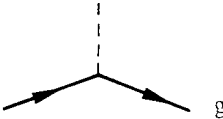
The purpose of this section is to spell out the precise form of the Feynman rules for a given Lagrangian. In principle, this is very straightforward: the propagators are defined by the quadratic part of the Lagrangian, and the rest is represented by vertices. As is well known, the propagators are minus the inverse of the operator found in the quadratic term, for example

$$\mathcal{L} = \frac{1}{2} \phi(\partial^2 - m^2)\phi \rightarrow (k^2 + m^2 - i\epsilon)^{-1},$$

$$\mathcal{L} = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \rightarrow (i\gamma k + m)^{-1} = \frac{-i\gamma k + m}{k^2 + m^2 - i\epsilon}.$$

Customarily, one derives this using commutation rules of the fields, etc. We will simply skip the derivation and define the propagator, including the  $i\epsilon$  prescription for the pole.

Similarly, vertices arise. For instance, if the interaction Lagrangian contains a term providing for the interaction of fermions and a scalar field one has

$$\mathcal{L}_I = g(\bar{\psi}\psi)\phi$$


In this and similar cases there is no difficulty in deriving the rules by the usual canonical formalism. If however derivatives, or worse non-local terms, occur in  $\mathcal{L}$ , then complications arise. Again we will short-circuit all difficulties and define our vertices, including non-local vertices, directly from the Lagrangian. Furthermore, we will allow sources that can absorb or emit particles. They are an important tool in the analysis. In the rest of this section we will try to define precisely the Feynman rules for the general case, including factors  $\pi$ , etc. Basically the recipe is the straightforward generalization of the simple cases shown above.

The most general Lagrangian to be discussed here is

$$\mathcal{L}(x) = \psi_i^*(x)V_{ij}\psi_j(x) + \frac{1}{2} \phi_i(x)W_{ij}\phi_j(x) + \mathcal{L}_I(\psi^*, \psi, \phi). \quad (2.1)$$

The  $\psi_i$  and  $\phi_i$  denote sets of complex and real fields that may be scalar, spinor, vector, tensor, etc., fields. The index  $i$  stands for any spinor, Lorentz, isospin, etc., index.  $V$  and  $W$  are matrix operators that may contain derivatives, and whose Fourier transform must have an inverse. Furthermore, these inverses must satisfy the Källén-Lehmann representation, to be discussed later. The interaction Lagrangian  $\mathcal{L}_I(\psi^*, \psi, \phi)$  is any polynomial in certain coupling constants  $g$  as well as the fields. This interaction Lagrangian is allowed to be non-local, i.e. not only depend on fields in the point  $x$ , but also on fields at other space time points  $x'$ ,  $x''$ , ... . The coefficients in the polynomial expansion may be functions of  $x$ . The explicit form of a general term in  $\mathcal{L}_I(x)$  is

$$\int d_4 x_1 d_4 x_2, \dots, \alpha_{i_1 i_2 \dots}(x, x_1, x_2, \dots) \quad (2.2)$$

$$\psi_{i_1}^*(x_1), \dots, \psi_{i_m}(x_m), \dots, \phi_{i_n}(x_n), \dots$$

The  $\alpha$  may contain any number of differential operators working on the various fields.

Roughly speaking propagators are defined to be minus the inverse of the Fourier transforms of  $V$  and  $W$ , and vertices as the Fourier transforms of the coefficients  $\alpha$  in  $\mathcal{L}_I$ .

The action  $S$  is defined by

$$iS = i \int d_4 x \mathcal{L}(x) . \quad (2.3)$$

In  $\mathcal{L}$  we make the replacement

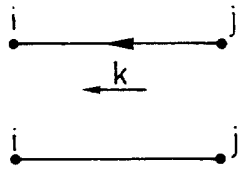
$$\begin{aligned} \psi_i(x) &= \int d_4 k a_i(k) e^{ikx} , \\ \psi_i^*(x) &= \int d_4 k b_i(k) e^{ikx} , \\ \phi_i(x) &= \int d_4 k c_i(k) e^{ikx} , \\ \alpha_{i_1 i_2 \dots}(x, x_1, x_2, \dots) &= \\ &= \int d_4 k d_4 k_1 d_4 k_2, \dots, e^{ikx + ik_1(x-x_1) + ik_2(x-x_2) + \dots} \bar{\alpha}_{i_1 \dots}(k, k_1, k_2, \dots) . \end{aligned}$$

The action times  $i$  takes the form

$$\begin{aligned} (2\pi)^4 i b(k) \bar{V}_{ij}(k) a_j(k) + \frac{1}{2} (2\pi)^4 i c_i(k) \bar{W}_{ij}(k) c_j(k) \\ + \dots + (2\pi)^4 i \delta_4(k + k_1 + \dots) \bar{\alpha}_i(k, k_1, k_2, \dots) \times \\ \times b_{i_1}(k_1), \dots, a_{i_m}(k_m), \dots, c_{i_n}(k_n) + \dots , \quad (2.4) \end{aligned}$$

each term integrated over the momenta involved. The  $\bar{V}$  and  $\bar{W}$  contain a factor  $ik_n$  (or  $-ik_n$ ) for every derivative  $\partial/\partial x_\mu$  acting to the right (left) in  $V$  and  $W$ , respectively. The  $\bar{\alpha}$  contain a factor  $ik_{j\mu}$  for every derivative  $\partial/\partial x_{j\mu}$  acting on a field with argument  $x_j$ .

The propagators are defined to be:

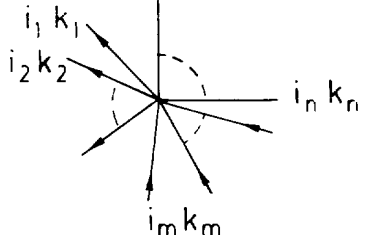


$$\Delta_{Fij}(k) = - \frac{1}{(2\pi)^4 i} \left[ \bar{V}^{-1}(k) \right]_{ij} , \quad (2.5)$$

$$\Delta_{Fij}(k) = - \frac{1}{(2\pi)^4 i} \left[ \frac{1}{2} \bar{W}(k) + \frac{1}{2} \tilde{\bar{W}}(-k) \right]_{ij}^{-1} .$$

Here  $\tilde{W}$  is  $\bar{W}$  reflected, i.e.  $\tilde{W}_{ij} = \bar{W}_{ji}$ . In the rare case of real fermions the propagator must be minus the inverse of the antisymmetric part of  $W$ . Furthermore, there is the usual  $i\epsilon$  prescription for the poles of these propagators. The momentum  $k$  in Eqs. (2.5) is the momentum flow in the direction of the arrow.

The definition of the vertices is:



$$(2\pi)^{4i} \sum_{\{1\dots m-1\}} \sum_{\{m\dots n-1\}} \sum_{\{n\dots\}} (-1)^P \times$$

$$\times \alpha_{i_1\dots i_n}(k, k_1, k_2, \dots) \delta_4(k + k_1 + \dots) \quad (2.6)$$

The summation is over all permutations of the indices and momenta indicated. The momenta are taken to flow inwards. Any field  $\psi^*$  corresponds to a line with an arrow pointing outwards; a field  $\psi$  gives an opposite arrow. The  $\phi$  fields give arrow-less lines. The factor  $(-1)^P$  is only of importance if several fermion fields occur. All fermion fields are taken to anticommute with all other fermion fields. There is a factor -1 for every permutation exchanging two fermion fields.

The coefficients  $\alpha$  will often be constants. Then the sum over permutations results simply in a factor. It is convenient to include such factors already in  $f_I$ ; for instance

$$f_I(x) = \frac{1}{3!6!} g \psi^*(x)^3 \psi(x)^6$$

gives as vertex simply the constant  $g$ .

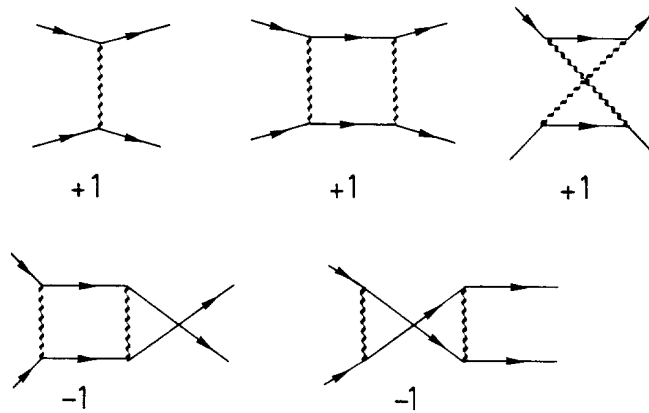
As indicated, the coefficients  $\alpha$  may be functions of  $x$ , corresponding to some arbitrary dependence on the momentum  $k$  in (2.6). This momentum is not associated with any of the lines of the vertex. If we have such a  $k$  dependence, i.e. the coefficient  $\alpha$  is non-zero for some non-zero value of  $k$ , then this vertex will be called a source. Sources will be indicated by a cross or other convenient notation as the need arises.

A diagram is obtained by connecting vertices and sources by means of propagators in accordance with the arrow notations. Any diagram is provided with a combinatorial factor that corrects for double counting in case identical particles occur. The computation of these factors is somewhat cumbersome; the recipe is given in Appendix A.

Further, if fermions occur, diagrams are provided with a sign. The rule is as follows:

- i) there is a minus sign for every closed fermion loop;
- ii) diagrams that are related to each other by the omission or addition of boson lines have the same sign;
- iii) diagrams related by the exchange of two fermion lines, internal or external, have a relative minus sign.

EXAMPLE 2.1.1 Electron-electron scattering in quantum-electrodynamics. Some diagrams and their relative sign are given in the figure. The fourth and fifth diagrams are really the same diagram and should not be counted separately:



This raises the question of the recognition of topologically identical diagrams. In Appendix A some considerations pertinent to the topology of quantum electrodynamics are presented.

## 2.2 A simple theorem; some examples

THEOREM Diagrams are invariant for the replacement

$$\psi_i^* \rightarrow \psi_k^* X_{ki}$$

in the Lagrangian (1.1), where X is any matrix that may include derivatives but must have an inverse.

The theorem is trivial to prove. Any oriented propagator obtains a factor  $X^{-1}$  that cancels against the factor X occurring in the vertices. It is left to the reader to generalize this theorem to include transformations of the  $\psi$ 's and  $\phi$ 's.

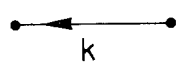
Some examples illustrating our definitions are in order.

EXAMPLE 2.2.1 Charged scalar particles with  $\psi^4$  interaction

$$\mathcal{L}(x) = \psi^*(\partial^2 - m^2)\psi + \frac{g}{2! 2!} (\psi^*\psi)^2 + J^*(x)\psi + \psi^*J(x) . \quad (2.7)$$

Only  $\psi^*(x)$  and  $\psi(x)$  in the same point x occur, and in accordance with the description given at the beginning of the previous section we have here a local Lagrangian. The functions  $J(x)$  and  $J^*(x)$  are source functions.

Propagator:



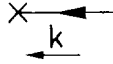
$$\Delta_F = \frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 - i\epsilon} .$$

Vertex:

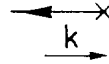


$(2\pi)^4 ig$  ( $\delta$  function for energy-momentum conservation is understood.)

Sources:



$(2\pi)^4 iJ^*(k)$



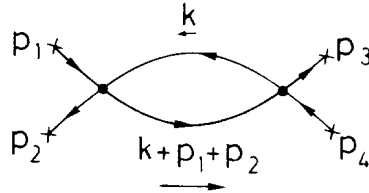
$(2\pi)^4 iJ(k)$

The functions  $J(k)$ ,  $J^*(k)$  are the Fourier transforms of  $J(x)$  and  $J^*(x)$

$$J(x) = \int d_4k e^{-ikx} J(k)$$

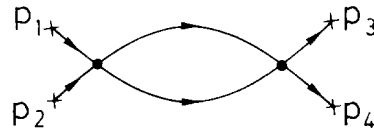
Note that  $k$  is the momentum flowing from the line into the source.

Some diagrams:



$$\frac{J(-p_1)}{p_1^2 + m^2 - i\epsilon} \frac{J^*(-p_2)}{p_2^2 + m^2 - i\epsilon} \frac{J^*(p_3)}{p_3^2 + m^2 - i\epsilon} \frac{J(p_4)}{p_4^2 + m^2 - i\epsilon} \delta_4(p_1 + p_2 - p_3 - p_4) \times$$

$$\times g^2 \int d_4k \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(k + p_1 + p_2)^2 + m^2 - i\epsilon}$$



$$\frac{J(-p_1)}{p_1^2 + m^2 - i\epsilon} \frac{J(-p_2)}{p_2^2 + m^2 - i\epsilon} \frac{J^*(p_3)}{p_3^2 + m^2 - i\epsilon} \frac{J^*(p_4)}{p_4^2 + m^2 - i\epsilon} \delta_4(p_1 + p_2 - p_3 - p_4) \times$$

$$\times \frac{1}{2} g^2 \int d_4k \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(k + p_1 + p_2)^2 + m^2 - i\epsilon}$$

The factor  $1/2$  in the second case is the combinatorial factor occurring because of the identical particles in the intermediate state.

EXAMPLE 2.2.2 Free real vector mesons

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu W_\nu - \partial_\nu W_\mu)^2 - \frac{1}{2} m^2 W_\mu^2 \quad (2.8)$$



In terms of four real fields  $\phi_\alpha \equiv W_\alpha$ ,  $\alpha = 1, \dots, 4$ ,

$$\mathcal{L} = \frac{1}{2} \phi_\alpha \left[ -(\vec{\partial}_\mu \vec{\partial}_\mu + m^2) \delta_{\alpha\beta} + \vec{\partial}_\alpha \vec{\partial}_\beta \right] \phi_\beta \equiv \frac{1}{2} \phi_\alpha V_{\alpha\beta} \phi_\beta .$$

The matrix  $V$  in momentum space is

$$-\delta_{\alpha\beta}(k^2 + m^2) + k_\alpha k_\beta .$$

The vector meson propagator becomes:

$$\begin{array}{c} i \quad \quad \quad j \\ \bullet \text{-----} \bullet \\ \quad \quad \quad k \end{array} \quad \Delta_{F\alpha\beta} = - \frac{1}{(2\pi)^4 i} (V^{-1})_{\alpha\beta} = \frac{1}{(2\pi)^4 i} \frac{\delta_{\alpha\beta} + k_\alpha k_\beta / m^2}{k^2 + m^2 - i\epsilon} . \quad (2.9)$$

Indeed

$$\begin{aligned} & \left[ \delta_{\alpha\beta}(k^2 + m^2) - k_\alpha k_\beta \right] \frac{\delta_{\beta\lambda} + k_\beta k_\lambda / m^2}{k^2 + m^2 - i\epsilon} = \\ & = \frac{1}{k^2 + m^2 - i\epsilon} \left[ \delta_{\alpha\lambda}(k^2 + m^2) - k_\alpha k_\lambda + k_\alpha k_\lambda \frac{k^2 + m^2}{m^2} - k_\alpha k_\lambda \frac{k^2}{m^2} \right] = \delta_{\alpha\lambda} . \end{aligned}$$

EXAMPLE 2.2.3 Electron in an external electromagnetic field  $A_\mu$

$$\mathcal{L} = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi + ieA_\mu \bar{\psi} \gamma^\mu \psi . \quad (2.10)$$

Note that  $\bar{\psi} = \psi^* \gamma^4$ . Because of our little theorem the matrix  $\gamma^4$  can be omitted in giving rules for the diagrams. Then:

$$\begin{array}{c} \bullet \text{-----} \bullet \\ \quad \quad \quad k \\ \longleftarrow \\ \quad \quad \quad p-k \\ \swarrow \quad \searrow \\ \times \quad \quad \quad p \\ \quad \quad \quad \searrow \end{array} \quad \Delta_F = \frac{1}{(2\pi)^4 i} \frac{-i\gamma k + m}{k^2 + m^2 - i\epsilon} . \quad (2.11)$$

$$-(2\pi)^4 e A_\mu(k) \gamma^\mu .$$

As a further application of our theorem we may substitute  $\psi \rightarrow (-\gamma^\nu \partial_\nu + m)\psi$  to give

$$\mathcal{L} = \bar{\psi}(\partial^2 - m^2)\psi + ieA_\mu \bar{\psi} \gamma^\mu (-\gamma^\nu \partial_\nu + m)\psi .$$

This gives the equivalent rules:

$$\begin{array}{c} \bullet \text{-----} \bullet \\ \swarrow \quad \searrow \\ \times \quad \quad \quad p \\ \quad \quad \quad \searrow \end{array} \quad \Delta_F = \frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 - i\epsilon}$$

$$-(2\pi)^4 e A_\mu(k) \gamma^\mu (-i\gamma^\nu p_\nu + m) .$$

### 2.3 Internal consistency

Two points need to be investigated. First, the separation in real and complex fields is really arbitrary, because for any complex field  $\phi$  one can always write

$$\phi = \frac{1}{\sqrt{2}} (A + iB) , \quad \phi^* = \frac{1}{\sqrt{2}} (A - iB) ,$$

where A and B are real fields. The question is whether the results will be independent of the representation chosen. This indeed is the case, and may be best explained by considering an example:

$$\begin{aligned} \mathcal{L} &= \phi_i^* V_{ij} \phi_j + J_i^* \phi_i + \phi_i^* J_i \\ \begin{array}{c} i \longleftarrow j \\ \times \longleftarrow i \\ \longleftarrow \times \end{array} & \begin{array}{l} - \frac{1}{(2\pi)^4 i} (V^{-1})_{ij} \\ (2\pi)^4 i J_i^* \\ (2\pi)^4 i J_i . \end{array} \end{aligned}$$

The diagram containing two sources is:

$$\begin{array}{c} i \longleftarrow j \\ \times \longleftarrow \times \end{array} \quad - (2\pi)^4 i J_i^* (V^{-1})_{ij} J_j .$$

On the other hand, let us write  $\phi = (1/\sqrt{2})(A + iB)$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} A_i V_{ij} A_j + \frac{1}{2} B_i V_{ij} B_j + \frac{i}{2} A_i V_{ij} B_j - \frac{i}{2} B_i V_{ij} A_j + \\ &+ \frac{1}{\sqrt{2}} (J_i^* A_i + i J_i^* B_i + A_i J_i - i B_i J_i) . \end{aligned}$$

Defining the real field  $X_i$

$$X = \begin{pmatrix} A \\ B \end{pmatrix} ,$$

we have

$$\mathcal{L} = \frac{1}{2} X_i W_{ij} X_j + F_i^* X_i + F_i X_i$$

with

$$W = \begin{pmatrix} V^s & iV^a \\ -iV^a & V^s \end{pmatrix} , \quad F^* = \frac{1}{\sqrt{2}} \begin{pmatrix} J_i^* \\ iJ_i^* \end{pmatrix} , \quad F = \frac{1}{\sqrt{2}} \begin{pmatrix} J_i \\ -iJ_i \end{pmatrix} ,$$

where the superscripts s and a denote the symmetrical and the antisymmetrical part, respectively.

Let  $Y$  now be the inverse of  $V$ . Thus

$$VY = 1, \quad YV = 1.$$

Writing  $V = V^S + V^a$ ,  $Y = Y^S + Y^a$  one finds, comparing the reflected  $VY$  with  $YV$

$$\begin{aligned} V^S Y^S + V^a Y^a &= 1, \\ V^S Y^a + V^a Y^S &= Y^a V^S + Y^S V^a = 0. \end{aligned}$$

The inverse of the matrix  $W$  is therefore

$$W^{-1} = \begin{pmatrix} Y^S & iY^a \\ -iY^a & Y^S \end{pmatrix},$$

where  $Y = V^{-1}$ . The source-source diagram becomes

$$-(2\pi)^4 i F^* W^{-1} F = -(2\pi)^4 i \frac{1}{2} 2J^* (Y^S + Y^a) J = -(2\pi)^4 i J^* V^{-1} J$$

as before. Clearly the separation into real and imaginary parts amounts to separation into symmetric and antisymmetric parts of the propagator.

The second point to be investigated is the question of separation of the quadratic, i.e. propagator defining, part in  $\mathcal{L}$ . Thus let there be given the Lagrangian

$$\mathcal{L} = \phi_i^* V_{ij} \phi_j + \phi_i^* V'_{ij} \phi_j.$$

One can either say that one has a propagator  $-(V + V')^{-1}$  for the  $\phi$ -field, or alternatively a propagator  $-(V)^{-1}$  and a vertex:



However, these two cases give the same result. Summing over all possible insertions of the vertex  $V'$  one finds:

$$\begin{aligned} & \bullet \leftarrow \bullet + \bullet \leftarrow \bullet \leftarrow \bullet + \dots \\ & -\frac{1}{V} + \left(-\frac{1}{V}\right) V' \left(-\frac{1}{V}\right) + \dots = -\frac{1}{V} \frac{1}{1 + V'/V} = -\frac{1}{V + V'}. \end{aligned}$$

This demonstrates the internal consistency of our scheme of definitions. We leave it as an exercise to the reader to verify that combinational factors check if one makes the replacement  $\phi \rightarrow (A + iB)/\sqrt{2}$ , for instance for the diagrams:



### 2.4 Definition of the Green's functions

Let there be given a general Lagrangian of the form (2.1). Corresponding to any field we introduce source terms (we will need many, but write only one for each field)

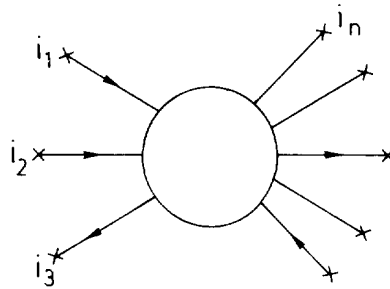
$$\mathcal{L} \rightarrow \mathcal{L} + J_i^* \psi_i + \psi_i^* J_i + \phi_i K_i .$$

According to the previous rules such terms give rise to the following vertices:

$$\begin{aligned} \times \longleftarrow & (2\pi)^4 iJ^*(k) , \\ \longrightarrow \times & (2\pi)^4 iJ(k) , \\ \times \longleftarrow & (2\pi)^4 iK(k) , \end{aligned}$$

where the  $k$  dependence implies Fourier transformation.

Remember now that the interaction Lagrangian was a polynomial in some coupling constants. For a given order in these coupling constants we may consider the sum of all diagrams connecting  $n$  sources. All  $n$  sources are to be taken different, because we want



to be able to vary the momenta independently. Each of these diagrams, and *a fortiori* their sum, will be of the form

$$J_{i_1}(k_1) J_{i_2}'(k_2) \dots K_{i_n}(k_n) G_{i_1 \dots i_n}(k_1, \dots, k_n) .$$

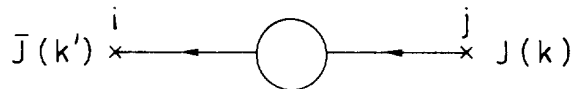
The function  $G_i$  will be called the  $n$ -point Green's function for the given external line configuration for the order in the coupling constants specified. Factors  $(2\pi)^4 i$  from the source vertices are included. The  $k_i$  denote the momenta flowing from the sources into the diagrams. The propagators that connect to the sources are included in this definition.

The first example of Section 2.2 shows some diagrams contributing to the second-order four-point Green's functions.

### 2.5 Definition of the S-matrix; some examples

Roughly speaking the S-matrix obtains from the Green's functions in two steps: (i) the momenta of the external line are put on mass shell, and (ii) the sources are normalized such that they correspond to emission or absorption of one particle. Both these statements are somewhat vague, and we must precise them, but they reflect the essential physical content of the reasoning below.

Consider the diagrams connecting two sources:



The corresponding expression is

$$\bar{J}_i(k') G_{ij}(k, k') J_j(k) . \quad (2.12)$$

The two-point Green's function will in general have a pole (or possibly many single poles) at some value  $-M^2$  of the squared four-momentum  $k_\nu$ . If there is no pole there will be no corresponding S-matrix element; such will be the case if a particle becomes unstable because of the interactions. At the pole the Green's function will be of the form

$$G_{ij}(k, k') = (2\pi)^4 i \delta_4(k + k') \frac{K_{ij}(k)}{k^2 + M^2} \quad \text{at} \quad k^2 \rightarrow -M^2 .$$

The matrix residue  $K_{ij}$  can be a function of the components  $k_\mu$ , with the restriction that  $k^2 = -M^2$ .

First we will treat the currents for emission of a particle, corresponding to incoming particles of the S-matrix. Define a new set of currents  $J_i^{(a)}$ , one for every non-zero eigenvalue of  $K$ , which are mutually orthogonal and eigenstates of the matrix  $K(k)$

$$\begin{aligned} J_i^{(a)*} J_i^{(b)} &= 0 \quad \text{if} \quad a \neq b , \\ K_{ij}(k) J_j^{(a)}(k) &= f^a(k) J_i^{(a)}(k) , \end{aligned} \quad (2.13)$$

and normalized such that

$$\left[ J_j^{(a)}(k) \right]^* K_{ji}(k) J_j^{(a)}(k) = \begin{cases} 1 & \text{for integer spin} \\ \frac{k_0}{m} & \text{for half-integer spin} . \end{cases} \quad (2.14)$$

This is possible only if all eigenvalues of  $K$  are positive. In the case of negative eigenvalues normalization is done with minus the right-hand side of Eq. (2.14).

The sources thus defined are the properly normalized sources for emission of a particle or antiparticle [the latter follows from considering  $\tilde{K}(-k)$ ]. The properly normalized sources for absorption of a particle or antiparticle follow by considering Eqs. (2.13) and (2.14) but with  $k$  replaced by  $-k$  in  $K$ .

The above procedure defines the currents up to a phase factor. We must take care that the phase factor for the emission of a certain particle agrees with that for absorption of that same particle. This is fixed by requiring that the two-point Green's function provided with such sources has precisely the residue 1 (or  $k_0/m$ ) for  $k^2 = -m^2$ .

The matrix elements of the matrix  $S'$  (almost, but not exactly the S-matrix, see below) for  $n$  ingoing particles and  $m$  outgoing particles are defined by

$$\begin{aligned} \langle p_1 b_1, \dots, p_m b_m | S' | k_1 a_1, \dots, k_n a_n \rangle &\equiv \prod_{r=1}^n \lim_{k_r^2 \rightarrow -M_r^2} J_{i_r}^{(a_r)}(k_r) (k_r^2 + M_r^2) \times \\ &\times \prod_{s=1}^m \lim_{p_s^2 \rightarrow -M_s^2} J_{j_s}^{(b_s)}(p_s) (p_s^2 + M_s^2) \times G_{i_1 \dots j_m}(k_1, \dots, k_n, -p_1, \dots, -p_m) . \end{aligned} \quad (2.15)$$

The energies  $k_{1_0}, \dots, k_{n_0}$  and  $p_{1_0}, \dots, p_{m_0}$  are all positive. The minus sign for the momenta  $p$  in the Green's function appears because in the matrix element the momenta of the outgoing particles are taken to be flowing out, while in our Green's functions the convention was that all momenta flow inwards.

EXAMPLE 2.5.1 Scalar particles. Near the physical mass pole the two-point Green's function will be such that

$$\frac{G(k, k')}{(2\pi)^4 i \delta_4(k - k')} = \frac{1}{Z^2(k^2 + M^2)} . \quad (2.16)$$

For any  $k$  the properly normalized external current is  $\tilde{J} = Z$ , and the prescription to find  $S'$ -matrix elements with external scalar particles is

$$\begin{aligned} \langle p_1, \dots, p_m | S' | k_1, \dots, k_n \rangle &= \prod_{r=1}^n \lim_{k_r^2 \rightarrow -M^2} Z(k_r^2 + M^2) \times \\ &\times \prod_{s=1}^m \lim_{p_s^2 \rightarrow -M^2} Z(p_s^2 + M^2) G(k_1, \dots, k_n, -p_1, \dots, -p_m) . \end{aligned} \quad (2.17)$$

EXAMPLE 2.5.2 Fermions as in QED. Near the physical mass pole the two-point Green's function will be such that

$$\frac{G_{ij}(k, k')}{(2\pi)^4 i \delta_4(k - k')} = \frac{1}{Z^2(i\gamma k + M)} = \frac{K_{ij}(k)}{k^2 + M^2} , \quad (2.18)$$

with

$$K_{ij}(k) = \frac{1}{Z^2} (-i\gamma k + M)_{ij} . \quad (2.19)$$

A set of eigenstates of this matrix is provided by the solutions  $u^a(k)$  of the Dirac equation

$$(i\gamma k + M)u = 0 , \quad u^{a*} u^a = 1 , \quad a = 1, 2 . \quad (2.20)$$

Because of the normalization condition Eq. (2.14), we must take for the currents  $J^{(a)}(k)$

$$J^{(a)}(k) = u^a(k) \frac{Z}{2M} \sqrt{2k_0} , \quad (2.21)$$

with the Dirac spinors normalized to 1, see Eq. (2.20). Note that the factor  $1/2M$  cancels upon multiplication of this source with the propagator numerator (2.19).

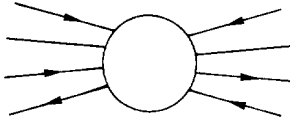
The antiparticle emission currents are obtained by considering  $\tilde{K}(-k)$ . The solutions are the antiparticle spinors

$$\bar{u}^a(k) , \quad a = 3, 4 .$$

For outgoing antiparticles and particles  $K(-k)$  and  $\tilde{K}(k)$  must be considered to obtain the proper absorption currents. The solutions are the spinors  $u^a(k)$ ,  $a = 3, 4$  and  $\bar{u}^a(k)$ ,  $a = 1, 2$ . The phase condition  $\bar{u}^a K u^a = 2M$  dictates a minus sign for the source for emission of an antiparticle (see Appendix A).

Let us now complete the S-matrix definition. The prescription given above results in zero when applied to the two-point function, because there will be two factors  $k^2 + M^2$  (one for every source). Thus we evidently do not obtain exactly the S-matrix that one has in such cases. This discrepancy is related to the treatment of the over-all  $\delta$  function of energy-momentum conservation, when passing from S-matrix elements to transition probabilities. Anyway, to get the S-matrix we must also allow lines where particles go through without any interaction, and associate a factor 1 with such lines. These particles must, of course, have a mass as given by the pole of their propagator.

Matrix elements of the S-matrix, including possible lines going straight through, will be denoted by graphs with external lines that have no terminating cross. The convention is: left are incoming particles, right outgoing. Energy flows from left to right. The direction of the arrows is of course not related to the direction of the energy flow.



We emphasize again that the above S-matrix elements include diagrams containing interaction free lines. For instance, the diagram shown is included in the 3-particle-in/3-particle-out S-matrix element.



The definition of the S-matrix given above applies if there is no gauge symmetry. For gauge theories some of the propagators correspond to "ghost" particles that are assumed not to have physical relevance. In defining the S-matrix, the sources must be restricted to emit or absorb only physical particles. Such sources will be called physical sources and have to be defined in the precise context of the gauge symmetry. To show in such cases that the S-matrix is unitary requires then special effort.

## 2.6 Definition of $S^\dagger$

The matrix elements of the matrix  $S^\dagger$  are defined as usual by

$$\langle \alpha | S^\dagger | \beta \rangle = \langle \beta | S | \alpha \rangle^* , \quad (2.22)$$

where the complex conjugation implies also the replacement  $i\epsilon \rightarrow -i\epsilon$  in the propagators.

The matrix elements of  $S^\dagger$  can also be obtained in another way. In addition to the Lagrangian  $\mathcal{L}$  defining  $S$ , consider the conjugated Lagrangian  $\mathcal{L}^\dagger$ . The conjugated Lagrangian  $\mathcal{L}^\dagger$  is obtained from  $\mathcal{L}$  by complex conjugation and reversal of the order

of the fields. The latter is only relevant for fermions. This  $\mathcal{L}^\dagger$  may be used to define another S-matrix; let  $\bar{S}$  denote the matrix obtained in the usual way from  $\mathcal{L}^\dagger$ , however with the opposite sign for the  $i\epsilon$  in the propagators and also the replacement  $i \rightarrow -i$  in the notorious factors  $(2\pi)^4 i$ . We claim that the matrix elements of  $\bar{S}$  are equal to those of  $S^\dagger$ . In formula we get

$$\langle \alpha | S(\mathcal{L}, i)^\dagger | \beta \rangle = \langle \alpha | S(\mathcal{L}^\dagger, -i) | \beta \rangle .$$

The proof rests mainly on the observation that an incoming particle source is obtained by considering [notation of Section 2.5, Eq. (2.13)]:

$$\times \longrightarrow \qquad K_{ij}(k) J_j(k)$$

and the complex conjugate of an outgoing particle source by study of:

$$\longrightarrow \times \qquad J_i(-k) K_{ij}^*(k) ,$$

or, equivalently,

$$\tilde{K}_{ij}^*(-k) J_j(k) ,$$

where  $\tilde{K}_{ij} = K_{ji}$ . This is indeed what corresponds to complex conjugation of the propagator defining part of  $\mathcal{L}^\dagger$

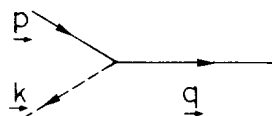
$$\left[ \phi_i^* V_{ij}(\partial) \phi_i \right]^* = \phi_i^* V_{ij}^\dagger(-\partial) \phi .$$

Note the change of sign of the derivative: what worked to the right works now to the left, which implies a minus sign.  $V^\dagger$  is obtained by transposition and complex conjugation of  $V$ . Clearly complex conjugation and exchange of incoming and outgoing states corresponds to the use of  $V^\dagger$  instead of  $V$ .

