



# **Turbulence Modeling for CFD**

by

**David C. Wilcox**

**DCW Industries, Inc.  
La Cañada, California**

*Dedicated to my Wife*

BARBARA

*my Children*

KINLEY and BOB

*and my Dad*

## **Turbulence Modeling for CFD**

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# Notation

This section includes the most commonly used notation in this book. In order to avoid departing too much from conventions normally used in literature on turbulence modeling and general fluid mechanics, a few symbols denote more than one quantity.

## English Symbols

Symbol	Definition
$a$	Speed of sound
$a_{ijkl}$	Rapid pressure-strain tensor
$A_n, B_n, C_n, D_n$	Coefficients in tridiagonal matrix equation
$A_o^+$	Van Driest damping constant
$A_{ij}$	Slow pressure-strain tensor
$b_{ij}$	Dimensionless Reynolds-stress anisotropy tensor
$B$	Additive constant in the law of the wall
$c_{b1}, c_{b2}$	Closure coefficients
$c_f$	Skin friction based on edge velocity, $\tau_w / (\frac{1}{2}\rho U_e^2)$
$c_{f\infty}$	Skin friction based on freestream velocity, $\tau_w / (\frac{1}{2}\rho U_\infty^2)$
$c_{w1}, c_{w2}, c_{w3}$	Closure coefficients
$C_1, C_2$	Closure coefficients
$C_{cp}, C_{wk}$	Closure coefficients
$C_D, C_E$	Closure coefficients
$C_{dif}, C_{Kleb}$	Closure coefficients
$C_K$	Kolmogorov constant
$C_{L1}, C_{L2}$	Closure coefficients
$C_p$	Specific heat at constant pressure; pressure coefficient
$C_s, C_\epsilon$	Closure coefficients
$C_S$	Smagorinsky constant
$C_v$	Specific heat at constant volume
$C_\delta$	Shear-layer spreading rate

$C_{\epsilon 1}, C_{\epsilon 2}, C_{\epsilon 3}$	Closure coefficients
$C_{\tau 1}, C_{\tau 2}$	Closure coefficients
$C_{\mu}$	Closure coefficient
$C_{ij}$	LES cross-term stress tensor
$C_{ijk}$	Turbulent transport tensor
$D$	Drag per unit body width
$D_{ij}$	Production tensor, $\tau_{im}\partial U_m/\partial x_j + \tau_{jm}\partial U_m/\partial x_i$
$e$	Specific internal energy ; small-eddy energy
$E$	Total energy; viscous damping function
$E(\kappa)$	Energy spectral density
$E(\eta)$	Dimensionless self-similar dissipation rate
$E_h$	Discretization error
$f_{\mu}, f_1, f_2, f_s$	Viscous damping functions
$\mathbf{f}, \mathbf{f}_v$	Turbulence flux vectors
$F(\eta)$	Dimensionless self-similar streamfunction
$F_{Kleb}(y; \delta)$	Klebanoff intermittency function
$\mathbf{F}, \mathbf{F}_v$	Mean-flow flux vectors
$G$	Amplitude factor in von Neumann stability analysis
$G(\mathbf{x} - \boldsymbol{\xi})$	LES filter
$h$	Specific enthalpy
$H$	Total enthalpy; channel height; shape factor, $\delta^*/\theta$
$\mathcal{H}(x)$	Heaviside step function
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors in $x, y, z$ directions
$I$	Unit (identity) matrix
$II, III$	Stress tensor invariants
$j$	Two-dimensional ( $j = 0$ ), axisymmetric ( $j = 1$ ) index
$J$	Specific momentum flux (flux per unit mass)
$k$	Kinetic energy of turbulent fluctuations per unit mass
$k_g$	Geometric progression ratio
$k_R$	Surface roughness height
$K$	Distortion parameter
$K(\eta)$	Dimensionless self-similar turbulence kinetic energy
$K_{\epsilon}, K_{\omega}$	Effective Kármán constant for compressible flows
$Kn$	Knudsen number
$\ell$	Turbulence length scale; characteristic eddy size
$\ell_{mfp}$	Mean free path
$\ell_{mix}$	Mixing length
$L$	Characteristic length scale
$L_{ij}$	Leonard stress tensor
$M$	Mach number
$M_{ijkl}$	Rapid pressure-strain tensor
$M_c$	Convective Mach number

