Chapter 2

The Closure Problem

Because turbulence consists of random fluctuations of the various flow properties, we use a statistical approach. Our purposes are best served by using the procedure introduced by Reynolds (1895) in which all quantities are expressed as the sum of mean and fluctuating parts. We then form the time average of the continuity and Navier-Stokes equations. As we will see in this chapter, the nonlinearity of the Navier-Stokes equation leads to the appearance of momentum fluxes that act as apparent stresses throughout the flow. These momentum fluxes are unknown a priori. We then derive equations for these stresses and the resulting equations include additional unknown quantities. This illustrates the issue of closure, i.e., establishing a sufficient number of equations for all of the unknowns.

2.1 Reynolds Averaging

We begin with the averaging concepts introduced by Reynolds (1895). In general, Reynolds averaging assumes a variety of forms involving either an integral or a summation. The three forms most pertinent in turbulence-model research are the **time average**, the **spatial average** and the **ensemble average**.

Time averaging is appropriate for **stationary turbulence**, i.e., a turbulent flow that, on the average, does not vary with time. For such a flow, we express an instantaneous flow variable as $f(\mathbf{x}, t)$. Its time average, $F_T(\mathbf{x})$, is defined by

$$F_T(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(\mathbf{x}, t)\, dt \tag{2.1}$$

Spatial averaging is appropriate for **homogeneous** turbulence, which is a turbulent flow that, on the average, is uniform in all directions. We average over all spatial coordinates by doing a volume integral. Calling the average F_V, we have

$$F_V(t) = \lim_{V \to \infty} \frac{1}{V} \iiint f(\mathbf{x}, t)\, dV \tag{2.2}$$

Ensemble averaging is the most general type of averaging. As an idealized example, in terms of measurements from N identical experiments where $f(\mathbf{x}, t) = f_n(\mathbf{x}, t)$ in the n^{th} experiment, the average is F_E, where

$$F_E(\mathbf{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\mathbf{x}, t) \tag{2.3}$$

For turbulence that is both stationary and homogeneous, we may assume that these three averages are all equal. This assumption is known as the **ergodic hypothesis**.

Because virtually all engineering problems involve **inhomogeneous turbulence**, time averaging is the most appropriate form of Reynolds averaging. The time-averaging process is most clearly explained for stationary turbulence. For such a flow, we express the instantaneous velocity, $u_i(\mathbf{x}, t)$, as the sum of a mean, $U_i(\mathbf{x})$, and a fluctuating part, $u_i'(\mathbf{x}, t)$, so that

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}) + u_i'(\mathbf{x}, t) \tag{2.4}$$

As in Equation (2.1), the quantity $U_i(\mathbf{x})$ is the time-averaged, or mean, velocity defined by

$$U_i(\mathbf{x}) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} u_i(\mathbf{x}, t)\, dt \tag{2.5}$$

The time average of the mean velocity is again the same time-averaged value, i.e.,

$$\overline{U_i}(\mathbf{x}) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} U_i(\mathbf{x})\, dt = U_i(\mathbf{x}) \tag{2.6}$$

where an overbar is shorthand for time average. The time average of the fluctuating part of the velocity is zero. That is, using Equation (2.6),

$$\overline{u_i'} = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} [u_i(\mathbf{x}, t) - U_i(\mathbf{x})]\, dt = U_i(\mathbf{x}) - \overline{U_i}(\mathbf{x}) = 0 \tag{2.7}$$

While Equation (2.5) is mathematically well defined, we can never truly realize infinite T in any physical flow. This is not a serious problem in practice however. In forming our time average, we just select a time T that is very long relative to the maximum period of the velocity fluctuations, T_1. In other words, rather than formally taking the limit $T \to \infty$, we do the indicated integration in Equation (2.5) with $T \gg T_1$. As an example, for flow at 10 m/sec in a 5 cm diameter pipe, an integration time of 20 seconds would probably be adequate. In this time the flow moves 4,000 pipe diameters.