EXAMPLE 2.6.1 Fermion coupled to complex scalar field

$$\mathcal{L} = -\bar{\psi}(\gamma\partial + m)\psi + \phi^*(\partial^2 - m^2)\phi + g\bar{\psi}(1 + \gamma^5)\psi\phi^* .$$

The lowest order S-matrix element is:



$$\langle q | S | p, k \rangle = (2\pi)^4 i g \delta_4(p + k - q) \bar{u}(q) (1 + \gamma^5) u(p) \sqrt{4q_0 p_0} .$$

According to Eq. (2.22)

$$\langle p, k | S^\dagger | q \rangle = -(2\pi)^4 i g^* \delta_4(p + k - q) \bar{u}(p) (1 - \gamma^5) u(q) \sqrt{4q_0 p_0} .$$

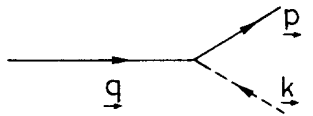
There is a minus sign because  $\gamma^5 \gamma^4 = -\gamma^4 \gamma^5$ .

Consider now  $\mathcal{L}^\dagger$

$$\mathcal{L}^\dagger = -\bar{\psi}(\gamma\partial + m)\psi + \phi^*(\partial^2 - m^2)\phi + g^*\bar{\psi}(1 - \gamma^5)\psi\phi .$$



There are minus signs because of  $\gamma^4, \gamma^5$  exchange and  $\partial \rightarrow -\partial$  changes except  $\partial_4 \rightarrow \partial_4$ . Including the sign change for  $(2\pi)^4 i$ , we obtain for  $\bar{S}$ :

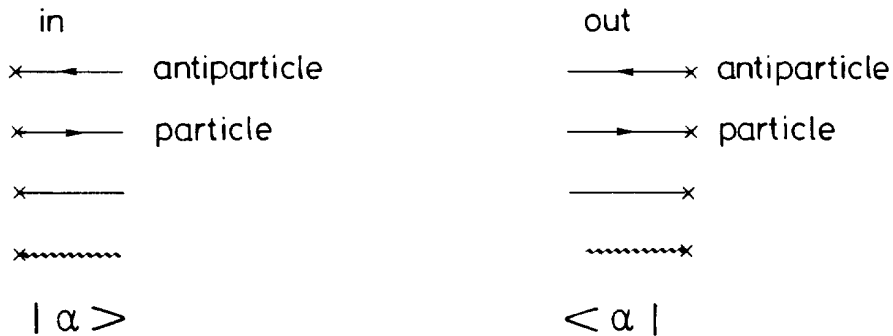


$$\langle p, k | \bar{S} | q \rangle = -(2\pi)^4 i g^* \delta^4(p+k-q) \bar{u}(p)(1-\gamma^5)u(q)\sqrt{4p_0q_0},$$

which equals the result for  $S^\dagger$  found above.

To summarize, the matrix elements of  $S^\dagger$  can be obtained either directly from their definition (2.22), or by the use of different Feynman rules. These new rules can be obtained from the old ones by reversing all arrows in vertices and propagators and replacing all vertex functions and propagators by their complex conjugate (for the propagators this means using the Hermitian conjugate propagators). Also, the factors  $(2\pi)^4 i$  and the  $i\epsilon$  terms in the propagators are to be complex conjugated. The in- and out-state source functions are defined by the usual procedure, involving now the Hermitian conjugate propagators.

Let us finally, for the sake of clarity, formulate the definition of ingoing and outgoing states in the diagram language for both the old and the new rules:



### 2.7 Definition of transition probabilities; cross-sections and lifetimes

The S-matrix elements are the transition amplitudes of the theory. The probability amplitudes are defined by the absolute value squared of these amplitudes. Conservation of probability requires that the S-matrix be unitary

$$\sum_{\beta} |\langle \beta | S | \alpha \rangle|^2 = \sum_{\beta} \langle \alpha | S^\dagger | \beta \rangle \langle \beta | S | \alpha \rangle = 1 . \quad (2.23)$$

This property will be true only if the diagrams satisfy certain conditions, and we will investigate this in Section 6.

In the usual way, lifetimes and cross-sections can be deduced from the transition probabilities.

Consider the decay of particle  $\alpha$  into particles 1, 2, 3, ..., n. The decay width  $\Gamma$  (= inverse lifetime  $\tau$ ) is

$$\frac{1}{\tau} = \Gamma = \int \frac{d_3 p_1}{2p_{10}(2\pi)^3}, \dots, \frac{d_3 p_n}{2p_{n0}(2\pi)^3} \frac{\delta_4(p_\alpha - p_1 - p_2 \dots - p_n)}{2p_{\alpha_0}(2\pi)^4} \times \\ \times \langle \alpha | M^\dagger | 1, 2, \dots, n \rangle \langle 1, 2, \dots, n | M | \alpha \rangle, \quad (2.24)$$

where the matrix elements of  $M$  (and  $M^\dagger$ ) are those of  $S$  (and  $S^\dagger$ ) without the energy-momentum conservation  $\delta$  function

$$\langle \beta | M | \alpha \rangle = \frac{\langle \beta | S | \alpha \rangle}{\delta_4(p_\alpha - p_\beta)}. \quad (2.25)$$

The  $p_{\alpha_0}, \dots, p_{n_0}$  are the energies of particles  $\alpha, 1, 2, \dots, n$ . Dividing  $\Gamma$  by  $\hbar = 6.587 \times 10^{22} \text{ MeV} \cdot \text{sec}$  gives  $\Gamma$  in  $\text{sec}^{-1}$ , provided all was computed in natural units ( $\hbar = c = 1$ ) with the MeV as the only unit left.

Next, consider the scattering of a particle  $\alpha$  on a particle  $\beta$  at rest giving rise to a final state with  $n$  particles. The cross-section  $\sigma$  for this process is

$$\sigma = \int \frac{d_3 p_1}{2p_{10}(2\pi)^3}, \dots, \frac{d_3 p_n}{2p_{n0}(2\pi)^3} \frac{p_{\alpha_0}}{|\vec{p}_\alpha|} \frac{\delta_4(p_\alpha + p_\beta - p_1 - p_2 \dots - p_n)}{4p_{\alpha_0}p_{\beta_0}(2\pi)^4} \times \\ \times \langle \alpha, \beta | M^\dagger | 1, 2, \dots, n \rangle \langle 1, 2, \dots, n | M | \alpha, \beta \rangle. \quad (2.26)$$

Here  $\vec{p}_\alpha$  is the three-momentum of the incoming particle  $\alpha$  in the rest system of the  $\beta$  particle (target particle). Multiplying by  $(\hbar c)^2 = (1.9732 \times 10^{-11} \text{ MeV} \cdot \text{cm})^2$ , one obtains  $\sigma$  in  $\text{cm}^2$ .

### 3. DIAGRAMS AND FUNCTIONAL INTEGRALS

In all proofs we will rely only on combinatorics of diagrams. However, the rules and definitions may look somewhat *ad hoc*, and the purpose of this section is to show a formal equivalence with certain integral expressions.

Imagine a world with only a finite number of space-time points  $x_\mu^a$ ,  $a = 1, \dots, N$ ;  $\mu = 1, \dots, 4$ . For simplicity, we consider only the case of real boson fields. The action  $S$  is now

$$S = \frac{1}{2} \sum_{a,b} \phi_i(x^a) W_{ij}^{ab} \phi_j(x^b) + \sum_a \mathcal{L}_I(x_a, \phi) , \quad (3.1)$$

where  $W$  is taken to be symmetric. Derivatives occur as differences. The diagram rules corresponding to this action are as given in Section 2.

Suppose now that  $S$  is real if  $\phi$  is real. The following theorem holds.

The rules defined in Section 2 for connected diagrams are precisely the same as the mathematical rules to obtain the function  $\Gamma$  defined by

$$e^{i\Gamma} = C \int e^{iS[\phi(x)]} \prod_a \prod_i d\phi_i(x^a) , \quad (3.2)$$

where the right-hand side is understood as a series expansion in terms of the coefficients in  $\mathcal{L}_I$ . Here  $C$  is an arbitrary constant not depending on the sources in  $\mathcal{L}_I$ . The set of all diagrams including disconnected ones are obtained by expanding  $e^{i\Gamma}$ .

Instead of Eq. (3.2) we will use the condensed notation

$$e^{i\Gamma} = C \int \mathcal{D}\phi e^{iS(\phi)} . \quad (3.3)$$

The theorem is easy to prove. First calculate the integral for free particles, using for the action the expression

$$S_0(\phi, J) = \frac{1}{2} \sum \phi_i(x^a) W_{ij}^{ab} \phi_j(x^b) + \sum J_i(x^a) \phi_i(x^a) \quad (3.4)$$

with arbitrary source functions  $J$ . Define

$$e^{i\Gamma_0(J)} = C_0 \int \mathcal{D}\phi e^{iS_0(\phi, J)} . \quad (3.5)$$

$\Gamma_0$  can be found by making a shift in the integration variables

$$\phi_i(x^a) = \phi'_i(x^a) - \sum_b (W^{-1})_{ij}^{ab} J_j(x^b) , \quad (3.6)$$

so that

$$S_0(\phi, J) = \frac{1}{2} \sum \phi' W \phi' - \frac{1}{2} \sum J_i(x^a) (W^{-1})_{ij}^{ab} J_j(x^b) . \quad (3.7)$$

The primed fields are not coupled to the J! We find

$$\Gamma_0(J) = -\frac{1}{2} \sum JW^{-1}J, \quad (3.8)$$

because the integral over the  $\phi'$  is now a source independent constant and can be absorbed in the constant C in Eq. (3.3). Note the factor  $1/2$  because of identical sources.

We see that  $\Gamma^0$  is nothing but the free particle propagator. Even the  $i\epsilon$  prescription can be introduced correctly if we introduce a smooth cut-off for the integral for large values of the fields  $\phi$

$$e^{1/2 i\phi W\phi} \rightarrow e^{1/2 i\phi W\phi} e^{-1/2 \epsilon \phi^2}. \quad (3.9)$$

This makes the integral well defined even in directions where  $\phi W\phi = 0$ . The result is the usual  $i\epsilon$  addition to W.

Let us now consider also interaction terms. The perturbation expansion is

$$e^{i\Sigma \ell_I(x^a)} = 1 + i \sum_a \ell_I(x^a, \phi) + \frac{i^2}{2!} \sum_{a,b} \ell_I(x^a, \phi) \ell_I(x^b, \phi) \dots \quad (3.10)$$

or

$$e^{iS} = e^{iS_0} \left( 1 + i \sum \ell_I + \dots \right), \quad (3.11)$$

where the sources are included in  $S_0$ . From the definition of  $S_0$  we have

$$\left[ i\phi_i(x^a) i\phi_j(x^b) \dots \right] e^{iS_0(\phi, J)} = \left[ \frac{\partial}{\partial J_i(x^a)} \frac{\partial}{\partial J_j(x^b)} \dots \right] e^{iS_0(\phi, J)}. \quad (3.12)$$

Consequently we can replace everywhere the fields  $\phi_i(x^a)$  by the derivative  $\delta/\delta J_i(x^a)$

$$\int \mathcal{D}\phi e^{iS(\phi)} = C \left[ 1 + i \sum_a \ell_I \left( x^a, \frac{\partial}{\partial J(x)} \right) + \dots \right] \int \mathcal{D}\phi e^{iS_0(\phi, J)}. \quad (3.13)$$

The remaining integral is precisely the one computed before and is equal to  $\exp(i\Gamma_0)$  with  $\Gamma_0$  given in Eq. (3.8). Expanding this exponential

$$e^{i\Gamma_0} = 1 - \frac{1}{2} i \sum JW^{-1}J + \frac{1}{8} i^2 (JW^{-1}J)^2 \dots, \quad (3.14)$$

we see that every term in Eq. (3.13) corresponds to a diagram with vertices and propagators as defined in Section 2. Diagrams not coupled to sources are called vacuum renormalizations and must be absorbed in the constant C. Note the combinatorial factors due to the occurrence of identical sources.

The functional integral notation (3.3) for the amplitude in the presence of sources is elegant and compact, and it is very tempting to write the amplitudes of relativistic field theories in this way. Many theorists can indeed not resist this temptation. Let us

see what is involved. In our definition (3.3) we restricted ourselves to a finite number of space-time points. In any realistic theory this number is infinite and summations are to be replaced by integrations. So a suitable limiting procedure must be defined, but unfortunately these definitions cannot always be given in a manner free of ambiguities. Generally speaking, difficulties set in about at the same point that difficulties appear in the usual operator formalism, in particular we mention higher order derivatives or worse non-localities in the Lagrangian. We shall, therefore, in this report not try to formulate such a definition but attach to the result of manipulations with functional integrals a heuristic value. Everything is to be verified explicitly by combinatorics of diagrams. But having verified certain basic algebraic properties we can happily manipulate these "path" integrals. One of the most interesting manipulations used with great advantage in connection with gauge theories is the change of field variables. Both local and non-local canonical transformations turn out to be correctly described by the path integral formulae, as we shall see later.

We may finally mention that the various sign prescriptions, in the case of fermion fields, cannot be obtained in a simple way. They can be corrected for by hand afterwards, or in a more sophisticated way an algebra of anticommuting variables can be introduced. Such work can be found in the literature and is quite straightforward.

#### 4. KALLEN-LEHMANN REPRESENTATION

The quantities  $-V^{-1}$  and  $-W^{-1}$  [see Eqs. (2.5)] that are defined as propagators in the theory are more precisely called the bare propagators. This in contrast to the two-point Green's function that when divided by  $-(2\pi)^8 \delta_4(k+k')$  is called the dressed propagator. Both the bare and dressed propagators are required to satisfy the Kallén-Lehmann representation, but it will turn out that the dressed propagators satisfy this automatically if the bare propagators do.

Consider any propagator and decompose it into invariant functions. For instance, for vector mesons

$$(2\pi)^4 i \Delta_{F\mu\nu}(k) = \delta_{\mu\nu} f_1(k^2) + k_\mu k_\nu f_2(k^2) . \quad (4.1)$$

The Kallén-Lehmann representation for the invariant functions is

$$f(-s) = \int_{a \geq 0}^{\infty} \frac{\rho(s')}{s' - s - i\epsilon} ds' . \quad (4.2)$$

The functions  $\rho(s')$  must be real. For bare propagators we will insist that the  $\rho(s')$  are a sum of  $\delta$  functions

$$\rho(s') = \sum_i a_i \delta(s' - m_i^2) \quad \text{for bare propagators} \quad (4.3)$$

with real coefficients  $a_i$ , and real positive  $m_i^2$ .

Now introduce the Fourier transform of the propagators

$$\Delta_{Fij}(x) = \int d_4 k e^{ikx} \Delta_{Fij}(k) . \quad (4.4)$$

Corresponding to the decomposition of  $\Delta_F(k)$  in invariant functions we will have a decomposition involving derivatives. For instance, for vector mesons

$$\Delta_{F\mu\nu}(x) = \delta_{\mu\nu} \Delta_1(x) - \partial_\mu \partial_\nu \Delta_2(x) , \quad (4.5)$$

where  $\Delta_1$  and  $\Delta_2$  are the Fourier transforms of  $f_1$  and  $f_2$  above. The statement that any function  $f(x)$  satisfies the Kallén-Lehmann representation is equivalent to the statement

$$f(x) = \theta(x_0) f^+(x) + \theta(-x_0) f^-(x) , \quad (4.6)$$

where  $f^+$  ( $f^-$ ) is a positive (negative) energy function

$$f^\pm(x) = \frac{1}{(2\pi)^3} \int_{a \geq 0}^{\infty} ds' \rho(s') \int d_4 k e^{ikx} \theta(\pm k_0) \delta(k^2 + s') . \quad (4.7)$$

The proof is very simple. Using the Fourier representation

$$\theta(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{e^{i\tau x_0}}{\tau - i\epsilon} , \quad (4.8)$$

one derives immediately the result. Of course this derivation is only correct if the integral in Eq. (4.2) exists, and this is in general the case. If not one must introduce regulators, to be discussed below. However, we need the representation (4.6) not only for the invariant functions, but also for the complete propagator such as in Eq. (4.5). Whether this is true depends on the convergence of the dispersion integrals. If the functions  $f^+$  and  $f^-$  go for  $x_0 = 0$  sufficiently smoothly over into one another, then the expression (4.6) has no particular singularity at  $x_0 = 0$  and one may ignore the action of derivatives on the  $\theta$  functions. Now from the above equation (4.7) it is clear that  $f^+(x_0) = f^-(-x_0)$ , so in any case  $f^+(0) = f^-(0)$ . This is enough to treat the case of one derivative such as occurring in the case of fermion propagators. Consider now

$$\partial_0 f^\pm = \frac{1}{(2\pi)^3} \int ds' \rho(s') \int d_4 k e^{ikx} (-ik_0) \theta(\pm k_0) \delta(k^2 + s') . \quad (4.9)$$

For  $x_0 = 0$  we can do the  $k_0$  integral

$$\partial_0 f^\pm \Big|_{x_0=0} = \frac{1}{(2\pi)^3} \int d_3 k e^{i\vec{k}x} \int ds' \rho(s') \left( \mp \frac{i}{2} \right) . \quad (4.10)$$

Clearly  $\partial_0 f^+ = \partial_0 f^-$  in  $x_0 = 0$  only if the dispersion integral is superconvergent

$$\int ds' \rho(s') = 0 . \quad (4.11)$$

For bare propagators, see Eq. (4.3), this can only be true if some of the coefficients  $a_i$  are negative, some positive. This implies the existence of negative metric particles, which in turn may lead to unphysical results, such as negative lifetimes or cross-sections, or lack of unitarity depending on how one defines things.

Let us now assume that the superconvergence equation (4.11) holds. Then one obtains indeed

$$\partial_\mu \partial_\nu [\theta(x_0) f^+ + \theta(-x_0) f^-] = \theta(x_0) \partial_\mu \partial_\nu f^+ + \theta(-x_0) \partial_\mu \partial_\nu f^- , \quad (4.12)$$

using

$$\delta(x_0) \partial_0 f^+ - \delta(x_0) \partial_0 f^- = 0 ,$$

as well as

$$\delta'(x_0) f^+ - \delta'(x_0) f^- = 0 .$$

## 5. INTERIM CUT-OFF METHOD: UNITARY REGULATORS

Because of divergencies, the definition of diagrams given in Section 2 may be meaningless. Moreover, the propagators may not satisfy the Källén-Lehmann representation because of lack of superconvergence (see Section 4). To avoid such difficulties we introduce what we will call unitary regulators. This regulator method works for any theory in the sense that it allows a proper definition of all diagrams and is moreover very suitable in connection with the proof of unitarity, causality, etc. It fails in the case of Lagrangians invariant under a gauge group, for which we will later introduce a more sophisticated method.

The prescription is exceedingly simple: construct things so that any propagator is replaced by a sum of propagators. The extra propagators correspond to heavy particles, and coefficients [the  $a_i$  in Eq. (4.3)] are chosen such that the high momentum behaviour is very good. This is achieved as follows. Introduce in the original Lagrangian  $\mathcal{L}$ , Eq. (2.1), sets of regulator fields  $\psi_i^\lambda$  and  $\phi_i^\lambda$  in the following way:

$$\begin{aligned} \mathcal{L}^R = & \psi_i^* V_{ij} \psi_j + \sum_{\lambda} a_{\lambda} \psi_i^{\lambda*} (V + M^{\lambda})_{ij} \psi_j^{\lambda} + \phi_i W_{ij} \phi_j + \sum_{\lambda} b_{\lambda} \phi_i^{\lambda} (W + m^{\lambda})_{ij} \phi_j^{\lambda} + \\ & + \mathcal{L}_I \left( \psi^* + \sum \psi^{\lambda*}, \psi + \sum \psi^{\lambda}, \phi + \sum \phi^{\lambda} \right). \end{aligned} \quad (5.1)$$

The coefficients  $a_{\lambda}$  and  $b_{\lambda}$  and the mass matrices  $M^{\lambda}$  and  $m^{\lambda}$  must be chosen so as to assure the proper high momentum behaviour.

The propagators for the regulating fields  $\psi^{\lambda}$  are

$$\Delta_{Fij}^{R\lambda} = -a_{\lambda}^{-1} (V + M^{\lambda})_{ij}^{-1}. \quad (5.2)$$

Because in the interaction Lagrangian  $\psi$  is everywhere replaced by  $\psi + \sum \psi^{\lambda}$  (similarly  $\psi^*$  and  $\phi$ ), only the following combination of propagators will occur in the diagrams,

$$-\frac{1}{V} - \sum_{\lambda} \frac{1}{a_{\lambda}} \frac{1}{V + M^{\lambda}}. \quad (5.3)$$

Choosing the appropriate coefficients and mass matrices all diagrams will become finite. If every eigenvalue of the mass matrix is made very large, diagrams that were finite before the introduction of regulators will converge to their unregulated value.

EXAMPLE 5.1 Charged scalar particles with  $\phi^4$  interaction

$$\mathcal{L}^R = \psi^* (\partial^2 - m^2) \psi - \psi^{1*} (\partial^2 - m^2 - M^2) \psi^1 + \frac{g}{4} (\psi^* + \psi^{1*})^2 (\psi + \psi^1)^2. \quad (5.4)$$

$$\psi \text{ propagator: } \frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2}. \quad (5.5)$$

$$\psi^1 \text{ propagator: } -\frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 + M^2}. \quad (5.6)$$



The combination occurring in the diagrams is

$$\frac{1}{(2\pi)^4 i} \frac{M^2}{(k^2 + m^2)(k^2 + m^2 + M^2)} . \quad (5.7)$$

This behaves as  $k^{-4}$  at high momenta. The diagrams shown in Example 2.2.1 are now finite for large but finite  $M^2$ .

## 6. CUTTING EQUATIONS

### 6.1 Preliminaries

In order to keep the work of this section transparent, we will suppress indices, derivatives, etc. In particular, for vertices we retain only a factor  $(2\pi)^4 ig$  in momentum space, that is  $ig$  in coordinate space. There is no difficulty whatsoever in reintroducing the necessary details.

It is assumed that diagrams are sufficiently regulated, so that no divergencies occur.

The starting point is the decomposition of the propagator into positive and negative energy parts

$$\Delta_{ij}(x) = \theta(x_0)\Delta_{ij}^+(x) + \theta(-x_0)\Delta_{ij}^-(x) , \quad (6.1)$$

$$\Delta_{ij}^\pm(x) = \frac{1}{(2\pi)^3} \int d_4 k e^{ikx} \theta(\pm k_0) \rho(k^2) , \quad (6.2)$$

with  $x = x_i - x_j$ . Here we used the notation  $\Delta_{ij}(x) = \Delta_{Fij}(x_i - x_j)$ .

In view of the reality of the spectral functions  $\rho$  we have  $\Delta_{ij}^\pm = (\Delta_{ji}^\mp)^*$ . Also

$$\Delta_{ij}^\pm = \Delta_{ji}^\mp . \quad (6.3)$$

Consequently

$$\Delta_{ij}^* = \theta(x_i - x_j)\Delta_{ij}^- + \theta(x_j - x_i)\Delta_{ij}^+ . \quad (6.4)$$

As usual

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{e^{i\tau x}}{\tau - i\epsilon} = \begin{cases} 1 & \text{if } x_0 > 0 \\ 0 & \text{if } x_0 < 0 \end{cases} \quad (6.5)$$

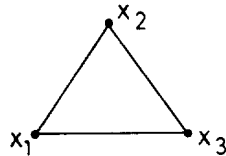
$$\theta(x) + \theta(-x) = 1 .$$

The representation (6.5) can be verified by choosing the integration contour with a large semi-circle in the upper (lower) complex plane for  $x_0$  positive (negative).

Consider now a diagram with  $n$  vertices. Such a diagram represents in coordinate space an expression containing many propagators depending on arguments  $x_1, \dots, x_n$ . We will denote such an expression by

$$F(x_1, x_2, \dots, x_n) .$$

For example, the triangle diagram represents the function:



$$F(x_1, x_2, x_3) = (ig)^3 \Delta_{31} \Delta_{23} \Delta_{12} . \quad (6.6)$$

Every diagram, when multiplied by the appropriate source functions and integrated over all  $x$  contributes to the S-matrix. The contribution to the T-matrix, defined by

$$S = 1 + iT \quad (6.7)$$

is obtained by multiplying by a factor  $-i$ . Unitarity of the S-matrix implies an equation for the imaginary part of the so defined T matrix

$$T - T^\dagger = iT^\dagger T . \quad (6.8)$$

The T-matrix, or rather the diagrams, are also constrained by the requirement of causality. As yet nobody has found a definition of causality that corresponds directly to the intuitive notions; instead formulations have been proposed involving the off-mass-shell Green's functions. We will employ the causality requirement in the form proposed by Bogoliubov that has at least some intuitive appeal and is most suitable in connection with a diagrammatic analysis. Roughly speaking Bogoliubov's condition can be put as follows: if a space-time point  $x_1$  is in the future with respect to some other space-time point  $x_2$ , then the diagrams involving  $x_1$  and  $x_2$  can be rewritten in terms of functions that involve positive energy flow from  $x_2$  to  $x_1$  only.

The trouble with this definition is that space-time points cannot be accurately pinpointed with relativistic wave packets corresponding to particles on mass-shell. Therefore this definition cannot be formulated as an S-matrix constraint. It can only be used for the Green's functions.

Other definitions refer to the properties of the fields. In particular there is the proposal of Lehmann, Symanzik and Zimmermann that the fields commute outside the light cone. Defining fields in terms of diagrams, this definition can be shown to reduce to Bogoliubov's definition. The formulation of Bogoliubov causality in terms of cutting rules for diagrams will be given in Section 6.4.

## 6.2 The largest time equation

Let us now consider a function  $F(x_1, x_2, \dots, x_n)$  corresponding to some diagram. We define new functions  $F$ , where one or more of the variables  $x_1, \dots, x_n$  are underlined. Consider

$$F(x_1, x_2, \dots, \underline{x_i}, \dots, \underline{x_j}, \dots, x_n) . \quad (6.9)$$

This function is obtained from the original function  $F$  by the following:

- i) A propagator  $\Delta_{ki}$  is unchanged if neither  $x_k$  nor  $x_i$  are underlined.
- ii) A propagator  $\Delta_{ki}$  is replaced by  $\Delta_{ki}^+$  if  $x_k$  but not  $x_i$  is underlined.
- iii) A propagator  $\Delta_{ki}$  is replaced by  $\Delta_{ki}^-$  if  $x_i$  but not  $x_k$  is underlined.
- iv) A propagator  $\Delta_{ki}$  is replaced by  $\Delta_{ki}^*$  if  $x_k$  and  $x_i$  are underlined.
- v) For any underlined  $x$  replace one factor  $i$  by  $-i$ . Apart from that, the rules for the vertices remain unchanged.

Equations (6.1) and (6.4) lead trivially to an important equation, the largest time equation. Suppose the time  $x_{i_0}$  is larger than any other time component. Then any function  $F$  in which  $x_i$  is not underlined equals minus the same function but with  $x_i$  now underlined

$$F(x_1, \dots, x_i, \dots, \underline{x_j}, \dots, x_n) = -F(x_1, \dots, \underline{x_i}, \dots, \underline{x_j}, \dots, x_n) . \quad (6.11)$$

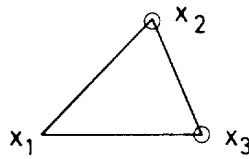
The minus sign is a consequence of point (v). In view of what follows it is useful to invent a diagrammatic representation of the newly-defined functions:

Any function  $F$  is represented by a diagram where any vertex corresponding to an underlined variable is provided with a circle.

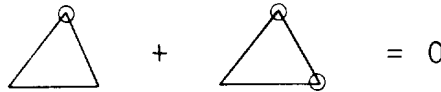
EXAMPLE 6.2.1 If  $F(x_1, x_2, x_3)$  is given by Eq. (6.6) then

$$F(x_1, \underline{x_2}, \underline{x_3}) = (ig)^3 \Delta_{31}^+ \Delta_{23}^* \Delta_{12}^- . \quad (6.12)$$

The corresponding diagram is:



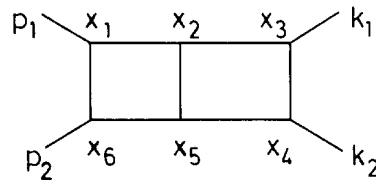
If the time component of  $x_3$  is largest we have, for instance:



From such a diagram it is impossible to see if a given line connecting a circled to an uncircled vertex corresponds to a  $\Delta^+$  or  $\Delta^-$  function. But due to Eq. (6.2) the result is the same anyway. The important fact is that energy always flows from the uncircled to the circled vertex, because of the  $\theta$  function in Eq. (6.2). Of course there is no restriction on the sign of energy flow for lines connecting two circled or two uncircled vertices.

### 6.3 Absorbive part

To obtain the contribution of a diagram to the S-matrix the corresponding function  $F(x_1, \dots, x_n)$  must be multiplied with the appropriate source functions for the ingoing and outgoing lines and integrated over all  $x_i$ . For instance, the function  $F(x_1, \dots, x_6)$  corresponding to the diagram:



must be multiplied by

$$e^{ip_1 x_1} e^{ip_2 x_6} e^{-ik_1 x_3} e^{-ik_2 x_4}$$

and subsequently integrated over  $x_1, \dots, x_6$ .

The restriction that Eq. (6.11) only holds if  $x_{i_0}$  is larger than the other time components makes it impossible to write down the analogue of Eq. (6.11) in momentum space.

We will now write down an equation which follows directly from the largest time equation, but holds whatever the time ordering of the various  $x_i$ . Thus consider a function  $F(x_1, \dots, x_n)$  corresponding to some diagram. We have

$$\sum_{\text{underlinings}} F(x_1, \dots, \underline{x_i}, \dots, \underline{x_j}, \dots, x_n) = 0 . \quad (6.13)$$

The summation goes over all possible ways that the variables may be underlined. For instance, there is one term without underlined variables,  $n$  terms with one underlined variable, etc. There is also one term, the last, where all variables are underlined. Under certain conditions, to be discussed in Section 7, that term is related to the first term

$$F(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}) = F(x_1, x_2, \dots, x_n)^* . \quad (6.14)$$

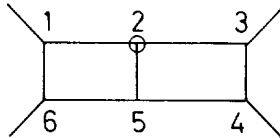
The proof of Eq. (6.13) is trivial. Let one of the  $x$ , say  $x_i$ , have the largest time component. Then on the left-hand side of Eq. (6.13) any diagram with  $x_i$  not underlined cancels against the term where in addition  $x_i$  is underlined, by virtue of the largest time equation.

Now we can multiply Eq. (6.13) by the appropriate source factors and integrate over all  $x$ . We obtain a set of functions  $\bar{F}$  depending on the various external momenta and further internal momenta (loop momenta). Here, the bar on  $F$  denotes the Fourier transform. The functions  $\bar{F}$  are composed of  $\Delta^\pm$ ,  $\Delta$  and  $\Delta^*$  functions

$$\Delta^\pm(k) = \frac{1}{(2\pi)^3} \theta(\pm k_0) \int_{a \geq 0} ds' \rho(s') \delta(k^2 + s') , \quad (6.15)$$

$$\Delta(k) = \frac{1}{(2\pi)^4 i} \int_{a \geq 0} ds' \rho(s') \frac{1}{k^2 + s' - i\epsilon} . \quad (6.16)$$

We now observe that in the resulting equation many terms will be zero, due to conflicting energy  $\theta$  functions. Take, for example, a diagram<sup>\*)</sup> with one underlined point which is not connected to an outgoing line:

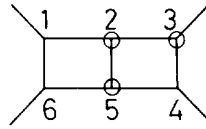


\*) In this and in the following diagrams, incoming particles are at the left, outgoing at the right (energy flows in at the left, out at the right).

Because of the  $\theta$  functions in the  $\Delta^\pm$  [see Eq. (6.15)], energy is forced to flow towards the circled vertex. Since energy conservation holds in that vertex this is impossible and we conclude that this diagram is zero. The same is true if vertex 5 is circled instead. Further, if the momenta  $p_1$  and  $p_2$  represent incoming particles, which implies energy flowing from the outside into vertices 1 and 6, then also the diagrams with vertex 1 and/or vertex 6 circled are zero. Also the diagram with 2 and 5 circled is zero, even if now energy may flow in either direction between 2 and 5, because all other lines ending in 2 or 5 force energy flow towards these vertices. We thus come to the following result.

