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"Relative State" Formulation of Quantum Mechanics*

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1. INTRODUCTION

THE task of quantizing general relativity raises serious questions about the meaning of the present formulation and interpretation of quantum mechanics when applied to so fundamental a structure as the space-time geometry itself. This paper seeks to clarify the foundations of quantum mechanics. It presents a reformulation of quantum theory in a form believed suitable for application to general relativity.

The aim is not to deny or contradict the conventional formulation of quantum theory, which has demonstrated its usefulness in an overwhelming variety of problems, but rather to supply a new, more general and complete formulation, from which the conventional interpretation can be *deduced*.

The relationship of this new formulation to the older formulation is therefore that of a metatheory to a theory, that is, it is an underlying theory in which the nature and consistency, as well as the realm of applicability, of the older theory can be investigated and clarified.

The new theory is not based on any radical departure from the conventional one. The special postulates in the old theory which deal with observation are omitted in the new theory. The altered theory thereby acquires a new character. It has to be analyzed in and for itself before any identification becomes possible between the quantities of the theory and the properties of the world of experience. The identification, when made, leads back to the omitted postulates of the conventional theory that deal with observation, but in a manner which clarifies their role and logical position.

We begin with a brief discussion of the conventional formulation, and some of the reasons which motivate one to seek a modification.

2. REALM OF APPLICABILITY OF THE CONVENTIONAL OR "EXTERNAL OBSERVATION" FORMULATION OF QUANTUM MECHANICS

We take the conventional or "external observation" formulation of quantum mechanics to be essentially the following[1]: A physical system is completely described by a state function ψ, which is an element of a Hilbert space, and which furthermore gives information only to the extent of specifying the probabilities of the results of various observations which can be made *on* the system *by* external observers. There are two fundamentally different ways in which the state function can change:

Process 1: The discontinuous change brought about by the observation of a quantity with eigenstates $\phi_1, \phi_2, \cdots$, in which the state ψ will be changed to the state ϕ_j with probability $|(\psi,\phi_j)|^2$.

Process 2: The continuous, deterministic change of state of an isolated system with time according to a wave equation $\partial\psi/\partial t = A\psi$, where A is a linear operator.

This formulation describes a wealth of experience. No experimental evidence is known which contradicts it.

Not all conceivable situations fit the framework of this mathematical formulation. Consider for example an isolated system consisting of an observer or measuring apparatus, plus an object system. Can the change with time of the state of the *total* system be described by Process 2? If so, then it would appear that no discontinuous probabilistic process like Process 1 can take place. If not, we are forced to admit that systems which contain observers are not subject to the same kind of quantum-mechanical description as we admit for all other physical systems. The question cannot be ruled out as lying in the domain of psychology. Much of the discussion of "observers" in quantum mechanics has to do with photoelectric cells, photographic plates, and similar devices where a mechanistic attitude can hardly be contested. For the following one can *limit himself to this class of problems*, if he is unwilling to consider observers in the more familiar sense on the same mechanistic level of analysis.

What mixture of Processes 1 and 2 of the conventional formulation is to be applied to the case where only an approximate measurement is effected; that is, where an apparatus or observer interacts only weakly and for a limited time with an object system? In this case of an

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[1] We use the terminology and notation of J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by R. T. Beyer (Princeton University Press, Princeton, 1955).

approximate measurement a proper theory must specify (1) the new state of the object system that corresponds to any particular reading of the apparatus and (2) the probability with which this reading will occur. von Neumann showed how to treat a special class of approximate measurements by the method of projection operators.[2] However, a general treatment of all approximate measurements by the method of projection operators can be shown (Sec. 4) to be impossible.

How is one to apply the conventional formulation of quantum mechanics to the space-time geometry itself? The issue becomes especially acute in the case of a closed universe.[3] There is no place to stand outside the system to observe it. There is nothing outside it to produce transitions from one state to another. Even the familiar concept of a proper state of the energy is completely inapplicable. In the derivation of the law of conservation of energy, one defines the total energy by way of an integral extended over a surface large enough to include all parts of the system and their interactions.[4] But in a closed space, when a surface is made to include more and more of the volume, it ultimately disappears into nothingness. Attempts to define a total energy for a closed space collapse to the vacuous statement, zero equals zero.

How are a quantum description of a closed universe, of approximate measurements, and of a system that contains an observer to be made? These three questions have one feature in common, that they all inquire about the *quantum mechanics* that is *internal to an isolated system*.

No way is evident to apply the conventional formulation of quantum mechanics to a system that is not subject to *external* observation. The whole interpretive scheme of that formalism rests upon the notion of external observation. The probabilities of the various possible outcomes of the observation are prescribed exclusively by Process 1. Without that part of the formalism there is no means whatever to ascribe a physical interpretation to the conventional machinery. But Process 1 is out of the question for systems not subject to external observation.[5]

3. QUANTUM MECHANICS INTERNAL TO AN ISOLATED SYSTEM

This paper proposes to regard pure wave mechanics (Process 2 only) as a complete theory. It postulates that a wave function that obeys a linear wave equation everywhere and at all times supplies a complete mathematical model for every isolated physical system without exception. It further postulates that every system that is subject to external observation can be regarded as part of a larger isolated system.

