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ИНТЕГРАЛЫ И РЯДЫ



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ИНТЕГРАЛЫ И РЯДЫ ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ



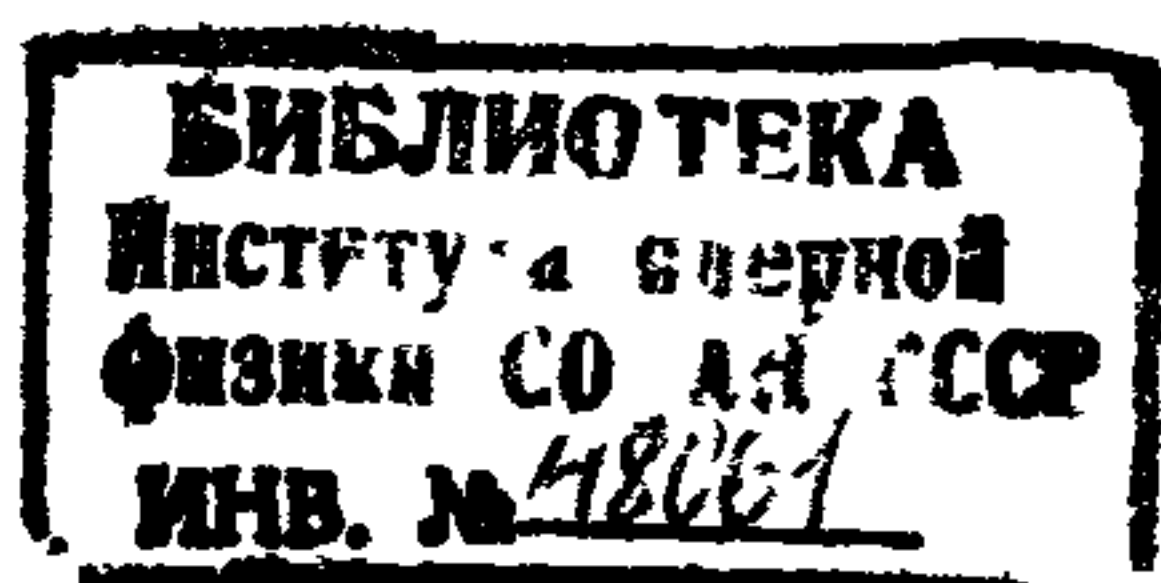
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Книга содержит неопределенные и определенные (в том числе кратные) интегралы, конечные суммы, ряды и произведения с элементарными функциями. Она является наиболее полным справочным руководством, включает результаты, изложенные в аналогичных изданиях, а также в научной и периодической литературе, опубликованной в последние годы. Некоторые результаты публикуются впервые.

Книга предназначена для широкого круга специалистов в различных областях знаний, а также для студентов вузов.



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ПРЕДИСЛОВИЕ

Книга содержит неопределенные и определенные (в том числе кратные) интегралы, конечные суммы, ряды и произведения с элементарными функциями. Она написана с целью удовлетворения запросов широкого круга специалистов в различных областях знаний и включает результаты, изложенные в аналогичных изданиях, а также в научной и периодической литературе, опубликованной в последнее время. Ряд результатов публикуется впервые. Отметим, что, интерпретируя некоторые интегралы и ряды как обобщенные функции, интегральные преобразования обобщенных функций или рассматривая их регуляризацию, можно придать им смысл в более широкой области изменения параметров.

Достаточно подробное оглавление позволяет ознакомиться с содержанием книги и отражает ее отличие от ранее выпущенных справочных руководств в этой области математического анализа.

Найти вполне оптимальную систему упорядочения материала книги вряд ли представляется возможным, но мы надеемся, что с помощью оглавления читатель сумеет отыскать нужные формулы.

Применяемые обозначения, как правило, общеприняты в математической литературе и приводятся в указателях в конце книги. При ссылках запись вида 5.1.4.3 обозначает формулу 3 из пункта 5.1.4; $k, l, m, n = 0, 1, 2, \dots$, если не указаны другие условия.

Для компактности изложения в ряде формул используется сокращенная запись. Например, формула

$$\int_0^1 x^{\alpha-1} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \frac{\pi}{2\alpha} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mp \frac{1}{\sqrt{\pi}} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \quad [\operatorname{Re} \alpha > -\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}]$$

представляет собой сокращенную запись двух формул:

$$\int_0^1 x^{\alpha-1} \arcsin x dx = \frac{\pi}{2\alpha} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \right) \quad [\operatorname{Re} \alpha > -1]$$

(берется только верхний знак и верхнее выражение в фигурных скобках) и

$$\int_0^1 x^{\alpha-1} \arccos x dx = \frac{\sqrt{\pi}}{2\alpha} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \quad [\operatorname{Re} \alpha > 0].$$

(берется только нижний знак и нижнее выражение в фигурных скобках).

Следует также отметить, что для некоторых интегралов имеется несколько различных представлений, которые записываются под разными номерами, но без повторения левой части.

Книга содержит три приложения, которые могут быть использованы при вычислении рядов и интегралов, а также при приведении их к виду, содержащемуся в книге.

При составлении настоящего справочного руководства использованы литературные источники, список которых приводится в конце книги.

Мы будем весьма признательны всем читателям, которые обратят наше внимание на неизбежные в работе такого объема недосмотры.

Глава 1. НЕОПРЕДЕЛЕННЫЕ ИНТЕГРАЛЫ

1.1. ВВЕДЕНИЕ

1.1.1. Предварительные сведения.

В этой главе содержатся неопределенные интегралы от элементарных функций; постоянная интегрирования для краткости опущена. Например, вместо

$$\int \sin x \, dx = -\cos x + C$$

пишем

$$\int \sin x \, dx = -\cos x.$$

В ряде формул, где при интегрировании возникает логарифм абсолютной величины, знак абсолютной величины для простоты опущен.

Некоторые формулы при определенных значениях параметров теряют смысл. Если эти значения следуют из структуры формулы, то соответствующие разъяснения опускаются. Выражения для интеграла при этих значениях параметров, как правило, даются в последующих формулах.

Ниже используются следующие обозначения:

x, t, s, u, v — независимые переменные,

f, g — функции,

P, Q, P_i, Q_i — многочлены,

$R = \frac{P}{Q}$ — рациональная функция,

a, b, c, d — действительные числа,

$p, q, r, \alpha, \lambda, \mu, \nu$ — комплексные числа,

$k, l, m, n = 0, 1, 2, 3, \dots$

В случаях, когда имеются ограничения на значения параметров, делаются специальные оговорки.

1.1.2. Основные интегралы.

$$1. \int x^\lambda dx = \frac{x^{\lambda+1}}{\lambda+1} \quad [\lambda \neq -1].$$

$$2. \int \frac{dx}{x} = \ln |x|. \quad 3. \int e^x dx = e^x.$$

$$4. \int a^x dx = \frac{a^x}{\ln a} \quad [a > 0, a \neq 1].$$

$$5. \int \sin x \, dx = -\cos x. \quad 6. \int \cos x \, dx = \sin x.$$

7. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$ 8. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$ Г
9. $\int \frac{\sin x}{\cos^2 x} dx = \sec x.$ 10. $\int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x.$
11. $\int \operatorname{tg} x dx = -\ln |\cos x|.$ 12. $\int \operatorname{ctg} x dx = \ln |\sin x|.$
13. $\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right|.$ 14. $\int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$
15. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a}$ [a ≠ 0]
16. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \operatorname{Arth} \frac{x}{a} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$ [a ≠ 0]
17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arcsin} \frac{x}{a}$ [a ≠ 0].
18. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{Arsh} \frac{x}{a} = \ln (x + \sqrt{x^2 + a^2})$ [a ≠ 0].
19. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{Arch} \frac{x}{a} = \ln |x + \sqrt{x^2 - a^2}|$ [a ≠ 0].
20. $\int \operatorname{sh} x dx = \operatorname{ch} x.$ 21. $\int \operatorname{ch} x dx = \operatorname{sh} x.$
22. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$ 23. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$
24. $\int \operatorname{th} x dx = \ln \operatorname{ch} x.$ 25. $\int \operatorname{cth} x dx = \ln |\operatorname{sh} x|.$
26. $\int \frac{dx}{\operatorname{sh} x} = \ln \left| \operatorname{th} \frac{x}{2} \right|.$ 27. $\int \frac{dx}{\operatorname{ch} x} = 2 \operatorname{arctg} e^x.$

1.1.3. Общие формулы

28. $\int (af(x) + bg(x)) dx = a \int f(x) dx + b \int g(x) dx.$
29. $\int \frac{d}{dx} f(x) dx = f(x).$
30. $\int \frac{df(x)}{dx} g(x) dx = f(x) g(x) - \int f(x) \frac{dg(x)}{dx} dx$ [интегрирование по частям].
31. $\int f(x) dx = \int f(g(y)) g'(y) dy$ [интегрирование подстановкой, $x = g(y)$].

1.2. СТЕПЕННАЯ И АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

1.2.1. Введение

В этом пункте кратко изложены некоторые сведения общего характера, которые могут быть использованы при вычислении неопределенных интегралов

1. Если $P_1(x)$ и $Q_1(x)$ — произвольные многочлены, то их отношение может быть представлено в виде

$$\frac{P_1(x)}{Q_1(x)} = P_2(x) + \frac{P(x)}{Q(x)},$$

где $P_2(x)$, $P(x)$, $Q(x)$ — многочлены, причем степень $P(x)$ меньше степени $Q(x)$. Пусть a_i , $i=1, \dots, m$, — корни многочлена $Q(x)$, n_i — соответствующие кратности корней. Тогда

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n_1} \frac{A_k^{(1)}}{(x-a_1)^k} + \sum_{k=1}^{n_2} \frac{A_k^{(2)}}{(x-a_2)^k} + \dots + \sum_{k=1}^{n_m} \frac{A_k^{(m)}}{(x-a_m)^k},$$

где

$$A_k^{(i)} = \frac{1}{(n_i - k)!} \frac{d^{n_i - k}}{dx^{n_i - k}} \left[\frac{P(x)}{Q(x)} (x - a_i)^{n_i} \right] \Big|_{x=a_i}.$$

Если $n_i = 1$ для всех i , то

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^m \frac{A^{(i)}}{x - a_i},$$

где $A^{(i)} = P(a_i)/Q'(a_i)$

При объединении пар слагаемых, соответствующих комплексно сопряженным корням, получаются дроби вида $\frac{Ax + B}{x^2 + bx + c}$, где трехчлен $x^2 + bx + c$ имеет действительные коэффициенты. Поэтому вычисление интеграла $\int \frac{P(x)}{Q(x)} dx$ сводится к интегрированию рациональных дробей $\int \frac{dx}{(x-a)^n}$, $\int \frac{Ax+B}{(x^2+bx+c)^n} dx$.

2. Интегралы вида $\int x^p (ax^r + b)^q dx$ (p, q, r — рациональные числа) выражаются в виде конечной комбинации элементарных функций только в следующих случаях

- а) q — целое число,
- б) $(p+1)/r$ — целое число; тогда

$$\int x^p (ax^r + b)^q dx = \frac{1}{r} \int t^{(p+1)/r - 1} (at + b)^q dt \quad [t = x^r],$$

- в) $\frac{p+1}{r} + q$ — целое число, тогда

$$\int x^p (ax^r + b)^q dx = \frac{1}{r} \int \left(\frac{at+b}{t} \right)^q t^{(p+1)/r + q - 1} dt \quad [t = x^r]$$

3. Интегралы вида $\int R \left(x, \left(\frac{ax+b}{cx+d} \right)^p, \left(\frac{ax+b}{cx+d} \right)^q, \dots \right) dx$ где $p, q, \dots$ — рациональные числа, приводятся к интегралам от рациональных функций подстановкой

$$\frac{ax+b}{cx+d} = t^m,$$

где m — общий знаменатель дробей $p, q, \dots$.

4. Интегралы вида

$$I = \int \frac{R_1(x) + R_2(x) \sqrt{P(x)}}{R_3(x) + R_4(x) \sqrt{P(x)}} dx,$$

где R_1, R_2, R_3, R_4 — рациональные функции, а $P(x)$ — многочлен 3-й или 4-й степени с действительными коэффициентами и простыми корнями, приводятся к виду

$$I = \int R_5(x) dx + \int \frac{R(x)}{\sqrt{P(x)}} dx,$$

где

$$R_5(x) = \frac{R_1(x) R_3(x) - R_2(x) R_4(x) P(x)}{R_3^2(x) - R_4^2(x) P(x)},$$

$$R(x) = \frac{(R_2(x) R_3(x) - R_1(x) R_4(x)) P(x)}{R_3^2(x) - R_4^2(x) P(x)}.$$

Эллиптический интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ при помощи замен переменных которые указаны ниже, может быть приведен к форме, содержащей рациональные функции от эллиптических функций Якоби. Интегралы от эллиптических функций Якоби см. в [24]

1) Пусть $P(x) = a_0(x^2 \pm a^2)(x^2 \pm b^2)$. Тогда интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации трех интегралов: $\int \frac{dx}{\sqrt{P(x)}}$, $\int \frac{x^2 dx}{\sqrt{P(x)}}$, $\int \frac{dx}{(x^2 - \rho^2) \sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\sqrt{(a^2 - x^2)(x^2 - b^2)}, \quad \sqrt{(a^2 - x^2)(b^2 - x^2)}, \quad \sqrt{(a^2 + x^2)(b^2 - x^2)},$$

$$\sqrt{(a^2 + x^2)(x^2 - b^2)}, \quad \sqrt{(x^2 + a^2)(x^2 + b^2)}, \quad \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)},$$

где a^2, b^2 — действительные числа, ρ^2 и $\bar{\rho}^2$ — комплексно сопряженные числа. В первых пяти случаях подстановка

$$x^2 = \frac{A_1 + A_2 \operatorname{sn}^2 u}{B_1 + B_2 \operatorname{sn}^2 u} \quad (0 \leq u \leq K),$$

где $\operatorname{sn} u$ — эллиптический синус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные, приводит интегралы к виду $\int R_6(\operatorname{sn} u) du$, где R_6 — рациональная функция.

В шестом случае следует использовать замену

$$x^2 = \frac{A_1 + A_2 \operatorname{cn} u}{B_1 + B_2 \operatorname{cn} u} \quad (0 \leq u \leq 2K),$$

где $\operatorname{cn} u$ — эллиптический косинус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные

2) Пусть $P(x) = a_0(x + a_1)(x + a_2)(x + a_3)$, где все a_i различны. Интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации трех интегралов: $\int \frac{dx}{\sqrt{P(x)}}$, $\int \frac{x dx}{\sqrt{P(x)}}$, $\int \frac{dx}{(x - \rho) \sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих пяти способов

$$\sqrt{(x - a)(x - b)(x - c)}, \quad \sqrt{(a - x)(x - b)(x - c)}, \quad \sqrt{(a - x)(b - x)(x - c)},$$

$$\sqrt{(x - a)[(x - b_1)^2 + b_2]}, \quad \sqrt{(a - x)[(x - b_1)^2 + b_2]},$$

где a, b, c, b_1, b_2 действительны и $a > b > c$. В первых трех случаях к интегралам от эллиптических функций Якоби приводит подстановка

$$x = \frac{A_1 + A_2 \operatorname{sn}^2 u}{B_1 + B_2 \operatorname{sn}^2 u} \quad (0 \leq u \leq K), \quad (*)$$

где A_1, A_2, B_1, B_2 — некоторые постоянные. Если два корня многочлена $P(x)$ комплексны, то следует сделать замену

$$x = \frac{A_1 + A_2 \operatorname{sn} u}{B_1 + B_2 \operatorname{sn} u} \quad (0 \leq u \leq 2K). \quad (**)$$

3) Пусть $P(x) = a_0(x+a_1)(x+a_2)(x+a_3)(x+a_4)$, где все a_i различны. Интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации четырех интегралов: $\int \frac{dx}{\sqrt{P(x)}}, \int \frac{x dx}{\sqrt{P(x)}}, \int \frac{x^2 dx}{\sqrt{P(x)}}, \int \frac{dx}{(x-p)\sqrt{P(x)}}.$

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\begin{aligned} & \sqrt{(x-a)(x-b)(x-c)(x-d)}, \quad \sqrt{(a-x)(x-b)(x-c)(x-d)}, \\ & \sqrt{(a-x)(b-x)(x-c)(x-d)}, \quad \sqrt{(x-a)(x-b)[(x-b_1)^2+b_2^2]}, \\ & \sqrt{(x-a)(b-x)[(x-b_1)^2+b_2^2]}, \quad \sqrt{[(x-b_1)^2+b_2^2][(x-b_3)^2+b_4^2]}, \end{aligned}$$

где $a, b, c, d, b_1, b_2, b_3, b_4$ действительны и $a > b > c > d$.

В первых трех случаях следует сделать замену (*), а в остальных — замену (**).

1.2.2. Интегралы вида $\int x^p (ax^r + b)^q dx$.

$$1. \int x^p (ax^r + b)^q dx = \frac{x^{p+1} (ax^r + b)^q}{qr + p + 1} + \frac{qrb}{qr + p + 1} \int x^p (ax^r + b)^{q-1} dx.$$

$$2. = -\frac{x^{p+1} (ax^r + b)^{q+1}}{(q+1)rb} + \frac{qr+r+p+1}{(q+1)rb} \int x^p (ax^r + b)^{q+1} dx.$$

$$3. = \frac{x^{p+1} (ax^r + b)^q}{p+1} - \frac{qra}{p+1} \int x^{p+r} (ax^r + b)^{q-1} dx.$$

$$4. = \frac{x^{p-r+1} (ax^r + b)^{q+1}}{(q+1)ra} - \frac{p-r+1}{(q+1)ra} \int x^{p-r} (ax^r + b)^{q+1} dx.$$

$$5. = \frac{x^{p-r+1} (ax^r + b)^{q+1}}{(qr+p+1)a} - \frac{(p-r+1)b}{(qr+p+1)a} \int x^{p-r} (ax^r + b)^q dx.$$

$$6. = \frac{x^{p+1} (ax^r + b)^{q+1}}{(p+1)b} - \frac{(qr+r+p+1)a}{(p+1)b} \int x^{p+r} (ax^r + b)^q dx.$$

$$7. \int x^p (ax^r + b)^n dx = \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^{p+kr+1}}{p+kr+1}.$$

$$8. \int \frac{x^p dx}{(ax^r + b)^q} = \frac{x^{p+1}}{(q-1)rb(ax^r + b)^{q-1}} + \frac{(q-1)r-p-1}{(q-1)rb} \int \frac{x^p dx}{(ax^r + b)^{q-1}}.$$

$$9. = \frac{-x^{p-r+1}}{(q-1)ra(ax^r + b)^{q-1}} + \frac{p-r+1}{(q-1)ra} \int \frac{x^{p-r} dx}{(ax^r + b)^{q-1}}.$$

$$10. \int \frac{x^{r-1} dx}{(ax^r + b)^q} = \frac{-1}{(q-1)ra(ax^r + b)^{q-1}}.$$

$$11. \int \frac{x^{r-1} dx}{ax^r + b} = \frac{1}{ra} \ln |ax^r + b|.$$

$$12. \int \frac{dx}{x^p (ax^r + b)^q} = \frac{1}{(q-1) r b x^{p-1} (ax^r + b)^{q-1}} + \\ + \frac{p + (q-1)r - 1}{(q-1) r b} \int \frac{dx}{x^p (ax^r + b)^{q-1}}.$$

$$13. \int \frac{dx}{x^p (ax^r + b)} = -\frac{1}{(p-1) b x^{p-1}} - \frac{a}{b} \int \frac{dx}{x^{p-r} (ax^r + b)}.$$

$$14. \int \frac{dx}{x (ax^r + b)} = \frac{1}{rb} \ln \left| \frac{x^r}{ax^r + b} \right|.$$

1.2.3. Интегралы вида $\int \frac{x^m dx}{x^n \pm a^n}$.

$$1. \int \frac{dx}{x^{2n} + a^{2n}} = -\frac{1}{2na^{2n-1}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n} \pi + a^2 \right) + \\ + \frac{1}{na^{2n-1}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n} \pi}{a \sin \frac{2k+1}{2n} \pi}.$$

$$2. \int \frac{dx}{x^{2n+1} + a^{2n+1}} = \frac{1}{(2n+1)a^{2n}} \ln |x+a| - \\ - \frac{1}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) + \\ + \frac{2}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

$$3. \int \frac{dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n-1}} \ln \left| \frac{x-a}{x+a} \right| + \\ + \frac{1}{2na^{2n-1}} \sum_{k=1}^{n-1} \cos \frac{k\pi}{n} \ln \left(x^2 - 2ax \cos \frac{k\pi}{n} + a^2 \right) - \\ - \frac{1}{na^{2n-1}} \sum_{k=1}^{n-1} \sin \frac{k\pi}{n} \operatorname{arctg} \frac{x - a \cos \frac{k\pi}{n}}{a \sin \frac{k\pi}{n}}.$$

$$4. \int \frac{dx}{x^{2n+1} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n}} \ln |x-a| - \\ - \frac{1}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 + 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) - \\ - \frac{2}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x + a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

$$5. \int \frac{x^m}{x^{2n} + a^{2n}} dx = \frac{1}{na^{2n-m-1}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n} \pi}{a \sin \frac{2k+1}{2n} \pi} -$$

$$- \frac{1}{2na^{2n-m-1}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n} \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n} \pi + a^2 \right)$$

[1 ≤ m ≤ 2(n-1)].

$$6. \int \frac{x^m dx}{x^{2n+1} + a^{2n+1}} = \frac{(-1)^m}{(2n+1)a^{2n-m}} \ln |x+a| -$$

$$- \frac{1}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n+1} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) +$$

$$+ \frac{2}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n+1} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}$$

[1 ≤ m ≤ 2n-1].

$$7. \int \frac{x^m dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n-m-1}} [\ln |x-a| + (-1)^{m-1} \ln |x+a|] +$$

$$+ \frac{1}{2na^{2n-m-1}} \sum_{k=1}^{n-1} \cos \frac{(m+1)k\pi}{n} \ln \left(x^2 - 2ax \cos \frac{k\pi}{n} + a^2 \right) -$$

$$- \frac{1}{na^{2n-m-1}} \sum_{k=1}^{n-1} \sin \frac{(m+1)k\pi}{n} \operatorname{arctg} \frac{x - a \cos \frac{k\pi}{n}}{a \sin \frac{k\pi}{n}}$$

[1 ≤ m ≤ 2(n-1)].

$$8. \int \frac{x^m dx}{x^{2n+1} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n-m}} \ln |x-a| +$$

$$+ \frac{(-1)^{m+1}}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n+1} \ln \left(x^2 + 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) +$$

$$+ \frac{2(-1)^{m+1}}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n+1} \operatorname{arctg} \frac{x + a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}$$

[1 ≤ m ≤ 2n-1].

1.2.4. Интегралы вида $\int \frac{x^p dx}{(x+a)^q}$.

$$1. \int \frac{x^p dx}{(x+a)^q} = -\frac{x^p}{(q-1)(x+a)^{q-1}} + \frac{p}{q-1} \int \frac{x^{p-1} dx}{(x+a)^{q-1}}.$$

$$2. \int_0^x \frac{x^{\lambda-1} dx}{x+a} = \frac{1}{a} x^{\lambda} \Phi \left(-\frac{x}{a}, 1, \lambda \right)$$

[Re λ > 0; x < a].

$$3. \int_0^x \frac{x^{\lambda-1} dx}{(x+a)^{\nu}} = \frac{1}{\lambda} a^{-\nu} x^{\lambda} {}_2F_1(\nu, \lambda; 1+\lambda; -\frac{x}{a}) \quad [\operatorname{Re} \lambda > 0].$$

$$4. \int_x^{\infty} \frac{x^{\lambda-1} dx}{(x+a)^{\nu}} = \frac{x^{\lambda-\nu}}{\nu-\lambda} {}_2F_1(\nu, \nu-\lambda; 1+\nu-\lambda; -\frac{a}{x}) \quad [\operatorname{Re} \lambda < \operatorname{Re} \nu].$$

$$5. \int \frac{x^m dx}{(x+a)^q} = \sum_{k=0}^m \binom{m}{k} \frac{(-a)^k (x+a)^{m-q-k+1}}{m-q-k+1};$$

если $k=m-q+1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1} (-a)^{m-q+1} \ln |x+a|$.

$$6. \int \frac{x^m dx}{(x+a)^q} = - \sum_{k=0}^m \binom{m}{k} \frac{(-a)^{m-k}}{(q-k-1)(x+a)^{q-k-1}};$$

если $k=q-1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1} (-a)^{m-q-1} \ln |x+a|$.

$$7. \int \frac{dx}{(x+a)^n} = -\frac{1}{(n-1)(x+a)^{n-1}}. \quad 8. \int \frac{dx}{x+a} = \ln |x+a|.$$

$$9. \int \frac{x dx}{(x+a)^n} = -\frac{1}{(n-2)(x+a)^{n-2}} + \frac{1}{(n-1)(x+a)^{n-1}}.$$

$$10. \int \frac{x^2 dx}{(x+a)^n} = -\frac{1}{(n-3)(x+a)^{n-3}} + \frac{2a}{(n-2)(x+a)^{n-2}} - \frac{a^2}{(n-1)(x+a)^{n-1}}.$$

$$11. \int \frac{x^3 dx}{(x+a)^n} = -\frac{1}{(n-4)(x+a)^{n-4}} + \frac{3a}{(n-3)(x+a)^{n-3}} - \frac{3a^2}{(n-2)(x+a)^{n-2}} + \frac{a^3}{(n-1)(x+a)^{n-1}}.$$

$$12. \int \frac{x^m dx}{x+a} = \sum_{k=0}^{m-1} \frac{(-1)^k a^k x^{m-k}}{m-k} + (-a)^m \ln |x+a|.$$

$$13. \int \frac{x dx}{x+a} = x - a \ln |x+a|. \quad 14. \int \frac{x^2 dx}{x+a} = \frac{x^2}{2} - ax + a^2 \ln |x+a|.$$

$$15. \int \frac{x^3 dx}{x+a} = \frac{x^3}{3} - \frac{ax^2}{2} + a^2 x - a^3 \ln |x+a|.$$

$$16. \int \frac{x^m dx}{(x+a)^2} = \sum_{k=1}^{m-1} (-1)^k \frac{ka^{k-1} x^{m-k}}{m-k} + \frac{(-1)^{m-1} a^m}{x+a} + (-1)^{m-1} m a^{m-1} \ln |x+a|.$$

$$17. \int \frac{x dx}{(x+a)^2} = \frac{a}{x+a} + \ln |x+a|.$$

$$18. \int \frac{x^2 dx}{(x+a)^2} = x - \frac{a^2}{x+a} - 2a \ln |x+a|.$$

$$19. \int \frac{x^3 dx}{(x+a)^2} = \frac{x^2}{2} - 2ax + \frac{a^2}{x+a} + 3a^2 \ln |x+a|.$$

$$20. \int \frac{x dx}{(x+a)^3} = -\frac{1}{x+a} + \frac{a}{2(x+a)^2}.$$

$$21. \int \frac{x^2 dx}{(x+a)^3} = \frac{2a}{x+a} - \frac{a^2}{2(x+a)^2} + \ln |x+a|.$$

$$22. \int \frac{x^3 dx}{(x+a)^3} = x - \frac{3a^2}{x+a} + \frac{a^3}{2(x+a)^2} - 3a \ln |x+a|.$$

1.2.5. Интегралы вида $\int \frac{dx}{x^p (x+a)^q}.$

$$1. \int \frac{dx}{x^p (x+a)^q} = \frac{-1}{(p-1) a x^{p-1} (x+a)^{q-1}} - \frac{p+q-2}{(p-1) a} \int \frac{dx}{x^{p-1} (x+a)^q}.$$

$$2. = \frac{1}{(q-1) a x^{p-1} (x+a)^{q-1}} + \frac{p+q-2}{(q-1) a} \int \frac{dx}{x^p (x+a)^{q-1}}.$$

$$3. \int \frac{dx}{x^m (x+a)^n} = \frac{1}{a^{m+n-1}} \sum_{k=0}^{m+n-2} (-1)^{k+1} \binom{m+n-2}{k} \frac{(x+a)^{m-k-1}}{(m-k-1) x^{m-k-1}};$$

если $k=m-1$, то вместо соответствующего члена в сумме следует взять $(-1)^m \binom{m+n-2}{m-1} \ln \left| \frac{x+a}{x} \right|$.

$$4. \int \frac{dx}{x (x+a)^n} = \sum_{k=1}^{n-1} \frac{1}{k a^{n-k} (x+a)^k} - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right|.$$

$$5. = \frac{1}{a^n} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n-1}{k} \left(\frac{x}{x+a} \right)^k - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right|.$$

$$6. \int \frac{dx}{x^2 (x+a)^n} = -\frac{1}{a x (x+a)^{n-1}} - \frac{n}{a} \left[\sum_{k=1}^{n-1} \frac{1}{k a^{n-k} (x+a)^k} - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right| \right].$$

$$7. \int \frac{dx}{x^3 (x+a)^n} = \frac{(n+1)x-a}{2a^2 x^2 (x+a)^{n-1}} + \frac{n(n+1)}{2a^2} \int \frac{dx}{x (x+a)^n}.$$

$$8. \int \frac{dx}{x^m (x+a)} = \sum_{k=1}^{m-1} \frac{(-1)^k}{(m-k) a^k x^{m-k}} + \frac{(-1)^m}{a^m} \ln \left| \frac{x+a}{x} \right|.$$

$$9. \int \frac{dx}{x (x+a)} = \frac{1}{a} \ln \left| \frac{x}{x+a} \right|.$$

$$10. \int \frac{dx}{x^2 (x+a)} = -\frac{1}{ax} + \frac{1}{a^2} \ln \left| \frac{x+a}{x} \right|.$$

$$11. \int \frac{dx}{x^3 (x+a)} = \frac{1}{a^2 x} - \frac{1}{2ax^2} - \frac{1}{a^3} \ln \left| \frac{x+a}{x} \right|.$$

$$12. \int \frac{dx}{x (x+a)^2} = \frac{1}{a} \left(\frac{1}{x+a} - \frac{1}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$13. \int \frac{dx}{x^2(x+a)^2} = -\frac{1}{a^2} \left(\frac{1}{x+a} + \frac{1}{x} - \frac{2}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$14. \int \frac{dx}{x^3(x+a)^2} = \frac{1}{a^3} \left(\frac{1}{x+a} + \frac{2}{x} - \frac{a}{2x^2} - \frac{3}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$15. \int \frac{dx}{x(x+a)^3} = \frac{1}{a^2(x+a)} + \frac{1}{2a(x+a)^2} - \frac{1}{a^3} \ln \left| \frac{x+a}{x} \right|.$$

$$16. \int \frac{dx}{x^2(x+a)^3} = -\frac{1}{2a^2(x+a)^2} - \frac{2}{a^3(x+a)} - \frac{1}{a^3x} + \frac{3}{a^4} \ln \left| \frac{x+a}{x} \right|.$$

$$17. \int \frac{dx}{x^3(x+a)^3} = \frac{1}{a^4} \left[\frac{3}{x+a} + \frac{a}{2(x+a)^2} + \frac{3}{x} - \frac{a}{2x^2} - \frac{6}{a} \ln \left| \frac{x+a}{x} \right| \right].$$

1.2.6. Интегралы вида $\int (x+c)^p \left(\frac{x+a}{x+b} \right)^q dx$.

$$1. \int x^p \left(\frac{x+a}{x+b} \right)^q dx = \frac{(x+a)(x+b)x^{p-1}}{p+1} \left(\frac{x+a}{x+b} \right)^q - \\ - \frac{q(b-a) + p(b+a)}{p+1} \int x^{p-1} \left(\frac{x+a}{x+b} \right)^q dx - \frac{(p-1)ab}{p+1} \int x^{p-2} \left(\frac{x+a}{x+b} \right)^q dx.$$

$$2. \int x \left(\frac{x+a}{x+b} \right)^q dx = \\ = \frac{(x+a)^q}{2(x+b)^{q-2}} + \frac{b(x+a)^q}{(q-1)(x+b)^{q-1}} + \frac{q[(q-1)(a-b) - 2b]}{2(q-1)} \int \left(\frac{x+a}{x+b} \right)^{q-1} dx.$$

$$3. = \frac{(x+a)^{q+1}}{2(x+b)^{q-1}} - \frac{q(b-a) + a+b}{2} \int \left(\frac{x+a}{x+b} \right)^q dx.$$

$$4. \int \frac{(x+a)^p}{(x+b)^q} dx = -\frac{(x+a)^p}{(q-1)(x+b)^{q-1}} + \frac{p}{q-1} \int \frac{(x+a)^{p-1}}{(x+b)^{q-1}} dx.$$

$$5. \int \frac{(x+a)^p}{(x+b)^{p+2}} dx = \frac{1}{(p+1)(b-a)} \left(\frac{x+a}{x+b} \right)^{p+1}.$$

$$6. \int \frac{(x+a)^p}{(x+b)^{p+3}} dx = \frac{(x+a)^{p+1} [x + (p+1)(b-a) + b]}{(p+1)(p+2)(b-a)^2(x+b)^{p+2}}.$$

$$7. \int \frac{1}{(x+c)^p} \left(\frac{x+a}{x+b} \right)^q dx = \frac{-1}{(p-1)(a-c)(b-c)} \left\{ \frac{1}{(x+c)^{p-1}} \cdot \frac{(x+a)^{q+1}}{(x+b)^{q-1}} + \right. \\ \left. + [(p-2)(a+b-2c) - q(b-a)] \int \frac{1}{(x+c)^{p-1}} \left(\frac{x+a}{x+b} \right)^q dx + \right. \\ \left. + (p-3) \int \frac{1}{(x+c)^{p-2}} \left(\frac{x+a}{x+b} \right)^q dx \right\}.$$

$$8. \int \frac{1}{x+c} \left(\frac{x+a}{x+b} \right)^n dx = \sum_{k=1}^{n-1} \frac{1}{n-k} \left[\left(\frac{a-c}{b-c} \right)^k - 1 \right] \left(\frac{x+a}{x+b} \right)^{n-k} - \\ - \left[\left(\frac{a-c}{b-c} \right)^n - 1 \right] \ln |x+b| + \left(\frac{a-c}{b-c} \right)^n \ln |x+c|.$$

$$9. \int \left(\frac{x+a}{x+b} \right)^n dx = \\ = - \sum_{k=0}^{n-2} \frac{n}{(n-k)(n-k-1)} \frac{(x+a)^{n-k}}{(x+b)^{n-k-1}} + n(x+a) - n(b-a) \ln |x+b|.$$

$$10. = - \sum_{k=1}^{n-1} \binom{n}{k+1} \frac{(a-b)^{k+1}}{k(x+b)^k} + x + n(a-b) \ln |x+b|.$$

$$11. \int \frac{x+a}{x+b} dx = x + (a-b) \ln |x+b|.$$

$$12. \int x \frac{x+a}{x+b} dx = \frac{x^2}{2} + (a-b)(x - b \ln |x+b|).$$

$$13. \int x^2 \frac{x+a}{x+b} dx = \frac{x^3}{3} + (a-b) \left(\frac{x^2}{2} - bx + b^2 \ln |x+b| \right).$$

$$14. \int \frac{x+a}{x(x+b)} dx = \frac{a}{b} \ln |x| - \frac{a-b}{b} \ln |x+b|.$$

$$15. \int \left(\frac{x+a}{x+b} \right)^2 dx = x - \frac{(a-b)^2}{x+b} + 2(a-b) \ln |x+b|.$$

$$16. \int x \left(\frac{x+a}{x+b} \right)^2 dx = \frac{x^2}{2} + 2(a-b)x + (a-b) \left[\frac{b(a-b)}{x+b} + (a-3b) \ln |x+b| \right].$$

$$17. \int x^2 \left(\frac{x+a}{x+b} \right)^2 dx = \frac{x^3}{3} + (a-b)x^2 + \frac{b(a-b)^2 x}{x+b} + (a-b) [(a-3b)(x+b) - 2b(a-2b) \ln |x+b|].$$

$$18. \int \frac{1}{x} \left(\frac{x+a}{x+b} \right)^2 dx = \frac{(a-b)^2}{b(x+b)} + \frac{a^2}{b^2} \ln |x| - \left(\frac{a^2}{b^2} - 1 \right) \ln |x+b|.$$

$$19. \int \frac{(x+a)^2}{x+b} dx = \frac{x^2}{2} + (2a-b)x + (a-b)^2 \ln |x+b|.$$

$$20. \int x \frac{(x+a)^2}{x+b} dx = \frac{x^3}{3} + (2a-b) \frac{x^2}{2} + (a-b)^2 (x - b \ln |x+b|).$$

$$21. \int x^2 \frac{(x+a)^2}{x+b} dx = \frac{x^4}{4} + (2a-b) \frac{x^3}{3} + (a-b)^2 \left(\frac{x^2}{2} - bx + b^2 \ln |x+b| \right).$$

$$22. \int \frac{(x+a)^2}{x(x+b)} dx = x + \frac{1}{b} (a^2 \ln |x| - (a-b)^2 \ln |x+b|).$$

$$23. \int \frac{x+a}{(x+b)^2} dx = -\frac{a-b}{x+b} + \ln |x+b|.$$

$$24. \int x \frac{x+a}{(x+b)^2} dx = x + \frac{b(a-b)}{x+b} + (a-2b) \ln |x+b|.$$

$$25. \int x^2 \frac{x+a}{(x+b)^2} dx = \frac{x^2}{2} + (a-2b)x - \frac{b^2(a-b)}{x+b} - b(2a-3b) \ln |x+b|.$$

$$26. \int \frac{x+a}{x(x+b)^2} dx = \frac{a-b}{b(x+b)} - \frac{a}{b^2} \ln \left| \frac{x+b}{x} \right|.$$

1.2.7. Интегралы вида $\int \frac{x^p dx}{(x+a)^q (x+b)^r}.$

$$1. \int \frac{x^p dx}{(x+a)^q (x+b)^r} = \frac{1}{p-q-r+1} \left\{ \frac{x^{p-1}}{(x+a)^{q-1} (x+b)^{r-1}} - [(p-r)a + (p-q)b] \int \frac{x^{p-1} dx}{(x+a)^q (x+b)^r} - (p-1)ab \int \frac{x^{p-2} dx}{(x+a)^q (x+b)^r} \right\}.$$

$$\begin{aligned}
 2. \int \frac{x^k dx}{(x+a)^m (x+b)^n} = & \\
 = - \sum_{p=1}^{m-1} \frac{(-1)^{m+k+p+1}}{p (x+a)^p} & \left[\sum_{q=0}^{m-p-1} \binom{k}{q} \binom{m+n-p-q-2}{n-1} \frac{a^{k-q}}{(b-a)^{m+n-p-q-1}} \right] - \\
 - \sum_{p=1}^{n-1} \frac{(-1)^m}{p (x+b)^p} & \left[\sum_{q=0}^{n-p-1} \binom{k}{q} \binom{m+n-p-q-2}{m-1} \frac{(-b)^{k-q}}{(b-a)^{m+n-p-q-1}} \right] + \\
 + (-1)^{m+k+1} \sum_{p=0}^{m-1} \binom{k}{p} \binom{m+n-p-2}{n-1} & \frac{a^{k-p}}{(b-a)^{m+n-p-1}} \ln |x+a| + \\
 + (-1)^m \sum_{p=0}^{n-1} \binom{k}{p} \binom{m+n-p-2}{m-1} & \frac{(-b)^{k-p}}{(b-a)^{m+n-p-1}} \ln |x+b|.
 \end{aligned}$$

$$\begin{aligned}
 3. \int \frac{dx}{(x+a)^q (x+b)^r} = & \\
 = - \frac{1}{(r-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} - & \frac{(q+r-2)}{(r-1)(a-b)} \int \frac{dx}{(x+a)^q (x+b)^{r-1}}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & = \frac{1}{(q-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} + \\
 & + \frac{q+r-2}{(q-1)(a-b)} \int \frac{dx}{(x+a)^{q-1}(x+b)^r}.
 \end{aligned}$$

$$\begin{aligned}
 5. \int \frac{dx}{(x+a)^m (x+b)^n} = & \\
 = \frac{(-1)^{m+n-2}}{(b-a)^{m+n-1}} \sum_{\substack{k=0 \\ k \neq n-1}}^{m+n-2} \frac{(-1)^k}{n-k-1} \binom{m+n-2}{k} & \left(\frac{x+a}{x+b} \right)^{n-k-1} + \\
 + \binom{m+n-2}{n-1} \frac{(-1)^{m-1}}{(b-a)^{m+n-1}} \ln \left| \frac{x+a}{x+b} \right|.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & = \sum_{k=1}^{m-1} \frac{(-1)^{m+k}}{k} \binom{m+n-k-2}{n-1} \frac{1}{(b-a)^{m+n-k-1} (x+a)^k} - \\
 - (-1)^m \sum_{k=1}^{n-1} \frac{1}{k} \binom{m+n-k-2}{m-1} & \frac{1}{(b-a)^{m+n-k-1} (x+b)^k} + \\
 + \frac{(-1)^{m+1}}{(b-a)^{m+n-1}} \binom{m+n-1}{n-1} \ln \left| \frac{x+a}{x+b} \right|.
 \end{aligned}$$

$$7. \int \frac{dx}{(x+a)(x+b)} = -\frac{1}{a-b} \ln \left| \frac{x+a}{x+b} \right|.$$

$$8. \int \frac{x dx}{(x+a)(x+b)} = \frac{1}{a-b} (a \ln |x+a| - b \ln |x+b|).$$

$$9. \int \frac{x^2 dx}{(x+a)(x+b)} = x + \frac{1}{a-b} (b^2 \ln |x+b| - a^2 \ln |x+a|).$$

$$10. \int \frac{dx}{x(x+a)(x+b)} = \frac{1}{ab} \ln |x| + \frac{1}{a(a-b)} \ln |x+a| - \frac{1}{b(a-b)} \ln |x+b|.$$

$$11. \int \frac{dx}{x^2(x+a)(x+b)} = -\frac{1}{abx} - \frac{a+b}{a^2b^2} \ln |x| - \\ - \frac{1}{a^2(a-b)} \ln |x+a| + \frac{1}{b^2(a-b)} \ln |x+b|.$$

$$12. \int \frac{dx}{(x+a)(x+b)^2} = \frac{-1}{(a-b)(x+b)} + \frac{1}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$

$$13. \int \frac{x dx}{(x+a)(x+b)^2} = \frac{b}{(a-b)(x+b)} - \frac{a}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$

$$14. \int \frac{x^2 dx}{(x+a)(x+b)^2} = \frac{-b^2}{(a-b)(x+b)} + \frac{a^2}{(a-b)^2} \ln |x+a| + \frac{b^2-2ab}{(a-b)^2} \ln |x+b|.$$

$$15. \int \frac{dx}{x(x+a)(x+b)^2} = \frac{1}{b(a-b)(x+b)} + \frac{1}{ab^2} \ln |x| - \\ - \frac{1}{a(a-b)^2} \ln |x+a| + \frac{2b-a}{b^2(a-b)} \ln |x+b|.$$

$$16. \int \frac{dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2} \left(\frac{1}{x+a} + \frac{1}{x+b} \right) + \frac{2}{(a-b)^3} \ln \left| \frac{x+a}{x+b} \right|.$$

$$17. \int \frac{x dx}{(x+a)^2(x+b)^2} = \frac{1}{(a-b)^2} \left(\frac{a}{x+a} + \frac{b}{x+b} \right) - \frac{a+b}{(a-b)^3} \ln \left| \frac{x+a}{x+b} \right|.$$

$$18. \int \frac{x^2 dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2} \left(\frac{a^2}{x+a} + \frac{b^2}{x+b} \right) + \frac{2ab}{(a-b)^3} \ln \left| \frac{x+a}{x+b} \right|.$$

1.2.8. Интегралы вида $\int R(x, ax^2+bx+c) dx$.

$$1. \int (ax^2+bx+c)^n dx = \frac{1}{2(2n+1)a} [(2ax+b)(ax^2+bx+c)^n - \\ - n(b^2-4ac) \int (ax^2+bx+c)^{n-1} dx].$$

$$2. = (-1)^n \frac{(n!)^2}{(2n+1)!} \frac{2ax+b}{2a} \sum_{k=0}^n (-1)^k \binom{2k}{k} \left(\frac{b^2-4ac}{a} \right)^{n-k} (ax^2+bx+c)^k.$$

$$3. \int \frac{(ax^2+bx+c)^n dx}{x^m} = -\frac{(ax^2+bx+c)^{n+1}}{(m-1)cx^{m-1}} + \\ + \frac{b(n-m+2)}{c(m-1)} \int \frac{(ax^2+bx+c)^n dx}{x^{m-1}} + \frac{a(2n-m+3)}{c(m-1)} \int \frac{(ax^2+bx+c)^n}{x^{m-2}} dx.$$

$$4. \int \frac{x^m dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} + \\ + \frac{(m-1)c}{(2n-m-1)a} \int \frac{x^{m-2} dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a} \int \frac{x^{m-1} dx}{(ax^2+bx+c)^n}.$$

$$5. \int \frac{x^{2n-1} dx}{(ax^2+bx+c)^n} = \frac{1}{a} \int \frac{x^{2n-3} dx}{(ax^2+bx+c)^{n-1}} - \\ - \frac{c}{a} \int \frac{x^{2n-3} dx}{(ax^2+bx+c)^n} - \frac{b}{a} \int \frac{x^{2n-2} dx}{(ax^2+bx+c)^n}.$$

$$6. \int \frac{x dx}{(ax^2 + bx + c)^n} = -\frac{1}{2(n-1)a(ax^2 + bx + c)^{n-1}} - \frac{b}{2a} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$7. = \frac{bx + 2c}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} + \frac{(2n-3)b}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$8. \int \frac{x^2 dx}{(ax^2 + bx + c)^n} = -\frac{(b^2 - 2ac)(2ax + b) - b(b^2 - 4ac)}{2(n-1)a^2(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} - \frac{2(2n-3)(b^2 - 2ac) - 2(n-1)(b^2 - 4ac)}{2(n-1)a(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$9. \int \frac{dx}{x^m(ax^2 + bx + c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2 + bx + c)^{n-1}} - \frac{(m+n-2)b}{(m-1)c} \int \frac{dx}{x^{m-1}(ax^2 + bx + c)^n} - \frac{(m+2n-3)a}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^2 + bx + c)^n}.$$

$$10. \int \frac{dx}{x(ax^2 + bx + c)^n} = \frac{1}{2(n-1)c(ax^2 + bx + c)^{n-1}} - \frac{b}{2c} \int \frac{dx}{(ax^2 + bx + c)^n} + \frac{1}{c} \int \frac{dx}{x(ax^2 + bx + c)^{n-1}}.$$

$$11. \int \frac{dx}{(ax^2 + bx + c)^n} = \frac{-2ax - b}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} - \frac{2(2n-3)a}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$12. = \frac{2ax + b}{2n-1} \times \\ \times \sum_{k=0}^{n-2} \frac{2^k(2n-1)(2n-3)\dots(2n-2k-1)a^k}{(n-1)(n-2)\dots(n-k-1)(4ac-b^2)^{k+1}(ax^2 + bx + c)^{n-k-1}} + \\ + \frac{(2n-3)!!(2a)^{n-1}}{(n-1)!(4ac-b^2)^{n-1}} \int \frac{dx}{ax^2 + bx + c}.$$

$$13. \int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| \quad [b^2 - 4ac > 0].$$

$$14. = \frac{2}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2ax + b}{\sqrt{4ac - b^2}} \quad [b^2 - 4ac < 0].$$

$$15. = -\frac{2}{2ax + b} \quad [b^2 = 4ac].$$

$$16. = \frac{1}{a(p-q)} \ln \left| \frac{x-p}{x-q} \right|,$$

где p, q — действительные корни многочлена $ax^2 + bx + c$.

$$17. \int \frac{dx}{(ax^2 + bx + c)^2} = \frac{2ax + b}{(4ac - b^2)(ax^2 + bx + c)} + \frac{2a}{4ac - b^2} \int \frac{dx}{ax^2 + bx + c}.$$

$$18. \int \frac{dx}{(ax^2+bx+c)^3} = \frac{2ax+b}{2(4ac-b^2)(ax^2+bx+c)^2} + \\ + \frac{3a(2ax+b)}{(4ac-b^2)^2(ax^2+bx+c)} + \frac{6a^2}{(4ac-b^2)^2} \int \frac{dx}{ax^2+bx+c}.$$

$$19. \int \frac{x dx}{ax^2+bx+c} = \frac{1}{2a} \ln |ax^2+bx+c| - \\ - \frac{b}{a\sqrt{|b^2-4ac|}} \begin{cases} \frac{1}{2} \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| & [b^2 > 4ac], \\ \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}} & [4ac > b^2] \end{cases}$$

$$20. \int \frac{x^2 dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2} \ln |ax^2+bx+c| + \frac{b^2-2ac}{2a^2} \int \frac{dx}{ax^2+bx+c}.$$

$$21. \int \frac{x^3 dx}{ax^2+bx+c} = \\ = \frac{ax^2-2bx}{2a^2} + \frac{b^2-ac}{2a^3} \ln |ax^2+bx+c| - \frac{b(b^2-3ac)}{2a^3} \int \frac{dx}{ax^2+bx+c}.$$

$$22. \int \frac{x dx}{(ax^2+bx+c)^2} = \frac{bx+2c}{(b^2-4ac)(ax^2+bx+c)} + \frac{b}{b^2-4ac} \int \frac{dx}{ax^2+bx+c}.$$

$$23. \int \frac{x^2 dx}{(ax^2+bx+c)^2} = \frac{(2ac-b^2)x-bc}{a(b^2-4ac)(ax^2+bx+c)} - \frac{2c}{b^2-4ac} \int \frac{dx}{ax^2+bx+c}.$$

$$24. \int \frac{x^3 dx}{(ax^2+bx+c)^2} = \frac{c(2ac-b^2)+b(3ac-b^2)x}{a^2(4ac-b^2)} + \\ + \frac{1}{2a^2} \ln |ax^2+bx+c| + \frac{b(6ac-b^2)}{2a^2(b^2-4ac)} \int \frac{dx}{ax^2+bx+c}.$$

$$25. \int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c} \ln \frac{x^2}{|ax^2+bx+c|} - \\ - \frac{b}{c\sqrt{|b^2-4ac|}} \begin{cases} \frac{1}{2} \ln \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} & [b^2 > 4ac], \\ \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}} & [b^2 < 4ac]. \end{cases}$$

$$26. \int \frac{dx}{x^2(ax^2+bx+c)} = -\frac{1}{cx} - \frac{b}{2c^2} \ln \frac{x^2}{|ax^2+bx+c|} + \frac{b^2-2ac}{2c^2} \int \frac{dx}{ax^2+bx+c}.$$

$$27. \int \frac{dx}{x^3(ax^2+bx+c)} = \\ = \frac{2bx-c}{2c^2x^2} + \frac{b^2-ac}{2c^3} \ln \frac{x^2}{|ax^2+bx+c|} + \frac{b(3ac-b^2)}{2c^3} \int \frac{dx}{ax^2+bx+c}.$$

$$28. \int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} \left[1 + \frac{b(b+2ax)}{b^2-4ac} \right] + \\ + \frac{1}{2c^2} \ln \frac{x^2}{|ax^2+bx+c|} - \frac{b}{2c^2} \left(1 - \frac{2ac}{b^2-4ac} \right) \int \frac{dx}{ax^2+bx+c}.$$

$$29. \int \frac{dx}{x^2 (ax^2 + bx + c)^2} = \frac{(b^2 - 3ac)(b + 2ax)}{c^2 (4ac - b^2) (ax^2 + bx + c)} - \frac{c + bx}{c^2 x (ax^2 + bx + c)} - \\ - \frac{b}{c^3} \ln \frac{x^2}{|ax^2 + bx + c|} + \frac{1}{b^2 - 4ac} \left(\frac{b^4}{c^3} - \frac{6b^2a}{c^2} + \frac{6a^2}{c} \right) \int \frac{dx}{ax^2 + bx + c}.$$

$$30. \int \frac{dx}{x^3 (ax^2 + bx + c)^2} = \\ = \frac{3bx - c}{2c^2 x^2 (ax^2 + bx + c)} + \frac{3b^2 - 2ac}{c^2} \int \frac{dx}{x (ax^2 + bx + c)^2} + \frac{9ab}{2c^2} \int \frac{dx}{(ax^2 + bx + c)^2}.$$

1.2.9. Интегралы вида $\int R(x+d, ax^2+bx+c) dx$.

$$1. \int (x+d)^m (ax^2+bx+c)^n dx = \frac{1}{(m+2n+1)a} \left[(x+d)^{m+1} (ax^2+bx+c)^{n+1} + \right. \\ \left. + (m+n)(2ad-b) \int (x+d)^{m-1} (ax^2+bx+c)^n dx - \right. \\ \left. - (m-1)(ad^2-bd+c) \int (x+d)^{m-2} (ax^2+bx+c)^n dx \right].$$

$$2. \int \frac{(ax^2+bx+c)^n}{(x+d)^m} dx = - \frac{(ax^2+bx+c)^n}{(m-2n-1)(x+d)^{m-1}} - \\ - \frac{2n(ad^2-bd+c)}{m-2n-1} \int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^m} dx - \frac{n(b-2ad)}{m-2n-1} \int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}} dx.$$

$$3. = - \frac{1}{(m-1)(ad^2-bd+c)} \left[\frac{(ax^2+bx+c)^{n+1}}{(x+d)^{m-1}} + \right. \\ \left. + (m-n-2)(b-2ad) \int \frac{(ax^2+bx+c)^n}{(x+d)^{m-1}} dx + (m-2n-3)a \int \frac{(ax^2+bx+c)^n}{(x+d)^{m-2}} dx \right].$$

$$4. = - \frac{(ax^2+bx+c)^n}{(m-1)(x+d)^{m-1}} + \\ + \frac{n(b-2ad)}{m-1} \int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}} dx + 2na \int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-2}} dx.$$

$$5. \int \frac{(ax^2+bx+c)^n}{x+d} dx = (ad^2-bd+c)^n \ln|x+d| + \\ + \int (ax+b-ad) \left[\sum_{k=0}^{n-1} (ad^2-bd+c)^{n-k-1} (ax^2+bx+c)^k \right] dx.$$

$$6. \int \frac{(x+d)^m dx}{(ax^2+bx+c)^n} = \frac{(2ad-b)(2ax+b) + (4ac-b^2)}{2(n-1)(4ac-b^2)(ad^2-bd+c)} \frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-1}} + \\ + \frac{2(m-2n+4)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)} \frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-2}} + \\ + \left[\frac{4(2n-3)a}{(n-1)(4ac-b^2)} - \frac{2(m-n+2)}{(n-1)(ad^2-bd+c)} \right] \int \frac{(x+d)^m dx}{(ax^2+bx+c)^{n-1}} - \\ - \frac{2(m-2n+4)(m-2n+5)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)} \int \frac{(x+d)^m dx}{(ax^2+bx+c)^{n-2}}.$$

$$7. = \frac{1}{(m-2n+1)a} \left[\frac{(x+d)^{m-1}}{(ax^2+bx+c)^{n-1}} + \right. \\ \left. + (m-n)(2ad-b) \int \frac{(x+d)^{m-1} dx}{(ax^2+bx+c)^n} - (m-1)(ad^2-bd+c) \int \frac{(x+d)^{m-2} dx}{(ax^2+bx+c)^n} \right].$$

$$8. = \frac{1}{(n-1)(4ac-b^2)} \left[\frac{(2ax+b)(x+d)^m}{(ax^2+bx+c)^{n-1}} - \right. \\ \left. - 2(m-2n+3)a \int \frac{(x+d)^m dx}{(ax^2+bx+c)^{n-1}} - (b-2ad)m \int \frac{(x+d)^{m-1} dx}{(ax^2+bx+c)^{n-1}} \right].$$

$$9. \int \frac{dx}{(x+d)^m (ax^2+bx+c)^n} = \frac{-1}{(m-1)(ad^2-bd+c)} \times \\ \times \left[\frac{1}{(x+d)^{m-1} (ax^2+bx+c)^{n-1}} + 2(m+n-2)(b-2ad) \times \right. \\ \left. \times \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} + (m+2n-3)a \int \frac{dx}{(x+d)^{m-2} (ax^2+bx+c)^n} \right].$$

$$10. = \frac{1}{2(ad^2-bd+c)} \left[\frac{1}{(n-1)(x+d)^{m-1} (ax^2+bx+c)^{n-1}} + 2(ad-b) \times \right. \\ \left. \times \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} + \frac{m+2n-3}{n-1} \int \frac{dx}{(x+d)^m (ax^2+bx+c)^{n-1}} \right].$$

$$11. = \frac{1}{2(m+n-1)(ad-b)} \left[\frac{1}{(x+d)^m (ax^2+bx+c)^{n-1}} + \right. \\ \left. + (m+2n-2)a \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} \right] \quad [ad^2-bd+c=0].$$

$$12. \int \frac{dx}{(x+d)(ax^2+bx+c)^n} = \frac{1}{ad^2-bd+c} \left[\frac{1}{2(n-1)(ax^2+bx+c)^{n-1}} + \right. \\ \left. + (ad-b) \int \frac{dx}{(ax^2+bx+c)^n} + \int \frac{dx}{(x+d)(ax^2+bx+c)^{n-1}} \right].$$

$$13. \int \frac{dx}{(x+d)(ax^2+bx+c)} = \frac{1}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \\ + \frac{2ad-b}{2(ad^2-bd+c)\sqrt{|b^2-4ac|}} \begin{cases} \operatorname{arctg} \frac{2ax+b}{\sqrt{|b^2-4ac|}} & [b^2-4ac < 0], \\ \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| & [b^2-4ac > 0]. \end{cases}$$

$$14. \int \frac{dx}{(x+d)^2 (ax^2+bx+c)} = \\ = -\frac{1}{ad^2-bd+c} \left[\frac{1}{x+d} + \frac{2ad-b}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \right. \\ \left. + \frac{2c(ad^2-bd+c)-(2ad-b^2)}{2(ad^2-bd+c)^2} \int \frac{dx}{ax^2+bx+c} \right].$$

1.2.10. Интегралы вида $\int \frac{x^m dx}{(x^2 \pm a^2)^n}$.

$$1. \int \frac{x^m dx}{(x^2 \pm a^2)^n} = \pm \frac{x^{m+1}}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \mp \frac{(m-2n+3)}{2(n-1)a^2} \int \frac{x^m dx}{(x^2 \pm a^2)^{n-1}}.$$

$$2. = \frac{x^{m-1}}{(m-2n+1)(x^2 \pm a^2)^{n-1}} \mp \frac{(m-1)a^2}{m-2n+1} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^n}.$$

$$3. = \frac{-x^{m-1}}{2(n-1)(x^2 \pm a^2)^{n-1}} + \frac{m-1}{2(n-1)} \int \frac{x^{m-2}}{(x^2 \pm a^2)^{n-1}}.$$

$$4. = \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n-1}} \mp a^2 \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^n}.$$

$$\begin{aligned}
 5. \int \frac{x^{2m} dx}{(x^2 + a^2)^n} &= \frac{x^{2m+1}}{2(n-1)a^2} \left[\frac{1}{(x^2 + a^2)^{n-1}} + \right. \\
 &+ \sum_{k=1}^{n-2} \frac{(2n-2m-3)(2n-2m-5) \dots (2n-2m-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \frac{1}{a^{2k} (x^2 + a^2)^{n-k-1}} \Big] + \\
 &+ (-1)^m a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5) \dots (3-2m)(1-2m)}{2^{n-1} (n-1)!} \times \\
 &\times \left[\sum_{k=1}^m \frac{(-1)^k}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + \operatorname{arctg} \frac{x}{a} \right].
 \end{aligned}$$

$$\begin{aligned}
 6. \int \frac{x^{2m} dx}{(x^2 - a^2)^n} &= -\frac{x^{2m+1}}{2(n-1)a^2} \left[\frac{1}{(x^2 - a^2)^{n-1}} - \right. \\
 &- \sum_{k=1}^{n-2} (-1)^k \frac{(2n-2m-3)(2n-2m-5) \dots (2n-2m-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \times \\
 &\times \frac{1}{a^{2k} (x^2 - a^2)^{n-k-1}} \Big] + \\
 &+ (-1)^{n+1} a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5) \dots (3-2m)(1-2m)}{2^{n-1} (n-1)!} \times \\
 &\times \left[\sum_{k=1}^m \frac{1}{2k-1} \left(\frac{x}{a} \right)^{2k-1} - \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right].
 \end{aligned}$$

$$7. \int \frac{x^{2m+1} dx}{(x^2 \pm a^2)^n} = - \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{a^{2k}}{2(n-m+k-1)(x^2 \pm a^2)^{n-m+k-1}}.$$

$$\begin{aligned}
 8. &= \frac{1}{2(\pm a^2)^{n-m-1}} \sum_{k=0}^{n-m-2} (-1)^k \binom{n-m-2}{k} \frac{1}{m+k+1} \times \\
 &\times \left(\frac{x^2}{x^2 \pm a^2} \right)^{m+k+1} \quad [n \geq m+2]
 \end{aligned}$$

$$9. \int \frac{x^{2m} dx}{x^2 \pm a^2} = \sum_{k=0}^{m-1} (-1)^k \frac{a^{2k} x^{2m-2k-1}}{2m-2k-1} + (-1)^m a^{2m-1} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{array} \right\}.$$

$$10. \int \frac{x^{2m+1} dx}{x^2 \pm a^2} = \frac{1}{2} \sum_{k=0}^{m-1} (-1)^k \frac{a^{2k} x^{2m-2k}}{m-k} + \frac{1}{2} (\mp a^2)^m \ln |x^2 \pm a^2|.$$

$$\begin{aligned}
 11. \int \frac{x^{2m} dx}{(x^2 \pm a^2)^2} &= \pm \frac{x^{2m+1}}{2a^2 (x^2 \pm a^2)} + \frac{1}{2} (1-2m) a^{2m-2} \times \\
 &\times \left[\pm \sum_{k=1}^m \frac{(-1)^{m+k}}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + (-1)^m \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\} \right].
 \end{aligned}$$

$$12. \int \frac{dx}{(x^2 \pm a^2)^n} = \pm \frac{x}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}}.$$

$$13. \int \frac{dx}{(x^2 \pm a^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} (\pm 1)^k \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k) a^{2k} (x^2 \pm a^2)^{n-k}} +$$

$$+ (\pm 1)^n \frac{(2n-3)!!}{2^{n-1} (n-1)! a^{2n-1}} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$14. \int \frac{dx}{x^2 \pm a^2} = \frac{1}{a} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{array} \right\}.$$

$$15. \int \frac{dx}{(x^2 \pm a^2)^2} = \frac{x}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^3} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$16. \int \frac{dx}{(x^2 \pm a^2)^3} = \pm \frac{x}{4a^2(x^2 \pm a^2)^2} + \frac{3x}{8a^4(x^2 \pm a^2)} + \frac{3}{8a^5} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{array} \right\}.$$

$$17. \int \frac{x dx}{x^2 \pm a^2} = \frac{1}{2} \ln |x^2 \pm a^2|.$$

$$18. \int \frac{x^2 dx}{x^2 \pm a^2} = x - a \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$19. \int \frac{x^3 dx}{x^2 \pm a^2} = \frac{x^2}{2} \mp \frac{a^2}{2} \ln |x^2 \pm a^2|.$$

$$20. \int \frac{x^4 dx}{x^2 \pm a^2} = \frac{x^3}{3} \mp a^2 x \pm \left\{ \begin{array}{l} a^3 \operatorname{arctg} \frac{x}{a} \\ \frac{a^3}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$21. \int \frac{x dx}{(x^2 \pm a^2)^2} = -\frac{1}{2(x^2 \pm a^2)}.$$

$$22. \int \frac{x^2 dx}{(x^2 \pm a^2)^2} = -\frac{x}{2(x^2 \pm a^2)} + \left\{ \begin{array}{l} \frac{1}{2a} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{4a} \ln \left| \frac{x-a}{x+a} \right| \end{array} \right\}.$$

$$23. \int \frac{x^3 dx}{(x^2 \pm a^2)^2} = \pm \frac{a^2}{2(x^2 \pm a^2)} + \frac{1}{2} \ln |x^2 \pm a^2|.$$

$$24. \int \frac{x^4 dx}{(x^2 \pm a^2)^2} = x \pm \frac{a^2 x}{2(x^2 \pm a^2)} - \left\{ \begin{array}{l} \frac{3a}{2} \operatorname{arctg} \frac{x}{a} \\ \frac{3a}{4} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

1.2.11. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^n}$.

$$1. \int \frac{dx}{x^{2m} (x^2 \pm a^2)^n} = \mp \frac{1}{(2m-1) a^2 x^{2m-1} (x^2 \pm a^2)^{n-1}} \mp \frac{2m+2n-3}{(2m-1) a^2} \int \frac{dx}{x^{2m-2} (x^2 \pm a^2)^n}.$$

$$2. = \frac{\pm 1}{(2m+2n-1) (x^2 \pm a^2)^{n-1}} \times \\ \times \sum_{k=1}^m \frac{(\mp 1)^k (2m+2n-1) (2m-2n-3) \dots (2m+2n-2k+1)}{(2m-1) (2m-3) \dots (2m-2k+1) a^{2k}} \frac{1}{x^{2m-2k+1}} + \\ + (\mp 1)^m \frac{(2n-1) (2n+1) \dots (2n+2m-3)}{(2m-1)!! a^{2m}} \int \frac{dx}{(x^2 \pm a^2)^n} \quad [\text{см. 1.2.10}].$$

$$3. \int \frac{dx}{x^{2m+1} (x^2 \pm a^2)^n} = \mp \frac{1}{2ma^2 x^{2m} (x^2 \pm a^2)^{n-1}} \mp \frac{m+n-1}{ma^2} \int \frac{dx}{x^{2m-1} (x^2 \pm a^2)^n}.$$

$$4. = \frac{1}{2(\mp a^2)^{m+n}} \sum_{k=0}^{m+n-1} (-1)^k \binom{m+n-1}{k} \frac{1}{k-n+1} \left(\frac{x^2 \pm a^2}{x^2} \right)^{k-n+1}.$$

$$5. \int \frac{dx}{x (x^2 \pm a^2)^n} = \frac{\pm 1}{2(n-1) a^2 (x^2 \pm a^2)^{n-1}} \pm \frac{1}{a^2} \int \frac{dx}{x (x^2 \pm a^2)^{n-1}}.$$

$$6. = \frac{1}{2} \sum_{k=1}^{n-1} \frac{(\pm 1)^k}{(n-k) a^{2k} (x^2 \pm a^2)^{n-k}} + \frac{(\pm 1)^n}{2a^{2n}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$7. \int \frac{dx}{x^{2m} (x^2 \pm a^2)} = \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k}}{(2k+1) a^{2(m-k)} x^{2k+1}} + \frac{(\mp 1)^m}{a^{2m+1}} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{array} \right\}.$$

$$8. \int \frac{dx}{x^{2m+1} (x^2 \pm a^2)} = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(\mp 1)^{k+1}}{(m-k) a^{2(k+1)} x^{2(m-k)}} + \frac{(\mp 1)^{m+1}}{2a^{2(m+1)}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$9. \int \frac{dx}{x^{2m} (x^2 \pm a^2)^2} = \pm \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k} (m-k)}{(2k+1) a^{2m-2k+2} x^{2k+1}} - \\ - \frac{(\mp 1)^{m+1} x}{2a^{2m+2} (x^2 \pm a^2)} + \frac{(\mp 1)^m (2m+1)}{2a^{2m+2}} \left\{ \begin{array}{l} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$10. \int \frac{dx}{x(x^2 \pm a^2)} = \pm \frac{1}{2a^2} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$11. \int \frac{dx}{x^2(x^2 \pm a^2)} = \mp \frac{1}{a^2 x} \mp \frac{1}{a^3} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. \left[\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right] \right\}.$$

$$12. \int \frac{dx}{x^3(x^2 \pm a^2)} = \mp \frac{1}{2a^2 x^2} - \frac{1}{2a^4} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$13. \int \frac{dx}{x^4(x^2 \pm a^2)} = \mp \frac{1}{3a^2 x^3} + \frac{1}{a^4 x} + \frac{1}{a^5} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. \left[\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right] \right\}.$$

$$14. \int \frac{dx}{x(x^2 \pm a^2)^2} = \frac{1}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^4} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$15. \int \frac{dx}{x^2(x^2 \pm a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2 \pm a^2)} - \frac{3}{2a^5} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. \left[\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right] \right\}.$$

$$16. \int \frac{dx}{x^3(x^2 \pm a^2)^2} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^4(x^2 \pm a^2)} \mp \frac{1}{a^6} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$17. \int \frac{dx}{x^4(x^2 \pm a^2)^2} = -\frac{1}{3a^4 x^3} \pm \frac{2}{a^6 x} \pm \frac{x}{2a^6(x^2 \pm a^2)} + \frac{5}{2a^7} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. \left[\frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right] \right\}.$$

$$18. \int \frac{dx}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2} \left[\ln |x+b| - \frac{1}{2} \ln(x^2+a^2) + \frac{b}{a} \operatorname{arctg} \frac{x}{a} \right].$$

$$19. \int \frac{dx}{(x+b)(x^2-a^2)} = \\ = -\frac{1}{a^2-b^2} \ln |x+b| + \frac{1}{2a(a+b)} \ln |x-a| + \frac{1}{2a(a-b)} \ln |x+a|.$$

1.2.12. Интегралы вида $\int \frac{x^{\pm m} dx}{(x^3+a^3)^n}.$

$$1. \int \frac{x^m dx}{(x^3+a^3)^n} = \frac{x^{m+1}}{3(n-1)a^3(x^3+a^3)^{n-1}} - \frac{m-3n+4}{3(n-1)a^3} \int \frac{x^m dx}{(x^3+a^3)^{n-1}}.$$

$$2. = \frac{x^{m-2}}{(m-3n+1)(x^3+a^3)^{n-1}} - \frac{(m-2)a^3}{m-3n+1} \int \frac{x^{m-3} dx}{(x^3+a^3)^n}.$$

$$3. \int \frac{x^m dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3 \int \frac{x^{m-3} dx}{x^3+a^3}.$$

$$4. \int \frac{dx}{(x^3+a^3)^n} = \frac{x}{3a^2(n-1)(x^3+a^3)^{n-1}} + \frac{3n-4}{3a^3(n-1)} \int \frac{dx}{(x^3+a^3)^{n-1}}.$$

$$5. \int \frac{x dx}{(x^3+a^3)^n} = \frac{x^2}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-5}{3a^3(n-1)} \int \frac{x dx}{(x^3+a^3)^{n-1}}.$$

6. $\int \frac{dx}{x^3 + a^3} = \frac{1}{6a^2} \ln \frac{(x+a)^2}{x^2 - ax + a^2} + \frac{1}{a^2 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
7. $\int \frac{x dx}{x^3 + a^3} = \frac{1}{6a} \ln \frac{x^2 - ax + a^2}{(x+a)^2} + \frac{1}{a \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
8. $\int \frac{x^2 dx}{x^3 + a^3} = \frac{1}{3} \ln |x^3 + a^3|.$
9. $\int \frac{x^3 dx}{x^3 + a^3} = x + \frac{a}{6} \ln \frac{x^2 - ax + a^2}{(x+a)^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
10. $\int \frac{dx}{x(x^3 + a^3)} = \frac{1}{3a^3} \ln \left| \frac{x^3}{x^3 + a^3} \right|.$
11. $\int \frac{dx}{x^2(x^3 + a^3)} = -\frac{1}{a^3 x} + \frac{1}{6a^4} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{1}{a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
12. $\int \frac{dx}{x^3(x^3 + a^3)} = -\frac{1}{2a^3 x^2} - \frac{1}{6a^5} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{1}{a^5 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
13. $\int \frac{dx}{(x^3 + a^3)^2} = \frac{x}{3a^3(x^3 + a^3)} + \frac{1}{9a^5} \ln \frac{(x+a)^2}{x^2 - ax + a^2} + \frac{2}{3a^5 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
14. $\int \frac{x dx}{(x^3 + a^3)^2} = \frac{x^2}{3a^3(x^3 + a^3)} + \frac{1}{18a^4} \ln \frac{x^2 - ax + a^2}{(x+a)^2} + \frac{1}{3a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
15. $\int \frac{x^2 dx}{(x^3 + a^3)^2} = -\frac{1}{3(x^3 + a^3)}.$
16. $\int \frac{dx}{x(x^3 + a^3)^2} = \frac{1}{3a^3(x^3 + a^3)} + \frac{1}{3a^5} \ln \left| \frac{x^3}{x^3 + a^3} \right|.$
17. $\int \frac{dx}{x^2(x^3 + a^3)^2} = \frac{1}{3a^3 x(x^3 + a^3)} - \frac{4}{3a^5 x} +$
 $+ \frac{2}{9a^7} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{4}{3a^7 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
18. $\int \frac{dx}{x^3(x^3 + a^3)^2} = -\frac{1}{3a^3 x^2(x^3 + a^3)} - \frac{5}{6a^5 x^2} -$
 $- \frac{5}{18a^7} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{5}{3a^7 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$

1.2.13. Интегралы вида $\int \frac{x^{\pm m} dx}{(x^4 \pm a^4)^n}, \int \frac{x^{\pm m} dx}{(ax^4 + bx^2 + c)^n}.$

1. $\int \frac{x^m dx}{(x^4 \pm a^4)^n} = \pm \frac{x^{m+1}}{4(n-1)a^4(x^4 \pm a^4)^{n-1}} \pm \frac{4n-m-5}{4(n-1)a^4} \int \frac{x^m dx}{(x^4 \pm a^4)^{n-1}}.$
2. $\int \frac{x^m dx}{(x^4 \pm a^4)^n} = \frac{x^{m-3}}{(m-4n+1)(x^4 \pm a^4)^{n-1}} \mp \frac{(m-3)a^4}{(m-4n+1)} \int \frac{x^{m-4} dx}{(x^4 \pm a^4)^n}.$
3. $\int \frac{x^m dx}{x^4 \pm a^4} = \frac{x^{m-3}}{m-3} \mp a^4 \int \frac{x^{m-4} dx}{x^4 \pm a^4}.$

$$4. \int \frac{dx}{x^m (x^4 \pm a^4)^n} = \mp \frac{1}{(m-1) a^4 x^{m-1} (x^4 \pm a^4)^{n-1}} \mp \frac{4n+m-5}{(m-1) a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

$$5. = \pm \frac{1}{a^4} \int \frac{dx}{x^m (x^4 \pm a^4)^{n-1}} - \frac{1}{a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

$$6. \int \frac{dx}{x^4 \pm a^4} = \left\{ \frac{1}{4a^3 \sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right. \\ \left. \frac{1}{4a^3} \ln \left| \frac{x+a}{x-a} \right| + \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a} \right\}.$$

$$7. \int \frac{x dx}{x^4 \pm a^4} = \left\{ \frac{1}{2a^2} \operatorname{arctg} \frac{x^2}{a^2} \right. \\ \left. \frac{1}{4a^2} \ln \frac{|x^2 - a^2|}{x^2 + a^2} \right\}.$$

$$8. \int \frac{x^2 dx}{x^4 \pm a^4} = \left\{ \frac{1}{4a \sqrt{2}} \ln \frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \frac{1}{2a \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right. \\ \left. - \frac{1}{4a} \ln \left| \frac{x+a}{x-a} \right| + \frac{1}{2a} \operatorname{arctg} \frac{x}{a} \right\}.$$

$$9. \int \frac{x^3 dx}{x^4 \pm a^4} = \frac{1}{4} \ln |x^4 \pm a^4|. \quad 10. \int \frac{dx}{x (x^4 \pm a^4)} = \pm \frac{1}{4a^4} \ln \frac{x^4}{|x^4 \pm a^4|}.$$

$$11. \int \frac{dx}{x^2 (x^4 \pm a^4)} = \mp \frac{1}{a^4 x} \mp \frac{1}{a^4} \int \frac{x^2 dx}{x^4 \pm a^4}.$$

$$12. \int \frac{dx}{x^3 (x^4 \pm a^4)} = \mp \frac{1}{2a^4 x^2} \mp \frac{1}{4a^6} \left\{ \begin{array}{l} 2 \operatorname{arctg} \frac{x^2}{a^2} \\ \ln \frac{|x^2 - a^2|}{x^2 + a^2} \end{array} \right\}.$$

$$13. \int \frac{x^m dx}{(x^4 \pm a^4)^2} = \frac{x^{m+1}}{4a^4 (a^4 \pm x^4)} \pm \frac{3-m}{4a^4} \int \frac{x^m dx}{x^4 \pm a^4}.$$

$$14. \int \frac{dx}{(x^4 \pm a^4)^2} = \frac{x}{4a^4 (a^4 \pm x^4)} \pm \\ \pm \left\{ \frac{3}{16a^7 \sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{3}{8a^7 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right. \\ \left. \frac{3}{16a^7} \ln \left| \frac{x+a}{x-a} \right| + \frac{3}{8a^7} \operatorname{arctg} \frac{x}{a} \right\}.$$

$$15. \int \frac{x dx}{(x^4 \pm a^4)^2} = \frac{x^2}{4a^4 (a^4 \pm x^4)} + \left\{ \frac{1}{4a^6} \operatorname{arctg} \frac{x^2}{a^2} \right. \\ \left. \frac{1}{8a^6} \ln \frac{x^2 + a^2}{|x^2 - a^2|} \right\}.$$

16. $\int \frac{x^2 dx}{(x^4 \pm a^4)^2} = \frac{x^3}{4a^4 (a^4 \pm x^4)} \pm$
 $\pm \left\{ \frac{1}{16a^5 \sqrt{2}} \ln \frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \frac{1}{8a^5 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right\}$
 $\pm \left\{ \frac{1}{16a^5} \ln \left| \frac{x-a}{x+a} \right| + \frac{1}{8a^5} \operatorname{arctg} \frac{x}{a} \right\}.$
17. $\int \frac{x^3 dx}{(x^4 \pm a^4)^2} = -\frac{1}{4(x^4 \pm a^4)}.$
18. $\int \frac{dx}{x(x^4 \pm a^4)^2} = \frac{1}{4a^4 (a^4 \pm x^4)} + \frac{1}{4a^8} \ln \frac{x^4}{|x^4 \pm a^4|}.$
19. $\int \frac{dx}{x^2 (x^4 \pm a^4)^2} = -\frac{1}{a^8 x} - \frac{x^3}{4a^8 (x^4 \pm a^4)} - \frac{5}{4a^8} \int \frac{x^2 dx}{x^4 \pm a^4}.$
20. $\int \frac{x^m dx}{(ax^4 + bx^2 + c)^n} = \frac{x^{m-3}}{(m-4n+1)a(ax^4 + bx^2 + c)^{n-1}} -$
 $-\frac{(m-3)c}{(m-4n+1)a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^n} - \frac{(m-2n-1)b}{(m-4n+1)a} \int \frac{x^{m-2} dx}{(ax^4 + bx^2 + c)^n}.$
21. $= \frac{1}{a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^{n-1}} - \frac{c}{a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^n} -$
 $-\frac{b}{a} \int \frac{x^{m-2} dx}{(ax^4 + bx^2 + c)^n}.$
22. $\int \frac{dx}{(ax^4 + bx^2 + c)^n} = \frac{abx^3 + (b^2 - 2ac)x}{2(n-1)c(b^2 - 4ac)(ax^4 + bx^2 + c)^{n-1}} +$
 $+\frac{(4n-7)ab}{2(n-1)c(b^2 - 4ac)} \int \frac{x^2 dx}{(ax^4 + bx^2 + c)^{n-1}} +$
 $+\frac{2(n-1)(b^2 - 4ac) + 2ac - b^2}{2(n-1)c(b^2 - 4ac)} \int \frac{dx}{(ax^4 + bx^2 + c)^{n-1}}.$
23. $\int \frac{dx}{x^m (ax^4 + bx^2 + c)^n} = \frac{-1}{(m-1)cx^{m-1}(ax^4 + bx^2 + c)^{n-1}} -$
 $-b \frac{m+2n-3}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^4 + bx^2 + c)^n} - a \frac{m+4n-5}{(m-1)c} \int \frac{dx}{x^{m-4}(ax^4 + bx^2 + c)^n}.$
24. $\int \frac{dx}{ax^4 + bx^2 + c} = \frac{2a}{\sqrt{b^2 - 4ac}} \left(\int \frac{dx}{2ax^2 + b - \sqrt{b^2 - 4ac}} - \right.$
 $\left. - \int \frac{dx}{2ax^2 + b + \sqrt{b^2 - 4ac}} \right) \quad [b^2 > 4ac].$
25. $= \frac{1}{3a \sin \alpha} \left[\sin \frac{\alpha}{2} \ln \frac{x^2 + 2x\sqrt{a/c} \cos(\alpha/2) + \sqrt{a/c}}{x^2 - 2x\sqrt{a/c} \cos(\alpha/2) + \sqrt{a/c}} + \right.$
 $\left. + 2 \cos \frac{\alpha}{2} \operatorname{arctg} \frac{x^2 - \sqrt{a/c}}{2x\sqrt{a/c} \sin(\alpha/2)} \right] \quad \left[b^2 < 4ac, \cos \alpha = -\frac{b}{2\sqrt{ac}} \right].$
26. $\int \frac{dx}{(ax^4 + bx^2 + c)^2} = \frac{abx^3 + (b^2 - 2ac)x}{2c(b^2 - 4ac)(ax^4 + bx^2 + c)} +$
 $+\frac{b^2 - 6ac}{2a(b^2 - 4ac)} \int \frac{dx}{ax^4 + bx^2 + c} + \frac{ab}{2c(b^2 - 4ac)} \int \frac{x^2 dx}{ax^4 + bx^2 + c}.$

$$27. \int \frac{x dx}{ax^4 + bx^2 + c} = \frac{1}{2\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax^2 + b - \sqrt{b^2 - 4ac}}{2ax^2 + b + \sqrt{b^2 - 4ac}} \right| \quad [b^2 > 4ac].$$

$$28. = \frac{1}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2ax^2 + b}{\sqrt{4ac - b^2}} \quad [b^2 < 4ac].$$

$$29. \int \frac{x^2 dx}{ax^4 + bx^2 + c} = \frac{b + \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b + \sqrt{b^2 - 4ac}} - \\ - \frac{b - \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b - \sqrt{b^2 - 4ac}} \quad [b^2 > 4ac].$$

$$30. = \frac{1}{4a(\sqrt{c/4a} - b/4a)^{1/2}} \left(\int \frac{x dx}{x^2 - 2(\sqrt{c/4a} - b/4a)^{1/2}x + 2\sqrt{c/4a}} - \right. \\ \left. - \int \frac{x dx}{x^2 + 2(\sqrt{c/4a} - b/4a)^{1/2}x + 2\sqrt{c/4a}} \right) \quad [b^2 < 4ac].$$

1.2.14. Интегралы вида $\int R(x, ax^{2k} + bx^k + c) dx$.

$$1. \int x^m (ax^{2k} + bx^k + c)^n dx = \frac{x^{m+1} (ax^{2k} + bx^k + c)^{n+1}}{(m+1)c} - \\ - \frac{(m+nk+k+1)b}{(m+1)c} \int x^{m+k} (ax^{2k} + bx^k + c)^n dx - \\ - \frac{(m+2nk+2k+1)a}{(m+1)c} \int x^{m+2k} (ax^{2k} + bx^k + c)^n dx.$$

$$2. = \frac{x^{m+1} (ax^{2k} + bx^k + c)^n}{m+1} - \frac{bnk}{m+1} \int x^{m+k} (ax^{2k} + bx^k + c)^{n-1} dx - \\ - \frac{2ank}{m+1} \int x^{m+2k} (ax^{2k} + bx^k + c)^{n-1} dx.$$

$$3. = \frac{x^{m-2k+1} (ax^{2k} + bx^k + c)^{n+1}}{(m+2nk+1)a} - \frac{(m-2k+1)c}{(m+2nk+1)a} \times \\ \times \int x^{m-2k} (ax^{2k} + bx^k + c)^n dx - \frac{(m+nk-k+1)b}{(m+2nk+1)a} \int x^{m-k} (ax^{2k} + bx^k + c)^n dx.$$

$$4. = \frac{x^{m+1} (ax^{2k} + bx^k + c)^n}{m+2nk+1} + \frac{2nkc}{m+2nk+1} \times \\ \times \int x^m (ax^{2k} + bx^k + c)^{n-1} dx + \frac{nkb}{m+2nk+1} \int x^{m+k} (ax^{2k} + bx^k + c)^{n-1} dx.$$

$$5. \int \frac{x^m dx}{(ax^{2k} + bx^k + c)^n} = \frac{x^{m-2k+1}}{(m-2nk+1)a (ax^{2k} + bx^k + c)^{n-1}} - \\ - \frac{(m-2k+1)c}{(m-2nk+1)a} \int \frac{x^{m-2k} dx}{(ax^{2k} + bx^k + c)^n} - \frac{(m-nk-k+1)b}{(m-2nk+1)a} \int \frac{x^{m-k} dx}{(ax^{2k} + bx^k + c)^n}.$$

$$6. \int \frac{dx}{x^m (ax^{2k} + bx^k + c)^n} = \frac{-1}{(m-1)c x^{m-1} (ax^{2k} + bx^k + c)^{n-1}} - \\ - \frac{(m+nk-k-1)b}{(m-1)c} \int \frac{dx}{x^{m-k} (ax^{2k} + bx^k + c)^n} - \\ - \frac{m+2nk-2k-1}{(m-1)c} \int \frac{dx}{x^{m-2k} (ax^{2k} + bx^k + c)^n}.$$

1.2.15. Интегралы вида $\int R(x^{1/2}, ax+b) dx$.

$$1. \int x^{m+1/2} (ax+b)^n dx = 2x^{m+3/2} \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^k}{2k+2m+3}.$$

$$2. \int \frac{(ax+b)^n}{x^{m+1/2}} dx = \frac{2}{x^{m-1/2}} \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^k}{2k-2m+1}.$$

$$3. \int \frac{x^{m+1/2} dx}{(ax \pm b)^n} = -\frac{x^{m+1/2}}{(n-1)a(ax \pm b)^{n-1}} + \frac{2m+1}{2(n-1)a} \int \frac{x^{m-1/2} dx}{(ax \pm b)^{n-1}}.$$

$$4. = \frac{2}{a^n} \int \frac{t^{2m+2} dt}{(t^2 \pm b/a)^n} \quad [t = \sqrt{x} \text{ см. 1.2.10}].$$

$$5. \int \frac{x^{m+1/2} dx}{ax \pm b} = 2 \sum_{k=0}^m \frac{(\mp 1)^k b^k x^{m-k+1/2}}{(2m-2k+1)a^{k+1}} + \left(\mp \frac{b}{a}\right)^{m-1} \int \frac{dx}{x^{1/2}(ax \pm b)}.$$

$$6. \int \frac{dx}{x^{m+1/2}(ax \pm b)^n} = \mp \frac{2}{(2m-1)b x^{m-1/2}(ax \pm b)^{n-1}} \mp \frac{(2m+2n-3)a}{(2m-1)b} \int \frac{dx}{x^{m-1/2}(ax \pm b)^n} = \frac{2}{a^n} \int \frac{dt}{t^{2m}(t^2 \pm b/a)^n}.$$

$$7. \int \frac{x^{1/2} dx}{ax \pm b} = \frac{2x^{1/2}}{a} \mp \left(\frac{b}{a^3}\right)^{1/2} \left\{ \begin{array}{l} 2 \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \\ \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \end{array} \right\}.$$

$$8. \int \frac{x^{3/2} dx}{ax \pm b} = \frac{2ax^{3/2} \mp 6bx^{1/2}}{3a^2} \pm \left(\frac{b^3}{a^5}\right)^{1/2} \left\{ \begin{array}{l} 2 \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \\ \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \end{array} \right\}.$$

$$9. \int \frac{x^{1/2} dx}{(ax \pm b)^2} = -\frac{x^{1/2}}{a(ax \pm b)} + \frac{1}{(a^3b)^{1/2}} \left\{ \begin{array}{l} \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \\ \frac{1}{2} \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \end{array} \right\}.$$

$$10. \int \frac{x^{3/2} dx}{(ax \pm b)^2} = \frac{2ax^{3/2} \pm 3bx^{1/2}}{a^2(ax \pm b)} - 3 \left(\frac{b}{a^5}\right)^{1/2} \left\{ \begin{array}{l} \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \\ \frac{1}{2} \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \end{array} \right\}.$$

$$11. \int \frac{dx}{x^{1/2}(ax \pm b)} = \pm \frac{1}{(ab)^{1/2}} \left\{ \begin{array}{l} 2 \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \\ \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \end{array} \right\}.$$

$$12. \int \frac{dx}{x^{3/2}(ax \pm b)} = \mp \frac{2}{bx^{1/2}} \mp \left(\frac{a}{b^3}\right)^{1/2} \left\{ 2 \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \right. \\ \left. \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right\}.$$

$$13. \int \frac{dx}{x^{1/2}(ax \pm b)^2} = \pm \frac{x^{1/2}}{b(ax \pm b)} + \frac{1}{(ab^3)^{1/2}} \left\{ \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \right. \\ \left. \frac{1}{2} \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \right\}.$$

$$14. \int \frac{dx}{x^{3/2}(ax \pm b)^2} = \frac{-3ax \mp 2b}{b^2 x^{1/2}(ax \pm b)} - 3 \left(\frac{a}{b^5}\right)^{1/2} \left\{ \operatorname{arctg} \left(\frac{ax}{b}\right)^{1/2} \right. \\ \left. \frac{1}{2} \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right\}.$$

1.2.16. Интегралы вида $\int R(x^{1/2}, x^2 \pm a^2) dx$.