A diagram containing circled vertices gives rise to a non-zero contribution if the circled vertices form connected regions that contain one or more outgoing lines. And also the uncircled vertices must form connected regions involving incoming lines.

Thus, for example:

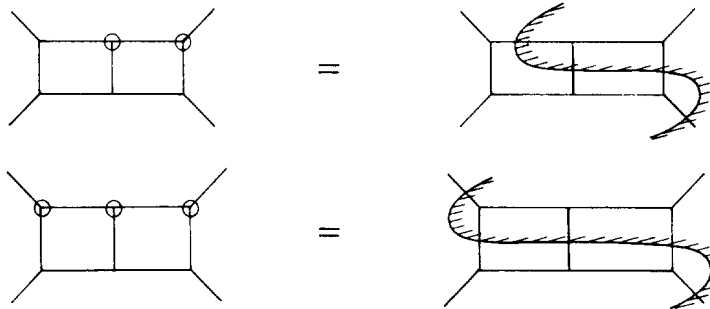


is zero because in vertex 4 we have a conflicting situation. Examples of non-zero diagrams are:

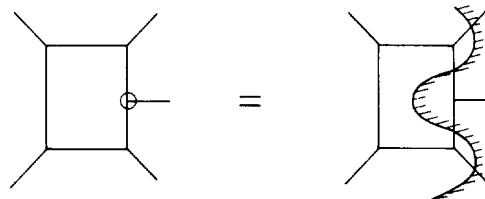


Note that an ingoing line may be attached to a circled region.

Since the circled vertices form connected regions we may drop the circles and indicate the region with the help of a boundary line:



Here is an example of another diagram:



Note that no special significance is attached to the cutting of an external line.

Taking the above into account, Eq. (6.13) now reduces to

$$\bar{F}(k_1, \dots, k_n) + \hat{\bar{F}}(k_1, \dots, k_n) = - \sum_{\text{cuttings}} \bar{F}_c(k_1, \dots, k_n) . \quad (6.17)$$

Here  $\bar{F}$  is the Fourier transform of the function  $F$  without underlinings,  $\hat{\bar{F}}$  the Fourier transform of the function  $F$  with all variables underlined. The functions  $F_c$  correspond to all non-zero diagrams containing both circled and uncircled vertices. They correspond to all possible cuttings of the original diagram with the prescription that for a cut line the propagator function  $\Delta(k)$  must be replaced by  $\Delta^\pm(k)$  with the sign such that energy is forced to flow towards the shaded region. Equation (6.17) is Cutkosky's cutting rule.

Remembering that the T-matrix is obtained by multiplying by  $-i$ , we see that Eq. (6.17) is of exactly the same structure as the unitarity equation (6.8). There is one notable difference: Eq. (6.17) holds for a single diagram, while unitarity is a property true for a transition amplitude, that is for the sum of diagrams contributing to a given process.

Equation (6.17) holds for any theory described by a Lagrangian, whether it is unitary or not. The Feynman rules for  $\hat{\bar{F}}$  are, however, different from those for  $T^\dagger$  (Section 2.6). Therefore, if Eq. (6.17) is truly to imply unitarity a number of properties must hold. This will be discussed later.

#### 6.4 Causality

Again consider any diagram, that represents a function  $F(x_1, \dots, x_n)$ . Take any two variables, say  $x_i$  and  $x_j$ . Let us suppose that the time component of  $x_j$  is larger than  $x_{i_0}$ . The following equation holds independently of the time ordering of the other time components

$$\sum_{\substack{\text{underlinings} \\ \text{except } x_i}} F(x_1, \dots, \underline{x_k}, \dots, x_n) = 0 \quad \text{if } x_{i_0} < x_{j_0} . \quad (6.18)$$

Again terms cancel in pairs. We do not need the diagrams where  $x_i$  is underlined, because we know for sure that  $x_{i_0}$  is never the largest time.

Equation (6.18), when multiplied by the appropriate source functions and integrated over all  $x$  except  $x_i$  and  $x_j$ , is the single diagram version of Bogoliubov's causality condition. His notation is

$$\frac{\delta^2 S}{\delta g(x_i) \delta g(x_j)} \hat{S} + \frac{\delta S}{\delta g(x_i)} \frac{\delta \hat{S}}{\delta g(x_j)} = 0 \quad \text{if } x_{i_0} < x_{j_0} . \quad (6.19)$$

Here the first term describes cut diagrams (including the case of no cut at all -- the unit part of  $\hat{S}$ ) with  $x_i$  and  $x_j$  not circled, and the second term denotes diagrams with  $x_j$  but not  $x_i$  circled.  $\hat{S}$  is the S-matrix obtained from the conjugate Feynman rules (i.e. all vertices underlined), and will often be equal to  $S^\dagger$ . Further  $g(x)$  is the coupling constant, made into a function of space-time.

Similarly we can consider the case where  $x_{i_0} > x_{j_0}$ . Then we have an equation where now  $x_j$  is never to be underlined. Separating off the term with no variable underlined we may combine equations, with the result

$$F(x_1, x_2, \dots, x_n) = -\theta(x_{j_0} - x_{i_0}) \sum_i' F(x_1, \dots, \underline{x_k}, \dots, x_n) - \theta(x_{i_0} - x_{j_0}) \sum_j' F(x_1, \dots, \underline{x_k}, \dots, x_n). \quad (6.20)$$

The prime indicates absence of the term without underlined variables. The index  $i$  implies absence of diagrams with  $x_i$  underlined.

The summations in Eq. (6.20) still contain many identical terms, namely those where neither  $x_i$  nor  $x_j$  are underlined. Also these may be taken together to give

$$F(x_1, \dots, x_n) = - \sum_{ij}' F(x_1, \dots, x_n) - \theta(x_{j_0} - x_{i_0}) \sum_{\substack{j \text{ underlined} \\ i \text{ not}}} F(x_1, \dots, x_n) - \theta(x_{i_0} - x_{j_0}) \sum_{\substack{i \text{ underlined} \\ j \text{ not}}} F(x_1, \dots, x_n). \quad (6.21)$$

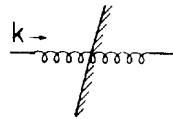
The first term on the right-hand side of Eq. (6.21) is a set of cut diagrams, with  $x_i$  and  $x_j$  always in the unshaded region. They represent the product  $\hat{S}\hat{S}$  with the restriction that  $x_i$  and  $x_j$  are vertices of  $S$ . We can now apply the same equation, with the same points  $x_i$  and  $x_j$ , to the diagrams for  $S$  in this product. Doing this as many times as necessary, the right-hand side of Eq. (6.21) can be reduced entirely to the sum of two terms, one containing a function  $\theta(x_{i_0} - x_{j_0})$  multiplying a function whose Fourier transforms contains  $\theta$  functions forcing energy flow from  $i$  to  $j$ , the other containing the opposite combination. This is precisely of the form indicated in Section 6.1.

Let us now return to Eq. (6.21). Introducing for  $\theta$  the Fourier representation Eq. (6.5), we can see  $\theta$  as another kind of propagator connecting the points  $x_i$  and  $x_j$ . Multiplying by the appropriate source functions and integrating over all  $x_i$  we obtain the following diagrammatic equation:

$$\text{Diagram 1} = - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} \quad (6.22)$$

The blob stands for any diagram or collection of diagrams. The points 1 and 2 indicate two arbitrarily selected vertices. The "self inductance" is the contribution due to the  $\theta$  function, and is obviously non-covariant:

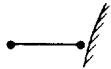
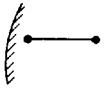





$$\frac{1}{(2\pi)^4 i} \frac{1}{-k_0 - i\epsilon} \delta_3(\vec{k}) . \quad (6.23)$$

Of course, in the diagrams on the right-hand side summation over all cuts with the points 1 and 2 in the position shown is intended.

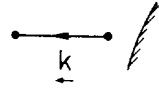
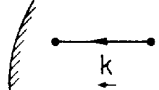
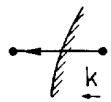
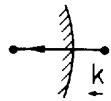
This is perhaps the right moment to summarize the Feynman rules for the cut diagrams. As an example we take the simple scalar theory:

	Propagator in unshaded region:	$\frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 - i\epsilon}$	} (6.24)
	Propagator in shadowed region:	$-\frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 + i\epsilon}$	
	Cut line:	$\frac{1}{(2\pi)^3} \theta(k_0) \delta(k^2 + m^2) .$	

Vertex in unshaded region:  $(2\pi)^4 ig .$

Vertex in shadowed region:  $-(2\pi)^4 ig .$

For a spin- $1/2$  particle everything obtains by multiplying with the factor  $-i\gamma k + m$ :

	$\frac{1}{(2\pi)^4 i} \frac{-i\gamma k + m}{k^2 + m^2 - i\epsilon}$	} (6.25)
	$-\frac{1}{(2\pi)^4 i} \frac{-i\gamma k + m}{k^2 + m^2 + i\epsilon}$	
	$\frac{1}{(2\pi)^3} (-i\gamma k + m) \theta(-k_0) \delta(k^2 + m^2)$	
	$\frac{1}{(2\pi)^3} (-i\gamma k + m) \theta(k_0) \delta(k^2 + m^2) .$	

The most simple application concerns the case of only one propagator connecting two sources. We will let these sources emit and absorb energy, but we will not put anything on mass-shell. Indeed, nowhere have mass-shell conditions been used in the derivations.

Thus consider:

$$\begin{array}{c} J_1 \\ \times \xrightarrow{\quad k \quad} \times J_2 \end{array} \qquad (2\pi)^4 i J_1(k) \frac{1}{k^2 + m^2 - i\epsilon} J_2(k)$$

with  $J_1$  and  $J_2$  non-zero only if  $k_0 > 0$ . The unitarity equation (6.17) reads:

$$\times \xrightarrow{\quad} \times + \left( \times \xrightarrow{\quad} \times \right)^* = - \times \text{ / } \times - \times \text{ / } \times$$

The complex conjugation does apply to everything except the sources  $J$ . The second term on the right-hand side is zero, because of the condition  $k_0 > 0$ . The equation becomes

$$J \left[ (2\pi)^4 i \frac{1}{k^2 + m^2 - i\epsilon} - (2\pi)^4 i \frac{1}{k^2 + m^2 + i\epsilon} \right] J = J \left[ + \frac{i^2 (2\pi)^8}{(2\pi)^3} \theta(k_0) \delta(k^2 + m^2) \right] J .$$

Note that the vertex in the shadowed region gives a factor  $-(2\pi)^4 i$ . With

$$\frac{1}{a - i\epsilon} = P\left(\frac{1}{a}\right) + i\pi\delta(a) ,$$

we see the equation holds true.

Also Eq. (6.22) can be verified:

$$\times \xrightarrow{\quad} \times = - \times \text{ / } \times - \times \text{ / } \times$$

We now obtain (note the minus sign for vertex in shadowed region)

$$(2\pi)^4 i \frac{1}{k^2 + m^2 - i\epsilon} = \frac{(2\pi)^8 i^2}{(2\pi)^3 2\pi i} \times \int_{-\infty}^{\infty} dp_0 \left\{ \frac{1}{-p_0 - i\epsilon} \theta(k_0 - p_0) \delta[(k - p)^2 + m^2] + \frac{1}{p_0 - i\epsilon} \theta(-k_0 + p_0) \delta[(k - p)^2 + m^2] \right\}_{\vec{p}=0} .$$

The four-vector  $p_0$  has zero space components [see expression (6.23)]. The  $p_0$  integration is trivial and gives the desired result.

### 6.5 Dispersion relations

Equations (6.22) are nothing but dispersion relations, valid for any diagram. Let  $\tau$  be the fourth component of the momentum flowing through the self-inductance. Let  $f^+(\tau)$  be the function corresponding to the cut diagrams excluding the  $\tau$ -line in the second term on the right-hand side of Eq. (6.22), with  $\tau$  directed from 1 to 2. Similarly for  $f^-(\tau)$ .

If  $f$  and  $f'$  represent the left-hand side and the first term on the right-hand side, respectively, we have

$$f = -f' - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{-\tau - i\epsilon} f^+(\tau) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i\epsilon} f^-(\tau) . \quad (6.26)$$

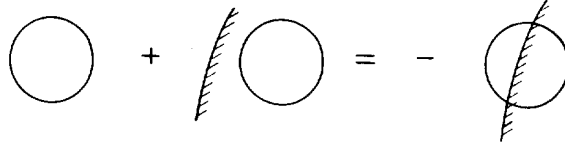
Of course, all the functions  $f$  depend on the various external momenta. The function  $f^+(\tau)$  will be zero for large positive  $\tau$ , namely as soon as  $\tau$  becomes larger than the total amount of energy flowing into the diagram in the unshadowed region. Similarly  $f^-(\tau)$  is zero for large negative  $\tau$ .

The dispersion relations Eq. (6.26) are very important in connection with renormalization. If all subdivergencies of a diagram have been removed by suitable counter terms, then all cut diagrams will be finite (involving products of subdiagrams with certain finite phase-space integrals). According to Eq. (6.26) the infinities in the diagram must then arise because of non-converging dispersion integrals. Suitable subtractions, i.e. counter terms, will make the integrals finite.

It may finally be noted that our dispersion relations are very different from those usually advertised. We do not disperse in some external Lorentz invariant, such as the centre-of-mass energy or momentum transfer.

7. UNITARITY

If the cutting equation (6.17), diagrammatically represented as:



corresponding to  $T - T^\dagger = iT^\dagger T$ , is to imply unitarity, the following must hold:

- i) The diagrams in the shadowed region must be those that occur in  $S^\dagger$ ;
- ii) The  $\Delta^\dagger$  functions must be equal to what is obtained when summing over intermediate states.

Referring to our discussion of the matrix  $S^\dagger$ , in Section 2.6, we note that point (i) will be true if the Lagrangian generating the S-matrix is its own conjugate.

Point (ii) amounts to the following. The two-point Green's function, on which the definition of the S-matrix sources was based, contained a matrix  $K_{ij}$  [see Eq. (2.12) and following]. In considering  $S^\dagger S$  one will encounter (particle-out of S, particle-in of  $S^\dagger$ ):

$$\begin{array}{c}
 S \quad \xrightarrow{\quad} \times \quad \xrightarrow{\quad} S^\dagger \\
 \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
 \quad \quad \quad k \quad \quad \quad k
 \end{array}
 \quad \sum_a K_{ij}^\dagger (-k) J_j^{*(a)}(k) J_\ell^a(k) K_{\ell m}(-k) \quad (7.1)$$

in the sum over intermediate states. The K are from the propagators attached to the sources. Because of  $\ell = (\ell)_{\text{conjugate}}$  we have  $K_{\ell m}^\dagger(-k) = K_{\ell m}(k)$ . Also if  $J(k)K(-k) \sim J(k)$  then  $K^\dagger(-k)J^* \sim J^*$ , showing that J and  $J^*$  are the appropriate eigen currents of S and  $S^\dagger$ . If unitarity is to be true we require that this sum (7.1) occurring in  $S^\dagger S$  equals the matrix  $K_{im}$  occurring when cutting a propagator.

The proof of this is trivial. Suppose  $K_{ij}$  is diagonal with diagonal elements  $\lambda_i$ . The current defining equations (2.13) and (2.14) imply that the currents are of the form

$$J^{(a)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\sqrt{\lambda_a} \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

There are no currents corresponding to zero eigenvalues. Obviously

$$\sum_a J^{(a)*} J^{(a)} = K^{-1} , \quad (7.2)$$

and this remains true if one provides the currents with phase factors, etc.

For spin- $\frac{1}{2}$  particles things are slightly more complicated, because of  $\gamma^4$  manipulations. For instance, one will have

$$K^\dagger(-k)\gamma^4 = \gamma^4 K(k) . \quad (7.3)$$

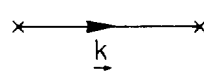
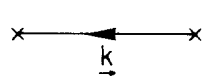
Also the normalization of the currents is different. One finds the correct expression when summing up particle-out/particle-in states, but a minus sign extra for antiparticle-out/antiparticle-in states. This factor is found back in the prescription -1 for every fermion loop. A few examples are perhaps useful.

EXAMPLE 7.1

$$\mathcal{L} = -\bar{\psi}(\gamma\partial + m)\psi + \frac{1}{2} \phi(\partial^2 - m^2)\phi + g\bar{\psi}\psi\phi .$$

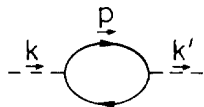
Propagators and vertices have been given before. The appropriate source functions and related equations are given in Appendix A.

There are four two-point Green's functions:

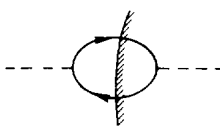
	$\bar{u}^a(k) \frac{-i\gamma k + m}{k^2 + m^2} u^a(k) \frac{2k_0}{4m^2} ; \quad a = 1, 2 ,$
	$-\bar{u}^a(k) \frac{i\gamma k + m}{k^2 + m^2} u^a(k) \frac{2k_0}{4m^2} ; \quad a = 3, 4 .$

Note the minus sign for the incoming antiparticle wave-function.

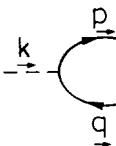
Scalar particle self-energy (we write also  $\delta$  functions):

	$-g^2 \delta_4(k - k') \int d_4p \frac{-i\gamma p + m}{p^2 + m^2 - i\epsilon} \frac{-i\gamma(p - k) + m}{(p - k)^2 + m^2 - i\epsilon} .$
-------------------------------------------------------------------------------------	------------------------------------------------------------------------------------------------------------------------------------------

Note the minus sign for the closed fermion loop. Cut diagram [remember  $-(2\pi)^4 i$  for vertex in shadowed region ]:

	$-(2\pi)^2 g^2 \delta_4(k - k') \int d_4p (-i\gamma p + m) \theta(p_0) \delta(p^2 + m^2) \times$ $\times [-i\gamma(p - k) + m] \theta(k_0 - p_0) \delta[(p - k)^2 + m^2] .$
-------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Decay of scalar into two fermions:

	$(2\pi)^4 i g \sqrt{4p_0 q_0} \bar{u}(p) u^{\alpha}(q) \delta_4(k - p - q) .$
-------------------------------------------------------------------------------------	-------------------------------------------------------------------------------

The superscript  $\alpha$  now indicates antiparticle spinor. The complex conjugate, but with  $k'$  instead of  $k$ , is

$$-(2\pi)^4 ig\sqrt{4p_0q_0} \bar{u}^\alpha(q)u(p)\delta_4(k' - p - q) .$$

The product of the two summed over intermediate states is

$$(2\pi)^8 g^2 4p_0q_0 \int \frac{d_3p}{(2\pi)^3 2p_0} \frac{d_3q}{(2\pi)^3 2q_0} \delta_4(k - p - q)\delta_4(k' - p - q) \frac{1}{2p_0} (-i\gamma p + m) \frac{-1}{2q_0} (i\gamma q + m) .$$

Note the minus sign for the  $q$ -spinor sum.

Since  $p_0 = \sqrt{\vec{p}^2 + m^2}$  we have

$$\int \frac{d_3p}{2p_0} = \int d_4p \theta(p_0) \delta(p^2 + m^2)$$

and similarly for  $q$ . The  $q$  integration can be performed

$$-(2\pi)^2 g^2 \delta_4(k - k') \int d_4p \theta(p_0) \delta(p^2 + m^2) \theta(k_0 - p_0) [-i\gamma(p - k) + m] \delta[(p - k)^2 + m^2] (-i\gamma p + m) ,$$

which indeed equals the result for the cut diagram. The minus sign for the closed fermion loop appears here as a minus sign in front of the antiparticle spinor summation. One may wonder what happens in the case where an antiparticle line is cut, but when there is no closed fermion loop. An example is provided by the antiparticle self-energy as compared to particle self-energy:



Somehow there must also be an extra minus sign for the first diagram. Indeed it is there, because the first diagram contains an incoming antiparticle whose wave-function has a minus sign.

All this demonstrates a tight interplay between statistics (minus sign for fermion loops) and the transformation properties under Lorentz transformations of the spinors. The latter requires the normalization to energy divided by mass as given before, and also relates to the minus sign for antiparticle source summation.

If an integer spin field is assigned Fermi statistics in the form of minus signs for line interchanges then unitarity will be violated.

8. INDEFINITE METRIC

If the numerator  $K$  of a propagator has a negative eigenvalue at the pole then unitarity cannot hold because Eq. (7.2) cannot hold. Unitarity can be restored if we introduce the convention that a minus sign is to be attached whenever such a state appears. It is said that the state has negative norm, and transition probabilities as well as cross-sections and lifetimes can now take negative values. This is, of course, physically unacceptable, and particles corresponding to these states are called ghosts. In theories with ghosts special mechanisms must be present to assure absence of unphysical effects. In gauge theories negative metric ghosts occur simultaneously with certain other particles with positive metric, in such a way that the transition probabilities cancel. Also the second type of particle, although completely decent, is called a ghost and has no physical significance.

9. DRESSED PROPAGATORS

The perturbation series as formulated up to now will in general be divergent in a certain region. Consider the case of a scalar particle interacting with itself and possibly other particles. Let  $\delta_4(k - k')\Gamma(k^2)$  denote the contribution of all self-energy diagrams that cannot be separated into two pieces by cutting one line. These diagrams are called irreducible self-energy diagrams. The two-point Green's function for this scalar particle is of the form

$$-(2\pi)^4 \delta_4(k - k') \bar{\Delta}_F(k) . \tag{9.1}$$

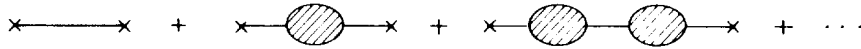
The function  $\bar{\Delta}_F$  is called the dressed propagator. This in contrast to the propagator of the scalar particle, called the bare propagator. If we denote this bare propagator by  $\Delta_F$  we find

$$\bar{\Delta}_F = \Delta_F + \Delta_F \Gamma \Delta_F + \Delta_F \Gamma \Delta_F \Gamma \Delta_F + \dots . \tag{9.2}$$

Summing this series

$$\bar{\Delta}_F = \frac{\Delta_F}{1 - \Gamma \Delta_F} \tag{9.3}$$

corresponding to the diagrams:



where the hatched blobs stand for the irreducible self-energy diagrams.

The function  $\Gamma$  is proportional to the coupling constant of the theory. It is clear that the perturbation series converges only if  $\Gamma \Delta_F < 1$ . But if  $\Delta_F$  has a pole for a certain value of the four-momentum then this series will certainly not converge near this pole, unless  $\Gamma$  happens to be zero there. And if we remember that the definition of the S-matrix involves precisely the behaviour of the propagators at the poles we see that this problem needs discussion.

There are two possible solutions to this difficulty. The simplest solution is to arrange things in such a way that indeed  $\Gamma$  is zero at the pole. This can be done by introducing a suitable vertex in the Lagrangian. This vertex contains two scalar fields and equals minus the value of  $\Gamma$  at the pole. For instance, suppose

$$\Delta_F = \frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m^2 - i\epsilon} . \tag{9.4}$$

The function  $\Gamma(k^2)$  can be expanded at the point  $k^2 + m^2 = 0$

$$\Gamma(k) = \Gamma_0 + (k^2 + m^2)\Gamma_1 + \Gamma_2(k) . \tag{9.5}$$

$\Gamma_0$  and  $\Gamma_1$  are constants,  $\Gamma_2$  is of order  $(k^2 + m^2)^2$ . Introduce in the Lagrangian a term that leads to the vertex:



$$\text{---}\bullet\text{---} = -[\Gamma_0 + (k^2 + m^2)\Gamma_1] . \quad (9.6)$$

Instead of  $\Gamma$  we will now have a function  $\Gamma'$ :

$$\Gamma' = \text{---}\textcircled{\text{hatched}}\text{---} + \text{---}\bullet\text{---}$$

$\Gamma'$  will be of order  $(k^2 + m^2)^2$  and the series (9.2) converges at the pole. Actually this reasoning is correct only in lowest order because the new vertex can also appear inside the hatched blob. Through the introduction of further higher order vertices of the type (9.6) the required result can be made accurate to arbitrary order.

This procedure, involving mass and wave-function renormalization corresponding to  $\Gamma_0$  and  $\Gamma_1$  type terms, respectively, embodies certain inconveniences. First of all, it may be that  $\Gamma_0$  and  $\Gamma_1$  contain imaginary parts. This occurs if the particle becomes unstable because of the interaction. Such truly physical effects are part of the content of the theory and the above neutralizing procedure cannot be carried through. Furthermore, in the case of gauge theories, the freedom in the choice of terms in the Lagrangian is limited by gauge invariance, and it is not sure that the procedure can be carried through without gauge invariance violation.

An alternative solution is to use directly the summed expression (9.3) for the propagators in the diagrams. These diagrams must then, of course, contain no further internal self-energy parts, that is they must be skeleton diagrams. The function  $\Gamma$  occurring in the propagator must then be calculated with a certain accuracy in the coupling constant. For instance, in lowest order for any Green's function the recipe is to compute tree diagrams using bare propagators. In the next order there are diagrams with one closed loop (no self-energy loops) and bare propagators, and tree diagrams with dressed propagators where  $\Gamma$  is computed by considering one-closed-loop self-energy diagrams.

It is clear that this recipe leads to all kinds of complications, which however do not appear to be very profound. Mainly, kinematics and the perturbation expansion have to be considered together. With respect to renormalization the complications are trivial, because then only the behaviour of the propagator for very large  $k^2$  is of importance. But then the series expansion Eq. (9.2) is permitted.

For the rest of this section we will consider the implications of the use of dressed propagators for cutting rules.

As a first step we note that the dressed propagator satisfies the Källén-Lehmann representation. This follows from the causality relation Eq. (6.21), or in picture Eq. (6.22) for the two-point Green's function, where  $x_1$  and  $x_2$  are taken to be the in-source and out-source vertices, respectively. The first term on the right-hand side of these equations is then zero. Indeed, this gives precisely Eq. (6.1), the decomposition of the (dressed) propagator into positive and negative frequency parts. The derivation of the cutting relations, which is based on this decomposition goes through unchanged.

However, with respect to unitarity there is a further subtlety. Using dressed propagators the prescription is to use only diagrams without self-energy insertions, that is skeleton diagrams. Now the  $\Delta^\dagger$  function corresponding to a dressed propagator is evidently obtained by cutting a dressed propagator. The dressed propagator is a series, and cutting gives the result (we use again the example of a scalar particle)

$$\text{Re } \bar{\Delta}_F = \bar{\Delta}_F(-\text{Re } \Gamma)\bar{\Delta}_F^* + \text{pole part} \quad (9.7)$$

or

$$\text{Re } \bar{\Delta}_F = \text{Re } \frac{1}{(2\pi)^4 i [k^2 + m^2 + i\Gamma/(2\pi)^4]} = \frac{-\text{Re } \Gamma}{(2\pi)^8 |k^2 + m^2 + i\Gamma/(2\pi)^4|^2} + \frac{\delta(k^2 + M^2)}{2(2\pi)^3 Z}, \quad (9.8)$$

where we used

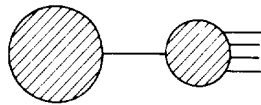
$$k^2 + m^2 + i\Gamma/(2\pi)^4 \sim Z(k^2 + M^2 - i\epsilon) + O[(k^2 + M^2)^2] \quad (9.9)$$

near the pole. Now  $\text{Re } \Gamma$  is obtained when cutting the irreducible self-energy diagrams. In diagrammatic form we have:

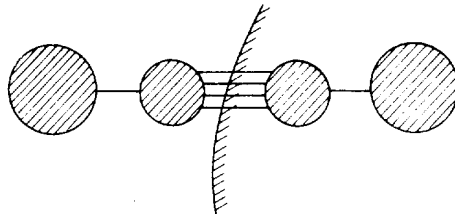
$$\text{Diagram with slash} = \text{Diagram with vertical slash (pole part)} + \text{Diagram with shaded circle and slash} \quad (9.10)$$

Note the occurrence of dressed propagators in the second term on the right.

The subtlety hinted at above is the following. The matrix  $S$  contains skeleton diagrams with dressed propagators. Cutting these diagrams apparently results in expressions obtained when cutting self-energy diagrams. Indeed, these arise automatically in  $S^\dagger S$ . If  $S$  contains skeleton diagrams of the type:



then  $S^\dagger S$  contains the type:



Even if  $S$  and  $S^\dagger$  contain no reducible diagrams, the product  $S^\dagger S$  nevertheless has self-energy structures. They correspond to what is obtained by cutting a dressed propagator.

## 10. CANONICAL TRANSFORMATIONS

### 10.1 Introduction

In this section we study the behaviour of the theory under field transformations. Fields by themselves are not very relevant quantities, from the physical point of view. The S-matrix is supposed to describe the physical content of the theory, and there is no direct relation between S-matrix and fields. Given the Green's functions fields may be defined; the Green's functions, however, can be considered as a rather arbitrary extension of the S-matrix to off-mass-shell values of the external momenta. Within the framework of perturbation theory it is possible, up to a point, to define the fields by considering those Green's functions that behave as smoothly as possible when going off mass-shell. For gauge theories this is still insufficient to fix the theory, there being many choices of fields (and Green's functions) that give the same physics (S-matrix) with equally smooth behaviour. For gravitation the situation is even more bewildering, up to the point of frustration.

In the study of field transformations path integrals have been of great heuristic value. The essential characteristics will be shown in the next section.

### 10.2 Path integrals

A few simple equations form the basis of all path integral manipulations, and will be listed here.

Let  $\alpha$  be a complex number with a positive non-zero imaginary part. Furthermore,  $z = x + iy$  is a complex variable. We have

$$\iint dz e^{i\alpha z^* z} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{i\alpha(x^2+y^2)} . \quad (10.1)$$

Introducing polar coordinates

$$= 2\pi \int_0^{\infty} e^{i\alpha r^2} r dr = \frac{i\pi}{\alpha} . \quad (10.2)$$

Incidentally, realizing that the expression (10.1) is a pure square

$$\int_{-\infty}^{\infty} dx e^{i\alpha x^2} = \sqrt{\frac{i\pi}{\alpha}} . \quad (10.3)$$

Let now  $z$  be a complex  $n$ -component vector, and  $A$  a complex  $n \times n$  matrix. The generalization of Eqs. (10.1), (10.2) is

$$\int dz_1 \dots dz_n e^{i(z^*, Az)} = \frac{\pi^{n,n}}{\det(A)} , \quad (10.4)$$

where

$$\int dz_j = \int_{-\infty}^{\infty} dx_j \int_{-\infty}^{\infty} dy_j , \quad z_j = x_j + iy_j . \quad (10.5)$$

Equation (10.4) follows trivially from Eqs. (10.1), (10.2) in the case where A is diagonal. Next note that the integration measure is invariant for unitary transformations U. To see this write

$$U = A + iB ,$$

where A and B are real matrices. The fact that  $UU^\dagger = 1$  implies

$$A\tilde{A} + B\tilde{B} = 1 , \quad B\tilde{A} - A\tilde{B} = 0 , \quad (10.6)$$

where the wigggle denotes reflection. If now  $z' = Uz$  then  $x' = Ax - By$  and  $y' = Ay + Bx$ . That is, the 2n dimensional vector x, y transforms as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The determinant of this  $2n \times 2n$  transformation matrix is 1. In fact, the matrix is orthogonal, because multiplication with its transpose gives 1, on account of the identities (10.6).

Because of the invariance of both the integration measure as well as the complex scalar product under unitary transformations, we conclude that Eq. (10.5) holds for any complex matrix A that can be diagonalized by means of a unitary transformation and where all diagonal matrix elements have a positive non-zero imaginary part.

Consider now a path integral involving a Lagrangian depending on a set of real fields  $A_i$  (see Section 3)

$$\int \mathcal{D}A_i e^{iS(A)} , \quad (10.7)$$

where the action S is given by

$$S(A) = \int d_4x f(A) . \quad (10.8)$$

Suppose we want to use other fields  $B_i$  that are related to the fields  $A_i$  as follows

$$A_i(x) = B_i(x) + f_i(x, B) . \quad (10.9)$$

The  $f_i$  are arbitrary functions [apart from the fact that Eq. (10.9) must be invertible]. They may depend on the fields B at any space-time point including the space-time point x.