The wave function is taken as the basic physical entity with *no a priori interpretation*. Interpretation only comes *after* an investigation of the logical structure of the theory. Here as always the theory itself sets the framework for its interpretation.[5]

For any interpretation it is necessary to put the mathematical model of the theory into correspondence with experience. For this purpose it is necessary to formulate abstract models for observers that can be treated within the theory itself as physical systems, to consider isolated systems containing such model observers in interaction with other subsystems, to deduce the changes that occur in an observer as a consequence of interaction with the surrounding subsystems, and to interpret the changes in the familiar language of experience.

Section 4 investigates representations of the state of a composite system in terms of states of constituent subsystems. The mathematics leads one to recognize the concept of the *relativity of states*, in the following sense: a constituent subsystem cannot be said to be in any single well-defined state, independently of the remainder of the composite system. To any arbitrarily chosen state for one subsystem there will correspond a unique *relative state* for the remainder of the composite system. This relative state will usually depend upon the choice of state for the first subsystem. Thus the state of one subsystem does not have an independent existence, but is fixed only by the state of the remaining subsystem. In other words, the states occupied by the subsystems are not independent, but *correlated*. Such correlations between systems arise whenever systems interact. In the present formulation all measurements and observation processes are to be regarded simply as interactions between the physical systems involved—interactions which produce strong correlations. A simple model for a measurement, due to von Neumann, is analyzed from this viewpoint.

Section 5 gives an abstract treatment of the problem of observation. This uses only the superposition principle, and general rules by which composite system states are formed of subsystem states, in order that the results shall have the greatest generality and be applicable to any form of quantum theory for which these principles hold. Deductions are drawn about the state of the observer relative to the state of the object system. It is found that experiences of the observer (magnetic tape memory, counter system, etc.) are in full accord with predictions of the conventional "external observer" formulation of quantum mechanics, based on Process 1.

Section 6 recapitulates the "relative state" formulation of quantum mechanics.

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[2] Reference 1, Chap. 4, Sec. 4.

[3] See A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1950), third edition, p. 107.

[4] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Press, Cambridge, 1951), p. 343.

[5] See in particular the discussion of this point by N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **12**, No. 8 (1933).

4. CONCEPT OF RELATIVE STATE

We now investigate some consequences of the wave mechanical formalism of composite systems. If a composite system S, is composed of two subsystems S_1 and S_2, with associated Hilbert spaces H_1 and H_2, then, according to the usual formalism of composite systems, the Hilbert space for S is taken to be the *tensor product* of H_1 and H_2 (written $H=H_1\otimes H_2$). This has the consequence that if the sets $\{\xi_i^{S_1}\}$ and $\{\eta_j^{S_2}\}$ are complete orthonormal sets of states for S_1 and S_2, respectively, then the general state of S can be written as a superposition:

$$\psi^S=\sum_{i,j}a_{ij}\xi_i^{S_1}\eta_j^{S_2}. \tag{1}$$

From (3.1) although S is in a definite state ψ^S, the subsystems S_1 and S_2 do not possess anything like definite states independently of one another (except in the special case where all but one of the a_{ij} are zero).

We *can*, however, for any choice of a state in one subsystem, *uniquely* assign a corresponding *relative* state in the other subsystem. For example, if we choose ξ_k as the state for S_1, while the composite system S is in the state ψ^S given by (3.1), then the corresponding *relative state* in S_2, $\psi(S_2;\, \text{rel}\xi_k,S_1)$, will be:

$$\psi(S_2;\, \text{rel}\xi_k,S_1)=N_k\sum_j a_{kj}\eta_j^{S_2} \tag{2}$$

where N_k is a normalization constant. This relative state for ξ_k is *independent* of the choice of basis $\{\xi_i\}$ $(i\neq k)$ for the orthogonal complement of ξ_k, and is hence determined uniquely by ξ_k alone. To find the relative state in S_2 for an arbitrary state of S_1 therefore, one simply carries out the above procedure using any pair of bases for S_1 and S_2 which contains the desired state as one element of the basis for S_1. To find states in S_1 relative to states in S_2, interchange S_1 and S_2 in the procedure.

In the conventional or "external observation" formulation, the relative state in S_2, $\psi(S_2;\, \text{rel}\phi,S_1)$, for a state ϕ^{S_1} in S_1, gives the conditional probability distributions for the results of all measurements in S_2, given that S_1 has been measured and found to be in state ϕ^{S_1}—i.e., that ϕ^{S_1} is the eigenfunction of the measurement in S_1 corresponding to the observed eigenvalue.

For any choice of basis in S_1, $\{\xi_i\}$, it is always possible to represent the state of S, (1), as a *single* superposition of pairs of states, each consisting of a state from the basis $\{\xi_i\}$ in S_1 and its relative state in S_2. Thus, from (2), (1) can be written in the form:

$$\psi^S=\sum_i\frac{1}{N_i}\xi_i^{S_1}\psi(S_2;\, \text{rel}\xi_i,S_1). \tag{3}$$

This is an important representation used frequently.