Условие: $a > 0$.

$$1. \int \frac{x^{m+1/2}}{(x^2 \pm a^2)^n} dx = \pm \frac{x^{m+3/2}}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{4n-2m-7}{4(n-1)a^2} \int \frac{x^{m+1/2}}{(x^2 \pm a^2)^{n-1}} dx.$$

$$2. \int \frac{x^{m+1/2}}{x^2 \pm a^2} dx = \frac{2}{2m-1} x^{m-1/2} \mp a^2 \int \frac{x^{m-3/2}}{x^2 \pm a^2} dx.$$

$$3. \int \frac{x^{2m+1/2}}{x^2 \pm a^2} dx = 2 \sum_{k=1}^m \frac{(\mp a^2)^{k-1}}{4m-4k+3} x^{2m-2k+3/2} + (\mp a^2)^m \int \frac{x^{1/2}}{x^2 \pm a^2} dx.$$

$$4. \int \frac{x^{2m+3/2}}{x^2 \pm a^2} dx = 2 \sum_{k=0}^{m-1} \frac{(\mp a^2)^k}{4m-4k+1} x^{2m-2k+1/2} + (\mp a^2)^m \int \frac{x^{3/2}}{x^2 \pm a^2} dx.$$

$$5. \int \frac{x^{1/2} dx}{x^2 + a^2} = \frac{1}{2\sqrt{2a}} \ln \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} - \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

$$6. \int \frac{x^{1/2} dx}{x^2 - a^2} = \frac{1}{2\sqrt{a}} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| - \frac{1}{\sqrt{a}} \operatorname{arctg} \sqrt{\frac{a}{x}}.$$

$$7. \int \frac{x^{3/2} dx}{x^2 + a^2} = 2\sqrt{x} + \frac{\sqrt{2a}}{4} \ln \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} + \frac{\sqrt{2a}}{2} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

$$8. \int \frac{x^{3/2} dx}{x^2 - a^2} = 2\sqrt{x} + \frac{\sqrt{a}}{2} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| + \sqrt{a} \operatorname{arctg} \sqrt{\frac{a}{x}}.$$

1.2.17. Интегралы вида $\int x^{\pm m} (ax+b)^{n+1/2} dx$.

$$1. \int x^m (ax+b)^{n+1/2} dx = \frac{2(ax+b)^{n+3/2}}{a^{m+1}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m+2n-2k+3} \binom{m}{k} (ax+b)^{m-k}.$$

$$2. \int (ax+b)^{n+1/2} dx = \frac{2}{(2n+3)a} (ax+b)^{n+3/2}.$$

$$3. \int x^m (ax+b)^{1/2} dx = \frac{2}{(2m+3)a} x^m (ax+b)^{3/2} - \\ - \frac{2mb}{(2m+3)a} \int x^{m-1} (ax+b)^{1/2} dx = \frac{2(ax+b)^{3/2}}{a^{m+1}} \sum_{k=0}^m \binom{m}{k} \frac{(-b)^{m-k}}{2k+3} (ax+b)^k.$$

$$4. \int \frac{(ax+b)^{n+1/2}}{x^m} dx = -\frac{(ax+b)^{n+3/2}}{(m-1)bx^{m-1}} + \\ + \frac{(2n-2m+5)a}{2(m-1)b} \int \frac{(ax+b)^{n+1/2}}{x^{m-1}} dx.$$

$$5. = \frac{2(ax+b)^{n+1/2}}{(2n-2m+3)x^{m-1}} + \frac{(2n+1)b}{2n-2m+3} \int \frac{(ax+b)^{n-1/2}}{x^m} dx.$$

$$6. \int \frac{(ax+b)^{n+1/2}}{x} dx = \frac{2(ax+b)^{n+1/2}}{2n+1} + b \int \frac{(ax+b)^{n-1/2}}{x} dx.$$

$$7. \int \frac{(ax+b)^{1/2}}{x^m} dx = -\frac{(ax+b)^{3/2}}{(m-1)bx^{m-1}} + \frac{(5-2m)a}{2(m-1)b} \int \frac{(ax+b)^{1/2}}{x^{m-1}} dx.$$

$$8. \int (ax+b)^{1/2} dx = \frac{2}{3a} (ax+b)^{3/2}.$$

$$9. \int x(ax+b)^{1/2} dx = \frac{2}{a^2} \left[\frac{ax+b}{5} - \frac{b}{3} \right] (ax+b)^{3/2}.$$

$$10. \int x^2(ax+b)^{1/2} dx = \frac{2}{a^3} \left[\frac{(ax+b)^2}{7} - \frac{2b(ax+b)}{5} + \frac{b^2}{3} \right] (ax+b)^{3/2}.$$

$$11. \int x^3(ax+b)^{1/2} dx = \\ = \frac{2}{a^4} \left[\frac{(ax+b)^3}{9} - \frac{3b(ax+b)^2}{7} + \frac{3b^2(ax+b)}{5} - \frac{b^3}{3} \right] (ax+b)^{3/2}.$$

$$12. \int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2}.$$

$$13. \int x(ax+b)^{3/2} dx = \frac{2}{a^2} \left[\frac{(ax+b)^2}{7} - \frac{b}{5} \right] (ax+b)^{5/2}.$$

$$14. \int x^2(ax+b)^{3/2} dx = \frac{2}{a^3} \left[\frac{(ax+b)^3}{9} - \frac{2b(ax+b)}{7} + \frac{b^2}{5} \right] (ax+b)^{5/2}.$$

$$15. \int x^3(ax+b)^{3/2} dx = \frac{2}{a^4} \left[\frac{(ax+b)^4}{11} - \frac{3b(ax+b)^3}{9} + \right. \\ \left. + \frac{3b^2(ax+b)}{7} - \frac{b^3}{5} \right] (ax+b)^{5/2}.$$

$$16. \int \frac{(ax+b)^{1/2}}{x} dx = 2(ax+b)^{1/2} + \begin{cases} b^{1/2} \ln \left| \frac{(ax+b)^{1/2} - b^{1/2}}{(ax+b)^{1/2} + b^{1/2}} \right| & [b > 0], \\ -2(-b)^{1/2} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$$

$$17. \int \frac{(ax+b)^{1/2}}{x^2} dx = -\frac{(ax+b)^{1/2}}{x} + \frac{a}{2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$18. \int \frac{(ax+b)^{1/2}}{x^3} dx = -\frac{(2b+ax)}{4bx^2} (ax+b)^{1/2} - \frac{a^2}{8b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$19. \int \frac{(ax+b)^{3/2}}{x} dx = \frac{2}{3} (ax+b)^{3/2} + 2b (ax+b)^{1/2} + b^2 \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$20. \int \frac{(ax+b)^{3/2}}{x^2} dx = -\frac{(ax+b)^{5/2}}{bx} + \frac{3a}{2b} \int \frac{(ax+b)^{3/2}}{x} dx.$$

$$21. \int \frac{(ax+b)^{3/2}}{x^3} dx = -\left(\frac{1}{2bx^2} + \frac{a}{4b^2x}\right) (ax+b)^{5/2} + \frac{3a^2}{8b^2} \int \frac{(ax+b)^{3/2}}{x} dx.$$

1.2.18. Интегралы вида $\int \frac{x^m dx}{(ax+b)^{n+1/2}}.$

$$1. \int \frac{x^m dx}{(ax+b)^{n+1/2}} = \frac{2}{a^{m+1} (ax+b)^{n-1/2}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m-2n-2k+1} \binom{m}{k} (ax+b)^{m-k}.$$

$$2. \int \frac{dx}{(ax+b)^{n+1/2}} = -\frac{2}{(2n-1)a(ax+b)^{n-1/2}}.$$

$$3. \int \frac{x^m dx}{(ax+b)^{1/2}} = \frac{2}{(2m+1)a} x^m (ax+b)^{1/2} - \frac{2mb}{(2m+1)a} \int \frac{x^{m-1} dx}{(ax+b)^{1/2}}.$$

$$4. = \frac{2(ax+b)^{1/2}}{a^{m+1}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m-2k+1} \binom{m}{k} (ax+b)^{m-k}.$$

$$5. \int \frac{dx}{(ax+b)^{1/2}} = \frac{2}{a} (ax+b)^{1/2}.$$

$$6. \int \frac{x dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2} (ax+b)^{1/2}.$$

$$7. \int \frac{x^2 dx}{(ax+b)^{1/2}} = \frac{2}{a^3} \left[\frac{(ax+b)^2}{5} - \frac{2b(ax+b)}{3} + b^2 \right] (ax+b)^{1/2}.$$

$$8. \int \frac{x^3 dx}{(ax+b)^{1/2}} = \frac{2}{a^4} \left[\frac{(ax+b)^3}{7} - \frac{3b(ax+b)^2}{5} + b^2(ax+b) - b^3 \right] (ax+b)^{1/2}.$$

$$9. \int \frac{dx}{(ax+b)^{3/2}} = -\frac{2}{a(ax+b)^{1/2}}.$$

$$10. \int \frac{x dx}{(ax+b)^{3/2}} = \frac{2(ax+2b)}{a^2(ax+b)^{1/2}}.$$

$$11. \int \frac{x^2 dx}{(ax+b)^{3/2}} = \frac{2}{a^3} \left[\frac{(ax+b)^2}{3} - 2b(ax+b) - b^2 \right] \frac{1}{(ax+b)^{1/2}}.$$

$$12. \int \frac{x^3 dx}{(ax+b)^{3/2}} = \frac{2}{a^4} \left[\frac{(ax+b)^3}{5} - b(ax+b)^2 + 3b^2(ax+b) + b^3 \right] \frac{1}{(ax+b)^{1/2}}.$$

1.2.19. Интегралы вида $\int \frac{dx}{x^m (ax+b)^{n+1/2}}$.

$$1. \int \frac{dx}{x^m (ax+b)^{n+1/2}} = -\frac{1}{(m-1)x^{m-1}(ax+b)^{n+1/2}} - \frac{(2n+1)a}{2(m-1)} \int \frac{dx}{x^{m-1}(ax+b)^{n+3/2}}.$$

$$2. = \frac{-2}{(2n-1)ax^m(ax+b)^{n-1/2}} - \frac{2m}{(2n-1)a} \int \frac{dx}{x^{m+1}(ax+b)^{n-1/2}}.$$

$$3. = 2a^{m-1} \int \frac{dt}{t^{2n}(t^2-b)^m} \quad [t=\sqrt{ax+b}, \text{ см. 1.2 10}].$$

$$4. \int \frac{dx}{x(ax+b)^{n+1/2}} = \frac{2}{(2n-1)b(ax+b)^{n-1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{n-1/2}}.$$

$$5. \int \frac{dx}{x^m(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{2(m-1)b} \int \frac{dx}{x^{m-1}(ax+b)^{1/2}}.$$

$$6. \int \frac{dx}{x(ax+b)^{1/2}} = \begin{cases} \frac{1}{b^{1/2}} \ln \left| \frac{(ax+b)^{1/2} - b^{1/2}}{(ax+b)^{1/2} + b^{1/2}} \right| & [b > 0], \\ \frac{2}{(-b)^{1/2}} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$$

$$7. \int \frac{dx}{x^2(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{bx} - \frac{a}{2b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$8. \int \frac{dx}{x^3(ax+b)^{1/2}} = \frac{3ax-2b}{4b^2x^2} (ax+b)^{1/2} + \frac{3a^2}{8b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$9. \int \frac{dx}{x(ax+b)^{3/2}} = \frac{2}{b(ax+b)^{1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$10. \int \frac{dx}{x^2(ax+b)^{3/2}} = \frac{-3ax-b}{b^2x(ax+b)^{1/2}} - \frac{3a}{2b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$11. \int \frac{dx}{x^3(ax+b)^{3/2}} = \left(-\frac{1}{2bx^2} + \frac{5a}{4b^2x} + \frac{15a^2}{4b^3} \right) \frac{1}{(ax+b)^{1/2}} + \frac{15a^2}{8b^3} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$12. \int \frac{(ax+b)^{1/2}+c}{(ax+b)^{1/2}+d} dx = x + 2 \frac{c-d}{a} [(ax+b)^{1/2} - d \ln |(ax+b)^{1/2}+d|].$$

1.2.20. Интегралы вида $\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$.

$$1. \int (ax+b)^{m+1/2} (cx+d)^{n+1/2} dx = \frac{(ax+b)^{m+1/2} (cx+d)^{n+3/2}}{(m+n+2)c} - \frac{(2m+1)(ad-bc)}{2(m+n+2)c} \int (ax+b)^{m-1/2} (cx+d)^{n+1/2} dx.$$

2.
$$= (cx+d)^{n+3/2} \sum_{k=0}^m \frac{(-1)^k (2m+1)(2m-1)\dots(2m-2k+3)(ad-bc)^k}{(m+n+2)(m+n+1)\dots(m+n-k+2)c^{k+1}} \times$$

$$\times (ax+b)^{m-k+1/2} + \frac{m!!}{(-2c)^{m+1}} \sqrt{ax+b} \times$$

$$\times \sum_{k=0}^n \frac{(2n+1)(2n-1)\dots(2n-2k+3)(ad-bc)^{m+k+1}}{2^k (m+n+2)(m+n+1)\dots(n-k+1)a^{k+1}} \times$$

$$\times (cx+d)^{n-k+1/2} + \frac{m!! n!! (ad-bc)^{m+n+2}}{(m+n+2)! (-2c)^{m+1} (2a)^{n+1}} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$
3.
$$\int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n+1/2}} dx = \frac{1}{(m-n+1)c} \cdot \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} -$$

$$- \frac{(2m+1)(ad-bc)}{2(m-n+1)c} \int \frac{(ax+b)^{m-1/2}}{(cx+d)^{n+1/2}} dx.$$
4.
$$= \frac{2}{(2n-1)(ad-bc)} \cdot \frac{(ax+b)^{m+3/2}}{(cx+d)^{n-1/2}} +$$

$$+ \frac{2(n-m-2)a}{(2n-1)(ad-bc)} \int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} dx.$$
5.
$$= (ax+b)^{m+3/2} \sum_{k=0}^{n-m-2} \frac{(n-m-2)(n-m-3)\dots(n-m-k-1)2^{k+1}a^k}{(2n-1)(2n-3)\dots(2n-2k-1)(ad-bc)^{k+1}} \times$$

$$\times \frac{1}{(cx+d)^{n-k-1/2}} \quad [n \geq m+2].$$
6.
$$\int \frac{dx}{(ax+b)^{m+1/2}(cx+d)^{n+1/2}} = \frac{-2}{(2m-1)(ad-bc)(ax+b)^{m-1/2}(cx+d)^{n-1/2}} -$$

$$- \frac{2(m+n-1)}{(2m-1)(ad-bc)} \int \frac{dx}{(ax+b)^{m-1/2}(cx+d)^{n+1/2}}.$$
7.
$$= \frac{1}{(cx+d)^{n-1/2}} \sum_{k=0}^{m-1} \frac{(m+n-1)(m+n-2)\dots(m+n-k)(-2)^{k+1}c^k}{(2m-1)(2m-3)\dots(2m-2k-1)(ad-bc)^{k+1}} \times$$

$$\times \frac{1}{(ax+b)^{m-k-1/2}} +$$

$$+ \frac{(-2c)^m}{(m-1)!!} \sqrt{ax+b} \sum_{k=0}^{n-1} \frac{(m+n-1)(m+n-2)\dots(n-k)2^{k+1}a^k}{(2n-1)(2n-3)\dots(2n-2k-1)(ad-bc)^{n+k+1}} \times$$

$$\times \frac{1}{(cx+d)^{n-k-1/2}}.$$
8.
$$\int \frac{dx}{\sqrt{(ax+b)(cx+d)}} = \frac{2}{\sqrt{ac}} \ln(\sqrt{c(ax+b)} + \sqrt{a(cx+d)})$$

$$[ac > 0; x > \max(-b/a, -d/c)].$$

$$9. \quad = -\frac{2}{\sqrt{ac}} \ln (\sqrt{-c(ax+b)} + \sqrt{-a(cx+d)})$$

[$ac > 0$; $x < \max(-b/a, -d/c)$].

$$10. \quad = -\frac{1}{\sqrt{-ac}} \arcsin \frac{c(ax+b) + a(cx+d)}{|ad-bc|}$$

[$ac < 0$].

$$11. \quad \int \frac{dx}{(x-p) \sqrt{(ax+b)(cx+d)}} =$$

$$= \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{(cp+d)(ax+b)} - \sqrt{(ap+b)(cx+d)}]^2}{|x-p|}$$

[$(ap+b)(cp+d) > 0$; $(cp+d)(ax+b) > 0$; $(ap+b)(cx+d) > 0$].

$$12. \quad = \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{-(cp+d)(ax+b)} - \sqrt{-(ap+b)(cx+d)}]^2}{|x-p|}$$

[$(ap+b)(cp+d) > 0$, $(cp+d)(ax+b) < 0$, $(ap+b)(cx+d) < 0$].

$$13. \quad = \frac{1}{\sqrt{-(ap+b)(cp+d)}} \arcsin \frac{(cp+d)(ax+b) + (ap+b)(cx+d)}{|ad-bc| |x-p|}$$

[$(ap+b)(cp+d) < 0$].

$$14. \quad \int \frac{x^n}{A \sqrt{ax+b} + B \sqrt{cx+d}} dx =$$

$$= \frac{A}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{ax+b}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{ax+b} dx \right) -$$

$$- \frac{B}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{cx+d}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{cx+d} dx \right)$$

[$p = \frac{-bA^2 + dB}{aA^2 - cB^2}$].

$$15. \quad = \frac{A}{bA^2 - dB^2} \int x^n \sqrt{ax+b} dx - \frac{B}{bA^2 - dB^2} \int x^n \sqrt{cx+d} dx$$

[$aA^2 - cB^2 = 0$].

$$16. \quad \int \frac{\sqrt{ax+b}}{x-p} dx = 2 \sqrt{ax+b} \mp$$

$$\mp \sqrt{|ap+b|} \begin{cases} \ln \left| \frac{\sqrt{ax+b} + \sqrt{ap+b}}{\sqrt{ax+b} - \sqrt{ap+b}} \right| & [ap+b > 0], \\ 2 \operatorname{arctg} \sqrt{\frac{|ap+b|}{ax+b}} & [ap+b < 0]. \end{cases}$$

$$17. \quad \int \frac{\alpha \sqrt{ax+b} + \beta \sqrt{cx+d}}{\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}} dx = \frac{\alpha \gamma a + \beta \delta c}{\gamma^2 a + \delta^2 c} x - \frac{\alpha \delta - \beta \gamma}{\gamma^2 a + \delta^2 c} \sqrt{(ax+b)(cx+d)} +$$

$$+ \frac{2\gamma\delta(\alpha\delta - \beta\gamma)(ad-bc)}{(\gamma^2 a - \delta^2 c)^2} \ln |\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}| -$$

$$- \frac{(\gamma^2 a + \delta^2 c)(\alpha\delta - \beta\gamma)(ad-bc)}{2(\gamma^2 a - \delta^2 c)^2} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

1.2.21. Интегралы вида $\int R(x, \sqrt[n]{ax+b}) dx$.

$$1. \int R(x, \sqrt[n]{ax+b}) dx = \frac{n}{a} \int R\left(\frac{t^n-b}{a}, t\right) t^{n-1} dt \quad [t = \sqrt[n]{ax+b}].$$

$$2. \int x^p (ax+b)^{1/n} dx = \frac{n}{(np+n+1)a} x^p (ax+b)^{1+1/n} - \frac{bnp}{(np+n+1)a} \int x^{p-1} (ax+b)^{1/n} dx.$$

$$3. \int x^m (ax+b)^{1/n} dx = \frac{1}{a^{m+1}} (ax+b)^{1+1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1+1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{1/n} dx = \frac{n}{(n+1)a} (ax+b)^{1+1/n}.$$

$$5. \int \frac{(ax+b)^{1/n}}{x^m} dx = -\frac{(ax+b)^{1+1/n}}{(m-1)bx^{m-1}} - \frac{(mn-2n-1)a}{(m-1)nb} \int \frac{(ax+b)^{1/n}}{x^{m-1}} dx.$$

$$6. \int \frac{(ax+b)^{1/n}}{x} dx = n(ax+b)^{1/n} + nb^{1/n} \int \frac{dt}{t^n-1} \quad [t = (ax/b + 1)^{1/n}].$$

$$7. \int \frac{x^m dx}{(ax+b)^{1/n}} = \frac{n}{(mn+n-1)a} x^m (ax+b)^{1-1/n} - \frac{mnb}{(mn+n-1)a} \int \frac{x^{m-1} dx}{(ax+b)^{1/n}}$$

$$8. = \frac{1}{a^{m+1}} (ax+b)^{1-1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1-1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$9. \int \frac{dx}{(ax+b)^{1/n}} = \frac{n}{(n-1)a} (ax+b)^{1-1/n}.$$

1.2.22. Интегралы вида $\int R[x, (ax+b)^{p/n}] dx$.

$$1. \int x^m (ax+b)^{p/n} dx = \frac{n}{(mn+n+p)a} x^m (ax+b)^{1+p/n} - \frac{mnb}{(mn+n+p)a} \int x^{m-1} (ax+b)^{p/n} dx.$$

$$2. = \frac{n}{a} (ax+b)^{1+p/n} \times \sum_{k=0}^m \frac{m(m-1)\dots(m-k+1)}{(mn+p+n)(mn+p)\dots(mn+n+p-kn)} \left(-\frac{bn}{a}\right)^k x^{m-k}.$$

$$3. = \frac{n}{a^{m+1}} (ax+b)^{1+p/n} \sum_{k=0}^m \frac{\binom{m}{k} (-b)^k}{(m-k+1)n+p} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{p/n} dx = \frac{n}{(n+p)a} (ax+b)^{1+p/n}.$$

$$5. \int \frac{(ax+b)^{p/n}}{(x-\alpha)^m} dx = -\frac{1}{(m-1)(a\alpha+b)} \frac{(ax+b)^{1+p/n}}{(x-\alpha)^{m-1}} - \frac{[(m-2)n-p]a}{(m-1)n(a\alpha+b)} \int \frac{(ax+b)^{p/n}}{(x-\alpha)^{m-1}} dx.$$

$$6. = (ax+b)^{1+p/n} \times \\ \times \sum_{k=1}^{n-1} (-1)^k \frac{(m-p-2n)(mn-p-3n) \dots (mn-p-nk) a^{k-1}}{(m-1)(m-2) \dots (m-k) n^{k-1} (a\alpha+b)} \frac{1}{(x-\alpha)^{m-k}} - \\ - \frac{p(p+1) \dots (p+m-1) a^{m-1}}{n^{m-1} (m-1)! (a\alpha+b)^{m-1}} \int \frac{(ax+b)^{p/n}}{x-\alpha} dx.$$

$$7. = \frac{-na^{m-1}}{(m-1)n-p} (ax+b)^{1-m+p/n} \quad [a\alpha+b=0].$$

$$8. \int \frac{(ax+b)^{p/n}}{x-\alpha} dx = n(a\alpha+b)^{p/n} \int \frac{y^{n+p-1}}{y^n-1} dy \\ \left[y = \left(\frac{ax+b}{a\alpha+b} \right)^{1/n}; \quad p = \pm 1, \pm 2, \dots; \text{ см } 1.2.3 \right].$$

1.2.23. Интегралы вида $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$.

$$1. \int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx = \frac{n(ad-bc)}{ac} \int R\left[\frac{-adt^n+bc}{ac(t^n-1)}, \sqrt[n]{\frac{a}{c}t}\right] \frac{t^{n-1}}{(t^n-1)^2} dt \\ \left[t = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}} \right].$$

$$2. \int \left(\frac{ax+b}{cx+d}\right)^{p/n} dx = \frac{ax+b}{a} \left(\frac{ax+b}{cx+d}\right)^{p/n} + \frac{p(ad-bc)}{ac} \left(\frac{a}{c}\right)^{p/n} \int \frac{t^{n+p-1}}{t^n-1} dt \\ \left[t = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}}, \text{ см } 1.2.3 \right].$$

$$3. \int x^m \left(\frac{ax+b}{cx+d}\right)^{p/n} dx = \frac{(ax+b)(cx+d)x^{m-1}}{(m+1)ac} \left(\frac{ax+b}{cx+d}\right)^{p/n} - \\ - \frac{(ad-bc)p+mn(ad+bc)}{(m+1)nac} \int x^{m-1} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx - \\ - \frac{(m-1)bd}{(m+1)ac} \int x^{m-2} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx.$$

$$4. \int x \left(\frac{ax+b}{cx+d}\right)^{p/n} dx = \frac{(ax+b)(cx+d)}{2ac} \left(\frac{ax+b}{cx+d}\right)^{p/n} - \\ - \frac{(ad-bc)p+(ad+bc)n}{2nac} \int \left(\frac{ax+b}{cx+d}\right)^{p/n} dx.$$

$$5. \int \frac{1}{(x-\alpha)^{m+1}} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx = A \frac{(ax+b)(cx+d)}{(x-\alpha)^m} \left(\frac{ax+b}{cx+d}\right)^{p/n} + \\ + B \int \frac{1}{(x-\alpha)^m} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx + D \int \frac{1}{(x-\alpha)^{m-1}} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx, \\ A = \frac{-1}{m(a\alpha+b)(c\alpha+d)}, \quad B = \frac{-n(m-1)(ad+bc+2a\alpha)+p(ad-bc)}{mn(a\alpha+b)(c\alpha+d)}, \\ D = \frac{-(m-2)ac}{m(a\alpha+b)(c\alpha+d)}.$$

$$6. \int \frac{1}{x-\alpha} \left(\frac{ax+b}{cx+d}\right)^{p/n} dx = n \left(\frac{a\alpha+b}{c\alpha+d}\right)^{p/n} \int \frac{t^{p-1}}{t^n-1} dt - n \left(\frac{a}{c}\right)^{p/n} \int \frac{s^{p-1}}{s^n-1} ds \\ \left[t = \sqrt[n]{\frac{(c\alpha+a)(ax+b)}{(a\alpha+b)(cx+d)}}, \quad s = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}}, \text{ см } 1.2.3 \right].$$

1.2.24. Интегралы вида $\int R(x^{1/n}, x^2 \pm a^2) dx$.

$$1. \int \frac{x^{p/n} dx}{(x^2 \pm a^2)^m} = \frac{\pm x^{1+p/n}}{2(m-1)a^2(x^2 \pm a^2)^{m-1}} \pm \frac{(2m-3)n-p}{2(m-1)a^2n} \int \frac{x^{p/n} dx}{(x^2 \pm a^2)^{m-1}}.$$

$$2. = x^{1+p/n} \sum_{k=1}^{m-1} \frac{(2mn-3n-p)(2mn-5n-p)\dots(2mn-2nk+n-p)}{(m-1)(m-2)\dots(m-k)n^{k-1}(\pm 2a^2)^k} \times \\ \times \frac{1}{(x^2 \pm a^2)^{m-k}} + \frac{(n-p)(3n-p)\dots(2mn-3n-p)}{(m-1)!(\pm 2a^2n)^{m-1}} \int \frac{x^{p/n} dx}{x^2 \pm a^2}.$$

$$3. \int \frac{x^{p/n} dx}{x^2 \pm a^2} = \frac{n}{p-n} x^{p/n-1} \mp a^2 \int \frac{x^{p/n-2} dx}{x^2 \pm a^2}.$$

$$4. \int \frac{dx}{x^{p/n}(x^2 \pm a^2)} = \frac{\pm nx^{1-p/n}}{(n-p)a^2} \mp \frac{1}{a^2} \int \frac{dx}{x^{p/n-2}(x^2 \pm a^2)}.$$

$$5. \int \frac{x^{p/n}}{x^2 \pm a^2} dx = \frac{1}{a^{1-p/n}} \sum_{k=0}^{n-1} \left(\sin \frac{4k+1}{2n} p\pi \ln |x^{1/n} - a^{1/n}| + \right. \\ \left. + \cos \frac{4k+1}{2n} p\pi \operatorname{arctg} \frac{x^{1/n} - a^{1/n} \cos \frac{4k+1}{2n} \pi}{a^{1/n} \sin \frac{4k+1}{2n} \pi} \right) \quad [p=1, 2, \dots, n-1; a>0].$$

$$6. \int \frac{x^{p/n}}{x^2 - a^2} dx = \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n - 1} dt - \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n + 1} dt \\ [t = (x/a)^{1/n}; p = \pm 1, \pm 2, \dots; a > 0; \text{ см. 1.2 3}].$$

1.2.25. Интегралы вида $\int_a^x R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $x > a > b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{x-a}{x-b}}, k = \sqrt{\frac{b-c}{a-c}}$.

$$1. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_a^x \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b)\Pi(\varphi, 1, k) + b^2 F(\varphi, k)].$$

$$3. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k).$$

$$4. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-c} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$5. \int_a^x \frac{dx}{(x-p) \sqrt{(x-a)(x-b)(x-c)}} =$$

$$= \frac{2}{(b-p)(a-p) \sqrt{a-c}} \left[(b-a) \Pi \left(\varphi, \frac{b-p}{a-b}, k \right) + (a-p) F(\varphi, k) \right] \quad [p \neq a].$$

$$6. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} =$$

$$= \frac{2}{(a-b)(b-c)^2 \sqrt{a-c}} [(2a-b-c) E(\varphi, k) - 2(a-b) F(\varphi, k)] +$$

$$+ \frac{2}{(a-c)(b-c)} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$7. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)}} dx = -2 \sqrt{a-c} E(\varphi, k) + 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$8. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)}} dx = \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k) - 2 \sqrt{a-c} E(\varphi, k) +$$

$$+ 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$9. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)}} dx = 2 \sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$10. \int_a^x \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2 \sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)].$$

$$11. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx =$$

$$= \frac{2 \sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2(a-b)}{(b-c) \sqrt{a-c}} F(\varphi, k) - 2 \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$12. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k) - 2 \frac{b-c}{a-c} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$13. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2 \sqrt{a-c}}{a-b} E(\varphi, k).$$

$$14. \int_a^x \sqrt{\frac{(x-a)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] +$$

$$+ \frac{2}{3} (x+2c-a-2b) \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$15. \int_a^x \sqrt{\frac{(x-a)(x-c)}{x-b}} dx = \frac{2}{3} \frac{\sqrt{(a-c)^3}}{b-c} [(a+c-2b) E(\varphi, k) - (a-b) F(\varphi, k)] + \\ + \frac{2(a-c)}{3(b-c)} (x+b-a-c) \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$16. \int_a^x \sqrt{\frac{(x-b)(x-c)}{x-a}} dx = \frac{2}{3} \sqrt{a-c} [2(a-b) F(\varphi, k) + (b+c-2a) E(\varphi, k)] + \\ + \frac{2}{3} (x+2a-2b-c) \sqrt{\frac{(x-a)(x-b)}{x-c}}.$$

1.2.26. Интегралы вида $\int_x^\infty R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $x \geq a > b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-c}{x-c}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].$$

$$3. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \\ - \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$4. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \\ = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b) F(\varphi, k) - (a+b-2c) E(\varphi, k)] + \\ + \frac{2x-a-b}{(a-b)^2 \sqrt{(x-a)(x-b)(x-c)}} \quad [x > a].$$

$$\begin{aligned}
 6. \int_x^{\infty} \frac{dx}{V(x-a)^3(x-b)(x-c)^3} &= \\
 &= \frac{2}{(a-b)(b-c)V(a-c)^3} [(a+c-2b)E(\varphi, k) - (a-b)F(\varphi, k)] + \\
 &\quad + \frac{2}{(a-b)(a-c)} V \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 7. \int_x^{\infty} \frac{dx}{V(x-a)(x-b)^3(x-c)^3} &= \\
 &= \frac{2}{(a-b)(b-c)^2 V a-c} [(2a-b-c)E(\varphi, k) - 2(a-b)F(\varphi, k)] - \\
 &\quad - \frac{2}{(a-b)(b-c)} V \sqrt{\frac{x-a}{(x-b)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_x^{\infty} \frac{dx}{V(x-a)^3(x-b)^3(x-c)^3} &= \frac{2}{(a-b)^2(b-c)^2 V(a-c)^3} \times \\
 &\times [(a-b)(2a-b-c)F(\varphi, k) - 2(a^2+b^2+c^2-ab-ac-bc)E(\varphi, k)] + \\
 &\quad + \frac{2[x(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)V(x-a)(x-b)(x-c)} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_x^{\infty} \frac{dx}{(x-p)V(x-a)(x-b)(x-c)} &= \\
 &= \frac{2}{(p-c)V a-c} \left[\Pi\left(\varphi, \frac{p-c}{a-c}, k\right) - F(\varphi, k) \right].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_x^{\infty} V \sqrt{\frac{x-c}{(x-a)^3(x-b)}} dx &= \\
 &= \frac{2}{V a-c} F(\varphi, k) - \frac{2V a-c}{a-b} E(\varphi, k) + \frac{2(a-c)}{a-b} V \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 11. \int_x^{\infty} V \sqrt{\frac{x-b}{(x-a)^3(x-c)}} dx &= \frac{2}{V a-c} [F(\varphi, k) - E(\varphi, k)] + 2 V \sqrt{\frac{x-b}{(x-a)(x-c)}} \\
 &\quad [x > a].
 \end{aligned}$$

$$12. \int_x^{\infty} V \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2V a-c}{a-b} E(\varphi, k) - 2 \frac{b-c}{a-b} V \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$13. \int_x^{\infty} V \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2V a-c}{b-c} [F(\varphi, k) - E(\varphi, k)] + 2 V \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$14. \int_x^{\infty} \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k).$$

$$15. \int_x^{\infty} \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\varphi, k)$$

1.2.27. Интегралы вида $\int_x^a R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $a > x \geq b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-x}{a-b}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\varphi, k) + 2 \frac{a}{b} \sqrt{a-c} E(\varphi, k).$$

$$3. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k) - \\ - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$4. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\varphi, k) - \\ - \frac{2}{(b-c)(a-c)} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$5. \int_x^a \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-p)\sqrt{a-c}} \Pi\left(\varphi, \frac{a-b}{a-p}, k\right) \quad [p \neq a].$$

$$6. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \\ = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\varphi, k) - 2(2a-b-c)E(\varphi, k)] + \\ + \frac{2(a-b-c+x)}{(a-b)(b-c)(a-c)} \sqrt{\frac{a-x}{(x-b)(x-c)}} \quad [x > b].$$

$$7. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)].$$

$$8. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\varphi, k).$$

$$9. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k).$$

$$10. \int_x^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{c-b} E(\varphi, k) + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$11. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \\ - \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$12. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-c} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$13. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$14. \int_x^a \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

$$15. \int_x^a \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(a+c-2b)E(\varphi, k) + (b-c)F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

$$16. \int_x^a \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

1.2.28. Интегралы вида $\int_b^x R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $a \geq x > b > c$.

Сбозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-b)}{(a-b)(x-c)}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_b^x \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} \{ (b-c) \Pi(\varphi, k^2, k) + c F(\varphi, k) \}.$$

$$3. \int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

$$4. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\varphi, k).$$

$$5. \int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \\ = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(b-c) F(\varphi, k) - (2b-a-c) E(\varphi, k)] + \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

$$6. \int_b^x \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}} = \\ = \frac{2}{(c-p)(b-p)\sqrt{a-c}} \left[(c-b) \Pi\left(\varphi, k^2 \frac{c-p}{b-p}, k\right) + (b-p) F(\varphi, k) \right] \quad [p \neq b].$$

$$7. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$8. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - \\ - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$9. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\varphi, k) - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$10. \int_b^x \sqrt{\frac{x-b}{(a-x)^3(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\varphi, k) + 2 \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

$$11. \int_b^x \sqrt{\frac{x-c}{(a-x)^3(x-b)}} dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k) - \left[-\frac{2\sqrt{a-c}}{a-b} E(\varphi, k) + 2\frac{a-c}{a-b} \sqrt{\frac{x-b}{(a-x)(x-c)}} \right] \quad [x < a].$$

$$12. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$13. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$14. \int_b^x \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] + \\ + \frac{2}{3} (x+c-a-b) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$15. \int_b^x \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(b-c)F(\varphi, k) + (a+c-2b)E(\varphi, k)] + \\ + \frac{2}{3} (2b-a-2c+x) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$16. \int_b^x \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} (b+2c-2a-x) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

1.2.29. Интегралы вида $\int_x^b R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}) dx$.

Условие: $a > b > x \geq c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c)\Pi(\varphi, k^2, k) + aF(\varphi, k)].$$

$$3. \int_a^b \frac{dx}{V(a-x)^3(b-x)(x-c)} = \frac{2}{(a-b)V(a-c)} E(\varphi, k).$$

$$4. \int_x^b \frac{dx}{V(a-x)(b-x)(x-c)} = \frac{2}{(b-c)V(a-c)} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} V \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x > c].$$

$$5. \int_a^b \frac{dx}{V(a-x)^2(b-x)(x-c)^3} = \\ = \frac{2}{(b-c)(a-b)V(a-c)} [(a-b)F(\varphi, k) + (2b-a-c)E(\varphi, k)] + \\ + \frac{2}{(b-c)(a-c)} V \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x > c].$$

$$6. \int_a^b \frac{dx}{(p-x)V(a-x)(b-x)(x-c)} = \\ = \frac{2}{(p-a)(p-b)V(a-c)} \left[(b-a)\Pi\left(\varphi, k^2 \frac{p-a}{p-b}, k\right) + (p-b)F(\varphi, k) \right] \quad [p \neq b].$$

$$7. \int_a^b V \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2V(a-c)E(\varphi, k) - 2V \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$8. \int_a^b V \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2V(a-c)E(\varphi, k) - \frac{2(a-b)}{V(a-c)} F(\varphi, k) - \\ - 2V \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$9. \int_x^b V \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2V(a-c)[F(\varphi, k) - E(\varphi, k)] + \\ + 2V \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$10. \int_a^b V \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2V(a-c)}{a-b} E(\varphi, k) - \frac{2}{V(a-c)} F(\varphi, k).$$

$$11. \int_a^b V \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{V(a-c)} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_c^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx =$$

$$= 2 \frac{a-b}{(b-c) \sqrt{a-c}} F(\varphi, k) - \frac{2 \sqrt{a-c}}{b-c} E(\varphi, k) + 2 \frac{a-c}{b-c} \sqrt{\frac{b-x}{(a-x)(x-c)}}.$$

$$13. \int_c^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\varphi, k) + 2 \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x > c].$$

$$14. \int_c^b \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] +$$

$$+ \frac{2}{3} (2c-2x-b+x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$15. \int_c^b \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(c-b) F(\varphi, k) + (2b-a-c) E(\varphi, k)] +$$

$$+ \frac{2}{3} (2a+c-2b-x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$16. \int_c^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-a) F(\varphi, k) + (2x-b-c) E(\varphi, k)] +$$

$$+ \frac{2}{3} (2c-b-x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

1.2.30. Интегралы вида $\int_c^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}) dx$,

Условие: $a > b > x > c$.

Собозначения: $\varphi = \arcsin \sqrt{\frac{x-c}{b-c}}, \quad k = \sqrt{\frac{b-c}{a-c}}.$

$$1. \int_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_c^x \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\varphi, k) - 2 \sqrt{a-c} E(\varphi, k).$$

$$3. \int_c^x \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b) \sqrt{a-c}} E(\varphi, k) -$$

$$- \frac{2}{(a-b)(a-c)} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$4. \int_c^x \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k) - \\ - \frac{2\sqrt{a-c}}{(a-b)(b-c)} L(\varphi, k) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

$$5. \int_c^x \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \\ = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\varphi, k) - (a+b-2c)E(\varphi, k)] + \\ + \frac{2[a^2+b^2-ac-bc-x(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \sqrt{\frac{x-c}{(a-x)(b-x)}} \quad [x < b].$$

$$6. \int_c^x \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)}} = \\ = \frac{2}{(p-c)\sqrt{a-c}} \Pi\left(\varphi, \frac{b-c}{p-c}, k\right) \quad [p \neq c].$$

$$7. \int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k).$$

$$8. \int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k).$$

$$9. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_c^x \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \\ = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$11. \int_c^x \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-c} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$12. \int_c^x \sqrt{\frac{a-x}{(b-x)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

$$13. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)^3}} dx = -\frac{2\sqrt{a-c}}{a-b} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{(1-x)(x-c)}{b-x}} \quad [a < b].$$

$$14. \int_c^x \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] + \\ + \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

$$15. \int_c^x \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c) E(\varphi, k) + (a-b) F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

$$16. \int_c^x \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c) E(\varphi, k) - 2(a-b) F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

1.2.31. Интегралы вида $\int_x^c R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}) dx$.

Условие. $a > b > c > x$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{c-x}{b-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)}} = \\ = \frac{2}{\sqrt{a-c}} [cF(\varphi, k) + (a-c) E(\varphi, k)] - 2 \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$3. \int_x^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \\ = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$4. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k).$$

$$5. \int_x^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] + \frac{2}{(a-b)(a-c)} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$6. \int_x^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(p-b)(p-c)\sqrt{a-c}} \times \\ \times \Pi\left(\varphi, \frac{p-b}{p-c}, k\right) + \frac{2}{(p-b)\sqrt{a-c}} F(\varphi, k) \quad (p \neq c).$$

$$7. \int \sqrt{\frac{a-x}{(b-x)(c-x)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$8. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)}} dx = \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - 2\sqrt{a-c} E(\varphi, k) + \\ + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$9. \int \sqrt{\frac{c-x}{(a-x)(b-x)}} dx = -2\sqrt{a-c} E(\varphi, k) + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$10. \int_x^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k) - \frac{2(a-b)}{a-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$11. \int_x^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \\ - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k) - 2\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$12. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)].$$

$$13. \int_x^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k).$$

$$14. \int_x^c \sqrt{\frac{(a-x)(b-x)}{c-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-c)F(\varphi, k) + \\ + (2c-a-b)E(\varphi, k)] + \frac{2}{3} (a+2b-2c-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$15. \int_x^c \sqrt{\frac{(a-x)(c-x)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} (a+c-b-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$16. \int_x^c \sqrt{\frac{(b-x)(c-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} (2b-2a+c-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

1.2.32. Интегралы вида $\int_{-\infty}^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}) dx$.

Условие: $a > b > c \geq x$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-c}{a-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$3. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \\ - \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$4. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \\ = \frac{2}{(c-b)\sqrt{a-c}} E(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

$$5. \int_{-\infty}^x \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)}} = \\ = \frac{2}{(a-p)\sqrt{a-c}} \left[\Pi\left(\varphi, \frac{a-p}{a-c}, k\right) - F(\varphi, k) \right].$$

$$\begin{aligned}
 6. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} &= \\
 &= \frac{2}{(b-c)(a-b)^2 \sqrt{a-c}} [(a+b+2c) E(\varphi, k) - 2(b-c) F(\varphi, k)] - \\
 &\quad - \frac{2}{(a-b)(b-c)} \sqrt{\frac{c-x}{(a-x)(b-x)}}.
 \end{aligned}$$

$$\begin{aligned}
 7. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} &= \\
 &= \frac{2}{(a-b)(b-c)^2 \sqrt{a-c}} [(b-c) F(\varphi, k) - (a+b-2c) E(\varphi, k)] + \\
 &\quad + \frac{2(b+c-2x)}{(b-c)^2 \sqrt{(a-x)(b-x)(c-x)}} \quad [x < c].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} &= \\
 &= \frac{2}{(a-b)(b-c) \sqrt{(a-c)^3}} [(2b-a-c) E(\varphi, k) - (b-c) F(\varphi, k)] + \\
 &\quad + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} &= \frac{2}{(a-b)^2(b-c)^2 \sqrt{(a-c)^3}} \times \\
 &\times [(b-c)(a+b-2c) F(\varphi, k) - 2(c^2+a^2+b^2-ab-ac-bc) E(\varphi, k)] + \\
 &\quad + \frac{2[c(a-c)+b(a-b)-x(2a-c-b)]}{(a-b)(a-c)(b-c)^2 \sqrt{(a-x)(b-x)(c-x)}} \quad [x < c].
 \end{aligned}$$

$$10. \int_{-\infty}^x \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k).$$

$$11. \int_{-\infty}^x \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k).$$

$$\begin{aligned}
 12. \int_{-\infty}^x \sqrt{\frac{a-x}{(b-x)(c-x)^3}} dx &= \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) + \\
 &\quad + \frac{2(a-c)}{b-c} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].
 \end{aligned}$$

$$13. \int_{-\infty}^x \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - 2 \frac{a-b}{b-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$14. \int_{-\infty}^x \sqrt{\frac{b-x}{(a-x)(c-x)^3}} dx =$$

$$= \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

$$15. \int_{-\infty}^x \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx =$$

$$= \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

1.2.33. Интегралы вида $\int_a^x R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условие: $x > a > b > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(x-a)}{(a-d)(x-b)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_a^x \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi\left(\varphi, \frac{a-d}{b-d}, k\right) + bF(\varphi, k) \right].$$

$$3. \int_a^x \frac{dx}{x \sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{ab \sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi\left[\varphi, \frac{b(a-d)}{a(b-d)}, k\right] + aF(\varphi, k) \right\}.$$

$$4. \int_a^x \frac{dx}{(p-x) \sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (a-b) \Pi\left[\varphi, \frac{(a-d)(p-b)}{(b-d)(p-a)}, k\right] + (p-a) F(\varphi, k) \right\}.$$

$$5. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{a-d}{b-d}, k\right) - F(\varphi, k) \right].$$

$$6. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{a-d}{b-d}, k\right).$$

$$7. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\varphi, \frac{a-d}{b-d}, k \right) + (b-c) F(\varphi, k) \right].$$

$$8. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\varphi, \frac{a-d}{b-d}, k \right) + (b-d) F(\varphi, k) \right].$$

$$9. \int_a^x \sqrt{\frac{x-a}{(x-b)^2(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^2(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \\ - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{c-d} \sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

$$11. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^2}} dx = \\ = \frac{-2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \frac{2}{c-d} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

$$12. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^2}} dx = \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}} + \\ + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) + 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\varphi, k).$$

$$13. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^2(x-d)}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(b-c)}{(a-c)(c-d)} \sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

$$14. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^2(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$15. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^2}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

$$16. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}} dx =$$

$$= \frac{2 \sqrt{(a-c)(b-a)}}{(a-b)(b-c)} E(\varphi, k) - \frac{2(c-d)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$17. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}} dx =$$

$$= \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

1.2.34. Интегралы вида $\int_a^x R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условия $a > x > b > c > d$

Обозначения $\varphi = \arcsin \sqrt{\frac{(b-d)(a-x)}{(a-b)(x-d)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(x-c)(b-d)}}$.

$$1. \int_x^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\varphi, \frac{b-a}{b-d}, k\right) + d F(\varphi, k) \right].$$

$$2. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$3. \int_x^a \frac{dx}{x \sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{ad \sqrt{(a-c)(b-d)}} \left\{ (d-a) \Pi\left[\varphi, \frac{d(b-a)}{a(b-d)}, k\right] + a F(\varphi, k) \right\}.$$

$$4. \int_x^a \frac{dx}{(p-x) \sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (a-d) \Pi\left[\varphi, \frac{(b-a)(p-d)}{(b-d)(p-a)}, k\right] + (p-a) F(\varphi, k) \right\} \quad [p \neq a]$$

$$5. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{b-a}{b-d}, k\right) - F(\varphi, k) \right].$$

$$6. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi\left(\varphi, \frac{b-a}{b-d}, k\right) - (b-d) F(\varphi, k) \right].$$

$$7. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-b)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{b-a}{b-d}, k \right) + (d-c) F(\varphi, k) \right].$$

$$8. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{b-a}{b-d}, k \right).$$

$$9. \int_x^a \sqrt{\frac{a-x}{(x-b)^2(x-c)(x-d)}} dx = \frac{2}{c-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}} \quad [x > b].$$

$$10. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^2(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \\ - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}$$

$$11. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^2(x-d)}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$13. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = \\ = 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\varphi, k) - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$14. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^2(x-d)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}}.$$

$$15. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$16. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k) -$$

$$- \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}} \quad [x > b].$$

$$17. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx =$$

$$= \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(c-d)}{(a-c)(b-c)} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

1.2.35. Интегралы вида $\int_b^x R \sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d} dx$.

Условие: $a > x > b > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-b)}{(a-b)(x-c)}}, \quad k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$

$$1. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_b^x \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi \left(\varphi, \frac{a-b}{a-c}, k \right) + c F(\varphi, k) \right].$$

$$3. \int_b^x \frac{dx}{x \sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{bc \sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left[\varphi, \frac{c(a-b)}{b(a-c)}, k \right] + b F(\varphi, k) \right\}.$$

$$4. \int_b^x \frac{dx}{(x-p) \sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(b-p)(p-c) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (b-c) \Pi \left[\varphi, \frac{(a-b)(p-c)}{(a-c)(p-b)}, k \right] + (p-b) F(\varphi, k) \right\} \quad [p \neq b].$$

$$5. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi \left(\varphi, \frac{a-b}{a-c}, k \right) + (a-c) F(\varphi, k) \right].$$

$$6. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{a-b}{a-c}, k \right) - F(\varphi, k) \right].$$

$$7. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{a-b}{a-c}, k\right),$$

$$8. \int_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi\left(\varphi, \frac{a-b}{a-c}, k\right) + (c-d) F(\varphi, k) \right].$$

$$9. \int_b^x \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}} \quad [x < a].$$

$$10. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx = \\ = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$11. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$12. \int_b^x \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}} dx = \\ = \frac{2}{d-a} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{a-d} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}}.$$

$$13. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$14. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d)E(\varphi, k) - (a-d)(b-c)F(\varphi, k)] - \frac{2}{a-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$15. \int_b^x \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}} dx = \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}} \quad [x < a].$$

$$16. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx =$$

$$= \frac{2}{a-d} \left[\sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{c-d}{b-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}} \right].$$

$$17. \int_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

1.2.36. Интегралы вида $\int_x^b R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}, \sqrt{x-d}) dx$

Условие: $a > b > x \geq c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_x^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + aF(\varphi, k) \right].$$

$$3. \int_x^b \frac{dx}{x \sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$

$$= \frac{2}{ab \sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi\left[\varphi, \frac{a(b-c)}{b(a-c)}, k\right] + bF(\varphi, k) \right\}.$$

$$4. \int_x^b \frac{cx}{(p-x) \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (b-a) \Pi\left[\varphi, \frac{(b-c)(p-a)}{(a-c)(p-b)}, k\right] + (p-b) F(\varphi, k) \right\} \quad |p \neq b|.$$

$$5. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{b-c}{a-c}, k\right).$$

$$6. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{b-c}{a-c}, k\right) - F(\varphi, k) \right].$$

$$7. \int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + (a-c) F(\varphi, k) \right].$$

$$8. \int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi \left(\varphi, \frac{b-c}{a-c}, k \right) + (a-d) F(\varphi, k) \right].$$

$$9. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)^2(x-d)}} dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) + \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$

$$10. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^2}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-d)}{(b-d)(c-d)} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$11. \int_x^b \sqrt{\frac{b-x}{(a-x)^2(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)^2(x-d)}} dx = \\ = \frac{2}{d-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{c-d} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$

$$13. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^2}} dx = \\ = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} [(a-c)(b-d)E(\varphi, k) - (a-b)(c-d)F(\varphi, k)] - \\ - \frac{2}{c-d} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$14. \int_x^b \sqrt{\frac{x-c}{(a-x)^2(b-x)(x-d)}} dx = \\ = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$15. \int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^2}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$16. \int_x^b \sqrt{\frac{x-d}{(a-x)^2(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

$$17. \int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$

1.2.37. Интегралы вида $\int_c^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условие: $a > b > x > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_c^x \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + dF(\varphi, k) \right].$$

$$3. \int_c^x \frac{dx}{x \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \\ = \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi\left[\varphi, \frac{d(b-c)}{c(b-d)}, k\right] + cF(\varphi, k) \right\}.$$

$$4. \int_c^x \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d) \sqrt{(a-c)(b-d)}} \times \\ \times \left\{ (c-d) \Pi\left[\varphi, \frac{(b-c)(p-d)}{(b-d)(p-c)}, k\right] + (p-c) F(\varphi, k) \right\} \quad [p \neq c].$$

$$5. \int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + (a-d) F(\varphi, k) \right].$$

$$6. \int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + (b-d) F(\varphi, k) \right].$$

$$7. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) - F(\varphi, k) \right].$$

$$8. \int_c^x \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{b-c}{b-d}, k\right).$$

$$9. \int_c^x \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}} \quad [x < b].$$

$$10. \int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$11. \int_c^x \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$12. \int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d) \sqrt{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d) E(\varphi, k) - (a-b)(c-d) F(\varphi, k)].$$

$$13. \int_c^x \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx = \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \\ - \frac{2(c-d)}{(a-d) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$14. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{b-a} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}}.$$

$$15. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$16. \int_c^x \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(a-d)}{(a-b)(a-c)} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$17. \int_c^x \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}} dx = \frac{2(c-d)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}} \quad [x < b].$$

1.2.38. Интегралы вида $\int_x^c R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}) dx$.

Условие: $a > b > c > x \geq d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(c-x)}{(c-d)(b-x)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_x^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \times \\ \times \left[(c-b) \Pi\left(\varphi, \frac{c-d}{b-d}, k\right) + b F(\varphi, k) \right].$$

$$3. \int_x^c \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \\ = \frac{2}{bc \sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi\left[\varphi, \frac{b(c-d)}{c(b-d)}, k\right] + c F(\varphi, k) \right\}.$$

$$4. \int_x^c \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c) \sqrt{(a-c)(b-d)}} \times \\ \times \left\{ (c-b) \Pi\left[\varphi, \frac{(c-d)(p-b)}{(b-d)(p-c)}, k\right] + (p-c) F(\varphi, k) \right\} \quad [p \neq c].$$

$$5. \int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi\left(\varphi, \frac{c-d}{b-d}, k\right) + (a-b) F(\varphi, k) \right].$$

$$6. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{c-d}{b-d}, k\right).$$

$$7. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{c-d}{b-d}, k\right) - F(\varphi, k) \right].$$

$$8. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi \left(\varphi, \frac{c-d}{b-d}, k \right) + (b-d) F(\varphi, k) \right].$$

$$9. \int_x^c \sqrt{\frac{a-x}{(b-x)^2(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$10. \int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{c-d} \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \quad [x > d].$$

$$11. \int_x^c \sqrt{\frac{b-x}{(a-x)^2(c-x)(x-d)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(a-b)}{(a-c)(a-d)} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$12. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}} dx = \\ = \frac{2}{(a-d)(c-d) \sqrt{(a-c)(b-d)}} [(b-c)(a-d)F(\varphi, k) - (a-c)(b-d)E(\varphi, k)] + \\ + \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \quad [x > d].$$

$$13. \int_x^c \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)}} dx = \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} F(\varphi, k) - \\ - \frac{2(b-c)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{a-d} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$14. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^2(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$15. \int_x^c \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)^3}} dx = \\ = \frac{2}{d-a} \left[\sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \right] \quad [x > d].$$

$$16. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$

$$= \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$17. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx =$$

$$= \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) - \frac{2(a-d)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$

1.2.39. Интегралы вида $\int_d^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}) dx$.

Условия: $a > b > c \geq x > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-d)}{(c-d)(a-x)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_d^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_d^x \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) + a F(\varphi, k) \right].$$

$$3. \int_d^x \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{ad \sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi\left[\varphi, \frac{a(d-c)}{d(a-c)}, k\right] + d F(\varphi, k) \right\}.$$

$$4. \int_d^x \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (d-a) \Pi\left[\varphi, \frac{(d-c)(p-a)}{(a-c)(p-d)}, k\right] + (p-d) F(\varphi, k) \right\} \quad [p \neq d].$$

$$5. \int_d^x \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{d-c}{a-c}, k\right).$$

$$6. \int_d^x \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) - (a-b) F(\varphi, k) \right].$$

7. $\int_d^x \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx =$
 $= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - (a-c) F(\varphi, k) \right].$
8. $\int_d^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$
 $= \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - F(\varphi, k) \right].$
9. $\int_d^x \sqrt{\frac{a-x}{(b-x)^2(c-x)(x-d)}} dx =$
 $= \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(x-d)(c-x)}{(a-x)(b-x)}}.$
10. $\int_d^x \sqrt{\frac{a-x}{(b-x)(c-x)^2(x-d)}} dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) -$
 $- 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) + 2 \frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$
11. $\int_d^x \sqrt{\frac{b-x}{(a-x)^2(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$
12. $\int_d^x \sqrt{\frac{b-x}{(a-x)(c-x)^2(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] +$
 $+ \frac{2}{c-d} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$
13. $\int_d^x \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)}} dx =$
 $= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k).$
14. $\int_d^x \sqrt{\frac{c-x}{(a-x)(b-x)^2(x-d)}} dx =$
 $= \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$
15. $\int_d^x \sqrt{\frac{x-d}{(a-x)^2(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$

$$16. \int_a^x \sqrt{\frac{x-d}{(a-x)(b-\lambda)(c-x)}} dx = \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) -$$

$$- \frac{2(a-d)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{(c-\lambda)(x-d)}{(a-\lambda)(b-\lambda)}}.$$

$$17. \int_a^x \sqrt{\frac{\lambda-d}{(a-x)(b-x)(c-\lambda)}} dx =$$

$$= \frac{-2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{(b-\lambda)(\lambda-d)}{(a-x)(c-\lambda)}}.$$

1.2.40. Интегралы вида $\int_x^d R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{d-x}) dx$.

Условие. $a > b > c > d > x$

Обозначения $\varphi = \arcsin \sqrt{\frac{(a-c)(d-x)}{(a-d)(c-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_x^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k)$$

$$2. \int_x^d \frac{\lambda dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[c \Pi \left(\varphi, \frac{a-d}{a-c}, k \right) - (c-d) F(\varphi, k) \right].$$

$$3. \int_x^d \frac{dx}{\lambda \sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$

$$= \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi \left[\varphi, \frac{c(a-d)}{d(a-c)}, k \right] + d F(\varphi, k) \right\}.$$

$$4. \int_x^d \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{(p-c)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (d-c) \Pi \left[\varphi, \frac{(a-d)(p-c)}{(a-c)(p-d)}, k \right] + (p-d) F(\varphi, k) \right\}$$

[$p \neq d$].

$$5. \int_x^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\varphi, \frac{a-d}{a-c}, k \right) + (a-c) F(\varphi, k) \right].$$

$$6. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\varphi, \frac{a-d}{a-c}, k \right) + (b-c) \Gamma(\varphi, k) \right].$$

$$7. \int_x^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{a-d}{a-c}, k \right).$$

$$8. \int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \\ = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{a-d}{a-c}, k \right) - F(\varphi, k) \right].$$

$$9. \int_x^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \\ = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$10. \int_x^d \sqrt{\frac{a-x}{(a-x)(c-x)^3(d-x)}} dx = \\ = \frac{2 \sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \frac{a-b}{b-c} \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k)$$

$$11. \int_x^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$12. \int_x^d \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

$$13. \int_x^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}} dx = \frac{2(-d)}{(a-d) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$14. \int_x^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(b-c)}{(a-b)(b-d)} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$15. \int_x^d \sqrt{\frac{d-x}{(a-x)^2(b-x)(c-x)}} dx = \\ = \frac{2}{b-a} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{a-b} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$16. \int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \frac{2}{(a-b)(b-c)} \sqrt{(a-c)(b-d)} E(\varphi, k) - \\ - \frac{2(c-d)}{(a-c)(b-d)} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$17. \int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

1.2.41. Интегралы вида $\int x^m (x^2 \pm a^2)^{n+1/2} dx$.

$$1. \int x^m (x^2 \pm a^2)^{n+1/2} dx = \\ = \frac{x^{m+1} (x^2 \pm a^2)^{n+3/2}}{m+2n+2} \mp \frac{(m-1)a^2}{m+2n+2} \int x^{m-2} (x^2 \pm a^2)^{n+1/2} dx.$$

$$2. \int x^{2m} (x^2 \pm a^2)^{n+1/2} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+2m+2} \times \\ \times \left[x^{2m-1} + \sum_{k=1}^{m-1} (-1)^k \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m+n)(m+n-1) \dots (m+n-k+1)} a^{2k} x^{2m-2k-1} \right] + \\ + (-1)^m \frac{a^{2m} (2m-1)!!}{2^m (n+m+1)(n+m) \dots (n+2)} \int (x^2 \pm a^2)^{n+1/2} dx.$$

$$3. \int x^{2m+1} (x^2 \pm a^2)^{n+1/2} dx = \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k a^{2m-k} (x^2 \pm a^2)^{n+3/2}}{2n+2k+3}.$$

$$4. \int x (x^2 \pm a^2)^{n+1/2} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+3}.$$

$$5. \int (x^2 \pm a^2)^{n+1/2} dx = \frac{x (x^2 \pm a^2)^{n+1/2}}{2(n+1)} \pm \frac{(2n+1)a^2}{2(n+1)} \int (x^2 \pm a^2)^{n-1/2} dx.$$

$$6. = \frac{x (x^2 \pm a^2)^{n+1/2}}{2n+2} \times \\ \times \left[(x^2 \pm a^2)^n + \sum_{k=1}^n (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2^k n(n-1) \dots (n-k+1)} a^{2k} (x^2 \pm a^2)^{n-k} \right] + \\ + \frac{(-1)^{n-1} a^{2n+2} (2n+1)!!}{2^{n+1} (n+1)!} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

7. $\int x^{2m} (x^2 \pm a^2)^{1/2} dx = \frac{(x^2 \pm a^2)^{3/2}}{2(m+1)} \times$
 $\times \left[x^{2m-1} + \sum_{k=1}^{m-1} (-1)^k \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k m (m-1) \dots (m-k+1)} a^{2k} x^{2m-2k-1} \right] +$
 $+ (-1)^m \frac{a^{2m} (2m-1)!!}{2^m (m+1)!} \int (x^2 \pm a^2)^{1/2} dx$
8. $\int (x^2 \pm a^2)^{1/2} dx = \frac{1}{2} x (x^2 \pm a^2)^{1/2} + \frac{a^2}{2} \ln |x + (x^2 \pm a^2)^{1/2}|.$
9. $\int x (x^2 \pm a^2)^{1/2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2}.$
10. $\int x^2 (x^2 \pm a^2)^{1/2} dx =$
 $= \frac{1}{4} x (x^2 \pm a^2)^{3/2} + \frac{a^2}{8} x (x^2 \pm a^2)^{1/2} - \frac{a^4}{8} \ln |x + (x^2 \pm a^2)^{1/2}|.$
11. $\int x^3 (x^2 \pm a^2)^{1/2} dx = \frac{1}{5} (x^2 \pm a^2)^{5/2} + \frac{a^2}{3} (x^2 \pm a^2)^{3/2}.$
12. $\int (x^2 \pm a^2)^{3/2} dx =$
 $= \frac{1}{4} x (x^2 \pm a^2)^{3/2} + \frac{3}{8} a^2 x (x^2 \pm a^2)^{1/2} + \frac{3}{8} a^4 \ln |x + (x^2 \pm a^2)^{1/2}|.$
13. $\int x (x^2 \pm a^2)^{3/2} dx = \frac{1}{5} (x^2 \pm a^2)^{5/2}.$
14. $\int x^3 (x^2 \pm a^2)^{3/2} dx =$
 $= \frac{1}{6} x (x^2 \pm a^2)^{5/2} + \frac{a^2}{24} x (x^2 \pm a^2)^{3/2} - \frac{a^4}{16} x (x^2 \pm a^2)^{1/2} +$
 $+ \frac{a^6}{16} \ln |x + (x^2 \pm a^2)^{1/2}|.$
15. $\int x^5 (x^2 \pm a^2)^{3/2} dx = \frac{1}{7} (x^2 \pm a^2)^{7/2} + \frac{a^2}{5} (x^2 \pm a^2)^{5/2}.$

1.2.42. Интегралы вида $\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^m} dx.$

1. $\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^m} dx = \mp \frac{(x^2 \pm a^2)^{n+3/2}}{(m-1) a^2 x^{m-1}} \pm \frac{2n-m+4}{(m-1) a^2} \int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{m-2}} dx.$
2. $= -\frac{(x^2 \pm a^2)^{n+1/2}}{(m-1) x^{m-1}} + \frac{2n-1}{m-1} \int \frac{(x^2 \pm a^2)^{n-1/2}}{x^{m-2}} dx.$
3. $\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2n}} dx = \mp \frac{(x^2 \pm a^2)^{n+3/2}}{(2m-1) a^2} \times$
 $\times \left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1} \frac{(-1)^k 2^k (m-n-2)(m-n-3) \dots (m-n-k-1)}{(2m-3)(2m-5) \dots (2m-2k-1) a^{2k}} \frac{1}{x^{2m-2k-1}} \right] +$
 $+ \frac{(-1)^m 2^m (m-n-2)(m-n-3) \dots (-n)(-n-1)}{a^{2m} (2m-1)!!} \int (x^2 \pm a^2)^{n+1/2} dx.$

$$4. \quad = \frac{(x^2 + a^2)^{n-3/2}}{(x^2 + a^2)^{m-n-1} x^{n+3}} \sum_{k=0}^{m-n-2} (-1)^k \binom{m-n-2}{k} \frac{1}{2n+2k+3} \left(\frac{x^2 + a^2}{x^2} \right)^k$$

[$m \geq n+2$].

$$5. \quad \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2n}} dx =$$

$$= - \sum_{k=0}^{n-1} \frac{(2n+1)(x^2 + a^2)^{n-k-1/2}}{[(2n-2k)^2 - 1] x^{2n-2k-1}} + (2n+1) \int (x^2 + a^2)^{1/2} dx.$$

$$6. \quad \int \frac{(x^2 + a^2)^{n-1/2}}{x^{2m-1}} dx =$$

$$= - \frac{(x^2 + a^2)^{n-3/2}}{2n-2m+1} \sum_{k=1}^m (-1)^k \frac{(2n-2m+1)(2n-2m+3)}{2^k m(m-1)} \frac{(2n-2m+2k-1)}{(m-k+1) a^{-k} x^{2m-2k+2}} +$$

$$+ (-1)^m \frac{(2n-2m+3)(2n-2m+5)}{2^m a^{-m} m!} \frac{(2n+1)}{x} \int \frac{(x^2 + a^2)^{n+1/2}}{x} dx.$$

$$7. \quad \int \frac{(x^2 + a^2)^{n-1/2}}{x} dx =$$

$$= \sum_{k=0}^n \frac{(x^2 + a^2)^k (x^2 + a^2)^{n-k+1/2}}{2n-2k+1} + (-1)^n a^{n+1} \left\{ \frac{1}{2} \ln \left| \frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a} \right| \right.$$

$$\left. \arcsin \left| \frac{a}{x} \right| \right\}.$$

$$8. \quad \int \frac{(x^2 + a^2)^{1/2}}{x} dx = (x^2 + a^2)^{1/2} - a \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$9. \quad \int \frac{(x^2 + a^2)^{1/2}}{x^3} dx = - \frac{(x^2 + a^2)^{1/2}}{2x^2} + \ln |x + (x^2 + a^2)^{1/2}|.$$

$$10. \quad \int \frac{(x^2 + a^2)^{1/2}}{x^3} dx = - \frac{(x^2 + a^2)^{1/2}}{2x^2} + \frac{1}{2a} \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$11. \quad \int \frac{(x^2 + a^2)^{3/2}}{x} dx =$$

$$= \frac{1}{3} (x^2 + a^2)^{3/2} \pm a^2 (x^2 + a^2)^{1/2} \mp a^3 \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$12. \quad \int \frac{(x^2 + a^2)^{3/2}}{x} dx =$$

$$= - \frac{(x^2 + a^2)^{3/2}}{x} + \frac{3}{2} x (x^2 + a^2)^{1/2} \pm \frac{3}{2} a^2 \ln |x + (x^2 + a^2)^{1/2}|.$$

$$13. \int \frac{(x^2 + a^2)^{3/2}}{x^3} dx =$$

$$= -\frac{(x^2 + a^2)^{3/2}}{2x^2} + \frac{3}{2} (x^2 + a^2)^{1/2} - \frac{3a}{2} \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| + \arccos \left| \frac{a}{x} \right| \right\}.$$

$$14. \int \frac{(x^2 + a^2)^{n+1/2}}{x^{n+2}} dx =$$

$$= - \sum_{k=0}^n \frac{1}{2n-2k+1} \left(\frac{x^2 + a^2}{x^2} \right)^{n-k+1/2} + \ln |x + \sqrt{x^2 + a^2}|.$$

$$15. \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2n+4}} dx = - \frac{(x^2 + a^2)^{n+3/2}}{(2n+3) a^2 x^{2n+3}}.$$

1.2.43. Интегралы вида $\int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}}$.

$$1. \int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{-x^{m-1}}{(2n-1)(x^2 \pm a^2)^{n+1/2}} + \frac{m-1}{2n-1} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$2. = \pm \frac{x^{m+1}}{(2n-1)a^2(x^2 \pm a^2)^{n+1/2}} + \frac{2n-m-2}{(2n-1)a^2} \int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$3. = \frac{x^{m-1}}{(m-2n)(x^2 \pm a^2)^{n+1/2}} + \frac{(m-1)a^2}{m-2n} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$4. \int \frac{x^{2m} dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{x^{2m+1} \sqrt{x^2 \pm a^2}}{(2n-1)a^2} \times$$

$$\times \left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (\pm 1)^k \frac{2^k (n-m-1)(n-m-2) \dots (n-m-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k}}{(x^2 \pm a^2)^{n-k}} \right] +$$

$$+ \frac{(\pm 1)^n 2^n (n-m-1)(n-m-2) \dots (-m+1)(-m)}{a^{2n} (2n-1)!!} \int \frac{x^{2m} dx}{\sqrt{x^2 \pm a^2}}$$

$$5. = (\pm 1)^n a^{2m-2n} \sum_{k=0}^{n-m-1} \frac{(-1)^k}{2m+2k+1} \binom{n-m-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{m+k+1/2}$$

[n ≥ m+1]

$$6. \int \frac{x^{2m} dx}{\sqrt{x^2 \pm a^2}} = \frac{\sqrt{x^2 \pm a^2}}{2m} \times$$

$$\times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(-1)^k (2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} a^{2k} x^{2m-2k-1} \right] +$$

$$+ (-1)^m a^{2m} \frac{(2m-1)!!}{2^m m!} \ln |x + \sqrt{x^2 \pm a^2}|.$$

$$7. \int \frac{x^{2n-2} dx}{(x^2 \pm a^2)^{n+1/2}} = \pm \frac{x^{2n-1}}{(2n-1)a^2(x^2 \pm a^2)^{n+1/2}}.$$

- $$8. \int \frac{x^{2m+1} dx}{(x^2 \pm a^2)^{n+1/2}} = \sum_{k=0}^m (-1)^{m+k-1} \binom{m}{k} \frac{a^{2m-2k}}{(2n-2k-1)(x^2 \pm a^2)^{n-k-1/2}}.$$
- $$9. \int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \pm \frac{x}{(2n-1)a^2(x^2 \pm a^2)^{n-1/2}} \pm \frac{2n-2}{(2n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1/2}}.$$
- $$10. \int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{x(x^2 \pm a^2)^{1/2}}{(2n-1)a^2} \times$$
- $$\times \left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (-1)^k \frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \cdot \frac{1}{a^{2k}(x^2 \pm a^2)^{n-k}} \right].$$
- $$11. = \frac{(-1)^n}{a^{2n}} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{k+1/2}.$$
- $$12. \int \frac{x dx}{(x^2 \pm a^2)^{n+1/2}} = -\frac{1}{(2n-1)(x^2 \pm a^2)^{n-1/2}}.$$
- $$13. \int \frac{dx}{(x^2 \pm a^2)^{1/2}} = \ln |x + (x^2 \pm a^2)^{1/2}|.$$
- $$14. \int \frac{x dx}{(x^2 \pm a^2)^{1/2}} = (x^2 \pm a^2)^{1/2}.$$
- $$15. \int \frac{x^2 dx}{(x^2 \pm a^2)^{3/2}} = \frac{x(x^2 \pm a^2)^{1/2}}{2} \mp \frac{a^2}{2} \ln |x + (x^2 \pm a^2)^{1/2}|.$$
- $$16. \int \frac{x^2 dx}{(x^2 \pm a^2)^{1/2}} = \frac{1}{3} (x^2 \pm a^2)^{3/2} \mp a^2 (x^2 \pm a^2)^{1/2}.$$
- $$17. \int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \pm \frac{x}{a^2(x^2 \pm a^2)^{1/2}}.$$
- $$18. \int \frac{a dx}{(x^2 \pm a^2)^{3/2}} = -\frac{1}{(x^2 \pm a^2)^{1/2}}.$$
- $$19. \int \frac{x^2 dx}{(x^2 \pm a^2)^{3/2}} = -\frac{x}{(x^2 \pm a^2)^{1/2}} + \ln |x + (x^2 \pm a^2)^{1/2}|.$$
- $$20. \int \frac{x^3 dx}{(x^2 \pm a^2)^{3/2}} = \frac{x^2 \pm 2a^2}{(x^2 \pm a^2)^{1/2}}.$$

1.2.44. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}}.$

- $$1. \int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}} =$$
- $$= \pm \frac{1}{(2n-1)a^2 x^{m-1} (x^2 \pm a^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1)a^2} \int \frac{dx}{x^m (x^2 \pm a^2)^{n-1/2}}.$$
- $$2. = \frac{-1}{(2n-1)x^{m-1} (x^2 \pm a^2)^{n-1/2}} - \frac{m+1}{2n-1} \int \frac{dx}{x^{m+1} (x^2 \pm a^2)^{n-1/2}}.$$

$$3. = \frac{-1}{(m-1)x^{m-1}(x^2 \pm a^2)^{n-1/2}} - \frac{2n-1}{m-1} \int \frac{dx}{x^{m-3}(x^2 \pm a^2)^{n-3/2}}.$$

$$4. \int \frac{dx}{x^{2m}(x^2 \pm a^2)^{n-1/2}} = \frac{(-1)^{m+n}}{a^{2n-2m}} \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2k-2m-1} \binom{m+n-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{m+1/2}.$$

$$5. \int \frac{dx}{x(x^2 \pm a^2)^{n-1/2}} = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-2k-1)a^{2k-2}(x^2 \pm a^2)^{n-k-1/2}} - \frac{(-1)^{n+1}}{a^{2n-1}} \left\{ \ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$6. \int \frac{dx}{x(x^2 \pm a^2)^{1/2}} = \frac{1}{a} \left\{ \ln \left| \frac{x}{a + (x^2 \pm a^2)^{1/2}} \right| \right. \\ \left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$7. \int \frac{dx}{x^2(x^2 \pm a^2)^{1/2}} = \mp \frac{(x^2 \pm a^2)^{1/2}}{a x}.$$

$$8. \int \frac{dx}{x(x^2 \pm a^2)^{1/2}} = \mp \frac{(x^2 \pm a^2)^{1/2}}{2a^2 x^2} + \frac{1}{2a^3} \left\{ \ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$9. \int \frac{dx}{x(x^2 \pm a^2)^{3/2}} = \pm \frac{1}{a^2(x^2 \pm a^2)^{1/2}} - \frac{1}{a^3} \left\{ \ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. \arccos \left| \frac{a}{x} \right| \right\}.$$

$$10. \int \frac{dx}{x^2(x^2 \pm a^2)^{3/2}} = -\frac{2x^2 \pm a^2}{a^4 x(x^2 \pm a^2)^{1/2}}.$$

$$11. \int \frac{dx}{x^3(x^2 \pm a^2)^{3/2}} = \frac{1}{2a^2 x^2(x^2 \pm a^2)^{1/2}} - \frac{3}{2a^4(x^2 \pm a^2)^{1/2}} + \frac{3}{2a^5} \left\{ \ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. \arccos \left| \frac{a}{x} \right| \right\}.$$

1.2.45. Интегралы вида $\int \frac{dx}{(x+b)^n \sqrt{x^2 \pm a^2}}, \int \frac{dx}{(x^2 \pm l^2) \sqrt{x^2 \pm a^2}}.$

$$1. \int \frac{dx}{(x+b)^n \sqrt{x^2 \pm a^2}} = \frac{-1}{(n-1)(b^2 \pm a^2)} \left| \frac{1}{(x+b)^{n-1}} \right. \\ \left. - (2n-3)l \int \frac{dx}{(x+b)^{n-1} \sqrt{x^2 \pm a^2}} + (n-2) \int \frac{dx}{(x+b)^{n-2} \sqrt{x^2 \pm a^2}} \right|.$$

$$2. \int \frac{dx}{(x+b)\sqrt{x^2+a^2}} = \frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{a^2 - bx - \sqrt{(a^2+b^2)(x^2+a^2)}}{x+b} \right|.$$

$$3. \int \frac{dx}{(x+b)\sqrt{x^2-a^2}} = \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{a^2 + bx + \sqrt{(b^2-a^2)(x^2-a^2)}}{x+b} \right|$$

[b^2 > a^2].

$$4. = \frac{1}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{\sqrt{(a^2-b^2)(x^2-a^2)}}{a^2+bx}$$

[a^2 > b^2].

$$5. = \frac{1}{\sqrt{a^2-b^2}} \arccos \frac{a^2+bx}{a(x+b)}$$

[a^2 > b^2].

$$6. \int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a} \sqrt{\frac{x-a}{x+a}}.$$

$$7. \int \frac{dx}{(x-a)\sqrt{x^2-a^2}} = -\frac{1}{a} \sqrt{\frac{x+a}{x-a}}.$$

$$8. \int \frac{dx}{(x^2+b^2)^m (x^2+a^2)^{n+1/2}} =$$

$$= \frac{x}{2(m-1)(a^2-b^2)b^2(x^2+b^2)^{m-1}(x^2+a^2)^{n-1/2}} +$$

$$+ \frac{(2m-3)a^2 - (2n+4m-6)b^2}{2(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-1}(x^2+a^2)^{n+1/2}} +$$

$$+ \frac{m+n-2}{(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-2}(x^2+a^2)^{n+1/2}}.$$

$$9. \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2-b^2)a^2(x^2+a^2)^{n-1/2}} +$$

$$+ \frac{(4n-3)a^2 - (2n-2)b^2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n-1/2}} -$$

$$- \frac{2n-2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n-3/2}}.$$

$$10. \int \frac{dx}{(x^2+b^2)\sqrt{x^2+a^2}} = \frac{1}{b\sqrt{b^2-a^2}} \ln \frac{|x\sqrt{b^2-a^2} + b\sqrt{x^2+a^2}|}{\sqrt{x^2+b^2}}$$

[b^2 > a^2].

$$11. = \frac{1}{b\sqrt{a^2-b^2}} \operatorname{arctg} \frac{x\sqrt{a^2-b^2}}{b\sqrt{x^2+a^2}}$$

[a^2 > b^2].

$$12. \int \frac{dx}{(x^2-b^2)\sqrt{x^2+a^2}} = \frac{1}{2b\sqrt{a^2+b^2}} \ln \left| \frac{x\sqrt{a^2+b^2} - b\sqrt{x^2+a^2}}{x\sqrt{a^2+b^2} + b\sqrt{x^2+a^2}} \right|.$$

$$13. \int \frac{dx}{(x^2+b^2)\sqrt{x^2-a^2}} = \frac{1}{b\sqrt{a^2+b^2}} \ln \frac{|x\sqrt{a^2+b^2} + b\sqrt{x^2-a^2}|}{\sqrt{x^2+b^2}}.$$

$$14. \int \frac{dx}{(x^2-b^2)\sqrt{x^2-a^2}} = \frac{1}{b\sqrt{b^2-a^2}} \ln \frac{|x\sqrt{b^2-a^2} - b\sqrt{x^2-a^2}|}{\sqrt{x^2-b^2}}$$

[b^2 > a^2].

$$15. \quad = \frac{1}{b \sqrt{a^2 - b^2}} \arcsin \frac{b \sqrt{x^2 - a^2}}{a \sqrt{x^2 - b^2}} \quad [a^2 > b^2].$$

$$16. \quad = \frac{1}{b \sqrt{a^2 - b^2}} \operatorname{arctg} \frac{b \sqrt{x^2 - a^2}}{x \sqrt{a^2 - b^2}} \quad [a^2 > b^2].$$

1.2.46. Интегралы вида $\int x^m (a^2 - x^2)^{n+1/2} dx$.

$$1. \quad \int x^n (a^2 - x^2)^{n+1/2} dx = -\frac{x^{m-1} (a^2 - x^2)^{n+3/2}}{m+2n+2} + \\ + \frac{(m-1)a^2}{m+2n+2} \int x^{m-2} (a^2 - x^2)^{n+1/2} dx.$$

$$2. \quad = \frac{x^{m+1} (a^2 - x^2)^{n+1/2}}{m+2n+2} + \frac{(2n+1)a^2}{m+2n+2} \int x^m (a^2 - x^2)^{n-1/2} dx.$$

$$3. \quad \int x^{2m} (a^2 - x^2)^{n+1/2} dx = -\frac{(a^2 - x^2)^{n+3/2}}{2n+2m+2} \times \\ \times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m+n)(m+n-1)\dots(m+n-k+1)} a^{2k} x^{2m-2k-1} \right] + \\ + \frac{a^{2n} (2m-1)!!}{2^n (m+n+1)(m+n)\dots(n+2)} \int (a^2 - x^2)^{n+1/2} dx.$$

$$4. \quad \int x^{2n+1} (a^2 - x^2)^{n+1/2} dx = \sum_{k=0}^n \frac{(-1)^{k+1} a^{2m-2k}}{2n+2k+3} \binom{m}{k} (a^2 - x^2)^{n+k+3/2}.$$

$$5. \quad \int x (a^2 - x^2)^{n+1/2} dx = -\frac{1}{2n+3} (a^2 - x^2)^{n+3/2}.$$

$$6. \quad \int (a^2 - x^2)^{n+1/2} dx = \frac{x}{2n+2} (a^2 - x^2)^{n+1/2} + \frac{(2n+1)a^2}{2n+2} \int (a^2 - x^2)^{n-1/2} dx.$$

$$7. \quad \int (a^2 - x^2)^{n+1/2} dx = \frac{x (a^2 - x^2)^{1/2}}{2n+2} \times \\ \times \left[(a^2 - x^2)^n + \sum_{k=1}^n \frac{(2n+1)(2n-1)\dots(2n-2k+3)}{2^k n(n-1)\dots(n-k+1)} a^{2k} (a^2 - x^2)^{n-k} \right] + \\ + \frac{(2n+1)!! a^{2n+2}}{2^{n+1} (n+1)!} \arcsin \frac{x}{|a|}.$$

$$8. \quad \int (a^2 - x^2)^{1/2} dx = \frac{x}{2} (a^2 - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{|a|}.$$

$$9. \quad \int x (a^2 - x^2)^{1/2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2}.$$

$$10. \quad \int x^2 (a^2 - x^2)^{1/2} dx = \frac{x}{8} (2x^2 - a^2) (a^2 - x^2)^{1/2} + \frac{a^4}{8} \arcsin \frac{x}{|a|}.$$

$$11. \quad \int x^3 (a^2 - x^2)^{1/2} dx = \frac{1}{5} (a^2 - x^2)^{5/2} - \frac{a^2}{3} (a^2 - x^2)^{3/2}.$$

$$12. \int (a^2 - x^2)^{3/2} dx = \frac{x}{4} (a^2 - x^2)^{3/2} + \frac{3a^2 x}{8} (a^2 - x^2)^{1/2} + \frac{3a^4}{8} \arcsin \frac{x}{a}.$$

$$13. \int x (a^2 - x^2)^{3/2} dx = -\frac{1}{5} (a^2 - x^2)^{5/2}.$$

$$14. \int x^2 (a^2 - x^2)^{3/2} dx = -\frac{x}{48} (8x^4 - 14a^2 x^2 + 3a^4) (a^2 - x^2)^{1/2} + \frac{a^6}{16} \arcsin \frac{x}{a}.$$

$$15. \int x^3 (a^2 - x^2)^{3/2} dx = \frac{1}{7} (a^2 - x^2)^{7/2} - \frac{a^2}{5} (a^2 - x^2)^{5/2}.$$

1.2.47. Интегралы вида $\int \frac{(a^2 - x^2)^{n+1/2}}{x^m} dx$.

$$1. \int \frac{(a^2 - x^2)^{n+1/2}}{x^m} dx = -\frac{(a^2 - x^2)^{n+3/2}}{(m-1)a^2 x^{m-1}} - \frac{2n-m+4}{(m-1)a^2} \int \frac{(a^2 - x^2)^{n+1/2}}{x^{m-2}} dx.$$

$$2. = -\frac{(a^2 - x^2)^{n+1/2}}{(m-1)x^{m-1}} - \frac{2n+1}{m-1} \int \frac{(a^2 - x^2)^{n-1/2}}{x^{m-2}} dx.$$

$$3. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2m}} dx = -\frac{(a^2 - x^2)^{n+3/2}}{(2m-1)a^2} \times \\ \times \left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1} \frac{2^k (m-n-2)(m-n-3) \dots (m-n-k-1)}{(2m-3)(2m-5) \dots (2m-2k-1)} \frac{a^{2k}}{x^{2m-2k-1}} \right] + \\ + \frac{2^n (m-n-2)(m-n-3) \dots (-n)(-n-1)}{a^{2m} (2m-1)!!} \int (a^2 - x^2)^{n+1/2} dx.$$

$$4. \int \frac{(a^2 - x^2)^{n+1/2}}{x} dx = \\ = \sum_{k=0}^n \frac{a^{2k}}{2n-2k+1} (a^2 - x^2)^{n-k+1/2} + \frac{a^{2n+1}}{2} \ln \left| \frac{a - (a^2 - x^2)^{1/2}}{a + (a^2 - x^2)^{1/2}} \right|.$$

$$5. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2n+2}} dx = \sum_{k=0}^n \frac{(-1)^{k+1}}{2n-2k+1} \left(\frac{a^2 - x^2}{x^2} \right)^{n-k+1/2} + (-1)^{n+1} \arcsin \frac{x}{a}.$$

$$6. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2n+4}} dx = -\frac{(a^2 - x^2)^{n+3/2}}{(2n+3)a^2 x^{2n+3}}.$$

$$7. \int \frac{(a^2 - x^2)^{1/2}}{x} dx = (a^2 - x^2)^{1/2} - a \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$8. \int \frac{(a^2 - x^2)^{1/2}}{x^2} dx = -\frac{(a^2 - x^2)^{1/2}}{x} - \arcsin \frac{x}{a}.$$

$$9. \int \frac{(a^2 - x^2)^{1/2}}{x^3} dx = -\frac{1}{2x^2} (a^2 - x^2)^{1/2} + \frac{1}{2a} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$10. \int \frac{(a^2 - x^2)^{3/2}}{x} dx = \frac{1}{3} (a^2 - x^2)^{3/2} + a^2 (a^2 - x^2)^{1/2} - a^3 \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$11. \int \frac{(a^2 - x^2)^{3/2}}{x^2} dx = -\frac{1}{x} (a^2 - x^2)^{3/2} - \frac{3x}{2} (a^2 - x^2)^{1/2} - \frac{3a^2}{2} \arcsin \frac{x}{a}.$$

$$12. \int \frac{(a^2 - x^2)^{3/2}}{x^3} dx = -\frac{(a^2 - x^2)^{3/2}}{2x^2} - \frac{3}{2} (a^2 - x^2)^{1/2} + \frac{3a}{2} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

1.2.48. Интегралы вида $\int \frac{x^m dx}{(a^2 - x^2)^{n+1/2}}$.

$$1. \int \frac{x^m dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{m-1}}{(2n-1)(a^2 - x^2)^{n-1/2}} - \frac{m-1}{2n-1} \int \frac{x^{m-2} dx}{(a^2 - x^2)^{n-1/2}}.$$

$$2. = \frac{x^{m-1}}{(2n-1)a^2(a^2 - x^2)^{n-1/2}} + \frac{2n-m-2}{(2n-1)a^2} \int \frac{x^m dx}{(a^2 - x^2)^{n-1/2}}.$$

$$3. = -\frac{x^{m-1}}{(m-2n)(a^2 - x^2)^{n-1/2}} + \frac{(m-1)a^2}{m-2n} \int \frac{x^{m-2} dx}{(a^2 - x^2)^{n+1/2}}.$$

$$4. \int \frac{x^{2m+1} dx}{(a^2 - x^2)^{n+1/2}} = \sum_{k=0}^m \frac{(-1)^k a^{2m-2k}}{2n-2k-1} \binom{m}{k} \frac{1}{(a^2 - x^2)^{n-k-1/2}}.$$

$$5. \int \frac{x^{2m} dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{2m+1} \sqrt{a^2 - x^2}}{(2n-1)a^2} \times \\ \times \left[\frac{1}{(a^2 - x^2)^n} + \sum_{k=1}^{n-1} \frac{2^k (n-m-1)(n-m-2) \dots (n-m-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k}}{(a^2 - x^2)^{n-k}} \right] + \\ + \frac{2^n (n-m-1)(n-m-2) \dots (-m+1)(-m)}{a^{2n} (2n-1)!!} \int \frac{x^{2m} dx}{\sqrt{a^2 - x^2}}.$$

$$6. \int \frac{x^{2m} dx}{(a^2 - x^2)^{n+1/2}} = \frac{1}{a^{2n-2m}} \sum_{k=0}^{n-m-1} \frac{1}{2m+2k+1} \binom{n-m-1}{k} \left(\frac{x^2}{a^2 - x^2} \right)^{m+k+1/2} \\ [n \geq m+1].$$

$$7. \int \frac{x^{2n-2} dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{2n-1}}{(2n-1)a^2(a^2 - x^2)^{n-1/2}}.$$

$$8. \int \frac{x^{2m} dx}{(a^2 - x^2)^{1/2}} = -\frac{(a^2 - x^2)^{1/2}}{2m} \times \\ \times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} a^{2k} x^{2m-2k-1} \right] + \\ + a^{2m} \frac{(2m-1)!!}{2^m m!} \arcsin \frac{x}{a}.$$

$$9. \int \frac{dx}{(a^2 - x^2)^{n+1/2}} = \frac{x(a^2 - x^2)^{1/2}}{2n-1} \times \\ \times \left[\frac{1}{(a^2 - x^2)^n a^2} + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k-2}}{(a^2 - x^2)^{n-k}} \right] \\ [n \geq 1].$$

$$10. = \frac{1}{a^{2n}} \sum_{k=0}^{n-1} \frac{1}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{a^2 - x^2} \right)^{k+1/2}.$$

$$11. \int \frac{x dx}{(a^2 - x^2)^{n+1/2}} = \frac{1}{(2n-1)(a^2 - x^2)^{n-1/2}}.$$

$$12. \int \frac{dx}{(a^2 - x^2)^{1/2}} = \arcsin \frac{x}{|a|}. \quad 13. \int \frac{x dx}{(a^2 - x^2)^{1/2}} = -(a^2 - x^2)^{1/2}.$$

$$14. \int \frac{x^2 dx}{(a^2 - x^2)^{1/2}} = -\frac{x}{2} (a^2 - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{|a|}.$$

$$15. \int \frac{x^3 dx}{(a^2 - x^2)^{1/2}} = \frac{1}{3} (a^3 - x^2)^{3/2} - a^2 (a^2 - x^2)^{1/2}.$$

$$16. \int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 (a^2 - x^2)^{1/2}}. \quad 17. \int \frac{x dx}{(a^2 - x^2)^{3/2}} = \frac{1}{(a^2 - x^2)^{1/2}}.$$

$$18. \int \frac{x^2 dx}{(a^2 - x^2)^{3/2}} = \frac{x}{(a^2 - x^2)^{1/2}} - \arcsin \frac{x}{|a|}.$$

$$19. \int \frac{x^3 dx}{(a^2 - x^2)^{3/2}} = \frac{2a^2 - x^2}{(a^2 - x^2)^{1/2}}.$$

1.2.49. Интегралы вида $\int \frac{dx}{x^m (a^2 - x^2)^{n+1/2}}.$

$$1. \int \frac{dx}{x^m (a^2 - x^2)^{n+1/2}} = \frac{1}{(2n-1) a^2 x^{m-1} (a^2 - x^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1) a^2} \int \frac{dx}{x^m (a^2 - x^2)^{n-1/2}}.$$

$$2. = \frac{-1}{(m-1) a^2 x^{m-1} (a^2 - x^2)^{n-1/2}} + \frac{m+2n-2}{(m-1) a^2} \int \frac{dx}{x^{m-2} (a^2 - x^2)^{n+1/2}}.$$

$$3. = \frac{1}{(2n-1) x^{m+1} (a^2 - x^2)^{n-1/2}} + \frac{m+1}{2n-1} \int \frac{dx}{x^{m+2} (a^2 - x^2)^{n-1/2}}.$$

$$4. \int \frac{dx}{x^{2m} (a^2 - x^2)^{n+1/2}} = \frac{-1}{a^{2n+2m}} \sum_{k=0}^{m+n-1} \frac{1}{2m-2k-1} \binom{m+n-1}{k} \left(\frac{a^2 - x^2}{x^2} \right)^{m-k-1/2}.$$

$$5. \int \frac{dx}{x (a^2 - x^2)^{n+1/2}} = \sum_{k=0}^{n-1} \frac{1}{(2n-2k-1) a^{2k+2} (a^2 - x^2)^{n-k-1/2}} - \frac{1}{a^{2n+1}} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$6. \int \frac{dx}{x (a^2 - x^2)^{1/2}} = -\frac{1}{a} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

- $$7. \int \frac{x}{x^2 (a^2 - x^2)^{1/2}} = -\frac{1}{a^2 x} (a^2 - x^2)^{1/2}.$$
- $$8. \int \frac{dx}{x^3 (a^2 - x^2)^{1/2}} = -\frac{1}{2a^2 x^2} (a^2 - x^2)^{1/2} - \frac{1}{2a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$
- $$9. \int \frac{dx}{x (a^2 - x^2)^{3/2}} = \frac{1}{a^2 (a^2 - x^2)^{1/2}} - \frac{1}{a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$
- $$10. \int \frac{dx}{x^2 (a^2 - x^2)^{3/2}} = \frac{2x^2 - a^2}{a^4 x (a^2 - x^2)^{1/2}}.$$
- $$11. \int \frac{dx}{x^3 (a^2 - x^2)^{3/2}} = -\frac{1}{2a^2 x^2 (a^2 - x^2)^{1/2}} +$$
- $$+ \frac{3}{2a^4 (a^2 - x^2)^{1/2}} - \frac{3}{2a^5} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

1.2.50. Интегралы вида $\int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}},$

$$\int \frac{dx}{(x^2 \pm b^2) (a^2 - x^2)^{n+1/2}}.$$

$$1. \int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}} = \frac{1}{(n-1)(b^2 - a^2)} \times$$

$$\times \left[\frac{\sqrt{a^2 - x^2}}{(x+b)^{n-1}} + (2n-3)b \int \frac{dx}{(x+b)^{n-1} \sqrt{a^2 - x^2}} - (n-2) \int \frac{dx}{(x+b)^{n-2} \sqrt{a^2 - x^2}} \right]$$

[$b \neq -a$].

$$2. = \frac{1}{(2n-1)a} \left[\frac{\sqrt{a^2 - x^2}}{(x-a)^n} - (n-1) \int \frac{dx}{(x-a)^{n-1} \sqrt{a^2 - x^2}} \right]$$

[$b = -a$].

$$3. \int \frac{dx}{(x+b) \sqrt{a^2 - x^2}} = \frac{-1}{\sqrt{a^2 - b^2}} \ln \left| \frac{bx + a^2 + \sqrt{(a^2 - b^2)(a^2 - x^2)}}{x+b} \right|$$

[$a^2 > b^2$].

$$4. = \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arccg} \frac{\sqrt{(b^2 - a^2)(a^2 - x^2)}}{a^2 + bx}$$

[$a^2 < b^2$].

$$5. = \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arcsin} \frac{a^2 + bx}{a|x+b|}$$

[$a^2 < b^2$].

$$6. \int \frac{dx}{(x+a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a-x}{a+x}}$$

[$a > 0$].

$$7. \int \frac{dx}{(x-a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a+x}{a-x}}$$

[$a > 0$].

$$8. \int \frac{-x}{(b^2 - x^2)^m (a^2 - x^2)^{n+1/2}} = \frac{x}{2(m-1)(a^2 - b^2)b^2 (b^2 - x^2)^{m-1} (a^2 - x^2)^{n-1/2}} +$$

$$+ \frac{(2m-3)a^2 - 2(m+2n-3)b^2}{2(m-1)(a^2 - b^2)b^2} \int \frac{dx}{(b^2 - x^2)^{m-1} (a^2 - x^2)^{n-1/2}} +$$

$$+ \frac{m+n-2}{(m-1)(a^2 - b^2)b^2} \int \frac{dx}{(b^2 - x^2)^{m-2} (a^2 - x^2)^{n+1/2}}.$$

9.
$$\int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2 - b^2)a^2(a^2 - x^2)^{n-1/2}} +$$

$$+ \frac{(4n-3)a^2 - 2(n-1)b^2}{(2n-1)(a^2 - b^2)a^2} \int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n-1/2}} -$$

$$- \frac{2(n-1)}{(2n-1)(a^2 - b^2)a^2} \int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n-3/2}}.$$
10.
$$\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 - b^2}} \ln \frac{|x\sqrt{a^2 - b^2} + b\sqrt{a^2 - x^2}|}{\sqrt{x^2 - b^2}} \quad [a^2 > b^2].$$
11.
$$= \frac{1}{b\sqrt{b^2 - a^2}} \operatorname{arctg} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}} \quad [a^2 < b^2].$$
12.
$$= \frac{1}{b\sqrt{b^2 - a^2}} \arcsin \frac{x\sqrt{b^2 - a^2}}{a\sqrt{b^2 - x^2}} \quad [a^2 < b^2].$$
13.
$$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \operatorname{arctg} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}.$$

1.2.51. Интегралы вида $\int x^{m-1} (ax^2 + bx + c)^{n+1/2} dx$.

