Turbulence Modeling for CFD

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by

David C. Wilcox

DCW Industries, Inc.
La Cañada, California

Dedicated to my Wife

BARBARA

my Children

KINLEY and BOB

and my Dad

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About the Author

Dr. David C. Wilcox, was born in Wilmington, Delaware. He did his undergraduate studies from 1963 to 1966 at the Massachusetts Institute of Technology, graduating with a Bachelor of Science degree in Aeronautics and Astronautics. From 1966 to 1967, he was an Engineer Scientist Specialist at McDonnell Douglas Aircraft Division in Long Beach, California, working for A. M. O. Smith. His experience with McDonnell Douglas was primarily in subsonic and transonic flow calculations. From 1967 to 1970, he attended the California Institute of Technology, graduating with a Ph.D. in Aeronautics. In 1970 he joined TRW Systems, Inc. in Redondo Beach, California, as a Member of the Technical Staff. He performed studies of both high- and low-speed fluid-mechanical and heat-transfer problems, such as turbulent hypersonic flow and thermal radiation from a flame. From 1972 to 1973, he was a staff scientist for Applied Theory, Inc., in Los Angeles, California, responsible for scientific-project management. He participated directly in many research efforts involving numerical computation and analysis of a wide range of fluid flows such as separated turbulent flow, transitional flow and hypersonic plume-body interaction. In 1973, he founded DCW Industries, Inc., a La Cañada, California firm engaged in engineering research, software development and publishing, for which he is currently the President. He has taught several fluid mechanics and applied mathematics courses at the University of Southern California and at the University of California, Los Angeles.

Dr. Wilcox has published many papers and reports on turbulence modeling, computational fluid dynamics, boundary-layer separation, boundary-layer transition, thermal radiation, and rapidly rotating fluids. He is an Associate Fellow of the American Institute of Aeronautics and Astronautics (AIAA) and has served as an Associate Editor for the AIAA Journal.

Contents

Notation

This section includes the most commonly used notation in this book. In order to avoid departing too much from conventions normally used in literature on turbulence modeling and general fluid mechanics, a few symbols denote more than one quantity.

English Symbols

Symbol	**Definition**
a	Speed of sound
a_{ijkl}	Rapid pressure-strain tensor
A_n, B_n, C_n, D_n	Coefficients in tridiagonal matrix equation
A_o^+	Van Driest damping constant
A_{ij}	Slow pressure-strain tensor
b_{ij}	Dimensionless Reynolds-stress anisotropy tensor
B	Additive constant in the law of the wall
c_{b1}, c_{b2}	Closure coefficients
c_f	Skin friction based on edge velocity, $\tau_w/(\frac{1}{2}\rho U_e^2)$
c_{f_∞}	Skin friction based on freestream velocity, $\tau_w/(\frac{1}{2}\rho U_\infty^2)$
c_{w1}, c_{w2}, c_{w3}	Closure coefficients
C_1, C_2	Closure coefficients
C_{cp}, C_{wk}	Closure coefficients
C_D, C_E	Closure coefficients
C_{dif}, C_{Kleb}	Closure coefficients
C_K	Kolmogorov constant
C_{L1}, C_{L2}	Closure coefficients
C_p	Specific heat at constant pressure; pressure coefficient
C_s, C_ϵ	Closure coefficients
C_S	Smagorinsky constant
C_v	Specific heat at constant volume
C_δ	Shear-layer spreading rate