Summarizing: There does not, in general, exist anything like a single state for one subsystem of a composite system. Subsystems do not possess states that are independent of the states of the remainder of the system, so that the subsystem states are generally correlated *with one another. One can arbitrarily choose a state for one subsystem, and be led to the relative state for the remainder. Thus we are faced with a fundamental* relativity of states, *which is implied by the formalism of composite systems. It is meaningless to ask the absolute state of a subsystem—one can only ask the state relative to a given state of the remainder of the subsystem.*

At this point we consider a simple example, due to von Neumann, which serves as a model of a measurement process. Discussion of this example prepares the ground for the analysis of "observation." We start with a system of only one coordinate, q (such as position of a particle), and an apparatus of one coordinate r (for example the position of a meter needle). Further suppose that they are initially independent, so that the combined wave function is $\psi_0^{S+A}=\phi(q)\eta(r)$ where $\phi(q)$ is the initial system wave function, and $\eta(r)$ is the initial apparatus function. The Hamiltonian is such that the two systems do not interact except during the interval $t=0$ to $t=T$, during which time the total Hamiltonian consists only of a simple interaction,

$$H_I=-i\hbar q(\partial/\partial r). \tag{4}$$

Then the state

$$\psi_t^{S+A}(q,r)=\phi(q)\eta(r-qt) \tag{5}$$

is a solution of the Schrödinger equation,

$$i\hbar(\partial\psi_t^{S+A}/\partial t)=H_I\psi_t^{S+A}, \tag{6}$$

for the specified initial conditions at time $t=0$.

From (5) at time $t=T$ (at which time interaction stops) there is no longer any definite independent apparatus state, nor any independent system state. The apparatus therefore does not indicate any definite object-system value, and nothing like process 1 has occurred.

Nevertheless, we *can* look upon the total wave function (5) as a *superposition* of pairs of subsystem states, each element of which has a definite q value and a correspondingly displaced apparatus state. Thus after the interaction the state (5) has the form:

$$\psi_T^{S+A}=\int\phi(q')\delta(q-q')\eta(r-qT)dq', \tag{7}$$

which is a superposition of states $\psi_{q'}=\delta(q-q')\eta(r-qT)$. Each of these elements, $\psi_{q'}$, of the superposition describes a state in which the system has the definite value $q=q'$, and in which the apparatus has a state that is displaced from its original state by the amount $q'T$. These elements $\psi_{q'}$ are then superposed with coefficients $\phi(q')$ to form the total state (7).

Conversely, if we transform to the representation where the *apparatus* coordinate is definite, we write (5) as

$$\psi_T^{S+A}=\int(1/N_{r'})\xi^{r'}(q)\delta(r-r')dr',$$

where

$$\xi^{r'}(q)=N_{r'}\phi(q)\eta(r'-qT) \tag{8}$$

and

$$(1/N_{r'})^2=\int\phi^*(q)\phi(q)\eta^*(r'-qT)\eta(r'-qT)dq.$$

Then the $\xi^{r'}(q)$ are the relative system state functions[6] for the apparatus states $\delta(r-r')$ of definite value $r=r'$.

If T is sufficiently large, or $\eta(r)$ sufficiently sharp (near $\delta(r)$), then $\xi^{r'}(q)$ is nearly $\delta(q-r'/T)$ and the relative system states $\xi^{r'}(q)$ are nearly eigenstates for the values $q=r'/T$.

We have seen that (8) is a superposition of states $\psi_{r'}$, *for each of which* the apparatus has recorded a definite value r', and the system is left in approximately the eigenstate of the measurement corresponding to $q=r'/T$. The discontinuous "jump" into an eigenstate is thus only a relative proposition, dependent upon the mode of decomposition of the total wave function into the superposition, and relative to a particularly chosen apparatus-coordinate value. So far as the complete theory is concerned all elements of the superposition exist simultaneously, and the entire process is quite continuous.

von Neumann's example is only a special case of a more general situation. Consider any measuring apparatus interacting with any object system. As a result of the interaction the state of the measuring apparatus is no longer capable of independent definition. It can be defined only *relative* to the state of the object system. In other words, there exists only a correlation between the states of the two systems. It seems as if nothing can ever be settled by such a measurement.

This indefinite behavior seems to be quite at variance with our observations, since physical objects always appear to us to have definite positions. Can we reconcile this feature wave mechanical theory built purely on Process 2 with experience, or must the theory be abandoned as untenable? In order to answer this question we consider the problem of observation itself within the framework of the theory.

5. OBSERVATION

We have the task of making deductions about the appearance of phenomena to observers which are considered as purely physical systems and are treated within the theory. To accomplish this it is necessary to identify some present properties of such an observer with features of the past experience of the observer. Thus, in order to say that an observer 0 has observed the event α, it is necessary that the state of 0 has become changed from its former state to a new state which is dependent upon α.

It will suffice for our purposes to consider the observers to possess memories (i.e., parts of a relatively permanent nature whose states are in correspondence with past experience of the observers). In order to make deductions about the past experience of an observer it is sufficient to deduce the present contents of the memory as it appears within the mathematical model.

As models for observers we can, if we wish, consider automatically functioning machines, possessing sensory apparatus and coupled to recording devices capable of registering past sensory data and machine configurations. We can further suppose that the machine is so constructed that its present actions shall be determined not only by its present sensory data, but by the contents of its memory as well. Such a machine will then be capable of performing a sequence of observations (measurements), and furthermore of deciding upon its future experiments on the basis of past results. If we consider that current sensory data, as well as machine configuration, is immediately recorded in the memory, then the actions of the machine at a given instant can be regarded as a function of the memory contents only, and all relavant experience of the machine is contained in the memory.