Обозначение: $X = ax^2 + bx + c$.

1.
$$\int x^m X^{n+1/2} dx = \frac{x^{m-1} X^{n+3/2}}{(m+2n+2)a} -$$

$$- \frac{(2m+2n+1)b}{2(m+2n+2)a} \int x^{m-1} X^{n+1/2} dx - \frac{(m-1)c}{(m+2n+2)a} \int x^{m-2} X^{n+1/2} dx.$$
2.
$$\int x X^{n+1/2} dx = \frac{X^{n+3/2}}{(2n+3)a} - \frac{b}{2a} \int X^{n+1/2} dx.$$
3.
$$\int X^{n+1/2} dx = \frac{2ax+b}{4(n+1)a} X^{n+1/2} + \frac{(2n+1)(4ac-b^2)}{8(n+1)a} \int X^{n-1/2} dx.$$
4.
$$= \frac{(2ax+b)X^{1/2}}{4(n+1)a} \times$$

$$\times \left[X^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)} \left(\frac{4ac-b^2}{a}\right)^{k+1} X^{n-k-1} \right] +$$

$$+ \frac{(2n+1)!!}{8^{n+1}(n+1)!} \left(\frac{4ac-b^2}{a}\right)^{n+1} \int \frac{dx}{X^{1/2}}.$$
5.
$$\int x^m X^{1/2} dx = \frac{x^{m-1} X^{3/2}}{(m+2)a} - \frac{(2m+1)b}{2(m+2)a} \int x^{m-1} X^{1/2} dx -$$

$$- \frac{(m-1)c}{(m+2)a} \int x^{m-2} X^{1/2} dx.$$
6.
$$\int X^{1/2} dx = \frac{2ax+b}{4a} X^{1/2} + \frac{4ac-b^2}{8a^{3/2}} \ln \left| \frac{2ax+b}{2\sqrt{a}} + X^{1/2} \right| \quad [a > 0].$$
7.
$$= \frac{2ax+b}{4a} X^{1/2} + \frac{b^2-4ac}{8a\sqrt{|a|}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \quad [a < 0; b^2 > 4ac].$$

$$8. \int xX^{1/2} dx = \frac{1}{2a} X^{3/2} - \frac{b}{2a} \int X^{1/2} dx.$$

$$9. \int x^2 X^{1/2} dx = \frac{6ax - 5b}{24a^2} X^{3/2} + \frac{5b^2 - 4ac}{16a^2} \int X^{1/2} dx.$$

$$10. \int x^3 X^{1/2} dx = \frac{48a^2x^2 - 42abx + 35b^2 - 32ac}{240a^2} X^{3/2} - \frac{7b^3 - 12abc}{32a^3} \int X^{1/2} dx.$$

$$11. \int \frac{X^{n+1/2}}{x^m} dx = -\frac{1}{(m-1)cx^{m-1}} X^{n+3/2} + \\ + \frac{(2n-2m+5)b}{2(m-1)c} \int \frac{X^{n+1/2}}{x^{m-1}} dx + \frac{(2n-m+4)a}{(m-1)c} \int \frac{X^{n+1/2}}{x^{m-2}} dx.$$

$$12. \int \frac{X^{n+1/2}}{x} dx = \frac{X^{n+1/2}}{2n+1} + \frac{b}{2} \int X^{n-1/2} dx + c \int \frac{X^{n-1/2}}{x} dx.$$

$$13. \int \frac{(ax^2+bx)^{n+1/2}}{x^m} dx = \frac{2}{(2n-2m+3)bx^m} (ax^2+bx)^{n+3/2} + \\ + \frac{2(m-2n-3)a}{(2n-2m+3)b} \int \frac{(ax^2+bx)^{n+1/2}}{x^{m-1}} dx.$$

$$14. \int \frac{X^{1/2}}{x} dx = X^{1/2} + \frac{b}{2} \int \frac{dx}{X^{1/2}} + c \int \frac{dx}{xX^{1/2}}.$$

$$15. \int \frac{X^{1/2}}{x^2} dx = -\frac{X^{1/2}}{x} + a \int \frac{dx}{X^{1/2}} + \frac{b}{2} \int \frac{dx}{xX^{1/2}}.$$

$$16. \int \frac{(ax^2+bx)^{1/2}}{x^2} dx = -\frac{2(ax^2+bx)^{1/2}}{x} + a \int \frac{dx}{(ax^2+bx)^{1/2}}.$$

$$17. \int \frac{X^{1/2}}{x^3} dx = -\left(\frac{1}{2x^2} + \frac{b}{4cx}\right) X^{1/2} + \left(\frac{a}{2} + \frac{b^2}{8c}\right) \int \frac{dx}{xX^{1/2}}.$$

$$18. \int \frac{(ax^2+bx)^{1/2}}{x^3} dx = -\frac{2}{3bx^3} (ax^2+bx)^{3/2}.$$

$$19. \int \frac{X^{3/2}}{x} dx = \frac{X^{3/2}}{3} + \frac{2abx + b^2 + 8ac}{8a} X^{1/2} + \\ + \frac{(12ac - b^2)b}{16a} \int \frac{dx}{X^{1/2}} + c^2 \int \frac{dx}{xX^{1/2}}.$$

$$20. \int \frac{X^{3/2}}{x^2} dx = -\frac{X^{5/2}}{cx} + \frac{ax+b}{c} X^{3/2} + \frac{3(2ax+3b)}{4} X^{1/2} + \\ + \frac{3(4ac+b^2)}{8} \int \frac{dx}{X^{1/2}} + \frac{3bc}{2} \int \frac{dx}{xX^{1/2}}.$$

$$21. \int \frac{(ax^2+bx)^{3/2}}{x^2} dx = \frac{(ax^2+bx)^{3/2}}{2x} + \frac{3b}{4} (ax^2+bx)^{1/2} + \frac{3b^2}{8} \int \frac{dx}{(ax^2+bx)^{1/2}}.$$

$$22. \int \frac{X^{3/2}}{x^3} dx = -\left(\frac{1}{2cx^2} + \frac{b}{4c^2x}\right) X^{5/2} + \\ + \frac{abx + 2ac + b^2}{4c^2} X^{3/2} + \frac{3(abx + 2ac + b^2)}{4c} X^{1/2} + \\ + \frac{3ab}{2} \int \frac{dx}{X^{1/2}} + \frac{3(4ac + b^2)}{8} \int \frac{dx}{xX^{1/2}}.$$

$$23. \int \frac{(ax^2 + bx)^{3/2}}{x^3} dx = \left(a - \frac{2b}{x}\right) (ax^2 + bx)^{1/2} + \frac{3ab}{2} \int \frac{dx}{(ax^2 + bx)^{1/2}}.$$

1.2.52. Интегралы вида $\int \frac{x^{m-1} dx}{(ax^2 + bx + c)^{n+1/2}}.$

Обозначение $X = ax^2 + bx + c.$

$$1. \int \frac{x^m dx}{X^{n+1/2}} = \frac{x^{m-1}}{(m-2n)aX^{n-1/2}} - \frac{(2m-2n-1)b}{2(m-2n)a} \int \frac{x^{m-1} dx}{X^{n-1/2}} - \\ - \frac{(m-1)c}{(m-2n)a} \int \frac{x^{m-2} dx}{X^{n+1/2}}.$$

$$2. = \frac{2x^m}{(2n-1)bX^{n-1/2}} + \frac{2(2n-m-1)}{(2n-1)b} \int \frac{x^{m-1} dx}{X^{n-1/2}} - \frac{2c}{b} \int \frac{x^{n-1} dx}{X^{n+1/2}}.$$

$$3. = \frac{1}{a} \int \frac{x^{m-2} dx}{X^{n-1/2}} - \frac{c}{a} \int \frac{x^{n-2} dx}{X^{n+1/2}} - \frac{b}{a} \int \frac{x^{n-1} dx}{X^{n+1/2}}.$$

$$4. \int \frac{x^{2n} dx}{X^{n-1/2}} = -\frac{x^{2n-1}}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{x^{2n-1} dx}{X^{n+1/2}} + \frac{1}{a} \int \frac{x^{2n-2} dx}{X^{n-1/2}}.$$

$$5. \int \frac{x dx}{X^{n+1/2}} = -\frac{1}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{dx}{X^{n+1/2}}.$$

$$6. \int \frac{dx}{X^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} + \frac{8(n-1)a}{(2n-1)(4ac-b^2)} \int \frac{dx}{X^{n-1/2}}.$$

$$7. = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} \times \\ \times \left[1 + \sum_{k=1}^{n-1} \frac{8^k (n-1)(n-2)}{(2n-3)(2n-5)} \frac{(n-k)}{(2n-2k-1)} \cdot \frac{a^k X^k}{(4ac-b^2)^k} \right].$$

$$8. \int \frac{dx}{X^{1/2}} = \frac{1}{\sqrt{a}} \ln \left| \frac{2ax+b}{2\sqrt{a}} + X \right| \quad [a > 0].$$

$$9. = \frac{1}{\sqrt{a}} \operatorname{Arsh} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad [a > 0; b^2 < 4ac].$$

$$10. = \frac{-1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \quad [a < 0; b^2 > 4ac].$$

$$11. = \frac{1}{\sqrt{a}} \ln(2ax+b) \quad [a > 0; b^2 = 4ac].$$

$$12. \int \frac{x dx}{X^{1/2}} = \frac{X^{1/2}}{a} - \frac{b}{2a} \int \frac{dx}{X^{1/2}}.$$

$$13. \int \frac{x^2 dx}{X^{1/2}} = \frac{2ax - 3b}{4a^2} X^{1/2} + \frac{3b^2 - 4ac}{8a^2} \int \frac{dx}{X^{1/2}}.$$

$$14. \int \frac{x^3 dx}{X^{1/2}} = \frac{8a^2x^2 - 10abx + 15b^2 - 16ac}{24a^3} X^{1/2} - \frac{5b^3 - 12abc}{16a^3} \int \frac{dx}{X^{1/2}}.$$

$$15. \int \frac{dx}{X^{3/2}} = \frac{2(2ax + b)}{(4ac - b^2) X^{1/2}}.$$

$$16. \int \frac{x dx}{X^{3/2}} = \frac{2(bx + 2c)}{(b^2 - 4ac) X^{1/2}}.$$

$$17. \int \frac{x^2 dx}{X^{3/2}} = \frac{(4ac - 2b^2)x - 2bc}{a(b^2 - 4ac) X^{1/2}} + \frac{1}{a} \int \frac{dx}{X^{1/2}}.$$

$$18. \int \frac{x^3 dx}{X^{3/2}} = \frac{a(4ac - b^2)x^2 + b(10ac - 3b^2)x + c(8ac - 3b^2)}{a^2(4ac - b^2) X^{1/2}} - \frac{3b}{2a^2} \int \frac{dx}{X^{1/2}}.$$

$$19. \int \frac{dx}{x^m X^{n+1/2}} = -\frac{1}{(m-1)cx^{m-1}X^{n-1/2}} - \frac{(2m+2n-3)b}{2(m-1)c} \int \frac{dx}{x^{m-1}X^{n+1/2}} - \frac{(2n+m-2)a}{(m-1)c} \int \frac{dx}{x^{m-2}X^{n-1/2}}.$$

$$20. \int \frac{dx}{xX^{n+1/2}} = \frac{1}{(2n-1)cX^{n-1/2}} - \frac{b}{2c} \int \frac{dx}{X^{n+1/2}} + \frac{1}{c} \int \frac{dx}{xX^{n-1/2}}.$$

$$21. \int \frac{dx}{x^m (ax^2 + bx)^{n+1/2}} = -\frac{2}{(2m+2n-1)bx^m (ax^2 + bx)^{n-1/2}} - \frac{2(m+2n-1)a}{(2m+2n-1)b} \int \frac{dx}{x^{m-1} (ax^2 + bx)^{n+1/2}}.$$

$$22. \int \frac{dx}{xX^{1/2}} = -\frac{1}{\sqrt{c}} \ln \left| \frac{2c + bx + 2\sqrt{c} X^{1/2}}{x} \right| \quad [c > 0].$$

$$23. = -\frac{1}{\sqrt{c}} \operatorname{Arth} \frac{2c + bx}{2\sqrt{c} X^{1/2}} \quad [c > 0].$$

$$24. = -\frac{1}{\sqrt{c}} \operatorname{Arsh} \frac{2c + bx}{x\sqrt{4ac - b^2}} \quad [c > 0, b^2 < 4ac].$$

$$25. = \frac{1}{\sqrt{c}} \ln \left| \frac{x}{2c + bx} \right| \quad [c > 0, b^2 = 4ac]$$

$$26. = \frac{1}{\sqrt{-c}} \operatorname{arctg} \frac{2c + bx}{2\sqrt{-c} X^{1/2}} \quad [c < 0].$$

$$27. = \frac{1}{\sqrt{-c}} \operatorname{arcsin} \frac{2c + bx}{x\sqrt{b^2 - 4ac}} \quad [c < 0, b^2 > 4ac]$$

$$28. = -\frac{2\sqrt{bx + ax^2}}{bx} \quad [c = 0, b \neq 0].$$

$$29. \int \frac{dx}{x^2 X^{1/2}} = -\frac{X^{1/2}}{cx} - \frac{b}{2c} \int \frac{d\lambda}{x X^{1/2}}.$$

$$30. \int \frac{dx}{x^2 (ax^2 + bx)^{1/2}} = \frac{2}{3} \left(-\frac{1}{bx^2} + \frac{2a}{b^2 x} \right) (ax^2 + bx)^{1/2}.$$

$$31. \int \frac{dx}{x^3 X^{1/2}} = \left(-\frac{1}{2cx^2} + \frac{3b}{4c^2 x} \right) X^{1/2} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c} \right) \int \frac{d\lambda}{\lambda X^{1/2}}.$$

$$32. \int \frac{dx}{x^3 (ax^2 + bx)^{1/2}} = \frac{2}{5} \left(-\frac{1}{bx^3} + \frac{4a}{3b^2 x^2} - \frac{8a^2}{3b^3 x} \right) (ax^2 + bx)^{1/2}.$$

$$33. \int \frac{dx}{x X^{3/2}} = \frac{2(abx - 2ac + b^2)}{c(b^2 - 4ac) X^{1/2}} + \frac{1}{c} \int \frac{d\lambda}{x X^{1/2}}.$$

$$34. \int \frac{dx}{x (ax^2 + bx)^{3/2}} = \frac{2}{3} \left(-\frac{1}{bx} + \frac{4a}{b^2} - \frac{8a^2 x}{b^3} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$$

$$35. \int \frac{d\lambda}{x^2 \lambda^{3/2}} = \left[-\frac{1}{cx} + \frac{2ab}{c(4ac - b^2)} + \frac{a(3b^2 - 3ac)\lambda}{c^2(4ac - b^2)} \right] \frac{1}{X^{1/2}} - \frac{3b}{2c^2} \int \frac{d\lambda}{x X^{1/2}}.$$

$$36. \int \frac{dx}{x^2 (ax^2 + bx)^{3/2}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{2a}{b^2 x} - \frac{8a^2}{b^3} - \frac{16a^3 x}{b^4} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$$

$$37. \int \frac{dx}{x^3 X^{3/2}} = \left[-\frac{1}{cx^2} + \frac{5b}{2c^2 x} + \frac{15b^4 - (2ab^2 + 24a^2 c^2)}{2c^3(b^2 - 4ac)} + \right. \\ \left. + \frac{ab(15b^2 - 52ac)x}{2c^3(b^2 - 4ac)} \right] \frac{1}{2X^{1/2}} + \frac{15b^2 - 12ac}{8c^3} \int \frac{dx}{x X^{1/2}}.$$

$$38. \int \frac{dx}{x^3 (ax^2 + bx)^{3/2}} = \frac{2}{7} \left(-\frac{1}{bx^3} + \frac{8a}{5b^2 x^2} - \frac{16a^2}{5b^3 x} + \frac{64a^3}{5b^4} + \right. \\ \left. + \frac{128a^4 x}{5b^5} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$$

1.2.53. Интегралы вида $\int R(x+p, ax^2+bx+c) dx$.

Обозначение $X = ax^2 + bx + c$.

$$1. \int \frac{dx}{(x+p)^m X^{n+1/2}} = \frac{-1}{(m-1)(ap^2 - bp + c)(x+p)^{m-1} X^{n-1/2}} - \\ - \frac{(2m+2n-3)(b-2ap)}{2(m-1)(ap^2 - bp + c)} \int \frac{dx}{(x+p)^{m-1} X^{n+1/2}} - \\ - \frac{(m+2n-2)a}{(m-1)(ap^2 - bp + c)} \int \frac{dx}{(x+p)^{m-2} X^{n+1/2}}.$$

$$2. = \frac{-2}{(2m+2n-1)(b-2ap)(x+p)^m X^{n-1/2}} - \\ - \frac{2(m+2n-1)a}{(2m+2n-1)(b-2ap)} \int \frac{dx}{(x+p)^{m-1} X^{n+1/2}} \\ [ap^2 - bp + c = 0; b - 2ap \neq 0].$$

3.
$$\int \frac{dx}{(x+p) X^{n-1/2}} = \frac{1}{(2n-1)(ap^2-bp+c) X^{n-1/2}} +$$

$$+ \frac{1}{ap^2-bp+c} \int \frac{dx}{(x+p) X^{n-1/2}} + \frac{2ap-b}{2(ap^2-bp+c)} \int \frac{dx}{X^{n+1/2}}.$$
4.
$$= \frac{-1}{(2n+1) X^{n+1/2}} + \frac{b-2ap}{2} \int \frac{dx}{X^{n+3/2}} \quad [ap^2-bp+c=0].$$
5.
$$\int \frac{X^{n-1/2}}{(x+p)^m} dx = \frac{-X^{n+3/2}}{(m-1)(ap^2-bp+c)(x+p)^{m-1}} +$$

$$+ \frac{(2n-2m+5)(b-2ap)}{2(m-1)(ap^2-bp+c)} \int \frac{X^{n+1/2}}{(x+p)^{m-1}} dx + \frac{(2n-m+4)a}{(m-1)(ap^2-bp+c)} \int \frac{X^{n+1/2}}{(x+p)^{m-2}}.$$
6.
$$= \frac{2X^{n+3/2}}{(2n-2m+3)(b-2ap)(x+p)^m} -$$

$$- \frac{2(2n-m+3)a}{(2n-2m+3)(b-2ap)} \int \frac{X^{n+1/2}}{(x+p)^{m-1}} dx$$

$$[ap^2-bp+c=0, b-2ap \neq 0].$$
7.
$$\int \frac{X^{n+1/2}}{x+p} dx = \frac{X^{n+1/2}}{2n+1} + \frac{b-2ap}{2} \int X^{n-1/2} dx + (ap^2-bp+c) \int \frac{X^{n-1/2}}{x+p} dx.$$
8.
$$\int \frac{X^{1/2} dx}{(x+p)^n} = -\frac{X^{1/2}}{(n-1)(x+p)^{n-1}} + \frac{1}{2(n-1)} \int \frac{(b+2ax) dx}{(x+p)^{n-1} X^{1/2}}.$$
9.
$$\int \frac{X^{1/2} dx}{x+p} = \int \frac{ax+b-ap}{X^{1/2}} dx + (ap^2-bp+c) \int \frac{dx}{(x+p) X^{1/2}}.$$
10.
$$\int \frac{dx}{(x+p)^n X^{1/2}} = \frac{-X^{1/2}}{(n-1)(ap^2-bp+c)(x+p)^{n-1}} -$$

$$- \frac{(2n-3)(b-2ap)}{2(n-1)(ap^2-bp+c)} \int \frac{dx}{(x+p)^{n-1} X^{1/2}} -$$

$$- \frac{(n-2)a}{(n-1)(ap^2-bp+c)} \int \frac{dx}{(x+p)^{n-2} X^{1/2}}.$$
11.
$$= \frac{2X^{1/2}}{(2n-1)(2ap-b)(x+p)^n} + \frac{2(n-1)a}{(2n-1)(2ap-b)} \int \frac{dx}{(x+p)^{n-1} X^{1/2}}$$

$$[ap^2-bp+c=0, 2p-b \neq 0]$$
12.
$$\int \frac{dx}{(x+p)^n X^{1/2}} = - \int \frac{t^{n-1} dt}{[a+(b-2ap)t+(ap^2-bp+c)t^2]^{1/2}} \quad \left[t = \frac{1}{x+p} \right].$$
13.
$$\int \frac{dx}{(x+p) X^{1/2}} = - \frac{1}{(ap^2-bp+c)^{1/2}} \ln \left| \frac{(ap^2-bp+c)^{1/2} + X^{1/2}}{x+p} + \right.$$

$$\left. + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}} \right| \quad [ap^2-bp+c > 0].$$
14.
$$= \frac{1}{2(ap^2-bp+c)^{1/2}} \ln \left| \frac{(ap^2-bp+c)^{1/2} - X^{1/2}}{x+p} + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}} \right|$$

$$[ap^2-bp+c > 0].$$
15.
$$= \frac{1}{(bp-ap^2-c)^{1/2}} \arcsin \frac{(b-2ap)x-bp+2c}{(x+p)(b^2-4ac)^{1/2}}$$

$$[ap^2-bp+c < 0; b^2 > 4ac].$$

$$16. \quad = -\frac{2X^{1/2}}{(b-2ap)(x+p)} \quad [ap^2 - bp + c = 0].$$

$$17. \quad \int \frac{dx}{(x^2 + px + q) X^{1/2}} = \frac{1}{(p^2 - 4q)^{1/2}} \left[\int \frac{dx}{\left(x + \frac{p - \sqrt{p^2 - 4q}}{2}\right) X^{1/2}} - \int \frac{dx}{\left(x + \frac{p + \sqrt{p^2 - 4q}}{2}\right) X^{1/2}} \right] \quad [p^2 > 4q].$$

$$18. \quad \int \frac{dx}{(x^2 + p^2) X^{1/2}} = \int \frac{(\alpha - \beta)(t+1) dt}{[(\alpha^2 + p^2)t^2 + (\beta^2 + p^2)][(a\alpha^2 + b\alpha + c)t^2 + \alpha\beta^2 + b\beta + c]^{1/2}} \\ \left[t = \frac{x - \beta}{\alpha - x}; \alpha \text{ и } \beta \text{ определяются из системы уравнений} \right. \\ \left. b(\alpha + \beta) + 2c - 2ap^2 = 0, \alpha\beta + p^2 = 0 \right].$$

$$19. \quad \int \frac{(\alpha v + \beta) dx}{(p + X)^n X^{1/2}} = \frac{\alpha}{a} \int \frac{du}{(p + u^2)^n} + \frac{2\beta a - \alpha b}{2a} \int \frac{(1 - av^2)^{n-1} dv}{(p + c - b^2/4a - av^2)^n} \\ \left[u = X^{1/2}; v = \frac{b + 2\alpha v}{2aX^{1/2}} \right].$$

1.2.54. Интегралы вида $\int f(x, \sqrt{x^2 - x + 1}) dx$.

Условие: $x > 1$.

Обозначение: $\varphi = \arcsin \frac{1}{\sqrt{x^2 - x + 1}}$.

$$1. \quad \int_x^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)}} = F\left(\varphi, \frac{\sqrt{3}}{2}\right) \quad [x \geq 1].$$

$$2. \quad \int_x^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2x-1)}{\sqrt{x(x-1)(x^2-x+1)}} - 4E\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$3. \quad \int_x^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3} \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) \right] \quad [x \geq 1].$$

$$4. \quad \int_x^\infty \frac{dx}{(2x-1)^2 \sqrt{x(x-1)(x^2-x+1)}} = \\ = \frac{4}{3} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{2}{2x-1} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$5. \quad \int_x^\infty \frac{dx}{(2x-1)^4 \sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \\ - \frac{1}{27} F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{8(5x^2-5x+2)}{9(2x-1)^3} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$6. \int_x^{\infty} \frac{(2x-1)^2 dx}{V x^3 (x-1)^3 (x^2-x+1)} = 4 \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) + \frac{2x-1}{2V x(x-1)(x^2-x+1)} \right]$$

$$7. \int_x^{\infty} \frac{(2x-1)^2 dx}{V x(x-1)(x^2-x+1)^3} = 4E\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$8. \int_x^{\infty} V \frac{x(x-1)}{(x^2-x+1)^3} dx = \frac{4}{3} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$9. \int \frac{1}{(2x-1)^2} V \frac{x(x-1)}{x^2-x+1} dx = \frac{1}{3} \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) \right] + \frac{1}{2(2x-1)} V \frac{x(x-1)}{x^2-x+1}.$$

$$10. \int_x^{\infty} \frac{1}{(2x-1)^2} V \frac{x^2-x+1}{x(x-1)} dx = E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2x-1)} V \frac{x(x-1)}{x^2-x+1}.$$

$$11. \int_0^x \frac{1}{[1+(\sqrt{3}+1)x]^2} V \frac{x^2-x+1}{x(x+1)} dx = \frac{1}{\sqrt{3}} E\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) \quad [x > 0].$$

$$12. \int_0^x V \frac{x(x+1)}{(x^2-x+1)^3} dx = \frac{1}{\sqrt[4]{27}} E\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) + \frac{2-\sqrt{3}}{\sqrt[4]{27}} F\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) - \frac{2(2+\sqrt{3})}{\sqrt{3}} \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x} V \frac{x(1+x)}{x^2-x+1} \quad [x > 0].$$

1.2.55. Интегралы вида $\int_0^x R(x, V x^2+a^2, V x^2+b^2) dx$.

• Условия: $x > 0, a > b > 0$.

Обозначения: $\varphi = \operatorname{arctg} \frac{x}{b}, k = \sqrt{\frac{a^2-b^2}{a}}$.

$$1. \int_0^x \frac{dx}{V(x^2+a^2)(x^2+b^2)} = \frac{1}{a} F(\varphi, k).$$

$$2. \int_0^x \frac{x^2 dx}{V(x^2+a^2)(x^2+b^2)} = x V \frac{x^2+a^2}{x^2+b^2} - a E(\varphi, k).$$

$$3. \int_0^x \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2+b^2)}} =$$

$$= \frac{1}{a(p+b^2)} \left[\frac{b^2}{p} \Pi\left(\varphi, \frac{p+b^2}{p}, k\right) + F(\varphi, k) \right] \quad [p \neq 0].$$

$$4. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} =$$

$$= \frac{1}{ab^2(a^2-b^2)^2} [(a^2+b^2)E(\varphi, k) - 2b^2F(\varphi, k)] - \frac{x}{a^2(a^2-b^2)\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$5. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} =$$

$$= \frac{1}{a(a^2-b^2)} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{a^2\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$6. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)} [a^2E(\varphi, k) - b^2F(\varphi, k)].$$

$$7. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)} [a^2E(\varphi, k) - b^2F(\varphi, k)] -$$

$$- \frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$8. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2} [F(\varphi, k) - E(\varphi, k)].$$

$$9. \int_0^x \sqrt{\frac{x^2+a^2}{x^2+b^2}} dx = a[F(\varphi, k) - E(\varphi, k)] + x \sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

$$10. \int_0^x \sqrt{\frac{x^2+b^2}{x^2+a^2}} dx = \frac{b^2}{a} F(\varphi, k) - aE(\varphi, k) + x \sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

$$11. \int_0^x \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}} dx = \frac{a}{b^2} E(\varphi, k).$$

$$12. \int_0^x \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}} dx = \frac{1}{a} E(\varphi, k) - \frac{a^2-b^2}{a^2} \frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$13. \int_0^x \sqrt{(x^2+a^2)(x^2+b^2)} dx = \frac{a}{3} [2b^2F(\varphi, k) - (a^2+b^2)E(\varphi, k)] +$$

$$+ \frac{x}{3} (x^2+a^2+2b^2) \sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

$$14. \int \frac{dx}{\sqrt{x(x^2+1)(x^2+a^2)}} =$$

$$= \frac{1}{2\sqrt{a(a+1)}} \left[F \left(\arccos \frac{(x+\sqrt{a})^2 - (a+1)x}{(x+\sqrt{a})^2 + (a+1)x}, \frac{(\sqrt{a}+1)^2}{2(a+1)} \right) \pm \right.$$

$$\left. \pm F \left(\arccos \frac{(x-\sqrt{a})^2 - (a+1)x}{(x-\sqrt{a})^2 + (a+1)x}, \frac{(\sqrt{a}-1)^2}{2(a+1)} \right) \right] \quad \left[x, a > 0; \begin{cases} 0 \leq x \leq \sqrt{a} \\ \sqrt{a} \leq x < \infty \end{cases} \right].$$

$$15. \int \frac{\sqrt{x} dx}{\sqrt{(x^2+1)(x^2+a^2)}} =$$

$$= \frac{-1}{2\sqrt{a+1}} \left[F \left(\arccos \frac{(x+\sqrt{a})^2 - (a+1)x}{(x+\sqrt{a})^2 + (a+1)x}, \frac{(\sqrt{a}+1)^2}{2(a+1)} \right) \mp \right.$$

$$\mp F \left(\arccos \frac{(x-\sqrt{a})^2 - (a+1)x}{(x-\sqrt{a})^2 + (a+1)x}, \frac{(\sqrt{a}-1)^2}{2(a+1)} \right) \right] \quad \left[x, a > 0; \begin{cases} 0 \leq x \leq \sqrt{a} \\ \sqrt{a} \leq x < \infty \end{cases} \right].$$

$$16. \int_0^x \frac{dx}{\sqrt{x[(x+a)^2+b^2]}} = \frac{1}{\sqrt{c}} F \left(2 \operatorname{arctg} \sqrt{\frac{x}{c}}, \sqrt{\frac{c-a}{2c}} \right)$$

$$[x > 0; c = \sqrt{a^2+b^2}].$$

$$17. \int_0^x \frac{dx}{\sqrt{x(x+a)[(x+b)^2+c^2]}} =$$

$$= \frac{1}{\sqrt{pq}} F \left(2 \operatorname{arctg} \sqrt{\frac{qx}{p(x+a)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2+a^2}{pq}} \right)$$

$$[x > 0, p^2 = b^2+c^2; q^2 = (b+a)^2+c^2].$$

$$18. \int_0^x \frac{dx}{\sqrt{x(a-x)[(x+b)^2+c^2]}} =$$

$$= \frac{1}{\sqrt{pq}} F \left(2 \operatorname{arccotg} \sqrt{\frac{q(a-x)}{px}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2+a^2}{pq}} \right)$$

$$[a > x > 0, p = (b-a)^2+c^2; q^2 = b^2+c^2].$$

1.2.56. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2+a^2}, \sqrt{x^2+b^2}) dx$.

Условия $x \geq 0, c > b > 0$.

Сбозначения: $\varphi = \operatorname{arccotg} \frac{x}{a}, k = \sqrt{\frac{a^2-b^2}{a}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{x^2 \sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{b^2 x} \sqrt{\frac{x^2+b^2}{x^2+a^2}} - \frac{1}{ab^2} E(\varphi, k) \quad [x > 0].$$

$$3. \int_x^{\infty} \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)} \left[\Pi \left(\varphi, \frac{a^2+p}{a^2}, k \right) - F(\varphi, 1) \right].$$

$$4. \int_x^{\infty} \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \\ = \frac{1}{ab^2(a^2-b^2)} [a^2 E(\varphi, k) - b^2 F(\varphi, k)] - \frac{x}{b^2 \sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$6. \int_x^{\infty} \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} = \\ = \frac{1}{ab^2(a^2-b^2)^2} [(a^2+b^2) E(\varphi, k) - 2b^2 F(\varphi, k)] - \frac{x}{b^2(a^2-b^2) \sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$7. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)} [a^2 E(\varphi, k) - b^2 F(\varphi, k)].$$

$$8. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$9. \int_x^{\infty} \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}} dx = \frac{a}{b^2} E(\varphi, k) - \frac{(a^2-b^2)x}{b^2 \sqrt{(x^2+a^2)(x^2+b^2)}}.$$

$$10. \int_x^{\infty} \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}} dx = \frac{1}{a} E(\varphi, k).$$

$$11. \int_x^{\infty} \sqrt{\frac{x^2+a^2}{x^2+b^2}} \frac{dx}{x} = \frac{1}{a} F(\varphi, k) - \frac{a}{b^2} E(\varphi, k) + \frac{a^2}{b^2 x} \sqrt{\frac{x^2+b^2}{x^2+a^2}} \quad [x > 0].$$

$$12. \int_x^{\infty} \sqrt{\frac{x^2+b^2}{x^2+a^2}} \frac{dx}{x^2} = \frac{1}{a} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{\frac{x^2+b^2}{x^2+a^2}} \quad [x > 0].$$

1 2 57. Интегралы вида $\int_b^x R(x, \sqrt{x^2+a^2}, \sqrt{x^2-b^2}) dx$.

Условие $x > b > 0$.

Обозначения $\varphi = \arccos \frac{b}{x}$, $k = \frac{a}{\sqrt{a^2+b^2}}$

$$1. \int_b^x \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k).$$

$$2. \int_b^x \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} =$$

$$= \frac{b^2}{\sqrt{a^2 + b^2}} F(\varphi, k) - \sqrt{a^2 + b^2} E(\varphi, k) + \frac{1}{x} \sqrt{(x^2 + a^2)(x^2 - b^2)}.$$

$$3. \int_b^x \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} [(a^2 + b^2) E(\varphi, k) - b^2 F(\varphi, k)].$$

$$4. \int_b^x \frac{dx}{(p - x^2) \sqrt{(x^2 + a^2)(x^2 - b^2)}} =$$

$$= \frac{1}{p(p - b^2) \sqrt{a^2 + b^2}} \left[b^2 \Pi\left(\varphi, \frac{p}{p - b^2}, k\right) + (p - b^2) F(\varphi, k) \right] \quad [p \neq b^2]$$

$$5. \int_b^x \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)}} =$$

$$= \frac{1}{a^2 \sqrt{a^2 + b^2}} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{(a^2 + b^2)x} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$6. \int_b^x \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\varphi, k) - \frac{a^2}{x(a^2 + b^2)} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$7. \int_b^x \sqrt{\frac{x^2 + a^2}{x^2 - b^2}} dx = \sqrt{a^2 + b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{(x^2 + a^2)(x^2 - b^2)}.$$

$$8. \int_b^x \frac{1}{x^2} \sqrt{\frac{x^2 + a^2}{x^2 - b^2}} dx = \frac{\sqrt{a^2 + b^2}}{b^2} E(\varphi, k).$$

$$9. \int_b^x \sqrt{\frac{x^2 - b^2}{x^2 + a^2}} dx = \frac{1}{x} \sqrt{(x^2 + a^2)(x^2 - b^2)} - \sqrt{a^2 + b^2} E(\varphi, k).$$

$$10. \int_b^x \frac{1}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} [F(\varphi, k) - E(\varphi, k)].$$

$$11. \int_b^x \sqrt{\frac{x^2 - b^2}{(x^2 + a^2)^3}} dx =$$

$$= \frac{\sqrt{a^2 + b^2}}{a^2} E(\varphi, k) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\varphi, k) - \frac{1}{x} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$12. \int_b^x \sqrt{(x^2 + a^2)(x^2 - b^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} [(b^2 - a^2) E(\varphi, k) - b^2 F(\varphi, k)] + \\ + \frac{x^2 + a^2 - b^2}{3x} \sqrt{(x^2 + a^2)(x^2 - b^2)}.$$

1.2.58. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2 + a^2}, \sqrt{x^2 - b^2}) dx$.

Условие. $x - b > 0$.

Обозначения. $\varphi = \arcsin \sqrt{\frac{a^2 + b^2}{x^2 + a^2}}$, $k = \frac{a}{\sqrt{a^2 + b^2}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \\ = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} [(a^2 + b^2) E(\varphi, k) - b^2 F(\varphi, k)] - \frac{1}{b^2 x} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$3. \int_x^\infty \frac{dx}{(x^2 - p) \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \\ = \frac{1}{(a^2 + p) \sqrt{a^2 + b^2}} \left[\Pi\left(\varphi, \frac{a^2 + p}{a^2 + b^2}, k\right) - F(\varphi, k) \right].$$

$$4. \int_x^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\varphi, k).$$

$$6. \int_x^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)^3}} = \frac{x}{b^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} - \frac{1}{b^2 \sqrt{a^2 + b^2}} E(\varphi, k).$$

$$7. \int_x^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)^3}} = \frac{1}{\sqrt{a^2 + b^2}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{x}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} \quad [x > b].$$

$$8. \int_x^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)^3}} = \frac{b^2 - a^2}{a^2 b^2 \sqrt{(a^2 + b^2)^3}} E(\varphi, k) - \\ - \frac{1}{a^2 \sqrt{(a^2 + b^2)^3}} F(\varphi, k) + \frac{x}{b^2 (a^2 + b^2) \sqrt{(x^2 + a^2)(x^2 - b^2)}} \quad [x > b].$$

$$9. \int_x^\infty \sqrt{\frac{x^2 + a^2}{(x^2 - b^2)^3}} dx = \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\varphi, k) + \\ + \frac{(a^2 + b^2)x}{b^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} \quad [x > b].$$

$$10. \int_x^\infty \frac{1}{x^2} \sqrt{\frac{x^2 + a^2}{x^2 - b^2}} dx = \frac{\sqrt{a^2 + b^2}}{b^2} E(\varphi, k) - \frac{a^2}{b^2 x} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$11. \int_x^\infty \frac{1}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{\frac{x^2 - b^2}{x^2 + a^2}}.$$

$$12. \int_x^\infty \sqrt{\frac{x^2 - b^2}{(x^2 + a^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\varphi, k) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\varphi, k).$$

1.2.5°. Интегралы вида $\int_0^x R(x, \sqrt{x^2 + a^2}, \sqrt{b^2 - x^2}) dx$.

Условие: $b \geq x > 0$.

Обозначения: $\varphi = \arcsin \frac{x}{b} \sqrt{\frac{a^2 + b^2}{x^2 + a^2}}$, $k = \frac{b}{\sqrt{a^2 + b^2}}$.

$$1. \int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k).$$

$$2. \int_0^x \frac{x^2 dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\varphi, k) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\varphi, k) - \\ - x \sqrt{\frac{b^2 - x^2}{x^2 + a^2}}.$$

$$3. \int_0^x \frac{dx}{(p - x^2) \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \\ = \frac{1}{p(p + a^2) \sqrt{a^2 + b^2}} \left\{ a \operatorname{II} \left[\rho, \frac{b^2(p + a^2)}{p(a^2 + b^2)}, k \right] + p F(\varphi, k) \right\} \quad [p \neq 0].$$

$$4. \int_0^x \frac{dx}{\sqrt{(x^2 + a^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\varphi, k).$$

$$5. \int_0^x \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} [F(\varphi, k) - E(\varphi, k)].$$

$$6. \int_0^x \frac{dx}{V(x^2+a^2)(b^2-x^2)^3} = \frac{1}{b^2 V(a^2+b^2)} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{x}{b^2 V(x^2+a^2)(b^2-x^2)} \quad [x < b]$$

$$7. \int_0^x \frac{x^2 dx}{V(x^2+a^2)(b^2-x^2)^3} = \frac{x}{V(x^2+a^2)(b^2-x^2)} - \frac{1}{V(a^2+b^2)} E(\varphi, k) \quad [x < b]$$

$$8. \int_0^x \frac{dx}{V(x^2+a^2)^3(b^2-x^2)^3} = \frac{1}{a^2 b^2 V(a^2+b^2)^3} [a^2 F(\varphi, k) - (a^2 - b^2) E(\varphi, k)] + \\ + \frac{x}{b^2(a^2+b^2)V(x^2+a^2)(b^2-x^2)} \quad [x < b]$$

$$9. \int_0^x V \frac{x^2+a^2}{b^2-x^2} dx = V(a^2+b^2) E(\varphi, k) - x V \frac{b^2-x^2}{x^2+a^2}.$$

$$10. \int_0^x V \frac{x^2+a^2}{(b^2-x^2)^3} dx = \frac{a^2}{b^2 V(a^2+b^2)} F(\varphi, k) - \frac{V(a^2+b^2)}{b^2} E(\varphi, k) + \\ + \frac{(a^2+b^2)x}{b^2 V(x^2+a^2)(b^2-x^2)} \quad [x < b]$$

$$11. \int_0^x V \frac{b^2-x^2}{x^2+a^2} dx = V(a^2+b^2) [F(\varphi, k) - E(\varphi, k)] + x V \frac{b^2-x^2}{x^2+a^2}.$$

$$12. \int_0^x V \frac{b^2-x^2}{(x^2+a^2)^3} dx = \frac{V(a^2+b^2)}{a^3} E(\varphi, k) - \frac{1}{V(a^2+b^2)} F(\varphi, k).$$

$$13. \int_0^x V(x^2+a^2)(b^2-x^2) dx = \frac{1}{3} V(a^2+b^2) [a^2 F(\varphi, k) - (a^2 - b^2) E(\varphi, k)] + \\ + \frac{x}{3} (x^2 + 2a^2 - b^2) V \frac{b^2-x^2}{x^2+a^2}.$$

1.2.60. Интегралы вида $\int_x^b R(x, V(x^2+a^2), V(b^2-x^2)) dx$.

Условие: $b > x \geq 0$.

Обозначения: $\varphi = \arccos \frac{x}{b}$, $k = \frac{b}{V(a^2+b^2)}$.

$$1. \int_x^b \frac{dx}{V(x^2+a^2)(b^2-x^2)} = \frac{1}{V(a^2+b^2)} F(\varphi, k).$$

$$2. \int_x^b \frac{x^2 dx}{V(x^2 + a^2)(b^2 - x^2)} = V a^2 + b^2 E(\varphi, k) - \frac{a^2}{V a^2 + b^2} F(\varphi, k).$$

$$3. \int_x^b \frac{dx}{x^2 V(x^2 + a^2)(b^2 - x^2)} = \frac{1}{a^2 b^2 V a^2 + b^2} [a^2 F(\varphi, k) - (a^2 + b^2) E(\varphi, k)] + \\ + \frac{1}{a^2 b^2 x} V(x^2 + a^2)(b^2 - x^2) \quad [x > 0].$$

$$4. \int_x^b \frac{dx}{(p - x^2) V(x^2 + a^2)(b^2 - x^2)} = \\ = \frac{1}{(p - b^2) V a^2 + b^2} \Pi\left(\varphi, \frac{b^2}{b^2 - p}, k\right) \quad [p \neq b^2].$$

$$5. \int_x^b \frac{dx}{V(x^2 + a^2)^3 (b^2 - x^2)} = \frac{1}{a^2 V a^2 + b^2} E(\varphi, k) - \frac{x}{a^2 (a^2 + b^2)} V \frac{b^2 - x^2}{x^2 + a^2}.$$

$$6. \int_x^b \frac{x^2 dx}{V(x^2 + a^2)^3 (b^2 - x^2)} = \frac{1}{V a^2 + b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{a^2 + b^2} V \frac{b^2 - x^2}{x^2 + a^2}.$$

$$7. \int_x^b V \frac{x^2 + a^2}{b^2 - x^2} dx = V a^2 + b^2 E(\varphi, k).$$

$$8. \int_x^b \frac{1}{x^2} V \frac{x^2 + a^2}{b^2 - x^2} dx = \frac{V a^2 + b^2}{b^2} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{V(b^2 - x^2)(x^2 + a^2)}{b^2 x} \quad [x > 0].$$

$$9. \int_x^b V \frac{b^2 - x^2}{x^2 + a^2} dx = V a^2 + b^2 [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_x^b \frac{1}{x^2} V \frac{b^2 - x^2}{x^2 + a^2} dx = \frac{V(b^2 - x^2)(x^2 + a^2)}{a^2 x} - \frac{V a^2 + b^2}{a^2} E(\varphi, k) \quad [x > 0].$$

$$11. \int_x^b V \frac{b^2 - x^2}{(x^2 + a^2)^3} dx = \frac{V a^2 + b^2}{a^2} E(\varphi, k) - \frac{1}{V a^2 + b^2} F(\varphi, k) - \frac{x}{a^2} V \frac{b^2 - x^2}{x^2 + a^2}.$$

$$12. \int_x^b V(x^2 + a^2)(b^2 - x^2) dx = \frac{1}{3} V a^2 + b^2 [a^2 F(\varphi, k) + 2(b^2 - a^2) E(\varphi, k)] + \\ + \frac{x}{3} V(x^2 + a^2)(b^2 - x^2).$$

1.2.61. Интегралы вида $\int_a^x R(x, \sqrt{x^2 - a^2}, \sqrt{x^2 - b^2}) dx$.

Условие: $x > c > b > 0$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}$.

$$1. \int_a^x \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_a^x \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = a \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + x \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$3. \int_a^x \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{a^2 x} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$4. \int_a^x \frac{dx}{(p - x^2) \sqrt{(x^2 - a^2)(x^2 - b^2)}} =$$

$$= \frac{1}{a(p - a^2)(p - b^2)} \left[(a^2 - b^2) \Pi\left(\varphi, \frac{p - b^2}{p - a^2}, \frac{b}{a}\right) + (p - a^2) F\left(\varphi, \frac{b}{a}\right) \right]$$

[$p \neq a^2, b^2$].

$$5. \int_a^x \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{a}{b^2(a^2 - b^2)} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{ab^2} F\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_a^x \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{a}{a^2 - b^2} E\left(\varphi, \frac{b}{a}\right).$$

$$7. \int_a^x \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} dx = x \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} - a E\left(\varphi, \frac{b}{a}\right).$$

$$8. \int_a^x \frac{1}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} dx = \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{a^2 - b^2}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{1}{x} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$9. \int_a^x \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$10. \int_a^x \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} dx = \frac{a^2 - b^2}{a} F\left(\varphi, \frac{b}{a}\right) - a E\left(\varphi, \frac{b}{a}\right) + x \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$11. \int_a^x \frac{1}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} dx = \frac{1}{a} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{x} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$12. \int_a^x \sqrt{(x^2 - a^2)(x^2 - b^2)} dx = \frac{a}{3} \left[(a^2 + b^2) E\left(\varphi, \frac{b}{a}\right) - \right. \\ \left. - (a^2 - b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} (x^2 - a^2 - 2b^2) \sqrt{\frac{x^2 - a^2}{x^2 - b^2}}.$$

$$13. \int \frac{dx}{\sqrt{x(x^2 - 1)(x^2 - a^2)}} = \\ = -\frac{1}{\sqrt{2a(a+1)}} \left[F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{x + \sqrt{a}}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \mp \right. \\ \left. \mp F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{|x - \sqrt{a}|}, \frac{\sqrt{a}+1}{\sqrt{2(a+1)}}\right) \right] \\ \left[a > 0; \begin{cases} 0 \leq x \leq \min(1, a) \\ \max(1, a) \leq x < \infty \end{cases} \right].$$

$$14. \int \frac{\sqrt{x} dx}{\sqrt{(x^2 - 1)(x^2 - a^2)}} = \frac{1}{\sqrt{2(a+1)}} \left[F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{x + \sqrt{a}}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \mp \right. \\ \left. \mp F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{|x - \sqrt{a}|}, \frac{\sqrt{a}+1}{\sqrt{2(a+1)}}\right) \right] \\ \left[a > 0; \begin{cases} 0 \leq x \leq \min(1, a) \\ \max(1, a) \leq x < \infty \end{cases} \right].$$

1.2.62. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2 - a^2}, \sqrt{x^2 - b^2}) dx$.

Условие: $x \geq a > b > 0$.

Обозначение: $\varphi = \arcsin \frac{a}{x}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_x^\infty \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$3. \int_x^\infty \frac{dx}{(x^2 - p) \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ap} \left[\Pi\left(\varphi, \frac{p}{a^2}, \frac{b}{a}\right) - F\left(\varphi, \frac{b}{a}\right) \right]$$

$[p \neq 0].$

$$4. \int_x^\infty \frac{dx}{\sqrt{(x^2 - a^2)^3(x^2 - b^2)}} = \frac{1}{a(b^2 - a^2)} \left[E\left(\varphi, \frac{b}{a}\right) - \frac{a}{x} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} \right]$$

$[x > a].$

$$5. \int_x^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3(x^2 - b^2)}} = \frac{a}{a^2 - b^2} \left[\frac{a}{x} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} - E\left(\varphi, \frac{b}{a}\right) \right] + \\ + \frac{1}{a} F\left(\varphi, \frac{b}{a}\right) \quad [x > a].$$

$$6. \int_x^\infty \frac{dx}{V(x^2 - a^2)(x^2 - b^2)^3} = \frac{1}{b^2(a^2 - b^2)} \left[aE\left(\varphi, \frac{b}{a}\right) - \frac{b^2}{x} V \frac{x^2 - a^2}{x^2 - b^2} \right] - \frac{1}{ab^2} F\left(\varphi, \frac{b}{a}\right).$$

$$7. \int_x^\infty \frac{x^2 dx}{V(x^2 - a^2)(x^2 - b^2)^3} = \frac{1}{a^2 - b^2} \left[aE\left(\varphi, \frac{b}{a}\right) - \frac{b^2}{x} V \frac{x^2 - a^2}{x^2 - b^2} \right].$$

$$8. \int_x^\infty \frac{dx}{V(x^2 - a^2)^3(x^2 - b^2)^3} = \frac{1}{ab^2(a^2 - b^2)} F\left(\varphi, \frac{1}{a}\right) - \frac{a^2 + b^2}{ab^2(a^2 - b^2)^2} E\left(\varphi, \frac{b}{a}\right) + \frac{1}{x(a^2 - b^2)V(x^2 - a^2)(x^2 - b^2)} \quad [x > a].$$

$$9. \int_x^\infty \frac{1}{x^2} V \frac{x^2 - a^2}{x^2 - b^2} dx = \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{a^2 - b^2}{ab^2} F\left(\varphi, \frac{b}{a}\right).$$

$$10. \int_x^\infty V \frac{x^2 - a^2}{(x^2 - b^2)^3} dx = \frac{a}{b^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{x} V \frac{x^2 - a^2}{x^2 - b^2}.$$

$$11. \int_x^\infty \frac{1}{x^2} V \frac{x^2 - b^2}{x^2 - a^2} dx = \frac{1}{a} E\left(\varphi, \frac{b}{a}\right).$$

$$12. \int_x^\infty V \frac{x^2 - b^2}{(x^2 - a^2)^3} dx = \frac{1}{a} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{x} V \frac{x^2 - b^2}{x^2 - a^2} \quad [x > a].$$

1.2.63. Интегралы вида $\int_x^a R(x, \sqrt{a^2 - x^2}, \sqrt{x^2 - b^2}) dx$.

Условие: $a > x \geq b > 0$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a^2 - x^2}{a^2 - b^2}}$, $k = \sqrt{\frac{a^2 - b^2}{a^2}}$.

$$1. \int_x^a \frac{dx}{V(a^2 - x^2)(x^2 - b^2)} = \frac{1}{a} F(\varphi, k).$$

$$2. \int_x^a \frac{x^2 dx}{V(a^2 - x^2)(x^2 - b^2)} = aE(\varphi, k).$$

$$3. \int_x^a \frac{dx}{x^2 V(a^2 - x^2)(x^2 - b^2)} = \frac{1}{ab^2} E(\varphi, k) - \frac{1}{a^2 b^2 x} V(a^2 - x^2)(x^2 - b^2).$$

$$4. \int_x^a \frac{dx}{(\lambda^2 - p) \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a(a^2 - p)} \Pi \left(\varphi, \frac{a^2 - b^2}{a^2 - p}, k \right) \quad [p \neq a^2].$$

$$5. \int_x^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} \left[b^2 F(\varphi, k) - a^2 E(\varphi, k) + ax \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} \right] \quad [x > b].$$

$$6. \int_x^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left[a F(\varphi, k) - a E(\varphi, k) + x \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} \right] \quad [x > b].$$

$$7. \int_x^a \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a [F(\varphi, k) - E(\varphi, k)].$$

$$8. \int_x^a \frac{1}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = \frac{a}{b^2} E(\varphi, k) - \frac{1}{a} F(\varphi, k) - \frac{\sqrt{(a^2 - x^2)(x^2 - b^2)}}{b^2 x}.$$

$$9. \int_x^a \sqrt{\frac{a^2 - x^2}{(x^2 - b^2)^3}} dx = \frac{x}{b^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} - \frac{a}{b^2} E(\varphi, k) \quad [x > b].$$

$$10. \int_x^a \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = a E(\varphi, k) - \frac{b^2}{a} F(\varphi, k).$$

$$11. \int_x^a \frac{1}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = \frac{1}{a} [F(\varphi, k) - E(\varphi, k)] + \frac{\sqrt{(a^2 - x^2)(x^2 - b^2)}}{a^2 x}.$$

$$12. \int_x^a \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} [(a^2 + b^2) E(\varphi, k) - 2b^2 F(\varphi, k)] - \frac{x}{3} \sqrt{(a^2 - x^2)(x^2 - b^2)}.$$

1.2.64. Интегралы вида $\int_b^x R(x, \sqrt{a^2 - x^2}, \sqrt{x^2 - b^2}) dx$.

Условие $a - \lambda > b > 0$

Обозначения $\varphi = \arcsin \frac{a}{x} \sqrt{\frac{x^2 - b^2}{a^2 - b^2}}, \quad k = \frac{\sqrt{a^2 - b^2}}{a}.$

$$1. \int_b^x \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\varphi, k).$$

$$2. \int_b^x \frac{x^2 dx}{V(a^2 - x^2)(x^2 - b^2)} = aE(\varphi, k) - \frac{1}{x} V(a^2 - x^2)(x^2 - b^2).$$

$$3. \int_b^x \frac{dx}{x^2 V(a^2 - x^2)(x^2 - b^2)} = \frac{1}{ab^2} E(\varphi, k).$$

$$4. \int_b^x \frac{dx}{(p - x^2) V(a^2 - x^2)(x^2 - b^2)} = \\ = \frac{1}{ap(p - b^2)} \left\{ b^2 \Pi \left[\varphi, \frac{p(a^2 - b^2)}{a^2(p - b^2)}, k \right] + (p - b^2) F(\varphi, k) \right\} \quad [p \neq b^2].$$

$$5. \int_b^x \frac{dx}{V(a^2 - x^2)^3(x^2 - b^2)} = \frac{1}{a(a^2 - b^2)} \left[F(\varphi, k) - E(\varphi, k) + \frac{a}{x} V \frac{x^2 - b^2}{a^2 - x^2} \right] \\ [x < a].$$

$$6. \int_b^x \frac{x^2 dx}{V(a^2 - x^2)^3(x^2 - b^2)} = \\ = \frac{1}{a(a^2 - b^2)} \left[b^2 F(\varphi, k) - a^2 E(\varphi, k) + \frac{a^3}{x} V \frac{x^2 - b^2}{a^2 - x^2} \right] \quad [x < a].$$

$$7. \int_b^x V \frac{a^2 - x^2}{x^2 - b^2} dx = a[F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} V(a^2 - x^2)(x^2 - b^2).$$

$$8. \int_b^x \frac{1}{x^2} V \frac{a^2 - x^2}{x^2 - b^2} dx = \frac{a}{b^2} E(\varphi, k) - \frac{1}{a} F(\varphi, k).$$

$$9. \int_b^x \frac{1}{x^2} V \frac{x^2 - b^2}{a^2 - x^2} dx = \frac{1}{a} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_b^x V \frac{x^2 - b^2}{a^2 - x^2} dx = aE(\varphi, k) - \frac{b^2}{a} F(\varphi, k) - \frac{1}{x} V(a^2 - x^2)(x^2 - b^2).$$

$$11. \int_b^x V \frac{x^2 - b^2}{(a^2 - x^2)^3} dx = \frac{1}{x} V \frac{x^2 - b^2}{a^2 - x^2} - \frac{1}{a} E(\varphi, k) \quad [x < a].$$

$$12. \int_b^x V(a^2 - x^2)(x^2 - b^2) dx = \frac{a}{3} [(a^2 + b^2) E(\varphi, k) - 2b^2 F(\varphi, k)] + \\ + \frac{x^2 - a^2 - b^2}{3x} V(a^2 - x^2)(x^2 - b^2).$$

1.2.65. Интегралы вида $\int_0^x R(x, \sqrt{a^2 - x^2}, \sqrt{b^2 - x^2}) dx$.

Условие: $a > b > x > 0$.

Обозначение: $\varphi = \arcsin \frac{x}{b}$.

$$1. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \left[\Gamma\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$3. \int_0^x \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ap} \Pi\left(\varphi, \frac{b^2}{p}, \frac{b}{a}\right) \quad [p \neq b].$$

$$4. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 (a^2 - b^2)} \left[a E\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$5. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 - b^2} \left[a E\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right] - \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{1}{b^2 (a^2 - b^2)} \left[a E\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} \right] \quad [x < b].$$

$$7. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{a^2 - b^2} \left[x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} - a E\left(\varphi, \frac{b}{a}\right) \right] \quad [x < b].$$

$$8. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} F\left(\varphi, \frac{b}{a}\right) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E\left(\varphi, \frac{b}{a}\right) + \frac{[a^4 + b^4 - (a^2 + b^2)x^2]x}{a^2 b^2 (a^2 - b^2)^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} \quad [x < b].$$

$$9. \int_0^x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a E\left(\varphi, \frac{b}{a}\right).$$

$$10. \int_0^x \sqrt{\frac{a^2 - x^2}{(b^2 - x^2)^3}} dx = \frac{a}{b^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{b^2} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} \quad [x < b].$$

$$11. \int_0^x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = aE\left(\varphi, \frac{b}{a}\right) - \frac{a^2 - b^2}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$12. \int_0^x \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) + \frac{x}{a} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$13. \int_0^x \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \left[(a^2 + b^2) E\left(\varphi, \frac{b}{a}\right) - (a^2 - b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} \sqrt{(a^2 - x^2)(b^2 - x^2)}.$$

1.2.66. Интегралы вида $\int_x^b R(x, \sqrt{a^2 - x^2}, \sqrt{b^2 - x^2}) dx$.

Условие $a > b > x \geq 0$.

Обозначение. $\varphi = \arcsin \frac{a}{b} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_x^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}.$$

$$3. \int_x^b \frac{dx}{x^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{b^2 x} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$$

[$x > 0$],

$$4. \int_x^b \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(b^2 - x^2)}} =$$

$$= \frac{1}{a(p - a^2)(p - b^2)} \left\{ (b^2 - a^2) \Pi \left[\varphi, \frac{b^2(p - a^2)}{a^2(p - b^2)}, \frac{b}{a} \right] + (p - b^2) F\left(\varphi, \frac{b}{a}\right) \right\}$$

[$p \neq b^2$].

$$5. \int_x^b \frac{dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{1}{a(a^2 - b^2)} E\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_x^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{a}{a^2 - b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$7. \int_x^b \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a \left[E\left(\varphi, \frac{b}{a}\right) - \frac{x}{a} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$8. \int_x^b \frac{1}{x^2} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = \frac{a^2 - b^2}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) + \frac{a^2}{b^2 x} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$$

[$x > 0$]

$$9. \int_x^b \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E\left(\varphi, \frac{b}{a}\right) - \frac{a^2 - b^2}{a} F\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$$

$$10. \int_x^b \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$11. \int_x^b \frac{1}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = \frac{1}{x} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} - \frac{1}{a} E\left(\varphi, \frac{b}{a}\right)$$

[$x > 0$].

$$12. \int_x^b \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \left[(a^2 + b^2) E\left(\varphi, \frac{b}{a}\right) - (a^2 - b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} (x^2 - 2a^2 - b^2) \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}.$$

1.2.67. Интегралы вида $\int R(x, \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}) dx$.

Обозначения: $\varphi = \arccos \frac{x^2 - \rho\bar{\rho}}{x^2 + \rho\bar{\rho}}$, $k = \frac{1}{2} \sqrt{-\frac{(\rho - \bar{\rho})^2}{\rho\bar{\rho}}}$, ρ и $\bar{\rho}$ — комплексно сопряженные числа.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{1}{\sqrt{\rho\bar{\rho}}} F(\varphi, k).$$

$$2. \int_x^\infty \frac{x^2 dx}{(x^2 - \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{2x \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}}{(\rho + \bar{\rho})^2 (x^4 - \rho^2 \bar{\rho}^2)} - \frac{1}{(\rho + \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E(\varphi, k).$$

$$3. \int_x^\infty \frac{x^2 dx}{(x^2 + \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = -\frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} [F(\varphi, k) - E(\varphi, k)].$$

$$4. \int_x^\infty \frac{x^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4 \sqrt{\rho\bar{\rho}}}{(\rho^2 - \bar{\rho}^2)^2} E(\varphi, k) + \frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\varphi, k) - \frac{2x(x^2 - \rho\bar{\rho})}{(\rho + \bar{\rho})^2 (x^2 + \rho\bar{\rho}) \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}}.$$

$$\begin{aligned}
 5. \int_x^\infty \frac{(x^2 - \rho\bar{\rho}) dx}{V(\lambda^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3} &= \\
 &= -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} [F(\varphi, k) - E(\varphi, k)] + \frac{2x(x^2 - \rho\bar{\rho})}{(x^2 + \rho\bar{\rho}) V(\lambda^2 + \rho^2)(\lambda^2 + \bar{\rho}^2)}. \\
 6. \int_x^\infty \frac{(x^2 - \rho\bar{\rho})^2 dx}{(\lambda^2 + \rho\bar{\rho})^2 V(x^2 + \rho^2)(x^2 + \bar{\rho}^2)} &= -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} E(\varphi, k) + \\
 &+ \frac{(\rho + \bar{\rho})^2}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\varphi, k). \\
 7. \int_x^\infty \frac{(x^2 + \rho\bar{\rho})^2 dx}{[(x^2 + \rho\bar{\rho})^2 - 4\rho^2\bar{\rho}x^2] V(x^2 + \rho^2)(x^2 + \bar{\rho}^2)} &= \frac{1}{V\rho\bar{\rho}} \Pi(\varphi, \rho^2, k). \\
 8. \int_x^\infty \frac{V(x^2 + \rho^2)(\lambda^2 + \bar{\rho}^2)}{(x^2 + \rho\bar{\rho})^2} dx &= \frac{1}{V\rho\bar{\rho}} F(\varphi, k).
 \end{aligned}$$

1.2.68. Интегралы вида $\int R(x, \sqrt{x^3+1}) dx$ *).

Условие: $x \geq -1$.

Обозначения: $\varphi = \frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}, \quad k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

$$\begin{aligned}
 1. \int \frac{x^n}{\sqrt{x^3+1}} dx &= \frac{2}{2n-1} x^{n-2} \sqrt{x^3+1} - \frac{2n-4}{2n-1} \int \frac{x^{n-3}}{\sqrt{x^3+1}} dx. \\
 2. \int \frac{dx}{\sqrt{x^3+1}} &= -\frac{1}{\sqrt[4]{3}} F(\varphi, k). \\
 3. \int \frac{x}{\sqrt{x^3+1}} dx &= \frac{2\sqrt{x^3+1}}{x+1+\sqrt{3}} - \frac{\sqrt{3}-1}{\sqrt[4]{3}} F(\varphi, k) + 2\sqrt[4]{3} E(\varphi, k). \\
 4. \int \frac{x^2}{\sqrt{x^3+1}} dx &= \frac{2}{3} \sqrt{x^3+1}. \\
 5. \int \frac{dx}{(x-\rho)\sqrt{x^3+1}} &= \frac{1}{\sqrt[4]{3}(\rho+1+\sqrt{3})} F(\varphi, k) - \\
 &- \frac{1}{2\sqrt[4]{3}(\rho+1)} \left[\frac{\rho+1-\sqrt{3}}{\rho+1+\sqrt{3}} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1-\rho_1 \sin^2 x) \sqrt{1-k^2 \sin^2 x}} \right] \\
 &\quad \left[\rho_1 = \frac{(\rho+1+\sqrt{3})^2}{4\sqrt[4]{3}(\rho+1)}; \rho \neq -1, -1-\sqrt{3}, \text{ см 1.5.33. 12-14} \right]. \\
 6. \int \frac{dx}{(\rho+1+\sqrt[4]{3}) \sqrt{x^3+1}} &= -\frac{\sqrt[4]{3}}{6} F(\varphi, k) + \frac{\sqrt[4]{3}}{6k} \arcsin(k \sin \varphi). \\
 7. \int \frac{dx}{(x+1)\sqrt{x^3+1}} &= \frac{-2\sqrt{x^3+1}}{\sqrt{3}(x+1)(x+1+\sqrt{3})} - \frac{\sqrt[4]{3}}{3} [F(\varphi, k) - 2E(\varphi, k)].
 \end{aligned}$$

*) См. также 1.2.73.

$$8. \int \frac{dx}{x \sqrt{x^3+1}} =$$

$$= \frac{1}{2} \frac{3}{3} F(\varphi, k) + \frac{2-\sqrt{3}}{2} \frac{1}{3} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right) - \frac{1}{4} \ln \frac{x^2+2+2\sqrt{x^3+1}}{x^2+2-2\sqrt{x^3+1}}.$$

1.2.69. Интегралы вида $\int R(x, \sqrt{x^3-1}) dx$ *).

Условие: $x > 1$.

Обозначения. $\varphi = \arccos \frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$, $\psi = \arccos \frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$,

$$k = \sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

$$1. \int_1^x \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k). \quad 2. \int_x^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\psi, k).$$

$$3. \int_1^x \frac{x dx}{\sqrt{x^3-1}} = \left(\sqrt[4]{3} + \frac{1}{\sqrt[4]{3}} \right) F(\varphi, k) - 2 \sqrt[4]{3} E(\varphi, k) + \frac{2 \sqrt{x^3-1}}{\sqrt{3}-1+x}.$$

$$4. \int_x^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}} [F(\psi, k) - 2E(\psi, k)] +$$

$$+ \frac{2}{\sqrt[4]{3}} \frac{\sqrt{x^3+x+1}}{(x-1+\sqrt{3})\sqrt{x-1}} \quad [x > 1].$$

$$5. \int_1^x \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\varphi, k).$$

$$6. \int_x^\infty \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\psi, k).$$

$$7. \int_1^x \frac{(x-1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\varphi, k) - E(\varphi, k)].$$

$$8. \int_x^\infty \frac{(x-1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\psi, k) - E(\psi, k)].$$

$$9. \int_1^x \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\varphi, k).$$

$$10. \int_x^\infty \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\psi, k).$$

*). См. также 1.2.73.

$$11. \int_1^x \frac{(x-1) dx}{(x^2+x+1) \sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}} E(\varphi, k) - \\ - \frac{2+\sqrt[4]{3}}{\sqrt[4]{27}} F(\varphi, k) - \frac{2-\sqrt[4]{3}}{\sqrt[4]{3}} \frac{2(x-1)(\sqrt[4]{3}+1-x)}{(\sqrt[4]{3}-1+x) \sqrt{x^3-1}}.$$

$$12. \int_1^x \frac{(x-1+\sqrt[4]{3})^2 dx}{[(x-1+\sqrt[4]{3})^2-4\sqrt[4]{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\varphi, p^2, k).$$

$$13. \int_x^\infty \frac{(x-1+\sqrt[4]{3})^2 dx}{[(x-1+\sqrt[4]{3})^2-4\sqrt[4]{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\psi, p^2, k).$$

1.2.70. Интегралы вида $\int R(x, \sqrt[4]{1-x^3}) dx$ *).

Условие: $x \leq 1$.

Обозначения: $\varphi = \arccos \frac{x-1+\sqrt[4]{3}}{x-1-\sqrt[4]{3}}, \quad \psi = \arccos \frac{\sqrt[4]{3}-1+x}{\sqrt[4]{3}+1-x},$

$$k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt[4]{3}}}{2}.$$

$$1. \int \frac{x^m dx}{\sqrt[4]{1-x^3}} = \frac{-2x^{m-2} \sqrt[4]{1-x^3}}{2m-1} + \frac{2(m-2)}{2m-1} \int \frac{x^{m-3} dx}{\sqrt[4]{1-x^3}}.$$

$$2. \int_{-\infty}^x \frac{dx}{\sqrt[4]{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k). \quad 3. \int_x^1 \frac{dx}{\sqrt[4]{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\psi, k).$$

$$4. \int_x^1 \frac{x dx}{\sqrt[4]{1-x^3}} = \left(\frac{1}{\sqrt[4]{3}} - \sqrt[4]{3} \right) F(\psi, k) + 2\sqrt[4]{3} E(\psi, k) - \frac{2\sqrt[4]{1-x^3}}{\sqrt[4]{3}+1-x}.$$

$$5. \int \frac{x^2 dx}{\sqrt[4]{1-x^3}} = -\frac{2}{3} \sqrt[4]{1-x^3}.$$

$$6. \int \frac{dx}{(x-\rho) \sqrt[4]{1-x^3}} = \frac{-1}{\sqrt[4]{3}(\rho-1-\sqrt[4]{3})} F(\varphi, k) + \\ + \frac{1}{2\sqrt[4]{3}(\rho-1)} \left[\frac{\rho-1+\sqrt[4]{3}}{\rho-1-\sqrt[4]{3}} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x) \sqrt[4]{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(\rho-1-\sqrt[4]{3})^2}{4\sqrt[4]{3}(\rho-1)}, \rho \neq 1, 1+\sqrt[4]{3}; \text{ см } 1.5.33, 12-14 \right].$$

$$7. \int \frac{dx}{(x-1-\sqrt[4]{3}) \sqrt[4]{1-x^3}} = -\frac{\sqrt[4]{3}}{6} F(\varphi, k) + \frac{\sqrt[4]{3}}{6k} \arcsin(k \sin \varphi).$$

*) См. также 1.2.74.

$$8. \int \frac{dx}{(x-1)\sqrt{1-x^3}} = \frac{2\sqrt{x^2+x+1}}{\sqrt{3}(x-1-\sqrt{3})\sqrt{1-x}} - \frac{1}{\sqrt[4]{27}} [F(\varphi, k) - 2E(\varphi, k)] \quad [x < 1].$$

$$9. \int \frac{dx}{x\sqrt{1-x^3}} = \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\varphi, k) + \frac{2-\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right) + \frac{1}{4} \ln \frac{x^2+2-2\sqrt{1-x^3}}{x^2+2+2\sqrt{1-x^3}}.$$

$$10. \int_0^x \frac{(x-1)dx}{(x-1+\sqrt{3})^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} \left[E(\varphi, k) - \frac{2\sqrt[4]{3}\sqrt{1-x^3}}{x^2-2x-2} \right].$$

$$11. \int_0^1 \frac{(x-1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[4]{27}} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_x^1 \frac{(x-1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[4]{27}} [F(\psi, k) - E(\psi, k)].$$

$$13. \int_{-\infty}^x \frac{(x^2+x+1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\varphi, k).$$

$$14. \int_x^1 \frac{(x^2+x+1)dx}{(x-1+\sqrt{3})^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\psi, k).$$

$$15. \int_x^1 \frac{(x-1-\sqrt{3})^2 dx}{[(x-1-\sqrt{3})^2 + 4\sqrt{3}p^2(x-1)]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\psi, p^2, k).$$

$$16. \int_{-\infty}^x \frac{(x-1-\sqrt{3})^2 dx}{[(x-1-\sqrt{3})^2 + 4\sqrt{3}p^2(x-1)]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\varphi, p^2, k).$$

$$17. \int_x^1 \sqrt{1-x^3} dx = \frac{1}{5} [\sqrt[4]{27} F(\psi, k) - 2x\sqrt{1-x^3}].$$

$$18. \int_0^x \sqrt{\frac{x}{1-x^3}} \frac{dx}{1-x} = \frac{1}{\sqrt[4]{27}} \left[F\left(\arccos \frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) - 2E\left(\arccos \frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) \right] + \frac{2}{\sqrt{3}} \frac{\sqrt{x(1+x+x^2)}}{\sqrt{1-x}(1+\sqrt{3}x-x)} \quad [x < 1].$$

1.2.71. Интегралы вида $\int R(x, \sqrt{x^4+1}) dx$.

Обозначения: $\varphi = \arccos \frac{x^2-1}{x^2+1}$, $\psi = \operatorname{arctg} \left[(1+\sqrt{2}) \frac{1-x}{1+x} \right]$,

$$k = \frac{1}{2} (3\sqrt{2}-1).$$

$$1. \int \frac{x^n dx}{\sqrt{x^4+1}} = \frac{x^{n-3}}{n-1} \sqrt{x^4+1} - \frac{n-3}{n-1} \int \frac{x^{n-4}}{\sqrt{x^4+1}} dx.$$

$$2. \int_0^x \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2} F \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) \quad [x \leq 1].$$

$$3. \int_x^\infty \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2} F \left(\varphi, \frac{\sqrt{2}}{2} \right) \quad [x \geq 1].$$

$$4. \int_x^1 \frac{dx}{\sqrt{x^4+1}} = (2-\sqrt{2}) F(\psi, k) \quad [0 \leq x < 1].$$

$$5. \int \frac{x dx}{\sqrt{x^4+1}} = \frac{1}{2} \ln(x^2 + \sqrt{x^4+1}).$$

$$6. \int_0^x \frac{x^2 dx}{\sqrt{x^4+1}} = \frac{x \sqrt{x^4+1}}{x^2+1} + \frac{1}{2} F \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) - E \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) \quad [0 \leq x \leq 1].$$

$$7. \int_1^x \frac{x^2 dx}{\sqrt{x^4+1}} = \frac{x \sqrt{x^4+1}}{x^2+1} - \frac{1}{2} F \left(\arccos \frac{x^2-1}{x^2+1}, \frac{\sqrt{2}}{2} \right) + E \left(\arccos \frac{x^2-1}{x^2+1}, \frac{\sqrt{2}}{2} \right) \quad [x \geq 1].$$

$$8. \int \frac{x^3 dx}{\sqrt{x^4+1}} = \frac{1}{2} \sqrt{x^4+1}. \quad 9. \int \frac{dx}{x \sqrt{x^4+1}} = \frac{1}{2} \ln \frac{x^2}{\sqrt{x^4+1}+1}.$$

$$10. \int_x^\infty \frac{dx}{x^2 \sqrt{x^4+1}} = \frac{1}{2} \left[F \left(\varphi, \frac{\sqrt{2}}{2} \right) - 2E \left(\varphi, \frac{\sqrt{2}}{2} \right) \right] + \frac{\sqrt{x^4+1}}{x(x^2+1)} \quad [x \geq 1].$$

$$11. \int_x^1 \frac{(1-x)^2 dx}{(x^2-x\sqrt{2}+1) \sqrt{x^4+1}} = \frac{1}{\sqrt{2}} [F(\psi, k) - E(\psi, k)] \quad [0 \leq x < 1].$$

$$12. \int_x^1 \frac{(1+x)^2 dx}{(x^2+x\sqrt{2}+1) \sqrt{x^4+1}} = \frac{3\sqrt{2}+4}{2} E(\psi, k) - \frac{3\sqrt{2}-4}{2} F(\psi, k) \quad [0 \leq x < 1].$$

13. $\int_x^1 \frac{(x^2 + x\sqrt{2} + 1) dx}{(x^2 - x\sqrt{2} + 1)\sqrt{x^4 + 1}} = (2 + \sqrt{2}) E(\psi, k)$ $[0 \leq x < 1].$
14. $\int_x^\infty \frac{x^2 dx}{(x^4 + 1)\sqrt{x^4 + 1}} = \frac{1}{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{1}{4} F\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{x(x^2 - 1)}{2(x^2 + 1)\sqrt{x^4 + 1}}$ $[x \geq 1].$
15. $\int_x^\infty \frac{x^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = \frac{1}{4} \left[F\left(\varphi, \frac{\sqrt{2}}{2}\right) - E\left(\varphi, \frac{\sqrt{2}}{2}\right) \right]$ $x \geq 1.$
16. $\int_x^\infty \frac{x^2 dx}{(x^2 - 1)^2 \sqrt{x^4 + 1}} = \frac{x\sqrt{x^4 + 1}}{2(x^4 - 1)} - \frac{1}{4} E\left(\varphi, \frac{\sqrt{2}}{2}\right)$ $[x > 1].$
17. $\int_x^\infty \frac{(x^2 + 1)^2 dx}{[(x^2 + 1)^2 - 4\rho^2 x^2] \sqrt{x^4 + 1}} = \frac{1}{2} \Pi\left(\varphi, \rho^2, \frac{\sqrt{2}}{2}\right)$ $[x \geq 1].$
18. $\int_x^\infty \frac{(x^2 - 1)^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = E\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{1}{2} F\left(\varphi, \frac{\sqrt{2}}{2}\right)$ $[x \geq 1].$
19. $\int_x^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 + 1)^2} = \frac{1}{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right)$ $[x \geq 1].$
20. $\int_x^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 - 1)^2} = \frac{1}{2} \left[F\left(\varphi, \frac{\sqrt{2}}{2}\right) - E\left(\varphi, \frac{\sqrt{2}}{2}\right) \right] + \frac{x\sqrt{x^4 + 1}}{x^4 - 1}$ $[x > 1].$

1.2.72. Интегралы вида $\int R(x, \sqrt{\pm x^4 \mp 1}) dx$.