According to well-known rules

$$\mathcal{D}A_i = \det \left( \frac{\partial A_i}{\partial B_j} \right) \mathcal{D}B_j \quad (10.10)$$

or

$$\mathcal{D}A_i = \det \left( \delta_{ij} + \frac{\delta f_i(B)}{\delta B_j} \right) \mathcal{D}B_j . \quad (10.11)$$



The explicitly written factor  $\frac{1}{2}$  is a combinatorial factor. It follows also by treating the path integral along the lines indicated in Section 3. In fact, it is easily established that in  $n^{\text{th}}$  order there will be a contribution of  $n$  times the first-order loop with a factor  $1/n!$ . Similarly for the two-Y loop. In this way, it is seen that the whole series adds up to an exponential, with in the exponent only single closed loops

$$\int \mathcal{D}\phi e^{iS(\phi)} = C \exp(\Gamma) \quad (10.16)$$

$$\Gamma = \text{[Diagram 1]} + \frac{1}{2} \text{[Diagram 2]} + \frac{1}{3} \text{[Diagram 3]} + \dots$$

This can be easily inverted simply by replacing  $\Gamma$  by  $-\Gamma$ , that is by the prescription that every closed  $\phi$ -loop must be given a minus sign. We so arrive at the equation

$$\int \mathcal{D}A_i e^{iS(A)} = \int \mathcal{D}B_i \int \mathcal{D}\phi_i e^{iS(\phi) + iS'(B)} \quad (10.17)$$

with the additional rule that every closed  $\phi$ -loop must be given a minus sign.  $S(\phi)$  is given in Eq. (10.14). Finally  $S'(B)$  follows by substituting the transformation Eq. (10.9) into the action for the A-field, Eq. (10.8)

$$S'(B) = S[B + f(B)] . \quad (10.18)$$

Summarizing, the theory remains unchanged if a field transformation is performed, provided closed loops of ghost particles (with a minus sign/loop) are also introduced. The vertices in the ghost loop are determined by the transformation law.

In the following sections we will derive this same result, however without the use of path integrals.

### 10.3 Diagrams and field transformations

In this section we will consider field transformations of the very simplest type. This we do in order to make the mechanism transparent.

Consider the Lagrangian

$$\mathcal{L}(\phi, \phi^*, A) = \frac{1}{2} \phi_i V_{ij} \phi_j + \frac{1}{2} A_i (\partial^2 - m^2) A_i + \mathcal{L}_I(\phi) + J_i \phi_i + H_i A_i . \quad (10.19)$$

We assume  $V$  to be symmetric.  $\mathcal{L}_I$  is any interaction Lagrangian not involving the A-fields; the latter are evidently free fields. The diagrams corresponding to this Lagrangian are well defined; if necessary use can be made of the previously-given regularization procedure. The  $\phi$ - and A-propagators are:

$$\begin{array}{c} i \\ \bullet \text{---} \bullet \\ j \end{array} \quad \frac{1}{(2\pi)^4 i} (-V^{-1})_{ij} , \tag{10.20}$$

$$\begin{array}{c} i \\ \bullet \text{---} \bullet \\ j \end{array} \quad \frac{1}{(2\pi)^4 i} \frac{\delta_{ij}}{k^2 + m^2 - i\epsilon} .$$

The vertices stemming from  $\mathcal{L}_I$  involve  $\phi$ -lines only.

We want to study another Lagrangian  $\mathcal{L}'$  obtained from  $\mathcal{L}$  by the replacement

$$\phi_i \rightarrow \phi_i + \alpha_{ij} A_j , \tag{10.21}$$

where  $\alpha$  is any matrix. The A remain unchanged.

THEOREM The Green's functions of the new Lagrangian  $\mathcal{L}'$  are equal to those of the original Lagrangian. In particular, A remains a free field, that is all Green's functions involving an A as external leg are zero, except for A-propagators connecting directly two A-sources.

The proof of this theorem is very simple, and consists mainly of expanding the quadratic part of  $\mathcal{L}'$ . We have (dropping all indices)

$$\mathcal{L}' = \frac{1}{2} \phi V \phi + \phi V \alpha A + \frac{1}{2} A \tilde{\alpha} V \alpha A + \frac{1}{2} A (\partial^2 - m^2) A + \mathcal{L}_I(\phi + \alpha A) + J(\phi + \alpha A) + H A . \tag{10.22}$$

Here the wiggle denotes reflection.

Rather than trying to invert the complete matrix in the quadratic part we will treat as the propagator part only the terms  $\phi V \phi$  and  $A(\partial^2 - m^2)A$ . The remaining terms are treated as interaction terms. We obtain the following A- $\phi$  vertices:

$$\begin{array}{c} i \\ \text{---} \bullet \text{---} \text{---} \text{---} \\ j \end{array} \quad (2\pi)^4 i (V\alpha)_{ij} , \tag{10.23}$$

$$\begin{array}{c} i \\ \text{---} \text{---} \bullet \text{---} \text{---} \\ j \end{array} \quad (2\pi)^4 i (\tilde{\alpha} V \alpha)_{ij} .$$

They will be called "special vertices". In addition to these there will be many vertices involving A-lines, because of the replacement (10.21) performed in the interaction Lagrangian.

Consider now any vertex containing an A-field (excluding the A-sources H). Because the A-line arises due to the substitution (10.21) there always exists a similar vertex with the A-line replaced by a  $\phi$ -line. But the A-line can be connected to that vertex also after transformation to a  $\phi$ -line via one of the special vertices. The sum of the two possibilities is zero.

EXAMPLE 10.3.1 In the transformed Lagrangian  $\mathcal{L}'$ , Eq. (10.22) we have a new vertex

$$J \times \text{---} \text{---} \text{---} \quad (2\pi)^4 i J \alpha$$

connecting an A-line with the source of the  $\phi$ -field. In addition, we have the original vertex:

$$J \times \text{-----} \quad (2\pi)^4 iJ$$

and then also the diagram involving one special vertex. The factor  $V$  in the vertex cancels:

$$J \times \text{-----} \begin{matrix} V\alpha \\ \bullet \\ -V^{-1} \end{matrix} \text{-----}$$

against the propagator, leaving a minus sign. The diagram cancels against the previous one.

EXAMPLE 10.3.2 For the two-point Green's function at the zero loop level we have:

$$\begin{array}{c} \bullet \text{-----} \bullet + \bullet \text{-----} \begin{matrix} \tilde{\alpha} V \alpha \\ \bullet \\ -V^{-1} \end{matrix} \text{-----} \bullet \\ + \bullet \text{-----} \begin{matrix} \tilde{\alpha} V \\ \bullet \\ -V^{-1} \end{matrix} \text{-----} \begin{matrix} V \alpha \\ \bullet \\ -V^{-1} \end{matrix} \text{-----} \bullet + \text{ terms with } \alpha^4, \text{ etc.} \end{array}$$

Only the first term survives; all others cancel in pairs.

Diagrams with loops involving A-lines also cancel since they involve vertices already discussed.

The above theorem can easily be generalized to the case of more complicated transformations, such as

$$\phi \rightarrow \phi + f(A)$$

with  $f$  any function of the A fields.

EXAMPLE 10.3.3 Consider the transformation

$$\phi_i \rightarrow \phi_i + \alpha_{ijkl} A_j A_k A_l.$$

The following special vertices are generated [we leave the factors  $(2\pi)^4 i$  as understood]:

$$\begin{array}{c} V\alpha \\ \bullet \\ \text{-----} \end{array} \quad \begin{array}{c} \text{-----} \\ \bullet \\ \alpha V \alpha \end{array}$$

Again, by the same trivial mechanism as before we have, for example:

$$\begin{array}{c} J\alpha \\ \times \\ \text{-----} \end{array} + \begin{array}{c} J \\ \times \\ \text{-----} \begin{matrix} V\alpha \\ \bullet \\ -V^{-1} \end{matrix} \text{-----} \end{array} = 0$$

$$\begin{array}{c} \alpha V \\ \bullet \\ \text{-----} \begin{matrix} V\alpha \\ \bullet \\ -V^{-1} \end{matrix} \text{-----} \end{array} + \begin{array}{c} \text{-----} \\ \bullet \\ \alpha V \alpha \end{array} = 0$$



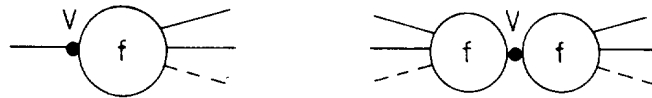
The situation becomes more complicated if we allow the function  $f$  also to depend on the  $\phi$ -fields, even in a non-local way.

#### 10.4 Local and non-local transformations

Starting again from the Lagrangian (10.19) we consider now the general substitution

$$\phi \rightarrow \phi + f(A, \phi) .$$

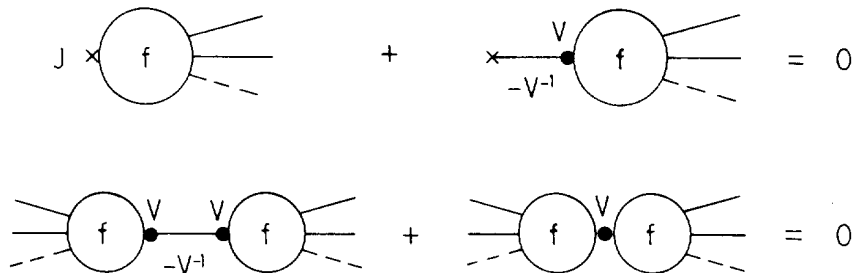
We will now have special vertices involving the unspecified function  $f$ . The function  $f$  will be represented as a blob with a certain number of  $A$ - and  $\phi$ -lines of which we will



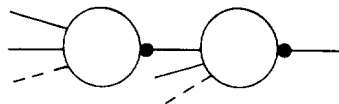
indicate only three explicitly. The propagator matrix  $V$  is always there as a factor and is indicated by a dot. The  $\phi$ -line attached at that point will be called the original  $\phi$ -line of the special vertex.

Now this original  $\phi$ -line may be connected to any of the old vertices of the theory or to another original  $\phi$ -line. Again the cancellation mechanism works.

##### EXAMPLE 10.4.1



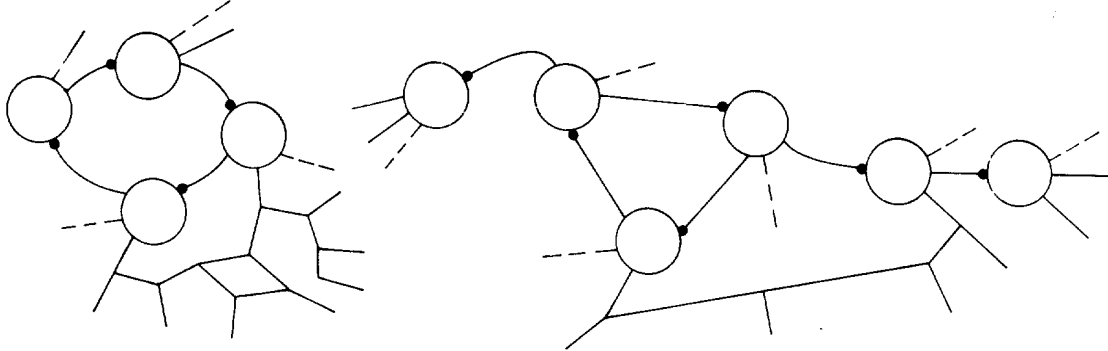
No new feature arises in these cases. However, if an original  $\phi$ -line is connected to any of the other  $\phi$ -lines of a special vertex (except the  $fVf$  vertex that is cancelled out already as shown above) no cancellation occurs. For example, there is nothing that cancels the following construction:



Thus we get a non-zero extra contribution only if the original  $\phi$ -line is connected to another  $\phi$ -line of the special vertex. All this means that the new Lagrangian contains all the contributions of the original Lagrangian plus a new kind of diagram where all the

original  $\phi$ -lines are connected to one of the other  $\phi$ -lines of the special vertex. Such diagrams contain at least one closed loop of special vertices with "wings" of old vertices as well as special vertices.

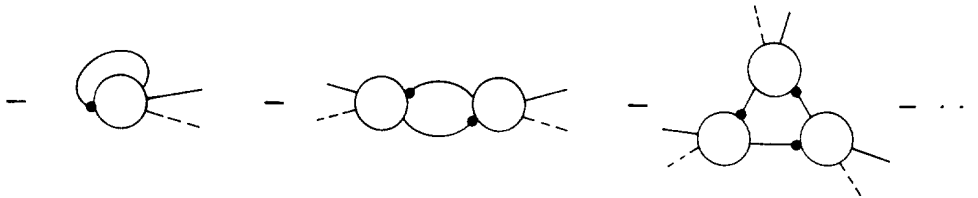
EXAMPLE 10.4.2 Some new diagrams:



Considering these diagrams one immediately notices that the factors  $V$  in the special vertices are always multiplied with the propagators  $-V^{-1}$ . This then gives as net result the simple propagator factor  $-1$  for the "ghost" connecting the  $f$  vertices. See Eq. (10.14). There is no momentum dependence in that propagator. If now the special vertices are local, i.e. only polynomial dependence on the momenta and no factors such as  $1/k^2$ , then these vertices can be represented by a point and the closed loop momentum integration becomes the integral over a polynomial. Within the regularization scheme to be introduced later such integrals are zero, and nothing survives.

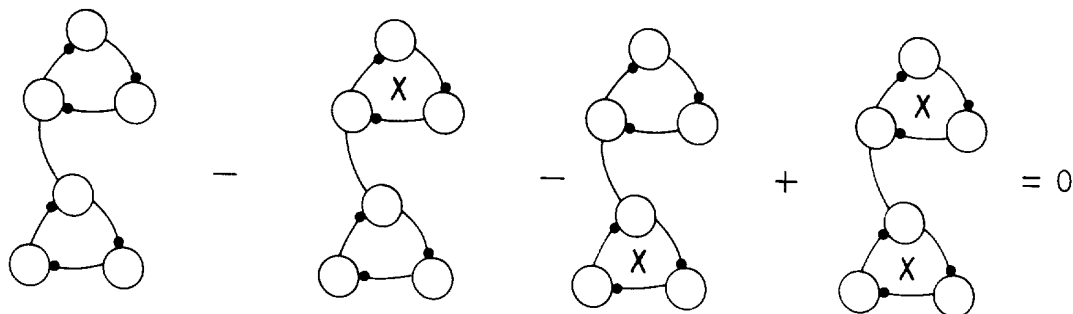
If, however, the field transformation is non-local these new diagrams survive and give an additional contribution with respect to what we had from the original Lagrangian.

How can we get back to the original Green's functions? At first sight it seems that we must simply subtract these diagrams, i.e. introduce new vertices in the transformed Lagrangian that produce precisely such diagrams, but with the opposite sign. This however ignores the possibility that the original  $\phi$ -line of a special vertex connects to any of these new vertices. The correct solution is quite simple: introduce vertices that reproduce the closed loops only, without the "wings", and give each of those closed loops a minus sign. Thus at the Lagrangian level the extra terms are as depicted below:



Now there is also the possibility that the original  $\phi$ -line of a special vertex connects to these counter loops. In this way counter "winged" diagrams arise automatically and need not to be introduced by hand.

EXAMPLE 10.4.3 The following cancellation occurs in case of a Lagrangian including counter closed loops:



The crosses denote counter closed loops.

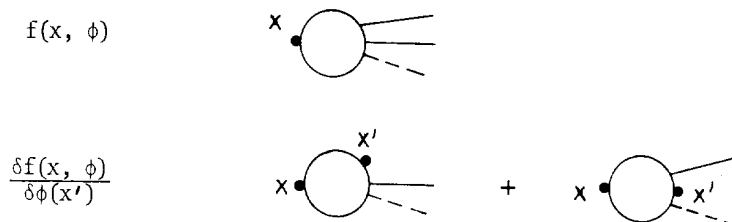
The solution may thus be summarized as follows. Start from a Lagrangian  $\mathcal{L}(\phi)$ . Perform the transformation  $\phi_i(x) \rightarrow \phi_i(x) + f_i(x, \phi)$ . Add a ghost Lagrangian  $\mathcal{L}_{\text{ghost}}$ . This ghost Lagrangian must be such that it gives rise to "wingless" closed loops of functions  $f$ , with one of the external lines removed, at which point another (or the same) function  $f$  is attached. The connection is by means of a propagator  $-1/(2\pi)^4 i$ . Of course, every function also carries a factor  $(2\pi)^4 i$ . Further there is a minus sign for every such closed loop.

As indicated, the vertices in such loops are the  $f$  with one line taken off.

Symbolically

$$\frac{\delta f_i(x, \phi)}{\delta \phi_i(x')} \cdot$$

Pictorially



This agrees indeed precisely with the result found with the help of path integrals.

It may be that the reader is somewhat worried about combinatorial factors in these cancellations. A well-known theorem states that combinatorial factors are impossible to explain; everyone must convince himself that the above ghost loop prescription leads exactly to the required cancellations. There is really nothing difficult here; we do not want to suggest that there is. A good guideline is always given by the path integral formulation. Another way is to convince oneself that, for every possibility of special lines and vertices connecting up, there is a similar possibility arising from the ghost Lagrangian. That is, the precise factor in front of any possibility is not relevant, as long as it is known that it is the same as found in the counterpart.

### 10.5 Concerning the rigour of the derivations

There is nothing mysterious about the previous derivations -- provided we remember that we are working within the context of perturbation theory. Thus the transformations must be such that the new Lagrangian is of the type as described at the beginning. In particular the quadratic part of the new Lagrangian must be such that propagators exist. For instance, a substitution of the form  $\phi \rightarrow \phi - \phi$  is clearly illegal. In short, the canonical transformation must be invertible.

There is a further problem connected with the  $i\epsilon$  prescription. The propagators are not strictly  $-V^{-1}$ , in the notation of the previous section, but have been modified through the  $i\epsilon$  addition. The factors  $V$  appearing in the special vertices must therefore be modified also such that the key-relation  $(-V^{-1})V = -1$  remains strictly true. In connection with gauge field theory the functions  $f$  themselves contain often factors  $V^{-1}$ ; careless handling of such factors may lead to errors. As we will see the correct  $i\epsilon$  prescription for Faddeev-Popov ghosts can be established by precise consideration of these circumstances.



## 11. THE ELECTROMAGNETIC FIELD

### 11.1 Lorentz gauge; Bell-Treiman transformations and Ward identities

The theorem about transformations of fields proved in the foregoing section applies to any Lagrangian  $\mathcal{L}$  and is quite general. We will now exploit its consequences in the case of the electromagnetic field Lagrangian, in order to derive in particular the so-called "generalized Ward identities" for the Green's functions of the theory.

We start considering the free electromagnetic field case, since it contains all the main features of the problems we want to study. This includes the case of interactions with other particles (e.g. electrons) provided these interactions are introduced in a gauge-invariant way.

The Lagrangian giving rise to the Maxwell equations is

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + A_\mu J_\mu . \quad (11.1)$$

As is well known, in trying to apply the canonical formalism to the quantization of this Lagrangian, many difficulties are encountered. Such difficulties are essentially due to some redundancy of the electromagnetic field variables, meaning that some combination of them is actually decoupled from the other ones and themselves.

Within the scheme defined in Section 2, to get rules for diagrams and then a theory from a given Lagrangian, such difficulties manifest themselves in the fact that the matrix  $V_{ij}$  that defines the propagator has no inverse in the case of the Lagrangian (11.1).

To avoid such problems conventionally one adds to the Lagrangian (11.1), with some motivations, a term  $-\frac{1}{2}(\partial_\mu A_\mu)^2$ . Then one obtains

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2} (\partial_\mu A_\mu)^2 = -\frac{1}{2} (\partial_\mu A_\nu)^2 + A_\mu J_\mu . \quad (11.2)$$

Now the propagator is well defined:

$$\mu \text{---} \overset{k}{\text{-----}} \text{---} \nu \quad \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{k^2 - i\epsilon}$$

and we have the vertex:

$$\text{---} \overset{k}{\text{-----}} \text{---} \quad (2\pi)^4 i \tilde{J}_\mu(k) .$$

This certainly defines a theory, but in fact we have no idea if the physical content is as that described by the Maxwell equations. This is mainly the subject we want to study now. In doing that we will encounter many other somewhat related problems which must be solved when constructing a theory for the electromagnetic field. The key to all these problems is provided by the generalized Ward identities.

In order to derive these identities we add to the Lagrangian (11.2) a free real scalar field  $B$ , coupled to a source  $J_B$ . We get

$$\mathcal{L}''(A, B) = -\frac{1}{2} (\partial_\mu A_\nu)^2 + J_\mu A_\mu + \frac{1}{2} B(\partial^2 - \mu^2)B + J_B B . \quad (11.3)$$

We will now perform a Bell-Treiman transformation. This is a canonical transformation that is in form also a gauge transformation. Here

$$A_\mu \rightarrow A_\mu - \epsilon \partial_\mu B \quad (11.4)$$

depending on a parameter  $\epsilon$ . The replacement (11.4) in  $\mathcal{L}''$ , Eq. (11.3), gives up to first order in this parameter

$$\mathcal{L}''(A_\mu - \epsilon \partial_\mu B, B) = \mathcal{L}''(A, B) + \epsilon (\partial_\mu A_\nu) \partial_\mu \partial_\nu B - \epsilon \partial_\mu B J_\mu . \quad (11.5)$$

As stated, the transformation (11.4) is in form just a gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda , \quad (11.6)$$

with  $\Lambda$  any function of  $x$ . Note that the first Lagrangian, Eq. (11.1), apart from the source term is invariant under such a transformation. On the contrary, the second Lagrangian (11.2), and *a fortiori* Eq. (11.3), is not gauge invariant because the added term  $-\frac{1}{2}(\partial_\mu A_\mu)^2$  is not. Precisely the source term and this  $-\frac{1}{2}(\partial_\mu A_\mu)^2$  term are responsible for the difference, in Eq. (11.5), between the original Lagrangian  $\mathcal{L}''(A_\mu, B)$  and the transformed one.

Let us come back to our local canonical transformation (11.4). The difference between the original and the new Lagrangian gives rise to the following extra vertices:

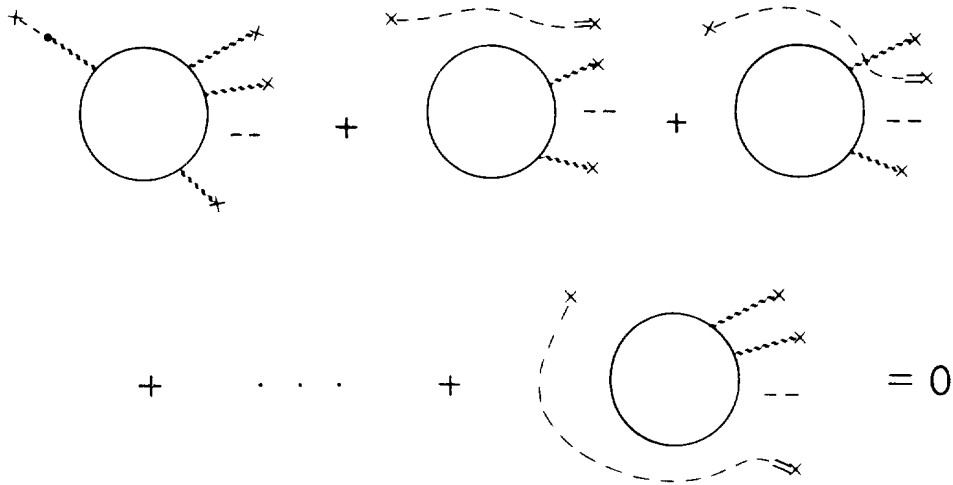
$$\begin{array}{l} \text{---} \frac{k}{\text{---}} \text{---} \bullet \text{---} \mu \qquad \qquad -\epsilon(2\pi)^4 k^2 k_\mu , \\ \text{---} \text{---} \text{---} \rightleftharpoons \qquad \qquad \epsilon(2\pi)^4 k_\mu \tilde{J}_\mu . \end{array} \quad (11.7)$$

Here the dotted lines denote the B-field, and the short double line at the sources denotes the gauge factor  $\partial_\mu$  appearing in the gauge transformation (11.6) in front of  $\Lambda$ .

[*Note:* A derivative  $\partial_\mu$ , acting on a field in an interaction term of the Lagrangian, gives  $i$  times the momentum of the field flowing towards the vertex.]

Of course the vertex term  $\epsilon(2\pi)^4 k_\mu J_\mu$  would give no contribution if the source of the electromagnetic field  $J_\mu$  were a "gauge-invariant source", namely if  $\partial_\mu J_\mu = 0$ . In the discussion which follows we want, however, to allow in general also "non-gauge-invariant sources", whose four-divergence is different from zero.

As a consequence of the general theorem proved in the foregoing section, the B-field remains a free field, just as in the theory described by the Lagrangian  $\mathcal{L}''$ . To first order in  $\epsilon$ , considering the n-point Green's function with one B source we have:



In the first diagram the  $B-A_\mu$  vertex is followed by a photon propagator. One has:

$$\begin{array}{c} \text{---} \xrightarrow{k} \text{---} \text{---} \mu \end{array} \quad -\epsilon(2\pi)^4 k^2 k_\mu \frac{1}{(2\pi)^4 i} \frac{1}{k^2} = i\epsilon k_\mu .$$

Apart from a sign this is precisely the same factor as occurring in the  $\partial_\mu B J_\mu$  vertex  $[\epsilon i k_\mu (2\pi)^4 i J_\mu]$  and we may use the same notation for it:

$$\text{---} \text{---} \text{---} \mu = \text{---} \text{---} \text{---} \mu$$

The resulting Ward identities are then:

Equation (11.8) shows a diagrammatic identity. On the left is a circle with a dashed line on the left and several dashed lines on the right. This is equal to a sum of two terms. The first term is a circle with a dashed line on the left and several dashed lines on the right, with a dashed line loop on top. The second term is a circle with a dashed line on the left and several dashed lines on the right, with a dashed line loop on the right. The equation is labeled (11.8).

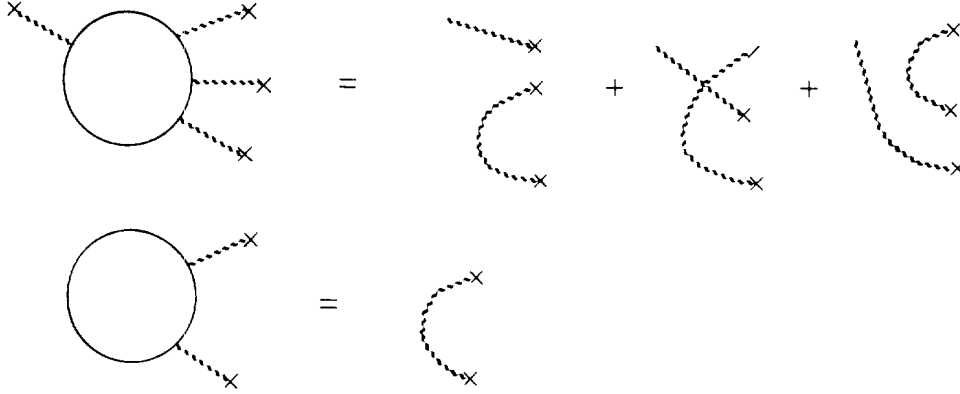
This result looks perhaps a little strange, since usually one tends to forget the lines that go straight through without interaction. As a matter of fact, this is allowed only in the case of gauge-invariant sources, for which, since then  $k_\mu J_\mu = 0$ , we have:

$$J_B \text{---} \text{---} \text{---} \mu = 0$$

Equation (11.8) is an equation for Green's functions. Of course in defining Green's functions one should not employ particular properties of sources, such as  $\partial_\mu J_\mu = 0$ . As we will see, the S-matrix will be defined using gauge-invariant sources for the electromagnetic field, and for such cases the right-hand side of Eq. (11.8) is zero.



We remark that the Ward identity, Eq. (11.8), is trivially true as it stands in the case of no interaction that we are considering. It is in fact sufficient to realize that here, for example for the four-point Green's function, we have:



It is worth while to note that the above derivation goes through also if electrons, or any other particles, are present, provided the photon is coupled to them in a gauge-invariant way. Let us consider in some details the introduction of the electrons into the theory. The Lagrangian to start from is

$$\mathcal{L}(A_\mu, \psi, \bar{\psi}, B) = \mathcal{L}''(A_\mu, B) - \bar{\psi}(\gamma\partial + m)\psi + ie\bar{\psi}\gamma^\mu\psi A_\mu + \bar{J}_e\psi + \bar{\psi}J_e,$$

where  $\mathcal{L}''(A_\mu, B)$  is given in Eq. (11.3). The new pieces of this Lagrangian involving the electron field, the source terms  $\bar{J}_e\psi + \bar{\psi}J_e$  excepted, are invariant under the transformation (11.4) if also the electron field is transformed

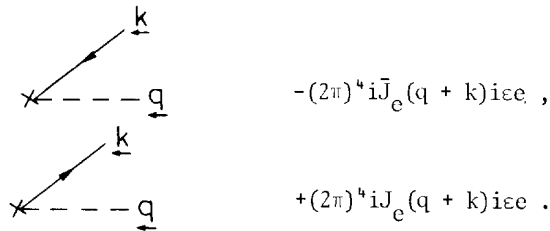
$$\psi \rightarrow e^{-i\epsilon e B}\psi \approx (1 - i\epsilon e B)\psi + O(\epsilon^2),$$

$$\bar{\psi} \rightarrow \bar{\psi}e^{+i\epsilon e B} \approx \bar{\psi}(1 + i\epsilon e B) + O(\epsilon^2).$$

Then, up to first order in  $\epsilon$ , we have

$$\begin{aligned} \mathcal{L}[A_\mu - \epsilon\partial_\mu B, (1 - i\epsilon e B)\psi, \bar{\psi}(1 + i\epsilon e B), B] &= \mathcal{L}(A_\mu, \psi, \bar{\psi}, B) + \\ &+ \epsilon(\partial_\mu A_\nu)\partial_\mu\partial_\nu B - \epsilon\partial_\mu B J_\mu - i\epsilon e \bar{J}_e\psi + i\epsilon e \bar{\psi}J_e. \end{aligned}$$

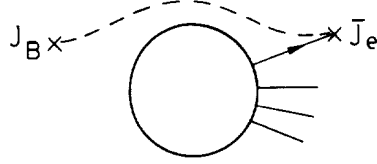
The extra vertices of the transformed Lagrangian are then the same as those of the free field case, given in (11.7), together with the following ones involving the electron sources:



$$-(2\pi)^4 i\bar{J}_e(q+k)i\epsilon e,$$

$$+(2\pi)^4 iJ_e(q+k)i\epsilon e.$$

Therefore, for Green's functions involving no external electron lines, the Ward identities are exactly the same as in Eq. (11.8), even if in this case the bubbles contain any number of closed electron loops. When external electron lines are present, the Ward identities receive additional contributions of the form:



### 11.2 Lorentz gauge: S-matrix and unitarity

Let us now investigate the S-matrix, keeping in mind for simplicity the free electromagnetic field Lagrangian  $\mathcal{L}$ , Eq. (11.2). The matrix K (see Sections 2 and 7) is here

$$K_{\mu\nu} = \delta_{\mu\nu} .$$

From the cutting equation, Section 7, we derive the fact that unitarity would hold if the sources were normalized according to Eq. (7.2). However, the complex conjugate of a four-vector is defined to have an additional minus sign in its fourth component (see the Introduction), so we are forced to attribute to the fourth component of the vector particle a negative metric (Section 8). The sources  $J_\mu^{(a)}$ ,  $a = 1, \dots, 4$  are chosen such that

$$\begin{aligned} \delta_{\mu\nu} J_\nu^{(a)} &= f^{(a)} J_\mu^{(a)} , \\ J_\mu^{*(a)} \delta_{\mu\nu} J_\nu^{(a)} &= \eta(a) = 1 \quad \text{for } a = 1, 2, 3 , \\ &= -1 \quad \text{if } a = 4 . \end{aligned} \tag{11.9}$$

On the other hand, it is well known that due to gauge invariance we do have a positive metric theory, but with only two photon polarizations. In the system where  $k_\mu = (0, 0, \kappa, ik_0)$ , we label the sources as follows

$$\begin{aligned} J_\mu^{(1)} &= (1, 0, 0, 0) , \\ J_\mu^{(2)} &= (0, 1, 0, 0) , \\ J_\mu^{(3)} &= (0, 0, 1, 0) , \\ J_\mu^{(4)} &= (0, 0, 0, i) . \end{aligned}$$

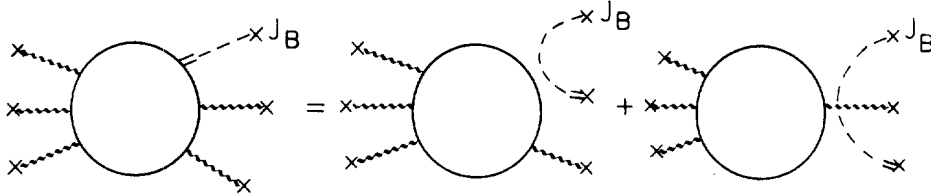
We now postulate that only the first two of these sources emit physical photons. In terms of a non-covariant object  $z_\mu = (0, 0, -\kappa, ik_0)$  obtained from  $k_\mu$  by space-reflection, we have

$$\sum_{a=1,2} J_\mu^{(a)} J_\nu^{(a)*} = \delta_{\mu\nu} + \frac{k_\mu z_\nu + z_\mu k_\nu}{(kz)} ,$$

whereas

$$K_{\mu\nu}^{-1} = \delta_{\mu\nu} .$$

Considering certain sets of cut diagrams we can apply the Ward identities to the left- and the right-hand side, to see that the terms proportional to  $k_\mu$  and  $k_\nu$ , respectively cancel among themselves. For instance, at the left-hand side of a cut diagram one can apply the Ward identity:



As before the double line indicates a factor  $ik_\mu$ . Note that all polarizations can occur at the intermediate states, i.e. the out-states for  $S$  and the in-states for  $S^\dagger$ , but the other lines are physical.