For such machines we are justified in using such phrases as "the machine has perceived A" or "the machine is aware of A" if the occurrence of A is represented in the memory, since the future behavior of the machine will be based upon the occurrence of A. In fact, all of the customary language of subjective experience is quite applicable to such machines, and forms the most natural and useful mode of expression when dealing with their behavior, as is well known to individuals who work with complex automata.

When dealing with a system representing an observer quantum mechanically we ascribe a state function, ψ^0, to it. When the state ψ^0 describes an observer whose memory contains representations of the events $A, B, \cdots, C$ we denote this fact by appending the memory sequence in brackets as a subscript, writing:

$$\psi^0{}_{[A,B,\cdots,C]}. \tag{9}$$

The symbols $A, B, \cdots, C$, which we assume to be ordered time-wise, therefore stand for memory configurations which are in correspondence with the past experience of the observer. These configurations can be regarded as punches in a paper tape, impressions on a magnetic reel, configurations of a relay switching circuit, or even configurations of brain cells. We require only that they be capable of the interpretation "The observer has experienced the succession of events $A, B, \cdots, C$." (We sometimes write dots in a memory sequence, $\cdots A, B, \cdots, C$, to indicate the possible presence of previous

[6] This example provides a model of an approximate measurement. However, the relative system states after the interaction $\xi^{r'}(q)$ cannot ordinarily be generated from the original system state ϕ by the application of *any* projection operator, E. Proof: Suppose on the contrary that $\xi^{r'}(q)=NE\phi(q)=N'\phi(q)\eta(r'-qt)$, where N, N' are normalization constants. Then

$$E(NE\phi(q))=NE^2\phi(q)=N''\phi(q)\eta^2(r'-qt)$$

and $E^2\phi(q)=(N''/N)\phi(q)\eta^2(r'-qt)$. But the condition $E^2=E$ which is necessary for E to be a projection implies that $N'/N''\,\eta(q)=\eta^2(q)$ which is generally false.

memories which are irrelevant to the case being considered.)

The mathematical model seeks to treat the interaction of such observer systems with other physical systems (observations), within the framework of Process 2 wave mechanics, and to deduce the resulting memory configurations, which are then to be interpreted as records of the past experiences of the observers.

We begin by defining what constitutes a "good" observation. A good observation of a quantity A, with eigenfunctions ϕ_i, for a system S, by an observer whose initial state is ψ^0, consists of an interaction which, in a specified period of time, transforms each (total) state

$$\psi^{S+0}=\phi_i\psi^0[\ldots] \tag{10}$$

into a new state

$$\psi^{S+0'}=\phi_i\psi^0[\ldots\alpha_i] \tag{11}$$

where α_i characterizes[7] the state ϕ_i. (The symbol, α_i, might stand for a recording of the eigenvalue, for example.) That is, we require that the system state, *if it is an eigenstate*, shall be unchanged, and (2) that the observer state shall change so as to describe an observer that is "aware" of which eigenfunction it is; that is, some property is recorded in the memory of the observer which characterizes ϕ_i, such as the eigenvalue. The requirement that the eigenstates for the system be unchanged is necessary if the observation is to be significant (repeatable), and the requirement that the observer state change in a manner which is different for each eigenfunction is necessary if we are to be able to call the interaction an observation at all. How closely a general interaction satisfies the definition of a good observation depends upon (1) the way in which the interaction depends upon the dynamical variables of the observer system—including memory variables—and upon the dynamical variables of the object system and (2) the initial state of the observer system. Given (1) and (2), one can for example solve the wave equation, deduce the state of the composite system after the end of the interaction, and check whether an object system that was originally in an eigenstate is left in an eigenstate, as demanded by the repeatability postulate. This postulate is satisfied, for example, by the model of von Neumann that has already been discussed.

From the definition of a good observation we first deduce the result of an observation upon a system which is *not* in an eigenstate of the observation. We know from our definition that the interaction transforms states $\phi_i\psi^0[\cdots]$ into states $\phi_i\psi^0[\cdots\alpha_i]$. Consequently these solutions of the wave equation can be superposed to give the final state for the case of an arbitrary initial system state. Thus if the initial system state is not an eigenstate, but a general state $\sum_i a_i\phi_i$, the final total state will have the form:

$$\psi^{S+0'}=\sum a_i\phi_i\psi^0[\cdots\alpha_i]. \tag{12}$$

This superposition principle continues to apply in the presence of further systems which do not interact during the measurement. Thus, if systems $S_1, S_2, \cdots, S_n$ are present as well as 0, with original states $\psi^{S_1}, \psi^{S_2}, \cdots, \psi^{S_n}$, and the only interaction during the time of measurement takes place between S_1 and 0, the measurement will transform the initial total state:

$$\psi^{S_1+S_2+\cdots+S_n+0}=\psi^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots] \tag{13}$$

into the final state:

$$\psi'^{S_1+S_2+\cdots+S_n+0}=\sum_i a_i\phi_i^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i] \tag{14}$$

where $a_i=(\phi_i^{S_1},\psi^{S_1})$ and $\phi_i^{S_1}$ are eigenfunctions of the observation.