Обозначения: $\varphi = \arccos \frac{1}{x}$, $\psi = \arccos x$.

1. $\int_1^x \frac{dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F\left(\varphi, \frac{\sqrt{2}}{2}\right)$ $[x > 1].$
2. $\int_1^x \frac{x^2 dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F\left(\varphi, \frac{\sqrt{2}}{2}\right) - \sqrt{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right) + \frac{1}{x} \sqrt{x^4 - 1}$ $[x > 1].$
3. $\int_1^x \frac{x^4 dx}{\sqrt{x^4 - 1}} = \frac{1}{3} F\left(\varphi, \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{3} x \sqrt{x^4 - 1}$ $[x > 1].$
4. $\int_x^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right)$ $[x < 1].$

$$5. \int_x^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \sqrt{2} E\left(\psi, \frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) \quad [x < 1].$$

$$6. \int_x^1 \frac{x^4 dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) + \frac{x}{3} \sqrt{1-x^4} \quad [x < 1].$$

1.2.73. Интегралы вида $\int f(x, \sqrt{x+x^4}) dx$.

Условие: $x \geq 0$ или $x \leq -1$.

Обозначения: $\varphi = \arccos \frac{(1-\sqrt{3})x+1}{(1+\sqrt{3})x+1}$, $k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

$$1. \int \frac{dx}{\sqrt{x^4+x}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

$$2. \int \frac{x dx}{\sqrt{x^4+x}} = \frac{1}{2} \ln \frac{2x^2+1+2\sqrt{x^4+x}}{2x-1} + \frac{1-\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) - \\ - \frac{2-\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right).$$

$$3. \int \frac{x^2 dx}{\sqrt{x^4+x}} = \frac{(1+\sqrt{3})\sqrt{x^4+x}}{x+1+\sqrt{3}x} + \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\varphi, k) - \sqrt[4]{3} E(\varphi, k).$$

$$4. \int \frac{dx}{(x-\rho)\sqrt{x^4+x}} = \frac{-1-\sqrt{3}}{\sqrt[4]{3}(1+\rho+\sqrt{3}\rho)} F(\varphi, k) - \\ - \frac{1}{2\sqrt[4]{3}(1+\rho)\rho} \left[\frac{1+\rho-\sqrt{3}\rho}{1+\rho+\sqrt{3}\rho} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x)\sqrt{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(1+\rho+\sqrt{3}\rho)^2}{-4\sqrt{3}\rho(1+\rho)}; \rho \neq 0, -1, \frac{1-\sqrt{3}}{2}; \text{см. 1.5.33, 12-14} \right].$$

$$5. \int \frac{dx}{(x+1)\sqrt{x^4+x}} = \frac{-2\sqrt{x^4+x}}{\sqrt{3}(x+1+\sqrt{3}x)(x+1)} + \\ + \frac{2\sqrt[4]{3}}{3} \left[\frac{\sqrt{3}-1}{2} F(\varphi, k) + E(\varphi, k) \right].$$

$$6. \int \frac{dx}{(2x+\sqrt{3}-1)\sqrt{x^4+x}} = \frac{\sqrt[4]{3}}{6} \left[F(\varphi, k) + \frac{2+\sqrt{3}}{k} \arcsin(k \sin \varphi) \right].$$

$$7. \int \frac{dx}{x\sqrt[4]{x^4+x}} = -\frac{2\sqrt{x^4+x}}{(x+1+\sqrt{3}x)x} + \frac{\sqrt{3}-1}{\sqrt[4]{3}} F(\varphi, k) - 2\sqrt[4]{3} E(\varphi, k).$$

1.2.74. Интегралы вида $\int R(x, \sqrt{x-x^4}) dx$.

Условие: $0 \leq x \leq 1$.

Обозначения: $\varphi = \arccos \frac{1-(\sqrt{3}+1)x}{1+(\sqrt{3}-1)x}$, $k = \frac{\sqrt{2-\sqrt{3}}}{2}$.

$$1. \int \frac{dx}{\sqrt{x-x^4}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

$$2. \int \frac{x dx}{\sqrt{x-x^4}} = -\operatorname{arctg} \frac{x-x^2}{\sqrt{x-x^4}} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) + \frac{2+\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3-2\sqrt{3}}{6}, k\right).$$

$$3. \int \frac{x^2 dx}{\sqrt{x-x^4}} = \frac{(1-\sqrt{3})\sqrt{x-x^4}}{\sqrt{3}x-x+1} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) + \sqrt[4]{3} E(\varphi, k).$$

$$4. \int \frac{dx}{(x-\rho)\sqrt{x-x^4}} = \frac{-\sqrt{3}+1}{\sqrt[4]{3}(1-\rho+\sqrt{3}\rho)} F(\varphi, k) - \\ - \frac{1}{2\sqrt[4]{3}\rho(1-\rho)} \left[\frac{1-\rho-\sqrt{3}\rho}{1-\rho+\sqrt{3}\rho} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x)\sqrt{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(1-\rho+\sqrt{3}\rho)^2}{-4\sqrt{3}\rho(1-\rho)}; \rho \neq 0, 1, -\frac{1+\sqrt{3}}{2}; \text{см. 1.5.33. 12-14} \right].$$

$$5. \int \frac{dx}{(x-1)\sqrt{x-x^4}} = \frac{2\sqrt{x-x^4}}{\sqrt{3}(x-1)(\sqrt{3}x-x+1)} - \\ - \frac{2\sqrt[4]{3}}{3} \left[\frac{\sqrt{3}+1}{2} F(\varphi, k) - E(\varphi, k) \right].$$

$$6. \int \frac{dx}{(2x+1+\sqrt{3})\sqrt{x-x^4}} = \frac{\sqrt[4]{3}}{6} \left[F(\varphi, k) + \frac{2-\sqrt{3}}{k} \arcsin(k \sin \varphi) \right].$$

$$7. \int \frac{dx}{x\sqrt{x-x^4}} = \frac{-2\sqrt{x-x^4}}{x(\sqrt{3}x-x+1)} + \frac{1+\sqrt{3}}{\sqrt[4]{3}} F(\varphi, k) - 2\sqrt[4]{3} E(\varphi, k).$$

1.2.75. Интегралы вида $\int R(x, \sqrt{x^4+2b^2x^2+a^4}) dx$.

Условия: $a > b > 0, x \geq 0$.

Обозначения: $\varphi = \arccos \frac{x^2-a^2}{x^2+a^2}, k = \frac{\sqrt{a^2-b^2}}{a\sqrt{2}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{x^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a^3} [F(\varphi, k) - 2E(\varphi, k)] + \frac{\sqrt{x^4+2b^2x^2+a^4}}{a^2x(x^2+a^2)}$$

$[x > 0].$

$$3. \int_x^\infty \frac{x^2 dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{4a(a^2-b^2)} [F(\varphi, k) - E(\varphi, k)].$$

$$4. \int_x^\infty \frac{x^2 dx}{(x^2-a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{x\sqrt{x^4+2b^2x^2+a^4}}{2(a^2+b^2)(x^4-a^4)} - \frac{1}{4a(a^2+b^2)} E(\varphi, k).$$

$[x > a].$

$$5. \int_x^\infty \frac{(x^2-a^2)^2 dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{a}{a^2-b^2} E(\varphi, k) - \frac{a^2+b^2}{2a(a^2-b^2)} F(\varphi, k).$$

$$6. \int_x^{\infty} \frac{(x^2 + a^2)^2 dx}{[(x^2 + a^2)^2 - 4a^2 p^2 x^2] \sqrt{x^4 + 2b^2 x^2 + a^4}} = \frac{1}{2a} \Pi(\varphi, p^2, k).$$

$$7. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \frac{a}{2(a^4 - b^4)} E(\varphi, k) - \frac{1}{4a(a^2 - b^2)} F(\varphi, k) - \\ - \frac{x(x^2 - a^2)}{2(a^2 + b^2)(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$8. \int_x^{\infty} \frac{(x^2 + a^2)^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\varphi, k) - \frac{(a^2 - b^2)x(x^2 - a^2)}{(a^2 + b^2)(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$9. \int_x^{\infty} \frac{(x^2 - a^2)^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \\ = \frac{a}{a^2 - b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{x(x^2 - a^2)}{(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$10. \int_x^{\infty} \frac{\sqrt{x^4 + 2b^2 x^2 + a^4}}{(x^2 + a^2)^2} dx = \frac{1}{2a} E(\varphi, k).$$

$$11. \int_x^{\infty} \frac{\sqrt{x^4 + 2b^2 x^2 + a^4}}{(x^2 - a^2)^2} dx = \frac{1}{2a} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{x^4 - a^4} \sqrt{x^4 + 2b^2 x^2 + a^4}.$$

$$12. \int_x^a \frac{dx}{\sqrt{x^4 + 2b^2 x^2 + a^4}} = \\ = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} F \left[\operatorname{arctg} \left(\frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a - x}{a + x} \right), \frac{2\sqrt{a}\sqrt{2}(a^2 - b^2)}{a\sqrt{2} + \sqrt{a^2 - b^2}} \right] \\ [0 \leq x < a].$$

1.2.76. Интегралы вида $\int R(x, \sqrt{x^6 + 1}) dx$.

Обозначения: $\varphi = \arccos \frac{(1 - \sqrt{3})x^2 + 1}{(1 + \sqrt{3})x^2 + 1}$, $\psi = \arccos \frac{x^2 + 1 - \sqrt{3}}{x^2 + 1 + \sqrt{3}}$,

$$k = \sin \frac{5\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

$$1. \int \frac{dx}{\sqrt{x^6 + 1}} = \frac{1}{2\sqrt[4]{3}} F(\varphi, k).$$

$$2. \int \frac{x}{\sqrt{x^6 + 1}} dx = \frac{-1}{2\sqrt[4]{3}} F(\psi, k).$$

$$3. \int \frac{x^2}{\sqrt{x^6 + 1}} dx = \frac{1}{3} \ln(x^3 + \sqrt{x^6 + 1}).$$

$$4. \int \frac{x^3}{\sqrt{x^6+1}} dx = \frac{\sqrt{x^6+1}}{x^2+1+\sqrt{3}} - \sqrt[4]{3} \left[\frac{3-\sqrt{3}}{6} F(\psi, k) - E(\psi, k) \right].$$

$$5. \int \frac{x^4}{\sqrt{x^6+1}} dx = \frac{(1+\sqrt{3})x\sqrt{x^6+1}}{2(1+\sqrt{3})x^2+2} + \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\varphi, k) - \frac{\sqrt[4]{3}}{2} E(\varphi, k).$$

$$6. \int \frac{x^5}{\sqrt{x^6+1}} dx = \frac{1}{3} \sqrt{x^6+1}.$$

$$7. \int \frac{dx}{(x-\rho)\sqrt{x^6+1}} = \frac{1}{2\sqrt[4]{3}} \left\{ \frac{1}{\rho^2+1+\sqrt{3}} F(\psi, k) - \right. \\ \left. - \frac{\rho^2+1-\sqrt{3}}{2(\rho^2+1)(\rho^2+1+\sqrt{3})} \Pi(\psi, -\rho_1, k) - \frac{1}{2(\rho^2+1)} I(\psi, \rho_1, k) \right\} + \\ + \frac{\rho}{2\sqrt[4]{3}} \left\{ \frac{-1-\sqrt{3}}{\rho^2+1+\sqrt{3}\rho^2} F(\varphi, k) - \frac{\rho^2+1-\sqrt{3}\rho^2}{2\rho^2(\rho^2+1)(\rho^2+1+\sqrt{3}\rho^2)} \Pi(\varphi, -\rho_1, k) - \right. \\ \left. - \frac{1}{2\rho^2(\rho^2+1)} I(\varphi, \rho_2, k) \right\} \\ \left[I(\varphi, \rho, k) = \int_0^\varphi \frac{\cos \varphi d\varphi}{(1+\rho \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}; x > 0, \rho \neq 0; \right. \\ \left. \rho_1 = \frac{-(\rho^2+1+\sqrt{3})^2}{4\sqrt{3}(\rho^2+1)}; \rho_2 = \frac{-(\rho^2+1+\sqrt{3}\rho^2)^2}{4\sqrt{3}\rho^2(\rho^2+1)}, \text{ см. 1.2 50} \right].$$

$$8. \int \frac{dx}{x\sqrt{x^6+1}} = \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\psi, k) + \frac{2-\sqrt{3}}{4\sqrt[4]{3}} \Pi\left(\psi, -\frac{3+2\sqrt{3}}{6}, k\right) - \\ - \frac{1}{8} \ln \frac{x^4+2+2\sqrt{x^6+1}}{x^4+2-2\sqrt{x^6+1}}.$$

1.2.77. Интегралы вида $\int R(x, \sqrt{1-x^6}) dx$.

Условие: $0 \leq x \leq 1$.

Обозначения: $\varphi = \arccos \frac{(1+\sqrt{3})x^2-1}{(-1+\sqrt{3})x^2+1}, k_1 = \sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2};$

$\Phi = \arccos \frac{x^2-1+\sqrt{3}}{x^2-1-\sqrt{3}}, k_2 = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$

$$1. \int \frac{dx}{\sqrt{1-x^6}} = \frac{-1}{2\sqrt[4]{3}} F(\varphi, k_1). \quad 2. \int \frac{x}{\sqrt{1-x^6}} dx = \frac{1}{2\sqrt[4]{3}} F(\psi, k_2).$$

$$3. \int \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3} \arcsin(x^3).$$

$$4. \int \frac{x^3}{\sqrt{1-x^6}} dx = \frac{-\sqrt{1-x^6}}{x^2-1-\sqrt{3}} - \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\psi, k_2) + \sqrt[4]{3} E(\psi, k_2).$$

$$5. \int \frac{x^4}{\sqrt{1-x^6}} dx = \frac{(1-\sqrt{3})x\sqrt{1-x^6}}{2(\sqrt{3}x^2-x^2+1)} + \frac{1+\sqrt{3}}{4\sqrt[4]{3}} F(\varphi, k_1) - \frac{\sqrt[4]{3}}{2} E(\varphi, k_1).$$

$$6. \int \frac{x^5}{\sqrt{1-x^6}} dx = \frac{-1}{3} \sqrt{1-x^6}.$$

$$7. \int \frac{dx}{(x-\rho) \sqrt{1-x^6}} = \frac{1}{2} \int \frac{dt}{(t-\rho^2) \sqrt{1-t^3}} + \frac{\rho}{2} \int \frac{dt}{(t-\rho^2) \sqrt{t-t^4}} \\ [t=x^2; \text{ см. 1.2.70 и 1.2.74}].$$

$$8. \int \frac{dx}{x \sqrt{1-x^6}} = \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\psi, k_2) + \frac{2-\sqrt{3}}{4\sqrt[4]{3}} \Pi\left(\psi, \frac{3+2\sqrt{3}}{6}, k_2\right) + \\ + \frac{1}{8} \ln \frac{x^4+2-2\sqrt{1-x^6}}{x^4+2+2\sqrt{1-x^6}}.$$

1.2.78. Интегралы вида $\int R(x, \sqrt[3]{x^2 \pm 1}) dx$.

$$1. \int R(x, \sqrt[3]{x^2+1}) dx = \\ = -2\sqrt[4]{27} \int R\left(2\sqrt[4]{27} \frac{\sin \varphi \sqrt{1-k^2 \sin^2 \varphi}}{(1-\cos \varphi)^2}, \frac{(\sqrt{3}-1) \cos \varphi + \sqrt{3}+1}{1-\cos \varphi}\right) \times \\ \times \frac{2+\cos \varphi - 2k^2 \sin^2 \varphi}{(1-\cos \varphi)^2 \sqrt{1-k^2 \sin^2 \varphi}} d\varphi \\ \left[\sin \varphi = \frac{2\sqrt[4]{3} (\sqrt[3]{x^2+1} - 1)^{1/2}}{\sqrt[3]{x^2+1} + \sqrt{3}-1}; \cos \varphi = \frac{\sqrt[3]{x^2+1} - \sqrt{3}-1}{\sqrt[3]{x^2+1} + \sqrt{3}-1}; k = \frac{\sqrt{2-\sqrt{3}}}{2} \right].$$

$$2. \int R(x, \sqrt[3]{x^2-1}) dx = \\ = -2\sqrt[4]{27} \int R\left(2\sqrt[4]{27} \frac{\sin \varphi \sqrt{1-k^2 \sin^2 \varphi}}{(1-\cos \varphi)^2}, \frac{(1+\sqrt{3}) \cos \varphi + \sqrt{3}-1}{1-\cos \varphi}\right) \times \\ \times \frac{2+\cos \varphi - 2k^2 \sin^2 \varphi}{(1-\cos \varphi)^2 \sqrt{1-k^2 \sin^2 \varphi}} d\varphi \\ \left[\sin \varphi = \frac{2\sqrt[4]{3} (\sqrt[3]{x^2-1} + 1)^{1/2}}{\sqrt[3]{x^2-1} + 1 + \sqrt{3}}; \cos \varphi = \frac{\sqrt[3]{x^2-1} + 1 - \sqrt{3}}{\sqrt[3]{x^2-1} + 1 + \sqrt{3}}; k = \frac{\sqrt{2+\sqrt{3}}}{2} \right].$$

1.2.79. Интегралы вида $\int R(x, \sqrt[4]{x^2 \pm 1}) dx, \int R(x, \sqrt[4]{1-x^2}) dx$.

Обозначения: $\varphi = \arccos \frac{1}{\sqrt[4]{x^2+1}}, \psi = \arccos \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}},$
 $\lambda = \arccos \sqrt[4]{1-x^2}.$

$$1. \int_0^x \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2x}{\sqrt[4]{x^2+1}} \quad [x > 0].$$

$$2. \int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2} F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > 0].$$

$$3. \int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \sqrt{2} F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > 0].$$

$$4. \int_0^x \frac{x^2 dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2x}{\sqrt[4]{x^2+1}} \quad [x > 0].$$

$$5. \int_1^x \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\psi, \frac{1}{\sqrt{2}}\right) - 2E\left(\psi, \frac{1}{\sqrt{2}}\right) + \frac{2x^4\sqrt{x^2-1}}{1+\sqrt{x^2-1}} \quad [x > 1].$$

$$6. \int_1^x \frac{dx}{\sqrt[4]{(x^2-1)^3}} = F\left(\psi, \frac{1}{\sqrt{2}}\right) \quad [x > 1].$$

$$7. \int_1^x \frac{dx}{x^2 \sqrt[4]{x^2-1}} = E\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1}{2} F\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}}{x} \quad [x > 1].$$

$$8. \int_1^x \frac{dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{x^2-1}} = \frac{1}{2} \left[F\left(\psi, \frac{1}{\sqrt{2}}\right) - E\left(\psi, \frac{1}{\sqrt{2}}\right) \right] \quad [x > 1].$$

$$9. \int_1^x \frac{x^2 dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{(x^2-1)^3}} = E\left(\psi, \frac{1}{\sqrt{2}}\right) \quad [x > 1].$$

$$10. \int_0^x \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] \quad [0 < x \leq 1].$$

$$11. \int_0^x \frac{x^2 dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] - \frac{2x}{5} \sqrt[4]{(1-x^2)^3} \quad [0 < x \leq 1].$$

$$12. \int_0^x \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} F\left(\lambda, \frac{1}{\sqrt{2}}\right) \quad [0 < x \leq 1].$$

$$13. \int_0^x \frac{x^2 dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\lambda, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} x \sqrt[4]{1-x^2} \quad [0 < x \leq 1].$$

$$14. \int_0^x \frac{dx}{(1+\sqrt{1-x^2})\sqrt[4]{1-x^2}} = \sqrt{2} \left[F\left(\lambda, \frac{1}{\sqrt{2}}\right) - E\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] + \frac{x \sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}} \quad [0 < x \leq 1].$$

$$15. \int_0^x \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] - \frac{2x \sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}} \quad [0 < x \leq 1].$$

1.2.80. Интегралы вида $\int R(\sqrt[4]{\pm(x-a)}, \sqrt[4]{(x-b)}) dx$.

Обозначения. $\varphi = \arccos \frac{a-b-2\sqrt{(x-a)(x-b)}}{a-b+2\sqrt{(x-a)(x-b)}}$,

$$\psi = \arccos \sqrt[4]{\frac{4(a-x)(x-b)}{(a-b)^3}}.$$

$$1. \int_a^x \frac{dx}{\sqrt[4]{(x-a)(x-b)}} = \sqrt{\frac{a-b}{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \\ + \frac{2(2x-a-b)\sqrt[4]{(x-a)(x-b)}}{a-b+2\sqrt{(x-a)(x-b)}} \quad [x > a > b].$$

$$2. \int_a^x \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > a > b].$$

$$3. \int_b^x \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b} \left\{ 2 \left[E\left(\frac{1}{\sqrt{2}}\right) + E\left(\psi, \frac{1}{\sqrt{2}}\right) \right] - \right. \\ \left. - K\left(\frac{1}{\sqrt{2}}\right) - F\left(\psi, \frac{1}{\sqrt{2}}\right) \right\} \quad [a \geq x > b].$$

$$4. \int_b^x \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}} \left[K\left(\frac{1}{\sqrt{2}}\right) + F\left(\psi, \frac{1}{\sqrt{2}}\right) \right] \quad [a \geq x > b].$$

1.2.81. Интегралы вида $\int R(\sqrt[4]{x^4+1}) dx$.

$$1. \int \frac{dx}{\sqrt[4]{x^4+1}} = \frac{1}{4} \ln \frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} - \frac{1}{2} \operatorname{arctg} \frac{\sqrt[4]{x^4+1}}{x}.$$

$$2. \int x \sqrt[4]{x^4+1} dx = \frac{1}{3} x^2 \sqrt[4]{x^4+1} + \frac{1}{3\sqrt{2}} F\left(\arccos \frac{1}{\sqrt[4]{x^4+1}}, \frac{1}{\sqrt{2}}\right).$$

1.3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

1.3.1. Интегралы вида $\int f(e^{ax}) dx$.

$$1. \int f(e^{ax}) dx = \frac{1}{a} \int \frac{f(t)}{t} dt \quad [t = e^{ax}].$$

$$2. \int f(b^x) dx = \int f(e^{x \ln b}) dx \quad [b > 0; b \neq 1].$$

$$3. \int e^{ax} dx = \frac{1}{a} e^{ax}.$$

$$4. \int b^x dx = \frac{b^x}{\ln b} \quad [b > 0; b \neq 1].$$

$$5. \int \frac{dx}{e^{ax}+b} = \frac{1}{ab} [ax - \ln(e^{ax}+b)].$$

$$6. \int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}} \operatorname{arctg} \left(e^{ax} \sqrt{\frac{b}{c}} \right) \quad [bc > 0].$$

$$= \frac{1}{2a\sqrt{-bc}} \ln \frac{c + e^{ax} \sqrt{-bc}}{c - e^{ax} \sqrt{-bc}} \quad [bc < 0].$$

$$7. \int \frac{e^{ax} - 1}{e^{ax} + 1} dx = \frac{2}{a} \ln (e^{ax/2} + e^{-ax/2}).$$

$$8. \int \frac{dx}{\sqrt{b + ce^{ax}}} = \frac{1}{a\sqrt{b}} \ln \frac{\sqrt{b + ce^{ax}} - \sqrt{b}}{\sqrt{b + ce^{ax}} + \sqrt{b}} \quad [b > 0]$$

$$= \frac{2}{a\sqrt{-b}} \operatorname{arctg} \frac{\sqrt{b + ce^{ax}}}{\sqrt{-b}} \quad [b < 0].$$

1.3.2. Интегралы вида $\int f(x, e^{ax}) dx$.

$$1. \int x^\lambda e^{ax} dx = \frac{1}{a} x^\lambda e^{ax} - \frac{\lambda}{a} \int x^{\lambda-1} e^{ax} dx.$$

$$2. \int \frac{1}{x^\lambda} e^{ax} dx = -\frac{e^{ax}}{(\lambda-1)x^{\lambda-1}} + \frac{a}{\lambda-1} \int \frac{e^{ax}}{x^{\lambda-1}} dx.$$

$$3. \int_0^x x^{\lambda-1} e^{-ax} dx = \frac{1}{a^\lambda} \gamma(\lambda, ax) \quad [\operatorname{Re} \lambda > 0].$$

$$4. \int_x^\infty x^{\lambda-1} e^{-ax} dx = \frac{1}{a^\lambda} \Gamma(\lambda, ax) \quad [\operatorname{Re} a > 0]$$

$$5. \int_x^\infty x^{\lambda-n-1} e^{-ax} dx = \frac{(-1)^n a^{n-1}}{(n-\lambda)(n-\lambda-1)\dots(1-\lambda)} \int_x^\infty x^{\lambda-1} e^{-ax} dx +$$

$$+ \frac{e^{-ax}}{x^{n-\lambda}} \sum_{k=0}^{n-1} \frac{(-1)^k (ax)^k}{(n-\lambda)(n-\lambda-1)\dots(n-\lambda-k)} \quad [\operatorname{Re} a > 0]$$

$$6. \int x^n e^{ax} dx = e^{ax} \left[\frac{x^n}{a} + \sum_{k=1}^n (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k} \right].$$

$$7. \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right).$$

$$8. \int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$$

$$9. \int P(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m \frac{(-1)^k}{a^k} \frac{d^k}{dx^k} P(x),$$

где $P(x)$ — многочлен степени m .

$$10. \int x^{n-1/2} e^{-ax} dx = (2n-1)!! \sqrt{\pi} 2^{-n} a^{-n-1/2} \operatorname{erf}(\sqrt{ax}) - \\ - \frac{1}{a} x^{n-1/2} e^{-ax} \sum_{k=0}^{n-1} \frac{(2n-1)!!}{(2n-2k-1)!!} (2ax)^{-k},$$

$$11. \int \frac{e^{ax}}{x^n} dx = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \operatorname{Ei}(ax).$$

$$12. \int \frac{e^{ax}}{x} dx = \operatorname{Ei}(ax) \quad [a \neq 0].$$

$$13. = \ln|x| + \sum_{k=1}^{\infty} \frac{(ax)^k}{k!k}.$$

$$14. \int_x^{\infty} \frac{e^{-ax}}{x} dx = -\operatorname{Ei}(-ax) \quad [a > 0].$$

$$15. \int \frac{e^{ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erfi}(\sqrt{ax}) \quad [a > 0].$$

$$16. \int \frac{e^{-ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erf}(\sqrt{ax}) \quad [a > 0].$$

$$17. \int \frac{e^{-ax}}{x^{3/2}} dx = -\frac{2}{\sqrt{x}} e^{-ax} - 2\sqrt{\pi a} \operatorname{erf}(\sqrt{ax}) \quad [a > 0].$$

$$18. \int_x^{\infty} x^{-n-1/2} e^{-ax} dx = \frac{(-2)^n a^{n-1/2} \sqrt{\pi}}{(2n-1)!!} \operatorname{erfc}(\sqrt{ax}) + \\ + 2 \frac{e^{-ax}}{x^{n-1/2}} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!! (-2ax)^k}{(2n-1)!!} \quad [a > 0].$$

$$19. \int_x^{\infty} \frac{e^{-ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erfc}(\sqrt{ax}) \quad [a > 0].$$

$$20. \int \frac{e^{ax}}{x^2} dx = -\frac{e^{ax}}{x} + a \operatorname{Ei}(ax).$$

$$21. \int \frac{e^{ax}}{x^3} dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2} \operatorname{Ei}(ax).$$

$$22. \int \frac{e^{ax} dx}{x+b} = e^{-ab} \operatorname{Ei}(ab+ax).$$

$$23. \int \frac{x^m e^{ax} dx}{(x+b)^n} = (-1)^m b^m e^{-ab} \sum_{k=0}^m \frac{(-1)^k}{b^k} \binom{m}{k} \int t^{k-n} e^{at} dt \quad [t = x+b].$$

$$24. \int \frac{x e^{ax} dx}{x+b} = \frac{e^{ax}}{a} - b e^{-ab} \operatorname{Ei}(ax+ab).$$

$$25. \int \frac{x^2 e^{ax} dx}{x+b} = \frac{ax-ab-1}{a^2} e^{ax} + b^2 e^{-ab} \operatorname{Ei}(ax+ab).$$

$$26. \int \frac{e^{ax} dx}{(x+b)^2} = -\frac{e^{ax}}{x+b} + a e^{-ab} \operatorname{Ei}(ax+ab).$$

$$27. \int \frac{x e^{ax} dx}{(x+b)^2} = \frac{b e^{ax}}{x+b} + (1-ab) e^{-ab} \operatorname{Ei}(ax+ab).$$

$$28. \int \frac{x^2 e^{ax} dx}{(x+b)^2} = \left(\frac{1}{a} - \frac{b^2}{x+b} \right) e^{ax} + (ab^2 - 2b) e^{-ab} \operatorname{Ei}(ax+ab).$$

$$29. \int \frac{e^{ix} dx}{x^2+a^2} = \frac{i}{2a} [e^a \operatorname{Ei}(-a+ix) - e^{-a} \operatorname{Ei}(a+ix)].$$

$$30. \int \frac{x e^{ix} dx}{x^2+a^2} = \frac{1}{2} [e^{-a} \operatorname{Ei}(a+ix) + e^a \operatorname{Ei}(-a+ix)].$$

$$31. \int \frac{e^x dx}{x^2+a^2} = \frac{1}{a} \operatorname{Im} [e^{ia} \operatorname{Ei}(x-ia)].$$

$$32. \int \frac{x e^x dx}{x^2+a^2} = \operatorname{Re} [e^{ia} \operatorname{Ei}(x-ia)].$$

$$33. \int_0^x \frac{e^x - 1}{x} dx = \operatorname{Ei}(x) - \mathbf{C} - \ln x$$

[$x > 0$].

$$34. \int_0^x \frac{(1 - e^{-ix}) dx}{x} = \mathbf{C} + \ln x - \operatorname{ci}(x) + i \operatorname{Si}(x).$$

1.3.3. Интегралы вида $\int f(x, e^{-a^2 x^2}) dx$.

$$1. \int_0^x e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erf}(ax).$$

$$2. \int_0^x e^{a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erfi}(ax).$$

$$3. \int_x^\infty e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erfc}(ax).$$

$$4. \int x^p e^{-ax^2} dx = -\frac{x^{p-1}}{2a} e^{-ax^2} + \frac{p-1}{2a} \int x^{p-2} e^{-ax^2} dx.$$

$$5. \int x^{2n} e^{-a^2 x^2} dx =$$

$$= -\frac{x^{2n-1} e^{-a^2 x^2} (2n-1)!!}{2a^2} \sum_{k=0}^{n-1} \frac{1}{2^k (2n-2k-1)!! (ax)^{2k}} +$$

$$+ \frac{\sqrt{\pi}}{2^{n+1} a^{2n+1}} \operatorname{erf}(ax).$$

6. $\int x^{2n+1} e^{-ax^2} dx = -\frac{x^{2n} e^{-ax^2}}{2a} \sum_{k=0}^n \frac{n!}{(n-k)! a^k x^{2k}}.$
7. $\int x e^{-ax^2} dx = -\frac{1}{2a} e^{-ax^2}.$
8. $\int x^2 e^{-a^2 x^2} dx = -\frac{1}{2a^2} x e^{-a^2 x^2} + \frac{\sqrt{\pi}}{4a^3} \operatorname{erf}(ax).$
9. $\int x^3 e^{-ax^2} dx = -\frac{ax^2 + 1}{2a^2} e^{-ax^2}.$
10. $\int \frac{e^{-ax^2}}{x^p} dx = -\frac{e^{-ax^2}}{(p-1)x^{p-1}} - \frac{2a}{p-1} \int \frac{e^{-ax^2}}{x^{p-2}} dx.$
11. $\int \frac{e^{-a^2 x^2}}{x^{2n}} dx = \frac{1}{2a^2 \Gamma(n+1/2)} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k \Gamma\left(n-k+\frac{1}{2}\right) a^{2k} x^{2k-2n-1} +$
 $+ \frac{(-1)^n \pi a^{2n-1}}{2\Gamma(n+1/2)} \operatorname{erf}(ax).$
12. $\int \frac{e^{-a^2 x^2}}{x^{2n+1}} dx = \frac{1}{2a^2 n!} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k (n-k)! a^{2k} x^{2k-2n-2} + \frac{(-a^2)^n}{2(n!)} \operatorname{Ei}(-a^2 x^2).$
13. $\int \frac{e^{-a^2 x^2}}{x} dx = \frac{1}{2} \operatorname{Ei}(-a^2 x^2).$
14. $\int \frac{e^{-a^2 x^2}}{x^2} dx = -\frac{1}{x} e^{-a^2 x^2} - \sqrt{\pi} a \operatorname{erf}(ax).$
15. $\int \frac{e^{-a^2 x^2}}{x^3} dx = -\frac{1}{2x^2} e^{-a^2 x^2} - \frac{a^2}{2} \operatorname{Ei}(-a^2 x^2).$
16. $\int f(x) e^{-(ax^2+bx+c)} dx =$
 $= \frac{1}{\sqrt{a}} e^{(b^2-4ac)/(4a)} \int f\left(\frac{2\sqrt{a}t-b}{2a}\right) e^{-t^2} dt \quad \left[t = \sqrt{a}\left(x + \frac{b}{2a}\right), a > 0\right].$
17. $\int e^{-(ax^2+bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)} \operatorname{erf}\left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right).$
18. $\int (x-a)^n e^{-c^2(x-b)^2} dx =$
 $= \sum_{k=0}^n \binom{n}{k} \frac{(b-a)^{n-k}}{c^{k+1}} \int t^k e^{-t^2} dt \quad [t = c(x-b), c \neq 0].$
19. $\int x^n e^{-a^2 x^2+bx} dx =$
 $= \frac{1}{a^{n+1}} e^{b^2/(4a^2)} \sum_{k=0}^n \binom{n}{k} \left(\frac{b}{2a}\right)^{n-k} \int t^k e^{-t^2} dt \quad \left[t = ax - \frac{b}{2a}\right].$

$$20. \int_0^x e^{-a^2 x^2 - b^2/x^2} dx = \frac{\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erf} \left(ax + \frac{b}{x} \right) + e^{-2ab} \operatorname{erf} \left(ax - \frac{b}{x} \right) - e^{2ab} + e^{-2ab} \right] \quad [\operatorname{Re} b^2 > 0].$$

1.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1.4.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

1. $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx = \int R\left(\frac{t^2-1}{2t}, \frac{t^2+1}{2t}, \frac{t^2-1}{t^2+1}, \frac{t^2+1}{t^2-1}\right) \frac{dt}{t} \quad [t = e^x].$
2. $\int R(\operatorname{sh} x, \operatorname{ch}^2 x) \operatorname{ch} x dx = \int R(t, 1+t^2) dt \quad [t = \operatorname{sh} x].$
3. $\int R(\operatorname{sh}^2 x, \operatorname{ch} x) \operatorname{sh} x dx = \int R(t^2-1, t) dt \quad [t = \operatorname{ch} x].$
4. $\int R(\operatorname{th} x, \operatorname{cth} x) dx = \int R\left(t, \frac{1}{t}\right) \frac{dt}{1-t^2} \quad [t = \operatorname{th} x].$

1.4.2. Интегралы вида $\int \operatorname{sh}^p x dx, \int \operatorname{ch}^p x dx$.

1. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx = \frac{1}{p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \mp \frac{p-1}{p} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p-2} dx.$
2. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2m} dx = (\mp 1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \frac{(\mp 1)^k}{2m-2k} \binom{2m}{k} \left\{ \frac{\operatorname{sh} (2m-2k)x}{\operatorname{ch} (2m-2k)x} \right\}.$
3. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2m+1} dx = \sum_{k=0}^m \frac{(\mp 1)^{m+k}}{2k+1} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k+1}.$
4. $= \frac{1}{2^{2m}} \sum_{k=0}^m \frac{(\mp 1)^k}{2m-2k+1} \binom{2m+1}{k} \left\{ \frac{\operatorname{ch} (2m-2k+1)x}{\operatorname{sh} (2m-2k+1)x} \right\}.$
5. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$
6. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = \frac{1}{4} \operatorname{sh} 2x \mp \frac{x}{2}.$
7. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \mp \frac{3}{4} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{1}{12} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} = \mp \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^3.$
8. $\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^4 dx = \frac{3}{8} x \mp \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x = \mp \frac{3}{8} x \mp \frac{3}{8} \operatorname{sh} x \operatorname{ch} x + \frac{1}{4} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$

$$\begin{aligned}
 9. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-p} dx &= \mp \frac{1}{p-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-p+1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \mp \frac{p-2}{p-1} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-p+2} dx. \\
 10. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m} dx &= \frac{1}{2m-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \left[\mp \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+1} + \right. \\
 &\quad \left. + \sum_{k=1}^{m-1} (\mp 1)^{k-1} \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k+1} \right]. \\
 11. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m-1} dx &= \frac{1}{2m} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \left[\mp \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m} + \right. \\
 &\quad \left. + \sum_{k=1}^{m-1} (\mp 1)^{k-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k} \right] + \\
 &\quad + (\mp 1)^m \frac{(2m-1)!!}{(2m)!!} \left\{ \ln \left| \operatorname{th} \frac{x}{2} \right| \right\} \left\{ \operatorname{arctg} \operatorname{sh} x \right\}.
 \end{aligned}$$

$$12. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = \left\{ \ln \left| \operatorname{th} \frac{x}{2} \right| \right\} \left\{ \operatorname{arctg} \operatorname{sh} x \right\} = \left\{ \frac{1}{2} \ln \frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1} \right\} \left\{ \frac{1}{2} \operatorname{arctg} e^x \right\}.$$

$$13. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = \mp \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\}.$$

$$14. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} dx = \mp \frac{1}{2} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} \mp \frac{1}{2} \left\{ \ln \left| \operatorname{th} \frac{x}{2} \right| \right\} \left\{ \operatorname{arctg} \operatorname{sh} x \right\}.$$

$$15. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-4} dx = -\frac{1}{3} \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\}^3 + \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\}.$$

1.4.3. Интегралы вида $\int \operatorname{sh}^p x \operatorname{ch}^q x dx$.

$$1. \int \operatorname{sh}^p x \operatorname{ch}^q x dx = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \operatorname{sh}^p x \operatorname{ch}^{q-2} x dx.$$

$$2. = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{p+q} - \frac{p-1}{p+q} \int \operatorname{sh}^{p-2} x \operatorname{ch}^q x dx.$$

$$3. = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{q+1} - \frac{p-1}{q+1} \int \operatorname{sh}^{p-2} x \operatorname{ch}^{q+2} x dx.$$

$$4. = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+1} - \frac{q-1}{p+1} \int \operatorname{sh}^{p+2} x \operatorname{ch}^{q-2} x dx.$$

$$5. = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{p+1} - \frac{p+q+2}{p+1} \int \operatorname{sh}^{p+2} x \operatorname{ch}^q x dx.$$

$$6. = -\frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \operatorname{sh}^p x \operatorname{ch}^{q+2} x dx.$$

$$7. \int \left\{ \frac{\operatorname{sh}^n x \operatorname{ch} x}{\operatorname{ch}^n x \operatorname{sh} x} \right\} dx = \frac{1}{n+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{n+1}.$$

- $$8. \int \frac{\{sh^p x \, ch^{2n} x\}}{\{ch^p x \, sh^{2n} x\}} dx = \frac{1}{2n+p} \left\{ \frac{sh \, x}{ch \, x} \right\}^{p+1} \left[\left\{ \frac{ch \, x}{sh \, x} \right\}^{2n-1} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-1)(2n-3) \dots (2n-2k+1)}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \left\{ \frac{ch \, x}{sh \, x} \right\}^{2n-2k-1} \right] + \\ + \frac{(\pm 1)^n (2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \left\{ \frac{sh \, x}{ch \, x} \right\}^p dx \quad [p \neq -2, -4, \dots, -2n].$$
- $$9. \int \frac{\{sh^p x \, ch^{2n+1} x\}}{\{ch^p x \, sh^{2n+1} x\}} dx = \frac{1}{2n+p+1} \left\{ \frac{sh \, x}{ch \, x} \right\}^{p+1} \left[\left\{ \frac{ch \, x}{sh \, x} \right\}^{2n} + \right. \\ \left. + \sum_{k=1}^n \frac{(\pm 1)^k 2^k n(n-1) \dots (n-k+1)}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \left\{ \frac{ch \, x}{sh \, x} \right\}^{2n-2k} \right] \\ [p \neq -1, -3, \dots, -(2n+1)].$$
- $$10. \int sh \, ax \, ch \, bx \, dx = \frac{ch \, (a+b)x}{2(a+b)} + \frac{ch \, (a-b)x}{2(a-b)}.$$
- $$11. \int sh \, ax \, ch \, ax \, dx = \frac{1}{4a} ch \, 2ax.$$
- $$12. \int \frac{\{sh \, x \, ch^p x\}}{\{ch \, x \, sh^p x\}} dx = \frac{1}{p+1} \left\{ \frac{ch \, x}{sh \, x} \right\}^{p+1}.$$
- $$13. \int sh^2 x \, ch^2 x \, dx = -\frac{x}{8} + \frac{1}{32} sh \, 4x.$$
- $$14. \int \frac{\{sh^2 x \, ch^3 x\}}{\{ch^2 x \, sh^3 x\}} dx = \frac{1}{5} \left\{ \frac{ch^2 x \, sh^3 x}{sh^2 x \, ch^3 x} \right\} \pm \frac{2}{15} \left\{ \frac{sh^3 x}{ch^3 x} \right\}.$$
- $$15. \int \frac{\{sh^2 x \, ch^4 x\}}{\{ch^2 x \, sh^4 x\}} dx = \mp \frac{x}{16} - \frac{1}{64} sh \, 2x \pm \frac{1}{64} sh \, 4x + \frac{1}{192} sh \, 6x.$$
- $$16. \int sh^3 x \, ch^3 x \, dx = \frac{1}{6} sh^6 x + \frac{1}{4} sh^4 x = \frac{1}{6} ch^6 x - \frac{1}{4} ch^4 x.$$
- $$17. \int \frac{\{sh^3 x \, ch^4 x\}}{\{ch^3 x \, sh^4 x\}} dx = \frac{1}{7} \left\{ \frac{sh \, x}{ch \, x} \right\}^2 \mp \frac{2}{35} \left\{ \frac{sh^2 x \, ch^5 x}{ch^2 x \, sh^5 x} \right\}.$$
- $$18. \int sh^4 x \, ch^4 x \, dx = \frac{3x}{128} - \frac{1}{128} sh \, 4x + \frac{1}{1024} sh \, 8x.$$
- $$19. \int sh \, ax \, sh \, bx \, sh \, cx \, dx = \frac{ch \, (a+b+c)x}{4(a+b+c)} - \frac{ch \, (-a+b+c)x}{4(-a+b+c)} - \\ - \frac{ch \, (a-b+c)x}{4(a-b+c)} - \frac{ch \, (a+b-c)x}{4(a+b-c)}.$$
- $$20. \int sh \, ax \, sh \, bx \, ch \, cx \, dx = \frac{sh \, (a+b+c)x}{4(a+b+c)} - \frac{sh \, (-a+b+c)x}{4(-a+b+c)} - \\ - \frac{sh \, (a-b+c)x}{4(a-b+c)} + \frac{sh \, (a+b-c)x}{4(a+b-c)}.$$
- $$21. \int sh \, ax \, ch \, bx \, ch \, cx \, dx = \frac{ch \, (a+b+c)x}{4(a+b+c)} - \frac{ch \, (-a+b+c)x}{4(-a+b+c)} + \\ + \frac{ch \, (a-b+c)x}{4(a-b+c)} + \frac{ch \, (a+b-c)x}{4(a+b-c)}.$$

$$22. \int \operatorname{ch} ax \operatorname{ch} bx \operatorname{ch} cx \, dx = \frac{\operatorname{sh} (a+b+c)x}{4(a+b+c)} + \frac{\operatorname{sh} (-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{sh} (a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh} (a+b-c)x}{4(a+b-c)}.$$

1.4.4. Интегралы вида $\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x} dx$, $\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x} dx$.

$$1. \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x} dx = \frac{1}{p-q} \frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} - \frac{p-1}{p-q} \int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^q x} dx.$$

$$2. = \frac{1}{q-1} \frac{\operatorname{sh}^{p+1} x}{\operatorname{ch}^{q-1} x} - \frac{p-q+2}{q-1} \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{q-2} x} dx.$$

$$3. = -\frac{1}{q-1} \frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} + \frac{p-1}{q-1} \int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^{q-2} x} dx.$$

$$4. \int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x} dx = \frac{1}{q-p} \frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{q-p} \int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^p x} dx.$$

$$5. = -\frac{1}{p-1} \frac{\operatorname{ch}^{q+1} x}{\operatorname{sh}^{p-1} x} + \frac{q-p+2}{p-1} \int \frac{\operatorname{ch}^q x}{\operatorname{sh}^{p-2} x} dx.$$

$$6. = -\frac{1}{p-1} \frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{p-1} \int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^{p-2} x} dx.$$

$$7. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-2n} x}{\operatorname{ch}^p x \operatorname{sh}^{-2n} x} \right\} dx = \pm \frac{1}{2n-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} \left[\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+1} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+2k+1} \right] + \\ + (\pm 1)^n \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx.$$

$$8. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-2n-1} x}{\operatorname{ch}^p x \operatorname{sh}^{-2n-1} x} \right\} dx = \pm \frac{1}{2n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} \left[\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+2k} \right] + \\ + (\pm 1)^n \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-1} x}{\operatorname{ch}^p x \operatorname{sh}^{-1} x} \right\} dx.$$

$$9. \int \left\{ \frac{\operatorname{sh}^{2m} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-1} + (\mp 1)^m \left\{ \frac{\operatorname{arctg} \operatorname{sh} x}{\ln |\operatorname{th} (x/2)|} \right\}.$$

$$10. \int \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k} + \\ + (\mp 1)^m \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\} = \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k} + (\mp 1)^m \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$11. \int \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n} x} \right\} dx = \sum_{k=0}^m \frac{(\mp 1)^{m+k}}{2k-2n+1} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n+1}.$$

$$12. \int \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n-1} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n-1} x} \right\} dx = \sum_{\substack{k=0 \\ k \neq n}}^m \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} +$$

$$+ (\mp 1)^{m+n} \binom{m}{n} \ln \left\{ \left| \frac{\operatorname{ch} x}{\operatorname{sh} x} \right| \right\} \quad [n \leq m].$$

$$13. = \sum_{k=0}^m \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} \quad [n > m].$$

$$14. \int \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-p} x}{\operatorname{ch} x \operatorname{sh}^{-p} x} \right\} dx = -\frac{1}{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-p+1}.$$

$$15. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-p-2} x}{\operatorname{ch}^p x \operatorname{sh}^{-p-2} x} \right\} dx = \pm \frac{1}{p+1} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{p+1}.$$

$$16. \int \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-1} x}{\operatorname{ch} x \operatorname{sh}^{-1} x} \right\} dx = \ln \left\{ \left| \frac{\operatorname{ch} x}{\operatorname{sh} x} \right| \right\}.$$

$$17. \int \left\{ \frac{\operatorname{sh}^2 x \operatorname{ch}^{-1} x}{\operatorname{ch}^2 x \operatorname{sh}^{-1} x} \right\} dx = \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{\mp} \left\{ \frac{\operatorname{arctg}(\operatorname{sh} x)}{\ln |\operatorname{th}(x/2)|} \right\}.$$

$$18. \int \left\{ \frac{\operatorname{sh}^3 x \operatorname{ch}^{-1} x}{\operatorname{ch}^3 x \operatorname{sh}^{-1} x} \right\} dx = \frac{1}{2} \operatorname{ch}^2 x \mp \ln \left\{ \left| \frac{\operatorname{ch} x}{\operatorname{sh} x} \right| \right\}.$$

$$19. \int \left\{ \frac{\operatorname{sh}^4 x \operatorname{ch}^{-1} x}{\operatorname{ch}^4 x \operatorname{sh}^{-1} x} \right\} dx = \frac{1}{3} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 \mp \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} + \left\{ \frac{\operatorname{arctg}(\operatorname{sh} x)}{\ln |\operatorname{th}(x/2)|} \right\}.$$

$$20. \int \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} dx = -\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1}.$$

$$21. \int \left\{ \frac{\operatorname{sh}^2 x \operatorname{ch}^{-2} x}{\operatorname{ch}^2 x \operatorname{sh}^{-2} x} \right\} dx = x - \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}.$$

$$22. \int \left\{ \frac{\operatorname{sh}^3 x \operatorname{ch}^{-2} x}{\operatorname{ch}^3 x \operatorname{sh}^{-2} x} \right\} dx = \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \pm \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1}.$$

$$23. \int \left\{ \frac{\operatorname{sh}^4 x \operatorname{ch}^{-2} x}{\operatorname{ch}^4 x \operatorname{sh}^{-2} x} \right\} dx = \mp \frac{3}{2} x + \frac{1}{4} \operatorname{sh} 2x \pm \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}.$$

$$24. \int \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-3} x}{\operatorname{ch} x \operatorname{sh}^{-3} x} \right\} dx = \pm \frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2.$$

$$25. \int \left\{ \frac{\operatorname{sh}^2 x \operatorname{ch}^{-3} x}{\operatorname{ch}^2 x \operatorname{sh}^{-3} x} \right\} dx = -\frac{1}{2} \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} + \frac{1}{2} \left\{ \frac{\operatorname{arctg}(\operatorname{sh} x)}{\ln |\operatorname{th}(x/2)|} \right\}.$$

$$26. \int \left\{ \frac{\operatorname{sh}^3 x \operatorname{ch}^{-3} x}{\operatorname{ch}^3 x \operatorname{sh}^{-3} x} \right\} dx = -\frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 + \ln \left\{ \left| \frac{\operatorname{ch} x}{\operatorname{sh} x} \right| \right\}.$$

$$27. \int \left\{ \frac{\operatorname{sh}^4 x \operatorname{ch}^{-3} x}{\operatorname{ch}^4 x \operatorname{sh}^{-3} x} \right\} dx = \pm \frac{1}{2} \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} + \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \mp \frac{3}{2} \left\{ \frac{\operatorname{arctg}(\operatorname{sh} x)}{\ln |\operatorname{th}(x/2)|} \right\}.$$

$$28. \int \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-4} x}{\operatorname{ch} x \operatorname{sh}^{-4} x} \right\} dx = -\frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3}.$$

$$29. \int \left\{ \frac{\operatorname{sh}^2 x \operatorname{ch}^{-4} x}{\operatorname{ch}^2 x \operatorname{sh}^{-4} x} \right\} dx = \pm \frac{1}{3} \left\{ \operatorname{th} x \right\}^3.$$

$$30. \int \left\{ \frac{\operatorname{sh}^3 x \operatorname{ch}^{-4} x}{\operatorname{ch}^3 x \operatorname{sh}^{-4} x} \right\} dx = - \left\{ \operatorname{ch} x \right\}^{-1} \pm \frac{1}{3} \left\{ \operatorname{sh} x \right\}^{-3}.$$

$$31. \int \left\{ \frac{\operatorname{sh}^4 x \operatorname{ch}^{-4} x}{\operatorname{ch}^4 x \operatorname{sh}^{-4} x} \right\} dx = - \frac{1}{3} \left\{ \operatorname{cth} x \right\}^3 - \left\{ \operatorname{cth} x \right\} + x.$$

1.4.5. Интегралы вида $\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x}$.

$$1. \int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x} = - \frac{1}{(p-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} - \frac{p+q-2}{p-1} \int \frac{dx}{\operatorname{sh}^{p-2} x \operatorname{ch}^q x}.$$

$$2. = \frac{1}{(q-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^{q-2} x}.$$

$$3. \int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2m-2k-1} \binom{m+n-1}{k} \operatorname{th}^{2k-2m+1} x.$$

$$4. \int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \operatorname{th}^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln |\operatorname{th} x|.$$

$$5. \int \left\{ \frac{\operatorname{sh}^{-2m} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{-2m} x \operatorname{sh}^{-1} x} \right\} dx = \\ = \sum_{k=1}^m \frac{(-1)^k}{2m-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k-1} + (-1)^m \left\{ \operatorname{arctg} (\operatorname{sh} x) \right\} \left\{ \ln |\operatorname{th} (x/2)| \right\}.$$

$$6. \int \left\{ \frac{\operatorname{sh}^{-2m-1} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{-2m-1} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(-1)^k}{2m-2k+2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k-2} + (-1)^m \ln |\operatorname{th} x|.$$

$$7. \int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln |\operatorname{th} x|.$$

$$8. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-2} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-2} x} \right\} dx = \pm \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} \pm \left\{ \ln |\operatorname{th} (x/2)| \right\} \left\{ \operatorname{arctg} (\operatorname{sh} x) \right\}.$$

$$9. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-3} x} \right\} dx = - \frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 + \ln \left| \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} \right|.$$

$$10. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-4} x} \right\} dx = \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} \pm \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3} + \left\{ \ln |\operatorname{th} (x/2)| \right\} \left\{ \operatorname{arctg} (\operatorname{sh} x) \right\}.$$

$$11. \int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2 \operatorname{cth} 2x.$$

$$12. \int \left\{ \frac{\operatorname{sh}^{-2} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-2} x \operatorname{sh}^{-3} x} \right\} dx = - \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} - \frac{1}{2} \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} - \frac{3}{2} \left\{ \ln |\operatorname{th} (x/2)| \right\} \left\{ \operatorname{arctg} (\operatorname{sh} x) \right\}.$$

$$13. \int \left\{ \frac{\operatorname{sh}^{-2} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-2} x \operatorname{sh}^{-4} x} \right\} dx = \pm \frac{1}{3} \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-3} x} \right\} \mp \frac{8}{3} \operatorname{cth} 2x.$$

$$14. \int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^3 x} = \frac{1}{2} \operatorname{th}^2 x - \frac{1}{2} \operatorname{cth}^2 x - 2 \ln |\operatorname{th} x|.$$

$$15. \int \left\{ \frac{\operatorname{sh}^{-3} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-3} x \operatorname{sh}^{-4} x} \right\} dx =$$

$$= -\frac{1}{2} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2} - \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3} - \frac{1}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-2} x}{\operatorname{sh} x \operatorname{ch}^2 x} \right\} - \frac{5}{2} \left\{ \ln \left| \operatorname{th} (x/2) \right| \right\}.$$

$$16. \int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^4 x} = 8 \operatorname{cth} 2x - \frac{8}{3} \operatorname{cth}^3 2x.$$

1.4.6. Интегралы вида $\int \operatorname{th}^p x dx$, $\int \operatorname{cth}^p x dx$.

$$1. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^p dx = -\frac{1}{p-1} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{p-1} + \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{p-2} dx.$$

$$2. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n} dx = -\sum_{k=1}^n \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n-2k+1} + x.$$

$$3. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n+1} dx = -\sum_{k=1}^n \frac{1}{2n-2k+2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n-2k+2} + \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$4. = -\frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2k} + \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$5. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} dx = \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$6. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 dx = x - \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}.$$

1.4.7. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx$.

$$1. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} dx =$$

$$= \frac{Bc - Cb + (Ac - Ca) \operatorname{ch} x + (Ab - Ba) \operatorname{sh} x}{(1-n)(a^2 - b^2 + c^2)(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \operatorname{ch} x - (n-2)(Ac - Ca) \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} dx$$

[$a^2 + c^2 \neq b^2$].

$$2. = \frac{Bc - Cb - Ca \operatorname{ch} x - Ba \operatorname{sh} x}{(n-1)a(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} +$$

$$+ \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2} \right] (c \operatorname{ch} x + b \operatorname{sh} x) \frac{(n-1)!}{(2n-1)!!} \times$$

$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)! a^k} \cdot \frac{1}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-k}}$$

[$a^2 + c^2 = b^2$].

$$3. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x + c \operatorname{sh} x} dx = \frac{Cb - Bc}{b^2 - c^2} \ln(a + b \operatorname{ch} x + c \operatorname{sh} x) +$$

$$+ \frac{Bb - Cc}{b^2 - c^2} x + \left(A - a \frac{Bb - Cc}{b^2 - c^2} \right) \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x}$$

[$b^2 \neq c^2$].

$$4. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x \pm b \operatorname{sh} x} dx = \frac{C \mp B}{2a} (\operatorname{ch} x \mp \operatorname{sh} x) +$$

$$+ \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2} \right] x + \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2} \right] \ln (a + b \operatorname{ch} x \pm b \operatorname{sh} x)$$

[ab ≠ 0]

$$5. \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \operatorname{arctg} \frac{(b-a) \operatorname{th} \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}}$$

[b² > a² + c²; a ≠ b]

$$6. = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \operatorname{th} \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \operatorname{th} \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}}$$

[b² < a² + c²; a ≠ b]

$$7. = \frac{1}{c} \ln \left(a + c \operatorname{th} \frac{x}{2} \right)$$

[a = b, c ≠ 0]

$$8. = \frac{2}{(a-b) \operatorname{th} \frac{x}{2} + c}$$

[b² = a² + c²]

$$9. \int \frac{dx}{(a \operatorname{ch} x + b \operatorname{sh} x)^n} = \frac{1}{(a^2 - b^2)^{n/2}} \int \frac{dx}{\operatorname{ch}^n \left(x + \operatorname{Arth} \frac{b}{a} \right)}$$

[a > |b|]

$$10. = \frac{1}{(b^2 - a^2)^{n/2}} \int \frac{dx}{\operatorname{sh}^n \left(x + \operatorname{Arth} \frac{a}{b} \right)}$$

[b > |a|]

$$11. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a \operatorname{ch} x + b \operatorname{sh} x} dx = \frac{A}{\sqrt{a^2 - b^2}} \operatorname{arctg} \left| \operatorname{sh} \left(x + \operatorname{Arth} \frac{b}{a} \right) \right| +$$

$$+ \frac{1}{a^2 - b^2} \left[(Ca - Bb) \ln \operatorname{ch} \left(x + \operatorname{Arth} \frac{b}{a} \right) + (Ba - Cb) x \right]$$

[a > |b|]

$$12. = \frac{A}{\sqrt{b^2 - a^2}} \ln \left| \operatorname{th} \frac{x + \operatorname{Arth} \frac{a}{b}}{2} \right| +$$

$$+ \frac{1}{b^2 - a^2} \left[(Cb - Ba) x + (Bb - Ca) \ln \operatorname{sh} \left(x + \operatorname{Arth} \frac{a}{b} \right) \right]$$

[b > |a|]

$$13. = \frac{1}{a} \left[\frac{B+C}{2} x - \frac{B-C}{4} e^{-2x} - Ae^{-x} \right]$$

[a = b]

$$14. = \frac{1}{a} \left[\frac{B-C}{2} x + \frac{B+C}{4} e^{2x} + Ae^x \right]$$

[a = -b]

$$15. \int \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a \operatorname{th} \frac{x}{2} - b + \sqrt{a^2 + b^2}}{a \operatorname{th} \frac{x}{2} - b - \sqrt{a^2 + b^2}}.$$

$$16. = \frac{2}{\sqrt{a^2 + b^2}} \operatorname{Arth} \frac{a \operatorname{th} \frac{x}{2} - b}{\sqrt{a^2 + b^2}}.$$

$$17. \int \frac{dx}{a + b \operatorname{ch} x} = \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \quad [b^2 > a^2; x < 0].$$

$$18. = -\frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \quad [b^2 > a^2; x > 0].$$

$$19. = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}}{a + b - \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}} \quad [a^2 > b^2].$$

$$20. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(1 \pm \operatorname{ch} x)^n} dx = \frac{B \operatorname{sh} x}{(1-n)(1 \pm \operatorname{ch} x)^n} \pm \frac{C}{1-n} \frac{1}{(1 \pm \operatorname{ch} x)^{n-1}} +$$

$$+ \left(\frac{n}{n-1} B \pm A \right) \frac{(n-1)!}{(2n-1)!!} \operatorname{sh} x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \frac{1}{(1 \pm \operatorname{ch} x)^{n-k}}.$$

$$21. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{1 \pm \operatorname{ch} x} dx = \pm Bx \pm C \ln |1 \pm \operatorname{ch} x| + (A \mp B) \frac{\operatorname{ch} x \mp 1}{\operatorname{sh} x}.$$

$$22. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x)(a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x)} dx =$$

$$= A_0 \ln \frac{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x} + A_1 \int \frac{dx}{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x} +$$

$$+ A_2 \int \frac{dx}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x},$$

где

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} b_1 & c_1 \\ B & C \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ C & A \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2} \cdot \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \right].$$

$$\begin{aligned}
 23. \quad \int \frac{A \operatorname{ch}^2 x + 2B \operatorname{sh} x \operatorname{ch} x + C \operatorname{sh}^2 x}{a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x} dx = \\
 = \frac{1}{4b^2 - (a+c)^2} \{ [4Bb - (A+C)(a+c)] x + \\
 + [(A+C)b - B(a+c)] \ln (a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x) + \\
 + [2(A-C)b^2 + 2Bb(a-c) + (Ca - Ac)(a+c)] f(x) \},
 \end{aligned}$$

где

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{th} x + b - \sqrt{b^2 - ac}}{c \operatorname{th} x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac],$$

$$= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{th} x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac],$$

$$= -\frac{1}{c \operatorname{th} x + b} \quad [b^2 = ac].$$

$$\begin{aligned}
 24. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (a + b \operatorname{sh} x)} dx = \\
 = \frac{1}{a} \left[A \ln \left| \operatorname{th} \frac{x}{2} \right| + B \ln \left| \frac{\operatorname{sh} x}{a + b \operatorname{sh} x} \right| + (Ca - Ab) \int \frac{dx}{a + b \operatorname{sh} x} \right].
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (a + b \operatorname{ch} x)} dx = \\
 = \frac{1}{a^2 - b^2} \left[(Aa + Bb) \ln \left| \operatorname{th} \frac{x}{2} \right| + (Ab - Ba) \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{sh} x} \right| \right] + C \int \frac{dx}{a + b \operatorname{ch} x}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (1 \pm \operatorname{ch} x)} dx = \\
 = \frac{B \pm A}{2} \ln \left| \frac{\operatorname{th} (x/2)}{\operatorname{cth} (x/2)} \right| \mp \frac{A \mp B}{4} \left\{ \frac{\operatorname{th} (x/2)}{\operatorname{cth} (x/2)} \right\}^2 + C \frac{\operatorname{ch} x \mp 1}{\operatorname{sh} x}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{ch} x (a + b \operatorname{sh} x)} dx = \\
 = \frac{1}{a^2 + b^2} \left[(Aa + Cb) \operatorname{arctg} (\operatorname{sh} x) + (Ab - Ca) \ln \left| \frac{a + b \operatorname{sh} x}{\operatorname{ch} x} \right| \right] + B \int \frac{dx}{a + b \operatorname{sh} x}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{ch} x (a + b \operatorname{ch} x)} dx = \\
 = \frac{1}{a} \left[A \operatorname{arctg} (\operatorname{sh} x) - C \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{ch} x} \right| - (Ab - Ba) \int \frac{dx}{a + b \operatorname{ch} x} \right].
 \end{aligned}$$

$$29. \quad \int \frac{dx}{a + b \operatorname{sh}^2 x} = \frac{1}{\sqrt{a(b-a)}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} - 1 \operatorname{th} x \right) \quad [b/a > 1].$$

$$\begin{aligned}
 30. \quad = \frac{1}{\sqrt{a(a-b)}} \operatorname{Arth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \\
 [0 < b/a < 1] \text{ или } [b/a < 0, \operatorname{sh}^2 x < -a/b].
 \end{aligned}$$

$$31. \quad = \frac{1}{\sqrt{a(a-b)}} \operatorname{Arcth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \quad [b/a < 0, \operatorname{sh}^2 x > -a/b].$$

$$32. \quad = \frac{1}{a} \operatorname{th} x \quad [a = b].$$

$$33. \quad = \frac{1}{a\sqrt{2}} \left\{ \operatorname{Arth} (\sqrt{2} \operatorname{th} x) \right\} \quad [a = -b, \operatorname{sh}^2 x \leq 1].$$

$$34. \quad \int \frac{dx}{a + b \operatorname{ch}^2 x} = \frac{1}{\sqrt{-a(a+b)}} \operatorname{arctg} \left(\sqrt{-\left(1 + \frac{b}{a}\right)} \operatorname{cth} x \right) \quad [b/a < -1].$$

$$35. \quad = \frac{1}{\sqrt{a(a+b)}} \operatorname{Arth} \left(\sqrt{1 + \frac{b}{a}} \operatorname{cth} x \right) \quad [-1 < b/a < 0, \operatorname{ch}^2 x > -a/b].$$

$$36. \quad = \frac{1}{\sqrt{a(a+b)}} \operatorname{Arcth} \left(\sqrt{1 + \frac{b}{a}} \operatorname{cth} x \right) \quad [b/a > 0] \text{ или } [-1 < b/a < 0, \operatorname{ch}^2 x < -a/b].$$

$$37. \quad = \frac{1}{a\sqrt{2}} \operatorname{Arcth} (\sqrt{2} \operatorname{cth} x) \quad [a = b].$$

$$38. \quad = \frac{1}{a} \operatorname{cth} x \quad [a = -b].$$

$$39. \quad \int \left(a + b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 \right)^{-2} dx = \\ = \frac{1}{2a(b+a)} \left[\pm b \operatorname{sh} x \operatorname{ch} x \left(a + b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 \right)^{-1} + (b \mp 2a) \int \left(a + b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 \right)^{-1} dx \right].$$

$$40. \quad \int \frac{dx}{a + b \operatorname{th}^2 x} = \frac{1}{a+b} \left[x + \sqrt{\frac{b}{a}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} \operatorname{th} x \right) \right] \quad [ab > 0, a+b \neq 0].$$

$$41. \quad = \frac{1}{a+b} \left[x + \frac{b}{2\sqrt{-ab}} \ln \frac{\sqrt{-ab} - b \operatorname{th} x}{\sqrt{-ab} + b \operatorname{th} x} \right] \quad [ab < 0, a+b \neq 0].$$

$$42. \quad = \frac{1}{2a} x + \frac{1}{4a} \operatorname{sh} 2x \quad [a+b=0].$$

1.4.8. Интегралы вида $\int R(\operatorname{sh}(ax+b), \operatorname{ch}(cx+d)) dx$.

$$1. \quad \int \left\{ \frac{\operatorname{sh}(ax+b) \operatorname{sh}(cx+d)}{\operatorname{ch}(ax+b) \operatorname{ch}(cx+d)} \right\} dx = \\ = \frac{1}{2(a+c)} \operatorname{sh} [(a+c)x + b + d] + \frac{1}{2(a-c)} \operatorname{sh} [(a-c)x + b - d].$$

$$2. \quad \int \operatorname{sh}(ax+b) \operatorname{ch}(cx+d) dx = \\ = \frac{1}{2(a+c)} \operatorname{ch} [(a+c)x + b + d] + \frac{1}{2(a-c)} \operatorname{ch} [(a-c)x + b - d].$$

$$3. \quad \int \left\{ \frac{\operatorname{sh}(ax+b) \operatorname{sh}(ax+d)}{\operatorname{ch}(ax+b) \operatorname{ch}(ax+d)} \right\} dx = \mp \frac{x}{2} \operatorname{ch}(b-d) + \frac{1}{4a} \operatorname{sh}(2ax + b + d).$$

$$4. \quad \int \operatorname{sh}(ax+b) \operatorname{ch}(ax+d) dx = \frac{x}{2} \operatorname{sh}(b-d) + \frac{1}{4a} \operatorname{ch}(2ax + b + d).$$

1.4.9. Интегралы вида $\int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx, \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx$.

$$1. \quad \int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{p+a} \left[\operatorname{sh}^p x \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - p \int \operatorname{sh}^{p-1} x \left\{ \frac{\operatorname{ch} (a-1)x}{\operatorname{sh} (a-1)x} \right\} dx \right].$$

$$2. \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{p+a} \left[\operatorname{ch}^p x \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} + p \int \operatorname{ch}^{p-1} x \left\{ \frac{\operatorname{sh} (a-1)x}{\operatorname{ch} (a-1)x} \right\} dx \right].$$

$$3. \int \operatorname{sh} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx = \left\{ \frac{2n}{(-1)^{n+1}} \right\} \left[\frac{1}{p+2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+2} + \right. \\ \left. + \sum_{k=1}^{n-1} (\pm 1)^k \frac{(4n^2-2^2)(4n^2-4^2)\dots(4n^2-4k^2)}{(2k+1)!(2k+p+2)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+2} \right].$$

[$p \neq -2, -4, \dots, -2n$].

$$4. \int \operatorname{ch} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx = (\pm 1)^n \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx + \\ + \sum_{k=1}^n (\pm 1)^k \frac{4n^2(4n^2-2^2)\dots[4n^2-(2k-2)^2]}{(2k)!} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p} dx.$$

$$5. \int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} (p+2)x}{\operatorname{ch} (p+2)x} \right\} dx = \frac{1}{p+1} \operatorname{sh}^{p+1} x \left\{ \frac{\operatorname{sh} (p+1)x}{\operatorname{ch} (p+1)x} \right\}.$$

$$6. \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} (p+2)x}{\operatorname{ch} (p+2)x} \right\} dx = \frac{1}{p+1} \operatorname{ch}^{p+1} x \left\{ \frac{\operatorname{ch} (p+1)x}{\operatorname{sh} (p+1)x} \right\}.$$

$$7. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{sh} (2n+1)x}{\operatorname{ch}^p x \operatorname{ch} (2n+1)x} \right\} dx = (\pm 1)^n (2n+1) \left[\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} dx + \right. \\ \left. + \sum_{k=1}^n (\pm 1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+1} dx \right].$$

$$8. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch} (2n+1)x}{\operatorname{ch}^p x \operatorname{sh} (2n+1)x} \right\} dx = \frac{(\pm 1)^n}{p+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} + \\ + \sum_{k=1}^n (\pm 1)^{n+k} \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times \\ \times \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+1} \quad [p \neq -3, -5, \dots, -(2n+1)].$$

$$9. \int \left\{ \frac{\operatorname{sh} (2n+1)x \operatorname{sh}^{-1} x}{\operatorname{ch} (2n+1)x \operatorname{ch}^{-1} x} \right\} dx = 2 \sum_{k=0}^{n-1} (\pm 1)^k \frac{\operatorname{sh} (2n-2k)x}{2n-2k} + (\pm 1)^n x.$$

$$10. \int \left\{ \frac{\operatorname{sh} (2n+1)x \operatorname{ch}^{-1} x}{\operatorname{ch} (2n+1)x \operatorname{sh}^{-1} x} \right\} dx = 2 \sum_{k=0}^{n-1} (\mp 1)^k \frac{\operatorname{ch} (2n-2k)x}{2n-2k} + (\mp 1)^n \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$11. \int \operatorname{sh} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = 2 \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \left\{ \frac{\operatorname{sh} (2n-2k-1)x}{\operatorname{ch} (2n-2k-1)x} \right\}.$$

$$12. \int \operatorname{ch} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = \\ = 2 \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \left\{ \frac{\operatorname{ch} (2n-2k-1)x}{\operatorname{sh} (2n-2k-1)x} \right\} + (\pm 1)^n \left\{ \ln |\operatorname{th} (x/2)| \right\}.$$

$$13. \int \operatorname{sh} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-n} dx = \frac{2}{2-n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2-n}.$$

$$14. \int \operatorname{sh} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = 2 \ln \left\{ \left| \frac{\operatorname{sh} x}{\operatorname{ch} x} \right| \right\}.$$

$$15. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = 2 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \pm \left\{ \ln |\operatorname{th} (x/2)| \right\}.$$

$$16. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = - \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} + 2x.$$

$$17. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} dx = - \frac{1}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-2} x}{\operatorname{sh} x \operatorname{ch}^{-2} x} \right\} + \frac{3}{2} \left\{ \ln |\operatorname{th} (x/2)| \right\}.$$

$$18. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-1} x}{\operatorname{ch} 3x \operatorname{ch}^{-1} x} \right\} dx = \operatorname{sh} 2x \pm x.$$

$$19. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-2} x}{\operatorname{ch} 3x \operatorname{ch}^{-2} x} \right\} dx = 4 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \pm 3 \left\{ \ln |\operatorname{th} (x/2)| \right\}.$$

$$20. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-3} x}{\operatorname{ch} 3x \operatorname{ch}^{-3} x} \right\} dx = 4x - 3 \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\}.$$

$$21. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-n} x}{\operatorname{ch} 3x \operatorname{sh}^n x} \right\} dx = \frac{4}{3-n} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{3-n} \mp \frac{1}{1-n} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{1-n}.$$

$$22. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-1} x}{\operatorname{ch} 3x \operatorname{sh}^{-1} x} \right\} dx = 2 \operatorname{sh}^2 x \mp \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$23. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-3} x}{\operatorname{ch} 3x \operatorname{sh}^{-3} x} \right\} dx = - \frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 + 4 \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

1.4.10. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, \sqrt{\operatorname{sh} 2ax}) dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arccos \frac{1 - \operatorname{sh} 2ax}{1 + \operatorname{sh} 2ax}$.

$$1. \int_0^x \sqrt{\operatorname{sh} 2ax} dx = \frac{1}{2a} \left[F \left(\varphi, \frac{1}{\sqrt{2}} \right) - 2E \left(\varphi, \frac{1}{\sqrt{2}} \right) \right] + \frac{1}{a} \frac{\sqrt{\operatorname{sh} 2ax (1 + \operatorname{sh}^2 2ax)}}{1 + \operatorname{sh} 2ax}.$$

$$2. \int_0^x \frac{\sqrt{\operatorname{sh} 2ax} dx}{(1 + \operatorname{sh} 2ax)^2} = \frac{1}{4a} \left[F \left(\varphi, \frac{1}{\sqrt{2}} \right) - E \left(\varphi, \frac{1}{\sqrt{2}} \right) \right].$$

$$3. \int_0^x \frac{dx}{\sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$4. \int_0^x \frac{\operatorname{ch}^2 2ax \, dx}{(1 + \operatorname{sh} 2ax)^2 \sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$5. \int_0^x \frac{(1 - \operatorname{sh} 2ax)^2 \, dx}{(1 + \operatorname{sh} 2ax)^2 \sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$6. \int_0^x \frac{(1 + \operatorname{sh} 2ax)^2 \, dx}{[(1 + \operatorname{sh} 2ax)^2 - 4p^2 \operatorname{sh} 2ax] \sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} \Pi\left(\varphi, p^2, \frac{1}{\sqrt{2}}\right).$$

1.4.11. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, \sqrt{\operatorname{ch} 2ax}) \, dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{\operatorname{ch} 2ax - 1}{\operatorname{ch} 2ax}}$.

$$1. \int_0^x \sqrt{\operatorname{ch} 2ax} \, dx = \frac{1}{a\sqrt{2}} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{\operatorname{sh} 2ax}{a\sqrt{\operatorname{ch} 2ax}}.$$

$$2. \int_0^x \frac{\sqrt{\operatorname{ch} 2ax} \, dx}{p^2 + (1 - p^2) \operatorname{ch} 2ax} = \frac{1}{a\sqrt{2}} \Pi\left(\varphi, p^2, \frac{1}{\sqrt{2}}\right).$$

$$3. \int_0^x \frac{dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$4. \int_0^x \frac{\operatorname{sh} 2ax \, dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{1}{a} [\sqrt{\operatorname{ch} 2ax} - 1].$$

$$5. \int_0^x \frac{\operatorname{sh}^2 2ax \, dx}{\sqrt{\operatorname{ch} 2ax}} = -\frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) + \frac{1}{3a} \operatorname{sh} 2ax \sqrt{\operatorname{ch} 2ax}.$$

$$6. \int_0^x \frac{\operatorname{th}^2 2ax \, dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{th} 2ax}{3a \sqrt{\operatorname{ch} 2ax}}.$$

$$7. \int_0^x \frac{dx}{\sqrt{\operatorname{ch}^3 2ax}} = \frac{1}{a\sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

1.4.12. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a+b \operatorname{sh} x}) dx$.

Условия: $a, b > 0, x > x_1$.

Обозначения: $\varphi = \arccos \frac{\sqrt{a^2+b^2}-a-b \operatorname{sh} x}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x}$,

$$k = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}}, \quad x_1 = -\operatorname{Arsh} \frac{a}{b}.$$

$$1. \int_{x_1}^x \sqrt{a+b \operatorname{sh} x} dx = \sqrt[4]{a^2+b^2} [F(\varphi, k) - 2E(\varphi, k)] + \frac{2b \operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x}.$$

$$2. \int_{x_1}^x \frac{\sqrt{a+b \operatorname{sh} x}}{\operatorname{ch}^2 x} dx = \sqrt[4]{a^2+b^2} E(\varphi, k) - \frac{\sqrt{a^2+b^2}-a}{2\sqrt[4]{a^2+b^2}} F(\varphi, k) - \\ - \frac{a+\sqrt{a^2+b^2}}{b} \frac{\sqrt{a^2+b^2}-a-b \operatorname{sh} x}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x} \frac{\sqrt{a+b \operatorname{sh} x}}{\operatorname{ch} x}.$$

$$3. \int \frac{\operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{p+q \operatorname{sh} x} dx = 2\sqrt{\frac{aq-bp}{q}} \operatorname{Arcth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{aq-bp}} \\ [b \operatorname{sh} x > 0; (aq-bp)/q > 0].$$

$$4. = 2\sqrt{\frac{aq-bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{aq-bp}} \\ [b \operatorname{sh} x < 0; (aq-bp)/q > 0].$$

$$5. = 2\sqrt{\frac{bp-aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{bp-aq}} \\ [(aq-bp)/q < 0].$$

$$6. \int_{x_1}^x \frac{\sqrt{a+b \operatorname{sh} x} dx}{[\sqrt{a^2+b^2}-a-b \operatorname{sh} x]^2} = \\ = -\frac{1}{\sqrt[4]{a^2+b^2}(\sqrt{a^2+b^2}-a)} E(\varphi, k) + \frac{b}{\sqrt{a^2+b^2}-a} \frac{\operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{a^2+b^2-(a+b \operatorname{sh} x)^2}.$$

$$7. \int_{x_1}^x \frac{dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{1}{\sqrt[4]{a^2+b^2}} F(\varphi, k).$$

$$8. \int_{x_1}^x \frac{\operatorname{ch} x dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{2}{b} \sqrt{a+b \operatorname{sh} x}.$$

$$9. \int_{x_1}^x \frac{\operatorname{ch}^2 x dx}{(\sqrt{a^2+b^2}+a+b \operatorname{sh} x)^2 \sqrt{a+b \operatorname{sh} x}} = \frac{1}{b^2 \sqrt[4]{a^2+b^2}} E(\varphi, k).$$

$$10. \int_{x_1}^x \frac{\operatorname{cth} x dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{2}{\sqrt{a}} \operatorname{Arcth} \sqrt{1+\frac{b}{a} \operatorname{sh} x}$$

$[b \operatorname{sh} x > 0; a > 0].$

$$11. \quad = \frac{2}{\sqrt{a}} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \quad [b \operatorname{sh} x < 0; a > 0].$$

$$12. \quad = \frac{2}{\sqrt{-a}} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{sh} x\right)} \quad [a < 0].$$

1.4.13. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{b \operatorname{ch} x - a}) dx$.

Условия: $b > a > 0, x > 0$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{b(\operatorname{ch} x - 1)}{b \operatorname{ch} x - a}}, k = \sqrt{\frac{a+b}{2b}}$.

$$1. \quad \int_0^x \sqrt{b \operatorname{ch} x - a} dx = (b-a) \sqrt{\frac{2}{b}} F(\varphi, k) - 2 \sqrt{2b} E(\varphi, k) + \frac{2b \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

$$2. \quad \int_0^x \frac{\sqrt{b \operatorname{ch} x - a}}{\operatorname{ch} x - 1} dx = \sqrt{a+b} [F(\varphi, k) - E(\varphi, k)].$$

$$3. \quad \int_0^x \frac{\sqrt{b \operatorname{ch} x - a} dx}{p^2 b - a + b(1-p^2) \operatorname{ch} x} = \sqrt{\frac{2}{b}} \Pi(\varphi, p^2, k).$$

$$4. \quad \int_0^x \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} F(\varphi, k).$$

$$5. \quad \int_0^x \frac{\operatorname{ch} x dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} [F(\varphi, k) - 2E(\varphi, k)] + \frac{2 \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

$$6. \quad \int_0^x \frac{\operatorname{sh} x dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{b} [\sqrt{b \operatorname{ch} x - a} - \sqrt{b-a}].$$

$$7. \quad \int_0^x \frac{dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{1}{b^2 - a^2} \sqrt{\frac{2}{b}} [2bE(\varphi, k) - (b-a)F(\varphi, k)].$$

$$8. \quad \int_0^x \frac{(\operatorname{ch} x + 1) dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{2}{b-a} \sqrt{\frac{2}{b}} E(\varphi, k).$$

1.4.14. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{b \operatorname{ch} x - a}) dx$.

Условия: $a > b > 0, x > x_1$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{b \operatorname{ch} x - a}{b(\operatorname{ch} x - 1)}}, k = \sqrt{\frac{2b}{a+b}}, x_1 = \operatorname{Arch} \frac{a}{b}$.

$$1. \quad \int_{x_1}^x \sqrt{b \operatorname{ch} x - a} dx = -2 \sqrt{a+b} E(\varphi, k) + 2 \operatorname{cth} \frac{x}{2} \sqrt{b \operatorname{ch} x - a}.$$

$$2. \int_{x_1}^x \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$3. \int_{x_1}^x \frac{\operatorname{cth}^2 \frac{x}{2} dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\varphi, k).$$

$$4. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x + 1) \sqrt{b \operatorname{ch} x - a}} = \frac{1}{\sqrt{a+b}} [F(\varphi, k) - E(\varphi, k)] + \frac{2\sqrt{b \operatorname{ch} x - a}}{(a+b) \operatorname{sh} x}.$$

$$5. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x - 1) \sqrt{b \operatorname{ch} x - a}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{1}{\sqrt{a+b}} F(\varphi, k).$$

$$6. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{b \operatorname{ch} x - a}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+2b) F(\varphi, k) - (a+3b) E(\varphi, k)] + \\ + \frac{\sqrt{b \operatorname{ch} x - a}}{3(a+b) \operatorname{sh} x} \left(2 \frac{a+3b}{a+b} - \operatorname{th}^2 \frac{x}{2} \right).$$

$$7. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x - 1)^2 \sqrt{b \operatorname{ch} x - a}} = \\ = \frac{1}{3(a-b)^2 \sqrt{a+b}} [(a+2b)(a-b) F(\varphi, k) + (3a-b)(a+b) E(\varphi, k)] + \\ + \frac{a+b}{6b(a-b)} \frac{\operatorname{ch}(x/2)}{\operatorname{sh}^3(x/2)} \sqrt{b \operatorname{ch} x - a}.$$

1.4.15. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a-b \operatorname{ch} x}) dx$.

Условия: $a > b > 0$, $0 < x < x_1$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-b \operatorname{ch} x}{a-b}}$, $k = \sqrt{\frac{a-b}{a+b}}$, $x_1 = \operatorname{Arch} \frac{a}{b}$.

$$1. \int_x^{x_1} \sqrt{a-b \operatorname{ch} x} dx = 2\sqrt{a+b} [F(\varphi, k) - E(\varphi, k)].$$

$$2. \int_x^{x_1} \frac{dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$3. \int_x^{x_1} \frac{\operatorname{sh} x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2}{b} \sqrt{a-b \operatorname{ch} x}.$$

$$4. \int_x^{x_1} \frac{\operatorname{ch} x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\varphi, k) - \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$5. \int_x^{x_1} \frac{\operatorname{ch}^2 x \, dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2(b-2a)}{3b \sqrt{a+b}} F(\varphi, k) + \frac{4a \sqrt{a+b}}{3b^2} E(\varphi, k) + \\ + \frac{2}{3b} \operatorname{sh} x \sqrt{a-b \operatorname{ch} x}.$$

$$6. \int_x^{x_1} \frac{dx}{\operatorname{ch} x \sqrt{a-b \operatorname{ch} x}} = \frac{2b}{a \sqrt{a+b}} \Pi\left(\varphi, \frac{a-b}{a}, k\right).$$

$$7. \int_x^{x_1} \frac{dx}{(1+\operatorname{ch} x) \sqrt{a-b \operatorname{ch} x}} = \frac{1}{\sqrt{a+b}} E(\varphi, k) - \frac{1}{a+b} \operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}.$$

$$8. \int_x^{x_1} \frac{dx}{(1+\operatorname{ch} x)^2 \sqrt{a-b \operatorname{ch} x}} = \frac{1}{3 \sqrt{(a+b)^3}} [(a+3b) E(\varphi, k) - b F(\varphi, k)] - \\ - \frac{1}{3(a+b)^2} \frac{\operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}}{\operatorname{ch} x + 1} [2a + 4b + (a+3b) \operatorname{ch} x].$$

$$9. \int_x^{x_1} \frac{dx}{(a-b-arp^2+bp^2 \operatorname{ch} x) \sqrt{a-b \operatorname{ch} x}} = \frac{2}{(a-b) \sqrt{a+b}} \Pi(\varphi, p^2, k).$$

1.4.16. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a+b \operatorname{ch} x}) dx$.

Условия: $a > b > 0, x > 0$.