In our case the right-hand side of this Ward identity is zero because the sources on the right-hand side absorb energy only, and therefore:

$$= 0$$

So, due to the Ward identities, one may replace the factor  $K_{\mu\nu}^{-1}$  in the intermediate states, found from the cutting equation, by  $J_\mu^{(1)} J_\nu^{(1)} + J_\mu^{(2)} J_\nu^{(2)}$ , which implies unitarity in a Hilbert space with only transverse photons.

We must make a slight distinction between physical sources and gauge-invariant sources. The combination

$$J_\mu = k_0 J_\mu^{(3)} + \kappa J_\mu^{(4)} = (0, 0, k_0, i\kappa)$$

is gauge invariant because  $\partial_\mu J_\mu = 0$ , but on mass shell this source is proportional to  $k_\mu$  and it gives no contribution due to the Ward identities, so despite its gauge invariance, it is unphysical, in the sense that it emits nothing at all, not even ghosts.

### 11.3 Other gauges: the Faddeev-Popov ghost

Before going on, let us come back for a moment and try to interpret the Ward identity we have proved. Starting from a gauge-invariant Lagrangian, Eq. (11.1), we added the term

$$-\frac{1}{2} (\partial_\mu A_\mu)^2$$

in order to define a propagator. The Ward identities follow by performing a Bell-Treiman transformation, i.e. a canonical transformation that is also a gauge transformation. As we already noted, only the above term (and the source term) gives rise to a contribution, namely  $\varepsilon(\partial_\mu A_\mu)(\partial^2 B)$ . This is the only coupling of the B-field formally appearing in the transformed Lagrangian. Now the fact that B remains a free field, i.e. gives zero when part of a Green's function, implies that  $\partial_\mu A_\mu$  is also free. This is actually the content of the Ward identity. Therefore we conclude that the addition of the term  $-\frac{1}{2}(\partial_\mu A_\mu)^2$  does not change the physics of the theory.

This defines our starting point. If, due to a gauge invariance, a Lagrangian is singular, i.e. the propagators do not exist, then a "good" Lagrangian can be obtained by adding a term  $-\frac{1}{2}C^2$ , where C behaves under gauge transformations as  $C \rightarrow C + \hat{t}\Lambda$ . Here  $\hat{t}$  is any field-independent quantity that may contain derivatives. The argument given above, showing that the addition of the term  $-\frac{1}{2}(\partial_\mu A_\mu)^2$  does not modify the physical content of the theory, can equally well be applied here. C will appear to be a free field, as can be seen by performing a Bell-Treiman transformation. However, for this simple recipe to be correct, as in the case explicitly considered  $C = \partial_\mu A_\mu$ , one needs  $\hat{t}$ , defined by the gauge transformation  $C \rightarrow C + \hat{t}\Lambda$ , not to depend on any fields.

The difficulty with Yang-Mills fields is that the gauge transformations are more complicated, so much so that no simple C with the required properties exists. [Actually there exists a choice of C for Yang-Mills fields that is acceptable from this point of view, namely  $C = \alpha A_3$ ,  $\alpha \rightarrow \infty$ . See R.L. Arnowitt and S.I. Fickler, Phys. Rev. 127, 1821 (1962).] For this reason we will now study quantum electrodynamics using a gauge function C that has more complicated properties under gauge transformations.

We take for understood the theory corresponding to the Lagrangian

$$\mathcal{L}(A_\mu) = \mathcal{L}_{\text{inv}}(A_\mu) - \frac{1}{2} C^2, \quad C = \partial_\mu A_\mu. \quad (11.10)$$

The sources will be taken to be gauge invariant and are included in  $\mathcal{L}_{\text{inv}}$ . Later we will consider also non-gauge-invariant sources.

Let us now suppose that we would like to have  $\partial_\mu A_\mu + \lambda A_\mu^2$  instead of  $\partial_\mu A_\mu$  for the function C. We can go from the above Lagrangian with  $C = \partial_\mu A_\mu$  to the case of  $C = \partial_\mu A_\mu + \lambda A_\mu^2$  by means of a non-local Bell-Treiman transformation

$$A_\mu \rightarrow A_\mu + \lambda \partial^{-2} \partial_\mu (A_\nu)^2,$$

or, more explicitly,

$$A_\mu \rightarrow A_\mu - i\lambda \partial_\mu \int d_4 x' \Delta(x - x') A_\nu^2(x'), \quad (11.11)$$

with

$$\Delta(x - x') = \frac{1}{(2\pi)^4 i} \int d_4 k e^{ik(x-x')} \frac{1}{k^2} .$$

Here we have a situation as described in Subsection 10.5. For the subsequent manipulations to be true we must in fact supply a  $-i\epsilon$  to the denominator. Note that

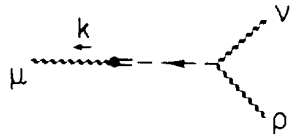
$$\partial_x^2 \Delta(x - x') = i\delta_4(x - x') .$$

$\mathcal{L}_{inv}$  is unchanged under this somewhat strange gauge transformation, and

$$\partial_\mu A_\mu \rightarrow \partial_\mu A_\mu + \lambda A_\mu^2 .$$

In view of the structure of our canonical transformation we may expect that a ghost Lagrangian must be added.

Performing the transformation (11.11) one gets to first order in  $\epsilon$  the following special vertex (see Section 10):

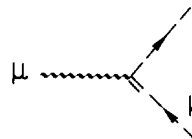


$$-2\lambda(-1)(2\pi)^4 ik^2 ik_\mu \frac{1}{k^2} \delta_{\nu\rho} .$$

Here, as usual, the dot indicates the factor  $V_{ij}$  which is minus the inverse photon propagator, and the short double line a factor  $ik_\mu$ . The dotted line represents the function  $\Delta(x - x')$ , which is just like a scalar massless particle propagator. The theorem proved in Section 10 shows that the theory described by the transformed Lagrangian remains unchanged if we also provide for ghost loops, constructed by connecting the original photon lines to one of the other photon lines of the special vertex. For example, we have:



Here we have cancelled the photon propagators against the dots, so that the new vertex:



$$(2\pi)^4 i2\lambda ik_\mu$$

appears which can be formed by introducing a massless complex field  $\phi$  interacting with the photon via the interaction Lagrangian  $\mathcal{L}_I = 2\lambda\phi^* A_\mu \partial_\mu \phi$ . All the closed loops constructed this way must have a minus sign in front.





with  $\mathcal{M}$  some operator, which in the actual case depends on the fields  $A_\mu$ . Then the ghost Lagrangian is simply

$$\phi^* \mathcal{M} \phi .$$

Here

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \Lambda , \\ C = \partial_\mu A_\mu + \lambda A_\mu^2 &\rightarrow C + \partial^2 \Lambda + 2\lambda A_\mu \partial_\mu \Lambda + O(\Lambda^2) , \end{aligned} \quad (11.13)$$

from which (11.12) follows.

The generality of this prescription can be understood by looking at our manipulations. The non-local transformation (11.11) was taken such that  $\partial_\mu A \rightarrow \partial_\mu A_\mu + \lambda A_\mu^2$ . If we consider a gauge transformation

$$\partial_\mu A_\mu \rightarrow \partial_\mu A_\mu + \partial^2 \Lambda ,$$

then we must solve the equation

$$\partial^2 \Lambda = e A_\mu^2 .$$

The fact that the propagator  $1/k^2$  appeared in our loops is thus simply related to the behaviour  $\partial_\mu A_\mu \rightarrow \partial_\mu A_\mu + \partial^2 \Lambda$ . Secondly, the special vertices are

$$A_\mu V_{\mu\nu} \partial_\nu \Lambda ,$$

where  $V_{\mu\nu}$  is in the usual way related to the photon propagator. Then the vertices appearing in the ghost loops arise from the prescription: remove one factor  $A_\mu$  from the expression  $\lambda A_\mu^2$  and replace it by  $\partial_\mu \Lambda$ . Diagrammatically:



A factor of 2 provides for the two possible ways of removing an  $A_\mu$ -field from  $\lambda A_\mu^2$ . This is indeed precisely what one obtains to first order in  $\Lambda$  by submitting  $\lambda A_\mu^2$  to a gauge transformation, see Eq. (11.13).

The beauty of this recipe to find the ghost Lagrangian is that there is no reference to the function  $C$  that we started from. To construct the ghost Lagrangian one needs only the  $C$  actually required. Also we need only infinitesimal gauge transformations.

EXAMPLE 11.3.2 The above prescription works also in the case of  $C = \partial_\mu A_\mu$  that we started from. In fact we have

$$\partial_\mu A_\mu \rightarrow \partial_\mu A_\mu + \partial^2 \Lambda ,$$

meaning that the ghost Lagrangian should be

$$\mathcal{L}_\phi = \phi^* \partial^2 \phi$$



with no coupling however of the  $\phi$ -field to the electromagnetic field. Therefore, as we already know, no ghost loop is to be added in this case.

Even if the above manipulations are quite solid one may have some doubts concerning the final prescription for the ghost Lagrangian. To establish correctness of the theory, as we did in the case of  $C = \partial_\mu A_\mu$ , Ward identities are needed. In the general case such identities are more complicated than those given before and we will call them Slavnov-Taylor identities.

#### 11.4 The Slavnov-Taylor identities

In this section we will derive the Slavnov-Taylor identities, using only local canonical transformations. One can always keep in mind the explicit case of  $C = \partial_\mu A_\mu + \lambda A_\mu^2$ , but we now start adapting a general notation anticipating the general case.

Let there be given a function  $C$ , that behaves under a general gauge transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$  as follows

$$C \rightarrow C + (\hat{m} + \hat{\ell})\Lambda + O(\Lambda^2) . \quad (11.14)$$

Here we split up the earlier factor  $\mathcal{M} = (\hat{m} + \hat{\ell})$  with the part  $\hat{\ell}$  depending on the field  $A_\mu$ . For  $C = \partial_\mu A_\mu + \lambda A_\mu^2$ , we have  $\hat{m} = \partial^2$  and  $\hat{\ell} = 2\lambda A_\mu \partial_\mu$ .

Our starting point will be the Lagrangian

$$\mathfrak{L}(A_\mu) = \mathfrak{L}_{\text{inv}}(A_\mu) - \frac{1}{2} C^2 + \phi^* \hat{m} \phi + \phi^* \hat{\ell} \phi + J_\mu A_\mu \quad (11.15)$$

with the -1/loop prescription for ghost loops. As before, we add to this Lagrangian a piece describing a free field  $B$

$$\frac{1}{2} B(\partial^2 - \mu^2)B + J_B B$$

and perform the local Bell-Treiman transformation

$$A_\mu \rightarrow A_\mu + \hat{\mathfrak{t}}_\mu B$$

with  $\hat{\mathfrak{t}}_\mu = \partial_\mu$ .

Working to first order in the field  $B$

$$\mathfrak{L}(A_\mu + \partial_\mu B) = \mathfrak{L}(A_\mu) - C(\hat{m} + \hat{\ell})B + \phi^*(\hat{\mathfrak{e}} + \hat{\mathfrak{d}})B\phi + J_\mu \partial_\mu B . \quad (11.16)$$

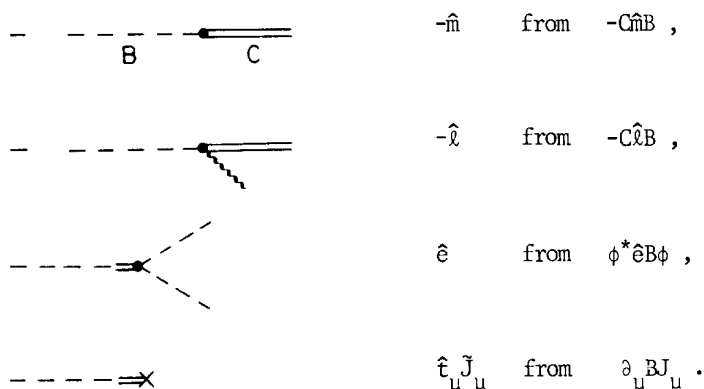
Here we used the transformation

$$\hat{\ell} \rightarrow \hat{\ell} + (\hat{\mathfrak{e}} + \hat{\mathfrak{d}})B + O(B^2) .$$

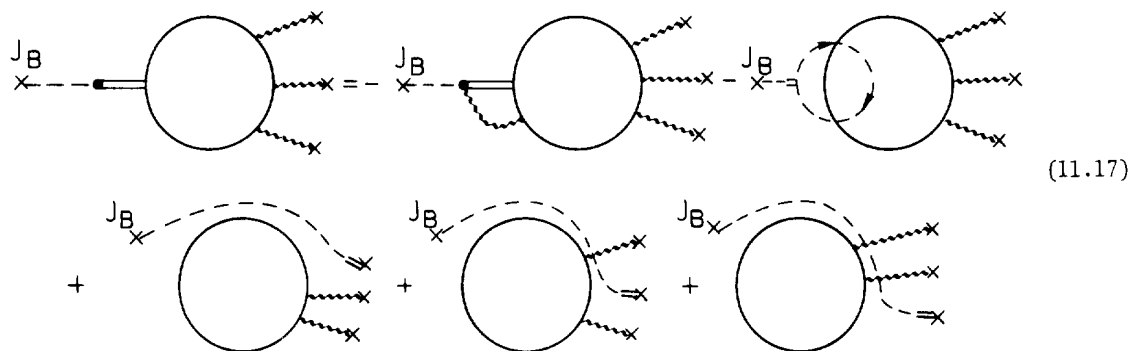
Again we distinguish the field-independent part  $\hat{\mathfrak{e}}$  and the field-dependent part  $\hat{\mathfrak{d}}$ . In the actual case of  $\hat{\ell} = \lambda A_\mu \partial_\mu$  we have

$$\hat{\mathfrak{e}}B = 2\lambda \partial_\mu B \partial_\mu , \quad \hat{\mathfrak{d}} = 0 .$$

The vertices involving the B-field in the transformed Lagrangian (11.16) are [the over-all factor  $(2\pi)^4 i$  is always left understood]:



The vertex corresponding to  $\phi^* \hat{d} B \phi$  is not drawn since in the actual case ( $C = \partial_\mu A_\mu + \lambda A_\mu^2$ )  $\hat{d}$  is zero. The vertices  $C\hat{m}B$  and  $C\hat{\ell}B$  appear with a minus sign in the Lagrangian: this sign is not included in the vertex definition and thus must be provided separately. The statement that B remains a free field becomes then to first order in the B-vertices:



In the second term at the right-hand side we exhibited explicitly the whole loop to which the B-line is attached, together with the associated minus sign. The blobs are built up from diagrams containing propagators and vertices as given in the previous section. The  $\phi$  particles go around in loops only, and such loops are included in the blobs.

In the above equation there is one crucial point. Consider the very simplest case, no interaction and only one photon source. Since  $C = \partial_\mu A_\mu + \lambda A_\mu^2$ , one has to zeroth order in  $\lambda$ :

$$\times \text{---} \bullet \text{---} \times \left( = - \times \text{---} \bullet \text{---} \hat{m} \text{---} \times \right) = \times \text{---} \text{---} \times$$

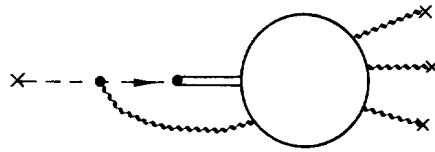
The first diagram (similarly to the second one) contains the B propagator pole, a photon propagator pole, the vertex factor  $(2\pi)^4 i \hat{m}$  and the factor  $\hat{t}_\mu$ . The last diagram has only

the B pole and a factor  $\hat{t}_\mu = \partial_\mu$ . The equation can be true only if  $\hat{m}$  is zero precisely on the photon pole. We had  $\hat{m} = \partial^2 + -k^2$ , and we must therefore always take  $(2\pi)^4 i \hat{m} = (2\pi)^4 i (\partial^2 + i\epsilon) \rightarrow -(2\pi)^4 i (k^2 - i\epsilon)$ . As will be seen, this  $\hat{m}$  is minus the inverse of the ghost propagator. In the general case, considering lowest order identities, one must check on the  $i\epsilon$  in the propagators in the manner described here.

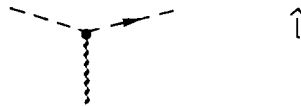
We will perform some manipulations on Eq. (11.17). Consider the first diagram on the right-hand side. Using:

$$\frac{\text{---}}{C} = - \frac{\text{---} \xrightarrow{\hat{m}} \text{---}}{-\hat{m}^{-1} C}$$

the contribution of this diagram becomes:



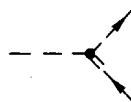
with a new vertex:



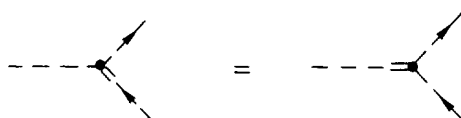
Treating this first vertex as a photon source we see that we can iterate, getting:

$$\begin{aligned} & \text{Diagram 1} = - \text{Diagram 2} + \\ & + \text{Diagram 3} + \sum_J \text{Diagram 4} + \text{Diagram 5} \end{aligned} \tag{11.18}$$

$\sum_J$  means sum over all the photon sources. The last diagram arises because the first vertex is treated as a source. Some new vertices enter, whose meaning is unambiguously defined. For example, we have:

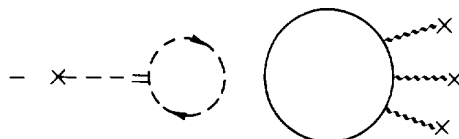


which correspond precisely to a vertex  $(2\pi)^4 i \hat{e}$  or to an interaction term  $\phi^* \partial_\mu B \partial_\mu \phi$ . In particular, since this term is invariant under the interchange  $B \rightarrow \phi$ , we have:



This equality, trivial in the case of quantum electrodynamics, becomes much more tricky in the case of non-Abelian gauge symmetries. Then a similar identity follows from the group structure, and is closely related to the Jacobi identity for the structure constants of the group.

The right-hand side of Eq. (11.18) can now be inserted in place of the first diagram on the right-hand side of the original Eq. (11.17). We note that some cancellations occur. The second diagram on the right-hand side of Eq. (11.17) contains also the configuration where the ghost has no further interaction:

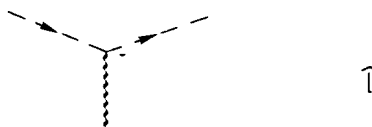


This cancels precisely the last diagram of the iterated equation, after insertion in Eq. (11.17).

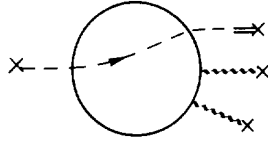
This process can be repeated indefinitely and one finally arrives at the equation:

The equation is labeled (11.19) on the right. It shows a diagrammatic expansion. On the left, a vertex labeled  $J_B$  with a dashed line and 'x' enters from the left into a solid circular loop. This loop has three wavy lines with 'x' marks extending to the right. This is equal to a sum over  $J$  of a vertex with a dashed line and 'x' entering from the left into a solid circular loop with a dashed line passing through it, plus a sum over  $J$  of a vertex with a dashed line and 'x' entering from the left into a solid circular loop with a dashed line passing through it and a wavy line with 'x' extending to the right.

The ghost line going through the diagram may have zero, one, two, etc., vertices of the type:



If one notes now that the mass of the B-field is an absolutely free parameter in our discussion, we can also give now the B-field the propagator  $-(2\pi)^4 i \hat{m}^{-1}$ . In this way we can simply reduce the first diagram on the right-hand side of Eq. (11.19) to:



and also include in it the last diagram of Eq. (11.19). Observing that:

$$\begin{array}{c} \times \\ \text{---} \\ \hat{m} \\ \text{---} \\ -\hat{m}^{-1} \\ \text{---} \\ \text{C} \end{array} \text{ (circle with solid arrow) } = - \text{ (circle with dashed arrow) } \begin{array}{c} \times \\ \text{---} \\ \text{C} \end{array}$$

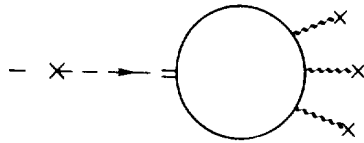
we get the identity:

$$0 = \begin{array}{c} \times \\ \text{---} \\ \text{C} \end{array} \text{ (circle) } + \begin{array}{c} \times \\ \text{---} \\ \hat{m} \\ \text{---} \\ -\hat{m}^{-1} \\ \text{---} \\ \text{C} \end{array} \text{ (circle with solid arrow) } + \begin{array}{c} \times \\ \text{---} \\ \text{C} \end{array} \text{ (circle with dashed arrow) } + \begin{array}{c} \times \\ \text{---} \\ \text{C} \end{array} \text{ (circle with dashed arrow) } \quad (11.20)$$

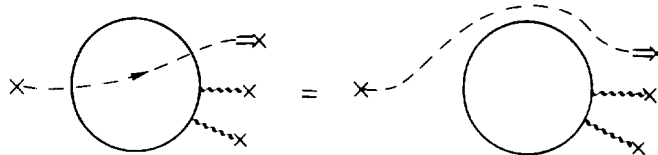
*Note:* In the case of  $C = \partial_\mu A_\mu$  this equation coincides with the Ward identity (11.8). One has indeed:

$$\begin{array}{c} \times \\ \text{---} \\ \underline{k} \\ \text{---} \\ \mu \end{array} = (2\pi)^4 i \tilde{J}_B \frac{1}{(2\pi)^4 i k^2} (-i k_\mu) = - \begin{array}{c} \times \\ \text{---} \\ \underline{k} \\ \text{---} \\ \mu \end{array}$$

so that the first diagram of Eq. (11.20) can also be drawn as:



Furthermore, in this case of  $C = \partial_\mu A_\mu$  the ghost does not interact and then:



Equation (11.20) can be generalized as follows. Suppose that a source is coupled to several electromagnetic fields, for instance

$$JA_{\mu}^2 \quad \text{or} \quad JA_{\mu}^4$$

or even more general to a local but otherwise arbitrary function  $R(A)$  of the  $A_{\mu}$ . Example:

$$J(\partial_{\mu}A_{\mu} + \lambda A_{\mu}^2 + \kappa A_{\mu}^4, \dots) .$$

The only way this comes into the above derivation is through the behaviour of this whole term under gauge transformations. Suppose one has in the Lagrangian (11.15) also the coupling

$$JR(A_{\mu}) .$$

Furthermore, under a gauge transformation

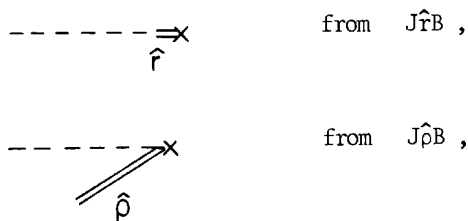
$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda ,$$

one has

$$R \rightarrow R + \hat{r}\Lambda + \hat{\rho}\Lambda .$$

$\hat{r}$  is the part independent of the  $A_{\mu}$  and  $\hat{\rho}$  contains the  $A_{\mu}$  dependent parts. Both  $\hat{r}$  and  $\hat{\rho}$  may contain derivatives.

Now, performing the Bell-Treiman transformation with the B-field one obtains the vertices:



where the double line denotes a collection of possibilities, including at least one photon line. The whole derivation can be carried through unchanged, and given many sources  $J^a$  coupled to many field functions  $R_a$  we get:

$$\text{Diagram 1} + \sum_a \text{Diagram 2} + \text{Diagram 3} = 0 \quad (11.21)$$

where  $\hat{r}_a$  and  $\hat{\rho}_a$  are defined by the behaviour of the functions  $R_a$  under gauge transformations

$$\text{if } A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda , \quad R_a \rightarrow R_a + \hat{r}_a\Lambda + \hat{\rho}_a(A_{\mu})\Lambda .$$

The above identities are the Slavnov-Taylor identities.

### 11.5 Equivalence of gauges

From the preceding sections we obtain the following prescription for handling a theory with gauge invariance. First choose a non-gauge-invariant function  $C$  and add  $-\frac{1}{2}C^2$  to the Lagrangian. Next consider the properties of  $C$  under infinitesimal gauge transformations, for instance

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \Lambda, \\ C &\rightarrow C + (\hat{m} + \hat{\ell})\Lambda. \end{aligned}$$

A ghost part must also be added to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \frac{1}{2} C^2 + \phi^* (\hat{m} + \hat{\ell})\psi,$$

with the prescription of providing a factor  $-1$  for every closed  $\phi$  loop. Clearly the choice of  $C$  is limited by the fact that the operator  $\hat{m}$ , defining the ghost propagator, must have an inverse. Moreover  $C$  must together with  $\mathcal{L}_{\text{inv}}$  define non-singular  $A$  propagators, but this is automatic if  $C$  breaks the gauge invariance.

Suppose we had taken a slightly different  $C$

$$C' = C + \epsilon R.$$

Now under a gauge transformation

$$C' \rightarrow C' + (\hat{m} + \hat{\ell})\Lambda + \epsilon(\hat{r} + \hat{\rho})\Lambda,$$

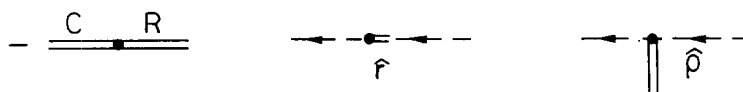
where  $\hat{r}$  and  $\hat{\rho}$  are, respectively, the field-independent and field-dependent parts resulting from a gauge transformation of  $R$ . For example, if  $R = A_\mu^2$  then  $R \rightarrow R + 2A_\mu \partial_\mu \Lambda$  and we have  $\hat{r} = 0$  and  $\hat{\rho} = 2A_\mu \partial_\mu$ .

The ghost Lagrangian must be changed accordingly, and we get

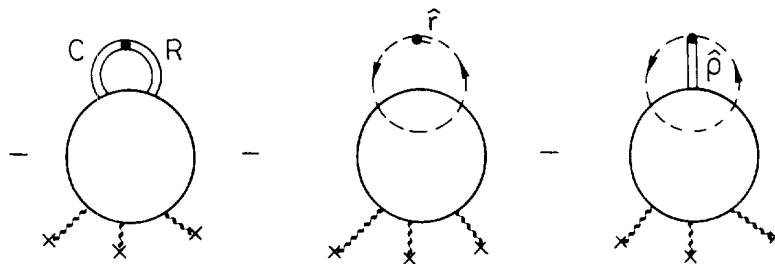
$$\mathcal{L}' = \mathcal{L}_{\text{inv}} - \frac{1}{2} (C + \epsilon R)^2 + \phi^* (\hat{m} + \hat{\ell} + \epsilon\hat{r} + \epsilon\hat{\rho})\phi. \quad (11.22)$$

We now prove that to first order in  $\epsilon$  the  $S$ -matrix generated by  $\mathcal{L}'$  equals the  $S$ -matrix generated by  $\mathcal{L}$ . This is clearly sufficient to have equivalence of any two gauges that can be connected by a series of infinitesimal steps.

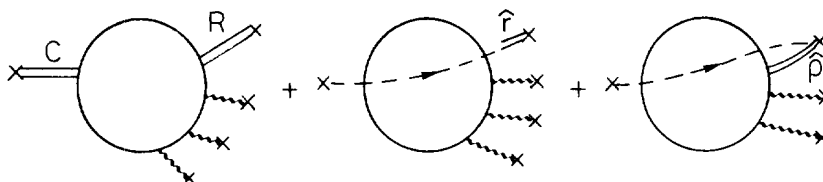
Let us compare the Green's functions of  $\mathcal{L}'$  with those of  $\mathcal{L}$ . The difference is given by Green's functions containing one  $\epsilon$ -vertex. From Eq. (11.22) these vertices are:



The difference between an  $\ell'$  and  $\ell$  Green's function with the same external legs is then:



We exhibited explicitly the minus sign associated with the ghost loops. Opening up the top vertex this difference is:



If now all the original sources are gauge invariant, namely  $\partial_\mu J_\mu = 0$ , we see that this difference is zero as a consequence of the Slavnov-Taylor identities, Eq. (11.21). The diagrams in Eq. (11.21), where the ghost line is attached to such currents, give no contribution, since for these currents we have  $\hat{r}_\mu = \partial_\mu$  (and  $\hat{r}_\mu J_\mu = 0$ ) and  $\hat{\rho} = 0$ . As the S-matrix is defined on the basis of gauge-invariant sources, we see that the S-matrix is invariant under a change of gauge as given above.

### 11.6 Inclusion of electrons; wave-function renormalization

The preceding discussion has been carried out in such a way that the inclusion of electrons changes practically nothing. The main difference is that we now must introduce sources that emit or absorb electrons and such sources are not gauge invariant. This complicates somewhat the discussion of the equivalence of gauges.

Thus consider electron source terms

$$J_e \psi + \bar{\psi} J_e .$$

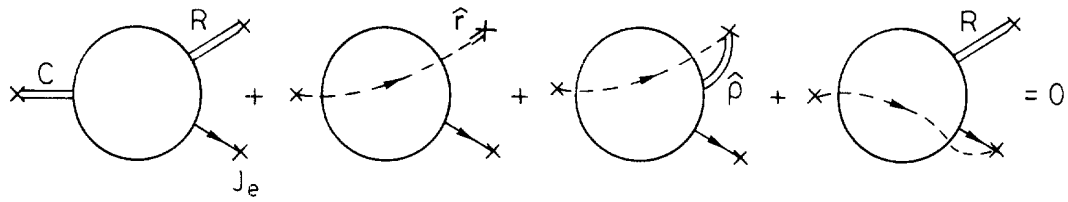
Under gauge transformations ( $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ )

$$\psi \rightarrow e^{ie\Lambda} \psi \simeq \psi + ie\Lambda \psi ,$$

$$\bar{\psi} \rightarrow \bar{\psi} - ie\bar{\psi} \Lambda .$$

Let us consider the difference of the Green's functions of two different gauges in the presence of an electron source. For simplicity we will only draw one of them explicitly. The Slavnov-Taylor identity is:



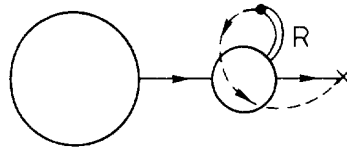


The first three terms are precisely those found in considering the difference of the Green's functions, or more precisely of objects obtained when unfolding a vertex in the difference of the Green's function (see preceding section).

Folding back the C and R source to obtain again the true difference of the Green's function and using the Slavnov-Taylor identity we get:

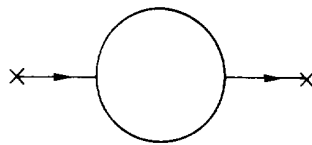
$$\mathcal{G}_{\mathcal{F}'} - \mathcal{G}_{\mathcal{F}} = \text{Diagram}$$

In general, such a type of diagram will have no pole as one goes to the electron mass-shell. Thus passing to the S-matrix the contribution of most diagrams will disappear. But included in the above set are also diagrams of the type:



which do have a pole. However, we must not forget that the S-matrix definition is based on the two-point function. The sources must be such that the residue of the two-point function including the sources is one.

Consider the two-point function for the electron:



Also this function will change, and for any of the two sources we will have precisely the same change as above:

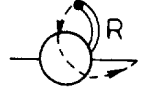
$$\Delta \left( \text{Diagram} \right) = \text{Diagram} + \text{Diagram}$$

In accordance with our definition of the S-matrix we must redefine our sources such that the residue of the two-point function remains one; that is

$$J' = J - \delta ,$$

where

$\delta$  = value at the pole of



It is seen that including the redefinition of the sources the S-matrix is unchanged under a change of gauge.

In conventional language, the above shows that the electron wave-function renormalization is gauge dependent, but that is of no consequence for the S-matrix.

