Appendix B

Rudiments of Perturbation Methods

When we work with perturbation methods, we are constantly dealing with the concept of **order of magnitude**. There are three conventional **order symbols** that provide a mathematical measure of the order of magnitude of a given quantity, viz., **Big O**, **Little o**, and $\sim$. They are defined as follows.

Big O : $f(\delta) = O[g(\delta)]$ as $\delta \to \delta_o$ if a neighborhood of δ_o exists and a constant M exists such that $|f| \leq M|g|$, i.e., $f(\delta)/g(\delta)$ is bounded as $\delta \to \delta_o$.

Little o : $f(\delta) = o[g(\delta)]$ as $\delta \to \delta_o$ if, given any $\epsilon > 0$, there exists a neighborhood of δ_o such that $|f| \leq \epsilon|g|$, i.e., $f(\delta)/g(\delta) \to 0$ as $\delta \to \delta_o$.

$\sim$: $f(\delta) \sim g(\delta)$ as $\delta \to \delta_o$ if $f(\delta)/g(\delta) \to 1$ as $\delta \to \delta_o$.

For example, the Taylor series for the exponential function is

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots \tag{B.1}$$

where "$\cdots$" is conventional shorthand for the rest of the Taylor series, i.e.,

$$\cdots = \sum_{n=4}^{\infty} \frac{(-1)^n x^n}{n!} \tag{B.2}$$

In terms of the ordering symbols, we can replace "$\cdots$" as follows.

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + o(x^3) \tag{B.3}$$

We define an **asymptotic sequence of functions** as a sequence $\phi_n(\delta)$, n = 1, 2, 3, ... satisfying the condition

$$\phi_{n+1}(\delta) = o[\phi_n(\delta)] \quad \text{as} \quad \delta \to \delta_o \tag{B.4}$$

Examples of asymptotic sequences are:

$$\left.\begin{array}{lcll} \phi_n(\delta) & = & 1, (\delta-\delta_o), (\delta-\delta_o)^2, (\delta-\delta_o)^3, \ldots & \delta \to \delta_o \\ \phi_n(\delta) & = & 1, \delta^{1/2}, \delta, \delta^{3/2}, \ldots & \delta \to 0 \\ \phi_n(\delta) & = & 1, \delta, \delta^2 \ell n \delta, \delta^2, \ldots & \delta \to 0 \\ \phi_n(x) & = & x^{-1}, x^{-2}, x^{-3}, x^{-4}, \ldots & x \to \infty \end{array}\right\} \tag{B.5}$$

We say that $g(\delta)$ is **transcendentally small** if $g(\delta)$ is $o[\phi_n(\delta)]$ for all n. For example,

$$e^{-1/\delta} = o(\delta^n) \quad \text{for all } n \tag{B.6}$$

An asymptotic expansion is the sum of the first N terms in an asymptotic sequence. It is the asymptotic expansion of a function $F(\delta)$ as $\delta \to \delta_o$ provided

$$F(\delta) = \sum_{n=1}^{N} a_n \phi_n(\delta) + o[\phi_N(\delta)] \tag{B.7}$$

The following are a few useful asymptotic expansions generated from simple Taylor series expansions, all of which are convergent as $\delta \to 0$.

$$\left.\begin{array}{lcl} (1+\delta)^n & \sim & 1 + n\delta + \frac{n(n-1)}{2}\delta^2 + O(\delta^3) \\ \ell n(1+\delta) & \sim & \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 + O(\delta^4) \\ (1-\delta)^{-1} & \sim & 1 + \delta + \delta^2 + O(\delta^3) \\ \cos\delta & \sim & 1 - \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4 + O(\delta^6) \\ \sin\delta & \sim & \delta - \frac{1}{6}\delta^3 + \frac{1}{120}\delta^5 + O(\delta^7) \\ \tan\delta & \sim & \delta + \frac{1}{3}\delta^3 + \frac{2}{15}\delta^5 + O(\delta^7) \end{array}\right\} \tag{B.8}$$

Not all asymptotic expansions are developed as a Taylor series, nor are they necessarily convergent. For example, consider the complementary error function, $erfc(x)$, i.e.,