Thus we arrive at the general rule for the transformation of total state functions which describe systems within which observation processes occur:

Rule 1: The observation of a quantity A, with eigenfunctions $\phi_i^{S_1}$, in a system S_1 by the observer 0, transforms the total state according to:

$$\psi^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots] \rightarrow\sum_i a_i\phi_i^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i] \tag{15}$$

where

$$a_i=(\phi_i^{S_1},\psi^{S_1}).$$

If we next consider a *second* observation to be made, where our total state is now a superposition, we can apply Rule 1 separately to each element of the superposition, since each element separately obeys the wave equation and behaves independently of the remaining elements, and then superpose the results to obtain the final solution. We formulate this as:

Rule 2: Rule 1 may be applied separately to each element of a superposition of total system states, the results being superposed to obtain the final total state. Thus, a determination of B, with eigenfunctions $\eta_j^{S_2}$, on S_2 by the observer 0 transforms the total state

$$\sum_i a_i\phi_i^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i] \tag{16}$$

into the state

$$\sum_{i,j} a_i b_j\phi_i^{S_1}\eta_j^{S_2}\psi^{S_3}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i,\beta_j] \tag{17}$$

where $b_j=(\eta_j^{S_2},\psi^{S_2})$, which follows from the application of Rule 1 to each element $\phi_i^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i]$, and then superposing the results with the coefficients a_i.

These two rules, which follow directly from the superposition principle, give a convenient method for determining final total states for any number of observation processes in any combinations. We now seek the *interpretation* of such final total states.

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[7] It should be understood that $\psi^0[\cdots\alpha_i]$ is a *different* state for each i. A more precise notation would write $\psi^0_i[\cdots\alpha_i]$, but no confusion can arise if we simply let the ψ_i^0 be indexed only by the index of the memory configuration symbol.

Let us consider the simple case of a single observation of a quantity A, with eigenfunctions ϕ_i, in the system S with initial state ψ^S, by an observer 0 whose initial state is $\psi^0[\cdots]$. The final result is, as we have seen, the superposition

$$\psi'^{S+0}=\sum_i a_i\phi_i\psi^0[\cdots\alpha_i]. \qquad (18)$$

There is no longer any independent system state or observer state, although the two have become correlated in a one-one manner. However, in each *element* of the superposition, $\phi_i\psi^0[\cdots\alpha_i]$, the object-system state is a particular eigenstate of the observation, and *furthermore the observer-system state describes the observer as definitely perceiving that particular system state.* This correlation is what allows one to maintain the interpretation that a measurement has been performed.

We now consider a situation where the observer system comes into interaction with the object system for a second time. According to Rule 2 we arrive at the total state after the second observation:

$$\psi''^{S+0}=\sum_i a_i\phi_i\psi^0[\cdots\alpha_i,\alpha_i]. \qquad (19)$$

Again, each element $\phi_i\psi^0[\cdots\alpha_i,\alpha_i]$ describes a system eigenstate, but this time also describes the observer as having obtained the *same result* for each of the two observations. Thus for every separate state of the observer in the final superposition the result of the observation was repeatable, even though different for different states. This repeatability is a consequence of the fact that after an observation the *relative* system state for a particular observer state is the corresponding eigenstate.

Consider now a different situation. An observer-system 0, with initial state $\psi^0[\cdots]$, measures the *same* quantity A in a number of separate, identical, systems which are initially in the same state, $\psi^{S_1}=\psi^{S_2}=\cdots=\psi^{S_n}=\sum_i a_i\phi_i$ (where the ϕ_i are, as usual, eigenfunctions of A). The initial total state function is then

$$\psi_0{}^{S_1+S_2+\cdots+S_n+0}=\psi^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots]. \qquad (20)$$

We assume that the measurements are performed on the systems in the order S_1, S_2, $\cdots S_n$. Then the total state after the first measurement is by Rule 1,

$$\psi_1{}^{S_1+S_2+\cdots+S_n+0}=\sum_i a_i\phi_i{}^{S_1}\psi^{S_2}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i{}^1] \qquad (21)$$

(where $\alpha_i{}^1$ refers to the first system, S_1).

After the second measurement it is, by Rule 2,

$$\psi_2{}^{S_1+S_2+\cdots+S_n+0}=\sum_{i,j}a_ia_j\phi_i{}^{S_1}\phi_j{}^{S_2}\psi^{S_3}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i{}^1,\alpha_j{}^2] \qquad (22)$$

and in general, after r measurements have taken place ($r\leq n$), Rule 2 gives the result:

$$\psi_r=\sum_{i,j,\ldots k}a_ia_j\cdots a_k\phi_i{}^{S_1}\phi_j{}^{S_2}\cdots\phi_k{}^{S_r}\psi^{S_{r+1}}\cdots\psi^{S_n}\psi^0[\cdots\alpha_i{}^1,\alpha_j{}^2,\cdots\alpha_k{}^r] \qquad (23)$$

We can give this state, ψ_r, the following interpretation. It consists of a superposition of states:

$$\psi'_{ij\ldots k}=\phi_i{}^{S_1}\phi_j{}^{S_2}\cdots\phi_k{}^{S_r}\times\psi^{S_{r+1}}\cdots\psi^{S_n}\psi^0[\alpha_i{}^1,\alpha_j{}^2,\cdots\alpha_k{}^r] \qquad (24)$$

each of which describes the observer with a definite memory sequence $[\alpha_i{}^1,\alpha_j{}^2,\cdots\alpha_k{}^r]$. Relative to him the (observed) system states are the corresponding eigenfunctions $\phi_i{}^{S_1}$, $\phi_j{}^{S_2}$, $\cdots$, $\phi_k{}^{S_r}$, the remaining systems, S_{r+1}, $\cdots$, S_n, being unaltered.

