Appendix A

Cartesian Tensor Analysis

The central point of view of tensor analysis is to provide a systematic way for transforming quantities such as vectors and matrices from one coordinate system to another. Tensor analysis is a very powerful tool for making such transformations, although the analysis generally is very involved. For our purposes, working with Cartesian coordinates is sufficient so that we only need to focus on issues of notation, nomenclature and some special tensors. This appendix presents rudiments of Cartesian tensor analysis.

We begin by addressing the question of notation. In Cartesian tensor analysis we make extensive use of subscripts. For consistency with general tensor analysis nomenclature we will use the terms subscript and index interchangeably. The components of an n-dimensional vector $\mathbf{x}$ are denoted as x_1, x_2, $\ldots$, x_n. For example, in three-dimensional space, we rewrite the coordinate vector $\mathbf{x}$ = (x, y, z) as $\mathbf{x} = (x_1, x_2, x_3)$. Now consider an equation describing a plane in three-dimensional space, viz.,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = c \tag{A.1}$$

where a_i and c are constants. This equation can be written as

$$\sum_{i=1}^{3} a_i x_i = c \tag{A.2}$$

In tensor analysis, we introduce the Einstein summation convention and rewrite Equation (A.2) in the shorthand form

$$a_i x_i = c \tag{A.3}$$

The Einstein summation convention is as follows:

> **Repetition of an index in a term denotes summation with respect to that index over its range.**

The **range** of an index i is the set of n integer values 1 to n. An index that is summed over is called a **dummy index**; one that is not summed is called a **free index**.

Since a dummy index simply indicates summation, it is immaterial what symbol is used. Thus, $a_i x_i$ may be replaced by $a_j x_j$, which is obvious if we simply note that

$$\sum_{i=1}^{3} a_i x_i = \sum_{j=1}^{3} a_j x_j \tag{A.4}$$

As an example of an equation with a free index, consider a unit normal vector $\mathbf{n}$ in three-dimensional space. If the unit normals in the x_1, x_2 and x_3 directions are $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, then the direction cosines α_1, α_2 and α_3 for the vector $\mathbf{n}$ are given by

$$\alpha_k = \mathbf{n} \cdot \mathbf{i}_k \tag{A.5}$$

There is no implied summation in Equation (A.5). Rather, it is a shorthand for the three equations defining the direction cosines. Because the length of a unit vector is one, we can take the dot product of $(\alpha_1, \alpha_2, \alpha_3)$ with itself and say that

$$\alpha_i \alpha_i = 1 \tag{A.6}$$

As another example, consider the total differential of a function of three variables, $p(x_1, x_2, x_3)$. We have

$$dp = \frac{\partial p}{\partial x_1} dx_1 + \frac{\partial p}{\partial x_2} dx_2 + \frac{\partial p}{\partial x_3} dx_3 \tag{A.7}$$

In tensor notation, this is replaced by

$$dp = \frac{\partial p}{\partial x_i} dx_i \tag{A.8}$$

Equation (A.8) can be thought of as the dot product of the gradient of p, namely ∇p, and the differential vector $d\mathbf{x} = (dx_1, dx_2, dx_3)$. Thus, we can also say that the i component of ∇p, which we denote as $(\nabla p)_i$, is given by

$$(\nabla p)_i = \frac{\partial p}{\partial x_i} = p_{,i} \tag{A.9}$$

where a comma followed by an index is tensor notation for differentiation with respect to x_i. Similarly, the divergence of a vector $\mathbf{u}$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = u_{i,i} \tag{A.10}$$

where we again denote differentiation with respect to x_i by ", i".

Thus far, we have dealt with scalars and vectors. The question naturally arises about how we might handle a matrix. The answer is we denote a matrix by using two subscripts, or indices. The first index corresponds to row number while the second corresponds to column number. For example, consider the 3 x 3 matrix $[A]$ defined by

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{A.11}$$

In tensor notation, we represent the matrix $[A]$ as A_{ij}. If we post-multiply an m x n matrix B_{ij} by an n x 1 column vector x_j, their product is an m x 1 column vector y_i. Using the summation convention, we write

$$y_i = B_{ij} x_j \tag{A.12}$$

Equation (A.12) contains both a free index (i) and a dummy index (j). The product of a square matrix A_{ij} and its inverse is the unit matrix, i.e.,

$$[A][A]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.13}$$

Equation (A.13) is rewritten in tensor notation as follows:

$$A_{ik}(A^{-1})_{kj} = \delta_{ij} \tag{A.14}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{A.15}$$

We can use the Kronecker delta to rewrite Equation (A.6) as

$$\alpha_i \delta_{ij} \alpha_j = 1 \tag{A.16}$$

This corresponds to pre-multiplying the 3 x 3 matrix δ_{ij} by the row vector $(\alpha_1, \alpha_2, \alpha_3)$ and then post-multiplying their product by the column vector $(\alpha_1, \alpha_2, \alpha_3)^T$, where superscript T denotes transpose.