$C_{\epsilon 1}$, $C_{\epsilon 2}$, $C_{\epsilon 3}$	Closure coefficients
$C_{\tau 1}$, $C_{\tau 2}$	Closure coefficients
C_μ	Closure coefficient
C_{ij}	LES cross-term stress tensor
C_{ijk}	Turbulent transport tensor
D	Drag per unit body width
D_{ij}	Production tensor, $\tau_{im}\partial U_m/\partial x_j + \tau_{jm}\partial U_m/\partial x_i$
e	Specific internal energy ; small-eddy energy
E	Total energy; viscous damping function
$E(\kappa)$	Energy spectral density
$E(\eta)$	Dimensionless self-similar dissipation rate
E_h	Discretization error
f_μ, f_1, f_2, f_s	Viscous damping functions
$\mathbf{f}$, $\mathbf{f}_v$	Turbulence flux vectors
$F(\eta)$	Dimensionless self-similar streamfunction
$F_{Kleb}(y;\delta)$	Klebanoff intermittency function
$\mathbf{F}$, $\mathbf{F}_v$	Mean-flow flux vectors
G	Amplitude factor in von Neumann stability analysis
$G(\mathbf{x}-\boldsymbol{\xi})$	LES filter
h	Specific enthalpy
H	Total enthalpy; channel height; shape factor, δ^*/θ
$\mathcal{H}(x)$	Heaviside step function
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	Unit vectors in x, y, z directions
I	Unit (identity) matrix
II, III	Stress tensor invariants
j	Two-dimensional ($j=0$), axisymmetric ($j=1$) index
J	Specific momentum flux (flux per unit mass)
k	Kinetic energy of turbulent fluctuations per unit mass
k_g	Geometric progression ratio
k_R	Surface roughness height
K	Distortion parameter
$K(\eta)$	Dimensionless self-similar turbulence kinetic energy
K_ϵ, K_ω	Effective Kármán constant for compressible flows
Kn	Knudsen number
ℓ	Turbulence length scale; characteristic eddy size
ℓ_{mfp}	Mean free path
ℓ_{mix}	Mixing length
L	Characteristic length scale
L_{ij}	Leonard stress tensor
M	Mach number
M_{ijkl}	Rapid pressure-strain tensor
M_c	Convective Mach number

M_t	Turbulence Mach number, $\sqrt{2k}/a$
M_{to}	Closure coefficient
$N(\eta)$	Dimensionless self-similar eddy viscosity
N_{CFL}	CFL number
N_ω	Constant in near-wall solution for ω
$\mathcal{N}(u_i)$	Navier-Stokes operator
p	Instantaneous static pressure
p_{ij}	Instantaneous momentum-flux tensor
P	Mean static pressure
P_{ij}	Production tensor, $\tau_{im}\partial U_j/\partial x_m + \tau_{jm}\partial U_i/\partial x_m$
P_k, P_ω, P_ϵ	Net production per unit dissipation of k, ω, ϵ
Pr_L, Pr_T	Laminar, turbulent Prandtl number
q_j	Heat-flux vector
q_w	Surface heat flux
q_{L_j}, q_{T_j}	Laminar, turbulent mean heat-flux vector
Q_{ij}	LES stress tensor, $C_{ij} + R_{ij}$
$\mathbf{Q}$	Dependent variable vector
r, θ, x	Cylindrical polar coordinates
R	Pipe radius; channel half height; perfect gas constant
R_{ij}	SGS Reynolds stress tensor
$R_{ij}(\mathbf{x},\mathrm{t};\mathbf{r})$	Two-point velocity correlation tensor
$\mathcal{R}$	Radius of curvature
$\mathcal{R}_{ij}(\mathbf{x},t;t')$	Autocorrelation tensor
R^+	Sublayer scaled radius or half height, $u_\tau R/\nu$
R_β, R_k, R_ω	Closure coefficients in viscous damping functions
Re_L	Reynolds number based on length L
Re_T	Turbulence Reynolds number, $k^{1/2}\ell/\nu$
Re_τ	Sublayer scaled radius or half height, R^+
Ri_T	Turbulence Richardson number
R_y	Near-wall turbulence Reynolds number, $k^{1/2}y/\nu$
s_{ij}	Instantaneous strain-rate tensor
$\mathbf{s}$, $\mathbf{S}$	Source-term vectors
S	Source term — production minus dissipation
S_{ij}	Mean strain-rate tensor
$\overset{\circ}{S}_{ij}$	Oldroyd derivative of S_{ij}
S_e, S_k, S_u, S_w	Source terms in a similarity solution
S_B	Dimensionless surface mass injection function
S_R	Dimensionless surface roughness function
t	Time
t_{ij}	Instantaneous viscous stress tensor
T	Temperature; characteristic time scale