$M_t$	Turbulence Mach number, $\sqrt{2k}/a$
$M_{to}$	Closure coefficient
$N(\eta)$	Dimensionless self-similar eddy viscosity
$N_{CFL}$	CFL number
$N_\omega$	Constant in near-wall solution for $\omega$
$\mathcal{N}(u_i)$	Navier-Stokes operator
$p$	Instantaneous static pressure
$p_{ij}$	Instantaneous momentum-flux tensor
$P$	Mean static pressure
$P_{ij}$	Production tensor, $\tau_{im}\partial U_j/\partial x_m + \tau_{jm}\partial U_i/\partial x_m$
$P_k, P_\omega, P_\epsilon$	Net production per unit dissipation of $k, \omega, \epsilon$
$Pr_L, Pr_T$	Laminar, turbulent Prandtl number
$q_j$	Heat-flux vector
$q_w$	Surface heat flux
$q_{Lj}, q_{Tj}$	Laminar, turbulent mean heat-flux vector
$Q_{ij}$	LES stress tensor, $C_{ij} + R_{ij}$
$\mathbf{Q}$	Dependent variable vector
$r, \theta, x$	Cylindrical polar coordinates
$R$	Pipe radius; channel half height; perfect gas constant
$R_{ij}$	SGS Reynolds stress tensor
$R_{ij}(\mathbf{x}, t; \mathbf{r})$	Two-point velocity correlation tensor
$\mathcal{R}$	Radius of curvature
$\mathcal{R}_{ij}(\mathbf{x}, t; t')$	Autocorrelation tensor
$R^+$	Sublayer scaled radius or half height, $u_\tau R/\nu$
$R_\beta, R_k, R_\omega$	Closure coefficients in viscous damping functions
$Re_L$	Reynolds number based on length $L$
$Re_T$	Turbulence Reynolds number, $k^{1/2}\ell/\nu$
$Re_\tau$	Sublayer scaled radius or half height, $R^+$
$Ri_T$	Turbulence Richardson number
$R_y$	Near-wall turbulence Reynolds number, $k^{1/2}y/\nu$
$s_{ij}$	Instantaneous strain-rate tensor
$\mathbf{s}, \mathbf{S}$	Source-term vectors
$S$	Source term — production minus dissipation
$S_{ij}$	Mean strain-rate tensor
$\overset{\circ}{S}_{ij}$	Oldroyd derivative of $S_{ij}$
$S_e, S_k, S_u, S_w$	Source terms in a similarity solution
$S_B$	Dimensionless surface mass injection function
$S_R$	Dimensionless surface roughness function
$t$	Time
$t_{ij}$	Instantaneous viscous stress tensor
$T$	Temperature; characteristic time scale

$T'$	Freestream turbulence intensity
$u, v, w$	Instantaneous velocity components in $x, y, z$ directions
$u_i$	Instantaneous velocity in tensor notation
$\mathbf{u}$	Instantaneous velocity in vector notation
$u', v', w'$	Fluctuating velocity components in $x, y, z$ directions
$u'_i$	Fluctuating velocity in tensor notation
$\mathbf{u}'$	Fluctuating velocity in vector notation
$\tilde{u}, \tilde{v}, \tilde{w}$	Favre-averaged velocity components in $x, y, z$ directions
$\tilde{u}_i$	Favre-averaged velocity in tensor notation
$\tilde{\mathbf{u}}$	Favre-averaged velocity in vector notation
$u'', v'', w''$	Favre fluctuating velocity components in $x, y, z$ directions
$u''_i$	Favre fluctuating velocity in tensor notation
$\mathbf{u}''$	Favre fluctuating velocity; fluctuating molecular velocity
$u_{rms}, v_{rms}$	RMS fluctuating velocity components in $x, y$ directions
$\overline{u'_i u'_j}$	Temporal average of fluctuating velocities
$u_\tau$	Friction velocity, $\sqrt{\tau_w / \rho_w}$
$\hat{\mathbf{u}}$	Velocity perturbation vector
$U, V, W$	Mean velocity components in $x, y, z$ directions
$U_i$	Mean velocity in tensor notation
$\mathbf{U}$	Mean velocity in vector notation
$U^+$	Dimensionless, sublayer-scaled, velocity, $U/u_\tau$
$U_m$	Maximum or centerline velocity
$\mathcal{U}(\eta)$	Dimensionless self-similar streamwise velocity
$v_{mix}$	Mixing velocity
$v_{th}$	Thermal velocity
$v_w$	Surface injection velocity
$\mathcal{V}(\eta)$	Dimensionless self-similar normal velocity
$W(\eta)$	Dimensionless self-similar specific dissipation rate
$x, y, z$	Rectangular Cartesian coordinates
$x_i$	Position vector in tensor notation
$\mathbf{x}$	Position vector in vector notation
$y^+$	Dimensionless, sublayer-scaled, distance, $u_\tau y / \nu$
$y_2^+$	$y^+$ at first grid point above surface
$y_m$	Inner/outer layer matching point