12. COMBINATORIAL METHODS

There are essentially three levels of sophistication with which one can do combinatorics. On the first level one simply uses identities of vertices that are a consequence of gauge invariance. Example: electrons interacting with photons. The part of the Lagrangian containing electrons is

$$-\bar{\psi}(\gamma\partial + m)\psi + ieA_{\mu}\bar{\psi}\gamma^{\mu}\psi .$$

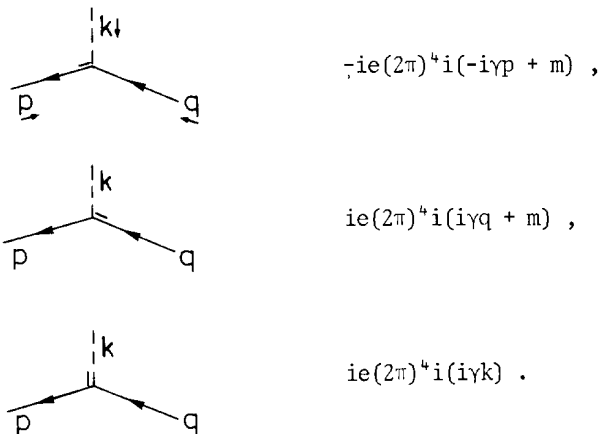
Now write

$$\psi \rightarrow \psi + ieB\psi , \quad \bar{\psi} \rightarrow \bar{\psi} - ieB\bar{\psi} , \quad A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}B .$$

Of course this Lagrangian remains invariant under these transformations, but we want to understand this in terms of diagrams. One has, not cancelling anything, as extra terms

$$-ie\bar{\psi}(\gamma\partial + m)B\psi + ie\bar{\psi}B(\gamma\partial + m)\psi + ie\partial_{\mu}B\bar{\psi}\gamma^{\mu}\psi .$$

These are simply vertices:



How does the cancellation manifest itself? In the Lagrangian one must write

$$\partial_{\mu}(B\psi) = \partial_{\mu}B\psi + B\partial_{\mu}\psi$$

to see it. Here we take the first vertex and write

$$p = -q - k .$$

Then one sees explicitly how the three vertices together give zero

$$-ie(2\pi)^4 i\{i\gamma q + i\gamma k + m - i\gamma q - m - i\gamma k\} = 0 ,$$

and this remains true if we replace m everywhere by m - iε. Only then can one say that the vertices contain factors which are inverse propagators; now we know for sure that the following Green's function identity holds (in lowest order):

The diagram shows an equality between a ghost loop and three ghost tree diagrams. On the left, a circle represents a ghost loop with an inverted propagator (short double line) on the left and two external lines with 'x' marks on the right. This is equal to the sum of three diagrams: 1) a tree with an inverted propagator on the left and two external lines on the right; 2) a tree with an inverted propagator on the left, a vertical internal line, and two external lines on the right; 3) a tree with an inverted propagator on the left, a vertical internal line, and two external lines on the right, with the internal line and one external line swapped.

where the short double line indicates the inverted propagator including  $i\epsilon$ . Exploiting this fact we obtain:

The diagram shows an equality between a ghost tree diagram and two other ghost tree diagrams. On the left, a tree with an inverted propagator on the left and two external lines on the right. This is equal to the sum of two diagrams: 1) a tree with a dashed line on the left, a vertical internal line, and two external lines on the right; 2) a tree with a dashed line on the left, a vertical internal line, and two external lines on the right, with the internal line and one external line swapped.

which is a Ward identity given before.

The above makes clear how the very first level combinatorics is the building block for the second-level combinatorics. This second-level combinatorics uses the fact that all terms cancel when one has gauge invariance. But this fact must be verified explicitly by means of first-level combinatorics to ascertain that the  $i\epsilon$  prescription is consistent with the gauge invariance. It is just by such a type of reasoning that the  $i\epsilon$  in the ghost propagator is fixed, as shown before.

The third level of combinatorics is that when one uses non-local canonical transformations. These can be used to derive the Slavnov-Taylor identities directly, as was done by Slavnov. One must be very careful about the  $i\epsilon$  prescription; our procedure whereby this identity was derived by means of first- and second-level combinatorics shows that the usual  $-i\epsilon$  prescription for the ghost propagator is the correct one.

### 13. REGULARIZATION AND RENORMALIZATION

#### 13.1 General remarks

As noted before, the regulation scheme introduced in the beginning is not gauge invariant. We need a better scheme that can also be used in case of gauge theories, Abelian or non-Abelian. An elegant method is the dimension regularization scheme which we will discuss now.

A good regularization scheme must be such that for certain values of some parameters (the masses  $\Lambda$  in the unitary cut-off method) the theory is finite and well defined. The physical theory obtains in a certain limit (masses becoming very big), and one requires that quantities that were already finite before regularization was introduced remain unchanged in this limit.

In order to obtain a finite physical theory it will be necessary to introduce counterterms in the Lagrangian. A theory is said to be renormalizable if by addition of a finite number of kinds of counterterms a finite physical theory results. This physical theory must of course not only be finite, but also unitary, causal, etc. With regard to a regularization and a renormalization procedure for a gauge theory, some problems arise which are peculiar to this kind of theory. As demonstrated before, the Ward identities (or more generally the Slavnov-Taylor identities) must hold, because they play a crucial role in proving unitarity. Since regularization implies the introduction of new rules for diagrams, and because new vertices, corresponding to the renormalization counterterms are added, it becomes problematic as to whether our final renormalized theory satisfies Ward identities. As far as the regularization procedure is concerned, a first step is to provide a scheme in which the Ward identities are satisfied for any value of the regularization parameters.

This goal is obtained in the dimensional regularization scheme. It must be stressed, however, that this does not guarantee that the counterterms satisfy Ward identities and indeed, in general, one has considerable difficulty in proving Ward identities for the renormalized theory. The point is this: in the unrenormalized theory there exist Ward identities, and an invariant cut-off procedure guarantees that the counterterms satisfy certain relations. From these relations one can derive new Ward identities satisfied by the renormalized theory. However, these Ward identities turn out to be different: one may speak of renormalized Ward identities. That is, one can prove that the renormalized theory has a certain symmetry structure (giving rise to certain Ward identities), but one has to show also that this symmetry is the same as that of the unrenormalized theory. Let us remark once more that Ward identities are needed for the renormalized theory because they are needed in proving unitarity. Indeed, Ward identities have nothing to do with renormalizability but everything to do with unitarity.

Another problem to be considered is the following one. If one admits the possibility that the renormalized symmetry is different from the unrenormalized one, then formally the following can happen. If one carries through a renormalization program, one must first make a choice of gauge in the unrenormalized theory. Perhaps then the symmetry of the renormalized Lagrangian depends on the initial choice of gauge. Given a gauge theory one must show explicitly that this is not the case, and that the various renormalized

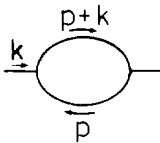
Lagrangians belonging to different gauge choices in the unrenormalized theory are related by a change of gauge with respect to the renormalized symmetry. In quantum electrodynamics this problem is stated as the gauge independence of the renormalized theory.

### 13.2 Dimensional regularization method: one-loop diagrams

In the dimensional regularization scheme a parameter  $n$  is introduced that in some sense can be visualized as the dimension of space time. For  $n \neq 4$  a finite theory results; the physical theory obtains in the limit  $n = 4$ .

As a first step, we define the procedure for one-loop diagrams. The example we will treat makes clear explicitly that the dimensional method in no way depends on the use of Feynman parameters. Actually, since for two or more closed loops ultra-violet divergencies may also be transferred from the momentum integrations to the Feynman parameter integrations the use of Feynman parameters in connection with the dimensional regularization scheme must be avoided, or at least be done very judiciously. The procedure for multiloop diagrams will be defined in subsequent subsections.

Consider a self-energy diagram with two scalar intermediate particles in  $n$  dimensions:



$$I_n = \int d_n p \frac{1}{(p^2 + m^2 - i\epsilon)[(p+k)^2 + M^2 - i\epsilon]} . \quad (13.1)$$

In the integrand the loop momentum  $p$  is an  $n$  component vector. This expression makes sense in one-, two-, and three-dimensional space; in four dimensions the integral is logarithmically divergent. To evaluate this integral we can go to the  $k$  rest-frame ( $k = 0, 0, 0, i\mu$ ). Next we can introduce polar coordinates in the remaining space dimensions

$$I_n = \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} \omega^{n-2} d\omega \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin \theta_2 \int_0^{\pi} d\theta_3 \sin^2 \theta_3 \dots \int_0^{\pi} d\theta_{n-2} \sin^{n-3} \theta_{n-2} \times$$

$$\times \frac{1}{(-p_0^2 + \omega^2 + m^2 - i\epsilon)[-(p_0 + \mu)^2 + \omega^2 + M^2 - i\epsilon]} . \quad (13.2)$$

Here  $\omega$  is the length of the vector  $p$  in the  $n - 1$  dimensional subspace. The integrand has no dependence on the angles  $\theta_1, \dots, \theta_{n-2}$ , and one can integrate using

$$\int_0^{\pi} \sin^m \theta d\theta = \sqrt{\pi} \frac{\Gamma(\frac{m}{2} + \frac{1}{2})}{\Gamma(\frac{m}{2} + 1)} . \quad (13.3)$$

These and other useful formulae will be given in Appendix B. The result is

$$I_n = \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} \omega^{n-2} d\omega \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \left\{ \frac{1}{(-p_0^2 + \omega^2 + m^2)[-(p_0 + \mu)^2 + \omega^2 + M^2]} \right\}. \quad (13.4)$$

This integral makes sense also for non-integer, in fact also for complex  $n$ . We can use this equation to define  $I_n$  in the region where the integral exists. And outside that region we define  $I_n$  as the analytic continuation in  $n$  of this expression.

Apparently this expression becomes meaningless for  $n \leq 1$ , for then the  $\omega$  integral diverges near  $\omega = 0$ , or for  $n \geq 4$  due to the ultraviolet behaviour. The lower limit divergence is not very serious and mainly a consequence of our procedure. Actually for  $n \rightarrow 1$  not only does the integral diverge, but also the  $\Gamma$  function in the denominator, so that  $I_1$  from Eq. (13.4) is an undetermined form  $\infty/\infty$ . Let us first fix  $n$  to be in the region where the above expression exists, for instance  $1.5 < n < 1.75$ . Next we perform a partial integration with respect to  $\omega^2$

$$d\omega \omega^{n-2} = \frac{1}{2} d\omega^2 (\omega^2)^{(n-3)/2} = \frac{1}{2} d\omega^2 \frac{2}{n-1} \frac{d}{d\omega^2} (\omega^2)^{(n-1)/2}.$$

For  $n$  in the given domain the surface terms are zero. Using  $z\Gamma(z) = \Gamma(z+1)$  and repeating this operation  $\lambda$  times we obtain

$$I_n = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + \lambda\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\omega \omega^{n-2+2\lambda} \left(-\frac{\partial}{\partial \omega^2}\right)^\lambda \left\{ \dots \right\}. \quad (13.5)$$

We have derived this equation for  $1 < n < 4$ . However it is meaningful also for  $1 - 2\lambda < n < 4$ . The ultraviolet behaviour is unchanged, but the divergence near  $\omega = 0$  is seen to cancel against the pole of the  $\Gamma$  function. Note that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z};$$

thus for  $z \rightarrow 0$  this behaves as  $1/z$ .

The last equation is analytic in  $n$  for  $1 - 2\lambda < n < 4$ . Since it coincides with the original  $I_n$  for  $1 < n < 4$  it must be equal to the analytic continuation of  $I_n$  outside  $1 < n < 4$ . Clearly, we now have an explicit expression for  $I_n$  for arbitrarily small values of  $n$  (taking  $\lambda$  sufficiently large). It is equally obvious that if the original expression which we started from had been convergent, then its value would have been equal to  $I_n$  for  $n \rightarrow 4$ . Moreover, cutting equations can be derived using only time and energy components, and as long as the last given expression for  $I_n$  exists, and  $p_0$  and  $\omega$  integrations can be exchanged (i.e. for  $n < 4$ ) these cutting equations can be established.

Let us now see what happens for  $n \geq 4$ . Again we will use the method of partial integrations to perform the analytic continuation. For simplicity we set  $\lambda = 0$ . First fix  $n$  in the region  $1 < n < 4$ . Next we insert

$$1 = \frac{1}{2} \left( \frac{dp_0}{dp_0} + \frac{d\omega}{d\omega} \right).$$

Next we perform partial integration with respect to  $p_0$  and  $\omega$ . Again in the given domain the surface terms are zero. Further

$$\begin{aligned} \frac{1}{2} \left( -p_0 \frac{d}{dp_0} - \omega \frac{d}{d\omega} \right) \omega^{n-2} \{ \dots \} &= \frac{1}{2} \left( -n + 2 + \frac{-2p_0^2 + 2\omega^2}{(-p_0^2 + \omega^2 + m^2)} + \frac{-2p_0(p_0 + \mu) + 2\omega^2}{[-(p_0 + \mu)^2 + \omega^2 + M^2]} \right) \omega^{n-2} \{ \dots \} = \\ &= \frac{1}{2} \left( -n + 6 + \frac{-2m^2}{(-p_0^2 + \omega^2 + m^2)} + \frac{2p_0\mu + 2\mu^2 - 2M^2}{[-(p_0 + \mu)^2 + \omega^2 + M^2]} \right) \omega^{n-2} \{ \dots \}. \end{aligned}$$

Inserting this in the right-hand side of Eq. (13.4) gives

$$I_n = \frac{-n + 6}{2} I_n - I'_n \quad (13.6)$$

or

$$I_n = -\frac{2}{n-4} I'_n$$

with

$$\begin{aligned} I'_n &= \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} \omega^{n-2} d\omega \times \\ &\times \left\{ \frac{2m^2}{(-p_0^2 + \omega^2 + m^2)^2 [-(p_0 + \mu)^2 + \omega^2 + M^2]} + \frac{-2p_0\mu - 2\mu^2 + 2M^2}{(-p_0^2 + \omega^2 + m^2) [-(p_0 + \mu)^2 + \omega^2 + M^2]^2} \right\}. \end{aligned} \quad (13.7)$$

The integral  $I'_n$  is convergent for  $1 < n < 5$ . Now the  $I_n$  given before and this expression are equal for  $1 < n < 4$ ; since the last expression is analytic for  $n < 5$  with a simple pole at  $n = 4$ , it must be equal to the analytic continuation of  $I_n$ .

The above procedure may be repeated indefinitely. One finds that  $I_n$  is of the form

$$I_n = \Gamma\left(\frac{4-n}{2}\right) f(n, \mu, m, M), \quad (13.8)$$

where the function  $f$  is well-behaved for arbitrarily large  $n$ . The  $\Gamma$  function shows simple poles at  $n = 4, 6, 8, \dots$ .

We see now why the limit  $n \rightarrow 4$  cannot be taken: there is a pole for  $n = 4$ . It is very tempting to say that one must introduce a counterterm equal to minus the pole and its residue. But if unitarity is to be maintained this counterterm may not have an imaginary part, i.e. it must be a polynomial in  $\mu, m$  and  $M$ . Thus we must find the form of the residue of the pole. It will turn out to be of the required form. To show that it is of the polynomial form in the general case of many loops necessitates use of the cutting equations. For the one-loop case at hand we will simply compute  $I_n$  using Feynman parameters. One has

$$I_n = \int_0^1 dx \int d_n p \frac{1}{[p^2 + 2pkx + k^2x + M^2x + m^2(1-x)]^2}.$$



Shifting integration variables ( $p' = p + kx$ ), making the Wick rotation, and introducing n-dimensional polar coordinates, one computes

$$I_n = \frac{i\pi^{n/2} \Gamma\left(2 - \frac{n}{2}\right)}{\Gamma(2)} \int_0^1 dx \frac{1}{[M^2x + m^2(1-x) + k^2x(1-x)]^{2-n/2}} .$$

In this way we have explicitly  $I_n$  in the form of Eq. (13.8). For  $n = 4, 6, 8, \dots$ , the integrand is a simple polynomial, for  $n = 4$  the integral gives simply 1. Thus the pole term is

$$PP(I_n) = \frac{i\pi^2}{1} \frac{2}{4-n} , \quad (13.9)$$

where PP stands for "pole part". Using the equation

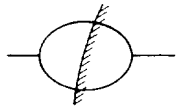
$$a^\epsilon = e^{\epsilon \ln a} = 1 + \epsilon \ln a + O(\epsilon^2) ,$$

one may compute

$$\lim_{n \rightarrow 4} [I_n - PP(I_n)] = -i\pi^2 \int_0^1 dx \ln [M^2x + m^2(1-x) + k^2x(1-x)] + C .$$

Here C is a constant related to the n dependence other than in the exponent of the denominator, containing for instance  $\ln \pi$ , from  $\pi^{n/2}$ . In general C is a polynomial just as the pole part of  $I_n$ . Since we could have taken as our starting point an  $I_n$  multiplied by  $b^{n-4}$ , where b is any constant, we see that C is undetermined. This is the arbitrariness that always occurs in connection with renormalization.

We now must do some work that will facilitate the treatment of the multiloop case. Let us consider the cutting equation. One has:



$$= f^+(k^2) = -(2\pi)^2 \int d_1 p \theta(-p_0) \delta(p^2 + m^2) \theta(p_0 + k_0) \delta[(p+k)^2 + M^2] .$$

In the rest frame

$$f^+(\mu) = -(2\pi)^2 \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + \lambda\right)} \int_0^\infty d\omega \omega^{n-2+2\lambda} \left(-\frac{\partial}{\partial \omega^2}\right)^\lambda \times \\ \times \int_{-\infty}^\infty dp_0 \theta(-p_0) \delta(-p_0^2 + \omega^2 + m^2) \theta(p_0 + \mu) \delta(-2p_0\mu + M^2 - m^2 - \mu^2) . \quad (13.10)$$

The function  $f^-(k^2)$  can be obtained by changing the signs of the arguments of the  $\theta$  functions.

In coordinate space one has

$$f(x) = \theta(x_0) f^+(x) + \theta(-x_0) f^-(x) .$$

The Fourier transform of this statement is

$$f(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{1}{\tau - i\epsilon} f^+(k + \tau) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{1}{\tau - i\epsilon} f^-(k - \tau) . \quad (13.11)$$

$\tau$  can be considered as an  $n$  vector with all components zero except the energy component. As long as the  $\omega$  integral is sufficiently convergent (i.e.  $n < 4$ ) one may exchange freely the  $\omega$  and  $\tau$  integration. Doing the  $p_0$  and  $\tau$  integrations one obtains of course the old result for  $I_n$ , which is not very interesting. Let us therefore leave the  $\tau$  integration in front of the  $\omega$  integral and compute the  $p_0$  integral. One finds for the  $p_0$  integral in  $f^+$

$$\int dp_0 \dots = \frac{1}{2|\mu|} \theta(\mu - M - m) \delta(\omega^2 - \kappa^2) ,$$

where

$$\kappa = \sqrt{\frac{(\mu^2 - m^2 - M^2)^2 - 4m^2M^2}{4\mu^2}} .$$

The  $\theta$  function expresses the fact that  $\mu$  must be positive and furthermore that  $\kappa^2$  must be positive in order for the  $\omega$  integration to give a non-zero result. For  $f^-$  one obtains the same, except for the change  $\theta(\mu - M - m) \rightarrow \theta(-\mu - M - m)$ .

Also the  $\omega$  integration can be done, and is of course independent of  $\lambda$  (this follows as usual by considering the integral for  $1 < n < 4$  and doing the necessary partial integrations). For  $\lambda = 0$  one finds

$$f^+(\mu) = -(2\pi)^2 \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\kappa^{n-3}}{4\mu} \theta(\mu - M - m) .$$

The complete function  $f(k) = I_n$  obtains as given above. We must study

$$\int d\tau \frac{1}{\tau - i\epsilon} f^+(\mu + \tau) \rightarrow \int_{M+m}^{\infty} \frac{d\tau}{\tau - \mu - i\epsilon} \frac{\kappa^{n-3}}{4|\tau| \Gamma\left(\frac{n-1}{2}\right)} \equiv g^+(\mu) ,$$

where

$$\kappa = \sqrt{\frac{(\tau^2 - \mu^2 - M^2)^2 - 4m^2M^2}{4\tau^2}} .$$

Since

$$\kappa^2(\tau, m, M) = (\tau - m - M)\sigma(\tau, m, M) , \quad (13.12)$$

where  $\sigma$  is positive and finite at threshold  $\tau \rightarrow m + M$ , this integral is not well defined for  $n \leq 1$ . This is how the divergence at  $\omega = 0$ , previously found in Eq. (13.4), manifests itself here. But this is again no problem, and really due to the fact that our derivation is correct only for  $n > 1$ . We will come back to that below. And there is no trouble in

constructing the analytic continuation to smaller values of  $n$ . This can be done by performing partial integrations with respect to the factor  $(\tau - M - m)$  in Eq. (13.12).

For  $n \geq 4$ , however, the integral diverges for large values of  $|\tau|$ . This can be handled as follows. The expression (13.12) for  $g^+(\mu)$  is nothing but a dispersion relation, and we may perform a subtraction

$$g^+(\mu) = g(0) + \mu \int_{M+m}^{\infty} \frac{d\tau}{\tau(\tau - \mu - i\epsilon)} \frac{\kappa^{n-3}}{4|\tau|\Gamma\left(\frac{n-1}{2}\right)},$$

$$g(0) = \int_{m+M}^{\infty} \frac{d\tau}{4\tau^2} \frac{\kappa^{n-3}}{\Gamma\left(\frac{n-1}{2}\right)}.$$

As before, by inserting  $d\tau/d\tau$  it may be shown that  $g(0)$  has a pole at  $n = 4$ . The remainder of  $g^+$ , however, is perfectly well-behaved for  $n < 5$ . Again we see that the pole terms (in  $n = 4$ ) have the proper polynomial behaviour; they are like subtractions in a dispersion relation.

We must now clear up a final point, namely the question of the behaviour of the  $\tau$  integral near threshold. Consider as an example the function

$$f(\tau) = \frac{(\tau - 1)^\alpha}{\tau^4}.$$

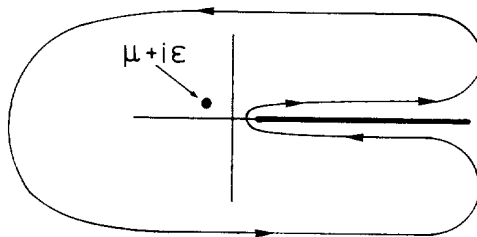
In the complex  $\tau$  plane we have a pole at  $\tau = 0$  and for non-integer  $\alpha$  a cut along the real axis from  $\tau = 1$  to  $\tau = \infty$ . Multiply this function with  $(\tau - \mu - i\epsilon)^{-1}$  and integrate over a small circle around the point  $\tau = \mu$ . One obtains

$$2\pi i f(\mu).$$

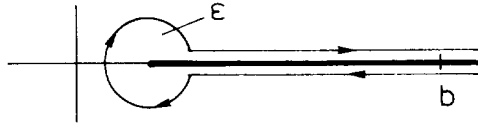
On the other hand the contour may be enlarged; we get

$$f(\mu) = \frac{1}{2\pi i} \int_C \frac{d\tau}{\tau - \mu - i\epsilon} \frac{(\tau - 1)^\alpha}{\tau^4} + \text{contribution of the origin},$$

where  $C$  is as in the the following diagram:



The circle at infinity may be ignored provided  $\alpha < 4$ . Since now the integrand has a quite singular behaviour at  $\tau = 1$ , this point must be treated carefully. The contour may be divided into a contribution of a small circle with radius  $\epsilon$  around this point, and the rest.



In considering the integral over the circle,  $\tau$  may be set to one except in the factor  $(\tau - 1)^\alpha$ . Moreover, we may introduce the change of variable  $\tau = \tau - 1$ . Writing  $\tau = \epsilon e^{i\phi}$ :

$$\int_{\epsilon} d\tau \tau^\alpha = i \int_{\pi}^{-\pi} d\phi \epsilon^{\alpha+1} e^{i(\alpha+1)\phi} = \frac{\epsilon^{\alpha+1}}{\alpha+1} \left\{ e^{-i(\alpha+1)\pi} - e^{i(\alpha+1)\pi} \right\}.$$

On the other hand, the contribution of the two contour lines from the circle to some point  $b$  above and below the cut contains a part

$$\int_{\epsilon}^b d\tau \tau^\alpha \left( e^{-i\pi\alpha} - e^{i\pi\alpha} \right).$$

The integrand is the jump across the cut. This can be integrated to give

$$\frac{1}{\alpha+1} \left( b^{\alpha+1} - \epsilon^{\alpha+1} \right) \left( e^{-i\pi\alpha} - e^{i\pi\alpha} \right).$$

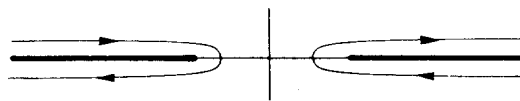
Together with the contribution from the circle

$$\frac{1}{\alpha+1} b^{\alpha+1} \left( e^{-i\pi\alpha} - e^{i\pi\alpha} \right).$$

This is independent of  $\epsilon$ , and the limit  $\epsilon \rightarrow 0$  can be taken. Note that the result is  $-2\pi i$  in the limit  $\alpha \rightarrow -1$ , as should be for a clockwise contour.

In the expression for  $f^+(\mu)$  [and  $f^-(\mu)$ ] we have not bothered about the precise behaviour at the start of the cut. This is in principle accounted for by the  $\theta$  function. This  $\theta$  function gives only the contribution along the cut; the small circle has been ignored. This is allowed only if  $n > 1$  (corresponding to  $\alpha > -1$ ). Otherwise one must carefully specify what happens at threshold in  $f^+$ ,  $f^-$ , and in the subsequent integrals over  $\tau$ .

To see in detail how this goes consider a function of  $\tau^2$  having a cut from  $\tau^2 = a^2$  to  $\tau^2 = \infty$ , but otherwise analytic (and going sufficiently fast to zero at infinity). In the  $\tau$  plane the function has cuts from  $-a$  to  $-\infty$  and  $+a$  to  $+\infty$ . The dispersion relation leads in the  $\tau$  plane to the following contour:



Consider the right-hand side contour. The cut starts at the point  $\tau = a$ . We may write

$$\int_{\leftarrow} \frac{1}{\tau - s - i\epsilon} f(\tau^2) + \int_{a-\delta}^{\infty} \frac{1}{\tau - s - i\epsilon} \bar{f}(\tau^2) ,$$

where  $\leftarrow$  stands for the circle at  $a$  with radius  $\delta$ , and  $\bar{f}(\tau^2)$  is the jump over the cut, i.e.

$$\bar{f}(\tau^2) = \lim_{\delta' \rightarrow 0} \left\{ f(\tau^2 + i\delta') - f(\tau^2 - i\delta') \right\} .$$

This is the precise equivalent of our  $f^+$ . It is essential to first take the limit  $\delta' = 0$  before the limit  $\delta = 0$ .

Let us now return to the question of subtractions. It is now possible to turn the reasoning around. We know that for  $I_n$  and its Fourier transforms the following properties hold:

- i)  $I_n(k)$  has poles for  $n = 4, 6, \dots$ , and the  $f_n^+(x)$  have no pole for  $n = 4$ .
- ii)  $I_n(x) = \theta(x_0) f_n^+(x) + \theta(-x_0) f_n^-(x)$  for  $n < 4$ .
- iii)  $I_n(x)$ ,  $f_n^+(x)$  are Lorentz-invariant.

The functions  $f_n^+$  and  $f_n^-$  are non-singular for  $n = 4$ ; they are the cut diagrams and these are not divergent for  $n = 4$ . Concerning point (ii), the derivation of the cutting equation requires  $n < 4$ . Of course it is possible to define things by analytic continuation, but our dispersion-like relations as exhibited above will hold only for  $n < 4$ . From (ii) and (iii) it follows immediately that the pole in  $n = 4$  of  $I_n(x)$  can at most be a  $\delta$  function. The precise reasoning is as follows. Applying a Lorentz transformation to (ii) and insisting on Lorentz invariance we find

$$I_n(x) = f_n^+(x) = f_n^-(x) \quad \text{for} \quad x_0 = 0, \quad \vec{x} \neq 0 .$$

Consider now the analytic continuation of (ii) to larger  $n$  values. If  $f^+$  and  $f^-$  have no pole for  $n = 4$ , then the only way that  $I_n(x)$  can have a pole for  $n = 4$  is in the point  $x_0 = 0, \vec{x} = 0$ . Remember now that we have shown that the Fourier transform of  $I_n$  is of the form

$$\frac{1}{n-4} \times \text{finite function for } n < 5 ,$$

i.e. the Fourier transform of  $I_n$  exists for  $n < 5$ . Since the Fourier transform of the pole part is a function which is non-zero only for  $x_0 = \vec{x} = 0$ , the only possibility is that this pole part is a polynomial in the external momenta (and the various masses in the problem), i.e. in coordinate space a  $\delta$  function and derivatives of  $\delta$  functions.

Thus we know now that the residue of  $I_n$  at the pole is a polynomial. Differentiating  $I_n$  with respect to the external momentum  $k$ , in the region  $n < 4$  where we can use a well-defined representation, see Eqs. (13.4) and (13.5), we see that this derivative is finite for  $n = 4$ . It follows that the pole part is independent of  $k$ , which is indeed what is found before, Eq. (13.9). Also, this is all we need to know for renormalization purposes.

### 13.3 Multiloop diagrams

We must now extend the method to diagrams containing arbitrarily many loops. This is quite straightforward. First note that external momenta (and sources) span at most a four-dimensional space. We split  $n$ -dimensional space into this four-dimensional space and the rest

$$\int d_{n^2} p_1 d_{n^2} p_2 \dots d_{n^2} p_k = \int d_{4^2} p_1 d_{4^2} p_2 \dots d_{4^2} p_k \int d_{n-4} P_1 \dots d_{n-4} P_k . \quad (13.13)$$

The  $\underline{p}_i$  are the components of the  $p_i$  in four-dimensional space. The integrand will depend on the scalar products of the  $\underline{p}_i$  with themselves and the external vectors, and furthermore on the scalar products of the  $P_i$  with themselves. Again, the integration over those angles that do not appear in the integrand may be performed. This is done as follows. Consider the integral

$$\int d_{n-4} P_i .$$

The argument of this integral is already integrated over  $P_{i+1} \dots P_k$ ; therefore the integral depends only on the scalar product of  $P_i$  with  $P_1 \dots P_{i-1}$ . These vectors span an  $i-1$  dimensional space, and we may write

$$\int d_{n-4} P_i = \int d_{i-1} \bar{P}_i \int d_{n-3-i} \bar{\bar{P}}_i . \quad (13.14)$$

Now the integrand will no longer depend on the direction of  $\bar{\bar{P}}$ , and introducing polar coordinates in  $\bar{\bar{P}}$  space

$$\int d_{n-3-i} \bar{\bar{P}}_i = \frac{2\pi^{(n-3-i)/2}}{\Gamma\left(\frac{n-3-i}{2}\right)} \int_0^\infty \omega_i^{n-4-i} d\omega_i . \quad (13.15)$$

The  $\omega_i$  represent the lengths of the vectors  $\bar{\bar{P}}_i$ . Introduce in  $k$ -dimensional space the vectors

$$q = \begin{pmatrix} \omega_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, q_2 = \begin{pmatrix} (P_2)_1 \\ \omega_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, q_k = \begin{pmatrix} (\bar{P}_k)_1 \\ \vdots \\ (\bar{P}_k)_{k-1} \\ \omega_n \end{pmatrix} .$$

Obviously

$$(q_i q_j) = (P_i P_j) .$$

Also, for  $i \neq k$ ,

$$\int d_k q_i = \int d_{i-1} \bar{q}_i \int d_{k-i+1} \bar{\bar{q}}_i = \int d_{i-1} \bar{P}_i \frac{2\pi^{(k-i+1)/2}}{\Gamma\left(\frac{k-i+1}{2}\right)} \int_0^\infty \omega_i^{k-i} d\omega_i , \quad (13.16)$$

but note

$$\int d_k q_k = \int d_{k-1} \bar{P}_k \int_{-\infty}^{\infty} d\omega_k, \quad (13.17)$$

with  $\omega_k$  running from  $-\infty$  to  $+\infty$ . We have therefore comparing Eqs. (13.14) and (13.15) with (13.16) and (13.17)

$$\int d_{n-4} P_i = \pi^{(n-k-4)/2} \frac{\Gamma\left(\frac{k-i+1}{2}\right)}{\Gamma\left(\frac{n-3-i}{2}\right)} \int d_k q_i \omega_i^{n-4-k} \quad \text{for } i \neq k,$$

$$\int d_{n-4} P_k = 2\pi^{(n-k-4)/2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-3-k}{2}\right)} \int d_k q_k \omega_k^{n-4-k} \theta(\omega_k),$$

where we used  $\Gamma(1/2) = \sqrt{\pi}$ .

Finally note

$$\omega_1 \omega_2 \dots \omega_k = \varepsilon_{i_1 \dots i_k} (q_1)_{i_1} \dots (q_k)_{i_k} \equiv \det q.$$

Since  $\omega_1, \omega_2, \dots$ , are positive we can write

$$\theta(\omega_k) = \theta(\det q).$$

We arrive thus at the equation

$$\int d_{n-4} P_1 \dots d_{n-4} P_k = 2 \prod_{i=1}^k \pi^{(n-k-4)/2} \frac{\Gamma\left(\frac{k-i+1}{2}\right)}{\Gamma\left(\frac{n-3-i}{2}\right)} \times \int d_k q_1 \dots d_k q_k (\det q)^{n-4-k} \theta(\det q). \quad (13.18)$$

Next we have

$$(\det q)^\alpha = \frac{1}{(\alpha+k) \dots (\alpha+1)} \left( \det \frac{\partial}{\partial q} \right) (\det q)^{\alpha+1}.$$

This equation can be arrived at by tedious work. Just write out the determinants with the help of the  $\varepsilon$  symbol with  $k$  indices. Note that for  $\alpha = 0$  the equation follows trivially because of

$$\varepsilon_{i_1 \dots i_k} \varepsilon_{i_1 \dots i_k} = k!$$

Also for  $k = 1$  the equation is trivial.