T'	Freestream turbulence intensity
u, v, w	Instantaneous velocity components in x, y, z directions
u_i	Instantaneous velocity in tensor notation
$\mathbf{u}$	Instantaneous velocity in vector notation
u', v', w'	Fluctuating velocity components in x, y, z directions
u'_i	Fluctuating velocity in tensor notation
$\mathbf{u}'$	Fluctuating velocity in vector notation
$\tilde{u}$, $\tilde{v}$, $\tilde{w}$	Favre-averaged velocity components in x, y, z directions
$\tilde{u}_i$	Favre-averaged velocity in tensor notation
$\tilde{\mathbf{u}}$	Favre-averaged velocity in vector notation
u'', v'', w''	Favre fluctuating velocity components in x, y, z directions
u''_i	Favre fluctuating velocity in tensor notation
$\mathbf{u}''$	Favre fluctuating velocity; fluctuating molecular velocity
u_{rms}, v_{rms}	RMS fluctuating velocity components in x, y directions
$\overline{u'_i u'_j}$	Temporal average of fluctuating velocities
u_τ	Friction velocity, $\sqrt{\tau_w/\rho_w}$
$\hat{\mathbf{u}}$	Velocity perturbation vector
U, V, W	Mean velocity components in x, y, z directions
U_i	Mean velocity in tensor notation
$\mathbf{U}$	Mean velocity in vector notation
U^+	Dimensionless, sublayer-scaled, velocity, U/u_τ
U_m	Maximum or centerline velocity
$\mathcal{U}(\eta)$	Dimensionless self-similar streamwise velocity
v_{mix}	Mixing velocity
v_{th}	Thermal velocity
v_w	Surface injection velocity
$\mathcal{V}(\eta)$	Dimensionless self-similar normal velocity
$W(\eta)$	Dimensionless self-similar specific dissipation rate
x, y, z	Rectangular Cartesian coordinates
x_i	Position vector in tensor notation
$\mathbf{x}$	Position vector in vector notation
y^+	Dimensionless, sublayer-scaled, distance, $u_\tau y/\nu$
y_2^+	y^+ at first grid point above surface
y_m	Inner/outer layer matching point

Greek Symbols

Symbol	**Definition**
α, α^*	Closure coefficients
$\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$	Closure coefficients
α_o, α_o^*	Closure coefficients in viscous damping functions