There are some flows for which the mean flow contains very slow variations with time that are not turbulent in nature. For instance, we might impose a slowly varying periodic pressure gradient in a duct or we might wish to compute flow over a helicopter blade. Clearly, Equations (2.4) and (2.5) must be modified to accommodate such applications. The simplest, but a bit more arbitrary, method is to replace Equations (2.4) and (2.5) with

$$u_i(\mathbf{x},t) = U_i(\mathbf{x},t) + u_i'(\mathbf{x},t) \tag{2.8}$$

and

$$U_i(\mathbf{x},t) = \frac{1}{T}\int_t^{t+T} u_i(\mathbf{x},t)\,dt, \qquad T_1 \ll T \ll T_2 \tag{2.9}$$

where T_2 is the time scale characteristic of the slow variations in the flow that we do not wish to regard as belonging to the turbulence. Figure 2.1 illustrates these concepts.

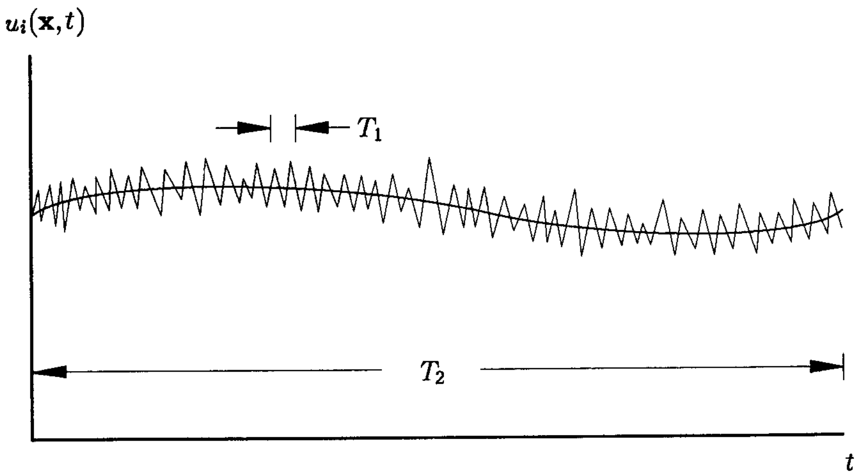

Figure 2.1: Time averaging for nonstationary turbulence.

A word of caution is in order regarding Equation (2.9). We are implicitly assuming that time scales T_1 and T_2 exist that differ by several orders of magnitude. Very few unsteady flows of engineering interest are guaranteed to satisfy this condition. We cannot use Equations (2.8) and (2.9) for such flows because there is no distinct boundary between our imposed unsteadiness and turbulent fluctuations. For such flows, the mean and fluctuating components are correlated, i.e., the time average of their product is non-vanishing. In meteorology, for example, this is known as the **spectral gap problem**. If the flow is periodic, **Phase Averaging** (see Problems) can be used; otherwise, ensemble averaging is necessary. For a rigorous approach, an alternative method such as Large Eddy Simulation (Chapter 8) will be required.

Clearly our time averaging process, involving integrals over time, commutes with spatial differentiation. Thus, for any scalar p and vector u_i,

$$\overline{p_{,i}} = P_{,i} \quad \text{and} \quad \overline{u_{i,j}} = U_{i,j} \tag{2.10}$$

Because we are dealing with definite integrals, time averaging is a linear operation. Thus if c_1 and c_2 are constants while a and b denote any two flow properties, then

$$\overline{c_1 a + c_2 b} = c_1 A + c_2 B \tag{2.11}$$

The time average of an unsteady term like $\partial u_i/\partial t$ is obviously zero for stationary turbulence. For nonstationary turbulence, we must look a little closer. We know that

$$\frac{1}{T}\int_t^{t+T} \frac{\partial}{\partial t}(U_i + u_i')\, dt = \frac{U_i(\mathbf{x}, t+T) - U_i(\mathbf{x}, t)}{T} + \frac{u_i'(\mathbf{x}, t+T) - u_i'(\mathbf{x}, t)}{T} \tag{2.12}$$