The determinant of a 3 x 3 matrix A_{ij} is

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{array}{l} A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\ -A_{11}A_{32}A_{23} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \end{array} \tag{A.17}$$

Tensor analysis provides a shorthand for this operation as well. Specifically, we replace Equation (A.17) by

$$det(A_{ij}) = |A_{ij}| = \epsilon_{rst}A_{r1}A_{s2}A_{t3} \tag{A.18}$$

where ϵ_{rst} is the **permutation tensor** defined by

$$\left.\begin{array}{l} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \\ \epsilon_{111} = \epsilon_{222} = \epsilon_{333} = \epsilon_{112} = \epsilon_{113} = \epsilon_{221} = \epsilon_{223} = \epsilon_{331} = \epsilon_{332} = 0 \end{array}\right\} \tag{A.19}$$

In other words, ϵ_{ijk} vanishes whenever the values of any two indices are the same; $\epsilon_{ijk} = 1$ when the indices are a permutation of 1, 2, 3; and $\epsilon_{ijk} = -1$ otherwise.

As can be easily verified, the cross product of two vectors **a** and **b** can be expressed as follows.

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}a_j b_k \tag{A.20}$$

In particular, the curl of a vector **u** is

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \epsilon_{ijk}u_{k,j} \tag{A.21}$$

The Kronecker delta and permutation tensor are very important quantities that appear throughout this book. They are related by the ϵ-δ identity, which is the following.

$$\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks} \tag{A.22}$$

All that remains to complete our brief introduction to tensor analysis is to define a tensor. Tensors are classified in terms of their rank. To determine the rank of a tensor, we simply count the number of indices.

The lowest rank tensor is rank zero which corresponds to a scalar, i.e., a quantity that has magnitude only. Thermodynamic properties such as pressure and density are scalar quantities. Vectors such as velocity, vorticity and pressure gradient are tensors of rank one. They have both magnitude and direction. Matrices are rank two tensors. The stress tensor is a good example for illustrating physical interpretation of a second rank tensor.

It defines a force per unit area that has a magnitude and two associated directions, the direction of the force and the direction of the normal to the plane on which the force acts. For a normal stress, these two directions are the same; for a shear stress, they are (by convention) normal to each other.

As we move to tensors of rank three and beyond, the physical interpretation becomes more difficult to ascertain. This is rarely an issue of great concern since virtually all physically relevant tensors are of rank 2 or less. The permutation tensor is of rank 3, for example, and is simply defined by Equation (A.19).

A tensor a_{ij} is **symmetric** if $a_{ij} = a_{ji}$. Many important tensors in mathematical physics are symmetric, e.g., stress, strain and strain-rate tensors, moment of inertia tensor, virtual-mass tensor. A tensor is **skew symmetric** if $a_{ij} = -a_{ji}$. The rotation tensor, $\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ is skew symmetric.

As a final comment, in performing tensor analysis operations with tensors that are not differential operators, we rarely have to worry about preserving the order of terms as we did in Equation (A.16). There is no confusion in writing $\delta_{ij}\alpha_i\alpha_j$ in place of $\alpha_i\delta_{ij}\alpha_j$. This is only an issue when the indicated summations actually have to be done. However, care should be exercised when differentiation occurs. As an example, $\nabla \cdot \mathbf{u} = \partial u_i/\partial x_i$ is a scalar number while $\mathbf{u}\cdot\nabla = u_i\partial/\partial x_i$ is a scalar differential operator.

Problems

A.1 Use the ϵ-δ identity to verify the well known vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

A.2 Show that, when i, j, k range over 1, 2, 3

(a) $\delta_{ij}\delta_{ji} = 3$

(b) $\epsilon_{ijk}\epsilon_{jki} = 6$

(c) $\epsilon_{ijk}A_j A_k = 0$

(d) $\delta_{ij}\delta_{jk} = \delta_{ik}$

A.3 Verify that $2S_{ij,j} = \nabla^2 u_i$ for incompressible flow, where S_{ij} is the strain-rate tensor, i.e., $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

A.4 Show that the scalar product $S_{ij}\Omega_{ji}$ vanishes identically if S_{ij} is a symmetric tensor and Ω_{ij} is skew symmetric.

A.5 If u_j is a vector, show that the tensor $\omega_{ik} = \epsilon_{ijk}u_j$ is skew symmetric.

A.6 Show that if A_{jk} is a skew-symmetric tensor, the unique solution of the equation $\omega_i = \frac{1}{2}\epsilon_{ijk}A_{jk}$ is $A_{mn} = \epsilon_{mni}\omega_i$.

A.7 The incompressible Navier-Stokes equation in a coordinate system rotating with constant angular velocity $\boldsymbol{\Omega}$ and with position vector $\mathbf{x} = x_k\mathbf{i}_k$ is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla\left(\frac{p}{\rho}\right) - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x} + \nu\nabla^2\mathbf{u}$$

(a) Rewrite this equation in tensor notation.

(b) Using tensor analysis, show that for $\boldsymbol{\Omega} = \Omega\mathbf{k}$ ($\mathbf{k}$ is a unit vector aligned with $\boldsymbol{\Omega}$), the centrifugal force per unit mass is given by

$$-\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x} = \nabla(\frac{1}{2}\Omega^2 x_k x_k) - [\mathbf{k} \cdot \nabla(\frac{1}{2}\Omega^2 x_k x_k)]\mathbf{k}$$

A.8 Using tensor analysis, prove the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})$$