α_T, σ_T, ω_T	Defect-layer similarity parameters
β, β^*	Closure coefficients
β_T	Equilibrium parameter, $(\delta^*/\tau_w)dP/dx$
γ	Specific heat ratio, C_p/C_v
δ	Boundary layer or shear layer thickness
δ^*	Displacement thickness, $\int_0^\delta \left(1 - \frac{\rho}{\rho_e}\frac{U}{U_e}\right) dy$
δ_v^*	Velocity thickness, $\int_0^\delta \left(1 - \frac{U}{U_e}\right) dy$
δ_x	Finite-difference matrix operator
δ_{ij}	Kronecker delta
Δ	LES filter width
$\Delta(x)$	Clauser thickness, $U_e\delta^*/u_\tau$
$\Delta\mathbf{Q}$, Δx, Δy	Incremental change in $\mathbf{Q}$, x, y
Δt	Timestep
ϵ	Dissipation per unit mass
ϵ_d	Dilatation dissipation
ϵ_s	Solenoidal dissipation
ϵ_{ij}	Dissipation tensor
ϵ_{ijk}	Permutation tensor
ζ	Second viscosity coefficient
η	Kolmogorov length scale; similarity variable
θ	Momentum thickness, $\int_0^\delta \frac{\rho}{\rho_e}\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dy$
κ	Kármán constant; thermal conductivity; wavenumber
κ_v	Effective Kármán constant for flows with mass injection
λ	Taylor microscale
λ_{max}	Largest eigenvalue
μ	Molecular viscosity
μ_T	Eddy viscosity
μ_{T_i}	Inner-layer eddy viscosity
μ_{T_o}	Outer-layer eddy viscosity
ν	Kinematic molecular viscosity, μ/ρ
ν_T	Kinematic eddy viscosity, μ_T/ρ
ξ	Dimensionless streamwise distance
ξ^*, $\hat{\xi}$	Closure coefficients
$\tilde{\pi}$	Coles' wake-strength parameter
Π_{ij}	Pressure-strain correlation tensor
ρ	Mass density
σ, σ^*	Closure coefficients
σ_k, σ_ϵ	Closure coefficients
σ_{L1}, σ_{L2}	Closure coefficients
σ_τ, $\sigma_{\tau1}$, $\sigma_{\tau2}$	Closure coefficients

$\sigma(x)$	Nonequilibrium parameter
σ_{ij}	Instantaneous total stress tensor
τ	Kolmogorov time scale; turbulence dissipation time
τ_{ij}	Reynolds stress tensor
$\tau_{turnover}$	Eddy turnover time
τ_{xy}	Reynolds shear stress
τ_{xx}, τ_{yy}, τ_{zz}	Normal Reynolds stresses
τ_w	Surface shear stress
υ	Kolmogorov velocity scale; closure coefficient
ϕ	Dimensionless parameter, $(\nu_w/\rho u_\tau^3)dP/dx$
χ	Free shear layer closure coefficient
ψ	Streamfunction
$\psi_k, \psi_\epsilon, \psi_\omega$	Parabolic marching scheme coefficients
ω	Specific dissipation rate; vorticity vector magnitude

Other

Symbol	Definition
$\partial\mathbf{f}/\partial\mathbf{q}$	Turbulence flux-Jacobian matrix
$\partial\mathbf{F}/\partial\mathbf{Q}$	Mean-flow flux-Jacobian matrix
$\partial\mathbf{s}/\partial\mathbf{q}$	Source-Jacobian matrix

Subscripts

Symbol	Definition
DNS	Direct Numerical Simulation
e	Boundary-layer-edge value
eq	Equilibrium value
LES	Large Eddy Simulation
o	Centerline value
v	Viscous
w	Wall (surface) value
∞	Freestream value

Superscripts

Symbol	Definition
$+$	Sublayer-scaled value

Preface

This book has been developed from the author's lecture notes used in presenting a post-graduate course on turbulence modeling at the University of Southern California. While several computational fluid dynamics (CFD) texts include some information about turbulence modeling, very few texts dealing exclusively with turbulence modeling have been written. As a consequence, turbulence modeling is regarded by many CFD researchers as "black magic," lacking in rigor and physical foundation. This book has been written to show that turbulence modeling can be done in a systematic and physically sound manner. This is not to say all turbulence modeling has been done in such a manner, for indeed many ill-conceived and ill-fated turbulence models have appeared in engineering journals. Even this author, early in his career, devised a turbulence model that violated Galilean invariance of the time-averaged Navier-Stokes equations! However, with judicious use of relatively simple mathematical tools, systematic construction of a well-founded turbulence model is not only possible but can be an exciting and challenging research project.

Thus, the primary goal of this book is to provide a systematic approach to developing a set of constitutive equations suitable for computation of turbulent flows. The engineer who feels no existing turbulence model is suitable for his or her needs and wishes to modify an existing model or to devise a new model will benefit from this feature of the text. A methodology is presented in Chapters 3 and 4 for devising and testing such equations. The methodology is illustrated in great detail for two-equation turbulence models. However, it is by no means limited to such models and is used again in Chapter 6 for a full Reynolds-stress model, but with less detail.