The second term on the right-hand side of Equation (2.12) vanishes because T effectively approaches ∞ on the time scale of the turbulent fluctuations. By contrast, T is very small relative to the time scale of the mean flow, so that the first term is the value corresponding to the limit $T \to 0$, i.e., $\partial U_i/\partial t$. Hence,

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial U_i}{\partial t} \tag{2.13}$$

Although it may seem a bit unusual to be taking the limit $T \to \infty$ and $T \to 0$ in the same equation, the process can be fully justified using the two-timing method from perturbation theory [see Kevorkian and Cole (1981)]. The notion is simply that we have a slow time scale and a fast time scale, similar to the case of small damping on a linear oscillator. In a perturbation analysis of such a problem, dependent variables become functions of two independent time variables (essentially t/T_1 and t/T_2). In the normal spirit of perturbation theory, the limit $t/T_1 \to \infty$ corresponds to the limit $t/T_2 \to 0$.

2.2 Correlations

Thus far we have considered time averages of linear quantities. When we time average the product of two properties, say ϕ and ψ, we have the following:

$$\overline{\phi\psi} = \overline{(\Phi + \phi')(\Psi + \psi')} = \overline{\Phi\Psi + \Phi\psi' + \Psi\phi' + \phi'\psi'} = \Phi\Psi + \overline{\phi'\psi'} \tag{2.14}$$

where we take advantage of the fact that the product of a mean quantity and a fluctuating quantity has zero mean. There is no a priori reason for the time average of the product of two fluctuating quantities to vanish. Thus, Equation (2.14) tells us the mean value of a product, $\overline{\phi\psi}$, differs from the product of the mean values, $\Phi\Psi$. The quantities ϕ' and ψ' are said to be **correlated** if $\overline{\phi'\psi'} \neq 0$. They are **uncorrelated** if $\overline{\phi'\psi'} = 0$.

Similarly, for a triple product, we find

$$\overline{\phi\psi\xi} = \Phi\Psi\Xi + \overline{\phi'\psi'}\Xi + \overline{\psi'\xi'}\Phi + \overline{\phi'\xi'}\Psi + \overline{\phi'\psi'\xi'} \tag{2.15}$$

Again, terms linear in ϕ', ψ' or ξ' have zero mean. As with terms quadratic in fluctuating quantities, there is no a priori reason for the cubic term, $\overline{\phi'\psi'\xi'}$, to vanish.

2.3 Reynolds-Averaged Equations

For simplicity we confine our attention to incompressible flow. Effects of compressibility will be addressed in Chapter 5. The equations for conservation of mass and momentum are

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.16}$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ji}}{\partial x_j} \tag{2.17}$$

The vectors u_i and x_i are velocity and position, t is time, p is pressure, ρ is density and t_{ij} is the viscous stress tensor defined by

$$t_{ij} = 2\mu s_{ij} \tag{2.18}$$

where μ is molecular viscosity and s_{ij} is the strain-rate tensor,

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.19}$$

To simplify the time-averaging process, we rewrite the convective term in conservation form, i.e.,

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_j u_i) - u_i \frac{\partial u_j}{\partial x_j} = \frac{\partial}{\partial x_j}(u_j u_i) \tag{2.20}$$

where we take advantage of Equation (2.16) in order to drop $u_i \partial u_j / \partial x_j$. Combining Equations (2.17) through (2.20) yields the Navier-Stokes equation in conservation form.

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu s_{ji}) \tag{2.21}$$

Time averaging Equations (2.16) and (2.21) yields the **Reynolds averaged equations of motion in conservation form**, viz.,

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{2.22}$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_j}\left(U_j U_i + \overline{u_j' u_i'}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji}) \tag{2.23}$$

The time-averaged conservation of mass, Equation (2.22), is identical to the instantaneous Equation (2.16) with the mean velocity replacing the instantaneous velocity. Subtracting Equation (2.22) from Equation (2.16) shows that the fluctuating velocity, u_i', also has zero divergence. Aside from replacement of instantaneous variables by mean values, the only difference between the time-averaged and instantaneous momentum equations is the appearance of the correlation $\overline{u_i' u_j'}$.