A typical element $\psi'_{ij\ldots k}$ of the final superposition describes a state of affairs wherein the observer has perceived an apparently random sequence of definite results for the observations. Furthermore the object systems have been left in the corresponding eigenstates of the observation. At this stage suppose that a redetermination of an earlier system observation (S_l) takes place. Then it follows that every element of the resulting final superposition will describe the observer with a memory configuration of the form $[\alpha_i{}^1,\cdots\alpha_j{}^l,\cdots\alpha_k{}^r,\alpha_j{}^l]$ in which the earlier memory coincides with the later—i.e., the memory states are *correlated.* It will thus *appear* to the observer, as described by a typical element of the superposition, that each initial observation on a system caused the system to "jump" into an eigenstate in a random fashion and thereafter remain there for subsequent measurements on the same system. Therefore—disregarding for the moment quantitative questions of relative frequencies—the probabilistic assertions of Process 1 *appear* to be valid to the observer described by a typical element of the final superposition.

We thus arrive at the following picture: Throughout all of a sequence of observation processes there is only one physical system representing the observer, yet there is no single unique *state* of the observer (which follows from the representations of interacting systems). Nevertheless, there is a representation in terms of a *superposition*, each element of which contains a definite observer state and a corresponding system state. Thus with each succeeding observation (or interaction), the observer state "branches" into a number of different states. Each branch represents a different outcome of the measurement and the *corresponding* eigenstate for the object-system state. All branches exist simultaneously in the superposition after any given sequence of observations.‡

‡ *Note added in proof.*—In reply to a preprint of this article some correspondents have raised the question of the "transition from possible to actual," arguing that in "reality" there is—as our experience testifies—no such splitting of observer states, so that only one branch can ever actually exist. Since this point may occur to other readers the following is offered in explanation.

The whole issue of the transition from "possible" to "actual" is taken care of in the theory in a very simple way—there is no such transition, nor is such a transition necessary for the theory to be in accord with our experience. From the viewpoint of the theory *all* elements of a superposition (all "branches") are "actual," none any more "real" than the rest. It is unnecessary to suppose that all but one are somehow destroyed, since all the

The "trajectory" of the memory configuration of an observer performing a sequence of measurements is thus not a linear sequence of memory configurations, but a branching tree, with all possible outcomes existing simultaneously in a final superposition with various coefficients in the mathematical model. In any familiar memory device the branching does not continue indefinitely, but must stop at a point limited by the capacity of the memory.

In order to establish quantitative results, we must put some sort of measure (weighting) on the elements of a final superposition. This is necessary to be able to make assertions which hold for almost all of the observer states described by elements of a superposition. We wish to make quantitative statements about the relative frequencies of the different possible results of observation—which are recorded in the memory—for a typical observer state; but to accomplish this we must have a method for selecting a typical element from a superposition of orthogonal states.

We therefore seek a general scheme to assign a measure to the elements of a superposition of orthogonal states $\sum_i a_i\phi_i$. We require a positive function m of the complex coefficients of the elements of the superposition, so that $m(a_i)$ shall be the measure assigned to the element ϕ_i. In order that this general scheme be unambiguous we must first require that the states themselves always be normalized, so that we can distinguish the coefficients from the states. However, we can still only determine the *coefficients*, in distinction to the states, up to an arbitrary phase factor. In order to avoid ambiguities the function m must therefore be a function of the amplitudes of the coefficients alone, $m(a_i)=m(|a_i|)$.

We now impose an additivity requirement. We can regard a subset of the superposition, say $\sum_{i=1}^{n} a_i\phi_i$, as a single element $\alpha\phi'$:

$$\alpha\phi'=\sum_{i=1}^{n} a_i\phi_i. \tag{25}$$

We then demand that the measure assigned to ϕ' shall be the sum of the measures assigned to the ϕ_i (i from 1 to n):

$$m(\alpha)=\sum_{i=1}^{n} m(a_i). \tag{26}$$

Then we have already restricted the choice of m to the square amplitude alone; in other words, we have $m(a_i)=a_i^*a_i$, apart from a multiplicative constant.