A secondary goal of this book is to provide a rational way for deciding how complex a model is needed for a given problem. The engineer who wishes to select an existing model that is sufficient for his or her needs will benefit most from this feature of the text. Chapter 3 begins with the simplest turbulence models and subsequent chapters chart a course leading to some of the most complex models that have been applied to a nontrivial turbulent flow problem. Two things are done at each level of complexity. First, the range of applicability of the model is estimated. Second, many of the applications are repeated for all of the models to illustrate how accuracy changes with complexity.

The methodology makes extensive use of tensor analysis, similarity solutions, singular perturbation methods, and numerical procedures. The text assumes the user has limited prior knowledge of these mathematical concepts and provides what is needed both in the main text and in the Appendices. For example, Appendix A introduces rudiments of tensor analysis to facilitate manipulation of the Navier-Stokes equation, which is done extensively in Chapter 2. Chapter 3 shows, in detail, the way a similarity solution is generated. Similarity solutions are then obtained for the turbulent mixing layer, jet and far wake. Appendix B presents elements of singular perturbation theory. Chapters 4, 5 and 6 use the methods to dissect model-predicted features of the turbulent boundary layer.

No book on turbulence-model equations is complete without a discussion of numerical solution methods. Anyone who has ever tried to obtain a numerical solution to a set of turbulence transport equations can attest to this. Often, standard numerical procedures just won't work and alternative methods must be found to obtain accurate converged solutions. Chapter 7 focuses on numerical methods and elucidates some of the commonly encountered problems such as stiffness, sharp turbulent-nonturbulent interfaces, and difficulties attending turbulence related time scales.

The concluding chapter presents a brief overview of new horizons including direct numerical simulation (DNS), large-eddy simulation (LES) and the interesting mathematical theory of chaos.

Because turbulence modeling is a key ingredient in CFD work, the text would be incomplete without companion software implementing numerical solutions to standard turbulence model equations. Appendices C and D describe several computer programs that are included on the floppy disk accompanying the book. The programs all have a similar structure and can be easily modified to include new turbulence models.

The material presented in this book is appropriate for a one-semester, first or second year graduate course, or as a reference text for a CFD course. Successful study of this material requires an understanding of viscous-flow and boundary-layer theory. Some degree of proficiency in solving partial differential equations is also needed. A knowledge of computer programming, preferably in FORTRAN, will help the reader gain maximum benefit from the companion software described in the Appendices.

I extend my thanks to Dr. L. G. Redekopp of USC for encouraging and supporting development of the course for which this book is intended. A friend of many years, Dr. P. Bradshaw, reviewed the entire manuscript as I wrote it, and taught me a lot through numerous discussions, comments and suggestions that greatly improved the final draft. Another long time friend, Dr. D. D. Knight, helped me understand why I had to write this book, reviewed the manuscript from cover to cover and offered a great deal of physical and computational insight in the process. My favorite mathematics teacher, Dr. D. S. Cohen, made sure I omitted the dot over every ι and crossed every z in Appendix B. Drs. F. R. Menter and C. C. Horstman were kind enough to provide results of several of their computations in digital form. Thanks are also due for the support and help of several friends and colleagues, most notably Drs. P. J. Roache, C. G. Speziale and R. M. C. So.

I thank the nine students who were the first to take the course that this book was written for. Their patience was especially noteworthy, particularly in regard to typographical errors in the homework problems! That outstanding group of young engineers is D. Foley, R. T. Holbrook, N. Kale, T.-S. Leu, H. Lin, T. Magee, S. Tadepalli, P. Taniguchi and D. Wallace.

Finally, I owe a lifelong debt to my loving wife Barbara for tolerating the hectic pace first in college and then in the business world. Without her, this book would not have been possible.

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