Herein lies the fundamental problem of turbulence for the engineer. In order to compute all mean-flow properties of the turbulent flow under consideration, we need a prescription for computing $\overline{u_i' u_j'}$.

Equation (2.23) can be written in its most recognizable form by using Equation (2.20) in reverse. There follows

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ji} - \rho \overline{u_j' u_i'}\right) \tag{2.24}$$

Equation (2.24) is usually referred to as the **Reynolds-averaged Navier-Stokes equation**. The quantity $-\rho \overline{u_i' u_j'}$ is known as the **Reynolds-stress tensor** and we denote it by τ_{ij}. Thus,

$$\tau_{ij} = -\rho \overline{u_i' u_j'} \tag{2.25}$$

By inspection, $\tau_{ij} = \tau_{ji}$ so that this is a symmetric tensor, and thus has six independent components. Hence, we have produced six unknown quantities as a result of Reynolds averaging. Unfortunately, we have gained no additional equations. So, for general three-dimensional flows, we have four unknown mean-flow properties, viz., pressure and the three velocity components. Along with the six Reynolds-stress components, we thus have ten unknowns. Our equations are mass conservation [Equation (2.22)] and the three components of Equation (2.24) for a grand total of four. This means our system is not yet **closed**. To close the system, we must find enough equations to solve for our unknowns.

2.4 The Reynolds-Stress Equation

In quest of additional equations, we can take moments of the Navier-Stokes equation. That is, we multiply the Navier-Stokes equation by a fluctuating property and time average the product. Using this procedure, we can derive a differential equation for the Reynolds-stress tensor. To illustrate the process, we introduce some special notation. Let $\mathcal{N}(u_i)$ denote the Navier-Stokes operator, viz.,

$$\mathcal{N}(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{2.26}$$

The viscous term has been simplified by noting from mass conservation (for incompressible flow) that $s_{ki,k} = u_{i,kk}$. Thus, the Navier-Stokes equation can be written symbolically as

$$\mathcal{N}(u_i) = 0 \tag{2.27}$$

In order to derive an equation for the Reynolds stress tensor, we form the following time average.

$$\overline{u_i'\mathcal{N}(u_j) + u_j'\mathcal{N}(u_i)} = 0 \tag{2.28}$$

Note that, consistent with the symmetry of the Reynolds stress tensor, the resulting equation is also symmetric in i and j. For the sake of clarity, we proceed term by term. Also, for economy of space, we use tensor notation for derivatives throughout the time averaging process. First, we consider the **unsteady term**.

$$\begin{aligned}
\overline{u_i'(\rho u_j)_{,t} + u_j'(\rho u_i)_{,t}} &= \overline{\rho u_i'(U_j + u_j')_{,t}} + \overline{\rho u_j'(U_i + u_i')_{,t}} \\
&= \overline{\rho u_i' U_{j,t}} + \overline{\rho u_i' u'_{j,t}} + \overline{\rho u_j' U_{i,t}} + \overline{\rho u_j' u'_{i,t}} \\
&= \overline{\rho u_i' u'_{j,t}} + \overline{\rho u_j' u'_{i,t}} \\
&= \overline{(\rho u_i' u_j')_{,t}} \\
&= -\frac{\partial \tau_{ij}}{\partial t}
\end{aligned} \tag{2.29}$$

