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**Borisov A. V., Mamaev I. S.**

Rigid Body Dynamics. — Izhevsk: NIC "Regular & Chaotic Dynamics", 2001, 384 p.

The book discusses main forms of equations of motion of a rigid body, including the motion in potential fields, in fluid (Kirchhoff's equations), and motion of a rigid body with cavities filled with fluid. The book contains conditions of the order reduction of these equations, and existence of cyclic variables. It collects almost all integrable cases presently known, and methods of their explicit integration. For the purpose of investigation, the computer techniques, allowing vivid representation of the motion picture, are widely used. The majority of results presented in the book belongs to the authors.

For students and graduate students of mechanical, mathematical and physical departments of universities, mathematical physicists, and specialists on dynamical systems.

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# Contents

# PREFACE

## I.

We have started writing this book two years ago with the aim of collecting all the integrable cases, known in rigid body dynamics. We felt that such a project could be realized rather fast, and the book was supposed to be published in 2000 — the year of 150-anniversary of S. V. Kowalevskaya. We also wanted to present comprehensive information concerning the case and the method she had discovered.

However, our plans were gradually extended, mainly because of the active application of numerical experiments and computer visualization methods together with analytical computations. In the end, we have developed a perfectly new viewpoint to one of the classical fields of mechanics, the one which allows generalization encompassing the whole dynamics.

In the preface we give honor to computer dynamics whose development and application to dynamic problems of top theory the reader will be meeting through the whole book. Computer investigations in dynamics, or just computer dynamics, is, in our opinion, a separate scientific field, establishing general regularities of the motion of real physical systems by means of a series of numerical methods and techniques. Each of these methods has its own peculiar features (stability and others) and possesses some internal parameters (like a pitch and precision). That's why results of such an investigation are related to the reality only indirectly. However, similar conclusions may be also made for the ordinary analytical (or purely mathematical) methods that demand rigorous proofs at each step of the process. At that a lot of physically evident facts may lead to unsolvable problems (these are especially numerous in nonlinear dynamics and mathematical theory of chaos). Here we are going to indicate only problems with the ergodicity proof, entropy computation, small parameter estimates, KAM-theory applicability and so on. Nevertheless, the solution of these problems, will not at the least advance our understanding of the remarkable regularities which we observe, following the evolution of chaos in particular systems.

In this book a classical branch of rigid body dynamics, dealing with the search of possible integrable cases, finds its natural conclusion. It's probable that other cases and integrals that may be found in the future, will never arise



the kind of attention that was aroused by the already found and cited here ones. Classics tried to use them to understand motion, and they achieved temporary successes. In rigid body dynamics the enthusiasm about geometric interpretations of motion, tracing back to Poinsot, every now and then was replaced by the analytical investigations, the majority of which, unfortunately, was unnecessary neither for physicists, nor for engineers, and soon became comprehensible for the narrow specialists only.

In this book we, probably, somewhat ignored proofs and precise formulations. We simultaneously used achievements of topology, analysis and computer experiments to receive sufficiently complete understanding of motion. It's not that easy to assert if we attained our aim, but one thing can't be doubted: even the classical cases (like Lagrange, Kowalevskaya, Goryachev – Chaplygin cases) have experienced in this approach the second birth, they have best the framework of dull computations and become rather tangible. Probably, it should be the ambition of mechanics — to present a certain algorithm, by means of which we can look into the whole variety of motions and clearly imagine each particular motion and its peculiarities.

In this book we try to revive traditions of mathematical literature of Euler times. According to Jacobi expression [183], "Euler himself in spite of considering only particular cases, selects them so well that the general method determined later adds to his results only little or nothing at all".

Thus, if we consider the laws of nature, leading to a certain system of differential equations, be established, then for its analysis the computer and analytical methods turn out to be complementary. Here we'd like to emphasize the difference between our viewpoint and the prevailing one that the "real science" is analytical, and the computer is capable of giving only illustrations to analytical methods and impulsion for statements of new theorems. That's certainly true, as well, but it's only a byproduct of computer investigations. The latter have their own inner logic and a system of descriptions of physical phenomena. Systematic development of computer investigations , revealing new areas of computer (or "virtual") dynamics, is the matter of the nearest future.

As a historical prospect, or rather as a funny thing, illustrating the excessive belief in the power of logical method, note that Leibnitz and Descartes in their papers, before developing proper mathematical methods, "proved" the existence of motion and even God.

## II.

In addition to the idea of computer dynamics, in the book we tried to show the most modern methods of Poisson dynamics and geometry, theories of Lie groups and algebras, which were only designated in our previous book *Poisson Structures and Lie Algebras in Hamiltonian Mechanics*, which, as we feel, was quite a success. Rigid body dynamics plays a special role in the development of these methods. In a certain sense it represents a ground for testing new mathematical means, and at the time being it's difficult to appreciate its significance, especially for the development of many sections of topology and nonlinear Poisson structures, nonholonomic geometry, theory of symmetries and tensor invariants.

We can even assert that, similar to the way the understanding of profound ideas of H. Poincaré, concerning the nonintegrability of dynamical systems, became possible due to the analysis of three body problem, results and techniques of Sophus Lie entered general mathematical culture because of their application to theory of tops, exemplifying the mechanical realization of the most natural Lie groups and algebras. Besides, unlike celestial mechanics and theory of oscillations, rigid body dynamics contains, on the one hand, a series of nontrivial integrable cases, and, on the other hand, on account of configurational space compactness it is mostly preferable for the analysis of chaotic motions.

## III.

While checking nearly all modern and classical integrable cases , we used the analytical computational system MAPLE. It happened so that some previously known results turned out to be not absolutely correct, and some others were considerably simplified.

Computer visualization of motion and numerical integration were carried out on the software complex "Computer dynamics" invented in the scientific-publishing center "Regular and Chaotic Dynamics".

The problems of stability of particular motions and the majority of applied and technical problems, whose thorough presentation requires a separate treatise, were left beyond the book. Nevertheless, even a physicist or an engineer may understand from the book general formalism of notation of main dynamical equations , and key aspects of regular and chaotic behavior in rigid body dynamics. In this aspect, the book can be considered as a reference book, where, nevertheless, we try to explain derivation of the main results, and sometimes produce complete proofs.



We decided to neglect sections, concerning nonholonomic systems, and also multidimensional generalizations of rigid body dynamics. They are rather extensive, and we'll try to explain them separately.

In the beginning of the book we gathered short historical accounts about the creators of rigid body dynamics. These assays allow to trace the evolution of ideas in this field, and, probably, to correct some historical discrepancies.

# INTRODUCTION

**1.** As an introduction we are going to present some short comments, concerning main stages of rigid body dynamics development. Integrable cases were the first to be studied. The most popular ones were found by Euler (1758) and Lagrange (1788) at the stage of formation and development of the main dynamical principles. At this point the basic system, used for approbations of various mathematical methods during next centuries, was the system of Euler – Poisson equations, describing motion of a heavy rigid body around a fixed point.

Substantially more difficult case of integrability of Euler – Poisson equations was discovered be S. V. Kowalevskaya in 1888. It has given an impulsion to new investigations in the field of integrable systems, This result was highly appreciated by the French Academy of Sciences. In 1888, S. V. Kowalevskaya was awarded with the Baurden Prize for the memoir on rotation of a rigid body around a fixed point. It should be mentioned that earlier the Academy of Sciences had announced about the competition on investigation of this problem twice, but nobody received the Prize. In spring of 1889 Kowalevskaya was honored with the Prize of the Swedish Royal Academy of Sciences for the second memoir on the problem of rigid body rotation.

The integrability of the Euler and Lagrange cases is stipulated by natural dynamical symmetries and preservation of the corresponding first integrals. S. V. Kowalevskaya has found her case of integrability, starting from nonevident analytical considerations and using theory of algebraic functions (whose particular case is elliptic functions), well developed at the time. She required uniqueness of the general solution on the complex plane of time, which in the future led to the beginnings of one of the most advanced methods of dynamic system analysis for integrability — the Painleve – Kowalevskaya test. As it is said, the Kowalevskaya integral doesn't have natural symmetry origin; its symmetries are hidden, and the problem of motion description and explicit integration itself is essentially more difficult in this case.

**2.** Starting from the middle of nineteenth and in the beginning of twentieth century in rigid body dynamics there were found integrable cases for various statements of problems on rigid body motion: motion of a body in fluid; motion of a body with cavities, filled with fluid; gyrostats; nonholonomic problems.



The study of these problems became possible due to the development of general dynamical formalism whose summit became the Poincaré equations, allowing to represent rigid body motion equations in terms of group variables.

Here we should also mention the progress of perfect fluid hydrodynamics and vortex theory whose foundations were laid by H. Helmholtz. That was the way to obtain equations for a vorticity vector, quite analogical to dynamical equations of kinetic moment, and Poincaré was the first to study precession of the equator axis, using, as the Earth model, a rigid body (a mantle), having cavities, filled with incompressible vortex fluid (core).

**3.** As it was already mentioned, in the classical period for various forms of equations it was considered of prime importance to find such cases (that could be fixed by restrictions of parameters and initial conditions) of explicit solvability of a problem in quadratures. In modern terminology these are called integrable cases.

The integrable cases are usually connected with the names of their discoverers. Among them are famous Western mathematics and mechanics: G. Kirchhoff, A. Clebsch, P. Appell, F. Brun, V. Volterra. Great achievements were made by Russian scientists A. M. Lyapunov, V. A. Steklov, N. E. Joukovskiy, S. A. Chaplygin. In this respect rigid body dynamics may be considered as a field, filled with interesting integrable problems, constituting the most valuable possession of modern dynamics.

In the classical period, except for the finding of first integrals, it was considered especially valuable to obtain explicit solutions in various classes of functions, mainly, elliptical ones. Particular successes were achieved here by S. V. Kowalevskaya, V. Volterra, G. Halphen, and up to this very day their technique remains unsurpassed.

**4.** In the first half of twentieth century the interest to integrable cases has, so to say, decreased. In many respects that was connected with understanding by the majority of mathematicians the results (obtained by H. Poincaré), concerning nonintegrability of a typical Hamiltonian dynamical system [144]. In the consciousness of mathematicians this fact depreciated many results of classics and led to the development of new methods of perturbation theory: the averaging principle, the KAM theory and others.

In the general case the main equations of rigid body dynamics are also nonintegrable. This means they have complicated and unpredictable behavior [144], whose study is a subject of a new field of investigations called determinate chaos. The effects of nonintegrability in rigid body dynamics find their systematic study in the treatise by V. V. Kozlov [92].

The book [92] is also important, because, unlike unnatural craving of classics for obtaining explicit solution, which allows to say but little about real motion of a system, it involves the question, concerning the qualitative analysis of integrable dynamical systems, and by using examples of the Kowalevskaya and Goryachev – Chaplygin tops the author makes general conclusions concerning the behavior of the line of nodes and proper rotation angles. The latter results were obtained by applying Liouville – Arnold theorem and Weyl theorem on uniform distribution.

**5.** The application of topological analysis methods to the integration of rigid body dynamics problems, namely the study of Liouville tori reconstructions under passing critical values, was for the first time offered by M. P. Harlamov [170] and developed in topological invariant theory, created to classify integrable Hamiltonian systems with two degrees of freedom. Nearly all known results, obtained by using this technique, are shown in the recently published book [25]. The complex methods, generally leading to the similar results, are advocated in the book by M. Audin [134].

**6.** The increase of interest to rigid body dynamics integrable problems in 1970–1990, having entailed the discovery of the whole series of new integrable cases, is connected with the isospectral deformation method development (Lax representations, $\mathbf{L}-\mathbf{A}$-pairs). As a rule, the majority of papers of that period concerns multidimensional generalizations of natural physical analogues already known. The development of this trend of researches is also associated with the penetration of ideas of Lie groups and algebras, and the analysis of general (nonlinear and degenerate) Poisson structures into dynamics. The present state of these problems may be found in our book [31].

It should also be noted that it turned out to be possible to extend many structures of the Lie algebraic approach and qualitative analysis methods to nonholonomic problems of rigid body dynamics, where within last decades several new integrable systems were added, as well [52, 36].

**7.** During last decades there appeared some more trends, concerning top dynamics. One appeared in quantum mechanics from the analysis of systems of interacting spins with anisotropy (a chain or $XYZ$-model of Heisenberg). Here the classical model is a foundation for understanding dynamics at a quantum level, and, in a certain sense, it can also be integrable and chaotic. The quantum chaos is only starting to be investigated, but in short time these researches will form a separate scientific branch, where the essential place will be given to quantum descriptions of tops. First of all, that is because the top model is a basic model in quantum theory of angular momentum, applied in quantum chemistry and molecular spectroscopy.

It is also interesting to know that the conditions of integrability and the integrals for a spin model, cited in the present day literature on quantum mechanics (see, for example, [259]), are but simplified results, obtained by classics (W. Frahm, F. Schottky) more than a hundred years ago. That is conditioned by the fact that many modern physicists who has gone far in the field of their abstract and intricate theories (like quantum field theory, gravitation theory), exhibit poor orientation in naturally originated questions, concerning dynamics of an ordinary toy top.



**8.** In a certain sense, even in the analysis of the integrable situation, for which the complete classification of all solutions is, in principle, possible, a computer has started a totally new era. If, earlier, in the investigation of integrable systems there prevailed analytical methods, making it possible to obtain explicit quadratures and geometric interpretations, which in many cases looked quite artificial (for instance, Joukovskiy interpretation of Kowalevskaya top motion [76]), the combination of ideas of topological analysis (bifurcational pattern), stability theory, phase section method and direct computer visualization of the “most remarkable” solutions is capable of representing an integrable situation and emphasizing the most characteristic features of motion. Such an investigation can provide a series of new results, even for a seemingly worn field (for example, for the Kowalevskaya top, Goryachev – Chaplygin top, Bobylev – Steklov solution). The point is that these results are very difficult to be detected in the cumbersome analytical expressions. It seems to be possible to obtain the proof of these facts analytically, as well, but only after their computer displaying. Here we should pay special attention to the analysis of motion in absolute space. Such an analysis was practically never carried out.

Some curious motions, exhibited by integrable tops, perhaps, are capable of evoking certain ideas concerning their practical application. Recall that, for example, the Kowalevskaya top (discovered more than a hundred years ago) is still out of the application, just because nothing at all was known about its motion, in spite of complete solution in elliptic functions.

We also cite some unstable periodic solutions, generating a family of doubly-asymptotic motions, whose behavior is most complicated and even in the presence of an additional integral looks chaotic. Under perturbation such solutions are the first to become destroyed, and near them, in phase space, there arise whole domains, full of “real” chaotic paths.

Computer researches make us “revise” many things and perceive the true meaning of analytical investigations. If some analytical results — like separation of variables — turn out to be quite useful for studying bifurcations and classical solutions, their further “development” up to obtaining explicit quadratures (in terms of $\theta$-functions) is practically of no use. These results are collected, for example, in the books [61, 72], but they are applicable as exercises on differential equations, rather than dynamical analysis methods.

**9.** As for the value of classical results in rigid body dynamics, it was somewhat doubted already in the seventies of the last century (K. Magnus [119]). The age of total belief in unlimited possibilities of computers generated the opinion that these results are of no use and the sufficiently powerful computer is capable of predicting motion at any interval of time with sufficient precision. However, the fact of the exponentially fast separation of paths (connected with the instability in the whole domains of phase space) in typical dynamical systems, which are nonintegrable, made such a computation at rather large time intervals physically meaningless, as far as the initial conditions for particular (applied) systems are never known with absolute precision.

It seems that one can hope for numerical methods only in the integrable case where this separation never happens. Nevertheless, it turns out that conservative systems preserve many elements of integrable dynamics even in the stochastic case. Under small perturbation of an integrable problem there continue to exist non-degenerate periodic orbits, and the majority of conditionally-periodical motions doesn't become destroyed (the KAM theory).

Under further increase of perturbation both periodic orbits, and invariant tori undergo various bifurcations, having some common regularities. They define the change of the whole structure of a phase flow, combining areas with regular and chaotic behavior, and provide scenarios of transition to chaos. In rigid body dynamics these investigations, which are incidentally impossible without highly precise computer simulation, were not carried out. In the present book we show only several examples of chaotic motion and hope that the nearest future will bring a lot of new and interesting results in this field.

# RIGID BODY DYNAMICS CREATORS

Here is a bit of information concerning the scientists, who obtained the main results, cited in the book. We meant to show their achievements in rigid body dynamics only, while many of them have also received well-known results in other fields of mathematics and mechanics. These brief sketches may be useful for understanding the evolution of principal ideas and methods, and also for elimination of some historical discrepancies.

All the sketches are in chronological order.

**Euler, Leonard (15.4.1707 – 18.9.1783)** — a great mathematician and mechanic. He was born in Switzerland, but the substantial part of his life he has spent in Russia (1727–41, 1766–83). Euler has contributed to nearly all branches of mathematics, his work is difficult to be surveyed and includes more than 865 essays.

L. Euler

In rigid body dynamics Euler has developed theory of moments of inertia and obtained the formula of velocity distribution in a rigid body. In 1750 he obtained the equations of motion in a fixed frame of reference, the ones which turned out to be of a little use in practice. In the works of 1758–1765 Euler, for the first time, introduced a moving frame of reference, attached to the body, and obtained the Euler – Poisson equations in the final form (the Poisson contribution, reflected in the name, seems to consist in their systematic account in his famous course on mechanics). These papers also contain Euler angles, kinematic relations, named after Euler, and an integrable case in the absence of a gravity field. As for the last case, Euler brings it up to quadratures and considers various particular solutions. In would be proper to mention the contribution Euler made into applied sciences — shipbuilding, artillery, turbine theory, strength of materials.

**Lagrange, Joseph Louis (25.1.1736 – 10.4.1813)** — a great French mathematician, mechanic, and astronomer. In his famous treatise *Analytical Mechanics* (in 2 volumes), along with the general formalism of dynamics,

J. L. Lagrange

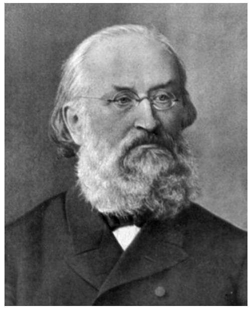
he has shown equations of rigid body motion in an arbitrary potential force field, using the frame of reference, attached to the body, angular momentum projections and direction cosines (volume II). There he also mentions an integrable case, characterized by the axial symmetry, which he reduced to quadratures. Following his principle of avoiding drawings, Lagrange doesn't give geometrical study of motion, so an apex behavior drawings, that were earlier included into nearly all textbooks on mechanics, appeared for the first time in the paper by Poisson (1815) who has investigated this problem as a totally new. Nevertheless, Poisson systematized notations that complicate understanding of treatises by D'Alembert, Euler and Lagrange, and considered various particular cases of motion (some textbooks refer to the Lagrange case as the Lagrange – Poisson case). In his turn, Lagrange has simplified the solution of the Euler case and has provided the direct proof of existence of third-degree equation real roots, defining the position of principal axes. We should note that Lagrange has also contributed into perturbation theory which enabled Jacobi to consider the problem about the Euler top perturbation and obtain the system of corresponding "osculating" variables.

L. Poinsot

**Poinsot, Louis (3.1.1777 – 5.12.1859)** — a French engineer, mechanic, and mathematician. He has given a geometric interpretation of the Euler case, introduced concepts of inertia ellipsoid, momentary axis of rotation, and the notions connected to it — polhode and herpolhode (1851). He has shown geometrical analysis of stability of rigid body rotation around the principal axis of inertia ellipsoid. Poinsot, in contrast to Lagrange, insisted that in mechanics geometric methods should be preferred to analytical ones — "in all these solutions we can see only computations without any clear picture of the body motion" [252]. Afterwards, the Poinsot ideas were supported and developed by N. E. Joukovskiy and S. A. Chaplygin. Poinsot also used a geometrical method for studying statics (*Statics Elements*, 1803).

**Kirchhoff, Gustav Robert (12.3.1824–17.10.1887)** — a German physicist and mechanic. In his *Lectures on Mathematical Physics* (1874–94, v. 1–4) he has laid the foundations of modern theory of elasticity, hydrodynamics, optics and electrodynamics. He has shown the analogy between the Euler – Poisson equations and equations of an elastic curve bending. Following the idea of Thomson and Tait, he reduced the problem of rigid body motion in perfect fluid, to the system of ordinary differential equations. He has found an integrable case, characterized by the axial symmetry, shown its solution in elliptic functions and considered various particular motions.



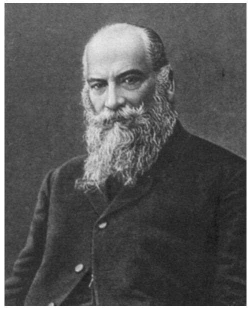


G. R. Kirchhoff

A. Clebsch

**Clebsch, Rudolph Fridrich Alfred (19.1.1833 – 7.11.1872)** — a German mathematician and mechanic. He has founded the journal *Mathematishe Annalen* which for sixty years was a leading mathematical journal. He was an expert in projective geometry and theory of invariants of algebraic forms. He offered a new form of notation for Kirchhoff's equations which is equivalent to the transition from Lagrangian to Hamiltonian description. For these equations he has shown a case of existence of an additional quadratic integral,which, as it turned out later, is identical to integrals of Brun and Tisserand.

**Joukovskiy, Nikolay Egorovitch (17.1.1847 – 17.3.1921)** — a Russian mechanic, mathematician, and engineer, as V. I. Lenin has put it, "the father of Russian aviation". In his master's thesis (1885) he laid the foundations of the theory of motion of a rigid body with cavities, completely filled with a perfect incompressible fluid. For multiconnected cavities he noticed the equivalence of the obtained form of equation with the equation of motion of a rigid body with a fly-wheel — a gyrostat. He introduced corresponding dynamical characteristics and carried out their computations for cavities of various shapes. He has indicated the case of integrability of a free gyrostat. The explicit solution for this case was obtained by V. Volterra by means of elliptic functions (1899).



N. E. Joukovskiy

He has investigated "planar" motions of a rigid body in Lobachevskiy space. He has offered a geometric interpretation and his own method of reducing to quadratures the Kowalevskaya case, investigating a certain auxiliary system of curvilinear coordinates. He has noted the pendulum nature of the mass center motion for the Hess case, having offered an interesting geometrical analysis for it. In view of his investigations in fluid dynamics he considered a series of model statements of problems concerning plane motion of plates under the action of lifting force conditioned by the circulation. In mechanics N. E. Joukovskiy considered a perfect solution to be geometrically clear and vivid picture of motion, similar to the Poinsot interpretation. However, it should be mentioned that the interpretations of gyrostat motion and the Kowalevskaya case, obtained by Joukovskiy personally, are rather difficult and not quite natural.

S. V. Kowalevskaya

**Kowalevskaya, Sophia Vasilievna (15.1.1850 – 10.2.1891)** — a famous Russian female-mathematician. In 1874 she presented her thesis in Göttingen and was awarded with a Philosophy Doctor degree; in 1884 she got a chair of mathematics in Stockholm University; in 1889 she was elected Corresponding Member of the St. Petersburg Academy of Sciences. She was in the editorial board of the journal *Acta Mathematica*. She was the first female professor of mathematics in the world.

For the discovery (after Euler and Lagrange) of the third case of integrability of the Euler – Poisson equations she was awarded with the Baurden Prize (1888), and for the second paper concerning rigid body rotation with the Prize of Swedish Royal Academy of Sciences. In these papers she has also offered so called Kowalevskaya method which is a widely used test for the integrability and concerns behavior of the general solution on a complex plane of time. She also obtained explicit quadratures, employing theta-functions of two variables. The transformations, carried out by Kowalevskaya, are still far from triviality and cannot be substantially simplified.

Kowalevskaya also dealt with general questions of integration of partial differential equations (the Cauchy – Kowalevskaya theorem), the stability of Saturn belts, the propagation of light within crystals.

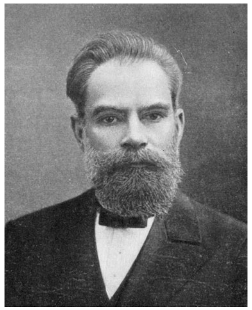


Being literary talented, Kowalevskaya has written several novels and memoirs which are read even nowadays.

**Poincaré, Henri Jules (29.4.1854 – 17.7.1912)** — a famous French mathematician, physicist, astronomer, and philosopher. In three volumes of his treatise *New Methods of Celestial Mechanics* he, using the three body problem as an example, laid the foundation of a new qualitative investigation of dynamical systems, indicated obstructions to existence of analytical integrals for a wide class of dynamical systems. He has expressed, but hasn't proved corresponding considerations about the Euler – Poisson equations. He has established a new form of dynamical equations in terms of group variables. This form classified the Euler and Lagrange particular results and turned out to be most useful for various problems of rigid body dynamics. The Hamiltonian variant of these equations was offered by N. G. Chetayev.

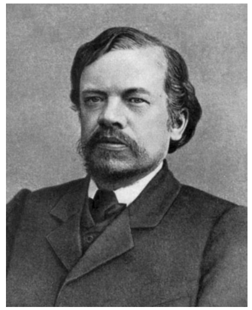
H. Poincaré

The group formalism being developed was applied by Poincaré to derive equations of a rigid body with cavities filled with perfect incompressible vortex fluid. For these equations he indicated an integrable case characterized by dynamical symmetry. He also obtained an elliptic quadrature which he used to explain various effects in the precession of the Earth he imagined as a hard shell (mantle) with a liquid core. He has also shown explicit formulae for small oscillation frequencies and obtained necessary conditions of stability.



A. M. Lyapunov

V. A. Steklov



S. A. Chaplygin

**Lyapunov, Alexander Mikhailovitch (6.6.1857 – 3.11.1918)** — a famous Russian mathematician and mechanic, the founder of the motion stability theory. He has discovered the case of integrability of Kirchhoff's equations for rigid body motion in fluid. In an extensive memoir of 1888 he indicated and investigated for stability helical motions of a rigid body in fluid. He made clear the question of correctness of the Kowalevskaya reasoning about uniqueness of the solution in integrable cases, by offering his own method, based on the introduction of a small parameter and investigation of the equation in variations — the Kowalevskaya – Lyapunov method.

**Steklov, Vladimir Andreevitch (9.1.1864 – 30.5.1926)** — a Russian mathematician and mechanic, the student of A. M. Lyapunov. In 1894 he presented his master thesis *On Rigid Body Motion in a Fluid* where he obtained the new case of integrability of Kirchhoff's equations and proved the theorem about impossibility of other cases, having an additional quadratic integral.

He has noticed the similarity between the Clebsch case and the Brun problem. In 1909 he indicated a new integrable family for the problem of motion of a rigid body with cavities, filled with the fluid (Poincaré – Joukovskiy equations). He has given two particular solutions of Euler – Poisson equations (one of them was given simultaneously with D. K. Bobylev).

**Chaplygin, Sergei Alexeyevitch (5.4.1869 – 8.10.1942)** — a Russian mathematician and mechanic, one of the creators of modern fluid dynamics. He has indicated a particular case of integrability of the Euler – Poisson equations at a zero area constant, having generalized a more particular solution given by D. N. Goryachev, and also more particular solutions, characterized by a system of linear invariant relations. For Kirchhoff's equations he has also obtained a similar case of particular integrability and its generalizations, investigated helical motions, and gave geometrical interpretation of various motions (in particular, for the Clebsch case). He has derived equations of heavy rigid body motion in a fluid and carried out a more detailed study of a case of planar and axially symmetric motion.

Chaplygin is especially famous by his works on nonholonomic mechanics where he has indicated a series of integrable problems of rigid body dynamics: rolling on axially symmetric solid plane, the "Chaplygin sphere", the Chaplygin sledge and so on. Similarly to N. E. Joukovskiy he was trying to introduce geometrical vividness in his masterly analytical computations.

**Kozlov, Valeriy Vasilievitch (born 1.01.1950)** — a Russian mathematician and mechanic, Member of the Russian Academy of Sciences (from 2000). In a series of works, combined in the treatise *Methods of Qualitative Analysis in*



*Rigid Body Dynamics* (MSU[1], 1980), he has proved nonexistence of analytical integrals of the Euler – Poisson equations, and has also indicated dynamical effects, preventing the integrability of these equations — splitting separatrices and appearance of a large number of non-degenerate periodical solutions. These investigations "have closed" the Poincaré problem, he has indicated in *New Methods of Celestial Mechanics* (v. 1), and started a new era in rigid body dynamics. In the foreground there are qualitative investigation methods, rather than search of particular solutions of a given algebraic structure.


V. V. Kozlov

V. V. Kozlov has also offered new techniques of analysis of integrable systems, based on the application of Liouville – Arnold geometrical theorem and Weyl theorem on uniform distribution. As a certain substantiation of the Kowalevskaya method V. V. Kozlov has proved a series of statements, connecting the general solution ramification on a complex plane of time with nonexistence of single-valued first integrals (Penleve – Golubev hypothesis). V. V. Kozlov was the first to apply variational techniques to obtain periodic solutions in rigid body dynamics.

[1]Moscow State University. — *Trans. Rem.*

# Chapter 1
# RIGID BODY MOTION EQUATIONS AND THEIR INTEGRATION

## § 1. Poisson Brackets and Hamiltonian Formalism

### 1. Poisson Manifolds

The majority of problems considered in the present book allows canonical Hamiltonian notation and has the first integral — the energy one. However, it's not rare a case when it's more convenient not to use the canonical form of these equations of motion, but a certain system of algebraic variables. Such a system is the most acceptable one for investigation: search for integrals, particular solutions, stability analysis and so on. The system expressed in terms of these variables will not only preserve a lot of ordinary Hamiltonian system properties, but acquire some specific distinctions being studied in the general theory of Poisson structures. The above-mentioned theory can be found in [31].

Here we are going to give a short account of principal definitions and results, necessary for solving of problems of rigid body dynamics. It should also be noted that evolution of theory of Poisson structures was stimulated in many respects by top dynamics, for the latter allows to turn abstract enunciations of many theories into more descriptive and natural ones.

Those who don't know differential and symplectic geometry very well (at this point we can recommend the books [75, 6, 7]), while reading this section, may envision all the results in the coordinate form and ignore mathematical terminology which is sometimes too formal. This terminology is founded on simple dynamic facts, but at first glance it can seem a bit alienated from them.

**Poisson brackets and their properties.** Ordinary *Hamiltonian form of dynamics equations* is represented by

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad H = H(\boldsymbol{q}, \boldsymbol{p}), \tag{1.1}$$

where canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ are determined on some even-dimensional manifold $(\boldsymbol{q}, \boldsymbol{p}) \in M^{2n}$ — *a phase space* $(\dim M = 2n)$. The function $H$ is called



*a Hamiltonian*. The quantity $n = \dfrac{\dim M}{2}$ is known as a number of degrees of freedom of Hamiltonian system (1.1).

The divergence of vector field (1.1) equals zero. This means that a phase flow is incompressible (*the Liouville theorem*).

If *a Poisson bracket* of two functions $F$ and $G$ be introduced according to the formula

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right), \tag{1.2}$$

then equations (1.1) could be rewritten in the form

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}. \tag{1.3}$$

Any differentiable function $F = F(\boldsymbol{q}, \boldsymbol{p})$ also evolves in accordance with the Hamiltonian law:

$$\dot{F} = \{F, H\}. \tag{1.4}$$

Equations (1.1) are not invariant with respect to arbitrary coordinate transformations. Moreover, if main equations of rigid body dynamics be written in the form (1.1), they would lose their algebraicity and acquire some peculiarities, far from the problem essence (see. § 4 s.2). Before introducing motion equations in a more acceptable form, preserving main properties of a canonical notation, let's dwell on the invariant statement of Hamiltonian mechanics.

Under invariant construction of Hamiltonian formalism (following *P. Dirac*) one proceeds from equations (1.3) and postulate properties of *Poisson brackets*[1], defined for functions given on a certain manifold $M$ with coordinates $\boldsymbol{x} = (x^1, \dots, x^n)$. These brackets should satisfy the following conditions:

1°. $\{\lambda F_1 + \mu F_2, G\} = \lambda\{F_1, G\} + \mu\{F_2, G\}, \quad \lambda, \mu \in \mathbb{R}$ — *bilinearity*,

2°. $\{F, G\} = -\{G, F\}$ — *skew-symmetry*,

3°. $\{F_1 F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$ — *the Leibnitz rule*,

4°. $\{\{H, F\}, G\} + \{\{G, H\}, F\} + \{\{F, G\}, H\} = 0$ — *the Jacobi identity*.

We'll be calling the Poisson bracket $\{\cdot, \cdot\}$ *the Poisson structure* as well, and the manifold $M$, on which it is given, *the Poisson manifold*.

In the above definition we abandoned the *non-degeneracy* requirement, (i. e., for any function $F(\boldsymbol{x}) \not\equiv \mathrm{const}$ there exist $G \not\equiv \mathrm{const}$, $\{F, G\} \not\equiv 0$), which

[1] Hereinafter we say both Poisson brackets and bracket, allowing a certain liberty of language.

are a fortiori satisfied for canonical structure (1.2). This permits, for example, to introduce a Poisson bracket for odd-dimensional systems. In our reasoning a Poisson structure may turn out to be *degenerate* and possess *Casimir functions* $F_k(\boldsymbol{x})$, commutating with all variables $x_i$ and, hence, with any functions $G(\boldsymbol{x})$ on $M$: $\{F_k, G\} = 0$. Casimir functions are also called *central functions, Casimirs or annihilators*.

Properties $1^\circ$– $4^\circ$ allow to write the Poisson bracket of functions $F$ and $G$ in the explicit coordinate form

$$\{F, G\} = \sum_{i,j} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}. \tag{1.5}$$

Basic brackets $J^{ij} = \{x^i, x^j\}$ are called *structural functions* of Poisson manifold $M$ with respect to the given, as a matter of fact, local frame of reference $\boldsymbol{x} = (x_1, \ldots, x_n)$ [7, 135]. They form *structural matrix* (*tensor*) $\mathbf{J} = \|J^{ij}\|$ of the size $n \times n$.

If

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix}, \qquad \mathbf{E} = \|\delta_i^j\|, \tag{1.6}$$

then we receive a canonical Poisson bracket, determined by formula (1.2).

A structural matrix $\mathbf{J}(\boldsymbol{x})$ meets the following conditions, resulting from $1^\circ$– $4^\circ$:

I. skew-symmetry:

$$J^{ij}(\boldsymbol{x}) = -J^{ji}(\boldsymbol{x}), \tag{1.7}$$

II. the Jacobi identity:

$$\sum_{l=1}^{n} \left( J^{il} \frac{\partial J^{jk}}{\partial x^l} + J^{kl} \frac{\partial J^{ij}}{\partial x^l} + J^{jl} \frac{\partial J^{ki}}{\partial x^l} \right) = 0. \tag{1.8}$$

Therefore, for instance, any constant skew-symmetrical matrix $\|J^{ij}\|$ specifies a Poisson structure.

An invariant object, determined by tensor $\mathbf{J}$, is a bivector (bivector field):

$$\mathbf{J}(dF, dG) = \sum J^{ij}(\boldsymbol{x}) \frac{\partial F}{\partial x^i} \wedge \frac{\partial G}{\partial x^j},$$

where $dF$ is a covector with components $\dfrac{\partial F}{\partial x^i}$.



On a manifold (of arbitrary dimensionality) the vector field $\boldsymbol{X}_H = \{\boldsymbol{x}, H\}$ determines the Hamiltonian system, which in the component notation may be written as

$$\dot{x}^i = X_H^i = \{x^i, H\} = \sum_j J^{ij}(x) \frac{\partial H}{\partial x^j}. \tag{1.9}$$

The function $H = H(\boldsymbol{x})$ is also called *Hamiltonian* (*Hamilton function*).

The commutator of vector fields and Poisson brackets are related as

$$[\boldsymbol{X}_H,\, \boldsymbol{X}_F] = -\boldsymbol{X}_{\{H,F\}}.$$

It's also easy to check that any Hamiltonian field gives rise to the transformation (phase flow), preserving Poisson brackets.

The function $F(\boldsymbol{x})$ is referred to as *the first integral* of a system if its derivative along the system equals zero: $\dot{F} = \boldsymbol{X}_H(F) = 0$. This condition is equivalent to $\{F, H\} = 0$.

The system of equations

$$F_1(\boldsymbol{x}) = 0,\, \ldots,\, F_k(\boldsymbol{x}) = 0 \tag{1.10}$$

sets *a system of invariant relations* (these relations define an *invariant manifold*) if $\{F_i,\, H\} = 0$ on the manifold defined by (1.10).

**Nondegenerate bracket. Symplectic structure.** If a Poisson bracket is nondegenerate, then it can be uniquely matched with a closed nondegenerate 2-form. Indeed, for any smooth function $F$ the operation $\boldsymbol{X}_F = \{F, \cdot\}$ is a differentiation which gives a certain tangent vector on $M$. Using $1^\circ$– $4^\circ$, in this case we can show that every tangent vector can be represented in such a form.

Let's define the 2nd form of $\omega^2$, using the formula

$$\omega^2(\boldsymbol{X}_G,\, \boldsymbol{X}_F) = \{F, G\}.$$

From axioms $1^\circ$- $4^\circ$ it follows that it is bilinear, skew-symmetric, nondegenerate and closed. This 2nd-form is referred to as *a symplectic structure*, and the manifold $M$ as *a symplectic manifold*.

In the coordinate representation the form of $\omega^2$ is written as $\sum\limits_{i,j} \omega_{ij} dx_i \wedge dx_j$, where $\|\omega_{ij}\| = \|J^{ij}\|^{-1}$, and in canonical case (1.6) $\omega^2 = \sum\limits_i dp_i \wedge dq_i$. Any symplectic structure can be locally reduced to such a form in accordance with the *Darboux theorem* [135]. In the section to follow we'll give a more general statement of this theorem.

**Symplectic foliation. Darboux theorem generalization.** If a Poisson bracket is degenerate, then the Poisson manifold (phase space) foliates into *symplectic fibres* (*leaves*). For these leaves Poisson structure contraction is no longer degenerate. As a rule, these leaves constitute the general level of all Casimir functions. Both the Darboux theorem and the canonical form of equations of motion are valid on a leave. However, reducing to such a system is not always necessary for applications, for it usually leads to the loss of algebraicity of differential equations and limitations in using geometry and topology research procedures.

Remark. To find integrals, particular solutions and to analyze stability, rigid-body dynamics generally uses the algebraic form of equations of motion. This form is also good for their numerical integration as a result of the canonical form having some peculiarities, related to local variables degenerating at some individual points, for example, Euler angles Poisson sphere poles, see §§ 2, 3.

For the matters of quality analysis and perturbation theory generation, the canonical form of notation is normally used, because it has these methods well developed and algorithmized.

*Poisson structure rank* at the point $\boldsymbol{x} \in M$ is a structure tensor rank at this point (it's obviously even). Poisson structure rank on the whole manifold $M$ is the maximum rank it has at a certain point $\boldsymbol{x} \in M$. For symplectic manifolds Poisson structure rank at any point is constant and maximum.

Let's state the general *Darboux theorem* for arbitrary Poisson manifolds [31, 135].

**Theorem 1.** *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson manifold of $n$-dimensionality and rank of the bracket $\{\cdot\,,\cdot\}$ at the point $\boldsymbol{x} \in M$ be locally constant and equal to $2r$ (Poisson structure rank). Then there exists a local system of (canonical) coordinates $x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_{n-2r}$, where Poisson brackets are written as*

$$\{x_i, x_j\} = \{y_i, y_j\} = \{x_i, z_k\} = \{y_i, z_k\} = \{z_k, z_l\} = 0,$$

$$\{x_i, y_j\} = \delta_{ij},$$

*where* $1 \leqslant i, j \leqslant r, \quad 1 \leqslant k, l \leqslant n - 2r.$

In the mentioned coordinates a symplectic leave is given by the equation $z_i = c_i$, $(c_i = \text{const})$, and a symplectic structure on it by the form $\omega = \sum\limits_i dx_i \wedge dy_i$.

*Singular symplectic leaves* pass through the points where the Poisson bracket rank is not maximum (less than $2r$) (for more details see [31]). Systems on singular symplectic leaves are also often found in mechanics [31, 141].



## 2. The Lie – Poisson Bracket

One of the most important examples of the Poisson structures concerns *Lie algebras*. Let $c_{ij}^k$ be structural constants of algebra $\mathfrak{g}$ in the basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. The *Lie – Poisson bracket* of the couple of functions $F, H$, given on some (or other) linear space $V$ with coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$ and basis $\omega^1, \ldots, \omega^n$, is defined by the formula

$$\{F, H\} = \sum_{i,j=1}^{n} J_{ij}(\boldsymbol{x}) \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial x_j}, \tag{1.11}$$

where $J_{ij}(\boldsymbol{x}) = \sum\limits_k c_{ij}^k x_k$ is a structural tensor, linear with respect to $x_k$. All the necessary identities 1°–4° (see s. 1) for a structural tensor can be obtained from the properties of structural constants of Lie algebra:

1. $c_{ij}^k = -c_{ji}^k$,
2. $\sum\limits_m (c_{im}^l c_{jk}^m + c_{km}^l c_{ij}^m + c_{jm}^l c_{ki}^m) = 0.$

As it's commonly known from the theory of Lie algebras, Lie – Poisson structure symplectic leaves represent orbits of the coadjoint presentation of the corresponding Lie group (see [6, 7, 135]). Formal statement and corresponding proof are given, for example, in [6]. the Hamilton equations for the Lie – Poisson structure in componentwise notation are written as

$$\dot{x}_i = \{x_i, H\} = \sum_{k,j} c_{ij}^k x_k \frac{\partial H}{\partial x^j}. \tag{1.12}$$

Remark. Equations (1.12) may be also written in a more invariant coordinate-free form

$$\dot{\boldsymbol{x}} = \mathrm{ad}^*_{dH}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathfrak{g}^*, \tag{1.13}$$

where $\mathrm{ad}^*_{\boldsymbol{\xi}}$, $(\boldsymbol{\xi} \in \mathfrak{g})$ is an operator of the coadjoint presentation of a Lie algebra $\mathfrak{g}$: $\mathrm{ad}^*_{\boldsymbol{\xi}} \colon \mathfrak{g}^* \to \mathfrak{g}^*$.

In rigid body dynamics the Lie – Poisson bracket is found very frequently, because the system configurational space is normally represented by a certain combination of natural Lie groups ($SO(3)$, $E(3), \ldots$). However, under reduction with respect to cyclic variables, nonlinear Poisson brackets may appear (see §§ 1, 2 ch. 4).

Let's now address to the derivation of rigid body motion equations from the main dynamic principles.

## § 2. Poincaré and Poincaré – Chetayev Equations

### 1. Poincaré Equations

The most natural forms of rigid body motion equations, convenient for investigations, may be obtained from general equations of dynamics in *quasi-coordinates*. The Lagrangian form of these equations was determined by *H. Poincaré* [255], and the Hamiltonian form by *N. G. Chetayev* [181]. Their possible generalizations for a nonholonomic case were examined in [91, 154]. In rigid body dynamics Poincaré – Chetayev equations result in Hamiltonian equations with a linear structural tensor, i. e., in the Lie – Poisson structure just examined (see § 1). We'll adduce here our own derivation of Poincaré and Poincaré – Chetayev equations, because the available literature lacks their discussion.

Let's consider equations of motion of a Lagrangian dynamic system determined by generalized redundant coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)$ (which are dependent, i. e., $m < n$ *holonomic* constraints of the type $f_j(\boldsymbol{q}) = 0,\ j = 1, \ldots, m$ are imposed in these coordinates) and by *quasi-velocity* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ expressed in terms of generalized velocities $\dot{q}_i$ according to the formulae

$$\dot{q}_i = \sum_{s=1}^{k} v_i^s(\boldsymbol{q})\omega_s, \qquad i = 1, \ldots, n. \tag{2.1}$$

Here it is supposed that every holonomic constraint is taken into account, i. e.,

$$(\nabla f_j, \dot{\boldsymbol{q}}) = \sum_{i,\,s} v_i^s(\boldsymbol{q})\omega_s \frac{\partial f_j}{\partial q_i} \equiv 0, \quad j = 1, \ldots m.$$

In case $k > n - m$, this condition leads to the fact that quasi-velocities are connected via relations, linear with respect to $\omega_i$.

The quantities $\omega_s$ are referred to as *Poincaré parameters* and represent the system velocity components in a nonholonomic basis of vector fields

$$\boldsymbol{v}^s = \sum_i v_i^s(\boldsymbol{q}) \frac{\partial}{\partial q_i}. \tag{2.2}$$

Suppose that vector fields form a closed system

$$[\boldsymbol{v}^i, \boldsymbol{v}^j] = c_{ij}^s(\boldsymbol{q})\boldsymbol{v}^s, \qquad i, j, s = 1, \ldots, k. \tag{2.3}$$

In case $k \leqslant n$, this condition is a consequence of the integrability of constraints [135]. If every $c_{ij}^s$ is constant, then fields $\boldsymbol{v}^s$ determine a certain finite Lie algebra. The equations of motion in variables $(q_1, \ldots, q_n,\ \omega_1, \ldots, \omega_k)$ in the Lagrangian form are written as:



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega_i}\right) = \sum_{r,s} c_{ri}^s \omega_r \frac{\partial L}{\partial \omega_s} + \boldsymbol{v}^i(L), \qquad i = 1, \ldots, k, \tag{2.4}$$

and are called *Poincaré equations*. Together with (2.1), they constitute the whole system of equations of motion. In formula (2.4) the differentiation along the vector field $\boldsymbol{v}^i$ is found by means of formula (2.2).

If a Lagrange function is a homogeneous quadratic form of its angular velocities (e. g. kinetic energy), then $\boldsymbol{v}_i(L) = 0$, and system (2.4), determining $\boldsymbol{\omega}$, is separated and integrated by itself. In this case equations (2.4) are referred to as *Euler – Poincaré equations*.

Poincaré obtained his equations, using the Hamilton variation principle [255]. We'll show the derivation of equations (2.4) directly from Euler – Lagrange equations for the case, when the number of components of the quasi-velocity $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ coincides with the dimensionality of configurational $M^k$-space, being defined by relations $f_j(\boldsymbol{q}) = 0$, $j = 1, \ldots, m$, e. g. $k = n - m$.

Introduce local coordinates $x_i$ on $M^k$, for which Euler – Lagrange equations may be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\frac{\partial L}{\partial x_i}\right) = 0, \quad i = 1, \ldots, k. \tag{2.5}$$

According to (2.1), (2.2), the following relations are valid

$$\begin{gathered} \omega_s = \sum_{i=1}^{k} a_s^i \dot{x}_i, \quad \dot{x}_i = \sum_{s=1}^{k} b_i^s \omega_s, \\ \boldsymbol{v}^s = \sum_{i=1}^{k} b_i^s \frac{\partial}{\partial x_i}, \quad i,\, s = 1, \ldots, k, \end{gathered} \tag{2.6}$$

where $\mathbf{A} = \|a_s^i\|$, $\mathbf{B} = \|b_i^s\|$ are reciprocal matrices ($\mathbf{AB} = \mathbf{E}$).

Now, represent the Lagrange function in terms of quasi-velocities as

$$\tilde{L}(\boldsymbol{x}, \boldsymbol{\omega}) = L(\boldsymbol{x}, \dot{\boldsymbol{x}}). \tag{2.7}$$

By means of (2.6) we obtain

$$\begin{gathered} \frac{\partial L}{\partial x_i} = \frac{\partial \tilde{L}}{\partial x_i} + \sum_{k,\,s} \dot{x}_k \frac{\partial \tilde{L}}{\partial \omega_s} \frac{\partial b_s^k}{\partial x_i}, \\ \frac{\partial L}{\partial \dot{x}_i} = \sum_s \frac{\partial \tilde{L}}{\partial \omega_s} b_s^i, \quad i = 1, \ldots, k. \end{gathered} \tag{2.8}$$

Substitute (2.8) into equations (2.5) and multiply them by the matrix $\mathbf{A}$, then in the obtained system make substitution (2.6) and use the following representation for structural coefficients (2.3):

$$c_{sp}^r(x) = \sum_{k,\,i} a_r^k \left( b_i^s \frac{\partial b_k^p}{\partial x_i} - b_i^p \frac{\partial b_k^s}{\partial x_i} \right).$$

Collecting similar terms will produce equations (2.4). ■

In case when the number of quasi-velocities exceeds configurational space dimensionality, the arguments become somewhat more complicated because the matrices $\mathbf{A}$, $\mathbf{B}$ are not quadratic and do not have reciprocal ones.

## 2. Poincaré – Chetayev Equations

*N. G. Chetayev* modified Poincaré equations (2.4), (2.1), having made use of *Legendre transformation*:

$$M_i = \frac{\partial L}{\partial \omega_i}, \qquad \sum_i \omega_i M_i - L\,|_{\boldsymbol{\omega} \to \boldsymbol{M}} = H(\boldsymbol{M}, \boldsymbol{q}). \tag{2.9}$$

The variables $M_i$ mean "*quasi-momenta*". In this case $\omega_i = \partial H / \partial M_i$, and equations (2.4) may be written as:

$$\dot{M}_i = \sum_{rs} c_{ri}^s \frac{\partial H}{\partial M_r} M_s - \boldsymbol{v}^i(H), \qquad i = 1, \ldots, k. \tag{2.10}$$

To obtain a closed system, equations (2.10) should be supplied with (2.1) in the form

$$\dot{q}_i = \sum_s v_i^s(\boldsymbol{q}) \frac{\partial H}{\partial M_s}, \qquad i = 1, \ldots, n. \tag{2.11}$$

System of equations (2.10), (2.11) is a Hamiltonian system with a Poisson degenerate bracket, being determined for arbitrary functions $f(\boldsymbol{M}, \boldsymbol{q})$, $g(\boldsymbol{M}, \boldsymbol{q})$ by the formula [181]

$$\{f, g\} = \sum_i \left( \frac{\partial g}{\partial M_i} v^i(f) - \frac{\partial f}{\partial M_i} v^i(g) \right) + \sum_{sij} c_{ij}^s \frac{\partial f}{\partial M_j} \frac{\partial g}{\partial M_i} M_s. \tag{2.12}$$

It is not a problem to check that this bracket satisfies all the necessary conditions 1°–4° (§ 1, s. 1). From relation (2.12) we can easily obtain the structural matrix $J^{ij}$:



$$\{M_i, M_j\} = \sum_s c_{ij}^s(\boldsymbol{q}) M_s, \qquad \{q_i, q_j\} = 0, \qquad \{q_i, M_j\} = v_i^j(\boldsymbol{q}). \tag{2.13}$$

**Historical comment.** For equations of dynamics in the form (2.10), (2.11), N. G. Chetayev [181] was also developing the integration theory, similar to the Hamilton – Jacobi method. However, if in the canonical case successful variables separation is connected with remarkable frames of reference on configurational space (like elliptic or sphero-conical coordinates), for the algebraic notation form (2.10), (2.11) only trivial symmetries (existing, for example, in the Lagrange case (see ch. 2)) may be examined this way.

This very fact caused ceasing the following development of his considerations about Routh theorem generalizations, concerning cyclic integral presence and order reduction. For Poincaré – Chetayev equations, in the presence of the first integrals (like cyclic ones), ch. 4, §§ 1, 2 offers a new reduction procedure which enables receiving the equations of the reduced system in the simplest algebraic form and in some cases leads to nonlinear Poisson brackets.

## 3. Equations on Lie Groups

In rigid body dynamics a configurational space is, as a rule, a certain natural Lie group. For example, when a rigid body rotates around a fixed point, it is a group $SO(3)$; under unrestricted motion of a rigid body it is $E(3) = SO(3) \otimes_s \mathbb{R}^3$, which is a semidirect product of a rotation algebra $SO(3)$ and a commutative translation algebra $\mathbb{R}^3$.

As a basis of vector fields $\boldsymbol{v}^s$ (2.2) it's convenient to take left-invariant (right-invariant) vector fields from its Lie algebra. Then, tensor $c_{ij}^k$ doesn't depend on coordinates and is defined by Lie algebra structural constants. Bracket (2.12) determines so called *canonical structure on a cotangent foliation* with a base: a Lie group [31].

If the Hamiltonian $H$ doesn't depend on $q_i$, i. e. $(\boldsymbol{v}_i(H) = 0)$, then the equations for quasi-momenta $M_1, \ldots, M_k$ become closed. That's the way to obtain Euler equations of rigid body inertial motion, when constants $c_{ij}^s$ are defined by the algebra $so(3)$. For an arbitrary algebra with structural constants $c_{ij}^s$ the equations of such kind with a quadratic Hamiltonian are also (as in sec. 1) called Euler – Poincaré equations.

If the Hamiltonian $H$ depends on coordinates, but one can choose redundant coordinates in such a way that all the components of left-invariant fields $v_r^s(\boldsymbol{q})$

are linear with respect to $\boldsymbol{q}$, then bracket (2.13) becomes an ordinary Lie–Poisson bracket, and all the geometric relations for redundant variables will be its Casimir functions or invariant relations. It can be achieved if Lie group matrix realization be used, and its matrix components be chosen as redundant coordinates. The Lie–Poisson structure, thus received, corresponds to the semidirect sum $\mathfrak{g} \oplus_s \mathbb{R}^{n^2}$, where $\mathbb{R}^{n^2}$ is a space of $n \times n$-matrices, $\mathfrak{g}$ is a Lie algebra of the given group. The above mentioned structure is called *a natural canonical structure of a cotangent foliation* to a Lie group. This technique can be used, for example, to obtain equations of motion of rigid body in terms of direction cosines and momenta (see § 4). Lie group matrix realization ia also applied in dynamics of multidimensional rigid body [24, 31].

Hamilton equations on a Lie group in a natural canonic structure for problems of rigid body dynamics (which has all the groups unimodular) always have a standard invariant measure. It's the analogue of the Liouville theorem about solenoidality of canonical Hamiltonian flow.

The detailed derivation of equations of motion of a rigid body in an arbitrary potential force field is studied in § 4. More complicated equations whose derivation requires using the main principles of hydrodynamics describing a rigid body motion in fluid and also a body with cavities filled with fluid, are considered in ch. 5, § 2.

#### 4. Comments

So, Poincaré and Poincaré–Chetayev equations represent only a convenient means for recording Lagrangian and Hamiltonian forms of equations of motion of a system in terms of an arbitrary system of variables, including redundant ones. At the same time, the possibility of such a representation is connected with the system having *a tensor invariant*: a Poisson structure whose coordinate notation depends on the choice of variables, the Poisson structure for redundant variables being admittedly degenerate. It should be mentioned that the Lagrangian system with the Lagrangian function, non-degenerate with respect to velocities, is admitted to have this tensor invariant.

It would be interesting to note that the majority of mechanics can comprehend the connection of the Lagrangian and Hamiltonian forms in the canonical notation only. Thus, in the book [21] the Hamiltonian form of rigid body dynamics equations is considered to be deliberately established from certain, not quite natural considerations, in particular, with the reference to the paper [133], where the author, without being fully aware of the general formalism of dynamic equations, in fact even rediscovers Euler angles and conjugate momenta. Further, in [21] the author is proving several strange theorems that the Lagrangian form can be obtained from the Hamiltonian one; and here we certainly have a kind of confusion, as far as the Poisson commutation of moment components with the momentum and direction cosines is identical, so that the same Kirchhoff's equations may be envisioned, on the one hand, as a part of momentum equations on the group $E(3)$ — Euler–Poincaré equations for $\boldsymbol{M}$, $\boldsymbol{p}$, which, in case of a zero potential, is separated from positional equations (for direction cosines), and, on the other hand — as Hamiltonian equations on $SO(3)$, in which connection the components of momentum force $\boldsymbol{p}$ are bound to be interpreted as direction cosines. Here, by the way, lies Steklov analogy [160] (see also § 4 and ch. 3, § 9).

The complicated coordinate form of the notation of Newton's equations of satellite dynamics is used in [11], where even an energy integral presence is not evident at all.

Even in the outstanding book [97] the author is proving the statement concerning "non-Hamiltonianity" of Euler–Poincaré equations (considered separately from positional variables), which is supposed to be connected with the absence of the invariant measure, having a certain analytical structure which, for example, solvable (non-unimodular) Lie groups lack.

Here we should also mention the book [249] and the works of the same style in general (J. Marsden, A. Weinstein and others), where, as a result of the excessive formalization of both forms of dynamic equations and reduction procedure, even rather simple problems require great intellectual efforts. As for a bit more complicated mechanical problems, they just remain beyond the framework of such an approach.

## § 3. Various Systems of Variables in Rigid Body Dynamics

Various systems of variables are used for describing rigid body motion. Each and every system has both advantages and disadvantages for each particular problem. So, the first integral obtaining and investigating some matters of stability and topological analysis require variables, in which the equations are polynomial (or even homogeneous). For numerical integration, apart from a simple system of differential equations, it's desirable to have the minimum order of the system. For high-quality examination and application of perturbation theory and nonlinear normalization methods there is a need for systems of canonical variables, which are the best for reflecting the specific character of the unperturbed problem. Here we adduce the main sets of variables used in rigid body dynamics. In practice, especially in applications to gyroscopic devices, various combinations and modifications of these systems, possessing more special properties, are used.

## 1. The Euler Angles

Consider a rigid body rotating in a potential force field around a fixed point $O$. Various systems of variables are used for its motion description. The configurational space constituting the set of all rigid body positions is a Lie group $SO(3)$. Then the *Euler angles* $\theta, \varphi, \psi$ [9] may be taken as coordinates defining a rigid body position.

For their introduction let's take the point $O$ as a place for the apices of two orthogonal trihedrons: fixed $OXYZ$ and moving $Oxyz$, rigidly bound with the rotating rigid body (fig. 1).

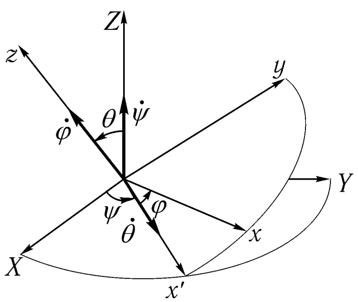



Figure 1. The Euler angles

The first rotation through the angle $\psi$ (*precession angle*) about the axis $OZ$ transfers the moving trihedron $Oxyz$ into the position $Ox'y'z'$. The second rotation through the angle $\theta$ (*nutation angle*) is made around the axis $Ox'$, referred to as *a node line*. The last rotation through the angle $\varphi$ (*a proper rotation angle*) around the axis $Oz$ aligns both trihedrons. Thus, three rotations defined by the Euler angles $\theta, \varphi, \psi$ allow to make the complete specification of the position of moving trihedron with respect to the fixed one. The projections of angular velocity $\boldsymbol{\omega}$ $\omega_1$, $\omega_2$, $\omega_3$ on the moving trihedron $Oxyz$ axes are expressed in terms of the Euler angles as follows:

$$\begin{aligned}\omega_1 &= \dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi,\\ \omega_2 &= \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi,\\ \omega_3 &= \dot\psi\cos\theta + \dot\varphi.\end{aligned} \tag{3.1}$$

These relations are called *Euler's kinematic formulae*. Using (3.1), it's easy to write the Lagrange function of the system $L = L(\varphi,\, \psi,\, \theta,\, \dot\varphi,\, \dot\psi,\, \dot\theta)$ (see § 6). It helps to define canonical momenta (by means of the Legendre transformation):

$$p_\varphi = \frac{\partial L}{\partial\dot\varphi}\,, \quad p_\psi = \frac{\partial L}{\partial\dot\psi}\,, \quad p_\theta = \frac{\partial L}{\partial\dot\theta}\,. \tag{3.2}$$



## 2. The Euler Variables. Momentum Components and Direction Cosines

Consider another system of variables $(\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where $\boldsymbol{M} = (M_1, M_2, M_3)$ are angular momentum components along the coordinate axes of the system $Oxyz$ attached to the body, and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are projections of the fixed space unit vectors onto the same axes. *The matrix of direction cosines* (a rotation matrix) defining the rigid body position in the fixed space

$$\mathbf{Q} = \begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}, \tag{3.3}$$

is an orthogonal one and belongs to the group $SO(3)$.

It's evident that

$$\begin{aligned}(\boldsymbol{\alpha},\,\boldsymbol{\alpha}) &= (\boldsymbol{\beta},\,\boldsymbol{\beta}) = (\boldsymbol{\gamma},\,\boldsymbol{\gamma}) = 1,\\ (\boldsymbol{\alpha},\,\boldsymbol{\beta}) &= (\boldsymbol{\alpha},\,\boldsymbol{\gamma}) = (\boldsymbol{\beta},\,\boldsymbol{\gamma}) = 0,\end{aligned}$$

where, and hereinafter, the round brackets mean an ordinary scalar product.

Taking these relations into account, we'll receive that the angular velocity projections onto the moving trihedron $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ may be represented as a skew-symmetrical matrix $\widetilde{\boldsymbol{\omega}} = \mathbf{Q}\dot{\mathbf{Q}}^T$, $\widetilde{\boldsymbol{\omega}} = \|\omega_{jk}\|$ with components $\omega_{ij} = -\varepsilon_{ijk}\omega_k$.

Similarly, projections of the angular velocity $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ onto the axes $OXYZ$ may be obtained from the matrix $\mathbf{Q}^T\dot{\mathbf{Q}}$.

The directions of vectors of angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ in the moving and fixed spaces specify conical surfaces, referred to by Poinsot as *loose and fixed axoids*. In this case the rigid body motion itself is represented as sliding-free rolling of a loose axoid over the fixed one, both making contact along the instantaneous axis of rotation at each moment of time. If free motion of a body (without a fixed point) be considered, then in the corresponding interpretation the motion will look like rolling of one axoid over the other together with the sliding along a certain axis defining instantaneous helical (spatially-rotational) motion. If instantaneous values of angular velocities be marked along axoid generatrices, then we correspondingly receive *loose and fixed hodographs*. In the general case they represent complicated spatial curves.

By means of the Lagrange function $L = L(\boldsymbol{\omega},\, \boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma})$ the *angular momentum* $\boldsymbol{M}$ is expressed in terms of angular velocity by the formula

$$\boldsymbol{M} = \frac{\partial L}{\partial\boldsymbol{\omega}}. \tag{3.4}$$

It is connected with the Euler variables $\varphi$, $\psi$, $\theta$, $p_\varphi$, $p_\psi$, $p_\theta$ by the following relations, which can be obtained from Euler kinematic equations (3.1), (3.2)

$$\begin{aligned} M_1 &= \frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) + p_\theta\cos\varphi, \\ M_2 &= \frac{\cos\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) - p_\theta\sin\varphi, \\ M_3 &= p_\varphi. \end{aligned} \tag{3.5}$$

Remark 1. Our terminology is somewhat different from the rigid body dynamics definition of the moment $\boldsymbol{M} = \sum \boldsymbol{r}_i \times m_i\boldsymbol{v}_i$, though both terms agree if $L = T$ is a kinetic energy. The difference arises in the presence of gyroscopic forces, which in the Lagrangian lead to the terms, linear with respect to generalized velocities. Then definition (3.4), originating from Chetayev transformation, is more convenient.

Remark 2. The connection of direction cosines (3.3) with the Euler angles is expressed in the matrix form

$$\mathbf{Q} = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

### 3. Rodrigue – Hamilton Quaternion Parameters

It was already noticed by *C. Gauss* that the rigid body position may be uniquely determined by the set of quaternions $\lambda = \lambda_0 + \boldsymbol{i}\lambda_1 + \boldsymbol{j}\lambda_2 + \boldsymbol{k}\lambda_3$ with the unit norm $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. They form the group $Sp(1)$, which is the universal covering group of $SO(3)$ ($SO(3) \approx Sp(1)/\pm 1$) [75]. The way to introduce such kind of redundant coordinates, referred to in mechanics as *Rodrigue – Hamilton parameters*, may be looked up, for example, in the treatise by Whittaker [167]. Let's clarify the geometrical meaning of parameters $\lambda_s$ [108, 167].

Kinematics says that if a rigid body has a fixed point $O$, then from any position of this body it's possible to move into the given position, making a rotation through the angle $\chi$ with respect to the axis $OL$, attached to the body (fig. 2). Let the orientation of the axis $OL$ be given by the unit vector $\boldsymbol{e}$. The position of any point of the body is defined by the position vector $\overrightarrow{OM} = \boldsymbol{r}$. Let the vector $\boldsymbol{r}$ take position $\overrightarrow{OM}' = \boldsymbol{r}'$ after the rotation. The vector

$$\boldsymbol{p} = \overrightarrow{OM}' - \overrightarrow{OM} = \boldsymbol{r}' - \boldsymbol{r}$$



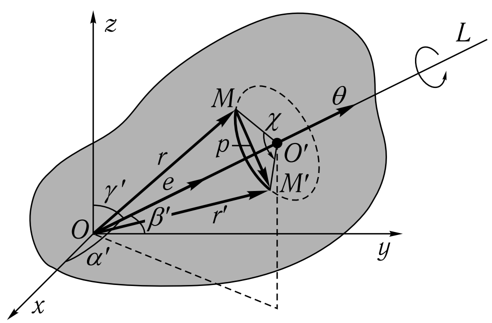



Figure 2. Rodrigue – Hamilton quaternion parameters.

may be expressed in terms of $\boldsymbol{r}, \boldsymbol{e}$ and $\chi$. The relation mentioned is determined by the Rodrigue formula

$$\boldsymbol{p} = \frac{1}{1 + \frac{1}{4}\theta^2}\boldsymbol{\theta} \times (\boldsymbol{r} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{r}), \tag{3.6}$$

where the vector

$$\boldsymbol{\theta} = 2\,\mathrm{tg}\,\frac{\chi}{2}\boldsymbol{e} \tag{3.7}$$

is called *a finite rotation vector*. This vector is directed along the axis of the unit vector $\boldsymbol{e}$ and its magnitude equals $2\,\mathrm{tg}(\chi/2)$.

Let

$$\boldsymbol{e} = \boldsymbol{i}\cos\alpha' + \boldsymbol{j}\cos\beta' + \boldsymbol{k}\cos\gamma', \tag{3.8}$$

where $\alpha', \beta', \gamma'$ are the angles formed by the vector $\boldsymbol{e}$ with the axes $x, y, z$.

These very quantities

$$\begin{aligned} \lambda_0 &= \cos\frac{\chi}{2}, & \lambda_1 &= \cos\alpha'\sin\frac{\chi}{2}, \\ \lambda_2 &= \cos\beta'\sin\frac{\chi}{2}, & \lambda_3 &= \cos\gamma'\sin\frac{\chi}{2} \end{aligned} \tag{3.9}$$

are *Rodrigue – Hamilton parameters*. A parameter $\lambda_0$ equals the cosine of the half-angle $\chi$, defining the body finite rotation. Remaining parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ are proportional to the sine of the half-angle $\chi$ multiplied by cosines of the angles between the axis $OL$ and coordinate axes.

There exists a certain relation between Rodrigue – Hamilton parameters and the Euler angles $\theta, \varphi, \psi$:

$$\begin{aligned} \lambda_0 &= \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, & \lambda_1 &= \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}, \\ \lambda_2 &= \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}, & \lambda_3 &= \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}. \end{aligned} \tag{3.10}$$

Direction cosines $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are quadratically related to quaternions. These relations specify the Cayley parametrization of the group $SO(3)$. Thus, we obtain the double-covering of $SO(3)$ by the three-dimensional sphere $S^3$. Thus, quaternions $\lambda_i$ and $-\lambda_i$ have one and the same corresponding element from $SO(3)$. A matrix of direction cosines (3.3) in quaternionic representation is written as:

$$\mathbf{Q} = \begin{pmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2+\lambda_1\lambda_3) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{pmatrix}. \tag{3.11}$$

In the index form for components of the matrix $\mathbf{Q} = \|Q_{ij}\|$ the expression

$$Q_{ij} = -2\Big(\lambda_i\lambda_j + \Big(\lambda_0^2 - \frac{1}{2}\Big)\delta_{ij} - \lambda_0\lambda_k\varepsilon_{ijk}\Big).$$

is valid.

Remark 3. The relation between projections of angular velocity $\boldsymbol{\omega}$ and Rodrigue – Hamilton parameters has the form

$$\begin{aligned} \omega_1 &= 2(\lambda_0\dot\lambda_1 + \lambda_3\dot\lambda_2 - \lambda_2\dot\lambda_3 - \lambda_1\dot\lambda_0), \\ \omega_2 &= 2(-\lambda_3\dot\lambda_1 + \lambda_0\dot\lambda_2 + \lambda_1\dot\lambda_3 - \lambda_2\dot\lambda_0), \\ \omega_3 &= 2(\lambda_2\dot\lambda_1 - \lambda_1\dot\lambda_2 + \lambda_0\dot\lambda_3 - \lambda_3\dot\lambda_0). \end{aligned}$$

Remark 4. Complex quantities $\alpha$, $\beta$, $\gamma$, $\delta$, satisfying the condition

$$\alpha\delta - \beta\gamma = 1,$$

and referred to as *Cayley – Klein parameters,* may be considered analogically to Rodrigue – Hamilton parameters. They may be regarded as components of the complex rotation matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$



with the unit determinant.

The relation between Cayley – Klein and Rodrigue – Hamilton parameters is expressed by formulae

$$\alpha = \lambda_0 + i\lambda_3, \quad \beta = -\lambda_2 + i\lambda_1, \quad \gamma = \lambda_2 + i\lambda_1, \quad \delta = \lambda_0 - i\lambda_3,$$

and their definition in terms of the Euler angles can be written as

$$\begin{aligned} \alpha &= \cos\frac{\theta}{2}e^{i\frac{\psi+\varphi}{2}}, & \beta &= i\sin\frac{\theta}{2}e^{i\frac{\psi-\varphi}{2}}, \\ \gamma &= i\sin\frac{\theta}{2}e^{-i\frac{\psi-\varphi}{2}}, & \delta &= \cos\frac{\theta}{2}e^{-i\frac{\psi+\varphi}{2}}. \end{aligned}$$

## 4. Andoyaer – Deprit Variables

Andoyaer – Deprit variables find the widest application in the perturbation theory and have a dynamic origin illustrated by fig. 3 (see also [71, 92, 31]).

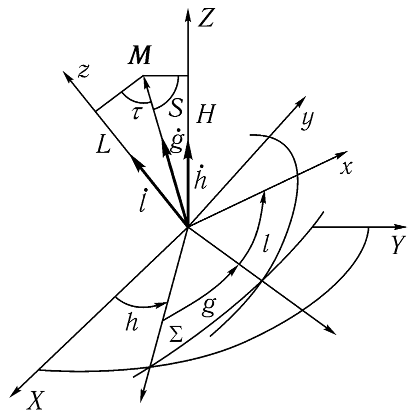



Figure 3. Andoyaer – Deprit Variables.

Here $OXYZ$ designates a fixed trihedron with the origin in the fixation point, $Oxyz$ is a moving frame of reference, rigidly bound to the body, $\Sigma$ is a plane, passing through the fixation point and perpendicular to the angular momentum vector of top $\boldsymbol{M}$ (3.5). In the agreed notations:

- $L$ is an angular momentum projection on the moving axis $Oz$;
- $G$ is an angular momentum magnitude;
- $H$ is an angular momentum projection on the fixed axis $OZ$;

$l$ is an angle formed by the axis $Ox$ and the intersection line of $\Sigma$ with the planes $Oxy$ and $OXY$;

$g$ is an angle formed by the intersection line of $\Sigma$ with planes $Oxy$ and $OXY$;

$h$ is an angle between the axis $OX$ and the intersection line of $\Sigma$ with plane $OXY$.

The expressions for angular momentum components in terms of variables $L$, $G$, $H$, $l$, $g$, $h$ have the form

$$M_1 = \sqrt{G^2-L^2}\sin l,\quad M_2 = \sqrt{G^2-L^2}\cos l,\quad M_3 = L,\quad G^2 = \boldsymbol{M}^2, \tag{3.12}$$

i. e., $L$, $l$ are cylindric coordinates on a two-dimensional sphere in the space of moments $M_1$, $M_2$, $M_3$.

For components of all direction cosines there exist the following expressions, which seem to be lacking in corpore in the literature available:

$$\begin{aligned}
\alpha_1 &= -\sin l\sin h\cos g\sin\tau\sin\zeta + \sin l\sin h\cos\tau\cos\zeta -\\
&\quad -\sin l\sin g\cos h\sin\tau - \cos l\sin h\sin g\sin\zeta + \cos l\cos g\cos h,\\
\alpha_2 &= \cos l\cos g\sin h\sin\tau\sin\zeta - \cos l\sin h\cos\tau\cos\zeta +\\
&\quad +\cos l\cos h\sin g\sin\tau - \sin l\sin g\sin\zeta\sin h + \sin l\cos h\cos g,\\
\alpha_3 &= \sin h\cos\tau\cos g\sin\zeta + \sin h\sin\tau\cos\zeta + \cos\tau\sin g\cos h,\\
\beta_1 &= -(\sin l\cos h\cos g\sin\tau\sin\zeta - \sin l\cos h\cos\zeta\cos\tau -\\
&\quad -\sin l\sin g\sin h\sin\tau + \cos l\cos h\sin g\sin\zeta + \cos l\cos g\sin h),\\
\beta_2 &= \cos l\cos h\sin\tau\cos g\sin\zeta - \cos l\cos h\cos\zeta\cos\tau -\\
&\quad -\cos l\sin g\sin h\sin\tau - \sin l\cos h\sin g\sin\zeta - \sin l\cos g\sin h,\\
\beta_3 &= -\sin h\cos\tau\sin g + \cos\tau\cos g\sin\zeta\cos h + \sin\tau\cos\zeta\cos h,\\
\gamma_1 &= (\sin\zeta\cos\tau + \sin\tau\cos\zeta\cos g)\sin l + \cos\zeta\sin g\cos l,\\
\gamma_2 &= (\sin\zeta\cos\tau + \sin\tau\cos\zeta\cos g)\cos l - \cos\zeta\sin g\sin l,\\
\gamma_3 &= \sin\zeta\sin\tau - \cos\tau\cos\zeta\cos g,
\end{aligned} \tag{3.13}$$

where $\sin\tau = \dfrac{L}{G}$, $\sin\zeta = \dfrac{H}{G}$.

Remark 5. The expression of direction cosines $\gamma_i$ in terms of Andoyaer – Deprit variables may be found in several sources [9, 92, 28]. In this case we can make an inversion and use formulae $L = M_3$, $G = \sqrt{(\boldsymbol{M}, \boldsymbol{M})}$, $l = \operatorname{arctg}\left(\dfrac{M_1}{M_2}\right)$, $g = \arcsin\left(\dfrac{M_2\gamma_1 - M_1\gamma_2}{\sqrt{M_1^2+M_2^2}}\right)$. The expressions for $\alpha_3$, $\beta_3$ may be obtained just from geometrical considerations. To receive the rest of direction cosines one needs to use commutative relations (4.16) given in the next section. The expressions for $\lambda_i$ parameters, indicated in the book [31], in terms of Andoyaer – Deprit variables are not correct. These expressions can be received from the following relations

$$\lambda_0^2 = \frac{1+\alpha_1+\beta_2+\gamma_3}{4},\quad \lambda_1^2 = \frac{1+\alpha_1-\beta_2-\gamma_3}{4},$$
$$\lambda_2^2 = \frac{1-\alpha_1+\beta_2-\gamma_3}{4},\quad \lambda_3^2 = \frac{1-\alpha_1-\beta_2+\gamma_3}{4},$$

and $\lambda_i$ themselves will be defined up to a sign.



### 5. Comments

The system of Andoyaer – Deprit variables cannot be divided into positional and purely momentum components, like the Euler angles and conjugate canonical momenta do. However, they are convenient for applying the perturbation theory technique, on account of their connection with angular momentum components. In two most popular integrable (unperturbed) problems of rigid body dynamics — the Euler and Lagrange cases — the variables $G$ and $L$ are corresponding integrals of motion. Similar systems of “osculating elements”, not necessarily canonical, were already used by Poisson, Charlie, Andoyaer and by Tisserand for constructing theories of physical libration of the Moon and rotary motion of planets in celestial mechanics. When in this century A. Deprit introduced these systems in his paper [71], his object was to clarify the phase geometry of the Euler case (see § 2 ch. 2), which helped to realize their universal meaning in rigid body dynamics: they were used for numerical investigations [28], and for the application of qualitative analysis methods in [92], where they are referred to as *special canonical variables*.

The systematic investigation of motion equations of a heavy gyroscope in Rodrigue – Hamilton (and also Cayley – Klein) parameters is developed in the remarkable book by F. Klein and A. Sommerfeld “On the Top Theory” [238] (it goes without saying that the main results here are received by F. Klein, see as well [237]). At that time the Hamiltonian structure of these equations (like equations on Lie algebra) was not known yet, but, nevertheless, these parameters turned out to be convenient both for explicit integration in elliptic functions, and for the analysis of various particular solutions. The system of redundant variables (a kind of Plücker coordinates), close to quaternions, was investigated by E. Studi in his book “Dy-name Geometry". He has also calculated rigid body kinetic energy in terms of these coordinates.