Обозначения: $\varphi = \arcsin\left(\operatorname{th} \frac{x}{2}\right), k = \sqrt{\frac{a-b}{a+b}}$.

$$1. \int_0^x \sqrt{a+b \operatorname{ch} x} \, dx = 2 \sqrt{a+b} [F(\varphi, k) - E(\varphi, k)] + 2 \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x}.$$

$$2. \int_0^x \frac{\sqrt{a+b \operatorname{ch} x}}{\operatorname{ch} x + 1} \, dx = \sqrt{a+b} E(\varphi, k).$$

$$3. \int \frac{\operatorname{sh} x \sqrt{a+b \operatorname{ch} x}}{p+q \operatorname{ch} x} \, dx = 2 \sqrt{\frac{aq-bp}{q}} \operatorname{Arcth} \sqrt{\frac{q(a+b \operatorname{ch} x)}{aq-bp}} \\ [(aq-bp)/q > 0].$$

$$4. = 2 \sqrt{\frac{bp-aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{ch} x)}{bp-aq}} \\ [(aq-bp)/q < 0].$$

$$5. \int_0^x \frac{dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$6. \int_0^x \frac{\operatorname{sh} x \, dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{b} (\sqrt{a+b \operatorname{ch} x} - \sqrt{a+b}).$$

$$7. \int_0^x \frac{\operatorname{ch} x dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k) - \frac{2\sqrt{a+b}}{b} E(\varphi, k) + \frac{2}{b} \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x}.$$

$$8. \int_0^x \frac{\operatorname{ch} x - 1}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{2}{b} \left[\operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x} - \sqrt{a+b} E(\varphi, k) \right].$$

$$9. \int_0^x \frac{(\operatorname{ch} x - 1)^2}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{4\sqrt{a+b}}{3b^2} [(a+3b) E(\varphi, k) - bF(\varphi, k)] + \\ + \frac{4}{3b^2} \left[b \operatorname{ch}^2 \frac{x}{2} - (a+3b) \right] \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x},$$

$$10. \int_0^x \frac{\operatorname{th} x dx}{\sqrt{a+b \operatorname{ch} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x}.$$

$$11. \int_0^x \frac{\operatorname{th}^2(x/2)}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{2\sqrt{a+b}}{a-b} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_0^x \frac{dx}{(\operatorname{ch} x + 1) \sqrt{a+b \operatorname{ch} x}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{2b}{(a-b)\sqrt{a+b}} F(\varphi, k).$$

$$13. \int_0^x \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{a+b \operatorname{ch} x}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} [b(5b-a) F(\varphi, k) + \\ + (a-3b)(a+b) E(\varphi, k)] + \frac{1}{6(a-b)} \frac{\operatorname{sh}(x/2)}{\operatorname{ch}^3(x/2)} \sqrt{a+b \operatorname{ch} x}.$$

$$14. \int_0^x \frac{(1 + \operatorname{ch} x) dx}{[1 + p^2 + (1 - p^2) \operatorname{ch} x] \sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} \Pi(\varphi, p^2, k).$$

1.4.17. Интегралы вида

$$\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a^2 \operatorname{sh}^2 x \pm b^2}, \sqrt{b^2 - a^2 \operatorname{sh}^2 x}, \sqrt{a^2 \operatorname{ch}^2 x \pm b^2}, \\ \sqrt{b^2 - a^2 \operatorname{ch}^2 x}) dx.$$

$$1. \int \frac{1}{\sqrt{\operatorname{sh}^2 x + a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{sh}^2 x + a^2} \right).$$

$$2. \int \frac{1}{\sqrt{\operatorname{sh}^2 x - a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{sh}^2 x - a^2} \right) \quad [\operatorname{sh}^2 x > a^2].$$

$$3. \int \frac{1}{\sqrt{a^2 - \operatorname{sh}^2 x}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \begin{cases} \arcsin [(\operatorname{ch} x)/\sqrt{a^2 + 1}] \\ \arcsin [(\operatorname{sh} x)/a] \end{cases} \quad [\operatorname{sh}^2 x < a^2].$$

$$4. \int \frac{1}{\sqrt{\operatorname{ch}^2 x + a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{ch}^2 x + a^2} \right).$$

$$5. \int \frac{1}{\sqrt{\operatorname{ch}^2 x - a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{ch}^2 x - a^2} \right) \quad [\operatorname{ch}^2 x > a^2].$$

$$6. \int \frac{1}{\sqrt{a^2 - \operatorname{ch}^2 x}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \begin{cases} \arcsin [(\operatorname{ch} x)/a] \\ \arcsin [(\operatorname{sh} x)/\sqrt{a^2 - 1}] \end{cases} \quad [\operatorname{ch}^2 x < a^2].$$

$$7. \int_0^x \frac{dx}{\sqrt{1 + k^2 \operatorname{sh}^2 x}} = F(\arcsin \operatorname{th} x, \sqrt{1 - k^2}) \quad [x > 0; k^2 < 1].$$

$$8. \int_0^x \frac{dx}{\sqrt{k^2 + (1 - k^2) \operatorname{ch}^2 x}} = F(\arcsin \operatorname{th} x, k) \quad [x > 0; k^2 < 1].$$

$$9. \int_0^x \frac{dx}{\sqrt{1 - k^2 \operatorname{ch}^2 x}} = F\left(\arcsin \frac{\operatorname{th} x}{\sqrt{1 - k^2}}, \sqrt{1 - k^2}\right) \quad [0 < x < \operatorname{ch}(1/\sqrt{1 - k^2}); k^2 < 1].$$

$$10. \int_x^\infty \frac{dx}{\sqrt{\operatorname{ch}^2 x - k^2}} = F(\arcsin \operatorname{sech} x, k) \quad [x > 0; k^2 < 1].$$

$$11. \int_x^\infty \frac{dx}{\sqrt{\operatorname{sh}^2 x + k^2}} = F(\arcsin \operatorname{sech} x, \sqrt{1 - k^2}) \quad [x > 0; k^2 < 1].$$

1.4.18. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a \operatorname{sh} x + b \operatorname{ch} x}) dx$.

Условия: $b > a > 0, x > x_1$.

Обозначения: $\varphi = \arccos \frac{\sqrt[4]{b^2 - a^2}}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}, x_1 = -\operatorname{Arsh} \frac{a}{\sqrt{b^2 - a^2}}$.

$$1. \int_{x_1}^x \sqrt{a \operatorname{sh} x + b \operatorname{ch} x} dx = \sqrt[4]{4(b^2 - a^2)} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2(a \operatorname{ch} x + b \operatorname{sh} x)}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}.$$

$$2. \int_{x_1}^x \frac{dx}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}} = \sqrt[4]{\frac{4}{b^2 - a^2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$3. \int_{x_1}^x \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = \sqrt[4]{\frac{4}{(b^2 - a^2)^3}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$4. \int_{x_1}^x \frac{(\sqrt{b^2 - a^2} + a \operatorname{sh} x + b \operatorname{ch} x) dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = 2 \sqrt[4]{\frac{4}{b^2 - a^2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

1.4.19. Интегралы вида $\int \sqrt{\operatorname{th} x} dx, \int \sqrt{\operatorname{cth} x} dx$.

$$1. \int \sqrt{\operatorname{th} x} dx = \operatorname{Arth} \sqrt{\operatorname{th} x} - \operatorname{arctg} \sqrt{\operatorname{th} x}.$$

$$2. \int \sqrt{\operatorname{cth} x} dx = \operatorname{Arcth} \sqrt{\operatorname{cth} x} - \operatorname{arctg} \sqrt{\operatorname{cth} x}.$$

1.4.20. Интегралы вида $\int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx$.

$$1. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx = \frac{1}{q} x^p \left\{ \frac{\operatorname{sh}^{q-1} x \operatorname{ch} x}{\operatorname{ch}^{q-1} x \operatorname{sh} x} \right\} - \frac{p}{q^2} x^{p-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q + \\ + \frac{p(p-1)}{q^2} \int x^{p-2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx + \frac{q-1}{q} \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{q-2} dx.$$

$$2. \int x^m \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = (-1)^n \binom{2n}{n} \frac{x^{m+1}}{2^{2n}(m+1)} + \\ + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \int x^m \operatorname{ch} (2n-2k)x dx.$$

$$3. \int x^m \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \int x^m \left\{ \frac{\operatorname{sh} (2n-2k+1)x}{\operatorname{ch} (2n-2k+1)x} \right\} dx.$$

$$4. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = \frac{1}{2n+1} \sum_{k=0}^{n-1} (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+1)}{2^{k+1} n(n-1) \dots (n-k)} \times \\ \times \left[\frac{-1}{2n-2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n-2k} + x \left\{ \frac{\operatorname{sh}^{2n-2k-1} x \operatorname{ch} x}{\operatorname{ch}^{2n-2k-1} x \operatorname{sh} x} \right\} \right] + (-1)^n \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2}.$$

$$5. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = \\ = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \frac{2^k (n+1)n \dots (n-k+1)}{(2n+1)(2n-1) \dots (2n-2k+1)} \left[\frac{-1}{2n-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n-2k+1} + \right. \\ \left. + x \left\{ \frac{\operatorname{sh}^{2n-2k} x \operatorname{ch} x}{\operatorname{ch}^{2n-2k} x \operatorname{sh} x} \right\} \right].$$

$$6. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - p \int x^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} dx.$$

$$7. \int x^{2n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (2n)! \left[\sum_{k=0}^n \frac{x^{2k}}{(2k)!} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \sum_{k=0}^{n-1} \frac{x^{2k+1}}{(2k+1)!} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$8. \int x^{2n+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (2n+1)! \sum_{k=0}^n \left[\frac{x^{2k+1}}{(2k+1)!} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \frac{x^{2k}}{(2k)!} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$9. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$10. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (x^2 + 2) \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$11. \int x^n \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = (-1)^n \frac{x^{n+1}}{2(n+1)} + \\ + \frac{n!}{4} \left[\sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \sum_{k=0}^{[(n-1)/2]} \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x \right].$$

$$12. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = \frac{x}{4} \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x + \frac{x^2}{4}.$$

$$13. \int_0^x x^{\alpha-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \frac{x^{\alpha+\delta}}{\alpha+\delta} {}_1F_2 \left(\frac{\alpha+\delta}{2}; \frac{\alpha+\delta}{2}+1, \frac{1}{2}+\delta; \frac{x^2}{4} \right)$$

$$[\operatorname{Re} \alpha > -\delta; \delta = \begin{cases} 1 \\ 0 \end{cases}]$$

$$14. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \operatorname{sh} 2x - \frac{x}{4} \operatorname{ch} 2x + \frac{x^3}{6}.$$

$$15. \int x^n \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \frac{n!}{4} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \left[\frac{1}{3^{2k+1}} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} + \right. \\ \left. + 3 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \right] - \frac{n!}{4} \sum_{k=0}^{[(n-1)/2]} \frac{x^{n-2k-1}}{(n-2k-1)!} \left[\frac{1}{3^{2k+2}} \left\{ \frac{\operatorname{sh} 3x}{\operatorname{ch} 3x} \right\} + 3 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$16. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \pm \frac{3}{4} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} - \frac{1}{36} \left\{ \frac{\operatorname{sh} 3x}{\operatorname{ch} 3x} \right\} + \frac{3}{4} x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{x}{12} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\}.$$

$$17. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \mp \left(\frac{3}{4} x^2 + \frac{3}{2} \right) \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \left(\frac{x^2}{12} + \frac{1}{54} \right) \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} + \\ + \frac{3}{2} x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} - \frac{x}{18} \left\{ \frac{\operatorname{sh} 3x}{\operatorname{ch} 3x} \right\}.$$

1.4.21. Интегралы вида $\int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx$.

$$1. \int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx = - \frac{1}{(p-1)x^{p-1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q - \frac{q}{(p-1)(p-2)x^{p-2}} \left\{ \frac{\operatorname{sh}^{q-1} x \operatorname{ch} x}{\operatorname{ch}^{q-1} x \operatorname{sh} x} \right\} + \\ + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{q-2} dx + \frac{q^2}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx.$$

$$2. \int \frac{1}{x^{2n}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = - \frac{1}{x(2n-1)!} \left[\sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right] + \frac{1}{(2n-1)!} \left\{ \frac{\operatorname{chi} x}{\operatorname{shi} x} \right\}.$$

$$3. \int \frac{1}{x^{2n+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = - \frac{1}{x(2n)!} \left[\sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right] + \frac{1}{(2n)!} \left\{ \frac{\operatorname{shi} x}{\operatorname{chi} x} \right\}.$$

$$4. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \operatorname{chi} (2n-2k) x + \frac{(-1)^n}{2^{2n}} \binom{2n}{n} \ln |x|.$$

$$5. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \left\{ \begin{matrix} \operatorname{shi} (2n-2k+1) x \\ \operatorname{chi} (2n-2k+1) x \end{matrix} \right\}.$$

$$6. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = -\frac{(-1)^n}{2^{2n} x} \binom{2n}{n} - \\ - \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \left[\frac{\operatorname{ch} (2n-2k) x}{x} - (2n-2k) \operatorname{shi} (2n-2k) x \right].$$

$$7. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = -\frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \times \\ \times \left[\frac{1}{x} \left\{ \frac{\operatorname{sh} (2n-2k+1) x}{\operatorname{ch} (2n-2k+1) x} \right\} - (2n-2k+1) \left\{ \begin{matrix} \operatorname{chi} (2n-2k+1) x \\ \operatorname{shi} (2n-2k+1) x \end{matrix} \right\} \right].$$

$$8. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \left\{ \begin{matrix} \operatorname{shi} x \\ \operatorname{chi} x \end{matrix} \right\}. \quad 9. \int_0^x \frac{\operatorname{sh} x}{x} dx = \operatorname{shi} x.$$

$$10. \int_0^x \frac{\operatorname{ch} x - 1}{x} dx = \operatorname{chi} x - \ln x - C.$$

$$11. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = -\frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} + \left\{ \begin{matrix} \operatorname{chi} x \\ \operatorname{shi} x \end{matrix} \right\}.$$

1.4.22. Интегралы вида $\int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^m dx$.

$$1. \int x^p \operatorname{th} x dx = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(2k+p) (2k)!} x^{2k+p} \quad [|x| < \pi/2; p \geq -1].$$

$$2. \int x^p \operatorname{cth} x dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k+p) (2k)!} x^{2k+p} \quad [|x| < \pi; p \geq 1].$$

$$3. \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n} dx = \frac{1}{p+1} x^{p+1} + \sum_{k=0}^{n-1} (-1)^{n+k} \binom{n}{k} \int x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} dx.$$

$$4. \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n+1} dx = \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} dx + \\ + \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{2n-2k} \binom{n}{k} \left[-x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} + p \int x^{p+1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} dx \right].$$

1.4.23. Интегралы вида $\int x^r \operatorname{sh}^p x \operatorname{ch}^q x dx$.

$$1. \int x^r \operatorname{sh}^p x \operatorname{ch}^q x dx = \frac{1}{(p+q)^2} [(p+q) x^r \operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x - \\ - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r+1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x dx + \\ + r p \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x dx + (q-1)(p+q) \int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x dx].$$

$$2. = \frac{1}{(p+q)^2} [(p+q) x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x - \\ - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r-1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x dx - \\ - r q \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x dx - (p-1)(p+q) \int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x dx].$$

$$3. \int x^p \left\{ \frac{\operatorname{sh}^{2m} x \operatorname{ch}^{-n} x}{\operatorname{ch}^{2m} x \operatorname{sh}^{-n} x} \right\} dx = \sum_{k=0}^m (\mp 1)^{m+k} \binom{m}{k} \int x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-n} dx.$$

$$4. \int x^p \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-n} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-n} x} \right\} dx = \sum_{k=0}^m (\mp 1)^{m+k} \binom{m}{k} \int x^p \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{2k-n} x}{\operatorname{ch} x \operatorname{sh}^{2k-n} x} \right\} dx.$$

$$5. \int x^p \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-n} x}{\operatorname{ch} x \operatorname{sh}^{-n} x} \right\} dx = -\frac{x^p}{n-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-n+1} + \frac{p}{n-1} \int x^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-n+1} dx.$$

$$6. \int x \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} dx = -x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} + \left\{ \operatorname{arctg} \operatorname{sh} x \right\}.$$

1.4.24. Интегралы вида $\int \frac{x^p}{\operatorname{sh}^q x} dx$, $\int \frac{x^p}{\operatorname{ch}^q x} dx$.

$$1. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q} dx = \mp \frac{p x^{p-1}}{(q-1)(q-2)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} \mp \frac{x^p}{q-1} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-q+1} x}{\operatorname{sh} x \operatorname{ch}^{-q+1} x} \right\} \mp \\ \pm \frac{p(p-1)}{(q-1)(q-2)} \int x^{p-2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx \mp \frac{q-2}{q-1} \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx.$$

$$2. \int \frac{x^p}{\operatorname{sh} x} dx = \sum_{k=0}^{\infty} \frac{(2-2^{2k}) B_{2k}}{(2k)! (p+2k)} x^{p+2k} \quad [|x| < \pi; p > 0].$$

$$3. \int \frac{x^p}{\operatorname{ch} x} dx = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)! (p+2k+1)} x^{p+2k+1} \quad [|x| < \pi/2; p \geq 0].$$

$$4. \int \frac{x^p}{\operatorname{sh}^2 x} dx = -x^p \operatorname{cth} x + p \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k-1) (2k)!} x^{p+2k-1} \quad [|x| < \pi; p > 1].$$

$$5. \int \frac{x^p}{\operatorname{ch}^2 x} dx = x^p \operatorname{th} x - p \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1) B_{2k}}{(p+2k-1) (2k)!} x^{p+2k-1} \quad [|x| < \pi/2; p > 1].$$

$$6. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2n} dx = \frac{1}{2n} \sum_{k=1}^{n-1} (-1)^k \frac{2n(2n-2) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+1)} \times \\ \times \left[x \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{2k-2n-1} x}{\operatorname{sh} x \operatorname{ch}^{2k-2n-1} x} \right\} + \frac{1}{2n-2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-2n} \right] + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx.$$

$$7. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2n-1} dx = \frac{1}{2n+1} \sum_{k=1}^n (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2n(2n-2) \dots (2n-2k+2)} \times \\ \times \left[x \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{2k-2n-2} x}{\operatorname{sh} x \operatorname{ch}^{2k-2n-2} x} \right\} + \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-2n-1} \right] + \\ + (-1)^n \frac{(2n-1)!!}{(2n)!!} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx.$$

$$8. \int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q} dx = \pm \frac{p}{(q-1)(q-2)x^{p+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} \mp \\ \mp \frac{1}{(q-1)x^p} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{q+1} x}{\operatorname{sh} x \operatorname{ch}^{q+1} x} \right\} - \frac{q-2}{q-1} \int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx \pm \\ \pm \frac{p(p+1)}{(q-1)(q-2)} \int \frac{1}{x^{p+2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx.$$

$$9. \int \frac{dx}{x^n \operatorname{sh} x} = -[1 + (-1)^n] \frac{2^{n-1}-1}{n!} B_n \ln |x| + \sum_{\substack{k=0 \\ k \neq n/2}}^{\infty} \frac{2-2^{2k}}{(2k)!(2k-n)} B_{2k} x^{2k-n} \\ [|x| < \pi; n \geq 1].$$

$$10. \int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0 \\ k \neq (n-1)/2}}^{\infty} \frac{E_{2k}}{(2k)!(2k-n+1)} x^{2k-n+1} + \\ + \frac{1}{2} [1 - (-1)^n] \frac{E_{n-1}}{(n-1)!} \ln |x| \quad [|x| < \pi/2].$$

$$11. \int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1 - (-1)^n] \frac{2^n n}{(n+1)!} B_{n+1} \ln |x| - \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k \neq (n+1)/2}}^{\infty} \frac{B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \quad [|x| < \pi].$$

$$12. \int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + [1 - (-1)^n] \frac{2^n (2^{n+1}-1)n}{(n+1)!} B_{n+1} \ln |x| + \\ + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(2^{2k}-1)B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \quad [|x| < \pi/2].$$

$$13. \int \frac{x dx}{\operatorname{sh} x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k)!(2k+1)} B_{2k} x^{2k+1} \quad [|x| < \pi].$$

$$14. \int \frac{x dx}{\operatorname{ch} x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k)! (2k+2)} \quad [|x| < \pi/2]$$

$$15. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = \mp x \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} \pm \ln \left\{ \left| \frac{\operatorname{ch} x}{\operatorname{ch} x} \right| \right\}.$$

$$16. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} dx = \mp \frac{x}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-2} x}{\operatorname{sh} x \operatorname{ch}^{-2} x} \right\} \mp \frac{1}{2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} \mp \frac{1}{2} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx.$$

$$17. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-4} dx = \\ = \mp \frac{x}{3} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-3} x}{\operatorname{sh} x \operatorname{ch}^{-3} x} \right\} \mp \frac{1}{6} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} \mp \frac{2}{3} x \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} - \frac{2}{3} \ln \left\{ \left| \frac{\operatorname{sh} x}{\operatorname{ch} x} \right| \right\}.$$

1.4.25. Интегралы вида $\int R \left(x^p, \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}, a + b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right) dx$.

$$1. \int \frac{x^p \operatorname{ch} x dx}{(a + b \operatorname{sh} x)^q} = - \frac{x^p}{(q-1)b(a + b \operatorname{sh} x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \operatorname{sh} x)^{q-1}}.$$

$$2. \int \frac{x^p \operatorname{sh} x dx}{(a + b \operatorname{ch} x)^q} = - \frac{x^p}{(q-1)b(a + b \operatorname{ch} x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \operatorname{ch} x)^{q-1}}.$$

$$3. \int \frac{x dx}{1 \pm \operatorname{ch} x} = x \left\{ \frac{\operatorname{th} (x/2)}{\operatorname{cth} (x/2)} \right\} - 2 \ln \left\{ \left| \frac{\operatorname{ch} (x/2)}{\operatorname{sh} (x/2)} \right| \right\}.$$

$$4. \int \frac{x \operatorname{sh} x dx}{(1 \pm \operatorname{ch} x)^2} = - \frac{x}{\operatorname{ch} x \pm 1} \pm \left\{ \frac{\operatorname{th} (x/2)}{\operatorname{cth} (x/2)} \right\}.$$

1.4.26. Интегралы вида $\int (bx + c)^{\pm n} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx$.

$$1. \int (bx + c)^n \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{b^n}{a^{n+1}} \left(\operatorname{ch} \frac{ac}{b} \int t^n \left\{ \frac{\operatorname{sh} t}{\operatorname{ch} t} \right\} dt - \operatorname{sh} \frac{ac}{b} \int t^n \left\{ \frac{\operatorname{ch} t}{\operatorname{sh} t} \right\} dt \right) \quad \left[\frac{a}{b} t = bx + c \right].$$

$$2. \int (bx + c) \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{a} (bx + c) \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{b}{a^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$3. \int (bx + c)^2 \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{a} \left[(bx + c)^2 + \frac{2b^2}{a^2} \right] \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{2b(bx + c)}{a^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$4. \int (bx + c)^3 \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{bx + c}{a} \left[(bx + c)^2 + \frac{6b^2}{a^2} \right] \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{3b}{a^2} \left[(bx + c)^2 + \frac{2b^2}{a^2} \right] \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$5. \int \frac{1}{(bx + c)^n} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{a^{n-1}}{b^n} \left(\operatorname{ch} \frac{ac}{b} \int \frac{1}{t^n} \left\{ \frac{\operatorname{sh} t}{\operatorname{ch} t} \right\} dt - \operatorname{sh} \frac{ac}{b} \int \frac{1}{t^n} \left\{ \frac{\operatorname{ch} t}{\operatorname{sh} t} \right\} dt \right) \quad \left[\frac{a}{b} t = bx + c \right].$$

$$6. \int \frac{1}{bx + c} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{b} \operatorname{ch} \frac{ac}{b} \left\{ \frac{\operatorname{shi} (ax + ac/b)}{\operatorname{chi} (ax + ac/b)} \right\} - \frac{1}{b} \operatorname{sh} \frac{ac}{b} \left\{ \frac{\operatorname{chi} (ax + ac/b)}{\operatorname{shi} (ax + ac/b)} \right\} = \\ = \frac{1}{2b} e^{-ac/b} \operatorname{Ei} (ax + ac/b) \mp \frac{1}{2b} e^{ac/b} \operatorname{Ei} \left(-ax - \frac{ac}{b} \right).$$

$$7. \int \frac{1}{(bx+c)^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{b(bx+c)} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} + \frac{a}{b} \int \frac{1}{bx+c} \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} dx.$$

$$8. \int \frac{1}{(bx+c)^3} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{2b(bx+c)^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} - \\ - \frac{a}{2b^2(bx+c)} \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} + \frac{a^2}{2b^2} \int \frac{1}{bx+c} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx.$$

1.4.27. Интегралы вида $\int R(x, e^{ax}, \operatorname{sh} bx, \operatorname{ch} bx) dx$.

$$1. \int e^{ax} \left\{ \frac{\operatorname{sh}(bx+c)}{\operatorname{ch}(bx+c)} \right\} dx = \frac{e^{ax}}{a^2-b^2} \left[a \left\{ \frac{\operatorname{sh}(bx+c)}{\operatorname{ch}(bx+c)} \right\} - b \left\{ \frac{\operatorname{ch}(bx+c)}{\operatorname{sh}(bx+c)} \right\} \right].$$

$$2. \int e^{ax} \left\{ \frac{\operatorname{sh}(ax+c)}{\operatorname{ch}(ax+c)} \right\} dx = \mp \frac{1}{2} x e^{-c} \pm \frac{1}{4a} e^{2ax+c}.$$

$$3. \int e^{-ax} \left\{ \frac{\operatorname{sh}(ax+c)}{\operatorname{ch}(ax+c)} \right\} dx = \frac{1}{2} x e^c \pm \frac{1}{4a} e^{-(2ax+c)}.$$

$$4. \int x^p e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} \int x^p e^{(a+b)x} dx \mp \frac{1}{2} \int x^p e^{(a-b)x} dx \quad [\text{см. 1.3.2}]$$

$$5. \int_0^x x^{\alpha-1} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [(a-b)^{-\alpha} \gamma(\alpha, ax-bx) \mp (a+b)^{-\alpha} \gamma(\alpha, ax+bx)] \\ \left[\operatorname{Re} \alpha > -\delta, \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$6. \int_x^\infty x^{\alpha-1} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [(a-b)^{-\alpha} \Gamma(\alpha, ax-bx) \mp (a+b)^{-\alpha} \Gamma(\alpha, ax+bx)] \\ [a > b > 0].$$

$$7. \int x e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{ax}}{a^2-b^2} \left[\left(ax - \frac{a^2+b^2}{a^2-b^2} \right) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} - \right. \\ \left. - \left(bx - \frac{2ab}{a^2-b^2} \right) \left\{ \frac{\operatorname{ch} bx}{\operatorname{sh} bx} \right\} \right].$$

$$8. \int x^2 e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{ax}}{a^2-b^2} \left[ax^2 - \frac{2'(a^2+b^2)}{a^2-b^2} x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2} \right] \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} - \\ - \frac{e^{ax}}{a^2-b^2} \left[bx^2 - \frac{4ab}{a^2-b^2} x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2} \right] \left\{ \frac{\operatorname{ch} bx}{\operatorname{sh} bx} \right\}.$$

$$9. \int x e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) \mp \frac{x^2}{4}.$$

$$10. \int x^2 e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{e^{2ax}}{4a} \left(x^2 \mp \frac{x}{a} \mp \frac{1}{2a^2} \right) \mp \frac{x^3}{6}.$$

$$11. \int \frac{1}{x} e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [\operatorname{Ei}(ax+bx) \mp \operatorname{Ei}(ax-bx)] \quad [a^2 \neq b^2].$$

$$12. \int_x^\infty \frac{1}{x} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = -\frac{1}{2} [\operatorname{Ei}(bx-ax) \mp \operatorname{Ei}(-ax-bx)] \quad [a > |b|].$$

$$13. \int_0^x \frac{1}{x} e^{-ax} \operatorname{sh} bx \, dx = \frac{1}{2} \ln \frac{a+b}{a-b} + \frac{1}{2} [\operatorname{Ei}(bx-ax) - \operatorname{Ei}(-ax-bx)] \quad [a > |b|].$$

$$14. \int \frac{1}{x} e^{ax} \left\{ \begin{matrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{matrix} \right\} dx = \frac{1}{2} [\operatorname{Ei}(2ax) \mp \ln x].$$

$$15. \int_0^x \frac{1}{x} e^{\pm x} \operatorname{sh} x \, dx = \pm \frac{1}{2} [\operatorname{Ei}(\pm 2x) - (C + \ln 2x)].$$

$$16. \int \frac{1}{x} e^{-ax} \left\{ \begin{matrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{matrix} \right\} dx = \frac{1}{2} [\ln x \mp \operatorname{Ei}(-2ax)].$$

$$17. \int \frac{1}{x^2} e^{ax} \left\{ \begin{matrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{matrix} \right\} dx = a \operatorname{Ei}(2ax) - \frac{1}{2x} (e^{2ax} \mp 1).$$

$$18. \int \frac{1}{x^2} e^{-ax} \left\{ \begin{matrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{matrix} \right\} dx = -\frac{1}{2x} \pm \left[\frac{1}{2x} e^{-2ax} + a \operatorname{Ei}(-2ax) \right].$$

$$19. \int \frac{x \, dx}{\operatorname{ch}^2 ax} = \frac{x}{a} \operatorname{th} ax - \frac{1}{a^2} \ln |\operatorname{ch} ax|.$$

$$20. \int_x^\infty \frac{1}{x} \left\{ \begin{matrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{matrix} \right\}^{-1} dx = -2 \sum_{k=0}^{\infty} (\pm 1)^k \operatorname{Ei}[-(2k+1)ax].$$

$$21. \int_0^x \frac{x \, dx}{\cos t - \operatorname{ch} ax} = \frac{2}{a^2 \sin t} \left[L\left(\theta - \frac{t}{2}\right) - L\left(\theta + \frac{t}{2}\right) + 2L\left(\frac{t}{2}\right) \right] \\ \left[\theta = \operatorname{arctg}\left(\operatorname{th} \frac{ax}{2} \operatorname{ctg} \frac{t}{2}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

$$22. \int_0^x \frac{x \operatorname{ch}(ax/2)}{\cos t - \operatorname{ch} ax} dx = \frac{2}{a^2 \sin(t/2)} \left[L\left(\frac{\theta-t}{4}\right) - L\left(\frac{\theta+t}{4}\right) - \right. \\ \left. - L\left(\pi - \frac{\psi+t}{4}\right) - L\left(\frac{\psi-t}{4}\right) + 2L\left(\frac{t}{4}\right) + 2L\left(\frac{2\pi-t}{4}\right) \right] \\ \left[\theta = 4 \operatorname{arctg}\left(\operatorname{th} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right), \quad \psi = 4 \operatorname{arctg}\left(\operatorname{cth} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

1.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.5.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

$$1. \int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) dx = \\ = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}, \frac{1-t^2}{2t}\right) \frac{2 \, dt}{1+t^2} \quad \left[t = \operatorname{tg} \frac{x}{2}\right].$$

Если

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

то

$$2. \int R(\sin x, \cos x) dx = - \int R(\sqrt{1-t^2}, t) \frac{dt}{\sqrt{1-t^2}} \quad [t = \cos x].$$

Если

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

то

$$3. \int R(\sin x, \cos x) dx = \int R(t, \sqrt{1-t^2}) \frac{dt}{\sqrt{1-t^2}} \quad [t = \sin x].$$

Если

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

то

$$4. \int R(\sin x, \cos x) = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right) \frac{dt}{1+t^2} \quad [t = \operatorname{tg} x].$$

1.5.2. Интегралы вида $\int \sin^p x dx$.

$$1. \int \sin^p x dx = -\frac{1}{p} \sin^{p-1} x \cos x + \frac{p-1}{p} \int \sin^{p-2} x dx.$$

$$2. \int \sin^{2n} x dx =$$

$$= -\frac{\cos x}{2n} \left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sin^{2n-2k-1} x \right] +$$

$$+ \frac{(2n-1)!!}{2^n n!} x.$$

$$3. = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}.$$

$$4. \int \sin^{2n+1} x dx =$$

$$= -\frac{\cos x}{2n+1} \left[\sin^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n (n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \sin^{2n-2k-2} x \right].$$

$$5. = (-1)^{n+1} \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n-2k+1)x}{2n-2k+1}.$$

$$6. = \sum_{k=0}^n \frac{(-1)^{k+1}}{2k+1} \binom{n}{k} \cos^{2k+1} x.$$

$$7. \int \sin x dx = -\cos x.$$

$$8. \int \sin^2 x dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$9. \int \sin^3 x dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x.$$

$$10. \int \sin^4 x dx = \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \\ = -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3}{8} x.$$

$$11. \int \sin^5 x dx = -\frac{5}{8} \cos x + \frac{5}{48} \cos 3x - \frac{1}{80} \cos 5x = \\ = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{15} \cos^3 x - \frac{4}{5} \cos x.$$

$$12. \int \sin^6 x dx = \frac{5}{16} x - \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{192} \sin 6x = \\ = -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x.$$

$$13. \int \frac{dx}{\sin^p x} = -\frac{\cos x}{(p-1) \sin^{p-1} x} + \frac{p-2}{p-1} \int \frac{dx}{\sin^{p-2} x}.$$

$$14. \int \frac{dx}{\sin^{2n} x} = \\ = -\frac{\cos x}{2n-1} \left[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right].$$

$$15. \int \frac{dx}{\sin^{2n+1} x} = \\ = -\frac{\cos x}{2n} \left[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x \right] + \\ + \frac{(2n-1)!!}{2^n n!} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$16. \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| = \ln \left| \frac{1 - \cos x}{\sin x} \right| = \ln \left| \frac{\sin x}{1 + \cos x} \right| = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right|.$$

$$17. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$18. \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$19. \int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3 \sin^3 x} - \frac{2}{3} \operatorname{ctg} x = -\frac{1}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

$$20. \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4 \sin^4 x} - \frac{3}{8} \frac{\cos x}{\sin^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$21. \int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \operatorname{ctg}^3 x - \frac{4}{5} \operatorname{ctg} x = -\frac{1}{5} \operatorname{ctg}^5 x - \frac{2}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

1.5.3. Интегралы вида $\int \cos^p x dx$.

$$1. \int \cos^p x dx = \frac{1}{p} \sin x \cos^{p-1} x + \frac{p-1}{p} \int \cos^{p-2} x dx.$$

$$2. \int \cos^{2n} x \, dx =$$

$$= \frac{\sin x}{2n} \left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \cos^{2n-2k-1} x \right] +$$

$$+ \frac{(2n-1)!!}{2^n n!} x$$

$$3. = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin (2n-2k)x}{2n-2k}.$$

$$4. \int \cos^{2n+1} x \, dx =$$

$$= \frac{\sin x}{2n+1} \left[\cos^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \cos^{2n-2k-2} x \right].$$

$$5. = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin (2n-2k+1)x}{2n-2k+1}.$$

$$6. = \sum_{k=0}^n \frac{(-1)^k}{2^{k+1}} \binom{n}{k} \sin^{2k+1} x.$$

$$7. \int \cos x \, dx = \sin x.$$

$$8. \int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$9. \int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x.$$

$$10. \int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x =$$

$$= \frac{3}{8} x + \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x.$$

$$11. \int \cos^5 x \, dx = \frac{5}{8} \sin x + \frac{5}{48} \sin 3x + \frac{1}{80} \sin 5x =$$

$$= \frac{4}{5} \sin x - \frac{4}{15} \sin^3 x + \frac{1}{5} \cos^4 x \sin x.$$

$$12. \int \cos^6 x \, dx = \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x =$$

$$= \frac{5}{16} x + \frac{5}{16} \sin x \cos x + \frac{5}{24} \sin x \cos^3 x + \frac{1}{6} \sin x \cos^5 x.$$

$$13. \int \frac{dx}{\cos^p x} = \frac{\sin x}{(p-1) \cos^{p-1} x} + \frac{p-2}{p-1} \int \frac{dx}{\cos^{p-2} x}.$$

$$14. \int \frac{dx}{\cos^{2n} x} = \frac{\sin x}{2n-1} \left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right].$$

$$15. \int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n} \left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right] + \frac{(2n-1)!!}{2^n n!} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| = \ln \left| \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|.$$

$$17. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$18. \int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$19. \int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \operatorname{tg} x = \frac{1}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

$$20. \int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$21. \int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \operatorname{tg}^3 x + \frac{4}{5} \operatorname{tg} x = \frac{1}{5} \operatorname{tg}^5 x + \frac{2}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

1.5.4. Интегралы вида $\int \sin^p x \cos^q x dx$.

$$1. \int \sin^p x \cos^q x dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x dx.$$

$$2. = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx.$$

$$3. = \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x dx.$$

$$4. = \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x dx.$$

$$5. = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x dx.$$

$$6. = -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x dx.$$

$$7. = \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left(\sin^2 x - \frac{q-1}{p+q-2} \right) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x dx.$$

$$8. \int \sin^p x \cos^{2n} x dx =$$

$$= \frac{\sin^{p+1} x}{2n+p} \left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \right] +$$

$$+ \frac{(2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \sin^p x dx \quad [p \neq -2, -4, \dots, -2n].$$

$$9. \int \sin^p x \cos^{2n+1} x dx =$$

$$= \frac{\sin^{p+1} x}{2n+p+1} \left[\cos^{2n} x + \sum_{k=1}^n \frac{2^k n (n-1) \dots (n-k+1) \cos^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right]$$

$$[p \neq -1, -3, \dots, -(2n+1)].$$

$$10. \int \cos^p x \sin^{2n} x dx =$$

$$= -\frac{\cos^{p+1} x}{2n+p} \left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \right] +$$

$$+ \frac{(2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \cos^p x dx \quad [p \neq -2, -4, \dots, -2n].$$

$$11. \int \cos^p x \sin^{2n+1} x dx =$$

$$= -\frac{\cos^{p+1} x}{2n+p+1} \left[\sin^{2n} x + \sum_{k=1}^n \frac{2^k n (n-1) \dots (n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right]$$

$$[p \neq -1, -3, \dots, -(2n+1)]$$

$$12. \int \sin x \cos^p x dx = -\frac{1}{p+1} \cos^{p+1} x$$

 $[p \neq -1].$

$$13. \int \sin^p x \cos x dx = \frac{1}{p+1} \sin^{p+1} x$$

 $[p \neq -1].$

$$14. \int \sin^2 x \cos^3 x dx = -\frac{1}{8} \left(\frac{1}{4} \sin 4x - x \right).$$

$$15. \int \sin^2 x \cos^3 x dx = -\frac{1}{16} \left(\frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right) =$$

$$= \frac{\sin^3 x}{5} \left(\cos^2 x + \frac{2}{3} \right) = \frac{\sin^3 x}{5} \left(\frac{5}{3} - \sin^2 x \right).$$

$$16. \int \sin^2 x \cos^4 x dx = \frac{x}{16} + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x.$$

$$17. \int \sin^3 x \cos^2 x dx = \frac{1}{16} \left(\frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right) = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x.$$

$$18. \int \sin^3 x \cos^3 x dx = \frac{1}{32} \left(\frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right).$$

$$19. \int \sin^3 x \cos^4 x dx = \frac{1}{7} \cos^5 x \left(-\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right).$$

$$20. \int \sin^4 x \cos^2 x dx = \frac{1}{16} x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x.$$

$$21. \int \sin^4 x \cos^3 x dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right).$$

$$22. \int \sin^4 x \cos^4 x dx = \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x.$$

$$23. \int \sin ax \sin bx \sin cx dx =$$

$$= -\frac{1}{4} \left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right].$$

$$24. \int \sin ax \cos bx \cos cx dx =$$

$$= -\frac{1}{4} \left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right].$$

$$25. \int \cos ax \sin bx \sin cx dx =$$

$$= \frac{1}{4} \left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right].$$

$$26. \int \cos ax \cos bx \cos cx dx =$$

$$= \frac{1}{4} \left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right].$$

1.5.5. Интегралы вида $\int \frac{\sin^p x}{\cos^q x} dx$.

$$1. \int \frac{\sin^p x}{\cos^q x} dx = -\frac{\sin^{p-1} x}{(p-q) \cos^{q-1} x} + \frac{p-1}{p-q} \int \frac{\sin^{p-2} x}{\cos^q x} dx.$$

$$2. = \frac{\sin^{p+1} x}{(q-1) \cos^{q-1} x} - \frac{p-q+2}{q-1} \int \frac{\sin^p x}{\cos^{q-2} x} dx.$$

$$3. = \frac{\sin^{p-1} x}{(q-1) \cos^{q-1} x} - \frac{p-1}{q-1} \int \frac{\sin^{p-2} x}{\cos^{q-2} x} dx.$$

$$4. \int \frac{\sin^p x}{\cos^{2n+1} x} dx = \frac{\sin^{p+1} x}{2n} \times$$

$$\times \left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right] +$$

$$+ \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\sin^p x}{\cos x} dx.$$

$$5. \int \frac{\sin^p x}{\cos^{2n} x} dx = \frac{\sin^{p+1} x}{2n-1} \times$$

$$\times \left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right] +$$

$$+ \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \sin^p x dx.$$

$$6. \int \frac{\sin^{2m+1} x}{\cos^{2n+1} x} dx = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq n}}^m (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k} + (-1)^{n+1} \binom{m}{n} \ln |\cos x|$$

[$n \leq m$].

$$7. = \frac{1}{2} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k}$$

[$n > m$].

$$8. \int \frac{\sin^{2m+1} x}{\cos^{2n} x} dx = \sum_{k=0}^m (-1)^{k+1} \binom{m}{k} \frac{\cos^{2k-2n+1} x}{2k-2n+1}.$$

$$9. \int \frac{\sin x}{\cos^q x} dx = \frac{1}{(q-1) \cos^{q-1} x}.$$

$$10. \int \frac{\sin^p x}{\cos^{p+2} x} dx = \frac{1}{p+1} \operatorname{tg}^{p+1} x.$$

$$11. \int \frac{\sin^{2n+1} x}{\cos x} dx = - \sum_{k=1}^n \frac{\sin^{2k} x}{2k} - \ln |\cos x|.$$

$$12. = \sum_{k=1}^n \frac{(-1)^{k+1}}{2k} \binom{n}{k} \cos^{2k} x - \ln |\cos x|.$$

$$13. \int \frac{\sin^{2n} x}{\cos x} dx = - \sum_{k=1}^n \frac{\sin^{2k-1} x}{2k-1} + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$14. \int \frac{\sin x}{\cos x} dx = - \ln |\cos x|.$$

$$15. \int \frac{\sin^2 x}{\cos x} dx = - \sin x + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{\sin^3 x}{\cos x} dx = - \frac{\sin^2 x}{2} - \ln |\cos x| = \frac{1}{2} \cos^2 x - \ln |\cos x|.$$

$$17. \int \frac{\sin^4 x}{\cos x} dx = - \frac{1}{3} \sin^3 x - \sin x + \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$18. \int \frac{\sin^p x}{\cos^2 x} dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-2} x dx.$$

$$19. \int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x}.$$

$$20. \int \frac{\sin^2 x}{\cos^2 x} dx = \operatorname{tg} x - x.$$

$$21. \int \frac{\sin^3 x}{\cos^2 x} dx = \cos x + \frac{1}{\cos x}.$$

$$22. \int \frac{\sin^4 x}{\cos^2 x} dx = \operatorname{tg} x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$23. \int \frac{\sin x}{\cos^3 x} dx = \frac{1}{2 \cos^2 x} = \frac{1}{2} \operatorname{tg}^2 x.$$

$$24. \int \frac{\sin^2 x}{\cos^3 x} dx = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$25. \int \frac{\sin^3 x}{\cos^3 x} dx = \frac{1}{2 \cos^2 x} + \ln |\cos x|.$$

$$26. \int \frac{\sin^4 x}{\cos^3 x} dx = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$27. \int \frac{\sin x}{\cos^4 x} dx = \frac{1}{3 \cos^3 x}. \quad 28. \int \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3} \operatorname{tg}^3 x.$$

$$29. \int \frac{\sin^3 x}{\cos^4 x} dx = -\frac{1}{\cos x} + \frac{1}{3 \cos^3 x}.$$

$$30. \int \frac{\sin^4 x}{\cos^4 x} dx = \frac{1}{3} \operatorname{tg}^3 x - \operatorname{tg} x + x.$$

1.5.6. Интегралы вида $\int \frac{\cos^q x}{\sin^p x} dx$.

$$1. \int \frac{\cos^q x}{\sin^p x} dx = \frac{\cos^{q-1} x}{(q-p) \sin^{p-1} x} + \frac{q-1}{q-p} \int \frac{\cos^{q-2} x}{\sin^p x} dx.$$

$$2. = -\frac{\cos^{q+1} x}{(p-1) \sin^{p-1} x} - \frac{q-p+2}{p-1} \int \frac{\cos^q x}{\sin^{p-2} x} dx.$$

$$3. = -\frac{\cos^{q-1} x}{(p-1) \sin^{p-1} x} - \frac{q-1}{p-1} \int \frac{\cos^{q-2} x}{\sin^{p-2} x} dx.$$

$$4. \int \frac{\cos^p x}{\sin^{2n+1} x} dx = -\frac{\cos^{p+1} x}{2n} [\operatorname{cosec}^{2n} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x] + \\ + \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\cos^p x}{\sin x} dx.$$

$$5. \int \frac{\cos^p x}{\sin^{2n} x} dx = -\frac{\cos^{p+1} x}{2n-1} [\operatorname{cosec}^{2n-1} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x] + \\ + \frac{(2n-p-2)(2n-p-4) \dots (2-p)(-p)}{(2n-1)!!} \int \cos^p x dx.$$

$$6. \int \frac{\cos^{2n+1} x}{\sin^{2m+1} x} dx = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq m}}^n (-1)^{k+1} \binom{n}{k} \frac{\sin^{2k-2m} x}{m-k} + (-1)^m \binom{n}{m} \ln |\sin x|.$$

$[m \leq n].$

$$7. \quad = \frac{1}{2} \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \frac{\sin^{2k-2m} x}{m-k} \quad [m > n].$$

$$8. \quad \int \frac{\cos^{2n+1} x}{\sin^{2m} x} dx = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\sin^{2k-2m+1} x}{2k-2m+1}.$$

$$9. \quad \int \frac{\cos x}{\sin^p x} dx = -\frac{1}{(p-1) \sin^{p-1} x}.$$

$$10. \quad \int \frac{\cos^q x}{\sin^{q+2} x} dx = -\frac{1}{q+1} \operatorname{ctg}^{q+1} x.$$

$$11. \quad \int \frac{\cos^{2n+1} x}{\sin x} dx = \sum_{k=1}^n \frac{\cos^{2k} x}{2k} + \ln |\sin x|.$$

$$12. \quad = \frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \sin^{2k} x + \ln |\sin x|.$$

$$13. \quad \int \frac{\cos^{2n} x}{\sin x} dx = \sum_{k=1}^n \frac{\cos^{2k-1} x}{2k-1} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$14. \quad \int \frac{\cos x}{\sin x} dx = \ln |\sin x|.$$

$$15. \quad \int \frac{\cos^2 x}{\sin x} dx = \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$16. \quad \int \frac{\cos^3 x}{\sin x} dx = \frac{\cos^2 x}{2} + \ln |\sin x|.$$

$$17. \quad \int \frac{\cos^4 x}{\sin x} dx = \frac{1}{3} \cos^3 x + \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$18. \quad \int \frac{\cos^p x}{\sin^2 x} dx = -\frac{\cos^{p-1} x}{\sin x} - (p-1) \int \cos^{p-2} x dx.$$

$$19. \quad \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}. \quad 20. \quad \int \frac{\cos^2 x}{\sin^2 x} dx = -\operatorname{ctg} x - x.$$

$$21. \quad \int \frac{\cos^3 x}{\sin^2 x} dx = -\sin x - \frac{1}{\sin x}.$$

$$22. \quad \int \frac{\cos^4 x}{\sin^2 x} dx = -\operatorname{ctg} x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$23. \quad \int \frac{\cos x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x}.$$

$$24. \quad \int \frac{\cos^2 x}{\sin^3 x} dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$25. \quad \int \frac{\cos^3 x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - \ln |\sin x|.$$

$$26. \int \frac{\cos^4 x}{\sin^3 x} dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$27. \int \frac{\cos x}{\sin^4 x} dx = -\frac{1}{3 \sin^3 x}. \quad 28. \int \frac{\cos^2 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^3 x.$$

$$29. \int \frac{\cos^3 x}{\sin^4 x} dx = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}.$$

$$30. \int \frac{\cos^4 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^3 x + \operatorname{ctg} x + x.$$

1.5.7. Интегралы вида $\int \frac{dx}{\sin^p x \cos^q x}$.

$$1. \int \frac{dx}{\sin^p x \cos^q x} = \frac{-1}{(p-1) \sin^{p-1} x \cos^{q-1} x} + \frac{p+q-2}{p-1} \int \frac{dx}{\sin^{p-2} x \cos^q x}.$$

$$2. = \frac{1}{(q-1) \sin^{p-1} x \cos^{q-1} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\sin^p x \cos^{q-2} x}$$

[см. также 1.5.4].

$$3. \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\operatorname{tg}^{2k-2m+1} x}{2k-2m+1}.$$

$$4. \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln |\operatorname{tg} x|.$$

$$5. \int \frac{dx}{\sin^{2m+1} x \cos x} = -\sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$6. \int \frac{dx}{\sin^{2m} x \cos x} = -\sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right|.$$

$$7. \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$8. \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$9. \int \frac{dx}{\sin x \cos x} = \ln |\operatorname{tg} x|. \quad 10. \int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$11. \int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2 \cos^2 x} + \ln |\operatorname{tg} x|.$$

$$12. \int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$13. \int \frac{dx}{\sin^2 x \cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| - \operatorname{cosec} x.$$

$$14. \int \frac{dx}{\sin^2 x \cos^2 x} = -2 \operatorname{ctg} 2x.$$

$$15. \int \frac{dx}{\sin^2 x \cos^3 x} = \left(\frac{1}{2 \cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$17. \int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln |\operatorname{tg} x|.$$

$$18. \int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x} \left(\frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$19. \int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2 \cos 2x}{\sin^2 2x} + 2 \ln |\operatorname{tg} x|.$$

$$20. \int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3 \cos^3 x} - \frac{\cos x}{2 \sin^2 x} + \frac{5}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$21. \int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3 \sin^3 x} + \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$22. \int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$23. \int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^2 x} + \frac{5}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$24. \int \frac{dx}{\sin^4 x \cos^4 x} = -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x.$$

1.5.8. Интегралы вида $\int \operatorname{tg}^p x dx$, $\int \operatorname{ctg}^p x dx$.

$$1. \int \frac{\operatorname{tg} x}{\operatorname{ctg} x} dx = \pm \frac{1}{p-1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-1} - \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-2} dx.$$

$$2. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+1} + (-1)^n x.$$

$$3. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n+1} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+2} \mp$$

$$\mp (-1)^n \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$4. = \pm \sum_{k=1}^n \frac{(-1)^{n+k}}{2k} \binom{n}{k} \left\{ \frac{\cos x}{\sin x} \right\}^{-2k} \mp (-1)^n \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$5. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx = \mp \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}. \quad 6. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = \pm \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} - x.$$

$$7. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^3 dx = \pm \frac{1}{2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 \pm \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$8. \int \sqrt{\operatorname{tg} x} dx = \frac{1}{\sqrt{2}} [\ln (\sin x + \cos x - \sqrt{\sin 2x}) + \arcsin (\sin x - \cos x)].$$

$$9. = \frac{1}{\sqrt{2}} \left[\frac{1}{2} \ln \frac{1 - \sqrt{2} \operatorname{tg} x + \operatorname{tg} x}{1 + \sqrt{2} \operatorname{tg} x + \operatorname{tg} x} + \operatorname{arctg} \frac{\sqrt{2} \operatorname{tg} x}{1 - \operatorname{tg} x} \right].$$

1.5.9. Интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) dx$.

$$1. \int \frac{A + B \cos x + C \sin x}{(a + b \cos x + c \sin x)^n} dx =$$

$$= \frac{(Bc - Cb) + (Ac - Ca) \cos x - (Ab - Ba) \sin x}{(n-1)(a^2 - b^2 - c^2)(a + b \cos x + c \sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba) \cos x - (Ac - Ca) \sin x]}{(a + b \cos x + c \sin x)^{n-1}} dx$$

[$n \geq 2, a^2 \neq b^2 + c^2$].

$$2. = \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2} \right) \times$$

$$\times (-c \cos x + b \sin x) \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)! a^k} \frac{1}{(a + b \cos x + c \sin x)^{n-k}}$$

[$n \geq 2, a^2 = b^2 + c^2$].

$$3. \int \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x} dx = \frac{Bc - Cb}{b^2 + c^2} \ln(a + b \cos x + c \sin x) +$$

$$+ \frac{Bb + Cc}{b^2 + c^2} x + \left(A - \frac{Bb + Cc}{b^2 + c^2} a \right) \int \frac{dx}{a + b \cos x + c \sin x}.$$

$$4. \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n} \quad [b = r \cos \alpha, c = r \sin \alpha].$$

$$5. \int \frac{dx}{a + b \cos x + c \sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \operatorname{arctg} \frac{(a-b) \operatorname{tg} \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}} \quad [a^2 > b^2 + c^2].$$

$$6. = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a-b) \operatorname{tg} \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a-b) \operatorname{tg} \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}} \quad [a^2 < b^2 + c^2].$$

$$7. = \frac{1}{c} \ln \left(a + c \operatorname{tg} \frac{x}{2} \right) \quad [a = b].$$

$$8. = \frac{-2}{c + (a-b) \operatorname{tg} \frac{x}{2}} \quad [a^2 = b^2 + c^2].$$

$$9. \int \frac{dx}{(a \cos x + b \sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{dx}{\sin^n \left(x + \operatorname{arctg} \frac{a}{b} \right)} \quad [\text{см. 1.5.2.}].$$

$$10. \int \frac{A + B \cos x + C \sin x}{a \cos x + b \sin x} dx = \frac{Ba + Cb}{a^2 + b^2} x +$$

$$+ \frac{Bb - Ca}{a^2 + b^2} \ln \sin \left(x + \operatorname{arctg} \frac{a}{b} \right) + \frac{A}{\sqrt{a^2 + b^2}} \ln \operatorname{tg} \left[\frac{1}{2} \left(x + \operatorname{arctg} \frac{a}{b} \right) \right].$$

$$11. \int \frac{1}{a \cos x + b \sin x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{a^2 + b^2} \left[\left\{ \frac{b}{a} \right\} x \mp \left\{ \frac{a}{b} \right\} \ln (a \cos x + b \sin x) \right].$$

$$12. \int \frac{dx}{(a \cos x + b \sin x)^2} = -\frac{1}{a^2 + b^2} \cdot \frac{a \sin x - b \cos x}{a \cos x + b \sin x}.$$

$$13. \int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{a \operatorname{tg} \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \quad [a^2 > b^2].$$

$$14. = \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \operatorname{tg} \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \operatorname{tg} \frac{x}{2} + b + \sqrt{b^2 - a^2}} \quad [a^2 < b^2].$$

$$15. \int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} \operatorname{tg} \frac{x}{2}}{a + b} \quad [a^2 > b^2].$$

$$16. = \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} + a + b}{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} - a - b} \quad [a^2 < b^2].$$

$$17. \int \frac{A + B \cos x + C \sin x}{(1 \pm \sin x)^n} dx = -\frac{1}{2^{n-1}} \left[2C \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \pm \right. \\ \left. \pm (A \mp C) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \right] \mp \frac{B}{n-1} \frac{1}{(1 \pm \sin x)^{n-1}} \quad [n \geq 2]$$

$$18. = \pm Cx + (A \mp C) \operatorname{tg} \left(\frac{\pi}{4} \mp \frac{x}{2} \right) \pm B \ln (1 \pm \sin x) \quad [n=1].$$

$$19. \int \frac{A + B \cos x + C \sin x}{(1 \pm \cos x)^n} dx = \frac{1}{2^{n-1}} \left[2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{\pm(2k+1)} \frac{x}{2}}{2k+1} \pm \right. \\ \left. \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{\pm(2k+1)} \frac{x}{2}}{2k+1} \right] \pm \frac{C}{n-1} \frac{1}{(1 \pm \cos x)^{n-1}} \quad [n \geq 2].$$

$$20. = \pm Bx \pm (A \mp B) \operatorname{tg}^{\pm 1} \frac{x}{2} \mp C \ln (1 \pm \cos x) \quad [n=1].$$

$$21. \int \frac{A + B \cos x + C \sin x}{\sin x (a + b \sin x)} dx = \\ = \frac{A}{a} \ln \operatorname{tg} \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} + \frac{Ca - Ab}{a} \int \frac{dx}{a + b \sin x}.$$

$$22. \int \frac{A + B \cos x + C \sin x}{\sin x (a + b \cos x)} dx = \\ = \frac{1}{a^2 - b^2} \left[(Aa - Bb) \ln \operatorname{tg} \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right] + C \int \frac{dx}{a + b \cos x}.$$

$$23. \int \frac{A + B \cos x + C \sin x}{\sin x (1 \pm \cos x)} dx = \frac{A \pm B}{2} \ln \operatorname{tg} \frac{x}{2} - \frac{B \mp A}{2} \frac{1}{1 \pm \cos x} \pm C \left(\operatorname{tg} \frac{x}{2} \right)^{\pm 1}.$$

$$24. \int \frac{A + B \cos x + C \sin x}{\cos x (a + b \sin x)} dx = \\ = \frac{1}{a^2 - b^2} \left[(Aa - Cb) \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - (Ab - Ca) \ln \frac{a + b \sin x}{\cos x} \right] + B \int \frac{dx}{a + b \sin x}.$$

$$25. \int \frac{A + B \cos x + C \sin x}{\cos x (1 \pm \sin x)} dx = \\ = \frac{A \pm C}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp C}{2} \cdot \frac{1}{(1 \pm \sin x)} + B \operatorname{tg} \left(\frac{\pi}{4} \pm \frac{x}{2} \right);$$

$$26. \int \frac{A + B \cos x + C \sin x}{\cos x (a + b \cos x)} dx = \\ = \frac{A}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{C}{a} \ln \frac{a + b \cos x}{\cos x} + \left(B - \frac{Ab}{a} \right) \int \frac{dx}{a + b \cos x}.$$

$$27. \int \frac{dx}{a + b \left\{ \frac{\sin x}{\cos x} \right\}^2} = \frac{\pm \operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg} \left(\sqrt{\frac{a+b}{a}} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right) \quad \left[\frac{b}{a} > -1 \right];$$

$$28. = \pm \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth} \left(\sqrt{-\frac{a+b}{a}} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right). \\ \left[\frac{b}{a} < -1; \left\{ \frac{\sin x}{\cos x} \right\}^2 < -\frac{a}{b} \right].$$

$$29. = \pm \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth} \left(\sqrt{-\frac{a+b}{a}} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right). \\ \left[\frac{b}{a} < -1; \left\{ \frac{\sin x}{\cos x} \right\}^2 > -\frac{a}{b} \right].$$

$$30. \int \frac{dx}{1 + \left\{ \frac{\sin x}{\cos x} \right\}^2} = \pm \frac{1}{\sqrt{2}} \operatorname{arctg} \left(\sqrt{2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right).$$

$$31. \int \frac{dx}{1 - \left\{ \frac{\sin x}{\cos x} \right\}^2} = \pm \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}.$$

$$32. \int \frac{dx}{\left(a + b \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b \left\{ \frac{\sin x}{\cos x} \right\}^2} \pm \frac{b \sin x \cos x}{a + b \left\{ \frac{\sin x}{\cos x} \right\}^2} \right].$$

$$33. \int \frac{dx}{\left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^2} = \frac{1}{4} \left[\frac{3}{\sqrt{2}} \operatorname{arctg} \left(\sqrt{2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right) \pm \frac{\sin x \cos x}{1 + \left\{ \frac{\sin x}{\cos x} \right\}^2} \right].$$

$$34. \int \frac{dx}{(a + b \sin^2 x)^3} = \frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg} (p \operatorname{tg} x) + \right. \\ \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{tg} x}{1 + p^2 \operatorname{tg}^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{tg}^2 x \right) \frac{2p \operatorname{tg} x}{(1 + p^2 \operatorname{tg}^2 x)^2} \right] \\ \left[p^2 = 1 + \frac{b}{a} > 0 \right].$$

15.9.]

$$\begin{aligned}
 35. \quad &= \frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{tg} x) + \right. \\
 &\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{tg} x}{1 - q^2 \operatorname{tg}^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \operatorname{tg}^2 x \right) \frac{2q \operatorname{tg} x}{(1 - q^2 \operatorname{tg}^2 x)^2} \right] \\
 &\quad \left[q^2 = -1 - \frac{b}{a} > 0, \sin^2 x < -\frac{a}{b}; \text{ при } \sin^2 x > -\frac{a}{b} \right. \\
 &\quad \left. \text{следует } \operatorname{Arth}(q \operatorname{tg} x) \text{ заменить на } \operatorname{Arcth}(q \operatorname{tg} x) \right].
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int \frac{dx}{(a + b \cos^2 x)^3} &= -\frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{ctg} x) + \right. \\
 &\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{ctg} x}{1 + p^2 \operatorname{ctg}^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{ctg}^2 x \right) \frac{2p \operatorname{ctg} x}{(1 + p^2 \operatorname{ctg}^2 x)^2} \right] \\
 &\quad \left[p^2 = 1 + \frac{b}{a} > 0 \right].
 \end{aligned}$$

$$\begin{aligned}
 37. \quad &= -\frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{ctg} x) + \right. \\
 &\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{ctg} x}{1 - q^2 \operatorname{ctg}^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \operatorname{ctg}^2 x \right) \frac{2q \operatorname{ctg} x}{(1 - q^2 \operatorname{ctg}^2 x)^2} \right] \\
 &\quad \left[q^2 = -1 - \frac{b}{a} > 0; \cos^2 x < -\frac{a}{b}; \text{ при } \cos^2 x > -\frac{a}{b} \right. \\
 &\quad \left. \text{следует } \operatorname{Arth}(q \operatorname{ctg} x) \text{ заменить на } \operatorname{Arcth}(q \operatorname{ctg} x) \right].
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \int \frac{A + B \cos x + C \sin x}{(a_1 + b_1 \cos x + c_1 \sin x)(a_2 + b_2 \cos x + c_2 \sin x)} dx &= \\
 &= A_0 \ln \frac{a_1 + b_1 \cos x + c_1 \sin x}{a_2 + b_2 \cos x + c_2 \sin x} + A_1 \int \frac{dx}{a_1 + b_1 \cos x + c_1 \sin x} + \\
 &\quad + A_2 \int \frac{dx}{a_2 + b_2 \cos x + c_2 \sin x},
 \end{aligned}$$

где

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} B & C \\ b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} C & B \\ c_2 & b_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2; \text{ см. 1.5.9.5} \right].$$

$$39. \int \frac{dx}{a \cos^2 x + b \sin x \cos x + c \sin^2 x} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \frac{2c \operatorname{tg} x + b - \sqrt{b^2 - 4ac}}{2c \operatorname{tg} x + b + \sqrt{b^2 - 4ac}} \quad [b^2 > 4ac].$$

$$40. = \frac{2}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2c \operatorname{tg} x + b}{\sqrt{4ac - b^2}} \quad [b^2 < 4ac].$$

$$41. = -\frac{2}{2c \operatorname{tg} x + b} \quad [b^2 = 4ac].$$

$$42. \int \frac{A \cos^2 x + 2B \sin x \cos x + C \sin^2 x}{a \cos^2 x + 2b \sin x \cos x + c \sin^2 x} dx = \frac{1}{4b^2 + (a-c)^2} \{ [4Bb + (A-C)(a-c)]x + [(A-C)b - B(a-c)] \times \\ \times \ln(a \cos^2 x + 2b \sin x \cos x + c \sin^2 x) + [2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)] f(x) \},$$

где

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{tg} x + b - \sqrt{b^2 - ac}}{c \operatorname{tg} x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac],$$

$$= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{tg} x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac],$$

$$= -\frac{1}{c \operatorname{tg} x + b} \quad [b^2 = ac].$$

$$43. \int \frac{dx}{(a \cos^2 x + b \sin x \cos x + c \sin^2 x)^2} = \frac{b \cos 2x + (c-a) \sin 2x}{(4ac - b^2)(a \cos^2 x + b \sin x \cos x + c \sin^2 x)} + \\ + \frac{2(a+c)}{4ac - b^2} \int \frac{dx}{a \cos^2 x + b \sin x \cos x + c \sin^2 x} \quad [b^2 \neq 4ac].$$

$$44. = 16a^2 \int \frac{dx}{(2a \cos x + b \sin x)^4} \quad [b^2 = 4ac].$$

$$45. \int \frac{\operatorname{tg} x dx}{\operatorname{tg} x + a} = \frac{1}{a^2 + 1} [x - a \ln(a \cos x + \sin x)].$$

$$46. \int \frac{\operatorname{tg} x - a}{\operatorname{tg} x + a} dx = \frac{1-a^2}{1+a^2} x - \frac{2a}{1+a^2} \ln \sin(x + \operatorname{arctg} a).$$

$$47. \int \frac{dx}{a^2 + \operatorname{tg}^2 x} = \frac{x}{a^2 - 1} - \frac{1}{(a^2 - 1)a} \operatorname{arctg} \frac{\operatorname{tg} x}{a}.$$

$$48. \int \frac{dx}{a^2 - \operatorname{tg}^2 x} = \frac{x}{a^2 + 1} - \frac{1}{2(a^2 + 1)a} \ln \left| \frac{\operatorname{tg} x - a}{\operatorname{tg} x + a} \right|.$$

$$49. \int \frac{dx}{1 + \operatorname{tg}^2 x} = \frac{x}{2} + \frac{1}{4} \sin 2x.$$

$$50. \int \frac{\operatorname{tg} x dx}{a^2 + \operatorname{tg}^2 x} = \frac{\ln(a^2 \cos^2 x + \sin^2 x)}{2(1-a^2)}.$$

1.5.10. Интегралы вида $\int \left\{ \frac{\sin(ax+b)}{\cos(ax+b)} \frac{\sin(cx+d)}{\cos(cx+d)} \right\} dx,$

$$\int \sin(ax+b) \cos(cx+d) dx.$$

$$1. \int \left\{ \frac{\sin(ax+b)}{\cos(ax+b)} \right\} dx = \mp \frac{1}{a} \left\{ \frac{\cos(ax+b)}{\sin(ax+b)} \right\}.$$

$$2. \int \left\{ \frac{\sin(ax+b)}{\cos(ax+b)} \frac{\sin(cx+d)}{\cos(cx+d)} \right\} dx = \\ = \frac{1}{2(a-c)} \sin[(a-c)x+b-d] \mp \frac{\sin[(a+c)x+b+d]}{2(a+c)}.$$

$$3. \int \sin(ax+b) \cos(cx+d) dx = \\ = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)}.$$

$$4. \int \left\{ \frac{\sin(ax+b)}{\cos(ax+b)} \frac{\sin(ax+d)}{\cos(ax+d)} \right\} dx = \frac{x}{2} \cos(b-d) \mp \frac{\sin(2ax+b+d)}{4a}.$$

$$5. \int \sin(ax+b) \cos(ax+d) dx = \frac{x}{2} \sin(b-d) - \frac{\cos(2ax+b+d)}{4a}.$$

1.5.11. Интегралы вида $\int \sin^p x \sin ax dx.$

$$1. \int \sin^p x \sin ax dx = \frac{-1 + \sin^p x \cos ax}{p+a} + \frac{p}{p+a} \int \sin^{p-1} x \cos(a-1)x dx.$$

$$2. = -\frac{\sin^p x \cos ax}{p+a} + \frac{p \sin^{p-1} x \sin(a-1)x}{(p+a)(p+a-2)} - \\ - \frac{p(p-1)}{(p+a)(p+a-2)} \int \sin^{p-2} x \sin(a-2)x dx.$$

$$3. \int \sin^p x \sin 2nx dx = \\ = 2n \left\{ \frac{\sin^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots [4n^2-(2k)^2]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x \right\} \\ [p \neq -2, -4, \dots, -2n]$$

$$4. \int \sin^p x \sin(2n+1)x dx = (2n+1) \left\{ \int \sin^{p+1} x dx + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \sin^{2k+p+1} x dx \right\}.$$

$$5. \int \sin^{p-1} x \sin(p+1)x dx = \frac{1}{p} \sin^p x \sin px.$$

$$6. \int \sin^{p-1} x \sin \left[(p+1) \left(\frac{\pi}{2} - x \right) \right] dx = \frac{1}{p} \sin^p x \cos p \left(\frac{\pi}{2} - x \right).$$

$$7. \int \frac{\sin ax}{\sin^p x} dx = 2 \int \frac{\cos(a-1)x}{\sin^{p-1} x} dx + \int \frac{\sin(a-2)x}{\sin^p x} dx.$$

$$8. \int \frac{\sin 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin (2k-1)x}{2k-1}.$$

$$9. \int \frac{\sin (2n+1)x}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + x.$$

$$10. \int \frac{\sin 2x}{\sin^n x} dx = -\frac{2}{(n-2) \sin^{n-2} x}.$$

$$11. \int \frac{\sin 2x}{\sin^2 x} dx = 2 \ln |\sin x|. \quad 12. \int \frac{\sin 3x}{\sin x} dx = x + \sin 2x.$$

$$13. \int \frac{\sin 3x}{\sin^2 x} dx = 3 \ln \left| \operatorname{tg} \frac{x}{2} \right| + 4 \cos x.$$

$$14. \int \frac{\sin 3x}{\sin^3 x} dx = -3 \operatorname{ctg} x - 4x.$$

1.5.12. Интегралы вида $\int \sin^p x \cos ax dx$.

$$1. \int \sin^p x \cos ax dx = \frac{\sin^p x \sin ax}{p+a} - \frac{p}{p+a} \int \sin^{p-1} x \sin (a-1)x dx.$$

$$2. = \frac{\sin^p x \sin ax}{p+a} + \frac{p \sin^{p-1} x \cos (a-1)x}{(p+a)(p+a-2)} - \\ - \frac{p(p-1)}{(p+a)(p+a-2)} \int \sin^{p-2} x \cos (a-2)x dx.$$

$$3. \int \sin^p x \cos 2nx dx = \int \sin^p x dx + \\ + \sum_{k=1}^n (-1)^k \frac{4n^2(4n^2-2^2) \dots [4n^2-(2k-2)^2]}{(2k)!} \int \sin^{2k+p} x dx.$$

$$4. \int \sin^p x \cos (2n+1)x dx = \frac{\sin^{p+1} x}{p+1} + \\ + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \sin^{2k+p+1} x$$

$[p \neq -1, -3, -5, \dots, -(2n+1)].$

$$5. \int \sin^{p-1} x \cos (p+1)x dx = \frac{1}{p} \sin^p x \cos px.$$

$$6. \int \sin^{p-1} x \cos \left[(p+1) \left(\frac{\pi}{2} - x \right) \right] dx = -\frac{1}{p} \sin^p x \sin p \left(\frac{\pi}{2} - x \right).$$

$$7. \int \frac{\cos ax}{\sin^p x} dx = -2 \int \frac{\sin (a-1)x}{\sin^{p-1} x} dx + \int \frac{\cos (a-2)x}{\sin^p x} dx.$$

$$8. \int \frac{\cos 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos (2k-1)x}{2k-1} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$9. \int \frac{\cos (2n+1) x}{\sin x} dx = \sum_{k=1}^n \frac{\cos 2kx}{k} + \ln |\sin x|.$$

$$10. \int \frac{\cos 2x}{\sin x} dx = 2 \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|. \quad 11. \int \frac{\cos 2x}{\sin^2 x} dx = -\operatorname{ctg} x - 2x.$$

$$12. \int \frac{\cos 2x}{\sin^3 x} dx = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|;$$

$$13. \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}.$$

$$14. \int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln |\sin x|.$$

$$15. \int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln |\sin x|.$$

1.5.13. Интегралы вида $\int \cos^p x \sin ax dx$.

$$1. \int \cos^p x \sin ax dx = -\frac{\cos^p x \cos ax}{p+a} + \frac{p}{p+a} \int \cos^{p-1} x \sin (a-1) x dx.$$

$$2. \int \cos^p x \sin 2nx dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots [4n^2-(2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\} \\ [p \neq -2, -4, \dots, -2n].$$

$$3. \int \cos^p x \sin (2n+1) x dx = (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times \right. \\ \left. \times \cos^{2k+p+1} x \right\} [p \neq -1, -3, -5, \dots, -(2n+1)].$$

$$4. \int \cos^{p-1} x \sin (p+1) x dx = -\frac{1}{p} \cos^p x \cos px.$$

$$5. \int \frac{\sin ax}{\cos^p x} dx = 2 \int \frac{\sin (a-1) x}{\cos^{p-1} x} dx - \int \frac{\sin (a-2) x}{\cos^p x} dx.$$

$$6. \int \frac{\sin (2n+1) x}{\cos x} dx = \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos 2kx}{k} + (-1)^{n+1} \ln |\cos x|.$$

$$7. \int \frac{\sin 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos (2k-1) x}{2k-1}.$$

$$8. \int \frac{\sin 2x}{\cos^n x} dx = \frac{2}{(n-2) \cos^{n-2} x}. \quad 9. \int \frac{\sin 2x}{\cos^2 x} dx = -2 \ln |\cos x|.$$

$$10. \int \frac{\sin 3x}{\cos^n x} dx = \frac{4}{(n-3) \cos^{n-3} x} - \frac{1}{(n-1) \cos^{n-1} x}.$$

$$11. \int \frac{\sin 3x}{\cos x} dx = 2 \sin^2 x + \ln |\cos x|.$$

$$12. \int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2 \cos^2 x} - 4 \ln |\cos x|.$$

1.5.14. Интегралы вида $\int \cos^p x \cos ax dx$.

$$1. \int \cos^p x \cos ax dx = \frac{\cos^p x \sin ax}{p+a} + \frac{p}{p+a} \int \cos^{p-1} x \cos (a-1)x dx.$$

$$2. \int \cos^p x \cos 2nx dx = (-1)^n \left\{ \int \cos^p x dx + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{4n^2 (2n^2 - 2^2) \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \cos^{2k+p} x dx \right\}.$$

$$3. \int \cos^p x \cos (2n+1)x dx = (-1)^n (2n+1) \left\{ \int \cos^{p+1} x dx + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \cos^{2k+p+1} x dx \right\}.$$

$$4. \int \cos^{p-1} x \cos (p+1)x dx = \frac{1}{p} \cos^p x \sin px.$$

$$5. \int \frac{\cos ax}{\cos^p x} dx = 2 \int \frac{\cos (a-1)x}{\cos^{p-1} x} dx - \int \frac{\cos (a-2)x}{\cos^p x} dx.$$

$$6. \int \frac{\cos 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin (2k-1)x}{2k-1} + (-1)^n \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$7. \int \frac{\cos (2n+1)x}{\cos x} dx = \sum_{k=1}^n (-1)^{n-k} \frac{\sin 2kx}{k} + (-1)^n x.$$

$$8. \int \frac{\cos 2x}{\cos x} dx = 2 \sin x - \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$9. \int \frac{\cos 2x}{\cos^2 x} dx = 2x - \operatorname{tg} x.$$

$$10. \int \frac{\cos 2x}{\cos^3 x} dx = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$11. \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x.$$

$$12. \int \frac{\cos 3x}{\cos^2 x} dx = 4 \sin x - 3 \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$13. \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \operatorname{tg} x.$$

1.5.15. Интегралы вида $\int \left\{ \frac{\sin x}{\cos x} \right\}^m \left\{ \frac{\sin nx}{\cos nx} \right\}^{-1} dx$.

$$1. \int \frac{\sin^m x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \left[\frac{2k+1}{2(2n+1)} \pi \right] \ln \left| \frac{\sin \left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[\frac{(k+n+1)\pi}{2(2n+1)} - \frac{x}{2} \right]} \right|$$

[$m \leq 2n$].

$$2. \int \frac{\sin^{2m} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln |\cos x| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left| \cos^2 x - \sin^2 \frac{k\pi}{2n} \right| \right\}$$

[$m \leq n$].

$$3. \int \frac{\sin^{2m+1} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left| \operatorname{tg} \left(\frac{n+k}{4n} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{n-k}{4n} \pi - \frac{x}{2} \right) \right| \right\}$$

[$m < n$].

$$4. \int \frac{\sin^{2m} x dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left| \operatorname{tg} \left(\frac{2n+2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{2n-2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \right| \right\}$$

[$m \leq n$].

$$5. \int \frac{\sin^{2m+1} x dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln |\cos x| + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left| \cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right| \right\}$$

[$m \leq n$].

$$6. \int \frac{\sin^m x dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left(\frac{2k+1}{4n} \pi \right) \times$$

$$\times \ln \left| \frac{\sin \left(\frac{2k-2n+1}{8n} \pi + \frac{x}{2} \right)}{\sin \left(\frac{2k+2n+1}{8n} \pi - \frac{x}{2} \right)} \right|$$

[$m < 2n$].

$$7. \int \frac{\cos^{2m+1} x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln |\sin x| + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left| \sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right| \right\}$$

[$m \leq n$].

$$8. \int \frac{\cos^{2m} x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln \left| \operatorname{tg} \frac{x}{2} \right| + \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n+2} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n+2} \right) \right| \right\} \quad [m \leq n].$$

$$9. \int \frac{\cos^{2m+1} x}{\sin 2nx} dx = \\ = \frac{1}{2n} \left\{ \ln \left| \operatorname{tg} \frac{x}{2} \right| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n} \right) \right| \right\} \\ [m < n].$$

$$10. \int \frac{\cos^{2m} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln |\sin x| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left| \sin^2 x - \sin^2 \frac{k\pi}{2n} \right| \right\} \\ [m \leq n].$$

$$11. \int \frac{\cos^m x}{\cos nx} dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n} \pi \ln \left| \frac{\sin \left(\frac{2k+1}{4n} \pi + \frac{x}{2} \right)}{\sin \left(\frac{2k+1}{4n} \pi - \frac{x}{2} \right)} \right| \\ [m \leq n].$$

1.5.16. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax}) dx$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{2 \sin ax}{1 + \sin ax + \cos cx}}$.

$$1. \int \sin^{m+1/2} 2x \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \pm 2^{m+3/2} \int \frac{t^{2(2n+m+2)}}{(1+t^4)^{n+m+2}} dt \\ [t = \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2}] \\ = \pm 2^{m+3/2} \int \frac{t^{2(m+1)} dt}{(1+t^4)^{n+m+2}} \quad [t = \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2}]$$

$$2. \int \sin^{m+1/2} 2x \left\{ \frac{\sec x}{\operatorname{cosec} x} \right\}^{2n+1} dx = \\ = \pm 2^{m+3/2} \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{1}{2m-4n+4k+1} \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{m-2n+2k} \\ [n \geq m+1]$$

$$3. \int \sin^{m+1/2} 2x \left\{ \frac{\sin x}{\cos x} \right\}^{2n} \left\{ \frac{\cos x}{\sin x} \right\} dx = \pm 2^{m+3/2} \int \frac{t^{2(m+2)} dt}{(1+t^4)^{n+m+2}} \\ [t = \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2}].$$

$$4. \int \sqrt{\sin 2x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \pm \frac{1}{2} \left\{ \frac{\cos x}{\sin x} \right\} \sqrt{\sin 2x} + \\ + \frac{1}{4} [\ln (\sin x + \cos x \pm \sqrt{\sin 2x}) + \arcsin (\sin x - \cos x)].$$

$$5. \int_0^x \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} F\left(\varphi, \frac{1}{\sqrt{2}}\right),$$

$$6. \int \frac{1}{\sin^{n+1/2} 2x} \left\{ \frac{\sin x}{\cos x} \right\}^{2m+1} dx =$$

$$= \pm 2^{-n+1/2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{3/2} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{1}{4m-2n+4k+3} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2m-n+2k}$$

[$n \geq m+1$].

$$7. \int \frac{1}{\sqrt{\sin 2x}} \left\{ \frac{\sin x}{\cos x} \right\} dx =$$

$$= \frac{1}{2} [\ln (\sin x + \cos x \mp \sqrt{\sin 2x}) + \arcsin (\sin x - \cos x)].$$

$$8. \int \frac{1}{\sin^{n+1/2} 2x} \left\{ \frac{\sec x}{\operatorname{cosec} x} \right\}^{2m+1} dx =$$

$$= \pm 2^{1/2-n} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{1/2} \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{1}{4m+2n-4k+1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2m+n-2k}.$$

$$9. \int \frac{\sin x}{\sin^{n+1/2} 2x} \cos^{\pm 2m} x dx =$$

$$= 2^{-n+1/2} \operatorname{tg}^{3/2} x \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{\operatorname{tg}^{2k-n} x}{4k-2n+3} \quad [n \geq m+1].$$

$$10. \int \frac{\cos x}{\sin^{n+1/2} 2x} \sin^{\pm 2m} x dx =$$

$$= -2^{-n+1/2} \operatorname{ctg}^{3/2} x \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{\operatorname{ctg}^{2k-n} x}{4k-2n+3} \quad [n \geq m+1].$$

$$11. \int \frac{\operatorname{tg}^{\pm l} x dx}{\sin^{2m+1} x \sin^{n+1/2} 2x} =$$

$$= -2^{-n+1/2} \operatorname{ctg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{ctg}^{2m+n-1-2k} x}{4m+2n+2l-4k+1}.$$

$$12. \int \frac{\operatorname{tg}^{\pm l} x dx}{\cos^{2m+1} x \sin^{n+1/2} 2x} = 2^{-n+1/2} \operatorname{tg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k+l-n} x}{4k \pm 2l - 2n + 1}.$$

$$13. \int_0^x \frac{\sin ax dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

14. $\int_0^x \frac{\sin ax \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[\sqrt{\operatorname{tg} ax} - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right]$
 $[ax \neq \pi/2].$
15. $\int_0^x \frac{(1 + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$
16. $\int_0^x \frac{(1 + \cos ax) \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \sqrt{\operatorname{tg} ax} \right]$
 $[ax \neq \pi/2].$
17. $\int_0^x \frac{(1 - \sin ax + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$
18. $\int_0^x \frac{(1 + \sin ax + \cos ax) \, dx}{[1 - \cos ax + (1 - 2r^2) \sin ax] \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$

1.5.17. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx.$

Условие: $0 < ax \leq \pi/4.$

Обозначение: $\varphi = \arcsin(\sqrt{2} \sin ax).$

1. $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{1 - 2 \sin^2 t}) \, dt$
 $[t = ax; \text{ см. 1.5.36}].$
2. $\int_0^x \sqrt{\cos 2ax} \, dx = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$
3. $\int \cos^{n+1/2} 2x \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \mp \frac{\sqrt{\cos 2x}}{2(n+1)} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \times$
 $\times \left[\cos^n 2x + \sum_{k=0}^{n-1} (\mp 1)^{k+1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{2^{k+1} n(n-1)\dots(n-k)} \cos^{n-k-1} 2x \right] +$
 $+ (-1)^n \frac{(2n+1)!!}{(2n+2)!! \sqrt{2}} \left\{ \ln \left(\cos x + \sqrt{\frac{\cos 2x}{2}} \right) \right\}_{\arcsin(\sqrt{2} \sin x)}.$
4. $\int \sqrt{\cos 2x} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \mp \frac{1}{2} \sqrt{\cos 2x} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \frac{1}{2\sqrt{2}} \left\{ \ln \left(\cos x + \sqrt{\frac{\cos 2x}{2}} \right) \right\}_{\arcsin(\sqrt{2} \sin x)}.$
5. $\int_0^x \frac{\sqrt{\cos 2ax}}{\cos^2 ax} \, dx = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax}.$

$$6. \int_0^x \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$7. \int \frac{1}{\cos^{n+1/2} 2x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \mp \frac{1}{(2n-1) \cos^{n-1/2} 2x} \left\{ \frac{\cos x}{\sin x} \right\} \times \\ \times \left[1 + \sum_{k=0}^{n-2} (\mp 1)^{k+1} \frac{2^{k+1} (n-1) (n-2) \dots (n-k-1)}{(2n-3) (2n-5) \dots (2n-2k-3)} \cos^{k+1} 2x \right].$$

$$8. = \frac{1}{\sqrt{\cos 2x}} \left\{ \frac{\cos x}{\sin x} \right\} \sum_{k=0}^{n-1} \frac{(\mp 1)^{n+k+1} 2^k}{2k+1} \binom{n-1}{k} \frac{1}{\cos^k 2x} \left\{ \frac{\cos x}{\sin x} \right\}^{2k}.$$

$$9. \int \frac{1}{\sqrt{\cos 2x}} \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{\sqrt{2}} \left\{ \ln \left(\cos x - \sqrt{\frac{\cos 2x}{2}} \right) \right. \\ \left. \arcsin(\sqrt{2} \sin x) \right\}.$$

$$10. \int_0^x \frac{\sin^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$11. \int_0^x \frac{\cos^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$12. \int_0^x \frac{\operatorname{tg}^2 ax}{\sqrt{\cos 2ax}} dx = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax}.$$

$$13. \int_0^x \frac{\operatorname{tg}^4 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{3a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}.$$

$$14. \int_0^x \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{tg} ax}{a} \sqrt{\cos 2ax}.$$

$$15. \int_0^x \frac{dx}{(1-2r^2 \sin^2 ax) \sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

$$16. \int_0^x \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} = \\ = \frac{2\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{(6 \cos^2 ax + 1) \sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}.$$

$$17. \int_0^x \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}.$$

$$18. \int_0^x \frac{\sin^2 ax}{\sqrt{\cos^3 2ax}} dx = \frac{\sin 2ax}{2a \sqrt{\cos 2ax}} - \frac{1}{a \sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

1.5.18. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx$.

Условие: $\pi/4 \leq ax < \pi/2$.