Turning to the **convective term**, we have

$$\begin{aligned}
\overline{\rho u_i' u_k u_{j,k} + \rho u_j' u_k u_{i,k}} &= \overline{\rho u_i'(U_k + u_k')(U_j + u_j')_{,k}} \\
&+ \overline{\rho u_j'(U_k + u_k')(U_i + u_i')_{,k}} \\
&= \overline{\rho u_i' U_k u'_{j,k}} + \overline{\rho u_i' u_k'(U_j + u_j')_{,k}} \\
&+ \overline{\rho u_j' U_k u'_{i,k}} + \overline{\rho u_j' u_k'(U_i + u_i')_{,k}} \\
&= U_k \overline{(\rho u_i' u_j')_{,k}} + \overline{\rho u_i' u_k'} U_{j,k} \\
&+ \overline{\rho u_j' u_k'} U_{i,k} + \overline{\rho u_k' (u_i' u_j')_{,k}} \\
&= -U_k \frac{\partial \tau_{ij}}{\partial x_k} - \tau_{ik} \frac{\partial U_j}{\partial x_k} - \tau_{jk} \frac{\partial U_i}{\partial x_k} \\
&+ \frac{\partial}{\partial x_k} \overline{(\rho u_i' u_j' u_k')}
\end{aligned} \tag{2.30}$$

In order to arrive at the final line of Equation (2.30), we use the fact that $\partial u_k'/\partial x_k = 0$. The **pressure gradient** term is straightforward.

$$\begin{aligned}
\overline{u_i' p_{,j} + u_j' p_{,i}} &= \overline{u_i'(P + p')_{,j}} + \overline{u_j'(P + p')_{,i}} \\
&= \overline{u_i' p'_{,j} + u_j' p'_{,i}} \\
&= \overline{u_i' \frac{\partial p'}{\partial x_j} + u_j' \frac{\partial p'}{\partial x_i}}
\end{aligned} \tag{2.31}$$

Finally, the **viscous term** yields

$$\begin{aligned}
\overline{\mu(u_i' u_{j,kk} + u_j' u_{i,kk})} &= \overline{\mu u_i'(U_j + u_j')_{,kk}} + \overline{\mu u_j'(U_i + u_i')_{,kk}} \\
&= \overline{\mu u_i' u'_{j,kk}} + \overline{\mu u_j' u'_{i,kk}} \\
&= \overline{\mu(u_i' u'_{j,k})_{,k}} + \overline{\mu(u_j' u'_{i,k})_{,k}} - \overline{2\mu u'_{i,k} u'_{j,k}} \\
&= \overline{\mu(u_i' u_j')_{,kk}} - \overline{2\mu u'_{i,k} u'_{j,k}} \\
&= -\nu \frac{\partial^2 \tau_{ij}}{\partial x_k \partial x_k} - 2\mu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}
\end{aligned} \tag{2.32}$$

Collecting terms, we arrive at the equation for the Reynolds stress tensor.

$$\begin{aligned}
\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} &= -\tau_{ik} \frac{\partial U_j}{\partial x_k} - \tau_{jk} \frac{\partial U_i}{\partial x_k} + 2\mu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} + \overline{u_i' \frac{\partial p'}{\partial x_j} + u_j' \frac{\partial p'}{\partial x_i}} \\
&+ \frac{\partial}{\partial x_k} \left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + \overline{\rho u_i' u_j' u_k'} \right]
\end{aligned} \tag{2.33}$$

We have gained six new equations, one for each independent component of the Reynolds-stress tensor. However, we have also generated 22 new unknowns! Specifically, accounting for all symmetries, we have the following.

$$\begin{aligned}
\overline{\rho u_i' u_j' u_k'} \quad &\rightarrow \quad 10 \text{ unknowns} \\
\overline{2\mu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \quad &\rightarrow \quad 6 \text{ unknowns} \\
\overline{u_i' \frac{\partial p'}{\partial x_j} + u_j' \frac{\partial p'}{\partial x_i}} \quad &\rightarrow \quad 6 \text{ unknowns}
\end{aligned}$$

With a little rearrangement of terms, we can cast the **Reynolds-stress equation** in its most recognizable form, viz.,

$$\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial U_j}{\partial x_k} - \tau_{jk} \frac{\partial U_i}{\partial x_k} + \epsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k} \left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk} \right] \quad (2.34)$$

where

$$\Pi_{ij} = \overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)} \quad (2.35)$$

$$\epsilon_{ij} = \overline{2\mu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \quad (2.36)$$

$$C_{ijk} = \overline{\rho u_i' u_j' u_k'} + \overline{p' u_i'} \delta_{jk} + \overline{p' u_j'} \delta_{ik} \quad (2.37)$$