To see this, note that the normality of ϕ' requires that $|\alpha|=(\sum a_i^*a_i)^{\frac{1}{2}}$. From our remarks about the dependence of m upon the amplitude alone, we replace the a_i by their amplitudes $u_i=|a_i|$. Equation (26) then imposes the requirement,

$$m(\alpha)=m(\sum a_i^*a_i)^{\frac{1}{2}}=m(\sum u_i^2)^{\frac{1}{2}}=\sum m(u_i)=\sum m(u_i^2)^{\frac{1}{2}}. \tag{27}$$

Defining a new function $g(x)$

$$g(x)=m(\sqrt{x}) \tag{28}$$

we see that (27) requires that

$$g(\sum u_i^2)=\sum g(u_i^2). \tag{29}$$

Thus g is restricted to be linear and necessarily has the form:

$$g(x)=cx \quad (c \text{ constant}). \tag{30}$$

Therefore $g(x^2)=cx^2=m(\sqrt{x^2})=m(x)$ and we have deduced that m is restricted to the form

$$m(a_i)=m(u_i)=cu_i^2=ca_i^*a_i. \tag{31}$$

We have thus shown that the only choice of measure consistent with our additivity requirement is the square amplitude measure, apart from an arbitrary multiplicative constant which may be fixed, if desired, by normalization requirements. (The requirement that the total measure be unity implies that this constant is 1.)

The situation here is fully analogous to that of classical statistical mechanics, where one puts a measure on trajectories of systems in the phase space by placing a measure on the phase space itself, and then making assertions (such as ergodicity, quasi-ergodicity, etc.) which hold for "almost all" trajectories. This notion of "almost all" depends here also upon the choice of measure, which is in this case taken to be the Lebesgue measure on the phase space. One could contradict the statements of classical statistical mechanics by choosing a measure for which only the exceptional trajectories had nonzero measure. Nevertheless the choice of Lebesgue measure on the phase space can be justified by the fact that it is the only choice for which the "conservation of probability" holds, (Liouville's theorem) and hence the only choice which makes possible any reasonable statistical deductions at all.

In our case, we wish to make statements about "trajectories" of observers. However, for us a trajectory is constantly branching (transforming from state to superposition) with each successive measurement. To

separate elements of a superposition individually obey the wave equation with complete indifference to the presence or absence ("actuality" or not) of any other elements. This total lack of effect of one branch on another also implies that no observer will ever be aware of any "splitting" process.

Arguments that the world picture presented by this theory is contradicted by experience, because we are unaware of any branching process, are like the criticism of the Copernican theory that the mobility of the earth as a real physical fact is incompatible with the common sense interpretation of nature because we feel no such motion. In both cases the argument fails when it is shown that the theory itself predicts that our experience will be what it in fact is. (In the Copernican case the addition of Newtonian physics was required to be able to show that the earth's inhabitants would be unaware of any motion of the earth.)

have a requirement analogous to the "conservation of probability" in the classical case, we demand that the measure assigned to a trajectory at one time shall equal the sum of the measures of its separate branches at a later time. This is precisely the additivity requirement which we imposed and which leads uniquely to the choice of square-amplitude measure. Our procedure is therefore quite as justified as that of classical statistical mechanics.

Having deduced that there is a unique measure which will satisfy our requirements, the square-amplitude measure, we continue our deduction. This measure then assigns to the $i, j, \cdots k$th element of the superposition (24),

$$\phi_i^{S_1}\phi_j^{S_2}\cdots\phi_k^{S_r}\psi^{S_{r+1}}\cdots\psi^{S_n}\psi^0{}_{[\alpha_i^1,\alpha_j^2,\cdots\alpha_k^r]} \tag{32}$$

the measure (weight)

$$M_{ij\ldots k}=(a_ia_j\cdots a_k)^*(a_ia_j\cdots a_k) \tag{33}$$

so that the observer state with memory configuration $[\alpha_i^1, \alpha_j^2, \cdots, \alpha_k^r]$ is assigned the measure $a_i^*a_ia_j^*a_j\cdots a_k^*a_k=M_{ij\ldots k}$. We see immediately that this is a product measure, namely,

$$M_{ij\ldots k}=M_iM_j\cdots M_k \tag{34}$$

where

$$M_l=a_l^*a_l$$

so that the measure assigned to a particular memory sequence $[\alpha_i^1, \alpha_j^2, \cdots, \alpha_k^r]$ is simply the product of the measures for the individual components of the memory sequence.

There is a direct correspondence of our measure structure to the probability theory of random sequences. *If we regard* the $M_{ij\ldots k}$ as probabilities for the sequences then the sequences are equivalent to the random sequences which are generated by ascribing to each term the *independent* probabilities $M_l=a_l^*a_l$. Now probability theory is equivalent to measure theory mathematically, so that we can make use of it, while keeping in mind that all results should be translated back to measure theoretic language.

Thus, in particular, if we consider the sequences to become longer and longer (more and more observations performed) *each* memory sequence of the final superposition will satisfy any given criterion for a randomly generated sequence, generated by the independent probabilities $a_l^*a_l$, except for a set of total measure which tends toward zero as the number of observations becomes unlimited. Hence all averages of functions over *any* memory sequence, including the special case of frequencies, can be computed from the probabilities $a_i^*a_i$, except for a set of memory sequences of measure zero. We have therefore shown that the statistical assertions of Process 1 will appear to be valid to the observer, *in almost all* elements of the superposition (24), in the limit as the number of observations goes to infinity.

While we have so far considered only sequences of observations of the same quantity upon identical systems, the result is equally true for arbitrary sequences of observations, as may be verified by writing more general sequences of measurements, and applying Rules 1 and 2 in the same manner as presented here.