Обозначение: $\varphi = \arcsin(\sqrt{2} \cos ax)$.

$$1. \int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{2 \sin^2 t - 1}) dt \quad [t = ax; \text{ см. 1.5.38}].$$

$$2. \int_x^{\pi/(2a)} \sqrt{-\cos 2ax} dx = \frac{1}{a \sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$3. \int_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$4. \int_x^{\pi/(2a)} \frac{\sin^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$5. \int_x^{\pi/(2a)} \frac{\cos^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right]$$

$$6. \int_x^{\pi/(2a)} \frac{\operatorname{ctg}^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax}.$$

$$7. \int_x^{\pi/(2a)} \frac{\cos^4 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a \sqrt{2}} \left[\frac{5}{2} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - 3F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}.$$

$$8. \int_x^{\pi/(2a)} \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax}.$$

$$9. \int_x^{\pi/(2a)} \frac{dx}{(1 - 2r^2 \cos^2 ax) \sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

$$10. \int_x^{\pi/(2a)} \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} = \frac{-2}{3a \sqrt{2}} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 6E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{\cos ax}{3a \sin^3 ax} (6 \sin^2 ax + 1) \sqrt{-\cos 2ax}.$$

$$11. \int_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}.$$

$$12. \int_x^{\pi/(2a)} \frac{\cos^2 ax \, dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}}.$$

1.5.19. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a \pm b \sin x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{1 - \sin x}{2}},$

$\psi = \arcsin \sqrt{\frac{a(1 - \sin x)}{a+b}}, \quad k = \sqrt{\frac{2b}{a+b}}.$

$$1. \int_x^{\pi/2} \sqrt{a+b \sin x} \, dx = 2\sqrt{a+b} E(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right].$$

$$2. = (a-b) \sqrt{\frac{2}{b}} F\left(\psi, \frac{1}{k}\right) + 2\sqrt{2b} E\left(\psi, \frac{1}{k}\right) \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$3. \int_x^{\pi/2} \frac{dx}{\sqrt{a+b \sin x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$4. = \sqrt{\frac{2}{b}} F\left(\psi, \frac{1}{k}\right) \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$5. \int_x^{\pi/2} \frac{\sin x \, dx}{\sqrt{a+b \sin x}} = \frac{2\sqrt{a+b}}{b} E(\varphi, k) - \frac{2a}{b\sqrt{a+b}} F(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$6. = \sqrt{\frac{2}{b}} \left\{ 2E\left(\psi, \frac{1}{k}\right) - F\left(\psi, \frac{1}{k}\right) \right\} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$7. \int_x^{\pi/2} \frac{\sin^2 x \, dx}{\sqrt{a+b \sin x}} = -\frac{4a\sqrt{a+b}}{3b^2} E(\varphi, k) + \frac{2(2a^2+b^2)}{3b^2\sqrt{a+b}} F(\varphi, k) + \frac{2}{3b} \cos x \sqrt{a+b \sin x} \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$8. = \sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{3b} F\left(\psi, \frac{1}{k}\right) - \frac{4a}{3b} E\left(\psi, \frac{1}{k}\right) \right\} + \frac{2}{3b} \cos x \sqrt{a+b \sin x} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$9. \int_x^{\pi/2} \frac{\operatorname{tg}^2 x \, dx}{\sqrt{a+b \sin x}} = \frac{-1}{\sqrt{a+b}} F(\varphi, k) - \frac{a}{(a-b)\sqrt{a+b}} E(\varphi, k) + \\ + \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < b < a, -\frac{\pi}{2} < x < \frac{\pi}{2}\right].$$

$$10. = -\sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a-b)} F\left(\psi, \frac{1}{k}\right) + \frac{ab}{a^2-b^2} E\left(\psi, \frac{1}{k}\right) \right\} + \\ + \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right].$$

$$11. \int_x^{\pi/2} \frac{dx}{(2-p^2+p^2 \sin x) \sqrt{a+b \sin x}} = \frac{1}{\sqrt{a+b}} \Pi(\varphi, p^2, k) \\ \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right].$$

$$12. \int_x^{\pi/2} \frac{dx}{(a+b-p^2b+p^2b \sin x) \sqrt{a+b \sin x}} = \frac{1}{a+b} \sqrt{\frac{2}{b}} \Pi\left(\psi, p^2, \frac{1}{k}\right) \\ \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right].$$

$$13. \int_x^{\pi/2} \frac{1-\sin x}{1+\sin x} \frac{dx}{\sqrt{a+b \sin x}} = \frac{2}{a-b} \left\{ -\sqrt{a+b} E(\varphi, k) + \right. \\ \left. + \operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right) \sqrt{a+b \sin x} \right\} \quad \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right].$$

$$14. \int_x^{\pi/2} \frac{dx}{\sqrt{(a+b \sin x)^3}} = \frac{-2b \cos x}{(a^2-b^2) \sqrt{a+b \sin x}} + \frac{2}{(a-b) \sqrt{a+b}} E'(\varphi, k) \\ \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right].$$

$$15. = \sqrt{\frac{2}{b}} \left\{ \frac{1}{a+b} F\left(\psi, \frac{1}{k}\right) - \frac{2b}{b^2-a^2} E\left(\psi, \frac{1}{k}\right) \right\} - \frac{2b}{b^2-a^2} \frac{\cos x}{\sqrt{a+b \sin x}} \\ \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right].$$

$$16. \int (1 \pm \sin ax)^{n+1/2} dx = \mp \frac{1}{a} \sqrt{1 \mp \sin x} \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k+1} \frac{(1 \mp \sin ax)^k}{2k+1}.$$

$$17. \int \frac{dx}{(1 \pm \sin ax)^{n+1/2}} = \\ = \frac{\cos ax}{na2^n (1 \pm \sin ax)^{n+1/2}} \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} \times \right. \\ \left. \times 2^{n-k-2} (1 \pm \sin ax)^{k+1} \right] \mp \frac{(2n-1)!!}{(2n)!!} \frac{1}{2^n \sqrt{2a}} \ln \frac{\sqrt{2} + \sqrt{1 \mp \sin ax}}{\sqrt{2} - \sqrt{1 \mp \sin ax}}.$$

1.5.20. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a \pm b \cos x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{b(1 - \cos x)}{a + b}}$,

$\psi = \arcsin \sqrt{\frac{(a + b)(1 - \cos x)}{2(a - b \cos x)}}$, $k = \sqrt{\frac{2b}{a + b}}$.

$$1. \int_0^x \sqrt{a + b \cos x} dx = 2 \sqrt{a + b} E\left(\frac{x}{2}, k\right) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$2. = \sqrt{\frac{2}{b}} \left\{ (a - b) F\left(\varphi, \frac{1}{k}\right) + 2b E\left(\varphi, \frac{1}{k}\right) \right\} \\ \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$3. \int_0^x \sqrt{a - b \cos x} dx = 2 \sqrt{a + b} E(\psi, k) - \frac{2b \sin x}{\sqrt{a - b \cos x}} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$4. \int_0^x \sqrt{\frac{a - b \cos x}{1 + p \cos x}} dx = \frac{2(a - b)}{(1 + p) \sqrt{a + b}} \Pi\left(\psi, \frac{2ap}{(a + b)(1 + p)}, k\right) \\ [a > b > 0; 0 \leq x \leq \pi; p \neq -1].$$

$$5. \int_0^x \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{\sqrt{a + b}} F\left(\frac{x}{2}, k\right) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$6. = \sqrt{\frac{2}{b}} F\left(\varphi, \frac{1}{k}\right) \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$7. \int_0^x \frac{dx}{\sqrt{a - b \cos x}} = \frac{2}{\sqrt{a + b}} F(\psi, k) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$8. \int_0^x \frac{\cos x dx}{\sqrt{a + b \cos x}} = \frac{2}{b \sqrt{a + b}} \left\{ (a + b) E\left(\frac{x}{2}, k\right) - a F\left(\frac{x}{2}, k\right) \right\} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$9. = \sqrt{\frac{2}{b}} \left\{ 2E\left(\varphi, \frac{1}{k}\right) - F\left(\varphi, \frac{1}{k}\right) \right\} \\ \left[b > |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$10. \int_0^x \frac{\cos x dx}{\sqrt{a - b \cos x}} = \frac{2}{b \sqrt{a + b}} \{ (b - a) \Pi(\psi, k^2, k) + a F(\psi, k) \} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$11. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, k\right) - 2a(a+b) E\left(\frac{x}{2}, k\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$12. = \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\varphi, \frac{1}{k}\right) - 4aE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$13. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F(\psi, k) - 2a(a+b) E(\psi, k) \right\} + \\ + \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} \quad [a > b > 0; 0 \leq x < \pi].$$

$$14. \int (1 \pm \cos ax)^{n+1/2} dx = \pm \sqrt{1 \mp \cos ax} \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k+1} \frac{(1 \mp \cos ax)^k}{2k+1}.$$

$$15. \int_0^x \frac{dx}{(2-p^2+p^2 \cos x) \sqrt{a+b \cos x}} = \frac{1}{\sqrt{a+b}} \Pi\left(\frac{x}{2}, p^2, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$16. \int_0^x \frac{dx}{(a+b-p^2b+p^2b \cos x) \sqrt{a+b \cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}} \Pi\left(\varphi, p^2, \frac{1}{k}\right) \\ \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$17. \int_0^x \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b \cos x}} = \frac{2}{a-b} \operatorname{tg} \frac{x}{2} \sqrt{a+b \cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$18. \int_0^x \frac{dx}{\sqrt{(a+b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E\left(\frac{x}{2}, k\right) - \frac{2b}{a^2-b^2} \frac{\sin x}{\sqrt{a+b \cos x}} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$19. = \frac{1}{a^2-b^2} \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\varphi, \frac{1}{k}\right) + 2bE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2b}{b^2-a^2} \frac{\sin x}{\sqrt{a+b \cos x}} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$20. \int_0^x \frac{dx}{\sqrt{(a-b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E(\psi, k) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$21. \int \frac{dx}{(1 \pm \cos ax)^{n+1/2}} = \pm \frac{\sin ax}{na2^n (1 \pm \cos ax)^{n+1/2}} \times$$

$$\times \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} 2^{n-k-2} (1 \pm \cos ax)^{n+1} \right] \pm$$

$$\pm \frac{(2n-1)!}{(2n)!} \cdot \frac{1}{a2^n \sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 \pm \cos ax}}{\sqrt{2} - \sqrt{1 \pm \cos ax}}.$$

1.5.21. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a+b \sin x+c \cos x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{\sqrt{b^2+c^2}-b \sin x-c \cos x}{2 \sqrt{b^2+c^2}}}$,

$$k = \sqrt{\frac{2 \sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}, \quad x_1 = \arcsin \frac{b}{\sqrt{b^2+c^2}}.$$

$$1. \int_x^{x_1} \sqrt{a+b \sin x+c \cos x} dx = 2 \sqrt{a+\sqrt{b^2+c^2}} E(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$2. = 2 \sqrt{2} \sqrt[4]{b^2+c^2} E(\varphi, k) - \frac{\sqrt{2} (\sqrt{b^2+c^2}-a)}{\sqrt[4]{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(\frac{-a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$3. \int_x^{x_1} \frac{dx}{\sqrt{a+b \sin x+c \cos x}} = \frac{2}{\sqrt{a+\sqrt{b^2+c^2}}} F(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$4. = \frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$5. \int_x^{x_1} \frac{\sin x dx}{\sqrt{a+b \sin x+c \cos x}} =$$

$$= \frac{\sqrt{2} b}{\sqrt[4]{(b^2+c^2)^3}} \{2E(\varphi, k) - F(\varphi, k)\} + \frac{2c}{b^2+c^2} \sqrt{a+b \sin x+c \cos x}$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$6. \int \frac{(b \cos x - c \sin x) dx}{\sqrt{a+b \sin x+c \cos x}} = 2 \sqrt{a+b \sin x+c \cos x}.$$

$$7. \int_x^{x_1} \frac{\sqrt{b^2+c^2}+b \sin x+c \cos x}{\sqrt{a+b \sin x+c \cos x}} dx = 2 \sqrt{a+\sqrt{b^2+c^2}} E(\varphi, k) -$$

$$- \frac{2(a-\sqrt{b^2+c^2})}{\sqrt{a+\sqrt{b^2+c^2}}} F(\varphi, k) \quad [0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$8. \int_x^{x_1} \frac{\sqrt{b^2 + c^2} + b \sin x + c \cos x}{\sqrt{a + b \sin x + c \cos x}} dx = 2 \sqrt{2} \sqrt[4]{b^2 + c^2} E(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2 + c^2}; x_1 = \arccos\left(-\frac{a}{\sqrt{b^2 + c^2}}\right) \leq x < x_1 \right].$$

1.5.22. Интегралы вида $\int R(\sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \Delta^n dx = \frac{n-1}{n} (2-k^2) \int \Delta^{n-2} dx -$$

$$- \frac{n-2}{n} (1-k^2) \int \Delta^{n-4} dx + \frac{k^2}{n} \Delta^{n-2} \sin x \cos x.$$

$$2. \int \frac{dx}{\Delta^{n+1}} =$$

$$= -\frac{k^2 \sin x \cos x}{(n-1)(1-k^2)\Delta^{n-1}} + \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{(n-1)(1-k^2)} \int \frac{dx}{\Delta^{n-3}}.$$

$$3. \int \Delta dx = E(x, k).$$

$$4. \int \Delta^3 dx = \frac{2}{3} (2-k^2) E(x, k) - \frac{1-k^2}{3} F(x, k) + \frac{k^2}{3} \Delta \sin x \cos x.$$

$$5. \int \frac{dx}{\Delta} = F(x, k). \quad 6. \int \frac{dx}{\Delta^3} = \frac{1}{1-k^2} E(x, k) - \frac{k^2}{1-k^2} \frac{\sin x \cos x}{\Delta}.$$

1.5.23. Интегралы вида $\int R(\sin x, \sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{\sin x dx}{\Delta^p} = \frac{-\cos x}{(p-2)(1-k^2)\Delta^{p-1}} + \frac{p-3}{(p-2)(1-k^2)} \int \frac{\sin x dx}{\Delta^{p-1}}.$$

$$2. \int \frac{\sin x dx}{\Delta^{2n+1}} = -\cos x \sum_{l=0}^{n-1} \frac{2^l (n-1)(n-2)\dots(n-l-1)}{(2n-1)(2n-3)\dots(2n-2l-1)(1-k^2)^{l+1} \Delta^{2n-2l-1}}.$$

$$3. = -\frac{1}{(1-k^2)^n} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-k^2)^l \cos^{2l+1} x}{2l+1 \Delta^{2l+1}}.$$

$$4. \int \frac{\sin^p x dx}{\Delta^{p+3}} = \frac{-\sin^{p-3} x \cos x}{(p+1)k^2(1-k^2)\Delta^{p+1}} +$$

$$+ \frac{2(p-1)k^2 - p + 2}{(p+1)k^2(1-k^2)} \int \frac{\sin^{p-2} x dx}{\Delta^{p+1}} + \frac{p-3}{(p+1)k^2(1-k^2)} \int \frac{\sin^{p-4} x dx}{\Delta^{p-1}}.$$

$$5. \int \Delta \sin x dx = -\frac{\Delta \cos x}{2} - \frac{1-k^2}{2k} \ln(k \cos x + \Delta).$$

$$6. \int \Delta \sin^2 x dx = -\frac{\Delta}{3} \sin x \cos x + \frac{1-k^2}{3k^2} F(x, k) + \frac{2k^2-1}{3k^2} E(x, k).$$

7. $\int \Delta \sin^3 x dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k \cos x + \Delta).$
8. $\int \Delta \sin^4 x dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x -$
 $-\frac{2(2k^4 - k^2 - 1)}{15k^4} F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x, k).$
9. $\int \Delta^3 \sin x dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3(1 - k^2)^2}{8k} \ln(k \cos x + \Delta).$
10. $\int \Delta^3 \sin^2 x dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x +$
 $+\frac{(1 - k^2)(3 - 4k^2)}{15k^2} F(x, k) - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k).$
11. $\int \Delta^3 \sin^3 x dx = \frac{8k^4 \sin^4 x + 2k^2(5k^2 - 7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x -$
 $-\frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln(k \cos x + \Delta).$
12. $\int \frac{\sin^n x dx}{\Delta} = \frac{\sin^{n-3} x}{(n-1)k^2} \Delta \cos x +$
 $+\frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} dx.$
13. $\int \frac{\sin x dx}{\Delta} = \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} = -\frac{1}{k} \ln(k \cos x + \Delta) = \frac{1}{k} \ln(\Delta - k \cos x).$
14. $\int \frac{\sin^2 x dx}{\Delta} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k).$
15. $\int \frac{\sin^3 x dx}{\Delta} = \frac{\Delta \cos x}{2k^2} - \frac{1+k^2}{2k^3} \ln(k \cos x + \Delta).$
16. $\int \frac{\sin^4 x dx}{\Delta} = \frac{\Delta \sin x \cos x}{3k^2} + \frac{2+k^2}{3k^4} F(x, k) - \frac{2(1+k^2)}{3k^4} E(x, k).$
17. $\int \frac{\sin x dx}{\Delta^3} = -\frac{\cos x}{(1-k^2)\Delta}.$
18. $\int \frac{\sin^2 x dx}{\Delta^3} = \frac{1}{k^2(1-k^2)} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{1-k^2} \frac{\sin x \cos x}{\Delta}.$
19. $\int \frac{\sin^3 x dx}{\Delta^3} = -\frac{\cos x}{k^2(1-k^2)\Delta} + \frac{1}{k^3} \ln(k \cos x + \Delta).$
20. $\int \frac{\sin^4 x dx}{\Delta^3} = \frac{2-k^2}{k^4(1-k^2)} E(x, k) - \frac{2}{k^4} F(x, k) - \frac{\sin x \cos x}{k^2(1-k^2)\Delta}.$
21. $\int \frac{\Delta dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln(k \cos x + \Delta).$
22. $\int \frac{\Delta dx}{\sin^2 x} = (1-k^2) F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$
23. $\int \frac{\Delta dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{1-k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$

$$24. \int \frac{\Delta dx}{\sin^4 x} = \\ = \frac{1}{3} [-\Delta \operatorname{ctg}^3 x + (k^2 - 3) \Delta \operatorname{ctg} x + 2(1 - k^2) F(x, k) + (k^2 - 2) E(x, k)].$$

$$25. \int \frac{dx}{\Delta \sin^n x} = \\ = -\frac{\Delta \cos x}{(n-1) \sin^{n-1} x} + \frac{(n-2)(1+k^2)}{n-1} \int \frac{dx}{\Delta \sin^{n-2} x} - \frac{(n-3)k^2}{n-1} \int \frac{dx}{\Delta \sin^{n-4} x}.$$

$$26. \int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} = \ln \frac{\Delta + \cos x}{\sin x}.$$

$$27. \int \frac{dx}{\Delta \sin^2 x} = \int \frac{1 + \operatorname{ctg}^2 x}{\Delta} dx = F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$$

$$28. \int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1+k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$29. \int \frac{dx}{\Delta \sin^4 x} = \\ = \frac{1}{3} [-\Delta \operatorname{ctg}^3 x - \Delta (2k^2 + 3) \operatorname{ctg} x + (k^2 + 2) F(x, k) - 2(k^2 + 1) E(x, k)].$$

1.5.24. Интегралы вида $\int R(\cos x, \sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{\cos x dx}{\Delta^p} = \frac{\sin x}{(p-2) \Delta^{p-2}} + \frac{p-3}{p-2} \int \frac{\cos x dx}{\Delta^{p-2}}.$$

$$2. \int \frac{\cos x dx}{\Delta^{2n+1}} = -\sin x \sum_{l=0}^{n-1} \frac{2^l (n-1)(n-2) \dots (n-l+1)}{(2n-1)(2n-3) \dots (2n-2l-1) \Delta^{2n-2l-1}}.$$

$$3. = \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{k^{2l}}{2l+1} \frac{\sin^{2l+1} x}{\Delta^{2l+1}}.$$

$$4. \int \Delta \cos x dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x).$$

$$5. \int \Delta \cos^2 x dx = \frac{\Delta}{3} \sin x \cos x - \frac{1-k^2}{3k^2} F(x, k) + \frac{k^2+1}{3k^2} E(x, k).$$

$$6. \int \Delta \cos^3 x dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x).$$

$$7. \int \Delta \cos^4 x dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x + \\ + \frac{2(1-k^2)(1-3k^2)}{15k^4} F(x, k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x, k).$$

$$8. \int \Delta^3 \cos x dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin(k \sin x).$$

- $$9. \int \Delta^3 \cos^2 x dx = \frac{-3k^2 \sin^2 x + k^2 + 6}{15} \Delta \sin x \cos x -$$
- $$- \frac{(1-k^2)(k^2+3)}{15k^2} F(x, k) - \frac{2k^4-7k^2-3}{15k^2} E(x, k).$$
- $$10. \int \Delta^3 \cos^3 x dx =$$
- $$= \frac{8k^4 \sin^4 x - 2k^2(6k^2+7) \sin^2 x + 30k^2 + 3}{48k^2} \Delta \sin x + \frac{6k^2-1}{16k^3} \arcsin(k \sin x).$$
- $$11. \int \frac{\cos^n x dx}{\Delta} = \frac{\cos^{n-2} x}{(n-1)k^2} \Delta \sin x + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} dx +$$
- $$+ \frac{n-3}{n-1} \frac{1-k^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} dx.$$
- $$12. \int \frac{\cos x dx}{\Delta} = \frac{1}{k} \arcsin(k \sin x) = \frac{1}{k} \operatorname{arctg} \frac{k \sin x}{\Delta}.$$
- $$13. \int \frac{\cos^2 x dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{1-k^2}{k^2} F(x, k).$$
- $$14. \int \frac{\cos^3 x dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{2k^2-1}{2k^3} \arcsin(k \sin x).$$
- $$15. \int \frac{\cos x dx}{\Delta^3} = \frac{\sin x}{\Delta}.$$
- $$16. \int \frac{\cos^2 x dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}.$$
- $$17. \int \frac{\cos^3 x dx}{\Delta^3} = -\frac{(1-k^2) \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin(k \sin x).$$
- $$18. \int \frac{\cos^4 x dx}{\Delta^3} = \frac{2-k^2}{k^4} E(x, k) - \frac{2(1-k^2)}{k^4} F(x, k) - \frac{(1-k^2) \sin x \cos x}{k^2 \Delta}.$$
- $$19. \int \frac{\cos^4 x dx}{\Delta} = \frac{\Delta \sin x \cos x}{3k^2} + \frac{4k^2-2}{3k^4} E(x, k) + \frac{3k^4-5k^2+2}{3k^4} F(x, k).$$
- $$20. \int \frac{\Delta dx}{\cos x} = \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} + k \arcsin(k \sin x).$$
- $$21. \int \frac{\Delta dx}{\cos^2 x} = F(x, k) - E(x, k) + \Delta \operatorname{tg} x.$$
- $$22. \int \frac{\Delta dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$$
- $$23. \int \frac{\Delta dx}{\cos^4 x} = \frac{1}{3} \left\{ \left[\operatorname{tg}^3 x - \frac{2k^2-3}{1-k^2} \operatorname{tg} x \right] \Delta + 2F(x, k) + \frac{k^2-2}{1-k^2} E(x, k) \right\}.$$
- $$24. \int \frac{dx}{\Delta \cos x} = -\frac{1}{2 \sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$$
- $$25. \int \frac{dx}{\Delta \cos^2 x} = F(x, k) - \frac{1}{1-k^2} E(x, k) + \frac{1}{1-k^2} \Delta \operatorname{tg} x.$$
- $$26. \int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2(1-k^2) \cos^2 x} + \frac{2k^2-1}{4(1-k^2)^{3/2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$$

$$27. \int \frac{dx}{\Delta \cos^4 x} =$$

$$= \frac{1}{3(1-k^2)} \left[\Delta \operatorname{tg}^3 x - \frac{5k^2-3}{1-k^2} \Delta \operatorname{tg} x - (3k^2-2) F(x, k) + \frac{2(2k^2-1)}{1-k^2} E(x, k) \right].$$

$$28. \int \frac{dx}{\Delta \cos^n x} =$$

$$= \frac{\Delta \sin x}{(n-1)(1-k^2) \cos^{n-1} x} - \frac{(n-2)(2k^2-1)}{(n-1)(1-k^2)} \int \frac{dx}{\Delta \cos^{n-2} x} +$$

$$+ \frac{(n-3)k^2}{(n-1)(1-k^2)} \int \frac{dx}{\Delta \cos^{n-4} x}.$$

1.5.25. Интегралы вида $\int \sin^m x \cos^n x \sqrt{1-k^2 \sin^2 x}^p dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \int \Delta^p \sin^m x \cos^n x dx = \frac{1}{(m+n+p)k^2} \left\{ \Delta^{p+2} \sin^{m-3} x \cos^{n+1} x + \right.$$

$$+ [m+n-2+(m+p-1)k^2] \int \Delta^p \sin^{m-2} x \cos^n x dx -$$

$$\left. - (m-3) \int \Delta^p \sin^{m-4} x \cos^n x dx \right\}.$$

$$2. = \frac{1}{(m+n+p)k^2} \left\{ \Delta^{p+2} \sin^{m+1} x \cos^{n-3} x + \right.$$

$$+ [(n+p-1)k^2 - (m+n-2)(1-k^2)] \int \Delta^p \sin^m x \cos^{n-2} x dx +$$

$$\left. + (n-3)(1-k^2) \int \Delta^p \sin^m x \cos^{n-4} x dx \right\} \quad [m+n+p \neq 0].$$

$$3. \int \Delta^p \sin x \cos^n x dx = -\frac{\Delta^{p+2} \cos^{n-1} x}{(n+p+1)k^2} - \frac{(n-1)(1-k^2)}{(n+p+1)k^2} \int \Delta^p \cos^{n-2} x \sin x dx.$$

$$4. \int \Delta^p \sin^m x \cos x dx = -\frac{\Delta^{p+2} \sin^{m-1} x}{(m+p+1)k^2} + \frac{m-1}{(m+p+1)k^2} \int \Delta^p \sin^{m-2} x \cos x dx.$$

$$5. \int \Delta^p \sin^3 x \cos^n x dx =$$

$$= \frac{(n+p+1)k^2 \cos^2 x + [(p+2)k^2 + n+1]}{(n+p+1)(n+p+3)k^4} \Delta^{p+2} \cos^{n-1} x -$$

$$- \frac{[(p+2)k^2 + n+1](n-1)(1-k^2)}{(n+p+1)(n+p+3)k^4} \int \Delta^p \cos^{n-2} x \sin x dx.$$

$$6. \int \Delta^p \sin^m x \cos^3 x dx =$$

$$= \frac{(m+p+1)k^2 \sin^2 x - [(p-m+1)k^2 + m+1]}{(m+p+1)(m+p+3)k^4} \Delta^{p+2} \sin^{m-1} x +$$

$$+ \frac{[(p-m+1)k^2 + m+1](m-1)}{(m+p+1)(m+p+3)k^4} \int \Delta^p \sin^{m-2} x \cos x dx.$$

$$7. \int \Delta \sin x \cos x dx = -\frac{\Delta^3}{3k^2}.$$

$$8. \int \Delta \sin x \cos^2 x dx = -\frac{2k^2 \cos^2 x + 1 - k^2}{8k^2} \Delta \cos x + \frac{(1-k^2)^2}{8k^3} \ln(k \cos x + \Delta).$$

$$9. \int \Delta \sin x \cos^3 x dx = -\frac{3k^4 \sin^4 x - k^2 (5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta.$$

$$10. \int \Delta \sin x \cos^4 x dx = \\ = \frac{-8k^4 \sin^4 x + 2k^2 (7k^2 + 1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x - \\ - \frac{(1 - k^2)^3}{16k^5} \ln (k \cos x + \Delta).$$

$$11. \int \Delta \sin^2 x \cos x dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin (k \sin x).$$

$$12. \int \Delta \sin^2 x \cos^2 x dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x - \\ - \frac{(1 - k^2)(2 - k^2)}{15k^4} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k).$$

$$13. \int \Delta \sin^2 x \cos^3 x dx = \\ = \frac{-8k^4 \sin^4 x + 2k^2 (6k^2 + 1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x + \\ + \frac{2k^2 - 1}{16k^5} \arcsin (k \sin x).$$

$$14. \int \Delta \sin^3 x \cos x dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$$

$$15. \int \Delta \sin^3 x \cos^2 x dx = \\ = \frac{8k^4 \sin^4 x - 2k^2 (k^2 + 1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x + \\ + \frac{(1 - k^2)^2 (k^2 + 1)}{16k^5} \ln (k \cos x + \Delta).$$

$$16. \int \Delta \sin^4 x \cos x dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin (k \sin x).$$

$$17. \int \Delta^3 \sin x \cos x dx = -\frac{\Delta^5}{5k^2}.$$

$$18. \int \Delta^3 \sin x \cos^2 x dx = \\ = \frac{-8k^4 \sin^4 x + 2k^2 (k^2 + 7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \Delta \cos x + \\ + \frac{(1 - k^2)^3}{16k^3} \ln (k \cos x + \Delta).$$

$$19. \int \Delta^3 \sin^2 x \cos x dx = \\ = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + \frac{1}{16k^3} \arcsin (k \sin x).$$

1.5.26. Интегралы вида $\int \frac{\sin^n x}{\cos^n x} \sqrt{1-k^2 \sin^2 x} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \frac{\sin x}{\cos x} \Delta dx = \int \Delta \operatorname{tg} x dx = -\Delta + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
2. $\int \frac{\sin x}{\cos^2 x} \Delta dx = \frac{\Delta}{\cos x} - k \ln (k \cos x + \Delta).$
3. $\int \frac{\sin x}{\cos^3 x} \Delta dx = \frac{\Delta}{2 \cos^2 x} + \frac{k^2}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
4. $\int \frac{\sin x}{\cos^4 x} \Delta dx = \frac{-(2k^2+1)k^2 \sin^2 x + 3k^4 - k^2 + 1}{3(1-k^2) \cos^3 x} \Delta.$
5. $\int \frac{\sin^2 x}{\cos x} \Delta dx =$
 $= -\frac{\Delta \sin x}{2} + \frac{2k^2-1}{2k} \arcsin (k \sin x) + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
6. $\int \frac{\sin^2 x}{\cos^2 x} \Delta dx = \int \Delta \operatorname{tg}^2 x dx = \Delta \operatorname{tg} x + F(x, k) - 2E(x, k).$
7. $\int \frac{\sin^2 x}{\cos^3 x} \Delta dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2-1}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - k \arcsin (k \sin x).$
8. $\int \frac{\sin^3 x}{\cos x} \Delta dx = -\frac{k^2 \sin x + 3k^2 - 1}{3k^2} \Delta + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
9. $\int \frac{\sin^3 x}{\cos^2 x} \Delta dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2-1}{2k} \ln (k \cos x + \Delta).$
10. $\int \frac{\sin^4 x}{\cos x} \Delta dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \Delta \sin x +$
 $+ \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin (k \sin x) + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
11. $\int \frac{\cos x}{\sin x} \Delta dx = \int \Delta \operatorname{ctg} x dx = \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$
12. $\int \frac{\cos^2 x}{\sin x} \Delta dx = \frac{\Delta \cos x}{2} + \frac{k^2+1}{2k} \ln (k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
13. $\int \frac{\cos^3 x}{\sin x} \Delta dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$
14. $\int \frac{\cos^4 x}{\sin x} \Delta dx = -\frac{2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \Delta \cos x +$
 $+ \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln (k \cos x + \Delta).$
15. $\int \frac{\cos x}{\sin^2 x} \Delta dx = -\frac{\Delta}{\sin x} - k \arcsin (k \sin x).$

$$16. \int \frac{\cos^2 x}{\sin^2 x} \Delta dx = \int \operatorname{ctg}^2 x \Delta dx = -\Delta \operatorname{ctg} x + (1-k^2) F(x, k) - 2E(x, k).$$

$$17. \int \frac{\cos^3 x}{\sin^2 x} \Delta dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2 + 1}{2k} \arcsin(k \sin x).$$

$$18. \int \frac{\cos x}{\sin^3 x} \Delta dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$$

$$19. \int \frac{\cos x}{\sin^4 x} \Delta dx = -\frac{\Delta^3}{3 \sin^3 x}.$$

$$20. \int \frac{\cos^2 x}{\sin^3 x} \Delta dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2 + 1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln(k \cos x + \Delta).$$

1.5.27. Интегралы вида $\int \frac{\sin^p x \cos^q x}{V(1-k^2 \sin^2 x)^r} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = V(1-k^2 \sin^2 x)$.

$$\begin{aligned} 1. \int \frac{\sin^p x \cos^q x}{\Delta^r} dx &= \\ &= \frac{-k^2 \sin^{p+1} x \cos^{q+1} x}{(r-2)(1-k^2) \Delta^{r-2}} + \frac{p-q-(2-k^2)(p-r+3)}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-2}} dx + \\ &\quad + \frac{p+q-r+4}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-4}} dx. \end{aligned}$$

$$2. = \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^r} dx - \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^{r-2}} dx.$$

$$3. = \frac{k^2-1}{k^2} \int \frac{\sin^p x \cos^{q-2} x}{\Delta^r} dx + \frac{1}{k^2} \int \frac{\sin^p x \cos^{q-2} x}{\Delta^{r-2}} dx.$$

$$4. \int \frac{\sin x \cos x}{\Delta^r} dx = \frac{1}{(r-2)k^2 \Delta^{r-2}}.$$

$$5. \int \frac{\sin x \cos x}{\Delta} dx = -\frac{\Delta}{k^2}.$$

$$6. \int \frac{\sin x \cos^2 x}{\Delta} dx = -\frac{\cos x \Delta}{2k^2} + \frac{1-k^2}{2k^3} \ln(k \cos x + \Delta).$$

$$7. \int \frac{\sin x \cos^3 x}{\Delta} dx = -\frac{1}{3k^4} (k^2 \cos^2 x - 2 + 2k^2) \Delta.$$

$$8. \int \frac{\sin x \cos^4 x}{\Delta} dx = \frac{3-5k^2+2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^5} \ln(k \cos x + \Delta).$$

$$9. \int \frac{\sin^2 x \cos x}{\Delta} dx = -\frac{\Delta \sin x}{2k^2} + \frac{\arcsin(k \sin x)}{2k^3}.$$

$$10. \int \frac{\sin^2 x \cos^2 x}{\Delta} dx = -\frac{\Delta \sin x \cos x}{3k^2} + \frac{2-k^2}{3k^4} E(x, k) + \frac{2k^2-2}{3k^4} F(x, k).$$

$$11. \int \frac{\sin^2 x \cos^3 x}{\Delta} dx = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \Delta \sin x + \frac{4k^2-3}{8k^5} \arcsin(k \sin x).$$

$$12. \int \frac{\sin^2 x \cos^4 x dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \Delta \sin x \cos x + \\ + \frac{9k^4 - 17k^2 + 8}{15k^6} F(x, k) - \frac{3k^4 - 13k^2 + 8}{15k^6} E(x, k).$$

$$13. \int \frac{\sin^3 x \cos x dx}{\Delta} = -\frac{1}{3k^4} (2 + k^2 \sin^2 x) \Delta.$$

$$14. \int \frac{\sin^3 x \cos^2 x dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \Delta \cos x - \frac{k^4 + 2k^2 - 3}{8k^5} \ln(k \cos x + \Delta).$$

$$15. \int \frac{\sin^3 x \cos^3 x dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4 - 4k^2) \sin^2 x - 10k^2 + 8}{15k^6} \Delta.$$

$$16. \int \frac{\sin^3 x \cos^4 x dx}{\Delta} = \\ = \frac{8k^4 \sin^4 x - 2k^2 (7k^2 - 5) \sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \Delta \cos x - \\ - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln(k \cos x + \Delta).$$

$$17. \int \frac{\sin^4 x \cos x dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \Delta \sin x + \frac{3}{8k^5} \arcsin(k \sin x).$$

$$18. \int \frac{\sin^4 x \cos^2 x dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \Delta \sin x \cos x + \\ + \frac{k^4 + 7k^2 - 8}{15k^6} F(x, k) - \frac{2k^4 + 3k^2 - 8}{15k^6} E(x, k).$$

$$19. \int \frac{\sin^4 x \cos^3 x dx}{\Delta} = \\ = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x - 18k^2 + 15}{48k^6} \Delta \sin x + \frac{6k^2 - 5}{16k^7} \arcsin(k \sin x).$$

$$20. \int \frac{\sin^m x \cos^n x dx}{\Delta^3} = \frac{\sin^{m-1} x \cos^{n-1} x}{k^2 \Delta} - \\ - \frac{m-1}{k^2} \int \frac{\sin^{m-2} x \cos^n x dx}{\Delta} + \frac{n-1}{k^2} \int \frac{\sin^m x \cos^{n-2} x dx}{\Delta} dx.$$

$$21. \int \frac{\sin x \cos x dx}{\Delta^3} = \frac{1}{k^2 \Delta}.$$

$$22. \int \frac{\sin x \cos^2 x dx}{\Delta^3} = \frac{\cos x}{k^2 \Delta} - \frac{1}{k^3} \ln(k \cos x + \Delta).$$

$$23. \int \frac{\sin x \cos^3 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}.$$

$$24. \int \frac{\sin x \cos^4 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3(1 - k^2)}{2k^5} \ln(k \cos x + \Delta).$$

$$25. \int \frac{\sin^2 x \cos x dx}{\Delta^3} = \frac{\sin x}{k^2 \Delta} - \frac{1}{k^3} \arcsin(k \sin x).$$

$$26. \int \frac{\sin^2 x \cos^2 x dx}{\Delta^3} = \frac{2 - k^2}{k^4} F(x, k) - \frac{2}{k^4} E(x, k) + \frac{\sin x \cos x}{k^2 \Delta}.$$

$$27. \int \frac{\sin^2 x \cos^3 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k \sin x).$$

$$28. \int \frac{\sin^3 x \cos x dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \Delta}.$$

$$29. \int \frac{\sin^3 x \cos^2 x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta).$$

$$30. \int \frac{\sin^4 x \cos x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k \sin x).$$

1.5.28. Интегралы вида $\int \frac{\sin^p x}{\cos^q x} \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)^r}},$

$$\int \frac{\cos^p x}{\sin^q x} \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)^r}}.$$

Условие: $0 < k^2 < 1.$

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}.$

$$1. \int \frac{\sin^p x dx}{\cos^q x \Delta} = \frac{\sin^{p-1} x}{(q-1)(1-k^2) \cos^{q-1} x \Delta^{r-2}} - \frac{p-q+2-(p-2q-r+5)k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x dx}{\cos^{q-2} x \Delta^r} - \frac{(p-q-r+4)k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x dx}{\cos^{q-4} x \Delta^r}.$$

$$2. \int \frac{\cos^q x dx}{\sin^p x \Delta^r} = \frac{-\cos^{q+1} x}{(p-1) \sin^{p-1} x \Delta^{r-2}} + \frac{p-q-2+(p+r-3)k^2}{p-1} \times \\ \times \int \frac{\cos^q x dx}{\sin^{p-2} x \Delta^r} + \frac{(p-q+r-4)k^2}{p-1} \int \frac{\cos^q x dx}{\sin^{p-4} x \Delta^r}.$$

$$3. \int \frac{\sin x dx}{\cos^{2m} x \Delta^{2n+1}} = \\ = (-1)^{m+n} \frac{k^{2(m+n-1)}}{(1-k^2)^{m+n}} \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{(-1)^l}{(2n-2l-1)2^l} \left(\frac{\cos x}{\Delta}\right)^{2n-2l-1}.$$

$$4. \int \frac{\cos x dx}{\sin^{2m} x \Delta^{2n+1}} = \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{k^{2(m+n-l-1)}}{2n-2l-1} \left(\frac{\sin x}{\Delta}\right)^{2n-2l-1}.$$

$$5. \int \frac{\sin x dx}{\cos x \Delta} = \int \operatorname{tg} x \frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$6. \int \frac{\sin x dx}{\cos^2 x \Delta} = \frac{\Delta}{(1-k^2) \cos x}.$$

$$7. \int \frac{\sin x dx}{\cos^3 x \Delta} = \int \operatorname{tg} x (1 + \operatorname{tg}^2 x) \frac{dx}{\Delta} = \\ = \frac{\Delta}{2(1-k^2) \cos^2 x} - \frac{k^2}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$8. \int \frac{\sin x dx}{\cos^4 x \Delta} = -\frac{2k^2 \cos^2 x - 1 + k^2}{3(1-k^2)^2 \cos^3 x} \Delta.$$

$$9. \int \frac{\sin^2 x dx}{\cos x \Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - \frac{1}{k} \arcsin(k \sin x).$$

$$10. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{tg}^2 x}{\Delta} dx = \frac{\Delta}{(1-k^2)} \operatorname{tg} x - \frac{1}{(1-k^2)} E(x, k).$$

$$11. \int \frac{\sin^2 x}{\cos^3 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2(1-k^2) \cos^2 x} - \frac{1}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$$

$$12. \int \frac{\sin^3 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$13. \int \frac{\sin^3 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{(1-k^2) \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$14. \int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \\ = \frac{\Delta \sin x}{2k^2} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - \frac{2k^2 + 1}{2k^3} \arcsin(k \sin x).$$

$$15. \int \frac{\cos x}{\sin x} \frac{dx}{\Delta} = \int \operatorname{ctg} x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$$

$$16. \int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$17. \int \frac{\cos^3 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta}.$$

$$18. \int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2 - 1}{2k^3} \ln(k \cos x + \Delta).$$

$$19. \int \frac{\cos x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x}.$$

$$20. \int \frac{\cos^2 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{ctg}^2 x}{\Delta} dx = -\Delta \operatorname{ctg} x - E(x, k).$$

$$21. \int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k} \arcsin(k \sin x).$$

$$22. \int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$$

$$23. \int \frac{\cos^2 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{1-k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$24. \int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta.$$

$$25. \int \operatorname{tg}^n x \frac{dx}{\Delta} = \frac{\operatorname{tg}^{n-3} x}{(n-1)(1-k^2) \cos^2 x} - \\ - \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-4} x}{\Delta} dx.$$

$$26. \int \operatorname{ctg}^n x \frac{dx}{\Delta} = -\frac{\operatorname{ctg}^{n-1} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1} (2-k^2) \times \\ \times \int \frac{\operatorname{ctg}^{n-2} x}{\Delta} dx - \frac{n-3}{n-1} (1-k^2) \int \frac{\operatorname{ctg}^{n-4} x}{\Delta} dx.$$

1.5.29. Интегралы вида $\int \frac{\sqrt{(1-k^2 \sin^2 x)^r}}{\sin^m x \cos^n x} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \frac{\Delta dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
2. $\int \frac{\Delta dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
3. $\int \frac{\Delta dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta} + \frac{2-k^2}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
4. $\int \frac{\Delta dx}{\sin x \cos^4 x} = \frac{(2k^2-3) \sin^2 x - 3k^2 + 4}{3(1-k^2) \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
5. $\int \frac{\Delta dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1+k^2}{2 \sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$
6. $\int \frac{\Delta dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{1-k^2} \operatorname{tg} x - \operatorname{ctg} x \right) \Delta + 2F(x, k) - \frac{2-k^2}{1-k^2} E(x, k).$
7. $\int \frac{\Delta dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2-3}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
8. $\int \frac{\Delta dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}} + \frac{k^2-2}{4} \ln \frac{1+\Delta}{1-\Delta}.$
9. $\int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2-3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$
10. $\int \frac{\Delta dx}{\sin^4 x \cos x} = -\frac{(3-k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$

1.5.30. Интегралы вида $\int \frac{dx}{\sin^m x \cos^n x \sqrt{(1-k^2 \sin^2 x)^r}}.$

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \frac{dx}{\Delta \sin x \cos x} =$
 $= \int (\operatorname{tg} x + \operatorname{ctg} x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{1}{2 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
2. $\int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{(1-k^2) \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$
3. $\int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\operatorname{ctg} x + 2 \operatorname{tg} x + \operatorname{tg}^3 x) \frac{dx}{\Delta} =$
 $= \frac{\Delta}{2(1-k^2) \cos^2 x} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta} + \frac{2-3k^2}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$

4. $\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2 - 3) \sin^2 x - 6k^2 + 4}{3(1 - k^2)^3 \cos^3 x} \Delta - \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
5. $\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta - \sqrt{1 - k^2} \sin x}{\Delta + \sqrt{1 - k^2} \sin x}.$
6. $\int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \left(\frac{\operatorname{tg} x}{1 - k^2} - \operatorname{ctg} x \right) \Delta + \frac{k^2 - 2}{1 - k^2} E(x, k) + 2F(x, k).$
7. $\int \frac{dx}{\Delta \sin^2 x \cos^3 x} =$
 $= \frac{(3 - 2k^2) \sin^2 x - 2(1 - k^2)}{2(1 - k^2) \sin x \cos^2 x} \Delta - \frac{4k^2 - 3}{4(1 - k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1 - k^2} \sin x}{\Delta - \sqrt{1 - k^2} \sin x}.$
8. $\int \frac{dx}{\Delta \sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$
9. $\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3 - k^2) \sin^2 x - 1 + k^2}{2(1 - k^2) \sin^2 x \cos x} \Delta + \frac{k^2 + 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$
10. $\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3 + 2k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta - \sqrt{1 - k^2} \sin x}{\Delta + \sqrt{1 - k^2} \sin x}.$

1.5.31. Интегралы вида

$$\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}, \sqrt{1 - q^2 \sin^2 x}) dx.$$

Условия: $0 < p^2 < q^2 < 1$, $0 < x \leq \pi/2$.

Обозначение: $\varphi = \arcsin \frac{\sqrt{1 - p^2} \sin x}{\sqrt{1 - p^2 \sin^2 x}}.$

1. $\int_0^x \frac{dx}{V(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)} = \frac{1}{V1 - p^2} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$
2. $\int_0^x \frac{\operatorname{tg}^2 x dx}{V(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)} =$
 $= \frac{\operatorname{tg} x \sqrt{1 - q^2 \sin^2 x}}{(1 - q^2) \sqrt{1 - p^2 \sin^2 x}} - \frac{1}{(1 - q^2) \sqrt{1 - p^2}} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$
3. $\int_0^x \frac{\sin^2 x dx}{V(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)^3} = \frac{\sqrt{1 - p^2}}{(1 - q^2)(q^2 - p^2)} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) -$
 $- \frac{1}{(q^2 - p^2) \sqrt{1 - p^2}} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) - \frac{\sin x \cos x}{(1 - q^2) V(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}.$
4. $\int_0^x \frac{\cos^2 x dx}{V(1 - p^2 \sin^2 x)^3(1 - q^2 \sin^2 x)} =$
 $= \frac{\sqrt{1 - p^2}}{q^2 - p^2} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) - \frac{1 - q^2}{(q^2 - p^2) \sqrt{1 - p^2}} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$

$$5. \int_0^x \frac{1}{1-p^2 \sin^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-p^2 \sin^2 x}} dx = \frac{1}{\sqrt{1-p^2}} E\left(\varphi, \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

$$6. \int_0^x \sqrt{\frac{1-p^2 \sin^2 x}{(1-q^2 \sin^2 x)^3}} dx = \\ = \frac{\sqrt{1-p^2}}{1-q^2} E\left(\varphi, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2} \frac{\sin x \cos x}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}}.$$

$$7. \int_0^x \frac{1}{1+(p^2 r^2 - p^2 - r^2) \sin^2 x} \sqrt{\frac{1-p^2 \sin^2 x}{1-q^2 \sin^2 x}} dx = \\ = \frac{1}{\sqrt{1-p^2}} \Pi\left(\varphi, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

1.5.32. Интегралы вида $\int \frac{(a + \sin x)^p}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \int \frac{(a + \sin x)^p}{\Delta} dx = \\ = \frac{1}{(p-1)k^2} \left[(a + \sin x)^{p-3} \cos x \Delta + 2(2p-3)ak^2 \int \frac{(a + \sin x)^{p-1}}{\Delta} dx + \right. \\ \left. + (p-2)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p-2}}{\Delta} dx - a(2p-5)(1+k^2-2a^2k^2) \times \right. \\ \left. \times \int \frac{(a + \sin x)^{p-3}}{\Delta} dx - (p-3)(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^{p-4}}{\Delta} dx \right] \\ \left[a^2 \neq 1, \frac{1}{k^2} \text{ при } p = -1, -2, \dots \right].$$

$$2. \int \frac{a + \sin x}{\Delta} dx = aF(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$3. \int \frac{(a + \sin x)^2}{\Delta} dx = \frac{1+k^2a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$4. \int \frac{dx}{(a + \sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^{n-1}} - \right. \\ \left. - (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a + \sin x)^{n-1} \Delta} - \right. \\ \left. - (n-2)(6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x)^{n-2} \Delta} - \right. \\ \left. - (10-4n)ak^2 \int \frac{dx}{(a + \sin x)^{n-3} \Delta} - (n-3)k^2 \int \frac{dx}{(a + \sin x)^{n-4} \Delta} \right].$$

$$5. \int \frac{dx}{(a + \sin x) \Delta} = \\ = \frac{1}{a} \Pi\left(x, \frac{1}{a^2}, k\right) + \frac{1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2} \Delta - \sqrt{1-k^2a^2} \cos x}{\sqrt{1-a^2} \Delta + \sqrt{1-k^2a^2} \cos x}.$$

- $$\begin{aligned}
 6. \int \frac{dx}{(a + \sin x)^2 \Delta} &= \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\Delta \cos x}{a + \sin x} - \right. \\
 &\quad \left. - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x) \Delta} - 2ak^2 \int \frac{(a + \sin x) dx}{\Delta} + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta} \right]. \\
 7. \int \frac{dx}{(a + \sin x)^3 \Delta} &= \\
 &= \frac{1}{2(1-a^2)(1-a^2k^2)} \left[-\frac{\Delta \cos x}{(a + \sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)^2 \Delta} - \right. \\
 &\quad \left. - (6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x) \Delta} + 2ak^2 F(x, k) \right]. \\
 8. \int \frac{dx}{(1 \pm \sin x)^n \Delta} &= \frac{1}{(2n-1)(1-k^2)} \left[\pm \frac{\Delta \cos x}{(1 \pm \sin x)^n} + \right. \\
 &\quad \left. + (n-1)(1-5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} + 2(2n-3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} - \right. \\
 &\quad \left. - (n-2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right]. \\
 9. \int \frac{dx}{(1 \pm \sin x) \Delta} &= \frac{\pm \Delta \cos x}{(1-k^2)(1 \pm \sin x)} + F(x, k) - \frac{1}{1-k^2} E(x, k). \\
 10. \int \frac{dx}{(1 \pm \sin x)^2 \Delta} &= \frac{1}{3(1-k^2)^2} \left[\pm \frac{(1-k^2) \Delta \cos x}{(1 \pm \sin x)^2} \pm \frac{(1-5k^2) \Delta \cos x}{1 \pm \sin x} + \right. \\
 &\quad \left. + (1-3k^2)(1-k^2) F(x, k) - (1-5k^2) E(x, k) \right]. \\
 11. \int \frac{dx}{(1 \pm k \sin x)^n \Delta} &= \\
 &= \frac{1}{(2n-1)(1-k^2)} \left[\pm \frac{k \Delta \cos x}{(1 \pm k \sin x)^n} + (n-1)(5-k^2) \int \frac{dx}{(1 \pm k \sin x)^{n-1} \Delta} - \right. \\
 &\quad \left. - 2(2n-3) \int \frac{dx}{(1 \pm k \sin x)^{n-2} \Delta} + (n-2) \int \frac{dx}{(1 \pm k \sin x)^{n-3} \Delta} \right]. \\
 12. \int \frac{dx}{(1 \pm k \sin x) \Delta} &= \pm \frac{k \Delta \cos x}{(1-k^2)(1 \pm k \sin x)} + \frac{1}{1-k^2} E(x, k). \\
 13. \int \frac{dx}{(1 \pm k \sin x)^2 \Delta} &= \frac{1}{3(1-k^2)^2} \left[\pm \frac{k(1-k^2) \Delta \cos x}{(1 \pm k \sin x)^2} \pm \right. \\
 &\quad \left. \pm \frac{k(5-k^2) \Delta \cos x}{1 \pm k \sin x} - 2(1-k^2) F(x, k) + (5-k^2) E(x, k) \right].
 \end{aligned}$$

1.5.33. Интегралы вида $\int \frac{(1+a \sin^2 x)^p}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

- $$\begin{aligned}
 1. \int \frac{(1+a \sin^2 x)^p}{\Delta} dx &= \frac{1}{(2p-1)k^2} \left[a^2(1+a \sin^2 x)^{p-2} \Delta \sin x \cos x + \right. \\
 &\quad + (2p-2)(3k^2+k^2a+a) \int \frac{(1+a \sin^2 x)^{p-1}}{\Delta} dx - \\
 &\quad - (2p-3)(3k^2+2ak^2+2a+a^2) \int \frac{(1+a \sin^2 x)^{p-2}}{\Delta} dx + \\
 &\quad \left. + 2(p-2)a(a+1)(k^2+a) \int \frac{(1+a \sin^2 x)^{p-3}}{\Delta} dx \right].
 \end{aligned}$$

$$2. \int \frac{1+a \sin^2 x}{\Delta} dx = \left(1 + \frac{a}{k^2}\right) F(x, k) - \frac{a}{k^2} E(x, k).$$

$$3. \int \frac{(1+a \sin^2 x)^2}{\Delta} dx = \frac{a^2}{3k^2} \Delta \sin x \cos x + \\ + \left[1 + \frac{2a}{k^2} + \frac{(2+k^2)a^2}{3k^4}\right] F(x, k) - \left[\frac{2a}{k^2} + \frac{2(k^2+1)a^2}{3k^4}\right] E(x, k).$$

$$4. \int \frac{dx}{(a + \sin^2 x)^n \Delta} = \frac{1}{2(a-1)(1+a)(a+k^2)} \left[\frac{\Delta \sin x \cos x}{(1+a \sin^2 x)^{n-1}} + \right. \\ + (2n-3)(a^2+2a+2ak^2+3k^2) \int \frac{dx}{(1+a \sin^2 x)^{n-1} \Delta} - \\ - (2n-4)(k^2a+a+3k^2) \int \frac{dx}{(1+a \sin^2 x)^{n-2} \Delta} + \\ \left. + (2n-5)k^2 \int \frac{dx}{(1+a \sin^2 x)^{n-3} \Delta} \right].$$

$$5. \int \frac{dx}{(1+a \sin^2 x) \Delta} = \Pi(x, -a, k).$$

$$6. \int \frac{dx}{(1+a \sin^2 x)^2 \Delta} = \\ = \frac{1}{2(1+a)(k^2+a^2)} \left[\frac{a^2 \Delta \sin x \cos x}{1+a \sin^2 x} + (a+k^2) F(x, k) + aE(x, k) + \right. \\ \left. + (a^2+2a+2ak^2+3k^2) \Pi(x, -a, k) \right].$$

$$7. \int_0^x \frac{\sin x dx}{(1+a \sin^2 x) \Delta} = \frac{1}{\sqrt{(1+a)(k^2+a)}} \ln \frac{\sqrt{1+a} \Delta - \sqrt{k^2+a} \cos x}{(\sqrt{1+a} - \sqrt{k^2+a}) \sqrt{1+a \sin^2 x}} \\ [a > -k^2].$$

$$8. = \frac{1}{2\sqrt{-(1+a)(k^2+a)}} \times \\ \times \left[\arcsin \frac{(1+a) \Delta^2 + (k^2+a) \cos^2 x}{(1-k^2)(1+a \sin^2 x)} - \arcsin \frac{1+2a+k^2}{1-k^2} \right] \quad [-1 < a < -k^2].$$

$$9. = -\frac{1}{\sqrt{(1+a)(k^2+a)}} \ln \frac{\sqrt{-(k^2+a)} \cos x - \sqrt{-(1+a)} \Delta}{[\sqrt{-(k^2+a)} - \sqrt{-(1+a)}] \sqrt{1+a \sin^2 x}} \\ [a < -1; 0 \leq \sin^2 x < -1/a].$$

$$10. \int_0^x \frac{\sin x dx}{(1-\sin^2 x) \Delta} = \frac{\Delta}{(1-k^2) \cos x} - \frac{1}{1-k^2}.$$

$$11. \int_0^x \frac{\sin x dx}{(1-k^2 \sin^2 x) \Delta} = -\frac{\cos x}{(1-k^2) \Delta} + \frac{1}{1-k^2}.$$

$$12. \int_0^x \frac{\cos x dx}{(1+a \sin^2 x) \Delta} = \frac{1}{\sqrt{k^2+a}} \operatorname{arctg} \frac{\sqrt{k^2+a} \sin x}{\Delta} \quad [a > -k^2].$$

$$13. \quad = \frac{1}{2\sqrt{-(k^2+a)}} \ln \frac{\Delta + \sqrt{-(k^2+a)} \sin x}{\Delta - \sqrt{-(k^2+a)} \sin x} \quad [a < -k^2; 0 \leq \sin^2 x < -1/a].$$

$$14. \quad \int_0^x \frac{\cos x \, dx}{(1-k^2 \sin^2 x) \Delta} = \frac{\sin x}{\Delta}.$$

1.5.34. Интегралы вида $\int \frac{(a + \cos x)^p}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \quad \int \frac{(a + \cos x)^p}{\Delta} dx = \frac{1}{(p-1)k^2} \left[(a + \cos x)^{p-3} \Delta \sin x + \right. \\ \left. + 2(2p-3)ak^2 \int \frac{(a + \cos x)^{p-1}}{\Delta} dx - (p-2)(1-2k^2+6a^2k^2) \times \right. \\ \left. \times \int \frac{(a + \cos x)^{p-2}}{\Delta} dx + (2p-5)a(1-2k^2+a^2k^2) \int \frac{(a + \cos x)^{p-3}}{\Delta} dx + \right. \\ \left. + (p-3)(1-a^2)(1-k^2a^2) \int \frac{(a + \cos x)^{p-4}}{\Delta} dx \right] \quad [p \neq 1; a^2 \neq 1, -(1-k^2)/k^2].$$

$$2. \quad \int \frac{a + \cos x}{\Delta} dx = aF(x, k) + \frac{1}{k} \arcsin(k \sin x).$$

$$3. \quad \int \frac{(a + \cos x)^2}{\Delta} dx = \frac{a^2k^2 + k^2 - 1}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2a}{k} \arcsin(k \sin x).$$

$$4. \quad \int \frac{dx}{(a + \cos x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-k^2+a^2k^2)} \left[\frac{\Delta \sin x}{(a + \cos x)^{n-1}} - \right. \\ \left. - (2n-3)(1-2k^2+2a^2k^2)a \int \frac{dx}{(a + \cos x)^{n-1} \Delta} - \right. \\ \left. - (n-2)(2k^2-1-6a^2k^2) \int \frac{dx}{(a + \cos x)^{n-2} \Delta} - \right. \\ \left. - (4n-10)ak^2 \int \frac{dx}{(a + \cos x)^{n-3} \Delta} + (n-3)k^2 \int \frac{dx}{(a + \cos x)^{n-4} \Delta} \right].$$

$$5. \quad = \frac{1}{(2n-1)a(1-2k^2+2a^2k^2)} \left[\frac{\sin x \Delta}{(a + \cos x)^n} + \right. \\ \left. + (n-1)(1-2k^2+6a^2k^2) \int \frac{dx}{(a + \cos x)^{n-1} \Delta} - \right. \\ \left. - 2(2n-3)k^2a \int \frac{dx}{(a + \cos x)^{n-2} \Delta} + (n-2)k^2 \int \frac{dx}{(a + \cos x)^{n-3} \Delta} \right] \\ [a = \pm 1, \pm \sqrt{1-k^2}/(ki)].$$

$$6. \quad \int \frac{dx}{(1 \pm \cos x) \Delta} = \frac{\pm 1 - \cos x}{\sin x} \Delta + F(x, k) - E(x, k).$$

$$7. \quad \int \frac{dx}{(1 \pm \cos x \, ki / \sqrt{1-k^2}) \Delta} = \\ = - \frac{k^2 \sin x \cos x (1 \mp \cos ki/x \sqrt{1-k^2})}{\Delta} \mp \frac{ki}{\sqrt{1-k^2}} \Delta \sin x + E(x, k).$$

$$\begin{aligned}
 8. \int \frac{dx}{(a + \cos x) \Delta} &= \frac{a}{a^2 - 1} \Pi \left(x, -\frac{1}{a^2 - 1}, k \right) + \\
 &+ \frac{1}{2 \sqrt{(1 - a^2)(1 + k^2 a^2 - k^2)}} \ln \frac{\sqrt{1 - a^2} \Delta + k \sqrt{1 + k^2 a^2 - k^2} \sin x}{\sqrt{1 - a^2} \Delta - k \sqrt{1 + k^2 a^2 - k^2} \sin x}. \\
 9. \int \frac{dx}{(a + \cos x)^2 \Delta} &= \frac{1}{(1 - a^2)(1 - k^2 + a^2 k^2)} \left[\frac{\sin x \Delta}{a + \cos x} - \right. \\
 &- (1 - 2k^2 + 2a^2 k^2) a \int \frac{dx}{(a + \cos x) \Delta} + 2ak^2 \int \frac{a + \cos x}{\Delta} dx - \\
 &\left. - k^2 \int \frac{(a + \cos x)^2}{\Delta} dx \right]. \\
 10. \int \frac{dx}{(a + \cos x)^3 \Delta} &= \frac{1}{2(1 - a^2)(1 - k^2 + a^2 k^2)} \left[\frac{\Delta \sin x}{(a + \cos x)^2} - \right. \\
 &- 3a(1 - 2k^2 + 2k^2 a^2) \int \frac{dx}{(a + \cos x)^2 \Delta} - \\
 &\left. - (2k^2 - 1 - 6a^2 k^2) \int \frac{dx}{(a + \cos x) \Delta} - 2ak^2 F(x, k) \right].
 \end{aligned}$$

1.5.35. Интегралы вида $\int \frac{(a + \operatorname{tg} x)^p}{\sqrt{1 - k^2 \sin^2 x}} dx$.

Условие. $0 < k^2 < 1$.

Обозначение. $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$\begin{aligned}
 1. \int \frac{(a + \operatorname{tg} x)^p}{\Delta} dx &= \\
 &= \frac{1}{(p - 1)(1 - k^2)} \left[\frac{(a + \operatorname{tg} x)^{p-3} \Delta}{\cos^2 x} + 2(2p - 3)a(1 - k^2) \int \frac{(a + \operatorname{tg} x)^{p-1}}{\Delta} dx - \right. \\
 &- (p - 2)(2 + 6a^2 - k^2 - 6k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-2}}{\Delta} dx + \\
 &+ (2p - 5)a(2 + 2a^2 - k^2 - 6k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-3}}{\Delta} dx - \\
 &\left. - (p - 3)(1 + a^2)(1 + a^2 - k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-4}}{\Delta} dx \right].
 \end{aligned}$$

$$2. \int \frac{a + \operatorname{tg} x}{\Delta} dx = aF(x, k) + \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}}.$$

$$\begin{aligned}
 3. \int \frac{(a + \operatorname{tg} x)^2}{\Delta} dx &= \\
 &= \frac{1}{1 - k^2} \Delta \operatorname{tg} x + a^2 F(x, k) - \frac{1}{1 - k^2} E(x, k) + \frac{a}{\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}}.
 \end{aligned}$$

$$\begin{aligned}
 4. \int \frac{dx}{(a + \operatorname{tg} x)^n \Delta} &= \frac{1}{(n - 1)(1 + a^2)(1 + a^2 - k^2 a^2)} \times \\
 &\times \left[-\frac{\Delta}{(a + \operatorname{tg} x)^{n-1} \cos^2 x} + (2n - 3)a(2 + 2a^2 - k^2 - 2k^2 a^2) \times \right. \\
 &\times \int \frac{dx}{(a + \operatorname{tg} x)^{n-1} \Delta} - (n - 2)(2 + 6a^2 - k^2 - 6k^2 a^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-2} \Delta} + \\
 &\left. + (4n - 10)a(1 - k^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-3} \Delta} - (n - 3)(1 - k^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-4} \Delta} \right] \\
 &\quad [n \neq 1, a^2 \neq -1, -1(1 - k^2)].
 \end{aligned}$$

- $$5. \int \frac{dx}{(a + \operatorname{tg} x) \Delta} = \frac{a}{1+a^2} F(x, k) + \frac{1}{a(1+a^2)} \Pi\left(x, \frac{1+a^2}{a^2}, k\right) -$$
- $$- \frac{1}{2\sqrt{(1+a^2)(1+a^2-k^2a^2)}} \ln \frac{\sqrt{1+a^2-k^2a^2} + \sqrt{1+a^2} \Delta}{\sqrt{1+a^2-k^2a^2} - \sqrt{1+a^2} \Delta}.$$
- $$6. \int \frac{dx}{(a + \operatorname{ctg} x) \Delta} = \frac{a}{1+a^2} \Pi\left(x, \frac{a^2}{1+a^2}, k\right) -$$
- $$- \frac{1}{\sqrt{(1+a^2)(a^2+1-k^2)}} \ln \frac{\sqrt{1+a^2} \Delta \cos x + a \sqrt{a^2+1-k^2}}{(\sqrt{1+a^2} \Delta + \sqrt{a^2+1-k^2}) \sqrt{1+a^2-k^2} \sin^2 x}$$
- [a > 1].
- $$7. \int \frac{dx}{(a + \operatorname{tg} x)^2 \Delta} = \frac{1}{(1+a^2)(1+a^2-k^2a^2)} \left[\frac{-\Delta}{(a + \operatorname{tg} x) \cos^2 x} + \right.$$
- $$+ a(2+2a^2-k^2-2k^2a^2) \int \frac{dx}{(a + \operatorname{tg} x) \Delta} -$$
- $$\left. - 2a(1-k^2) \int \frac{a + \operatorname{tg} x}{\Delta} dx + (1-k^2) \int \frac{(a + \operatorname{tg} x)^2}{\Delta} dx \right].$$
- $$8. \int \frac{dx}{(a + \operatorname{tg} x)^3 \Delta} = \frac{1}{2(1+a^2)(1+a^2-k^2a^2)} \left[\frac{-\Delta}{(a + \operatorname{tg} x)^2 \cos^2 x} + \right.$$
- $$+ 3a(2+2a^2-k^2-2k^2a^2) \int \frac{dx}{(a + \operatorname{tg} x)^2 \Delta} -$$
- $$\left. - (2+6a^2-k^2-6k^2a^2) \int \frac{dx}{(a + \operatorname{tg} x) \Delta} + 2a(1-k^2) F(x, k) \right].$$
- $$9. \int \frac{(a + \operatorname{tg}^2 x)^p}{\Delta} dx = \frac{1}{(2p-1)(1-k^2)} \left[\frac{(a + \operatorname{tg}^2 x)^{p-2} \operatorname{tg} x \Delta}{\cos^2 x} + \right.$$
- $$+ (2p-2)(2-3a-k^2+3k^2a) \int \frac{(a + \operatorname{tg} x)^{p-1}}{\Delta} dx +$$
- $$+ (2p-3)[1-4a+3a^2+(2a-3a^2)k^2] \int \frac{(a + \operatorname{tg}^2 x)^{p-2}}{\Delta} dx +$$
- $$\left. + 2(p-2)a(1-a)(1-a+k^2a) \int \frac{(a + \operatorname{tg}^2 x)^{p-3}}{\Delta} dx \right].$$
- $$10. \int \frac{a + \operatorname{tg}^2 x}{\Delta} dx = \frac{\Delta}{1-k^2} \operatorname{tg} x + aF(x, k) - \frac{1}{1-k^2} E(x, k).$$
- $$11. \int \frac{dx}{(a + \operatorname{tg}^2 x) \Delta} = \frac{1}{a-1} F(x, k) + \frac{1}{a(1-a)} \Pi\left(x, \frac{a-1}{a}, k\right).$$

1.5.36. Интегралы вида $\int R(\sin x, \cos x, \sqrt{1-p^2 \sin^2 x}) dx$

Условие: $p^2 > 1$.

Обозначение: $\varphi = \arcsin(p \sin x)$.

$$1. \int \sqrt{1-p^2 \sin^2 x} dx = pE\left(\varphi, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\varphi, \frac{1}{p}\right).$$

$$2. \int \frac{dx}{\sqrt{1-p^2 \sin^2 x}} = \frac{1}{p} F\left(\varphi, \frac{1}{p}\right).$$

$$3. \int \frac{dx}{(1-r^2 \sin^2 x) \sqrt{1-p^2 \sin^2 x}} = \frac{1}{p} \Pi \left(\varphi, \frac{r^2}{p^2}, \frac{1}{p} \right)$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{1-p^2 \sin^2 x}) dx$$

при $p^2 > 1$ можно пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

$$1) F(x, k) \rightarrow \frac{1}{p} F\left(\varphi, \frac{1}{p}\right);$$

$$2) E(x, k) \rightarrow pE\left(\varphi, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\varphi, \frac{1}{p}\right)$$

$$3) k \rightarrow p.$$

1.5.37. Интегралы вида $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$.

Обозначение: $\varphi = \arcsin \frac{\sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x}}$.

$$1. \int \sqrt{1+p^2 \sin^2 x} dx = \sqrt{1+p^2} E\left(\varphi, \frac{1}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}.$$

$$2. \int \frac{\sqrt{1+p^2 \sin^2 x} dx}{1+(p^2-r^2 p^2-r^2) \sin^2 x} = \frac{1}{\sqrt{1+p^2}} \Pi\left(\varphi, r^2, \frac{p}{\sqrt{1+p^2}}\right).$$

$$3. \int \frac{dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(\varphi, \frac{p}{\sqrt{1+p^2}}\right).$$

$$4. \int \frac{\sin x dx}{\sqrt{1+p^2 \sin^2 x}} = -\frac{1}{p} \arcsin \left(\frac{p \cos x}{\sqrt{1+p^2}} \right).$$

$$5. \int \frac{\cos x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

$$6. \int \frac{dx}{\sin x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{\sqrt{1+p^2 \sin^2 x} - \cos x}{\sqrt{1+p^2 \sin^2 x} + \cos x}.$$

$$7. \int \frac{dx}{\cos x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2} \sin x}.$$

$$8. \int \frac{\operatorname{tg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2}}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2}}.$$

$$9. \int \frac{\operatorname{ctg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{1 - \sqrt{1+p^2 \sin^2 x}}{1 + \sqrt{1+p^2 \sin^2 x}}.$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$$

можно также пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

$$1) k^2 \rightarrow -p^2;$$

$$2) F(x, k) \rightarrow \frac{1}{\sqrt{1+p^2}} F\left(\varphi, \frac{p}{\sqrt{1+p^2}}\right);$$

$$3) E(x, k) \rightarrow \sqrt{1+p^2} E\left(\varphi, -\frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}};$$

$$4) \frac{1}{k} \ln(k \cos x + \Delta) \rightarrow \frac{1}{p} \arcsin \frac{p \cos x}{\sqrt{1+p^2}};$$

$$5) \frac{1}{k} \arcsin(k \sin x) \rightarrow \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

1.5.38. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx$.

Условие: $a^2 > 1$.

Обозначение: $\varphi = \arcsin \frac{a \cos x}{\sqrt{a^2 - 1}}$.

$$1. \int \sqrt{a^2 \sin^2 x - 1} dx = \frac{1}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right) - a E\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right).$$

$$2. \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right).$$

$$3. \int \frac{\sin x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\varphi}{a}.$$

$$4. \int \frac{\cos x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln(a \sin x + \sqrt{a^2 \sin^2 x - 1}).$$

$$5. \int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}}.$$

$$6. \int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} \sin x - \sqrt{a^2 \sin^2 x - 1}}.$$

$$7. \int \frac{dx}{(1 - r^2 \sin^2 x) \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2 - 1)} \Pi\left(\varphi, \frac{r^2(a^2 - 1)}{a^2(r^2 - 1)}, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [r^2 > 1]$$

$$8. \int \frac{\operatorname{tg} x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}}.$$

$$9. \int \frac{\operatorname{ctg} x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin \frac{1}{a \sin x}.$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx \quad (a^2 > 1)$$

можно также воспользоваться формулами пп. 1.5.22—1.5.30. Для этого надо произвести в них следующие замены [5]:

$$1) F(x, k) \rightarrow -\frac{1}{ia} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right);$$

$$2) E(x, k) \rightarrow \frac{i}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right) - ia E\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right);$$

$$3) \frac{1}{k} \ln(k \cos x + \Delta) \rightarrow -\frac{\varphi}{ia};$$

$$4) \frac{1}{k} \arcsin (k \sin x) \rightarrow \frac{1}{ia} \ln (a \sin x + \sqrt{a^2 \sin^2 x - 1});$$

$$5) \frac{1}{-2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} \rightarrow -\frac{1}{i} \operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}};$$

$$6) \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} \rightarrow \frac{1}{2i\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} \sin x - \sqrt{a^2 \sin^2 x - 1}};$$

$$7) \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} \rightarrow -\frac{1}{i} \arcsin \frac{1}{a \sin x};$$

$$8) \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}} \rightarrow \frac{1}{2i\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2 \sin^2 x - 1}};$$

$$9) \text{ Оставшиеся } \Delta \rightarrow i \sqrt{a^2 \sin^2 x - 1};$$

$$10) k \rightarrow a;$$

$$11) \text{ умножить полученные равенства на } i.$$

1.5.39. Интегралы вида $\int f(\operatorname{tg} x) dx$, $\int f(\operatorname{ctg} x) dx$.

Обозначение: $t(x) = \begin{cases} \operatorname{tg} x \\ \operatorname{ctg} x \end{cases}$.

$$1. \int R(\sin^2 x, \cos^2 x, \operatorname{tg} x, \operatorname{ctg} x, \sqrt{a^2 \pm b^2 \operatorname{tg}^2 x}) dx = \int R\left(\frac{1}{1+\tau^2}, \frac{\tau^2}{1+\tau^2}, \tau, \frac{1}{\tau}, \sqrt{a^2 \pm b^2 \tau^2}\right) \frac{d\tau}{1+\tau^2} \quad [\tau = \operatorname{tg} x].$$

$$2. \int R(\sin^2 x, \cos^2 x, \operatorname{tg} x, \operatorname{ctg} x, \sqrt{a^2 \pm b^2 \operatorname{ctg}^2 x}) dx = \int R\left(\frac{\tau^2}{1+\tau^2}, \frac{1}{1+\tau^2}, \frac{1}{\tau}, \tau, \sqrt{a^2 \pm b^2 \tau^2}\right) \frac{d\tau}{1+\tau^2} \quad [\tau = \operatorname{ctg} x].$$

$$3. \int [a^2 + b^2 t^2(x)]^{n+1/2} t(x) dx = \pm \sqrt{a^2 + b^2 t^2(x)} \sum_{k=0}^n (a^2 - b^2)^k \frac{[a^2 + b^2 t^2(x)]^{n-k}}{2n - 2k + 1} + \begin{cases} \pm \frac{1}{2} (a^2 - b^2)^{n+1/2} \ln \frac{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2 t^2(x)}} & [a^2 > b^2], \\ (-1)^{n+1} (b^2 - a^2)^{n+1/2} \begin{cases} \operatorname{arctg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \\ \operatorname{arccotg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \end{cases} & [a^2 < b^2]. \end{cases}$$

$$\begin{aligned}
 4. \int [a^2 - b^2 t^2(x)]^{n+1/2} t(x) dx = \\
 = \pm \sqrt{a^2 - b^2 t^2(x)} \sum_{k=0}^n (a^2 + b^2)^k \frac{[a^2 - b^2 t^2(x)]^{n-k}}{2n - 2k + 1} \pm \\
 \pm \frac{1}{2} (a^2 + b^2)^{n+1/2} \ln \frac{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2 t^2(x)}}.
 \end{aligned}$$

$$\begin{aligned}
 5. \int \frac{t(x) dx}{[a^2 + b^2 t^2(x)]^{n+1/2}} = \\
 = \pm \frac{1}{\sqrt{a^2 + b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1) (a^2 - b^2)^{n-k} [a^2 + b^2 t^2(x)]^k} + \\
 + \begin{cases} \pm \frac{1}{2 (a^2 - b^2)^{n+1/2}} \ln \frac{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2 t^2(x)}} & [a^2 > b^2], \\ \frac{(-1)^n}{(b^2 - a^2)^{n+1/2}} \begin{cases} \operatorname{arctg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \\ \operatorname{arccotg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \end{cases} & [a^2 < b^2]. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 6. \int \frac{t(x) dx}{[a^2 - b^2 t^2(x)]^{n+1/2}} = \\
 = \pm \frac{1}{\sqrt{a^2 - b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1) (a^2 + b^2)^{n-k} [a^2 - b^2 t^2(x)]^k} \pm \\
 \pm \frac{1}{2 (a^2 + b^2)^{n+1/2}} \ln \frac{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2 t^2(x)}}.
 \end{aligned}$$

$$7. \int \frac{dx}{\sqrt{a^2 + b^2 t^2(x)}} = \frac{1}{2 \sqrt{b^2 - a^2}} \ln \frac{\sqrt{b^2 - a^2} t(x) \pm \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2} t(x) \mp \sqrt{a^2 + b^2 t^2(x)}} \quad [a^2 < b^2].$$

$$8. = \frac{1}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} t(x)}{\sqrt{a^2 + b^2 t^2(x)}} \quad [a^2 > b^2].$$

$$9. \int \frac{dx}{\sqrt{a^2 - b^2 t^2(x)}} = \frac{1}{\sqrt{a^2 + b^2}} \begin{cases} \operatorname{arctg} \frac{\sqrt{a^2 + b^2} t(x)}{\sqrt{a^2 - b^2 t^2(x)}} \\ \operatorname{arccotg} \frac{\sqrt{a^2 + b^2} t(x)}{\sqrt{a^2 - b^2 t^2(x)}} \end{cases}.$$

$$10. \int \frac{dx}{\sqrt{b^2 t^2(x) - a^2}} = \frac{1}{2 \sqrt{a^2 + b^2}} \ln \frac{\sqrt{a^2 + b^2} t(x) \pm \sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2} t(x) \mp \sqrt{b^2 t^2(x) - a^2}}.$$

$$11. \int \frac{t(x) dx}{\sqrt{a^2 + b^2 t^2(x)}} = \frac{1}{\sqrt{b^2 - a^2}} \begin{cases} \operatorname{arctg} \frac{\sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2}} \\ \operatorname{arccotg} \frac{\sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2}} \end{cases} \quad [a^2 < b^2].$$

$$12. \quad = \frac{1}{2 \sqrt{a^2 - b^2}} \ln \frac{\sqrt{a^2 - b^2} \mp \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} \pm \sqrt{a^2 + b^2 t^2(x)}} \quad [a^2 > b^2].$$

$$13. \quad \int \frac{t(x) dx}{\sqrt{a^2 - b^2 t^2(x)}} = \frac{1}{2 \sqrt{a^2 + b^2}} \ln \frac{\sqrt{a^2 + b^2} \mp \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} \pm \sqrt{a^2 - b^2 t^2(x)}}.$$

$$14. \quad \int \frac{t(x) dx}{\sqrt{b^2 t^2(x) - a^2}} = \frac{1}{\sqrt{a^2 + b^2}} \left\{ \begin{array}{l} \operatorname{arctg} \frac{\sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2}} \\ \operatorname{arctg} \frac{\sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2}} \end{array} \right\}.$$

$$15. \quad \int \frac{t(x) dx}{[a^2 \pm b^2 t^2(x)]^{n+1/2}} = \\ = \frac{1}{\sqrt{a^2 \pm b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1)(a^2 \mp b^2)^{n-k}(a^2 \pm b^2 t^2(x))^k} + \\ + \frac{1}{(a^2 - b^2)^n} \int \frac{t(x) dx}{\sqrt{a^2 \pm b^2 t^2(x)}}.$$

1.5.40. Интегралы вида $\int x^p \left\{ \frac{\sin x}{\cos x} \right\}^q dx$.

$$1. \quad \int x^p \left\{ \frac{\sin x}{\cos x} \right\}^q dx = \frac{x^{p-1}}{q^2} \left\{ \frac{\sin x}{\cos x} \right\}^{q-1} \times \\ \times \left[p \left\{ \frac{\sin x}{\cos x} \right\} \mp qx \left\{ \frac{\cos x}{\sin x} \right\} \right] + \frac{q-1}{q} \int x^p \left\{ \frac{\sin x}{\cos x} \right\}^{q-2} dx - \frac{p(p-1)}{q^2} \int x^{p-2} \left\{ \frac{\sin x}{\cos x} \right\}^q dx.$$

$$2. \quad \int x^m \left\{ \frac{\sin x}{\cos x} \right\}^{2n} dx = \binom{2n}{n} \frac{x^{m+1}}{2^{2n}(m+1)} + \\ + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (\mp 1)^{n+k} \binom{2n}{k} \int x^m \cos(2n-2k)x dx.$$

$$3. \quad \int x^m \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^n (\mp 1)^{n+k} \binom{2n+1}{k} \int x^m \left\{ \frac{\sin(2n-2k+1)x}{\cos(2n-2k+1)x} \right\} dx.$$

$$4. \quad \int x^n \left\{ \frac{\sin x}{\cos x} \right\} dx = \mp \sum_{k=0}^n k! \binom{n}{k} x^{n-k} \left\{ \frac{\cos(x+k\pi/2)}{\sin(x+k\pi/2)} \right\}.$$

$$5. \quad \int x^{2n} \left\{ \frac{\sin x}{\cos x} \right\} dx = \\ = (2n)! \left[\mp \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \left\{ \frac{\cos x}{\sin x} \right\} + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \left\{ \frac{\sin x}{\cos x} \right\} \right].$$

$$6. \quad \int x^{2n+1} \left\{ \frac{\sin x}{\cos x} \right\} dx = \\ = (2n+1)! \left[\mp \sum_{k=0}^n (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \left\{ \frac{\cos x}{\sin x} \right\} + \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \left\{ \frac{\sin x}{\cos x} \right\} \right].$$

7. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^{2n} dx =$

$$= \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2} + \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1) \dots (2n-2k+1)}{2^{k+1} n(n-1) \dots (n-k)} \times$$

$$\times \left[\frac{1}{2n-2k} \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k} \mp x \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k-1} \left\{ \frac{\cos x}{\sin x} \right\} \right].$$
8. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \frac{1}{n+1} \sum_{k=0}^n \frac{2^k (n+1) n \dots (n-k+1)}{(2n+1)(2n-1) \dots (2n-2k+1)} \times$

$$\times \left[\frac{1}{2n-2k+1} \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k+1} \mp x \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k} \left\{ \frac{\cos x}{\sin x} \right\} \right].$$
9. $\int x \left\{ \frac{\sin x}{\cos x} \right\} dx = \left\{ \frac{\sin x}{\cos x} \right\} \mp x \left\{ \frac{\cos x}{\sin x} \right\}.$
10. $\int x^2 \left\{ \frac{\sin x}{\cos x} \right\} dx = 2x \left\{ \frac{\sin x}{\cos x} \right\} \mp (x^2 - 2) \left\{ \frac{\cos x}{\sin x} \right\}.$
11. $\int x^3 \left\{ \frac{\sin x}{\cos x} \right\} dx = (3x^2 - 6) \left\{ \frac{\sin x}{\cos x} \right\} \mp (x^3 - 6x) \left\{ \frac{\cos x}{\sin x} \right\}.$
12. $\int x^n \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^{n+1}}{2(n+1)} \pm$

$$\pm \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k} (n-2k)!} \sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1} (n-2k-1)!} \cos 2x \right\}.$$
13. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^2}{4} \mp \frac{x}{4} \sin 2x \mp \frac{1}{8} \cos 2x.$
14. $\int x^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^3}{6} \mp \left(\frac{x^2}{4} - \frac{1}{8} \right) \sin 2x \mp \frac{x}{4} \cos 2x.$
15. $\int x^3 \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^4}{8} \mp \left(\frac{x^3}{4} - \frac{3x}{8} \right) \sin 2x \mp \left(\frac{3x^2}{8} - \frac{3}{16} \right) \cos 2x$
16. $\int x^n \left\{ \frac{\sin x}{\cos x} \right\}^3 dx =$

$$= \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left[\frac{1}{3^{2k+1}} \left\{ \frac{\cos 3x}{\sin 3x} \right\} \mp 3 \left\{ \frac{\cos x}{\sin x} \right\} \right] \mp \right.$$

$$\left. \mp \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k x^{n-2k-1}}{(n-2k-1)!} \left[\frac{1}{3^{2k+2}} \left\{ \frac{\sin 3x}{\cos 3x} \right\} \mp 3 \left\{ \frac{\sin x}{\cos x} \right\} \right] \right\}.$$
17. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^3 dx = \frac{3}{4} \left\{ \frac{\sin x}{\cos x} \right\} \mp \frac{1}{36} \left\{ \frac{\sin 3x}{\cos 3x} \right\} \mp \frac{3}{4} x \left\{ \frac{\cos x}{\sin x} \right\} + \frac{x}{12} \left\{ \frac{\cos 3x}{\sin 3x} \right\}.$
18. $\int x^\mu \left\{ \frac{\sin x}{\cos x} \right\} dx = \mp x^\mu \left\{ \frac{\cos x}{\sin x} \right\} \pm \mu \int x^{\mu-1} \left\{ \frac{\cos x}{\sin x} \right\} dx.$

$$19. \int_0^x x^{\mu-1} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [(ia)^{-\mu} \gamma(\mu, iax) \mp (-ia)^{-\mu} \gamma(\mu, -iax)]$$

$$\left[\operatorname{Re} \mu > \left\{ \begin{matrix} -1 \\ 0 \end{matrix} \right\}; x > 0 \right].$$

$$20. \int_x^\infty x^{\mu-1} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \frac{1}{2a^\mu} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [\mp e^{i\mu\pi/2} \Gamma(\mu, -iax) + e^{-i\mu\pi/2} \Gamma(\mu, iax)]$$

$$[\operatorname{Re} \mu < 1; x > 0].$$

$$21. \int_x^\infty x^{\mu-1} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = \left\{ \begin{matrix} S(x, \mu) \\ C(x, \mu) \end{matrix} \right\}.$$

1.5.41. Интегралы вида $\int \frac{1}{x^p} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^q dx$.

$$1. \int \frac{1}{x^p} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^q dx =$$

$$= -\frac{1}{(p-1)x^{p-1}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^q \mp \frac{q}{(p-1)(p-2)x^{p-2}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{q-1} \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} +$$

$$+ \frac{q(q-1)}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{q-2} dx - \frac{q^2}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^q dx.$$

$$2. \int \frac{1}{x^p} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = -\frac{1}{(p-1)x^{p-1}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \pm \frac{1}{p-1} \int \frac{1}{x^{p-1}} \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} dx.$$

$$3. = -\frac{1}{(p-1)x^{p-1}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \mp \frac{1}{(p-1)(p-2)x^{p-2}} \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} -$$

$$- \frac{1}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx.$$

$$4. \int \frac{1}{x^{2n}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = \frac{(-1)^{n+1}}{x(2n-1)!} \left[\pm \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} + \right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right] \pm \frac{(-1)^{n+1}}{(2n-1)!} \left\{ \begin{matrix} \operatorname{ci}(x) \\ \operatorname{si}(x) \end{matrix} \right\}.$$

$$5. \int \frac{1}{x^{2n+1}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = \frac{(-1)^{n+1}}{x(2n)!} \left[\pm \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} + \right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right] \pm \frac{(-1)^n}{(2n)!} \left\{ \begin{matrix} \operatorname{si}(x) \\ \operatorname{ci}(x) \end{matrix} \right\}.$$

$$6. \int \frac{1}{x} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{2m} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (\mp 1)^{m+k} \binom{2m}{k} \operatorname{ci}[(2m-2k)x].$$

$$7. \int \frac{1}{x} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{2m+1} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (\mp 1)^{m+k} \binom{2m+1}{k} \left\{ \begin{matrix} \operatorname{si}(2m-2k+1)x \\ \operatorname{ci}(2m-2k+1)x \end{matrix} \right\}.$$

8. $\int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\}^{2m} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} \pm$
 $\pm \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (\mp 1)^{m+k+1} \binom{2m}{k} \left[\frac{\cos (2m-2k)x}{x} + (2m-2k) \operatorname{si} [(2m-2k)x] \right].$
9. $\int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\}^{2m+1} dx = \pm \frac{1}{2^{2m}} \sum_{k=0}^m (\mp 1)^{m+k+1} \binom{2m+1}{k} \times$
 $\times \left[\frac{1}{x} \left\{ \frac{\sin (2m-2k+1)x}{\cos (2m-2k+1)x} \right\} \mp (2m-2k+1) \left\{ \frac{\operatorname{ci} (2m-2k+1)x}{\operatorname{si} (2m-2k+1)x} \right\} \right].$
10. $\int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}.$ 11. $\int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}.$
12. $\int_0^x \frac{\sin x}{x} dx = \operatorname{Si}(x).$ 13. $\int_0^x \frac{1-\cos x}{x} dx = C + \ln x - \operatorname{ci}(x).$
14. $\int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} \pm \left\{ \frac{\operatorname{ci}(x)}{\operatorname{si}(x)} \right\}.$
15. $\int \frac{1}{x^3} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \frac{1}{2x^2} \left\{ \frac{\sin x}{\cos x} \right\} \mp \frac{1}{2x} \left\{ \frac{\cos x}{\sin x} \right\} - \frac{1}{2} \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}.$
16. $\int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{1}{2} \ln |x| \mp \frac{1}{2} \operatorname{ci}(2x).$
17. $\int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\}^3 dx = \frac{3}{4} \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\} \mp \frac{1}{4} \left\{ \frac{\operatorname{si}(3x)}{\operatorname{ci}(3x)} \right\}.$

1.5.42. Интегралы вида $\int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^q dx.$

1. $\int x^p \operatorname{tg} x dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k} (2^{2k}-1)}{(p+2k)(2k)!} B_{2k} x^{p+2k} \quad [p \geq -1; |x| < \pi/2].$
2. $\int x^p \operatorname{ctg} x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq 1; |x| < \pi].$
3. $\int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n} dx = (-1)^n \frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \int x^p \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} dx$
[см. 1.5.44, 1.5.45].
4. $\int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n+1} dx = \pm \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} \binom{n}{k} x^p \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} +$
 $+ (-1)^n \int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx \mp p \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} \binom{n}{k} \int x^{p-1} \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} dx$
[см. 1.5.44, 1.5.45].

$$5. \int x \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = \pm x \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} + \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\} - \frac{x^2}{2}.$$

1.5.43. Интегралы вида $\int x^r \sin^p x \cos^q x dx$.

$$1. \int x^r \sin^p x \cos^q x dx = \frac{1}{(p+q)^2} \left\{ (p+q) x^r \sin^{p+1} x \cos^{q-1} x + \right. \\ \left. + r x^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x dx - \right. \\ \left. - r p \int x^{r-1} \sin^{p-1} x \cos^{q-1} x dx + (q-1)(p+q) \int x^r \sin^p x \cos^{q-2} x dx \right\}.$$

$$2. = \frac{1}{(p+q)^2} \left[- (p+q) x^r \sin^{p-1} x \cos^{q+1} x + \right. \\ \left. + r x^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x dx + \right. \\ \left. + r q \int x^{r-1} \sin^{p-1} x \cos^{q-1} x dx + (p-1)(p+q) \int x^r \sin^{p-2} x \cos^q x dx \right].$$

$$3. \int x \sin^p x \cos^q x dx = \frac{1}{p+q} \left[x \sin^{p+1} x \cos^{q-1} x - \right. \\ \left. - \int \sin^{p+1} x \cos^{q-1} x dx + (q-1) \int x \sin^p x \cos^{q-2} x dx \right].$$

$$4. = \frac{1}{p+q} \left[- x \sin^{p-1} x \cos^{q+1} x + \int \sin^{p-1} x \cos^{q+1} x dx + \right. \\ \left. + (p-1) \int x \sin^{p-2} x \cos^q x dx \right].$$

$$5. \int x^p \frac{\sin^{2m} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\cos^{n-2k} x} \quad [\text{см. 1.5.45}].$$

$$6. \int x^p \frac{\sin^{2m+1} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x} dx \quad [\text{см. 1.5.46}].$$

$$7. \int x^p \frac{\sin x}{\cos^n x} dx = \frac{x^p}{(n-1) \cos^{n-1} x} - \frac{p}{n-1} \int \frac{x^{p-1}}{\cos^{n-1} x} dx \quad [\text{см. 1.5.45}].$$

$$8. \int x^p \frac{\cos^{2m} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p}{\sin^{n-2k} x} dx \quad [\text{см. 1.5.44}].$$

$$9. \int x^p \frac{\cos^{2m+1} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x} dx \quad [\text{см. 1.5.46}].$$

$$10. \int x^p \frac{\cos x}{\sin^n x} dx = - \frac{x^p}{(n-1) \sin^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}}{\sin^{n-1} x} dx \quad [\text{см. 1.5.44}].$$

$$11. \int \frac{x \cos x}{\sin^2 x} dx = - \frac{x}{\sin x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$12. \int \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x} - \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

1.5.44. Интегралы вида $\int \frac{x^p dx}{\sin^q x}$.

$$1. \int \frac{x^p dx}{\sin^q x} = -\frac{x^{p-1} [p \sin x + (q-2) x \cos x]}{(q-1)(q-2) \sin^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{x^p dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\sin^{q-2} x}.$$

$$2. \int \frac{x^p dx}{\sin x} = \frac{x^p}{p} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(p+2k)(2k)!} B_{2k} x^{p+2k} \quad [|x| < \pi; p > 0].$$

$$3. \int \frac{x^p dx}{\sin^2 x} = -x^p \operatorname{ctg} x + \frac{p}{p-1} x^{p-1} + p \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{p+2k-1}}{(p+2k-1)(2k)!} B_{2k} \\ [|x| < \pi; p > 1].$$

$$4. \int \frac{x dx}{\sin^q x} = -\frac{x \cos x}{(q-1) \sin^{q-1} x} - \frac{1}{(q-1)(q-2) \sin^{q-2} x} + \frac{q-2}{q-1} \int \frac{x dx}{\sin^{q-2} x}.$$

$$5. \int \frac{x dx}{\sin^{2n} x} = \\ = -\frac{1}{2n} \sum_{k=1}^{n-1} \frac{2n(2n-2) \dots (2n-2k+2) [\sin x + (2n-2k) x \cos x]}{(2n-1)(2n-3) \dots (2n-2k+3)(2n-2k+1)(2n-2k) \sin^{2n-2k+1} x} + \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (\ln |\sin x| - x \operatorname{ctg} x).$$

$$6. \int \frac{x dx}{\sin^{2n+1} x} = \\ = -\frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1) \dots (2n-2k+1) [\sin x + (2n-2k-1) x \cos x]}{2n(2n-2) \dots (2n-2k+2)(2n-2k)(2n-2k-1) \sin^{2n-2k} x} + \\ + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\sin x}.$$

$$7. \int \frac{dx}{x^p \sin^q x} = \frac{p}{(q-1)(q-2) x^{p+1} \sin^{q-2} x} - \frac{\cos x}{(q-1) x^p \sin^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{dx}{x^p \sin^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)} \int \frac{dx}{x^{p+2} \sin^{q-2} x}.$$

$$8. \int \frac{dx}{x^n \sin x} = -\frac{1}{nx^n} - [1 + (-1)^n] (-1)^{n/2} \frac{2^{n-1}-1}{n!} B_n \ln x - \\ - \sum_{\substack{k=1 \\ k \neq n/2}}^{\infty} (-1)^k \frac{2(2^{2k-1}-1)}{(2k)!(2k-n)} B_{2k} x^{2k-n} \quad [n > 1; |x| < \pi].$$

$$9. \int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} - \\ - [1 - (-1)^n] (-1)^{(n+1)/2} \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)! (2k-n-1)} B_{2k} \quad [|x| < \pi].$$

$$10. \int \frac{x dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k}-1)}{(2k+1)!} B_{2k} x^{2k+1}.$$

$$11. \int \frac{x^2 dx}{\sin x} = \frac{x^2}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k}-1}{(2k)! (k+1)} B_{2k} x^{2k+2}.$$

$$12. \int \frac{x dx}{\sin^2 x} = -x \operatorname{ctg} x + \ln |\sin x|.$$

$$13. \int \frac{x dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x dx}{\sin x}.$$

$$14. \int \frac{x dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \operatorname{ctg} x + \frac{2}{3} \ln |\sin x|.$$

1.5.45. Интегралы вида $\int \frac{x^p dx}{\cos^q x}$.

$$1. \int \frac{x^p dx}{\cos^q x} = -\frac{x^{p-1} [p \cos x - (q-2) x \sin x]}{(q-1)(q-2) \cos^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{x^p dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cos^{q-2} x}.$$

$$2. \int \frac{x^p dx}{\cos x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{p+2k+1}}{(p+2k+1)(2k)!} \quad [|x| < \pi/2; p > 0].$$

$$3. \int \frac{x^p dx}{\cos^2 x} = x^p \operatorname{tg} x + p \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} (2^{2k}-1) x^{p+2k-1}}{(p+2k-1)(2k)!} B_{2k} \\ [p > 1; |x| < \pi/2]$$

$$4. \int \frac{x dx}{\cos^{2n} x} = \\ = \frac{2n+1}{(2n+2)2n} \sum_{k=0}^{n-1} \frac{(2n+2)2n \dots (2n-2k+2) [(2n-2k)x \sin x - \cos x]}{(2n+1)(2n-1) \dots (2n-2k+3)(2n-2k+1)(2n-2k) \cos^{2n-2k+1} x} + \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (x \operatorname{tg} x + \ln |\cos x|).$$

$$5. \int \frac{x dx}{\cos^{2n+1} x} = \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1) [(2n-2k+1)x \sin x - \cos x]}{2n(2n-2)\dots(2n-2k+2)(2n-2k)(2n-2k-1) \cos^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\cos x}.$$

$$6. \int \frac{dx}{x^p \cos^q x} = \frac{p}{(q-1)(q-2)x^{p+1} \cos^{q-2} x} + \frac{\sin x}{(q-1)x^p \cos^{q-1} x} + \frac{q-2}{q-1} \int \frac{dx}{x^p \cos^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)} \int \frac{dx}{x^{p+2} \cos^{q-2} x}.$$

$$7. \int \frac{dx}{x^n \cos x} = \frac{1}{2} [1 - (-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq (n-1)/2}}^{\infty} \frac{|E_{2k}| x^{2k-n+1}}{(2k-n+1)(2k)!} \quad [|x| < \pi/2].$$

$$8. \int \frac{dx}{x^n \cos^2 x} = \frac{\operatorname{tg} x}{x^n} - [1 - (-1)^n] (-1)^{(n+1)/2} \frac{2^n n}{(n+1)!} (2^{n+1} - 1) B_{n+1} \ln x - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(-1)^k (2^{2k} - 1) (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \quad [|x| < \pi/2].$$

$$9. \int \frac{x dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}. \quad 10. \int \frac{x dx}{\cos^2 x} = x \operatorname{tg} x + \ln |\cos x|.$$

$$11. \int \frac{x dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x dx}{\cos x}.$$

$$12. \int \frac{x dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \operatorname{tg} x + \frac{2}{3} \ln |\cos x|.$$

1.5.46. Интегралы вида $\int R(x^p, \sin x, \cos x, a + b \sin x + c \cos x) dx$.

$$1. \int \frac{x^p \cos x dx}{(a + b \sin x)^q} = - \frac{x^p}{(q-1)b(a + b \sin x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \sin x)^{q-1}}.$$

$$2. \int \frac{x^p \sin x dx}{(a + b \cos x)^q} = \frac{x^p}{(q-1)b(a + b \cos x)^{q-1}} - \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \cos x)^{q-1}}.$$

$$3. \int \frac{x dx}{1 \pm \sin x} = \mp x \left\{ \operatorname{tg} [(\pi - 2x)/4] \right\} + 2 \ln \left| \frac{\cos [(\pi - 2x)/4]}{\sin [(\pi - 2x)/4]} \right|.$$

$$4. \int \frac{x dx}{1 \pm \cos x} = \pm x \left\{ \operatorname{tg} (x/2) \right\} + 2 \ln \left| \frac{\cos (x/2)}{\sin (x/2)} \right|.$$

$$5. \int \frac{x \cos x dx}{(1 \pm \sin x)^2} = \mp \frac{x}{1 \pm \sin x} + \left\{ \operatorname{tg} [(2x - \pi)/4] \right\}.$$

$$6. \int \frac{x \sin x dx}{(1 \pm \cos x)^2} = \pm \frac{x}{1 \pm \cos x} - \left\{ \operatorname{tg} (x/2) \right\}.$$

$$7. \int \frac{x dx}{(\cos x + a \sin x)^2} = \frac{1}{a^2 + 1} \ln (\cos x + a \sin x) + x \frac{\operatorname{tg} x - a}{(a^2 + 1)(1 + a \operatorname{tg} x)}.$$

$$8. \int \frac{x^2 dx}{[(ax - b) \sin x + (a + bx) \cos x]^2} = \frac{x \sin x + \cos x}{b [(ax - b) \sin x + (a + bx) \cos x]}.$$

$$9. \int \frac{\cos^2 x dx}{[a \cos x + (ax + b) \sin x]^2} = \frac{\operatorname{tg} x}{a [a + (ax + b) \operatorname{tg} x]}.$$

1.5.47. Интегралы вида $\int \frac{x \sin^m x \cos^n x}{\sqrt{(1 - k^2 \sin^2 x)^r}} dx$.