This exercise illustrates the closure problem of turbulence. Because of the nonlinearity of the Navier-Stokes equation, as we take higher and higher moments, we generate additional unknowns at each level. At no point will this procedure balance our unknowns/equations ledger. On physical grounds, this is not a particularly surprising situation. After all, such operations are strictly mathematical in nature, and introduce no additional physical principles. The function of turbulence modeling is to devise approximations for the unknown correlations in terms of flow properties that are known so that a sufficient number of equations exists. In making such approximations, we close the system.

Problems

2.1 Suppose we have a velocity field that consists of: (i) a slowly varying component $U(t) = U_0 e^{-t/\tau}$ where U_0 and τ are constants and (ii) a rapidly varying component $u' = aU_0 \cos(2\pi t/\epsilon^2 \tau)$ where a and ϵ are constants with $\epsilon \ll 1$. We want to show that by choosing $T = \epsilon\tau$, the limiting process in Equation (2.9) makes sense.

(a) Compute the exact time average of $u = U + u'$.

(b) Replace T by $\epsilon\tau$ in the slowly varying part of the time average of u and let $t_f = \epsilon^2\tau$ in the fluctuating part of u to show that

$$\overline{U + u'} = U(t) + O(\epsilon)$$

where $O(\epsilon)$ denotes a quantity that goes to zero linearly with ϵ as $\epsilon \rightarrow 0$.

(c) Repeat Parts (a) and (b) for du/dt.

2.2 For an imposed periodic mean flow, a standard way of decomposing flow properties is to write

$$u(\mathbf{x}, t) = U(\mathbf{x}) + \hat{u}(\mathbf{x}, t) + u'(\mathbf{x}, t)$$

where $U(\mathbf{x})$ is the mean-value, $\hat{u}(\mathbf{x}, t)$ is the organized response component due to the imposed organized unsteadiness, and $u'(\mathbf{x}, t)$ is the turbulent fluctuation. $U(\mathbf{x})$ is defined as in Equation (2.5). We also use the **Phase Average** defined by

$$< u(\mathbf{x}, t) > \equiv \lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=0}^{N-1} u(\mathbf{x}, t + n\tau)$$

where τ is the period of the imposed excitation. Then, by definition,

$$< u(\mathbf{x}, t) > = U(\mathbf{x}) + \hat{u}(\mathbf{x}, t), \quad \overline{< u(\mathbf{x}, t) >} = U(\mathbf{x}), \quad < \hat{u}(\mathbf{x}, t) > = \hat{u}(\mathbf{x}, t)$$

Verify the following.

(a) $< U > = U$

(b) $\overline{\hat{u}} = 0$

(c) $\overline{u'} = 0$

(d) $< u' > = 0$

(e) $\overline{\hat{u}v'} = 0$

(f) $< \hat{u}v > = \hat{u} < v >$

(g) $< \hat{u}v' > = 0$

(h) $< Uv > = U < v >$

2.3 For an incompressible flow, we have an imposed freestream velocity given by

$$u(x,t) = U_0(1 - ax) + U_0 ax \sin 2\pi f t$$

where a is a constant of dimension 1/length, U_0 is a constant reference velocity, and f is frequency. Integrating over one period, compute the average pressure gradient, dP/dx, for $f = 0$ and $f \neq 0$ in the freestream where the inviscid Euler equation holds, i.e.,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

2.4 Compute the difference between the Reynolds average of a quadruple product $\phi\psi\xi\upsilon$ and the product of the means, $\Phi\Psi\Xi\Upsilon$.

2.5 Consider the Reynolds stress equation as stated in Equation (2.34).

(a) Show how Equation (2.34) follows from Equation (2.33).

(b) Contract Equation (2.34), i.e., set $i = j$ and perform the indicated summation, to derive a differential equation for the kinetic energy of the turbulence per unit mass defined by $k \equiv \frac{1}{2}\overline{u_i' u_i'}$.