## § 4. Equations of Motion in Various Forms

### 1. Equations of Motion of a Rigid Body with a Fixed Point

We are going to show the most important forms of rigid body dynamics equations in various systems of variables. The previous section remarks also hold true in their respect. Their application is determined by the purpose of investigation and depends on specific problem statement.

**Euler – Poincaré equations on group $SO(3)$.** Consider motion of a rigid body, one of whose points remains fixed in space (in some inertial frame of reference). In this case a configurational space is a group $SO(3)$. We'll use its representation in terms of orthogonal matrices of direction cosines (3.3) (see § 3, s. 2)

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in SO(3), \tag{4.1}$$

where, as above, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are fixed space unit vector projections on the axes, attached to the body.

Projections of the rigid body angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ on the same axes may be determined from Poisson equations

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \tag{4.2}$$

These equations show that vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are constant in the absolute space. Rewriting (4.2) in the matrix form, we'll obtain

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q}, \qquad \widetilde{\boldsymbol{\omega}} = \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} = -\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}, \tag{4.3}$$

where

$$\widetilde{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

From the group perspective, angular velocity projections $\omega_i$ in the coordinate system attached to the body correspond to velocity components of the point on the group $SO(3)$ in the basis of left-invariant vector fields. Analogically, the



angular velocity projection in the space $\Omega_i$ have corresponding velocity components in the basis of right-invariant vector fields

$$\boldsymbol{\omega} = \sum_k \omega_k \boldsymbol{\xi}_k, \qquad \boldsymbol{\xi}_k = -\sum_{ij} \varepsilon_{kij} \Big(\alpha_i \frac{\partial}{\partial \alpha_j} + \beta_i \frac{\partial}{\partial \beta_j} + \gamma_i \frac{\partial}{\partial \gamma_j}\Big). \tag{4.4}$$

To determine the fields $\boldsymbol{\xi}_k$ we'll write the derivative with respect to time taking into account (4.3)

$$\frac{df}{dt} = \mathrm{Tr}\Big(\dot{\mathbf{Q}}^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{Q}}\Big) = \mathrm{Tr}\Big((\widetilde{\boldsymbol{\omega}}\mathbf{Q})^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{Q}}\Big), \quad \frac{\partial f}{\partial \mathbf{Q}} = \Big\| \frac{\partial f}{\partial Q_{ij}} \Big\|, \tag{4.5}$$

and grouping terms in $\omega_i$, we receive vector fields $\boldsymbol{\xi}_i$ (4.4).

Commutational relations for vector fields $\boldsymbol{\xi}_k$ are written as

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \varepsilon_{ijk} \boldsymbol{\xi}_k, \tag{4.6}$$

where $\varepsilon_{i,j,k}$ are Levi-Civita symbols.

Substituting (4.4) and (4.6) into Euler – Poincaré equations (2.4), we'll receive equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial L}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \tag{4.7}$$

which, combined with (4.2), constitute the complete system of equations of motion of a rigid body with a fixed point. System (4.2), (4.7) was obtained by J. Lagrange in the second volume of his celebrated "Analytical Mechanics" [110].

Remark 1. We'll show the matrix form of motion equations (4.2), (4.7), as well, which allows a simple generalization for the multidimensional case

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}}\right) = \Big[\widetilde{\boldsymbol{\omega}}, \frac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}}\Big] + \frac{\partial L}{\partial \mathbf{Q}}\mathbf{Q}^{\mathrm{T}} - \left(\frac{\partial L}{\partial \mathbf{Q}}\right)^{\mathrm{T}}\mathbf{Q}, \quad \dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q},$$

where $\dfrac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}} = \Big\| \dfrac{\partial L}{\partial \widetilde{\omega}_{ij}} \Big\|$, $\dfrac{\partial L}{\partial \mathbf{Q}} = \Big\| \dfrac{\partial L}{\partial Q_{ij}} \Big\|$, and $[\cdot\,,\cdot]$ is an ordinary matrix commutator.

**Motion equations in terms of angular velocities and quaternions.** In addition to matrix realization (4.1) in § 3 we've also shown quaternion parametrization of the group $SO(3)$, for which vector fields (4.4) are linear functions of coordinates, as well. Indeed, we can show that on the unit sphere

$\lambda_0^2+\boldsymbol{\lambda}^2=1$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ angular velocity components (4.3) and vector fields (4.4) have the form [97, 108]

$$
\begin{aligned}
\omega_1&=2(\lambda_0\dot{\lambda_1}-\lambda_1\dot{\lambda_0}+\lambda_3\dot{\lambda_2}-\lambda_2\dot{\lambda_3}),\\
\omega_2&=2(\lambda_0\dot{\lambda_2}-\lambda_2\dot{\lambda_0}+\lambda_1\dot{\lambda_3}-\lambda_3\dot{\lambda_1}),\\
\omega_3&=2(\lambda_0\dot{\lambda_3}-\lambda_3\dot{\lambda_0}+\lambda_2\dot{\lambda_1}-\lambda_1\dot{\lambda_2}),\\
\boldsymbol{\xi}_1&=\frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_1}-\lambda_1\frac{\partial}{\partial\lambda_0}+\lambda_3\frac{\partial}{\partial\lambda_2}-\lambda_2\frac{\partial}{\partial\lambda_3}\right),\\
\boldsymbol{\xi}_2&=\frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_2}-\lambda_2\frac{\partial}{\partial\lambda_0}+\lambda_1\frac{\partial}{\partial\lambda_3}-\lambda_3\frac{\partial}{\partial\lambda_1}\right),\\
\boldsymbol{\xi}_3&=\frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_3}-\lambda_3\frac{\partial}{\partial\lambda_0}+\lambda_2\frac{\partial}{\partial\lambda_1}-\lambda_1\frac{\partial}{\partial\lambda_2}\right).
\end{aligned}
\tag{4.8}
$$

Commutational relations for fields $\boldsymbol{\xi}_k$ also have the form (4.6).

If (4.8) is taken into account, Poincaré equations (2.4) are written as

$$
\begin{gathered}
\frac{d}{dt}\left(\frac{\partial L}{\partial\boldsymbol{\omega}}\right)=\frac{\partial L}{\partial\boldsymbol{\omega}}\times\boldsymbol{\omega}+\frac{1}{2}\lambda_0\frac{\partial L}{\partial\boldsymbol{\lambda}}-\frac{1}{2}\boldsymbol{\lambda}\frac{\partial L}{\partial\lambda_0}+\frac{1}{2}\frac{\partial L}{\partial\boldsymbol{\lambda}}\times\boldsymbol{\lambda},\\
\dot{\lambda_0}=-\frac{1}{2}(\boldsymbol{\omega},\boldsymbol{\lambda}),\qquad \dot{\boldsymbol{\lambda}}=\frac{1}{2}\lambda_0\boldsymbol{\omega}+\frac{1}{2}\boldsymbol{\lambda}\times\boldsymbol{\omega}.
\end{gathered}
\tag{4.9}
$$

**Kinetic energy of a rigid body with a fixed point** in a vector and matrix form may be represented as

$$
T=\frac{1}{2}(\boldsymbol{\omega},\mathbf{I}\boldsymbol{\omega})=-\frac{1}{2}\operatorname{Tr}(\widetilde{\boldsymbol{\omega}}\mathbf{J}\widetilde{\boldsymbol{\omega}}).
\tag{4.10}
$$

Here $\mathbf{I}=\|I_{ij}\|$ is *a tensor of inertia* of a rigid body relatively to a fixed point of the body. The tensor components are determined by means of the expression

$$
I_{ij}=\int\limits_{\tau}(\boldsymbol{y}^2\delta_{ij}-y_iy_j)\rho(\boldsymbol{y})\,d^3\boldsymbol{y},
\tag{4.11}
$$

where integration is carried out over all the points $\boldsymbol{y}$ of the body $\tau$, and $\rho(\boldsymbol{y})$ is its density in the point $\boldsymbol{y}$.

The tensor $\mathbf{J}=\|J_{ij}\|$ is also called a tensor of inertia, but now it's determined by the formula

$$
J_{ij}=\int\limits_{\tau}y_iy_j\rho(\boldsymbol{y})\,d^3\boldsymbol{y};
\tag{4.12}
$$



this tensor is usually used for multidimensional generalizations.

The connection between $\mathbf{I}$ and $\mathbf{J}$ is represented by relations

$$
\mathbf{J}=\frac{1}{2}(\operatorname{Tr}\mathbf{I})\mathbf{E}-\mathbf{I},\qquad \mathbf{I}=(\operatorname{Tr}\mathbf{J})\mathbf{E}-\mathbf{J}.
\tag{4.13}
$$

In the system of axes, attached to the body, tensors $\mathbf{I}$ and $\mathbf{J}$ represent constant symmetrical matrices (in fixed space $\mathbf{I}$, $\mathbf{J}$ depend on coordinates). As a result of commutativity ($\mathbf{IJ}=\mathbf{JI}$), these matrices may be simultaneously reduced to the diagonal form. The corresponding frame of reference in the body is referred to as *principal*, and its axes — as *principal axes of* (*inertia*).

## 2. Hamiltonian Form of Equations of Motion for Various Systems of Variables

**Motion equations in the algebraic form.** The Hamiltonian form of equations (4.2), (4.7) may be represented by means of the Legendre transformation

$$
\boldsymbol{M}=\frac{\partial L}{\partial\boldsymbol{\omega}},\quad H=(\boldsymbol{M},\boldsymbol{\omega})-L|_{\boldsymbol{\omega}\to\boldsymbol{M}}.
\tag{4.14}
$$

For a natural system with kinetic energy (4.10) and potential energy $U(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ we obtain

$$
\boldsymbol{M}=\mathbf{I}\boldsymbol{\omega},\quad H=\frac{1}{2}(\boldsymbol{M},\mathbf{A}\boldsymbol{M})+U(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),
\tag{4.15}
$$

where $\mathbf{A}=\mathbf{I}^{-1}$, $\boldsymbol{M}$ are the projections of angular momentum components on the moving axes; $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are direction cosine components.

Proceeding from general formulae (2.13), and also from (4.6), we'll obtain that the Poisson bracket is defined by algebra $so(3)\oplus_s(\mathbb{R}^3\oplus\mathbb{R}^3\oplus\mathbb{R}^3)$, which is a semi-direct sum of the rotation algebra and three translation algebras

$$
\begin{gathered}
\{M_i,M_j\}=-\varepsilon_{ijk}M_k,\qquad \{M_i,\alpha_j\}=-\varepsilon_{ijk}\alpha_k,\\
\{M_i\beta_j\}=-\varepsilon_{ijk}\beta_k,\qquad \{M_i,\gamma_j\}=-\varepsilon_{ijk}\gamma_k,\\
\{\alpha_i,\alpha_j\}=\{\beta_i,\beta_j\}=\{\gamma_i,\gamma_j\}=\{\alpha_i,\beta_j\}=\{\alpha_i,\gamma_j\}=\{\beta_i,\gamma_j\}=0.
\end{gathered}
\tag{4.16}
$$

Hamiltonian equations of motion in the explicit form are written as

$$
\begin{gathered}
\dot{\boldsymbol{M}}=\boldsymbol{M}\times\frac{\partial H}{\partial\boldsymbol{M}}+\boldsymbol{\alpha}\times\frac{\partial H}{\partial\boldsymbol{\alpha}}+\boldsymbol{\beta}\times\frac{\partial H}{\partial\boldsymbol{\beta}}+\boldsymbol{\gamma}\times\frac{\partial H}{\partial\boldsymbol{\gamma}},\\
\dot{\boldsymbol{\alpha}}=\boldsymbol{\alpha}\times\frac{\partial H}{\partial\boldsymbol{M}},\qquad \dot{\boldsymbol{\beta}}=\boldsymbol{\beta}\times\frac{\partial H}{\partial\boldsymbol{M}},\qquad \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma}\times\frac{\partial H}{\partial\boldsymbol{M}},\\
H=\frac{1}{2}(\boldsymbol{M},\mathbf{A}\boldsymbol{M})+U(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}).
\end{gathered}
\tag{4.17}
$$

The form (4.17) may be also used for representing equations of the rigid body motion in a generalized potential field, for example, in a magnetic one; in this case Hamiltonian $H$ involves terms, linear with respect to $\boldsymbol{M}$ (see further).

Poisson bracket (4.16) is degenerate and possesses six Casimir's functions

$$\begin{aligned} &f_1 = (\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad f_2 = (\boldsymbol{\beta}, \boldsymbol{\beta}), \quad f_3 = (\boldsymbol{\gamma}, \boldsymbol{\gamma}), \\ &f_4 = (\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad f_5 = (\boldsymbol{\alpha}, \boldsymbol{\gamma}), \quad f_6 = (\boldsymbol{\beta}, \boldsymbol{\gamma}). \end{aligned} \tag{4.18}$$

A nonspecial symplectic leave, homeomorphic to a cotangent foliation of a three-dimensional sphere $T^*S^3$, has six dimensions. In consequence of fulfillment of orthonormal relations, the symplectic leave is defined by the conditions: $f_1 = f_2 = f_3 = 1$, $f_4 = f_5 = f_6 = 0$. It happens because the symplectic leave has six dimensions, and system (4.17) three degrees of freedom.

In a fixed frame of reference the rigid body position and velocity can be characterized by projections of unit vectors, bound to the body, on the fixed axes, expressed in terms of rows of matrix $\mathbf{Q}$, and angular momentum vector projections on the same axes

$$\begin{gathered} \boldsymbol{e}_1 = (\alpha_1, \beta_1, \gamma_1), \quad \boldsymbol{e}_2 = (\alpha_2, \beta_2, \gamma_2), \quad \boldsymbol{e}_3 = (\alpha_3, \beta_3, \gamma_3), \\ N_1 = (\boldsymbol{M}, \boldsymbol{\alpha}),\ N_2 = (\boldsymbol{M}, \boldsymbol{\beta}),\ N_2 = (\boldsymbol{M}, \boldsymbol{\gamma}). \end{gathered} \tag{4.19}$$

It's easy to show that variables $\boldsymbol{N}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ also form the Lie – Poisson structure, which differs from (4.16) only by sign

$$\begin{gathered} \{N_i, N_j\} = \varepsilon_{ijk} N_k, \\ \{N_i, e_{1j}\} = \varepsilon_{ijk} e_{1k}, \quad \{N_i, e_{2j}\} = \varepsilon_{ijk} e_{2k}, \quad \{N_i, e_{3k}\} = \varepsilon_{ijk} e_{3k}, \\ \{e_{ki}, e_{lj}\} = 0. \end{gathered} \tag{4.20}$$

Thus, for example, a spherical pendulum in a potential field may be represented just by means of variables $\boldsymbol{N}, \boldsymbol{e}_3$, where $\boldsymbol{e}_3$ is a unit vector, directed from the center of attaching to the bob, $\boldsymbol{N} = ml^2\boldsymbol{\omega}$, $\boldsymbol{\omega} = \boldsymbol{e}_3 \times \dot{\boldsymbol{e}}_3$ is an angular velocity, and $l$ is a pendulum length. Besides, the relation $(\boldsymbol{N}, \boldsymbol{e}_3) = 0$ is valid; it is a zero orbit of $e(3)$.

The Hamiltonian may be written as follows

$$H = \frac{1}{2ml^2}\boldsymbol{N}^2 + U(\boldsymbol{e}_3). \tag{4.21}$$

So, a spherical pendulum may be represented as a spherical top on a zero orbit of algebra $e(3)$.

It's also convenient to use these basic elements for description of reduction in the presence of the Lagrange integral $F = M_3 = \text{const}$ (see §§ 1, 2 ch. 4).



**Quaternion representation of equations of motion.** For practical computations redundancy of equations (4.17) is very inconvenient, because, for example, under numerical integration of these equations the orthonormal relations are rapidly violated. This drawback is absent in the quaternion form of representation of equations of motion. This form is indicated by the authors in [30, 31]. The direction cosine matrix in quaternion representation has the form (3.11), and corresponding commutational relations are

$$\begin{aligned} &\{M_i, M_j\} = -\varepsilon_{ijk} M_k, && \{M_i, \lambda_0\} = \frac{1}{2}\lambda_i, \\ &\{M_i, \lambda_j\} = -\frac{1}{2}(\varepsilon_{ijk}\lambda_k + \delta_{ij}\lambda_0), && \{\lambda_\mu, \lambda_\nu\} = 0. \end{aligned} \tag{4.22}$$

A Lie algebra, specifying them, is a semi-direct sum of rotation algebra $so(3)$ and translation algebra $\mathbb{R}^4$: $l(7) \approx so(3) \oplus_s \mathbb{R}^4$.

Bracket (4.22) is a degenerate one and possesses the only Casimir's function

$$F(\lambda) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{4.23}$$

A nonspecial symplectic leave is also homeomorphic to cotangent foliation of a three-dimensional sphere $T^*S^3$; the leave has six dimensions. The equations of motion can be written as

$$\begin{gathered} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{\lambda}} + \frac{1}{2}\frac{\partial H}{\partial \lambda_0}\boldsymbol{\lambda} - \frac{1}{2}\lambda_0\frac{\partial H}{\partial \boldsymbol{\lambda}}, \\ \dot{\lambda}_0 = -\frac{1}{2}\left(\boldsymbol{\lambda}, \frac{\partial H}{\partial \boldsymbol{M}}\right), \quad \dot{\boldsymbol{\lambda}} = \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\lambda_0\frac{\partial H}{\partial \boldsymbol{M}}. \end{gathered} \tag{4.24}$$

To be able to integrate them, we also need two integrals in involution.

Remark 2. For real systems, descending from rigid body dynamics, a Hamiltonian $H$ is a single-valued function on the group $SO(3)$, and as a result of its double covering by quaternions (3.11) the Hamiltonian function depends only on quadratic combinations $\lambda_i\lambda_j$. Nevertheless, systems containing the Hamiltonian, arbitrarily depending on quaternions, can be found in other sections of mechanics: celestial mechanics in a curved space, the Leggette system, quantum mechanics of spins (see ch. 3, 4). It's possible that the form (4.24) is more significant just for quantum mechanics, where some effects essentially concern additional spin variables.

**Canonical equations in Euler angles and Andoyaer – Deprit variables.** In the Euler angles $(\theta,\ \varphi,\ \psi)$ and corresponding canonical momenta $p_\theta,\ p_\varphi,\ p_\psi$ the equations of motion have the ordinary Hamiltonian form

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \boldsymbol{q} = (\theta,\ \varphi,\ \psi), \quad \boldsymbol{p} = (p_\theta,\ p_\varphi,\ p_\psi). \tag{4.25}$$

This form can be obtained from Lagrangian formalism in terms of variables $(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi})$ by means of the ordinary Legendre transformation

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \qquad H(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}, \boldsymbol{q})\Big|_{\dot{\boldsymbol{q}}, \boldsymbol{q} \to \boldsymbol{p}, \boldsymbol{q}}.$$

Here $L$ is a Lagrange function, which in case of a natural system has the form $L = T - U(\theta, \varphi, \psi)$, where the Lagrangian is determined by formulae (3.1). Kinetic energy of a rigid body does not depend on $\psi$ and is written as

$$\begin{aligned} T = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) = \frac{1}{2}\Big[a_1\Big(\frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) + p_\theta\cos\varphi\Big)^2 + \\ + a_2\Big(\frac{\cos\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) - p_\theta\sin\varphi\Big)^2 + a_3 p_\varphi^2\Big]. \end{aligned} \tag{4.26}$$

The rigid body motion in a potential field is described by means of a natural system, and the Hamiltonian has the form:

$$H = T + U(\theta, \varphi, \psi). \tag{4.27}$$

If the potential energy does not depend on $\psi$ $\left(\frac{\partial U}{\partial \psi} = 0\right)$, which corresponds to the force field invariance with respect to rotation around a vertical axis, fixed in space, then variable $\psi$ is cyclic, and generalized momentum $p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma})$ is conserved. Applying the Routh reduction with respect to precession angle $\psi$, we obtain the system, describing motion of a point over a sphere $\boldsymbol{\gamma}^2 = 1$ (where $\gamma_1 = \sin\theta\sin\varphi$, $\gamma_2 = \sin\theta\cos\varphi$, $\gamma_3 = \cos\theta$), called *a Poisson sphere*. In case $p_\psi \neq 0$, the Hamiltonian contains terms, linear with respect to velocities (gyroscopic members). They can't be removed by means of coordinate transformations and correspond to motion in a generalized potential field. The impossibility of removing results from the global effect of the "monopole" appearance. Its value can be calculated as integral of the form of gyroscopic force over the Poisson sphere (see [133]). P. Dirac was the first to pay attention to the "monopole" problem in view of the problem of quantization of a particle motion on a sphere. When $p_\psi = 0$, the reduced system is again natural.

In the presence of dynamical symmetry $a_1 = a_2$, kinetic energy (4.26) is somewhat simplified and doesn't depend on angle $\varphi$

$$T = \frac{1}{2}\left(a_1\left(p_\theta^2 + \frac{(p_\psi - p_\varphi\cos\theta)^2}{\sin^2\theta}\right) + a_3 p_\varphi^2\right). \tag{4.28}$$



If potential $U$ doesn't depend on $\varphi$ (i. e. $\frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial \varphi} = 0$), in other words $U = U(\theta) = U(\gamma_3)$, then there exists one more cyclic integral $p_\varphi = M_3 = c_2 = \text{const}$ — the Lagrange integral, corresponding to the system invariancy relatively to rotations around the dynamic symmetry axis. After the reduction we obtain an integrable unidegree system (for more details see § 3 ch. 2). In case $p_\psi = c_1 \neq 0$, but $p_\varphi = c_2 = 0$, equations describe a *spherical pendulum* motion.

In Andoyaer – Deprit variables the equations of motion also appear as (4.25), where $\boldsymbol{q} = (l, g, h)$, $\boldsymbol{p} = (L, G, H)$. As far as variables $L, G, H, l, g, h$ don't contain purely positional coordinates, which uniquely describe the body position, i. e., in cotangent foliation $TS^3$ they "mix" variational bases and fibres, then in the general case potential $U$ depends on a whole set of variables $U = U(L, G, H, l, g, h)$.

Kinetic energy $T$ is written as

$$T = \frac{1}{2}\big[(G^2 - L^2)(a_1\sin^2 l + a_2\cos^2 l) + a_3 L^2\big]. \tag{4.29}$$

And again it's easy to obtain that

1) if $\frac{\partial U}{\partial h} = 0$, then there exists an area integral $H = p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const}$,

2) if $a_1 = a_2$ and $\frac{\partial U}{\partial l} = 0$, then there exists the Lagrange integral $L = c_2 = \text{const}$.

The peculiarity of kinetic energy representation in the form (4.29) is its independence of the variable $g$. It allows immediate integration of the Euler problem — a free top motion, for which $U \equiv 0$ (see § 1 ch. 2). The corresponding cyclic integral is $G = \text{const}$. It represents the angular momentum magnitude $G^2 = \boldsymbol{M}^2$. This fact makes Andoyaer – Deprit variables useful for geometric interpretation and perturbed case analysis. The phase portrait of the Euler case on a sphere cylindrical development is shown at fig. 5. At superposition of perturbation, for example, a gravity field, the phase-plane portrait shows chaotic motions near separatrices, connecting unstable uniform rotations (fig. 6). Let us dwell on phase flow visualization methods.

### 3. The Poincaré Cross-Section and Chaotic Motions

To visualize chaotic motions of bidegree systems it's helpful to use *the Poincaré map* ( *the Poincaré cross-section, the phase cross-section*), reducing phase flow to a discrete two-dimensional map of the plane onto itself.

We'll describe this map construction technique for rigid body dynamics specifically. In this case it's convenient to use Andoyaer – Deprit variables, a secant plane, for the first time introduced in [215], and some other secant planes, making clear various aspects of motion.

At first, we make the level of energy $\mathscr{H}(L, G, H, l, g) = E = \text{const}$ fixed. If we have an axially symmetric field, then variable $h$ is cyclic and isn't involved in the Hamiltonian, and the conjugate variable $H$, representing an area constant, may be considered as a parameter. Thus, at the energy level we have three-dimensional phase flow. Choose a secant plane $g = g_0 \bmod 2\pi$, $g_0 = \text{const}$ (hereinafter we'll also use $l = l_0 \bmod 2\pi$, $l_0 = \text{const}$) and consider sequential intersections of the individual path of this plane $\dots, x_{n-1}, x_n, x_{n+1}, \dots$ in one and the same direction, i. e. $\operatorname{sgn} \dot{g}(x_n) = \operatorname{sgn} \dot{g}(x_{n+1})$ (fig. 4).

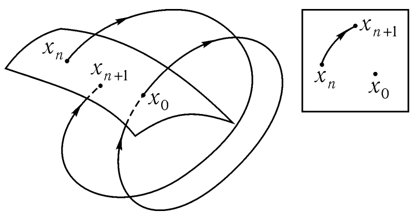



Figure 4

Remark 3. The last condition results from the need for periodic orbits, crossing the plane $g = g_0$, generally, in two points, to be fixed points of a point mapping $x_n = x_0$, $n = 1, \dots$ (see fig. 4).

The Poincaré map aligns each point $x_n$ with its sequential iteration $x_{n+1}$, belonging to the same phase path. Generally speaking, this mapping is defined locally, near a certain periodic solution, since under the phase flow action the point may leave the secant plane and never come back again. Nevertheless, this mapping is of great use because it illustrates various effects, concerning returning paths. It's usually referred to as *the first return map*.

It's also useful to consider the Poincaré maps in the global sense, selecting such phase plane domains, for which the Poincaré map is defined. They are called possible motion domains (PMD). They are usually determined from the existence of solution for the energy equation $\mathscr{H}(\boldsymbol{p}, \boldsymbol{q}) = E$, $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^4$, $q = q_0 = \text{const}$ (in our case $(\boldsymbol{p}, \boldsymbol{q}) = (L, G, l, g), q_0 = g_0$). If the energy level is compact, then the Poincaré theorem about return is valid, and the point will cross the chosen plane again, and infinitely many times. It's evident that the path touches the secant plane at the PMD boundary, i. e., we have intersection transversality loss. The Poincaré global maps are poorly studied yet.

In rigid body dynamics, from considerations of compactness, we further choose coordinates $l \bmod 2\pi$, $L/G$ on the secant plane since $|L/G| \leqslant 1$ (see [215, 28]). We determine mapping iterations by numerically integrating equations of motion in terms of variables $(\boldsymbol{M}, \boldsymbol{\gamma})$, and at bringing them onto the secant plane we transform them into variables $(L, G, l, g)$ according to formulae (3.12), (3.13)



$$
\begin{aligned}
M_1 &= \sqrt{G^2 - L^2} \sin l, \quad M_2 = \sqrt{G^2 - L^2} \cos l, \quad M_3 = L, \\
\gamma_1 &= \left( \frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \sin l + \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \cos l, \\
\gamma_2 &= \left( \frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \cos l - \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \sin l, \\
\gamma_3 &= \left(\frac{H}{G}\right) \left(\frac{L}{G}\right) - \sqrt{1 - \left(\frac{L}{G}\right)^2} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g.
\end{aligned}
\tag{4.30}
$$

This involves achieving the necessary accuracy of numerical integration and reducing computation time. We should also mention that the last versions of our software also use quaternion equations in terms of variables $(\boldsymbol{M}, \lambda)$, allowing to achieve even higher degree of accuracy, and at the same time to define absolute motion of a rigid body, necessary for visualizing paths of various points of the body.

If for integrable systems sequential iterations of mapping lay on invariant curves, consisting of periodic or quasi-periodic motions (see § 7) and defined by an additional integral (fig. 5), then in the general (nonintegrable) situation the path may chaotically fill whole domains in a phase space (at the level $H = h$, fig. 6).

The Poincaré map has appeared and is constantly used in theory of non-integrability and determinative chaos. It's also useful for investigating integrable problems since it vividly represents the mutual position of various particular solutions in the phase space. Among these solutions exist particularly notable and important ones (see ch. 2).

splitting to the periodic solutions.

For the Euler case the Poincaré map shows a familiar picture (see fig. 5). Incidentally, while introducing variables $L$, $G$, $H$, $l$, $g$, $h$ in [71] A. Deprit regarded the vivid interpretation of the Euler problem solutions, which can adequately replace Poincot geometric interpretation (§ 2 ch. 2), to be their main advantage. Further on, we are using the above-mentioned construction for studying both cases: integrable and non-integrable.

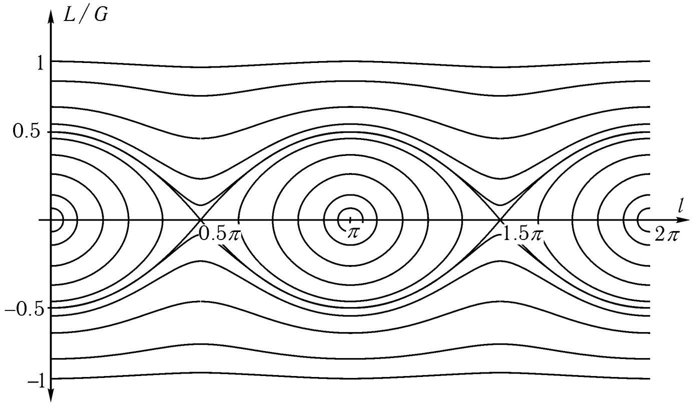



Figure 5. The Euler problem phase portrait. Stable fixed points and straight lines $|L| = G$ correspond to stable permanent rotations relatively to a longer and a shorter axes; unstable points correspond to rotations around the mean axis of inertia, separatrices are formed by double-asymptotic paths, connecting unstable permanent rotations.

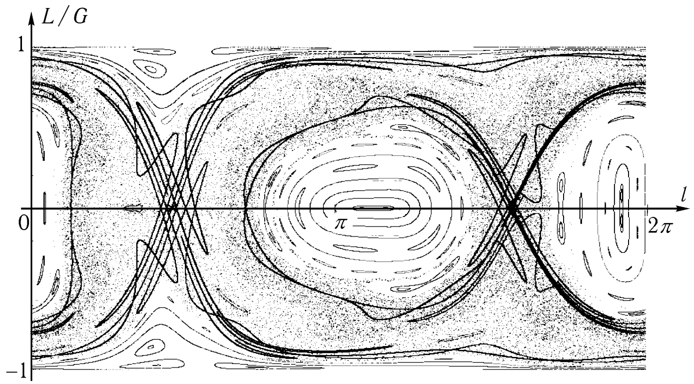



Figure 6. A phase portrait (the section by plane $g = \frac{\pi}{2}$) for the Euler – Poisson equation under $h = 1.5$, $c = 1$ and the following body parameters: $\mathbf{I} = \mathrm{diag}(1.5; 1.2)$, $\boldsymbol{r} = (0.5,\ 0,\ 0)$. One can see doubling of the period of the orbit, born from the permanent rotations near points $(\pi, 0)$ and $(2\pi, 0)$ at fig. 5, and separatrices, born from permanent rotations at points $(\frac{\pi}{2}, 0)$ and $(\frac{3}{2}\pi, 0)$ at fig. 5,

## § 5. Equations of Rigid Body Motion in Euclidian Space

### 1. Lagrange Formalism and Poincaré Equations on a Group $E(3)$

Let a rigid body be moving in Euclidian space $\mathbb{R}^3$, let its configurational space coincide with a group $E(3)$. In a matrix form group elements may be



represented as

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{Q}^{\mathrm{T}} & \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \\ \hline 0 & 1 \end{array}\right) \in E(3),$$

where $\mathbf{Q} \in SO(3)$ is a matrix of direction cosines (3.3), and $\boldsymbol{x}$ is a position vector of a certain point $C$, fixed inside the body, in projections on fixed axes (see fig. 7).

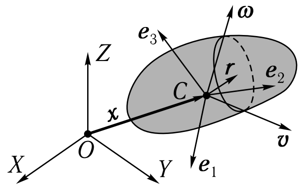



Figure 7. A free rigid body.

We'll write equations of motion for projections of angular velocity $\boldsymbol{\omega} = (\omega_1,\ \omega_2,\ \omega_3)$ and absolute velocity $\boldsymbol{v} = (v_1,\ v_2,\ v_3)$ of the center-of-mass on axes, attached to the body. Similarly to (4.3), we'll write down the following evident geometric relations

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q}, \qquad \boldsymbol{v} = \mathbf{Q}\dot{\boldsymbol{x}}. \tag{5.1}$$

Now we'll determine corresponding basic left-invariant fields on the group $E(3)$. To do that, we'll find time derivative in view of equations (5.1)

$$\frac{df}{dt} = \mathrm{Tr}\Big(\dot{\mathbf{Q}}^T\frac{\partial f}{\partial \mathbf{Q}}\Big) + \Big(\frac{\partial f}{\partial \boldsymbol{x}},\ \dot{\boldsymbol{x}}\Big) = \mathrm{Tr}\Big((\widetilde{\boldsymbol{\omega}}\mathbf{Q})^T\frac{\partial f}{\partial \mathbf{Q}}\Big) + \Big(\mathbf{Q}\frac{\partial f}{\partial \boldsymbol{x}},\ \boldsymbol{v}\Big),$$
$$\frac{\partial f}{\partial \mathbf{Q}} = \left\|\frac{\partial f}{\partial Q_{ij}}\right\|; \quad \frac{\partial f}{\partial \boldsymbol{x}} = \left(\frac{\partial f}{\partial x_1},\ \frac{\partial f}{\partial x_2},\ \frac{\partial f}{\partial x_3}\right).$$

Combining terms in $\omega_i$, $v_i$, we obtain

$$\begin{aligned} \boldsymbol{\omega} &= \omega_k\boldsymbol{\xi}_k, & \boldsymbol{\xi}_k &= -\sum_{ij}\varepsilon_{kij}\Big(\alpha_i\frac{\partial}{\partial\alpha_j} + \beta_i\frac{\partial}{\partial\beta_j} + \gamma_i\frac{\partial}{\partial\gamma_j}\Big), \\ \boldsymbol{v} &= v_i\boldsymbol{\zeta}_i, & \boldsymbol{\zeta}_i &= \alpha_i\frac{\partial}{\partial x_1} + \beta_i\frac{\partial}{\partial x_2} + \gamma_i\frac{\partial}{\partial x_3}. \end{aligned} \tag{5.2}$$

Basic field commutators $\boldsymbol{\xi}_i$, $\boldsymbol{\zeta}_j$ have the form

$$[\boldsymbol{\xi}_i,\ \boldsymbol{\xi}_j] = \varepsilon_{ijk}\boldsymbol{\xi}_k, \qquad [\boldsymbol{\xi}_i,\ \boldsymbol{\zeta}_j] = \varepsilon_{ijk}\boldsymbol{\zeta}_k, \qquad [\boldsymbol{\zeta}_i,\ \boldsymbol{\zeta}_j] = 0. \tag{5.3}$$

Taking into account (5.2) and (5.3), we can write Poincaré equations of motion (2.4) for free rigid body dynamics

$$\begin{gathered}\frac{d}{dt}\Big(\frac{\partial L}{\partial \boldsymbol{\omega}}\Big)=\frac{\partial L}{\partial \boldsymbol{\omega}}\times\boldsymbol{\omega}+\frac{\partial L}{\partial \boldsymbol{v}}\times\boldsymbol{v}+\frac{\partial L}{\partial \boldsymbol{\alpha}}\times\boldsymbol{\alpha}+\frac{\partial L}{\partial \boldsymbol{\beta}}\times\boldsymbol{\beta}+\frac{\partial L}{\partial \boldsymbol{\gamma}}\times\boldsymbol{\gamma},\\ \frac{d}{dt}\Big(\frac{\partial L}{\partial \boldsymbol{v}}\Big)=\frac{\partial L}{\partial \boldsymbol{v}}\times\boldsymbol{\omega}+\frac{\partial L}{\partial x_1}\boldsymbol{\alpha}+\frac{\partial L}{\partial x_2}\boldsymbol{\beta}+\frac{\partial L}{\partial x_3}\boldsymbol{\gamma},\\ \dot{\boldsymbol{\alpha}}=\boldsymbol{\alpha}\times\boldsymbol{\omega},\qquad \dot{\boldsymbol{\beta}}=\boldsymbol{\beta}\times\boldsymbol{\omega},\qquad \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma}\times\boldsymbol{\omega},\\ \dot{x}_1=\Big(\boldsymbol{\alpha},\,\frac{\partial L}{\partial \boldsymbol{v}}\Big),\qquad \dot{x}_2=\Big(\boldsymbol{\beta},\,\frac{\partial L}{\partial \boldsymbol{v}}\Big),\qquad \dot{x}_3=\Big(\boldsymbol{\gamma},\,\frac{\partial L}{\partial \boldsymbol{v}}\Big).\end{gathered}\tag{5.4}$$

## 2. Kinetic Energy of a Rigid Body in $\mathbb{R}^3$

Let's represent a position vector of every point of a rigid body in a fixed frame of reference as $\boldsymbol{q}=\mathbf{Q}^{\mathrm{T}}\boldsymbol{y}+\boldsymbol{x}$, where $\boldsymbol{y}$ is a position vector of the given point, constant in the frame of reference, attached to the body. Differentiating with respect to time $\dot{\boldsymbol{q}}=\dot{\mathbf{Q}}^{\mathrm{T}}\boldsymbol{y}+\dot{\boldsymbol{x}}$ and integrating with respect to $\boldsymbol{y}$, we'll obtain kinetic energy both in the vector, and in the matrix form

$$\begin{aligned}T&=\frac{1}{2}(\boldsymbol{\omega},\,\mathbf{I}\boldsymbol{\omega})+m(\boldsymbol{v},\,\boldsymbol{r}\times\boldsymbol{\omega})+\frac{1}{2}m\boldsymbol{v}^2=\\ &=-\frac{1}{2}\,\mathrm{Tr}(\widetilde{\boldsymbol{\omega}}\mathbf{J}\widetilde{\boldsymbol{\omega}})+m(\boldsymbol{v},\,\widetilde{\boldsymbol{\omega}}\boldsymbol{r})+\frac{1}{2}m\boldsymbol{v}^2,\end{aligned}\tag{5.5}$$

where $m=\int\limits_{\tau}\rho(\boldsymbol{y})\,d^3\boldsymbol{y}$ is a total mass of the body and $\boldsymbol{r}=\frac{1}{m}\int\boldsymbol{y}\rho(\boldsymbol{y})\,d^3y$ is a position vector of the center-of-mass in the system of axes bound to the body, $\rho(\boldsymbol{y})$ is a mass density of the body, and $\mathbf{I}$, $\mathbf{J}$ are defined by relations (4.11), (4.12).

If we are to choose the origin of the frame of reference, bound to the body, *in the gravity center*, then $\boldsymbol{r}=0$, and kinetic energy is divided into translational energy and the energy of rotation around the gravity center. This statement constitutes the celebrated *Bernoulli theorem*.

Remark 1. For a rigid body motion in a perfect incompressible fluid (Kirchhoff's equations), in the general case, the kinetic energy cannot be resolved into rotational and translational components.



## 3. The Hamiltonian Form of Equations of Motion of a Rigid Body in $\mathbb{R}^3$

To make transition to the Hamiltonian formalism (the Poincaré – Chetayev equations) we'll carry out Legendre transformation according to the formulae

$$\boldsymbol{M}=\frac{\partial L}{\partial \boldsymbol{\omega}},\quad \boldsymbol{p}=\frac{\partial L}{\partial \boldsymbol{v}},\qquad H=(\boldsymbol{M},\boldsymbol{\omega})+(\boldsymbol{p},\boldsymbol{v})-L|_{\boldsymbol{\omega},\boldsymbol{v}\to\boldsymbol{M},\boldsymbol{p}}.\tag{5.6}$$

Here $\boldsymbol{M}$ is an angular momentum, $\boldsymbol{p}$ is a body momentum in projections on the axes, bound to the body.

The Poisson bracket of variables $\boldsymbol{M}$, $\boldsymbol{p}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{x}$ can be obtained by formula (2.12). It is completely defined by the form of fields (5.2) and their commutators (5.3) and does not depend on the particular form of the Lagrange function. The only limitation is the condition of the Lagrange function nondegeneracy with respect to velocities.

Finally, we receive following (non-zero) Poisson brackets

$$\begin{gathered}\{M_i,\,M_j\}=-\varepsilon_{ijk}M_k,\quad \{M_i,\,p_j\}=-\varepsilon_{ijk}p_k,\\ \{M_i,\,\alpha_j\}=-\varepsilon_{ijk}\alpha_k,\ \ \{M_i,\,\beta_j\}=-\varepsilon_{ijk}\beta_k,\ \ \{M_i,\,\gamma_j\}=-\varepsilon_{ijk}\gamma_k,\\ \{p_i,\,x_1\}=-\alpha_i,\quad \{p_i,\,x_2\}=-\beta_i,\quad \{p_i,\,x_3\}=-\gamma_i.\end{gathered}\tag{5.7}$$

As it was remarked in § 2, s. 3 under such a matrix realization we obtain the Lie – Poisson bracket, corresponding to a semi-direct sum $e(3)\oplus_s\mathbb{R}^{12}$.

Remark 2. As it follows from relations (5.7), under quaternion parameterization of group of rotations the Poisson structure in variables $\boldsymbol{M},\lambda_0,\boldsymbol{\lambda},\boldsymbol{p},\boldsymbol{x}$ will contain quadratic brackets, because direction cosines quadratically depend on quaternions.

In the vector form the Hamiltonian equations of motion can be written as follows

$$\begin{gathered}\dot{\boldsymbol{M}}=\boldsymbol{M}\times\frac{\partial H}{\partial \boldsymbol{M}}+\boldsymbol{p}\times\frac{\partial H}{\partial \boldsymbol{p}}+\boldsymbol{\alpha}\times\frac{\partial H}{\partial \boldsymbol{\alpha}}+\boldsymbol{\beta}\times\frac{\partial H}{\partial \boldsymbol{\beta}}+\boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}},\\ \dot{\boldsymbol{p}}=\boldsymbol{p}\times\frac{\partial H}{\partial \boldsymbol{M}}-\frac{\partial H}{\partial x_1}\boldsymbol{\alpha}-\frac{\partial H}{\partial x_2}\boldsymbol{\beta}-\frac{\partial H}{\partial x_3}\boldsymbol{\gamma},\\ \dot{\boldsymbol{\alpha}}=\boldsymbol{\alpha}\times\frac{\partial H}{\partial \boldsymbol{M}},\qquad \dot{\boldsymbol{\beta}}=\boldsymbol{\beta}\times\frac{\partial H}{\partial \boldsymbol{M}},\qquad \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{M}},\\ \dot{x}_1=\Big(\boldsymbol{\alpha},\,\frac{\partial H}{\partial \boldsymbol{p}}\Big),\qquad \dot{x}_2=\Big(\boldsymbol{\beta},\,\frac{\partial H}{\partial \boldsymbol{p}}\Big),\qquad \dot{x}_3=\Big(\boldsymbol{\gamma},\,\frac{\partial H}{\partial \boldsymbol{p}}\Big).\end{gathered}\tag{5.8}$$

A free rigid body motion in a potential field in a center-of-mass system ($\boldsymbol{r}=0$ in equation (5.5)) is described by a natural mechanical system with a Hamiltonian function of the form

$$H = \frac{1}{2}(\boldsymbol{M},\, \boldsymbol{AM}) + \frac{1}{2m}\boldsymbol{p}^2 + U(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{x}), \tag{5.9}$$

where $\mathbf{A} = \mathbf{I}^{-1}$, and variables $\boldsymbol{M}$, $\boldsymbol{p}$ are expressed in terms of body velocities by the formulae

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega}, \qquad \boldsymbol{p} = m\boldsymbol{v}.$$

Remark 3. If the potential energy in (5.9) may be represented as

$$U = U_1(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) + U_2(\boldsymbol{x}),$$

then some equations from (5.8) become separated to constitute the system for variables $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, describing the body rotation around the center-of-mass. If, instead of the momentum projections on moving axes $\boldsymbol{p} = m\boldsymbol{v}$, the momentum in a fixed space $\boldsymbol{P} = m\dot{\boldsymbol{x}}$ be used, we also obtain a separate system, describing the center-of-mass motion in the canonical form

$$\dot{\boldsymbol{P}} = -\frac{\partial H_{\text{c.m.}}}{\partial \boldsymbol{x}}, \qquad \dot{\boldsymbol{x}} = \frac{\partial H_{\text{c.m.}}}{\partial \boldsymbol{P}}, \qquad H_{\text{c.m.}} = \frac{1}{2m}\boldsymbol{P}^2 + U_2(\boldsymbol{x}). \tag{5.10}$$

That is, the Poisson structure in terms of variables $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{P}, \boldsymbol{x}$ is not given by the Lie–Poisson bracket because the bracket between variables $\boldsymbol{P}, \boldsymbol{x}$ is canonical.

## § 6. Examples and Related Problem Statements

### 1. The Motion of a Rigid Body with a Fixed Point in the Superposition of Permanent Uniform Force Fields

As it is shown in [31], in this case any number of fields can be reduced to three mutually perpendicular fields. The Hamiltonian function has the form

$$H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) - (\boldsymbol{r}_1,\, \boldsymbol{\alpha}) - (\boldsymbol{r}_2,\, \boldsymbol{\beta}) - (\boldsymbol{r}_3,\, \boldsymbol{\gamma}), \tag{6.1}$$

where $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ are vectors, constant in the frame of reference attached to the body. They specify three different centers of reduction: gravity center analogues. When $\boldsymbol{r}_1 = \boldsymbol{r}_2 = 0$, the equations of motion for $\boldsymbol{M}$, $\boldsymbol{\gamma}$ are separated and referred to as the *Euler–Poisson equations*.



### 2. A Free Rigid Body in a Quadratic Potential

Let a rigid body be moving in a single field with a quadratic potential

$$\varphi(\boldsymbol{q}) = -\frac{1}{2}(\boldsymbol{q},\, \mathbf{B}\boldsymbol{q}) - (\boldsymbol{g},\, \boldsymbol{q}), \tag{6.2}$$

here $\mathbf{B}$ is a constant symmetrical matrix, $\boldsymbol{g}$ is a constant vector. Potential (6.2) arises, for example, under the second order expansion of the gravity potential near the Earth surface and also Coulomb potential of a charged body.

Substituting the position vector of a point in a fixed space as $\boldsymbol{q} = \mathbf{Q}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{x}$ ($\boldsymbol{y}$ is a position vector of the point within the body) and taking volume integral, we obtain the potential energy in the form to follow (see also [21])

$$\begin{gathered} U = \frac{1}{2}\operatorname{Tr}(\mathbf{Q}^{\mathrm{T}}\mathbf{I_1}\mathbf{Q}\mathbf{B}) - \frac{1}{2}\mu_0(\boldsymbol{x},\, \mathbf{B}\boldsymbol{x}) - \mu_0(\boldsymbol{g},\, \boldsymbol{x}) - \\ - \frac{1}{2}\mu_0(\mathbf{Q}\boldsymbol{g},\, \boldsymbol{r}_1) - \mu_0(\mathbf{Q}\mathbf{B}\boldsymbol{x},\, \boldsymbol{r}_1). \end{gathered} \tag{6.3}$$

Here $\mu_0 = \int\limits_\tau \mu(\boldsymbol{y})\, d^3\boldsymbol{y}$ is a total "charge" of the body in a given field, $\mu(\boldsymbol{y})$ is its density, $\boldsymbol{r}_1 = \frac{1}{\mu_0}\int\limits_\tau \boldsymbol{y}\mu(\boldsymbol{y})\, d^3\boldsymbol{y}$ is a position vector of the field reduction center, $I_{1ij} = \int (\delta_{ij}\boldsymbol{y}^2 - y_i y_j)\mu(\boldsymbol{y})\, d^3\boldsymbol{y}$.

For the gravity field $\mu(\boldsymbol{y})$ is a mass density, $\mu_0 = m$ is a body mass, $r_1 = \boldsymbol{r}$ is a center-of-mass position vector, $\mathbf{I}_1 = \mathbf{I}$ is a tensor of moments of inertia. In this case, while choosing in fixed space principal axes, corresponding to eigenvectors of matrix $\mathbf{B}$, in the center-of-mass system the Hamiltonian may be written as

$$\begin{gathered} H = \frac{1}{2}(\boldsymbol{M},\, \boldsymbol{AM}) + \frac{1}{2m}\boldsymbol{p}^2 + \\ + \frac{1}{2}(b_1(\boldsymbol{\alpha},\, \mathbf{I}\boldsymbol{\alpha}) + b_2(\boldsymbol{\beta},\, \mathbf{I}\boldsymbol{\beta}) + b_3(\boldsymbol{\gamma},\, \mathbf{I}\boldsymbol{\gamma})) - \frac{1}{2}m(\boldsymbol{x},\, \mathbf{B}\boldsymbol{x}) - m(\boldsymbol{g},\, \boldsymbol{x}), \\ \mathbf{B} = \operatorname{diag}(b_1,\, b_2,\, b_3). \end{gathered} \tag{6.4}$$

Thus, translational and rotational motions are separated, so that both systems can be integrated in quadratures [21] (ch. 3, § 12) (which is admittedly realized, when inertial mass equals gravitational one, i. e. for a gravity field). It should also be mentioned that rotation and translation are separated for an arbitrary field, if the field reduction center coincides with the center-of-mass.

**3. The Motion of a Body with a Fixed Point in a Rotating Frame of Reference**

Let a rigid body be moving so that one of its points is rotating uniformly with an angular velocity $\Omega$ along circumference of radius $R$. Choose three frames of reference:

1) fixed in space (inertial) frame of reference $OXYZ$ having its origin in the circumference center $O$;
2) a frame of reference, uniformly rotating along the circumference and having its center at the point $C$ and reference vectors: $\boldsymbol{e}_\tau$, which is a vector tangent to the circumference, $\boldsymbol{e}_n$, which is a vector normal to the circumference plane, and $\boldsymbol{e}_R$, which is a vector directed from the point $C$ to the center of circumference;
3) a system of axes, rigidly attached to the body $Cx_1x_2x_3$, with the origin at the point $C$ (see fig. 8).

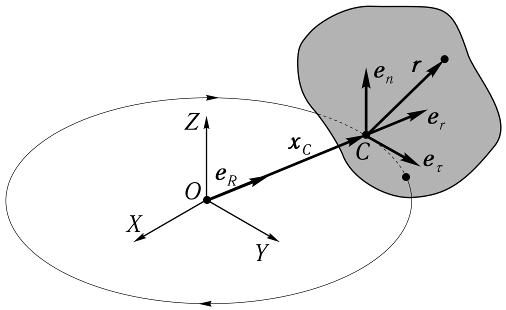



Figure 8. The motion of a body with a fixed point in a rotating frame of reference.

Configurational space of the system is a group $SO(3)$, which may be represented by matrices of transition $\mathbf{Q} \in SO(3)$ from the system of axes of a rigid body to the rotating frame of reference. They have the form (4.1), where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are projections of vectors $\boldsymbol{e}_\tau$, $\boldsymbol{e}_n$, $\boldsymbol{e}_R$ on the axes, bound to the body. Now we introduce one more matrix $\mathbf{B}$ of transition from the rotating frame of reference to the fixed one (columns of the matrix $\mathbf{B}$ are projections of unit vectors of fixed space on vectors $\boldsymbol{e}_\tau$, $\boldsymbol{e}_n$, $\boldsymbol{e}_R$).

The position of the rigid body point with the position vector $\boldsymbol{y}$ inside the body in a fixed space is given by vector

$$\boldsymbol{q} = \mathbf{B}^{\mathrm{T}}(t)(\mathbf{Q}^{\mathrm{T}}(t)\boldsymbol{y} + \boldsymbol{x}), \tag{6.5}$$



where $\boldsymbol{x}$ is a position vector of the point $C$ in the rotating frame of reference. Differentiating with respect to time $\dot{\boldsymbol{q}} = \dot{\mathbf{B}}^{\mathrm{T}}(t)(\mathbf{Q}^{\mathrm{T}}(t)\boldsymbol{y} + \boldsymbol{x}_c) + \mathbf{B}^{\mathrm{T}}(t)\dot{\mathbf{Q}}^{\mathrm{T}}(t)\boldsymbol{y}$ and integrating $\frac{1}{2}m(\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ over the body volume, we obtain kinetic energy in the form

$$T = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + \Omega(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\beta}) - \mu(\boldsymbol{\omega}, \boldsymbol{r} \times \boldsymbol{\alpha}) + \frac{1}{2}\Omega^2(\boldsymbol{\beta}, \mathbf{I}\boldsymbol{\beta}) - \mu\Omega(\boldsymbol{r}, \boldsymbol{\gamma}), \tag{6.6}$$

where $\mu = 2mR\Omega$, $\mathbf{I}$ is a tensor of inertia of the body with respect to point $C$, $\boldsymbol{r}$ is a position vector of the body center-of-mass in projections on the body axes.

The motion of a body in a potential field is described by the Lagrange function

$$L = T(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) - U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \tag{6.7}$$

where $T$ is a kinetic energy (6.6), and $U$ is a potential energy. The equations of motion of system (6.7) are determined by Poincaré equations (4.2), (4.7).

Carrying out the Legendre transformation for system (6.7), we find

$$\begin{gathered}
\boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \mathbf{I}(\boldsymbol{\omega} + \Omega\boldsymbol{\beta}) - \mu\boldsymbol{r} \times \boldsymbol{\alpha}, \\
H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) - \Omega(\boldsymbol{M}, \boldsymbol{\beta}) + \mu(\boldsymbol{M}, \mathbf{A}(\boldsymbol{r} \times \boldsymbol{\alpha})) + \\
+ \frac{1}{2}\mu^2 (\boldsymbol{r} \times \boldsymbol{\alpha}, \mathbf{A}(\boldsymbol{r} \times \boldsymbol{\alpha})) + U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}),
\end{gathered} \tag{6.8}$$

here $\mathbf{A} = \mathbf{I}^{-1}$. The equations of motion have the form (4.17).

System (6.8) is the one, to which the following classical problems of rigid body dynamics are reduced.

**A gyroscope and a Foucault's pendulum.** In this case $U = r_3\beta_3$, the body is axially symmetric

$$a_1 = a_2 = 1, \qquad r_1 = r_2 = 0,$$

and the Hamiltonian may be represented as

$$H = \frac{1}{2}(M_1^2 + M_2^2 + a_3M_3^2) - \Omega(\boldsymbol{M}, \boldsymbol{\beta}) - \mu r_3(M_1\alpha_2 - M_2\alpha_1) - \frac{1}{2}\mu^2 r_3^2\alpha_3^2. \tag{6.9}$$

In this case it's convenient to use variables of fixed space (4.19). If we choose the corresponding units to measure length and mass and denote the vector $\boldsymbol{e}_3 = \boldsymbol{l}$, the Hamiltonian can be written as

$$H = \frac{1}{2}\boldsymbol{N}^2 - \Omega N_2 + \mu(N_2l_3 - N_3l_2) - \frac{1}{2}\mu^2 l_1^2 - \mu l_3. \tag{6.10}$$

The system on a zero constant of integral $(\boldsymbol{N}, \boldsymbol{l}) = M_3 = 0$ corresponds to the gyroscope without proper rotation and is referred to as *Foucault's pendulum*.

**A satellite at the Earth circular orbit.** The center-of-masses coincides with the origin of the rotating system coordinates, i. e. $\boldsymbol{r} = 0$. Newton potential in quadratic approximation (under expansion by the ratio of satellite dimensions to the orbit radius) has the form

$$U = \frac{3}{2}\Omega^2(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma}), \quad \Omega^2 = \frac{GM}{R^3},$$

where $G$ is a gravitational constant, $M$ is a mass of the Earth, $R$ is a radius of the orbit. Thus,

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) - \Omega(\boldsymbol{M}, \boldsymbol{\beta}) + \frac{3}{2}\Omega^2(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma}). \tag{6.11}$$

One can use the book [11] to get acquainted with various dynamic effects of the satellite motion along the circular orbit.

## 4. Relative Motion of a Rigid Body with a Fixed Point

Let a rigid body with an attached point $O$ be moving in a frame of reference with the origin at its center $O$, which, in its turn, is moving and rotating according to the given law.

Denoting angular and linear velocities of the moving frame of reference in projections on axes, bound to the body, by $\boldsymbol{\Omega}$ and $\boldsymbol{V}$, correspondingly, we'll write the Lagrange function of a potential system in the form

$$L = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + (\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\Omega}) - m(\boldsymbol{W}, \boldsymbol{r}) - U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}). \tag{6.12}$$

Here $\boldsymbol{\omega}$ is an angular velocity of the body, $\boldsymbol{W} = \frac{d}{dt}\boldsymbol{V}$ is an acceleration of a reference point of the moving system, $\mathbf{I}$ is a tensor of the body inertia relatively to the point $O$, $m$ is a total mass, $\boldsymbol{r}$ is a position vector of the center-of-mass, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are unit vectors of the moving frame of reference. All the vectors mentioned are projected on the body axes, and $\boldsymbol{\Omega}$, $\boldsymbol{V}$ can be considered as given functions of time.

Angular momentum and Hamiltonian of system (6.12) are determined as follows

$$\begin{gathered} \boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \mathbf{I}(\boldsymbol{\omega} + \boldsymbol{\Omega}), \\ H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) - (\boldsymbol{M}, \boldsymbol{\Omega}) + m(\boldsymbol{W}, \boldsymbol{r}) + U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \end{gathered} \tag{6.13}$$

then the equations of motion have the form (5.8). Such systems may be exemplified by gyroscopes and pendants, placed in the aircraft and artificial satellites, being in the specified motion.



## 5. Rigid Body Motion on a Smooth Plane

Except for the Euler – Poisson equations, an interesting mechanical example, where the equations, describing the evolution of vectors $\boldsymbol{\omega}, \boldsymbol{\gamma}$ (or $\boldsymbol{M}, \boldsymbol{\gamma}$), get separated, is represented by the problem, concerning the rigid body motion on a smooth plane with the potential, depending on the distance to this plane.

Generally speaking, the system in absolute motion possesses five degrees of freedom, but in view of the fact that the reaction of the plane at perfect sliding is perpendicular to it, two projections of the system momentum on this plane are preserved. Choosing the frame of reference, rigidly bound to the body and having its origin in the center-of-mass (thereby eliminating its horizontal uniform rectilinear displacement) we'll obtain the Lagrange function, for motion in a potential field $U(\boldsymbol{\gamma})$

$$L = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + \frac{1}{2}m(\boldsymbol{\omega}, \boldsymbol{r} \times \boldsymbol{\gamma})^2 - U(\boldsymbol{\gamma}), \tag{6.14}$$

where $\mathbf{I}$ is a tensor of the body inertia relatively to the center-of-mass, $m$ is a mass of the body, $\boldsymbol{\omega}$ are projections of the angular velocity on the axes, bound to the body, $\boldsymbol{\gamma}$ is a vector normal to the plane in the same system of axes, and $\boldsymbol{r}$ is a vector directed from the point of contact to the body center-of-mass (see fig. 9).

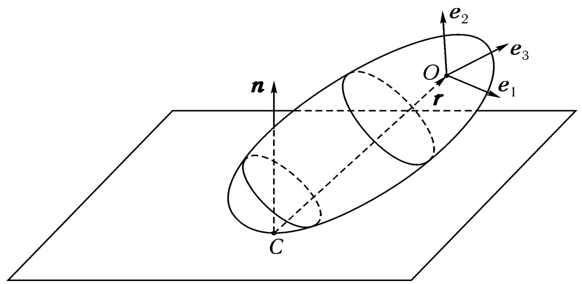



Figure 9. A rigid body on a smooth plane.

If a body is convex in every point and always touches the plane only at one point, then vector $\boldsymbol{r}$ is uniquely expressed in terms of vector $\boldsymbol{\gamma}$ by means of the Gauss projection of the body surface on the unit sphere

$$\boldsymbol{\gamma} = -\frac{\operatorname{grad} F(\boldsymbol{r})}{|\operatorname{grad} F(\boldsymbol{r})|}, \tag{6.15}$$

where $F(\boldsymbol{r}) = 0$ is an equation, specifying the body surface. For nonconvex bodies equation (6.15) allows several solutions of $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\gamma})$ and, as a rule, requires consideration of additional equations of impact.

For the ellipsoid with the principal semi-axes $b_1$, $b_2$, $b_3$ it's easy to obtain that

$$\boldsymbol{r} = k(b_1^2\gamma_1,\, b_2^2\gamma_2,\, b_3^2\gamma_3), \quad k = (b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2)^{-1/2}. \tag{6.16}$$

Applying the Legendre transformation, from (6.14) we obtain

$$\begin{gathered} \boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega}, \quad \mathbf{J} = \mathbf{I} + m\boldsymbol{a}\otimes\boldsymbol{a}, \\ H = \tfrac{1}{2}(\mathbf{I}\mathbf{A}\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + \tfrac{1}{2}m(\boldsymbol{a},\, \mathbf{A}\boldsymbol{M})^2 + U(\boldsymbol{\gamma}), \end{gathered} \tag{6.17}$$

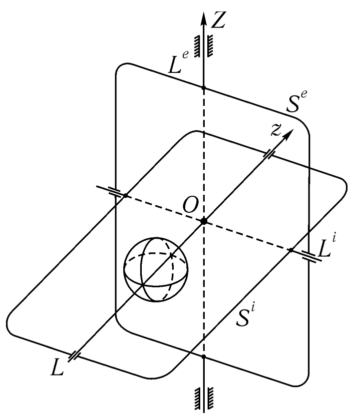



Figure 10. A gyroscope in a gimbal. The external frame of the gimbal $S^e$ rotates around axis $L^e$, fixed in space. The rotation axis $L^i$ of an internal frame $S^i$ is fastened to the axis $L^e$. The axis of rotation of the rigid body (gyroscope) $L$ is fixed on the internal frame.

where $\boldsymbol{a} = \boldsymbol{r}\times\boldsymbol{\gamma}$, $\mathbf{A} = \mathbf{J}^{-1}$. According to (4.16), the Poisson bracket of variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$ is determined by algebra $e(3)$.

For the gravity field the potential energy of the body can be represented as

$$U(\boldsymbol{\gamma}) = mg(\boldsymbol{r},\, \boldsymbol{\gamma}),$$

$g$ being a free fall acceleration. It doesn't make a problem, as well, to generalize the system by adding to the body a rotor with gyrostatic moment $\boldsymbol{K}$, then Hamiltonian (6.17) acquires terms, linear with respect to $\boldsymbol{M}$.

If a body is a sphere with an arbitrary ellipsoid of inertia, but the center-of-mass coincides with the geometric center, then we obtain either Euler system (in case $\boldsymbol{K} = 0$) (see § 2 ch. 2), or Joukovskiy–Volterra system (in case $\boldsymbol{K} \neq 0$) (see § 7 ch. 2).

### 6. A Gyroscope in a Gimbal

*A gyroscope in a gimbal* is a system of several bodies, connected by means of cylindrical joints (see fig. 10) [119].

Consider the case, which is very frequent in engineering, when axes $L_e$ and $L_i$, $L$ and $L_i$ are mutually perpendicular and cross at the single point $O$ [119]. Choose a fixed frame of reference with the origin at the point $O$ and the axis $OZ$, directed along the axis of rotation $L^e$; attach to the body a moving frame of reference with the origin at the point $O$ and the axis $Oz$, directed along the axis $L$. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ be projections of the unit vectors of a fixed space on the axes bound to the body, $\boldsymbol{\gamma}$ being a vector corresponding to the axis $OZ$.



The Lagrange function of a gyroscope in a potential field can be written as

$$\begin{gathered} L = \frac{1}{2}(\boldsymbol{\omega},\, \mathbf{I}\boldsymbol{\omega}) + \frac{1}{2}I^e\left(\frac{\omega_1\gamma_1 + \omega_2\gamma_2}{\gamma_1^2+\gamma_2^2}\right)^2 + \\ + \frac{1}{2(\gamma_1^2+\gamma_2^2)}\left[I_1^i(\omega_1\gamma_2 - \omega_2\gamma_1)^2 + (\omega_1\gamma_1+\omega_2\gamma_2)^2\Big(I_2^i + I_3^i\frac{\gamma_3^2}{\gamma_1^2+\gamma_2^2}\Big)\right] - U(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}), \end{gathered} \tag{6.18}$$

where $\boldsymbol{\omega} = (\omega_1,\, \omega_2,\, \omega_3)$ are projections of the angular velocity on the axes, bound to the body, $\mathbf{I}$ is a tensor of the moments of inertia of the frame $S^e$ relatively to the point $O$, $I^e$ is a moment of inertia of the frame $S^e$ with respect to the axis $L^e$, $I_1^i$, $I_2^i$, $I_3^i$ are principal moments of inertia of the internal frame.

The Hamiltonian form of system (6.18) can be obtained by means of Legendre transformation (4.14). However, the Hamiltonian function of the system is too cumbersome in the general case; we can show its form, provided that the body is dynamically symmetrical with respect to the axis $L$ ($I_1 = I_2$):

$$\begin{gathered} H = \frac{1}{2}a_3M_3^2 + \frac{1}{2}a_1k(M_1^2+M_2^2) + \\ + \frac{1}{2}a_1^2k\left[I_1^i(M_1\gamma_1 + M_2\gamma_2)^2 + \left(I^e + (I_3^i - I_2^i)\frac{\gamma_3^2}{\gamma_1^2+\gamma_2^2}\right)(M_1\gamma_2 - M_2\gamma_1)^2\right] + \\ + U(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}), \\ k = \left((1 + a_1I_1^i)\left(1 + a_1I^e + a_1(I_3^i - I_2^i)\frac{\gamma_3^2}{\gamma_1^2+\gamma_2^2}\right)\right)^{-1}, \end{gathered} \tag{6.19}$$

where $\mathbf{A} = \mathbf{I}^{-1} = \operatorname{diag}(a_1,\, a_2,\, a_3)$.

We can show the equations of two more remarkable problems, connected with the motion of a rigid body in fluid. Their systematical study we postpone till ch. 3. The detailed derivation can be found in § 2 ch. 5.

**Historical comment.** The Foucault pendulum and gyroscope were offered by the famous French physicist Léon Foucault (1819–1868) as the instruments for observing the Earth rotation with respect to the absolute space.

An idea of the pendulum turned out to be more fruitful and is used as a demonstration in the secondary school physics course. Nevertheless, the complete analysis of a nonlinear model — only little swings are usually considered — is still absent. It cannot be integrated. Kamerlingh-Onnes, who has discovered superconductivity, was one of the first, who tried to take into account the swing amplitude finiteness.

Experiments with a gyroscope, conducted by Foucault (1852), did not bring any satisfactory results: due to friction, the gyroscope was losing its velocity too rapidly, and there arose chaotic precession of the rotation axis. The idea was that the symmetrical gyroscope axis should remain constant in fixed space. It would allow to measure the Earth rotation. Nevertheless, Foucault, while creating his gyroscope, offered a series of technical innovations. For example, it was using *a gimbal*, which, by the way, was known before D. Cardano (1501–1576) to the French architect U. de Goncourt in thirteenth century. Foucault also noticed that if the gyroscope is deprived of one degree of freedom, then its rotation axis tends to coincide with the angular velocity of the transportation rotation of the gimbal bottom, connected with the angular velocity of the Earth rotation. This makes it possible to determine the direction to the North pole and the latitude of the instrument position.

Having analyzed two characteristic positions of the gyroscope, possessing two degrees of freedom, relatively to the surface of the rotating Earth, Foucault has invented two new devices — *a gyrocompass* and *a gyrolatitude*, which found their technical realization only at the end of the nineteenth and the beginning of the twentieth century (Obris, Sperry, Anscshütz and others) in the mechanisms of torpedo and aircraft control. L. Foucault has also invented the name — a gyroscope, which literally means “observation of rotation”. Various applications of the gyroscope are discussed in more details in the books by R. Grammel [66] and K. Magnus [119].

### 7. Rigid Body Motion in a Perfect Incompressible Fluid (Kirchhoff’s Equations)

In this case the Hamiltonian of the system is (see § 2 ch. 5)

$$H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + (\boldsymbol{M},\, \mathbf{B}\boldsymbol{p}) + \frac{1}{2}(\boldsymbol{p},\, \mathbf{C}\boldsymbol{p}) + U(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}, \boldsymbol{x}). \tag{6.20}$$

Here $\mathbf{A}$, $\mathbf{C}$ are symmetric matrices (associated moments of inertia and masses, defined by the geometry and inertial properties of the body), $\mathbf{B}$ is an arbitrary matrix, which can be chosen to be equal to zero for the body, possessing three mutually perpendicular planes of symmetry, intersecting at the body center-of-mass. The equations of motion have the form (5.8). It should be mentioned that usually Kirchhoff’s equations are referred to particular case (6.20), when $U(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}, \boldsymbol{x}) \equiv 0$, i. e., the case of inertial motion. In this case the system of equations for $(\boldsymbol{M}, \boldsymbol{p})$ becomes closed (these are the Euler – Poincaré equations on $e(3)$) and the analysis is in many respects similar to the Euler – Poincaré equations (for more details see § 9 ch. 3).



### 8. A Heavy Body Fall in Fluid, the Chaplygin Equations

Consider the motion of a body, whose three mutually perpendicular planes of the symmetry intersect at the center-of-mass [176], in fluid in the uniform gravity field. It’s easy to show that in this case the center-of-mass of the body coincides with the center-of-mass of the fluid volume, being displaced. The Hamiltonian of the system is

$$H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}(\boldsymbol{p},\, \mathbf{C}\boldsymbol{p}) - \mu(\boldsymbol{x},\, \boldsymbol{\gamma}). \tag{6.21}$$

As we can show from equations (5.8), the total momentum of the system is defined by

$$\boldsymbol{p} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + (P_3 - \mu t)\boldsymbol{\gamma},$$

where $\boldsymbol{P} = (P_1,\, P_2,\, P_3) = \mathrm{const}$ is an incentive momentum, which is a vector integral of motion.

Let an incentive impulse be equal to zero: $\boldsymbol{P} = 0$. In this case the system of equations, describing the evolution of variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$, becomes separated, and the Hamiltonian of such a reduced system will explicitly depend on time

$$H^* = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}\mu^2 t^2(\boldsymbol{\gamma},\, \mathbf{C}\boldsymbol{\gamma}), \tag{6.22}$$

where, as it’s clear from the explanation above, $\mathbf{A}$ is a tensor of associated moments of inertia, and $\mathbf{C}$ is a tensor of associated masses (see also [95]).

The equations of motion of system (6.22) are called *the Chaplygin equations* [176].

There exist two particular cases of system (6.22), for which equations of motion may be reduced to the pendulum-type equation ($\ddot{x} = at^2 \sin x$). The first case corresponds to plane-parallel motion of a body in fluid of a plate, and the second to motion of an axially symmetric rigid body. The latter case is discussed in detail in § 1 ch. 3.

Nonintegrability of system (6.22), both in a general, and in axially symmetric and plane cases is shown in the paper [96].

**Comments. 1.** For the first time, the equations of system (6.22) were obtained by S. A. Chaplygin in his student paper (1890), which was published much more later in his collected works (1933, v. 1). It's highly probable that Chaplygin refrained from publishing his results at once, because he could not integrate these equations explicitly. Besides, V. A. Steklov obtained these equations independently and published them in his well-known book [160] (1893), where he also mentioned some qualitative results about the body behavior.

**2.** In the paper [175] S. A. Chaplygin has also indicated the case, when the gravity force is balanced by the Archimedian force (the average density of the body equals the density of fluid), but the center-of-mass of the body does not coincide with the center-of-mass of the fluid volume, having been displaced. The body is acted on by a couple, and its total momentum in absolute space is preserved:

$$\boldsymbol{P} = P_1\boldsymbol{\alpha} + P_2\boldsymbol{\beta} + P_3\boldsymbol{\gamma},$$

where $\boldsymbol{P} = (P_1, P_2, P_3) = \mathrm{const}$. If "the initial impulse" is taken along the vertical axis: $\boldsymbol{P} = P\boldsymbol{\gamma}$, then the evolution of vectors $\boldsymbol{M}$, $\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}$ is directed along the gravity field) is described by means of a system on $e(3)$ with the Hamilton function

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}P^2(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mu(\boldsymbol{r}, \boldsymbol{\gamma}), \tag{6.23}$$

where $\boldsymbol{r}$ is a vector, connecting the center-of-mass of the body with the center-of-mass of the fluid volume, having been displaced. It also holds true in the general case, when a symmetrical body is moving in fluid under the action of balanced forces (there are only moments of forces): the equations of motion of a rigid body with a fixed point (the center-of-mass) become separated.

## § 7. Theorems about Integrability and Techniques of Integration

Differential equations, including the Hamiltonian ones, are usually divided into integrable and nonintegrable. At the same time, as it was noticed by *G. Birkhoff* [13], *"however, if we try to define integrability exactly, we may encounter with the possibility of many various definitions, each one having a certain theoretical interest"*. This statement by Birkhoff, who considered a dynamical problem to be solved, if we are presented with a certain algorithm for description of behavior of all its paths, contains the indication of the connection of integrability with a special regular character of motion in a phase space.



Such a regularity may be achieved, if the system possesses the sufficient number of conservation laws — first integrals, fields of symmetries or other tensor invariants.

Here we are going to state several principal approaches to the integrability of Hamiltonian and general differential equations. They concern determination of the system solutions in *quadratures*. Solving a system in quadratures is equivalent to representing its solution by means of the finite number of "algebraic" operations (including inversion of functions) and "*quadratures*" — computing integrals of the known functions. Various aspects of integrability are illustrated in the reviews [74, 136, 8] (see also [97]).

### 1. Hamiltonian Systems. The Liouville – Arnold Theorem

The following theorem connects integrability of the Hamiltonian system in quadratures with the presence of a sufficiently large set of its first integrals.

**Theorem 2.** *Suppose that on a symplectic manifold* $M^{2n} = (\boldsymbol{p}, \boldsymbol{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ *there are given* $n$ *functions in involution*

$$F_1, \ldots, F_n\colon \{F_i, F_j\} \equiv 0,\ i, j = 1, \ldots, n.$$

*Then, suppose that on* $M_f$*, which is a manifold of the level of integrals* $\{\boldsymbol{x} \in M^{2n} : F_i = c_i,\ i = 1, \ldots, n\}$*,* $n$ *functions* $F_i$ *are independent. Then:*

1. $M_f$ *is a smooth manifold, invariant with respect to the phase flow with the Hamiltonian function* $H = F_1$.

2. *If the manifold* $M_f$ *is both connected and compact, then it's diffeomorphic to an* $n$*-dimensional torus* (fig. 11)

$$T^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$$

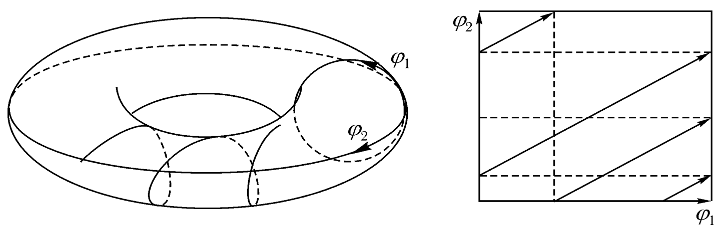



Figure 11. Quasiperiodic motion on torus and on its development.

3. *The phase flow with the Hamilton function $H$ defines conditionally periodic motion on $M_f$, i. e., in terms of certain angular coordinates $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ we have equations*

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}(c_1, \ldots, c_n) = (\omega_1, \ldots, \omega_n).$$

4. *Canonical equations with the Hamilton function $H$ are integrated in quadratures.*

Remark. The simplified version of this theorem (where only integrability in quadratures is asserted) was formulated by Bur and generalized by G. Liouville. Its classical proof is given, for example, in treatise by E. Whittaker [167]. The reduced statement of the theorem belongs to V. I. Arnold [6].

In this case the Hamiltonian system is called integrable according to Liouville (or completely integrable). One can show that for such a system in the vicinity of each torus there exist *variables*, called "*action-angle*" $(\boldsymbol{I}, \boldsymbol{\varphi} \bmod 2\pi) = (I_1, \ldots, I_n, \varphi_1 \mod 2\pi, \ldots, \varphi_n \mod 2\pi)$, where Hamiltonian $H(\boldsymbol{I})$ doesn't depend on angular variables $\boldsymbol{\varphi} \mod 2\pi$, and the equations of motion have the form

$$\dot{\boldsymbol{I}} = -\frac{\partial H}{\partial \boldsymbol{\varphi}} = 0, \qquad \dot{\boldsymbol{\varphi}} = \frac{\partial H}{\partial \boldsymbol{I}} = \boldsymbol{\omega}(\boldsymbol{I}),$$

Hence, $\boldsymbol{I}(t) = \boldsymbol{I}_0,\ \boldsymbol{\omega}(\boldsymbol{I}) = \boldsymbol{\omega}(\boldsymbol{I}_0) = (\omega_1, \ldots, \omega_n)$.