## Greek Symbols

Symbol	Definition
$\alpha, \alpha^*$	Closure coefficients
$\hat{\alpha}, \hat{\beta}, \hat{\gamma}$	Closure coefficients
$\alpha_o, \alpha_o^*$	Closure coefficients in viscous damping functions

$\alpha_T, \sigma_T, \omega_T$	Defect-layer similarity parameters
$\beta, \beta^*$	Closure coefficients
$\beta_T$	Equilibrium parameter, $(\delta^*/\tau_w)dP/dx$
$\gamma$	Specific heat ratio, $C_p/C_v$
$\delta$	Boundary layer or shear layer thickness
$\delta^*$	Displacement thickness, $\int_0^\delta \left(1 - \frac{\rho}{\rho_e} \frac{U}{U_e}\right) dy$
$\delta_v^*$	Velocity thickness, $\int_0^\delta \left(1 - \frac{U}{U_e}\right) dy$
$\delta_x$	Finite-difference matrix operator
$\delta_{ij}$	Kronecker delta
$\Delta$	LES filter width
$\Delta(x)$	Clouser thickness, $U_e \delta^*/u_\tau$
$\Delta \mathbf{Q}, \Delta x, \Delta y$	Incremental change in $\mathbf{Q}, x, y$
$\Delta t$	Timestep
$\epsilon$	Dissipation per unit mass
$\epsilon_d$	Dilatation dissipation
$\epsilon_s$	Solenoidal dissipation
$\epsilon_{ij}$	Dissipation tensor
$\epsilon_{ijk}$	Permutation tensor
$\zeta$	Second viscosity coefficient
$\eta$	Kolmogorov length scale; similarity variable
$\theta$	Momentum thickness, $\int_0^\delta \frac{\rho}{\rho_e} \frac{U}{U_e} \left(1 - \frac{U}{U_e}\right) dy$
$\kappa$	Kármán constant; thermal conductivity; wavenumber
$\kappa_v$	Effective Kármán constant for flows with mass injection
$\lambda$	Taylor microscale
$\lambda_{max}$	Largest eigenvalue
$\mu$	Molecular viscosity
$\mu_T$	Eddy viscosity
$\mu_{T_i}$	Inner-layer eddy viscosity
$\mu_{T_o}$	Outer-layer eddy viscosity
$\nu$	Kinematic molecular viscosity, $\mu/\rho$
$\nu_T$	Kinematic eddy viscosity, $\mu_T/\rho$
$\xi$	Dimensionless streamwise distance
$\xi^*, \hat{\xi}$	Closure coefficients
$\tilde{\pi}$	Coles' wake-strength parameter
$\Pi_{ij}$	Pressure-strain correlation tensor
$\rho$	Mass density
$\sigma, \sigma^*$	Closure coefficients
$\sigma_k, \sigma_\epsilon$	Closure coefficients
$\sigma_{L1}, \sigma_{L2}$	Closure coefficients
$\sigma_\tau, \sigma_{\tau 1}, \sigma_{\tau 2}$	Closure coefficients

$\sigma(x)$	Nonequilibrium parameter
$\sigma_{ij}$	Instantaneous total stress tensor
$\tau$	Kolmogorov time scale; turbulence dissipation time
$\tau_{ij}$	Reynolds stress tensor
$\tau_{turnover}$	Eddy turnover time
$\tau_{xy}$	Reynolds shear stress
$\tau_{xx}, \tau_{yy}, \tau_{zz}$	Normal Reynolds stresses
$\tau_w$	Surface shear stress
$\nu$	Kolmogorov velocity scale; closure coefficient
$\phi$	Dimensionless parameter, $(\nu_w/\rho u_\tau^3)dP/dx$
$\chi$	Free shear layer closure coefficient
$\psi$	Streamfunction
$\psi_k, \psi_\epsilon, \psi_\omega$	Parabolic marching scheme coefficients
$\omega$	Specific dissipation rate; vorticity vector magnitude