Now the integrand is a function of the scalar products  $a_{ij} = (q_i, q_j)$ . Let us consider the operation on such a function of the operator

$$\det \frac{\partial}{\partial q}.$$

One has

$$\frac{\partial}{\partial (q_i)_j} = \sum_s (q_s)_j \left( \frac{\partial}{\partial a_{si}} + \frac{\partial}{\partial a_{is}} \right)$$

$$\det \left( \frac{\partial}{\partial q} \right) = (\det q) \det \left( \frac{\partial}{\partial a_{ij}} + \frac{\partial}{\partial a_{ji}} \right) = 2^k (\det q) \det \left( \frac{1}{2} \frac{\partial}{\partial a_{ij}} + \frac{1}{2} \frac{\partial}{\partial a_{ji}} \right).$$

With these equations one can construct the analytic continuation of the above equation to small values of  $n$

$$\begin{aligned} \int dq_1 \dots dq_k (\det q)^{n-4-k} &= \int dq_1 \dots dq_k \frac{1}{(n-4)(n-5)\dots(n-3-k)} \left( \det \frac{\partial}{\partial q} \right) (\det q)^{n-3-k} \\ &= \frac{1}{(n-4)\dots(n-3-k)} \int dq_1 \dots dq_k (\det q)^{n-3-k} \left( \det - \frac{\partial}{\partial q} \right) \\ &= \frac{2^k}{(n-4)\dots(n-3-k)} \int dq_1 \dots dq_k (\det q)^{n-2-k} \left( \det - \frac{1}{2} \frac{\partial}{\partial a_{ij}} - \frac{1}{2} \frac{\partial}{\partial a_{ji}} \right). \end{aligned}$$

Performing this operation  $\lambda$  times gives

$$\begin{aligned} \int d_{n-4}^{P_1} \dots d_{n-4}^{P_k} &= 2 \prod_{i=1}^k \pi^{(n-k-4)/2} \frac{\Gamma\left(\frac{k-i+1}{2}\right)}{\Gamma\left(\frac{n-3-i}{2} + \lambda\right)} \times \\ &\times \int d_k q_1 \dots d_k q_k \theta(\det q) (\det q)^{n-4-k+2\lambda} \left( \det - \frac{1}{2} \frac{\partial}{\partial a_{ij}} - \frac{1}{2} \frac{\partial}{\partial a_{ji}} \right)^\lambda. \end{aligned} \tag{13.19}$$

Rather than worrying whether this derivation is correct or not we simply take this equation as the definition of the  $k$ -loop diagrams for non-integer  $n$ . It is not very difficult to verify that for finite diagrams the result coincides with the above equation for  $n = 4$  (see below). In doing such work it is often advantageous to go back to the special coordinate system with which we started; the expression in terms of the  $q_i$  is mainly useful for invariance considerations. For instance shifts of integration variables such as

$$q_1 \longrightarrow q_1 + q_2$$

leave both  $d_k q_1$  and  $d_k q_2$  as well as  $\det q$  unchanged as long as the integrals converge (which they do for sufficiently small  $n$ ).

Equation (13.19) is much too complicated for practical work, and we will instead use the notation

$$\int d_n p_1 \dots \int d_n p_n.$$

All the above work is to show how things can be defined for non-integer  $n$ .



The final step consists of showing that the algebraic properties necessary for diagrammatic analysis hold for this in the same way as they do for integer  $n$ . We have seen already that this is so for shifts of integration variables. Furthermore, clearly

$$\int d_{n^1} p_1 \int d_{n^2} p_2 = \int d_{n^2} p_2 \int d_{n^1} p_1$$

(always assuming that  $n$  is chosen such that the integrals converge).

Next there is the following problem. It seems that the definition depends on the number of loops. Then in considering cutting equations one has relations involving all kinds of possibilities on both sides of the cut, and it may seem hard to understand what happens. The answer to this is that the definition involves for  $k$  loops a  $k$ -dimensional  $q$ -space, but nothing prevents us from using a larger space. Thus in any equation one can use for  $k$  simply the largest number of closed loops that occur in any one term. And there is no difficulty in decreasing the  $k$  used if the number of loops is smaller than this  $k$ : simply perform the integrations over those directions that do not appear in the integrand.

Let us now consider the limit  $n \rightarrow 4$  in the case where the original integral exists. This means that there are no difficulties with the ultraviolet behaviour. Using Eq. (13.19) with  $\lambda$  sufficiently large we may first do the  $q_1$  integration. Using coordinates as described in the beginning one arrives at an integral of the type as encountered in the one-loop case

$$\frac{1}{\Gamma\left[\frac{1}{2}(n-4) + \lambda\right]} \int_0^\infty d\omega_1 \omega_1^{n-5+2\lambda} \left(-\frac{\partial}{\partial \omega_1^2}\right)^\lambda f(\omega_1^2) .$$

Keeping  $n$  in the neighbourhood of four it is possible to perform partial integration with respect to  $\omega_1$  only

$$\frac{1}{\Gamma\left[\frac{1}{2}(n-4) + 1\right]} \int_0^\infty d\omega_1 \omega_1^{n-3} \left(-\frac{\partial}{\partial \omega_1^2}\right) f(\omega_1^2) .$$

Next we write  $\omega_1^{n-3} = \omega_1 \cdot \omega_1^\varepsilon = \omega_1 [1 + \varepsilon \ln \omega_1 + O(\varepsilon^2)]$ , with  $\varepsilon = n - 4$ . Since  $\omega_1 \cdot d\omega_1 = \frac{1}{2} d\omega_1^2$ , we find

$$\frac{1}{\Gamma\left[\frac{1}{2}(n-4) + 1\right]} \left\{ \frac{1}{2} f(0) + \frac{n-4}{2} \int_0^\infty d\omega_1^2 \ln \omega_1 \left(-\frac{\partial}{\partial \omega_1^2}\right) f(\omega_1^2) + O[(n-4)^2] \right\} .$$

We see that the integral reduces to what one would have with  $q_1 = 0$ , plus a finite amount proportional to  $n - 4$ . Since  $q_1$  was the "unphysical" part of the momentum  $p_1$  we see that we recover in this way the expression that we started from. Moreover, if the original expression was finite, then the result for values of  $n$  close to four differs by finite terms proportional to  $n - 4$ .

If the integral does not exist for  $n = 4$ , then we are interested in constructing an analytic continuation to larger  $n$  values. This can be done as follows. Select a number of closed loops in the diagram. Call the loop-momenta associated with these loops  $p_1 \dots p_s$  ( $s$  loops selected). Next insert

$$1 = \frac{1}{s \cdot n} \sum_{j=1}^n \sum_{i=1}^s \frac{d(p_i)_j}{d(p_i)_j},$$

which is symbolic for

$$\frac{1}{4k} \sum_{j=1}^4 \sum_{i=1}^k \frac{dp_{ij}}{p_{ij}} + \frac{1}{k^2} \sum_{j=1}^k \sum_{i=1}^k \frac{dq_{ij}}{dq_{ij}}$$

and perform partial integrations. The result is the original integral  $I$  plus an integral  $I'$  that is better convergent with respect to the loops selected. Solving the equation with respect to  $I$  we get

$$I = \frac{1}{sn - \lambda} I'$$

in analogy with Eq. (13.6), obtained in the one-loop case. Here  $(-\lambda)$  is the number obtained by counting the powers of the momenta  $p_1 \dots p_s$  in the integrand. That is the very nice thing about this method: there is a direct relation between power counting and the location of the poles in the complex  $n$  plane.

Here we find a pole for  $n = \lambda/s$ . If  $4s - \lambda = 2, 1, 0$ , we have quadratic, linear, or logarithmic divergencies with respect to the  $p_1 \dots p_s$  integrations.

It is clear that we now can have poles for  $n = \lambda/s$ , with  $\lambda$  and  $s$  integers. Diagrams with many closed loops give many poles in the complex  $n$  plane. The first pole would be at  $n = 4 - 2/s, 4 - 1/s, 4$  for quadratic, etc., divergent integrals, the next is one (generally however two) units  $1/s$  further in the direction of increasing  $n$ , etc.

#### 13.4 The algebra of n-dimensional integrals

It is now very important to know how the previously discussed combinatorics survives all the definitions. In doing vector algebra one manipulates vectors and Kronecker  $\delta$  symbols according to the rules

$$\begin{aligned} \delta_{\mu\nu} p_\nu &= p_\mu, \\ p_\mu p_\mu &= p^2, \\ \delta_{\mu\nu} \delta_{\nu\alpha} &= \delta_{\mu\alpha}, \\ \delta_{\mu\mu} &= n. \end{aligned}$$

The only place where the dimension comes in is in the trace of the  $\delta$  symbol. This  $\delta$  symbol also appears naturally when performing integrals; for instance (see Appendix B)

$$\int d_n p \frac{p_\mu p_\nu}{(p^2 + 2pk + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha-n/2}} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma\left(\alpha - \frac{n}{2}\right) k_\mu k_\nu + \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \delta_{\mu\nu} (m^2 - k^2) \right\}.$$

The indices  $\mu, \nu$  are supposedly contracted with indices  $\mu, \nu$  of external quantities (such quantities are zero if the value of the index is larger than four) or other internal quantities. In doing combinatorics one will certainly meet identities of the form

$$\delta_{\mu\nu} p_\mu p_\nu = p^2 .$$

Now note

$$\int d_n p \frac{p^2}{(p^2 + 2pk + m^2)} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha-n/2}} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma\left(\alpha - \frac{n}{2}\right) k^2 + \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \frac{n}{2} (m^2 - k^2) \right\} .$$

All these integrals are computed according to the previously given recipes. The indices  $\mu, \nu$  simply specify two extra dimensions, the two-dimensional space spanned by the objects with which  $\mu$  and  $\nu$  specify contractions.

Clearly the two equations are consistent only if we use the rule  $\delta_{\mu\nu} \delta_{\mu\nu} = n$ .

All this can also be rephrased as follows. If we take the rule  $\delta_{\mu\mu} = n$ , then the algebra of the integrals is the same as that of the integrands.

The situation is somewhat more complicated if there are fermions and  $\gamma$  matrices. Now  $\gamma$  matrices never occur in final answers; only traces occur. The only relevant rules are

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2\delta_{\mu\nu} \mathbb{1} \quad (\mathbb{1} = \text{unit matrix}) , \\ \text{tr} (\gamma^\mu \gamma^\nu) &= 4\delta_{\mu\nu} . \end{aligned}$$

The numbers 2 and 4 are not directly related to the dimensionality of space-time, and they play no role in combinatorial relations. The  $\delta_{\mu\nu}$  must of course be treated as indicated above.

Some considerations are in order now to establish that our regularized diagrams satisfy Ward identities for any value of the parameter  $n$ . The combinatorial proof of Ward identities involves (i) vector algebra and (ii) shifting of integration variables. We have already shown in Subsection 13.3 that shifting of integration variables is actually allowed because of the invariance of  $\det q$  in Eq. (13.19). It is also easy to see now that, in the sense defined above, the vector algebra goes through unchanged for any  $n$ .

Difficulties arise as soon as in the Ward identities there appear quantities that have the desired properties only in four-dimensional space, like  $\gamma^5$  or the completely anti-symmetric tensor  $\epsilon_{\mu\nu\rho\sigma}$ . Then the scheme breaks down, and there are violations of the Ward identities proportional to  $n - 4$ . If there are infinities (i.e. poles for  $n = 4$ ) there may be finite violations of the Ward identities in the limit  $n = 4$ . This is what happens in the case of the famous Bell-Jackiw-Adler anomalies.

### 13.5 Renormalization

Ever since the invention of relativistic quantum electrodynamics, work has been devoted to the problem of renormalization. Mainly, there is the line of Bogoliubov-Parasiutik-Hepp-Zimmerman and the line of Stueckelberg-Petermann-Bogoliubov-Epstein-Glaser.

Of course, both treatments have a lot in common; the BPHZ method however seems at a disadvantage in the sense that there seem to be unnecessary complications. Unfortunately these complications are such as to inhibit greatly the treatment of gauge theories.

The SPBEG method, on the other hand, can be taken over unchanged, and also accomodates nicely with the dimensional regularization scheme. The fundamental ingredients are unitarity and causality precisely in the form of the cutting equations. The only complication that remains in the case of gauge theories is the problem of dressed/bare propagators discussed in Section 9. It seems unlikely that this is a fundamental difficulty.

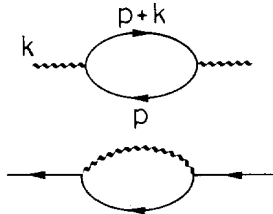
We will sketch in a rough way how renormalization proceeds in the SPBEG method. For more details we refer to the works of these authors.

In the foregoing a definition for diagrams also for non-integer  $n$  (= number of dimensions) has been given. This definition is such that:

- i) cutting equations hold for all  $n$ ;
- ii) Ward identities hold for all  $n$ , provided the  $\epsilon$  tensor and  $\gamma^5$  do not play a role in the combinatorics of these Ward identities;
- iii) divergencies manifest themselves as poles for  $n = 4$  in the complex  $n$  plane.

The problem is now to add counterterms to the Lagrangian such that the poles cancel out. The essential point is that this has to be done order by order in perturbation theory, and since this is precisely the origin of a lot of trouble in connection with gauge theories we will focus attention on that point.

Consider one-loop self-energy diagrams in quantum electromagnetics. They are:



The Feynman rules are:

	$\frac{1}{(2\pi)^4 i} \frac{-i\gamma p + m}{p^2 + m^2 - i\epsilon}$	$\frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{k^2 - i\epsilon}$
	$(2\pi)^4 i \cdot ie\gamma^\mu$	

They follow from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}(\gamma\partial + m)\psi + ieA_\mu \bar{\psi}\gamma^\mu\psi - \frac{1}{2} (\partial_\mu A_\mu)^2 .$$

The first diagram gives

$$I_n^Y = -e^2 \int d_n p \frac{\text{tr} \left\{ \gamma^\mu \left[ -i\gamma(p+k) + m \right] \gamma^\nu (-i\gamma p + m) \right\}}{(p^2 + m^2 - i\epsilon) [(p+k)^2 + m^2 - i\epsilon]} .$$

We must work out the trace using only  $\{\gamma^\mu \gamma^\nu\} = 2\delta_{\mu\nu} \mathbb{1}$  and  $\text{tr}(\gamma^\mu \gamma^\nu) = 4\delta_{\mu\nu}$ . The technique is to reduce a trace of  $\lambda$  matrices to traces of  $\lambda - 2$  matrices. For four matrices

$$\begin{aligned} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta) &= -\text{tr}(\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\beta) + 8\delta_{\mu\alpha} \delta_{\nu\beta} = \dots = -\text{tr}(\gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu) + \\ &+ 8(\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\nu} \delta_{\alpha\beta} + \delta_{\mu\beta} \delta_{\alpha\nu}) \end{aligned}$$

or

$$\text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta) = 4\delta_{\mu\alpha} \delta_{\nu\beta} - 4\delta_{\mu\nu} \delta_{\alpha\beta} + 4\delta_{\mu\beta} \delta_{\alpha\nu} .$$

This gives

$$I_n^Y = -4e^2 \int d_n p \frac{m^2 \delta_{\mu\nu} - (p+k)_\mu p_\nu - p_\mu (p+k)_\nu + \delta_{\mu\nu} (p^2 + pk)}{(p^2 + m^2 - i\epsilon) [(p+k)^2 + m^2 - i\epsilon]} .$$

Using Feynman parameters and the equations of Appendix B this integral can be computed giving

$$I_n = -\frac{4ie^2 \pi^{n/2}}{\Gamma(2)} \int dx \frac{\Gamma\left(\frac{4-n}{2}\right) 2x(1-x) (k_\mu k_\nu - k^2 \delta_{\mu\nu})}{[m^2 + k^2 x(1-x)]^{2-n/2}} .$$

The quadratic divergence (pole at  $n = 2$ ) has cancelled out through  $(1 - n/2)\Gamma(1 - n/2) = \Gamma(2 - n/2)$ .

The second diagram gives

$$I_n^e = -e^2 \int d_n p \frac{\gamma^\mu (-i\gamma p + m) \gamma^\mu}{(p^2 + m^2 - i\epsilon) [(p+k)^2 - i\epsilon]} .$$

From the anticommutation rules, and the rule  $\delta_{\mu\mu} = n$

$$\gamma^\mu \gamma^\mu = n \mathbb{1} .$$

Further

$$\gamma^\mu \gamma p \gamma^\mu = -\gamma^\mu \gamma^\mu \gamma p + 2\gamma p = (2 - n)\gamma p$$

(if  $n = 4$  this reduces to the well-known Chisholm rule. Such rules are not valid in  $n$ -dimensional space). The integral may now be worked out

$$I_n^e = - \frac{ie^2 \pi^{n/2}}{\Gamma(2)} \Gamma\left(\frac{4-n}{2}\right) \int_0^1 dx \frac{\mathbb{I} nm + (2-n)i\gamma kx}{[m^2(1-x) + k^2 x(1-x)]^{2-n/2}} .$$

Both  $I_n^Y$  and  $I_n^e$  have a pole at  $n = 4$ . The residues are easily computed

$$PP(I_n^Y) = \frac{8}{3} i\pi^2 e^2 \frac{1}{n-4} (k_\mu k_\nu - k^2 \delta_{\mu\nu}) ,$$

$$PP(I_n^e) = 2i\pi^2 e^2 \frac{1}{n-4} (4m\mathbb{I} - i\gamma k) .$$

If we introduce in the Lagrangian [remember that vertices and terms in the Lagrangian differ by a factor  $(2\pi)^4 i$ ] the counterterms

$$\frac{e^2}{24\pi^2(n-4)} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{e^2}{8\pi^2(n-4)} \bar{\psi}(\gamma\partial - 4m)\psi ,$$

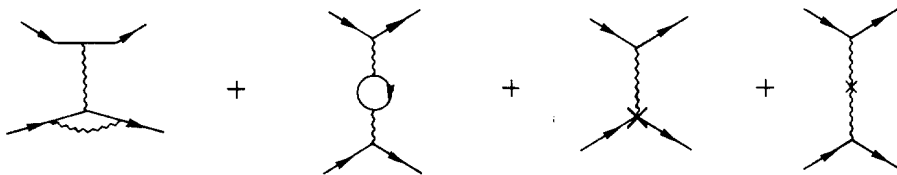
then the two-point functions up to order  $e^2$  will be free of poles for  $n = 4$ , and one can take the limit  $n = 4$ . To have all one-loop diagrams finite a counterterm to cancel vertex divergencies must be introduced.

Performing a similar calculation (simplified very much from the beginning if only the pole part is needed, see Appendix B) for the vertex diagram leads to the counterterm

$$- \frac{ie^3}{8\pi^2(n-4)} \bar{\psi} \gamma^\nu \psi A_\nu .$$

The remarkable fact is that the counterterms are gauge invariant by themselves. This would not have been so if the coefficients of  $\bar{\psi} \gamma \partial \psi$  and  $-iA_\nu \bar{\psi} \gamma^\nu \psi$  had been different. It will become clear that this is special to this gauge; in the case of a gauge where the ghost loops are non-zero the result is different, and the counterterms are in general not gauge invariant by themselves.

There are now two separate questions to be discussed. Including the counterterms, all results up to a certain order are finite. This order is such that one has one closed loop but no "countervertex", or a tree diagram including at most one countervertex. An example is electron-electron scattering to order  $e^4$  in amplitude. Some examples of contributing diagrams are:



The crosses denote counterterms. The first and third diagrams (and the second and fourth) together are finite.

The question is now whether it is possible to make the theory finite up to the next order in  $e^2$  by introducing further counterterms of the type shown above. Next one may ask how to understand in more detail the gauge structure of the renormalized theory, i.e. the Lagrangian including the counterterms. Both questions will be investigated now.

### 13.6 Overlapping divergencies

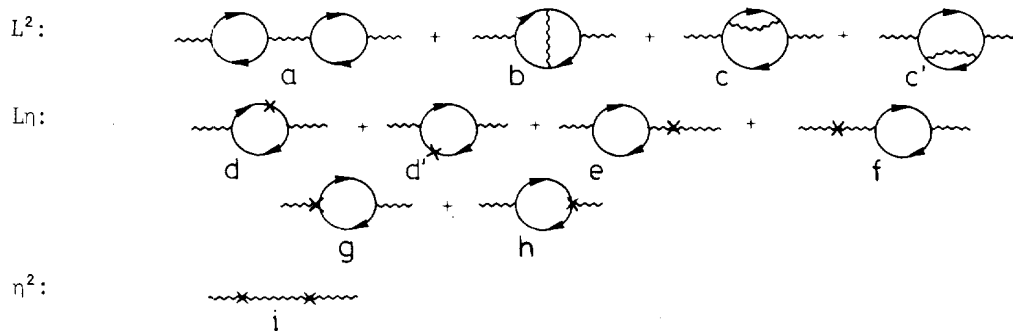
The problem is the following. Can the renormalization procedure be carried through order by order. First we must state more precisely what we mean, because in the Lagrangian we have counterterms of order  $e^2$  and  $e^3$ . To this purpose we introduce a parameter  $\eta$ ; and all counterterms found from the analysis of one-loop diagrams get this parameter as coefficient. Thus we have now

$$\mathcal{L}_{\text{counter}} = \eta \frac{1}{n-4} \left\{ \frac{e^2}{24\pi^2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{e^2}{8\pi^2} \bar{\psi}(\gamma\partial - 4m)\psi - \frac{ie^3}{8\pi^2} A_\mu \bar{\psi} \gamma^\mu \psi \right\}.$$

If now in a diagram the number of closed loops is  $i$  than we may associate with each diagram a factor  $L^i$ . The S matrix is finite up to first order in  $(L + \eta)$ , in the limit  $\eta = 1$ ,  $L = 1$ . Thus at most one closed loop no  $\eta$  vertex, or no closed loop one  $\eta$  vertex.

It may perhaps be noted that if the number of ingoing and outgoing lines is given, then specifying  $L + \eta$  is equivalent to specifying a certain order in  $e$ .

Let us now consider diagrams of order  $(L + \eta)^2$ , for instance photon self-energy diagrams. There are diagrams of order  $L^2$ , i.e. two closed loops, of order  $L\eta$  and of order  $\eta^2$ :



It is now necessary to introduce a classification. We divide all diagrams into two sets:

- i) the set of all diagrams of order  $L^2$ ,  $L\eta$  or  $\eta^2$  that can be disconnected by removing one propagator; here (a, e, f, and i);
- ii) the rest, called the overlapping diagrams.

Set (i) are the non-overlapping diagrams. No new divergences occur, as can be verified readily. In fact, on both sides of the propagator in question one finds precisely what has been made finite previously:

$$a + e + f + i = \left\{ \text{diagram 1} + \text{diagram 2} \right\} - \left\{ \text{diagram 3} + \text{diagram 4} \right\}$$

The overlapping diagrams may contain new divergencies, and we must try to prove that these new divergencies behave as local counterterms. In principle, the proof is very simple and based on the use of cutting equations. The first observation is that all cut diagrams [always taking together diagrams of a given order in  $(L + \eta)$  on each side of the cut] are finite, that is there is no pole for  $n = 4$ . The proof of this is easy; these cut diagrams are of the structure of a product of diagrams of lower order in  $(L + \eta)$ , which are supposedly finite, and an integration over intermediate states. Since for given energy the available phase space is finite the result follows. This assumes that the diagrams have no non-integrable singularities in phase space; the latter would correspond to infinite transition probabilities in lower order, which we take not to exist.

Now let us number the vertices to which the external lines are connected. Here there are only two such vertices, and we call them 1 and 2. According to our cutting relations we have, integrating over all  $x$  except over  $x_1$  and  $x_2$

$$f(x) = \theta(x_0)\Delta^+ + \theta(-x_0)\Delta^-$$

with  $x = x_2 - x_1$ . Here  $\Delta^+$  contains all cut diagrams with  $x_1$  in the unshadowed region, and  $\Delta^-$  all cut diagrams with  $x_2$  in the unshadowed region. This is of precisely the same structure as discussed before; we know that if  $\Delta^\pm$  are free of poles for  $n = 4$ , then  $f(x)$  can only have a pole part that is a  $\delta$  function or derivative of a  $\delta$  function. In other words, the pole part in  $n = 4$  must be a polynomial in the external momentum. For arbitrary diagrams this must be true for any combination of "external" vertices (= vertices that have at least one external line), and thus for any of the external momenta. Due to the fact that power counting and the existence of poles for  $n = 4$  are in a one-one relationship it can easily be established that the polynomials are at most of a certain degree; here, photon self energies, at most of degree 2. To do all this properly it is necessary to distinguish between over-all divergencies and subdivergencies (the former disappear if any of the propagators is opened), and show that there are no subdivergencies if all the lower order counterterms are included. Next it must be shown that after a certain number of differentiations with respect to the external momenta the over-all divergence disappears. All this is trivial: opening up a propagator in a self-energy diagram is equivalent to considering diagrams with four external lines of lower order in  $(L + \eta)$ , thus already finite. Over-all divergencies correspond to over-all power counting (partial differentiation with respect to all loop momenta), and differentiation with respect to any external momentum lowers this power by one.

### 13.7 The order of the poles

Renormalization may now be performed order by order in  $(L + \eta)$ . One starts with diagrams with  $L^1$  (one closed loop) and finds the necessary counterterms. They get a factor  $\eta$  in the Lagrangian. Next one considers diagrams of order  $L^2$  (two closed loops) or  $(L\eta)^1$  (one closed loop, one counter vertex) or  $\eta^2$  (tree diagrams, two counter vertices).



The total can be made finite by adding further counterterms. Such terms get a factor  $\eta^2$  in the Lagrangian. In this way one can go on and obtain a counter Lagrangian in the form of a power series in  $\eta$ . For  $\eta = 1$  the complete theory is finite (there are no poles for  $n = 4$ ).

We have already seen that the terms of order  $\eta$  contain simple poles only. We now want to indicate that the terms of order  $\eta^2$  contain poles of the form  $(n - 4)^{-2}$ , and furthermore, the coefficient of such a pole is determined by the order  $\eta$  terms.

First of all, it is trivial to see that there are no quadratic poles at order  $\eta^2$  if there are no poles at order  $\eta$ . If there are no poles at order  $\eta$  then there are no subdivergencies in two-loop diagrams. But over-all divergencies of diagrams of order  $L^2$  are simple poles (this follows from partial integration), i.e. there are no quadratic poles.

To see that there are in general quadratic poles consider the sum of photon self-energy diagrams (c) and (d). The sub-integral in (c) may be done and gives rise to an expression of the form

$$\Gamma\left(2 - \frac{n}{2}\right) (p)^{n-3} ,$$

where  $p$  is the momentum going into the sub-loop. This equation is symbolic in so far as that we have indicated only the momentum dependence with respect to power counting. The above expression can then be obtained from dimensional arguments. The sum of diagrams (c) and (d) is of the form

$$\int d_n p (p)^{-3} \left\{ \Gamma\left(2 - \frac{n}{2}\right) (p)^{n-3} - \frac{2}{4-n} (p)^1 \right\} .$$

Simplification can go too far; we write  $p = (p^2 + m^2)^{1/2}$ . Next

$$\int d_n p (p^2 + m^2)^\lambda = \Gamma\left(-\lambda - \frac{n}{2}\right) (m^2)^{n/2+\lambda}$$

and the integral becomes

$$\Gamma(-n + 3) \Gamma\left(2 - \frac{n}{2}\right) (m^2)^{n-3} - \Gamma\left(1 - \frac{n}{2}\right) \frac{2}{4-n} (m^2)^{n/2-1} .$$

Since

$$\Gamma(3 - n) = \frac{1}{3-n} \Gamma(4 - n) , \quad \Gamma\left(1 - \frac{n}{2}\right) = \frac{2}{2-n} \Gamma\left(2 - \frac{n}{2}\right) ,$$

we find for the double pole

$$\frac{1}{3-n} \frac{1}{4-n} \frac{2}{4-n} m^2 - \frac{2}{2-n} \frac{2}{4-n} \frac{2}{4-n} m^2 \rightarrow \left(\frac{1}{4-n}\right)^2 2m^2 .$$

If we write

$$(m^2)^{n-3} = m^2 + m^2(n-4) \ln m^2 ,$$

$$(m^2)^{n/2-1} = m^2 + m^2 \frac{n-4}{2} \ln m^2 ,$$

we discover that the  $\ln m^2$  term has no pole

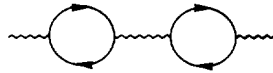
$$\frac{m^2 \ln m^2}{(4-n)^2} \left\{ \frac{2}{3-n} (n-4) - \frac{8}{2-n} \frac{n-4}{2} \right\} = \frac{m^2 \ln m^2}{4-n} \left\{ \frac{2}{3-n} - \frac{4}{2-n} \right\}.$$

The residue of the pole for  $n = 4$  is zero. This is as should be because a term  $\ln m^2$  is non-local, and not admissible as counterterm. Remember that  $m^2$  stands for invariants made up from external momenta, masses, etc.

The above argument is very general and in fact based only on loop and power counting. It can be extended to arbitrary order, and a precise relation between the coefficients of the various higher order poles can be found. (G.'t Hooft, CERN preprint, May 1973.)

### 13.8 Order-by-order renormalization

Consider the following two-loop diagram:



Suppose the self-energy bubble is a function of  $n$  and the momentum  $k$  of the form:

$$\text{bubble}(k^2) = \frac{1}{n-4} f_1(k^2) + f_2(k^2) + (n-4)f_3(k^2).$$

We have omitted a factor  $k^2 \delta_{\mu\nu} - k_\mu k_\nu$ . The two-loop diagram gives

$$[f(k^2)]^2 = \left( \frac{1}{n-4} \right)^2 f_1^2 + 2 \frac{f_1 f_2}{n-4} + f_2^2 + 2f_1 f_3 + O(n-4).$$

If we simply throw away the pole parts the result is for  $n = 4$

$$f_2^2 + 2f_1 f_3.$$

However, this result is wrong because it violates unitarity. Suppose now we do order-by-order subtraction. Then one has after the treatment of one closed loop:

$$\text{bubble}(k^2) + \text{cut bubble}(k^2) = f_2 + (n-4)f_3.$$

The physical result is the limit  $n = 4$  and is equal to  $f_2(k^2)$ . Cutting two-loop diagrams should therefore not involve  $f_3$ . Next include the proper counterterms and consider:

$$\text{two-loop} + \text{cut two-loop} + \text{cut bubble} + \text{cut cut bubble} = f_2^2 + O(n-4).$$

Indeed one obtains the correct result consistent with unitarity. This demonstrates that order-by-order renormalization is not equivalent to throwing away poles and their residues of the unrenormalized S-matrix. In a way that is a pity, because the last method is so much easier. But there is no escape, one must do things step by step, that is order by order in the parameter  $\eta$ .

### 13.9 Renormalization and Slavnov-Taylor identities

It has been shown that the one-loop counterterms are gauge invariant by themselves, in the case of Lorentz gauge quantum electrodynamics. The question is whether we can understand this, and to what extent this is a general phenomenon.

As a first step we note that combinatorial proofs can be read backwards, at least if they are of the local variety. In other words, Ward identities can be translated backwards into a symmetry of the Lagrangian. Therefore, if the Ward identities of the Lagrangian including counterterms are identical to those without counterterms, then both Lagrangians satisfy the same gauge invariance. Let us consider the S-T identities of the unrenormalized theory including electron sources:

We have drawn one photon and one electron source explicitly, leaving other sources understood.

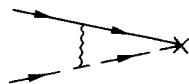
Since these identities are true for any value of the dimensional parameter  $n$ , they are in particular true for the pole parts.

Consider now one-loop diagrams. The pole parts in the above identity are:

- i) the pole parts found in the Green's functions themselves;
- ii) the pole part arising from a closed loop, involving the electron-ghost vertex shown in the last diagram.

Now this last vertex is defined by the behaviour of the Lagrangian under gauge transformations, but it is not present in the Lagrangian itself. In  $\mathcal{L}$  one only finds  $\bar{J}\psi$ ; if we introduce counterterms in the Lagrangian that make the Green's functions finite (including Green's functions with ingoing and outgoing ghost lines) then the above S-T identity can remain true only if we throw away the pole part of the type (ii). That is, for the renormalized Lagrangian, the S-T identities take the form:

Here the double cross stands for minus the pole part of any diagram such as:



Clearly we can understand this identity as an S-T identity if we redefine the behaviour of the term  $\bar{J}\psi$  under gauge transformations. If we say that under a renormalized gauge

transformation  $\psi$  transforms as

$$\psi \rightarrow \psi + ie\Lambda\psi + i \frac{Z}{n-4} \Lambda\psi ,$$

where  $Z$  is minus the residue of the pole part of the type (ii), then the above identity is again precisely an S-T identity. We conclude that the Lagrangian including counterterms is invariant for renormalized gauge transformations.