Variables action $\boldsymbol{I}$ "number" invariant tori $\mathbb{T}^n = M_f$ in $M^{2n}$, and variables angle $\boldsymbol{\varphi}$ are uniformly changing on them with $n$ various frequencies $\omega_1, \ldots, \omega_n$. Such a motion is called *quasi-periodic*. Variables action-angle are very important in the perturbation theory.

In some cases the number of independent variables may exceed $n = \frac{1}{2}\dim M^{2n}$. And not all of them are in involution and lead to *noncommutative integrability of the system*. In this case the invariant manifold $M_f$, if it's compact, is a less than $n$-dimensional torus [132].

Rigid body dynamics contains both commutative and noncommutative sets of integrals. The latter are used for degenerate systems, possessing redundant symmetries (dynamically symmetrical and spherical tops). In these cases it is also said that the system is *superintegrable*.

Remark 1. The Jacobi theorem, according to which the Poisson bracket of two integrals is also an integral, tells us that their complete family forms a certain, as a matter of fact, infinite Lie algebra. One of such examples is considered in the appendix.



Investigating algebra of integrals is also necessary for various methods of the system reduction, which consists in reduction to the less number of degrees of freedom (§ 1 ch. 4). The connection of noncommutative integrability with the Dirac reduction is discussed in the book [31] (see also [32]).

Remark 2. The supposition of compactness and connectedness of $M_f$ usually holds true in rigid body dynamics because of compactness of a configurational space, for example, the one being the group $SO(3)$, and limitations, imposed by the energy integral on the momenta.

Remark 3. If the integrals on $M_f$ become dependent, their general level, is not a smooth manifold. In the space of constant first integrals $(c_1, \ldots, c_n)$ these values form *bifurcation surfaces*, whose explicit form has been studied for the most of known integrable systems [25] (see ch. 2).

From the theoretic perspective the integrability of the Hamiltonian system in quadratures may not necessarily be connected with the presence of necessary quantity of the first integrals. It may be stipulated by the fields of symmetries, various invariant forms and other tensor laws of conservation [31, 83]. However, substantial examples refer only to particular combinations of such tensor invariants. Now we are going to consider one more typical example.

## 2. The Last Multiplier Theory. The Euler – Jacobi Theorem

Many problems of rigid body dynamics may be integrated also by means of another method, tracing back to Euler and Jacobi. We are speaking about the last multiplier theory, where, in order to obtain the integrability in quadratures, except for the sufficient quantity of the first integrals, it's necessary to determine the existence of a certain *invariant measure*. The advantage of this method is the possibility of its application not only to Hamiltonian systems, but, as a matter of fact, to arbitrary systems, for example, to nonholonomic ones. A series of nonholonomic systems, possessing nontrivial measure and integrated according to the last multiplier theory, was indicated by S. A. Chaplygin [179]. Though we don't consider them in the present book, we'd like to emphasize the fact that in nineteenth century the integrability of the most problems of rigid body dynamics was understood exactly as the Euler – Jacobi integrability, because the Hamiltonian structure, for example, of Euler – Poisson equations (see § 1 ch. 2), was not understood thoroughly well. We are going to stop here and discuss this method in more details.

Consider an arbitrary autonomous system of differential equations in $\mathbb{R}^n$

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n. \tag{7.1}$$

Let $g^t$ be its phase flow. In the general case for the integrability of this system one needs to know not less than $n-1$ first integrals. However, if equation (7.1) has an integral invariant with the smooth density $\mu(\boldsymbol{x})$, that is for any measurable region $D \subset \mathbb{R}^n$ and for all $t$ the equality

$$\int\limits_{g^t(D)} \mu(\boldsymbol{x})\, d\boldsymbol{x} = \int\limits_{D} \mu(\boldsymbol{x})\, d\boldsymbol{x},$$

is satisfied, then for the integrability of system (7.1) it's sufficient to know $n-$ $-2$ first integrals. Let's remember that according to the Liouville theorem the smooth function $\mu\colon \mathbb{R}^n \to \mathbb{R}$ is the density of the integral invariant $\int \mu(\boldsymbol{x})\, d\boldsymbol{x}$ if and only if

$$\operatorname{div}(\mu \boldsymbol{v}) = 0. \tag{7.2}$$

If $\mu(\boldsymbol{x}) > 0$ under all $\boldsymbol{x}$, then formula (7.2) defines a certain *measure* in $\mathbb{R}^n$, invariant with respect to the operation $g^t$. The presence of measure makes it easier to integrate differential equations. Euler has called $\mu$ *an integrating multiplier* (it's called *the last multiplier* as well).

The following statement is valid: *the Euler–Jacobi theorem* concerning the last multiplier [8, 91].

**Theorem 3.** *Suppose that system* (7.1) *of equations with the integrating multiplier* $\mu$ *possesses* $n-2$ *first integrals* $F_1, \ldots, F_{n-2}$. *Let functions* $F_1, \ldots, F_{n-2}$ *be independent on an invariant manifold* $M_c = \{\boldsymbol{x} \in \mathbb{R}^n \colon F_s(\boldsymbol{x}) = c_s,\ 1 \leqslant s \leqslant n-2\}$. *Then*

1. *the solutions of equation* (7.1)*, situated on* $M_c$*, are in quadratures .*

*If* $L_c$ *is a connected compact component and* $\boldsymbol{v}(\boldsymbol{x}) \neq 0$ *on* $L_c$*, then*

2. $L_c$ *is a smooth manifold, diffeomorphic to 2-dimensional torus,*

3. *on* $L_c$ *one may choose angular coordinates* $\varphi_1, \varphi_2 \mod 2\pi$ *such that equation* (7.1)*, expressed in terms of these variables, on* $L_c$ *would have the form*

$$\dot{\varphi}_1 = \frac{\lambda_1}{\Phi(\varphi_1, \varphi_2)}, \qquad \dot{\varphi}_2 = \frac{\lambda_2}{\Phi(\varphi_1, \varphi_2)},$$

*where* $\lambda_1, \lambda_2 = \mathrm{const}$*, and* $\Phi$ *is a smooth positive function with a period* $2\pi$ *with respect to* $\varphi_1$ *and* $\varphi_2$.

The function $\Phi(\varphi_1, \varphi_2)$, giving an invariant measure, is not reduced to a constant, and motion on torus, though occurring along rectilinear windings (fig. 11), is not uniform. We can point that in case of a Hamiltonian system such a reduction is always possible. This is the consequence of the Liouville–Arnold theorem.



For general systems (7.1), for example dissipative ones, the measure is, as a rule, absent, and the question of their integrability constitutes a separate problem (§ 1 ch. 5). Since here is no general technique, the system may behave differently, depending on certain sets of conservation laws (tensor invariants), which are not autonomous.

### 3. Separation of Variables. The Hamilton–Jacobi Method

The explicit solution of Hamiltonian equations in canonical form in most cases can be obtained by means of *method of separation of variables* [183]. In this case the problem of integration of $n$-power Hamiltonian system is reduced to obtaining solution of *the Hamilton–Jacobi equation* in partial derivatives

$$H\Big(\frac{\partial S}{\partial \boldsymbol{q}}, \boldsymbol{q}\Big) = \alpha_1, \tag{7.3}$$

which depends on $n$ constants $S(\boldsymbol{q}, \alpha_1, \ldots, \alpha_n)$ and satisfies the condition of nondegeneracy

$$\det\Big\|\frac{\partial^2 S}{\partial q_i \partial \alpha_j}\Big\| \neq 0.$$

Consider the function $S(\boldsymbol{q}, \alpha_1, \ldots, \alpha_n)$, which in this case is called *a complete integral* of equation (7.3), as a generating function of a canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{\beta}, \boldsymbol{\alpha})$:

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}, \qquad \boldsymbol{\beta} = \frac{\partial S}{\partial \boldsymbol{\alpha}}. \tag{7.4}$$

For new canonical variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, according to (7.3), we'll obtain equations of motion in the form [183, 128]

$$\dot{\alpha}_i = -\frac{\partial H}{\partial \beta_i} = 0, \qquad \dot{\beta}_i = \frac{\partial H}{\partial \alpha_i} = \delta_{1i}, \qquad i = 1, \ldots, n,$$

where $\delta_{ij}$ is a Kroneker symbol. These equations can be easily integrated:

$$\alpha_i = \alpha_i^0, \quad \beta_i = \delta_{1i} t + \beta_i^0, \tag{7.5}$$

where $\alpha_i^0, \beta_i^0 = \mathrm{const}$. Thus, (7.5), together with (7.4), gives the solution of canonical equations $\boldsymbol{q}(t), \boldsymbol{p}(t)$ in the form of system of algebraic equations.

*Variables separate* if one succeeds in selecting such coordinates on a configurational space, which have the complete integral represented in the form

$$S(\boldsymbol{q}, \boldsymbol{\alpha}) = \sum_{k=1}^{n} S_k(q_k, \alpha_1, \ldots, \alpha_n). \tag{7.6}$$

According to Jacobi, the method of separation of variables consists in finding for a problem such a system of coordinates (generally speaking, curvilinear ones), in terms of which (7.6) is valid. Jacobi also found one remarkable substitution, which led him to elliptic coordinates and allowed to integrate the problem about geodesics on ellipsoid — even in case of many dimensions. He also suggested that we should reverse the situation and "having found some remarkable substitution, look for the problems, where it can be successfully applied" [183].

Remark. Degenerate systems (with the redundant set of integrals), for example, a harmonic oscillator, the Kepler problem and others, may have several coordinate systems, where variables separate.

As examples, consider classical problems: the Jacobi proplem about geodesics on a three-axial ellipsoid, and the Neumann problem about motion of the point on a sphere in a quadratic potential. They are associated with two various, but mutual, integrable Clebsch cases in Kirchhoff's equations (see § 9 ch. 3). Their integration, and also elliptic and sphero-conical coordinates, appearing in the process, are universal in the theory of integrable systems. All the known problems, allowing the separation of variables (on a configurational space), are solved by using these coordinates or their degenerations.

**Geodesic flow on an ellipsoid (Jacobi problem) [183].** Let an ellipsoid in a three-dimensional space $\mathbb{R}^3$ with Cartesian coordinates $x_1$, $x_2$, $x_3$ be given by the equation

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\frac{x_3^2}{a_3}=1, \tag{7.7}$$

where $a_1>a_2>a_3>0$ are squares of the principal semi-axes.

Elliptic coordinates $\lambda_1$, $\lambda_2$, $\lambda_3$ in $\mathbb{R}^3$ are defined as roots of the cubic equation

$$f(\lambda)=\frac{x_1^2}{a_1-\lambda}+\frac{x_2^2}{a_2-\lambda}+\frac{x_3^2}{a_3-\lambda}=1, \tag{7.8}$$

where $\lambda_3<a_3<\lambda_2<a_2<\lambda_1<a_1$.

Cartesian coordinates are expressed in terms of elliptic ones by means of residues of function $f(\lambda)$ (7.8) at the points $a_1$, $a_2$, $a_3$, according to the formulae

$$x_1^2=\frac{(a_1-\lambda_1)(a_1-\lambda_2)(a_1-\lambda_3)}{(a_2-a_1)(a_3-a_1)},\quad x_2^2=\frac{(a_2-\lambda_1)(a_2-\lambda_2)(a_2-\lambda_3)}{(a_1-a_2)(a_3-a_2)},$$
$$x_3^2=\frac{(a_3-\lambda_1)(a_3-\lambda_2)(a_3-\lambda_3)}{(a_1-a_3)(a_2-a_3)}.$$



In terms of new variables ellipsoid (7.7) is given by the equation $\lambda_3=0$, and $\lambda_1$, $\lambda_2$ define the system of orthogonal coordinates on it. Rewriting the Hamiltonian of free motion of a unit mass point on ellipsoid (7.7) in these coordinates, we obtain

$$\begin{gathered}H=\frac{2}{\lambda_1-\lambda_2}\Big(g(\lambda_1)p_1^2+g(\lambda_2)p_2^2\Big),\\ g(\lambda)=(a_1-\lambda)(a_2-\lambda)(a_3-\lambda),\end{gathered} \tag{7.9}$$

which means that variables separate.

Using the expression for canonical momenta

$$p_1=(\lambda_1-\lambda_2)\frac{\lambda_1\dot\lambda_1}{4(a_1-\lambda_1)(a_2-\lambda_1)(a_3-\lambda_1)},$$
$$p_2=(\lambda_2-\lambda_1)\frac{\lambda_2\dot\lambda_2}{4(a_1-\lambda_2)(a_2-\lambda_2)(a_3-\lambda_3)},$$

we obtain the equations of motion in the form

$$\begin{gathered}\frac{d\lambda_1}{\sqrt{R(\lambda_1)}}=\frac{dt}{\lambda_1-\lambda_2},\quad \frac{d\lambda_2}{\sqrt{R(\lambda_2)}}=\frac{dt}{\lambda_2-\lambda_1},\\ R(\lambda)=-\frac{(\lambda-\alpha_1)(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}{\lambda},\end{gathered} \tag{7.10}$$

where $\alpha_1$ is a separation constant, satisfying inequalities $a_3<\alpha_1<a_1$. Equations (7.10) are associated with Abel, Jacobi and Kowalevskaya, who used them for integration in elliptic functions.

Remark. *Abel–Jacobi equations* (7.10) are also written in a bit different form

$$\frac{d\lambda_1}{\sqrt{R(\lambda_1)}}+\frac{d\lambda_2}{\sqrt{R(\lambda_2)}}=0,\quad \frac{\lambda_1\,d\lambda_1}{\sqrt{R(\lambda_1)}}+\frac{\lambda_2\,d\lambda_2}{\sqrt{R(\lambda_2)}}=dt. \tag{7.11}$$

In this case solutions for the Hamilton–Jacobi equations are represented in the form

$$S(\lambda_1,\lambda_2,\alpha_1,\alpha_2)=\sqrt{\frac{\alpha_2}{2}}\Big(\int\frac{\lambda_1-\alpha_1}{\sqrt{R(\lambda_1)}}\,d\lambda_1+\int\frac{\lambda_2-\alpha_1}{\sqrt{R(\lambda_2)}}d\lambda_2\Big), \tag{7.12}$$

where $\alpha_2=h$ is energy. The path and the law of motion can be determined from algebraic equations:

$$\frac{\partial S}{\partial\alpha_1}=\beta_1,\quad \frac{\partial S}{\partial\alpha_2}=t+\beta_2,\quad \beta_1,\beta_2=\text{const}. \tag{7.13}$$

**The system with a quadratic potential on a sphere (the Neumann problem) [251].** Let the equation

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{7.14}$$

define a sphere in three-dimensional space, and the potential energy of a unit mass particle in Cartesian coordinates have the form

$$U(\boldsymbol{x}) = \frac{1}{2}(a_1x_1^2 + a_2x_2^2 + a_3x_3^2), \quad 0 < a_3 < a_2 < a_1. \tag{7.15}$$

Sphero-conical coordinates $\lambda_1$, $\lambda_2$ on sphere (7.14) are defined as roots of the quadratic equation

$$f(\lambda) = \frac{x_1^2}{a_1 - \lambda} + \frac{x_2^2}{a_2 - \lambda} + \frac{x_3^2}{a_3 - \lambda} = 0, \tag{7.16}$$

satisfying inequalities $a_3 < \lambda_2 < a_2 < \lambda_1 < a_1$. The Cartesian coordinates are expressed in terms of sphero-conical coordinates as follows

$$x_1^2 = \frac{(a_1-\lambda_1)(a_1-\lambda_2)}{(a_2-a_1)(a_3-a_1)},\ x_2^2 = \frac{(a_2-\lambda_1)(a_2-\lambda_2)}{(a_1-a_2)(a_3-a_2)},\ x_3^2 = \frac{(a_3-\lambda_1)(a_3-\lambda_2)}{(a_1-a_3)(a_2-a_3)}. \tag{7.17}$$

The Hamiltonian of a particle, possessing potential (7.15) in terms of sphero-conical coordinates, has the form

$$\begin{aligned} H = \Big(\frac{2(\lambda_1 - a_1)(\lambda_1 - a_2)(\lambda_1 - a_3)}{\lambda_1 - \lambda_2}p_1^2 + \lambda_1\Big) + \\ + \Big(\frac{2(\lambda_2 - a_1)(\lambda_2 - a_2)(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}p_2^2 + \lambda_2\Big), \end{aligned} \tag{7.18}$$

where momenta $p_1$, $p_2$, canonically conjugate with variables $\lambda_1$, $\lambda_2$, are expressed in terms of velocities, according to the formulae

$$\begin{aligned} p_1 &= \frac{(\lambda_1 - \lambda_2)\dot{\lambda}_1}{4(\lambda_1 - a_1)(\lambda_1 - a_2)(\lambda_1 - a_3)}, \\ p_2 &= \frac{(\lambda_2 - \lambda_1)\dot{\lambda}_2}{4(\lambda_2 - a_1)(\lambda_2 - a_2)(\lambda_2 - a_3)}. \end{aligned} \tag{7.19}$$

Separating the variables, we obtain the Abel–Jacobi equations, specifying the evolution of $\lambda_1$, $\lambda_2$:

$$\frac{d\lambda_1}{\sqrt{R(\lambda_1)}} = \frac{dt}{\lambda_1 - \lambda_2}, \quad \frac{d\lambda_2}{\sqrt{R(\lambda_2)}} = \frac{dt}{\lambda_2 - \lambda_1},$$
$$R(\lambda) = -(\lambda^2 + 2\alpha_2\lambda + 2\alpha_1)(\lambda - a_1)(\lambda - a_2)(\lambda - a_3),$$



where $\alpha_1$, $\alpha_2 = h$ are constants of separation. The Hamilton–Jacobi equation solution is as follows

$$\begin{aligned} S(\lambda_1, \lambda_2, \alpha_1, \alpha_2) = \int \sqrt{-\frac{\lambda_1^2 + 2\alpha_2\lambda_1 + \alpha_1}{4(\lambda_1 - a_1)(\lambda_1 - a_2)(\lambda_1 - a_3)}}d\lambda_1 + \\ + \int \sqrt{-\frac{\lambda_2^2 + 2\alpha_2\lambda_2 + \alpha_1}{4(\lambda_2 - a_1)(\lambda_2 - a_2)(\lambda_2 - a_3)}}d\lambda_2. \end{aligned}$$

**Comments. 1.** As a rule, though not universally, the equations of motion for separating variables may be represented in the Abel–Jacobi form (7.11). It's known that any solution of such equations can be represented in terms of a theta-function (to speak more formally: linearized by means of the Abelian transformation on the Jacobian of the hyperelliptic curve). S. V. Kowalevskaya was the first to carry out such kind of linearization for the case she had discovered. To do that, she applied the theory of theta-functions of two variables, just developed by Rosenhine and Königsberger. Such a linearization resulted in a remarkable fact that the general solution of a system extends to single-valued holomorphic functions into the complex domain of time, i. e. the solution has only poles as singularities.

The expression of the general solution for the majority of integrable problems of rigid body dynamics in single-valued elliptic (in a complex sense) functions of time is conditioned by the fact that the general level of the first integrals, representing the intersection of rather simple algebraic surfaces, like quadric ones, allows extension into the complex domain to the Abelian manifolds (Abelian tori), allowing parameterization by means of theta-functions. It is studied in projective and algebraic geometry, and the systems, as they are, are called *algebraically integrable*. However, the general solution may be single-valued not on the complex plane of time, but on its finitely sheeted covering (see the Goryachev–Chaplygin case, § 5 ch. 2).

**2.** The problem about the separation of variables distinctly stated by K. Jacobi in his "Lectures on Dynamics" (1842–43) [183], is still under serious investigations. J. Liouville and P. Shtekkel found more general forms of Hamiltonians, allowing separation of variables. It turned out that if only configurational space transformations (point transformations) be used, the separation of variables is closely connected with the presence of the complete set of first integrals, quadratic with respect to momenta. For the first time, the results of such kind for natural systems with two degrees of freedom were shown by Darboux, Whittaker and Birkhoff [167, 13]. From the modern viewpoint they are discussed in [137].

It should be mentioned that the similar result for the system containing integrals, linear with respect to momenta, is connection of these integrals with existence of the group of symmetries in a configurational space and with a cyclic variable. In this case, although locally, the corresponding reduction of order is always possible.

If not only coordinate, but also momentum transformations (i. e. general transformations of the phase space) be used, then, in a certain sense, the problem becomes always solvable: according to the Liouville – Arnold theorem, in the neighborhood of the non-singular level of the first integrals there always exist variables of the type action–angle, which are the separating ones. Quite another matter is that these variables are usually different for various domains of the phase space, separated by the singular (critical) invariant tori, so that their creation (shown while proving the theorem) is not constructive. In practice, as a rule, the contrary thing happens: variables action–angle are constructed, in case some separating variables are found (see § 8 ch. 5).

Separated variables, obtained by means of the extended phase transformation, are known for the Kowalevskaya and Goryachev – Chaplygin cases (see §§ 4, 5 ch. 2, § 8 ch. 5). By the way, in these cases an additional integral has, correspondingly, third and fourth degree in momenta.

If for natural Hamiltonian systems, having two degrees of freedom and possessing an additional quadratic integral, there exist general considerations (see, for example, Whittaker [167], Birkhoff [13]), allowing constructive creation of separating variables, for unnatural systems with two degrees of freedom, and also for systems, possessing an additional integral with higher ($> 2$) degree in momenta, the separation of variables is a kind of an art. For multidimensional systems the matter of separation is even more difficult. In this case several multidimensional generalizations of systems with two degrees of freedom (a kind of the Jacobi and Neumann problems), for which there exist analogues of elliptic and sphero-conical coordinates, are practically known). The questions of separation of variables on $S^n$ is considered in more details in [18, 283].

**3.** The special analytical techniques, having allowed to find separation of variables for a series of problems of rigid body dynamics, including nonholonomic systems, were mastered by S. A. Chaplygin. The famous works by S. V. Kowalevskaya [86, 87] also present an example of the unsurpassed analytical skill. In twentieth century the technique of precise integration of determination of separating transformations was partially lost and replaced by the general procedure of integration by means of methods of inverse dissipation problem and determination of the Lax representations. In this procedure it is considered that the problem is solved, in case the Lax commutation representation (see [31]) with a spectrum parameter, allowing "in principle" to obtain the general solution in theta-functions, is shown. From the viewpoint of algebraic geometry we are talking here about possible linearization of the flow on *the Prime (Jacobi) manifolds* and, proceeding from the analysis of *pole expansions* (*divisors*), we can conclude the possibility of representation of the solution in functions of Riemann, Baker – Ahieser and others.



It should be mentioned that such general assertions, cited in the majority of papers, dedicated to the determination of the Lax pairs [21, 136, 146], although true, in a certain sense, are empty, because there exists no algorithm of construction of such a solution; anyway, the problem is no less difficult. It's also useful to notice that such kind of papers [146, 262] are exceedingly formalized and overloaded with complex algebraic geometry jargon (see also the recent book [134]), their curious result being that they don't clarify, but only complicate classical ideas. Here were shown no new separating transformations.

**4.** At the same time the determination of separating variables in an integrable system is of a great use for studying its dynamics. It allows to study solutions, most simply arranged (degeneracy cases or the Appelrot classes of "specially remarkable" motions of the Kowalevskaya top), carry out bifurcational (topological) and qualitative analysis [92, 170], explicitly construct the corresponding set of variables of the action–angle type. The last thing is especially important for the perturbed situation analysis and for the quantization purposes (for example, in quasi-classical approximation).

In conclusion, the explicit integration and corresponding separation of variables for the majority of problems of rigid body dynamics were found by classics at the end of nineteenth – beginning of the twentieth century. Nearly all of them, in a somewhat modified form, we cite in § 8 ch. 5. The question of separation of variables for many newer systems (gyrostatic generalizations, multidimensional tops) is still open. It's probable that for the solution of this problem it's necessary to modify the Jacobi method ideology itself and make his scheme not that "hard". As an additional information, which can be of use, one should apparently apply the topological analysis and complex methods.

Really, for the known integrated problems, critical levels of the integral set can be found as from the condition of the order of roots in a characteristic polynomial of the Abel – Jacobi equations, so directly from the condition of the integral manifold rank fall, which apparently allows to restore separating transformation a bit arbitrarily. Complex methods, based on the study of the Laurant full-parametric expansions, also seem to be effective [243]. Like the Lax spectrum representation, they are capable of representing a spectrum curve in a hyperelliptic case; here it's possible to restore separating transformations uniquely and obtain the Abel – Jacobi equations (M. Adler, P. van Moerbeke [186, 188], P. Vanaecke [279]). However, such an approach didn't help to integrate any new system.

Chapter 2

# EULER–POISSON EQUATIONS AND THEIR GENERALIZATIONS

## § 1. The Euler–Poisson Equations and Integrable Cases

### 1. A Rigid Body with a Fixed Point

*The Euler–Poisson equations*, describing rigid body motion around a fixed point in a uniform gravity field, are written as

$$\begin{cases} \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mu \boldsymbol{r} \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \end{cases} \tag{1.1}$$

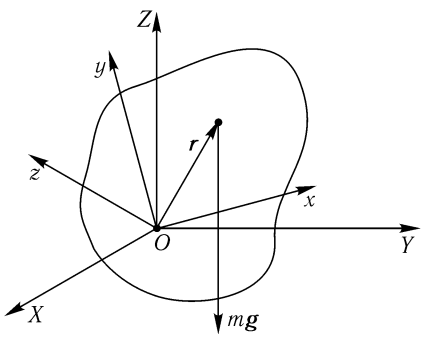


Figure 12. A rigid body with a fixed point in a gravity field.

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, $\boldsymbol{r} = (r_1, r_2, r_3)$ $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ are components of the angular velocity, of the position vector of the center-of-mass, and of the vertical unit vector in the system of principal axes, rigidly bound to the rigid body and passing through the fixation point, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is a tensor of inertia with respect to the point of fixation in these very axes, $\mu = mg$ is the body weight (fig. 12).

By means of projections of the vector of angular momentum $\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega}$ onto the same axes, equations (1.1) can be represented in the Hamiltonian form

$$\dot{M}_i = \{M_i, H\}, \quad \dot{\gamma}_i = \{\gamma_i, H\}, \quad i = 1, 2, 3, \tag{1.2}$$

with the Poisson bracket, corresponding to algebra $e(3)$

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{1.3}$$



and with the Hamiltonian — the total energy of the body

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) - \mu(\boldsymbol{r}, \boldsymbol{\gamma}), \tag{1.4}$$

where $\mathbf{A} = \mathbf{I}^{-1}$.

Remark 1. The equations of motion in the form (1.1) were known even to Euler (1758), who also found the simplest case of integrability when a rigid body coasts ($\boldsymbol{r} = 0$). The integrability of an axially symmetric top with the center of gravity disposed at the symmetry axis was determined by Lagrange and a bit later by Poisson, whose name appeared in the name of general equations (1.1).

Lie–Poisson bracket (1.3) is a degenerate one, it possesses two Casimir's functions, commutating in structure (1.3) with any function of $\boldsymbol{M}$, $\boldsymbol{\gamma}$,

$$F_1 = (\boldsymbol{M}, \boldsymbol{\gamma}), \quad F_2 = \boldsymbol{\gamma}^2. \tag{1.5}$$

In the vector form equations (1.2) can be written as

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \dfrac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \dfrac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \dfrac{\partial H}{\partial \boldsymbol{M}}. \end{cases} \tag{1.6}$$

The form of equations (1.2), (1.6) is obtained from the Poincaré–Chetayev equations, written on a group $SO(3)$ (see § 2, ch. 1).

Functions $F_1$ and $F_2$ are integrals of equations (1.6) with any Hamiltonian $H$. For the Euler–Poisson equations they have natural physical and geometrical origin. The integral $F_1$ is an angular momentum projection on the fixed vertical axis and in rigid body dynamics is referred to as *an area integral*. It is connected with symmetry with respect to rotations around a fixed vertical axis. The integral $F_2 = \text{const}$ is of the purely geometrical origin: it is a squared module of the vertical unit vector. For real motions this integral constant equals unity $F_2 = \boldsymbol{\gamma}^2 = 1$.

When bracket (1.3) is bounded to the the combined level set of integrals $F_1$ and $F_2$, it becomes nondegenerate and, according to the Darboux theorem, (§ 1 ch. 1) can be represented in an ordinary canonical form in some symplectic coordinates. For various purposes one can use both the Euler canonical variables $(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$ and the Andoyaer–Deprit variables $(L, G, H, l, g, h)$. In both cases, on a symplectic leave, being determined by $p_\psi = \text{const}$ (correspondingly, $H = \text{const}$), there arises a canonical system with two degrees of freedom.

## 2. Kirchhoff's Analogy for an Elastic Thread

There exists an analogy between the Euler – Poisson equations and the equations, describing equilibrium of an infinitely thin elastic cylinder — a thread, for the first time discovered by G. Kirchhoff [85]. In a certain sense, this analogy allows spatial interpretation of rigid body dynamics, replacing the time evolution of a system by the analysis of an elastic thread shape, or, to put it more precisely, of a position connected with the reference curve in the absolute space.

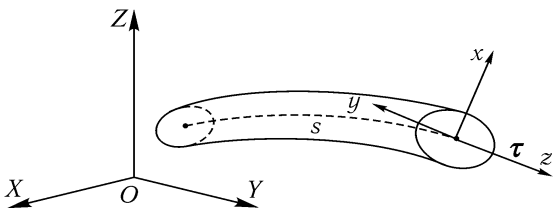



Figure 13. Kirchhoff's analogy for an elastic thread.

Let's consider an elastic bar, whose end points have *a constant force* and *a torque* applied. Let $s$ be an arc length and $ds$ — a given element of the bar. To each cross-section of the bar we attach its own frame of reference and designate the force and moment vectors, projected on these axes, as $\boldsymbol{P}$, $\boldsymbol{M}$ (see fig. 13). Then the equation of equilibrium , expressing the connection between the end force and the torque in each cross-section, is written as

$$\frac{d\boldsymbol{M}}{ds} = \boldsymbol{M} \times \boldsymbol{\omega} + \boldsymbol{P} \times \boldsymbol{\tau}. \tag{1.7}$$

Here $\boldsymbol{\omega}$ is a vector of "an angular velocity of rotation" of a frame of reference, bound to the cross-section, i. e. the velocity of rotation of a frame of reference, bound to the cross-section, which depends on the length of an arc $s$, and $\boldsymbol{\tau}$ is a unit vector, tangent to the bar axis. Designating the unit vector, directed along the axis of action of an end force $\boldsymbol{P}$, as $\boldsymbol{\gamma}$, we can write "a kinematic equation of equilibrium", expressing invariance of a force in an "absolute" system of axes

$$\frac{d\boldsymbol{\gamma}}{ds} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \tag{1.8}$$

When deformations are small, [85] the Hooke's law, expressing the connection between the rotation of the bar element and elastic moments, acting in this cross-



section, holds true in the form

$$\boldsymbol{\omega} = \mathbf{A}\boldsymbol{M},$$

where $\mathbf{A}$ is a constant matrix.

If in each cross-section we choose axes such, that one of them is directed along the tangent to the bar axis (the axis $Oz$), and two others — along principal moments of inertia of the cross-section (see fig. 13), we shall obtain

$$\mathbf{A} = \operatorname{diag}(a_1,\, a_2,\, a_3), \qquad \boldsymbol{\tau} = (0,\, 0,\, 1).$$

Equations (1.7), (1.8) can be represented in the Hamiltonian form (1.2) with the Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + \mu(\boldsymbol{\tau},\, \boldsymbol{\gamma}), \tag{1.9}$$

where $\mu = |\boldsymbol{P}|$ is an end force module. System (1.6) with Hamiltonian (1.9) is referred to as *the Euler – Kirchhoff equations*. But if the variables are given another meaning, then these are exact Euler – Poisson equations. Here lies the essence of Kirchhoff's analogy, which allows to investigate the results of rigid body dynamics for elastic system analysis.

The investigation of possible shapes of an elastic thread in the Kowalevskaya case (see Table 2.1) can be found in the paper [219].

## 3. Integrable Cases

To be integrable according to Liouville (see § 7, ch. 1), systems (1.1) and (1.6) should possess not only Hamiltonian (1.4), which is also the first integral of the system, but one more additional integral. The outstanding mathematicians, especially in the nineteenth century, spared no effort to find such an integral, but its general form wasn't obtained.

It turns out that there exist fundamental dynamic effects, obstructing the integrability of these equations in the general case. We are going to show the integrable cases, presently known. In table 2.1 the cases of Euler, Lagrange and Kowalevskaya are the general integrable cases, i. e. the additional first integral exists under the given limitations of parameters (matrix $\mathbf{A}$ and vector $\boldsymbol{r}$) for any initial conditions. The Goryachev – Chaplygin case is a special integrable case: here, for the existence of an additional integral, it's necessary to require that the area constant be equal to zero: $F_1 = 0$. The Hess case also concerns the existence of an invariant relation: $F = 0$, linear with respect to $\boldsymbol{M}$, for which $\dot{F} = \lambda F = 0$. In this chapter we are going to give the detailed consideration to the first four cases, carrying out their qualitative and computer

Table 2.1. The integrable cases of the Euler – Poisson equations.

| | The author | Additional conditions, the Hamiltonian | The first integral |
|---|---|---|---|
| 1 | Euler (1758) | $\boldsymbol{r} = 0, \quad H = \frac{1}{2}(\mathbf{A}\boldsymbol{M},\, \boldsymbol{M})$ | $F = \boldsymbol{M}^2 = \mathrm{const}$ |
| 2 | Lagrange (1788) | $a_1 = a_2 = 1,\ r_1 = r_2 = 0,$ $H = \frac{1}{2}(M_1^2 + M_2^2 + a_3 M_3^2) - r_3\gamma_3$ | $F = M_3 = \mathrm{const}$ |
| 3 | Kowalevskaya (1888) | $a_1 = a_2 = 1,\ a_3 = 2,\ r_2 = r_3 = 0,\ r_1 = x,$ $H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - x\gamma_1$ | $F = \left(\frac{M_1^2 - M_2^2}{2} + x\gamma_1\right)^2 + (M_1M_2 + x\gamma_2)^2$ |
| 4 | Goryachev, Chaplygin (1903) | $a_1 = a_2 = 1,\ \frac{a_3}{a_1} = 4, r_2 = r_3 = 0, r_1 = x,$ $H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - x\gamma_1,$ $(\boldsymbol{M},\, \boldsymbol{\gamma}) = 0$ | $F = M_3(M_1^2 + M_2^2) + xM_1\gamma_3$ |
| 5 | Hess (1890) | $r_2 = 0,\ r_1\sqrt{a_3 - a_2} \pm r_3\sqrt{a_2 - a_1} = 0,$ $H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + r_1\gamma_2 + r_3\gamma_3$ | $F = r_1M_1 \mp r_3M_3 = 0$ |

investigation. The Hess case analysis will be postponed till chapter 4, where we'll show its symmetrical origin and its connection with the order reduction, and also consider various generalizations.

There exist some more special solutions, usually representing certain periodic and asymptotic motions. Later we shall consider the most interesting ones, possessing a subtle mechanical meaning. Except for these solutions, more than 200 years old history of the search of an additional integral in the Euler – Poisson equations has given us a mass of doubtful, wrong, formal and complicated investigations, whose consideration was given in the book by the Donetsk[1] authors [61]. But having shown a series of mistakes in earlier papers, the Donetsk school has added some of its own solutions, whose notation can occupy several pages and whose meaning is absolutely vague. The results of such kind and the techniques (like invariant relations, hodographs etc.), having appeared on their way, are not related with the modern understanding of the problem of investigation of the Euler – Poisson equations. This ptoblem is largely concerned with the qualitative and computer-aided analysis combined with the study of the nonintegrable case.

[1]Donetsk is an Ukranian city, the scientific center of the Ukranian Academy of Sciences. — *Trans. rem.*



Here it's appropriate to cite K. Magnus [119]: "Around 1900 the search for the new integrable cases of inviting nonlinear equations of a heavy gyroscope motion almost turned into a kind of sports for mathematicians. However, the investigators often left the essence of the physical problem and dedicated their expansive investigations to the cases, that could not be realized either physically — due to violation of inequalities, connecting $I_1, I_2, I_3$, — or geometrically — due to omission of the condition $(\boldsymbol{\gamma}^2 = 1)$. We cannot consider these works here."

The main results of nonintegrability of the Euler – Poisson equations belong to V. V. Kozlov, S. L. Ziglin, S. V. Bolotin. They are discussed in the books [92, 97], and concern the splitting of asymptotic surfaces, the ramification of solutions on the complex plane of time, the birth of a large number of nondegenerate periodic solutions. This trend summit would be the theorem that the general cases of existence of the additional real analytical integral are just the cases of Euler, Lagrange, Kowalevskaya, and, for the particular integrals, the Goryachev – Chaplygin case should be added. Unfortunately, this hypothesis has not been totally proved up to the present day, in spite of separate and rather substantial achievements [97].

Algebraic integrability of the Euler – Poisson equations was investigated even by Husson (1906) [230] (see also [9]) who has shown that the problem cannot have any other algebraic integrals, except for the cases of Euler, Lagrange and Kowalevskaya.

The most complete computer-aided investigations of stochasticity in the Euler – Poisson equations are given in [28]. Here, the transversal intersection of perturbed separatrices may serve as "the computer proof" of nonexistence an additional real analytical integral. The paper [35] discovers an infinite period-doubling cascade of perturbed permanent rotations of the Euler – Poinsot problem, indicating the possibility of transition to chaos according to the Feigenbaum scenario.

Remark 2. For integrability of system, (1.1) according to the last multiplier theory (the Euler – Jacobi theory, see § 7 ch. 1), we also lack one more additional first integral. In fact, system under investigation (1.1) possesses three first integrals and a standard invariant measure $\rho = \mathrm{const}$. However, it should be noted that natural generalizations of equations (1.1) (see § 12 ch. 3) cannot be integrated by means of this technique. For such systems the integrability is determined by means of the Hamiltonian formalism and the Liouville theorem (§ 7 ch. 1).

### 4. Absolute Motion

Equations (1.6) describe dynamics of a reduced system such that the rigid body precession around a vertical line is ignored. To determine absolute motion, it's necessary to implement an additional quadrature

$$\dot{\psi} = \frac{\omega_1\gamma_1 + \omega_2\gamma_2}{\gamma_1^2 + \gamma_2^2} \tag{1.10}$$

or to integrate corresponding Poisson equations for direction cosines

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}. \tag{1.11}$$

Both these techniques are not optimal for numerical investigations. In the first case the problems arise because of the singularity near the poles $\gamma_3 = \pm 1$, in the second case — due to the loss of orthonormality of vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, caused by numerical method dissipation. To obtain the majority of computer-based illustrations, shown in the book, we used the quaternion form of notation equation of motion, shown in § 4 ch. 1. This system describes absolute dynamics of rigid body, it doesn't have any singularities, and is not a redundant one. Consequentey, it is irreplaceable for numerical investigations. In § 12 ch. 3 we have considered its applications to dynamics investigation in the superposition of potential field.

For Euler – Poisson equations (1.6), which, if the area constant is given, determine dynamics of a point on the Poisson sphere in a generalized potential field[1] (see § 1 ch. 4), only some families of periodic and asymptotic solutions are known. Nearly all of these solutions, the majority of which can be traced back to classics, are given below. Let us discuss typical situations, from the viewpoint of absolute motion, in greater detail.

**Fixed points on the Poisson sphere,** which determine the Staude solution and relative equilibria, correspond to uniform rotations of a body around a vertical line.

[1] When this system is written in canonical coordinates on a cotangent foliation to the Poisson sphere, in the Hamiltonian there arise terms, linear with respect to momenta — a magnetic monopole [133].



**Periodic solutions on the Poisson sphere** in an absolute space are, generally speaking, not periodic. To obtain such a periodicity, it's necessary (and sufficient) to have the commensurability of a period $T$ of reduced motion with the quantity

$$\tau = \int_t^{t+T} \frac{\omega_1\gamma_2 + \omega_2\gamma_2}{\gamma_1^2 + \gamma_2^2}\, dt,$$

computed from (1.10) along periodic motion $\boldsymbol{\omega}(t)$, $\boldsymbol{\gamma}(t)$. In the general case, $\tau$ and $T$ are not commensurable, and the absolute motion is a quasiperiodic and double-frequency one, i. e. the motion in the absolute space may look rather complicated.

**Quasiperiodic (double-frequency) paths** of a reduced system, in the general case, define triple-frequency quasiperiodic motions in the absolute space, which can look rather intricately. Nevertheless, these motions are regular, unlike chaotic ones, generated by chaotic paths of a reduced system. In the latter case the irregular behavior of a body requires probabilistic description.

**Regular precessions.** One more class of periodic solutions, that can be traced back to classical investigations of the Lagrange top dynamics, doesn't directly concern reduced system dynamics. These are regular precessions, which, in the general case, as it was noticed by Grioli (§ 6 ch. 2), are possible around a nonvertical axis. For such kind of motions the periodicity is required for a certain axis within the body, which should rotate around the axis, fixed in space. Absolute motion may turn out to be nonperiodic since the proper rotation around the axis within the body is not necessarily commensurable with motion of this axis in space. This is observed, for example, for regular precessions in the Lagrange case.

In some cases (for example, in the Hess case § 6 ch. 2) one of the axes of a body may be in a rather simple motion in the absolute space (according to the spherical pendulum law). Nevertheless, dynamics of a reduced system and of general absolute motion may be very complicated (the rest of phase variables in the Hess case varies asymptotically).

Further on, we give some of the most interesting reduced and absolute motions, obtained by means of computer simulation.

**Absolute motion: integrable and nonintegrable cases.** Here we shall dwell on the general principles of the experimental (both full-scale and computer-based) study of rigid body motion around a fixed point. To get any understanding of rigid body motion, a usual procedure is to observe motion of some characteristic points of the body, or, generally speaking, some variables that have the simplest natural regularities of the time evolution. If a certain variable $x$ varies under the motion, following the law $x(t, x_0)$, then it's possible to construct its frequency spectrum $\omega$, which can be determined from the Fourier transformation

$$\widehat{x}(\omega) = \lim_{T\to\infty} \int_{-T}^{T} x(t, x_0)\, e^{i\omega t} dt.$$

For the integrable systems, and also for motions of nonintegrable systems, lying on the invariant tori (and not in the stochastic layer), the variable $x$ is (according to the Liouville – Arnold theorem) a certain quasiperiodic function. In the general case of $n$ rationally independent frequencies ($n$ is a number of degrees of freedom) for chaotic motions the spectrum is continuous when the motion itself may show both regular portions and distinct chaotic pulsations.

Remark 3. The frequencies $\omega_1, \ldots, \omega_k$ are referred to as independent ones if the equality $n_1\omega_1 + \ldots + n_k\omega_k = 0$, where $n_i \in \mathbb{Z}$, holds true only when $n_1^2 + \ldots + n_k^2 = 0$.

For the rigid body motion in integrable cases $n = 3$, the absolute motion is, generally speaking, triple-frequency. The reduced system motion, in the presence of a linear integral (like an area integral), is double-frequency. In the latter case the third frequency is obtained from the quadrature for the precession angle in the process of transition to the absolute motion. Below we are discussing integrable cases of the Euler – Poisson equations.

For different variables in the absolute space the number of frequencies may decrease on account of the following reasons:

**a)** The integral algebra is redundant, as, for example, in the Euler – Poinsot problem.

Really, in the Euler case there exist three integrals of an area kind $N_1 = (\boldsymbol{M}, \boldsymbol{\alpha})$, $N_2 = (\boldsymbol{M}, \boldsymbol{\beta})$, $N_3 = (\boldsymbol{M}, \boldsymbol{\gamma})$, constituting the algebra $so(3)$: $\{N_i, N_j\} = \varepsilon_{ijk} N_k$. In this case the absolute phase space is foliated into two-dimensional tori, and not three-dimensional ones. It should also be noted that dynamics of a reduced system (on the Poisson sphere) is also double-frequency, i. e., a frequency is lost at the expense of realization of a quadrature for $\psi$.

The Euler case also has variables in a phase space — components $M_i$, which make single-frequency, i. e., periodic, motions. However, these variables are practically impossible to be measured.



**b)** Some coordinates of characteristic points may vary particularly easily.

For example, in case of the Lagrange top a nutation angle varies periodically. At the same time the absolute motion is triple-frequency, and the dynamics of a reduced system is (with respect to $\varphi$ or to $\psi$) double-frequency. An interesting fact is that on the zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, corresponding to a spherical pendulum, the absolute motion is double-frequency.

**c)** The motions correspond to particular (critical) tori, defining bifurcational curves.

Stable and unstable one-dimensional, and also asymptotic, invariant surfaces of a reduced system usually set double-frequency motions in the absolute space. This fact is vividly illustrated by the Kowalevskaya case and by the Goryachev – Chaplygin case. In the latter case, for the special solution of Goryachev, for small energies there occurs even greater degeneration, and the motion in the absolute space becomes periodic (see 5): the body makes curious pendular motions in space. It should also be noted that the Kowalevskaya top in a reduced phase space has a set of three variables $z_1, z_2, z_3$, in whose space there occurs a periodic motion on a certain ellipse (see § 4). These variables are highly unobvious and are produced as from components of the moment $\boldsymbol{M}$ , so from the components of the unit vector $\boldsymbol{\gamma}$ .

Finally, we shall dwell on the solutions of Grioli and Hess. In the absolute space the Grioli motion is strongly degenerate, and the paths of all points are periodic. The Hess top motion is characterized by simple behavior of the center-of-mass. It is governed by the spherical pendulum law and is a double-frequency one. At small energies and $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ the body makes periodic (single-frequency) motion, the center-of-mass oscillating in one plane, following the physical pendulum law, and the mean axis apex moving (periodically) along the loxodromic segment. However, as soon as $(\boldsymbol{M}, \boldsymbol{\gamma}) \neq 0$, the situation becomes an ordinary one, i. e., the body is making triple-frequency motion. At large energies the motion is double-frequency, and for the system, reduced with respect to $\psi$, the path is situated on a special torus, filled with double asymptotic motion ( for more details see § 3 ch. 4).

In conclusion, it should be mentioned that in the general nonintegrable case the body is making both complex chaotic motions, whose investigation seems to require not only frequency analysis, but also more subtle statistic characteristics (like correlational functions), and various periodic and quasiperiodic motions, whose definition in phase space constitutes one of the major problems of modern dynamics.

## § 2. The Euler Case

### 1. The Geometric Interpretation by Poinsot

In this section we shall discuss the most famous analytical and geometrical results concerning the Euler case, for which the body is moving without influence of a field ($\boldsymbol{r} = 0$), and the Hamiltonian (the kinetic energy) and an additional integral, which is square of the angular momentum module, are represented by

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}), \quad F_3 = \boldsymbol{M}^2 = f = \text{const.} \tag{2.1}$$

The intersection of a set of constant energy $H = h$ with sphere (2.1) in space of moments $\boldsymbol{M}$ represents closed spatial curves — *polhodes*. How they look on the set of energy $H = h$ is shown at fig. 14.

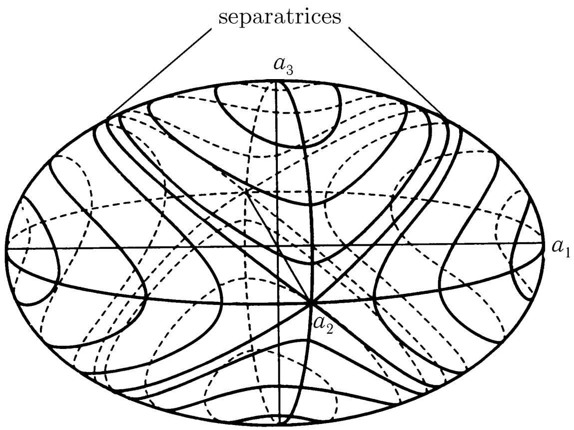



Figure 14. **Polhodes ($a_3 < a_2 < a_1$).** In the general case polhodes represent spatial algebraic curves of the fourth order. In two special cases they represent intersecting ellipses — separatrices, which are filled with double asymptotic motions to unstable permanent rotations around the mean axis of the inertia ellipsoid, corresponding to the points of intersection of ellipses. Polhodes degenerate into points for permanent rotations around the stable minor and major axes of the inertia ellipsoid.



The geometric interpretation of the Euler case was given by *L. Poinsot* in 1851 [257]. According to this interpretation, the inertia ellipsoid (the energy ellipsoid) with a fixed center $\frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = h$ rolls without slipping on the plane, fixed in the absolute space and perpendicular to the angular momentum vector (fig. 15). In this interpretation the position vector of the contact point serves as an instantaneous axis of rotation, and the angular velocity is proportional to the position vector length. The contact point on the ellipsoid draws *polhodes* (fig. 14), and on the plane — *herpolhodes* (see fig. 16, 19).

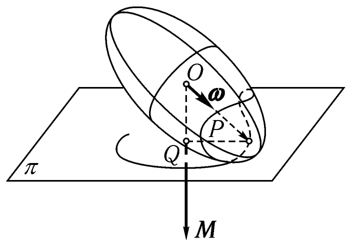



Figure 15. The Poinsot interpretation.

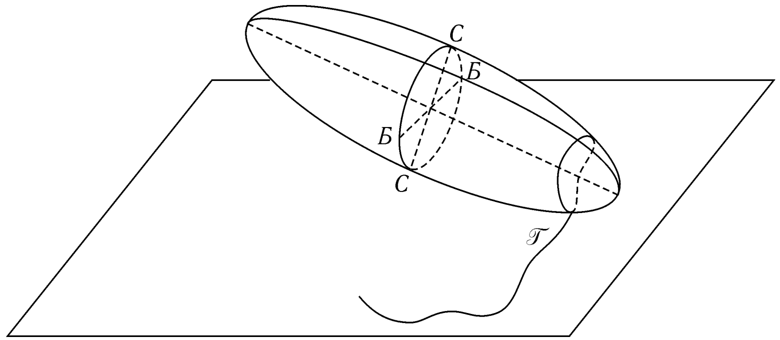



Figure 16. **Herpolhode.** As it was indicated by Hess[1], a herpolhode cannot have inflection points. In the general case a herpolhode is an open curve. This (wrong) figure, given in many books, belongs to Poinsot himself.

### 2. Explicit Integration and Bifurcational Analysis

Explicit integration of the Euler case is easy to obtain by means of the Andoyaer – Deprit variables, in which integral (2.1) is a cyclic one (for more details see § 3, ch. 1, where the phase portrait of the Euler case is also shown). We shall give expressions for moments $\boldsymbol{M}$ in one of the four domains, separated by separatrices, on the energy ellipsoid (see fig. 14). These expressions trace back to Jacobi, Kirchhoff, Greenhill (see, for example, [124, 145])

$$M_1 = \sqrt{\frac{(f-2hI_3)I_1}{I_1-I_3}}\,\mathrm{cn}(\tau,\,k), \quad M_2 = \sqrt{\frac{(f-2hI_3)I_2}{I_2-I_3}}\,\mathrm{sn}(\tau,\,k),$$
$$M_3 = \sqrt{\frac{(2hI_1-f)I_3}{I_1-I_3}}\,\mathrm{dn}(\tau,\,k), \tag{2.2}$$

where

$$k^2 = \frac{(I_1-I_2)(f-2hI_3)}{(I_2-I_3)(2hI_1-f)}, \quad \tau = \sqrt{\frac{(I_2-I_3)(2hI_1-f)}{I_1I_2I_3}}(t-t_0).$$

Under the transition to other domains it's necessary to change corresponding signs and replace dn by cn and vice versa, and also transform the expressions for the module of elliptic functions and uniformizing parameter $\tau$ [124].

**The motion in an absolute space.** The explicit time dependence of direction cosines can be obtained in the following way. Let's choose a fixed frame of reference, in which one of the unit vectors is directed along the angular momentum vector, fixed in absolute space, and two others are perpendicular to it

$$\boldsymbol{M} = \sqrt{f}\boldsymbol{\alpha}, \quad (\boldsymbol{M},\boldsymbol{\beta}) = (\boldsymbol{M},\boldsymbol{\gamma}) = 0, \tag{2.3}$$

where $f$ is a constant of integration (2.1). Let's find the time dependence of two vectors $\boldsymbol{\alpha},\boldsymbol{\gamma}$, and the vector $\boldsymbol{\beta}$ can be added according to the orthogonality ($\boldsymbol{\beta} = \boldsymbol{\gamma}\times\boldsymbol{\alpha}$).

Due to the constancy of the quantity $f$ the vector $\boldsymbol{\alpha}$ is determined from relations (2.2). Quadratures for $\boldsymbol{\gamma}$ may be easily obtained by means of sphero-conical coordinates on the Poisson sphere $\boldsymbol{\gamma}^2 = 1$. Really, in the chosen frame of referance the area constant equals zero, and the Hamiltonian in the Euler case coincides with the additional integral of the Neumann problem with a zero potential. Hence, in sphero-conical coordinates variables separate, and one can use formulae (7.17) ch. 1 ( in greater detail see § 7 ch. 1).

Remark. Using the Poisson equation $\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\mathbf{A}\boldsymbol{M}$, it's easy to express $\boldsymbol{M}$ in terms of $\boldsymbol{\gamma}$ and $\dot{\boldsymbol{\gamma}}$ under the condition that $(\boldsymbol{M},\boldsymbol{\gamma}) = 0$:

$$\boldsymbol{M} = \mathbf{A}^{-1}\frac{\dot{\boldsymbol{\gamma}}\times\mathbf{A}^{-1}\boldsymbol{\gamma}}{(\boldsymbol{\gamma},\,A^{-1}\boldsymbol{\gamma})}.$$

[1] In various textbooks this result is also attributed to de Sparr and Lecornu.



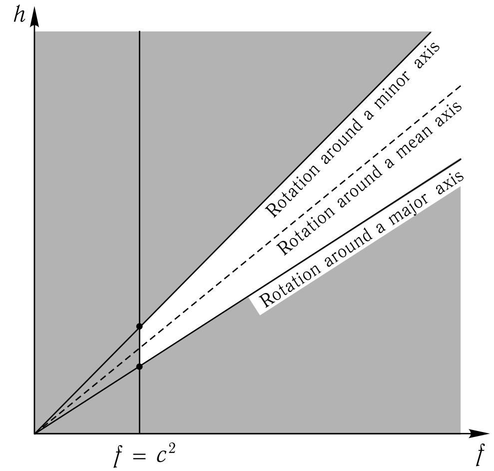



Figure 17. Bifurcational pattern of the Euler – Poinsot case on a plane $(h,\,f)$. The domains, where motion is impossible, are marked with grey colour. (The domain of intervals $f < c^2$, $c = (\boldsymbol{M},\,\boldsymbol{\gamma})$ is also inaccessible).

Bifurcational pattern of the Euler – Poinsot case on a plane of values of integrals $h$, $f$ is shown at fig. 17. It consists of three branches, specified by the equation $h = \frac{1}{2}a_i f$, $i = 1, 2, 3$, and corresponding to the rotations around three principal axes of inertia. The unstable rotations around a mean axis are shown by dotted lines; in this case a mean axis of inertia of the body in a fixed space describes *a loxodromic spiral* (a loxodrome) on the sphere, rotating through 180° (see fig. 18). We should remind that the loxodrome constitutes the same angle with all the meridians. Double asymptotic motions of a body in the Euler case are considered in greater details in § 9 ch. 5.

**A herpolhode.** We shall give the differential equation of a herpolhode. Darboux was the first to attempt to study this curve. The above mentioned equation is easy to obtain if one uses parametric representation of a polhode (where the parameter is replaced by squared distance $r^2$ from the polhode point to the ellipsoid center) and equations of motion. Here we omit the corresponding computations, and show the final result only (in greater detail see in [113]). In polar coordinates $\rho$, $\varphi$ with the center at the intersection of the momentum vector with a fixed plane at the point $Q$ (see fig. 15), the herpolhode equation iswritten as

$$\frac{d\zeta}{d\varphi} = \frac{2h}{f}\frac{\zeta+k}{\zeta\sqrt{P(\zeta)}}, \quad \zeta = \rho^2, \tag{2.4}$$

where $k = \frac{1}{D}\prod_i(1-a_iD)$, $D = \frac{f}{2h}$. The function $P(\zeta)$ represents a cubic polynomial:

$$P = 8hl^2 \prod_{i,jk}\Big[\frac{a_i^2}{(a_j-a_i)(a_k-a_i)}(\zeta-(a_j+a_k-Da_ja_k))\Big], \tag{2.5}$$

where $l = \sum_{i,jk}\frac{(a_i-a_j)a_k^2}{a_1a_2a_3}$. The solution of equation (2.4) can be obtained in elliptic quadratures; qualitative investigations of a polynomial $P(\zeta)$ result in the conclusion that the whole polhode is enclosed between two boundary concentric circles, which it touches alternately so that the moments of contact correspond to the transition of the vector $\boldsymbol{\omega}$ through the principal planes of inertia ellipsoid. A herpolhode does not have inflection points or cuspidal points (Hess). For separatrix motions it represents a spiral, infinitely twisting around the center, but, at the same time, having a finite length, equal to the length of the corresponding polhode arc. Typical paths of herpolhode are shown at fig. 19. Their detailed investigation can be found in the treatise by Grammel [66], where, depending on the position on a bifurcational pattern, the epicycloidal and pericycloidal Poinsot motions can be distinguished.

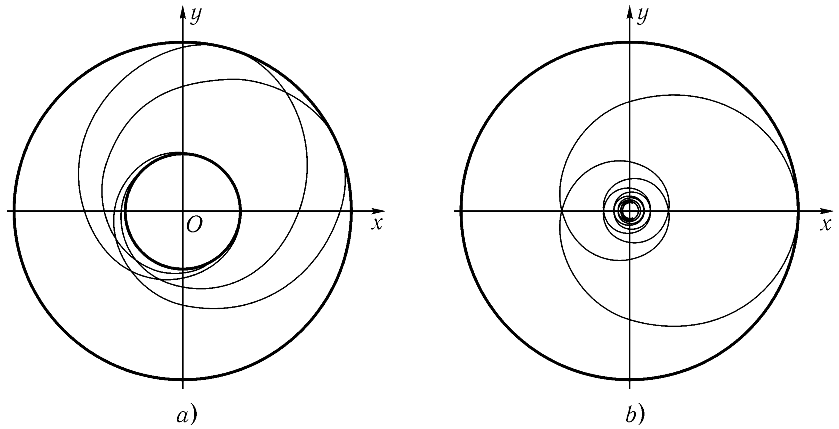



Figure 19. Herpolhodes for the Euler case. *a*) A herpolhode of a general kind — a quasiperiodic open curve. *b*) A herpolhode, corresponding to a separatrix — a curve, infinitely winding to the center.

## 3. Comments



Various techniques for the explicit integration of equations of motion of a free top can be found in works by Euler, Lagrange, Kirchhoff, Caley, Greenhill, and Jacobi. The works of the latter [231, 232] have a special interest. In these papers Jacobi has obtained explicit expressions in terms of elliptic functions not only for the angular velocity components, but for all (nine) direction cosines. Applying his results to perturbed motion, he constructed (in series) a system of osculating variables, similar to variables of the action-angle type. Their modern introduction, belonging to Yu. A. Sadov [9, 92], includes phase interpretation and the Andoyaer – Deprit variables. The first application of elliptic functions for the Euler case integration was ascribed by G. Lamb in his famous textbook [112] to a Rueb (1834 [264]).

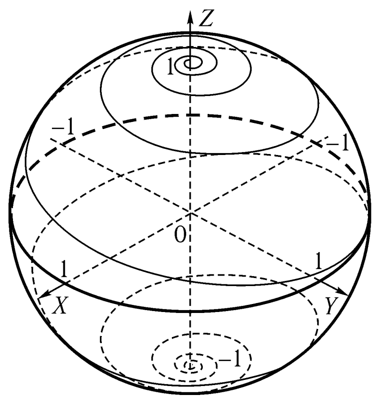



Figure 18. The loxodromic path of apex of a mean axis of inertia, of the body when the body is moving along a separatrix. Parameters: $\mathbf{A} = \mathrm{diag}(1, 1.02, 2.0)$.

Lagrange has also given his solution of the Euler problem in the *Analytical Mechanics*: “I introduced in this solution the clarity, and, if I can put it this way, the elegancy that could ever be imparted to this solution.” But at the same time Lagrange considered this case to be simple: “. . . thus, I flatter myself with the hope that I shall not be reproached for reconsideration of the present problem.” His solution is remarkable because he was the first to show explicitly the existence of three principal axes of inertia, which an arbitrary rigid body has (the reducibility of a symmetrical matrix to the diagonal form) — though the latter does not bear any relation to the Euler case itself. The Lagrange solution also has elliptic integrals, but it lacks the idea of their inversion — which appears in the works by Jacobi and achieves its perfection and a certain completeness in the papers by Weierstrass, Hermite and Halphen.

Poinsot himself was trying to improve the described geometric interpretation of motion that has become the pattern in mechanics, though it was not that clear for other integrable cases. He offered the second geometrical interpretation, taking account of time. According to the latter interpretation the cone, attached to the body, rolls over the plane, perpendicular to the angular momentum vector and rotating with the constant angular velocity. Darboux and Koenigs, on the basis of the second interpretation, have constructed a device, which they called *a herpolograph*, designed for the inertial motion demonstration. The improvements of the Poinsot interpretation were also offered by Jacobi, Sylvester, MacCullagh. These interpretations, in spite of their generality, are too far from the natural ones. They can be found in the books [113, 61, 163, 120] and others. Presently these results are of purely historical value.

To confirm the regularities of free motion, Maxwell has invented the model of a top, named after him. The experiments with this top are described in the book by Webster on mathematical physics [46], where the author gives a special place to the top theory: "That is a question of the extraordinary practical importance, especially for engineers, but those who study physics often try to escape it. Even Maxwell was drawing the attention of physicists to this matter and has invented a remarkable device to demonstrate corresponding phenomena." The earlier device, demonstrating free top motion, belongs to Bonnenberger (1817).

By means of Maxwell top one can become assured in stability or instability of rotations with respect to the principal axes. One can also become certain that motions, close to rotations with respect to the mean axis, i.e., to separatrices, are very complicated and seem to be irregular and chaotic. In fact, the "real" chaos in such motions arises at the introduction of perturbation, for example, a gravity field.

## § 3. The Lagrange Case

Let us consider the following integrable case, which is of the substantial interest from the viewpoint of both classical mechanics and technical applications. The main regularities of the Lagrange top motion constitute the content of the approximate (applied) theory of a gyroscope.

For the case being considered the body is dynamically symmetrical $a_1 = a_2$, and the center-of-mass lies on the axis of dynamical symmetry $r_1 = r_2 = 0$. An additional integral has the form $F_3 = M_3 = \mathrm{const}$.

**The reduction to one degree of freedom.** The Lagrange case is most easily integrated by means of the Euler angles $\theta, \varphi, \psi$ and conjugate canonical momenta $p_\theta, p_\varphi, p_\psi$. Actually, writing down the Lagrange function (see (4.28) ch. 1), we obtain that variables $\varphi, \psi$ are cyclic ones, and the corresponding momenta are integrals of motion:

$$p_\varphi = M_3 = \mathrm{const}, \quad p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma}) = \mathrm{const}.$$

Eliminating cyclic variables from the energy integral, we determine

$$h = \frac{I_1}{2}\dot\theta^2 + \frac{p_\varphi^2}{2I_3} + \frac{(p_\psi - p_\varphi\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta, \tag{3.1}$$



where values of cyclic integrals $p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma})$, $p_\varphi = M_3$ may be considered as parameters.

Without any limitation of generality let us assume that $I_1 = 1$, $mgl = 1$ (it can be achieved by choice of the length and time units), then from expression (3.1) we can find the quadrature for the nutation angle

$$\dot\theta^2 = 2h - ap_\varphi^2 - \frac{(p_\psi - p_\varphi\cos\theta)^2}{\sin^2\theta} - 2\cos\theta, \quad a = I_3^{-1}. \tag{3.2}$$

From relation (3.2) for the variable $u = \gamma_3 = \cos\theta$ we obtain an elliptic quadrature (see also [119]).

$$\begin{gathered}\dot u = \pm\sqrt{R(u)},\\ R(u) = 2(h' - u)(1 - u^2) - (p_\psi - p_\varphi u)^2,\\ h' = h - \frac{ap_\varphi^2}{2} = \mathrm{const}.\end{gathered} \tag{3.3}$$

The dependency $u(t)$ is expressed in terms of elliptic functions. The function $f(u)$ is referred to as *a gyroscopic function* and represents a cubic polynomial. In the general case it is of the type shown at fig. 20. The similar quadrature with the polynomial $R(u)$, probably, of higher degrees, also exists for various generalizations of the Lagrange case, allowing the integral $M_3 = \mathrm{const}$.

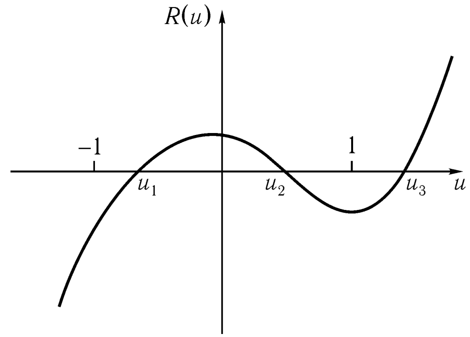



Figure 20. The Lagrange top gyroscopic function. It is easy to show that $u_3 \geqslant 1$.

**Complete system dynamics.** To determine complete system motion, according to the known law of motion $u(t)$, it is necessary to fulfill quadratures for the precession angle $\psi$ and the proper rotation angle $\varphi$:

$$\dot{\psi} = \frac{p_\psi - p_\varphi u}{1 - u^2}, \quad \dot{\varphi} = (a-1)p_\varphi + \frac{p_\varphi - p_\psi u}{1 - u^2}. \tag{3.4}$$

The motion of *an apex* — a point, lying at the dynamical symmetry axis, — is described in terms of spherical angles $\theta$, $\psi$. The apex path is always enclosed between two parallels, whose latitude is determined by gyroscopic function (3.3) (see fig. 20), and belongs to one of three types shown at fig. 21, 22, 23.

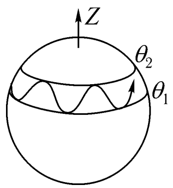



Fig. 21. $\dot{\psi}$ does not change its sign while moving and never vanishes.

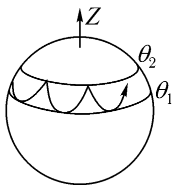



Fig. 22. $\dot{\psi}$ preserves its sign, vanishing periodically.

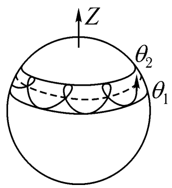



Fig. 23. $\dot{\psi}$ changes its sign.

At certain integral values a gyroscopic function is tangent to the horizontal axis at the point $u = 1$. That is the case of asymptotic (aperiodic) motion of the Lagrange top, when the axis of symmetry tends to take vertical position at $t \to \pm\infty$ (see fig. 24). The explicit formulae for this case are given in § 9 ch. 5.

### 1. Bifurcational Pattern and Geometrical Analysis of Motion

The Lagrange top motion type (the path form) is completely determined by values of integrals of motion $h$, $p_\varphi$, $p_\psi$. Further on, for the integral constants we shall also use following designations: $p_\varphi = p$, $p_\psi = c$.

We shall consider three-dimensional space, whose points are values of the first integrals $(h, p_\varphi, p_\psi)$. As far as the Lagrange top path form is completely determined by values of integrals $h$, $p_\varphi$, $p_\psi$, space may be divided into various domains, each one having its own type of motion. Thus, the domains of "allowed values" of integrals is constituted by those points of space, for which corresponding gyroscopic function (3.3) has positive values at the interval $u \in [-1, 1]$ (see fig. 20). This domain boundary is "*a set of regular precessions*"; at these integral values gyroscopic function (3.3) is tangent to the axis $Ou$ from below at the interval $[-1, 1]$. In this case the top performs *a regular precession*: the apex uniformly rotates around the vertical line, preserving a constant angle of slope, and the body uniformly rotates around its own axis. The general appearance of



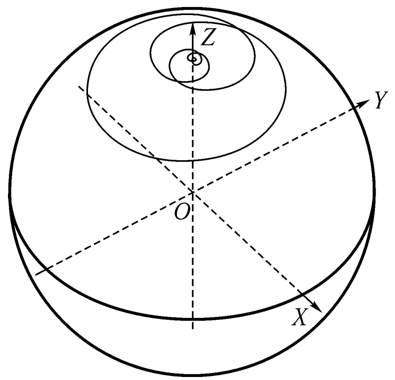



Figure 24. The motion of the apex of the Lagrange top center-of-mass in a fixed space for the asymptotic motion. For the first time this motion was indicated by Klein and Sommerfeld [238].

the set of regular precessions and of its section by various planes $p_\varphi = \text{const}$ and $p_\psi = \text{const}$ are shown at fig. 25, 27, 28, 26. At figures it is easy to notice two edges in the planes $p_\varphi \pm p_\psi = 0$ (one of the edges $p_\varphi - p_\psi = 0$ does not reach the origin of coordinates and originates from the point $p_\varphi = p_\psi = 2$). At small deviations from the condition of the root, having order higher than the first, there appears *pseudoregular precession*, which also has the nutation of the dynamical symmetry axis.

### 2. Various Reduced Systems (with respect to $\psi$ and to $\varphi$)

The Lagrange top apex motion in absolute space (fig. 21) can be obtained from canonical equations of the top motion in terms of Euler angles after the Routh reduction with respect to the proper rotation angle $\varphi$, which is a cyclic one. The reduction of the same equations with respect to the precession angle $\psi$ gives the Euler – Poisson equations. They describe the vertical unit vector evolution in the frame of reference, rigidly bound to the top, and have an independent interest from the physical viewpoint. In other words, this system describes motion of the "apex" of absolute space in the noninertial system, attached to the moving top. The apex paths on the corresponding sphere (it is the ordinary Poisson sphere) are quite similar to the paths of dynamical symmetry axis apex in fixed space (fig. 21, 22, 23). However, in case of particular motion, the behavior of various apices is usually different; this difference showing itself in the presence or absence of apex path loops (see fig. 21, 23). To put it more precisely,

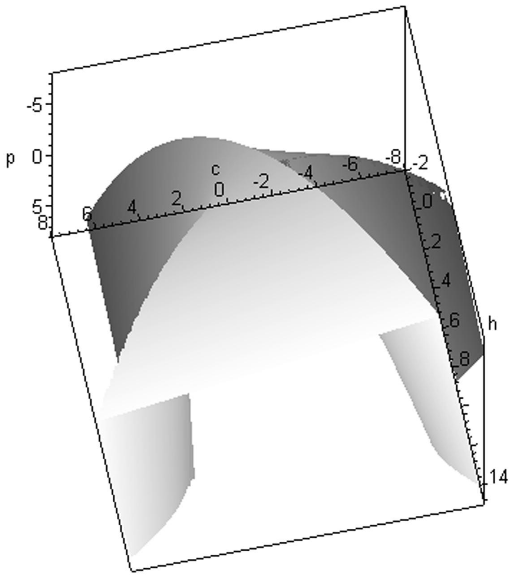



Figure 25. The surface of regular precessions in the space of energy integrals — $h$, area integrals — $c$, and Lagrange integrals — $p$ ($I_1 = 1$, $I_3 = 5$).

the situations, shown at fig. 26, are possible. The indicated picture of motion in the frame of reference, attached to the body, is useful for understanding the proper rotation angle, excluded from the majority of mechanical courses.

To justify the standard analysis incompleteness, one can cite the famous *conception of Hertz* [58] who offered to consider a proper rotation angle to be unobservable, and to connect the presence of corresponding cyclic motion with the "latent masses and parameters", leading to the changes in potential energy. If the same viewpoint be held for a noninertial observer, then the corresponding latent masses and parameters should be ascribed to absolute space.

## 3. Conjugate Poisson Structures

The Lagrange case is characterized by the additional symmetry: for this case there exists the second compatible Poisson structure [31] (that is, the system is a bi-Hamiltonian one). Really, the equations of motion can be obtained if the Hamilton function is defined in the form



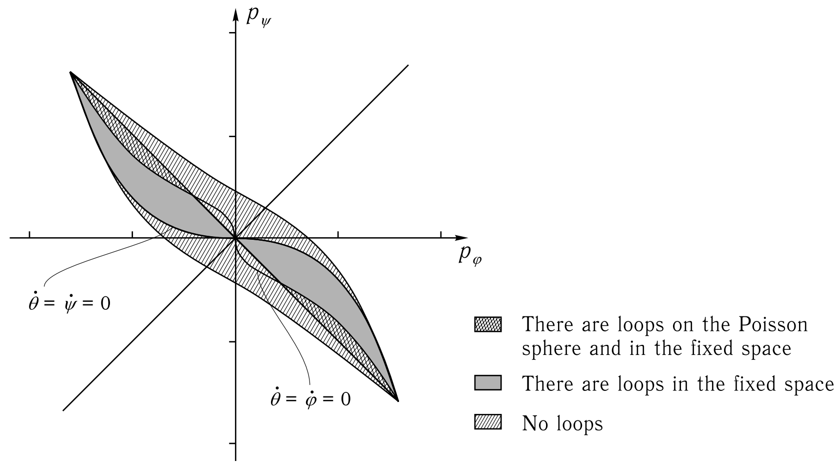



Figure 26. This is a section of the set of regular precessions (fig. 25) by the plane $h = 0$, where the lines of the cuspidal points of the apex of dynamical symmetry axis ($\dot\theta = \dot\psi = 0$) and the apex on the Poisson sphere ($\dot\theta = \dot\varphi = 0$) are also shown. In a narrow shaded domain the path has loops as on the Poisson sphere, so in the absolute space. In the rest of domains the loops are possible only in the absolute space or not possible at all.

$$H = (a-1)M_3\left(\frac{1}{2}(M_1^2 + M_2^2) + \gamma_3\right) + (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3), \quad (3.5)$$
$$a = \text{const}$$

and the Poisson bracket in the form

$$\{\gamma_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{M_1, M_2\} = 1, \quad \{M_i, \gamma_j\} = 0, \quad (3.6)$$

the latter one having annihilators $F_1 = M_3$, $F_2 = (\boldsymbol{\gamma}, \boldsymbol{\gamma})$.

Equations in the Hamiltonian form with bracket (3.5) and Hamiltonian (3.6) may be of use, while studying the Lagrange top perturbations by means of generalized potential and dissipative effects.

## 4. Historical Comments

Lagrange has indicated his case of integrability in the second volume of the *Analytical Mechanics*, where he has also stated the general outline of its integration. A bit later (1815) this very problem was also solved by Poisson who has added to the analytical computations of Lagrange of figures the apex motion path (similar to fig. 21–23), which further on were cited in nearly all textbooks on mechanics.

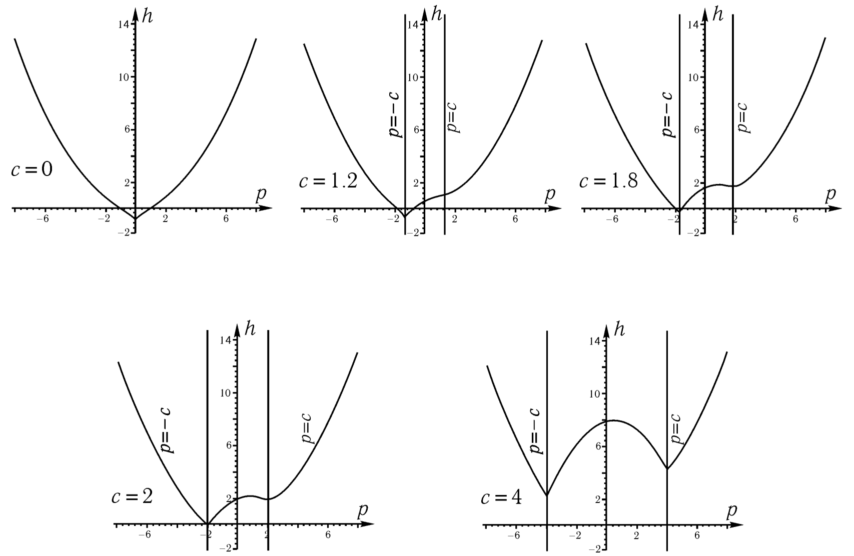



Figure 27. A bifurcational pattern on the plane $(h,\, p)$ at various constant values $c$ ($I_1 = 1$, $I_3 = 5$).