Условие: $0 < k^2 < 1$.

Сбозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{x \sin x \cos x}{\Delta} dx = -\frac{x\Delta}{k^2} + \frac{1}{k^2} E(x, k).$$

$$2. \int \frac{x \sin^3 x \cos x}{\Delta} dx = \frac{1 - k^2}{9k^4} F(x, k) + \frac{2k^2 + 5}{9k^4} E(x, k) - \\ - \frac{1}{9k^4} [3(3 - \Delta^2)x + k^2 \sin x \cos x] \Delta.$$

$$3. \int \frac{x \sin x \cos^3 x}{\Delta} dx = -\frac{1 - k^2}{9k^4} F(x, k) + \frac{7k^2 - 5}{9k^4} E(x, k) - \\ - \frac{1}{9k^4} [3(\Delta^2 - 3 + 3k^2)x - k^2 \sin x \cos x] \Delta.$$

$$4. \int \frac{x \sin x}{\Delta^3} dx = -\frac{x \cos x}{(1 - k^2)\Delta} + \frac{1}{k(1 - k^2)} \arcsin(k \sin x).$$

$$5. \int \frac{x \cos x}{\Delta^3} dx = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$6. \int \frac{x \sin x \cos x}{\Delta^3} dx = \frac{x}{k^2 \Delta} - \frac{1}{k^2} F(x, k).$$

$$7. \int \frac{x \sin^3 x \cos x}{\Delta^3} dx = x \frac{2 - k^2 \sin^2 x}{k^4 \Delta} - \frac{1}{k^4} [E(x, k) + F(x, k)].$$

$$8. \int \frac{x \sin x \cos^3 x}{\Delta^3} dx = x \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta} + \frac{1 - k^2}{k^4} F(x, k) + \frac{1}{k^4} E(x, k).$$

1.5.48. Интегралы вида $\int (x + b)^{\pm n} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx$.

$$1. \int (x + b)^n \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \\ = \frac{n!}{a} \left[\sum_{k=0}^{[n/2]} (-1)^k \left(\frac{1}{a} \right)^{2k} \frac{(x + b)^{n-2k}}{(n-2k)!} \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} + \right. \\ \left. + \sum_{k=0}^{[(n-1)/2]} (-1)^k \left(\frac{1}{a} \right)^{2k+1} \frac{(x + b)^{n-2k-1}}{(n-2k-1)!} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} \right].$$

$$2. \int (x + b) \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \mp \frac{1}{a} (x + b) \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} + \frac{1}{a^2} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\}.$$

$$3. \int (x+b)^2 \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \mp \frac{1}{a} \left[(x+b)^2 - \frac{2b}{a^2} \right] \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} + \frac{2}{a^2} (x+b) \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\}.$$

$$4. \int (x+b)^3 \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \mp \frac{x+b}{a} \left[(x+b)^2 - \frac{6}{a^2} \right] \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} + \\ + \frac{3}{a^2} \left[(x+b)^2 - \frac{2}{a^2} \right] \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\}.$$

$$5. \int \frac{1}{(x+b)^n} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = a^{n-1} \left(\cos ab \int \frac{1}{t^n} \left\{ \begin{matrix} \sin t \\ \cos t \end{matrix} \right\} dt \mp \right. \\ \left. \mp \sin ab \int \frac{1}{t^n} \left\{ \begin{matrix} \cos t \\ \sin t \end{matrix} \right\} dt \right) \quad [t = a(x+b)].$$

$$6. \int_x^\infty \frac{1}{(x+b)^n} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \\ = \sum_{k=1}^{n-1} \frac{(k-1)!}{(n-1)!} (x+b)^{-k} (-a)^{n-k-1} \left\{ \begin{matrix} \cos [\pi(n-k)/2 - ax] \\ \sin [\pi(n-k)/2 - ax] \end{matrix} \right\} - \\ - \frac{(-a)^{n-1}}{(n-1)!} \left[\text{ci}(ab+ax) \left\{ \begin{matrix} \cos(ab+\pi n/2) \\ \sin(ab+\pi n/2) \end{matrix} \right\} \pm \text{si}(ab+ax) \left\{ \begin{matrix} \sin(ab+\pi n/2) \\ \cos(ab+\pi n/2) \end{matrix} \right\} \right].$$

$$7. \int \frac{1}{x+b} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = \cos ab \left\{ \begin{matrix} \text{si}(ax+ab) \\ \text{ci}(ax+ab) \end{matrix} \right\} \mp \sin ab \left\{ \begin{matrix} \text{ci}(ax+ab) \\ \text{si}(ax+ab) \end{matrix} \right\}.$$

$$8. \int \frac{1}{(x+b)^2} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = -\frac{1}{x+b} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} \pm a \int \frac{1}{x+b} \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} dx.$$

$$9. \int \frac{1}{(x+b)^3} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx = -\frac{1}{2(x+b)^2} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} \mp \frac{a}{2(x+b)} \left\{ \begin{matrix} \cos ax \\ \sin ax \end{matrix} \right\} - \\ - \frac{a^2}{2} \int \frac{1}{x+b} \left\{ \begin{matrix} \sin ax \\ \cos ax \end{matrix} \right\} dx.$$

$$10. \int_x^\infty \frac{1}{x(x+b)} \sin ax \, dx = \frac{1}{b} [\cos ab \, \text{si}(ab+ax) - \sin ab \, \text{ci}(ab+ax) - \text{si}(ax)].$$

1.5.49. Интегралы вида $\int e^{ax} \sin^p x \cos^q x \, dx$.

$$1. \int e^{ax} \sin^p x \cos^q x \, dx = \\ = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^p x \cos^{q-1} x [a \cos x + (p+q) \sin x] - \right. \\ \left. - pa \int e^{ax} \sin^{p-1} x \cos^{q-1} x \, dx + (q-1)(p+q) \int e^{ax} \sin^p x \cos^{q-2} x \, dx \right\}.$$

$$2. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] + \right. \\ \left. + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x \, dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\}.$$

$$3. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - \right. \\ \left. - p \cos^2 x] + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x \, dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\}.$$

$$4. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + \right. \\ \left. + q \sin^2 x - p \cos^2 x) + q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx - \right. \\ \left. - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}.$$

$$5. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - \right. \\ \left. - p \cos^2 x) + p(p-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx + \right. \\ \left. + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx \right\}.$$

$$6. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^p dx = \\ = \frac{a}{a^2 + p^2 b^2} e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^p \mp \frac{pb}{a^2 + p^2 b^2} e^{ax} \left\{ \frac{\cos bx}{\cos bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{p-1} + \\ + \frac{p(p-1)b^2}{a^2 + p^2 b^2} \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{p-2} dx.$$

$$7. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n} dx = \\ = \sum_{k=0}^{n-1} \frac{(2n)! b^{2k} e^{ax}}{(2n-2k)! [a^2 + (2n)^2 b^2] [a^2 + (2n-2)^2 b^2] \dots [a^2 + (2n-2k)^2 b^2]} \times \\ \times \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k} \mp (2n-2k)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k-1} \right] + \\ + \frac{(2n)! b^{2n} e^{ax}}{[a^2 + (2n)^2 b^2] [a^2 + (2n-2)^2 b^2] \dots [a^2 + 4b^2] a}.$$

$$8. = \binom{2n}{n} \frac{e^{ax}}{2^{2n} a} + \\ + \frac{e^{ax}}{2^{2n-1}} \sum_{k=1}^n (\mp 1)^k \binom{2n}{n-k} \frac{1}{a^2 + 4b^2 k^2} (a \cos 2bkx + 2bk \sin 2bkx).$$

$$9. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n+1} dx = \\ = \sum_{k=0}^n \frac{(2n+1)! b^{2k} e^{ax}}{(2n-2k+1)! [a^2 + (2n+1)^2 b^2] [a^2 + (2n-1)^2 b^2] \dots [a^2 + (2n-2k+1)^2 b^2]} \times \\ \times \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k+1} \mp (2n-2k+1)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k} \right].$$

$$10. = \frac{e^{ax}}{2^{2n}} \sum_{k=0}^n \frac{(\mp 1)^k}{a^2 + (2k+1)^2 b^2} \binom{2n+1}{n-k} \times \\ \times \left[a \left\{ \frac{\sin (2k+1) bx}{\cos (2k+1) bx} \right\} \mp (2k+1)b \left\{ \frac{\cos (2k+1) bx}{\sin (2k+1) bx} \right\} \right].$$

$$11. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{e^{ax}}{a^2 + b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right].$$

$$12. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^2 dx = \frac{e^{ax}}{2a} \mp \frac{e^{ax}}{a^2 + 4b^2} \left(\frac{a}{2} \cos 2bx + b \sin 2bx \right).$$

$$13. \int e^{ax} \sin bx \cos cx dx = \\ = \frac{e^{ax}}{2} \left[\frac{a \sin (b+c)x - (b+c) \cos (b+c)x}{a^2 + (b+c)^2} + \frac{a \sin (b-c)x - (b-c) \cos (b-c)x}{a^2 + (b-c)^2} \right].$$

$$14. \int e^{ax} \sin^2 bx \cos cx dx = \\ = \frac{e^{ax}}{4} \left[2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos (2b+c)x + (2b+c) \sin (2b+c)x}{a^2 + (2b+c)^2} - \right. \\ \left. - \frac{a \cos (2b-c)x + (2b-c) \sin (2b-c)x}{a^2 + (2b-c)^2} \right].$$

$$15. \int e^{ax} \sin bx \cos^2 cx dx = \\ = \frac{e^{ax}}{4} \left[2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin (b+2c)x - (b+2c) \cos (b+2c)x}{a^2 + (b+2c)^2} + \right. \\ \left. + \frac{a \sin (b-2c)x - (b-2c) \cos (b-2c)x}{a^2 + (b-2c)^2} \right].$$

$$16. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p} dx = \\ = - \frac{e^{ax}}{(p-1)(p-2)b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+2} \pm (p-2)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+1} \right] + \\ + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+2} dx.$$

$$17. \int e^{-x^2} \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \frac{\sqrt{\pi}}{2} e^{-a^2/4} \left\{ \begin{matrix} \text{Im erf}(x - ia/2) \\ \text{Re erf}(x - ia/2) \end{matrix} \right\}.$$

1.5.50. Интегралы вида $\int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^p dx$.

$$1. \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^p dx = \\ = \pm \frac{e^{ax}}{p-1} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^{p-1} \mp \frac{a}{p-1} \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^{p-1} dx - \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^{p-2} dx.$$

$$2. \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\} dx = \frac{e^{ax}}{a} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\} \mp \frac{1}{a} \int e^{ax} \left\{ \frac{\cos x}{\sin x} \right\}^{-2} dx.$$

$$3. \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\}^2 dx = - \frac{e^{ax}}{a} \pm e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\} \mp a \int e^{ax} \left\{ \frac{\text{tg } x}{\text{ctg } x} \right\} dx.$$

1.5.51. Интегралы вида $\int R(x, e^{ax}, \sin bx, \cos bx) dx$.

Обозначения: $\sin \varphi = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \varphi = \frac{a}{\sqrt{a^2 + b^2}}.$

$$1. \int x^p e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{x^p e^{ax}}{a^2 + b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right] - \\ - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right] dx.$$

$$2. \quad = \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \left\{ \begin{matrix} \sin (bx + \varphi) \\ \cos (bx + \varphi) \end{matrix} \right\} - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \left\{ \begin{matrix} \sin (bx + \varphi) \\ \cos (bx + \varphi) \end{matrix} \right\} dx.$$

$$3. \quad \int_0^x x^{\lambda-1} e^{-ax} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [(a+ib)^{-\lambda} \gamma(\lambda, ax+ibx) \mp \\ \mp (a-ib)^{-\lambda} \gamma(\lambda, ax-ibx)] \quad \left[\operatorname{Re} \lambda > \left\{ \begin{matrix} -1 \\ 0 \end{matrix} \right\} \right].$$

$$4. \quad \int_x^\infty x^{\lambda-1} e^{-ax} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [(a+ib)^{-\lambda} \Gamma(\lambda, ax+ibx) \mp \\ \mp (a-ib)^{-\lambda} \Gamma(\lambda, ax-ibx)].$$

$$5. \quad \int x^n e^{ax} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \left\{ \begin{matrix} \sin (bx + \varphi k) \\ \cos (bx + \varphi k) \end{matrix} \right\}.$$

$$6. \quad \int x e^{ax} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} \mp \left(bx - \frac{2ab}{a^2 + b^2} \right) \left\{ \begin{matrix} \cos bx \\ \sin bx \end{matrix} \right\} \right].$$

$$7. \quad \int x^2 e^{ax} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \frac{e^{ax}}{a^2 + b^2} \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} \mp \\ \mp \frac{e^{ax}}{a^2 + b^2} \left[bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \left\{ \begin{matrix} \cos bx \\ \sin bx \end{matrix} \right\}.$$

$$8. \quad \int x^n e^{ax} \left\{ \begin{matrix} \sin (bx + c) \\ \cos (bx + c) \end{matrix} \right\} dx = \\ = \frac{e^{ax}}{n+1} \sum_{k=1}^{n+1} (-1)^{k-1} \frac{(n+1)n \dots (n-k+2)}{(a^2 + b^2)^{k/2}} x^{n-k+1} \left\{ \begin{matrix} \sin (bx + c + k\varphi) \\ \cos (bx + c + k\varphi) \end{matrix} \right\}.$$

$$9. \quad \int x e^{ax} \left\{ \begin{matrix} \sin (bx + c) \\ \cos (bx + c) \end{matrix} \right\} dx = \\ = \frac{1}{\sqrt{a^2 + b^2}} x e^{ax} \left\{ \begin{matrix} \sin (bx + c + \varphi) \\ \cos (bx + c + \varphi) \end{matrix} \right\} - \frac{1}{a^2 + b^2} e^{ax} \left\{ \begin{matrix} \sin (bx + c + 2\varphi) \\ \cos (bx + c + 2\varphi) \end{matrix} \right\}.$$

$$10. \quad \int_0^x \frac{1}{x} e^{ix} \sin x dx = \operatorname{Si}(2x) + i [\operatorname{Ci}(2x) - \operatorname{ci}(2x)].$$

1.5.52. Интегралы вида $\int \left\{ \begin{matrix} \operatorname{sh}(ax+b) \\ \operatorname{ch}(ax+b) \end{matrix} \right\}^m \left\{ \begin{matrix} \sin(cx+d) \\ \cos(cx+d) \end{matrix} \right\}^n dx.$

$$1. \quad \int \operatorname{sh}(ax+b) \left\{ \begin{matrix} \sin(cx+d) \\ \cos(cx+d) \end{matrix} \right\} dx = \frac{a}{a^2 + c^2} \operatorname{ch}(ax+b) \left\{ \begin{matrix} \sin(cx+d) \\ \cos(cx+d) \end{matrix} \right\} \mp \\ \mp \frac{c}{a^2 + c^2} \operatorname{sh}(ax+b) \left\{ \begin{matrix} \cos(cx+d) \\ \sin(cx+d) \end{matrix} \right\}.$$

$$2. \int \operatorname{ch}(ax+b) \left\{ \begin{matrix} \sin(cx+d) \\ \cos(cx+d) \end{matrix} \right\} dx = \frac{a}{a^2+c^2} \operatorname{sh}(ax+b) \left\{ \begin{matrix} \sin(cx+d) \\ \cos(cx+d) \end{matrix} \right\} \mp \\ \mp \frac{c}{a^2+c^2} \operatorname{ch}(ax+b) \left\{ \begin{matrix} \cos(cx+d) \\ \sin(cx+d) \end{matrix} \right\}.$$

$$3. \int \operatorname{sh} x \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = \frac{1}{2} \operatorname{ch} x \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \mp \frac{1}{2} \operatorname{sh} x \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\}.$$

$$4. \int \operatorname{ch} x \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} dx = \frac{1}{2} \operatorname{sh} x \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \mp \frac{1}{2} \operatorname{ch} x \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\}.$$

$$5. \int \operatorname{sh}^{2m}(ax+b) \sin^{2n}(cx+d) dx = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \\ + \frac{(-1)^{m+n}}{2^{2m+2n-1}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} \binom{2n}{k} \sin[(2n-2k)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \}.$$

$$6. \int \operatorname{sh}^{2m}(ax+b) \sin^{2n+1}(cx+d) dx = \\ = \frac{(-1)^{m+1}}{2^{2m+2n}} \binom{2m}{m} \sum_{k=0}^n \frac{(-1)^k}{(2n-2k+1)c} \binom{2n+1}{k} \cos[(2n-2k+1)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \sin[(2n-2k+1)(cx+d)] - \\ - (2n-2k+1)c \operatorname{ch}[(2m-2j)(ax+b)] \cos[(2n-2k+1)(cx+d)] \}.$$

$$7. \int \operatorname{sh}^{2m+1}(ax+b) \sin^{2n}(cx+d) dx = \\ = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{(-1)^j \binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{ch}[(2m-2j+1)(ax+b)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j+1)a \operatorname{ch}[(2m-2j+1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{sh}[(2m-2j+1)(ax+b)] \sin[(2n-2k)(cx+d)] \}.$$

$$8. \int \operatorname{sh}^{2m+1}(ax+b) \sin^{2n+1}(cx+d) dx =$$

$$= \frac{(-1)^n}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^{j+k} \binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1)(ax+b)] \sin [(2n-2k+1)(cx+d)] - \\ - (2n-2k+1) c \operatorname{sh} [(2m-2j+1)(ax+b)] \cos [(2n-2k+1)(cx+d)] \}.$$

$$9. \int \operatorname{sh}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x +$$

$$+ \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m}{j}}{(2m-2j)a} \operatorname{sh} [(2m-2j)(ax+b)] + \\ + \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin [(2n-2k)(cx+d)] + \\ + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k)(cx+d)] + \\ + (2n-2k) c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k)(cx+d)] \}.$$

$$10. \int \operatorname{sh}^{2m}(ax+b) \cos^{2n+1}(cx+d) dx =$$

$$= \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{\binom{2n+1}{k}}{(2n-2k+1)c} \sin [(2n-2k+1)(cx+d)] + \\ + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^j \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k+1)(cx+d)] + \\ + (2n-2k+1) c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k+1)(cx+d)] \}.$$

$$11. \int \operatorname{sh}^{2m+1}(ax+b) \cos^{2n}(cx+d) dx =$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{(-1)^j \binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{ch} [(2m-2j+1)(ax+b)] + \\ + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1)(ax+b)] \cos [(2n-2k)(cx+d)] + \\ + (2n-2k) c \operatorname{sh} [(2m-2j+1)(ax+b)] \sin [(2n-2k)(cx+d)] \}.$$

$$\begin{aligned}
 12. \int \operatorname{sh}^{2m+1}(ax+b) \cos^{2n+1}(cx+d) dx = \\
 = \frac{1}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^j \binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1)(ax+b)] \cos [(2n-2k+1)(cx+d)] + \\
 + (2n-2k+1) c \operatorname{sh} [(2m-2j+1)(ax+b)] \sin [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 13. \int \operatorname{ch}^{2m}(ax+b) \sin^{2n}(cx+d) dx = \\
 = \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \sin [(2n-2k)(cx+d)] + \\
 + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh} [(2m-2j)(ax+b)] + \\
 + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k) c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 14. \int \operatorname{ch}^{2m+1}(ax+b) \sin^{2n}(cx+d) dx = \\
 = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{\binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{sh} [(2m-2j+1)(ax+b)] + \\
 + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j+1) a \operatorname{sh} [(2m-2j+1)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k) c \operatorname{ch} [(2m-2j+1)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 15. \int \operatorname{ch}^{2m}(ax+b) \sin^{2n+1}(cx+d) dx = \\
 = \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{(-1)^{k+1} \binom{2n+1}{k}}{(2n-2k+1)c} \cos [(2n-2k+1)(cx+d)] + \\
 + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^k \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \sin [(2n-2k+1)(cx+d)] - \\
 - (2n-2k+1) c \operatorname{ch} [(2m-2j)(ax+b)] \cos [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \int \operatorname{ch}^{2m+1}(ax+b) \sin^{2n+1}(cx+d) dx = \\
 = \frac{(-1)^n}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^k \binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j+1)a \operatorname{sh}[(2m-2j+1)(ax+b)] \sin[(2n-2k+1)(cx+d)] - \\
 - (2n-2k+1)c \operatorname{ch}[(2m-2j+1)(ax+b)] \cos[(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \int \operatorname{ch}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \\
 = \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] + \\
 + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh}[(2m-2j)(ax+b)] + \\
 + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
 + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \int \operatorname{ch}^{2m+1}(ax+b) \cos^{2n}(cx+d) dx = \\
 = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{\binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{sh}[(2m-2j+1)(ax+b)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{\binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j+1)a \operatorname{sh}[(2m-2j+1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
 + (2n-2k)c \operatorname{ch}[(2m-2j+1)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \int \operatorname{ch}^{2m}(ax+b) \cos^{2n+1}(cx+d) dx = \\
 = \frac{\binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{\binom{2n+1}{k}}{(2n-2k+1)c} \sin[(2n-2k+1)(cx+d)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{\binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k+1)(cx+d)] + \\
 + (2n-2k+1)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 20. \int \operatorname{ch}^{2m+1}(ax+b) \cos^{2n+1}(cx+d) dx = \\
 = \frac{1}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{\binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j+1) a \operatorname{sh} [(2m-2j+1)(ax+b)] \cos [(2n-2k+1)(cx+d)] + \\
 + (2n-2k+1) c \operatorname{ch} [(2m-2j+1)(ax+b)] \sin [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

1.5.53. Интегралы вида $\int x^p \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx$.

$$1. \int x^p \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \mp \frac{x^{p-1}}{2} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\} \pm \frac{p-1}{2} \int x^{p-2} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\} dx.$$

$$\begin{aligned}
 2. \int x^n \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = (n-1)!! \left[\pm \sum_{k=0}^{[n/4]} (-1)^k \left(\frac{x^{n-4k+3}}{2^{2k-1} (n-4k+3)!!} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\} \mp \right. \right. \\
 \left. \left. \mp \frac{x^{n-4k+1}}{2^{2k} (n-4k+1)!!} \right) + \frac{(-1)^{[n/4]}}{2^{2[n/4]} (n-4[n/4]-1)!!} \int x^{n-4[n/4]} \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx \right].
 \end{aligned}$$

$$3. \int \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \sqrt{\frac{\pi}{2}} \left\{ \begin{smallmatrix} S(x^2) \\ C(x^2) \end{smallmatrix} \right\}. \quad 4. \int x \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \mp \frac{1}{2} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\}.$$

$$5. \int x^2 \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \mp \frac{x}{2} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\} \pm \frac{1}{2} \sqrt{\frac{\pi}{2}} \left\{ \begin{smallmatrix} C(x^2) \\ S(x^2) \end{smallmatrix} \right\}.$$

$$6. \int x^3 \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \mp \frac{x^2}{2} \left\{ \begin{smallmatrix} \cos x^2 \\ \sin x^2 \end{smallmatrix} \right\} + \frac{1}{2} \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\}.$$

$$7. \int \frac{1}{x} \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = \frac{1}{2} \left\{ \begin{smallmatrix} \operatorname{Si}(x^2) \\ \operatorname{Ci}(x^2) \end{smallmatrix} \right\}.$$

$$8. \int \frac{1}{x^2} \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} dx = -\frac{1}{x} \left\{ \begin{smallmatrix} \sin x^2 \\ \cos x^2 \end{smallmatrix} \right\} \pm \sqrt{2\pi} \left\{ \begin{smallmatrix} C(x^2) \\ S(x^2) \end{smallmatrix} \right\}.$$

$$\begin{aligned}
 9. \int \left\{ \begin{smallmatrix} \sin(ax^2+bx+c) \\ \cos(ax^2+bx+c) \end{smallmatrix} \right\} dx = \sqrt{\frac{\pi}{2a}} \left[\cos \frac{4ac-b^2}{4a} \left\{ \begin{smallmatrix} S[(2ax+b)^2/(4a)] \\ C[(2ax+b)^2/(4a)] \end{smallmatrix} \right\} \pm \right. \\
 \left. \pm \sin \frac{4ac-b^2}{4a} \left\{ \begin{smallmatrix} C[(2ax+b)^2/(4a)] \\ S[(2ax+b)^2/(4a)] \end{smallmatrix} \right\} \right] \quad [a > 0].
 \end{aligned}$$

1.6. ЛОГАРИФИЧЕСКАЯ ФУНКЦИЯ

1.6.1. Интегралы вида $\int x^p \ln^q x dx$.

$$1. \int f(x) \ln g(x) dx = F(x) \ln g(x) - \int \frac{F(x) g'(x)}{g(x)} dx \quad [F(x) = \int f(x) dx].$$

$$2. \int \ln g(x) dx = x \ln g(x) - \int \frac{x g'(x)}{g(x)} dx.$$

$$3. \int f(x) \ln^n x dx = F(x) \ln^n x - n \int \frac{F(x)}{x} \ln^{n-1} x dx \quad [F(x) = \int f(x) dx].$$

$$4. \int x^p \ln^q x dx = \frac{x^{p+1} \ln^{q+1} x}{q+1} - \frac{p+1}{q+1} \int x^p \ln^{q+1} x dx.$$

$$5. \quad = \frac{x^{p+1} \ln^q x}{p+1} - \frac{q}{p+1} \int x^p \ln^{q-1} x dx. \quad [p, q \neq -1].$$

$$6. \quad \int x^p \ln^n x dx = \frac{x^{p+1}}{n+1} \sum_{k=0}^n (-1)^k (n+1) n (n-1) \dots (n-k+1) \frac{\ln^{n-k} x}{(p+1)^{k+1}}.$$

$$7. \quad = \frac{\ln^{n+1} x}{n+1} \quad [p = -1].$$

$$8. \quad \int \frac{x^p}{\ln x} dx = \operatorname{li}(x^{p+1}) \quad [p \neq -1].$$

$$9. \quad = \ln |\ln x| + \sum_{k=1}^{\infty} \frac{(p+1)^k}{k! k} \ln^k x.$$

$$10. \quad \int \frac{\ln^q x}{x^p} dx = -\frac{\ln^q x}{(p-1)x^{p-1}} + \frac{q}{p-1} \int \frac{\ln^{q-1} x}{x^p} dx \quad [p, -q \neq 1].$$

$$11. \quad \int \frac{\ln^q x}{x} dx = \frac{\ln^{q+1} x}{q+1}.$$

$$12. \quad \int \frac{\ln^n x}{x^m} dx = \frac{-1}{(n+1)x^{m-1}} \sum_{k=0}^n \frac{(n+1)n \dots (n-k+1)}{(m-1)^{k+1}} \ln^{n-k} x \quad [m \neq 1].$$

$$13. \quad \int \ln^q x dx = x \ln^q x - q \int \ln^{q-1} x dx \quad [q \neq -1].$$

$$14. \quad \int \ln^n x dx = x \sum_{k=0}^n (-1)^k \binom{n}{k} k! \ln^{n-k} x.$$

$$15. \quad \int \ln x dx = x \ln x - x.$$

$$16. \quad \int \ln^2 x dx = x \ln^2 x - 2x \ln x + 2x.$$

$$17. \quad \int \ln^3 x dx = x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x.$$

$$18. \quad \int x^p \ln x dx = x^{p+1} \left[\frac{\ln x}{p+1} - \frac{1}{(p+1)^2} \right].$$

$$19. \quad \int x^p \ln^2 x dx = x^{p+1} \left[\frac{\ln^2 x}{p+1} - \frac{2 \ln x}{(p+1)^2} + \frac{2}{(p+1)^3} \right].$$

$$20. \quad \int x^p \ln^3 x dx = x^{p+1} \left[\frac{\ln^3 x}{p+1} - \frac{3 \ln^2 x}{(p+1)^2} + \frac{6 \ln x}{(p+1)^3} - \frac{6}{(p+1)^4} \right].$$

$$21. \quad \int \frac{\ln x}{x} dx = \frac{\ln^2 x}{2}. \quad 22. \quad \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x}.$$

$$23. \quad \int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}. \quad 24. \quad \int \frac{\ln^2 x}{x} dx = \frac{\ln^3 x}{3}.$$

$$25. \quad \int \frac{\ln^2 x}{x^2} dx = -\frac{\ln^2 x}{x} - \frac{2 \ln x}{x} - \frac{2}{x}.$$

$$26. \quad \int \frac{\ln^2 x}{x^3} dx = -\frac{\ln^2 x}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2}.$$

1.6.2. Интегралы вида $\int \frac{x^p dx}{\ln^q x}$.

$$1. \int \frac{x^p dx}{\ln^q x} = \frac{-x^{p+1}}{(q-1) \ln^{q-1} x} + \frac{p+1}{q-1} \int \frac{x^p dx}{\ln^{q-1} x}.$$

$$2. \int \frac{x^m dx}{\ln^n x} = \frac{-x^{m+1}}{n} \sum_{k=1}^{n-1} \frac{(m+1)^{k-1}}{(n-1)(n-2)\dots(n-k) \ln^{n-k} x} + \frac{(m+1)^{n-1}}{(n-1)!} \operatorname{li}(x^{m+1}).$$

$$3. \int \frac{x^m dx}{\ln x} = \operatorname{li}(x^{m+1}).$$

$$4. \int \frac{dx}{x^p \ln^q x} = \frac{-1}{(q-1) x^{p-1} \ln^{q-1} x} - \frac{p-1}{q-1} \int \frac{dx}{x^p \ln^{q-1} x}.$$

$$5. \int \frac{dx}{x^m \ln^n x} = \frac{1}{nx^{m-1}} \sum_{k=1}^{n-1} \frac{(-1)^k (m-1)^{k-1}}{(n-1)(n-2)\dots(n-k) \ln^{n-k} x} +$$

$$+ (-1)^{n-1} \frac{(m-1)^{n-1}}{(n-1)!} \operatorname{li}\left(\frac{1}{x^{m-1}}\right) \quad [n > 1].$$

$$6. \int \frac{dx}{x^p \ln x} = \ln |\ln x| + \sum_{k=1}^{\infty} (-1)^k \frac{(p-1)^k}{k! k} \ln^k x.$$

$$7. \int \frac{dx}{x \ln x} = \ln |\ln x|. \quad 8. \int \frac{dx}{x \ln^q x} = -\frac{1}{(q-1) \ln^{q-1} x}.$$

1.6.3. Интегралы вида $\int (x+a)^q \ln x dx$.

$$1. \int (x+a)^m \ln x dx = \frac{1}{m+1} (x+a)^{m+1} \ln x - \frac{1}{m+1} \int \frac{(x+a)^{m+1}}{x} dx.$$

$$2. = \frac{1}{m+1} [(x+a)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^m \binom{m}{k} \frac{a^{m-k} x^{k+1}}{(k+1)^2}.$$

$$3. \int (x+a) \ln x dx = \frac{1}{2} (x^2 + 2ax) \ln x - ax - \frac{x^2}{4}.$$

$$4. \int (x+a)^2 \ln x dx = \frac{1}{3} (x^3 + 3ax^2 + 3a^2x) \ln x - \frac{x^3}{9} - \frac{ax^2}{2} - a^2x.$$

$$5. \int (x+a)^3 \ln x dx = \frac{1}{4} [(x+a)^4 - a^4] \ln x - \frac{x^4}{16} - \frac{ax^3}{3} - \frac{3a^2x^2}{4} - a^3x.$$

$$6. \int \frac{\ln^n x dx}{(x+a)^m} = \frac{x \ln^n x}{(m-1)a(x+a)^{m-1}} - \frac{n}{(m-1)a} \int \frac{\ln^{n-1} x dx}{(x+a)^{m-1}} +$$

$$+ \frac{m-2}{(m-1)a} \int \frac{\ln^n x dx}{(x+a)^{m-1}}.$$

$$7. \int \frac{\ln x dx}{(x+a)^n} =$$

$$= -\frac{\ln x}{(n-1)(x+a)^{n-1}} + \frac{1}{(n-1)a^{n-1}} \left[\ln \frac{x}{x+a} + \sum_{k=1}^{n-2} \frac{(-1)^k}{k} \binom{n-2}{k} \left(\frac{x}{x+a}\right)^k \right].$$

8. $\int \frac{\ln x dx}{x+a} = \ln x \cdot \ln \frac{x+a}{a} + \text{Li}_2\left(-\frac{x}{a}\right).$
9. $\int \frac{\ln x dx}{(x+a)^2} = -\frac{\ln x}{x+a} + \frac{1}{a} \ln \frac{x}{x+a}.$
10. $\int \frac{\ln x dx}{(x+a)^3} = -\frac{\ln x}{2(x+a)^2} + \frac{1}{2a(x+a)} + \frac{1}{2a^2} \ln \frac{x}{x+a}.$
11. $\int \frac{\ln x dx}{(x+a)^{1/2}} = 2 \left[(\ln x - 2) (x+a)^{1/2} + a^{1/2} \ln \frac{(x+a)^{1/2} + a^{1/2}}{(x+a)^{1/2} - a^{1/2}} \right] \quad [a > 0].$
12. $= 2 \left[(\ln x - 2) (x+a)^{1/2} + 2(-a)^{1/2} \arctg \left(\frac{x+a}{-a} \right)^{1/2} \right] \quad [a < 0].$

1.6.4. Интегралы вида $\int \frac{x^m \ln x}{(a^2 \pm x^2)^{n/2}} dx.$

Условие: $a > 0.$

1. $\int \frac{\ln x}{x^2+a^2} dx = \frac{1}{a} \ln x \cdot \arctg \frac{x}{a} - \frac{1}{a} \text{Im Li}_2 \left(\frac{ix}{a} \right).$
2. $\int_0^x \frac{x \ln x}{x^2+a^2} dx = \frac{1}{4} \text{Li}_2 \left(-\frac{x^2}{a^2} \right) + \frac{1}{2} \ln x \cdot \ln \left(1 + \frac{x^2}{a^2} \right).$
3. $\int \frac{x^2 \ln x}{x^2+a^2} dx = -\frac{a}{2} \text{Cl}_2 \left(2 \arctg \frac{x}{a} \right) -$
 $-\frac{a}{2} \text{Cl}_2 \left(\pi - 2 \arctg \frac{x}{a} \right) + x \ln a - a \ln a \cdot \arctg \frac{x}{a}.$
4. $\int \frac{\ln x}{x^2-a^2} dx = \frac{1}{2a} \left[\ln x \cdot \ln \left| \frac{x-a}{x+a} \right| + \text{Li}_2 \left(\frac{x}{a} \right) - \text{Li}_2 \left(-\frac{x}{a} \right) \right].$
5. $\int_0^x \frac{x \ln x}{x^2-a^2} dx = -\frac{1}{4} \text{Li}_2 \left(1 - \frac{x^2}{a^2} \right) + \frac{1}{2} \ln a \cdot \ln (a^2 - x^2) - \ln^2 a + \frac{\pi^2}{24}.$
6. $\int_0^x \frac{\ln x}{\sqrt{a^2-x^2}} dx = \ln \frac{a}{2} \arcsin \frac{x}{a} - \frac{1}{2} \text{Cl}_2 \left(2 \arcsin \frac{x}{a} \right).$
7. $\int_x^a \frac{\ln x}{\sqrt{a^2-x^2}} dx = \ln \frac{a}{2} \cdot \arccos \frac{x}{a} + \frac{1}{2} \text{Cl}_2 \left(\pi - 2 \arccos \frac{x}{a} \right).$
8. $\int \frac{\ln x}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} \text{Li}_2 \left[\pm \frac{1}{a^2} (x - \sqrt{x^2 \pm a^2})^2 \right] +$
 $+\frac{1}{2} \ln^2 \frac{x + \sqrt{x^2 \pm a^2}}{2a} + \ln a \cdot \ln \sqrt{x^2 \pm a^2}.$

1.6.5. Интегралы вида $\int x^p \ln(ax+b) dx.$

1. $\int f(x) \ln^m(ax+b) dx = \frac{1}{a} \int f\left(\frac{t-b}{a}\right) \ln^m t dt \quad [t=ax+b].$

2. $\int x^{\alpha-1} \ln(ax+b) dx = \frac{x^{\alpha}}{\alpha} \ln(ax+b) + \frac{ax^{\alpha+1}}{\alpha(\alpha+1)b} {}_2F_1\left(1, \alpha; \alpha+1; \frac{-ax}{b}\right).$
3. $\int x^m \ln(ax+b) dx = \frac{1}{m+1} \left(x^{m+1} + (-1)^m \frac{b^{m+1}}{a^{m+1}} \right) \ln(ax+b) +$

$$+ \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} b^{k-1}}{(m-k+2) a^{k-1}}.$$
4. $\int \ln(ax+b) dx = \frac{1}{a} (ax+b) \ln(ax+b) - x.$
5. $\int x \ln(ax+b) dx = \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax+b) - \frac{1}{2} \left(\frac{x^2}{2} - \frac{bx}{a} \right).$
6. $\int x^2 \ln(ax+b) dx = \frac{1}{3} \left(x^3 + \frac{b^3}{a^3} \right) \ln(ax+b) - \frac{1}{3} \left(\frac{x^3}{3} - \frac{bx^2}{2a} + \frac{b^2x}{a^2} \right).$
7. $\int \frac{\ln(ax+b)}{x^p} dx = -\frac{\ln(ax+b)}{(p-1)x^{p-1}} + \frac{a}{p-1} \int \frac{dx}{(ax+b)x^{p-1}}.$
8. $\int \frac{\ln(ax+b)}{x^m} dx =$

$$= \frac{-1}{m-1} \frac{\ln(ax+b)}{x^{m-1}} + \frac{1}{m-1} \left(\frac{-a}{b} \right)^{m-1} \left[\ln \frac{ax+b}{x} + \sum_{k=1}^{m-2} \frac{1}{k} \left(\frac{-b}{ax} \right)^k \right].$$
9. $\int \frac{\ln(ax+b)}{x} dx = \ln|b| \cdot \ln|x| - \text{Li}_2\left(\frac{-ax}{b}\right) \quad [ab \neq 0].$
10. $= \ln|x| \cdot \ln(ax+b) - a \int \frac{\ln|x| dx}{ax+b}.$
11. $= \ln|b| \ln|x| + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left(\frac{ax}{b} \right)^k \quad [|ax| < |b|].$
12. $= \frac{\ln^2|ax|}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \left(\frac{b}{ax} \right)^k \quad [|ax| > |b|].$
13. $\int \frac{\ln(ax+b)}{x^2} dx = \frac{a}{b} \ln|x| - \left(\frac{1}{x} + \frac{a}{b} \right) \ln(ax+b).$
14. $\int \frac{\ln(ax+b)}{x+c} dx = \ln(b-ac) \ln\left(\frac{x+c}{c}\right) - \text{Li}_2\left[\frac{a(x+c)}{ac-b}\right] + \text{Li}_2\left(\frac{ac}{ac-b}\right).$
15. $\int \frac{\ln(x+a)}{(x+b)^2+c^2} dx = \frac{1}{2c} \text{arctg} \frac{cx}{b^2+c^2+bx} \cdot \ln[(a-b)^2+c^2] -$

$$- \frac{1}{2c} \text{Cl}_2(2\theta+2\varphi) + \frac{1}{2c} \text{Cl}_2(2\theta_0+2\varphi) - \frac{1}{2c} \text{Cl}_2(\pi-2\theta) +$$

$$+ \frac{1}{2c} \text{Cl}_2(\pi-2\theta_0) \quad \left[\text{tg } \theta = \frac{x+b}{c}, \text{tg } \theta_0 = \frac{b}{c}, \text{tg } \varphi = \frac{a-b}{c} \right].$$

1.6.6. Интегралы вида $\int x^{\pm m} \ln \frac{x+a}{x-a} dx$.

$$1. \int x^p \ln \frac{x+a}{x-a} dx = \frac{1}{p+1} x^{p+1} \ln \frac{x+a}{x-a} + \frac{2a}{p+1} \int \frac{x^{p+1}}{x^2-a^2} dx.$$

$$2. \int x^m \ln \frac{x+a}{x-a} dx = \frac{1}{m+1} [x^{m+1} - (-a)^{m+1}] \ln(x+a) - \\ - \frac{1}{m+1} (x^{m+1} - a^{m+1}) \ln(x-a) + \frac{a^{m+1}}{m+1} \sum_{k=1}^{m+1} \frac{1 - (-1)^{m-k+1}}{k} \left(\frac{x}{a}\right)^k.$$

$$3. \int \ln \frac{x+a}{x-a} dx = (x+a) \ln(x+a) - (x-a) \ln(x-a) = x \ln \frac{x+a}{x-a} + a \ln(x^2-a^2).$$

$$4. \int x \ln \frac{x+a}{x-a} dx = \frac{1}{2} (x^2-a^2) \ln \frac{x+a}{x-a} + ax.$$

$$5. \int x^2 \ln \frac{x+a}{x-a} dx = \frac{1}{3} x^3 \ln \frac{x+a}{x-a} + \frac{1}{3} a^3 \ln(x^2-a^2) + \frac{ax^2}{3}.$$

$$6. \int x^3 \ln \frac{x+a}{x-a} dx = \frac{x^4-a^4}{4} \ln \frac{x+a}{x-a} + \frac{ax^3}{6} + \frac{a^3x}{2}.$$

$$7. \int \frac{1}{x^m} \ln \frac{x+a}{x-a} dx = \frac{1}{(m-1)x^{m-1}} \ln \frac{x-a}{x+a} + \\ + \frac{1}{(m-1)a^{m-1}} \left[(-1)^{m-1} \ln \frac{x+a}{x} - \ln \frac{x-a}{x} \sum_{k=1}^{m-2} \frac{1 - (-1)^{m-k-1}}{k} \left(\frac{a}{x}\right)^k \right].$$

$$8. \int \frac{1}{x} \ln \frac{a+x}{a-x} dx = \text{Li}_2\left(\frac{x}{a}\right) - \text{Li}_2\left(-\frac{x}{a}\right) \quad [a \neq 0].$$

$$9. = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{x}{a}\right)^{2k+1}.$$

$$10. \int \frac{1}{x^2} \ln \frac{x+a}{x-a} dx = \frac{1}{x} \ln \frac{x-a}{x+a} - \frac{1}{a} \ln \frac{|x^2-a^2|}{x^2}.$$

1.6.7. Интегралы вида $\int x^{\pm m} \ln(x^2 \pm a^2) dx$.

$$1. \int x^{2n} \ln |x^2 \pm a^2| dx = \frac{1}{2n+1} \left[x^{2n+1} \ln |x^2 \pm a^2| - \right. \\ \left. - 2 \sum_{k=0}^n \frac{(\mp 1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right] \pm \left\{ \begin{array}{l} (-1)^n 2a^{2n+1} \text{arctg} \frac{x}{a} \\ a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$2. \int x^{2n+1} \ln |x^2 \pm a^2| dx = \\ = \frac{1}{2n+1} \left\{ [x^{2n+2} - (\mp 1)^{n+1} a^{2n+2}] \ln |x^2 \pm a^2| - \sum_{k=1}^{n+1} \frac{(\mp 1)^{n-k+1}}{k} a^{2n-2k+2} x^{2k} \right\}.$$

$$3. \int \ln |x^2 \pm a^2| dx = x \ln |x^2 \pm a^2| - 2x + \left\{ \begin{array}{l} 2a \operatorname{arctg} \frac{x}{a} \\ a \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$4. \int x \ln |x^2 \pm a^2| dx = \frac{1}{2} [(x^2 \pm a^2) \ln |x^2 \pm a^2| - x^2].$$

$$5. \int x^2 \ln |x^2 \pm a^2| dx = \frac{1}{3} x^3 \ln |x^2 \pm a^2| - \frac{2}{9} x^3 \pm \frac{2}{3} a^2 x + \frac{1}{3} \left\{ \begin{array}{l} 2a^3 \operatorname{arctg} \frac{x}{a} \\ a^3 \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$6. \int x^3 \ln |x^2 \pm a^2| dx = \frac{1}{4} \left[(x^4 - a^4) \ln |x^2 \pm a^2| - \frac{x^4}{2} \pm a^2 x^2 \right].$$

$$7. \int \frac{\ln |x^2 \pm a^2|}{x^{2m}} dx = \frac{-\ln |x^2 \pm a^2|}{(2m-1)x^{2m-1}} + \\ + \frac{(-1)^{m+1}}{(2m-1)a^{2m-1}} \sum_{k=0}^{m-2} \frac{(-1)^k}{2k+1} \left(\frac{a}{x}\right)^{2k+1} - \frac{(-1)^m}{(2m-1)a^{2m-1}} \left\{ \begin{array}{l} 2 \operatorname{arctg} \frac{x}{a} \\ \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$8. \int \frac{\ln |x^2 \pm a^2|}{x^{2m+1}} dx = \\ = \frac{-\ln |x^2 \pm a^2|}{2mx^{2m}} + \frac{(-1)^m}{2ma^{2m}} \left[\ln \frac{|x^2 \pm a^2|}{x^2} + \sum_{k=1}^{m-1} \frac{(-1)^k}{k} \left(\frac{a}{x}\right)^{2k} \right].$$

$$9. \int \frac{\ln |x^2 \pm a^2|}{x} dx = \ln^2 x + \frac{1}{2} \operatorname{Li}_2 \left(\mp \frac{a^2}{x^2} \right) \quad \left[\left\{ \begin{array}{l} a \neq 0 \\ x^2 > a^2 \end{array} \right\} \right].$$

$$10. = \ln a^2 \cdot \ln x - \frac{1}{2} \operatorname{Li}_2 \left(\mp \frac{x^2}{a^2} \right) \quad \left[\left\{ \begin{array}{l} a \neq 0 \\ x^2 < a^2 \end{array} \right\} \right].$$

$$11. \int \frac{\ln |x^2 \pm a^2|}{x^2} dx = -\frac{1}{x} \ln |x^2 \pm a^2| \pm \frac{1}{a} \left\{ \begin{array}{l} 2 \operatorname{arctg} \frac{x}{a} \\ \ln \left| \frac{x+a}{x-a} \right| \end{array} \right\}.$$

$$12. \int \frac{\ln |x^2 \pm a^2|}{x^3} dx = -\frac{1}{2a^2} \frac{a^2 \pm x^2}{x^2} \ln |x^2 \pm a^2| \pm \frac{1}{a^2} \ln x.$$

$$13. \int \frac{\ln(1+x^2)}{1-x} dx = \frac{1}{4} \operatorname{Li}_2(-x^2) + \frac{1}{2} \operatorname{Li}_2\left(\frac{2x}{1+x^2}\right) - \operatorname{Li}_2(x) + \\ + \frac{1}{4} \ln^2(1+x^2) - \ln(1-x) \ln(1+x^2).$$

$$14. \int \frac{\ln(x^2 + a^2)}{x^2 + a^2} dx = \\ = \frac{2 \ln a}{a} \operatorname{arctg} \frac{x}{a} + \frac{1}{2a} \ln 4 (x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{1}{a} \operatorname{Im} \operatorname{Li}_2 \left(\frac{a+ix}{2a} \right).$$

$$15. \int \frac{(x+a) \ln [(x+a)^2 + b^2]}{(x+a)^2 + c^2} dx = \\ = \frac{1}{2} \operatorname{Li}_2 \left[\frac{c^2 - b^2}{(x+a)^2 + c^2} \right] + \frac{1}{4} \ln \left[\frac{(x+a)^2 + c^2}{c^2} \right] \ln [c^2 (x+a)^2 + c^4].$$

$$\begin{aligned}
 16. \int \frac{\ln [(x+a)^2 + b^2]}{(x+c)^2 + d^2} dx &= \frac{\theta}{d} \ln 2 [(a-c)^2 + b^2 + d^2] - \frac{1}{d} \text{Cl}_2 (\pi - 2\theta) + \\
 &+ \frac{(\theta - \varphi)}{d} \ln \frac{1 - \sqrt{1 - \varepsilon^2}}{2} - \frac{\eta}{d} \ln \left(\frac{1 - \sqrt{1 - \varepsilon^2}}{\varepsilon} \right) + \frac{1}{2d} \text{Cl}_2 (2\eta) + \\
 &+ \frac{1}{2d} \text{Cl}_2 (4\theta - 4\varphi - 2\eta) - \frac{1}{2d} \text{Cl}_2 (4\theta - 4\varphi) \\
 &\left[\varepsilon = \sqrt{1 - \left[\frac{bd}{(a-c)^2 + b^2 + d^2} \right]^2}, \quad \text{tg } 2\varphi = \frac{2d(a-c)}{b^2 + (a-c)^2 - d^2}, \right. \\
 &\left. \text{tg } \theta = \frac{x+c}{d}, \quad \text{tg } \eta = \frac{\varepsilon \sin (2\theta - 2\varphi)}{1 + \varepsilon \cos (2\theta - 2\varphi) - \sqrt{1 - \varepsilon^2}} \right].
 \end{aligned}$$

1.6.8. Интегралы вида $\int x^{\pm m} \ln (x + \sqrt{x^2 \pm a^2}) dx$.

$$\begin{aligned}
 1. \int f(x) \ln (x + \sqrt{x^2 + a^2}) dx &= \\
 &= -2a \ln a \int f\left(\frac{2at}{t^2 - 1}\right) \frac{t^2 + 1}{(t^2 - 1)^2} dt - 2a \int f\left(\frac{2a(y-1)}{y(y-2)}\right) \frac{y^2 - 2y + 2}{y^2(y-2)^2} \ln y dy + \\
 &+ 2a \int f\left(\frac{2a(z+1)}{z(z+2)}\right) \frac{z^2 + 2z + 2}{z^2(z+2)^2} \ln z dz \\
 &\left[x = \frac{2at}{t^2 - 1}; \quad t = y - 1 = z + 1 = \frac{a + \sqrt{x^2 + a^2}}{x}; \quad a > 0 \right].
 \end{aligned}$$

$$\begin{aligned}
 2. \int f(x) \ln (x + \sqrt{x^2 - a^2}) dx &= \\
 &= -4a \ln a \int f\left(\frac{a(t^2 + 1)}{t^2 - 1}\right) \frac{t}{(t^2 - 1)^2} dt - 4a \int f\left(\frac{a(y^2 - 2y + 2)}{y(y-2)}\right) \frac{y-1}{y^2(y-2)^2} \ln y dy + \\
 &+ 4a \int f\left(\frac{a(z^2 + 2z + 2)}{z(z+2)}\right) \frac{z+1}{z^2(z+2)^2} \ln z dz \\
 &\left[x = a \frac{t^2 + 1}{t^2 - 1}; \quad t = y - 1 = z + 1 = \sqrt{\frac{x+a}{x-a}}; \quad a > 0 \right].
 \end{aligned}$$

$$3. \int x^p \ln (x + \sqrt{x^2 \pm a^2}) dx = \frac{x^{p+1}}{p+1} \ln (x + \sqrt{x^2 \pm a^2}) - \frac{1}{p+1} \int \frac{x^{p+1} dx}{\sqrt{x^2 \pm a^2}}.$$

$$4. \int \ln (x + \sqrt{x^2 \pm a^2}) dx = x \ln (x + \sqrt{x^2 \pm a^2}) - \sqrt{x^2 \pm a^2}.$$

$$5. \int x \ln (x + \sqrt{x^2 \pm a^2}) dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4} \right) \ln (x + \sqrt{x^2 \pm a^2}) - \frac{x}{4} \sqrt{x^2 \pm a^2}.$$

$$\begin{aligned}
 6. \int x^2 \ln (x + \sqrt{x^2 \pm a^2}) dx &= \\
 &= \frac{x^3}{3} \ln (x + \sqrt{x^2 \pm a^2}) - \frac{(x^2 \pm a^2)^{3/2}}{9} \pm \frac{a^2}{3} \sqrt{x^2 \pm a^2}.
 \end{aligned}$$

$$\begin{aligned}
 7. \int x^3 \ln (x + \sqrt{x^2 \pm a^2}) dx &= \left(\frac{x^4}{4} - \frac{3a^4}{32} \right) \ln (x + \sqrt{x^2 \pm a^2}) - \\
 &- \frac{x^3}{16} \sqrt{x^2 \pm a^2} \pm \frac{3}{32} a^2 x \sqrt{x^2 \pm a^2}.
 \end{aligned}$$

$$8. \int \frac{\ln (x + \sqrt{x^2 \pm a^2})}{x^p} dx = -\frac{\ln (x + \sqrt{x^2 \pm a^2})}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{dx}{x^{p-1} \sqrt{x^2 \pm a^2}}.$$

9.
$$\int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^{2n}} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{(2n-1)x^{2n-1}} \mp \frac{\sqrt{x^2 \pm a^2}}{(2n-1)(2n-2)} \times$$

$$\times \left[\frac{1}{a^2 x^{2n-2}} \pm \frac{1}{a^{2n}} \sum_{k=1}^{n-2} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{2^k (n-2)(n-3)\dots(n-k-1)} \left(\frac{a}{x}\right)^{2n-2k} \right] +$$

$$+ \frac{(-1)^n}{(2n-1)a^{2n-1}} \frac{(2n-3)!!}{(2n-2)!!} \left\{ \ln \left| \frac{a + \sqrt{x^2 \pm a^2}}{x} \right| \right.$$

$$\left. \mp \operatorname{arctg} \frac{\sqrt{x^2 - a^2}}{a} \right\}.$$
10.
$$\int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^{2n+1}} dx = -\frac{1}{2nx^{2n}} \ln(x + \sqrt{x^2 \pm a^2}) \mp \frac{\sqrt{x^2 \pm a^2}}{(2n-1)2n} \times$$

$$\times \left[\frac{1}{a^2 x^{2n-1}} + \frac{1}{a^{2n+1}} \sum_{k=1}^{n-1} (-1)^k \frac{2^k (n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \left(\frac{a}{x}\right)^{2n-2k-1} \right].$$
11.
$$\int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x} dx = \ln|a| \cdot \ln|x| \pm \frac{1}{2} \ln^2 \left| \frac{2x}{a} \right| \pm$$

$$\pm \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k-2)!! (2k)^3} \left(\frac{a}{x}\right)^{2k} \quad [\pm x/a > 1].$$
12.
$$= \ln|a| \cdot \ln|x| + \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{(2k)!! (2k+1)^3} \left(\frac{x}{a}\right)^{2k+1} \quad [|x| < |a|].$$
13.
$$\int \frac{\ln(x + \sqrt{x^2 - a^2})}{x} dx = \ln|a| \cdot \ln|x| + \frac{1}{2} \ln^2 \left| \frac{2x}{a} \right| +$$

$$+ \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k-2)!! (2k)^3} \left(\frac{a}{x}\right)^{2k} \quad [|x| > |a|].$$
14.
$$\int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^2} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{x} \mp \frac{1}{a} \left\{ \ln \left| \frac{a + \sqrt{x^2 \pm a^2}}{x} \right| \right.$$

$$\left. \mp \operatorname{arctg} \frac{\sqrt{x^2 - a^2}}{a} \right\}.$$
15.
$$\int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^3} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{2x^2} \mp \frac{\sqrt{x^2 \pm a^2}}{2a^2 x}.$$

1.6.9. Интегралы вида $\int \frac{x^p}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx,$

$\int \ln^p(x + \sqrt{x^2 \pm 1}) dx.$

1.
$$\int \frac{x^p}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \frac{x^{p-1}}{p} \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) -$$

$$- \frac{x^p}{p^2} \mp \frac{p-1}{p} \int \frac{x^{p-2}}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx.$$

$$2. \int \frac{1}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \frac{1}{2} \ln^2(x + \sqrt{x^2 \pm 1}).$$

$$3. \int \frac{x}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) - x.$$

$$4. \int \ln^p(x + \sqrt{x^2 \pm 1}) dx = x \ln^p(x + \sqrt{x^2 \pm 1}) - \\ - p \sqrt{x^2 \pm 1} \ln^{p-1}(x + \sqrt{x^2 \pm 1}) + p(p-1) \int \ln^{p-2}(x + \sqrt{x^2 \pm 1}) dx.$$

$$5. \int \ln^n(x + \sqrt{x^2 \pm 1}) dx = \\ = \frac{1}{n+1} \sum_{k=0}^{[n/2]} (n+1)n \dots (n-2k+1) x \ln^{n-2k}(x + \sqrt{x^2 \pm 1}) - \\ - \sum_{k=0}^{[n/2]} n(n-1) \dots (n-2k) \sqrt{x^2 \pm 1} \ln^{n-2k-1}(x + \sqrt{x^2 \pm 1}).$$

$$6. \int \ln^2(x + \sqrt{x^2 \pm 1}) dx = x \ln^2(x + \sqrt{x^2 \pm 1}) - 2 \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) + 2x.$$

1.6.10. Интегралы вида $\int f(x, \ln x, e^{ax}, \sin x, \cos x, \dots) dx$.

$$1. \int x^{\alpha-1} e^{-ax} \ln x dx = \\ = a^{-\alpha} \gamma(\alpha, ax) \ln x - \alpha^{-2} x^{\alpha} {}_2F_2(\alpha, \alpha; \alpha+1, \alpha+1; -ax) \quad [a, \operatorname{Re} \alpha > 0].$$

$$2. \int_0^x e^{-ax} \ln x dx = \frac{1}{a} [\operatorname{Ei}(-ax) - e^{-ax} \ln x - \ln a - C].$$

$$3. \int_x^{\infty} e^{-ax} \ln x dx = \frac{1}{a} [e^{-ax} \ln x - \operatorname{Ei}(-ax)] \quad [\operatorname{Re} \alpha > 0].$$

$$4. \int_x^{\infty} \frac{1}{x} e^{-ax} \ln x dx = \frac{1}{2} [(C + \ln ax)^2 + \zeta(2) - 2 \ln x \operatorname{Ei}(-ax)] + \sum_{k=0}^{\infty} \frac{(-ax)^{k+1}}{k! (k+1)^2}.$$

$$5. \int_0^x e^{-ax} \ln(x+b) dx = \\ = \frac{1}{a} [\ln b - e^{-ax} \ln(x+b) - e^{ab} \operatorname{Ei}(-ab) + e^{ab} \operatorname{Ei}(-ax-ab)].$$

$$6. \int_0^x e^{-ax} \ln(b-x) dx = \\ = \frac{1}{a} [\ln b - e^{-ax} \ln(b-x) + e^{-ab} \operatorname{Ei}(ab-ax) - e^{-ab} \operatorname{Ei}(ab)] \quad [b > a].$$

$$7. \int_0^x \ln(b^2 - x^2) e^{-ax} dx = \frac{1}{a} \{2 \ln b - e^{-ax} \ln(b^2 - x^2) - \\ - e^{-ab} [\operatorname{Ei}(ab) - \operatorname{Ei}(ab-ax)] - e^{ab} [\operatorname{Ei}(-ab) - \operatorname{Ei}(-ab-ax)]\}.$$

$$8. \int_0^x \ln x \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} dx = \pm \frac{1}{a} \left[\begin{Bmatrix} \text{ci}(ax) \\ \text{si}(ax) \end{Bmatrix} - \ln x \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} \right] - \frac{1}{a} \begin{Bmatrix} \ln a + \mathbf{C} \\ \pi/2 \end{Bmatrix}.$$

$$9. \int_0^x \ln(b-x) \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} dx = \\ = \mp \frac{1}{a} \ln(b-x) \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} \pm \frac{1}{a} [\text{ci}(ab-ax) - \text{ci}(ab)] \begin{Bmatrix} \cos ab \\ \sin ab \end{Bmatrix} + \\ + \frac{1}{a} [\text{Si}(ab-ax) - \text{Si}(ab)] \begin{Bmatrix} \sin ab \\ \cos ab \end{Bmatrix} + \frac{1}{a} \begin{Bmatrix} \ln b \\ 0 \end{Bmatrix}.$$

$$10. \int_0^x \frac{1}{x} \ln(1+ax^n) dx = -\frac{1}{n} \text{Li}_2(-ax^n).$$

$$11. \int_0^x \begin{Bmatrix} \ln \text{sh } x \\ \ln \text{ch } x \end{Bmatrix} dx = \frac{1}{2} \text{Li}_2(\pm e^{2x}) - x \ln 2 + \frac{x^2}{2} \mp \frac{\pi^2}{24} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}.$$

$$12. \int \ln(\sin x) dx = -\frac{i}{2} \text{Li}_2(e^{-2ix}) - x \ln 2 + \frac{i}{2} \left(\frac{\pi}{2} - x \right)^2.$$

$$13. = x \ln x - x + \sum_{k=1}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)! 4k} (2x)^{2k+1} \quad [|x| < \pi].$$

$$14. \int \ln(\cos x) dx = \frac{i}{2} \text{Li}_2(-e^{2ix}) - x \ln 2 - \frac{i}{2} x^2.$$

$$15. = \sum_{k=1}^{\infty} (-1)^k \frac{(2^{2k}-1) B_{2k}}{(2k+1)! 4k} (2x)^{2k+1} \quad [|x| < \pi/2].$$

$$16. \int_0^x \begin{Bmatrix} \ln \sin x \\ \ln \cos x \end{Bmatrix} dx = -x \ln 2 \mp \frac{1}{2} \begin{Bmatrix} \text{Cl}_2(2x) \\ \text{Cl}_2(\pi-2x) \end{Bmatrix}.$$

$$17. \int_a^x \ln(\cos a - \cos x) dx = \text{Cl}_2(2a) - \text{Cl}_2(x+a) - \text{Cl}_2(x-a) - (x-a) \ln 2.$$

$$18. \int_a^x \ln(\sin a + \sin x) dx = -i \text{Li}_2(e^{-i(x+a)}) - i \text{Li}_2(e^{-i(x-a+\pi)}) + \\ + i \text{Li}_2(e^{-ia}) + i \text{Li}_2(e^{i(a-\pi)}) - x \ln 2 + \frac{i}{2} x(x-\pi).$$

$$19. \int_0^x \ln(\cos a + \cos x) dx = \text{Cl}_2(\pi+a-x) + \text{Cl}_2(\pi-a-x) - x \ln 2.$$

$$20. \int_0^x \ln(\text{tg } x + \text{tg } a) dx = \\ = -x \ln \cos a - \frac{1}{2} \text{Cl}_2(2x+2a) + \frac{1}{2} \text{Cl}_2(2a) - \frac{1}{2} \text{Cl}_2(\pi-2x).$$

$$21. \int_0^x \ln(1 + \sin a \cos x) dx = x \ln \sin^2 \frac{a}{2} - \eta \ln \operatorname{tg}^2 \frac{a}{2} - \\ - \operatorname{Cl}_2(2x) + \operatorname{Cl}_2(2x - 2\eta) + \operatorname{Cl}_2(2\eta) \quad [\operatorname{tg} \eta = \sin x / (\operatorname{tg}(a/2) + \cos x)].$$

$$22. \int_0^x \ln(1 - 2a \cos x + a^2) dx = \operatorname{Cl}_2(2x + 2\omega) - \operatorname{Cl}_2(2x) - \operatorname{Cl}_2(2\omega) - 2\omega \ln a \\ [\operatorname{tg} \omega = a \sin x / (1 - a \cos x)].$$

$$23. \int_0^x \ln(a \cos x + b \sin x) dx = \\ = x \ln \frac{\sqrt{a^2 + b^2}}{2} - \frac{i}{2} [\operatorname{Li}_2(-e^{2i(\varphi-x)}) - \operatorname{Li}_2(-e^{2i\varphi}) - x(x-2\varphi)] \\ [\operatorname{tg} \varphi = b/a; |\varphi| \leq \pi/2; |\varphi - x| \leq \pi/2].$$

$$24. \int_0^x \ln(1 - a^2 \sin^2 x) dx = (\pi - 2\theta) \ln[(1 + \sqrt{1 - a^2})/a] + 2x \ln(a/2) - \\ - \frac{\pi}{2} \ln 2 + L(\theta + x) - L(\theta - x) + L(\pi/2 - 2x) \\ [|x| < \pi/2; \operatorname{ctg} \theta = \operatorname{tg} x \sqrt{1 - a^2}; |a| < 1].$$

$$25. = x \ln a - L[\pi/2 + x - \arcsin(1/a)] + L[\pi/2 - x - \arcsin(1/a)] \\ [|x| < \pi/2; a|\sin x| < 1; a > 0].$$

$$26. \int_0^x \frac{\sin 2x \ln \operatorname{tg} x}{1 - a^2 \sin^2 2x} dx = \\ = -\frac{1}{4a\sqrt{1-a^2}} \left[\frac{\pi}{2} \ln 2 + L(\theta - \arccos a) - L(\theta + \arccos a) - L(\pi/2 - 2 \arccos a) \right] \\ [\operatorname{tg} \theta = a \cos 2x / \sqrt{1 - a^2}; 0 < x < \pi/2; |a| < 1].$$

$$27. \int \left\{ \frac{\sin \ln x}{\cos \ln x} \right\} dx = \frac{x}{2} (\sin \ln x \mp \cos \ln x).$$

$$28. \int \frac{dx}{(\ln x + a)^n} = -nx \sum_{k=1}^{n-1} \frac{1}{n(n-1) \dots (n-k+1) (\ln x + a)^{n-k}} + \frac{e^{-a}}{(n-1)!} \operatorname{li}(xe^a).$$

$$29. \int \frac{dx}{\ln x + a} = e^{-a} \operatorname{li}(xe^a).$$

1.6.11. Интегралы вида $\int \frac{\ln(ax+b) \ln(cx+d)}{fx+g} dx.$

$$1. \int_0^x \frac{\ln^2(1-x)}{x} dx = \\ = \ln x \ln^2(1-x) + 2 \ln(1-x) \operatorname{Li}_2(1-x) - 2 \operatorname{Li}_3(1-x) + 2 \operatorname{Li}_3(1).$$

$$2. \int_0^x \frac{\ln^2(1+x)}{x} dx = \ln x \ln^2(1+x) - \frac{2}{3} \ln^3(1+x) - \\ - 2 \ln(1+x) \operatorname{Li}_2\left(\frac{1}{1+x}\right) - 2 \operatorname{Li}_3\left(\frac{1}{1+x}\right) + 2 \operatorname{Li}_3(1).$$

$$3. \int_0^x \frac{\ln x \ln(1-x)}{x} dx = \operatorname{Li}_3(x) - \ln x \operatorname{Li}_2(x).$$

$$4. \int_1^x \frac{\ln x \ln(x-1)}{x} dx = \frac{1}{3} \ln^3 x + \ln x \operatorname{Li}_2(1/x) - 2 \operatorname{Li}_3(1/x) + 2 \operatorname{Li}_3(1).$$

$$5. \int_0^x \frac{\ln^2 x \ln(1-x)}{x} dx = -2 \operatorname{Li}_4(x) + 2 \ln x \operatorname{Li}_3 x - \ln^2 x \operatorname{Li}_2(x).$$

$$6. \int_0^x \frac{\ln^3 x}{1-x} dx = -6 \operatorname{Li}_4(x) + 6 \ln x \operatorname{Li}_3(x) - 3 \ln^2 x \operatorname{Li}_2(x) - \ln^3 x \ln(1-x).$$

$$7. \int_0^x \frac{\ln^2 x \ln(1-x)}{1-x} dx = 2 \left[\operatorname{Li}_4(1-x) - \operatorname{Li}_4(x) - \operatorname{Li}_4\left(\frac{-x}{1-x}\right) - \operatorname{Li}_4(1) \right] + \\ + 2 \left[\ln(1-x) \operatorname{Li}_3 x - \ln x \operatorname{Li}_3(1-x) + \operatorname{Li}_3(1) \ln\left(\frac{x}{1-x}\right) \right] + \\ + 2 \ln x \ln(1-x) \operatorname{Li}_2(1-x) + \\ + \frac{1}{12} \ln^3(1-x) [6 \ln^2 x + 4 \ln x \ln(1-x) - \ln^2(1-x) - 2\pi^2].$$

$$8. \int_0^x \frac{\ln(1-x) \ln(1-ax)}{x} dx = \operatorname{Li}_3\left(\frac{1-ax}{1-x}\right) - \operatorname{Li}_3\left(\frac{1-ax}{a(1-x)}\right) - \operatorname{Li}_3(1-x) - \\ - \operatorname{Li}_3(1-ax) + \operatorname{Li}_3(1/a) + \operatorname{Li}_3(1) + \ln(1-ax) [\operatorname{Li}_2(1/a) - \operatorname{Li}_2(x)] + \\ + \ln(1-x) [\operatorname{Li}_2(1-ax) - \operatorname{Li}_2(1/a) + \pi^2/6] + \frac{1}{2} \ln a \ln^2(1-x).$$

$$9. \int \frac{\ln(ax+b) \ln(cx+d)}{x+g/f} dx = \ln \frac{f}{df-cg} \operatorname{Li}_2(y) + \ln \frac{f}{bf-ag} \operatorname{Li}_2(hy) + \\ + \ln \frac{f}{df-cg} \ln \frac{f}{bf-ag} \ln y + \int \ln(1-y) \ln(1-hy) \frac{dy}{y} \\ [h = (c/a)(bf-ag)/(df-cg); y = a(fx+g)/(ag-bf); ag-bf \neq 0; cg-df \neq 0].$$

$$10. = \ln \frac{af}{c(ag-bf)} \operatorname{Li}_2(y) + \ln \frac{af}{c(ag-bf)} \ln \frac{f}{bf-ag} \ln y - \\ - \frac{1}{2} \ln \frac{f}{ag-bf} \ln^2 y + \int \ln y \ln(1-y) \frac{dy}{y} \quad [ag-bf \neq 0; cg-df=0].$$

$$11. \quad = \ln \frac{cf}{a(cg-df)} \operatorname{Li}_2(z) + \ln \frac{af}{a(cg-df)} \ln \frac{f}{df-cg} \ln z - \\ - \frac{1}{2} \ln \frac{f}{df-cg} \ln^2 z + \int \ln z \ln(1-z) \frac{dz}{z} \\ \left[z = \frac{c(fx+g)}{cg-df}; ag-bf=0; cg-df \neq 0 \right].$$

$$12. \quad = \frac{1}{2} \ln \frac{c}{a} \ln^2 v + \frac{1}{3} \ln^3 v \quad [v = a(fx+g)/g; ag-bf=0; cg-df=0].$$

1.7. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.7.1. Введение.

Интегралы от обратных тригонометрических функций сводятся к интегралам от тригонометрических функций при помощи следующих формул:

$$1. \quad \int f\left(x, \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt \quad \left[x = \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}\right].$$

$$2. \quad \int f\left(x, \begin{Bmatrix} \operatorname{arctg} x \\ \operatorname{arccotg} x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-2} dt \quad \left[x = \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}\right].$$

$$3. \quad \int f\left(x, \sqrt{x^2+a^2}, \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix}\right) dx = \\ = \pm a \int f\left(a \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}, a \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-1}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-2} dt \quad \left[x = a \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}\right].$$

$$4. \quad \int f(x) \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix} dx = F(x) \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix} \mp a \int \frac{F(x)}{x^2+a^2} dx \\ [F(x) = \int f(x) dx].$$

$$5. \quad \int f\left(x, \sqrt{1-x^2}, \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}, \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt \\ \left[x = \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}\right].$$

$$6. \quad \int f(x) \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = F(x) \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \mp \int \frac{F(x)}{\sqrt{1-x^2}} dx \quad [F(x) = \int f(x) dx].$$

1.7.2. Интегралы вида $\int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n dx$.

$$1. \quad \int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n dx = x \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n \pm \\ \pm n \sqrt{a^2-x^2} \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-1} - n(n-1) \int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2} dx.$$

$$2. \quad = x \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2k)! \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2k} \mp \\ \mp \sqrt{a^2-x^2} \sum_{k=1}^{[(n+1)/2]} (-1)^k \binom{n}{2k-1} (2k-1)! \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2k+1}.$$

$$3. \quad \int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix} dx = x \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix} \pm \sqrt{a^2-x^2}.$$

$$4. \int \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}^2 dx = x \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}^2 \pm 2\sqrt{a^2-x^2} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} - 2x.$$

$$5. \int \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}^3 dx = x \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}^3 \pm \\ \pm 3\sqrt{a^2-x^2} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}^2 - 6x \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \mp 6\sqrt{a^2-x^2}.$$

1.7.3. Интегралы вида $\int x^{\pm n} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx$.

$$1. \int x^n \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \frac{x^{n+1}}{n+1} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \mp \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2-x^2}}.$$

$$2. \int x^{2n} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \frac{x^{2n+1}}{2n+1} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \\ \pm \frac{\sqrt{a^2-x^2}}{(2n+1)^2} \left[x^{2n} + a^{2n} \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \left(\frac{x}{a} \right)^{2n-2k-2} \right].$$

$$3. \int x^{2n+1} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \frac{x^{2n+2}}{2n+2} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \\ \pm \frac{\sqrt{a^2-x^2}}{(2n+2)^2} \left[x^{2n+1} + a^{2n+1} \sum_{k=1}^n \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2^k n(n-1) \dots (n-k+1)} \left(\frac{x}{a} \right)^{2n-2k+1} \right] - \\ - \frac{(2n+1)!!}{(2n+2) 2^{n+1} (n+1)!} a^{2n+2} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\}.$$

$$4. \int x \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \frac{x}{4} \sqrt{a^2-x^2}.$$

$$5. \int x^2 \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \frac{x^3}{3} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2-x^2}.$$

$$6. \int x^3 \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \left(\frac{x^4}{4} - \frac{3a^4}{32} \right) \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \frac{1}{32} (2x^3 + 3xa^2) \sqrt{a^2-x^2}.$$

$$7. \int \frac{1}{x^n} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = \\ = \frac{-1}{(n-1)x^{n-1}} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \pm \frac{1}{n-1} \int \frac{dx}{x^{n-1} \sqrt{a^2-x^2}}.$$

$$8. \int \frac{1}{x^{2n}} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} dx = -\frac{1}{(2n-1)x^{2n-1}} \left\{ \begin{matrix} \arcsin(x/a) \\ \arccos(x/a) \end{matrix} \right\} \mp \\ \mp \frac{\sqrt{a^2-x^2}}{(2n-1)(2n-2)} \left[\frac{1}{a^2 x^{2n-2}} + \frac{1}{a^{2n}} \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5) \dots (2n-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \times \right. \\ \left. \times \left(\frac{a}{x} \right)^{2n-2k-2} \right] \mp \frac{(2n-3)!!}{(2n-1) 2^{n-1} a^{2n-1} (n-1)!} \ln \left| \frac{a + \sqrt{a^2-x^2}}{x} \right| \quad [n \geq 2].$$

$$9. \int \frac{1}{x^{2n+1}} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} dx = -\frac{1}{2nx^{2n}} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} \mp$$

$$\mp \frac{\sqrt{a^2 - x^2}}{2n(2n-1)} \left[\frac{1}{a^2 x^{2n-1}} + \frac{1}{a^{2n+1}} \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \left(\frac{a}{x} \right)^{2n-2k-1} \right].$$

$$10. = -\frac{1}{2nx^{2n}} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} \mp$$

$$\mp \frac{1}{2na^{2n}} \sum_{k=0}^{n-1} \frac{1}{2n-2k-1} \binom{n-1}{k} \left(\frac{a^2 - x^2}{x^2} \right)^{n-k-1/2}.$$

$$11. \int \frac{1}{x} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} dx =$$

$$= \ln x \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} \pm \ln \frac{a^2 + a\sqrt{a^2 - x^2}}{2} \operatorname{arctg} \frac{a + \sqrt{a^2 - x^2}}{x} \mp$$

$$\mp 2 \operatorname{Im} \operatorname{Li}_2 \left(i \frac{a + \sqrt{a^2 - x^2}}{2x} \right) \pm 2 \operatorname{Im} \operatorname{Li}_2 \left(\frac{x + ia + i\sqrt{a^2 - x^2}}{2x} \right).$$

$$12. = \pm \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)^2} \left(\frac{x}{a} \right)^{2k+1} + \frac{\pi}{2} \left\{ \begin{matrix} 0 \\ \ln x \end{matrix} \right\}.$$

$$13. \int_0^x \frac{\arcsin x}{x} dx = \frac{1}{2} \operatorname{Cl}_2(2 \arcsin x) + \arcsin x \ln 2x.$$

$$14. \int \frac{1}{x^2} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} dx = -\frac{1}{x} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} \mp \frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|.$$

$$15. \int \frac{1}{x^3} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} dx = -\frac{1}{2x^2} \left\{ \arcsin(x/a) \right\} \left\{ \arccos(x/a) \right\} \mp \frac{1}{2a^2 x} \sqrt{a^2 - x^2}.$$

1.7.4. Интегралы вида $\int (1 \pm x)^{\pm n + 1/2} \left\{ \arcsin x \right\} \left\{ \arccos x \right\} dx.$

$$1. \int (1+x)^{n+1/2} \left\{ \arcsin x \right\} \left\{ \arccos x \right\} dx = \frac{2}{2n+3} (1+x)^{n+3/2} \left\{ \arcsin x \right\} \left\{ \arccos x \right\} \pm$$

$$\pm \frac{2^{n+3} \sqrt{1-x}}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(1-x)^k}{2^k (2k+1)}.$$

$$2. \int (1-x)^{n+1/2} \left\{ \arcsin x \right\} \left\{ \arccos x \right\} dx = -\frac{2}{2n+3} (1-x)^{n+3/2} \left\{ \arcsin x \right\} \left\{ \arccos x \right\} \pm$$

$$\pm \frac{2^{n+3} \sqrt{1+x}}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(1+x)^k}{2^k (2k+1)}.$$

$$\begin{aligned}
 3. \quad & \int \frac{1}{(1+x)^{n+1/2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = -\frac{2}{(2n-1)(1+x)^{n-1/2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \mp \frac{\sqrt{1-x}}{(2n-1)(2n-2)2^{n-3}(1+x)^{n-1}} \left[2^{n-2} + \right. \\
 & \quad \left. + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{(2n-4)(2n-6)\dots(2n-2k-2)} 2^{n-2-k}(1+x)^k \right] \pm \\
 & \quad \pm \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{1}{(2n-1)2^{n-3}\sqrt{2}} \ln \frac{\sqrt{2}-\sqrt{1-x}}{\sqrt{1+x}}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int \frac{1}{(1-x)^{n+1/2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = \frac{2}{(2n-1)(1-x)^{n-1/2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \mp \frac{\sqrt{1+x}}{(2n-1)(2n-2)2^{n-3}(1-x)^{n-1}} \left[2^{n-2} + \right. \\
 & \quad \left. + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{(2n-4)(2n-6)\dots(2n-2k-2)} 2^{n-2-k}(1-x)^k \right] \pm \\
 & \quad \pm \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{1}{(2n-1)2^{n-3}\sqrt{2}} \ln \frac{\sqrt{2}-\sqrt{1+x}}{\sqrt{1-x}}.
 \end{aligned}$$

$$5. \quad \int \frac{\arcsin x}{(ax+b)^2} dx = -\frac{\arcsin x}{a(ax+b)^2} - \frac{2}{a\sqrt{b^2-a^2}} \operatorname{arctg} \sqrt{\frac{(b-a)(1-x)}{(a+b)(1+x)}} \quad [a^2 < b^2].$$

$$6. \quad = -\frac{\arcsin x}{a(ax+b)} - \frac{1}{a\sqrt{a^2-b^2}} \ln \frac{\sqrt{(a+b)(1+x)} + \sqrt{(a-b)(1-x)}}{\sqrt{(a+b)(1+x)} - \sqrt{(a-b)(1-x)}} \quad [a^2 > b^2].$$

$$7. \quad \int \frac{x \arcsin x}{(1+cx^2)^2} dx = \frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{2c\sqrt{c+1}} \operatorname{arctg} \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \quad [c > -1].$$

$$8. \quad = -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \quad [c < -1].$$

1.7.5. Интегралы вида $\int x^p (1-x^2)^{q+1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}^r dx$.