We can therefore summarize the situation when the sequence of observations is arbitrary, when these observations are made upon the same or different systems in any order, and when the number of observations of each quantity in each system is very large, with the following result:

> Except for a set of memory sequences of measure nearly zero, the averages of any functions over a memory sequence can be calculated approximately by the use of the independent probabilities given by Process 1 for each initial observation, on a system, and by the use of the usual transition probabilities for succeeding observations upon the same system. In the limit, as the number of all types of observations goes to infinity the calculation is exact, and the exceptional set has measure zero.

This prescription for the calculation of averages over memory sequences by probabilities assigned to individual elements is precisely that of the conventional "external observation" theory (Process 1). Moreover, these predictions hold for almost all memory sequences. Therefore all predictions of the usual theory will appear to be valid to the observer in amost all observer states.

In particular, the uncertainty principle is never violated since the latest measurement upon a system supplies all possible information about the relative system state, so that there is no direct correlation between any earlier results of observation on the system, and the succeeding observation. Any observation of a quantity B, between two successive observations of quantity A (all on the same system) will destroy the one-one correspondence between the earlier and later memory states for the result of A. Thus for alternating observations of different quantities there are fundamental limitations upon the correlations between memory states for the same observed quantity, these limitations expressing the content of the uncertainty principle.

As a final step one may investigate the consequences of allowing several observer systems to interact with (observe) the same object system, as well as to interact with one another (communicate). The latter interaction can be treated simply as an interaction which correlates parts of the memory configuration of one observer with another. When these observer systems are investigated, in the same manner as we have already presented in this section using Rules 1 and 2, one finds that in *all elements* of the final superposition:

1. When several observers have separately observed the same quantity in the object system and then communicated the results to one another they find that they

are in agreement. This agreement persists even when an observer performs his observation *after* the result has been communicated to him by another observer who has performed the observation.

2. Let one observer perform an observation of a quantity A in the object system, then let a second perform an observation of a quantity B in this object system which does not commute with A, and finally let the first observer repeat his observation of A. Then the memory system of the first observer will *not* in general show the same result for both observations. The intervening observation by the other observer of the noncommuting quantity B prevents the possibility of any one to one correlation between the two observations of A.

3. Consider the case where the states of two object systems are correlated, but where the two systems do not interact. Let one observer perform a specified observation on the first system, then let another observer perform an observation on the second system, and finally let the first observer repeat his observation. Then it is found that the first observer always gets the same result both times, and the observation by the second observer has no effect whatsoever on the outcome of the first's observations. Fictitious paradoxes like that of Einstein, Podolsky, and Rosen[8] which are concerned with such correlated, noninteracting systems are easily investigated and clarified in the present scheme.

Many further combinations of several observers and systems can be studied within the present framework. The results of the present "relative state" formalism agree with those of the conventional "external observation" formalism in all those cases where that familiar machinery is applicable.

In conclusion, the continuous evolution of the state function of a composite system with time gives a complete mathematical model for processes that involve an idealized observer. When interaction occurs, the result of the evolution in time is a superposition of states, each element of which assigns a different state to the memory of the observer. Judged by the state of the memory in almost all of the observer states, the probabilistic conclusion of the usual "external observation" formulation of quantum theory are valid. In other words, pure Process 2 wave mechanics, without any initial probability assertions, leads to all the probability concepts of the familiar formalism.

6. DISCUSSION

The theory based on pure wave mechanics is a conceptually simple, causal theory, which gives predictions in accord with experience. It constitutes a framework in which one can investigate in detail, mathematically, and in a logically consistent manner a number of sometimes puzzling subjects, such as the measuring process itself and the interrelationship of several observers. Objections have been raised in the past to the conventional or "external observation" formulation of quantum theory on the grounds that its probabilistic features are postulated in advance instead of being derived from the theory itelf. We believe that the present "relative-state" formulation meets this objection, while retaining all of the content of the standard formulation.

While our theory ultimately justifies the use of the probabilistic interpretation as an aid to making practical predictions, it forms a broader frame in which to understand the consistency of that interpretation. In this respect it can be said to form a *metatheory* for the standard theory. It transcends the usual "external observation" formulation, however, in its ability to deal logically with questions of imperfect observation and approximate measurement.

The "relative state" formulation will apply to all forms of quantum mechanics which maintain the superposition principle. It may therefore prove a fruitful framework for the quantization of general relativity. The formalism invites one to construct the formal theory first, and to supply the statistical interpretation later. This method should be particularly useful for interpreting quantized unified field theories where there is no question of ever isolating observers and object systems. They all are represented in a *single* structure, the field. Any interpretative rules can probably only be deduced in and through the theory itself.

Aside from any possible practical advantages of the theory, it remains a matter of intellectual interest that the statistical assertions of the usual interpretation do not have the status of independent hypotheses, but are deducible (in the present sense) from the pure wave mechanics that starts completely free of statistical postulates.

[8] Einstein, Podolsky, and Rosen, Phys. Rev. **47**, 777 (1935). For a thorough discussion of the physics of observation, see the chapter by N. Bohr in *Albert Einstein, Philosopher-Scientist* (The Library of Living Philosophers, Inc., Evanston, 1949).