After the theory of Abelian functions had been created and the Euler case had been integrated, Jacobi made an attempt to receive similar quadratures for the Lagrange top. However, his work remained uncompleted. Various forms of general solution (i. e., expressions for angular velocities and for all direction cosines, or the Euler angles) in terms of theta-functions can be found in the books by F. Klein and A. Sommerfeld [238], E. Whittaker [167], A. S. Domogarov [73], W. D. MacMillan [120]. It seems that A. G. Greenhill [220] was one of the first to obtain the general solution. However, all the quadratures found are very complicated and practically are not used.

Jacobi was also trying to give a complete geometrical picture of motion similar to the Poinsot interpretation of the Euler case. He has formulated the proof-free statement that the Lagrange top motion can be resolved into two motions of the Poinsot type — direct and backward. In 1882 this statement was proved by E. Lottner, who published posthumous works of Jacobi. We do not discuss this result and its improvements, offered by Darboux, Halphen and Hess, because they are extremely complicated and artificial [120, 163]. Similarly to the analytical expressions, they are incapable of giving clear impression about the picture of motion.



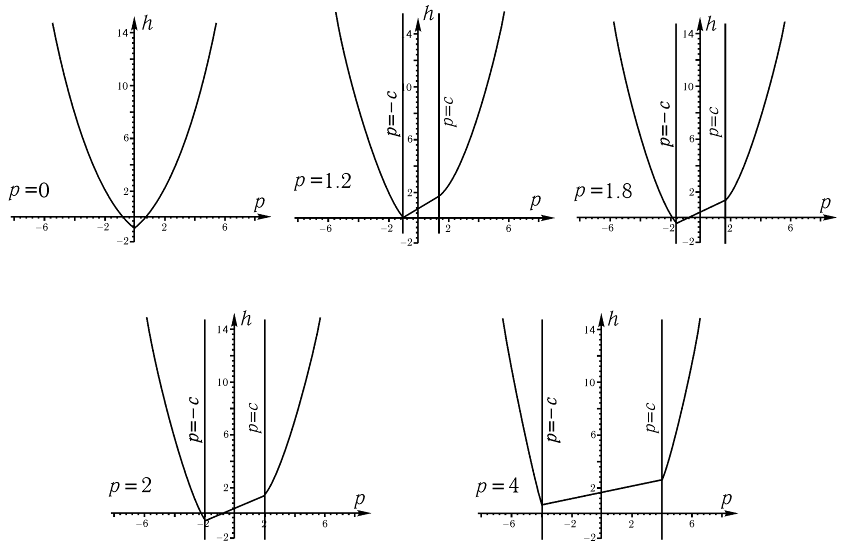



Figure 28. A bifurcational pattern on the plane $(h,\, c)$ at various constant values $p$ ($I_1 = 1$, $I_3 = 5$).

The most complete and physically clear description of the Lagrange top motion is given in the books by K. Magnus [119] and R. Grammel [66]. Here we have made this discussion more invariant and added vivid three-dimensional illustrations. In a certain sense, they show real complexity in classification of various motions of an axially symmetric top.

It should also be noted that the papers [134, 165] show bifurcational curves, not every of which coincides with ours. But if in the paper [165] this happened because of the brevity of exposition, when there was no task of making the complete analysis of motion, the book by M. Audin [134] shows some curves, which seem to be not absolutely right. The final conclusions are difficult to be made here, because the book [134], in spite of its name, is dedicated not to the real motion of tops, but rather to the "explanation" of the well known facts with extra overloading — the complex algebraic geometry formalism.

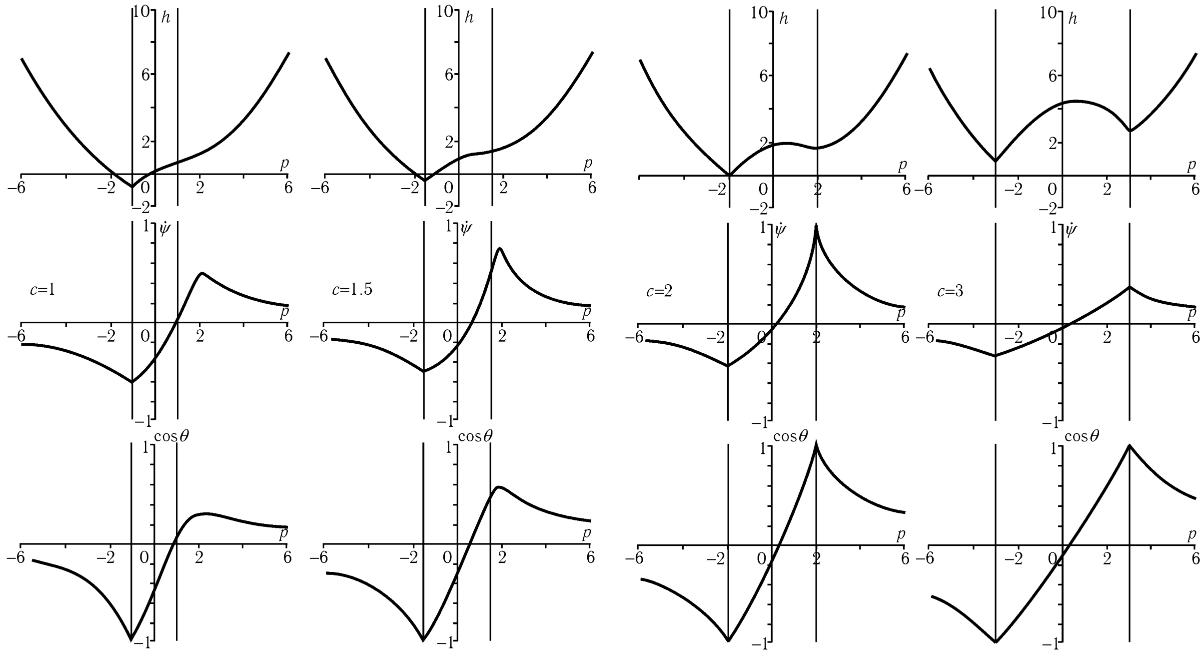



Figure 29. The figure shows sections of the set of regular precessions (see fig. 25) by various planes $c = \mathrm{const}$. It is also indicated here how, depending on the constant of the Lagrange integral $p$, the angular velocity $\dot{\psi}(p)$ and the latitude $\theta(p)$ of regular precessions are changing. In the presence of small friction there occurs the decrease (dissipation) of the constant of integral $p$ — the moment of the proper rotation of top, and the figure shows that, at first, the top tends to take vertical position $\cos\theta \to 1$, and if $c \geqslant 2$, it attains this position at $p = c$; then sometimes it is said that the top "falls asleep". Under further decrease of $p$ the precession direction (the sign of $\dot{\psi}$) changes its sign, and this can also be observed in experiments with a top.

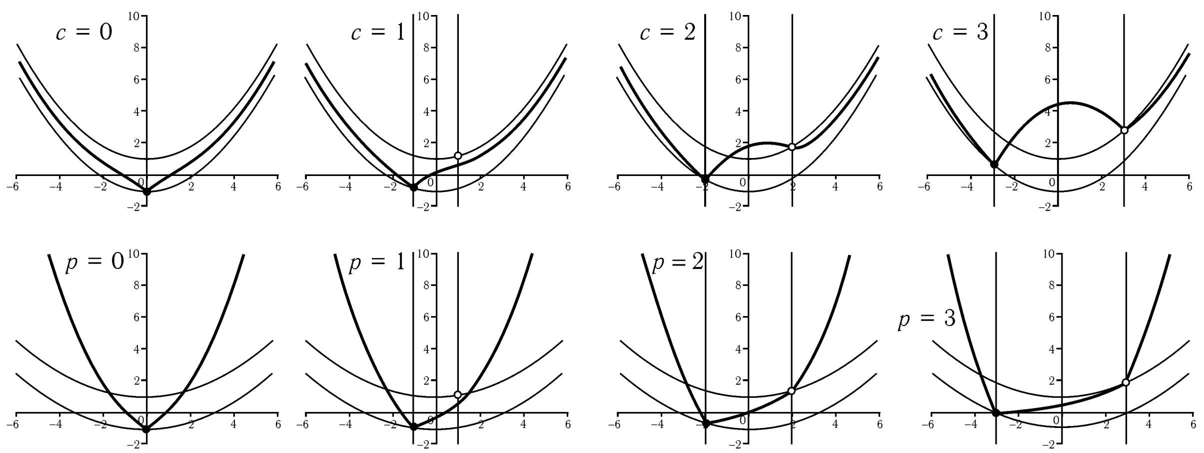



Figure 30. For the Lagrange top there exist two simple particular periodic solutions, for which the top axis is vertical, and the center-of-mass is either higher or lower than the fixation point: these are upper and lower solutions correspondingly. Conventionally, a top, rotating in the upper position, is referred to as "*an asleep top*". In the space of integrals these solutions are specified by the equations: $p = c$, $h = \frac{1}{2}ap^2 + 1$ — an upper solution, $p = -c$, $h = \frac{1}{2}ap^2 - 1$ — a lower solution. The figure shows various sections of the possible motion domains (PMD) and indicates corresponding particular solutions. One can see that the lower solution lies on the PMD boundary and, hence, is always stable. The upper solution, at $p = c < 2$, lies within PMD, and is, as one can show [124], unstable; when $p = c \geqslant 2$, this solution exceeds the PMD boundary, whereby it becomes stable. Here lies *the famous Maeyevskiy condition* of an asleep top stability.

# § 4. The Kowalevskaya Case

Additional integrals in the Euler and Lagrange cases are of a natural physical origin. In the first case the integral is a squared angular momentum module, in the second case its projection on the dynamical symmetry axis. In the non-integrable case, found by S. V. Kowalevskaya (1888), an additional integral does not originate from geometric symmetry. It was discovered almost a hundred years later and is much more complicated from the viewpoint of both explicit integration and qualitative analysis of motion.

In this case the body possesses dynamical symmetry: $a_1 = a_2$, and the center-of-mass lies in the equatorial plane of the inertia ellipsoid $r_3 = 0$. The relation $\frac{a_3}{a_1} = \frac{I_1}{I_3} = 2$ also holds true. The Hamiltonian and the additional integral, found by Kowalevskaya, have the form:

$$\begin{gathered} H = \frac{1}{2}\big(M_1^2 + M_2^2 + 2M_3^2\big) - x\gamma_1, \\ F_3 = \left(\frac{M_1^2 - M_2^2}{2} + x\boldsymbol{\gamma}_1\right)^2 + (M_1M_2 + x\boldsymbol{\gamma}_2)^2 = k^2, \end{gathered} \tag{4.1}$$

where the position vector of the center-of-mass has components $\boldsymbol{r} = (x, 0, 0)$, and the weight is $\mu = 1$ (without any limitation of generality).

### 1. Explicit Integration. The Kowalevskaya Variables

Apart from the additional integral, S. V. Kowalevskaya has found remarkable variables, transforming equations of motion (1.1) into the Abel–Jacobi form (see § 7, ch. 1). In the presence of such a form further integration in terms of theta-functions (of two variables) can be carried out according to a certain general pattern (see [86]). Here we shall show the corresponding replacement only.

The Kowalevskaya variables $s_1$, $s_2$ are determined by means of the formulae

$$\begin{gathered} s_1 = \frac{R - \sqrt{R_1R_2}}{2(z_1 - z_2)^2}, \quad s_2 = \frac{R + \sqrt{R_1R_2}}{2(z_1 - z_2)^2}, \\ z_1 = M_1 + iM_2, \quad z_2 = M_1 - iM_2, \\ R = R(z_1,\, z_2) = \frac{1}{4}z_1^2z_2^2 - \frac{h}{2}(z_1^2 + z_2^2) + c(z_1 + z_2) + \frac{k^2}{4} - 1, \\ R_1 = R(z_1,\, z_1), \qquad R_2 = R(z_2,\, z_2), \end{gathered} \tag{4.2}$$



where $F_1 = (M, \gamma) = c$, $H = h$. To simplify computations we assume $x = 1$ in all cases.

The equations of motion take the form

$$\frac{ds_1}{\sqrt{P(s_1)}} = \frac{dt}{s_1 - s_2}, \qquad \frac{ds_2}{\sqrt{P(s_2)}} = \frac{dt}{s_2 - s_1}, \tag{4.3}$$

where

$$P(s) = \left(\left(2s + \frac{h}{2}\right)^2 - \frac{k^2}{16}\right)\left(4s^3 + 2hs^2 + \left(\frac{h^2}{4} - \frac{k^2}{16} + \frac{1}{4}\right)s + \frac{c^2}{16}\right).$$

On the account of polynomial $P(s)$ having the fifth degree, the quadrature for (4.3) is referred to as *ultra-elliptic* (*hyperelliptic*).

### 2. A Bifurcational Pattern and the Appelrot Classes

The values of integrals $h$, $c$, $k$, at which polynomial $P(s)$ has high order roots, determine in the space of these integrals *a bifurcational pattern* — a set of two-dimensional surfaces, where transformation of the motion type takes place (see fig. 31). The ultra-elliptic quadratures in (4.3) are reduced to elliptic ones, and the corresponding (most remarkable) motions are referred to as *the Appelrot classes* [4]. Various branches of the bifurcational pattern correspond to various Appelrot classes.

It is a common fact, which is easy to show, that the Appelrot classes, determined from the high order of roots of polynomial $P(s) = 0$, coincide with the set of *special Liouville tori*, where integrals $H$, $F_1$, $F_2$, $F_3$ are dependent, i. e., the Jacobi matrix rank $\left\|\frac{\partial(H,\, F_2,\, F_3,\, F_4)}{\partial(\boldsymbol{M},\, \boldsymbol{\gamma})}\right\|$ decreases [170]. It is evident that these special tori define stable and unstable periodic motions and asymptotic paths to the latter in the phase space of a reduced system (i. e., for the Euler–Poisson equations).

The bifurcational pattern with the indication of branch stability is shown at fig. 31. Combined with Poincaré phase sections in canonical variables (for instance, the Andoyaer–Depritones, fig. 32, 33), it is of great use for dynamics since it allows vivid representation of all other paths of an integrable system in a phase space.

Explicit solutions for the Appelrot classes may be obtained directly without any use of equations (4.3). Their construction, concerning non-obvious manipulations, was started by G. Appelrot [4] himself, but the most complete form of

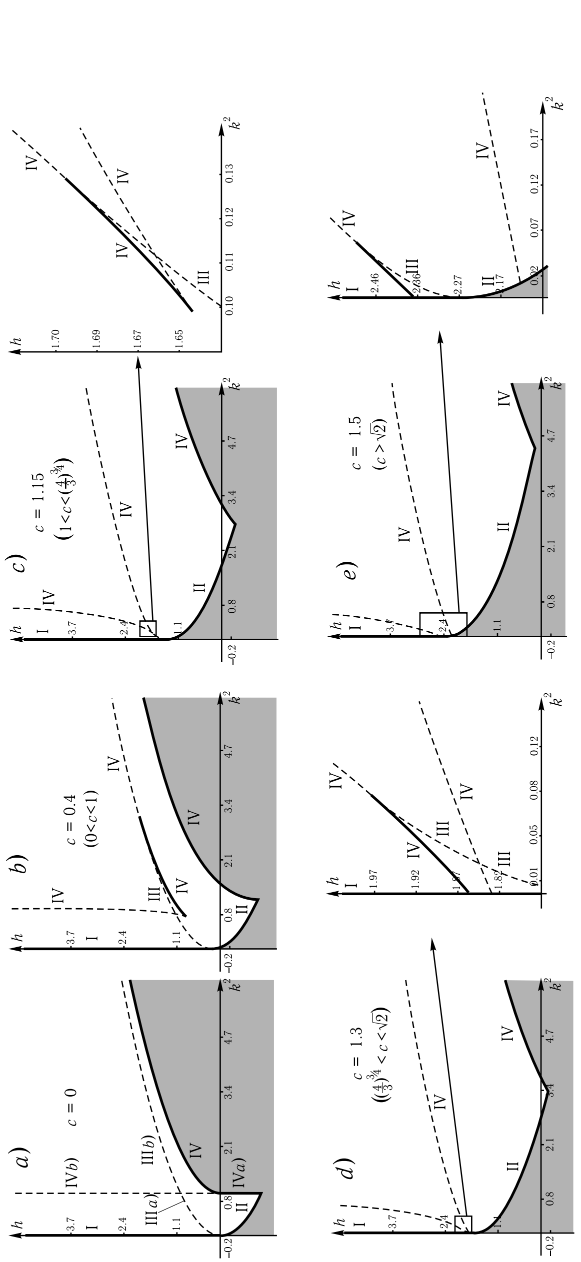


Figure 31. The bifurcational pattern of the Kowalevskaya case at various $c$. The Roman figures stand for the Appelrot classes. Continuous curves correspond to stable periodic solutions, the dotted ones to unstable solutions and separatrices.



this construction was given by the Donetsk mechanic A. I. Dokshevich [72]. We shall show some of his results, mainly connected with periodic and asymptotic motions (which are of the highest importance for dynamics) and shall try to clarify their mechanical meaning.

There exist four Appelrot classes.

**I. The Delauney solution [70]** : for this solution $k^2 = 0$, $h > c^2$, and there appears two invariant relations

$$\frac{M_1^2 - M_2^2}{2} + x\gamma_1 = 0, \quad M_1M_2 + x\gamma_2 = 0, \tag{4.4}$$

determining the Euler – Poisson periodic solution.

It turns out that in this case, when the area constant equals zero $c = 0$, the motion is periodic not for the reduced system (on the Poisson sphere) only, *but in the absolute space*, [60] as well (see fig. 36–39).

To obtain the explicit quadrature, on the level of integrals and invariant relations (4.4) we shall express all variables in terms of $M_1$

$$\begin{gathered} M_2^2 = 2z - M_1^2, \quad M_3^2 = h - M_1^2, \\ x\gamma_1 = -M_1^2 + z, \quad x\gamma_1 = -M_1(2z - M_1)^{1/2}, \quad x\gamma_3 = (x^2 - z^2)^{1/2}, \\ z = \frac{M_1^2 + M_2^2}{2} = (\gamma_1^2 + \gamma_2^2)^{1/2} = x\frac{-cM_1 \pm \sqrt{(h - c^2)(h - M_1^2)}}{h}. \end{gathered} \tag{4.5}$$

Then, for $M_1$ we obtain the quadrature

$$\dot{M}_1 = M_2M_3 = \left((h - M_1^2)(2z - M_1^2)\right)^{1/2}, \tag{4.6}$$

which is elliptic at $h = c^2$. At $c = 0$ it is also possible to obtain a simpler explicit solution if instead of $M_1$ a variable $M_3$ be used.

From figure 31 it follows that when $c$ is increased up to $c = \left(\frac{3}{4}\right)^{3/4}$, the branch of the fourth Appelrot class "cuts" into the Delauney solution, and, under further increase of $c$ up to $c^2 < 2$, breaks it into three parts. When $c^2 = 2$ at a point $h = 2$, $k^2 = 0$, the branches of all four Appelrot classes intersect. Their intersection point is assigned to an unstable fixed point on the Poisson sphere (*the Staude rotation*) (see § 6 ch. 2) and one-dimensional motion (asymptotic to this point), which is easily computed from (4.6) in terms of elementary functions

$$M_1 = \sqrt{2x}\,\frac{3 + \operatorname{ch}^2 u \pm 4\operatorname{ch} u}{9 - \operatorname{ch}^2 u}, \quad u = 2\sqrt{x}t. \tag{4.7}$$

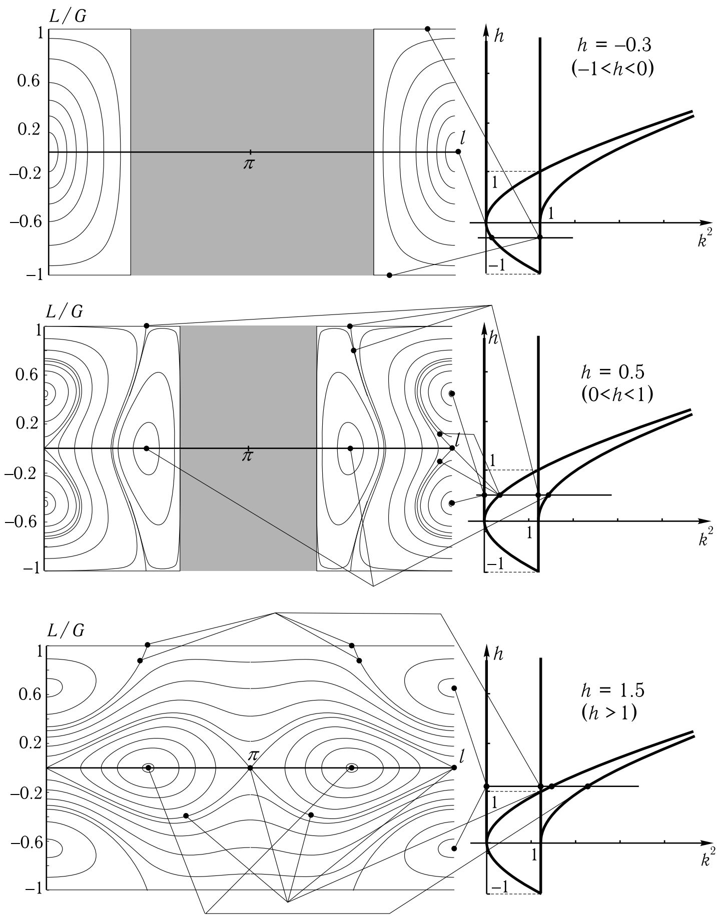
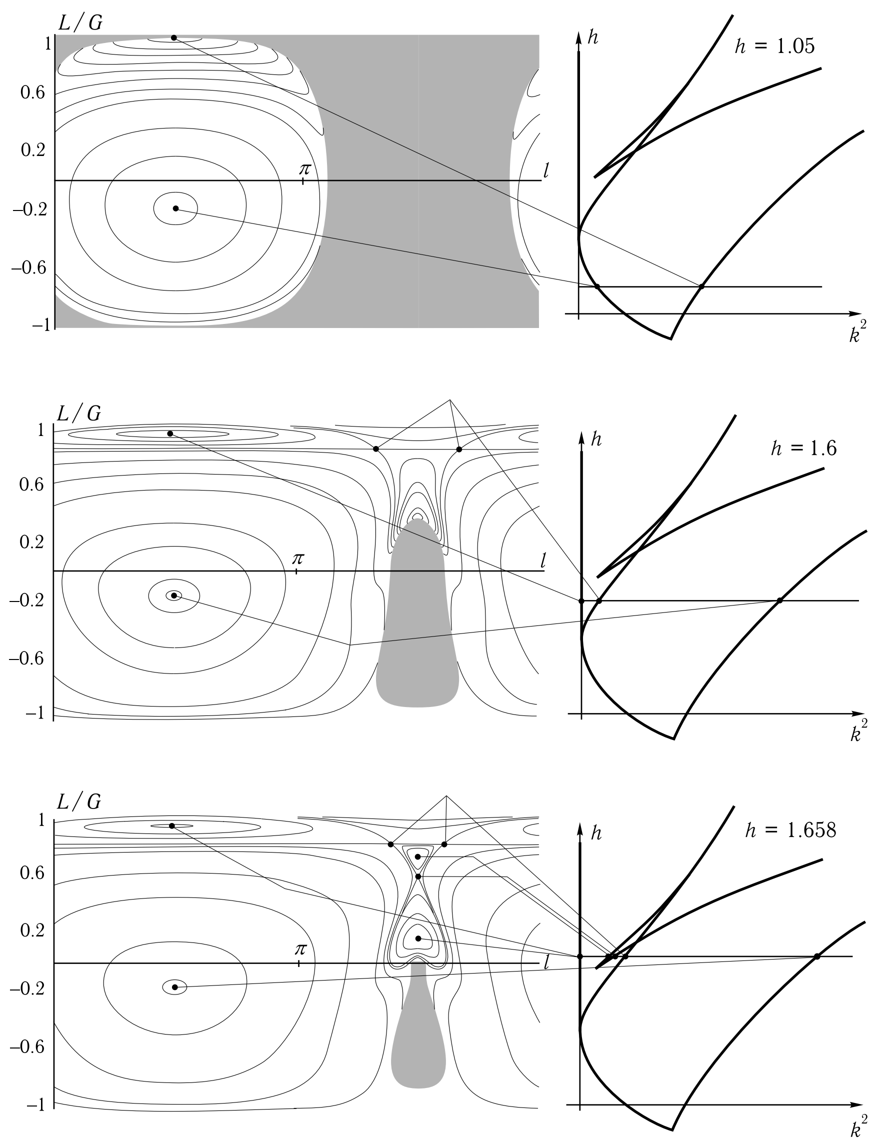



Figure 32. A phase portrait (the section by the plane $g = \pi/2$) for the Kowalevskaya case when the area constant equals zero $c = 0$. Three qualitatively different phase portraits are presented. The pictures vividly show what kind of portrait transformations and periodical solution bifurcations takes place, when critical energy levels $h = 0$ and $h = 1$ are intersecting. (The grey color shows nonphysical domain of values $l$, $L/G$ at the values of integrals $h$, $c$ given.)

At $c^2 > 2$ one branch of the fourth class also "cuts" in the Delauney solution, and its other branch intersects the part of parabola, corresponding to the second class.





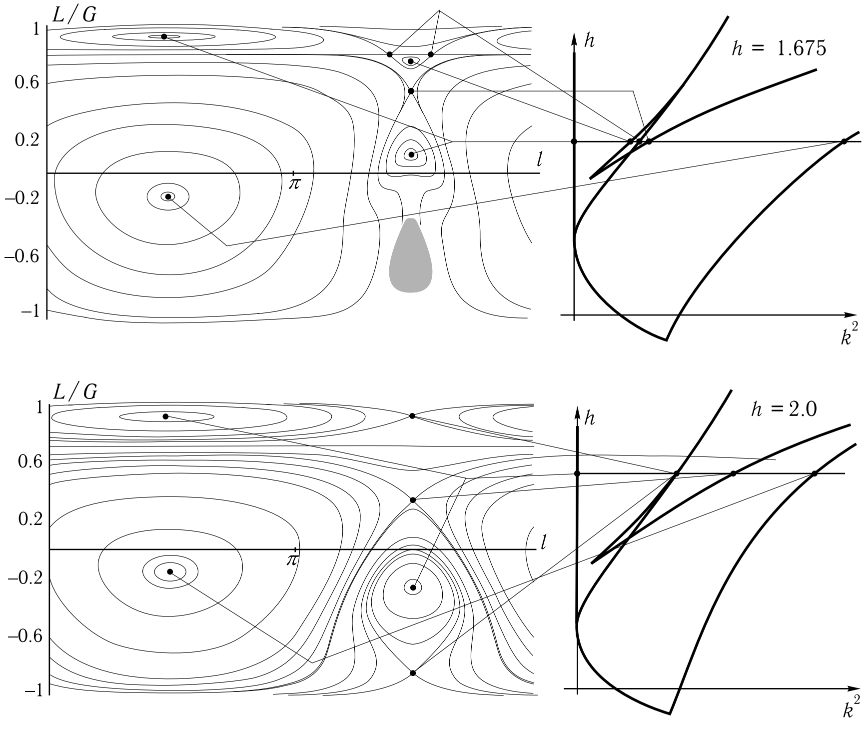



Figure 33. A phase portrait (the section by the plane $g = \pi$) for the Kowalevskaya case at $c = 1.15$ and fixed energy values $h$, which have phase portraits of qualitatively different type corresponding. The variables $l$ and $L/G$ correspond to the cylindrical development of a sphere, and the phase portrait is symmetrical with respect to meridian $l = \pi/2$, $\frac{3}{4}\pi$. (The bifurcational pattern at the right picture is shown as a scheme, without adhering to a scale.)

**II.** The second class solutions lie on the lower branch of parabola $(h - c^2)^2 = k^2$, and $\frac{1}{2}c^2 - 1 \leqslant h \leqslant c^2$. At $c = 0$ this class possesses stable periodic paths, and the body oscillates in a meridional plane, passing through the center-of-mass, and the conditions $M_1 = M_3 = 0$, $\gamma_2 = 0$ hold true.

At $c \neq 0$ there exist additional invariant relations

$$M_3 = c\gamma_3, \quad M_1^2 + M_2^2 + \frac{M_1}{c} = k, \tag{4.8}$$



and the explicit integration is carried our in [72]. Starting from $c > \sqrt{2}$, the branches of the second and fourth classes start intersecting.

**III.** This class is assigned to the branch of parabola above the point of contact with the axis $k^2 = 0$, which satisfies conditions

$$(h - c^2)^2 = k^2, \quad c^2 \leqslant h \leqslant c^2 + \frac{1}{2c^2}. \tag{4.9}$$

At $c = 0$ these conditions specify the whole upper branch of parabola, and at $c \neq 0$ this branch is limited from above by one of the IV class branches.

From the physical point of view, the third class corresponds to unstable periodic motions and motions, which are asymptotic to them. At $c = 0$ periodic motion for the part of the branch III a) represents oscillations of a physical pendulum in a meridional plane, passing through the center-of-mass, and for the part III b) — rotations in the same plane. These solutions meet at the point $h = {}$ $= 1$, which is an upper unstable position of equilibrium. Its instability can be rigorously proved in various ways [152]. Further on, this proof will be obtained by explicit construction of an asymptotic solution.

We shall be using the following parametrization of the general level of integrals of motion, corresponding to the third Appelrot class at a zero area constant $c = 0$ [72]

$$\begin{gathered} M_1 = \sqrt{M_1^2 + M_3^2}\sin\varphi, \quad M_3 = \sqrt{M_1^2 + M_3^2}\cos\varphi, \\ k_1 = k\cos 2\theta, \qquad k_2 = k\sin 2\theta, \end{gathered} \tag{4.10}$$

where $k_1 = \gamma_1 + \dfrac{M_1^2 - M_2^2}{2}$, $k_2 = \gamma_2 + M_1M_2$ (at $x = 1$), the Kowalevskaya integral having the form $k_1^2 + k_2^2 = k^2$.

Differentiating $\varphi$ with respect to time, we shall obtain

$$\dot\varphi = M_2 - \frac{M_1k_2}{M_1^2 + M_3^2}. \tag{4.11}$$

After one more differentiation (4.11) and elimination of $M_2$ by means of (4.11), taking into account $h = k > 0$, we have

$$2\ddot\varphi\cos\varphi + \dot\varphi\sin\varphi = 2h\cos^2\varphi\sin\varphi. \tag{4.12}$$

Having multiplied (4.12) by $\dfrac{\dot\varphi}{\cos^2\varphi}$ and having integrated it with respect to time, we obtain

$$\frac{\dot\varphi^2}{\cos\varphi} + 2h\cos\varphi = c_1 = \text{const}.$$

The constant of integration is found from the condition $\varphi = 0$, at which $M_1 = 0$, $\dot\varphi = M_2$, and thus $c_1^2 = 4x^2$. So,

$$\dot\varphi^2 = 2(x - k\cos\varphi)\cos\varphi, \quad k > 0. \tag{4.13}$$

Remark. At $c \neq 0$ for a similar (but a bit different) angular variable equation [72]

$$\dot\varphi^2 = 2(x - (k + c^2)\cos\varphi)\cos\varphi$$

is obtained.

For the angle $\theta$ we obtain equation

$$\dot\theta = -M_3 = -\sqrt{M_1^2 + M_3^2}\cos\varphi,$$

which after taking into account the energy integral $M_1^2 + M_3^2 - k_1 = h$ and the condition $h = k$, leading to the equality $\sqrt{M_1^2 + M_3^2} = \pm\sqrt{2k}\cos\theta$, is reduced to the following one

$$\dot\theta = \sqrt{2k}\cos\varphi\cos\theta.$$

After replacement $\cos\theta = (\operatorname{ch} u)^{-1}$ it can be written in the form

$$\dot u = \sqrt{2k}\cos\varphi.$$

So, the complete system of equations, specifying asymptotic paths of the third Appelrot class under conditions $c = 0$, $h = k > 0$, is reduced to

$$\begin{gathered} 2\dot\zeta = (1 - \zeta^2)(x - k + (x + k)\zeta^2), \quad \zeta = \operatorname{tg}\frac{\varphi}{2}, \\ \dot u = \sqrt{2k}\cos\varphi, \quad \operatorname{ch} u = (\cos\theta)^{-1}. \end{gathered} \tag{4.14}$$

Its solutions have the form

$$\begin{aligned} &1.\ \ k < x, \quad \zeta = \operatorname{cn}(\sqrt{x}t,\ k_0), && k_0^2 = \frac{x + k}{2x}, \\ &2.\ \ k > x, \quad \zeta = \operatorname{dn}\Big(\sqrt{\frac{x + k}{2}}t,\ k_0\Big), && k_0^2 = \frac{2x}{x + k}, \\ &3.\ \ k = x, \quad \zeta = (\operatorname{ch}\sqrt{x}t)^{-1}, \end{aligned}$$

where $k_0$ is a module of the corresponding elliptic functions of Jacobi.

Using 1–3, one can show that $\dot u$ is a function of constant signs, i. e. these solutions in case 1–2 specify asymptotic motions to the periodic solution, and in case 3 — to a fixed point. (Analytical quadratures in case $c \neq 0$ are more cumbersome [72].)



**IV.** This class consists of two branches (see fig. 31), one corresponding to stable periodic motions, and the other — to unstable motions and separatrices. At $c = 0$ these branches meet at the point $k^2 = x^2 = 1$, $h = 0$.

At $c \neq 0$ parametric equations of branches have the form

$$k^2 = 1 + tc + \frac{t^4}{4}, \qquad h = \frac{t^2}{2} - \frac{c}{t}, \tag{4.15}$$

$$\begin{aligned} &t \in (-\infty,\ 0) \cup (c,\ +\infty), && \text{with} \quad c > 0, \\ &t \in (-\infty,\ +\infty) \setminus \{0\}, && \text{with} \quad c < 0, \end{aligned}$$

at $c = 0$

$$\begin{aligned} &1.\ \ k^2 = x^2, \quad h < 0, \quad h^2 = k^2 + x^2 && \text{(the branch IVa);} \\ &2.\ \ k^2 = x^2, \quad h > 0 && \text{(the branch IVb).} \end{aligned}$$

Stable and unstable periodic motions for the fourth Appelrot class in the Kowalevskaya case (and also in the more general case, when the tensor of inertia has the form $\mathbf{I} = \operatorname{diag}(1,\ a,\ 2)$, $a = \text{const}$, and the solution itself does not depend on $a$) were found by D. K. Bobylev [15] and V. A. Steklov [161] (see also § 6).

**Bobylev – Steklov solution.** For this solution the following relations always hold true

$$M_2 = 0, \quad M_1 = m = \text{const};$$

these relations allow to express $\boldsymbol{\gamma}$ in terms of $M_3$

$$\gamma_1 = \frac{c}{m} - M_3^2, \quad \gamma_2 = \Big(k^2 - \Big(\frac{1}{2}m^2 - \frac{c}{m} + M_3^2\Big)^2\Big)^{1/2}, \quad \gamma_3 = mM_3$$

and obtain elliptic quadrature for $M_3$

$$\dot M_3 = -\Big(k^2 - \Big(\frac{1}{2}m^2 - \frac{c}{m} + M_3^2\Big)^2\Big)^{1/2}. \tag{4.16}$$

Here, $h$ and $k^2$ are specified by the parametric equation

$$h = \frac{1}{2}m^2 - \frac{c}{m}, \quad k^2 = 1 + \frac{1}{2}m^4 + cm,$$

i. e., they coincide with (4.15). At $c = 0$ in the fourth class there appear motions, corresponding to oscillations and rotations according to the law of physical pendulum in the equatorial plane of the ellipsoid of inertia. For these solutions

$$M_1 = m = 0, \quad \gamma_3 = 0, \quad \dot M_3 = -(1 - (h - M_3^2)^2)^{1/2}.$$

Asymptotic solutions for arbitrary values of $c \neq 0$ are determined in [72], but are very cumbersome. We shall indicate these solutions under additional conditions
$$k^2 = x^2, \quad h > 0, \quad c = 0. \tag{4.17}$$
To do that, we shall use a curious involution transformation, found by A. I. Dokshevich: $(\boldsymbol{M}, \boldsymbol{\gamma}) \mapsto (\boldsymbol{L}, \boldsymbol{s})$ (the square of which is identical):
$$\begin{aligned} L_1 &= -\frac{M_1}{M_1^2+M_2^2}, & s_1 &= -\gamma_1 + 2x\gamma_3^2\frac{M_1^2-M_2^2}{(M_1^2+M_2^2)^2}, \\ L_2 &= -\frac{M_2}{M_1^2+M_2^2}, & s_2 &= -\gamma_2 + 4x\gamma_3^2\frac{M_1M_2}{(M_1^2+M_2^2)^2}, \\ L_3 &= M_3 + 2x\gamma_3\frac{M_1}{M_1^2+M_2^2}, & s_3 &= \frac{\gamma_3}{M_1^2+M_2^2}. \end{aligned} \tag{4.18}$$
In terms of new variables $(\boldsymbol{L}, \boldsymbol{s})$ the equations of motion are written as
$$\begin{aligned} \dot L_1 &= L_2L_3, & \dot s_1 &= 2L_3s_2 - 4(k^2-x^2)s_3L_2, \\ \dot L_2 &= -L_1L_3 - xs_3, & \dot s_2 &= -2L_3s_1 + 4(k^2-x^2)s_1L_3, \\ \dot L_3 &= -2xcL_2 + xs_2, & \dot s_3 &= s_1L_2 - s_2L_1. \end{aligned} \tag{4.19}$$
Under condition (4.17) in system (4.19) the equations for $L_3$, $s_1$, $s_2$ separate and are reduced to quadratures
$$\begin{gathered} s_2 = (1-s_1^2)^{1/2}, \quad L_3 = (h+xs_1)^{1/2}, \\ \dot s_1 = 2\sqrt{(h+xs_1)(1-s_1^2)}. \end{gathered} \tag{4.20}$$
To obtain the solution of complete system (4.19), it is sufficient to find solution of the second order linear equation with coefficients, explicitly depending on time
$$\begin{gathered} L_1 = s_1^{-1}\Big(-L_3s_3 \mp s_2\sqrt{hs_3^2 - \frac{1}{4x}s_1}\Big), \quad L_2 = \sqrt{hs_3^2 - \frac{1}{4x}s_1}, \\ \ddot s_3 = -x(1+2s_1)s_3. \end{gathered} \tag{4.21}$$
Equations (4.20), (4.21) specify asymptotic solutions to periodic motions under conditions (4.17) (see fig. 45).



At $h = x$, which corresponds to the energy of the upper unstable equilibrium position, we shall obtain one more (in addition to the third class) solution, asymptotic to it, in terms of elementary functions
$$s_1 = 1 - 2\,\mathrm{th}\,u, \quad s_2 = 2\frac{\mathrm{th}\,u}{\mathrm{ch}\,u}, \quad L_3 = -\frac{\sqrt{2x}}{\mathrm{ch}\,u}, \quad u = \sqrt{2x}t.$$
Appelrot classes specify the simplest motions both in the reduced, and in the absolute phase space. The rest of the Kowalevskaya top motions have quasiperiodic character and depend on the corresponding domain of a bifurcational pattern. Under the Kowalevskaya case perturbation in the vicinity of unstable solutions and their separatrices there arises a stochastic layer (fig. 63). Unfortunately, the (asymptotic) solutions, presented in this section, did not make it possible yet to make any progress in the analytical investigation of nonintegrability of the perturbed Kowalevskaya top (via variational methods, at $c = 0$, nonintegrability has been proved in [22]).

### 3. Phase Portrait and Visualization of the Most Remarkable Solutions

For each fixed value of the area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = c$, specifying various types of the bifurcational patterns on the plane $(k^2, h)$, there exists its own set of phase portraits. By fixing the energy level $h$, we shall obtain several various types of phase portraits, specified by the intersections of a straight line $h = \mathrm{const}$ with a bifurcational pattern. Here we present two series of phase portraits, corresponding to the simplest (at $c = 0$, fig. 32) bifurcational pattern and to the most complex (at $1 < c < \left(\frac{4}{3}\right)^{3/4}$, fig. 33) one. Further, we shall also show a series of the "most remarkable" solutions on the Poisson sphere and in the absolute space.

Remark. The invariant tori topology is also investigated by means of the Poincaré sections in [205], in terms of other variables and without clarification of the mechanical meaning of different motions (in particular, the stability analysis).

**Phase portrait at $c = 0$.** In this case the bifurcational pattern consists of two parabola parts and two straight lines (see fig. 31 *a*). The physical meaning of branches, corresponding to the parabola $h^2 = k^2$ and to the straight line $k^2 = 1$, is especially simple and is described above. On the parabola there lie solutions, specifying planar oscillations and rotations of a rigid body in the meridional plane (around the axis $Oy$, perpendicular to the axis $Ox$, where the center-of-mass is situated), and on the straight line there lie planar oscillations and rotations in the equatorial plane (around the axis $Oz$). On two other branches

$k^2 = 0$ and $h^2 = k^2 - 1$ there are solutions of Delauney and Bobylev – Steklov, correspondingly.

Above we have given phase portraits with the indication of the place of the bifurcational pattern, where they are situated. As it follows from the figure 31 *a*), there exist three intervals for the constant energy $h$: $(-1, 0)$, (0,1), $(1, \infty)$, each one being assigned to qualitatively different types of phase portraits (see fig. 32).

**Phase portrait at $c = 1.15$ $\left(1 < c < \left(\frac{4}{3}\right)^{3/4}\right)$.** By means of the bifurcational diagram (fig. 31 *c*) one can determine that there exist five intervals of energy, each one being assigned to its own type of phase portrait (see fig. 33). In this case periodical solutions, corresponding to the bifurcational pattern branches, do not look so simple, as at $c = 0$, though they are approaching these solutions at $h \gg c$.

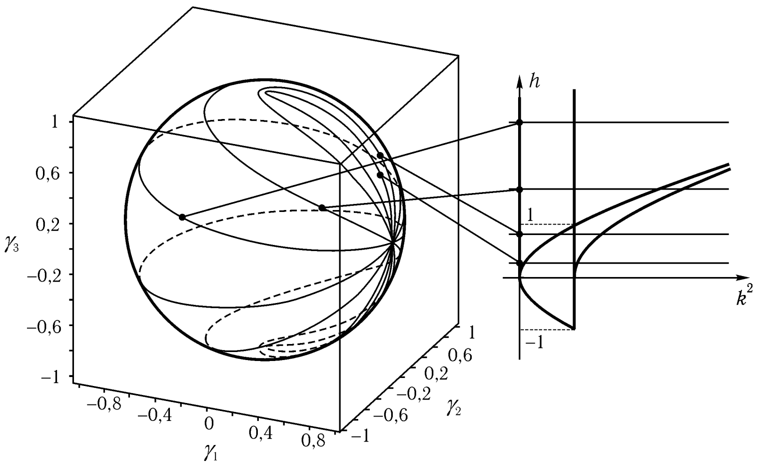



Figure 34. Delauney solution. The motion of the unit vector $\boldsymbol{\gamma}$ at a zero area constant $(c = 0)$ and various values of energy.

Remark. To construct phase portraits, we use the Poincaré sections in terms of Andoyaer – Deprit variables, described in § 3 ch. 1. For $c = 0$ we choose the secant plane in the form $g = \frac{\pi}{2}$, and for $c = 1.15$ we choose $g = \pi$. That is because in this case not every periodic solution intersects the plane $g = \frac{\pi}{2}$. We should also note the various type of phase portrait symmetry on a sphere $(l, L/G)$: at $g = \frac{\pi}{2}$ the portrait is symmetrical



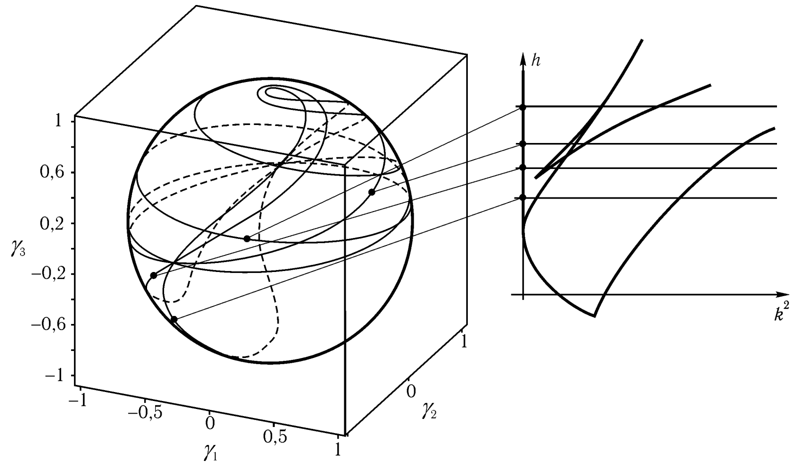



Figure 35. Delauney solution. The motion of the unit vector $\boldsymbol{\gamma}$ at a non-zero area constant $(c = 1.15)$ and various values of energy $h$.

with respect to the equator (of the axis $L/G = 0$), and at $g = \pi$ it is symmetrical with respect to the meridional plane $(l = \frac{\pi}{2}, \frac{3}{2}\pi)$.

Let us now turn to visualization of the most interesting motions of a rigid body in the reduced and absolute space.

**The Delauney solution ($k^2 = 0$).** In this case the path of the vertical unit vector $\boldsymbol{\gamma}$ on the Poisson sphere is represented by the figure-of-eight curves (see fig. 34, 35), at $c = 0$ (fig. 34), the self-intersection points of these "figure-of-eights" coinciding and having coordinates $\boldsymbol{\gamma} = (1, 0, 0)$. This point specifies the lower position of the body center-of-mass. Under increase of $c$ on the Poisson sphere there also appear irregular "figure-of-eights", all intersecting in two points on the Poisson sphere equator(see fig. 35).

It is known that, under $c = 0$, the Delauney solution specifies periodic motions not only in the reduced system, but in the absolute space as well [61]. At $c \neq 0$, it is not valid already, and the body motion in the absolute space is quasiperiodic. Figures 36–39 show paths of three rigid body apices at $c = 0$ and various values of energy. All the figures have fixed axes $OXYZ$ directed arbitrarily to have better exposition of the paths obtained.

**The Bobylev – Steklov solution.** Bobylev – Steklov solution is situated on the lower right branch of the bifurcational pattern (see fig. 31), and is assigned to the stable periodic solution on the Poisson sphere (see fig. 40, 41).

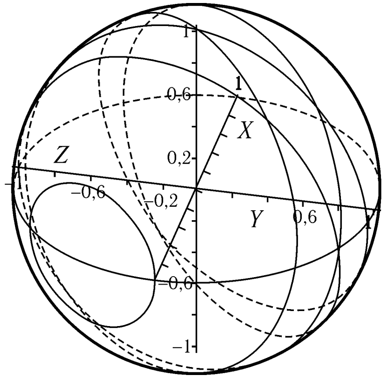



Figure 36. The Delauney solution. Motion of apices in a fixed frame of reference a zero area constant ($c = 0$).

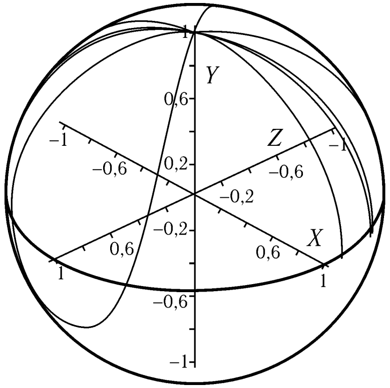



Figure 37. The Delauney solution. Motion of an apex of the center-of-mass at $c = 0$ and various $h$.

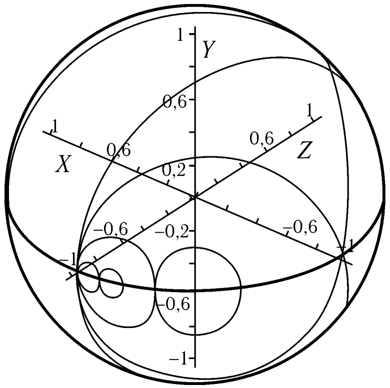



Figure 38. The Delauney solution. Motion of an apex, situated in the equatorial plane perpendicular to the position vector of the center-of-mass at $c = 0$ and various $h$.

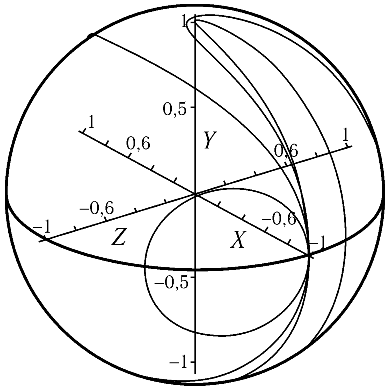



Figure 39. The Delauney solution. Motion of a dynamical symmetry axis apex at $c = 0$ and various $h$.

Fig. 40 vividly shows that at $c = 0$ all the paths on the Poisson sphere pass through the equator points $(0, 1, 0)$ and $(0, -1, 0)$ without intersecting meridional plane $\gamma_1 = 0$. This is assigned to the a remarkable motion of the center-of-mass in an absolute space — *it describes curves having cuspidal points, lying on the equator at any values of energy* (see fig. 42).



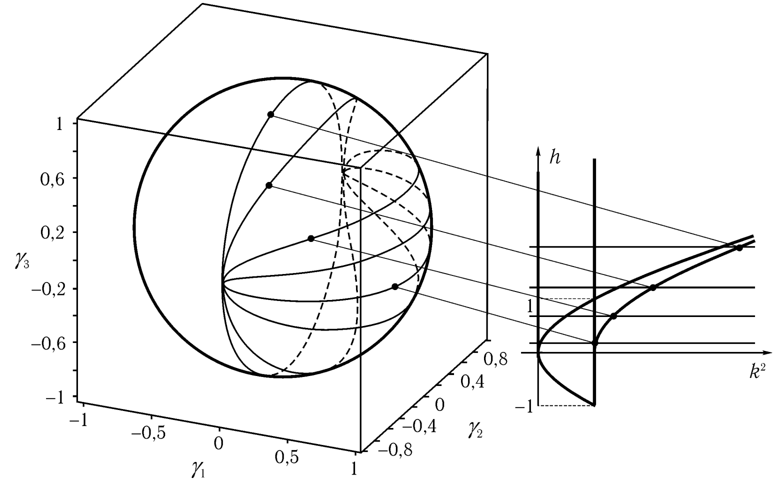



Figure 40. The Bobylev – Steklov solution. The vertical unit vector motion on the Poisson sphere at $c = 0$ and various values of energy.

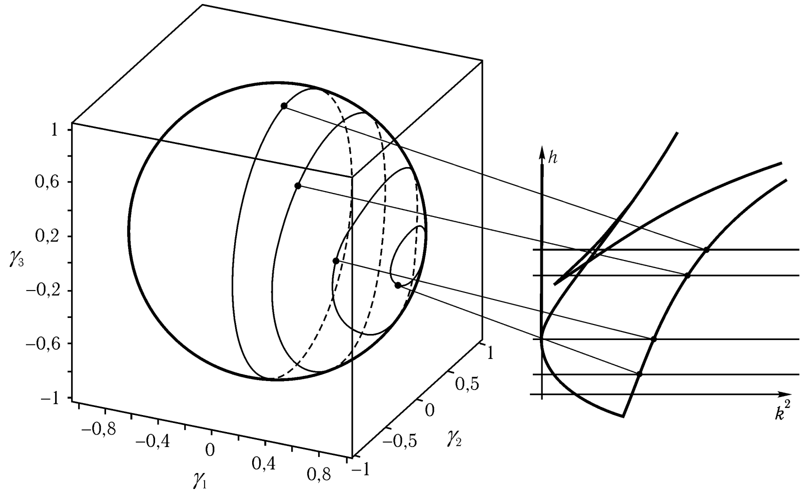



Figure 41. Bobylev – Steklov solution. The vertical unit vector motion on the Poisson sphere at $c \neq 0$ ($c = 1.15$) and various values of energy.

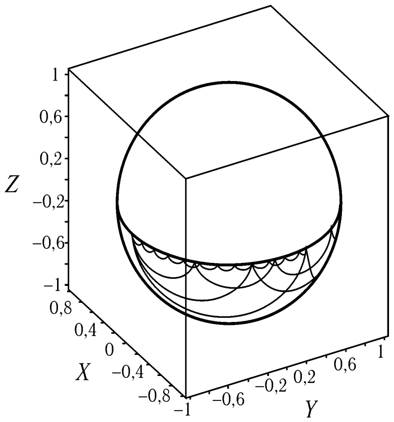



Figure 42. The Bobylev – Steklov solution. Motion of an apex, passing through the center-of-mass in a fixed space at $c = 0$ and various $h$.

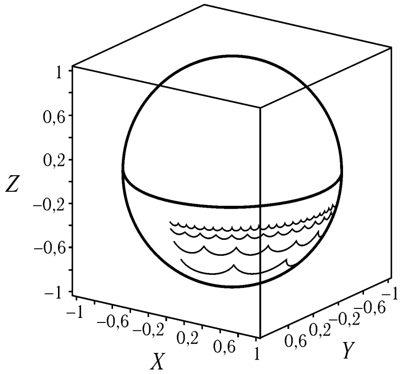



Figure 43. The Bobylev – Steklov solution. The motion of an apex, passing through the center-of-mass in a fixed space at $c \neq 0$ ($c = 1.15$) and various $h$.

At $c \neq 0$ the paths on the Poisson sphere are shown at fig. 41. In this case the apex of the center-of-mass describes curves with cuspidal points, lying at the same latitude, which depends on the constant of energy $h$, in a fixed space (see fig. 43). Physically, the Bobylev – Steklov solution may be implemented in the following way — the body is being twisted around the axis, passing through the center-of-mass and arbitrary in the absolute space, and let gone without any initial impulse.

Remark. The motion of the rest of apices in the fixed space is rather intricate, and we omit it.

**Unstable periodic solutions and separatrices** for the Kowalevskaya case look rather intricately both on the Poisson sphere and in the fixed space. Fig. 45 shows motion paths, corresponding to separatrices at $c \neq 0$ ($c = 1.15$) and the same value of energy $h = 2$. It is clearly seen that the path spends most of its time in the vicinity of the periodic solution; this fact is shown at the figure by darker shade in this domain.

In a certain sense, these paths represent all the complexity of the integrable Kowalevskaya case, some motions in which have visually chaotic character (in the absolute space the motion looks even more irregular).

Remark 1. We shall indicate one more representation of the Kowalevskaya integral



as a sum of squares. To do that, we shall use the moment projections on the semistationary axes

$$S_1 = M_1\gamma_1 + M_2\gamma_2, \quad S_3 = M_1\gamma_2 - M_2\gamma_1.$$

One can show that the Kowalevskaya integral allows the notation in the form

$$F = \left(\frac{M_1^2 + M_2^2}{2}\right)^2 + x(M_1S_1 + M_2S_2) + x^2(\gamma_1^2 + \gamma_2^2).$$

Assuming $\boldsymbol{S} = (S_1, S_2)$ and $\widetilde{\boldsymbol{M}} = (M_1, M_2)$ to be two-dimensional vectors, we shall designate the angle between these vectors as $\lambda$ (see fig. 44). Taking into account that $\gamma_1^2 + \gamma_2^2 = \sin^2\theta$, where $\theta$ is an angle between the vertical line and the axis of symmetry of the inertia ellipsoid, we shall write the Kowalevskaya integral in the form

$$F = \frac{1}{4}G^4\sin^2\lambda + \left(\frac{G^2\cos\lambda}{2} + x\sin\theta\right)^2 = k^2, \quad G^2 = \boldsymbol{M}^2.$$

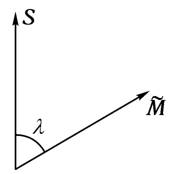



Figure 44

Remark 2. We shall also indicate a curious nonlinear transformation, preserving the structure of algebra $so(3)$:

$$K_1 = \frac{M_1^2 - M_2^2}{2\sqrt{M_1^2 + M_2^2}}, \quad K_2 = \frac{M_1M_2}{\sqrt{M_1^2 + M_2^2}}, \quad K_3 = \frac{1}{2}M_3.$$

It can be shown that from the viewpoint of Andoyaer – Deprit canonical variables it corresponds to the canonical transformation of the kind $(L, l) \mapsto \left(\frac{L}{2}, 2l\right)$.

Remark 3. The papers [224, 268] indicate a family of systems on the sphere $S^2$, allowing the fourth-degree integral in terms of the moments, which cannot be reduced to the Kowalevskaya case (or to its generalization, shown by Goryachev). The paper [267] offers the analogical structure for systems with the third-degree integral. It should only be noted that these papers do not have a single explicit form of the additional integral, and the corresponding family results from the solution of a certain differential equation, for which the theorems of existence are being established.

## 4. Historical Comments

**The Kowalevskaya method.** S. V. Kowalevskaya has found the general case of integrability, following not some physical considerations, but developing the ideas of K. Weierstrass, P. Painleve, and H. Poincaré about the investigation of analytical expansion of solutions of a system of ordinary differential equations to the complex plane of time. S. V. Kowalevskaya has supposed that in integrable cases the general solution on the complex plane does not have any singularities, other than the poles.

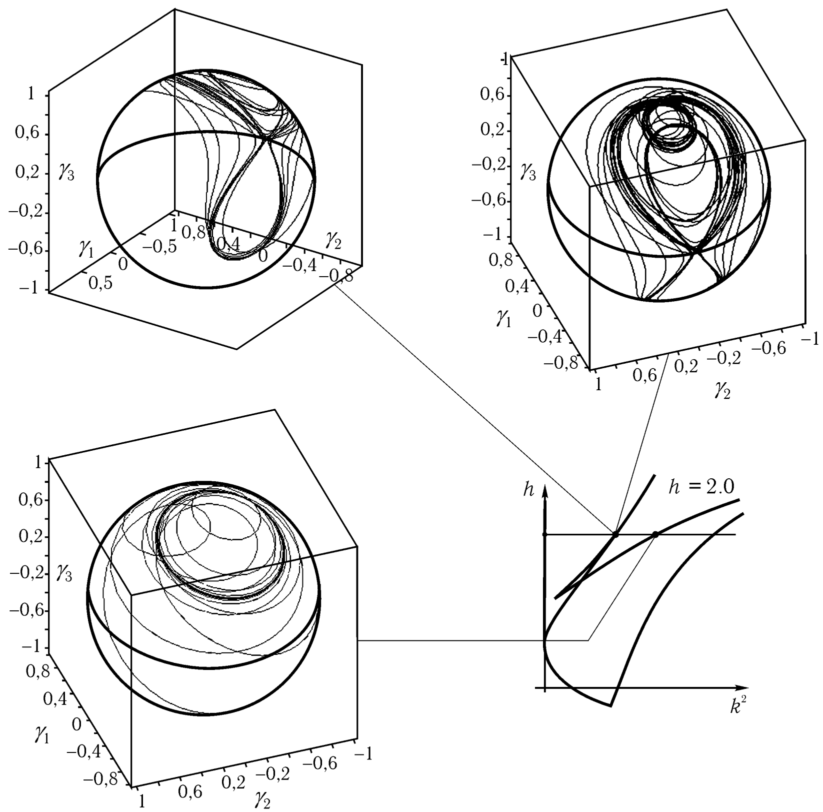



Figure 45. Paths on the Poisson sphere for solutions, asymptotic to unstable periodic motions.

This gave the possibility of finding the conditions, at which an additional integral exists. Except for the determination of the first integral itself, S. V. Kowalevskaya has found not quite obvious system of variables, which gives the Abel – Jacobi form of equations, and has also obtained explicit solution in terms of theta-functions. The reduction of the Kowalevskaya case to quadratures is still considered to be very complicated and cannot be substantially simplified.

A. M. Lyapunov in his paper [116] has refined the Kowalevskaya analysis (G. G. Appelrot [3] was engaged in this analysis, as well, in response to the critique of Kowalevskaya papers by the Academician A. A. Markov), having required



for the integrability to have uniqueness (meromorphy) of a general solution as a complex function of time, while studying the equation solutions in variations. The Lyapounov method is somewhat different from the Kowalevskaya approach, which was further developed in the papers by M. Adler, P. van Moerbeke, who has connected the presence of full parametric family of single-valued Laurent (pole) expansions with the *algebraic* integrability of a system (in a certain narrow sense [186, 187]). The most complete analysis of the full parametric expansions in the Euler – Poisson equations may be found in the book [243]. The classical exposition of the results of Kowalevskaya and Lyapounov can be looked up in several textbooks [9, 59].

The considerations of Kowalevskaya have established the foundation for a new technique of the analysis of a system for integrability, and at the same time they became the first pattern of search of obstacles to integrability, which have recently developed into the separate investigation trend [97]. It should also be noted that, in spite of some rigorous results, connecting the general solution branching with the first integral nonexistence [97], the Kowalevskaya technique still remains the test for integrability; it is ambiguous in many respects, and its application to various problems requires a certain art and additional considerations. In the literature on physics this technique is usually referred to as *the Painleve – Kowalevskaya test*.

**The Kowalevskaya case, its analysis and generalizations.** The geometrical interpretation of the Kowalevskaya case, which is, however, not sufficiently natural, and his own way of reducing the Kowalevskaya case to quadratures was offered by N. E. Joukovskiy [76]. He has also used Kowalevskaya variables to construct some curvilinear coordinates, corresponding to separating variables of the Kowalevskaya top, on the plane (the plane $M_1$, $M_2$). His reasoning was simplified by W. Tannenberg and G. K. Suslov [163, 274].

F. Kötter has also somewhat simplified the Kowalevskaya case explicit integration technique [233, 235] and has offered to investigate motion in ta frame of reference, uniformly rotating around vertical axis. From the modern perspective introduction of the Kowalevskaya variables and reduction to the Abel equations are discussed in [92]. The qualitative analysis of the dynamical symmetry axis motion is shown in [92]; the topological and bifurcational analyses are present in [170]. The variables action-angle for the Kowalevskaya top are constructed in [54] (see also [106, 204]). We give them in § 8, ch. 5. N. I. Mertsalov has carried out full-scale experiments, but has not found any singularities in the top motion [69].

G. V. Kolosov has integrated the Kowalevskaya case, having reduced it by means of nonlinear transformation of variables and time to the problem of motion of a point on a plane in potential, allowing the separation of variables. It is a well known analogy of Kolosov; its classical variant and new generalizations are discussed in § 8 ch. 5. It should also be noted that in the paper [103] G. V. Kolosov was studying path of the angular momentum vector end, having shown its regular singularities.

The configuration of complex tori by means of the algebraic geometry methods is investigated in [212, 134]. Bifurcational patterns for the Kowalevskaya case in connection with the Kolosov analogy are discussed in [217].

The Kowalevskaya top quantization is also a question, being considered from the time of the quantum mechanics origination (Laporte, 1933), but it is not absolutely clear even nowadays [106, 258]. The paper [204] contains the Picard – Fuchs equation, arising in the process of integration of the Kowalevskaya case. The first Lax representation for the $n$-dimensional Kowalevskaya case without a spectral parameter was constructed by A. M. Perelomov [142]. The representation, having the spectral parameter in a general statement (at motion in two uniform fields), was offered by A. G. Reyman, M. A. Semenov-Tian-Shansky [147]. This generalization of the Kowalevskaya case is still little studied (in particular, it is not integrated in terms of quadratures; the topological and qualitative analyses are also absent).

## § 5. The Goryachev – Chaplygin Case

Let us consider the particular integrable case of Goryachev – Chaplygin, for which the angular momentum vector lies in the horizontal plane, i. e., $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$. It is implemented under nearly the same limitations of dynamical parameters, as the Kowalevskaya case, but the ratio of moments of inertia now equals not two, but four — $\frac{a_3}{a_1} = 4$. The Hamiltonian and the additional integral a written as:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - x\gamma,$$
$$F = M_3(M_1^2 + M_2^2) + xM_1\gamma_3.$$

### 1. Explicit Integration

Variables of the Kowalevskaya type, reducing a system to the Abel – Jacobi equations were given by S. A. Chaplygin [174]. They are specified by formulae

$$M_1^2 + M_2^2 = 4uv, \qquad M_3 = u - v \tag{5.1}$$



and satisfy equations

$$\begin{gathered} \frac{du}{\sqrt{P_1(u)}} - \frac{dv}{\sqrt{P_2(v)}} = 0, \\ \frac{2u\,du}{\sqrt{P_1(u)}} + \frac{2v\,dv}{\sqrt{P_2(v)}} = dt, \end{gathered} \tag{5.2}$$

$$P_1(u) = -\Big(u^3 - \frac{1}{2}(h - x)u - \frac{1}{4}f\Big)\Big(u^3 - \frac{1}{2}(h + x)u - \frac{1}{4}f\Big),$$
$$P_2(v) = -\Big(v^3 - \frac{1}{2}(h - x)v + \frac{1}{4}f\Big)\Big(v^3 - \frac{1}{2}(h + x)v + \frac{1}{4}f\Big),$$

where $h$, $f$ are constants of the energy integral and the Chaplygin integral ($H = h$, $F = f$).

Remark. Introducing variables $u$, $v$, Chaplygin has actually constructed the system of Andoyaer – Deprit variables; to put it more precisely, the system of variables, connected with the above mentioned ones by relations $L = u - v$, $G = u + v$ [92]. In § 8 ch. 5 the Goryachev – Chaplygin case generalization constructed, and corresponding separating variables are found by means of the Andoyaer – Deprit variables analysis for the bundle of Poisson brackets , including algebras $so(4)$, $e(3)$, $so(3, 1)$.

### 2. A Bifurcational Pattern and a Phase Portrait

Using functions $P_1(u)$, $P_2(v)$, from the condition of multiplicity of these polynomials one can easily construct a bifurcational pattern [170]. On a plane $(f, h)$ it consists of three branches (fig. 46):

I. $f = 0, \quad h > -1,$

II. $h = \frac{3}{2}t^2 + 1, \qquad f = t^3, \quad t \in (-\infty, +\infty),$

III. $h = \frac{3}{2}t^2 - 1, \qquad f = t^3, \quad t \in (-\infty, +\infty).$

The first class (I) has to do with three periodic solutions:

1) rotations and oscillations in the equatorial plane of the inertia ellipsoid ($M_1 = M_3 = 0$, $\gamma_2 = 0$);
2) rotations and oscillations in the meridional plane of the inertia ellipsoid ($M_1 = M_2 = 0$, $\gamma_3 = 0$);
3) Goryachev particular solutions, corresponding to $f = 0$.

Unfortunately, solutions, lying on branches II, III, are practically not studied at all. Phase portraits, corresponding to various values of energy, are shown at fig. 47, 48.

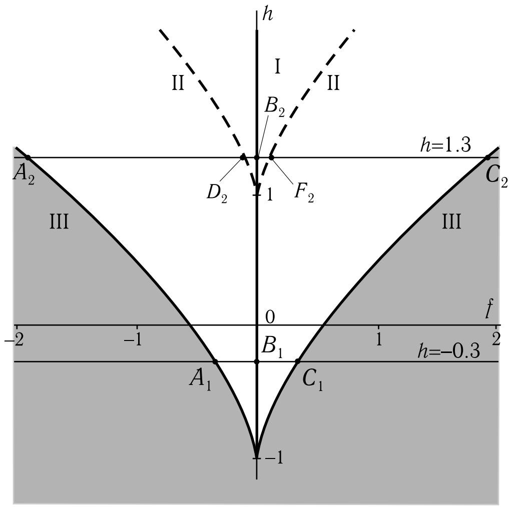



Figure 46. A bifurcational pattern of the Goryachev – Chaplygin case. Nonphysical domain of integrals is shaded with the grey color. The figure also shows two energy levels, for which phase portraits are constructed (see fig. 47, 48). The letters $A_i$, $B_i$, $C_i$, ... stand for periodic solutions and separatrices, which are indicated at phase portraits in the similar way.

Remark 1. The absence of explicit analytical expressions for asymptotic solutions also prevents perturbed system investigation. It should be noted that N. I. Mertsalov, in the paper [126], made an assertion concerning integrability of the Goryachev – Chaplygin top equations at $c = (\boldsymbol{M}, \boldsymbol{\gamma}) \neq 0$. However, as the computer-aided experiments, represented at fig. 49, show, this assertion is wrong, and near unstable manifolds at $c \neq 0$ there arises the stochastic layer leading to nonintegrability.

## 3. Visualizing the Most Remarkable Solutions

Among periodic solutions of the Goryachev – Chaplygin problem the special place is given to the *Goryachev solution*. At the bifurcational pattern it lies on a straight line $f = 0$; this line also contains periodic solutions of the Euler – Poisson equations, corresponding to oscillations (at $h < 1$) and rotations (at $h > 1$) of a rigid body in planes $Oxy$ and $Oxz$, which obey the physical pendulum law. Let us dwell on the Goryachev solution and solutions, situated on branches II and III (see fig. 46).



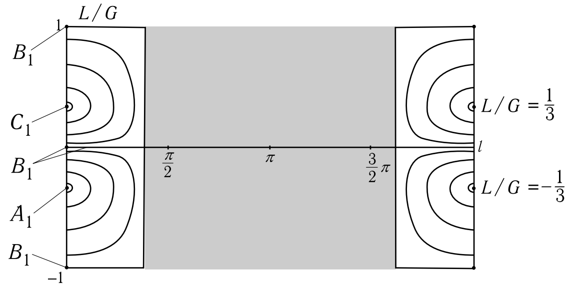



Figure 47. A phase portrait of the Goryachev – Chaplygin case at $h = 0.3$ (the section by a plane $g = \pi/2$). The letters $A_1$, $B_1$, $C_1$ stand for periodic solutions, placed on branches of a bifurcational pattern (fig. 46). The point $B_1$ on the bifurcational pattern, for which $f = 0$, is assigned to, first, two pendular periodic solutions (on the phase portrait they are placed at the poles of the sphere $L/G = \pm 1$ and at the point $l = 0$, $L/G = 0$) and, second, to the whole straight line $L/G = 0$, $l \neq 0$, also filled with the periodic solutions (the Goryachev solution) of the pendular type (see also s. 3).

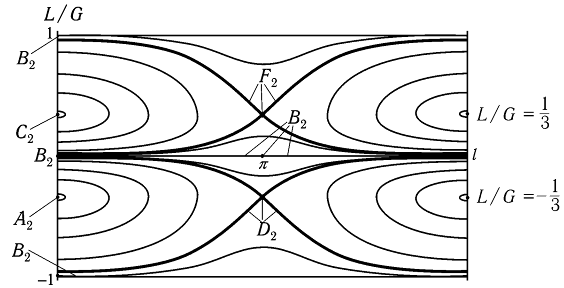



Figure 48. A phase portrait of the Goryachev – Chaplygin case at $h = 1.3$ (the section by a plane $g = \pi/2$). The letters $A_2$, $B_2$, $C_2$, $D_2$, $F_2$ stand for periodic solutions, placed on branches of a bifurcational pattern (fig. 46). In comparison with the previous portrait, unstable solutions (and separatrices to these solutions) — $D_2$ and $F_2$ — have appeared. Like the previous case, the point $B_2$ on the bifurcational pattern is assigned to four rotational periodic solutions (rotations in equatorial and meridional plane with taking into account the direction): these are the points $L/G = \pm 1$ and $l = 0$, $\pi$, $L/G = 0$, and a straight line $L/G = 0$, completely filled with the periodic solutions (the Goryachev solutions) of a reduced system (see s. 3).

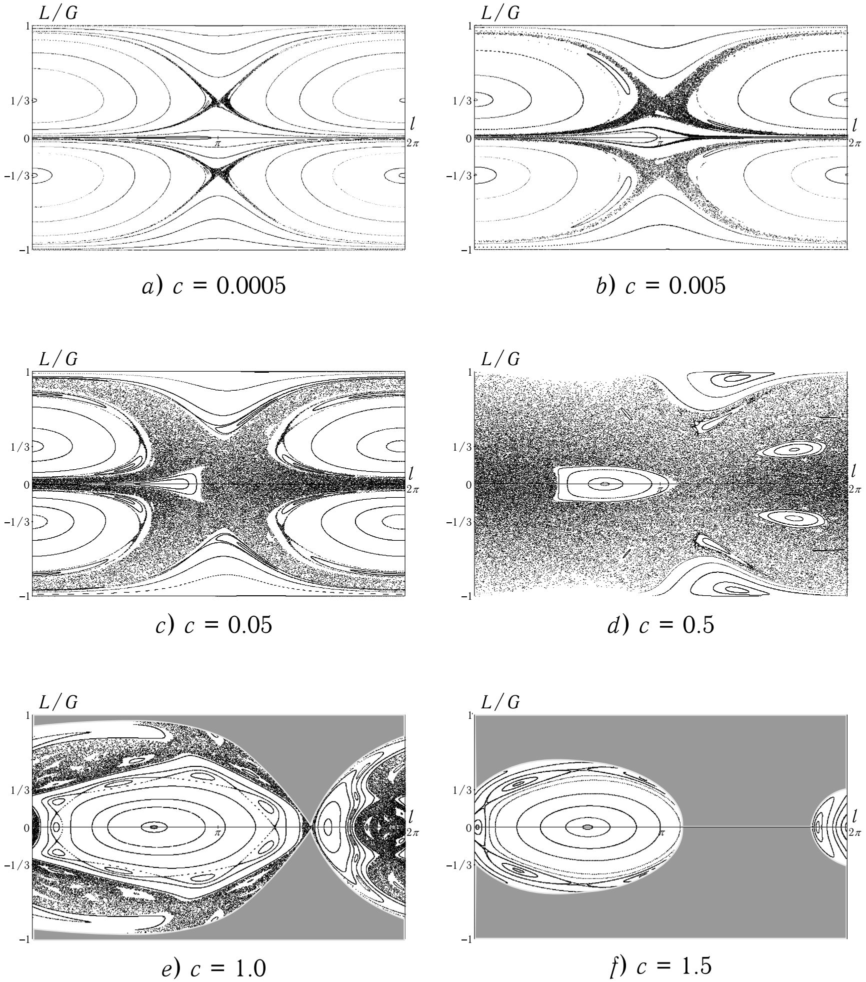
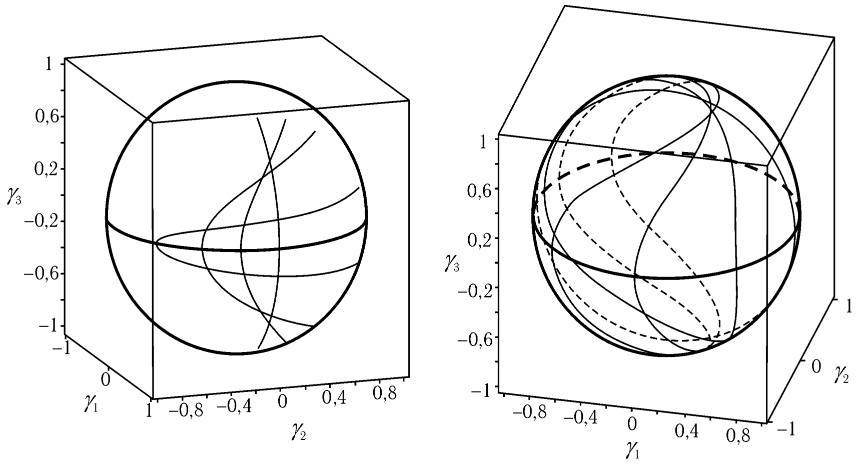



Figure 49. The perturbation of the Goryachev – Chaplygin case at a fixed energy ($h = 1.5$) and the increase of area constant (the section by a plane $g = \pi/2$ is shown; the domains of impossibility of motion are inked by grey). The figures show that in the vicinity of separatrices there arises a stochastic layer, which, at first, increases, and then decreases together with the possible motion domain. It is an interesting fact that under the following increase of $c$, the PMD decreases together with the stochastic layer until it has completely disappeared.






a) Motion $\gamma$ at $f = 0$ and $h = -0.5$ (various types of pendular motion)

a) Motion $\gamma$ at $f = 0$ and $h = -0.5$ (various types of rotational motion)

Figure 50. "The Goryachev solution" represents the complete torus, filled with periodic solutions of a reduced system $\boldsymbol{M}, \boldsymbol{\gamma}$ (so called resonance 1 : 1); at $h < 1$ (fig. a) these are pendular type solutions, and at $h > 1$ (fig. b) these are rotational type solutions. This figure and the following one contain paths on the Poisson sphere, corresponding to various solutions on this torus.