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \tag{B.9}$$

We can generate an asymptotic expansion using a succession of integration by parts operations. (To start the process, for example, multiply and divide the integrand by t so that $t\exp(-t^2)$ becomes integrable in closed form.) The expansion is:

$$erfc(x) \sim \frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(1)(3)\ldots(2n-1)}{2^{n+1}x^{2n+1}} \quad \text{as} \quad x \to \infty$$

$$\sim \frac{2}{\sqrt{\pi}}e^{-x^2}\left\{\frac{x^{-1}}{2} - \frac{x^{-3}}{4} + O(x^{-5})\right\} \tag{B.10}$$

A simple ratio test shows that this series is divergent for all values of x. However, if we define the remainder after the first N terms of the series as $R_N(x)$, there are two limits we can consider, viz.,

$$\lim_{x\to\infty}|R_N(x)|_{Fixed\ N} = 0 \quad \text{and} \quad \lim_{N\to\infty}|R_N(x)|_{Fixed\ x} = \infty \tag{B.11}$$

Thus this divergent series gives a good approximation to $erfc(x)$ provided we don't keep too many terms! This is often the case for an asymptotic series.

Part of our task in developing a perturbation solution is to determine the appropriate asymptotic sequence. It is usually obvious, but not always. Also, more than one set of $\phi_n(\delta)$ may be suitable, i.e., we are not guaranteed uniqueness in perturbation solutions. These problems, although annoying from a theoretical viewpoint, by no means diminish the utility of perturbation methods. Usually, we have physical intuition to help guide us in developing our solution. This type of mathematical approach is, after all, standard operating procedure for the engineer. We are, in essence, using the methods Prandtl and von Kármán used before perturbation analysis was given a name.

A **singular perturbation problem** is one in which no single asymptotic expansion is uniformly valid throughout the field of interest. For example, while $\delta/x^{1/2} = O(\delta)$ as $\delta \to 0$, the singularity as $x \to 0$ means this expression is not uniformly valid. Similarly, $\delta \ell n x = O(\delta)$ as $\delta \to 0$ and is not uniformly valid as $x \to 0$ and as $x \to \infty$. The two most common situations that lead to a singular perturbation problem are:

(a) the coefficient of the highest derivative in a differential equation is very small;

(b) difficulties arise in behavior near boundaries.

Case (b) typically arises in analyzing the turbulent boundary layer where logarithmic behavior of the solution occurs close to a solid boundary. The following second-order ordinary differential equation illustrates Case (a).

$$\delta\frac{d^2F}{ds^2} + \frac{dF}{ds} + F = 0; \qquad 0 \le s \le 1 \tag{B.12}$$

We want to solve this equation subject to the following boundary conditions.

$$F(0) = 0 \quad \text{and} \quad F(1) = 1 \tag{B.13}$$

We also assume that δ is very small compared to 1, i.e.,

$$\delta \ll 1 \tag{B.14}$$

This equation is a simplified analog of the Navier-Stokes equation. The second-derivative term has a small coefficient just as the second-derivative term in the Navier-Stokes equation, in nondimensional form, has the reciprocal of the Reynolds number as its coefficient. An immediate consequence is that only one boundary condition can be satisfied if we set $\delta = 0$. This is similar to setting viscosity to zero in the Navier-Stokes equation, which yields Euler's equation, and the attendant consequence that only the normal velocity surface boundary condition can be satisfied. That is, we cannot enforce the no-slip boundary condition for Euler-equation solutions.

The exact solution to this equation is

$$F(s;\delta) = \frac{e^{1-s} - e^{1-s/\delta}}{1 - e^{1-1/\delta}} \tag{B.15}$$

which clearly satisfies both boundary conditions. If we set $\delta = 0$ in Equation (B.12), we have the following first-order equation:

$$\frac{dF}{ds} + F = 0 \tag{B.16}$$

and the solution, $F(s;0)$, is

$$F(s;0) = e^{1-s} \tag{B.17}$$

where we use the boundary condition at $s = 1$. However, the solution fails to satisfy the boundary condition at $s = 0$ because $F(0;0) = e = 2.71828\cdots$. Figure B.1 illustrates the solution to our simplified equation for several values of δ.