To make the statement more precise we must add on a factor  $\eta$  to the pole term in the renormalized gauge transformation. Since we allow as a first step only one closed loop the statement is only true up to first order in  $\eta$ . It is unfortunately somewhat complicated to extend this work to arbitrary order in  $\eta$ , but we will do it in the next section.

In quantum electrodynamics things are never really complicated, because in the usual gauges (Lorentz, Landau, etc.) there is no ghost vertex, ergo there are no pole parts of the type (ii). Then the renormalized Lagrangian is invariant with respect to gauge transformations that transform an electron source coupling as before, i.e.

$$\psi \rightarrow \psi + ie\Lambda\psi$$

with the same  $e$  as in the unrenormalized Lagrangian. The result is

$$\begin{aligned} \mathcal{L}_{\text{unren}} & \text{ is invariant under the transformation } (\Lambda), \\ \mathcal{L}_{\text{unren}} + \mathcal{L}_{\text{counter}} & \text{ is invariant under the transformation } (\Lambda), \end{aligned}$$

therefore

$$\mathcal{L}_{\text{counter}} \quad \text{alone is invariant.}$$

In the general case this is not true.

### 13.10 Higher order counterterms

Doing things order by order we suspect (and have shown this to be true up to first order in  $\eta$ ) that  $\mathcal{L}_{\text{ren}}$  is of the form

$$\begin{aligned} \mathcal{L}_{\text{ren}}(\eta) &= \mathcal{L}_{\text{unren}} + \mathcal{L}_{\text{counter}} , \\ \mathcal{L}_{\text{counter}} &= \eta\mathcal{L}_1 + \eta^2\mathcal{L}_2 + \dots , \end{aligned}$$

where  $\mathcal{L}_1, \mathcal{L}_2, \dots$ , contain factors  $1/(n-4)$ . For  $\eta = 1$  the theory is finite. The renormalized Lagrangian  $\mathcal{L}(\eta)$  is invariant under gauge transformations of the form

$$\begin{aligned} A_\mu &\rightarrow A_\mu + (t_0 + t_1\eta + t_2\eta^2 + \dots)\Lambda , \\ \psi &\rightarrow \psi + (t'_0 + t'_1\eta + t'_2\eta^2 + \dots)\Lambda\psi , \end{aligned}$$

where the  $t$  and  $t'$  contain factors  $1/(n-4)$ .

Let us suppose this latter statement to be true up to order  $k$  in  $\eta$ . Consider now:

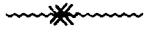
- i) diagrams containing only vertices of the unrenormalized Lagrangian with  $k+1$  closed loops; such diagrams exhibit poles up to degree  $k+1$  and satisfy the unrenormalized S-T identities;

- ii) diagrams containing counterterms (corresponding to vertices of  $\ell_1 \dots \ell_k, t_0 \dots t_k, t'_0 \dots t'_k$ ) but with at least one closed loop;
- iii) diagrams of order  $\eta^{k+1}$  (containing thus no closed loop).

Example for two closed loop photon self-energy diagrams:

Set (i) = diagrams (a), (b), (c).

Set (ii) = diagrams (d) to (h).

Set (iii) = diagram (i) plus a new diagram of the form  that is needed to remove the divergencies of the overlapping diagrams. Now (i) + (ii) + (iii) give a finite theory as  $n \rightarrow 4$  and  $\eta \rightarrow 1$ . That is simply how they were defined. Next (i) + (ii) satisfy Ward identities (the theory was assumed to be invariant up to order  $\eta^k$ ). Therefore the residues of the pole of set (iii) satisfy the S-T identities. Because the diagrams (iii) are tree diagrams the dimension  $n$  appears nowhere except in the pole factors  $1/(n - 4)$ . Therefore the diagrams (iii) satisfy the S-T identities, which means invariance of the renormalized Lagrangian under the renormalized gauge transformation considering terms of order  $\eta^{k-1}$  only. This completes the proof by induction.

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$$\left. \begin{aligned}
 \text{Incoming particle wave-function:} & \quad u^a(k)\sqrt{2k_0} ; \quad a = 1, 2 \\
 \text{Incoming antiparticle wave-function:} & \quad -\bar{u}^a(k)\sqrt{2k_0} ; \quad a = 3, 4 \\
 \text{Outgoing particle wave-function:} & \quad \bar{u}^a(k)\sqrt{2k_0} ; \quad a = 1, 2 \\
 \text{Outgoing antiparticle wave-function:} & \quad u^a(k)\sqrt{2k_0} ; \quad a = 3, 4
 \end{aligned} \right\} . \quad (\text{A.9})$$

Note the minus sign for the incoming antiparticle. The momentum  $k$  is directed inwards for incoming particles and outwards for outgoing particles. As usual  $\bar{u} = u^* \gamma^4$ .

The spinors are solutions of the Dirac equation (note that  $k_0 = +\sqrt{k^2 + m^2}$ )

$$\begin{aligned}
 (i\gamma^\mu k_\mu + m)u^a(k) &= 0, \quad a = 1, 2, \\
 (-i\gamma^\mu k_\mu + m)u^a(k) &= 0, \quad a = 3, 4.
 \end{aligned} \quad (\text{A.10})$$

In the  $4 \times 4$  representation, with  $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$

$$\begin{aligned}
 \gamma^0 &= \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \\
 \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
 \end{aligned} \quad (\text{A.11})$$

these solutions are given in the following table:

	$\uparrow u^1(k)$	$\downarrow u^2(k)$	$\downarrow u^3(k)$	$\uparrow u^4(k)$
$\sqrt{\frac{m+k_0}{2k_0}} \times$	1	0	$-\frac{k_3}{m+k_0}$	$\frac{k_1 - ik_2}{m+k_0}$
	0	1	$-\frac{k_1 + ik_2}{m+k_0}$	$-\frac{k_3}{m+k_0}$
	$\frac{k_3}{m+k_0}$	$\frac{k_1 - ik_2}{m+k_0}$	-1	0
	$\frac{k_1 + ik_2}{m+k_0}$	$-\frac{k_3}{m+k_0}$	0	1

The arrows denote spin up/down assignments in the  $k$  rest frame. Normalization:

$$\sum_{\alpha=1}^4 u_\alpha^{*i}(k)u_\alpha^j(k) = \delta_{ij}. \quad (\text{A.12})$$



Spin summations:

$$\begin{aligned} \sum_{i=1}^2 u_{\beta}^i(k) \bar{u}_{\alpha}^i(k) &= \frac{1}{2k_0} (-i\gamma k + m)_{\beta\alpha} , \\ \sum_{i=3}^4 u_{\beta}^i(k) \bar{u}_{\alpha}^i(k) &= -\frac{1}{2k_0} (i\gamma k + m)_{\beta\alpha} . \end{aligned} \quad (\text{A.13})$$

In connection with parity P, charge conjugation C, and time-reversal T, the following matrices and transformation properties are of relevance.

The matrix  $\gamma^4$  is the transformation matrix connected with space reflection

$$\begin{aligned} u^{\alpha}(-\vec{k}, k_0) &= -\gamma^4 u^{\alpha}(\vec{k}, k_0) , & \text{particle} & \quad \alpha = 1, 2 , \\ u^{\alpha}(-\vec{k}, k_0) &= -\gamma^4 u^{\alpha}(\vec{k}, k) , & \text{antiparticle} & \quad \alpha = 3, 4 . \end{aligned}$$

$$\gamma^4 \gamma^{\mu} \gamma^4 = \begin{cases} -\gamma^{\mu} & \mu = 1, 2, 3 \\ \gamma^{\mu} & \mu = 4 . \end{cases} \quad (\text{A.14})$$

The matrix C transforms an incoming particle into an incoming antiparticle, etc.

$$\begin{aligned} u^{\alpha}(k) &= -C \bar{u}^{\beta}(k) & \begin{cases} \alpha = 4, \beta = 1 \\ \alpha = 3, \beta = 2 \end{cases} \\ \bar{u}^{\alpha}(k) &= u^{\beta}(k) C^{-1} & \begin{cases} \alpha = 1, \beta = 4 \\ \alpha = 2, \beta = 3 \end{cases} \\ C^{-1} \gamma^{\alpha} C &= -\tilde{\gamma}^{\alpha} & (\sim = \text{transpose}) \\ C^{-1} \gamma^5 C &= \tilde{\gamma}^5 , \end{aligned} \quad (\text{A.15})$$

where  $C = \gamma^2 \gamma^4$ ,  $\tilde{C} = C^{-1} = -C$ .

In connection with time-reversal we have the matrix D

$$\begin{aligned} u^{\alpha}(-\vec{k}, k_0) &= -D \tilde{\gamma}^4 \bar{u}^{\beta}(\vec{k}, k_0) & \alpha = 2, \beta = 1 \\ &= D \tilde{\gamma}^4 \bar{u}^{\beta}(\vec{k}, k_0) & \alpha = 1, \beta = 2 \\ \bar{u}^{\alpha}(-\vec{k}, k_0) &= -u^{\beta}(\vec{k}, k_0) \tilde{\gamma}^4 D^{-1} & \alpha = 3, \beta = 4 \\ &= u^{\beta}(\vec{k}, k_0) \tilde{\gamma}^4 D^{-1} & \alpha = 4, \beta = 3 . \end{aligned} \quad (\text{A.16})$$

This changes spin, direction of three-momentum and furthermore exchanges in- and out-states;

$$D = -\gamma^5 C , \quad D^{-1} \gamma^{\mu} D = \tilde{\gamma}^{\mu} , \quad \mu = 1, 2, 3, 4, 5 .$$

Explicitly

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (\text{A.17})$$

Spin-1 particles

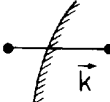
Propagator:

$$\frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 + m^2 - i\epsilon} .$$


In shadowed region:

$$- \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 + m^2 + i\epsilon} .$$


Cut propagator:

$$\frac{1}{(2\pi)^3} (\delta_{\mu\nu} + k_\mu k_\nu / m^2) \theta(k_0) \delta(k^2 + m^2) .$$


Particle wave-functions:

$$e_\mu(k) \quad \text{with} \quad e_\mu^*(k) e_\mu(k) = 1 \quad \text{and} \quad k_\mu e_\mu = 0 .$$

There are only three wave-functions, because the propagator matrix has one eigenvalue zero. In the  $k$  rest system the various assignments are

	<u>Incoming</u>	<u>Outgoing</u>
Spin-up:	$e_\mu = \frac{1}{\sqrt{2}} (1, i, 0, 0)$	$e_\mu = \frac{1}{\sqrt{2}} (1, -i, 0, 0)$
Spin-down:	$e_\mu = \frac{1}{\sqrt{2}} (1, -i, 0, 0)$	$e_\mu = \frac{1}{\sqrt{2}} (1, i, 0, 0)$
Spin-z component zero:	$e_\mu = (0, 0, 1, 0)$	$e_\mu = (0, 0, 1, 0)$

For outgoing particles we must take these expressions to have correct phases. Indeed, the residue of the two-point spin-up/spin-up amplitude is equal to one.

Combinatorial factors

These are best explained by considering a few examples. Let the interaction Lagrangian for a scalar field  $\phi$  be

$$\mathcal{L}_I = \frac{\alpha}{3!} \phi^3 + \frac{\beta}{4!} \phi^4 .$$

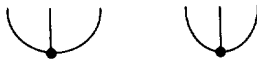
The vertices are:



The lowest order self-energy diagram is:



Draw two points  $x_1$  and  $x_2$  and draw in each of these points the  $\alpha$  vertex:



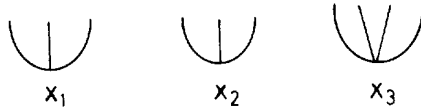
Now count in how many ways the lines can be connected with the same topological result. External line 1 can be attached in six, after that line 2 in three ways. After that there are two ways to connect the remaining lines such that the desired diagram results. Thus there are altogether  $6 \times 3 \times 2$  combinations. Now divide by the permutational factors of the vertices (here  $3!$  for each  $\alpha$  vertex). Finally divide by the number of permutations of the points  $x$  that have identical vertices. Here  $2!$ . The total result is

$$\frac{6 \times 3 \times 2}{3! \cdot 3! \cdot 2!} = \frac{1}{2}.$$

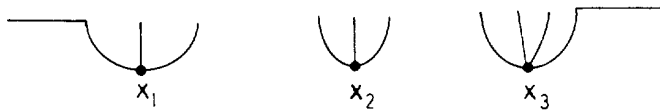
As another example consider the diagram:



There are three  $x$  points:



Line 1: six ways. Line 2: four ways. Then we have for instance:



There are  $6 \times 3 \times 2$  ways to connect the rest such as to get the desired topology. We must divide by vertex factors ( $3!$   $3!$   $4!$ ) and by  $2!$  (permutation of the identical vertex points  $x_1$  and  $x_2$ ). The result is

$$\frac{6 \times 4 \times 6 \times 3 \times 2}{3! \cdot 3! \cdot 4! \cdot 2!} = \frac{1}{2}.$$

Final example: two identical sources connected by a scalar line:

$$J \times \text{---} \times J \quad \text{Factor: } \frac{1}{2!} .$$

For two non-identical sources the factor is 1:

$$J_1 \times \text{---} \times J_2 \quad \text{Factor: } 1 .$$

Topology of quantum electrodynamics

We now show that the vertices of any diagram of quantum electrodynamics can be numbered in a unique way. Let the external momenta be  $k_1 \dots k_n$ .

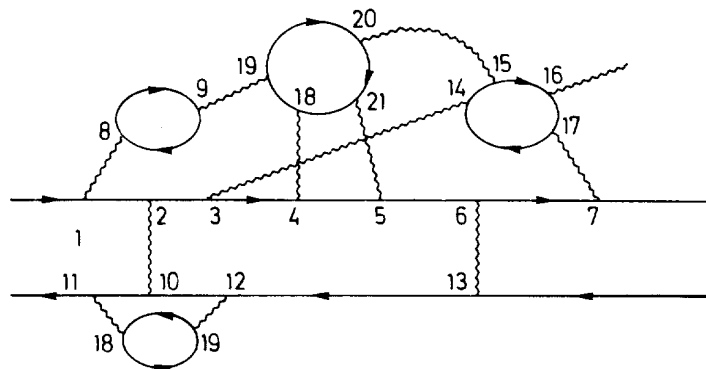
Step 1

Start at the electron line with the lowest momentum index. Follow the arrow of the line. Number the vertices consecutively.

Step 2

Go back to the first vertex (or lowest numbered vertex of which the photon line was not exploited). Follow the photon line. We arrive then either at a vertex that is already numbered, at an external photon line, or at another electron line. In the first two cases, take the next vertex along the electron line of step 1 and restart at step 1. When arriving at a new vertex, number again consecutively following the electron line along the arrow. When hitting the end, or an already numbered vertex, go back against the arrow and number all the vertices on the electron line before the vertex that was the entrance point of that line. After that restart 2.

Example:



APPENDIX B

SOME USEFUL FORMULAE

$$\int d_n x f(x) = \int f(x) r^{n-1} dr \sin^{n-2} \theta_{n-1} d\theta_{n-1} \sin^{n-3} \theta_{n-2} d\theta_{n-2} \dots d\theta_1 \quad (B.1)$$

with  $0 \leq \theta_1 \leq \pi$ , except  $0 \leq \theta_1 \leq 2\pi$ . If  $f(x)$  depends only on  $r = \sqrt{x_1^2 + \dots + x_n^2}$  one may perform the integration over angles using

$$\int_0^\pi \sin^m \theta d\theta = \sqrt{\pi} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \quad (B.2)$$

leading to

$$\int d_n x f(r) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int f(r) r^{n-1} dr, \quad (B.3)$$

$$\int_0^\infty dx \frac{x^\beta}{(x^2 + M^2)^\alpha} = \frac{1}{2} \frac{\Gamma\left(\frac{\beta+1}{2}\right) \Gamma\left(\alpha - \frac{\beta+1}{2}\right)}{\Gamma(\alpha) (M^2)^{\alpha - (\beta+1)/2}}. \quad (B.4)$$

Keeping the prescriptions and definitions of Section 13 in mind, the following equations hold for arbitrary  $n$

$$\int d_n p \frac{1}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha - n/2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)}, \quad (B.5)$$

$$\int d_n p \frac{p_\mu}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha - n/2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} (-k_\mu), \quad (B.6)$$

$$\int d_n p \frac{p^2}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha - n/2}} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma\left(\alpha - \frac{n}{2}\right) k^2 + \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \frac{n}{2} (m^2 - k^2) \right\}, \quad (B.7)$$

$$\int d_n p \frac{p_\mu p_\nu}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha - n/2}} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma\left(\alpha - \frac{n}{2}\right) k_\mu k_\nu + \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \frac{1}{2} \delta_{\mu\nu} (m^2 - k^2) \right\}, \quad (B.8)$$

$$\int d_n p \frac{p_\mu p_\nu p_\lambda}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha-n/2}} \times$$

$$\times \frac{1}{\Gamma(\alpha)} \left\{ -\Gamma\left(\alpha - \frac{n}{2}\right) k_\mu k_\nu k_\lambda - \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \frac{1}{2} (\delta_{\mu\nu} k_\lambda + \delta_{\mu\lambda} k_\nu + \delta_{\nu\lambda} k_\mu) (m^2 - k^2) \right\},$$

(B.9)

$$\int d_n p \frac{p^2 p_\mu}{(p^2 + 2kp + m^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2 - k^2)^{\alpha-n/2}} \times$$

$$\times \frac{1}{\Gamma(\alpha)} (-k_\mu) \left\{ \Gamma\left(\alpha - \frac{n}{2}\right) k^2 + \Gamma\left(\alpha - \frac{n}{2} - 1\right) \frac{n+2}{2} (m^2 - k^2) \right\}.$$

(B.10)

The above equations contain indices  $\mu, \nu, \lambda$ . These indices are understood to be contracted with arbitrary  $n$ -vectors  $q_1, q_2$ , etc. In computing the integrals one first integrates over the part of  $n$ -space orthogonal to the vectors  $k, q_1, q_2$ , etc., using Eqs. (B.1) to (B.4). After that the expressions are meaningful also for non-integer  $n$ . Note that formally Eqs. (B.6) to (B.10) may be obtained from Eq. (B.5) by differentiation with respect to  $k$ , or by using  $p^2 = (p^2 + 2pk + m^2) - 2pk - m^2$ .

To show that integrals over polynomials give zero within the dimensional regularization scheme is very simple. Consider, for example,

$$I(\alpha) = \int d_n p (p^2)^\alpha,$$

where  $\alpha$  is some integer greater than or equal to zero. According to Eq. (13.19) in the case of only one loop, we have

$$I(\alpha) = \int d^4 \underline{p} \int_0^\infty d\omega \omega^{n-5} \frac{2\pi^{n/2-2}}{\Gamma\left(\frac{n-4}{2}\right)} (\underline{p}^2 + \omega^2)^\alpha.$$

By partial integrations (see Subsection 13.2) we get now

$$I(\alpha) = \frac{2\pi^{n/2-2}}{\Gamma\left(\frac{n-4}{2} + \lambda\right)} \int d^4 \underline{p} \int_0^\infty d\omega \omega^{n-5+2\lambda} \left(-\frac{\partial}{\partial \omega^2}\right)^\lambda (\underline{p}^2 + \omega^2)^\alpha,$$

which gives zero for  $\lambda > \alpha$ .

A nice example, suggested by B. Lautrup is the following. Consider the following integral

$$I_\mu = \int d^4 k \frac{k_\mu}{(k^2 + m^2)^2},$$

which gives zero, because of symmetric integration, if one regularizes, for example, as follows

$$I_{\mu} \rightarrow \int d^4k k_{\mu} \left[ \frac{1}{(k^2 + m^2)^2} - \frac{1}{(k^2 + \Lambda^2)^2} \right].$$

It is also zero in the dimensional cut-off scheme according to Eq. (B.6).

Let us now shift the integration variable, forgetting about regulators

$$I_{\mu} = \int d^4k \frac{k_{\mu} + p_{\mu}}{[(k + p)^2 + m^2]^2}. \quad (\text{B.11})$$

Expanding the denominators, we get

$$I_{\mu} = \int \frac{d^4k}{(k^2 + m^2)^2} \left[ k_{\mu} + p_{\mu} - 4 \frac{k_{\mu}(k \cdot p)}{k^2 + m^2} \right] + O(p^2), \quad (\text{B.12})$$

which by symmetric integration ( $k_{\mu} \rightarrow 0$ ,  $k_{\mu}k_{\nu} \rightarrow \frac{1}{4}\delta_{\mu\nu}k^2$ ) gives

$$I_{\mu} = m^2 p_{\mu} \int \frac{d^4k}{(k^2 + m^2)^3} + O(p^2) \neq 0.$$

Using dimensional regularization, which means  $k_{\mu} \rightarrow 0$  but  $k_{\mu}k_{\nu} \rightarrow (\delta_{\mu\nu}/n)k^2$ , we get from Eq. (B.12)

$$I_{\mu}^n = p_{\mu} \int d^n k \frac{k^2 \left(1 - \frac{4}{n}\right) + m^2}{(k^2 + m^2)^3} + O(p^2).$$

From Eqs. (B.5) and (B.7)

$$\int d^n k \frac{1}{(k^2 + m^2)^3} = i\pi^{n/2} (m^2)^{n/2-3} \frac{\Gamma\left(3 - \frac{n}{2}\right)}{\Gamma(3)},$$

$$\int d^n k \frac{k^2}{(k^2 + m^2)^3} = i\pi^{n/2} (m^2)^{n/2-2} \frac{n}{2} \frac{\Gamma\left(2 - \frac{n}{2}\right)}{\Gamma(3)}.$$

Then

$$I_{\mu}^n = p_{\mu} i \frac{\pi^{n/2}}{2} (m^2)^{n/2-2} \left[ \left(1 - \frac{4}{n}\right) \frac{n}{2} \Gamma\left(2 - \frac{n}{2}\right) + \Gamma\left(3 - \frac{n}{2}\right) \right] + O(p^2).$$

In the limit  $n \rightarrow 4$ , remembering that

$$\Gamma(z) \stackrel{z \rightarrow -n}{\approx} \frac{(-1)^n}{n!} \frac{1}{z + n},$$

the coefficient of the  $p_\mu$  term turns out to be exactly zero. Of course, from Eqs. (B.5) to (B.7),  $I_\mu$  in Eq. (B.11) gives zero to any order in  $p$ .

In computing pole parts it is very advantageous to develop denominators. Take Eq. (B.5). For  $\alpha = 2$  we find the pole part

$$\text{PP}[(B.5), \alpha = 2] = \frac{-2i\pi^2}{n-4} = Z_0 . \quad (B.13)$$

This  $Z_0$  is a basic factor. Every logarithmically divergent integral has this factor, and further vectors,  $\delta$  functions, etc.

$$\text{PP} \int d_n p \frac{p_\alpha p_\beta}{(p^2 + m^2)^3} = Z_0 \frac{1}{4} \delta_{\alpha\beta} , \quad (B.14)$$

$$\text{PP} \int d_n p \frac{p_\alpha p_\beta p_\mu p_\nu}{(p^2 + m^2)^4} = Z_0 \frac{1}{24} (\delta_{\alpha\beta} \delta_{\mu\nu} + \delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu}) . \quad (B.15)$$

The four on the right-hand side follows from symmetry considerations; the coefficient follows because multiplication with  $\delta_{\alpha\beta}$  gives the previous integral. We leave it to the reader to find the general equation.

For other than logarithmically divergent integrals the denominator must be developed. For instance

$$\text{PP} \int d_n p \frac{p_\alpha}{(p^2 + 2pk + m^2)^2} = -k_\alpha Z_0 , \quad (B.16)$$

where we used Eq. (B.14) together with

$$\frac{1}{(p^2 + 2pk + m^2)^2} \simeq \frac{1}{p^4} \left\{ 1 - 4 \frac{pk}{p^2} + O(p^{-2}) \right\} .$$

Linearly, quadratically, etc., divergent integrals that have no dependence on masses or external momenta can be put equal to zero.

The result Eq. (B.16) coincides with what can be deduced from Eq. (B.6).



APPENDIX C

DEFINITION OF THE FIELDS FOR DRESSED PARTICLES

Also the matrix elements of fields and products of fields (such as encountered in currents) can be defined in terms of diagrams. It is then possible to derive, or rather verify, the equations of motion for the fields. This provides for the link between diagrams and the canonical operator formalism.

Since things tend to be technically complicated we will limit ourselves to a simple case, namely three real scalar fields interacting in the most simple way. The Lagrangian is taken to be

$$\mathcal{L} = \frac{1}{2} A(\partial^2 - m_A^2)A + \frac{1}{2} B(\partial^2 - m_B^2)B + \frac{1}{2} C(\partial^2 - m_C^2)C + g_{ABC} + J_A A + J_B B + J_C C . \quad (C.1)$$

The bare propagators will be denoted by the symbols  $\Delta_{FA}^b$ ,  $\Delta_{FB}^b$  and  $\Delta_{FC}^b$ , the dressed propagators by  $\bar{\Delta}_{FA}$ ,  $\bar{\Delta}_{FB}$  and  $\bar{\Delta}_{FC}$ . For example

$$\Delta_{FA}^b = \frac{1}{(2\pi)^4 i} \frac{1}{k^2 + m_A^2 - i\epsilon} , \quad \bar{\Delta}_{FA} = \frac{1}{(2\pi)^4 i} \frac{1}{Z_A^2(k^2 + M_A^2) - \Gamma_{1A}(k^2) - i\epsilon} . \quad (C.2)$$

The pole part of the dressed propagator plays an important role and will be denoted by  $\Delta_F$

$$\Delta_{FA} = \frac{1}{(2\pi)^4 i} \frac{1}{Z_A^2(k^2 + M_A^2) - i\epsilon} . \quad (C.3)$$

The result (C.2) has been obtained as follows [see Section 9, in particular Eq. (9.3)]. The function  $\Gamma_A(k^2)$  is the sum of all irreducible self-energy diagrams for the A-field. The dressed propagator is of the form  $(k^2 + m_A^2 - \Gamma_A)^{-1}$ . This expression will have a pole for some value of  $k^2$ , say for  $k^2 = -M_A^2$ . Then we can expand  $\Gamma_A$  around the point  $k^2 = -M_A^2$

$$\Gamma_A(k^2) = \delta m_A^2 + (k^2 + M_A^2)F_A + \Gamma_{1A}(k^2) , \quad Z_A^2 \equiv 1 - F_A , \quad \delta m_A^2 = m_A^2 - M_A^2 , \quad (C.4)$$

where  $\Gamma_{1A}$  is of order  $(k^2 + M_A^2)^2$ . Insertion of this expression leads to Eq. (C.2).

Next to the propagators we define external line factors  $N_A(k^2)$ , etc. They are the ratio of the dressed propagators and their pole parts

$$N_A(k^2) = \frac{\bar{\Delta}_{FA}}{Z_A \Delta_{FA}} . \quad (C.5)$$

In the limit  $k^2 = -M_A^2$  this is precisely the factor occurring in external lines when passing from Green's function to S-matrix. Finally we have the important  $\Delta^+$  and  $\Delta^-$  functions

$$\Delta_A^\pm = \frac{1}{(2\pi)^3} \frac{1}{Z_A^2} \theta(\pm k_0) \delta(k^2 + M_A^2) .$$



Remember that  $\Gamma_A$  starts with a B- and a C-line, and that  $N_A$  is attached to a B-C vertex [see Eq. (C.7)]. We see that the right-hand side of Eq. (C.7) consists of all skeleton diagrams starting with an amputated B-C vertex, apart from the  $\delta$ -correction.

We now define the product of this object and the matrix  $S^\dagger$ . It is obtained by connecting diagrams of  $\delta S/\delta A$  to diagrams of  $S^\dagger$  by means of  $\Delta^+$  functions:

$$\left. \begin{array}{c} \text{Diagram of } \delta S/\delta A \text{ connected to } S^\dagger \\ \text{via a shaded region} \end{array} \right\} \beta \equiv \langle \beta | S^\dagger \frac{\delta S}{\delta A(x)} | \alpha \rangle . \quad (\text{C.10})$$

This is a collection of cut diagrams, with  $S^\dagger$  corresponding to the part in the shadowed region. This definition of the product is the same as that encountered in the expression  $S^\dagger S$ .

It turns out that the differentiation symbol  $\delta/\delta$  in  $\delta S/\delta A$  has more than formal meaning. With the help of the cutting equations it is easy to show that

$$S^\dagger \frac{\delta S}{\delta A(x)} + \frac{\delta S^\dagger}{\delta A(x)} S = 0 . \quad (\text{C.11})$$

The second term has the point  $x$  to the right of the cutting line. This can be expressed formally by writing  $\delta(S^\dagger S)/\delta A(x) = 0$ , which is what is expected if unitarity holds,  $S^\dagger S = 1$ . One may speak of generalized unitarity, because the A-line is off mass-shell.

The A-field current  $j_A(x)$  is defined by

$$j_A(x) = i S^\dagger \frac{\delta S}{\delta A(x)} . \quad (\text{C.12})$$

By virtue of Eq. (C.11) it follows that  $j_A(x)$  is Hermitian.

To define the matrix elements of the field A consider the equation of motion

$$(\partial^2 - M_A^2)A(x) = j_A(x) . \quad (\text{C.13})$$

This is not directly the equation of motion that one would write down given the Lagrangian (C.1), because we have the mass  $M_A^2$  (defined by the location of the pole of the dressed propagator) instead of  $m_A^2$ .

The equation of motion (C.13) can be rewritten as an integral equation

$$\frac{1}{Z_A} A(x) = \frac{1}{Z_A} A_{\text{in}}(x) - i Z_A \int d_4 x' \Delta_{RA}(x - x') J_A(x') . \quad (\text{C.14})$$

The retarded  $\Delta$  function is

$$\Delta_{\text{RA}}(x) = \frac{1}{(2\pi)^4 i} \frac{1}{Z_A^2} \int d_4 k e^{ikx} \frac{1}{k^2 + M_A^2 - i\epsilon k_0} . \quad (\text{C.15})$$

This function is zero unless  $x_0 > 0$ . In fact

$$\Delta_{\text{R}}(x) = \theta(x_0) \{ \Delta^+(x) - \Delta^-(x) \} . \quad (\text{C.16})$$

In passing, we note the identities

$$\begin{aligned} \Delta_{\text{R}} - \Delta^+ &= -\Delta_{\text{F}}^* , \\ \Delta_{\text{R}} + \Delta^- &= \Delta_{\text{F}} . \end{aligned} \quad (\text{C.17})$$

Equation (C.14) defines the A-field in terms of diagrams. It satisfies the weak or asymptotic definition

$$\lim_{x_0 \rightarrow -\infty} \langle \alpha | A(x) | \beta \rangle = \lim_{x_0 \rightarrow -\infty} \langle \alpha | A_{\text{in}}(x) | \beta \rangle .$$

The field  $A_{\text{in}}(x)$  is a free field satisfying the equation of motion (C.13) with  $j = 0$ .

We will write Eq. (C.14) in terms of diagrams, and to that purpose we must introduce the "ordered product". We write

$$\frac{1}{Z_A} A_{\text{in}}(x) = \frac{1}{Z_A} A_{\text{in}}(x) S^\dagger S = \frac{1}{Z_A} : A_{\text{in}}(x) S^\dagger : S + Z_A \int d_4 x' \Delta_A^+(x-x') \frac{\delta S'}{\delta A(x')} S . \quad (\text{C.18})$$

The double dots imply that  $A_{\text{in}}$  is not to be connected by a  $\Delta^+$ -line to  $S^\dagger$ . Below this will be shown diagrammatically. The function  $\Delta_A^+$  has been defined before. Due to the presence of this  $\Delta^+$  only the mass-shell value of the factor  $N$  is required in  $\delta S^\dagger / \delta A$ , and this is  $1/Z_A$ . Using Eq. (C.18) we can rewrite the integral equation of motion (C.14) in the form

$$\frac{1}{Z_A} A(x) = \left[ \frac{1}{Z_A} : A_{\text{in}}(x) S^\dagger : - i Z_A \int d_4 x' \{ \Delta_{\text{RA}}(x-x') - \Delta_A^+(x-x') \} (-i) \frac{\delta S^\dagger}{\delta A(x')} \right] S . \quad (\text{C.19})$$

Keeping in mind Eq. (C.17) as well as Eq. (C.5) we see that  $A(x)/Z_A$  can be pictured as follows:

$$\frac{1}{Z_A} \left[ \text{Diagram 1} + \text{Diagram 2} \right] \quad (\text{C.20})$$

This then is the diagrammatic expression for the matrix elements of the field A. Similar expressions can be derived for the fields B and C. All kinds of relations from canonical field theory can be derived using these expressions. For instance

$$j_A(x) = - \frac{g}{Z_A Z_B Z_C} B(x)C(x) - \frac{\delta m_A^2}{Z_A^2} A(x) - \frac{F_A (\partial^2 - M_A^2)}{Z_A^2} A(x) \quad (C.21)$$

with  $F_A$  and  $\delta m$  from Eq. (C.4).