**The Goryachev solution [65].** For this solution there are two additional invariant relations [72]

$$M_1^2 + M_2^2 = bM_1^{2/3}, \quad f = M_3(M_1^2 + M_2^2) + M_1\gamma_3 = 0, \quad (b > 0). \quad (5.3)$$

These relations contain an arbitrary constant $b$, which parametrizes the whole family of periodic solutions: in the phase space it is a degenerate torus, filled with periodic solutions. Relations (5.3) were shown by D. N. Goryachev, whereupon S. A. Chaplygin understood at once that the condition $f = 0$ is too rigorous, and obtained solution of (5.2) in the generally accepted form. Under $h < 1$ and under the change of $b$ from 0 to $b_{\max}$, the solution turns from oscillation in the equatorial plane to the oscillation in the meridional plane (fig. 50). At the phase portrait (see fig. 47) these are the straight line $L/G = 0$ and the meridian, connecting it with the poles. Under $h > 1$ and under the change of $b$ from 0 to $b_{\max}$, the solution turns from one rotation in the equatorial plane into another one (in the opposite direction, fig. 48).

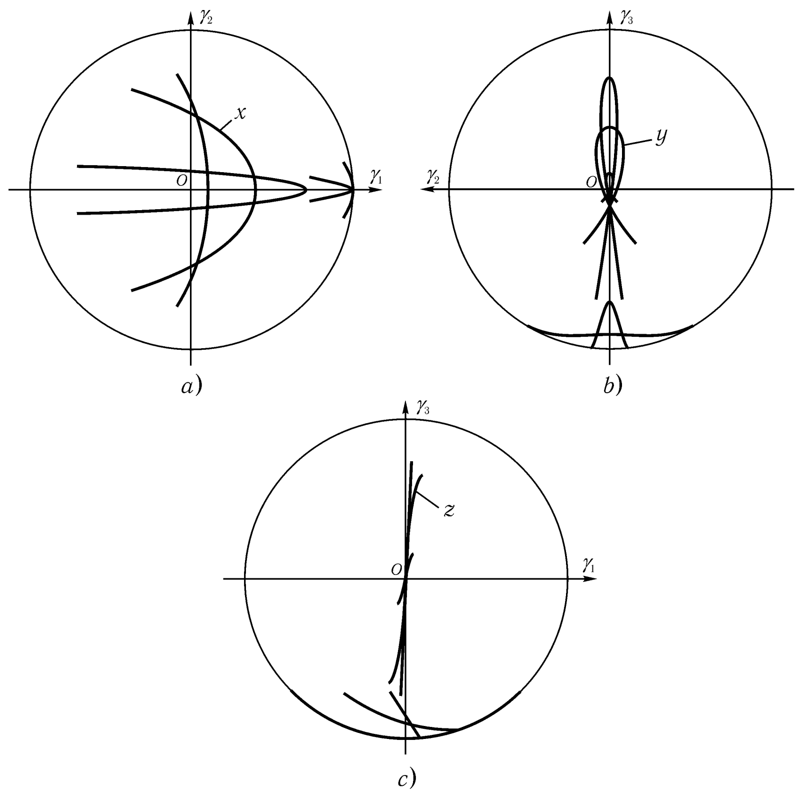



Figure 51. This figure illustrates behavior of rigid body principal axes in a fixed frame of reference for Goryachev solutions at fixed energy $h < 1$ ($h = -0.7$). It is clearly seen that these are *periodic* solutions in an absolute space. Under the change of parameter $b$, these solutions turn from oscillations in the plane $Oxy$ to oscillations in the plane $Oxz$. (The letters $x, y, z$ stand for axes, attached to the body.)

The apex motion on the Poisson sphere is shown at fig. 50. The remarkable phenomenon, that was not mentioned before, is the fact that for the Goryachev solution in the absolute space at $h < 1$ the motion is periodic and of the oscillatory type (see fig. 51). However, at $h > 1$ the corresponding motion is quasiperiodic and double-frequency one (fig. 52).

All the above mentioned facts are practically impossible to be seen directly from the analytic solution, which, for the first time, was obtained by Goryachev



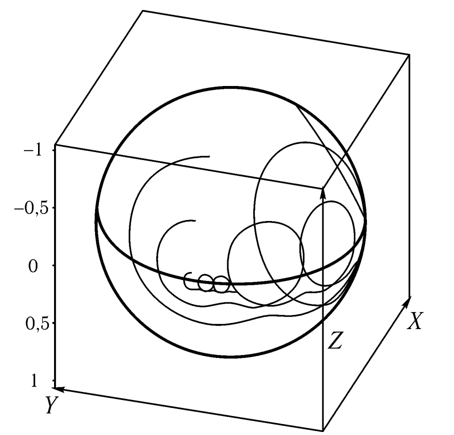



Figure 52. The figure, illustrating quasiperiodic motion in an absolute space (the motion of the principal axis $Oy$ is shown) for the Goryachev solution at $h > 1$ ($h = 1.7$).

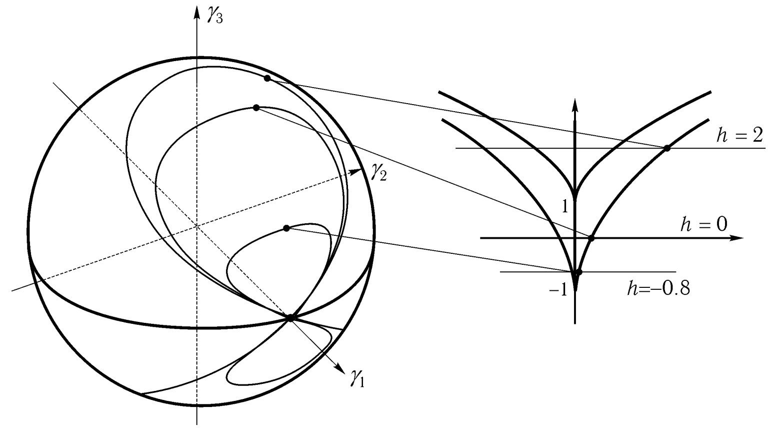



Figure 53. Motion of the vertical unit vector $\boldsymbol{\gamma}$ on the Poisson sphere for stable periodic motion in the Goryachev – Chaplygin case at various values of energy.

in a very cumbersome form [65]. In spite of some simplifications, appearing, for example, in [72], explicit formulae allow to give only rough ideas about motions found by means of the computer.

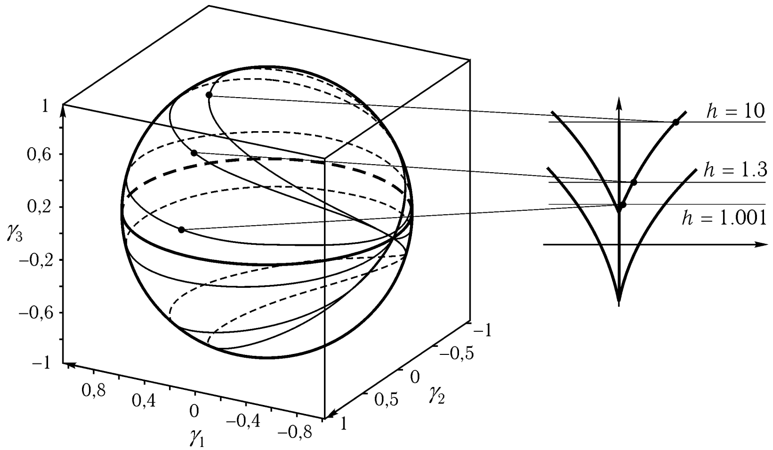



Figure 54. Motion of the vertical unit vector $\boldsymbol{\gamma}$ on the Poisson sphere for unstable periodic motion in the Goryachev – Chaplygin case at various values of energy.

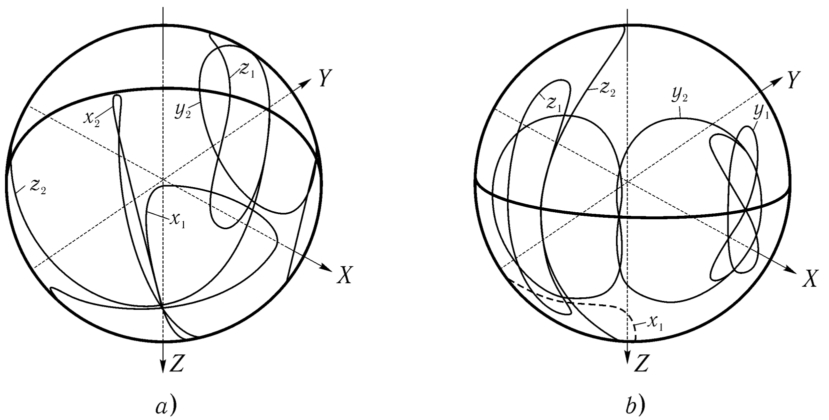



Figure 55. Motion of apices of the body principal axes in a fixed space in the Goryachev – Chaplygin case for the stable periodic solution, situated on branch III fig. 46, at two various values of energy $h_1, h_2$ from different viewpoints. The letters $x_i$, $y_i$, $z_i$, $i = 1, 2$ stand for paths of the corresponding axes, related to one and the same energy.

**Stable and unstable periodic solutions** of the Euler – Poisson equations for the Goryachev – Chaplygin case are situated, at the bifurcational pattern, on branches III and II, correspondingly (see fig. 46, 53–56). Numerical investigations show that the complete system motions in an absolute space, which correspond to these solutions, are *also periodic at any values of energy* (see fig. 55, 56). Earlier, this fact seems to be omitted in the literature, but it mirrors the specific character of rigid body dynamics on zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ (compare to the Delauney solution for the Kowalevskaya case, § 4 s. 3). Instead of the formal proof we give a series of figures, vividly verifying this statement. They show both system paths on the Poisson sphere, and paths of apices in the absolute space; the majority of these paths is rather complicated.



*The general conclusion for the Goryachev – Chaplygin case is the observation that in the process of its analysis we deal with curious oscillatory (rotational) motions in the absolute space, i. e., we can speak of a certain complicated pendulum. However, the application area of this kind of oscillations in not completely clear yet. We'd like to note the relative simplicity of motions of the Goryachev – Chaplygin top , as compared to the Kowalevskaya top. The sparse analytical results, obtained in the process of the Goryachev – Chaplygin case investigation, are incapable of giving a vivid representation of motion. The computer-aided investigation of motions, on the contrary, reveals its remarkable qualities, which are also typical for the related integrable systems.*

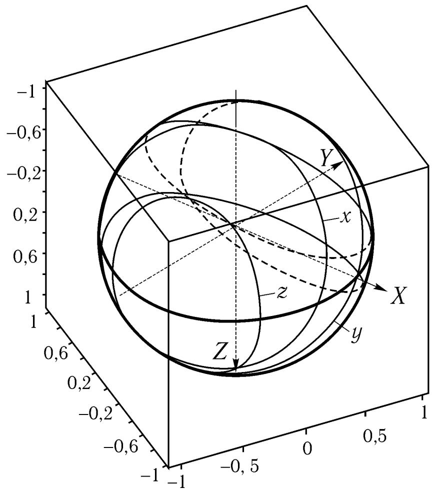



Figure 56. Motion of apices of the body principal axes in a fixed space in the Goryachev – Chaplygin case for the unstable periodic solution, situated on branch II fig. 46, at one value of energy. The letters $x$, $y$, $z$ stand for paths of the corresponding axes. (Motions at other values of energy do not have a qualitative difference, on which account we do not give them.)

## § 6. Particular Solutions

### 1. Hess Solution [228]

The Hess case is, in a certain sense, even more particular. Unlike

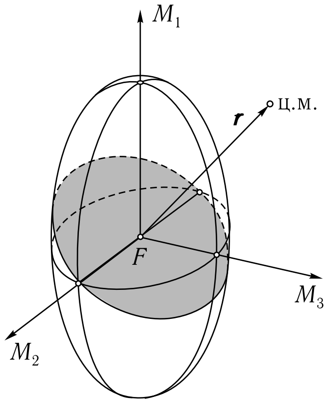



Figure 57. Gyration ellipsoid and position of the center-of-mass for the Hess case.

the previous cases (see §§ 2–5), it determines only a single parametric family of particular solutions, specified by the invariant relation (see table 2.1)

$$r_1 M_1 + r_3 M_3 = 0, \tag{6.1}$$

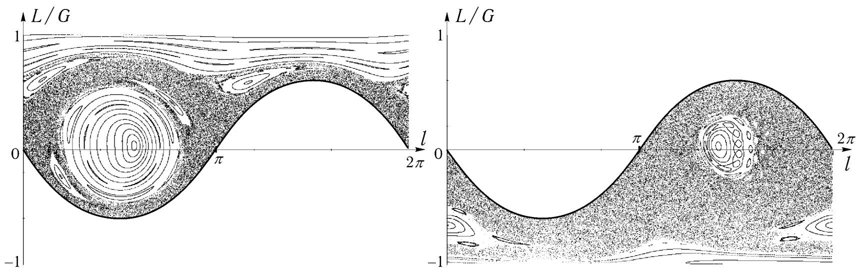
i. e., an isolated invariant manifold in a phase space (see fig. 58).

The physical meaning of boundaries on the parameters in the Hess case

$$\begin{gathered} r_1\sqrt{a_3 - a_2} \pm r_3\sqrt{a_2 - a_1} = 0, \\ r_2 = 0 \end{gathered} \tag{6.2}$$

lies in the following. Let us consider *a gyration ellipsoid* — a set of a kinetic energy level in the space of moment $\boldsymbol{M}$ (see fig. 57)

$$\frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) = \text{const.} \tag{6.3}$$




*a*) *b*)

Figure 58. A phase portrait (the section by the plane $g = \pi/2$) for the Hess case under conditions $\mathbf{I} = \operatorname{diag}(1, 0.625, 0.375)$, $\boldsymbol{r} = (3, 0, 4)$, $\mu = 1.995$ for the constants of integrals $h = 50.0$, $c = 5.0$. Two stochastic layers are clearly seen, they are separated by the coinciding Hess separatrices — the points from one layer do not penetrate into the other. At fig. b) one can also see *a meander torus*, arising under these conditions (see fig. 59).



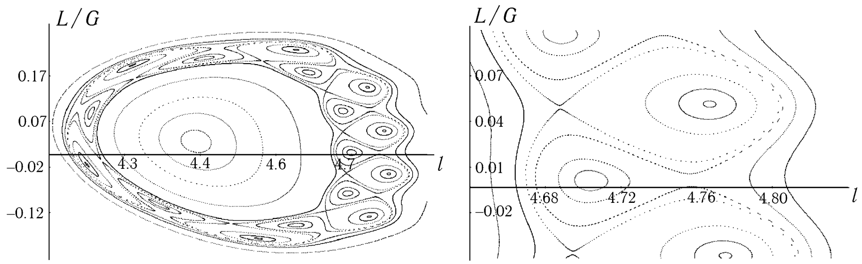



Figure 59. Meander tori, originating at the phase portrait in the Hess case (for parameters, see fig. 58).

As far as all the eigenvalues of matrix $\mathbf{A}$ are different, the gyration ellipsoid has two circular sections, passing through the mean axis. Conditions (6.2) mean that the center-of-mass lies on the axis, perpendicular to one of the circular sections of ellipsoid (6.3). Hess linear integral (to put it more precisely, invariant relation) (6.1) means that the moment projection on this axis equals zero.

The detailed analysis of this case is given in chapter 4; § 3, here we shall only mention that the Hess relation may specify the pair of coinciding separatrices at the phase portrait (see fig. 58). It is interesting to note that in the phase space for the Hess case there appears *a meander torus* (see fig. 59), though it does not look as a specific feature of this very case.

## 2. The Staude Permanent Rotations

Let us consider positions of relative equilibrium (i. e., positions of equilibrium of a reduced system on the Poisson sphere) for the Hamilton equations on algebra $e(3)$ with an arbitrary potential, depending on $\boldsymbol{\gamma}$:

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M},\, \boldsymbol{M}) + V(\boldsymbol{\gamma}). \tag{6.4}$$

From the condition of relative equilibrium $\dot{\boldsymbol{M}} = \dot{\boldsymbol{\gamma}} = 0$ and the area integral $(\boldsymbol{M},\, \boldsymbol{\gamma}) = c$ we find $\mathbf{A}\boldsymbol{M} = \lambda\boldsymbol{\gamma}$, $\lambda = \dfrac{c}{(\mathbf{A}^{-1}\boldsymbol{\gamma},\, \boldsymbol{\gamma})}$, and equations of motion give the relation

$$\begin{gathered} c^2(\mathbf{A}^{-1}\boldsymbol{\gamma} \times \boldsymbol{\gamma}) + \Big(\boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}\Big)(\mathbf{A}^{-1}\boldsymbol{\gamma},\, \boldsymbol{\gamma})^2 = 0, \\ \boldsymbol{\gamma}^2 = 1. \end{gathered} \tag{6.5}$$

This result was obtained by O. Staude in 1894 [271]. In the moving system the first equation (6.5) defines a certain cone, referred to as *the Staude cone*. With respect to each cone generatrix the body is uniformly rotating around the force field symmetry axis (in case of gravity field it is a vertical axis) with the angular velocity $|\boldsymbol{\omega}| = \dfrac{|c|}{(\mathbf{A}^{-1}\boldsymbol{\gamma},\, \boldsymbol{\gamma})}$.

One can also obtain equations (6.5), if one considers critical points of *a reduced potential*

$$V_c(\boldsymbol{\gamma}) = V(\boldsymbol{\gamma}) + \frac{c^2}{(\mathbf{A}^{-1}\boldsymbol{\gamma},\, \boldsymbol{\gamma})}. \tag{6.6}$$

In the modern terminology, created by S. Smale, the Staude rotations, defined by the reduced potential extremes, specify, on the plane of values of the first integrals $H = h,\, c^2$, a bifurcational pattern (*the Smale pattern*), separating domains with various topological type of foliation into three-dimensional invariant manifolds and corresponding types of possible motion domains (PMD).

Using the planar problem of three bodies as an example, Smale has offered a general technique for the investigation of integral manifold transformations under the transition across bifurcational curves. With reference to the Euler – Poisson equations (the linear potential) bifurcational curve transformations are qualitatively studied by S. B. Katok, Ya. V. Tatarinov and R. P. Kuzmina [84, 164, 109]. Let us show more precise numerical constructions of the Smale pattern bifurcational curves in the case of dynamic dissymmetry and under various positions of the unit vector of the center-of-mass (fig. 60).

As compared to the Euler case, whose bifurcational curves (permanent rotations) are marked by the dotted line, in the presence of a gravity field branches of permanent rotations split, the splitting being observed for the branches, corresponding to rotations around the axes, along which there is a nonzero component of the shift of the position vector of the center-of-mass.

It should be noted that the analysis of stability of Staude rotations is available in the extensive literature, which, unfortunately, is difficult for review. Nevertheless, these investigations do not give the final solution of the problem. The elementary investigation is given in the books by R. Grammel [66] and K. Magnus [119].

We should also note that the study of Staude rotations is important for investigation of stochasticity in a general nonintegrable situation; in a certain sense, they specify some basic periodic solutions, (both stable, and unstable solutions), whose expension with respect to the parameter defines the general scenario of transition to chaos.



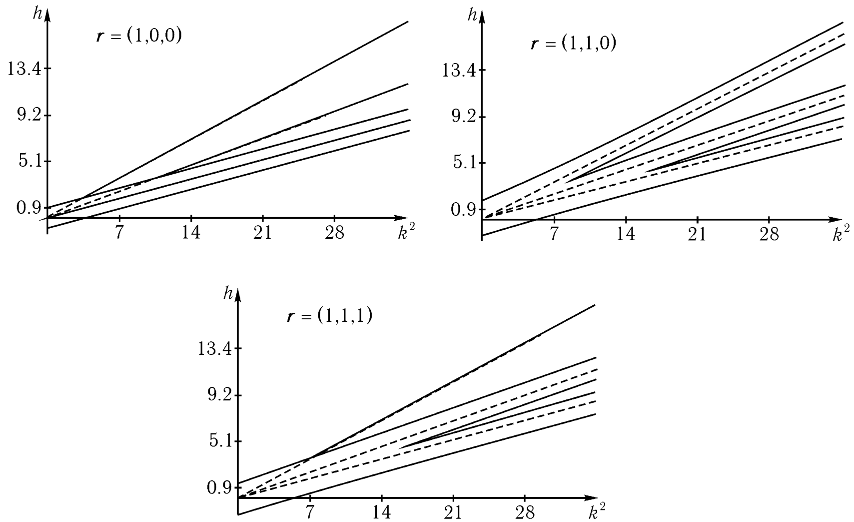



Figure 60. Smale patterns for various position of the center-of-mass $\boldsymbol{r}$ ($\mathbf{I} = \mathrm{diag}(2,\, 1.5,\, 1)$).

Remark 1. In case of the uniform gravity field the Staude cone represents an ordinary second order cone. Under particular assumptions with respect to parameters $a_i,\, r_i$ this cone may degenerate into the pair of planes (different or coincident) or become indefinite. It is an obvious thing that five following straight lines completely define the whole cone:

1) three principal axes of inertia with respect to the fixation point,
2) a straight line, connecting the fixation point with the center-of-mass,
3) a straight line, specified by the vector $\mathbf{A}\boldsymbol{r}$, along which $\boldsymbol{\omega}$ is directed, if the vector $\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega}$ is directed along the straight line, specified by the vector $\boldsymbol{r}$.

Remark 2. As it was noticed by W. van der Woude [284], the Staude cone (just for the uniform gravity field) represents a cone of straight lines in the body, originating from the fixation point and being principal axes of the inertia ellipsoid at least for one of its points. This very cone was considered by A. M. Ampére [189], when he analyzed the geometry of mass in a rigid body without taking into account the gravity force. For other potentials in (6.4) this result is, obviously, not valid any longer.

### 3. The Grioli Regular Precessions

The Staude solutions represent regular precessions around a vertical axis. These solutions are realized under any distribution of mass within the body.

The more general determination of the regular precessions proposes that, under such motions, there exist two special axes: one is fixed in space, and the other within the body, the angle between these axes being constant. For example, for the Lagrange top precessions of the dynamical symmetry axis apex around the vertical line (see § 3) are possible. It turns out that, as it was shown by Italian mechanic D. Grioli in 1947 [221], for the Euler–Poisson equations "non-vertical" precessions, which, however, exist under additional restrictions of the moments of inertia and the position of the center-of-mass, are possible.

For these precessions the center-of-mass lies on the perpendicular, drawn from the fixed point to the circular section of the *inertia ellipsoid*, and, in this sense, the Grioli case is mutual to the Hess case, in which the center-of-mass lies on the perpendicular, drawn from the fixed point to the circular section of the *gyration ellipsoid*. Such a connection with the energy ellipsoid also conditions the fact that all reasoning for the Grioloi solution is easier to perform for angular velocities $\boldsymbol{\omega}$, rather than for the angular momentum $\boldsymbol{M}$.

The easiest way to obtain explicit analytical expressions for the Grioli case, which failed to find their correct presentation anywhere [221, 61, 72] (Grioli himself uses somewhat intricate reasoning with the Euler angles), is to use non-principal moving frame of reference with the axis $Oz$, passing through the body center-of-mass.

In the chosen frame of reference the Hamiltonian $H$ has the form

$$\begin{gathered} H=\frac{1}{2}(\boldsymbol{M},\,\mathbf{A}\boldsymbol{M})-x\gamma_3=\frac{1}{2}(\mathbf{I}\boldsymbol{\omega},\,\boldsymbol{\omega})-x\gamma_3,\\ \boldsymbol{M}=\mathbf{I}\boldsymbol{\omega},\qquad \mathbf{A}=\mathbf{I}^{-1}, \end{gathered} \tag{6.7}$$

where the tensor of inertia is

$$\mathbf{I}=\begin{pmatrix} I_1 & 0 & I_{13}\\ 0 & I_1 & 0\\ I_{13} & 0 & I_3 \end{pmatrix},\quad x=\text{const}.$$

We shall look for the conditions, under which the angular velocity projection on the position vector of the center-of-mass is constant: $\omega_3=\text{const}$. Differentiating this correlation system (6.7) along, we shall obtain four independent additional invariant relations, which determine desired periodic solutions in a reduced phase space



$$\begin{gathered} \omega_1^2+\omega_2^2=\omega_3^2,\\ x\gamma_1+I_{13}(\omega_1^2-\omega_3^2)+I_3\omega_1\omega_3=0,\\ x\gamma_2+\omega_2(I_{13}\omega_1+I_3\omega_3)=0,\\ x\gamma_3-I_{13}\omega_1\omega_3=0. \end{gathered} \tag{6.8}$$

From relations (6.8) it follows that $\boldsymbol{\omega}^2=2\omega_3^2=\text{const}$, and, besides, $\omega_2=$ $=\omega_3\sin\tau$, $\omega_2=\omega_3\cos\tau$, $\tau=\omega_3(t-t_0)$. Expressions for constants of first integrals can also be obtained in terms of $\omega_3$, using (6.8)

$$H=\frac{1}{2}(I_1+I_3)\omega_3^2=h,\quad (\boldsymbol{M},\,\boldsymbol{\gamma})=\frac{I_{13}^2-I_1I_3}{x}\omega_3^2=c, \tag{6.9}$$

the $\omega_3$ itself being determined from the equation

$$\boldsymbol{\gamma}^2=\frac{I_3^2+I_{13}^2}{x^2}\omega_3^4=1.$$

*Thus, for given parameters of the body* $(I_1,\,I_3,\,I_{13},\,x)$ *there exists only one* (*accurate to a sign*) *value of* $\omega_3$ *and other integral constants, specifying the Grioli solution.*

After the quantities $\omega_1$, $\omega_2$, $\omega_3$ have been found as explicitly depending on time, it does not make a problem to obtain all direction cosines $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$: by doing this we shall define the rigid body motion in an absolute space. From the Poisson kinematic equations for the center-of-mass, having the coordinates $(\alpha_3,\,\beta_3,\,\gamma_3)$, it is easy to obtain $\alpha_3''=-\alpha_3$, $\beta_3''=-\beta_3$ (where double prime means double differentiating with respect to $\tau$). After these expressions have been integrated, we shall find $\alpha_3=\cos\tau$, $\beta_3=\dfrac{I_3}{\sqrt{I_3^2+I_{13}^2}}\sin\tau$. From relations (6.8) we shall also obtain $\gamma_2=\dfrac{I_{13}}{\sqrt{I_3^2+I_{13}^2}}\sin\tau$.

So, the general Grioloi solution is a periodic one (in an absolute space and with respect to all apices — in this sense such a regular precession is strongly degenerate), and the center-of-mass performs a uniform motion around the circumference of a big circle, perpendicular to the axis, inclined to the vertical line at the angle $\theta_0$, determined from the equality $\operatorname{tg}\theta_0=\dfrac{\gamma_3}{\beta_3}=\dfrac{I_{13}}{I_3}$ (fig. 61). In this sense, the Grioli solution is closer to rotational motions of a pendular type, rather than regular precessions.

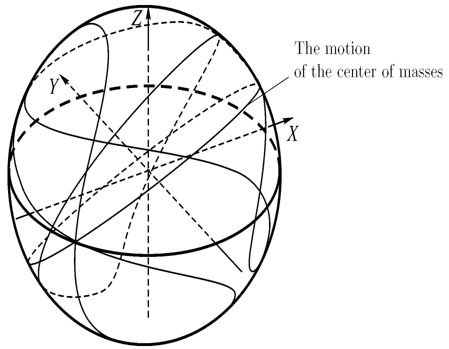



Figure 61. Motion of the principal axes of a body and of the center-of-mass for the Grioli solution at $I_1 = 1$, $I_3 = \frac{1}{2}$, $I_{13} = 0{,}4$, $\boldsymbol{r} = (0, 0, -1)$ (the center-of-mass is moving around a big circle).

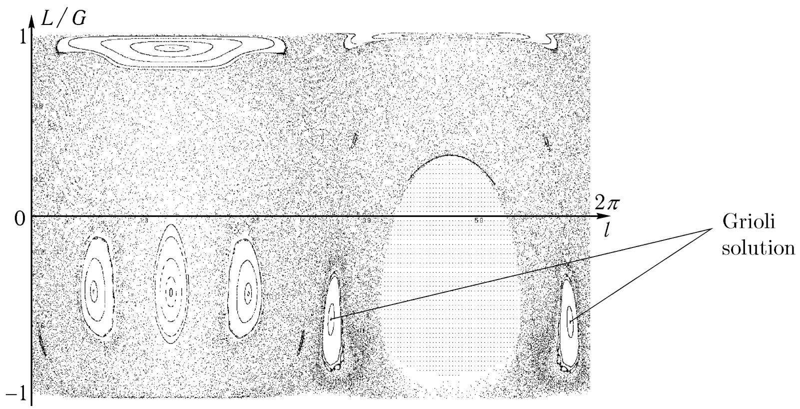



Figure 62. The phase portrait (the section by the plane $g = \pi$) under the conditions of existence of the Grioli solution (see the subscription under fig. 61). In this section the Grioli solution is represented by a fixed point (of the period 2).

At the phase portrait (fig. 62), which under conditions (6.9) is chaotic, the Grioli solution is determined by a fixed point of a stable type (we are not aware if the stability of these solutions was investigated analytically). Visualization of several (closed) characteristic paths of apices is shown at fig. 61.



Remark 3. Some authors (see, for example, [66]) consider the precession to be such a motion of the body, under which for a certain fixed axis within the body its line of nodes is rotating uniformly.

### 4. The Bobylev – Steklov Solution (1896) [15, 161]

We shall show one more particular solution, which is obtained in terms of elliptic quadratures and, under an additional condition, coincides with the Kowalevskaya top special solution, specified by the fourth Appelrot class. For this solution the Hamiltonian $H$ is written as

$$H = \frac{1}{2}(M_1^2 + aM_2^2 + 2M_3^2) + r\gamma_1, \quad a = \text{const}, \quad r = \text{const},$$

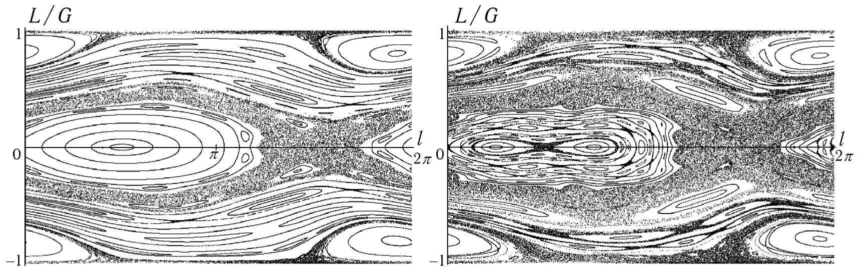
i. e., unlike the Kowalevskaya case, conditions of rotational symmetry of the inertia ellipsoid ($a \neq 1$) are not required. Assuming $M_2 = 0$, $M_1 = m = \text{const}$, it is easy to obtain $M_3 = -\frac{r}{m}\gamma_3$, and, using the area integral and the geometric integral, — an elliptic quadrature for $\gamma_3$:

$$\dot{\gamma}_3 = -m\sqrt{1 - \gamma_3^2 - \left(\frac{cm + r\gamma_3^2}{m^2}\right)^2},$$

where $c = (M, \gamma) = \text{const}$.

As it is shown at fig. 63, under the increase of $a$ this solution loses its stability and bifurcates: one stable periodic solution gives birth to two stable solutions and one unstable. Near the unstable solution, having general features of dynamics, shown at fig. 43, the stochastic layer is formed, which, expanding with the increase of $a$, determines general chaotization of a phase flow . The more detailed computer investigations are left beyond the present book. It is a curious fact that a very small deviation from the dynamic symmetry (i. e., from the Kowalevskaya case) — about one percent — leads to the appreciable chaotization of the portrait. This illustrates a certain "instability" of this case integrability, because it is technologically difficult to conditions of meet the exact dynamic symmetry. By the way, N. I. Mertsalov in his full-scale experiments had a very low accuracy both in production of the top itself, and in initial data specification. Thus, it was a natural result that his pictures could not clarify anything [69].

**The stability of particular solutions.** As for the investigation of stability of various particular solutions in rigid body dynamics (both in the integrable, and general cases) we can recommend the books [82, 152]. The stability of planar oscillations and rotations in the Kowalevskaya cases was investigated not




*a*) $a = 1{,}03$ *b*) $a = 1{,}07$

Figure 63. The Kowalevskaya integrable case instability. The phase portrait (section by the plane $g = \pi/2$) of the Kowalevskaya case perturbation under small deviation from the dynamic symmetry $\mathbf{A} = \mathrm{diag}(1, a, 2)$. The Bobylev – Steklov periodic solution is preserved at any value of $a$; at the phase portrait it is assigned to a fixed point $l = \pi/2$, $L/G = 0$. The energy and area integral values are $h = 4$, $c = 1$. (The period doubling bifurcation is observable.)

long ago by A. P. Markeyev [122, 123] by means of the Birkhoff's normal forms.

## § 7. Equations of Motion of a Heavy Gyrostat

### 1. A Gyrostat

Euler – Poisson equations (1.6) may be generalized if constant gyrostatic moment be introduced. We can simulate this moment, for example, by means of a balanced rotor, which rotates with constant angular velocity around an axis, fixed within a rigid body. Such a system is referred to as *a balanced gyrostat*. The similar moment arises, when we consider motion of a rigid body with multi-connected cavities, containing perfect incompressible fluid, and allowing the possibility of appearance of nonvanishing circulation [78] (see § 2 ch. 5).

Under such a generalization equations (1.6) remain unchanged, but Hamiltonian (1.4) acquires the term, linear with respect to moments:

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) - (\boldsymbol{r}, \boldsymbol{\gamma}) - (\boldsymbol{k}, \boldsymbol{M}), \tag{7.1}$$

where $\boldsymbol{k}$ is a certain constant vector caused by the presence of rotor.



Table 2.2

| Generalization of the case | The author | A Hamiltonian and an integral |
|---|---|---|
| of Euler – Poinsot | Joukovskiy (1885), Volterra (1899) | $H = \frac{1}{2}(\boldsymbol{M} - \boldsymbol{k}, \mathbf{A}(\boldsymbol{M} - \boldsymbol{k}))$ |
| | | $F = \boldsymbol{M}^2$ |
| of Lagrange | | $H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + r_3\gamma_3 + k_3M_3$ |
| | | $F = M_3$ |
| of Kowalevskaya | Yehia (1987), Komarov (1987) | $H = \frac{1}{2}\Big(M_1^2 + M_2^2 + 2\Big(M_3 - \frac{\lambda}{2}\Big)^2\Big) + r_1\gamma_1$ |
| | | $F = (M_1^2 - M_2^2 - 2r_1\gamma_1)^2 + (2M_1M_2 - 2r_1\gamma_2)^2 + 4\lambda(M_3 - \lambda)(M_1^2 + M_2^2) - 8r_1\lambda M_1\gamma_3$ |
| of Goryachev – Chaplygin | Sretenskiy (1963) | $H = \frac{1}{2}\Big(M_1^2 + M_2^2 + 4\Big(M_3 - \frac{k}{2}\Big)^2\Big) + r_1\gamma_1$ |
| | | $F = (M_3 - k)(M_1^2 + M_2^2) - r_1M_1\gamma_3$ |
| of Hess | Sretenskiy (1963) | $H = \frac{1}{2}\big(a_1(M_1+k_1)^2 + a_2M_2^2 + a_3(M_3+k_3)^2\big) + r_1\gamma_1 + r_3\gamma_3$, $r_1\sqrt{a_3 - a_2} = r_3\sqrt{a_2 - a_1}$ |
| | | $F = \sqrt{(a_2 - a_1)(a_3 - a_2)}(r_1M_1 + r_3M_3) + r_1a_3k_3 - r_3a_1k_1 = 0$, $\dot{F}\big\vert_{F=0} = 0$ |

Remark 1. The equations of motion of a gyrostat, representing the Hamilton equations on algebra $e(3)$ with Hamiltonian (7.1), can be physically obtained from the angular momentum theorem, which is applied for the total moment of the whole system

$$\mathfrak{M} = \boldsymbol{M} + \boldsymbol{k}, \quad \frac{\tilde{d}\mathfrak{M}}{dt} = \frac{d\mathfrak{M}}{dt} + \boldsymbol{\omega} \times \mathfrak{M} = \boldsymbol{F}, \tag{7.2}$$

where $\boldsymbol{M}$ is an angular momentum of a rigid body without a rotor, $\boldsymbol{k}$ is a rotor angular momentum, which, in the general case, depends on time $\boldsymbol{k} = \boldsymbol{k}(t)$, $\boldsymbol{F}$ is an external force moment, and $\dfrac{\widetilde{d}}{dt}$, $\dfrac{d}{dt}$ are vector derivatives in a fixed and moving frames of reference. The dependence $\boldsymbol{k}(t)$ is maintained forcedly (for example, by means of electrical motors), which does not violate conditions of applicability of theorem (7.2) . The same cannot be said about the energy conservation theorem, on the account of the inflow of external energy, providing the forced rotation. In the present case $\boldsymbol{F} = \boldsymbol{r} \times \boldsymbol{\gamma}$, and the rotor rotates with constant velocity: $\boldsymbol{k} = \mathbf{I}\boldsymbol{\omega}_0 = \mathrm{const}$. The more detailed discussion of gyrostats — systems with the internal cyclic motions, is available in the books [113, 57].

It turns out that all the cases from Table 2.1 ch. 3, § 2 may be generalized by the integrable way under additional reatrictions of the vector $\boldsymbol{k}$, i. e., the position of a gyrostat within the rigid body (see Table 2.2).

## 2. The Joukovskiy – Volterra Case

Let us consider in greater details dynamics of a rigid body with a gyrostat in the absence of a field. In this case equations for $\boldsymbol{M}$ separated and are integrate independently. We shall represent them in the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}(\boldsymbol{M} - \boldsymbol{k}).$$

The Hamiltonian and the additional integral also do not depend on configurational variables and can be represented in the form

$$H = \frac{1}{2}(\boldsymbol{M} - \boldsymbol{k},\, \mathbf{A}(\boldsymbol{M} - \boldsymbol{k})) = h, \quad F = \boldsymbol{M}^2 = f \tag{7.3}$$

(Hamiltonian (7.3) differs from Hamiltonian (7.1) by the substitution $\boldsymbol{k} \to \mathbf{A}\boldsymbol{k}$ and the constant term). Thus, the path in space $(M_1, M_2, M_3)$ represents intersection of a sphere with an ellipsoid, whose centers do not coincide. These curves are direct generalization of polhodes of the Euler problem (§ 2 ch. 2), but look much more complicated (fig. 64).

For the sake of convenience, branches of a bifurcational pattern on plane of integrals (7.3) $(h, f)$ may have a parametric representation. It is easily obtained from the condition of dependence of integrals (7.3) [170]

$$\begin{gathered} h = \frac{t^2}{2}\left(\frac{a_1k_1^2}{(a_1 - t)^2} + \frac{a_2k_2^2}{(a_2 - t)^2} + \frac{a_3k_3^2}{(a_3 - t)^2}\right), \\ f = \frac{a_1^2k_1^2}{(a_1 - t)^2} + \frac{a_2^2k_2^2}{(a_2 - t)^2} + \frac{a_3^2k_3^2}{(a_3 - t)^2}. \end{gathered} \tag{7.4}$$



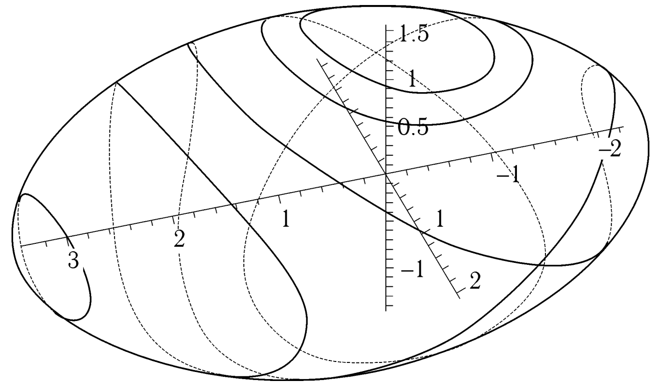



Figure 64. The Joukovskiy – Volterra problem polhodes.

Let $a_1 > a_2 > a_3 > 0$, then, when $t$ changes from $-\infty$ to $+\infty$, the bifurcational curve splits into four branches, corresponding to the change of $t$ in the following intervals (see fig. 65):

I. $t \in (-\infty, a_3)$ — the lower branch,
II. $t \in (a_1, a_2)$ — the branch, second from the bottom, with a cuspidal point,
III. $t \in (a_2, a_1)$ — the branch, third from the bottom, also with a cuspidal point,
IV. $t \in (a_1, \infty)$ — the upper branch.

The upper and the lower branches continuously converge at the point $t = \infty$.

Fig. 65 a) vividly shows, how, under the tendency $\boldsymbol{k} \to 0$, the pattern transforms into the pattern of the Euler – Poinsot case (see fig. 17).

If we consider the equations of motion only for variables $M_1$, $M_2$, $M_3$ (which separate) independently of positional variables $\boldsymbol{\gamma}$, then a bifurcational pattern will contain the above mentioned branches only. For the complete system of variables $(\boldsymbol{M}, \boldsymbol{\gamma})$ the pattern has a vertical line $f = c^2$ added, where $c = (\boldsymbol{M}, \boldsymbol{\gamma})$, the motion being possible only under condition $f > c^2$.

The straight line $f = c^2$ contains rigid body motions, for which the body moment in a fixed space is vertical:

$$\boldsymbol{M} = c\boldsymbol{\gamma}. \tag{7.5}$$

From this relation it follows that for the vector $\boldsymbol{\gamma}$ the path on the Poisson sphere also represents polhodes, congruent to the ones, shown at fig. 64, and

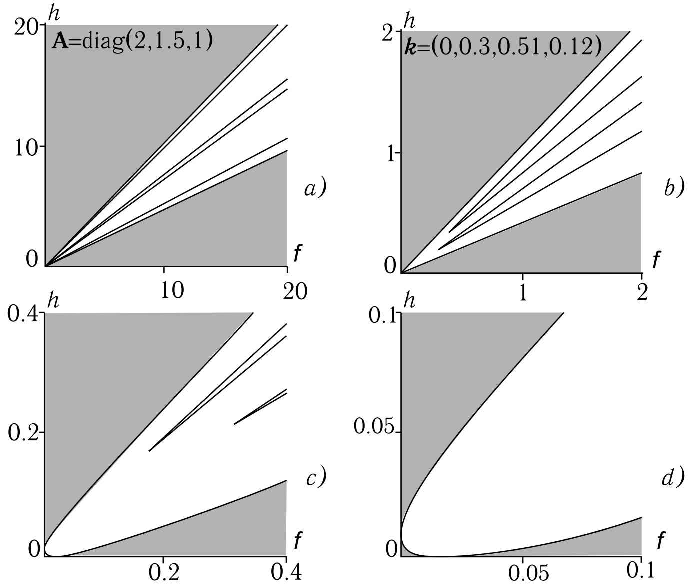



Figure 65. A bifurcational pattern of the Joukovskiy – Volterra case on plane of integrals $h = H$ and $f = \boldsymbol{M}^2$ in various scales. The domain of nonphysical values of integrals is shaded. Apart from the indicated curves, on the left the possible motion domain is limited by the vertical line $f = c^2$, $c = (\boldsymbol{M}, \boldsymbol{\gamma})$, so that $f > c^2$. Stable branches are shown by continuous lines, the unstable — by the dotted ones.

resulting from the intersection of a sphere with an ellipsoid:

$$\begin{gathered}\frac{1}{2}\left(\boldsymbol{\gamma} - \frac{\boldsymbol{k}}{c}, \mathbf{A}(\boldsymbol{\gamma} - \frac{\boldsymbol{k}}{c})\right) = \frac{h}{c^2},\\ \boldsymbol{\gamma}^2 = 1.\end{gathered} \tag{7.6}$$

If vector $\boldsymbol{k}$ lies in one of the principal planes, then the corresponding pair of branches at the bifurcational pattern intersects (see fig. 66), but if $\boldsymbol{k}$ is directed along the principal axis of the inertia ellipsoid, then two pairs of branches intersect.

The stability of pattern branches is shown at fig. 65; in the linear approximation it was investigated already by V. Volterra, but the most complete results have been obtained in [150, 57]. A somewhat general conclusion on stability is the fact that the rotor introduction leads to the double increase in both stable



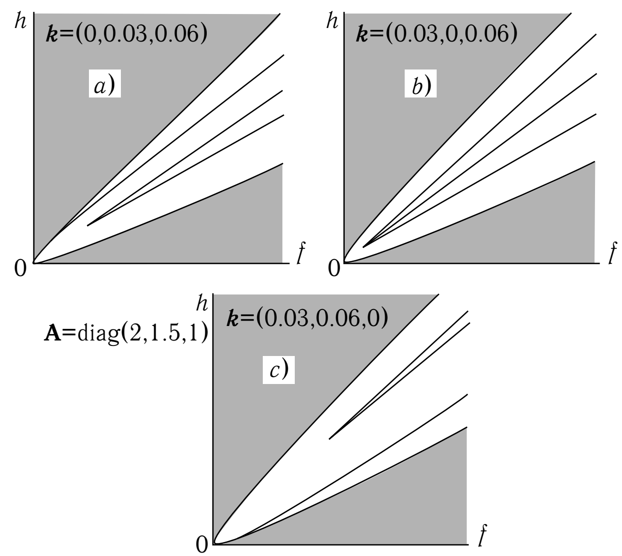



Figure 66. A bifurcational pattern for cases, when the gyrostatic moment vector lies in the principal plane.

a) $k_1 = 0$, in this case the upper part of branch III intersects with branch IV, (i. e., branch III starts from the “middle” of branch IV),

b) $k_2 = 0$: two middle parts of branches II and III intersect,

c) $k_3 = 0$: lower branches intersect (similarly to the case a).

stationary motions, and unstable ones. The unstable solutions vanish at small $h$, $c$, corresponding to fast rotation of rotor.

**The separation of variables for Joukovskiy – Volterra case.** The Joukovskiy – Volterra case was integrated in terms of elliptic functions by V. Volterra in [280] (see also [57]). N. E. Joukovskiy has indicated an additional integral only and investigated various mechanical statements of a problem [78] (see also [129]). The most simple separation can be done in terms of the *Andoyaer – Deprit variables* [80], as far as Hamiltonian (7.1) at $r = 0$ has the form

$$\begin{gathered}H = \frac{1}{2}(L^2 + \delta(G^2 - L^2)\cos^2 l) -\\ -\lambda_1\sqrt{G^2 - L^2}\sin l - \lambda_2\sqrt{G^2 - L^2}\cos l - \lambda_3 L,\end{gathered} \tag{7.7}$$

where $\delta = \frac{a_2 - a_1}{a_3 - a_1}$, $\lambda_i = \frac{a_i k_i}{a_3 - a_1}$, $i = 1, 2, 3$.

As it follows from (7.7), variable $g$ is a cyclic one. The explicit solution is reduced to a quadrature, containing a polynomial, which should be expressed in terms of standard elliptic integrals. In § 8 ch. 5 the more geometric procedure of the explicit solution is discussed.

Remark 2. The free gyrostat equations were discussed at the dawn of the quantum mechanics in connection with the molecular spectrum problem. Thus, in the book by M. Born [39] it is assumed that "an adequate model of a molecule is not just a top, but a rigid body, in which the flywheel with strong bearings seems to be embedded." Here the rigid body plays a role of the system of nuclei, and the flywheel — a role of the electron momentum. Kramers and Pauli, using this model, were trying (though they were not completely successful in doing this) to construct the theory of spectrum of molecules, having arbitrarily placed electron momentum.

**The explicit solution by V. Volterra.** To obtain the explicit solution in terms of elliptic functions, V. Volterra used projective coordinates

$$M_i = \frac{z_i}{z_4}, \quad i = 1, 2, 3 \tag{7.8}$$

and linear nondegenerate transformation

$$z_r = \sum_{s=1}^{4} C_{rs}\xi_s, \quad r = 1, 2, 3, 4, \tag{7.9}$$

which brings the equations of motion to the form

$$\begin{aligned} \xi_4\dot{\xi}_i - \xi_i\dot{\xi}_4 &= (\lambda_k - \lambda_j)\xi_j\xi_k/C, \\ \dot{\xi}_i\xi_j - \dot{\xi}_j\xi_i &= (\lambda_k - \lambda_4)\xi_k\xi_4/C, \end{aligned} \tag{7.10}$$

where $C = \det ||C_{rs}||$, and coefficients $C_{rs}$ are determined as solutions of the fourth-order equation, containing integral constants.

System (7.10) has the same structure as that of the differential relations for four Weierstrass sigma-functions $\sigma_1(u)$, $\sigma_2(u)$, $\sigma_3(u)$, $\sigma_4(u)$ of a complex argument $u$, $\lambda_i$, being expressed in terms of parameters of the differential equation for $\wp$-function of Weierstrass.

This consideration is the key one in the work of V. Volterra [280]. We are not going to give here any detailed computations, but content ourselves with the drawbacks of such an "explicit" solution. The fourth-order equation for coefficients of a matrix $C_{rs}$, specifying transformation (7.9), is not solved explicitly.



As a result, all further reasoning has only formal complex character, similar to the existence theorems. In practice, the solution itself does not give any useful dynamical derivations. All the results, obtained after Volterra (in stability, topological analysis and others) [57, 150], do not use his explicit quadratures. It seems that here one is not completely right, when one states the problem of reducing (in spite of any difficulties) to elliptic functions, which are little fit to problems of such kind. The similar difficulties arise with the Kötter [234, 236] "solutions" for the cases of Clebsch and Steklov. Although one has to refer to these solutions while writing papers, they are completely useless for dynamics and are practically not used. Generally, the unreasonable craving for complex methods can produce superdifficult and unsolvable problems of algebraic geometry [134] out of very natural mechanical problems.

### 3. The Explicit Integration of Other Cases

In generalizations of the Kowalevskaya and Goryachev – Chaplygin cases the gyrostatic moment is directed along the dynamical symmetry axis. The separation of variables for the Sretenskiy case (Goryachev – Chaplygin generalization) is indicated in [158, 159]. In § 7 ch. 5 we obtained it in other way and on the whole bunch of Poisson brackets. So, the Yehia – Komarov gyrostatic generalization of the Kowalevskaya case has not been integrated in terms of quadratures up to this day. In § 7 ch. 5 we extend this case to the bunch of Poisson brackets and show corresponding additional integrals.

The second Sretenskiy case, generalizing the Hess integral, can be integrated by means of a common scheme, which we are discussing further (§ 3 ch. 3). We should also note the result of L. Gavrilov [216], stating that the general integrable cases, given in Table (2.2), exhaust all the possibilities of existence of an additional algebraic integral of motion for system (7.1).

## § 8. Connected Systems of Rigid Bodies, a Rotator

We shall also give statements of various problems about motion of a connected system of two (in the general case— several) rigid bodies, whose particular case is a gyrostat, described above.

**The connected system of two tops.** Let us consider a system, consisting of a *lifting* body $\tau_0$ with a fixed point $O$ and a *lifted* body $\tau_1$, fixed in the lifting one in one its point $O_1$ (see fig. 67). Mass distribution in the system is changing at rotations of the lifted body.

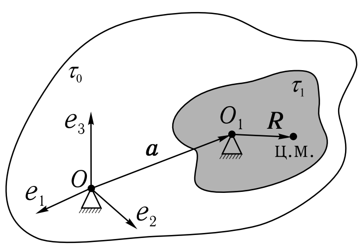



Figure 67.

Let us designate the vector, connecting projections of points $O$ and $O_1$ on the axes, bound to the lifting body, as $\boldsymbol{a} = (a_1,\, a_2,\, a_3)$, and the vector from the point $O_1$ to the center-of-mass of a lifted body, in projections on the same axes, as $\boldsymbol{R} = (R_1,\, R_2,\, R_3)$ (see fig. 67).

The system kinetic energy can be represented in the form

$$
\begin{gathered}
T = \frac{1}{2}(\boldsymbol{\omega},\, \mathbf{U}\boldsymbol{\omega}) + (\boldsymbol{\omega},\, \mathbf{V}\boldsymbol{\omega}_1) + \frac{1}{2}(\boldsymbol{\omega}_1,\, \mathbf{W}\boldsymbol{\omega}_1), \\
\mathbf{U} = \mathbf{I}_0 + \mathbf{I}_a + \mathbf{I}_1 + \frac{1}{2}(\mathbf{I}_2 + \mathbf{I}_2^T), \quad \mathbf{V} = \mathbf{I}_1 + \mathbf{I}_2, \quad \mathbf{W} = \mathbf{I}_1,
\end{gathered}
\tag{8.1}
$$

where $\boldsymbol{\omega}$ is an angular velocity of the body $\tau_0$, $\boldsymbol{\omega}_1$ is an angular velocity of a lifted body with respect to the lifting one, $\mathbf{I}_0$ is a tensor of inertia of a lifting body with respect to the point $O$, $\mathbf{I}_1$ is a tensor of inertia of a lifted body with respect to point $O_1$, $\mathbf{I}_a = ||\delta_{ij}\boldsymbol{a}^2 - a_i a_j||$, $\mathbf{I}_2 = ||\delta_{ij}(\boldsymbol{a},\, \boldsymbol{R}) - a_i R_j||$.

After the Legendre transformation

$$
\begin{gathered}
\boldsymbol{M} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \mathbf{U}\boldsymbol{\omega} + \mathbf{V}\boldsymbol{\omega}_1, \quad \boldsymbol{M}_1 = \frac{\partial T}{\partial \boldsymbol{\omega}_1} = \mathbf{V}\boldsymbol{\omega} + \mathbf{W}\boldsymbol{\omega}_1, \\
H = (\boldsymbol{M},\, \boldsymbol{\omega}) + (\boldsymbol{M}_1,\, \boldsymbol{\omega}_1) - T\Big|_{\boldsymbol{\omega},\, \boldsymbol{\omega}_1 \to \boldsymbol{M},\, \boldsymbol{M}_1}
\end{gathered}
\tag{8.2}
$$

we shall obtain Hamiltonian in the form of a homogeneous quadratic function of moments $\boldsymbol{M}$, $\boldsymbol{M}_1$

$$
H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) + (\boldsymbol{M},\, \mathbf{B}\boldsymbol{M}) + \frac{1}{2}(\boldsymbol{M}_1,\, \mathbf{C}\boldsymbol{M}_1). \tag{8.3}
$$

Unlike the Poincaré – Joukovskiy equations (considered further), describing motion of a body with a cavity, filled with vortex fluid (see ch. 3, § 10), matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ depend on positional variables, which determine the position of the lifted body with respect to the lifting one, specified by the element of a group



$SO(3)$. For such variables one can choose the Euler angles, direction cosines or any other frame of reference on the group $SO(3)$.

In the absence of an external field positional variables of a lifting body $\tau_0$ are not included in Hamiltonian (8.3). Choosing direction cosines $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ as variables, specifying position of the lifted body, we can write equations of motion of system (8.3) in the Hamiltonian form with the bracket, specified by algebra $so(3) \oplus (so(3) \oplus_s \mathbb{R}^9)$; the first term corresponds to the moment $\boldsymbol{M}$, the second — to $\boldsymbol{M}_1$, and the third — to positional variables of the body $\tau_1$.

In the coordinate notation the Poisson bracket has the form

$$
\begin{gathered}
\{M_i,\, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_{1i},\, M_{1j}\} = -\varepsilon_{ijk} M_{1k}, \\
\{M_i,\, \alpha_j\} = -\varepsilon_{ijk}\alpha_k, \quad \{M_{1i},\, \beta_i\} = -\varepsilon_{ijk}\beta_k, \quad \{M_{1i},\, \gamma_j\} = -\varepsilon_{ijk}\gamma_k.
\end{gathered}
$$

Other brackets are zero. This system possesses four degrees of freedom (in an external field there are six degrees of freedom).

**The body with a rotator** represents a system, consisting of a lifting body $\tau_0$ with a fixed point $O$, and a lifted body — *a rotator*, which makes free rotation around an axis, fixed in a lifting body. The angle of rotation of a lifted body around its axis will be designated as $\beta$.

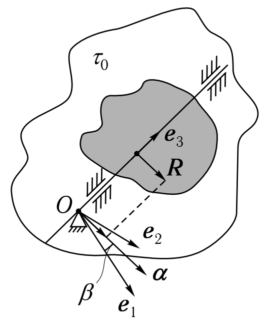



Figure 68. A body with a rotator.

Let us consider a particular case of such a system, when the rotator axis passes through the fixation point (see fig. 68); the more general statement is available in [81]. The kinetic energy can be represented in the form

$$
T = \frac{1}{2}(\boldsymbol{\omega},\, \mathbf{I}_0\boldsymbol{\omega}) + \frac{1}{2}(\boldsymbol{\omega} + \dot{\beta}\boldsymbol{n},\, \mathbf{I}_1(\boldsymbol{\omega} + \dot{\beta}\boldsymbol{n})), \tag{8.4}
$$

where $\boldsymbol{\omega}$ is an angular velocity of a lifting body, $\boldsymbol{n}$ is a unit vector, directed along the rotator axis, $\mathbf{I}_0$ is a tensor of a lifting body inertia, and $\mathbf{I}_1$ is a tensor of the rotator inertia. If the rotator is unbalanced, then $\mathbf{I}_1$ depends on the angle of rotation $\beta$.

Let us choose a frame of reference, bound to the lifting body in such a way that the axis $Ox_3$ is directed along the axis of rotator $\boldsymbol{n}$. Let a unit vector $\boldsymbol{\alpha} = (\alpha_1,\, \alpha_2,\, 0)$ define the direction of projection of the rotator center-of-mass onto the plane $x_1x_2$ (fig. 68). We shall determine moments and a Hamiltonian by means of the Legendre transformation

$$\boldsymbol{M} = \frac{\partial T}{\partial \boldsymbol{\omega}} = (\mathbf{I}_0 + \mathbf{I}_1)\boldsymbol{\omega} + \mathbf{I}_1\dot{\beta}\boldsymbol{n}, \quad L = \frac{\partial T}{\partial \dot{\beta}} = (\boldsymbol{n}, \mathbf{I}_1(\boldsymbol{\omega} + \dot{\beta}\boldsymbol{n})),$$

$$H = \frac{1}{2}(\boldsymbol{M} - L\boldsymbol{m}, \mathbf{J}^{-1}(\boldsymbol{M} - L\boldsymbol{m})) + \frac{L^2}{2(\boldsymbol{n}, \mathbf{I_1}\boldsymbol{n})}, \tag{8.5}$$

$$\boldsymbol{m} = \frac{\mathbf{I}_1\boldsymbol{n}}{(\mathbf{I}_1\boldsymbol{n}, \boldsymbol{n})}, \quad \mathbf{J} = \mathbf{I}_0 + \mathbf{I}_1 - (\boldsymbol{n}, \mathbf{I}_1\boldsymbol{n})\boldsymbol{m} \otimes \boldsymbol{m}.$$

Conditions of commutation between components $\boldsymbol{M}, L, \boldsymbol{\alpha}$ look rather natural

$$\begin{gathered}\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{L, \alpha_1\} = \alpha_2, \quad \{L, \alpha_2\} = -\alpha_1, \\ \{M_i, L\} = \{M_i, \alpha_1\} = \{M_i, \alpha_2\} = \{\alpha_1, \alpha_2\} = 0.\end{gathered} \tag{8.6}$$

Bracket (8.6) corresponds to the direct sum of algebras $so(2) \oplus e(2)$ and possesses two Casimir functions

$$F_1 = M_1^2 + M_2^2 + M_3^2, \quad F_2 = \alpha_1^2 + \alpha_2^2 = 1.$$

So, we have the system with two degrees of freedom. In the general case it is nonintegrable.

Remark 1. In the Hamiltonian form this system is often expressed in terms of other variables. Instead of positional variables of vector $\boldsymbol{\alpha}$ the rotator position in a lifting body is characterized by the angle $\beta$. Designating $p_\beta = L$, we shall write the Poisson bracket for these basic elements:

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{\beta, p_\beta\} = 1, \quad \{M_i, \beta\} = \{M_i, p_\beta\} = 0. \tag{8.7}$$

This corresponds to the direct sum $so(3) \oplus C_2$, where $C_2$ is a a constant bracket of canonical variables $\beta, p_\beta$. Poisson structure (8.7) possesses one Casimir's function $F = \boldsymbol{M}^2$.

Let us consider two famous integrable cases of system (8.5) with a cyclic integral, which were found by E. A. Ivin in [81] and represent various mechanical implementations of the Joukovskiy – Volterra system already known.

**1.** If $\mathbf{I}_1$ does not depend on $\boldsymbol{\alpha}$ (*a balanced rotator*), then $\beta$ is a cyclic variable, and $L$ is a cyclic integral.

In this case Hamiltonian (8.5) determines on the algebra $so(3) = \{(M_1, M_2, M_3)\}$ the Joukovskiy – Volterra system with the vector of gyrostatic moment $\boldsymbol{k} = L\boldsymbol{m}$.



**2.** *Let a lifting body be dynamically symmetrical and have a symmetry axis, coinciding with the rotator axis* — $\mathbf{I}_0 = \operatorname{diag}(I_1, I_1, I_3)$. In this case the cyclic integral is written as

$$F = M_3 - L, \tag{8.8}$$

it corresponds to the cyclic variable $\beta - \varphi$, where $\varphi$ is a proper rotation angle (the precession angle $\psi$ is also cyclic).

Let us consider a system of variables, commutating with integral (8.8),

$$K_1 = M_1\alpha_1 + M_2\alpha_2, \quad K_2 = -M_1\alpha_2 + M_2\alpha_1, \quad K_3 = M_3. \tag{8.9}$$

They form an algebra $so(3)$:

$$\{K_i, K_j\} = -\varepsilon_{ijk}K_k. \tag{8.10}$$

Variables (8.9) have a subtle mechanical meaning: they are projections of a shell angular momentum onto the axes, rigidly bound to the rotator. Expressing Hamiltonian (8.5) in terms of new variables (8.9), we shall obtain

$$H = \frac{1}{2}\Big(\boldsymbol{K} - F\boldsymbol{e}_3, (\widetilde{\mathbf{I}}_1^0)^{-1}(\boldsymbol{K} - F\boldsymbol{e}_3)\Big) + \frac{1}{2I_3}F^2, \quad \boldsymbol{e}_3 = (0, 0, 1), \tag{8.11}$$

$$\widetilde{\mathbf{I}}_1^0 = \begin{pmatrix} x_1 + I_1 & 0 & z_1 \\ 0 & x_2 + I_1 & z_2 \\ z_1 & z_2 & x_3 \end{pmatrix},$$

where $x_i$, $z_i$ are components of tensor of the rotator inertia with respect to the point $O$ in the chosen frame of reference of rotator. Thus, we again come to the Joukovskiy – Volterra problem, which is specified by Hamiltonian (8.11).

**Comments.** In the paper by E. A. Ivin [81] and in his thesis the integrable cases, discussed in this section, are given in the cumbersome and unclear form. That happened because of the absence of acceptable algebraization of the equations of motion of rotator. We obtained them by means of the Poisson structure general formalism [31]. Such an algebraization allows to see the connection with the Joukovskiy – Volterra problem, which, actually was not shown explicitly. It should also be noted that dynamics of a connected system of rigid bodies is still studied little.

**The Liouville equations** describe motion of a free rigid body, whose dynamical parameters are the given functions of time. They were obtained by J. Liouville in his paper [244] and received more detailed consideration in the treatise by F. Tisserand [275], where their possible physical applications to the problem of motion of celestial bodies, whose parameters change periodically (on the account of the glacier melting, flowing factors and others), are also indicated. The equations of motion of such a system have the form (7.2), where $\boldsymbol{k}(t)$ are the known functions of time, i. e., they represent the particular case of equations of a gyrostat, whose rotor is unbalanced, but performs a specified motion (i. e., the degrees of freedom connected with the rotator, are not added) within the body.

In the Hamiltonian form they represent equations on the algebra $so(3)$

$$\{M_i,\, M_j\} = -\varepsilon_{ijk} M_k \tag{8.12}$$

with the Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}(t)\boldsymbol{M}) - (\boldsymbol{k}(t),\, \boldsymbol{M}), \tag{8.13}$$

where a matrix $\mathbf{A}(t)$ and a vector $\boldsymbol{k}(t)$ are the known functions of time. For periodic functions the conditions of integrability were studied in [26], where it is shown that, under small perturbations of the Euler case, the only possible integrable case (with the accuracy up to the substitution of time) is the Joukovskiy – Volterra problem. In [29] analytic conditions of the adiabatic chaos origination are established, and the adiabatic invariant diffusion at the transition through the separatrix, under the slow and periodic change of functions $\mathbf{A}(t)$ and $\boldsymbol{k}(t)$, is studied.

Chapter 3

# Related Problems of Rigid Body Dynamics

## § 9. Kirchhoff's Equations

### 1. Equations of Motion and Physical Interpretations

**Rigid body dynamics in fluids** If a rigid body is moving in perfect incompressible fluid, possessing a single-valued potential of velocities and resting on infinity, the equations of motion of the rigid body (i. e., the system of six ordinary differential equations) become separated from partial differential equations for fluid motion [85] (for detailed derivation see § 2 ch. 5).

It was *G. Kirchhoff* who obtained and examined the Hamiltonian form of equations of motion of a rigid body under these conditions. The equations can be written on an algebra $e(3) = so(3) \oplus_s \mathbb{R}^3$ (see relations (1.3) ch. 2); under the proper denotation of variables their form is analogical to the one of the Euler – Poisson equations (§ 1 ch. 2)

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \dfrac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \dfrac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \dfrac{\partial H}{\partial \boldsymbol{M}}, \end{cases} \tag{9.1}$$

Here $\boldsymbol{M}$, $\boldsymbol{\gamma}$ represent three-dimensional vectors of "*impulsive moment*" and "*impulsive force*"; to be precise, the projections of these vectors on axes rigidly bound to the rigid body [85] (see also [31]).

A Hamiltonian $H$, which representing a kinetic energy of the system "body+fluid", is a positively determined quadratic form of variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M},\, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M},\, \boldsymbol{\gamma}) + \frac{1}{2}(\mathbf{C}\boldsymbol{\gamma},\, \boldsymbol{\gamma}), \tag{9.2}$$

where matrices $\mathbf{A}$, $\mathbf{C}$ are symmetrical, and a matrix $\mathbf{B}$ is arbitrary. The form (9.1–9.2) of Kirchhoff's equations was obtained by *A. Clebsch* [201].

Remark 1. G. Kirchhoff obtained equations (9.1) in the Lagrangian form (see § 2 ch. 5):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{v}} \times \boldsymbol{v},$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = \frac{\partial L}{\partial \boldsymbol{v}} \times \boldsymbol{\omega}.$$

A Lagrangian $L$, representing kinetic energy, is also a quadratic form of linear $\boldsymbol{v}$ and angular $\boldsymbol{\omega}$ velocity components. Kirchhoff has slightly modified Thomson's arguments which were very close to the final derivation [276].

Equations (9.1) always possess following integrals

$$F_1 = (\boldsymbol{M}, \boldsymbol{\gamma}) = c_1, \quad F_2 = \boldsymbol{\gamma}^2 = c_2, \quad F_3 = H = h. \tag{9.3}$$

Functions $F_1$ and $F_2$, called *integrals of impulsive moment and impulsive force,* correspondingly, are Casimir's functions. They fix a symplectic leaf (in further text we use the Euler – Poisson equations analogy and refer to an integral $F_1$ as an area integral). A Hamiltonian system (with Hamiltonian (9.2)) arising on the leaf lacks one more additional integral to be integrable (this follows from the last multiplier theory, as well, owing to the presence of a standard invariant measure). In the general case Kirchhoff's equations are not integrable. Their nonintegrability and stochasticity is discussed, for example, in [31].

Unlike the Euler – Poisson equations, the constant $c_2$ (in the integral $F_2$), expressing invariability of impulsive force value, does not necessarily equals 1.

The physical sense of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is explained in § 2 ch. 5; they concern associated masses and moments of inertia of a body in fluid. By choosing a frame of reference attached to a rigid body (see § 2 ch. 5), the matrix $\mathbf{A}$ can be reduced to the diagonal form, and the matrix $\mathbf{B}$ to a symmetrical form. Further on, this reduction is considered to be done, thus allowing to reduce the total number of parameters of system (9.2) to 15.

Since a zero vector field corresponds to an arbitrary linear combination of Casimir's functions $\alpha F_1 + \beta F_2$, this combination can be added to the Hamiltonian without changing equations of motion. This allows to reduce the number of parameters in the Hamiltonian by 2. In particular, conditions $\mathbf{B} = \lambda \mathbf{E}$ and $\mathbf{B} = 0$ ($\mathbf{C} = \lambda \mathbf{E}$ and $\mathbf{C} = 0$, as well), where $\lambda = \mathrm{const}$, are equivalent. One parameter can be eliminited by the substitution of time $t \to t/\alpha$. This entails multiplication of the Hamiltonian by an arbitrary constant $H \to \alpha H$, $\alpha = \mathrm{const}$. Thus, the number of parameters, defining family (9.2), equals 12.

**Brun's problem.** The problem about rigid body motion around a fixed point in a linear force field can be represented in the form (9.1) with quadratic



Hamiltonian (9.2). In a linear force field the force, acting on each particle of a body, is proportional to the distance from a certain plane. It is easy to show that the Hamiltonian $H$ in this case is written as

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \mu(\mathbf{I}\boldsymbol{\gamma}, \boldsymbol{\gamma}), \qquad \mathbf{A} = \mathbf{I}^{-1}, \tag{9.4}$$

where $\mathbf{I}$ is a tensor of inertia.

This problem was considered by Brun [198]. F. Tisserand examined the same problem in connection with motion of a rigid body under the action of Newton's gravity center [275]. In this case a quadratic potential in (9.4) appears as a quadropole approximation in Newton's potential expansion with respect to the ratio of the body dimensions to increasing distance from Newton's center. It turns out that Brun's problem is equivalent to Clebsch's integrable case in terms of Kirchhoff's equations (see § 12 ch. 3). This similarity (9.4) was noticed by *V. A. Steklov* [272].

**Grioli's problem.** It is the problem about motion of a charged rigid body with stationary distribution of charges (of a dielectric) around a fixed point in a permanent magnetic field [10, 191, 222, 223]. The Hamiltonian of the system contains terms which are cross (generalized potential) in $\boldsymbol{M}$ and $\boldsymbol{\gamma}$. It is written as

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) - \frac{1}{2}(\mathbf{J}\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{M}), \qquad \mathbf{A} = \mathbf{I}^{-1},$$

where $\mathbf{I}$ is a tensor of inertia, $\mathbf{J}$ is a symmetrical tensor of distribution of charges.

We can consider a more general force field, being a superposition of gyroscopic and potential forces quadratic in $\boldsymbol{M}$, $\boldsymbol{\gamma}$; equations of motion of such a system are also reduced to Kirchhoff's equations. The analogy between these problems is shown in several sources [21, 281]. However, the analogy is very complicated, since it is established on the level of equations of motion, not on the level of Hamiltonians and corresponding Poisson brackets. That was the natural way of establishing this analogy in [10].

**Neumann's system [251].** The classical integrable problem of C. Neumann about motion of a material point on a sphere in a field of forces with a quadratic potential $U = \frac{1}{2}(\mathbf{B}\boldsymbol{q}, \boldsymbol{q})$, $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$ is described by equations

$$\begin{gathered} \ddot{q}_i = b_i q_i + \lambda q_i, \quad i = 1, 2, 3, \\ \lambda = -(\mathbf{B}\boldsymbol{q}, \boldsymbol{q}) - \dot{\boldsymbol{q}}^2, \end{gathered} \tag{9.5}$$

where $q_i$ are redundant Cartesian coordinates of the point on a sphere $\boldsymbol{q}^2 = 1$, $\lambda$ is an indefinite constraint multiplier. Passing to variables $\boldsymbol{M} = \boldsymbol{q} \times \dot{\boldsymbol{q}}$, $\boldsymbol{\gamma} = \boldsymbol{q}$, equations (9.5) can be rewritten as

$$\dot{\boldsymbol{M}} = \boldsymbol{\gamma} \times \mathbf{B}\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{M}, \tag{9.6}$$

i. e., represented as a system on an algebra $e(3)$ with a Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{M}) + \frac{1}{2}(\mathbf{B}\boldsymbol{\gamma}, \boldsymbol{\gamma})$$

on a level $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$; this follows from the definition of $\boldsymbol{M}$, $\boldsymbol{\gamma}$ for this problem.

This Hamiltonian corresponds to Clebsch's case (see further) under an additional condition $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, i. e., a zero level of a Casimir function $F_2$ (9.3) is fixed. The shown analogy between motion of a point on a sphere and a rigid body motion is preserved in $n$-dimensional situation, as well (see [195]). The connection of Neumann's problem and Clebsch's case with invariant under translation solutions of Landau – Lifshitz equations is considered in § 6 ch. 5.

**Jacobi's problem about geodesics on an ellipsoid [183].** Let an ellipsoid in three dimensional space be given by the equation

$$(\boldsymbol{q}, \mathbf{B}\boldsymbol{q}) = 1, \quad \mathbf{B} = \operatorname{diag}(b_1, b_2, b_3).$$

Dynamics of a free particle on this ellipsoid is described by equations

$$\ddot{\boldsymbol{q}} = \lambda \mathbf{B}\boldsymbol{q}, \quad \lambda = -\frac{(\dot{\boldsymbol{q}}, \mathbf{B}\boldsymbol{q})}{(\mathbf{B}\boldsymbol{q}, \mathbf{B}\boldsymbol{q})}\mathbf{B}\boldsymbol{q}. \tag{9.7}$$

Passing to new variables

$$\boldsymbol{\gamma} = \mathbf{B}^{1/2}\boldsymbol{q}, \quad \boldsymbol{M} = (\mathbf{A}\dot{\boldsymbol{\gamma}}) \times \boldsymbol{\gamma}, \quad \mathbf{A} = \mathbf{B}^{-1},$$

equations of motion (9.7) can be represented in the Hamiltonian form (9.1) on an algebra $e(3)$ with the Hamiltonian

$$H = \frac{1}{2}\det \mathbf{B}\frac{(\boldsymbol{M}, \mathbf{A}\boldsymbol{M})}{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})} \tag{9.8}$$

on a zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$.



After the substitution of time $\dfrac{\det \mathbf{B}}{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})}\,dt = d\tau$ system (9.8) on the energy level $H = c\dfrac{\det \mathbf{B}}{2}$ is reduced to Clebsch's system (see further) with the Hamiltonian

$$H' = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) - \frac{1}{2}c(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}),$$

on the zero energy level $H' = 0$ (V. V. Kozlov [88]). Here $c$ is an arbitrary constant.

This analogy is preserved in the multidimensional case [195], as well. In the book [31] this isomorphism is discussed in details for the case of quaternion equation of rigid body dynamics.

Remark. C. Jacobi has shown that it is also possible to integrate the problem on motion of a material point on an ellipsoid in a field with a quadratic potential

$$U(\boldsymbol{q}) = \frac{1}{2}k\boldsymbol{q}^2, \tag{9.9}$$

i. e., the point is attached to the ellipsoid center by Hook's spring [183].

For the Hamiltonian form (9.1) in this case the Hamiltonian may be represented as

$$H = \frac{1}{2}\det \mathbf{B}\frac{(\boldsymbol{M}, \mathbf{A}\boldsymbol{M})}{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})} + \frac{1}{2}k(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma}). \tag{9.10}$$

After the change of time on the level $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ we obtain a new integrable system with the fourth degree potential

$$H' = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})(k'(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma}) - c), \quad k' = \frac{k}{\det \mathbf{B}}, \tag{9.11}$$

On the level $H' = 0$ the new system is isomorphic to system (9.10) on the level $H = \frac{1}{2}c\det \mathbf{B}$.

Integrable system (9.11) appears also in the investigation of separation of variables for polynomial potential on a sphere [18, 283].

Remark 2. In the paper [49] the author shows connection between $n$-dimensional Jacobi problem about geodesics and a stable zero position of equilibrium of Hill's type linear equation with periodic coefficients. It turns out that the number of resonance zones is finite and does not exceed the ellipsoid dimensionality whenever a periodic function $R(t)$ in the equation $\ddot{x} = -R(t)x$ is Lagrange's multiplier for a certain geodesic on an ellipsoid (to be precise, $R(t) = -\lambda(t)$).