As shown, the smaller the value of δ, the more closely $F(s;0)$ represents the solution throughout the region $0 < s \leq 1$. Only in the immediate vicinity of $s = 0$ is the solution inaccurate. The thin layer where $F(s;0)$ departs from the exact solution is called a boundary layer, in direct analogy to its fluid-mechanical equivalent.

To solve this problem using perturbation methods, we seek a solution that consists of two separate asymptotic expansions, one known as the

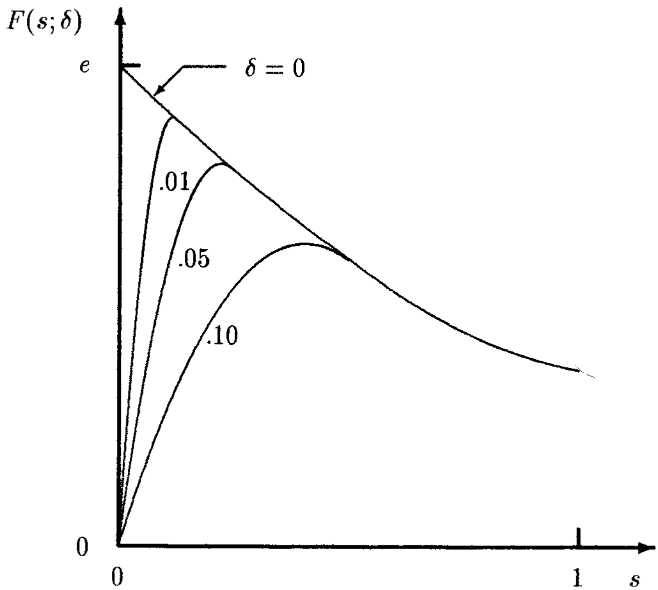

Figure B.1: Solutions to the model equation for several values of δ.

outer expansion and the other as the **inner expansion**. For the outer expansion, we assume a solution of the form

$$F_{outer}(s;\delta) \sim \sum_{n=0}^{N} F_n(s)\phi_n(\delta) \tag{B.18}$$

where the asymptotic sequence functions, $\phi_n(\delta)$, will be determined as part of the solution. Substituting Equation (B.18) into Equation (B.12) yields the following.

$$\sum_{n=0}^{N} \left\{ \frac{d^2 F_n}{ds^2} \delta\phi_n(\delta) + \frac{dF_n}{ds}\phi_n(\delta) + F_n\phi_n(\delta) \right\} = 0 \tag{B.19}$$

Clearly, if we select

$$\phi_n(\delta) = \delta^n \tag{B.20}$$

we, in effect, have a power-series expansion. Equating like powers of δ, the **leading-order** ($n = 0$) problem is Equation (B.16), while the second-derivative term makes its first appearance in the **first-order** ($n = 1$) problem. Our perturbation solution yields the following series of problems for the **outer expansion**.

$$\left.\begin{aligned}\frac{dF_0}{ds} + F_0 &= 0\\ \frac{dF_1}{ds} + F_1 &= -\frac{d^2F_0}{ds^2}\\ \frac{dF_2}{ds} + F_2 &= -\frac{d^2F_1}{ds^2}\\ &\vdots\end{aligned}\right\} \tag{B.21}$$

Provided we solve the equations in sequence starting at the lowest order ($n = 0$) equation, the right-hand side of each equation is known from the preceding solution and serves simply to make each equation for $n \geq 1$ non-homogeneous. Consequently, to all orders, the equation for $F_n(s)$ is of first order. Hence, no matter how many terms we include in our expansion, we can satisfy only one of the two boundary conditions. As in the introductory remarks, we elect to satisfy $F(1) = 1$. In terms of our expansion [Equations (B.18) and (B.20)], the boundary conditions for the F_n are

$$F_0(1) = 1 \quad \text{and} \quad F_n(1) = 0 \quad \text{for} \quad n \geq 1 \tag{B.22}$$

The solution to Equations (B.21) subject to the boundary conditions specified in Equation (B.22) is as follows.

$$\left.\begin{aligned}F_0(s) &= e^{1-s}\\ F_1(s) &= (1-s)e^{1-s}\\ &\vdots\end{aligned}\right\} \tag{B.23}$$