## Other

Symbol	Definition
$\partial \mathbf{f} / \partial \mathbf{q}$	Turbulence flux-Jacobian matrix
$\partial \mathbf{F} / \partial \mathbf{Q}$	Mean-flow flux-Jacobian matrix
$\partial \mathbf{s} / \partial \mathbf{q}$	Source-Jacobian matrix

## Subscripts

Symbol	Definition
<i>DNS</i>	Direct Numerical Simulation
<i>e</i>	Boundary-layer-edge value
<i>eq</i>	Equilibrium value
<i>LES</i>	Large Eddy Simulation
<i>o</i>	Centerline value
<i>v</i>	Viscous
<i>w</i>	Wall (surface) value
$\infty$	Freestream value

## Superscripts

Symbol	Definition
+	Sublayer-scaled value

# Preface

This book has been developed from the author's lecture notes used in presenting a post-graduate course on turbulence modeling at the University of Southern California. While several computational fluid dynamics (CFD) texts include some information about turbulence modeling, very few texts dealing exclusively with turbulence modeling have been written. As a consequence, turbulence modeling is regarded by many CFD researchers as "black magic," lacking in rigor and physical foundation. This book has been written to show that turbulence modeling can be done in a systematic and physically sound manner. This is not to say all turbulence modeling has been done in such a manner, for indeed many ill-conceived and ill-fated turbulence models have appeared in engineering journals. Even this author, early in his career, devised a turbulence model that violated Galilean invariance of the time-averaged Navier-Stokes equations! However, with judicious use of relatively simple mathematical tools, systematic construction of a well-founded turbulence model is not only possible but can be an exciting and challenging research project.

Thus, the primary goal of this book is to provide a systematic approach to developing a set of constitutive equations suitable for computation of turbulent flows. The engineer who feels no existing turbulence model is suitable for his or her needs and wishes to modify an existing model or to devise a new model will benefit from this feature of the text. A methodology is presented in Chapters 3 and 4 for devising and testing such equations. The methodology is illustrated in great detail for two-equation turbulence models. However, it is by no means limited to such models and is used again in Chapter 6 for a full Reynolds-stress model, but with less detail.

A secondary goal of this book is to provide a rational way for deciding how complex a model is needed for a given problem. The engineer who wishes to select an existing model that is sufficient for his or her needs will benefit most from this feature of the text. Chapter 3 begins with the simplest turbulence models and subsequent chapters chart a course leading to some of the most complex models that have been applied to a nontrivial



turbulent flow problem. Two things are done at each level of complexity. First, the range of applicability of the model is estimated. Second, many of the applications are repeated for all of the models to illustrate how accuracy changes with complexity.

The methodology makes extensive use of tensor analysis, similarity solutions, singular perturbation methods, and numerical procedures. The text assumes the user has limited prior knowledge of these mathematical concepts and provides what is needed both in the main text and in the Appendices. For example, Appendix A introduces rudiments of tensor analysis to facilitate manipulation of the Navier-Stokes equation, which is done extensively in Chapter 2. Chapter 3 shows, in detail, the way a similarity solution is generated. Similarity solutions are then obtained for the turbulent mixing layer, jet and far wake. Appendix B presents elements of singular perturbation theory. Chapters 4, 5 and 6 use the methods to dissect model-predicted features of the turbulent boundary layer.

No book on turbulence-model equations is complete without a discussion of numerical solution methods. Anyone who has ever tried to obtain a numerical solution to a set of turbulence transport equations can attest to this. Often, standard numerical procedures just won't work and alternative methods must be found to obtain accurate converged solutions. Chapter 7 focuses on numerical methods and elucidates some of the commonly encountered problems such as stiffness, sharp turbulent-nonturbulent interfaces, and difficulties attending turbulence related time scales.

The concluding chapter presents a brief overview of new horizons including direct numerical simulation (DNS), large-eddy simulation (LES) and the interesting mathematical theory of chaos.

Because turbulence modeling is a key ingredient in CFD work, the text would be incomplete without companion software implementing numerical solutions to standard turbulence model equations. Appendices C and D describe several computer programs that are included on the floppy disk accompanying the book. The programs all have a similar structure and can be easily modified to include new turbulence models.

The material presented in this book is appropriate for a one-semester, first or second year graduate course, or as a reference text for a CFD course. Successful study of this material requires an understanding of viscous-flow and boundary-layer theory. Some degree of proficiency in solving partial differential equations is also needed. A knowledge of computer programming, preferably in FORTRAN, will help the reader gain maximum benefit from the companion software described in the Appendices.

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*David C. Wilcox*