## 2. Integrable Cases

Table 3.1 shows all the known cases of Kirchhoff's equations integrability. Cases 1, 2, 3, 4 are general cases of integrability, and 5, 6 are particular cases. In the latter case, except for limitations for the system parameters, it is also

Table 3.1. Integrable cases of Kirchhoff's equation

| | Author | Conditions on parameters and the first integral $F$ |
|---|---|---|
| 1 | Kirchhoff (1870) [85] | $\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3)$, $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$,<br>$\mathbf{C} = \mathrm{diag}(c_1, c_2, c_3)$, $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$<br>$F = M_3$ (Lagrange case analogue) |
| 2 | Clebsch (1871) [201] | I $\begin{cases} \mathbf{A} = \mathrm{diag}(a_1, a_2, a_3),\ \mathbf{B} = 0,\ \mathbf{C} = \mathrm{diag}(c_1, c_2, c_3), \\ \dfrac{c_2 - c_3}{a_1} + \dfrac{c_3 - c_1}{a_2} + \dfrac{c_1 - c_2}{a_3} = 0,\ a_1 \neq a_2 \neq a_3 \neq a_1 \\ F = \boldsymbol{M}^2 - (\mathbf{A}\boldsymbol{\gamma}, \boldsymbol{\gamma}) \end{cases}$<br>II $\begin{cases} \mathbf{B} = 0,\ a_1 = a_2 = a_3, \quad \mathbf{C} - \text{arbitrary} \\ F = (\boldsymbol{M}, \mathbf{C}\boldsymbol{M}) - \mu(\boldsymbol{\gamma}, \mathbf{C}^{-1}\boldsymbol{\gamma}), \quad \mu = \mathrm{const} \end{cases}$ |
| 3 | Steklov (1893) [160] | $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$, $\mathbf{C} = \mathrm{diag}(c_1, c_2, c_3)$,<br>$b_j = \mu \dfrac{a_1 a_2 a_3}{a_j} + \nu, \quad c_k = \mu^2 a_k (a_i - a_j)^2 + \nu', \mu, \nu, \nu' = \mathrm{const}$<br>$F = \sum\limits_j (M_j^2 - 2\mu(a_j + \nu) M_j \gamma_j) + \mu^2((a_2 - a_3)^2 + \nu')\gamma_1^2 + \ldots$ |
| 4 | Lyapunov (1893) [115] | $\mathbf{A} = \mathbf{E} = \\|\delta_{ij}\\|$, $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$, $\mathbf{C} = \mathrm{diag}(c_1, c_2, c_3)$,<br>$b_j = -2\mu(d_j + \nu), \quad c_k = (d_i - d_j)^2 + \nu', \ldots, \quad d_i = \mathrm{const}$,<br>$F = \sum \left( d_i M_i^2 + \left( 2\mu \dfrac{d_1 d_2 d_3}{d_i} + \nu \right) M_i \gamma_i \right) +$<br>$+ \mu^2 d_1 (d_2 - d_3)^2 \gamma_1^2 + \ldots$ |
| 5 | Chaplygin (1902) [178] | $\mathbf{B} = 0$, $\mathbf{C} = \mathrm{diag}(c, -c, 0)$, $\mathbf{A} = \mathrm{diag}(a, a, 2a)$,<br>$F = (M_1^2 - M_2^2 + \frac{c}{a}\gamma_3^2)^2 + 4M_1^2 M_2^2, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ |
| 6 | Chaplygin (1897) [178] Kozlov, Onischenko (1982) [98] | $\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3)$, $b_i = b_{ii}$, $c_i = c_{ii}$,<br>$b_{13}\sqrt{a_2 - a_1} \mp (b_2 - b_1)\sqrt{a_3 - a_2} = 0$, $b_{12} = 0$,<br>$b_{13}\sqrt{a_3 - a_2} \pm (b_3 - b_2)\sqrt{a_2 - a_1} = 0$, $b_{23} = 0$,<br>$c_{13}\sqrt{a_2 - a_1} \mp (c_2 - c_1)\sqrt{a_3 - a_2} = 0$, $c_{12} = 0$,<br>$c_{13}\sqrt{a_3 - a_2} \pm (c_3 - c_2)\sqrt{a_2 - a_1} = 0$, $c_{23} = 0$<br>$F = M_1\sqrt{a_2 - a_1} \mp M_3\sqrt{a_3 - a_2} = 0$ |
| 7 | Sokolov (2001) [157] | $\mathbf{A} = \mathrm{diag}(1, 1, 2)$, $b_{13} = \alpha$, $b_{11} = b_{22} = b_{33} = b_{12} = b_{23} = 0$,<br>$c_{22} = 2\alpha^2$, $c_{33} = -2\alpha^2$, $c_{11} = c_{12} = c_{13} = c_{23} = 0$,<br>$F = (M_3 - \alpha\gamma_1)\big[(M_3 - \alpha\gamma_1)(\boldsymbol{M}^2 + 4\alpha(M_3\gamma_1 - M_1\gamma_3) +$<br>$+ 4\alpha^2(\gamma_1^2 + \gamma_3^2)) + 6\alpha(M_1 - 2\alpha\gamma_3)(\boldsymbol{M}, \boldsymbol{\gamma})\big]$ |



necessary to introduce additional limitations for values of integrals, i. e., initial conditions.

Necessary and sufficient conditions of integrability of Kirchhoff's equation are discussed in the paper [10].

As it was shown by *V. A. Steklov* [10, 27], *if Hamiltonian* (9.2) *is a positively definite form* (*it is deliberately fulfilled when a body moves in fluid*), *then cases* 1, 2, 3, 4 *from Table* 3.1 *exhaust the possibility of Kirchhoff's equations with an additional independent integral in linear and quadratic form of* $\boldsymbol{M}$, $\boldsymbol{\gamma}$. For the proof of this claim see, for example, [151].

Just before the book went into printing, we learnt about the results by V. V. Sokolov [157], who found a new integrable case of Kirchhoff's equation with the fourth degree integral (see Table 3.1). This result allowed to construct an analogical new case in terms of Poincaré–Joukovskiy equations (see § 10). We describe these cases in more details in § 12 ch. 5. They turned out unusual and remarkable, but need further investigations.

**Comments.** **1.** When in Hamiltonian (9.2) matrices $\mathbf{B}$ and $\mathbf{C}$ are not diagonal, the question of algebraic integrability was examined by *Roger Liouville* [245] (whom should not be interfered with the famous mathematician of nineteenth century — Joseph Liouville). In this paper *R. Liouville* indicates conditions of an additional integral existence when $b_{ij} \neq 0$ at $i \neq j$. However, the explicit form of this integral is absent. The numerical experiment done by the authors showed chaotic behavior of the system under Liouville's general conditions; this indicates a certain looseness of derivations in the paper [245].

**2.** Besides the cases of integrability shown in Table 3.1, there exists one more general case of integrability with an additional quadratic integral. It is implemented at $\mathbf{A} \equiv 0$; this cannot be assigned to a real situation. An additional integral $F = (\mathbf{B}\boldsymbol{\gamma}, \boldsymbol{\gamma})$ allows to reduce the system to quadratures. This reduction is easily fulfilled by means of a motor calculus (motorrechnung) (*A. A. Burov*, *V. N. Rubanovskiy* [44]).

**3.** Particular solutions of equations of rigid body motion in fluid were studied by A. M. Lyapunov [117], V. A. Steklov [162] and S. A. Chaplygin [173].

A. M. Lyapunov paid special attention to the questions of stability, V. A. Steklov to explicit integration, S. A. Chaplygin to geometric interpretation. A lot of their results is presently of the pure historical interest. The simplest particular solutions (in particular — planar motions) and their physical interpretation are discussed in the treatise by H. Lamb [111].

### 3. The Case of Axial Symmetry

This case was indicated by G. Kirchhoff for a dynamically symmetrical body of revolution, moving in perfect fluid. Kirchhoff has also integrated equations of motions in elliptic functions.

This case of integrability is similar to Lagrange's case in the Euler–Poisson equations (§ 3 ch. 2), and an additional integral $F = M_3$ is connected with the presence of a cyclic coordinate (a proper rotation angle). In § 1 ch. 4 we show reduction to one degree of freedom and explicit integration.

Plane motions and partial solutions (like helical ones) for a rigid body under Kirchhoff's conditions of integrability were studied in the book by Lamb [111].

### 4. Clebsch's case

A. Clebsch has found two related cases of integrability from conditions of an additional quadratic integral existence. One of them is mutual to the other, i. e., the Hamiltonian of one case can be taken for the integral in the other case. In fact, they form a single integrable family of quadratic Hamiltonians without cross terms ($\mathbf{B} = 0$).

The table presents classical forms of notation of Clebsch's integrals. Nevertheless, this integrable family may be represented in the more symmetrical form

$$\begin{aligned}\widetilde{G}_1 &= \mu\gamma_1^2 + \frac{M_3^2}{\lambda_1^2 - \lambda_2^2} + \frac{M_2^2}{\lambda_1^2 - \lambda_3^2},\\ \widetilde{G}_2 &= \mu\gamma_2^2 + \frac{M_3^2}{\lambda_2^2 - \lambda_1^2} + \frac{M_1^2}{\lambda_2^2 - \lambda_3^2},\\ \widetilde{G}_3 &= \mu\gamma_3^2 + \frac{M_2^2}{\lambda_3^2 - \lambda_1^2} + \frac{M_1^2}{\lambda_3^2 - \lambda_2^2}.\end{aligned} \tag{9.12}$$

We may consider following relations valid

$$\sum_{i=1}^{3} \widetilde{G}_i = \mu\boldsymbol{\gamma}^2, \quad \sum_{i=1}^{3} \lambda_i \widetilde{G}_i = H_I, \quad \sum_{i=1}^{3} \lambda_i^2 \widetilde{G}_i = H_{II},$$

where $H_I$ and $H_{II}$ are Hamiltonians of two mutual Clebsch's cases correspondingly.

Such a form of motion integrals (9.12), allows generalization to the multidimensional case [128]. It was shown by *K. Uhlenbeck* [278] in 1975 (one can also see these integrals in the paper by Devaney [203]) in the process of examining Neumann's problem. In 1859 *C. Neumann* integrated it by separation of variables (see § 7 ch. 1). For a three dimensional case integrals (9.12) were already known to *H. Weber* [282] (1878).

In § 10 ch. 3 Clebsch's integrable cases are extended to the bundle of Poisson brackets (in particular, it is a Shottky–Manakov system). The retraction and linear isomorphism of these cases are also shown. Clebsch's integrable family allows two different Lax representations with a spectral parameter. These are cited in the book [31].

Remark 3. F. Kötter showed Clebsch's integrable family in the symmetrical form, containing an arbitrary (spectral) parameter [236]

$$Q(s) = \sum_{i=1}^{3} \left(\sqrt{s - D_i} M_i + \sqrt{(s - D_j)(s - D_k)}\gamma_i\right)^2,$$

where $D_i$ are arbitrary constants, $s$ is the parameter. The connection of this representation with the existence of $\mathbf{L} - \mathbf{A}$-pair on Lie bundles is studied in [31].

Remark 4. In the paper [250] *H. Minkowski* showed the analogy of Clebsch's case to Jacobi's problem about geodesics, thereby giving his own way of its integration. This analogy is developed above in subsection 1 of this section (see also [195]).

Remark 5. S. A. Chaplygin [173], was trying to give geometrical interpretation of motion in Clebsch's case at $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$. He represented the motion as sliding-free rolling of a certain hyperboloid on a helical surface. In the paper [172] E. I. Harlamova showed that at $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ the corresponding motion can be obtained as the more natural generalization of Poinsot's interpretation: an ellipsoid of inertia rolls without sliding on the surface of an elliptic cylinder (fixed in space) whose axis is directed along vector $\boldsymbol{\gamma}$ and passes through a fixed point of the body.

### 5. The Steklov–Lyapunov Family

A. Clebsch has not analyzed conditions of existence of quadratic integrals to the full extent. V. A. Steklov has corrected his reasoning in his own master thesis (which in 1893 appeared as a separate book [160]) He indicated the case with a quadratic integral whose Hamiltonian contains terms cross in $\boldsymbol{M}, \boldsymbol{\gamma}$. The mutual case was shown by A. M. Lyapunov [115] who now corrected computations of Steklov whose research supervisor he was. Table 3.1 presents the classical form of Hamiltonians and integrals; this was the form shown by V. A. Steklov and A. M. Lyapunov.

The Steklov–Lyapunov family can also be written in a symmetrical form by means of three integrals in involutive

$$\widetilde{G}_i = \sum_{j\neq i}^{3} \frac{\left(M_{ij} + \frac{1}{2}(\lambda_i - \lambda_j)P_{ij}\right)^2}{\lambda_i - \lambda_j}, \quad i = 1, 2, 3, \tag{9.13}$$

where components of matrices $\mathbf{M} = \|M_{ij}\|$, $\mathbf{P} = \|P_{ij}\|$ are connected with vectors $\boldsymbol{M}, \boldsymbol{\gamma}$ by formulae

$$M_{ij} = -\varepsilon_{ijk} M_k, \quad P_{ij} = -\varepsilon_{ijk}\gamma_k.$$

For example, $\widetilde{G}_1$ is written as (other $\widetilde{G}_i$ are obtained by cyclic permutation)

$$\widetilde{G}_1 = \frac{\left(M_3 + \frac{1}{2}(\lambda_1 - \lambda_2)\gamma_3\right)^2}{\lambda_1 - \lambda_2} + \frac{\left(M_2 + \frac{1}{2}(\lambda_1 - \lambda_3)\gamma_2\right)^2}{\lambda_1 - \lambda_3}.$$

For functions (9.13) the expression

$$\sum_{i=1}^{3} \widetilde{G}_i = 2(\boldsymbol{M}, \boldsymbol{\gamma})$$

is valid.

Remark 6. For Lyapunov's case the Hamiltonian (see Table 3.1) can be obtained as

$$H_L = \frac{1}{2}\sum_{i=1}^{3} \lambda_i \widetilde{G}_i = \frac{1}{2}\boldsymbol{M}^2 + (\boldsymbol{M}, \mathbf{B}\boldsymbol{\gamma}) + \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}),$$

where $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$, $\mathbf{C} = \mathrm{diag}\left((b_2 - b_3)^2, (b_3 - b_1)^2, (b_1 - b_2)^2\right)$ and $b_k = \frac{1}{2}(\lambda_i + \lambda_j)$, $i, j, k = 1, 2, 3$.

For Steklov's case the Hamiltonian is obtained from the following relation

$$H = \frac{1}{2}\sum_{i=1}^{3} \lambda_i^2 \widetilde{G}_i = \frac{1}{2}\sum_{\text{cycle}}((\lambda_i^2 + \lambda_j^2)M_k^2 + (\lambda_i^2 + \lambda_j^2)M_k\gamma_k) + \frac{1}{4}((\lambda_i^2 - \lambda_j^2)^2\gamma_k^2)$$

with further substitution $a_k = \frac{1}{2}(\lambda_i + \lambda_j)$.

Representation (9.13) is the most symmetrical parametrization of *Steklov's* and *Lyapunov's* cases (see also § 10, ch. 3) and seems not to be indicated earlier. It may be obtained (in the multidimensional case, as well) by using $\mathbf{L} - \mathbf{A}$-pair with a hyperelliptic spectral parameter; this pair was shown in [23, 31].

In § 10 ch. 3 this family is extended to the Poincaré – Joukovskiy equations. Besides, this family, unlike Clebsch's case, allows addition of terms linear with respect to $\boldsymbol{M}$, $\boldsymbol{\gamma}$ (gyroscopic additions) (see further).



Remark 7. In the paper [100] *G. V. Kolosov* showed the fourth integral of equations of motion of the system with Hamiltonian

$$H = \frac{1}{2}\left(c_1 M_1^2 + c_2 M_2^2 + c_3 M_3^2 + 2b_1 M_1\gamma_1 + 2b_2 M_2\gamma_2 + 2b_3 M_3\gamma_3 + a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2\right)$$

at the following conditions for constants $a_i$, $b_i$, $c_i$ ($i$=1, 2, 3)

$$\begin{gathered}\frac{c_1(c_2 - c_3)}{b_3 - b_2} = \frac{c_2(c_3 - c_1)}{b_1 - b_3} = \frac{c_3(c_1 - c_2)}{b_2 - b_1},\\ a_1 - \frac{(b_2 - b_3)^2}{c_1} = a_2 - \frac{(b_3 - b_1)^2}{c_2} = a_3 - \frac{(b_1 - b_2)^2}{c_3}\end{gathered} \tag{9.14}$$

in the form

$$F = \frac{b_3 - b_1}{c_2}\left(M_1 - \frac{b_3 - b_2}{c_1}\gamma_1\right)^2 + \frac{b_3 - b_2}{c_1}\left(M_2 - \frac{b_3 - b_1}{c_2}\gamma_2\right)^2$$

He also showed that particular cases of conditions (9.14) are *Steklov's and Lyapunov's cases*. In this way he included them into a single integrable family whose particular representatives are also integrals (9.13). Sometimes this family is called "Lyaponov – Steklov – Kolosov case".

**Comments.** To investigate Clebsch's and Steklov – Lyapunov's cases, starting from the moment of their discovery and following the general ideology of the time, many mathematicians were trying to integrate them in terms of elliptic functions. This subject was of interest for H. Weber, G. Halphen, F. Kötter. H. Weber has integrated the second Clebsch's case [282] at $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, i. e., essentially Neumann's problem. G. Halphen [227] gave detailed consideration to the dynamical symmetry case whose integration in terms of elliptic functions is carried out similarly to the Lagrange top. F. Kötter offered his own integration method for two Clebsch's cases at $(\boldsymbol{M}, \boldsymbol{\gamma}) \neq 0$ [236]. In the second paper [234] he announced about explicit integration of Steklov's and Lyapunov's cases. Kötter's papers caused incomprehension even of his contemporaries (S. A. Chaplygin, V. A. Steklov, M. A. Tichomandritskiy) owing to their ambiguity and impossibility of verifying their results. Besides, the paper [234], published in the Proceedings of Prussian Royal Academy of Sciences, is too short and cannot be verified explicitly, even by using modern systems of analytical computations. The book [209] contains some geometrical arguments allegedly explaining the idea of Kötter's substitutions. However, they are far from sufficient. Besides, independently of righteousness of papers [234, 236], we should note that they lack explicit expression for characteristic polynomials in terms of constants of integrals in Abel – Jacobi equations. Such an "implicit" solution practically makes it useless because it prevents construction of bifurcational patterns, highlighting of especially remarkable solutions (see ch. 2) and the like.

Nevertheless, it should be noted that in his method of integration of Steklov–Lyapunov case Kötter actually obtained $\mathbf{L}-\mathbf{A}$-pair with a spectral parameter (see [31]) and a symmetrical one-parameter representation of integrals

$$Q(s)=\sum_{i=1}^{3}(s-b_i)(z_i+s\gamma_i)^2,$$

where $2z_i=M_i-(d_j+d_k)\gamma_i$, $d_i=\mathrm{const}$.

The methods of reducing Clebsch's case to quadratures, given in the books [9, 61] and obtained by Kobb and E. I. Harlamova, do not offer a real possibility of obtaining a general solution. Kobb wrote the Hamiltonian of a system in terms of Euler angles, and E. I. Harlamova [172] in terms of spheroconical coordinates. But in terms of both types of coordinates Clebsch's case cannot be separated for the non-zero area constant. It should also be noted that in unpublished manuscripts [180] S. A.Ċhaplygin also used Hamilton–Jacobi method for integration of two Clebsch's cases in terms of spheroconical coordinates. He also offered a similar procedure for integration of complete (i. e., for $\boldsymbol{M},\boldsymbol{\gamma}$) system of equations for the Euler–Poinsot case.

## 6. Chaplygin's case (I)

S. A. Chaplygin indicated a particular case of integrability on the zero area constant $(\boldsymbol{M},\boldsymbol{\gamma})=0$ with the fourth degree integral. A Hamiltonian and an integral may be represented as

$$\begin{aligned} H&=\frac{1}{2}\big(M_1^2+M_2^2+2M_3^2\big)+\frac{1}{2}c(\gamma_1^2-\gamma_2^2),\\ F&=\big(M_1^2-M_2^2+c\gamma_3^2\big)^2+4M_1^2M_2^2. \end{aligned} \tag{9.15}$$

This system is related to the Kowalevskaya case in Euler–Poisson equations. Its explicit integration was also carried out by S. A. Chaplygin [175].

O. I. Bogoyavlenskiy extended Chaplygin's case to Poincaré–Joukovskiy equations (see § 10 ch. 3); at that, Hamiltonian (9.15) suffers from dynamical symmetry breaking. § 8 ch. 5 contains generalization of these cases on the bundle of brackets and their explicit integration.

The connection of this system with rigid body dynamics in a uniform field superposition is shown in § 1 ch. 4. This section also informs how this case may be extended to the general integrable case in quaternion equations. The latter case is a direct generalization of the Kowalevskaya case. Chaplygin's case allows the addition of a gyrostat along the axis of dynamical symmetry (§ 1 ch. 4). Besides, a certain system is integrated on a zero area constant. A potential energy of this system is the superposition of the Chaplygin and Kowalevskaya cases (§ 7 ch. 5).



## 7. Chaplygin's Case (II)

This case is similar to the Hess case in Euler–Poisson equations and is connected with the presence of an invariant relation of the form

$$M_1\sqrt{a_2-a_1}\mp M_3\sqrt{a_3-a_2}=0, \qquad a_1<a_2<a_3. \tag{9.16}$$

The cumbersome conditions, cited in Table 3.1, are very simple from the geometrical viewpoint. Using the analogy with Euler–Poisson equations, we suppose that a dynamically asymmetrical rigid body is moving in a generalized potential field, i. e., $\boldsymbol{\gamma}$ are some positional variables. Then the existence of relation (9.16) is conditioned by the symmetry of potential and generalized potential of system (9.2) with respect to rotations around the perpendicular to the circular section of a gyration ellipsoid (compare to § 6 ch. 2).

S. A. Chaplygin indicated both conditions and technique of explicit integration of this case in his master thesis (1897) [178]. However, he failed to show its connection with the Hess case explicitly. In 1982 V. V. Kozlov and D. A. Onischenko [98], independently found it from the condition of splitting of separatrices. It turned out that in this case, like in the Hess case, one pair of separatrices of the reduced system (described by relation (9.16)) is doubled and define one-parameter family of double-asymptotic motions.

The connection of this case with the presence of a cyclic variable (the angle of rotation around the perpendicular to the circular section of a gyration ellipsoid) on level (9.16), and the possible reduction with respect to this variable is discussed in details in §§ 3, 4 ch. 4.

## 8. Integrable Generalizations with Linear Terms in a Hamiltonian

In the integrable cases considered above Hamiltonian (9.2) is a homogenous quadratic form of variables $\boldsymbol{M},\boldsymbol{\gamma}$. However, there are some cases (which are interesting from physical and mechanical viewpoints) when the terms, linear with respects to $\boldsymbol{M},\boldsymbol{\gamma}$ are added in the Hamiltonian. For various statements, given in 1, these additions have different interpretations. So, for dynamics of a rigid body in fluid these terms may be conditioned by multiple connections of the rigid body (see § 2 ch. 5); for Brun's system by presence of rotor and a uniform permanent force field; for the dynamics of a point on a sphere by the presence of a permanent electrical (magnetic) field.

**The equations of motion of a multiconnected body.** If a body moving under Kirchhoff's conditions has holes, i. e., is multiconnected, then its equations also have the form (9.1). However, the Hamiltonian obtains terms, linear with respect to $\boldsymbol{M}, \boldsymbol{\gamma}$ [111, 171]

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\mathbf{B}\boldsymbol{M}, \boldsymbol{\gamma}) + \frac{1}{2}(\mathbf{C}\boldsymbol{\gamma}, \boldsymbol{\gamma}) + (\boldsymbol{a}, \boldsymbol{M}) + (\boldsymbol{b}, \boldsymbol{\gamma}). \quad (9.17)$$

where $\boldsymbol{a}, \boldsymbol{b}$ are constant vectors. They are linearly expressed in terms of circulations of fluid velocities along the outlines of holes in the body (see § 2 ch. 5). The conditions of integrability of such systems were studied in [149, 148].

Here a trivial generalization allows the integrable cases of Kirchhoff and Chaplygin (II) (see Table 3.1); see also § 7 ch. 2, §§ 1, 2 ch. 4). In this situation a constant gyrostatic moment along the corresponding axis is added (for Kirchhoff it is the axis of dynamical symmetry, and for Chaplygin (II) it is the perpendicular to the circular section of a gyration ellipsoid).

The integrable generalization of Clebsch's case is unknown; Steklov – Lyapunov's family was generalized by V. N. Rubanovskiy [149] and the corresponding Lax representation is shown in the paper [208]. A gyrostatic generalization of Chaplygin's case (I) was obtained by H. Yehia [285] (given in § 7 ch. 5). Here we show Rubanovskiy's family in the most symmetrical form and generalization of the first Chaplygin case.

**Rubanovskiy's generalization of Steklov – Lyapunov integrable family.** In the paper [149] it was shown that Steklov – Lyapunov integrable family allows integrable generalization. In case of such a generalization the Hamiltonian has linear terms added. Let us write this integrable case in the form of family of three integrals in involution. These integrals have the form

$$\widetilde{J}_s = \widetilde{G}_s + \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\Bigg(\sum_{\text{cycl. var. } ijk} r_k\big(M_k + \frac{1}{2}(\lambda_i-\lambda_j)\gamma_k\big) + \\ + \frac{1}{2}r_s(2\lambda_s-\lambda_m-\lambda_n)\gamma_s\Bigg). \\ s, m, n = 1, 2, 3,$$



For example, $\widetilde{J}_1$ is written as (others are obtained by the cyclic permutation of indices)

$$\widetilde{J}_1 = \widetilde{G}_1 + \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\left(r_1\left(M_1 - \frac{\lambda_2+\lambda_3-2\lambda_1}{2}\gamma_1\right) + \right. \\ \left. + r_2\left(M_2 - \frac{\lambda_3-\lambda_1}{2}\gamma_2\right) + r_3\left(M_3 - \frac{\lambda_2-\lambda_3}{2}\gamma_3\right)\right), \quad (9.18)$$

where $\widetilde{G}_i$, $i = 1, 2, 3$ are Steklov – Lyapunov integrals (9.13). For integrals $\widetilde{J}_i$ the relation

$$\sum_{i=1}^{3} \widetilde{J}_i = 2(\boldsymbol{M}, \boldsymbol{p}).$$

is valid.

The Hamiltonian and the integral, found by V. N. Rubanovskiy [149], may be obtained from $\widetilde{J}_i$ as follows

$$H = \frac{1}{2}\sum_{i=1}^{3}\lambda_i^2\widetilde{J}_i = \frac{1}{2}\sum_{\text{cycle}}\big((\lambda_i^2+\lambda_j^2)M_k^2 + (\lambda_i^2+\lambda_j^2)M_k\gamma_k + \\ + \frac{1}{4}\big((\lambda_i^2-\lambda_j^2)^2\gamma_k^2\big) + \frac{1}{2}(\lambda_i+\lambda_j+2\lambda_k)r_k\gamma_k\big) + (\boldsymbol{r}, \boldsymbol{M}), \quad (9.19)$$

$$F = \frac{1}{2}\sum_{i=3}^{3}\lambda_i\widetilde{J}_i = \frac{1}{2}\boldsymbol{M}^2 + (\boldsymbol{M}, \mathbf{B}\boldsymbol{\gamma}) + \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}) + (\boldsymbol{r}, \boldsymbol{\gamma})$$

with further substitution $a_k = \frac{1}{2}(\lambda_i + \lambda_j)$, where $\mathbf{B} = \operatorname{diag}(b_1, b_2, b_3)$, $\mathbf{C} = \operatorname{diag}(b_2 - b_3, b_3 - b_1, b_1 - b_2)$ and $b_k = \frac{1}{2}(\lambda_i + \lambda_j)$, $i, j, k = 1, 2, 3$.

The Lax representation for this integrable case is shown in [208].

Remark 8. Integrals (9.18) can also be obtained by means of retraction from similar generalization (10.26) on $so(4)$ (§ 10 ch. 3).

**The Generalization of Chaplygin's Case (I).** This particular case of integrability can be generalized by means of addition of a constant gyroscopic moment along the axis of dynamical symmetry (H. Yehia [285]). The Hamiltonian and the integral may be represented in the form

$$\begin{aligned}
H &= \frac{1}{2}\Big(M_1^2 + M_2^2 + 2\Big(M_3 - \frac{\lambda}{2}\Big)^2\Big) + \frac{c}{2}(\gamma_1^2 - \gamma_2^2),\\
F &= \big(M_1^2 - M_2^2 + c\gamma_3^2\big) + 4M_1^2M_2^2 + \\
&\quad + 4\lambda\big(M_3(M_1^2 + M_2^2) - c\gamma_3(M_1\gamma_1 - M_2\gamma_2)\big) - \\
&\quad - 4\lambda^2(M_1^2 + M_2^2).
\end{aligned} \tag{9.20}$$

§ 7 ch. 5 discusses the extension of this case to the bundle of Poisson brackets and to the case of addition of terms, linear in $\gamma_i$, to Hamiltonian $H$ (9.20).

## § 10. Poincaré – Joukovskiy Equations

### 1. Equations of Motion and Their Physical Interpretation

**Poisson's structure and equations of motion.** First of all, let us consider a formally Hamiltonian system on an algebra $so(4)$ and premise some Lie algebra corollaries to a dynamical description. Depending on dynamical origin of the equations considered, it is convenient to use various systems of variables (denoted as $(\boldsymbol{M}, \boldsymbol{p})$ or $(\boldsymbol{K}, \boldsymbol{S})$), being in simple relations

$$\boldsymbol{M} = \frac{\boldsymbol{K} + \boldsymbol{S}}{2}, \quad \boldsymbol{p} = \frac{\boldsymbol{K} - \boldsymbol{S}}{2}.$$

**Variables** $(\boldsymbol{M}, \boldsymbol{p})$ correspond to "standard" (matrix) representation of $so(4)$ and commutative relations have the form

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, p_j\} = -\varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = -\varepsilon_{ijk}M_k. \tag{10.1}$$

Structure (10.1) has following Casimir's functions

$$F_1 = \boldsymbol{M}^2 + \boldsymbol{p}^2, \quad F_2 = (\boldsymbol{M}, \boldsymbol{p}). \tag{10.2}$$

Level sets of these integrals are diffeomorphic to $S^2 \times S^2$. It becomes evident if the integrals are expressed in terms of variables $(\boldsymbol{K}, \boldsymbol{S})$ (see (10.6)).

The equations of motion can be represented in the vector form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = \boldsymbol{p} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{p}}. \tag{10.3}$$

This system of variables is also suitable for description of *a linear bundle of Poisson structures* $\mathscr{L}_x$ (see §§ 3, 4 ch. 4, §§ 7, 8 ch. 5). This bundle includes



algebras $so(4)$, $e(3)$, $so(3,1)$, as well. Commutative relations for this bundle are as follows

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, p_j\} = -\varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = -x\varepsilon_{ijk}M_k, \tag{10.4}$$

where $x = \text{const}$ is an arbitrary constant. For $x > 0$ commutative relations (10.4) define algebra $so(4)$, for $x = 0$ — algebra $e(3)$, and for $x < 0$ — algebra $so(3,1)$. Really, at $|x| \neq 1$, $x \neq 0$, transformation $\boldsymbol{M} \to \boldsymbol{M}$, $\boldsymbol{p} \to |x|^{1/2}\boldsymbol{p}$ leads to the form (10.1) or the similar form for $so(3,1)$.

Under limited transition $x \to 0$ in commutational relations (10.4) we obtain algebra $e(3)$. This procedure is referred to as*retraction* (*contraction*) of Lie algebras. In some cases it allows to connect integrable cases, existing for the equations on various representatives of the bundle $\mathscr{L}_x$ (see also [133]).

**Variables** $(\boldsymbol{K}, \boldsymbol{S})$ correspond to "canonical" expansion of an algebra $so(4)$ into a direct sum $so(3) \oplus so(3)$. In algebra it is a well known fact that

$$\{K_i, K_j\} = -\varepsilon_{ijk}K_k, \quad \{K_i, S_j\} = 0, \quad \{S_i, S_j\} = -\varepsilon_{ijk}S_k. \tag{10.5}$$

In terms of new variables Casimir's functions have the form

$$F_1 = (\boldsymbol{K}, \boldsymbol{K}), \quad F_2 = (\boldsymbol{S}, \boldsymbol{S}). \tag{10.6}$$

The equations of motion are

$$\dot{\boldsymbol{K}} = \boldsymbol{K} \times \frac{\partial H}{\partial \boldsymbol{K}}, \qquad \dot{\boldsymbol{S}} = \boldsymbol{S} \times \frac{\partial H}{\partial \boldsymbol{S}}. \tag{10.7}$$

In case of quadratic Hamiltonian $H$ they may be interpreted as equations of two connected three-dimensional Euler tops (on $so(3)$).

**Poincaré – Joukovskiy equations.** These equations represent a Hamiltonian system on $so(4)$ with a quadratic Hamiltonian (Euler – Poincaré equations on $so(4)$, see § 2 ch. 1). In terms of vectors the Hamilton function may be represented as either

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + (\boldsymbol{M}, \mathbf{B}\boldsymbol{p}) + \frac{1}{2}(\boldsymbol{p}, \mathbf{C}\boldsymbol{p}) \tag{10.8}$$

or

$$H = \frac{1}{2}(\boldsymbol{K}, \mathbf{A}'\boldsymbol{K}) + (\boldsymbol{K}, \mathbf{B}'\boldsymbol{S}) + \frac{1}{2}(\boldsymbol{S}, \mathbf{C}'\boldsymbol{S}), \tag{10.9}$$

where $\mathbf{A}, \mathbf{A}', \mathbf{C}, \mathbf{C}'$ are certain symmetrical, and $\mathbf{B}, \mathbf{B}'$ — arbitrary matrices. They are related by an evident dependency

$$\mathbf{A} = \mathbf{A}' + \mathbf{B}' + \mathbf{B}'^{\mathrm{T}} + \mathbf{C}', \quad \mathbf{B} = \mathbf{A}' - \mathbf{B}' + \mathbf{B}'^{\mathrm{T}} - \mathbf{C}',$$
$$\mathbf{C} = \mathbf{A}' - \mathbf{B}' - \mathbf{B}'^{\mathrm{T}} + \mathbf{C}'.$$

To be certain, we shall use these notations henceforward.

Hamiltonian (10.8), (10.9) depends on 21 parameters. There exist three types of elementary transformations modifying (in particular, eliminating) parameters in Hamiltonian without changing equations of motion. The first type include group transformations $SO(3) \times SO(3)$. By means of these, matrices $\mathbf{A}'$ and $\mathbf{C}'$ in representation (10.9) may be simultaneously reduced to the diagonal form. Two more parameters can be eliminated if we add an arbitrary linear combination of Casimir's functions (which are homogeneous quadratic functions) $F_1, F_2$ to the Hamiltonian. Multiplication of the Hamiltonian by an arbitrary constant $H \to \alpha H$ with the substitution of time $t \to \frac{1}{\alpha} t$ allows further reduction of the number of parameters. *Thus, a quadratic family of Hamiltonians* (10.8) (*or* (10.9)) *is defined by twelve parameters.*

Further on, while evaluating the number of parameters in integrable families, we usually eliminate time substitution and get one more parameter.

**Historical comments. 1.** System (10.8) is connected with H. Poincaré and N. E. Joukovskiy because they obtained it considering the problem of motion of a body with cavities filed with vortex fluid. This problem is presented in the next subsection and the detailed derivation of equations of motion on the basis of fundamental principles of hydrodynamics is given in § 2 ch. 5 (in his well-known treatise [111] H. Lamb also gives the derivation and some results on stability by H. Poincaré). Then it turned out that the very same equations describe other mechanical and physical systems. We decided not to change the name of equations depending on such physical analogues.

**2.** In his paper [256] H. Poincaré has given rather modern derivation of equations (10.3), (10.8), using the formalism of general equations of motion on Lie groups. He has also explicitly indicated reduction to elliptic quadratures for the axially symmetric case and considered stability of regular precessions. Of a special interest here is his discussion with W. Kelvin about behavior of frequency and stability of precession of a body with a cavity filled with fluid. Poincaré applies system (10.7) to describe motion of the Earth representing a rigid shell (a mantle) and a liquid core. Further on this model was also studied by V. A. Steklov who presented in the paper [273] the integrable cases he had discovered.

**3.** N. E. Joukovskiy obtained equations (10.7) in his master's thesis [78] from simpler mechanical and hydrodynamical considerations. Then he concentrated his efforts on computation of dynamical characteristics for cavities of various geometry. Considering multiconnected cavities which allow circulational flows, N. E. Joukovskiy has found the case of integrability which a bit later was integrated in elliptic functions [280] by V. Volterra (see § 7 ch. 2, § 8 ch. 5). Circulational flows within cavities lead to the appearance of linear terms in Hamiltonian (10.8).



**Dynamics of a rigid body with a cavity containing fluid.** *Poincaré–Joukovskiy equations* (10.7), (10.9) describe motion around a fixed point of a rigid body having an ellipsoidal cavity filled with homogeneous perfect incompressible fluid being in vortex motion [111, 125, 129]. The detailed derivation of these equations is given in § 2 ch. 5.

Choose a frame of reference rigidly bound to the shell whose axes are parallel to the principal axes of the cavity. In terms of $(\boldsymbol{K}, \boldsymbol{S})$ vector $\boldsymbol{S}$ is proportional to *the vorticity of fluid* $\boldsymbol{\Omega} = \frac{1}{2} \operatorname{rot} \boldsymbol{v}$. In the frame of reference of the shell its components are

$$S_1 = \frac{2}{5} m_0 d_2 d_3 \Omega_1, \quad S_2 = \frac{2}{5} m_0 d_1 d_3 \Omega_2, \quad S_1 = \frac{2}{5} m_0 d_1 d_2 \Omega_3,$$

where $d_1, d_2, d_3$ are cavity semiaxes, and $m_0$ is a fluid mass. The vector evolution is defined by *Helmholtz hydrodynamical equations* [111].

Vector $\boldsymbol{K}$ can be regarded as a angular momentum of the system "body+fluid" and equals

$$\boldsymbol{K} = \mathbf{I}\boldsymbol{\omega} + \mathbf{J}\boldsymbol{\Omega},$$

where $\mathbf{I}$ is a tensor of inertia of the system "body+fluid", and components of a matrix $\mathbf{J} = \operatorname{diag}(J_1, J_2, J_3)$ have the form

$$J_i = \frac{4}{5} m_0 \varepsilon_{ijk} \frac{d_j^2 d_k^2}{d_j^2 + d_k^2},$$

where $\boldsymbol{\omega}$ is an angular velocity of a rigid shell.

Hamiltonian represents kinetic energy expressed in terms of variables $(\boldsymbol{K}, \boldsymbol{S})$ [129]

$$H = \frac{1}{2} (\boldsymbol{K}, \mathbf{A}'\boldsymbol{K}) + (\boldsymbol{K}, \mathbf{B}'\boldsymbol{S}) + \frac{1}{2} (\boldsymbol{S}, \mathbf{C}'\boldsymbol{S}), \qquad (10.10)$$

where $\mathbf{A}' = \mathbf{I}^{-1}$, $\mathbf{B}' = -\mathbf{D}\mathbf{I}^{-1}$, $\mathbf{C}' = \mathbf{D}(\mathbf{I}^{-1} + \mathbf{J}^{-1})\mathbf{D}$ and

$$\mathbf{D} = \operatorname{diag}\left(\frac{2d_2d_3}{d_2^2 + d_3^2}, \frac{2d_1d_3}{d_1^2 + d_3^2}, \frac{2d_1d_2}{d_1^2 + d_2^2}\right).$$

*Function* (10.10) *depends on nine parameters*: six moments of inertia of the shell, relations of principal semiaxes of the cavity, and mass of fluid.

Remark 1. The generalization of Poincaré – Joukovskiy equations for the case of the force field presence was considered in [56]. We obtain the Hamiltonian system on the direct sum $e(3) \oplus so(3)$. [56] contains some necessary conditions (without proof) of existence of additional analytical and polynomial integrals and shows trivial analogue of the Lagrange case which similar systems presumably possess.

**Rigid body dynamics in $\mathbb{R}^4$: a four-dimensional Euler top.** Equations of motion about a fixed point of a free four-dimensional rigid body in the frame of reference bound to the body have similar, but less general form. From this viewpoint the problem was considered in nineteenth century by *W. Frahm* (1875) and *F. Schottky* (1891) [21, 211, 265] (see § 2 ch. 5). Statement of the problem about motion of a four-dimensional rigid body goes back to A. Caley.

Let us choose a system of principal axes of the body. In such a system tensor of moments of inertia $\mathbf{J} = \|J_{\mu\nu}\| = \| \int x_\mu x_\nu dm\|$ has the diagonal form $\mathbf{J} = \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. Hamiltonian may be represented as

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}(\boldsymbol{p}, \mathbf{C}\boldsymbol{p}), \tag{10.11}$$

where

$$\begin{aligned} \mathbf{A} &= \operatorname{diag}\left(\frac{1}{\lambda_2+\lambda_3}, \frac{1}{\lambda_1+\lambda_3}, \frac{1}{\lambda_1+\lambda_2}\right), \\ \mathbf{C} &= \operatorname{diag}\left(\frac{1}{\lambda_0+\lambda_1}, \frac{1}{\lambda_0+\lambda_2}, \frac{1}{\lambda_0+\lambda_3}\right). \end{aligned} \tag{10.12}$$

Matrix $\mathbf{X} \in so^*(4)$ of the angular momentum of a rigid body is connected with its angular velocity $\boldsymbol{\Omega} \in so(4)$ by the formula

$$\mathbf{X} = \frac{1}{2}(\mathbf{J}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{J}),$$

where

$$\begin{gathered} \mathbf{X} = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ -p_1 & 0 & -M_3 & M_2 \\ -p_2 & M_3 & 0 & -M_1 \\ -p_3 & -M_2 & M_1 & 0 \end{pmatrix}, \\ \boldsymbol{\Omega} = \begin{pmatrix} 0 & \dfrac{p_1}{\lambda_0+\lambda_1} & \dfrac{p_2}{\lambda_0+\lambda_2} & \dfrac{p_3}{\lambda_0+\lambda_3} \\ -\dfrac{p_1}{\lambda_0+\lambda_1} & 0 & -\dfrac{M_3}{\lambda_1+\lambda_2} & \dfrac{M_2}{\lambda_1+\lambda_3} \\ -\dfrac{p_2}{\lambda_0+\lambda_2} & \dfrac{M_3}{\lambda_1+\lambda_2} & 0 & -\dfrac{M_1}{\lambda_2+\lambda_3} \\ -\dfrac{p_3}{\lambda_0+\lambda_3} & -\dfrac{M_2}{\lambda_1+\lambda_3} & \dfrac{M_1}{\lambda_2+\lambda_3} & 0 \end{pmatrix}. \end{gathered} \tag{10.13}$$



As it will be shown below, this system is integrable (the Schottky – Manakov case). System (10.11) also describes integrable geodesic flow of a certain metric on a group $SO(4)$ [5].

**A rigid body in curved space.** Forms (10.3) and (10.8) can also be used to express equations of free motion of three-dimensional rigid body in space of the constant of positive curvature: $S^3$ [31]. This is the consequence of analogy of this problem with motion of a four-dimensional rigid body. This analogy can be easily understood for the case of motion of a "planar" rigid body (a plate) in $S^2$. Really, one can consider a plate on a sphere to be equivalent to a rigid body in $\mathbb{R}^3$ with a fixed point in the center of a sphere. The center is connected with the plate by "massless" basic elements.

Remark 2. The development of kinematics, statics and dynamics of rigid body (and a system of material points, as well) in curved spaces goes back to W. K. Clifford, R. S. Ball, R. S. Heath (see [107]) who were developing the theory of screws, motors and biquaternions. In general, these investigations did not result in real things, and presently they are of purely historical interest.

**A rigid body in $S^3$ in fluid.** If we consider a free motion of a rigid body in curved space $S^3$ (a three-dimensional sphere) in a homogeneous incompressible perfect fluid (an analogue of Kirchhoff's equations (9.1) on $e(3)$), then Hamiltonian has a more general form in comparison with (10.11)

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + (\boldsymbol{M}, \mathbf{B}\boldsymbol{p}) + \frac{1}{2}(\boldsymbol{p}, \mathbf{C}\boldsymbol{p}) \tag{10.14}$$

with arbitrary matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ whose coefficients depend on associated masses and moments of inertia of the body. Quadratic form (10.14) represents kinetic energy and is positively determined. Similarly, one can write out equations of motion of a rigid body in fluid (or vacuum) in Lobachevskiy's space $L^3$. This problem was investigated by *G. Birkhoff* in his book [12], and by N. E. Joukovskiy — for the two-dimensional case [77]. Here one obtains a Hamiltonian system on algebra $so(3,1)$ with Hamiltonian (10.14). Our book [31] contains the derivation of equations of motion of a rigid body with a gyrostat in curved space. At that, Hamiltonian (10.14) gets terms linear with respect to $(\boldsymbol{M}, \boldsymbol{p})$. There one can also find the derivation of corresponding equations for motion in Lobachevskiy's space.

**A system of interacting spins.** Classical dynamics of two interacting *spins* (*spherical rotators*), corresponding to vector representation of group of rotations, is also described by a Hamiltonian system on $so(4)$ [247, 210, 269, 270]. Passing from spin operators $\widehat{\boldsymbol{S}}_1$, $\widehat{\boldsymbol{S}}_2$ to classical vectors $\boldsymbol{K} = \boldsymbol{S}_1$, $\boldsymbol{S} = \boldsymbol{S}_2$, we obtain dynamical system (10.7). Hamiltonians of type (10.9) correspond to various classical spin systems. In this case cross terms describe so called *exchange spin interaction*. For spins in the external magnetic field, linear terms should be added to the Hamiltonian

The most general of two-spin systems being considered is described by the Hamiltonian [247]

$$H = -(\mathbf{B}'\boldsymbol{K},\, \boldsymbol{S}) + \frac{1}{2}(\mathbf{A}'\boldsymbol{K},\, \boldsymbol{K}) + \frac{1}{2}(\mathbf{A}'\boldsymbol{S},\, \boldsymbol{S}), \tag{10.15}$$

where $\mathbf{A}'$, $\mathbf{B}'$ are certain diagonal matrices.

The case $\mathbf{A}' = 0$ corresponds to so called *two-spin $XYZ$ model*, the case $a'_{33} = b'_{33} = 0$ corresponds to *a generalized two-spin $XY$ model* (Heisenberg's model, see. § 6 ch. 5).

System (10.15) also describes dynamics of two connected classical tops (connection of two bodies, see § 8 ch. 2), whose interaction energy depends only on components of angular momentums and does not depend on positional variables.

## 2. Integrable Cases

As far as Poisson's structure (10.1), (10.5) has two main functions, for integration of corresponding equations of motion one more first integral is necessary. In the general case it does not exist, and phase space contains regions with chaotic behavior.

Integrable cases of system (10.8), (10.9) known up to date are presented in Table 3.2.

The general integrable case (7) discovered by the authors with V. V. Sokolov is considered in more details in § 12 ch. 5.

As far as algebra $so(4)$ allows both standard and canonical representations, in table 3.2 contains conditions for parameters only for the representation which makes them simpler.

Remark 3. Not every integrable case shown in Table 3.2, possesses physical content. It happens because coefficients of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ for Poincaré – Joukovskiy equations are not arbitrary and have rather limited region of variation.

Remark 4. The case of integrability of equations on algebra $e(3)$, like Lagrange's and Hess' cases for Euler – Poisson equations or Kirhchhoff's and Chaplygin's (II) cases for Kirchhoff's equations (when the additional integral depends on variables $\boldsymbol{M}$ only), are naturally transferred to systems on bundle (10.4), containing algebra $so(4)$ at $x = 1$.



This concerns the fact that equations for $\dot{\boldsymbol{M}}$ for all brackets of the bundle coincide (see below).

Remark 5. Relative equilibria of system (10.3), which have $\dot{\boldsymbol{K}} = \dot{\boldsymbol{S}} = \dot{\boldsymbol{M}} = \dot{\boldsymbol{\gamma}} = 0$, may be interpreted differently, depending on physical statements of problems. If we consider motion of a body with cavities filled with vortex fluid, these equilibria determine particular solutions. For these solutions the body motion is a uniform rotation around a certain axis, and vorticity vector is "frozen" within the body. Of special interest is the study of stationary configurations for the model of connected tops. This model defines dynamics of a chain of spins. Such configurations, specifying a certain coherent state, are of great importance in quantum physics. We consider them in ch. 5 for both finite-dimensional and infinite-dimensional cases.

## 3. The Case of Axial Symmetry (of H. Poincaré)

It is the simplest integrable case, for which the pair of eigenvalues of diagonal matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (or $\mathbf{A}', \mathbf{B}', \mathbf{C}'$) coincide, i. e., $a_{11} = a_{22}$, $b_{11} = b_{22}$, $c_{11} = c_{22}$. The Hamiltonian (after elimination of (10.2)) may be represented as

$$H = \frac{1}{2}\bigl(M_1^2 + M_2^2 + aM_3^2\bigr) + b\bigl(M_1p_1 + M_2p_2\bigr) + \frac{1}{2}c(p_1^2 + p_2^2).$$

An additional integral can be written as $M_3 = \text{const}$ or $K_3 + S_3 = \text{const}$. This is integral of Lagrange's type. Reducing to quadratures, which results in elliptic functions, was done by H. Poincaré [256] (see also § 2 ch. 4).

## 4. The Schottky – Manakov Case

In 1891 in his paper [265] *F. Schottky* showed the first integrable case of system (10.8) and noticed its connection with the Clebsch case in Kirchhoff's equations. Here $\mathbf{B} = 0$, and a Hamiltonian is given by formula (10.11), coefficients of matrices $\mathbf{A}, \mathbf{C}$ satisfy relations (10.12) with arbitrary parameters $\lambda_\mu$, $\mu = 0, \ldots, 3$. This case is also connected with the name of *S. V. Manakov* who has shown integrability of its $n$-dimensional analogue (1976, [121]).

In representation (10.9) for the Schottky – Manakov case the equality $\mathbf{A}' = \mathbf{C}'$ is valid. Under these limitations this case of integrability is unique in the class of systems with a quadratic integral. Really, the following statement (see, for example, [50])is valid.

*If $\mathbf{A}' = \mathbf{C}'$, if eigenvalues of the matrix $\mathbf{A}'$ differ, and if the matrix $\mathbf{B}'$ is nondegenerate, system* (10.9) *allows a quadratic integral then and only then* *when the following conditions are met*

$$\begin{gathered}\mathbf{A}' = \operatorname{diag}(a'_1, a'_2, a'_3), \quad \mathbf{B}' = \operatorname{diag}(b'_1, b'_2, b'_3),\\ b_1'^2(a'_2 - a'_3) + b_2'^2(a'_3 - a'_1) + b_3'^2(a'_1 - a'_2) +\\ +(a'_1 - a'_2)(a'_2 - a'_3)(a'_3 - a'_1)k^2 = 0, \quad k^2 = 1.\end{gathered} \tag{10.16}$$

The very same conditions of integrability are indicated in the physical papers [247, 269], dedicated to dynamics of two-spin model (10.15). Besides, the authors [247] independently discovered the Schottky case known for more than a century before them.

Remark 6. The algebraic integrability of system (10.3), (10.8) was investigated in the paper [226]; by means of the Kowalevskaya method this system was investigated by *M. Adler* and *P. van Moerbeke* [187, 186, 185], where the authors also discuss integrability. The paper [185] contains a new integrable case. The conditions of solution meromorphicity on a complex plane of time are obtained in the papers [37, 38]. They claim that in equality (10.16) $k$ should be an odd integer. The necessary conditions of algebraic integrability are also obtained in the paper [206], $k$ being rational.

We know the following symmetrical representation of integrals in involution (see, for example, [254]). By means of skew-symmetrical matrix $\mathbf{X} = ||X_{\mu\nu}||$ (10.13) they can be written as

$$G_\mu = \sum_{\nu=0}^{3} \frac{X_{\mu\nu}^2}{\lambda_\mu^2 - \lambda_\nu^2}, \quad \mu = 0, \ldots, 3. \tag{10.17}$$

The following equations are valid

$$\sum_{\mu=0}^{3} G_\mu = 0, \quad \sum_{\mu=0}^{3} \lambda_\mu G_\mu = 2H, \quad \sum_{\mu=0}^{3} \lambda_\mu^2 G_\mu = F_1, \tag{10.18}$$

where $F_1 = \boldsymbol{M}^2 + \boldsymbol{p}^2$ is Casimir's function, $H$ is the Schottky Hamiltonian (10.11).

*Thus, a linear combination of four integrals* (10.17) *on every level set of Casimir's functions specifies a five-parameter family of integrable quadratic Hamiltonians* (*three parameters are differences $\lambda_\mu^2 - \lambda_\nu^2$, two parameters are coefficients of the linear combination after elimination of Casimir's functions*). *The Hamiltonian for the Schottky–Manakov case belongs to family* (10.17).



F. Schottky [265] has written quadratic family (10.17) in a somewhat different form with an arbitrary parameter. It is

$$Q(s) = \sum_{\mathrm{cycle}\, i,j,k} \left( \sqrt{(s-a_i)(s-a_4)}M_i + \sqrt{(s-a_j)(s-a_k)}p_i \right)^2.$$

This family was rediscovered by *L. Haine* in the paper [226].

For the first time Lax representation with a rational spectral parameter (for the general $n$-dimensional case) was found in the paper by S. V. Manakov [121]. *Equations of motion* (10.3) *with Hamiltonian* (10.11) *allow Lax representation on matrices, linearly depending on an arbitrary parameter $\lambda$:*

$$\frac{d}{dt}(\mathbf{X} + \lambda\mathbf{J}^2) = [\mathbf{X} + \lambda\mathbf{J}^2,\ \boldsymbol{\Omega} + \lambda\mathbf{J}],$$

*where* $\mathbf{X}$, $\boldsymbol{\Omega}$ *are defined by formula* (10.13), *and* $\mathbf{J}$ *is a tensor of inertia* (*a symmetrical, positively determined matrix*).

Motion integrals in involution are obtained by the expansion of functions $P_k(\lambda) = \operatorname{tr}(\mathbf{X} + \lambda\mathbf{J}^2)^k$ in terms of $\lambda$. This family is complete, i. e., there is enough integrals for it to be integrable. Other Lax representations with a spectral parameter, and also their connection with bi-Hamiltonianity are considered in [24, 31, 168].

By means of retraction an algebra $so(4)$ transfers to an algebra $e(3)$; the Schottky–Manakov case transforms into Clebsch's case (S. P. Novikov, [133]). Really, let us make following substitutions of variables and parameters

$$\boldsymbol{p} \to \frac{\boldsymbol{\gamma}}{\sqrt{x}}, \quad \lambda_0 \to -\frac{1}{\sqrt{cx}}, \tag{10.19}$$

where $c$ is a certain constant. Consider a limiting transition $x \to 0$ in commutative relations being obtained and in corresponding integrals (10.4). Under given parametrization integrals (10.17) take the form (9.12).

Eliminating infinite (at $x \to 0$) term, proportional to the Casimir function according to the rule $H \to H + \frac{\sqrt{c}}{2\sqrt{x}}(\boldsymbol{\gamma}^2 + x\boldsymbol{M}^2)$, from the Hamiltonian (shown in Table 3.2), we obtain the Hamiltonian function for the Clebsch case (see. § 9 ch. 3) on $e(3)$ in the form

$$H = \frac{1}{2}\left(\frac{M_1^2}{\lambda_2 + \lambda_3} + \frac{M_2^2}{\lambda_3 + \lambda_1} + \frac{M_3^2}{\lambda_1 + \lambda_2}\right) + \frac{c}{2}\left(\lambda_1\gamma_1 + \lambda_2\gamma_2 + \lambda_3\gamma_3\right).$$

Table 3.2. Cases of Integrability of Poincaré – Joukovskiy Equations.

| | Author | Conditions for parameters and the first integral |
|---|---|---|
| 1 | Poincaré<br>(1910) [256] | $\mathbf{A}' = \mathrm{diag}(a'_1, a'_2, a'_3), \quad \mathbf{B}' = \mathrm{diag}(b'_1, b'_2, b'_3), \quad \mathbf{C}' = \mathrm{diag}(c'_1, c'_2, c'_3),$<br>$a'_1 = a'_2, \quad b'_1 = b'_2, \quad c'_1 = c'_2$<br>$F = M_3$ |
| 2 | Schottky<br>(1891) [265]<br>Manakov<br>(1976) [121] | $\mathbf{A} = \mathrm{diag}\Big(\frac{1}{\lambda_2+\lambda_3}, \frac{1}{\lambda_3+\lambda_1}, \frac{1}{\lambda_1+\lambda_2}\Big),$<br>$\mathbf{B} = 0, \quad \mathbf{C} = \mathrm{diag}\Big(\frac{1}{\lambda_0+\lambda_1}, \frac{1}{\lambda_0+\lambda_2}, \frac{1}{\lambda_0+\lambda_3}\Big)$<br>$F = M_1^2 + M_2^2 + M_3^2 + \Big(\frac{\lambda_1^2}{\lambda_1^2-\lambda_0^2}p_1^2 + \frac{\lambda_2^2}{\lambda_2^2-\lambda_0^2}p_2^2 + \frac{\lambda_3^2}{\lambda_3^2-\lambda_0^2}p_3^2\Big)$ |
| 3 | Steklov<br>(1909) [273] | $\mathbf{A}' = \mathrm{diag}(\lambda_2^2\lambda_3^2, \lambda_1^2\lambda_3^2, \lambda_1^2\lambda_2^2)$<br>$\mathbf{B}' = \mathrm{diag}(\lambda_2\lambda_3(\lambda_2^2+\lambda_3^2), \lambda_1\lambda_3(\lambda_1^2+\lambda_3^2), \lambda_1\lambda_2(\lambda_1^2+\lambda_2^2))$<br>$\mathbf{C}' = \mathrm{diag}((\lambda_2^2+\lambda_3^2)^2, (\lambda_1^2+\lambda_3^2)^2, (\lambda_1^2+\lambda_2^2)^2))$<br>$F = (\lambda_2^2+\lambda_3^2)S_1\Big(S_1 + \frac{2\lambda_2\lambda_3}{\lambda_2^2-\lambda_3^2}K_1\Big) +$<br>$+ (\lambda_1^2+\lambda_3^2)S_2\Big(S_2 + \frac{2\lambda_1\lambda_3}{\lambda_1^2-\lambda_3^2}K_2\Big) + (\lambda_1^2+\lambda_2^2)S_3\Big(S_3 + \frac{2\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}K_3\Big)$ |
| 4 | Adler, van Moerbeke,<br>1982, [187] | $\mathbf{A}' = \mathrm{diag}(a'_1, a'_2, a'_3),\ \mathbf{B}' = \mathrm{diag}(b'_1, b'_2, b'_3),\ \mathbf{C}' = \mathrm{diag}(c'_1, c'_2, c'_3),\ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$<br>$a'_i = -\frac{2}{3}\varepsilon_{ijk}\lambda_j^2\lambda_k^2,\ b'_i = \varepsilon_{ijk}(\lambda_i^4 + \lambda_j^2\lambda_k^2 - (\lambda_j+\lambda_k)^2(\lambda_j^2+\lambda_k^2))$<br>$c'_i = \frac{2}{3}\varepsilon_{ijk}(\lambda_i^4 - \lambda_j\lambda_k(\lambda_j^2+\lambda_k^2+\frac{5}{4}\lambda_j\lambda_k)), \quad \lambda_1+\lambda_2+\lambda_3 = 0$<br>$F = \frac{1}{2}\boldsymbol{K}^2\sum_k\Big((\lambda_i\lambda_j - \frac{1}{3}\lambda_k^2)S_k^2 + (\lambda_i\lambda_j - \lambda_k^2)K_kS_k\Big) +$<br>$+ \frac{1}{18}\boldsymbol{S}^2\sum_k\Big(\frac{5}{3}(\lambda_i\lambda_j - \lambda_k^2)S_k^2 + (7\lambda_i\lambda_j - 4\lambda_k^2)K_kS_k\Big) +$<br>$+ \frac{1}{2}(\boldsymbol{\lambda}, \boldsymbol{\lambda})\Big(\boldsymbol{K}^2 + \frac{1}{3}\boldsymbol{S}^2\Big)(\boldsymbol{K}, \boldsymbol{S}) - \frac{1}{9}\sum_{i<j}(\lambda_i-\lambda_j)^2(K_iS_iS_j^2 + K_jS_jS_i^2)$ |
| 5 | Bogoyavlenskiy<br>[21] | $I$: $\mathbf{A} = \mathrm{diag}(\alpha_2, \alpha_1, \alpha_1+\alpha_2),\ \alpha_1 = 1-a_1,\ \alpha_2 = 1-a_2,$<br>$\mathbf{B} = 0, \quad \mathbf{C} = \mathrm{diag}(a_2-a_1, a_1-a_2, 0),$<br>$F = \big(\alpha_1M_1^2 - \alpha_2M_2^2 - (a_1-a_2)p_3^2\big)^2 + 4\alpha_1\alpha_2M_1^2M_2^2, \quad (\boldsymbol{M}, \boldsymbol{p}) = 0,$<br>$II$: $\mathbf{B} = 0, \quad \mathbf{A} = \mathrm{diag}(2a_1, 2a_2, 2a_3), \quad \mathbf{C} = \mathrm{diag}(a_2+a_3, a_1+a_3, a_1+a_2)$<br>$F = \big((a_2-a_3)p_1^2 + (a_3-a_1)p_2^2 - (a_1-a_2)p_3^2\big)^2 + 4(a_3-a_2)(a_3-a_1)p_1^2p_2^2,\ (\boldsymbol{M}, \boldsymbol{p}) = 0$ |
| 6 | Borisov, Mamaev,<br>2000 | $\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3),\ \mathbf{B} = \lVert b_{ij}\rVert,\ \mathbf{C} = \lVert c_{ij}\rVert$<br>$b_{13}\sqrt{a_2-a_1} \mp (b_{22}-b_{11})\sqrt{a_3-a_2} = 0, \quad b_{12} = 0,$<br>$b_{13}\sqrt{a_3-a_2} \pm (b_{33}-b_{22})\sqrt{a_2-a_1} = 0, \quad b_{23} = 0,$<br>$c_{13}\sqrt{a_2-a_1} \mp (c_{22}-c_{11})\sqrt{a_3-a_2} = 0, \quad c_{12} = 0,$<br>$c_{13}\sqrt{a_3-a_2} \pm (c_{33}-c_{22})\sqrt{a_2-a_1} = 0, \quad c_{23} = 0,$<br>$F = M_1\sqrt{a_2-a_1} \mp M_3\sqrt{a_3-a_2} = 0$ |
| 7 | Borisov, Mamaev,<br>Sokolov, 2001 | $H = \frac{1}{2}\big(M_1^2 + M_2^2 + 2M_3^2\big) + \alpha M_3\gamma_1 - \alpha^2\gamma_3^2, \quad \alpha = \mathrm{const}$<br>$F = M_3\big[M_3\big(\boldsymbol{M}^2 + \alpha^2M_2^2 + 2\alpha(M_3\gamma_1 - M_1\gamma_3) + \alpha^2(\gamma_1^2+\gamma_3^2)\big) + 2\alpha(M_1 - \alpha\gamma_3)(\boldsymbol{M}, \boldsymbol{\gamma})\big]$ |

In this case the Clebsch integral is represented as

$$K = M_1^2 + M_2^2 + M_3^2 + c\left(\lambda_1^2\gamma_1^2 + \lambda_2^2\gamma_2^2 + \lambda_3^2\gamma_3^2\right).$$

Remark 7. Using substitution (10.19) one obtains the family of integrals, depending on the parameter $G_\mu(x)$, $\mu = 0, \ldots, 3$ and remaining in involution at an arbitrary $x$, i.,e., on the whole bundle $\mathscr{L}_x$.

It turns out that the Schottky – Manakov case and the Clebsch case are connected *not only via retraction, but are linearly isomorphic* (A. I. Bobenko, [14]). The corresponding transformation is written as

$$\begin{gathered} M_1 = \sqrt{(\lambda_2^2-\lambda_0^2)(\lambda_3^2-\lambda_0^2)}L_1, \quad M_2 = \sqrt{(\lambda_1^2-\lambda_0^2)(\lambda_3^2-\lambda_0^2)}L_2, \\ M_3 = \sqrt{(\lambda_1^2-\lambda_0^2)(\lambda_2^2-\lambda_0^2)}L_3, \\ p_1 = \sqrt{\lambda_1^2-\lambda_0^2}\gamma_1, \quad p_2 = \sqrt{\lambda_2^2-\lambda_0^2}\gamma_2, \quad p_3 = \sqrt{\lambda_3^2-\lambda_0^2}\gamma_3. \end{gathered} \tag{10.20}$$

The equations of motion for variables $(\boldsymbol{L}, \boldsymbol{\gamma})$ correspond to the Clebsch case on an algebra $e(3)$ with the following Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{L}, \mathbf{A}\boldsymbol{L}) + \frac{1}{2}(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}),$$

$$\mathbf{A} = \operatorname{diag}\left(-\frac{(\lambda_2-\lambda_0)(\lambda_3-\lambda_0)}{\lambda_2+\lambda_3}, -\frac{(\lambda_1-\lambda_0)(\lambda_3-\lambda_0)}{\lambda_1+\lambda_3}, -\frac{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}{\lambda_1+\lambda_2}\right),$$

$$\mathbf{C} = \left(\frac{1}{\lambda_1+\lambda_0}, \frac{1}{\lambda_2+\lambda_0}, \frac{1}{\lambda_3+\lambda_0}\right).$$

As it was shown by *A. V. Bolsinov* [23], a linear isomorphism also exists for multidimensional analogues of the problems being considered. Although explicit transformation (10.20) was shown in [14], at the level of similarity of motion integrals of both systems it was implicitly used already by F. Schottky (1891) [265]. Topological analysis and bifurcational patterns are present in the paper by A. A. Oshemkov [140] (see also the book [25]). Due to linear isomorphism with the Clebsch case, this analysis results are equivalent to the ones obtained in the paper [143].

### 5. Steklov's Case

Another integrable case, for which Hamiltonian (10.8) contains cross terms (i. e., a matrix $\mathbf{B} \not\equiv 0$), is discovered in the paper by V. A. Steklov [273]. Necessary and sufficient conditions of quadratic integrability were announced by A. P. Veselov [50].



*Let a Hamiltonian $H$ be given in representation* (10.10)*; where eigenvalues of both matrices* $\mathbf{A}'$ *and* $\mathbf{C}'$ *are different, and the matrix* $\mathbf{B}'$ *is nondegenerate* ($\det \mathbf{B}' \neq 0$). *Then for the existence of an additional motion integral, independent of $H$, $F_1$, $F_2$, it is necessary and sufficient to satisfy following conditions*:

$$\begin{gathered} \mathbf{A}' = \operatorname{diag}(a_1', a_2', a_3'), \quad \mathbf{B}' = \operatorname{diag}(b_1', b_2', b_3'), \quad \mathbf{C}' = \operatorname{diag}(c_1', c_2', c_3'), \\ b_1'^2(a_2'-a_3') + b_2'^2(a_3'-a_1') + b_3'^2(a_1'-a_2') + \\ +(a_1'-a_2')(a_2'-a_3')(a_3'-a_1') = 0, \\ b_1'^2(c_2'-c_3') + b_2'^2(c_3'-c_1') + b_3'^2(c_1'-c_2') + \\ +(c_1'-c_2')(c_2'-c_3')(c_3'-c_1') = 0, \end{gathered} \tag{10.21}$$

To describe the family of integrals in involution, similar to (10.17), let us introduce skew-symmetrical matrices $\mathbf{K}, \mathbf{S}, \mathbf{M}, \mathbf{P}$, corresponding to vectors $\boldsymbol{K}, \boldsymbol{S}, \boldsymbol{M}, \boldsymbol{P}$ whose components can be determined by the formulae

$$\begin{gathered} K_{ij} = -\varepsilon_{ijk}K_k, \quad S_{ij} = -\varepsilon_{ijk}S_k, \\ M_{ij} = -\varepsilon_{ijk}M_k, \quad P_{ij} = -\varepsilon_{ijk}p_k. \end{gathered} \tag{10.22}$$

Integrals may be represented in terms of components of these matrices in the form [24, 31]

$$G_i = \sum_{j\neq i}^{3} \frac{(\lambda_j K_{ij} + \lambda_i S_{ij})^2}{\lambda_i^2 - \lambda_j^2}, \quad i = 1, 2, 3, \tag{10.23}$$

For instance, $G_1$ is written as (others are obtained by cyclic permutation of indices)

$$G_1 = \frac{(\lambda_2 K_3 + \lambda_1 S_3)^2}{\lambda_1^2 - \lambda_2^2} + \frac{(\lambda_3 K_2 + \lambda_1 S_2)^2}{\lambda_1^2 - \lambda_3^2}.$$

For functions $G_i$ the relation

$$\sum_{i=1}^{3} G_i = -\boldsymbol{K}^2 + \boldsymbol{S}^2. \tag{10.24}$$

is valid. Consequently, set (10.23) determines the two-dimensional family of integrable cases. This family is specified by two parameters (in case $\lambda_3 \neq 0$, one can use, for example, relations $\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}$).

The Steklov case is usually connected with the Hamiltonian of the form (similarly to Kirchhoff's equations, see § 9 ch. 3)

$$H = \frac{1}{2}\sum_{i=1}^{3} \lambda_i^4 G_i = \frac{1}{2} \sum_{\text{cycl.var.}\, i,j,k} \left(\lambda_i\lambda_j K_k + (\lambda_i^2 + \lambda_j^2) S_k\right)^2 .$$

It is obtained from family (10.23) by summation and by the following substitution of parameters.

For *Steklov's family*, similarly to the Schottky – Manakov case, *one can make retraction to the Steklov – Lyapunov integrable family for Kirchhoff's equations* (9.1). To show this we'll make the following substitution of variables, parameters and integrals (10.23)

$$\boldsymbol{p} \to \frac{\boldsymbol{\gamma}}{\sqrt{x}}, \qquad (10.25)$$
$$\lambda_i \to 1 + \sqrt{x}\lambda_i, \quad G_i \to \sqrt{x} G_i, \quad i = 1, 2, 3.$$

*The family of obtained integrals, depending on a parameter $G_i(x)$, remains in involution over the whole bundle* (10.4)*; at $x \to 0$ they take form* (9.13) *indicated in the previous section*.

Remark 8. The paper [14] shows a somewhat different retraction of integrable cases. The author uses a symmetrical form of parametrization of Steklov's cases on $so(4)$ by means of elliptic functions.

A linear substitution of variables

$$\mathbf{M} = \mathbf{JKJ} + \frac{1}{2}\left(\mathbf{J}^2\mathbf{S} + \mathbf{S}\mathbf{J}^2\right), \quad \mathbf{P} = \mathbf{S},$$
$$\mathbf{J} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$$

transfers family (10.23) into Steklov – Lyapunov family (9.13) of Kirchhoff's equations [24, 31], i. e., similarly to Clebsch's and Schottky – Manakov's cases, they are linearly isomorphic.

The other, less symmetrical representation of the family of integrals in involution in Steklov's case was found by O. I. Bogoyavlenskiy [21] (see also [19, 20])



$$\begin{aligned}
I_1 &= \left(\sqrt{\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}}K_1 + \sqrt{\frac{\beta_1-\beta_3}{\beta_1-\beta_2}}S_1\right)^2 + \left(\sqrt{\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_2}}K_2 + \sqrt{\frac{\beta_2-\beta_3}{\beta_1-\beta_2}}S_2\right)^2,\\
I_2 &= \left(\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}K_1 + \sqrt{\frac{\beta_1-\beta_2}{\beta_1-\beta_3}}S_1\right)^2 - \left(\sqrt{\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}}K_3 + \sqrt{\frac{\beta_2-\beta_3}{\beta_1-\beta_3}}S_3\right)^2,\\
I_3 &= \left(\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_2-\alpha_3}}K_2 + \sqrt{\frac{\beta_1-\beta_2}{\beta_2-\beta_3}}S_2\right)^2 + \left(\sqrt{\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_3}}K_3 + \sqrt{\frac{\beta_1-\beta_3}{\beta_2-\beta_3}}S_3\right)^2,
\end{aligned} \qquad (10.26)$$

where $\alpha_i, \beta_i$ are arbitrary parameters.

### 6. The Integrable Case with Fourth Degree Integral (M. Adler, P. van Moerbeke)

The general integrable case, found by M. Adler and P. van Moerbeke [185], is still the most complicated and least investigated one in rigid body dynamics. It does not have any analogues for Kirchhoff's equations. Moreover, under retraction Hamiltonian degenerates into Casimir's function of an algebra $e(3)$.

The original paper [185] contains an additional fourth degree integral in a very cumbersome and asymmetrical form. A bit later A. Reyman and M. Semenov-Tian-Shansky showed that this case had Lax spectral representation constructed on a special algebra $\mathfrak{g}_2$ [260]. In the paper [24] the analogical $\mathbf{L}-\mathbf{A}$-pair was obtained in a more natural way, but the corresponding construction is also connected with the algebra $\mathfrak{g}_2$ and the presence of the compatible Poisson structure. However, the integral being obtained from $\mathbf{L}-\mathbf{A}$-pair requires additional and nontrivial simplifications which we have done. After these actions we obtained the form shown in Table 3.2.

It should be noted that in the paper [127] A. S. Mischenko and A. T. Fomenko had all the possibilities to find this case by means of argument shift method they were developing, but they seem to be prevented from doing this by their extra formalized and general reasoning. Curiously enough, in his latter books A. T. Fomenko (see, for example, [166]), when citing this case, refers to the paper [260] still failing to see the connection with his construction.

The separating variables for the Adler-van Moerbeke case are still unknown. Its topological analysis has not been carried out yet. In many respects its existence is connected with a special symmetry $so(4)$, allowing real representation in the form of a direct sum $so(3) \oplus so(3)$. This case is absent on $so(3,1)$ and does not allow multidimensional generalizations.

### 7. Particular Cases at $(M, p) = 0$

O. I. Bogoyavlenskiy has indicated two particular cases of integrability of the Poincaré – Joukovskiy equations with the fourth degree integral [16].

**The first case of Bogoyavlenskiy** transfers into the Chaplygin (I) case of Kirchhoff's equations under retraction. To make this connection more evident we write a Hamiltonian and an integral on the bundle $\mathscr{L}_x$ (10.4)

$$\begin{gathered} H = \frac{1}{2}\left(\alpha_2 M_1^2 + \alpha_1 M_2^2 + (\alpha_1+\alpha_2)M_3^2\right) - \frac{1}{2}(a_1 - a_2)\left(p_1^2 - p_2^2\right), \\ F = \left(\alpha_1 M_1^2 - \alpha_2 M_2^2 - (a_1 - a_2)p_3^2\right)^2 + 4\alpha_1\alpha_2 M_1 M_2, \\ \alpha_1 = 1 - x a_1, \quad \alpha_2 = 1 - x a_2, \quad a_1, a_2 = \text{const}. \end{gathered} \tag{10.27}$$

At $x = 1$, we obtain the integrable case shown in Table 3.2. It should be noted that at $x \neq 0$ Hamiltonian (10.27) describes motion of a dynamically asymmetrical body whose moments of inertia are governed by the relation $(I_1^{-1} + I_2^{-1} = = I_3^{-1})$.

§ 7 ch. 5 contains a more general family of particular integrable cases on the bundle $\mathscr{L}_x$. Among these particular cases are the Kowalevskaya case of the Euler – Poisson equations, the Chaplygin (I) case of Kirchhoff's equations, the Bogoyavlenskiy (I) case of the Poincaré – Joukovskiy equations, and various gyrostatic generalizations, as well.

**The second case of Bogoyavlenskiy** is given by the Hamiltonian of the form

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{4}\left((a_2 + a_3)p_1^2 + (a_1 + a_3)p_2^2 + (a_1 + a_2)p_3^2\right). \tag{10.28}$$

In this case the system allows three different integrals of the kind

$$\begin{gathered} F_i = \left((a_i - a_j)p_k^2 - (a_i - a_k)p_j^2 - (a_j - a_k)p_i^2\right) + 4(a_i - a_j)(a_i - a_k)p_i^2 p_k^2, \\ i, j, k = 1, 2, 3, \qquad i \neq j \neq k \neq i. \end{gathered}$$

Among these only one integral is independent. Actually, it is easily shown that they are linearly related by the expression

$$\alpha F_1 + \beta F_2 + \gamma F_3 = 0, \quad \text{under the condition that} \quad \alpha + \beta + \gamma = 0.$$

To make retraction on $e(3)$ we suppose $p_i \to \dfrac{p_i}{\sqrt{x}}$, $H \to xH$, and at $x \to 0$ we obtain an integrable case of Kirchhoff's equations with linear integrals, the kinetic energy being equal to zero.