Hence, our **outer expansion** assumes the following form.

$$F_{outer}(s;\delta) \sim e^{1-s}\left[1 + (1-s)\delta + O(\delta^2)\right] \tag{B.24}$$

In general, for singular perturbation problems, we have no guarantee that continuing to an infinite number of terms in the outer expansion yields a solution that satisfies both boundary conditions. That is, our expansion may or may not be convergent. Hence, we try a different approach to resolve the region near $s = 0$. We now generate an **inner expansion** in which we **stretch** the s coordinate. That is, we define a new independent variable σ as follows.

$$\sigma = \frac{s}{\mu(\delta)} \tag{B.25}$$

We assume an inner expansion in terms of a new set of asymptotic-sequence functions, $\psi_n(\delta)$, i.e.,

$$F_{inner}(\sigma;\delta) \sim \sum_{n=0}^{N} f_n(\sigma)\psi_n(\delta) \tag{B.26}$$

To best illustrate how we determine the appropriate stretching function, $\mu(\delta)$, consider the leading-order terms in the original differential equation, viz.,

$$\frac{d^2 f_0}{d\sigma^2}\left(\frac{\delta\psi_0}{\mu^2}\right) + \frac{df_0}{d\sigma}\left(\frac{\psi_0}{\mu}\right) + f_0\psi_0 = O\left(\frac{\delta\psi_1}{\mu^2}, \frac{\psi_1}{\mu}, \psi_1\right) \tag{B.27}$$

First of all, we must consider the three possibilities for the order of magnitude of $\mu(\delta)$, viz., $\mu \gg 1$, $\mu \sim 1$ and $\mu \ll 1$. If $\mu \gg 1$, inspection of Equation (B.27) shows that $f_0 = 0$ which is not a useful solution. If $\mu \sim 1$, we have the outer expansion. Thus, we conclude that $\mu \ll 1$.

We are now faced with three additional possibilities: $\delta\psi_0/\mu^2 \gg \psi_0/\mu$; $\delta\psi_0/\mu^2 \sim \psi_0/\mu$; and $\delta\psi_0/\mu^2 \ll \psi_0/\mu$. Using the boundary condition at $s = 0$, assuming $\delta\psi_0/\mu^2 \gg \psi_0/\mu$ yields $f_0 = A\sigma$ where A is a constant of integration. While this solution might be useful, we have learned nothing about the stretching function, $\mu(\delta)$. At the other extreme, $\delta\psi_0/\mu^2 \ll \psi_0/\mu$, we obtain the trivial solution, $f_0 = 0$, which doesn't help us in our quest for a solution. The final possibility, $\delta\psi_0/\mu^2 \sim \psi_0/\mu$, is known as the **distinguished limit**, and this is the case we choose. Thus,

$$\mu(\delta) = \delta \tag{B.28}$$

Again, the most appropriate choice for the $\psi_n(\delta)$ is

$$\psi_n(\delta) = \delta^n \tag{B.29}$$

The following sequence of equations and boundary conditions define the **inner expansion**.

$$\left.\begin{aligned} \frac{d^2 f_0}{d\sigma^2} + \frac{df_0}{d\sigma} &= 0 \\ \frac{d^2 f_1}{d\sigma^2} + \frac{df_1}{d\sigma} &= -f_0 \\ \frac{d^2 f_2}{d\sigma^2} + \frac{df_2}{d\sigma} &= -f_1 \\ &\vdots \end{aligned}\right\} \tag{B.30}$$

$$f_n(0) = 0 \quad \text{for all } n \geq 0 \tag{B.31}$$

Solving the leading, or **zeroth**, order problem ($n = 0$) and the **first** order problem ($n = 1$), we find

$$\left.\begin{aligned} f_0(\sigma) &= A_0(1 - e^{-\sigma}) \\ f_1(\sigma) &= (A_1 - A_0\sigma) - (A_1 + A_0\sigma)e^{-\sigma} \\ &\vdots \end{aligned}\right\} \quad \text{(B.32)}$$

where A_0 and A_1 are constants of integration. These integration constants arise because each of Equations (B.30) is of second order and we have used only one boundary condition.