### 8. The Hess Case Generalization

As shown above (see § 2), for the Poincaré – Joukovskiy equations the set of three equations for vector $\boldsymbol{M}$ coincides with the analogical set in Kirchhoff's equations. Consequently, there also exists an invariant relation of the Hess type

$$\sqrt{a_3 - a_2}M_3 \pm \sqrt{a_2 - a_1}M_1 = 0, \quad a_1 < a_2 < a_3. \tag{10.29}$$

Moreover, here, like in Kirchhoff's equations, $\boldsymbol{p}$-containing terms in Hamiltonian (10.8) should be invariant to rotations around the perpendicular to a circular section of a gyration ellipsoid. Invariant relation (10.29) defines a marked torus in phase space where the solution may be obtained in quadratures. The integration procedure can be carried out by means of results of §§ 1, 3 ch. 4. This generalization of the Hess case seems not to be indicated earlier, in spite of its natural origin.

### 9. Integrable Generalizations with Linear Terms in Hamiltonian

Similarly to Kirchhoff's equations, equations (10.3), (10.7) may have mechanical meaning if the Hamiltonian $H$ contains not only quadratic terms, but linear ones, as well. Depending on physical statements described in the first section, they may be interpreted in various ways. So, for dynamics of a rigid body with fluid these terms are conditioned by the presence of multiconnected cavities in the body, for the Euler four-dimensional top by addition of a balanced four-dimensional gyrostat (for the corresponding derivation see § 2 ch. 5), for a rigid body on $S^3$ in fluid by multiple connections of the rigid body moving in fluid, for a spin chain by a permanent external magnetic field where this spin chain is placed.

Analogically to Kirchhoff's equations, here we can also indicate integrable cases, generalizing those shown in Table 3.2. Cases 1 and 6 connected with rotational symmetry are generalized very easily: we just add a gyrostat along the axis of symmetry (for more details see § 1 and § 3 ch. 4). For the cases of Schottky, Adler-van Moerbeke and Bogoyavlenskiy (II) similar generalizations have not been found.

The generalization of Steklov's case leads to the integrable case, similar to the Rubanovskiy case for Kirchhoff's equations. For the first time it was shown by O. I. Bogoyavlenskiy [21] in a vague form. Here we present the most symmetrical expression.

**The analogue of the Rubanovskiy case on $so(4)$.** The symmetrical form of the corresponding family of integrals in involution may be represented as

$$J_s = G_s + \frac{1}{(\lambda_s^2-\lambda_m^2)(\lambda_s^2-\lambda_n^2)} \sum_{\text{cycle var. } ijk} r_k(\lambda_i\lambda_j K_k + \lambda_s^2 S_k), \qquad (10.30)$$

$$s, m, n = 1, 2, 3,$$

where $r_i$, $i = 1, 2, 3$ are three additional arbitrary parameters. For example, an integral $J_1$ is explicitly expressed as follows (other integrals are obtained by cyclic permutation)

$$J_1 = G_1 + \frac{r_1(\lambda_2\lambda_3K_1 + \lambda_1^2S_1) + r_2(\lambda_3\lambda_1K_2 + \lambda_1^2S_2) + r_3(\lambda_1\lambda_2K_3 + \lambda_1^2S_3)}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}.$$

For these integrals relation (10.24) is also valid if we take into account the substitution $G_i \to J_i$, $i = 1, 2, 3$. The Hamiltonian and the additional integral can be represented as

$$H = \sum_i \lambda_i J_i = \sum_{\text{cycle}} \Big( (\lambda_i + \lambda_j)S_k^2 - \frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}(S_k+K_k)^2 - -\lambda_i\lambda_j r_k(S_k+K_k) + (\lambda_i+\lambda_j)^2 r_k S_k \Big),$$

$$F = \sum_i \lambda_i^2 J_i = \sum_{\text{cycle}} \big( (\lambda_i^2+\lambda_j^2)S_k^2 + 2\lambda_i\lambda_j S_k K_k \big) + (\boldsymbol{r}, \boldsymbol{S}).$$

The Lax representation for this case is given in [208].

**The generalization of the first case of Bogoyavlenskiy.** Integrable system (10.27) also allows generalization when a constant gyrostatic moment $\lambda$ is added along the axis $OM_3$, though in this case it is not the axis of symmetry. The Hamiltonian and the integral are written as (to make things clear we represent them on the bundle $\mathscr{L}_x$)

$$H = \frac{1}{2}\big(\alpha_2 M_1^2 + \alpha_1 M_2^2 + (\alpha_1+\alpha_2)M_3^2\big) - \lambda M_3 - \frac{1}{2}(a_1 - a_2)(p_1^2 - p_2^2),$$

$$F = \big(\alpha_1 M_1^2 - \alpha_2 M_2^2 - (a_1-a_2)\gamma_3^2\big)^2 + 4\alpha_1\alpha_2 M_1^2 M_2^2 + + 4\lambda\big(M_3(\alpha_1 M_1^2 + \alpha_2 M_2^2) + (a_1-a_2)p_3(M_1p_1 - M_2p_2)\big) - 4\lambda^2(M_1^2+M_2^2),$$



where $\alpha_1 = 1 - xa_1$, $\alpha_2 = 1 - xa_2$, $a_1, a_2 = \text{const}$. The reduced form of this generalization seems to be pointed out by the authors of [34, 197]. In § 7 ch. 5 this case is included into the family which is even more abundant with arbitrary parameters. This family is defined on the bundle $\mathscr{L}_x$ of Poisson brackets, as well.

## § 11. The Remarkable Boundary Case of the Poincaré–Joukovskiy Equations. The Countable Family of First Integrals

**1.** In this section we consider equations of motion of a rigid body when in Poincaré–Joukovskiy equations (10.3) a boundary transition was made. It differs from the similar transition applied at retraction, and leads to the loss of Hamiltonianity. Really, if in equations (10.3) with Hamiltonian (10.8) the substitution $\boldsymbol{\gamma} \to \mu\boldsymbol{\gamma}$ is made and $\mu$ is tended to zero, commutating relations (10.1) have a singularity. Nevertheless, on the level of equations of motion we obtain

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \mathbf{B}\boldsymbol{M},$$
$$\mathbf{A} = \operatorname{diag}(a_1, a_2, a_3), \quad \mathbf{B} = \operatorname{diag}(b_1, b_2, b_3). \qquad (11.1)$$

System (11.1) describes the body rotation when the intensity of fluid vortex in a cavity is small in comparison with angular momentum (or vice versa). One can also make other interpretations of this boundary transition if one uses various physical statements of the problem (see § 10), specified by equations (10.3).

The first vector equation in (11.1) is integrated independently and represents the ordinary Euler case with integrals

$$I_1 = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}), \quad I_2 = (\boldsymbol{M}, \boldsymbol{M}).$$

The second equation in (11.1) is related to the Poisson kinematic equation. After the substitution of the already known function $\boldsymbol{M}(t)$ in this equation, it becomes a linear Hamilton system on $so(3)$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{B}\boldsymbol{M}(t), \quad \{\gamma_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k \qquad (11.2)$$

with periodic coefficients and linear Hamiltonian $H = (\mathbf{B}\boldsymbol{M}(t), \boldsymbol{\gamma})$. Equations (11.2), like system (11.1), possess a geometrical integral $I_3 = (\boldsymbol{\gamma}, \boldsymbol{\gamma})$. To be integrable, system (11.2) lacks one more integral with coefficients $I_4^*(\boldsymbol{\gamma}, t)$ periodic with respect to $t$. Thus, for integrability of system (11.1) one more first integral $I_4(\boldsymbol{M}, p)$ is needed. It also follows from the last multiplier theory.

**2.** The additional integral of systems (11.1), (11.2) always exists in real and analytical class of functions. This happens because equation (11.2) defines the linear mapping of a two-dimensional sphere for a period. This mapping also preserves a measure. Such mappings are integrable.

Is is easily seen that the additional integral is linear with respect to $\boldsymbol{\gamma}$:

$$I_4 = I_4(\boldsymbol{M}, \boldsymbol{\gamma}) = (\boldsymbol{\Omega}(\boldsymbol{M}), \boldsymbol{\gamma}),$$
$$I_4^* = I_4^*(\boldsymbol{\gamma}, t) = (\boldsymbol{\Omega}^*(t), \boldsymbol{\gamma}), \quad \boldsymbol{\Omega}^*(t) = \boldsymbol{\Omega}(\boldsymbol{M}(t)).$$

It turns out that for the function $\boldsymbol{\Omega}^*(t)$ we obtain equations, similar to (11.2) (!). So, to find the first integral $I_4$ or $I_4^*$, it is sufficient to find particular solutions of equations (11.2).

However, in the general case neither particular solutions (11.2), nor corresponding integrals (11.1), (11.2) can be obtained in the closed algebraic form. Such a solution is possible only in the form of an infinite series, it is ambiguous in the complex sense (and it is not algebraic) [97].

Remark. In the paper [37] authors computed the Kowalevskaya indices for system (11.1). They equal

$$\rho_1 = -1, \quad \rho_2 = \rho_3 = 2, \quad \rho_4 = 1, \quad \rho_{5,6} = 1 \pm n,$$
$$n = \left[\frac{b_1^2 a_{32} + b_2^2 a_{13} + b_3^2 a_{21}}{a_{23} a_{21} a_{13}}\right], \tag{11.3}$$
$$a_{ij} = a_i - a_j.$$

General solution (11.1) has finite-leave ramification on the complex plane of time under the condition $n = p/q$, $p, q \in \mathbb{Z}$. This condition is also necessary for the existence of an additional algebraic integral [206].

Physically, the absence of a “sufficiently” good (algebraic, polynomial) additional integral for system (11.1) is connected with the loss of symmetries conditioned by the Hamiltonianity (a Poisson structure is a tensor invariant). Nevertheless, the behavior of paths (11.1), (11.2) is always regular, Lyapunov indices equal zero, and a real-analytical integral formally exists.

**3.** Let us indicate conditions for the explicit finding of an additional integral. In other cases analogical constructions are hardly possible. For this it is necessary that in (11.3) $n = 2k+1$, $k \in \mathbb{Z}$. Additional integrals at different $k$ involved in a certain iteration process, for the first time were indicated without proofs by A. V. Borisov and A. V. Tsygvintsev in [37, 38]. Here we shall show the natural procedure of construction of these integrals whose existence presently seems a bit mysterious.



**4.** After the substitution of time $d\tau = M_1 M_2 M_3\, dt$ in terms of new variables $u_i = M_i^2$, $s_i = \dfrac{\gamma_i}{M_i}$ system (11.1) is written as

$$\begin{gathered}\dot{\boldsymbol{u}} = \boldsymbol{c}, \quad \dot{\boldsymbol{s}} = \mathbf{U}^{-1}\mathbf{A}_b \dot{\boldsymbol{s}},\\ \boldsymbol{u} = (u_1, u_2, u_3), \quad \boldsymbol{s} = (s_1, s_2, s_3), \quad \boldsymbol{c} = (-2a_{23}, -2a_{31}, -2a_{12}),\\ a_{ij} = a_i - a_j, \quad \mathbf{U} = \operatorname{diag}(u_1, u_2, u_3),\\ \mathbf{A}_b = \begin{pmatrix} a_{23} & b_3 & -b_2 \\ -b_3 & a_{31} & b_1 \\ b_2 & -b_1 & a_{12} \end{pmatrix},\end{gathered} \tag{11.4}$$

where point denotes differentiation with respect to $\tau$.

Integral of system (11.4) we shall be looking for in the form

$$F = \sum_i s_i u_i f_i(\boldsymbol{u}). \tag{11.5}$$

It leads for the vector $\boldsymbol{f} = (f_1, f_2, f_3)$ having the system of differential equations in terms of the first order partial derivatives

$$\mathbf{U}\widehat{D}\boldsymbol{f} = -\mathbf{A}_b \boldsymbol{f}, \tag{11.6}$$

where a scalar differential operator $\widehat{D}$ is written as

$$\widehat{D} = -2a_{23}\frac{\partial}{\partial u_1} - 2a_{31}\frac{\partial}{\partial u_2} - 2a_{12}\frac{\partial}{\partial u_3}. \tag{11.7}$$

To find the integral it is necessary to determine particular solution (11.6).

**Theorem.** *If conditions*

$$\begin{aligned} &1.\ n^2 a_{23} a_{31} a_{12} + b_1^2 a_{23} + b_2^2 a_{31} + b_3^2 a_{12} = 0,\\ &2.\ n = 2k+1, \quad k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \end{aligned} \tag{11.8}$$

*are met, equation* (11.6) *allows a polynomial particular solution* (*which can be written in the explicit form*).

Proof is given by induction.

Really, at $n=1$, ($k=0$) the particulal solution is easily determined. It corresponds to the constant eigenvector $\boldsymbol{f}_1$ of matrix $\mathbf{A}_b$. This vector is defined by a zero eigenvalue, since $\det \mathbf{A}_b = a_{23}a_{31}a_{12} + b_1^2 a_{23} + b_2^2 a_{31} + b_3^2 a_{12} = 0$. This exactly corresponds to conditions (11.8) at $n=1$.

At $n \neq 1$, we shall look for solution (11.6) in the form

$$\boldsymbol{f} = \mathbf{B}_1 \mathbf{U} \widetilde{\boldsymbol{f}}, \quad \text{where} \quad \mathbf{B}_1 = \mathbf{A}_b^{-1} \tag{11.9}$$

(since $n \neq 1$, $\det \mathbf{A}_b \neq 0$). Bearing in mind that $\widehat{D}\mathbf{U} = \mathbf{C}$, $\mathbf{C} = \operatorname{diag}(-2a_{23}, -2a_{31}, -2a_{12})$, from (11.6) we obtain an algebraic vector equation

$$\mathbf{U}\mathbf{B}_1\mathbf{C}\widetilde{\boldsymbol{f}} + \mathbf{U}\mathbf{B}_1\mathbf{U}\widehat{D}\widetilde{\boldsymbol{f}} = -\mathbf{U}\widetilde{\boldsymbol{f}},$$

which, after the multiplication of the left side by matrix $\mathbf{A}_b\mathbf{U}^{-1}$ and collection of the similar terms, may be written in the form, similar to (11.6)

$$\mathbf{U}\widehat{D}\widetilde{\boldsymbol{f}} = -(\mathbf{A}_b - \mathbf{C})\widetilde{\boldsymbol{f}} = -\mathbf{A}_{b,3}\widetilde{\boldsymbol{f}},$$

where

$$\mathbf{A}_{b,n} = \begin{pmatrix} na_{23} & b_3 & -b_2 \\ -b_3 & na_{31} & b_1 \\ b_2 & -b_1 & na_{12} \end{pmatrix}.$$

Owing to the fact that $\det \mathbf{A}_{b,n} = n(n^2 a_{23}a_{31}a_{12} + b_1^2 a_{23} + b_2^2 a_{31} + b_3^2 a_{12})$, the inductive process can be continued up to the necessary $n$ if we represent $\widetilde{\boldsymbol{f}}$ in the same form (11.9), and the solution can be obtained in the form

$$\boldsymbol{f} = \mathbf{A}_{b,1}^{-1}\mathbf{U}\mathbf{A}_{b,3}^{-1}\mathbf{U}\ldots\mathbf{A}_{b,n-2}^{-1}\mathbf{U}\boldsymbol{f}_n, \tag{11.10}$$

where $\boldsymbol{f}_n$ is an eigenvector of matrix $\mathbf{A}_{b,n}$. This vector corresponds to the zero eigenvalue $\mathbf{A}_{b,n}\boldsymbol{f}_n = 0$. ■

**5.** Consider the particular case $\mathbf{B}=n\mathbf{A}$, $n=2k+1$, $k \in \mathbb{Z}$ in more details. At $n > 0$ we have

$$\mathbf{A}_{b,n} = n\mathbf{A}_a = n \begin{pmatrix} a_{23} & a_3 & -a_2 \\ -a_3 & a_{31} & a_1 \\ a_2 & -a_1 & a_{12} \end{pmatrix},$$

and the eigenvector $\boldsymbol{f}_n$ for all values of $n$ is the same and equals to $\boldsymbol{f}_n = \boldsymbol{v}_+ = (1, 1, 1)$.



At $n < 0$ we have

$$\mathbf{A}_{b,n} = |n|\mathbf{A}_{-a} = |n| \begin{pmatrix} a_{23} & -a_3 & a_2 \\ a_3 & a_{31} & -a_1 \\ -a_2 & a_1 & a_{12} \end{pmatrix},$$

and the eigenvector $\boldsymbol{f}_n$ is also the same for all $n$ and may be written as

$$\boldsymbol{f}_n = \boldsymbol{v}_- = (a_1^{-1} - a_2^{-1} - a_3^{-1}, -a_1^{-1} + a_2^{-1} - a_3^{-1}, -a_1^{-1} - a_2^{-1} + a_3^{-1}).$$

Let us indicate explicit expressions of the integral for even more particular cases

1) $n = 1$.

   $I_4 = (\boldsymbol{M}, \boldsymbol{\gamma})$ is an ordinary area integral for the Euler case (see § 2 ch. 2).

2) $n = -1$.

$$I_4 = (\boldsymbol{\Omega}\boldsymbol{M}, \boldsymbol{\gamma}), \quad \boldsymbol{\Omega} = \mathbf{E} - \frac{2\mathbf{A}^{-1}}{\operatorname{Tr}\mathbf{A}^{-1}} \tag{11.11}$$

   is a certain analogue of the area integral. As we shall show further (see s. 6), at $n = -1$ equations (11.1) are reduced to the case when $n = 1$ by means of a linear transformation.

3) $n = \pm 3$.

   The solution (11.10) may be represented as

$$\boldsymbol{f} = (\mathbf{B}^s \pm \mathbf{B}^a)\mathbf{U}\boldsymbol{v}_\pm,$$

   where $\mathbf{B}^s$ and $\mathbf{B}^a$ represent symmetrical and skew-symmetrical matrices with components

$$b_{ij}^s = \begin{cases} 9a_i a_j, & i \neq j, \\ 9a_i^2 - \varepsilon_{ilm} a_{il} a_{im}, & i = j, \end{cases}$$
$$b_{ij}^a = -3a_{ij}a_k, \quad i \neq j \neq k.$$

   The vectors $\boldsymbol{v}_\pm$ are defined in s. 5. With respect to variables $\boldsymbol{M}, \boldsymbol{\gamma}$ the additional integral $I_4$ will have the fourth degree; in the general case this degree equals $2|k| + 2$.

**6.** It turns out [36] that in case $n = -1$, $\mathbf{B} = -\mathbf{A}$ the integral $I_4$ (11.11) exists without any changes for a more general system

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M} + \varepsilon\boldsymbol{\gamma} \times \mathbf{A}^{-1}\boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = n\boldsymbol{\gamma} \times \mathbf{A}\boldsymbol{M}, \end{cases} \tag{11.12}$$

corresponding to the addition of the Brun problem field (see § 1 ch. 2) with the potential $V = \frac{1}{2}\varepsilon(\boldsymbol{\gamma},\, \mathbf{A}^{-1}\boldsymbol{\gamma})$ to (11.1). The measure for equations (11.12) remains standard, but the integrals $I_1$, $I_2$ need some modification

$$\begin{gathered} I_1 = \frac{1}{2}(\boldsymbol{M},\, \mathbf{A}\boldsymbol{M}) - \frac{1}{2}\varepsilon(\boldsymbol{\gamma},\, \mathbf{A}^{-1}\boldsymbol{\gamma}), \\ I_2 = (\boldsymbol{M},\, \boldsymbol{M}) + \varepsilon \det \mathbf{A}^{-1}(\boldsymbol{\gamma},\, \mathbf{A}\boldsymbol{\gamma}). \end{gathered} \tag{11.13}$$

It is easy shown that the integrals of system (11.12), similar to $I_1$, $I_2$, exist at $\mathbf{B} = n\mathbf{A}$ for any value $n$, but the integral of the type $I_4$ will exist at $n = \pm 1$ only. Its generalization at $\varepsilon \neq 0$ for other $n = 2k + 1 = \pm 3, \pm 5, \ldots$ does not seem to be possible.

Remark. In terms of variables $\boldsymbol{u}$, $\boldsymbol{s}$ system (11.12) can be represented in the form

$$\dot{u}_i = c_i + \varepsilon\lambda_k b_{kj} s_k s_j, \quad \dot{\boldsymbol{s}} = \mathbf{U}^{-1}\mathbf{A}_b\boldsymbol{s},$$

$\lambda_k = a_k^{-1}$, $b_{kj} = b_k - b_j$. However, the reasoning of s. 3 shows that at $n \neq \pm 1$ the corresponding induction is impossible.

The case of $n = -1$ in (11.12), as it is noticed in [36], is reduced to $n = 1$ which corresponds to the Brun problem (or the Clebsch case) by means of linear transformation

$$\boldsymbol{W} = \boldsymbol{\Omega}\boldsymbol{M}, \quad \boldsymbol{\Omega} = \mathbf{E} - \frac{2\mathbf{A}^{-1}}{\operatorname{Tr}\mathbf{A}^{-1}}. \tag{11.14}$$

After this transformation system (11.12) obtains the form

$$\begin{cases} \dot{\boldsymbol{W}} = \mathbf{J}^{-1}\boldsymbol{W} \times \boldsymbol{W} - \varepsilon\boldsymbol{\gamma} \times \mathbf{J}\boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = \mathbf{J}^{-1}\boldsymbol{W} \times \boldsymbol{\gamma}, \quad \mathbf{J} = \boldsymbol{\Omega}\mathbf{A}^{-1}, \end{cases} \tag{11.15}$$

similar to (11.12) at $n = 1$ with the accuracy to the substitution $t \to -t$.

It should be noted that the addition of a constant gyrostatic moment to system (11.1), i. e., the construction of the Joukovskiy – Volterra problem generalization, does not lead to a new integrable system even at $n = -1$. Actually, the question of other possible generalizations of the countable family of integrals $I_4$ (for example, on $so(4)$, a gyrostat and others) remains open. It is possible that they just do not exist at all.



Remark 1. In the general case for arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$ in (11.1) the general integral is not single-valued and has branches on the complex plane of time. Under the conditions $b_1 = b_2 = 0$, $b_3 \neq 0$ it can be written explicitly

$$\begin{gathered} I_4 = \gamma_1 \sin\varphi + \gamma_2 \cos\varphi, \\ \varphi = \frac{b_3}{\sqrt{a_{13}a_{32}}} \ln(\sqrt{a_{13}}M_1 + \sqrt{a_{32}}M_2). \end{gathered} \tag{11.16}$$

The existence of such complicated integrals for system (11.1) is also connected with the loss of Hamiltonianity, though the last fact is not well substantiated.

Remark 2. Except for the cases $n = \pm 1$ for system (11.1) in the presence of integrals (11.5), the general solution is still not obtained in quadratures; it is also unclear if it is expressed in terms of elliptic functions. The topology of the corresponding levels of set of integrals is not investigated, as well.

**7.** System (11.1) may be also obtained under investigation of a nonholonomic problem about sliding-free rolling of a dynamically asymmetrical balanced ball (*the Chaplygin ball*) on the surface of a sphere (fig. 69).

In the absence of a force field equations of motion are written as [36]

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}, \\ \dot{\boldsymbol{\gamma}} = \dfrac{R}{R-a}\boldsymbol{\gamma} \times \boldsymbol{\omega}, \end{cases} \tag{11.17}$$

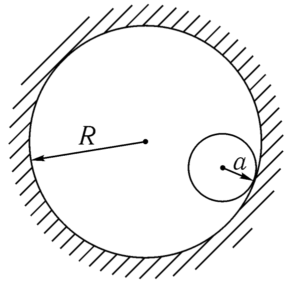



Figure 69

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}), \quad D = ma^2,$$

where $m$ is the mass of a ball, $\mathbf{I}$ is a tensor of inertia with respect to the geometric center.

Here we shall not dwell upon the study of integrability of system (11.17), we just note that when the parameter of nonholonomity $D$ tends to zero, we obtain system (11.1) with the matrix $\mathbf{B} = \lambda\mathbf{A}$, $\lambda = \frac{R}{R-a}$. Once again this indicates the necessity of study of equations (11.1), and also allows to generalize integrals $I_1$, $I_2$, $I_3$, $I_4$ to equations (11.17). Presently such a generalization, found in [36], is known only at $\lambda = \pm 1$. Moreover, in both cases we can consider more general situation (11.12), corresponding to the Brun field addition.

The case $\lambda = 1$, $R = \infty$ is reduced to the classical Chaplygin problem about rolling of a ball on a horizontal plane [179].

The case$\lambda = -1$, $a = 2R$ corresponds to the so called *spherical suspend*, when a dynamically asymmetrical sphere of a doubled radius rolls around a stationary ball. This integrable problem and its generalizations for the Brun field were found by A. V. Borisov [36].

**8.** At $\mathbf{B} = \lambda\mathbf{A}$, and at the coincidence of two eigenvalues of the matrix $\mathbf{A}$, for example, at $a_1 = a_2$, system (11.1), and the more general system

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M} + \boldsymbol{\gamma} \times \dfrac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} = \lambda\boldsymbol{\gamma} \times \mathbf{A}\boldsymbol{M}, \quad V = V(\gamma_3) \end{cases} \tag{11.18}$$

are algebraically integrable. The complete set of integrals is written as

$$\begin{aligned} &I_1 = \frac{1}{2}(M_1^2 + M_2^2) + (\lambda a_1)^{-1}V(\gamma_3), \\ &I_2 = M_3, \quad I_3 = \boldsymbol{\gamma}^2, \\ &I_4 = M_1\gamma_1 + M_2\gamma_2 + \frac{a_1 - a_3 + \lambda a_3}{\lambda a_1}M_3\gamma_3. \end{aligned} \tag{11.19}$$

It provides the Euler–Jacobi integrability (system (11.18) also possesses a standard invariant measure). Integrals (11.19) are similar to integrals of the Lagrange case, and they allow corresponding nonholonomic generalizations [196].

## § 12. A Rigid Body in an Arbitrary Potential Field

As it is shown in § 4 ch. 1, dynamics of a rigid body with a fixed point in an arbitrary potential field with potential $V$ is defined by a Hamiltonian system with three degrees of freedom (4.17) (or (4.24)) (§ 4 ch. 1). The Hamiltonian function has the form

$$\begin{gathered} H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + V, \qquad \mathbf{A} = \mathbf{I}^{-1}, \\ V \equiv V(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \equiv V(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \equiv V(\theta, \varphi, \psi). \end{gathered} \tag{12.1}$$

However, for the complete integrability (the Liouville integrability) two more independent integrals in involution are necessary.

Integrable cases for system (12.1) are known for three kinds of potentials $V(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \equiv V(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ (here $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are direction cosines, $\lambda$ are Rodrigue–Hamilton parameters).



**1)** The potential $V$ is linear with respect to components $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ (and quadratic with respect to $\lambda$). For the particular kind of $V$ in the presence of force field axial symmetry one gets the Euler–Poisson equations; thus, in the general case the system is referred to as *generalized Euler–Poisson equations*.

**2)** The potential $V$ is quadratic with respect to $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ (and has the fourth degree with respect to quaternions). This problem was considered by Brun and Goryachev.

**3)** The potential $V$ is linear with respect to $\lambda$. Although in a certain sense this case is easier than the previous ones, we put it at the last place due to the fact that it was not considered earlier. This might be connected with the absence of its reasonable mechanical interpretation. We called this case *quaternion Euler–Poisson equations*.

Let us consider these three cases in sequence and show all the known conditions of integrability characterized by necessary additional restrictions on arbitrary parameters. The motion is regular; the paths, in the nonspecial case, are quasiperiodic windings of three dimensional tori — joint level sets of first integrals.

### 1. Generalized Euler–Poisson Equations

First of all, it should be noted that any quantity of linear force fields is reduced to three mutually perpendicular force fields of unitary intensity. The *force centers* of these fields (the analogues of the center-of-mass for a gravity field) are placed within the body in an arbitrary way [31].

The Hamilton function is written as

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{r}_1, \boldsymbol{\alpha}) + (\boldsymbol{r}_2, \boldsymbol{\beta}) + (\boldsymbol{r}_3, \boldsymbol{\gamma}), \tag{12.2}$$

where $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ are position vectors of force centers of different nature. In case of a single field they are reduced to an ordinary gravity center.

Let us show the main results on reduction of potential energy of system (12.2) to the simplest form for various disposition of force centers $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$, taking into account the body geometry, as well; for details see [31].

1) *The centers of application of all fields are on a single axis.*

By means of the proper choice of fixed axes in space, the potential energy can be reduced to the form

$$V = \sqrt{a^2 + b^2 + c^2}\alpha_1,$$

where $a, b, c$ are distances between force centers and the point of fixation.

Thus, this case is reduced to a single force field, its force center $\boldsymbol{r}_1$ lying on the above mentioned axis.

2) *The centers of application of all fields are in one plane.*

By choosing the fixed axes we reduce the potential to the form

$$V = u\alpha_1 + v\alpha_2 + w\beta_2.$$

It means that the system of forces is reduced to two mutually orthogonal fields. In the general case the position vectors $\boldsymbol{r}_1 = (u, v, 0)$, $\boldsymbol{r}_2 = (0, w, 0)$ of their force centers are nonorthogonal.

3) *The centers of application of fields are arbitrary, but the tensor of inertia of a body is spherical* $(a_1 = a_2 = a_3)$.

In this case, if we choose moving principal axes in an even more arbitrary way, we can reduce the potential energy to the form

$$U = a\alpha_1 + b\beta_2 + c\gamma_3.$$

Depending on the disposition of force centers within a rigid body and restrictions on the moments of inertia, there exists the possibility of the following integrable cases, generalizing the corresponding ones in the Euler–Poisson equations.

**The Euler case.** In Hamiltonian (12.2) we should specify $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \boldsymbol{r}_3 = 0$. Additional integrals are projections of the angular momentum on fixed axes. These projections form a vector integral $\boldsymbol{N} = (N_1, N_2, N_3)$

$$N_1 = (\boldsymbol{M}, \boldsymbol{\alpha}), \qquad N_2 = (\boldsymbol{M}, \boldsymbol{\beta}), \qquad N_3 = (\boldsymbol{M}, \boldsymbol{\gamma}). \tag{12.3}$$

They also form an algebra $so(3)$: $\{N_i, N_j\} = \varepsilon_{ijk}N_k$. Consequently, the integrability is noncommutative (for more details see § 2 ch. 2).

**The generalized Lagrange case.** The body is dynamically symmetrical, and all three force centers lie on the dynamical symmetry axis. According to reduction results, this case is reduced to an ordinary Lagrange top in a single field with corresponding integrals $F_1 = (\boldsymbol{M}, \boldsymbol{\gamma})$, $F_2 = M_3$ (§ 3 ch. 2).

**The generalized Kowalevskaya case.** The ellipsoid of inertia is an ellipsoid of rotation. The moments of inertia are related as $a_1 = a_2 = \frac{1}{2}a_3$, $(a_i = I_i^{-1})$. Three force centers are arbitrarily situated in the equatorial plane of the ellipsoid of inertia. As it is shown above, here we may consider only two force centers.



The corresponding complete set of independent integrals in involution is shown by A. G. Reyman and M. A. Semenov-Tian-Shansky [147, 261, 194]. One integral is quadratic, the other (the analogue of the Kowalevskaya integral) has the fourth degree with respect to moments.

In this case, like in the ordinary Kowalevskaya case, the system allows generalization when a constant gyrostatic moment is added along the axis of dynamical symmetry. The Hamiltonian and integrals are written as [31, 261]

$$\begin{aligned}
H &= \frac{1}{2}(M_1^2 + M_2^2 + 2(M_3+\lambda)^2) - (\boldsymbol{r}_1, \boldsymbol{\alpha}) - (\boldsymbol{r}_2, \boldsymbol{\beta}),\\
F_1 &= (N_1\boldsymbol{r}_1 + N_2\boldsymbol{r}_2)^2 + 2N_3(\boldsymbol{r}_1\times\boldsymbol{r}_2, \boldsymbol{M}) +\\
&\quad + 2(\boldsymbol{r}_1\times\boldsymbol{r}_2, \boldsymbol{r}_2\times\boldsymbol{\alpha} - \boldsymbol{r}_1\times\boldsymbol{\beta}),\\
F_2 &= \Big(\frac{M_1^2 - M_2^2}{2} + g_\alpha\alpha_1 - h_\alpha\alpha_2 + g_\beta\beta_1 - h_\beta\beta_2\Big)^2 +\\
&\quad + (M_1M_2 + g_\alpha\alpha_2 + h_\alpha\alpha_1 + g_\beta\beta_2 + h_\beta\beta_1)^2 -\\
&\quad - 2\lambda(M_3 + 2\lambda)(M_1^2 + M_2^2) - 4\lambda\left(\alpha_3(\boldsymbol{M}, \boldsymbol{r}_1) + \beta_3(\boldsymbol{M}, \boldsymbol{r}_2)\right).
\end{aligned} \tag{12.4}$$

where $\boldsymbol{r}_1 = (g_\alpha, h_\alpha, 0)$, $\boldsymbol{r}_2 = (g_\beta, h_\beta, 0)$, $\lambda = \mathrm{const}$ is a gyrostatic moment, $N_i$ is specified by expressions (12.3).

Neither explicit integration, nor qualitative or topological analysis of this case were carried out up to this day.

Remark 1. A. G. Reyman and M. A. Semenov-Tian-Shansky have shown this integrable case in the $n$-dimensional situation, but under additional restrictions: the centers of reduction $\boldsymbol{r}_1, \boldsymbol{r}_2$ of mutually perpendicular fields are situated *at equal* distances from a fixed point, but not necessarily at right angle (or we may think that $\boldsymbol{r}_1 \perp \boldsymbol{r}_2$, and the fields $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are nonperpendicular). As the reduction results show, these restrictions are nonessential.

At $\boldsymbol{r}_2 = 0$ (or $\boldsymbol{r}_1 = 0$), integral $F_1$ (12.4) transforms into the area integral $(\boldsymbol{M}, \boldsymbol{\alpha}) = 0$ (correspondingly $(\boldsymbol{M}, \boldsymbol{\beta})$). In this case the precession angle $\psi$ is a cyclic variable. The reduction with respect to this angle gives the ordinary Kowalevskaya case in the Euler–Poisson equations (§ 4 ch. 2). The analogical reduction is possible in case $\boldsymbol{r}_1 \parallel \boldsymbol{r}_2$.

At $\boldsymbol{r}_1 \perp \boldsymbol{r}_2$, for example, one can choose $g_\alpha = h_\beta$, $h_\alpha = g_\beta = 0$ or $h_\alpha = g_\beta$, $g_\alpha = h_\beta = 0$, then instead of $F_1$ there appears a linear integral $M_3 \pm N_3 = M_3 \pm (\boldsymbol{M}, \boldsymbol{\gamma})$, a cyclic variable $\varphi \mp \psi$. The corresponding reduction and associated isomorphism with the integrable Chaplygin case in Kirchhoff's equations are considered in details in § 1 ch. 4. This integrable case was shown by H. Yehia [184] before the papers [147, 261] appeared.

**The generalization of the Delauney case.** Besides the reduction of order with respect to cyclic variables for generalized Kowalevskaya top (12.4), one more case of reduction is possible. This is the reduction to two degrees of freedom by using the Dirac reduction [31]. For that purpose let us consider Kowalevskaya integral $F_2$ (12.4) under the conditions $\lambda = 0$, $F_2 = z_1^2 + z_2^2 = 0$, which define *the generalized Delauney case* (O. I. Bogoyavlensky [19]). It is easy to see that the system is well bounded according to Dirac on invariant relations

$$\begin{aligned} z_1 &= \frac{M_1^2 - M_2^2}{2} + g_\alpha \alpha_1 - h_\alpha \alpha_2 + g_\beta \beta_1 - h_\beta \beta_2 = 0, \\ z_2 &= M_1 M_2 + g_\alpha \alpha_2 + h_\alpha \alpha_1 + g_\beta \beta_2 + h_\beta \beta_1 = 0, \end{aligned} \tag{12.5}$$

which are central functions of the Dirac structure [31]. On a four-dimensional symplectic leave of the Dirac bracket there exist two integrals (12.4), allowing complete integration of the system to be done.

On the level of invariant relations (12.5) there also exists an additional integral of the fourth degree

$$\begin{aligned} F_3 = \{z_1, z_2\} = -M_3(M_1^2 + M_2^2) + \\ + 2\alpha_3(M_1 g_\alpha + M_2 h_\alpha) + 2\beta_3(M_1 g_\beta + M_2 h_\beta). \end{aligned} \tag{12.6}$$

Really,

$$\begin{aligned} \{F_3, H\} = 2z_1(-g_\alpha \alpha_2 - h_\alpha \alpha_1 - g_\beta \beta_2 - h_\beta \beta_1) - \\ - 2z_2(-g_\alpha \alpha_1 + h_\alpha \alpha_2 - g\beta \beta_1 + h_\beta \beta_2), \end{aligned} \tag{12.7}$$

though in the general case the Jacobi theorem (stating that the commutator of two integrals is also an integral) is not generalized to invariant relations.

By means of (12.5) and (12.7) the integrability of the generalized Kowalevskaya top for the Delauney case can be established even without using integral $F_1$ (12.4). It turns out that the complete set of integrals, involving $F_1, z_1, z_2, F_3$, is already dependent. It is also curious that in the case of a single force field ($g_\alpha = g_\beta = h_\beta = 0$) integral (12.6), cubic with respect to moments, possesses a structure, almost analogical to the particular Goryachev – Chaplygin integral for the Euler – Poisson equations (see § 5 ch. 2).

**The generalized spherical top.** Here $a_1 = a_2 = a_3$, and under any disposition of the centers of reduction $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ within the body the system remains integrable. Besides, due to the invariance of kinetic energy with respect to the choice of axes within the body, the potential energy may have the form

$$V = x\alpha_1 + y\beta_2 + z\gamma_3.$$



In quaternion representation it can be regarded as an arbitrary quadratic form $V = \sum\limits_{i=0}^{3} b_{ij}\lambda_i\lambda_j$. Consequently, according to the analogy discussed in [31] (see also § 3 ch. 5), this case is isomorphic to the Neumann problem about motion of a point on a three-dimensional sphere $S^3$. The involution set of its (quadratic) integrals can be looked up in the paper by J. Moser [128]. This paper contains the separation of variables for the Neumann system on $S^n$. This separation was done by Rosochatius [263], in the nineteenth century. This author has also added curious singular terms whose mechanical meaning is discussed in § 11 ch. 5. Let us represent integrals in terms of necessary variables and in the most symmetrical form

$$\begin{gathered} H = 4\boldsymbol{M}^2 + \frac{1}{4}(a_0^2 + a_1^2 - a_2^2 - a_3^2)\alpha_1 + \\ + \frac{1}{4}(a_0^2 - a_1^2 + a_2^2 - a_3^2)\beta_2 + \frac{1}{4}(a_0^2 - a_1^2 - a_2^2 + a_3^2)\gamma_3, \\ F_1 = (\boldsymbol{M} + \boldsymbol{N}, \mathbf{A}(\boldsymbol{M} + \boldsymbol{N})) + (\boldsymbol{M} - \boldsymbol{N}, \mathbf{B}(\boldsymbol{M} - \boldsymbol{N})) + \\ + \frac{1}{4}(a_0 + a_1 - a_2 - a_3)\alpha_1 + \frac{1}{4}(a_0 - a_1 + a_2 - a_3)\beta_3 + \frac{1}{4}(a_0 - a_1 - a_2 + a_3)\gamma_3, \\ \mathbf{A} = \operatorname{diag}\Big(\frac{1}{a_0 + a_1}, \frac{1}{a_0 + a_2}, \frac{1}{a_0 + a_3}\Big), \quad \mathbf{B} = \operatorname{diag}\Big(\frac{1}{a_2 + a_3}, \frac{1}{a_1 + a_3}, \frac{1}{a_1 + a_2}\Big), \\ F_2 = a^2(\boldsymbol{M} + \boldsymbol{N})^2 - 4(\mathbf{C}\boldsymbol{M}, \boldsymbol{N}) + \frac{1}{4}(a_0^4 + a_1^4 - a_2^4 - a_3^4)\alpha_1 + \\ + \frac{1}{4}(a_0^4 - a_1^4 + a_2^4 - a_3^4)\beta_2 + \frac{1}{4}(a_0^4 - a_1^4 - a_2^4 + a_3^4)\gamma_3, \\ \mathbf{C} = \operatorname{diag}\big(a_2^2 + a_3^2, a_1^2 + a_3^2, a_1^2 + a_2^2\big), \end{gathered}$$

where $\boldsymbol{N}$ is defined by formulae (12.3), and $a^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Remark 2. Regarding $F_1$ as a Hamiltonian and using quaternion representation (see § 3 ch. 5), we obtain the integrable problem about motion of a four-dimensional rigid body in quadratic potential of the special form. This system can also be considered as the generalization of Clebsch's case (§ 9 ch. 3).

**The Hess case analogue.** If the ellipsoid of inertia of a rigid body is asymmetrical with respect to the fixation point, and the centers of reduction of all fields $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ are situated on the perpendicular to its circular section (passing through the mean axis), then, as it is said above, the potential is reduced to the case of a single field with the reduction center placed at the same axis. Thus, we obtain the ordinary Hess case of motion of a rigid body in the gravity field

(§ 6 ch. 2). The invariant relation for this case is written as

$$\sqrt{a_2 - a_1} M_1 \pm \sqrt{a_3 - a_2} M_3 = 0, \qquad a_1 < a_2 < a_3.$$

As it was shown by N. E. Joukovskiy [79], in this case the gravity center moves according to the spherical pendulum law.

The detailed investigation of the Hess case for a linear field and for more general type of fields can be looked up in §§ 3, 4 ch. 4, where we also show its connection with existence of a cyclic variable and with the Lagrange case.

## 2. The Brun System

Consider the case when the potential $V(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is quadratically related to direction cosines. This problem was studied by F. Brun in the nineteenth century [198], but the most complete results were obtained not long ago [18, 19, 20, 21, 146]. Brun has found two independent integrals of motion, but failed to establish integrability. To do that, one should use a Hamiltonian structure of equations of motion, the Liouville theorem (instead of the last multiplier theory which was generally used for integration in rigid body dynamics in nineteenth century), and the involution property of two missing first integrals. Although the integrability of a top in the $n$-dimensional case in quadratic potential was formally studied in [146] (A. G. Reyman, M. A. Semenov-Tian-Shansky), the most complete results are presented in the papers by O. I. Bogoyavlenskiy [18, 21]. These papers also contain various physical interpretations of this problem.

Remark 3. In the small book [62] D. N. Goryachev studied systems with quadratic potential. He obtained general conditions of existence of an additional linear integral and a quadratic integral for such a system. Independently of Brun, Goryachev showed the integrable case in the presence of a single field and found possibilities of a single quadratic integral for two force fields (for one particular case he pointed out the second necessary integral, as well). All these integrals can be obtained from the more general system considered below.

**The Lax representation and first integrals ([21, 31]).** Consider a Hamiltonian system in terms of variables $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. This system is determined by equations (4.17), by commutation conditions (4.16) ch. 1 and by Hamiltonian

$$H = \frac{1}{2}(\mathbf{I}^{-1}\boldsymbol{M}, \boldsymbol{M}) - x(\mathbf{I}\boldsymbol{\alpha}, \boldsymbol{\alpha}) - y(\mathbf{I}\boldsymbol{\beta}, \boldsymbol{\beta}) - z(\mathbf{I}\boldsymbol{\gamma}, \boldsymbol{\gamma}), \tag{12.8}$$

where $x, y, z \in \mathbb{R}, \mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is a tensor of inertia of the body. Hamiltonian (12.8) is obtained from (6.4) ch. 1, at $x = 0$. Consequently, for such



a system the corresponding physical conclusions are valid: such a potential is obtained from the Newtonian one at the decomposition near the gravitating body.

Let us identify three-dimensional vectors $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ with skew-symmetrical matrices $\mathbf{M}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\gamma}}$ according to formulae

$$M_{ij} = \varepsilon_{ijk} M_k, \quad \widetilde{\alpha}_{ij} = \varepsilon_{ijk}\alpha_k, \quad \widetilde{\beta}_{ij} = \varepsilon_{ijk}\beta_k, \quad \widetilde{\gamma}_{ij} = \varepsilon_{ijk}\gamma_k \tag{12.9}$$

and also define a symmetrical matrix

$$\mathbf{u} = x\widetilde{\boldsymbol{\alpha}}^2 + y\widetilde{\boldsymbol{\beta}}^2 + z\widetilde{\boldsymbol{\gamma}}^2, \tag{12.10}$$

where $x, y, z$ are determined in (12.8). Formulae (12.9) and (12.10) define embedding of the phase space of a system into space $L^9$ of $3 \times 3$-matrices, since any matrix $\mathbf{l}$ can be represented in the form $\mathbf{l} = \mathbf{M} + \mathbf{u}$. In this space commutative relations (5.7) of ch. 1 specify the Lie algebra structure, corresponding to a semi-direct sum $L^9 = so(3) \oplus_s \mathbb{R}^6$, where $so(3)$ is an algebra of $\mathbf{M}$, and $\mathbb{R}^6$ is a space of symmetrical $\mathbf{u}$-matrices whose commutator should be equal to zero. In the matrix form commutative relations for $\mathbf{l}_1 = \mathbf{M}_1 + \mathbf{u}_1$, and $\mathbf{l}_2 = \mathbf{M}_2 + \mathbf{u}_2$ can be written as

$$[\mathbf{M}, \mathbf{u}] = \mathbf{M}\mathbf{u} - \mathbf{u}\mathbf{M} \in \mathbb{R}^6, \quad [\mathbf{M}_1, \mathbf{M}_2] = \mathbf{M}_1\mathbf{M}_2 - \mathbf{M}_2\mathbf{M}_1 \in so(3),$$
$$[\mathbf{u}_1, \mathbf{u}_2] = 0. \tag{12.11}$$

Remark 4. A standard matrix commutator for $gl(3)$ defines commutative relations which differ from (12.11) in the inequality $[\boldsymbol{u}_1, \boldsymbol{u}_2] \neq 0$. These two sets of commutative relations are compatible and specify the bundle of Poisson brackets (for more details see [31]).

Poisson structure (12.11), corresponding to the algebra $L^9$, possesses Casimir's functions

$$F_1 = \operatorname{Tr}(\mathbf{u}), \quad F_2 = \operatorname{Tr}(\mathbf{u}^2), \quad F_3 = \operatorname{Tr}(\mathbf{u}^3),$$

and under the restriction on a six-dimensional manifold $M^6$ defined by these Casimir's functions becomes nondegenerate. For the Liouville integrability of the system it lacks two more additional integrals in involution. These integrals specify three-dimensional tori, bearing quasiperiodic motions.

Hamiltonian (12.8) in terms of variables $\mathbf{M}, \mathbf{u}$ has the form

$$H = -\operatorname{Tr}\left(\frac{1}{4}\mathbf{M}\boldsymbol{\omega} + \mathbf{u}\mathbf{I}\right),$$

the equations themselves can be written as

$$\dot{\mathbf{M}} = [\mathbf{M}, \boldsymbol{\omega}] + \left[\mathbf{u}, \frac{\partial H}{\partial \mathbf{u}}\right], \quad \dot{\mathbf{u}} = [\mathbf{u}, \boldsymbol{\omega}], \tag{12.12}$$

where $\boldsymbol{\omega} = \|\omega_{ij}\|$ is a skew-symmetrical matrix, corresponding to the angular velocity with components $\omega_{ij} = \dfrac{\partial H}{\partial M_{ij}} = I_k^{-1} M_{ij}$, and $\dfrac{\partial H}{\partial \mathbf{u}} = \left\|\dfrac{\partial H}{\partial u_{ij}}\right\| = -\mathbf{I}$.

Equations (12.12) can also be represented in the form of the Lax pair with a spectral parameter $\lambda$ involved in this representation in a rational way

$$\begin{gathered} \dot{\mathbf{L}} = [\mathbf{L}, \mathbf{A}], \\ \mathbf{L} = \lambda \mathbf{M} + \mathbf{u} + \lambda^2 \mathbf{B}, \quad \mathbf{A} = \boldsymbol{\omega} - \lambda \mathbf{I}, \end{gathered} \tag{12.13}$$

where $\mathbf{B} = (\det \mathbf{I})\mathbf{I}^{-1}$.

Two necessary independent motion integrals in involution are obtained as coefficients of $\lambda^k$ in the traces of degrees of matrix $\mathbf{L}$

$$G_1 = \operatorname{Tr}\Big(\frac{1}{2}\mathbf{M}^2 + \mathbf{B}\mathbf{u}\Big), \qquad G_2 = \operatorname{Tr}(\mathbf{M}^2\mathbf{u} + \mathbf{B}\mathbf{u}^2).$$

With the accuracy up to Casimir's functions they can be explicitly represented as

$$G_1 = \frac{1}{2}\boldsymbol{M}^2 + \det \mathbf{I}\left(x(\boldsymbol{\alpha}, \mathbf{I}^{-1}\boldsymbol{\alpha}) + y(\boldsymbol{\beta}, \mathbf{I}^{-1}\boldsymbol{\beta}) + z(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})\right),$$

$$G_2 = (x + y + z)\boldsymbol{M}^2 + x(\boldsymbol{M}, \boldsymbol{\alpha})^2 + y(\boldsymbol{M}, \boldsymbol{\beta})^2 + z(\boldsymbol{M}, \boldsymbol{\gamma})^2 + V,$$

$$V = \det \mathbf{I}\left[I_1^{-1}(\boldsymbol{p}, \mathbf{C}\boldsymbol{p}) + I_2^{-1}(\boldsymbol{q}, \mathbf{C}\boldsymbol{q}) + I_3^{-1}(\boldsymbol{r}, \mathbf{C}\boldsymbol{r})\right],$$

where $\mathbf{C} = \operatorname{diag}(2yz - x^2, 2xz - y^2, 2xy - z^2)$, $\boldsymbol{p} = (\alpha_1, \beta_1, \gamma_1)$, $\boldsymbol{q} = (\alpha_2, \beta_2, \gamma_2)$, $\boldsymbol{r} = (\alpha_3, \beta_3, \gamma_3)$.

An integrable system with a Hamiltonian $H = G_1$ can be represented as a problem about motion of a spherical top or a material point on $S^3$ in a force field with fourth degree potential (according to Rodrigue–Hamilton parameters or redundant variables correspondingly) [18, 89] (see § 3, § 2 ch. 5).

An integrable system with a Hamiltonian $H = G_2$. After the introduction of vector $\boldsymbol{N} = (N_1, N_2, N_3)$ (12.3), representing angular momentum projections on fixed axes, this system may be considered as a certain system on algebra $e(4)$ (see § 3 ch. 5). This system is integrable on a singular orbit defined by variables



$\boldsymbol{N} = (N_1, N_2, N_3)$, $\boldsymbol{p} = (\alpha_1, \beta_1, \gamma_1)$, $\boldsymbol{q} = (\alpha_2, \beta_2, \gamma_2)$, $\boldsymbol{r} = (\alpha_3, \beta_3, \gamma_3)$. Really, one can easily see that the algebra of variables $\boldsymbol{N}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is isomorphic to the algebra of variables $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. But due to the fact that $\boldsymbol{M}^2 = \boldsymbol{N}^2$ the integral $G_2$ on the algebra of $\boldsymbol{N}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is similar to Hamiltonian $H$ (12.8) on the algebra of $\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. In this sense, integrals $H$ and $G_2$ are mutual. *The Hamiltonians they define specify one and the same integrable system in various systems of variables connected with a moving and a fixed frames of reference.*

Remark 5. In the paper [17] the integrable problem which was considered becomes the basis for obtaining integrable cases for special systems of connected rigid bodies. However, these systems cannot be regarded as principally new dynamical problems since their dynamics is reduced to equations (12.12).

Remark 6. The paper [45] presents hydrodynamical interpretation of system (12.12). Here we may think that a free rigid body being linearly magnetized is moving in a uniform magnetic field (or a nonconducting rigid body being polarized is moving freely in a uniform electric field). The conditions of existence of two additional integrals indicated in [45], altogether with the integrals are also present in the general *Brun system*. Other physical interpretations of the general Brun system are collected in the book [21].

**The dynamical symmetry case.** Consider system (12.8) under the condition of dynamical symmetry ($I_1 = I_2 = 1$). It is reduced to two degrees of freedom and to the Neumann system. In this case Hamiltonian (12.8) of the system can be represented in the form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) - \frac{a-1}{a}(x\alpha_3^2 + y\beta_3^2 + z\gamma_3^2), \tag{12.14}$$

where $x, y, z,\ a = I_3^{-1} \in \mathbb{R}$. From equations of motion (12.12) it follows that the component $M_3$ is an integral of motion.

Moreover, as it follows from direct computations, the projections of moments on axes bound to absolute space $\boldsymbol{N}$ (12.3), and also projections on the same axes of unit vector directed along the dynamical symmetry axis (with components $(p_1, p_2, p_3) = (\alpha_3, \beta_3, \gamma_3)$) form a Lie algebra $e(3)$

$$\{N_i, N_j\} = \varepsilon_{ijk}N_k, \quad \{N_i, p_j\} = \varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = 0, \tag{12.15}$$

We have already mentioned this commutation in § 4 ch. 1. Eliminating the integral $M_3 = \mathrm{const}$ being the Casimir function of construction (12.15), we can write Hamiltonian (12.14) in terms of variables $N_i, p_j$ (making use of the fact that $\boldsymbol{M}^2 = \boldsymbol{N}^2$)

$$H = \frac{1}{2}\boldsymbol{N}^2 - \frac{a-1}{a}(xp_1^2 + yp_2^2 + zp_3^2). \tag{12.16}$$

Equations of motion with Hamiltonian (12.16) coincide with equations of motion of a point on a two-dimensional sphere in a force field with quadratic potential (the Neumann problem). This analogy was noticed in [18] without using equations on algebra of brackets (12.15) (see [31]).

**The Brun problem in a single field** is most famous. In this case equations of motion have the form of a Hamiltonian system on $e(3)$ with a Hamiltonian

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \frac{1}{2}\mu(\mathbf{A}^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})$$

and an additional integral

$$F = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{M}) - \frac{\mu}{2\det\mathbf{A}}(\mathbf{A}\boldsymbol{\gamma}, \boldsymbol{\gamma}).$$

This problem turns out to be equivalent to many other integrable dynamic systems, arising in various branches of mechanics and physics. For instance, we may mention the Clebsch case in Kirchhoff's equations, § 9 ch. 3.

### 3. Quaternion Euler – Poisson Equations

Consider the last and the least natural case of equations of motion of a rigid body with a potential, linear not with respect to direction cosines, but with respect to Rodrigue – Hamilton parameters

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \sum_{i=0}^{3} r_i\lambda_i, \quad r_i = \text{const}. \tag{12.17}$$

We suppose that equations of motion have the form (4.24) ch. 1. As we have already mentioned, one will fail to find such kind of potentials in mechanics. It happens because the dependence of such a potential on the body position is ambiguous (it has two values). To substantiate consideration of such equations one can refer to problems of quantum mechanics, of point mass dynamics in a curved space $S^3$ [31], and to some formal techniques of construction of $\mathbf{L} - \mathbf{A}$-pairs [31] (see § 4 ch. 5). It also turns out that reducing the order of system (12.17), we obtain the ordinary Euler – Poisson equations with additional terms. These terms have various physical interpretations (§ 1 ch. 4).

System (12.17) has a curious peculiarity: by means of transformations, linear with respect to $\lambda_i$, the general form of potential

$$V = \sum_{i=0}^{3} r_i\lambda_i \tag{12.18}$$



can be reduced to the form

$$V = r_0\lambda_0. \tag{12.19}$$

Really, linear transformations of quaternion space $\lambda_i$ (which do not change commutative relations and norm of quaternion) of the form

$$\begin{aligned}
\widetilde{\lambda}_0 &= R^{-1}(r_0\lambda_0 + r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3),\\
\widetilde{\lambda}_1 &= R^{-1}(r_0\lambda_1 - r_1\lambda_0 - r_2\lambda_3 + r_3\lambda_2),\\
\widetilde{\lambda}_2 &= R^{-1}(r_0\lambda_2 + r_1\lambda_3 - r_2\lambda_0 - r_3\lambda_1),\\
\widetilde{\lambda}_3 &= R^{-1}(r_0\lambda_3 - r_1\lambda_2 + r_2\lambda_1 - r_3\lambda_0),\\
R^2 &= r_0^2 + r_1^2 + r_2^2 + r_3^2
\end{aligned} \tag{12.20}$$

reduce potential (12.18) to the form (12.19). This linear transformation existence is a remarkable peculiarity of quaternion variables and bracket (4.22) of ch. 1. This transformation does not have any analogues for brackets of algebra $e(3)$ and $so(4)$.

In the general, the dynamically asymmetrical case when $a_1 \neq a_2 \neq a_3 \neq = a_1$, system (12.17) does not seem to be integrable, and there does not exist one of two necessary additional integrals. However, this fact was not proved anywhere, and the proof itself is not natural on the basis of various reasons. It should be noted that even application of the Kowalevskaya method for system (12.17) is not quite analogical to the classical Euler – Poisson problem.

At $a_1 = a_2$ there always exists a linear integral

$$\begin{aligned}
F_1 = M_3(r_0^2 + r_1^2 + r_2^2 + r_3^2) + N_3(r_1^2 + r_2^2 - r_0^2 - r_3^2) +\\
+ 2N_2(r_1r_0 - r_3r_2) - 2N_1(r_1r_2 - r_0r_3),
\end{aligned} \tag{12.21}$$

where $N_i$ are angular momentum projections on fixed axes. Under conditions $r_1 = r_2 = r_3 = 0$ this integral takes the natural form

$$F_1 = M_3 - N_3. \tag{12.22}$$

This (linear) integral corresponds to the cyclic variable $\varphi + \psi$. This fact is considered in every detail in § 1 ch. 4 dedicated to the order reduction. The Routh reduction, carried out with respect to this cyclic variable (for more details see § 1 ch. 4), leads to the Hamiltonian system on an algebra $e(3)$ with a zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ and a Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + a_3M_3^2) + c(a_3 - 1)M_3 + r_0\gamma_2 + \frac{1}{2}\frac{c^2}{\gamma_3^2}, \tag{12.23}$$

where $c$ is a constant of integral (12.22). Hamiltonian (12.23) corresponds to addition of two terms to the ordinary Euler – Poisson equations. These terms are gyrostatic member, linear with respect to $\boldsymbol{M}$, and a singular terms $\dfrac{c^2}{2\gamma_3^2}$, whose physical meaning is discussed in ch. 4.

Let us show integrable cases of system (12.17). They turn out to be equivalent to integrable cases of system (12.23).

**A spherical top ($\boldsymbol{a_1 = a_2 = a_3}$).** A Hamiltonian has the form

$$H = \frac{1}{2}\boldsymbol{M}^2 + r_0\lambda_0,$$

and, as it is shown in [31], the system is equivalent to the problem of motion of a material point on a three-dimensional sphere $S^3$. Due to the dependency of potential on $\lambda_0$ only, we can suppose that the material point is moving in the field of a fixed center placed in the north (south) pole, and the interaction force depends only on the distance from this center (the analogue of the problem about motion in a central field for $\mathbb{R}^3$). Like in the planar case, the vector of angular momentum of a particle is preserved:

$$\boldsymbol{L} = \frac{1}{2}(\boldsymbol{N} - \boldsymbol{M}) = \text{const}, \tag{12.24}$$

where $\boldsymbol{N}$ is a angular momentum vector in the system of fixed axes.

The components of vector $\boldsymbol{L}$ form an algebra $so(3)$: $\{L_i, L_j\} = \varepsilon_{ijk}L_k$. *The integrability is noncommutative*. It is said that such a system is *superintegrable*, and its three-dimensional tori are foliated into two-dimensional ones.

**"The Kowalevskaya case".** A Hamiltonian and an additional integral (of fourth degree) in involution to $F_1$ have the form

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + r_0\lambda_0, \\ F_2 &= (M_1N_1 + M_2N_2 + 2r_0\lambda_0)^2 + (N_1M_2 - N_2M_1 - 2r_0\lambda_3)^2 + \\ &+ (N_3 - M_3)\big(M_3(\boldsymbol{M}^2 - M_3N_3) + 2r_0(M_2\lambda_1 - M_1\lambda_2 + \frac{\lambda_0}{2}(M_3 - N_3))\big). \end{aligned} \tag{12.25}$$

Under reduction to system (12.23) we obtain an integrable case which can be included into the generalized Kowalevskaya family found by Goryachev and Yehia (see § 7 ch. 5, and also § 1 ch. 4).



**"The Goryachev – Chaplygin case".** A Hamiltonian and an additional integral are written as

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + r_0\lambda_0, \\ F_2 &= M_3(M_1^2 + M_2^2) + r_0(M_2\lambda_1 - M_1\lambda_2). \end{aligned} \tag{12.26}$$

Under reduction to system (12.23) this case is included into the generalized family shown in § 1 ch. 4.

Remark 7. There is a somewhat unexpected fact that the Lagrange and Hess cases are not generalized to system (12.17).

Remark 8. If in (12.25) and (12.26) we add a constant gyrostatic moment along the dynamical symmetry axis, we obtain integrable cases, corresponding to the generalized cases of Yehia and Sretenskiy in Euler – Poisson equations. The integrals for these cases are easily obtained from (12.23) by means of the lifting technique described in ch. 4.

In conclusion, it should be noted that for the quaternion Euler – Poisson equations both "the Kowalevskaya case" and "the Goryachev – Chaplygin case" are general cases of integrability. Thus, we are allowed to use them for some algebraic constructions (**L** – **A**-pair construction, etc.) and for establishing certain nontrivial interrelations and analogues for the corresponding integrable cases in classical Euler – Poisson equations (§ 7 ch. 5).

## Chapter 4
# Cyclic Integrals and Order Reduction

## § 1. Linear Integrals in Rigid Body Dynamics

This chapter is dedicated to the questions of existence of first integrals for various forms of equations of motion of a rigid body, considered in § 4 ch. 1. The first integrals are linear with respect to moments $\boldsymbol{M}$ (or, which is equivalent, with respect to angular velocities $\boldsymbol{\omega}$, generalized momenta $p_\theta, p_\varphi, p_\psi$, etc.). As it is known from Hamiltonian mechanics [6, 8], linear integrals are connected with the presence of cyclic variable and the possibility of order reduction. For the canonical and Lagrangian forms of notation the order reduction technique was developed by E. Routh (and is often referred to as *the Routh reduction*). In the book [31] we offered a more specialized algorithm of reduction in the presence of linear integrals. It allows to escape the canonical form in the reduction process and preserves the algebraical form of equations of motion. Moreover, in the reduced system of variables not only Hamiltonian changes its form, but the Poisson bracket does the same thing. The latter can become nonlinear. In some cases, shown below, the reduced system turns out to be equivalent to absolutely different system, as it may seem. It means that here we have a certain method of finding isomorphic problems in dynamics. This method is also transferred to the corresponding integrable problems.

In this section we state several theorems about order reduction for three various typical linear integrals and corresponding cyclic variables. Further, we concentrate our efforts on the inverse technique. It is connected with the transportation of results for the reduced system to general equations. By means of this scheme from integrable families for the reduced system (with two degrees of freedom) we can obtain integrable cases of more general equations of motion of a rigid body in a potential field (see § 12 ch. 3), i. e., for the system with three degrees of freedom. Besides, using this technique we can understand the meaning of different additions of singular character, like $\frac{a}{\gamma_3^2}$, $a = \mathrm{const}$, in generalizations of integrable cases. They were introduced by D. N. Goryachev while investigating and generalizing the cases of Goryachev – Chaplygin and Kowalevskaya. Actually, for the long time their mechanical meaning remained



unclear in spite of some "quantum mechanical" explanations. In the paper [31] they were interpreted as the reduction results. Except for this chapter, the related questions can be looked up in ch. 3 (§ 12), ch. 5 (§ 7).

It should be noted that linear integrals in general equations of rigid body dynamics around a fixed point were studied by D. N. Goryachev in the paper [62]. The paper contains three typical possibilities considered below. In a certain sense, they are the only possibilities (the proof of the latter statement does not seem to be easy). In § 3, § 4 corresponding reductions are applied to linear invariant relations introduced into dynamics by T. Levi-Civita. He also tried to use them in rigid body dynamics (together with celestial mechanics) [113]. However, Levi-Cevita's ideas are developed most explicitly when we consider invariant relations of the Hess type. It turns out that such relations exist for many related problems of rigid body dynamics. In this case there also exists a certain cyclic variable; the order reduction is possible, and we are presented with the analogy of the Lagrange case and its generalizations. In particular, this analogy gives some qualitative peculiarities of motion of the generalized Hess cases, typical for motion of a heavy symmetrical gyroscope. (For instance, it is the observation of N. E. Joukovskiy concerning the center-of-mass motion according to the spherical pendulum law in the Hess case.)

It is a well known fact that first integrals exist in the presence of a certain field of symmetries and under the possibility of order reduction, at least, a local one. It is a famous *Noether theorem*, whose using for Hamiltonian systems with integrals, linear with respect to momenta, requires some simplifications. For simplicity we consider the canonical case, though the reasoning can easily be applied to the general Poincaré – Chetayev equations, and, in particular, to the equations of rigid body dynamics in matrix realizations of Lie groups (specifying configurational spaces).

*Really*, for systems on cotangent foliation $TM$ with canonical structure $\{q_i, p_j\} = \delta_{ij}$ the presence of integral

$$F = \sum_i v_i(\boldsymbol{q}) p_i, \qquad \{F, H\} = 0, \tag{1.1}$$

linear with respect to momenta, leads to the phase flow given by Hamiltonian $F$

$$\frac{d\boldsymbol{q}}{ds} = \frac{\partial F}{\partial \boldsymbol{p}} = v(\boldsymbol{q}), \qquad \frac{d\boldsymbol{p}}{ds} = -\frac{\partial F}{\partial \boldsymbol{q}} \tag{1.2}$$

and defining the action of a single-parametric group of symmetries of Hamiltonian $H$. At that, the system of equations on the configurational space $M$

becomes separated

$$\frac{d\boldsymbol{q}}{ds} = \boldsymbol{v}(\boldsymbol{q}). \tag{1.3}$$

In the vicinity of a nonspecial point, field (1.3) can be rectified and represented in terms of some coordinates $Q_1, \ldots, Q_{n-1}, Q_n$ in the form

$$\frac{dQ_1}{ds} = \ldots = \frac{dQ_{n-1}}{ds} = 0, \quad \frac{dQ_n}{ds} = 1.$$

It is evident that canonical momentum $P_n$, corresponding to the coordinate $Q_n$, coincides with integral (1.1) $F = P_n$, and due to the relation $\{H, P_n\} = \dfrac{\partial H}{\partial Q_n} = 0$, the coordinate $Q_n$ is a cyclic one. It means that the order reduction is achieved. ■

Order reductions described below are carried out in the global and algebraical way, and we act accordingly to the almost analogical scheme. Having linear integral, we can write systems (1.2) and (1.3). Since system (1.3) becomes separated, it is easy to show its first integrals and integrals of associated system (1.3), as well.

Further on, *we are guided by the idea of using this set of integrals (which is usually redundant) as new variables for the initial system*. If the algebra of new variables with respect to Poisson brackets is closed (but nonlinear) and the Hamiltonian is expressed in terms of these variables only, we obtain a new Hamiltonian system. For this system cyclic integral (1.1) is a Casimir function, the rank of Poisson brackets is decreased by 2, i. e., the system is reduced. The advantages of this procedure of reduction, preserving the system algebraicity and its various dynamical applications, are described in our book [31]. Here we shall only dwell on its using in rigid body dynamics in three different variants described by theorems given below.

Consider motion of a rigid body around a fixed point in a generalized potential field. In such a field, except for potential forces, there also exist the gyroscopic ones. The latter are described by vector potential and result in terms, linear with respect to $\boldsymbol{M}$, in Hamiltonian

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}) + U, \tag{1.4}$$

where functions $\boldsymbol{U}$, $\boldsymbol{W} = (W_1, W_2, W_3)$, defining generalized potential, are supposed to be dependent on all variables $\boldsymbol{q}$, specifying the rigid body position. These can be the Euler angles $\theta, \varphi, \psi$, direction cosines $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and Rodrigue–Hamilton parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. Depending on a system of variables, we use a corresponding system of equations, defining motion (see § 3 ch. 1). For the higher degree of generality we also suppose that $\mathbf{A} = \mathbf{A}(\boldsymbol{q})$. This condition is necessary to the study sliding of a body on a plane and a gyroscope in a gimbal. Here and henceforward, $\boldsymbol{N} = (N_1, N_2, N_3)$ are projections of the angular momentum vector on fixed axes.



### 1. Classical Area Integral $N_3 = (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \mathrm{const}$

Symmetries, giving such an integral, are natural; they are connected with invariance of the generalized potential with respect to rotations around some fixed axis. Such axially symmetric fields include uniform ones, in particular, a gravity field.

The precession angle $\psi$ is a cyclic variable, and equations of motion can be represented on algebra $e(3)$.

In this case Hamiltonian (1.4) can be written if we choose variables $M_1, M_2, M_3$ and $\gamma_1, \gamma_2, \gamma_3$, defining a field symmetry unit vector in a fixed space, as basic elements

$$\begin{aligned} H &= \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}(\theta, \varphi)) + U(\theta, \varphi) = \\ &= \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}(\boldsymbol{\gamma})) + U(\boldsymbol{\gamma}), \end{aligned} \tag{1.5}$$

where

$$\gamma_1 = 2(\lambda_1\lambda_3 - \lambda_0\lambda_2), \quad \gamma_2 = 2(\lambda_0\lambda_1 + \lambda_2\lambda_3), \quad \gamma_3 = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2.$$

The Poisson bracket in terms of variables $(\boldsymbol{M}, \boldsymbol{\gamma})$ is given by an algebra $e(3)$ (see § 1 ch. 2). A symplectic leave of the algebra $e(3)$: $\{\boldsymbol{\gamma}^2 = 1, (\boldsymbol{M}, \boldsymbol{\gamma}) = c\}$ is diffeomorphic to a cotangent foliation to a two-dimensional sphere $S^2 = \{\boldsymbol{\gamma}^2 = 1\}$. This sphere is a configurational space of the reduced (with respect to $\psi$) system and is referred to as *the Poisson sphere.*

At $c \neq 0$, under order reduction there appear additional gyroscopic terms with a certain singularity. This singularity can be interpreted as *a monopole*. In the paper [133] the author regards the monopole introduction as a noncanonical distortion of the Poisson bracket.

Here we shall carry out the reduction to a zero area constant in the algebraic form without changing the bracket. In this case the singularity, corresponding to a monopole, will appear only in the Hamiltonian.

**Theorem 4.** *Equations of motion of a body with Hamiltonian* (1.5) *on the level of integral* $N_3 = c$ *are equivalent to the Hamilton equations on* $e(3)$ *on a* *zero area constant* $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ *with a Hamiltonian*

$$\begin{aligned} H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}) + c\frac{a_1\gamma_1 M_1 + a_2\gamma_2 M_2}{\gamma_1^2+\gamma_2^2} + \\ + \frac{c^2}{2}\frac{a_1\gamma_1^2 + a_2\gamma_2^2}{(\gamma_1^2+\gamma_2^2)^2} + \frac{c(W_1\gamma_1 + W_2\gamma_2)}{\gamma_1^2+\gamma_2^2} + U(\boldsymbol{\gamma}). \end{aligned} \tag{1.6}$$

Proof.

It is sufficient to carry out a transportation $(\boldsymbol{M}, \boldsymbol{\gamma}) \to (\boldsymbol{M}, \boldsymbol{\gamma})$. It *preserves a structure of algebra* $e(3)$ and transfers integral $(\boldsymbol{M}, \boldsymbol{\gamma}) = c$ into $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$. This transformation is written as

$$\begin{gathered} M_1 \to M_1 - c\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}, \quad M_2 \to M_2 - c\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}, \quad M_3 \to M_3, \\ \boldsymbol{\gamma} \to \boldsymbol{\gamma}. \end{gathered} \tag{1.7}$$

■

Remark 1. At $c = 0$ and $W \equiv 0$, the reduced system is natural, there is no monopole.

Remark 2. For the first time, we have shown transformation (1.7) in the book [31]. We have been trying to improve the transformation $\boldsymbol{M} \to \boldsymbol{M} + c\boldsymbol{\gamma}$, $c = (\boldsymbol{M}, \boldsymbol{\gamma})$, applied in [133] for reduction to a zero area constant. This transformation did not preserve the structure of $e(3)$.

The proved statement has a dynamical meaning: unlike classical and well known local reduction of order by means of the Routh technique with respect to the precession angle, we obtain all the necessary terms, arising under reduction, in the algebraic form. Due to $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, the reduced system with two degrees of freedom defines motion of a certain representing point (specified by a vertical unit vector) on the Poisson sphere in a generalized potential field (even at $\boldsymbol{W} \equiv 0$, and $c \neq 0$), in a metric determined by the form of kinetic energy. To carry out transition to canonical variables in Hamiltonian (1.6) one should make the following substitution

$$\begin{gathered} M_1 = -p_\varphi \operatorname{ctg}\theta \sin\varphi + p_\theta \cos\varphi, \ M_2 = -p_\varphi \operatorname{ctg}\theta\cos\varphi - p_\theta \sin\varphi, \ M_3 = p_\varphi, \\ \gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta, \end{gathered}$$

for which $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, and $p_\theta$, $p_\varphi$, $\theta$, $\varphi$ are related via canonical rules of commutation. $\theta$ and $\varphi$ represent spherical coordinates on the Poisson sphere. Unlike canonical form of notation, the algebraic form (1.6) and its analogues for other cyclic variables allow to notice various analogies among problems, show connection between integrable cases, get a deeper understanding of algebraic nature of the corresponding first integrals.



Remark 3. At $a_1 = a_2$, Hamiltonian (1.6) is simplified (with taking into account $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$)

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}) + U(\boldsymbol{\gamma}) - \frac{cM_3\gamma_3}{\gamma_1^2+\gamma_2^2} + \frac{c^2}{2(\gamma_1^2+\gamma_2^2)} + \frac{c(W_1\gamma_1 + W_2\gamma_2)}{\gamma_1^2+\gamma_2^2}. \tag{1.8}$$

## 2. Integral $N_3 - M_3 = (\boldsymbol{M}, \boldsymbol{\gamma}) - M_3 = c = \mathrm{const}$

This integral corresponds to the cyclic variable $\psi - \varphi$ (analogically, we can consider $N_3 + M_3$ $\psi + \varphi$); for the first time it was investigated by D. N. Goryachev [62]. Corresponding symmetries are not quite physically natural any longer and are connected with both the force field in space and dynamical characteristics of a rigid body. This body is not dynamically symmetrical. In this case (1.4) Hamiltonian can be written as

$$\begin{aligned} H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + 2\frac{M_1\lambda_1 + M_2\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}W_1(\lambda_0, \lambda_3) + \\ + 2\frac{M_2\lambda_1 - M_1\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}W_2(\lambda_0, \lambda_3) - 2M_3W_3(\lambda_0, \lambda_3) + U(\theta, \varphi + \psi), \end{aligned} \tag{1.9}$$

where $a$ is a certain arbitrary constant. Introduce a new system of variables

$$\begin{gathered} K_1 = 2\frac{M_1\lambda_1 + M_2\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}, \quad K_2 = 2\frac{M_2\lambda_1 - M_1\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}, \quad K_3 = -2M_3, \\ s_1 = \lambda_3, \quad s_2 = \lambda_0, \quad s_3 = \sqrt{\lambda_1^2+\lambda_2^2}, \end{gathered} \tag{1.10}$$

commutating in the following way

$$\begin{gathered} \{K_3, K_1\} = K_2, \qquad \{K_2, K_3\} = K_1, \qquad \{K_1, K_2\} = K_3 + \frac{s_3(\boldsymbol{s}, \boldsymbol{K})}{s_3^2}, \\ \{K_i, s_j\} = \varepsilon_{ijk}s_k, \qquad \{s_i, s_j\} = 0 \end{gathered} \tag{1.11}$$

and forming a closed algebra with respect to the nonlinear Poisson bracket. This bracket is born by relations (1.11). It is degenerate and possesses the Casimir functions

$$F_1 = s_3(\boldsymbol{s}, \boldsymbol{K}) = (\boldsymbol{M}, \boldsymbol{\gamma}) - M_3 = c, \qquad F_2 = (\boldsymbol{s}, \boldsymbol{s}) = 1.$$

One-parametric transformation

$$\boldsymbol{L} = \boldsymbol{K} - \alpha\frac{\boldsymbol{s}}{s_3}, \quad \alpha = \mathrm{const} \tag{1.12}$$

preserves bracket (1.11). Moreover,

$$s_3(\boldsymbol{L}, \boldsymbol{s}) = s_3(\boldsymbol{K}, \boldsymbol{s}) - \alpha. \tag{1.13}$$

Remark 4. Transformation (1.13), like system (1.10), was also shown in our book [31] and in our paper [30].

If we fix integral $(\boldsymbol{K}, \boldsymbol{s})s_3 = c$ and choose $\alpha = c$, due to $s_3(\boldsymbol{L}, \boldsymbol{s}) = 0$, this will result in dissipation of nonlinear terms in bracket (1.11), presently defined by algebra $e(3)$. Then Hamiltonian (1.11) will have the form

$$\begin{gathered} H = \frac{1}{8}(L_1^2 + L_2^2 + aL_3^2) + (\boldsymbol{L}, \boldsymbol{W}(\boldsymbol{s})) + U(\boldsymbol{s}) + \\ + c\frac{(\boldsymbol{s}, \boldsymbol{W}(\boldsymbol{s}))}{s_3^2} + \frac{1}{2}\frac{c^2}{s_3^2} + c(a-1)L_3. \end{gathered} \tag{1.14}$$

This is the way to prove

**Theorem 5.** *System with three degrees of freedom* (1.9) *on the level of cyclic integral* $N_3 - M_3 = c$ *under the order reduction transforms into system on* $e(3)$ *with a zero area constant* $(\boldsymbol{L}, \boldsymbol{s}) = 0$ *and Hamiltonian function* (1.14)*.*

It should be noted that under this reduction Hamiltonian (1.14) at $c \neq 0$ obtains additional terms. One of these can be interpreted as a gyrostatic moment directed along the dynamical symmetry axis, another as a singular term introduced in dynamics by D. N. Goryachev [63, 64].

If in rigid body dynamics the origin of integral $F = N_3 \pm M_3$ from symmetries is not evident, its meaning is easily understood from the analogy of celestial mechanics of curved space, to be more precise, of motion of a material point on spheres $S^2$, $S^3$ (see § 11 ch. 5). This integral exactly corresponds to the projection of angular momentum of the particle on a fixed axis, in whose respect the potential preserves axial symmetry.

### 3. Integral $M_3 = c = \mathrm{const}$ (the Lagrangian integral)

In this case we deal with a dynamically symmetrical body, and a force field, invariant with respect to the dynamical symmetry axis. A corresponding cyclic variable is the angle. The Hamiltonian is written in terms of direction cosines $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$

$$\begin{gathered} H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + (M_1\alpha_1 + M_2\alpha_2)W_1^{(\alpha)}(\theta, \psi) + \\ + (M_1\alpha_2 - M_2\alpha_1)W_2^{(\alpha)}(\theta, \psi) + \ldots + M_3W_3(\theta, \psi) + U(\theta, \psi), \end{gathered} \tag{1.15}$$



where we omit terms in $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, linear with respect to $M_1$, $M_2$. Introduce new variables, commutating with $M_3$ and specifying a reduced system

$$\begin{gathered} N_1 = (\boldsymbol{M}, \boldsymbol{\alpha}), \qquad N_2 = (\boldsymbol{M}, \boldsymbol{\beta}), \qquad N_3 = (\boldsymbol{M}, \boldsymbol{\gamma}), \\ \boldsymbol{p} = (\alpha_3, \beta_3, \gamma_3). \end{gathered} \tag{1.16}$$

The geometrical meaning of these variables is evident: vector $\boldsymbol{N}$ is composed of angular momentum components in a fixed frame of reference, and $\boldsymbol{p}$ are symmetry axis vector components in the same frame of reference.

Commutating relations for basic elements (1.16) correspond to algebra $e(3)$ (see § 3 ch. 1). Its Casimir's function has the form

$$F_1 = \boldsymbol{p}^2 = 1, \qquad F_2 = (\boldsymbol{N}, \boldsymbol{p}) = \boldsymbol{p}^2 M_3 = c.$$

The terms, linear with respect to $\boldsymbol{M}$, in Hamiltonian (1.15) can be determined from the following relations

$$\begin{aligned} &M_1\alpha_1 + M_2\alpha_2 = N_1 - p_1M_3, \quad M_1\alpha_2 - M_2\alpha_1 = p_2N_3 - p_3N_2, \\ &M_1\beta_1 + M_2\beta_2 = N_2 - p_2M_3, \quad M_1\beta_2 - M_2\beta_1 = p_3N_1 - p_1N_3, \\ &M_1\gamma_1 + M_2\gamma_2 = N_3 - p_3M_3, \quad M_1\gamma_2 - M_2\gamma_1 = p_1N_2 - p_2N_1. \end{aligned}$$

Eliminating constant terms, we can write the Hamiltonian of the reduced system as follows

$$\begin{gathered} H = \frac{1}{2}\boldsymbol{N}^2 + \big(\boldsymbol{N}, \boldsymbol{W}^{(1)}\big) + \big(\boldsymbol{p} \times \boldsymbol{N}, \boldsymbol{W}^{(2)}\big) + \\ + c\big(W_3(\boldsymbol{p}) - \big(\boldsymbol{p}, \boldsymbol{W}^{(1)}\big)\big) + U(\boldsymbol{p}), \\ \boldsymbol{W}^{(1)}(\boldsymbol{p}) = (W_1^{(\alpha)}, W_1^{(\beta)}, W_1^{(\gamma)}), \quad \boldsymbol{W}^{(2)}(\boldsymbol{p}) = (W_2^{(\alpha)}, W_2^{(\beta)}, W_2^{(\gamma)}). \end{gathered} \tag{1.17}$$

The reduction with respect to integral $M_3 = \mathrm{const}$ and variables (1.16) were already used in § 12 ch. 3 to establish the relation between the Brun problem under the dynamical symmetry condition and the integrable Clebsch case of Kirchhoff's equations.

### 4. Lifting of Integrable Systems

Of the greatest interest is an inverse problem: the obtaining of new integrable cases of system (1.4) with three degrees of freedom from the available integrable cases of Hamiltonian equations on $e(3)$, defining the reduced system with two degrees of freedom. Here we'll also show how the integrable systems on a zero area constant $(\boldsymbol{L}, \boldsymbol{s}) = 0$ of an algebra $e(3)$ can be lifted to general integrable system (1.4), possessing linear integral $M_3 \pm N_3$. At first, let us formulate one general result. It can be proved by means of direct examination.

**Theorem 6.** 1. *Let an integrable (at $(\boldsymbol{L}, \boldsymbol{s}) = 0$) system with the Hamiltonian function*

$$H = \frac{1}{2}(L_1^2 + L_2^2 + aL_3^2) + (\boldsymbol{L}, \boldsymbol{W}) + U(\boldsymbol{s})$$

*be given on an algebra $e(3)$ (i. e., there exists a particular additional first integral).*

*Then, by means of transformation*

$$\boldsymbol{L} = \boldsymbol{K} - \alpha\frac{\boldsymbol{s}}{s_3} \qquad (1.18)$$

*and substitution* (1.10)*, we obtain the system on quaternion algebra of brackets of variables $\boldsymbol{M}, \lambda_0, \lambda_1, \lambda_2, \lambda_3$ (§3 ch. 1 formula (3.11)). This system is integrable on the fixed level of integral $N_3 - M_3 = \alpha$ with the Hamiltonian function*

$$H' = H - \alpha\frac{\lambda_3 W_1 + \lambda_0 W_2 + \sqrt{\lambda_1^2 + \lambda_2^2}W_3}{\lambda_1^2 + \lambda_2^2} - \frac{1}{2}\frac{\alpha^2}{\lambda_1^2 + \lambda_2^2} + \alpha(a-1)M_3. \quad (1.19)$$

2. *If constants of the Hamiltonian $H$ can be chosen in such a way that $H'$ does not depend on $\alpha$, then system* (1.19) *is integrable at an arbitrary value of the linear integral $N_3 - M_3$. In the additional integral after transformations* (1.18) *and substitution* (1.10) *we need to assume that $\alpha = (M_1\lambda_1 + M_2\lambda_2 + M_1\lambda_2 - M_2\lambda_1 - \sqrt{\lambda_1^2 + \lambda_2^2}M_3)$.*

Remark. The integrable cases can be analogically lifted by means of linear integrals $M_3 = \text{const}$ and $N_3 = \text{const}$. However, the obtained generalizations of integrable cases contain terms, linear with respect to velocities. These terms do not have direct physical interpretation.

Let us consider two examples, illustrating Theorem 6.

**Generalization of the Yehia – Kowalevskaya family.** The paper [285] by H. Yehia contains the particular case of integrability $(\boldsymbol{L}, \boldsymbol{s}) = 0$, generalizing the Kowalevskaya case with a Hamiltonian

$$\begin{gathered}H = \frac{1}{2}\left(L_1^2 + L_2^2 + 2(L_3 + \xi)^2\right) + \\ + a\frac{\boldsymbol{s}^2}{s_3^2} + c_1 s_1 + c_2 s_2 + 2b_1 s_1 s_2 + b_2(s_2^2 - s_1^2).\end{gathered} \qquad (1.20)$$



and an additional integral, reduced in §7 ch. 5, see formula (7.4). After the transformation

$$\boldsymbol{L} = \boldsymbol{K} - \alpha\frac{\boldsymbol{s}}{s_3}$$

we obtain the Hamiltonian in the form

$$\begin{gathered}H = \frac{1}{2}(K_1^2 + K_2^2 + 2K_3^2) + c_1 s_1 + c_2 s_2 + 2b_1 s_1 s_1 + b_2(s_2^2 - s_1^2) + \\ + (2\xi - \alpha)K_3 + \frac{1}{2}\frac{2a\boldsymbol{s}^2 - 2\alpha s_3(\boldsymbol{K}, \boldsymbol{s}) + \alpha^2\boldsymbol{s}^2}{s_3^2}.\end{gathered} \qquad (1.21)$$

This Hamiltonian defines an integrable system with linear bracket (1.11) on a symplectic leave specified by the relation

$$s_3(\boldsymbol{K}, \boldsymbol{s}) = \alpha, \qquad \boldsymbol{s}^2 = 1.$$

Besides, since the structure of Hamiltonian (1.21) did not change (the least term nominator contains the Casimir function, equivalent to a constant), we can conclude that Hamiltonian (1.20) define the general case of integrability on nonlinear bracket (1.11). To obtain the integral on an arbitrary leave, let us redetermine constants in the Hamiltonian and in the integral according to the rule

$$\xi \to \xi + \frac{\alpha}{2}, \qquad a \to a + \frac{\alpha^2}{2}, \qquad (1.22)$$

and assume that

$$\alpha = \frac{s_3(\boldsymbol{K}, \boldsymbol{s})}{\boldsymbol{s}^2}.$$

This results in an additional integral of the form

$$\begin{gathered}F_2 = \left(K_1^2 - K_2^2 - 2\frac{K_1 s_1 - K_2 s_2}{s_3}F_0 - 2a\frac{s_1^2 - s_2^2}{s_3^2} - 2c_1 s_1 + 2c_2 s_2 - 2b_2 s_3^2\right)^2 + \\ + 4\left(K_1 K_2 - F_0\frac{M_1 s_2 + M_2 s_1}{s_3} - 2a\frac{s_1 s_2}{s_3^2} - c_1 s_2 - c_2 s_1 + b_1 s_3^2\right)^2 + \\ + 4(2\xi + F_0)\Big[-(K_3 + 2\xi)\Big(K_1^2 + K_2^2 + 2F_0 K_3 + 2a\frac{s_1^2 + s_2^2 + 2s_3^2}{s_3^2}\Big) - \\ - 2F_0(c_1 s_1 + c_2 s_2 + 2b_1 s_1 s_2 - b_2(s_1^2 - s_2^2)) + \\ + 2s_3(c_1 K_1 + c_2 K_2 + b_1(K_1 s_2 + K_2 s_1) - b_2(K_1 s_1 - K_2 s_2))\Big], \\ F_0 = \frac{s_3(\boldsymbol{K}, \boldsymbol{s})}{\boldsymbol{s}^2}.\end{gathered} \qquad (1.23)$$

This integral is already general.

Substituting the expressions $(\boldsymbol{K},\ \boldsymbol{s})$ in terms of moments $\boldsymbol{M}$ and quaternion parameters $\lambda$ (1.10) in (1.20) and (1.23), we obtain a general integrable system on a quaternion bracket in terms of variables $\boldsymbol{M},\ \lambda$ with a linear integral $F_1 = (\boldsymbol{M},\ \boldsymbol{\gamma}) - M_3$ and fourth degree linear integral (1.23). The particular case of system (1.20) and corresponding integral (1.23) in a simplified form are given in § 4 ch. 4.

Remark 5. Integral (1.23) can be represented in terms of direction cosines by means of relations

$$\begin{gathered}\frac{K_1 s_1 - K_2 s_2}{s_3} = 2\frac{M_1\alpha_3 + M_2\beta_3}{1-\gamma_3}, \quad \frac{K_1 s_2 + K_2 s_1}{s_3} = 2\frac{M_1\beta_3 - M_2\alpha_3}{1-\gamma_3},\\ K_1^2 - K_2^2 = \frac{(\alpha_1-\beta_2)(M_1^2 - M_2^2) + 2(\alpha_2+\beta_1)M_1M_2}{1-\gamma_3},\\ 2K_1K_2 = \frac{(\alpha_2+\beta_1)(M_1^2 - M_2^2) - 2(\alpha_1-\beta_2)M_1M_2}{1-\gamma_3}.\end{gathered} \tag{1.24}$$

**The generalized Goryachev – Chaplygin family.** Consider the analogical generalization of the integrable Goryachev – Chaplygin case on a zero leave with singular terms [63] (see § 7 ch. 5). A Hamiltonian has the form

$$H = \frac{1}{2}(L_1^2 + L_2^2 + 4L_3^2) + \xi L_3 + \frac{a\boldsymbol{s}^2}{s_3^2} + b_1 s_1 + b_2 s_2. \tag{1.25}$$

After transformations $\boldsymbol{L} = \boldsymbol{K} - \alpha\frac{\boldsymbol{s}}{s_3}$, $\alpha = \mathrm{const}$ and elimination of unessential constants we shall obtain a transformed Hamiltonian of the form

$$\begin{gathered}H = \frac{1}{2}(K_1^2 + K_2^2 + 4K_3^2) + b_1 s_1 + b_2 s_2 +\\ +(\xi - 3\alpha)K_3 + \frac{1}{2}\frac{2a\boldsymbol{s}^2 - 2\alpha s_3(\boldsymbol{K},\ \boldsymbol{s}) + \alpha^2\boldsymbol{s}^2}{s_3^2}.\end{gathered} \tag{1.26}$$

Analogically to the previous case, we obtain that system (1.25) defines the general case of integrability on a nonlinear (and, correspondingly, quaternion in terms of variables $\boldsymbol{M},\ \lambda$) bracket with the third degree integral of the form

$$F = (K_3 + \frac{1}{2}\xi)\left(K_1^2 + K_2^2 + 2a\frac{\boldsymbol{s}^2}{s_3^2}\right) - s_3(b_1K_1 + b_2K_2).$$



It is interesting to note that the integral does not change its form comparing to the form for the algebra $e(3)$ (see § 5 ch. 2).

In conclusion, we should say that the reducing and lifting techniques, described in this section, are used in § 12 ch. 3 and § 4 ch. 5 for the analysis of the quaternion Euler – Poisson equations and their integrable cases.

## § 2. Dynamical Symmetry and Lagrange's Integral

In this section we shall give a unique consideration to dynamical problems, possessing the analogue of Lagrange's integral, existing in the Euler – Poisson equations. It should be recalled that it was connected with the presence of two cyclic coordinates: $\psi$-angle of precession and $\varphi$-angle of proper rotation. The latter coordinate conditioned the presence of Lagrange's integral $M_3 = \mathrm{const}$, $\omega_3 = \mathrm{const}$ and preservation of the projection of angular velocity and angular momentum on the axis of dynamical symmetry. This integral is connected with the system invariant with respect to rotations about the dynamical symmetry axis.

It turns out that the integral of Lagrange's type exists for almost all theoretically interesting problems of rigid body dynamics. Moreover, the presence of this integral results in the integrability of cases which, as a rule, have great applied meaning. For example, the Lagrange case analogue for Kirchhoff's equations was shown by Kirchhoff who also integrated it and indicated the simplest motions. For the Poincaré – Joukovskiy equations (on $so(4)$) the Lagrange case analogue was shown by Poincaré. He wanted to substantiate his theoretical conclusions concerning the precession of the Earth rotation axis. As in these cases, so in the classical Lagrange problem, we can obtain an explicit (elliptic) quadrature for the nutation angle $\theta$. This quadrature is determined by a gyroscopic function. We can also use all the results of qualitative analysis of motion, given in § 3 ch. 2.

### 1. An Explicit Quadrature of the Generalized Lagrange Case. The Conditions of Integral Existence

Here we are going to present an explicit quadrature for the Lagrange case in the most general form. We suppose that the rigid body motion is described by the Hamiltonian

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M},\ \boldsymbol{M}) + (\boldsymbol{M},\ \boldsymbol{W}(\boldsymbol{\gamma})) + U(\boldsymbol{\gamma}), \tag{2.1}$$

where $\mathbf{A}$ is a constant, but not necessarily diagonal matrix. We also assume that we are given the Poisson structure defined by the bundle $\mathscr{L}_x$

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k,\ \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k,\ \{\gamma_i, \gamma_j\} = -\varepsilon_{ijk}xM_k, \quad (2.2)$$

where $x$ is a bundle parameter. Further, we shall consider corresponding conditions for the more general situation $\mathbf{A} = \mathbf{A}(\boldsymbol{\gamma})$, when the body slides on the plane or moves in a gimbal.

By means of explicit computations we can prove the validity of the following statement.

**Theorem 7.** *System* (2.1) *with bracket* (2.2) *allows linear integral of the form*

$$F = M_3 = c, \quad c = \text{const}, \quad (2.3)$$

*if the following conditions*

$$\begin{gathered}\mathbf{A} = \operatorname{diag}(a_1,\, a_1,\, a_3),\\ U(\boldsymbol{\gamma}) = U\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right), \qquad W_3(\boldsymbol{\gamma}) = W_3\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right),\\ \gamma_1\frac{\partial W_1}{\partial \gamma_2} - \gamma_2\frac{\partial W_1}{\partial \gamma_1} + W_2 = 0, \qquad \gamma_1\frac{\partial W_2}{\partial \gamma_2} - \gamma_2\frac{\partial W_2}{\partial \gamma_1} - W_1 = 0\end{gathered} \quad (2.4)$$

*are satisfied.*

Hamiltonian (2.1) under conditions (2.4) can be written as

$$\begin{gathered}H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + M_3W_3\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right) + U\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right) +\\ + \frac{M_1\gamma_1 + M_2\gamma_2}{\sqrt{\gamma_1^2+\gamma_2^2}}\widetilde{W}_1\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right) + \frac{M_1\gamma_2 - M_2\gamma_1}{\sqrt{\gamma_1^2+\gamma_2^2}}\widetilde{W}_2\left(\gamma_3, \sqrt{\gamma_1^2+\gamma_2^2}\right).\end{gathered} \quad (2.5)$$

System (2.5) on the level $M_3 = c$ in the general algebraic form can be reduced to a system with a single degree of freedom. Let us show this reduction in the explicit form.

The corresponding reduced variables are represented as

$$\begin{gathered}K_1 = \frac{M_1\gamma_1 + M_2\gamma_2}{\sqrt{\gamma_1^2+\gamma_2^2}}, \qquad K_2 = \frac{M_1\gamma_2 - M_2\gamma_1}{\sqrt{\gamma_1^2+\gamma_2^2}},\\ \sigma_1 = \sqrt{\gamma_1^2+\gamma_2^2}, \qquad \sigma_2 = \gamma_3.\end{gathered} \quad (2.6)$$



It is evident that $M_1^2 + M_2^2 = K_1^2 + K_2^2$. We should note that the analogical system of variables was used by Poincaré when he investigated the integrable case (shown by him) in the Poincaré – Joukovskiy equations.

The Poisson structure for variables (2.6) has the form

$$\begin{aligned}&\{K_1,\, K_2\} = -c + K_1\frac{\sigma_2}{\sigma_1}, && \{\sigma_1,\, \sigma_2\} = xK_2,\\ &\{K_1,\, \sigma_1\} = xc\frac{K_2}{\sigma_1}, && \{K_1,\, \sigma_2\} = -x\frac{K_1K_2}{\sigma_1},\\ &\{K_2,\, \sigma_1\} = -\sigma_2 - xc\frac{K_1}{\sigma_1}, && \{K_2,\, \sigma_2\} = -\sigma_1 + x\frac{K_1^2}{\sigma_1},\end{aligned} \quad (2.7)$$

all other brackets equal zero. The rank of bracket (2.7) equals two. Its Casimir's functions are

$$F_1 = \boldsymbol{\sigma}^2 + x\boldsymbol{K}^2 + xc = c_1, \qquad F_2 = K_1\sigma_1 + c\sigma_2 = c_2,$$

where $\boldsymbol{K} = (K_1,\, K_2)$, $\boldsymbol{\sigma} = (\sigma_1,\, \sigma_2)$, $\widetilde{\boldsymbol{W}} = (\widetilde{W}_1,\, \widetilde{W}_2)$, and the Hamiltonian is written as

$$H = \frac{1}{2}\boldsymbol{K}^2 + (\boldsymbol{K},\ \widetilde{\boldsymbol{W}}(\boldsymbol{\sigma})) + U(\boldsymbol{\sigma}) + cW_3(\boldsymbol{\sigma}). \quad (2.8)$$

This system with one degree of freedom is easily reduced to quadratures. In fact, on the level of Casimir's functions and the integral of energy $H = h$ we obtain

$$\begin{gathered}\dot\sigma_2 = K_2\left(\sigma_1 - x\frac{\partial U_*}{\partial \sigma_1}\right), \qquad U_*(\boldsymbol{\sigma}) = U(\boldsymbol{\sigma}) + cW_3(\boldsymbol{\sigma}),\\ K_2^2 = 2(h - U_*) - \left(\frac{c_2 - c\sigma_2}{\sigma_1}\right)^2.\end{gathered} \quad (2.9)$$

Eliminating $\sigma_1$ from combined equation for energy (2.8) and Casimir's functions $F_1$

$$\boldsymbol{\sigma}^2 - 2xU_* = c_1 - x(c + 2h),$$

we obtain quadrature for $\sigma_2$. For the Euler – Poisson equations the geometrical meaning of variable $\sigma_2$ is evident: it is cosine of the nutation angle. For the equations on $so(4)$ the angle cannot be interpreted that easily.

For a homogeneous quadratic potential energy $U_* = r_1\sigma_1^2 + r_2\sigma_2^2$ from (2.9) we obtain an elliptic quadrature of the form

$$\begin{gathered}\dot\sigma_2^2 = 2\left((h - r_2\sigma_2^2)(1 - 2xr_1) - a_1(c' - (1 - 2xr_2)\sigma_2^2)\right)(c' - (1 - 2xr_2)\sigma_2^2) -\\ - (1 - 2xr_1)^2(c_2 - c\sigma_2)^2 = f(\sigma_2), \qquad c' = c_1 - x(2h + c).\end{gathered} \quad (2.10)$$

Expressions (2.9), (2.10) generalize the known quadrature for the Lagrange case in rigid body dynamics [119]. The function $f(\sigma_2)$ is also called a gyroscopic function.

At $x = 0$, (2.10) gives a gyroscopic function of the Kirchhoff's case; at $x = 1$, it gives a gyroscopic function of the Poincaré case. For the classical Lagrange case, corresponding to $x = 0$, $W_3 = 0$, $U = -r\sigma_2$, the equation for $\sigma_2$ has the form

$$\dot{\sigma_2}^2 = -2r\sigma_2^3 - (2h + c)\sigma_2^2 + (2cc_2 - 2rc_1)\sigma_2 - c_2^2 + 2hc_1. \tag{2.11}$$

To obtain absolute motion of the dynamical symmetry axis one should carry out the quadrature for a precession angle $\psi$. For $x = 0$ it has the form $\dot{\psi} = = a_1 \dfrac{M_1\gamma_1 + M_2\gamma_2}{\gamma_1^2 + \gamma_2^2} = a_1 \dfrac{K_1}{\sigma_1}$, i. e., it is defined by the evolution of the reduced system variables. The similar conclusion holds for the quadrature of a proper rotation angle $\varphi$. We shall not dwell on achieving the general solution in absolute space; for such a solution most results, given in § 3 ch. 2, are valid.

We shall only emphasize that the solution for system (2.9) in elliptic functions can be obtained only under the condition of linear and quadratic dependence of potential (or generalized potential) on components $\boldsymbol{\gamma}$ (correspondingly, $\boldsymbol{M}$, $\boldsymbol{\gamma}$). In other cases a gyroscopic function is a polynomial of degree higher than the fourth, and the solution on the complex plane of time is already ramified. However, the qualitative analysis methods, discussed in ch. 2, can provide quite complete description of motion. This once again emphasizes the uselessness of explicit integration of such systems in theta-functions (including the classical Lagrange top, as well). This integration is not capable of giving anything for investigation of real motions.

### 2. A Top on a Smooth Plane in a Gravity Field

This top differs from reduced systems by the fact that matrix $\mathbf{A}$ depends on positional variables (see ch. 1, § 4). If the body is dynamically symmetrical $I_1 = I_2$ and is bounded by axial symmetrical surface, axes of dynamical and geometrical symmetry coinciding, the Hamiltonian can be represented as (see ch. 1, § 6)

$$\begin{gathered} H = \frac{1}{2}a_1 f\left(M_1^2 + M_2^2 + ma_1(\gamma_3 g_1 - g_2)^2(M_1\gamma_1 + M_2\gamma_2)^2\right) + \\ + \frac{1}{2}a_3 M_3^2 + \mu\left((\gamma_1^2 + \gamma_2^2)g_1 + \gamma_3 g_2\right), \end{gathered} \tag{2.12}$$

$$f^{-1} = 1 + ma_1(\gamma_1^2 + \gamma_2^2)(\gamma_3 g_1 - g_2)^2, \quad \mathbf{I}^{-1} = \operatorname{diag}(a_1,\, a_1,\, a_3),$$



where $\mathbf{I}$ is a tensor of inertia of the body with respect to the center-of-mass; $\mu$ is the body weight; $g_1 = g_1(\gamma_3)$, $g_2 = g_2(\gamma_3)$ are some functions, depending on the body geometry and specified by the equations

$$\boldsymbol{\gamma} = -\frac{\operatorname{grad} F(\boldsymbol{r})}{|\operatorname{grad} F(\boldsymbol{r})|}, \quad \boldsymbol{r} = (g_1(\gamma_3)\gamma_1,\, g_1(\gamma_3)\gamma_2,\, g_2(\gamma_3)). \tag{2.13}$$

In formula (2.13) the equation $F(\boldsymbol{r}) = 0$ specifies the body surface; due to the axial symmetry, $F = F(r_1^2 + r_2^2,\, r_3)$. System (2.12) is also reduced to one degree of freedom by means of variables (2.6); the quadrature for cosine of the nutation angle $\gamma_3 = \cos\theta$ can be obtained in the form

$$\begin{gathered} \dot{\gamma}_3^2 = a_1 f(1 - \gamma_3^2)\Bigl(2\bigl(h - \mu((1 - \gamma_3^2)g_1 + \gamma_3 g_2)\bigr) - \frac{a_1(c - M_3\gamma_3)}{1 - \gamma_3^2} - a_3 M_3^2\Bigr), \\ M_3 = \text{const}, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const}. \end{gathered} \tag{2.14}$$

**Comments.** The cases when an axially symmetric body rests on the plane at one point (a foot) or at a circumference (like a hoop or a coin disk) were studied the most thoroughly. In the first case, referred to as *the Lagrange top on a smooth plane*, or a toy top, the motion analysis can be carried out similarly to § 3 ch. 2. Under explicit integration of (2.14) we obtain a hyperelliptic quadrature (whose study was already done by Klein [237, 238]). However, after an unambiguous substitution of time, eliminating the denominator in (2.14), it is easily shown that all bifurcational patterns, given in § 3 ch. 2, remain practically unchanged. Moreover, the top foot on the plane will draw curves, similar to those, drawn by the Lagrange top apex on a fixed sphere. They are presented, for instance, in the book by Grammel [66].

Due to friction of the foot top at the plane, its general evolution is reduced to the situation when the dynamical symmetry axis (at the proper winding of top) quickly becomes vertical, and for some time the top "falls asleep". Various generalizations of this effect are given in [46, 66, 82, 122, 145].

In case of the disk motion the most extensive study was devoted to regular precessions and their stability [122]. The book [122] also investigates stability of vertical planar motions of a heavy elliptic disk whose equations, in the general case, are not integrable. It should also be mentioned that in the absolute absence of sliding (in the classical nonholonomic statement) the equations of disk rolling are also integrable (the problem of Chaplygin, Appell, Corteweg [2, 122]). However, dynamics they describe is substantially more complicated.

Nonintegrability of the problem about motion of a rigid body on a smooth plane was studied in [43] by means of the separatrix splitting method. However, the results obtained in [43] are not sufficient, and up to this time they did not allow to establish any nontrivial cases of integrability.

### 3. A Gyroscope in a Gimbal in an Axially Symmetric Field

Using variables (2.6), we shall obtain the following quadrature for the nutation angle cosine

$$\dot{\sigma_2}^2 = a_1 f(1 + a_1 g\sigma_1^2)\left(2(\widehat{h} - U(\sigma_2))\sigma_1^2 - a_1 f(c_2 - c\sigma_2)^2(1 + a_1 I_1^i\sigma_1^2)\right),$$
$$f^{-1} = (1 + a_1 I_1^i)\left(1 + a_1\left(I^e + (I_3^i - I_2^i)\frac{\sigma_2^2}{\sigma_1^2}\right)\right), \qquad (2.15)$$
$$g = I^e + (I_3^i - I_2)\frac{\sigma_2^2}{\sigma_1^2}, \quad \sigma_1^2 = 1 - \sigma_2^2,$$

where $\widehat{h} = h - \frac{1}{2}a_3c^2$; the meaning of parameters $I^e$, $I_k^i$, $I_k$ is explained in § 4 ch. 1. The analysis of motion of system (2.15) can be looked up in [119]. Since the external ring is present, the angular momentum vector has secular drift in space even in the absence of external forces. This drift, referred to as *the Magnus effect*, is explained by the appearance of moments of external ring reactions, perpendicular to the axis of its rotation. In the general case the equations of an asymmetrical gyroscope in a gimbal are not integrable [40].

### 4. The Axial Symmetry Case in Chapligin's Equations

As it was shown in § 7, ch. 1, dynamics of a rigid body in fluid in a gravity field without any initial type, can be described by a Hamiltonian system on $e(3)$ with a Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}\mu^2 t^2(\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}). \qquad (2.16)$$

Under the axial symmetry conditions Hamiltonian (2.16) can be represented in the form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + \frac{1}{2}\mu^2 t^2\gamma_3^2. \qquad (2.17)$$

An additional integral also has the form $F = M_3$.

For reduction one can use system of variables (2.6). However, it is more convenient to write the second order equation for the nutation angle. Really, for



$\gamma_3 = \cos\theta$, taking into account the relations $(M_1^2 + M_2^2)(\gamma_1^2 + \gamma_2^2) = (M_1\gamma_2 + M_2\gamma_2)^2 + (M_1\gamma_2 - M_2\gamma_1)^2$, $\dot{\gamma}_3 = M_2\gamma_1 - M_1\gamma_2$, we obtain

$$-\sin\theta\ddot{\theta} = \frac{M_3 - c\cos\theta}{\sin^2\theta} - \mu t^2\sin^2\theta\cos\theta, \quad c = (\boldsymbol{M}, \boldsymbol{\gamma}). \qquad (2.18)$$

If a body falls from the state of rest, then $M_3 = 0$, $c = 0$, and for the nutation angle we obtain a nonautonomous equation of the pendulum type [174]

$$\ddot{\theta} = \mu t^2\sin\theta\cos\theta. \qquad (2.19)$$

Other angles are given by equations

$$\dot{\varphi} = (a-1)M_3 + \frac{M_3 - c\cos\theta}{\sin^2\theta}, \quad \dot{\psi} = \frac{M_3 - c\cos\theta}{\sin^2\theta}. \qquad (2.20)$$

**Comments.** For the first time, equations (2.18), (2.19) were obtained by S. A. Chaplygin in his student paper and published in the complete set of his works (1933, v. 1, [177]). It is possible that Chaplygin decided not to publish his result at once because he failed to integrate these equations explicitly. Besides, V. A. Steklov obtained equations (2.18), (2.19) independently and published them in his celebrated book [160], where he had also given some qualitative results concerning the body behavior. The more detailed qualitative analysis of equations (2.19) was carried out by V. V. Kozlov [93]. He showed that (without initial impulse) under almost all initial conditions a plate tended to fall with uniform acceleration with its wider side below, and vibrated about a horizontal axis with an increasing frequency and decreasing amplitude. The asymptotical solution of equations (2.19) for larger periods of time can be looked up in the paper [202]. If the initial impulse does not equal zero, then the behavior of the solutions of equation (2.18) is practically unknown.

### 5. The Analogy between the Lagrange Top and the Leggette System

Up to now we discussed the reduction techniques (and the corresponding systems of variables) for those problems of rigid body dynamics which allow one linear integral. At the same time, there exists a series of systems when the problem possesses a redundant set of linear integrals which are not commutative. In this case the sequential application of the described reduction is not always possible, since the involution set formed by linear integrals, usually contains nonlinear integrals. In this case, following the scheme described in § 1, we can at once reduce the order by two degrees of freedom. This is achieved by the choice of the proper set of reduced (algebraic) variables.

In the paper [248] the authors considered explicit integration of one variant of the Leggette system, describing the behavior of a spin of atom of liquid helium $He^3$ in $\beta$-phase in the presence of a magnetic field. If we consider quaternion dynamical equations (see § 4 ch. 1), the Hamiltonian of such a system can be written as

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + bM_3 + U(\lambda_0), \tag{2.21}$$

where $U = C\Big(4\lambda_0^2 - \frac{3}{2}\Big)^2$, $b$, $c = \mathrm{const}$. Such a form of the Hamiltonian also occurs in problems about motion of a material point in curved space $S^3$ (see. § 2 ch. 5).

A system of the form (2.21) always possesses a cyclic integral $F = (\boldsymbol{M}, \boldsymbol{\gamma}) - M_3 = \mathrm{const}$. One more additional integral appears under the condition $b = 0$ (the absence of a magnetic field). In terms of variables (1.8) it has the form

$$F = K_2^2(s_1^2 + s_3^2) + (K_1 s_1 + K_2 s_3)^2.$$

The integration of this system in [248] is too complicated. At the same time, as it was shown in [31], this system is, in fact, one of generalizations of the Lagrange case (after the proper reduction). Really, in this case equations (2.21) possess a vector integral of motion

$$\boldsymbol{L} = ((\boldsymbol{M}, \boldsymbol{\alpha}) - M_1, (\boldsymbol{M}, \boldsymbol{\beta}) - M_2, (\boldsymbol{M}, \boldsymbol{\gamma}) - M_3)),$$

whose components form an algebra $so(3)$. Let us choose new variables which commutate simultaneously with all components of vector $\boldsymbol{L}$:

$$\begin{gathered} K_1 = \frac{\sqrt{(\boldsymbol{M} \times \boldsymbol{\lambda})^2}}{\sqrt{\boldsymbol{\lambda}^2}}, \qquad K_2 = \frac{(\boldsymbol{M}, \boldsymbol{\lambda})}{\sqrt{\boldsymbol{\lambda}^2}}, \\ \sigma_1 = \sqrt{\boldsymbol{\lambda}^2}, \qquad \sigma_2 = \lambda_0, \\ \boldsymbol{\lambda}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \end{gathered} \tag{2.22}$$

They form a nonlinear algebra

$$\begin{aligned} &\{K_2, K_1\} = \frac{p_1\sigma_2}{2\sigma_1}, \quad \{K_2, \sigma_1\} = -\frac{\sigma_2}{2}, \\ &\{K_2, \sigma_2\} = \frac{\sigma_1}{2}, \qquad \{K_1, \sigma_1\} = \{K_1, \sigma_2\} = 0 \end{aligned} \tag{2.23}$$

with Casimir's functions

$$F_1 = \boldsymbol{\sigma}^2, \qquad F_2 = K_1\sigma_1 = \mathrm{const},$$



where $\boldsymbol{\sigma} = (\sigma_1 \sigma_2)$, $\boldsymbol{K} = (K_1, K_2)$. (Less convenient basic elements are used in [133, 218])

Since the rank of bracket (2.22) equals zero, any Hamiltonian system on this bracket is integrable, including system (2.21) at $b = 0$. The Hamiltonian of this system in terms of new variables can be represented as

$$H = \frac{1}{2}\boldsymbol{K}^2 + U(\sigma_2).$$

The analogical representation was obtained for the Lagrange case generalization (see (2.7), (2.8)) at $c = 0$, $\boldsymbol{W} = 0$, $x = 0$. This analogy can be established directly (for example, in terms of the Euler angles). However, the algebraical approach to the questions of order reduction, developed in [31], is especially vivid and simple.

## § 3. The Hess Case: Geometry, Cyclic Variable, and Explicit Integration

In two sections to follow we shall sequentially discuss some questions of existence, qualitative analysis and explicit integration of systems of rigid body dynamics, allowing invariant relation of the Hess type. This relation is linear with respect to moments $\boldsymbol{M}$, and the analysis of conditions of its existence is close to the Lagrange case generalizations considered in § 2. It also turns out that these dynamical problems are similar in motion of certain, characteristic points of a body, and in questions of reduction. Further on, we shall also give an explicit quadrature for different variables, characterizing motion. In this section we shall consider a classical situation of Hamiltonian equations on an algebra $e(3)$ (of the Euler–Poisson type). In § 4 we shall give some generalizations connected with the superposition of several fields, and we shall also consider more complicated problems on algebra $e(3)$: sliding of a body on a plane, motion in a gimbal, and the Chaplygin equations.

### 1. A Potential System on Algebra $e(3)$. A Cyclic Coordinate

Let a Hamiltonian of the form

$$H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + U(\boldsymbol{\gamma}), \tag{3.1}$$

where $\mathbf{A} = \mathbf{I}^{-1} = \mathrm{diag}(a_1, a_2, a_3)$, be given on algebra $e(3)$. Consider the level set of kinetic energy in the space of moments — *a gyration ellipsoid*

$$(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) = \mathrm{const}. \tag{3.2}$$

Assume that $a_1 < a_2 < a_3$, then ellipsoid (3.2) has two circular sections, passing through the mean axis. Designate a direction vector, perpendicular to the circular section, by $\boldsymbol{n}$. Then the following unambiguous remark is valid.

*If a potential energy $U(\boldsymbol{\gamma})$ is invariant with respect to rotations of a body around the axis $\boldsymbol{n}$, then the equation $(\boldsymbol{M}, \boldsymbol{n}) = 0$ specifies the Hess invariant relation of system* (3.1).

In the explicit form this condition can be represented as

$$\left(\sqrt{a_2 - a_1}\left(\gamma_2\frac{\partial}{\partial\gamma_3} - \gamma_3\frac{\partial}{\partial\gamma_2}\right) \pm \sqrt{a_3 - a_2}\left(\gamma_1\frac{\partial}{\partial\gamma_2} - \gamma_2\frac{\partial}{\partial\gamma_1}\right)\right)U(\boldsymbol{\gamma}) = 0. \tag{3.3}$$

Correspondingly, the Hess integral has the form

$$\sqrt{a_2 - a_1}M_1 \pm \sqrt{a_3 - a_2}M_3 = 0. \tag{3.4}$$

The opposite signs correspond to different circular sections of ellipsoid (3.2).

In many respects the Hess case is similar to the Lagrange case. It concerns the fact that system (3.1) has a cyclic variable (an explicit symmetry of a Hamiltonian with respect to rotations) on one of the levels of a certain "cyclic" integral. To show this explicitly, let us represent Hamiltonian (3.1) in the frame of reference when one of the axes $Ox_3$ coincides with the axis, perpendicular to the circular section of ellipsoid (3.2) (see fig. 57, ch. 2)

$$H = \frac{1}{2}(a_1'(M_1^2 + M_2^2) + a_3'M_3^2 + 2bM_3M_1) + U(\gamma_3). \tag{3.5}$$

Such a frame of reference is not principal any longer. The matrix of transition to new coordinates (from the system of principal axes) can be expressed in terms of components of a matrix $\mathbf{A}$ according to formula

$$\mathbf{U} = \begin{pmatrix} \sqrt{\dfrac{a_3 - a_2}{a_3 - a_1}} & 0 & \mp\sqrt{\dfrac{a_2 - a_1}{a_3 - a_1}} \\ 0 & 1 & 0 \\ \pm\sqrt{\dfrac{a_2 - a_1}{a_3 - a_1}} & 0 & \sqrt{\dfrac{a_3 - a_2}{a_3 - a_1}} \end{pmatrix}. \tag{3.6}$$

Hess integral (3.4) has the form

$$M_3 = 0. \tag{3.7}$$

It is easily seen that Hamiltonian (3.5) on the level $M_3 = 0$ coincides with the Lagrange Hamiltonian § 1, 2 ch. 3. So, to describe the reduced system, defining



dynamics of the nutation angle of an apex of axis $\boldsymbol{n}$, perpendicular to the circular section of gyration ellipsoid, we can use variables (2.6) $\boldsymbol{K} = (K_1, K_2)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$,

$$K_1 = \frac{M_1\gamma_1 + M_2\gamma_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}, \quad K_2 = \frac{M_1\gamma_2 - M_2\gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}},$$
$$\sigma_1 = \sqrt{\gamma_1^2 + \gamma_2^2}, \quad \sigma_2 = \gamma_3.$$

On the level $M_3 = 0$ these variables form a closed system of equations

$$\begin{aligned} \dot K_1 &= \frac{1}{\sigma_1}a_1'K_1K_2\sigma_2, & \dot K_2 &= -\frac{1}{\sigma_1}a_1'K_1^2\sigma_2 - \sigma_1\frac{\partial U}{\partial\sigma_2}, \\ \dot\sigma_1 &= a_1'K_2\sigma_2, & \dot\sigma_2 &= a_1'K_2\sigma_1. \end{aligned} \tag{3.8}$$

Hamiltonian (3.5) can now be written in the form

$$H = \frac{1}{2}\boldsymbol{K}^2 - \mu\sigma_2 + \frac{1}{2}M_3(a_3'M_3 + 2bM_1).$$

Quadrature for $\sigma_2$ is given by equation (2.11) (at $c = 0$). It is interesting to note that in this case (like in the Lagrange case, see ch. 3, § 1), the precession angle $\psi$ is completely defined by the solution of the reduced system

$$\dot\psi = a_1'\frac{K_1}{\sigma_1},$$

and does not depend on quadrature for the proper rotation angle $\varphi(t)$. This fact was used by N. E. Joukovskiy to describe motion of the center-of-mass in the ordinary Hess case (see further).

Remark 1. The reduction of order in the presence of invariant relations, linear with respect to momenta, was extensively studied by T. Levi-Civita. His main results are contained in the famous book [113]. However, while applying his results to rigid body dynamics, he did not pay any attention to the Hess case. He concentrated his efforts on the more particular class of invariant relations defined by the Staude rotation. Levi-Civita and Libman have also investigated the question of existence of linear integrals in the case when a body moves in a potential field.

## 2. The Classical Hess Case

Let us give a more detailed consideration to the Hess case in the Euler–Poisson equations. In (3.5) let us assume $U = \widetilde{\mu}\gamma_3$, $\widetilde{\mu} = \text{const}$, $a_1' = 1$, $a_3' = a_3$, $b = a_{13}$. Dynamics of the complete system on the joint level of the

Hess relation $M_3 = 0$, area constant $M_1\gamma_1 + M_2\gamma_2 = c$, and energy $H = h$ is described by sequential quadratures

$$\dot{\gamma}_3^2 = 2(1-\gamma_3^2)\Big(h - \widetilde{\mu}\gamma_3 - \frac{c^2}{1-\gamma_3^2}\Big), \quad \dot{\psi} = \frac{c}{1-\gamma_3^2},$$
$$\dot{\varphi} = a_{13}M_1 + \frac{c}{1-\gamma_3^2}, \quad \dot{l} = -a_{13}K\sin l + \widetilde{\mu}\frac{c}{K^2}, \quad K = 2(h-\widetilde{\mu}\gamma_3), \tag{3.9}$$

where $K^2 = M_1^2 + M_2^2$, and $l$ is one of the Andoyaer–Deprit variables; it is defined by relations $M_1 = K\sin l$, $M_2 = K\cos l$.

Using first two quadratures, N. E. Joukovskiy [79] showed that the center-of-mass of a rigid body moves according to the spherical pendulum law. The last two equations in (3.9) show that to find the proper rotation angle $\varphi$, we have to solve the equation for $l$ with coefficients, explicitly depending on time. Such a method of solution does not seem to be given earlier. Usually, following P. A. Nekrasov [131], the proper rotation definition is reduced to the solution of equation of the Rikatti type.

Really, for the complex variable $z = M_1 + iM_2$ it is easy to obtain

$$e^{-i\varphi} = -\frac{\dot{\gamma}_3 + ic}{\sqrt{1-\gamma_3^2}}\frac{z}{K^2}.$$

This leads to the first order nonlinear equation for $z$:

$$\dot{z} + \frac{ia_{13}}{2}z^2 + \mu\frac{\dot{\gamma}_3 + ic}{K^2}z + \frac{1}{2}ia_{13}K = 0. \tag{3.10}$$

To justify our solution, we should notice its simpler form (3.9) in comparison with (3.10). Moreover, at $c = 0$, it becomes even simpler.

In [79] Joukovskiy also showed some more geometrical facts, concerning complete system dynamics in the Hess case.

It turns out that at each moment of time the motion path of gyration ellipsoid mean axis form a constant angle $\theta$ with the circular section plane

$$\sin\theta = \frac{a_2}{\sqrt{a_2(a_1+a_3) - a_1a_3}}. \tag{3.11}$$

This result helps to show that at the zero area constant $c = 0$ the mean axis of inertia moves along a loxodrome. Because of such a characteristic motion, Joukovskiy introduced the name of *loxodromic pendulum* (of Hess), showed



practical conditions of implementation of such a motion, and made a mechanical model for its observation [79].

Consider the case of a loxodromic pendulum ($c = 0$) in more detail (see fig. 70). From relations (3.9) we obtain

$$\dot{\gamma}_3^2 = 2(h - \widetilde{\mu}\gamma_3)(1-\gamma_3^2), \quad \dot{\psi} = 0, \quad \ln\Big(\operatorname{tg}\frac{l}{2}\Big) = \pm a_{13}K, \tag{3.12}$$

and also two corresponding cases:

$\boldsymbol{h > \widetilde{\mu}}$. The center-of-mass rotates along the principal circle (since $\psi = \text{const}$). The mean axis moves all over loxodrome, see fig. 18. In this case on a phase portrait (fig. 70 e,f), containing chaotical paths, the Hess solution separates two "nonmixing" stochastic layers (see also fig. 58). Actually, in this case the Hess solution cannot be realized; due to instability, the path drops into either one or the other layer.

At $h \to \infty$ (or $\widetilde{\mu} \to 0$) everything is reduced to the ordinary Euler case, the Hess solution tending to a separatrix of permanent rotation around the mean axis [92].

$\boldsymbol{h < \widetilde{\mu}}$. The center-of-mass makes planar oscillations according to the physical pendulum law, and the mean axis moves along the loxodrome segment according to (3.11). The solution is periodic in the absolute space (with a single frequency, like the Goryachev solution, § 5 ch. 2). On a phase portrait (see fig. 70 a,b,c) the Hess relation specifies an invariant curve filled with fixed points. This curve is situated within regular foliation.

At $c \neq 0$, the motion investigation is substantially difficult and cannot be done analytically. Fig. 71 shows a series of phase portraits, illustrating the effect of separating stochastic layers (at decreasing energy $h$) near the Hess solution.

Dynamics of absolute motion for small energies possesses three frequencies; under increasing energy motion with respect to one variable will have asymptotic character with only two frequencies remained.

Remark 2. If we consider perturbations of the Euler–Poinsot problem under the Hess conditions, we find that a pair of separatrices, arising from unstable permanent rotations, does not split at perturbation [92] (see fig. 70 f, 71 h). Integral (3.4) defines a special torus filled with double asymptotic paths, approaching some unstable periodic solutions, which at $\widetilde{\mu} \to 0$ transfer into permanent rotations around the mean axis. Such a description of reduced system dynamics does not contradict the Joukovskiy result about quasiperiodic motion of the center-of-mass of body (3.9), since the system, describing the center-of-mass motion, is obtained by reduction with respect not to the precession angle, but to the angle of proper rotation around the axis, perpendicular to the circular section.

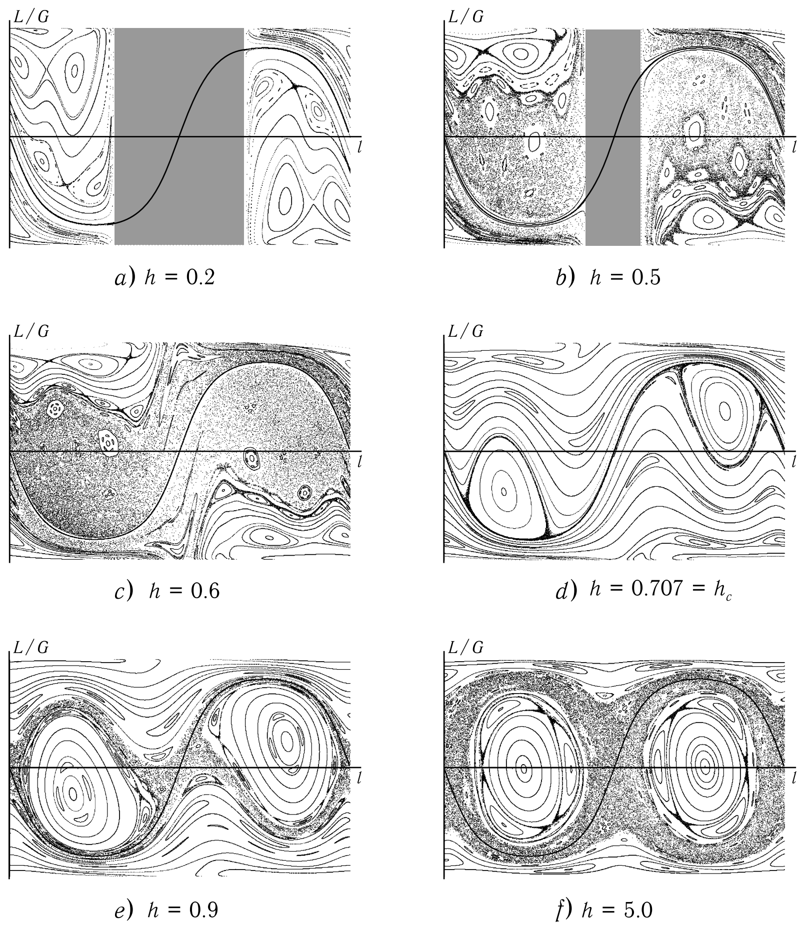



Figure 70. A phase portrait under the Hess conditions and a zero area constant ($H = \frac{1}{2}(M_1^2 + \frac{2}{3}M_2^2 + \frac{1}{2}M_3^2) + \frac{1}{\sqrt{3}}\gamma_1 + \frac{1}{\sqrt{6}}\gamma_3, \widetilde{\mu} = h_c$). Figures vividly show that the torus, corresponding to the Hess integral at small energies, is situated in a regular foliation. Grey color designates a physically impossible domain of values of variables.

**Historical comment.** Hess has obtained his integral when he was looking for various forms of equations of motion of a heavy rigid body. These equations were supposed to have various advantages comparing to the Euler – Poisson equations [228]. Hess also seems to give an idea of using not principal axes. The



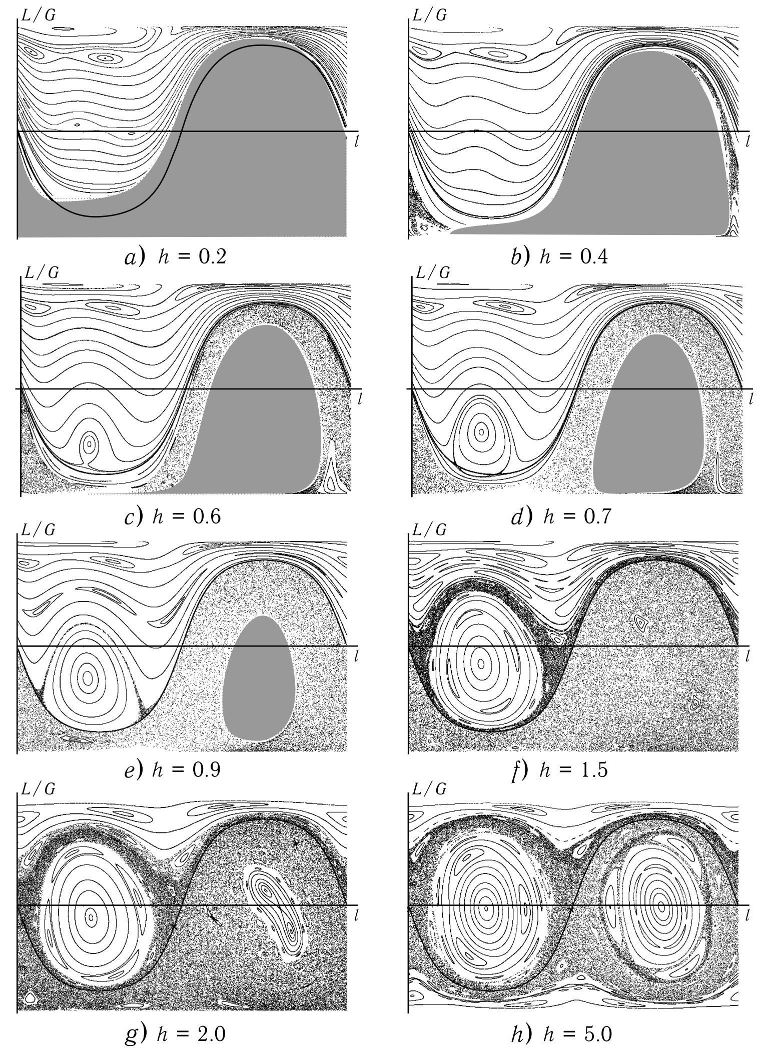



Figure 71. A phase portrait under the Hess conditions and nonzero area constant $c = 1$ ($H = \frac{1}{2}(M_1^2 + \frac{2}{3}M_3^2 + \frac{1}{2}M_3^2) + \frac{1}{\sqrt{3}}\gamma_1 + \frac{1}{\sqrt{6}}\gamma_3$). Like above, at large $h$ the Hess solution separates two stochastic layers, and at small $h$ it lies in a regular foliation.

equations in the Hess form can be looked up in the books [9, 59]. Conditions of the Hess integral existence were also obtained by G. G. Appelrot, who was trying to fill the gaps in the Kowalevskaya analysis [3]. Kowalevskaya herself missed this case in her investigation, though it did not spoil her results: under the Hess conditions the solution ramifies on a complex plane of time [3]. Nevertheless, the Hess case is sometimes referred to as the Hess – Appelrot case. As it was already noticed, the geometrical analysis and simulation of the Hess top were offered by Joukovskiy [79]. The extensive analytical memoir on its explicit solution (reduction to the Rikatti equations) belongs to Nekrasov [131]. The connection of the Hess invariant relations with the pair of unsplitted separatrices of the perturbed Euler – Poinsot problem was indicated by V. V. Kozlov [92].

In the Kirchhoff's equations the Hess case analogue (see the section to follow) was noticed by Chaplygin [178] (who used nonprincipal axes at once), and from the condition of splitting of separatrices it was obtained in [98]. The majority of geometrical and analytical dynamical derivations, shown for the ordinary Hess case, are valid for this analogue.

## § 4. The Hess Case Generalizations

We shall follow the general scheme of investigation. First, we shall indicate general dynamical conditions leading to the existence of invariant relation of the Hess type. Then we shall illustrate them using various mechanical systems [33].

First of all, we shall formulate even more general statement about the existence of an invariant relation of the form

$$M_3 - c = 0, \quad c = \text{const} \tag{4.1}$$

(which at an arbitrary $c$ will provide conditions of existence of integral of the Lagrange type; we have shown them in § 1 formula (1.13)) for the most complete form of a generalized potential system with the Hamiltonian

$$\begin{gathered} H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}'\boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{W}(\boldsymbol{q})) + U(\boldsymbol{q}), \\ \boldsymbol{q} \approx (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \approx (\theta, \varphi, \psi) \approx (\lambda_0, \lambda_1, \lambda_2, \lambda_3), \end{gathered} \tag{4.2}$$

where $\mathbf{A}'$ is a constant, not necessarily diagonal matrix (the system of nonprincipal axes is supposed to be used). Hamiltonian (4.2) also describes motion of a rigid body with a fixed point in the superposition of force fields (see § 12 ch. 3, where we discuss a series of generalizations of the Lagrange and Hess cases to



the special forms of potential $U(\boldsymbol{q})$, $W(\boldsymbol{q}) \equiv 0$). By means of direct computations we can show that conditions of existence of relation (4.1) are represented by

$$\begin{gathered} a'_{11} = a'_{22}, \qquad a'_{12} = 0, \\ (\widehat{L}_\alpha + \widehat{L}_\beta + \widehat{L}_\gamma)(U + cW_3) = 0, \\ (\widehat{L}_\alpha + \widehat{L}_\beta + \widehat{L}_\gamma)W_1 + W_2 + ca'_{23} = 0, \\ (\widehat{L}_\alpha + \widehat{L}_\beta + \widehat{L}_\gamma)W_2 - W_1 - ca'_{13} = 0, \end{gathered} \tag{4.3}$$

where $\widehat{L}_\alpha$, $\widehat{L}_\beta$, $\widehat{L}_\gamma$ are differential operators of the form:

$$\hat{L}_\alpha = \alpha_1\frac{\partial}{\partial\alpha_2} - \alpha_2\frac{\partial}{\partial\alpha_1}, \quad \hat{L}_\beta = \beta_1\frac{\partial}{\partial\beta_2} - \beta_2\frac{\partial}{\partial\beta_1}, \quad \hat{L}_\gamma = \gamma_1\frac{\partial}{\partial\gamma_2} - \gamma_2\frac{\partial}{\partial\gamma_1}.$$

Remark 1. In the system of principal axes operators $\widehat{L}_\alpha$, $\widehat{L}_\beta$, $\widehat{L}_\gamma$ have the form (3.3) § 3; they are transformed by means of matrix (3.6) § 3.

**Linear and quadratic potentials.** Let us explicitly show conditions of existence of Hess integral (4.1) for particular form of vector and scalar potentials $\boldsymbol{W}, U$ in (4.2), assuming that

$$\begin{gathered} \boldsymbol{W} = \boldsymbol{K} + \sum_{i=1}^{3} \mathbf{B}^{(i)}\boldsymbol{\alpha}_i, \\ U = \sum_{i=1}^{3}(\boldsymbol{r}^{(i)}, \boldsymbol{\alpha}_i) + \frac{1}{2}\sum_{i=1}^{3}(\boldsymbol{\alpha}_i, \mathbf{C}^{(i)}\boldsymbol{\alpha}_i), \end{gathered} \tag{4.4}$$

where $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}_2 = \boldsymbol{\beta}$, $\boldsymbol{\alpha}_3 = \boldsymbol{\gamma}$, $\boldsymbol{K}$, $\boldsymbol{r}_i$ are constant vectors, $\mathbf{C}^{(i)}$ are symmetrical, and $\mathbf{B}^{(i)}$ are arbitrary $3 \times 3$-matrices; $(i = 1, 2, 3)$.

Conditions of the Hess integral existence for some (more particular) cases of system (4.4) are given in the papers [98, 45] (see below).

Using relations (4.1) and (4.3), we obtain

$$\boldsymbol{K} = (-ca'_{13}, -ca'_{23}, k_3a'_{33}),$$

$k_3$ is an arbitrary constant,

$$\begin{gathered} b^{(i)}_{11} = b^{(i)}_{22}, \qquad b^{(i)}_{12} = -b^{(i)}_{12}, \qquad b^{(i)}_{13} = b^{(i)}_{23} = 0, \\ r^{(i)}_1 = cb^{(i)}_{31}, \qquad r^{(i)}_2 = cb^{(i)}_{32}, \\ \mathbf{C}^{(i)} = \operatorname{diag}(c^{(i)}_{11}, c^{(i)}_{11}, c^{(i)}_{33}). \end{gathered}$$

The Hamiltonian can be represented in the form

$$\begin{gathered} H=\frac{1}{2}(a'_{11}(M_1^2+M_2^2)+a'_{33}(M_3+k_3)^2)+\\ +(M_3-c)(a'_{13}M_1+a'_{23}M_2)+b_{11}^{(1)}(M_1\alpha_1+M_2\alpha_2)+\\ +b_{12}^{(1)}(M_1\alpha_2-M_2\alpha_1)+b_{33}^{(1)}M_3\alpha_3+(M_3-c)(b_{31}^{(1)}\alpha_1+b_{32}^{(1)}\alpha_2)+\\ +\frac{1}{2}(c_{11}^{(1)}(\alpha_1^2+\alpha_2^2)+c_{33}^{(1)}\alpha_3^2)+r_3^{(1)}\alpha_3+\ldots, \end{gathered}\tag{4.5}$$

where the analogical terms, containing $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, are omitted.

Consider a mutual system of variables: projections of the angular momentum vector on fixed axes $\boldsymbol{N}=(N_1,\,N_2,\,N_3)=((\boldsymbol{M},\boldsymbol{\alpha}),(\boldsymbol{M},\boldsymbol{\beta}),(\boldsymbol{M},\boldsymbol{\gamma}))$, and vector $\boldsymbol{p}=(\alpha_3,\beta_3,\gamma_3)$. Their commutation forms an algebra $e(3)$ (§ 4 ch. 1). In terms of this variables under conditions (4.3), (4.4) Hamiltonian (4.2) has the form

$$\begin{gathered} H=\frac{1}{2}a'_{11}\boldsymbol{N}^2+(\boldsymbol{b}_1,\,\boldsymbol{N})+(\boldsymbol{b}_2\times\boldsymbol{p},\,\boldsymbol{N})+(\boldsymbol{r}+c\boldsymbol{b}_3-c\boldsymbol{b}_1,\,\boldsymbol{p})+\frac{1}{2}(\boldsymbol{p},\,\mathbf{C}\mathbf{p})+\\ +(M_3-c)f(\boldsymbol{M},\,\boldsymbol{\alpha},\,\boldsymbol{\beta},\,\boldsymbol{\gamma}), \end{gathered}\tag{4.6}$$

where $\boldsymbol{b}_1=(b_{11}^{(1)},\,b_{11}^{(2)},\,b_{11}^{(3)})$, $\boldsymbol{b}_2=(b_{12}^{(1)},\,b_{12}^{(2)},\,b_{13}^{(3)})$, $\boldsymbol{b}_3=(b_{33}^{(1)},\,b_{33}^{(2)},\,b_{33}^{(3)})$, $\boldsymbol{r}=(r_3^{(1)},\,r_3^{(2)},\,r_3^{(3)})$, $\mathbf{C}=\operatorname{diag}(c_{33}^{(1)}-c_{11}^{(1)},\,c_{33}^{(2)}-c_{11}^{(2)},\,c_{33}^{(3)}-c_{11}^{(3)})$. However, function $f(\boldsymbol{M},\,\boldsymbol{\alpha},\,\boldsymbol{\beta},\,\boldsymbol{\gamma})$ cannot be expressed in terms of variables $\boldsymbol{N}$, $\boldsymbol{p}$. (Otherwise, we would obtain a top of the Lagrange type.)

Since $\boldsymbol{N}$, $\boldsymbol{p}$ commutate with $M_3$, the equations of motion for them on the level $M_3=c$ separate and are described by a Hamiltonian system on $e(3)$ with Hamiltonian (4.6) taken under the condition $M_3-c=0$, i. e., by a system with two degrees of freedom.

*Thus, we obtain a flow as a reduced system. This flow is isomorphic to equations of motion of a spherical ball in a generalized potential field on the fixed level of the area constant* $(\boldsymbol{N},\,\boldsymbol{p})=M_3=c$.

In the general case system (4.6) is not integrable, i. e., existence of the Hess invariant relation for system (4.2) is not equivalent to complete integrability. As we have already seen in § 1, the analogical remark holds true about the general Lagrange integral presence. This integral only provides the possibility of order reduction by one degree of freedom.

Show additional conditions for integrability of system (4.6). Really, at $b_1=b_2=b_3=r=0$, we have the integrable Clebsch case (at $c=0$ the Neumann system), and, at $b_1=b_2=b_3=0$, $c=0$, the Lagrange case for a single field.



Remark 2. Up to this moment we used a special frame of reference whose axes do not coincide with principal axes of the body, and $\mathbf{A}$ is not diagonal. In the frame of reference, for which a tensor of inertia is diagonal $\mathbf{A}=\operatorname{diag}(a_1,\,a_2,\,a_3)$, integral of the Hess type (4.1) has the form

$$\begin{gathered} F=\sqrt{a_2-a_1}\sqrt{a_3-a_2}(M_1\sqrt{a_2-a_1}\pm M_3\sqrt{a_3-a_2})-\\ -(K_1\sqrt{a_3-a_2}\pm K_3\sqrt{a_2-a_1})=0, \end{gathered}\tag{4.7}$$

where $\boldsymbol{K}$ is a constant vector in (4.4). The transformation to the system of principal axes of the body is done by means of matrix $\mathbf{U}$ (3.6).

Remark 3. The Hess integral, like the Lagrange integral, is present in a more complicated system with five degrees of freedom [41]: a body, suspended at a massless *rigid bar* (*a string*), moves in a gravity field [153]. To be integrable, even in the presence of the above mentioned integrals, this system needs three more integrals in involution. They are unknown, and the only integrable case concerns complete separation of motions when the point of body fixation at the string coincides with the center-of-mass.

**Known integrable cases.** In the case of a single field, axially symmetric in space, i. e., at $U=U(\boldsymbol{\gamma})$, $\boldsymbol{W}=\boldsymbol{W}(\boldsymbol{\gamma})$, different authors noticed the following analogues of the Hess integral which in this case is sufficient for complete integrability:

1. $U(\boldsymbol{\gamma})=\mu\gamma_3$, $\boldsymbol{W}(\boldsymbol{\gamma})=0$: the classical Hess case for the Euler – Poisson equations [228].
2. $U(\boldsymbol{\gamma})=\mu\gamma_3$, $\boldsymbol{W}=(ca'_{13},\,ca'_{23},\,k_3)$, $k_3=\mathrm{const}$: a gyrostatic generalization of the Hess case. It was shown by L. N. Stretenskiy [159].
3. $U(\boldsymbol{\gamma})=(\boldsymbol{\gamma},\,\mathbf{C}\boldsymbol{\gamma})$, $\mathbf{C}=\operatorname{diag}(c_1,\,c_1,\,c_3)$, $\boldsymbol{W}=(b_{11}\gamma_1+b_{12}\gamma_2,-b_{12}\gamma_1+b_{11}\gamma_2,b_{31}\gamma_1+b_{32}\gamma_2+b_{33}\gamma_3)$: a particular case of integrability (S. A. Chaplygin [178], V. V. Kozlov, D. A. Onischenko [98]) of the Kirchhoff equations.

Cases 1, 2, 3 satisfy general conditions (4.3), (4.5).

Remark 4. All generalizations of the Hess case, shown on algebra $e(3)$, can be naturally transferred to the case of a bundle $\mathscr{L}_x$, due to the fact that equations for $\boldsymbol{M}$ are the same on the whole bundle. Here the Hess invariant relation does not depend on the bundle parameters.

In the case of two force fields system (4.4) was considered in the paper [45] (A. A. Burov, G. I. Subhankulov), though in it potentials (4.4) are interpreted from the hydrodynamical viewpoint. The paper [45] contains two particular cases of existence of the Hess integral $M_3 = 0$ for a system. However, the question of complete integrability is not discussed. It turns out that in one of these cases the system is integrable, and in the other is not.

*The first case*

$$U = \frac{1}{2}\left(c_{11}^{(1)}(\alpha_1^2 + \alpha_2^2) + c_{33}^{(1)}\alpha_3^2\right) + \frac{1}{2}\left(c_{11}^{(2)}(\beta_1^2 + \beta_2^2) + c_{33}^{(2)}\beta_3^2\right).$$

The Hamiltonian of reduced system (4.6) can be represented in the form

$$H = \frac{1}{2}a'_{11}\boldsymbol{N}^2 + \frac{1}{2}(c_{33}^{(1)} - c_{11}^{(1)})p_1^2 + \frac{1}{2}(c_{33}^{(2)} - c_{11}^{(2)})p_2^2.$$

Due to relation $(\boldsymbol{N}, \boldsymbol{p}) = M_3 = 0$, this case is *isomorphic to the Neumann system* which is integrable.

*The second case*

$$U = r_3\alpha_3 + \frac{1}{2}\left(c_{11}(\beta_1^2 + \beta_2^2) + c_{33}\beta_3^2\right).$$

The reduced system has the form

$$H = \frac{1}{2}a'_{11}\boldsymbol{N}^2 + r_3 p_1 + \frac{1}{2}(c_{33} - c_{11})p_2^2, \quad (\boldsymbol{N}, \boldsymbol{p}) = 0.$$

This Hamiltonian corresponds to a spherical pendulum in a gravity field and in the Brun field, perpendicular to it. Such a system seems to be nonintegrable. Now, consider the application of these general conditions to three contiguous problems of rigid body dynamics.

**A rigid body on a smooth plane.** The paper [42] (A. A. Burov) contains the Hess invariant relation for rigid body dynamics with a gyrostat on a smooth horizontal plane. However, due to using angular velocities $\boldsymbol{\omega}$ and a system of principal axes, this relation has a bit unexpected form. Here we give conditions of existence of the Hess integral for equations in the Hamiltonian form on an algebra $e(3)$ in case when the force field potential is symmetrical with respect to rotations around a vertical line.

As it is shown in ch. 1, the equations of motion can be written in the Hamiltonian form on algebra $e(3)$ with the Hamilton function

$$\begin{gathered} H = \frac{1}{2}(\mathbf{A}(\boldsymbol{M} - \boldsymbol{K}), \mathbf{I}\mathbf{A}(\boldsymbol{M} - \boldsymbol{K})) + \frac{1}{2}m(\boldsymbol{a}, \mathbf{A}(\boldsymbol{M} - \boldsymbol{K})) + U(\boldsymbol{\gamma}), \\ \boldsymbol{a} = \boldsymbol{r} \times \boldsymbol{\gamma}, \qquad \mathbf{A} = (\mathbf{I} + m\boldsymbol{a} \otimes \boldsymbol{a})^{-1}, \end{gathered} \tag{4.8}$$



where $\boldsymbol{K}$ is a gyrostatic moment vector, constant in the system of axes bound to the body; $\boldsymbol{\gamma}$ is a vector normal to the plane; $\boldsymbol{M}$ is a vector of angular momentum with respect to the contact point. It is connected with the angular velocity by formula

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + m\boldsymbol{a}(\boldsymbol{a}, \boldsymbol{\omega}). \tag{4.9}$$

Here $\mathbf{I}$ is a tensor of inertia with respect to the center-of-mass, $m$ is a mass of the body.

Vector $\boldsymbol{r}(\boldsymbol{\gamma})$ is found from the condition

$$\boldsymbol{\gamma} = -\frac{\operatorname{grad} F(\boldsymbol{r})}{|\operatorname{grad} F(\boldsymbol{r})|},$$

where $F(\boldsymbol{r}) = 0$ specifies the equation of (everywhere convex) surface of the body.

*Let a body be bounded by an axially symmetric surface whose symmetry axis is perpendicular to the circular section of a gyration ellipsoid of the form*

$$(\boldsymbol{M}, \mathbf{I}^{-1}\boldsymbol{M}) = \text{const}.$$

*Choose a frame of reference such that one of its axes $Ox_3$ is perpendicular to the circular section, and the other $Ox_2$ is directed along the mean axis of inertia, then if $U$ depends only on $\gamma_3$ and the relation*

$$K_2 = 0, \qquad a_{11}^{(0)}K_1 + a_{13}^{(0)}K_3 = ca_{13}^{(0)},$$

*where $\mathbf{A}^{(0)} = \|a_{ij}^{(0)}\| = \mathbf{I}^{-1}$, is valid, the Hess invariant relation takes the form*

$$M_3 - c = 0.$$

Proof of this statement is direct examination done by any means of analytical computations.

The Hamiltonian in the Hess case differs from Hamiltonian in the Lagrange case on a smooth plane (2.12) (§ 1 ch. 3) by the presence of an additional term of the form $(M_3 - c)f(\boldsymbol{M}, \boldsymbol{\gamma})$. It turns to zero on the Hess integral level, where it is also possible to transfer to a reduced system defined by variables (2.6) (§ 1 ch. 3).

Remark 5. In the chosen frame of reference

$$\mathbf{A}^{(0)} = \begin{pmatrix} a_{11}^{(0)} & 0 & a_{13}^{(0)} \\ 0 & a_{11}^{(0)} & 0 \\ a_{13}^{(0)} & 0 & a_{33}^{(0)} \end{pmatrix},$$

the equation of the body surface has the form

$$F = F(x_1^2 + x_2^2,\, x_3) = 0,$$

and vector $\boldsymbol{a}$ in (4.8) can be represented as

$$\boldsymbol{a} = (-f(\gamma_3)\gamma_2,\, f(\gamma_3)\gamma_1,\, 0),$$

where $f$ depends on $\gamma_3$ only.

Remark 6. For motion of a rigid body on the absolutely rough plane (a nonholonomic system) the Hess case analogue has not been found up to this day. Nevertheless, the generalization of the Lagrange problem about rolling of an axially symmetric body on the plane exists, and was integrated by S. A. Chaplygin [122].

**A gyroscope in a gimbal.** In this problem kinetic energy also depends on positional variables, causing additional difficulties. Here it is also convenient to use the Hamiltonian form of system notation (for details see § 4 ch. 1). Since the obtained expressions are cumbersome, we shall give here only final result in the absence of a gyrostatic moment.

*Let a dynamically asymmetrical rigid body be fixed in a gimbal so that the axis of fixation at the internal frame* (*see fig.* 10 *ch.* 1) *coincides with the perpendicular to the circular section of a gyration ellipsoid, and potential energy of the body in the external field is invariant with respect to rotations of the body on the internal frame axis. Then there exists an invariant relation of the Hess type which in the frame of reference, one of whose axes* $(Ox_3)$ *coincides with the perpendicular to the circular section, has the form*

$$M_3 = 0. \tag{4.10}$$

For the first time, this case seems to be mentioned by the authors of [33].

For the case of a gravity field the body center-of-mass should lie at the internal frame axis.

Remark 7. It is easy to show generalization of this result to the case of a gyrostat. Then relation (4.10) will take the form $M_3 = c$, where $c$ is a fixed constant, depending on a gyrostatic moment.

**The Hess integral in the Chaplygin equations.** Show one more case of existence of the Hess invariant relation for a nonautonomous system, describing a rigid body fall in fluid without initial impulse § 7 ch. 1. The surface, bounding the body, is axially symmetric, and the symmetry axis is perpendicular to the circular section of a gyration ellipsoid. A Hamiltonian can be represented in the form



$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2 + 2a_{13}M_1M_3) + \frac{1}{2}\mu t^2\gamma_3^2. \tag{4.11}$$

In the chosen frame of reference an invariant relation has the form

$$M_3 = 0. \tag{4.12}$$

Here it is possible to obtain the equation for a nutation angle which does not differ from the axially symmetric case (see § 2)

$$-\sin\theta\ddot{\theta} = -\frac{c\cos\theta}{\sin^2\theta} - \mu t^2\sin^2\theta\cos\theta, \quad c = (\boldsymbol{M},\, \boldsymbol{\gamma}). \tag{4.13}$$

The proper rotation angle in this case is specified by the system

$$\dot{\varphi} = -\frac{c\cos\theta}{\sin^2\theta} + a_{13}M_1, \quad \dot{M}_1 = q_3M_1M_2 + \mu t^2\gamma_2\gamma_3,$$

where $\gamma_3 = \cos\theta$, $\gamma_1 = \sin\theta\sin\varphi$, $\gamma_2 = \sin\theta\cos\varphi$, and $M_2$ can be found from relation

$$M_1^2 + M_2^2 = \frac{c^2 + \dot{\gamma}_3^2}{1 - \gamma_3^2}.$$

It should be noted that, due to the validity of equations (4.13), the qualitative analysis of motion carried out in [93] holds true for the Hess case at $c = 0$.