To complete the solution, we perform an operation known as **matching**. To motivate the matching procedure, note that on the one hand, the boundary $s = 1$ is located at $\sigma = 1/\delta \rightarrow \infty$ as $\delta \rightarrow 0$. Hence, we need a boundary condition for $F_{inner}(\sigma; \delta)$ valid as $\sigma \rightarrow \infty$. On the other hand, the independent variable in the outer expansion is related to σ by $s = \delta\sigma$. Thus, for any finite value of σ, the inner expansion lies very close to $s = 0$. We **match** these two asymptotic expansions by requiring that

$$\lim_{\sigma \rightarrow \infty} F_{inner}(\sigma; \delta) = \lim_{s \rightarrow 0} F_{outer}(s; \delta) \quad \text{(B.33)}$$

The general notion is that on the scale of the outer expansion, the inner expansion is valid in an infinitesimally thin layer. Similarly, on the scale of the inner expansion, the outer expansion is valid for a region infinitely distant from $s = 0$. For the problem at hand,

$$\lim_{\sigma \rightarrow \infty} f_0(\sigma) = A_0 \quad \text{and} \quad \lim_{s \rightarrow 0} F_0(s) = e \quad \text{(B.34)}$$

Thus, we conclude that

$$A_0 = e \quad \text{(B.35)}$$

Equivalently, we can visualize the existence of an **overlap region** between the inner and outer solutions. In the overlap region, we stretch the s coordinate according to

$$s^* = \frac{s}{\nu(\delta)}; \qquad \delta \ll \nu(\delta) \ll 1 \quad \text{(B.36)}$$

In terms of this intermediate variable, for any finite value of s^*,

$$s \rightarrow 0 \quad \text{and} \quad \sigma \rightarrow \infty \quad \text{as} \quad \nu(\delta) \rightarrow 0 \quad \text{(B.37)}$$

Using this method, we can match to as high an order as we wish. For example, matching to n^{th} order, we perform the following limit operation.

$$\lim_{\delta \rightarrow 0} \left[\frac{F_{inner} - F_{outer}}{\delta^n} \right] = 0 \quad \text{(B.38)}$$

For the problem at hand, the independent variables s and σ become

$$s = \nu(\delta)s^* \quad \text{and} \quad \sigma = \frac{\nu(\delta)s^*}{\delta} \tag{B.39}$$

Hence, replacing $e^{-\nu(\delta)s^*}$ by its Taylor series expansion, we find

$$F_{outer} \sim e\left\{1 - \nu(\delta)s^* + \delta + O[\delta\nu(\delta)]\right\} \tag{B.40}$$

Similarly, noting that $e^{-\nu(\delta)s^*/\delta}$ is transcendentally small as $\delta \to 0$, we have

$$F_{inner} \sim A_0 - A_0\nu(\delta)s^* + A_1\delta + O(\delta^2) \tag{B.41}$$

Thus, holding s^* constant,

$$\lim_{\delta\to 0}\left[\frac{F_{inner} - F_{outer}}{\delta}\right] \sim \frac{(A_0 - e)(1 - \nu(\delta)s^*) + (A_1 - e)\delta + o(\delta)}{\delta} \tag{B.42}$$

Clearly, **matching to zeroth and first orders** can be achieved only if

$$A_0 = A_1 = e \tag{B.43}$$

In summary, the **inner and outer expansions** are given by

$$\left.\begin{array}{l} F_{outer}(s;\delta) \sim e^{1-s}\left[1 + (1-s)\delta + O(\delta^2)\right] \\ F_{inner}(\sigma;\delta) \sim e\left\{(1 - e^{-\sigma}) + [(1-\sigma) - (1+\sigma)e^{-\sigma}]\delta + O(\delta^2)\right\} \\ \sigma = s/\delta \end{array}\right\} \tag{B.44}$$

Finally, we can generate a single expansion, known as a **composite expansion**, that can be used throughout the region $0 \leq s \leq 1$. Recall that in the matching operations above, we envisioned an **overlap region**. In constructing a composite expansion, we note that the inner expansion is valid in the inner region, the outer expansion is valid in the outer region, and both are valid in the overlap region. Hence, we define

$$F_{composite} \equiv F_{inner} + F_{outer} - F_{cp} \tag{B.45}$$

where F_{cp} is the **common part**, i.e., the part of the expansions that cancel in the matching process. Again, for the case at hand, comparison of Equations (B.40) and (B.41) with A_0 and A_1 given by Equation (B.43) shows that

$$F_{cp} \sim e\left[1 + (1-\sigma)\delta + O(\delta^2)\right] \tag{B.46}$$

where we use the fact that $\nu(\delta)s^* = \delta\sigma$. Hence, the **composite expansion** is

$$F_{composite} \sim \left[e^{1-s} - e^{1-s/\delta}\right] + \left[(1-s)e^{1-s} - (1+s/\delta)e^{1-s/\delta}\right]\delta + O(\delta^2) \tag{B.47}$$

Retaining just the zeroth order term of the composite expansion yields an approximation to the exact solution that is accurate to better than 3% for δ as large as 0.2! This is actually a bit fortuitous since the leading term in Equation (B.47) and the exact solution differ by a transcendentally small term. What we have done is combine two non-uniformly valid expansions to achieve a **uniformly valid approximation** to the exact solution.

For the obvious reason, perturbation analysis is often referred to as the theory of **matched asymptotic expansions**. The discussion here, although sufficient for our needs, is brief and covers only the bare essentials of the theory. For additional information, see the excellent books by Van Dyke (1964), Bender and Orszag (1978) or Kevorkian and Cole (1981) on this powerful mathematical theory.

Problems

B.1 Consider the polynomial

$$x^3 - x^2 + \delta = 0$$

(a) For nonzero $\delta < 4/27$ this equation has three real and unequal roots. Why is this a singular perturbation problem in the limit $\delta \to 0$?

(b) Use perturbation methods to solve for the first two terms in the expansions for the roots.

B.2 The following is an example of a perturbation problem that is singular because of nonuniformity near a boundary. Consider the following first-order equation in the limit $\epsilon \to 0$.

$$x^3 \frac{dy}{dx} = \epsilon y^2; \qquad y(1) = 1$$

The solution is known to be finite on the closed interval $0 \le x \le 1$.

(a) Solve for the first two terms in the outer expansion and show that the solution has a singularity as $x \to 0$.

(b) Show that there is a boundary layer near $x = 0$ whose thickness is of order $\epsilon^{1/2}$.

(c) Solve for the first two terms of the inner expansion. Note that the algebra simplifies if you do the zeroth-order matching before attempting to solve for the next term in the expansion.

B.3 Generate the first two terms of the inner and outer expansions for the following boundary-value problem. Also, construct a composite expansion.

$$\delta \frac{d^2y}{dx^2} + \frac{dy}{dx} - xy = 0; \qquad \delta \ll 1$$

$$y(0) = 0 \quad \text{and} \quad y(1) = e^{1/2}$$

B.4 Generate the first two terms of the inner and outer expansions for the following boundary-value problem. Also, construct a composite expansion.

$$\delta\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{2}x^2; \qquad \delta \ll 1$$

$$y(0) = 1 \quad \text{and} \quad y(1) = 1/6$$

B.5 This problem demonstrates that the **overlap region** is not a layer in the same sense as the boundary layer. Rather, its thickness depends upon how many terms we retain in the matching process. Suppose we have solved a boundary-layer problem and the first three terms of the inner and outer expansions valid as $\epsilon \to 0$ are:

$$y_{outer}(x;\epsilon) \sim 1 + \epsilon e^{-x^2} + \epsilon^2 e^{-2x^2} + O(\epsilon^3)$$

$$y_{inner}(x;\epsilon) \sim A(1-e^{-\xi}) + \epsilon B(1-e^{-\xi}) + \epsilon^2 C(1-\xi^2) + O(\epsilon^3)$$

where

$$\xi \equiv \frac{x}{\epsilon^{1/2}}$$

Determine the coefficients A, B and C. Explain why the thickness of the overlap region, $\nu(\epsilon)$, must lie in the range

$$\epsilon^{1/2} \ll \nu(\epsilon) \ll \epsilon^{1/4}$$

as opposed to the normally assumed range $\epsilon^{1/2} \ll \nu(\epsilon) \ll 1$.