Интегралы этого пункта выражаются через элементарные функции в следующих случаях:

- 1) $p, r=0, 1, 2, \dots$; $q=-1, 0, 1, 2, \dots$;
- 2) $r=1$; $p=0, 1, 2, \dots$; $q=-1, -2, \dots$;
- 3) $p=-2, -4, \dots$;
- 4) $p=0$; $q=-1$.

$$\begin{aligned}
 1. \quad & \int x^m (1-x^2)^{n+1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = -\frac{x^{m-1} (1-x^2)^{n+3/2}}{m+2n+2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \pm \frac{1}{m+2n+2} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{m+2k}}{m+2k} + \\
 & \quad + \frac{m-1}{m+2n+2} \int x^{m-2} (1-x^2)^{n+1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int x (1-x^2)^{n+1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = -\frac{(1-x^2)^{n+3/2}}{2n+3} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \pm \frac{1}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k+1}}{2k+1}.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int (1-x^2)^{p+1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \frac{x (1-x^2)^{p+1/2}}{2p+2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \pm \\
 & \pm \frac{(1-x^2)^{p+1}}{4(p+1)^2} + \frac{2p+1}{2p+2} \int (1-x^2)^{p-1/2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx.
 \end{aligned}$$

$$4. \quad \int \frac{1}{\sqrt{1-x^2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}^p dx = \pm \frac{1}{p+1} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}^{p+1}.$$

$$5. \quad \int \frac{1}{\sqrt{1-x^2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}^{-1} dx = \pm \ln \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}.$$

$$\begin{aligned}
 6. \quad & \int \frac{\sqrt{1-x^2}}{x} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = \sqrt{1-x^2} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \mp x + \int \frac{1}{x \sqrt{1-x^2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int \frac{(1-x^2)^{n+1/2}}{x^m} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \\
 & = -\frac{(1-x^2)^{n+3/2}}{(m-1)x^{m-1}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \pm \frac{1}{m-1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k-m+2}}{2k-m+2} - \\
 & \quad - \frac{2n-m+4}{m-1} \int \frac{(1-x^2)^{n+1/2}}{x^{m-2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & = -\frac{(1-x^2)^{n+1/2}}{(m-1)x^{m-1}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \pm \\
 & \pm \frac{1}{m-1} \sum_{k=0}^n (-1)^k \frac{x^{2k-m+2}}{2k-m+2} - \frac{2n+1}{m-1} \int \frac{(1-x^2)^{n-1/2}}{x^{m-2}} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx.
 \end{aligned}$$

Если $k=m/2-1$, то соответствующий член в сумме следует заменить на $(-1)^{m/2-1} \frac{1}{m-1} \binom{n+1}{m/2-1} \ln x$.

9.
$$\int \frac{(1-x^2)^{n+1/2}}{x} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{(1-x^2)^{n+1/2}}{2n+1} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k+1}}{2k+1} + \int \frac{(1-x^2)^{n-1/2}}{x} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
10.
$$\int \frac{x}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = -\sqrt{1-x^2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm x.$$
11.
$$\int \frac{1}{(1-x^2)^{p+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x}{(2p-1)(1-x^2)^{p-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{(2p-1)(2p-2)(1-x^2)^{p-1}} + \frac{2p-2}{2p-1} \int \frac{1}{(1-x^2)^{p-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
12.
$$\int \frac{1}{(1-x^2)^{3/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \ln \sqrt{1-x^2}.$$
13.
$$\int \frac{x^m}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = -\frac{1}{m} x^{m-1} \sqrt{1-x^2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \frac{x^m}{m^2} +$$

$$+ \frac{m-1}{m} \int \frac{x^{m-2}}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
14.
$$\int \frac{x}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= \frac{1}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp \frac{1}{2n-1} \int \frac{dx}{(1-x^2)^n}.$$
15.
$$\int \frac{x^m}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x^{m-1}}{(2n-m)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2n-m} \int \frac{x^{m-1} dx}{(1-x^2)^n} - \frac{m-1}{2n-m} \int \frac{x^{m-2}}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
16.
$$\int \frac{x^m}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= \frac{x^{m-1}}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp \frac{1}{2n-1} \int \frac{x^{m-1} dx}{(1-x^2)^n} -$$

$$- \frac{m-1}{2n-1} \int \frac{x^{m-2}}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
17.
$$\int \frac{1}{x^m (1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \frac{1}{m-1} \int \frac{dx}{x^{m-1}(1-x^2)^n} +$$

$$+ \frac{2n+m-2}{m-1} \int \frac{1}{x^{m-2}(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$

$$18. \int \frac{1}{x(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= \frac{1}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp \frac{1}{2n-1} \int \frac{dx}{(1-x^2)^n} +$$

$$+ \int \frac{1}{x(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$

$$19. \int \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\}^p dx = x \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\}^p \pm$$

$$\pm p \sqrt{1-x^2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\}^{p-1} - p(p-1) \int \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\}^{p-2} dx.$$

$$20. \int \frac{dx}{\arcsin x} = \ln(\arcsin x) + \sum_{k=1}^{\infty} (-1)^k \frac{\arcsin^{2k} x}{2k(2k)!}.$$

$$21. \int \frac{dx}{\arccos x} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\arccos^{2k+1} x}{(2k+1)(2k+1)!}.$$

1.7.6. Интегралы вида $\int x^p \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} dx.$

$$1. \int x^p \operatorname{arcsec} \frac{x}{a} dx = \frac{x^{p+1}}{p+1} \operatorname{arcsec} \frac{x}{a} \mp \frac{a}{p+1} \int \frac{x^p dx}{\sqrt{x^2-a^2}}$$

$$\left[\left\{ \begin{matrix} 0 \\ \pi/2 \end{matrix} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ \pi \end{matrix} \right\} \right].$$

$$2. \int x^p \operatorname{arccosec} \frac{x}{a} dx = \frac{x^{p+1}}{p+1} \operatorname{arccosec} \frac{x}{a} \pm \frac{a}{p+1} \int \frac{x^p dx}{\sqrt{x^2-a^2}}$$

$$\left[\left\{ \begin{matrix} 0 \\ -\pi/2 \end{matrix} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ 0 \end{matrix} \right\} \right].$$

$$3. \int x^{2n} \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} dx = \frac{x^{2n+1}}{2n+1} \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} \mp$$

$$\mp \frac{ax^{2n-1} \sqrt{x^2-a^2}}{2n(2n+1)} \left[1 + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} \left(\frac{a}{x} \right)^{2k+2} \right] \mp$$

$$\mp \frac{1}{2n+1} \frac{(2n-1)!!}{(2n)!!} \ln \left| x + \sqrt{x^2-a^2} \right| \left[0 < \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} < \frac{\pi}{2} \right].$$

$$4. \int x^{2n+1} \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} dx = \frac{x^{2n+2}}{2n+2} \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} \mp$$

$$\mp \frac{ax^{2n} \sqrt{x^2-a^2}}{(2n+1)(2n+2)} \left[1 + \sum_{k=0}^{n-1} \frac{2n(2n-2) \dots (2n-2k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \left(\frac{a}{x} \right)^{2k+2} \right].$$

$$5. = \frac{x^{2n+2}}{2n+2} \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} \mp \frac{a^{2n+1} \sqrt{x^2-a^2}}{2n+2} \sum_{k=0}^n \binom{n}{k} \frac{1}{2k+1} \left(\frac{x^2-a^2}{a^2} \right)^k$$

$$\left[0 < \left\{ \begin{matrix} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{matrix} \right\} < \frac{\pi}{2} \right].$$

$$6. \int \frac{1}{x^{2n}} \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} dx = -\frac{1}{(2n-1)x^{2n-1}} \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} \pm$$

$$\pm \frac{\sqrt{x^2-a^2}}{(2n-1)a^{2n-1}x} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \left(\frac{x^2-a^2}{x^2} \right)^k \quad \left[0 < \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} < \frac{\pi}{2} \right].$$

$$7. \int \frac{1}{x^{2n+1}} \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} dx = -\frac{1}{2nx^{2n}} \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} \pm$$

$$\pm \frac{\sqrt{x^2-a^2}}{(2n)^2 ax^{2n}} \left[1 + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3)\dots(2n-2k-1)}{(2n-2)(2n-4)\dots(2n-2k-2)} \left(\frac{x}{a} \right)^{2k+2} \right] +$$

$$+ \frac{(2n-1)!!}{(2n)!! 2n} \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} \quad \left[0 < \left\{ \begin{array}{l} \operatorname{arcsec}(x/a) \\ \operatorname{arccosec}(x/a) \end{array} \right\} < \frac{\pi}{2} \right].$$

$$8. \int \operatorname{arcsec} \frac{x}{a} dx = x \operatorname{arcsec} \frac{x}{a} \mp a \ln |x + \sqrt{x^2-a^2}|$$

$$\left[\left\{ \begin{array}{l} 0 \\ \pi/2 \end{array} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ \pi \end{array} \right\} \right].$$

$$9. \int \operatorname{arccosec} \frac{x}{a} dx = x \operatorname{arccosec} \frac{x}{a} \pm a \ln |x + \sqrt{x^2-a^2}|$$

$$\left[\left\{ \begin{array}{l} 0 \\ -\pi/2 \end{array} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ 0 \end{array} \right\} \right].$$

$$10. \int x \operatorname{arcsec} \frac{x}{a} dx = \frac{x^2}{2} \operatorname{arcsec} \frac{x}{a} \mp \frac{a}{2} \sqrt{x^2-a^2}$$

$$\left[\left\{ \begin{array}{l} 0 \\ \pi/2 \end{array} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ \pi \end{array} \right\} \right].$$

$$11. \int x \operatorname{arccosec} \frac{x}{a} dx = \frac{x^2}{2} \operatorname{arccosec} \frac{x}{a} \pm \frac{a}{2} \sqrt{x^2-a^2}$$

$$\left[\left\{ \begin{array}{l} 0 \\ -\pi/2 \end{array} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ 0 \end{array} \right\} \right].$$

$$12. \int x^2 \operatorname{arcsec} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arcsec} \frac{x}{a} \mp \frac{ax}{6} \sqrt{x^2-a^2} \mp$$

$$\mp \frac{a^3}{6} \ln |x + \sqrt{x^2-a^2}| \quad \left[\left\{ \begin{array}{l} 0 \\ -\pi/2 \end{array} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ \pi \end{array} \right\} \right].$$

$$13. \int x^2 \operatorname{arccosec} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arccosec} \frac{x}{a} \pm \frac{ax}{6} \sqrt{x^2-a^2} \pm$$

$$\pm \frac{a^3}{6} \ln |x + \sqrt{x^2-a^2}| \quad \left[\left\{ \begin{array}{l} 0 \\ -\pi/2 \end{array} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{array}{l} \pi/2 \\ 0 \end{array} \right\} \right].$$

$$14. \int \frac{1}{x} \operatorname{arcsec} \frac{x}{a} dx = \frac{\pi}{2} \ln |x| + \frac{a}{x} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)^2} \left(\frac{a}{x} \right)^{2k+1}$$

$$[0 < \operatorname{arcsec}(x/a) < \pi/2].$$

$$15. \int \frac{1}{x} \operatorname{arccosec} \frac{x}{a} dx = -\frac{a}{x} - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)^2} \left(\frac{a}{x} \right)^{2k+1}$$

$$[-\pi/2 < \operatorname{arccosec}(x/a) < \pi/2].$$

$$16. \int \frac{1}{x^2} \operatorname{arcsec} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arcsec} \frac{x}{a} \pm \frac{\sqrt{x^2 - a^2}}{ax} \quad \left[\left\{ \begin{matrix} 0 \\ \pi/2 \end{matrix} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ \pi \end{matrix} \right\} \right].$$

$$17. \int \frac{1}{x^2} \operatorname{arccosec} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arccosec} \frac{x}{a} \mp \frac{\sqrt{x^2 - a^2}}{ax} \quad \left[\left\{ \begin{matrix} 0 \\ -\pi/2 \end{matrix} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ 0 \end{matrix} \right\} \right].$$

$$18. \int \frac{1}{x^3} \operatorname{arcsec} \frac{x}{a} dx = -\frac{1}{2x^2} \operatorname{arcsec} \frac{x}{a} \pm \frac{\sqrt{x^2 - a^2}}{4ax^2} \pm \frac{1}{4a^2} \arccos \left| \frac{a}{x} \right| \quad \left[\left\{ \begin{matrix} 0 \\ \pi/2 \end{matrix} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ \pi \end{matrix} \right\} \right].$$

$$19. \int \frac{1}{x^3} \operatorname{arccosec} \frac{x}{a} dx = -\frac{1}{2x^2} \operatorname{arccosec} \frac{x}{a} \mp \frac{\sqrt{x^2 - a^2}}{4ax^2} \mp \frac{1}{4a^2} \arccos \left| \frac{a}{x} \right| \quad \left[\left\{ \begin{matrix} 0 \\ -\pi/2 \end{matrix} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \begin{matrix} \pi/2 \\ 0 \end{matrix} \right\} \right].$$

1.7.7. Интегралы вида $\int (x \pm 1)^{\pm n + 1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx$.

Условие: $0 < \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} < \frac{\pi}{2}$.

$$1. \int (x+1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \frac{2}{2n+3} (x+1)^{n+3/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \mp \\ \mp \frac{4}{2n+3} \operatorname{arctg} \sqrt{x-1} \mp \frac{4}{2n+3} \sum_{k=1}^{n+1} \binom{n+1}{k} \int (y^2+1)^{k-1} dy \quad [y = \sqrt{x-1}].$$

$$2. \int (x-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \frac{2}{2n+3} (x-1)^{n+3/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{2(-1)^n}{2n+3} \ln \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \mp \frac{4(-1)^{n+1}}{2n+3} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int (y^2-1)^k dy \\ [y = \sqrt{x+1}].$$

$$3. \int \frac{1}{(x+1)^{n+1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = -\frac{2}{(2n-1)(x+1)^{n-1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{4}{2n-1} \operatorname{arctg} \sqrt{x-1} \mp \frac{4}{2n-1} \sum_{k=0}^{n-1} \int \frac{dy}{(y^2+2)^{n-k}} \quad [y = \sqrt{x-1}].$$

$$4. \int \frac{1}{(x-1)^{n+1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = -\frac{2}{(2n-1)(x-1)^{n-1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{2(-1)^n}{2n-1} \ln \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \mp \frac{4(-1)^n}{2n-1} \sum_{k=0}^{n-1} (-1)^k \int \frac{dy}{(y^2-2)^{n-k}} \quad [y = \sqrt{x+1}].$$

1.7.8. Интегралы вида $\int x^m (x^2-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx$.

Условие: $0 < \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} < \frac{\pi}{2}$.

Интегралы $\int x^m (x^2-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}^l dx$ при целых m, n, l с помощью подстановок $y = \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}$ сводятся к интегралам вида $\int y^l \left\{ \begin{matrix} \sin y \\ \cos y \end{matrix} \right\}^{2n+2} \times \times \left\{ \begin{matrix} \cos y \\ \sin y \end{matrix} \right\}^{-(2n+m+3)} dy$. Результат выражается в виде конечной комбинации элементарных функций в следующих случаях [18]:

- 1) $l \geq 1, \quad n \geq -1, \quad m \leq -2n-3$;
- 2) $l=1, \quad n < -1, \quad m \leq -2n-3$;
- 3) $l=1, \quad m$ — нечетное, $m \geq -2n-3$;
- 4) $m=-1, \quad n=-1$

$$1. \int \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} (x^2-1)^{n+1/2} dx = \frac{(x^2-1)^{n+1/2} x}{2(n+1)} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}^{\mp} \\ \mp \frac{(-1)^n}{2(n+1)} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k+1}}{2k+1} - \frac{2n+1}{2n+2} \int \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} (x^2-1)^{n-1/2} dx.$$

$$2. \int x^m (x^2-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \\ = \frac{x^{m-1} (x^2-1)^{n+3/2}}{m+2n+2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}^{\mp} + \frac{(-1)^{n+1}}{m+2n+2} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{m-2k-1}}{m-2k-1} + \\ + \frac{m-1}{m+2n+2} \int x^{m-2} (x^2-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx.$$

$$3. \int x (x^2-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \frac{(x^2-1)^{n+3/2}}{2n+3} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}^{\pm} \\ \pm \frac{(-1)^n}{2n+3} \ln x + \frac{(-1)^{n+1}}{2(2n+3)} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k}}{k}.$$

$$4. \int \frac{(x^2-1)^{n+1/2}}{x^m} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \\ = \frac{(x^2-1)^{n+3/2}}{(m-1)x^{m-1}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\}^{\mp} + \frac{(-1)^{n+1}}{m-1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k-m+1}}{2k-m+1} - \\ - \frac{2n-m+4}{m-1} \int \frac{(x^2-1)^{n+1/2}}{x^{m-2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx.$$

$$5. = -\frac{(x^2-1)^{n+1/2}}{(m-1)x^{m-1}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} - \frac{(-1)^n}{m-1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k-m+1}}{2k-m+1} + \\ + \frac{2n+1}{m-1} \int \frac{(x^2-1)^{n-1/2}}{x^{m-2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx.$$

- $$\begin{aligned}
 6. \int \frac{(x^2-1)^{n+1/2}}{x} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{(x^2-1)^{n+1/2}}{2n+1} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \mp \\
 &\mp \frac{(-1)^n}{2n+1} \ln x \mp \frac{(-1)^n}{2(2n+1)} \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{x^{2k}}{k} - \int \frac{(x^2-1)^{n-1/2}}{x} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx. \\
 7. \int \frac{\sqrt{x^2-1}}{x} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \mp \frac{1}{2} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\}^2 + \sqrt{x^2-1} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \mp \ln x. \\
 8. \int \frac{x^m}{\sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{1}{m} x^{m-1} \sqrt{x^2-1} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \mp \\
 &\mp \frac{x^{m-1}}{m(m-1)} + \frac{m-1}{m} \int \frac{x^{m-2}}{\sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx. \\
 9. \int \frac{x}{(x^2-1)^{n+1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{-1}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \\
 &\pm \frac{(-1)^n}{2(2n-1)} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n-1}{k} \left(\frac{x^2}{x^2-1} \right)^k \mp \frac{(-1)^n}{2n-1} \ln \frac{\sqrt{x^2-1}}{x}. \\
 10. \int \frac{1}{x^m (x^2-1)^{n+1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{1}{(m-1)x^{m-1}(x^2-1)^{n-1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \mp \\
 &\mp \frac{1}{m-1} \int \frac{dx}{x^m (x^2-1)^n} + \frac{2n+m-2}{m-1} \int \frac{1}{x^{m-2}(x^2-1)^{n+1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx. \\
 11. \int \frac{1}{x^m \sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{\sqrt{x^2-1}}{(m-1)x^{m-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \\
 &\pm \frac{1}{(m-1)^2 x^{m-1}} + \frac{m-2}{m-1} \int \frac{1}{x^{m-2} \sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx. \\
 12. \int \frac{1}{x^2 \sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \frac{\sqrt{x^2-1}}{x} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \frac{1}{x}. \\
 13. \int \frac{x}{\sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \sqrt{x^2-1} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \mp \ln x. \\
 14. \int \frac{1}{(x^2-1)^{n+1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= \mp \frac{x}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \\
 &\pm \frac{1}{2n-1} \int \frac{dx}{(x^2-1)^n} - \frac{2n-2}{2n-1} \int \frac{1}{(x^2-1)^{n-1/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx. \\
 15. \int \frac{1}{(x^2-1)^{3/2}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx &= -\frac{x}{\sqrt{x^2-1}} \left\{ \begin{array}{c} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \frac{1}{2} \ln \frac{x-1}{x+1}.
 \end{aligned}$$

$$16. \int \frac{x^m}{(x^2-1)^{n+1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx =$$

$$= -\frac{x^{m-1}}{(2n-m)(x^2-1)^{n-1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \frac{1}{2n-m} \int \frac{x^{m-2} dx}{(x^2-1)^n} -$$

$$- \frac{m-1}{2n-m} \int \frac{x^{m-2}}{(x^2-1)^{n+1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx.$$

$$17. = -\frac{x^{m-1}}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \frac{1}{2n-1} \int \frac{x^{m-2} dx}{(x^2-1)^n} +$$

$$+ \frac{m-1}{2n-1} \int \frac{x^{m-2}}{(x^2-1)^{n-1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx.$$

$$18. \int \frac{1}{x(x^2-1)^{n+1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx =$$

$$= -\frac{1}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} \pm \frac{1}{2n-1} \int \frac{dx}{x(x^2-1)^n} -$$

$$- \int \frac{1}{x(x^2-1)^{n-1/2}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx.$$

$$19. \int \frac{1}{x\sqrt{x^2-1}} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\} dx = \pm \frac{1}{2} \left\{ \begin{array}{l} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{array} \right\}^2.$$

1.7.9. Интегралы вида $\int x^p \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx.$

$$1. \int x^p \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{x^{p+1}}{p+1} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp \frac{a}{p+1} \int \frac{x^{p+1} dx}{x^2+a^2}.$$

$$2. \int x^{2n} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{x^{2n+1}}{2n+1} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp$$

$$\mp \frac{a}{2(2n+1)} \left[n! \sum_{k=0}^{n-1} \frac{(-a^2)^k (x^2+a^2)^{n-k}}{(n-k)! k! (n-k)} + (-a^2)^n \ln (x^2+a^2) \right].$$

$$3. \int x^{2n+1} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{x^{2n+2}}{2n+2} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} +$$

$$+ \frac{a^{2n+2}}{2n+2} \left[\sum_{k=1}^{n+1} \frac{(-1)^{n+k}}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + (-1)^n \operatorname{arctg} \frac{x}{a} \right].$$

$$4. \int \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = x \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp \frac{a}{2} \ln (x^2+a^2).$$

$$5. \int x \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{x^2+a^2}{2} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp \frac{1}{2} ax.$$

$$6. \int x^3 \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{x^3}{3} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp \frac{ax^2}{6} \pm \frac{a^3}{6} \ln (x^2+a^2).$$

$$7. \int x^3 \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} dx = \frac{1}{4} (x^4-a^4) \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arccotg} (x/a) \end{array} \right\} \mp \frac{ax^3}{12} \pm \frac{a^3x}{4}.$$

$$8. \int \frac{1}{x^n} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = \frac{-1}{(n-1)x^{n-1}} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \pm \frac{a}{n-1} \int \frac{dx}{x^{n-1}(x^2+a^2)}.$$

$$9. \int \frac{1}{x^{2n}} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = -\frac{1}{(2n-1)x^{2n-1}} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \mp \\ \mp \frac{1}{2(2n-1)a^{2n-1}} \left[\sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{k+1} \left(\frac{a}{x} \right)^{2k+2} + (-1)^{n-1} \ln \frac{x^2+a^2}{x^2} \right].$$

$$10. \int \frac{1}{x^{2n+1}} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = -\frac{1}{2nx^{2n}} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \pm \\ \pm \frac{1}{2na^{2n}} \left[\sum_{k=1}^n \frac{(-1)^k}{2n-2k+1} \left(\frac{a}{x} \right)^{2n-2k+1} + (-1)^n \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \right].$$

$$11. \int \frac{1}{x} \operatorname{arctg} \frac{x}{a} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{x}{a} \right)^{2k+1} \quad [|x| < |a|].$$

$$12. = \pm \frac{\pi}{2} \ln |x| + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [\pm x/a > 1].$$

$$13. \int \frac{1}{x} \operatorname{arcc tg} \frac{x}{a} dx = \frac{\pi}{2} \ln |x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{x}{a} \right)^{2k+1} \quad [|x| < |a|].$$

$$14. = - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [x/a > 1].$$

$$15. = \pi \ln |x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [x/a < -1].$$

$$16. \int \frac{1}{x} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = \pm \operatorname{Im} \operatorname{Li}_2 \left(\frac{ix}{a} \right) + \frac{\pi}{2} \left\{ \begin{array}{l} 0 \\ \ln |x| \end{array} \right\}.$$

$$17. \int \frac{1}{x^2} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = \frac{-1}{x} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \pm \frac{1}{2a} \ln \frac{x^2}{x^2+a^2}.$$

$$18. \int \frac{1}{x^3} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = -\frac{1}{2} \left(\frac{1}{x^2} + \frac{1}{a^2} \right) \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} \mp \frac{1}{2ax}.$$

$$19. \int \frac{1}{(x+b)^2} \left\{ \begin{array}{l} \operatorname{arctg} (x/a) \\ \operatorname{arcc tg} (x/a) \end{array} \right\} dx = \\ = \pm \frac{a}{a^2+b^2} \left[\ln \frac{x+b}{\sqrt{x^2+a^2}} - \frac{a^2-bx}{a(x+b)} \operatorname{arctg} \frac{x}{a} \right] - \frac{\pi}{2(x+b)} \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right\}.$$

$$20. \int \frac{\operatorname{arctg} (ax+b)}{cx+d} dx = \theta \ln \frac{\sin (\theta+\varphi)}{\cos \theta} + \frac{1}{2} \operatorname{Cl}_2 (2\varphi+2\theta) + \frac{1}{2} \operatorname{Cl}_2 (\pi-2\theta) \\ [\operatorname{tg} \varphi = ad-bc; \operatorname{tg} \theta = ax+b].$$

1.7.10. Интегралы вида $\int x^p (x^2 + a^2)^q \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\}^r dx$.

Интегралы этого пункта выражаются через элементарные функции в следующих случаях [18]:

- 1) $p, r = 1, 2, 3, \dots$ и $q < -\frac{p}{2} - 1$;
- 2) $r = 1, p$ — четное;
- 3) $r = 1, 2, p$ — нечетное, $q = -1, 0, 1, 2, \dots$;
- 4) $p = 0, q = -1$.

$$1. \int x^m (x^2 + a^2)^n \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx = \\ = \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} \sum_{k=0}^n \binom{n}{k} \frac{a^{2n-2k} x^{m+2k+1}}{m+2k+1} \mp \sum_{k=0}^n \frac{a^{2n-2k+1}}{m+2k+1} \binom{n}{k} \int \frac{x^{m+2k+1}}{1+x^2} dx.$$

$$2. \int x^{2m} (x^2 + a^2) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx = x^{2m+1} \left(\frac{x^2}{2m+3} + \frac{a^2}{2m+1} \right) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} \mp \\ \mp \frac{ax^{2m+2}}{(2m+2)(2m+3)} \mp \frac{(-1)^m}{(2m+1)(2m+3)} \sum_{k=1}^m \frac{(-1)^k a^{2m-2k+3}}{k} \binom{m}{k} (x^2 + a^2)^k \mp \\ \mp \frac{(-1)^m a^{2m+3}}{(2m+1)(2m+3)} \ln (x^2 + a^2).$$

$$3. \int x^{2m+1} (x^2 + a^2) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx = \\ = \frac{1}{2} \left[\frac{a^2 x^{2m+2}}{m+1} + \frac{x^{2m+4}}{m+2} + \frac{a^{2m+4}}{(m+1)(m+2)} \right] \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} \mp \\ \mp \frac{ax^{2m+3}}{2(m+2)(2m+3)} \mp \frac{a^3}{2(m+1)(m+2)} \sum_{k=0}^m (-1)^k \frac{a^{2k} x^{2m-2k+1}}{2m-2k+1}.$$

$$4. \int \frac{(x^2 + a^2)^n}{x^m} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx = \\ = \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} \sum_{k=0}^n \binom{n}{k} \frac{a^{2n-2k} x^{2k-m+1}}{2k-m+1} \mp \sum_{k=0}^n \frac{a^{2n-2k+1}}{2k-m+1} \binom{n}{k} \int \frac{x^{2k-m+1}}{x^2 + a^2} dx.$$

Если $k = (m-1)/2, k = 0, 1, \dots, n$, то соответствующие слагаемые в суммах заменяются на $a^{2n-m+1} \binom{n}{k} \ln \frac{x}{a} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\}$ и $a^{2n-m+2} \binom{n}{k} \int \frac{\ln (x/a)}{x^2 + a^2} dx$.

$$5. \int \frac{x^m}{(x^2 + a^2)^n} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx = \\ = -\frac{x^{m-1}}{2(n-1)(x^2 + a^2)^{n-1}} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} \pm \frac{a}{2(n-1)} \int \frac{x^{m-1} dx}{(x^2 + a^2)^n} + \\ + \frac{m-1}{2(n-1)} \int \frac{x^{m-2}}{(x^2 + a^2)^{n-1}} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arcctg} (x/a) \end{matrix} \right\} dx.$$

$$6. \int \frac{1}{(x^2 + a^2)^n} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$

$$= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \pm \frac{1}{4(n-1)^2 a (x^2 + a^2)^{n-1}} +$$

$$+ \frac{2n-3}{(2n-2)a^2} \int \frac{1}{(x^2 + a^2)^{n-1}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx.$$

$$7. \int \frac{1}{(x^2 + a^2)^n} \operatorname{arctg} \frac{x}{a} dx =$$

$$= \operatorname{arctg} \frac{x}{a} \left[\sum_{k=1}^{n-1} \frac{(2n-2k-2)!! (2n-3)!!}{(2n-2)!! (2n-2k-1)!!} \frac{x}{a^{2k} (x^2 + a^2)^{n-k}} + \frac{1}{2a^{2n-1}} \frac{(2n-3)!!}{(2n-2)!!} \right] +$$

$$+ \frac{a}{2} \sum_{k=1}^{n-1} \frac{(2n-3)!! (2n-2k-2)!!}{(2n-2)!! (2n-2k-1)!! (n-k)} \frac{1}{a^{2k} (x^2 + a^2)^{n-k}}.$$

$$8. \int \frac{x}{(x^2 + a^2)^n} \left\{ \operatorname{arctg} x \right\} dx =$$

$$= \frac{1}{2(n-1)} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \left[\frac{(2n-3)!!}{(2n-2)!! a^{2n-2}} - \frac{1}{(x^2 + a^2)^{n-1}} \right] \pm$$

$$\pm \frac{x}{4(n-1)^2 a (x^2 + a^2)^{n-1}} \left[1 + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5) \dots (2n-2k-1)}{(2n-4)(2n-6) \dots (2n-2k-2)} \left(\frac{x^2 + a^2}{a^2} \right)^k \right].$$

$$9. \int \frac{x^n}{x^2 + a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$

$$= \int x^{n-2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx - a^2 \int \frac{x^{n-2}}{x^2 + a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx.$$

$$10. \int \frac{x^{2m}}{x^2 + a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \pm \frac{(-1)^m a^{2m-1}}{2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2 +$$

$$+ \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \sum_{k=1}^m (-1)^{m-k} \frac{a^{2m-2k} x^{2k-1}}{2k-1} \mp$$

$$\mp \sum_{k=1}^m \frac{(-1)^{m-k} a^{2m-2k+1}}{2k-1} \int \frac{x^{2k-1}}{x^2 + a^2} dx.$$

$$11. \int \frac{x^{2m+1}}{1+x^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$

$$= \frac{1}{2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \sum_{k=1}^m (-1)^{m-k} \frac{a^{2m-2k} x^{2k}}{k} \mp$$

$$\mp \frac{1}{2} \sum_{k=1}^m \frac{(-1)^{m-k} a^{2m-2k-1}}{k} \int \frac{x^{2k} dx}{a^2 + x^2} + (-1)^m 2^m \int \frac{x}{a^2 + x^2} \left\{ \operatorname{arctg} x \right\} dx.$$

$$12. \int \frac{1}{x^2 + a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \pm \frac{1}{2a} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2.$$

- $$\begin{aligned}
 13. \quad & \int \frac{x}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \frac{1}{2} \ln \frac{\sqrt{x^2 + a^2}}{2a} \left\{ \operatorname{arctg} (x/a) \right\} \mp \\
 & \mp \frac{i}{4} \left[\operatorname{Li}_2 \left(\frac{a + ix}{2a} \right) - \operatorname{Li}_2 \left(\frac{a - ix}{2a} \right) \right] + \frac{\pi}{8} \left\{ \ln (x^2 + a^2) \right\}^0. \\
 14. \quad & = \frac{1}{2} \ln (x^2 + a^2) \left\{ \operatorname{arctg} (x/a) \right\} \mp \frac{a}{2} \int \frac{\ln (x^2 + a^2)}{x^2 + a^2} dx. \\
 15. \quad & \int \frac{x^2}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \\
 & = x \left\{ \operatorname{arctg} (x/a) \right\} \mp \frac{a}{2} \ln (x^2 + a^2) \mp \frac{a}{2} \left\{ \operatorname{arctg} (x/a) \right\}^2. \\
 16. \quad & \int \frac{x^3}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \\
 & = -\frac{ax}{2} + \frac{1}{2} (x^2 + a^2) \left\{ \operatorname{arctg} (x/a) \right\} \mp a^2 \int \frac{x}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx. \\
 17. \quad & \int \frac{1}{(x^2 + a^2)^2} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \\
 & = \frac{x}{2a^2 (x^2 + a^2)} \left\{ \operatorname{arctg} (x/a) \right\} \mp \frac{1}{4a^3} \left\{ \operatorname{arctg} (x/a) \right\}^2 \mp \frac{1}{4a} \frac{1}{(x^2 + a^2)}. \\
 18. \quad & \int \left\{ \operatorname{arctg} (x/a) \right\}^2 \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \\
 & = x \left\{ \operatorname{arctg} (x/a) \right\}^2 - a \ln \frac{\sqrt{x^2 + a^2}}{2a} \operatorname{arctg} \frac{x}{a} - a \operatorname{Im} \operatorname{Li}_2 \left(\frac{a + ix}{2a} \right) + \frac{\pi a}{2} \left\{ \ln (x^2 + a^2) \right\}^0. \\
 19. \quad & \int \frac{1}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\}^r \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \pm \frac{1}{(r+1)a} \left\{ \operatorname{arctg} (x/a) \right\}^{r+1}. \\
 20. \quad & \int \frac{1}{x^2 + a^2} \left\{ \operatorname{arctg} (x/a) \right\}^{-1} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \pm \frac{1}{a} \ln \left\{ \operatorname{arctg} (x/a) \right\}. \\
 21. \quad & \int \frac{x}{\sqrt{a^2 - x^2}} \left\{ \operatorname{arctg} (x/a) \right\} \left\{ \operatorname{arcc tg} (x/a) \right\} dx = \\
 & = -\sqrt{a^2 - x^2} \left\{ \operatorname{arctg} (x/a) \right\} \mp a \arcsin \frac{x}{a} + \sqrt{2} a \left\{ \operatorname{arctg} [\sqrt{2} x (a^2 - x^2)^{-1/2}] \right\} \left\{ \operatorname{arcc tg} [\sqrt{2} x (a^2 - x^2)^{-1/2}] \right\}. \\
 22. \quad & \int \frac{\operatorname{arctg} x}{(ax^2 + b)^{3/2}} dx = \frac{x \operatorname{arctg} x}{b (ax^2 + b)^{1/2}} - \frac{1}{b \sqrt{a-b}} \operatorname{arctg} \sqrt{\frac{ax^2 + b}{a-b}} \quad [a > b]. \\
 23. \quad & = \frac{x \operatorname{arctg} x}{b \sqrt{ax^2 + b}} - \frac{1}{2b \sqrt{b-a}} \ln \frac{\sqrt{ax^2 + b} - \sqrt{b-a}}{\sqrt{ax^2 + b} + \sqrt{b-a}} \quad [b > a].
 \end{aligned}$$

1.8. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

Неопределенные интегралы от обратных гиперболических функций вычисляются по формулам раздела 1.6 после следующих замен:

$$\operatorname{Arsh} x = \ln (x + \sqrt{x^2 + 1}),$$

$$\operatorname{Arch} x = \pm \ln (x + \sqrt{x^2 - 1})$$

$[x \geq 1].$

$$\operatorname{Arth} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad [|x| < 1],$$

$$\operatorname{Arcth} x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad [|x| > 1],$$

$$\operatorname{Arsech} x = \pm \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) \quad [0 < x \leq 1],$$

$$\operatorname{Arcosech} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad [x > 0].$$

Г л а в а 2. ОПРЕДЕЛЕННЫЕ ИНТЕГРАЛЫ

2.1. ВВЕДЕНИЕ

2.1.1. Предварительные сведения.

В этой главе содержатся определенные интегралы от элементарных функций. Интегралы, у которых один или оба предела интегрирования являются переменными, в случаях, когда подынтегральная функция от них не зависит (в частности, интегралы от иррациональных функций, сводящиеся к эллиптическим интегралам), помещены в главе 1.

Указанные в формулах условия сходимости интегралов обеспечивают их существование либо в обычном смысле, либо как несобственных интегралов, либо в смысле главного значения [20].

В начале соответствующих разделов помещены интегралы общего вида, зависящие от нескольких параметров и позволяющие получать более общие результаты, чем содержащиеся в последующих пунктах. Эти интегралы выражаются через функции $V_j^\pm$, V_j , приведенные в приложении III; параметры, входящие в них, указаны в каждой формуле в фигурных скобках.

В качестве примера вычислим интеграл

$$\int_0^a \frac{x^{\alpha-1} dx}{V \sqrt{a^2 - x^2} (bx + c)} = I \quad [a, b, c > 0].$$

В силу формулы 2.2.2.5

$$I = \frac{a^{\alpha-2}}{b} \begin{cases} V_1^+ + V_2^+ \\ V_1^- \end{cases} \quad \begin{matrix} [|y| \leq 1], \\ [|y| \geq 1]. \end{matrix}$$

где $y = c/(ab)$, $r = 2$, $\beta = 1/2$, $\rho = 0$. Пусть, например, $|y| \geq 1$ ($c \geq ab$); тогда $I = \frac{a^{\alpha-2}}{b} V_1^-$. Подставляя значения r , β и ρ в параметры E_1^- , и, a_3 , b_3 , c_3 , z функции V_1^- и учитывая, что $p = 2$, $q = 1$, получаем

$$\begin{aligned} V_1^- &= \sum_{h=0}^1 \frac{(-1)^h}{h!2} \Gamma \left[\begin{matrix} 1/2, (\alpha+h)/2, 1+h \\ 1, (1+\alpha+h)/2 \end{matrix} \right] y^{-h-1} \times \\ &\quad \times {}_4F_3 \left(\begin{matrix} 1, \Delta(2, 1+h), \Delta(1, (\alpha+h)/2); (-y)^{-2} \\ \Delta(2, 1+h), \Delta(1, (1+\alpha+h)/2) \end{matrix} \right) = \\ &= \frac{\sqrt{\pi}}{2y} \Gamma \left[\frac{\alpha/2}{(1+\alpha)/2} \right] {}_2F_1 \left(1, \frac{\alpha}{2}; \frac{1+\alpha}{2}; y^{-2} \right) - \\ &\quad - \frac{\sqrt{\pi}}{2y^2} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] {}_2F_1 \left(1, \frac{\alpha+1}{2}; 1+\frac{\alpha}{2}; y^{-2} \right). \end{aligned}$$

Таким образом,

$$\int_0^a \frac{x^{\alpha-1} dx}{V a^2 - x^2 (bx + c)} = \frac{a^{\alpha-1}}{2c} V \pi \Gamma \left[\frac{\alpha/2}{(1+\alpha)/2} \right] {}_2F_1 \left(1, \frac{\alpha}{2}; \frac{\alpha+1}{2}; \frac{a^2 b^2}{c^2} \right) - \\ - \frac{V \pi a^{\alpha} b}{2c^2} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] {}_2F_1 \left(1, \frac{\alpha+1}{2}; \frac{\alpha}{2} + 1; \frac{a^2 b^2}{c^2} \right) \\ \left[\operatorname{Re} \alpha > 0, a > 0, \left| \arg \frac{c}{ab} \right| < \pi \right].$$

Функции $V_1^+ + V_2^+$ и V_1^- в формуле 2.2.2.5 аналитически продолжают друг друга, поэтому полученная формула справедлива не только при $|y| = \left| \frac{c}{ab} \right| \geq 1$, но и при $\left| \arg \frac{c}{ab} \right| < \pi$. Подставив, в частности, в эту формулу $\alpha = 1$ и используя соотношения $\Gamma(1/2) = V \pi$, $\Gamma(3/2) = V \pi/2$, ${}_2F_1(a, b; b; z) = (1-z)^{-a}$, ${}_2F_1\left(1, 1; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z \cos z}$, получим

$$\int_0^a \frac{dx}{V a^2 - x^2 (bx + c)} = \frac{\pi}{2c} \left(1 - \frac{a^2 b^2}{c^2} \right)^{-1/2} - \frac{1}{c} \left(1 - \frac{a^2 b^2}{c^2} \right)^{-1/2} \arcsin \frac{ab}{c}$$

[см. 1.2.50.5].

2.1.2. Общие формулы.

Некоторые свойства определенного интеграла:

1. $\int_a^b f(x) dx = - \int_b^a f(x) dx.$
2. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$

Формула Ньютона—Лейбница:

$$3. \int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a) \quad [F'(x) = f(x)].$$

Интегрирование подстановкой:

$$4. \int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) g'(t) dt \\ [a = g(\alpha), b = g(\beta), g(t), g'(t) \text{ непрерывны, } g'(t) \neq 0 \text{ и } a < x = g(t) < b \text{ при } \alpha < t < \beta].$$

Интегрирование по частям:

$$5. \int_a^b f(x) g'(x) dx = f(x) g(x) \Big|_a^b - \int_a^b g(x) f'(x) dx.$$

Теоремы о среднем:

$$6. \int_a^b f(x) g(x) dx = f(\xi) \int_a^b g(x) dx \quad [f \text{ непрерывна на } [a, b], g(x) > 0, a \leq \xi \leq b].$$

$$7. \int_a^b f(x) g(x) dx = f(a) \int_a^{\xi} g(x) dx \quad [f \geq 0, \text{ монотонна и не возрастает; } a \leq \xi \leq b].$$

$$8. \quad = f(b) \int_{\xi}^b g(x) dx \quad [f \geq 0, \text{ монотонна и не убывает, } a \leq \xi \leq b].$$

$$9. \int_a^b f(x) g(x) dx = f(a) \int_a^{\xi} g(x) dx + f(b) \int_{\xi}^b g(x) dx \quad [f \text{ монотонна; } [a \leq \xi \leq b].$$

$$10. \quad = A \int_a^{\xi} g(x) dx + B \int_{\xi}^b g(x) dx$$

$[A \geq f(a+0), B \leq f(b-0), \text{ если } f \text{ монотонно убывает; } A \leq f(a+0), B \geq f(b-0), \text{ если } f \text{ монотонно возрастает, } a \leq \xi \leq b].$

Дифференцирование интеграла по параметру:

$$11. \frac{d}{dp} \int_a^b f(x, p) dx = f(b, p) \frac{db}{dp} - f(a, p) \frac{da}{dp} + \int_a^b \frac{\partial}{\partial p} f(x, p) dx.$$

$$12. \frac{d}{db} \int_a^b f(x) dx = f(b). \quad 13. \frac{d}{da} \int_a^b f(x) dx = -f(a).$$

Некоторые формулы замены переменной:

$$14. \int_a^b f(x) dx = \frac{b-a}{h} \int_0^h f\left(a + \frac{b-a}{h} x\right) dx = \frac{b-a}{d-c} \int_c^d f\left(\frac{cd-bc+(b-a)x}{d-c}\right) dx.$$

$$15. \int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$$

$$16. \int_a^b f(x) dx = (b-a) \int_0^{\infty} f\left(\frac{a+bx}{1+x}\right) \frac{dx}{(1+x)^2}.$$

$$17. \int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx = 0 \quad [f(-x) = -f(x)].$$

$$18. \quad = 2 \int_0^a f(x) dx \quad [f(-x) = f(x)].$$

$$19. \int_0^a f(x) dx = \int_0^a f(a-x) dx = \frac{1}{2} \int_0^a [f(x) + f(a-x)] dx.$$

$$20. \int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx = 0 \quad [f(2a-x) = -f(x)].$$

$$21. \quad = 2 \int_0^a f(x) dx \quad [f(2a-x)=f(x)].$$

$$22. \quad \int_0^1 \frac{f(x^2) dx}{V(1-x^2)(1-k^2x^2)} = \int_0^1 f\left(\frac{1-y^2}{1-k^2y^2}\right) \frac{dy}{V(1-y^2)(1-k^2y^2)}.$$

$$23. \quad \int_a^b f(x, Vx-a, Vb-x) dx = \\ = \int_0^1 f\left[a + \frac{4(b-a)y^2}{(1+y^2)^2}, \frac{2Vb-ay}{1+y^2}, \frac{Vb-a(1-y^2)}{1+y^2}\right] \frac{8(b-a)y(1-y^2)}{(1+y^2)^3} dy,$$

$$24. \quad = (b-a) \int_0^{\pi/2} f[a + (b-a) \sin^2 \varphi, Vb-a \sin \varphi, Vb-a \cos \varphi] \sin 2\varphi d\varphi.$$

$$25. \quad = (b-a) \int_0^\infty f\left[\frac{a+by}{1+y}, V\frac{(b-a)y}{1+y}, V\frac{b-a}{1+y}\right] \frac{dy}{(1+y)^2}.$$

$$26. \quad \int_a^b f(x, V(x-a)(b-x)) dx = (b-a) \int_0^\infty f\left(\frac{a+by}{1+y}, \frac{(b-a)Vy}{1+y}\right) \frac{dy}{(1+y)^2}.$$

$$27. \quad \int_a^b f(x, Vx^2-c^2) dx = 4c \int_\alpha^\beta f\left(\frac{a(1+y^2)}{1-y^2}, \frac{2cy}{1-y^2}\right) \frac{y dy}{(1-y^2)^2} \\ \left[0 < c \leq a < b, \alpha = \frac{V a^2 - c^2}{a+c}, \beta = \frac{V b^2 - c^2}{b+c}\right].$$

$$28. \quad = c \int_u^v f\left(\frac{c}{\cos \varphi}, c \operatorname{tg} \varphi\right) \frac{\sin \varphi}{\cos^2 \varphi} d\varphi \\ [0 < c \leq a < b, \cos u = c/a, \cos v = c/b].$$

$$29. \quad \int_a^b f(x, Vc^2-x^2) dx = 2c \int_\alpha^\beta f\left(\frac{2cy}{1+y^2}, \frac{c(1-y^2)}{1+y^2}\right) \frac{1-y^2}{(1+y^2)^2} dy \\ [-c \leq a < b \leq c; \alpha = a^{-1}(c - Vc^2 - a^2); \beta = b^{-1}(c - Vc^2 - b^2)].$$

$$30. \quad = c \int_u^v f(c \sin \varphi, c \cos \varphi) \cos \varphi d\varphi \\ [-c \leq a < b \leq c; \sin u = a/c; \sin v = b/c].$$

$$31. \quad \int_0^\infty \frac{f(x, r)}{x^2 + a^2} dx = \frac{\pi}{2a} F(re^{-a}) \\ [f(x, r) = \operatorname{Re} F(z), z = r(\cos x + i \sin x), F(z) \text{ аналитична при } |z| \leq r].$$

$$32. \quad \int_0^\infty \frac{xg(x, r)}{x^2 + a^2} dx = \frac{\pi}{2} [F(re^{-a}) - F(0)] \quad [g(x, r) = \operatorname{Im} F(z); \text{ см. 2.1.2.31}].$$

$$33. \int_0^{\infty} \frac{g(x, r)}{x} dx = \frac{\pi}{2} [F(r) - F(0)] \quad [\text{см. 2.1.2.31}].$$

$$34. \int_0^{\infty} \frac{g(x, r)}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} [F(r) - F(re^{-a})] \quad [\text{см. 2.1.2.31}].$$

$$35. \int_0^{\infty} f(x) \left\{ \frac{2x+y+2\sqrt{x^2+xy}}{2y+x+2\sqrt{y^2+xy}} \right\}^{\sigma} dx = 2 \int_0^{\infty} xf(x^2) \left(\left\{ \frac{x}{V_y} \right\} + V_{x^2+y} \right)^{2\sigma} dx.$$

$$36. \int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\pi/2} f(x) \left\{ \frac{1}{\cos x} \right\} dx \quad [f(x+\pi) = \pm f(x), f(-x) = f(x)].$$

$$37. \int_0^{\infty} \frac{f(px) - f(qx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{q}{p}$$

[$p, q > 0$; $f(x)$ непрерывна на $[0, \infty)$, $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$ — конечная величина].

$$38. = f(0) \ln \frac{q}{p}$$

[$p, q > 0$; $f(x)$ непрерывна на $[0, \infty)$, и существует $\int_R^{\infty} x^{-1} f(x) dx$, $R > 0$].

$$39. \int_0^{\pi/2} [f(x) - f(2x)] \operatorname{ctg} x dx = f(0) \ln 2 \quad [f(\pi - x) = -f(x)].$$

$$40. \int_0^{\infty} f(x) e^{-(ax^2 + 2bx + c)} dx = \frac{1}{V_a} e^{(b^2 - ac)/a} \int_{b/V_a}^{\infty} f\left(\frac{y}{V_a} - \frac{b}{a}\right) e^{-y^2} dy$$

[$a > 0$].

$$41. \int_{-\infty}^{\infty} f(x) e^{-\alpha(ax^2 + 2bx + c)} dx = \frac{1}{V_a} e^{\alpha(b^2 - ac)/a} \int_{-\infty}^{\infty} f\left(\frac{y}{V_a} - \frac{b}{a}\right) e^{-\alpha y^2} dy$$

[$\alpha = 1$ или i ; $a > 0$].

$$42. \int_0^{\infty} x^{k-1} f\left(ax + \frac{b}{x}\right) dx = \frac{2}{(2a)^k} \int_{2\sqrt{ab}}^{\infty} \frac{x^k f(x)}{V_{x^2 - 4ab}} dx \quad [k = 0 \text{ или } 1; a, b > 0].$$

$$43. \int_0^{\infty} x^{\pm 1/2} f\left(ax + \frac{b}{x}\right) dx = a^{-(2 \pm 1)/2} \int_0^{\infty} y^{-1/2} (y + V_{ab})^{(1 \pm 1)/2} f(y + 2V_{ab}) dy$$

[$a, b > 0$].

$$44. \int_0^{\infty} x^{\pm 3/2} f\left(ax + \frac{b}{x}\right) dx = \left\{ \frac{a^{-5/2}}{b^{-1/2}} \right\} \int_0^{\infty} y^{-1/2} (y^2 + 3V_{ab}y + ab)^{\delta} f(y + 2V_{ab}) dy$$

[$a, b > 0$; $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$].

$$45. \int_0^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right) \ln x \, dx = \ln a \int_0^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right) dx \quad [a > 0].$$

$$46. \int_0^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right) \operatorname{arctg} x \, dx = \frac{\pi}{4} \int_0^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right) dx.$$

$$47. \int_0^{\infty} \frac{1}{x} f\left(\frac{x^p}{a} + \frac{a}{x^p}\right) \ln x \, dx = \frac{\ln a}{p} \int_0^{\infty} \frac{1}{x} f\left(\frac{x^p}{a} + \frac{a}{x^p}\right) dx \quad [a, p > 0].$$

$$48. \int_0^{\infty} f(x^p + x^{-p}) \frac{\ln x}{1+x^2} \, dx = 0.$$

$$49. \int_0^{\infty} x^n f\left(\left[ax - \frac{b}{x}\right]^2\right) dx =$$

$$= \frac{1}{(2a)^{n+1}} \int_0^{\infty} f(y^2) \left[(V y^2 + 4ab - y)^n \left(1 - \frac{y}{V y^2 + 4ab}\right) + \right.$$

$$\left. + (V y^2 + 4ab + y)^n \left(1 + \frac{y}{V y^2 + 4ab}\right) \right] dy$$

$$\left[a, b > 0; ax - \frac{b}{x} = \mp y \text{ при } \begin{cases} 0 \leq x \leq V b/a \\ V b/a \leq x < \infty \end{cases} \right].$$

$$50. \int_0^{\infty} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \frac{1}{a} \int_0^{\infty} f(y^2) dy \quad [a, b > 0].$$

$$51. \int_0^{\infty} x^{\pm 2} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \left\{ \begin{matrix} a^{-3} \\ b^{-1} \end{matrix} \right\} \int_0^{\infty} (y^2 + ab)^{\delta} f(y^2) dy$$

$$[a, b > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$52. \int_0^{\infty} x^{\pm 4} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \left\{ \begin{matrix} a^{-5} \\ b^{-3} \end{matrix} \right\} \int_0^{\infty} \frac{y^4 + 3aby^2 + a^2b^2}{y^2 + ab} f(y^2) dy$$

$$[a, b > 0].$$

$$53. \int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f(x) dx \quad [ab > 0].$$

$$54. \int_{-\infty}^{\infty} f(a \operatorname{sh} x + b \operatorname{ch} x) \operatorname{sh} x \, dx = 2a \int_0^{\infty} f'(\operatorname{sgn} b \sqrt{b^2 - a^2} \operatorname{ch} x) \operatorname{sh}^2 x \, dx.$$

$$55. = \int_{-\infty}^{\infty} \frac{x}{V 1+x^2} f(ax + b V 1+x^2) \, dx.$$

$$56. \int_a^b f(\sin x, \cos x) dx = \int_{a+2\pi n}^{b+2\pi n} f(\sin x, \cos x) dx.$$

$$57. = \int_{a+(2n+1)\pi}^{b+(2n+1)\pi} f(-\sin x, -\cos x) dx.$$

$$58. = \int_{a+(n+1/2)\pi}^{b+(n+1/2)\pi} f((-1)^{n+1} \cos x, (-1)^n \sin x) dx \quad [n=0, \pm 1, \pm 2, \dots].$$

$$59. \int_{(2n+1/2)\pi}^{(2n+1)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(\sin x, -\cos x) dx.$$

$$60. \int_{(2n+1)\pi}^{(2n+3/2)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(-\sin x, -\cos x) dx.$$

$$61. \int_{(2n+3/2)\pi}^{(2n+2)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(-\sin x, \cos x) dx.$$

$$62. \int_0^{n\pi} x f(\sin x) dx = \pi n^2 \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = f(x)].$$

$$63. = \pi n (-1)^{n-1} \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = -f(x)].$$

$$64. \int_0^{\pi/2} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(\cos x, \sin x) dx.$$

$$65. = \int_0^1 f(x, \sqrt{1-x^2}) \frac{dx}{\sqrt{1-x^2}}.$$

$$66. = \int_0^1 f(\sqrt{x}, \sqrt{1-x}) \frac{dx}{2\sqrt{x(1-x)}}.$$

$$67. \int_0^{\pi/2} f(\sin 2x) \cos x dx = \int_0^{\pi/2} f(\cos^2 x) \cos x dx.$$

$$68. \int_0^{\pi/2} f(\sin^2 x, \cos^2 x) dx = \frac{1}{2} \int_0^{\pi} f(\sin^2 x, \cos^2 x) dx.$$

$$69. = \frac{1}{2} \int_{-\infty}^{\infty} f\left(\frac{y^2}{1+y^2}, \frac{1}{1+y^2}\right) \frac{dy}{1+y^2}.$$

$$70. \int_0^{\pi/2} f(\sin^2 x, \cos^2 x) \cos^{\alpha-2} x \cos \alpha x dx = \frac{1}{2} \int_{-\infty}^{\infty} f\left(\frac{y^2}{1+y^2}, \frac{1}{1+y^2}\right) \frac{dy}{(1+iy)^\alpha}.$$

$$\begin{aligned}
 71. \quad \int_0^{\pi/2} [f(e^{ix} \cos x) + f(e^{-ix} \cos x)] dx = \\
 = \frac{1}{2} \int_0^{\pi} [f(e^{ix} \cos x) + f(e^{-ix} \cos x)] dx = \frac{1}{2} \int_{-\pi}^{\pi} f(e^{ix} \cos x) dx = \pi f\left(\frac{1}{2}\right) \\
 \left[f\left(\frac{z+1}{2}\right) \text{ аналитична при } |z| \leq 1 \right].
 \end{aligned}$$

$$\begin{aligned}
 72. \quad \int_0^{\pi/2} \frac{f(e^{ix} \cos x) + f(e^{-ix} \cos x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{1}{2} \int_0^{\pi} \frac{f(e^{ix} \cos x) + f(e^{-ix} \cos x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \\
 = \frac{1}{2} \int_{-\pi}^{\pi} \frac{f(e^{ix} \cos x) dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{ab} f\left(\frac{b}{a+b}\right) \\
 [a, b > 0; f((z+1)/2) \text{ аналитична при } |z| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 73. \quad \int_0^{\pi} \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2b \cos x + b^2} dx = \frac{2\pi}{1-b^2} f(a+b) \\
 [|b| < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 74. \quad \int_0^{\pi} \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2b \cos x + b^2} (1-b \cos x) dx = \pi [f(a+b) + f(a)] \\
 [|b| < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 75. \quad \int_0^{\pi} \frac{f(a+e^{ix}) - f(a+e^{-ix})}{1-2b \cos x + b^2} \sin x dx = \frac{\pi i}{b} [f(a+b) - f(a)] \\
 [|b| < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$76. \quad \int_0^{\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} [f(\sin x, \cos x) + f(\sin x, -\cos x)] dx.$$

$$77. \quad \int_0^{\pi} f(\sin x) dx = 2 \int_0^{\pi/2} f(\sin x) dx.$$

$$78. \quad \int_0^{\pi} f^{(n)}(\cos x) \sin^{2n} x dx = (2n-1)!! \int_0^{\pi} f(\cos x) \cos nx dx.$$

$$\begin{aligned}
 79. \quad \int_0^{\pi} f\left(\frac{\sin^2 x}{1+2b \cos x + b^2}\right) dx &= \int_0^{\pi} f(\sin^2 x) dx & [|b| \geq 1]. \\
 &= \int_0^{\pi} f\left(\frac{\sin^2 x}{b^2}\right) dx & [0 < |b| < 1]
 \end{aligned}$$

$$80. \quad \int_{-\pi/2}^{\pi/2} f(\sin x, \cos x) dx = \int_0^{\pi/2} [f(\sin x, \cos x) + f(-\sin x, \cos x)] dx.$$

$$81. \int_{-\pi/2}^{\pi/2} f(\cos x) dx = 2 \int_0^{\pi/2} f(\cos x) dx.$$

$$82. \int_0^{2\pi} f(\sin x, \cos x) dx = \int_0^{\pi} [f(\sin x, \cos x) + f(-\sin x, \cos x)] dx.$$

$$83. = \int_0^{\pi/2} [f(\sin x, \cos x) + f(\sin x, -\cos x) + \\ + f(-\sin x, \cos x) + f(-\sin x, -\cos x)] dx.$$

$$84. \int_0^{2\pi} f(a \sin x + b \cos x) dx = \int_0^{2\pi} f(\sqrt{a^2 + b^2} \sin x) dx.$$

$$85. = 2 \int_0^{\pi} f(\sqrt{a^2 + b^2} \cos x) dx.$$

$$86. \int_0^{\infty} \frac{f(\sin x)}{x} dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin x} dx \quad [f(-x) = -f(x)].$$

$$87. \int_0^{\infty} \frac{f(\sin x)}{x^2} dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin^2 x} dx \quad [f(-x) = f(x)].$$

$$88. \int_0^{\infty} \frac{f(\sin x)}{x} \cos x dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin x} \cos^2 x dx \quad [f(-x) = -f(x)].$$

$$89. \int_0^{\infty} \frac{f(\sin x)}{x} \operatorname{tg} x dx = \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = f(x)].$$

$$90. \int_0^{\infty} \frac{f(\sin x)}{x^2 + z^2} dx = \frac{\operatorname{sh} 2z}{2z} \int_0^{\pi/2} \frac{f(\sin x)}{\operatorname{ch}^2 z - \cos^2 x} dx \quad [f(-x) = f(x)].$$

$$91. \int_0^{\infty} \frac{f(\cos x)}{x^2 + z^2} dx = \frac{\operatorname{sh} z}{z} \int_0^{\infty} \frac{f(\cos x) \cos x dx}{\operatorname{ch}^2 z - \cos^2 x} \quad [f(-x) = -f(x)].$$

$$92. \int_0^{\infty} \frac{x f(\sin x) \cos^{\delta} x}{x^2 + z^2} dx = \left\{ \frac{1}{\operatorname{ch} z} \right\} \int_0^{\pi/2} \frac{f(\sin x) \sin x \cos^{2\delta} x}{\operatorname{sh}^2 z + \sin^2 x} dx \\ [f(-x) = -f(x); \delta = \left\{ \frac{1}{0} \right\}].$$

$$93. \int_a^b f(x) [g(x)]^{\lambda} \ln^{\eta} g(x) dx = \left(\frac{d}{d\lambda} \right)^{\eta} \int_a^b f(x) [g(x)]^{\lambda} dx.$$

$$94. \int_a^b f(x) \ln^n g(x) dx = \left(\frac{d}{d\lambda} \right)^n \int_a^b f(x) [g(x)]^\lambda dx \Big|_{\lambda=0}.$$

$$95. \int_a^b f(x) \ln^n x dx = \left(\frac{d}{d\lambda} \right)^n \int_a^b x^\lambda f(x) dx \Big|_{\lambda=0}.$$

$$96. \int_0^1 f(x) \ln(\sin \pi x) dx = (1+a) \int_0^{1/2} f(x) \ln(\sin \pi x) dx \quad [f(1-x) = af(x)].$$

2.2. СТЕПЕННАЯ И АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

2.2.1. Введение.

Интегралы от степенной и алгебраических функций представляют собой широкий класс интегралов, к которым с помощью замен переменных сводятся многие интегралы, содержащие различные элементарные и специальные функции.

2.2.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие степенную и алгебраические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций $V_j^\pm$, V_j , $j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. Значения интегралов этого пункта при $|y|=1$ получаются предельным переходом при $|y| \rightarrow 1$. Кроме того, $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$1. \int_0^{\min(1, y)} x^{\alpha-1} (1-x^r)^{\beta-1} (y-x)^{\gamma-1} dx = \begin{cases} V_1^+ \\ V_1^- \end{cases} \quad \begin{matrix} [0 < y \leq 1], \\ [y \geq 1] \end{matrix}$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} \beta, \gamma, rj+\alpha \\ \beta-j, \gamma+rj+\alpha \end{matrix} \right], E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, (\alpha+h)/r \\ \beta+(\alpha+h)/r, \gamma-h \end{matrix} \right], \\ s=rj+\alpha+\gamma-1, u=\gamma-h-1, a_1=rj+\alpha, b_1=1-\beta+j, \\ c_1=\gamma+rj+\alpha, a_3=1-\gamma+h, b_3=(\alpha+h)/r, c_3=\beta+(\alpha+h)/r, z=y^p\} \\ [\operatorname{Re} \alpha > 0 \text{ и } \operatorname{Re} \gamma > 0 \text{ (при } 0 < y < 1), \text{ или } \operatorname{Re} \beta > 0 \text{ (при } y > 1), \text{ или } \operatorname{Re}(\beta+\gamma) > 1 \\ \text{(при } y=1)].$$

$$2. \int_{\max(1, y)}^{\infty} x^{\alpha-1} (x^r-1)^{\beta-1} (x-y)^{\gamma-1} dx = \begin{cases} V_2^+ \\ V_2^- \end{cases} \quad \begin{matrix} [0 < y \leq 1], \\ [y \geq 1] \end{matrix}$$

$$\{E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, 1-\beta+r^{-1}(1+h-\alpha-\gamma) \\ 1+r^{-1}(1+h-\alpha-\gamma), \gamma-h \end{matrix} \right], \\ E_2^- = \Gamma \left[\begin{matrix} \beta, \gamma, 1-\gamma-\alpha+r(1-\beta+j) \\ \beta-j, 1-\alpha+r(1-\beta+j) \end{matrix} \right], \\ t=h, v=\gamma+\alpha-1-r(1-\beta+j), a_2=1-\gamma+h, b_2=1-\beta+r^{-1}(1-\gamma-\alpha+h), \\ c_2=1+r^{-1}(1-\gamma-\alpha+h), a_4=1-\gamma-\alpha+r(1-\beta+j), b_4=1-\beta+j, \\ c_4=1-\alpha+r(1-\beta+j), z=y^p\} \\ [\operatorname{Re}(\alpha+\gamma+r\beta) < 1+r \text{ и } \operatorname{Re} \beta > 0 \text{ (при } 0 < y < 1), \text{ или } \operatorname{Re} \gamma > 0 \text{ (при } y > 1), \\ \text{или } \operatorname{Re}(\beta+\gamma) > 1 \text{ (при } y=1)].$$

$$3. \int_y^1 x^{\alpha-1} (1-x^r)^{\beta-1} (x-y)^{\gamma-1} dx = V_1^+ + V_2^+$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} \beta, \gamma, 1-\gamma-\alpha-rj \\ \beta-j, 1-\alpha-rj \end{matrix} \right], E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, r^{-1}(\gamma+\alpha-1-h) \\ \beta-r^{-1}(1-\gamma-\alpha+h), \gamma-h \end{matrix} \right],$$

$$s=rj+\alpha+\gamma-1, t=h, a_1=\alpha+rj, b_1=1-\beta+j, c_1=\gamma+\alpha+rj,$$

$$a_2=1-\gamma+h, b_2=1-\beta+r^{-1}(1-\gamma-\alpha+h), c_2=1+r^{-1}(1-\gamma-\alpha+h), z=y^p\}$$

$$[0 < y < 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$4. \int_1^y x^{\alpha-1} (x^r-1)^{\beta-1} (y-x)^{\gamma-1} dx = V_1^- + V_2^-$$

$$\{E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, 1-\beta-(\alpha+h)/r \\ 1-(\alpha+h)/r, \gamma-h \end{matrix} \right], E_2^- = \Gamma \left[\begin{matrix} \beta, \gamma, \alpha-r(1-\beta+j) \\ \beta-j, \gamma+\alpha-r(1-\beta+j) \end{matrix} \right],$$

$$u=\gamma-h-1, v=\gamma+\alpha-r(1-\beta+j)-1, a_3=1-\gamma+h, b_3=(\alpha+h)/r,$$

$$c_3=\beta+(\alpha+h)/r, a_4=1-\alpha-\gamma+r(1-\beta+j), b_4=1-\beta+j,$$

$$c_4=1-\alpha+r(1-\beta+j), z=y^p\}$$

$$[y > 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$5. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} (y+x)^{\rho-1} dx = \begin{cases} V_1^+ + V_2^+ & [y' \leq 1], \\ V_1^- & [|y| \geq 1] \end{cases}$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} \beta, rj+\alpha, 1-\rho-\alpha-rj \\ 1-\rho, \beta-j \end{matrix} \right], E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, r^{-1}(\alpha+\rho-h-1), 1-\rho+h \\ 1-\rho, \beta+r^{-1}(\alpha+\rho-h-1) \end{matrix} \right],$$

$$E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, (\alpha+h)/r, 1-\rho+h \\ 1-\rho, \beta+(\alpha+h)/r \end{matrix} \right], s=rj+\alpha+\rho-1,$$

$$t=h, u=\rho-h-1, a_1=rj+\alpha, b_1=1-\beta+j, c_1=\alpha+rj+\rho,$$

$$a_2=1-\rho+h, b_2=1-\beta+r^{-1}(1-\rho-\alpha+h), c_2=1+r^{-1}(1-\rho-\alpha+h),$$

$$a_3=1-\rho+h, b_3=(\alpha+h)/r, c_3=\beta+(\alpha+h)/r, z=(-y)^p\}$$

$$[|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$6. \int_0^\infty x^{\alpha-1} (1+x^r)^{\sigma-1} (y+x)^{\rho-1} dx = \begin{cases} V_1^+ + V_2^+ & [|y| \leq 1], \\ V_1^- + V_2^- & [|y| \geq 1] \end{cases}$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} 1-\sigma+j, \alpha+rj, 1-\rho-\alpha-rj \\ 1-\sigma, 1-\rho \end{matrix} \right],$$

$$E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} (\alpha+\rho-h-1)/r, 1-\sigma+r^{-1}(1-\alpha-\rho+h), 1-\rho+h \\ 1-\sigma, 1-\rho \end{matrix} \right],$$

$$E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} (\alpha+h)/r, 1-\sigma-(\alpha+h)/r, 1-\rho+h \\ 1-\sigma, 1-\rho \end{matrix} \right],$$

$$E_2^- = \Gamma \left[\begin{matrix} 1-\sigma+j, \alpha-r(1-\sigma+j), 1-\rho-\alpha+r(1-\sigma+j) \\ 1-\sigma, 1-\rho \end{matrix} \right],$$

$$s=rj+\alpha+\rho-1, t=h, u=\rho-1-h, v=\rho+\alpha-1-r(1-\sigma+j),$$

$$a_1=rj+\alpha, b_1=1-\sigma+j, c_1=\rho+\alpha+rj, a_2=1-\rho+h,$$

$$b_2=1-\sigma+r^{-1}(1+h-\alpha-\rho), c_2=1+r^{-1}(1+h-\alpha-\rho), a_3=1-\rho+h,$$

$$b_3=(\alpha+h)/r, c_3=\sigma+(\alpha+h)/r, a_4=1-\rho-\alpha+r(1-\sigma+j), b_4=1-\sigma+j,$$

$$c_4=1-\alpha+r(1-\sigma+j), z=(-1)^{p+q} y^p\}$$

$$[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha+\sigma r+\rho) < 1+r].$$

$$7. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} \frac{dx}{y-x} = \begin{cases} V_3^+ + V_4^+ & [0 < y \leq 1], \\ V_3^- & [y \geq 1] \end{cases}$$

$$\{E_3^+ = \pi \operatorname{ctg}[(\alpha + rj)\pi] \Gamma\left[\begin{matrix} \beta \\ \beta - j \end{matrix}\right], \quad E_4^+ = -\frac{1}{r} B\left(\beta, \frac{\alpha - h - 1}{r}\right),$$

$$E_3^- = \frac{1}{r} B\left(\beta, \frac{\alpha + h}{r}\right),$$

$$s = rj + \alpha - 1, \quad t = h, \quad u = -h - 1, \quad a_1 = 1 - \beta + j, \quad a_2 = 1 - \beta + (1 + h - \alpha)/r,$$

$$b_2 = 1 + (1 + h - \alpha)/r, \quad a_3 = (\alpha + h)/r, \quad b_3 = \beta + (\alpha + h)/r, \quad z = y^p\}$$

$$[\operatorname{Re} \alpha > 0 \text{ и } \operatorname{Re} \beta > 0 \text{ (при } y \neq 1), \text{ или } \operatorname{Re} \beta > 1 \text{ (при } y = 1)].$$

$$8. \int_0^\infty x^{\alpha-1} (1+x^r)^{\rho-1} \frac{dx}{y-x} = \begin{cases} V_3^+ + V_4^+ & [0 < y \leq 1], \\ V_3^- + V_4^- & [y \geq 1] \end{cases}$$

$$\{E_3^+ = \pi \operatorname{ctg}[(rj + \alpha)\pi] \Gamma\left[\begin{matrix} 1 - \rho + j \\ 1 - \rho \end{matrix}\right],$$

$$E_4^+ = -\frac{1}{r} B\left(\frac{\alpha - 1 - h}{r}, 1 - \rho + \frac{1 + h - \alpha}{r}\right), \quad E_3^- = \frac{1}{r} B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right),$$

$$E_4^- = \pi \operatorname{ctg}[(\alpha - r + r\rho)\pi] \Gamma\left[\begin{matrix} 1 - \rho + j \\ 1 - \rho \end{matrix}\right],$$

$$s = \alpha + rj - 1, \quad t = h, \quad u = -h - 1, \quad v = \alpha - 1 - r(1 - \rho),$$

$$a_1 = 1 - \rho + j, \quad a_2 = 1 - \rho + \frac{1 - \alpha + h}{r}, \quad b_2 = 1 + \frac{1 - \alpha + h}{r}, \quad a_3 = \frac{\alpha + h}{r},$$

$$b_3 = \rho + \frac{\alpha + h}{r}, \quad a_4 = 1 - \rho + j, \quad z = (-1)^q y^p\}$$

$$[y > 0; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha + r\rho) < 1 + r].$$

$$9. \int_0^\infty x^{\alpha-1} \frac{dx}{(1-x^r)(y-x)} =$$

$$= \frac{\pi}{1-y^p} \left[\sum_{j=0}^{q-1} \operatorname{ctg}[(rj + \alpha)\pi] y^{rj+\alpha-1} + \frac{1}{r} \sum_{h=0}^{p-1} \operatorname{ctg}\left(\frac{1+h-\alpha}{r}\pi\right) y^h \right]$$

$$[0 < y < 1].$$

$$= \frac{\pi}{y^p - 1} \left[\sum_{j=0}^{q-1} \operatorname{ctg}[(r + rj - \alpha)\pi] y^{p+\alpha-r(1+j)-1} + \right.$$

$$\left. + \frac{1}{r} \sum_{h=0}^{p-1} \operatorname{ctg}\left(\frac{\alpha + h}{r}\pi\right) y^{p-h-1} \right] \quad [y > 1], [0 < \operatorname{Re} \alpha < 1 + r].$$

$$10. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} (\sqrt{y} + \sqrt{y+x})^\sigma dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [|y| \leq 1], \\ V_5^- & [|y| \geq 1] \end{cases}$$

$$\{E_5^+ = -2^{\sigma+2\alpha+2jr} \sigma \Gamma\left[\begin{matrix} \beta, jr + \alpha, 1 - \sigma - 2\alpha - 2jr \\ \beta - j, 1 - \sigma - \alpha - jr \end{matrix}\right],$$

$$\begin{aligned}
 E_6^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\begin{matrix} h-\sigma/2, & 1/2-h \\ 1-\sigma/2-h \end{matrix}\right], \\
 E_7^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\sigma+2\alpha-2h-1}{2r}\right) \Gamma\left[\begin{matrix} (1-\sigma)/2+h, & -1/2-h \\ (1-\sigma)/2-h \end{matrix}\right], \\
 E_5^- &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\beta, \frac{\alpha+h}{r}\right) \Gamma\left[\begin{matrix} 2h-\sigma \\ 1+h-\sigma \end{matrix}\right], \quad s=\sigma/2+\alpha+jr, \\
 t=h, \quad t_1=1/2+h, \quad u=\sigma/2-h, \quad a_1=jr+\alpha, \quad b_1=\sigma+jr+\alpha, \\
 c_1=1-\beta+j, \quad d_1=1+\sigma/2+jr+\alpha, \quad e_1=(1+\sigma)/2+jr+\alpha, \\
 a_2=h-\sigma/2, \quad b_2=\sigma/2+h, \quad c_2=1-\beta+(2h-2\alpha-\sigma)/(2r), \quad d_2=1/2+h, \\
 e_2=1+(2h-2\alpha-\sigma)/(2r), \quad a_3=(1-\sigma)/2+h, \quad b_3=(1+\sigma)/2+h, \\
 c_3=1-\beta+(2h-2\alpha-\sigma+1)/(2r), \quad d_3=3/2+h, \quad e_3=1+(2h-2\alpha-\sigma+1)/(2r), \\
 a_4=h-\sigma/2, \quad b_4=(1-\sigma)/2+h, \quad c_4=(\alpha+h)/r, \quad d_4=1-\sigma+h, \\
 e_4=\beta+(\alpha+h)/r, \quad z=(-y)^p \quad [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$11. \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (\sqrt{x} + \sqrt{y+x})^{\sigma} dx = \begin{cases} V_5^+ + V_6^+ & [|y| \leq 1] \\ V_5^- + V_6^- & [|y| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_5^+ &= -2^{-2\alpha-2jr} \sigma \Gamma\left[\begin{matrix} \beta, & 2\alpha+2jr, & -\sigma/2-\alpha-jr \\ \beta-j, & 1-\sigma/2+\alpha+jr \end{matrix}\right], \\
 E_6^+ &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\beta, \frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\begin{matrix} 2h-\sigma \\ 1-\sigma+h \end{matrix}\right], \\
 E_5^- &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\alpha+h}{r}\right) \Gamma\left[\begin{matrix} 1/2-h, & h-\sigma/2 \\ 1-\sigma/2-h \end{matrix}\right], \\
 E_6^- &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{2\alpha+2h+1}{2r}\right) \Gamma\left[\begin{matrix} -1/2-h, & (1-\sigma)/2+h \\ (1-\sigma)/2-h \end{matrix}\right], \\
 s &= \sigma/2+\alpha+jr, \quad t=h, \quad u=\sigma/2-h, \quad u_1=(\sigma-1)/2-h, \quad a_1=jr+\alpha, \\
 b_1 &= 1/2+jr+\alpha, \quad c_1=1-\beta+j, \quad d_1=1+\sigma/2+jr+\alpha, \quad e_1=1-\sigma/2+jr+\alpha, \\
 a_2 &= h-\sigma/2, \quad b_2=(1-\sigma)/2+h, \quad c_2=1-\beta+(2h-2\alpha-\sigma)/(2r), \quad d_2=1-\sigma+h, \\
 e_2 &= 1+(2h-2\alpha-\sigma)/(2r), \quad a_4=h-\sigma/2, \quad b_4=\sigma/2+h, \\
 c_4 &= (\alpha+h)/r, \quad d_4=1/2+h, \quad e_4=\beta+(\alpha+h)/r, \quad a_5=(1+\sigma)/2+h, \\
 b_5 &= (1-\sigma)/2+h, \quad c_5=(2\alpha+2h+1)/(2r), \quad d_5=3/2+h, \\
 e_5 &= \beta+(2\alpha+2h+1)/(2r), \quad z=(-y)^p \quad [|\arg y| < \pi, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$12. \int_0^{\infty} x^{\alpha-1} (1+x)^{\rho-1} (\sqrt{y} + \sqrt{y+x})^{\sigma} dx = \begin{cases} V_6^+ + V_6^+ + V_7^+ & [|y| \leq 1] \\ V_5^- + V_7^- & [|y| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_5^+ &= -2^{\sigma+2\alpha+2jr} \sigma \Gamma\left[\begin{matrix} 1-\rho+j, & jr+\alpha, & -2jr-2\alpha-\sigma \\ 1-\rho, & 1-\sigma-\alpha-jr \end{matrix}\right], \\
 E_6^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\frac{\sigma+2\alpha-2h}{2r}, 1-\rho-\frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\begin{matrix} h-\sigma/2, & 1/2-h \\ 1-\sigma/2-h \end{matrix}\right], \\
 E_7^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\frac{\sigma-1+2\alpha-2h}{2r}, 1-\rho-\frac{\sigma-1+2\alpha-2h}{2r}\right) \times \\
 &\quad \times \Gamma\left[\begin{matrix} (1-\sigma)/2+h, & -1/2-h \\ (1-\sigma)/2-h \end{matrix}\right], \\
 E_5^- &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\frac{\alpha+h}{r}, 1-\rho-\frac{\alpha+h}{r}\right) \Gamma\left[\begin{matrix} -\sigma+2h \\ 1-\sigma+h \end{matrix}\right],
 \end{aligned}$$

$$E_7^- = -2^{\sigma+2\alpha-2r(1-\rho+j)} \sigma \Gamma \left[\begin{matrix} 1-\rho+j, \alpha+r(\rho-j-1), 2r(1-\rho+j)-2\alpha-\sigma \\ 1-\rho, 1-\sigma-\alpha+r(1+j-\rho) \end{matrix} \right],$$

$$s = \sigma/2 + \alpha + jr, t = h, t_1 = 1/2 + h, u = \sigma/2 - h, v = \sigma/2 + \alpha - r(1-\rho+j),$$

$$a_1 = \alpha + jr, b_1 = \sigma + \alpha + jr, c_1 = 1 - \rho + j, d_1 = 1 + \sigma/2 + \alpha + jr,$$

$$e_1 = (1 + \sigma)/2 + \alpha + jr, a_2 = h - \sigma/2, b_2 = \sigma/2 + h, c_2 = 1 - \rho + (2h - \sigma - 2\alpha)/(2r),$$

$$d_2 = 1/2 + h, e_2 = 1 + (2h - \sigma - 2\alpha)/(2r), a_3 = (1 - \sigma)/2 + h, b_3 = (1 + \sigma)/2 + h,$$

$$c_3 = 1 - \rho + (2h - \sigma - 2\alpha + 1)/(2r), d_3 = 3/2 + h, e_3 = 1 + (2h - \sigma - 2\alpha + 1)/(2r),$$

$$a_4 = h - \sigma/2, b_4 = (1 - \sigma)/2 + h, c_4 = (\alpha + h)/r, d_4 = 1 - \sigma + h,$$

$$e_4 = \rho + (\alpha + h)/r, a_5 = r(1 - \rho + j) - \alpha - \sigma/2, b_5 = r(1 - \rho + j) - \alpha + (1 - \sigma)/2,$$

$$c_5 = 1 - \rho + j, d_5 = 1 - \alpha + r(1 - \rho + j), e_5 = 1 - \sigma - \alpha + r(1 - \rho + j),$$

$$z = (-1)^{q+p} y^p \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha + \rho r + \sigma/2) < r].$$

$$13. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\bar{y} + x)^\sigma}{1-x^r} dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [0 < y \leq 1], \\ V_5^- + V_7^- & [y \geq 1] \end{cases}$$

$$\{E_5^+ = (-1)^{j+1} 2^{\sigma+2\alpha+2jr} \sigma j! \Gamma \left[\begin{matrix} \alpha + jr, -\sigma - 2\alpha - 2jr \\ 1 - \sigma - \alpha - jr \end{matrix} \right],$$

$$E_6^+ = \frac{\sigma V\bar{\pi}}{2r} \operatorname{ctg} \left(\frac{2h - 2\alpha - \sigma}{2r} \pi \right) \Gamma \left[\begin{matrix} h - \sigma/2, 1/2 - h \\ 1 - \sigma/2 - h \end{matrix} \right],$$

$$E_7^+ = \frac{\sigma V\bar{\pi}}{2r} \operatorname{ctg} \left(\frac{2h - 2\alpha - \sigma + 1}{2r} \pi \right) \Gamma \left[\begin{matrix} (1 - \sigma)/2 + h, -1/2 - h \\ (1 - \sigma)/2 - h \end{matrix} \right],$$

$$E_5^- = -\frac{\sigma \pi}{r} 2^{\sigma-2h} \operatorname{ctg} \left(\frac{\alpha + h}{r} \pi \right) \Gamma \left[\begin{matrix} 2h - \sigma \\ 1 - \sigma + h \end{matrix} \right],$$

$$E_7^- = (-1)^j j! \sigma \pi 2^{\sigma+2\alpha-2r(1+j)} \Gamma \left[\begin{matrix} -\sigma - 2\alpha + 2r(1+j), \alpha - r(1+j) \\ 1 - \sigma - \alpha + r(1+j) \end{matrix} \right],$$

$$s = \sigma/2 + \alpha + jr, t = h, t_1 = 1/2 + h, u = \sigma/2 - h, v = \sigma/2 + \alpha - r(1+j),$$

$$a_1 = \alpha + jr, b_1 = \sigma + \alpha + jr, c_1 = 1 + j, d_1 = 1 + \sigma/2 + \alpha + jr,$$

$$e_1 = (1 + \sigma)/2 + \alpha + jr, a_2 = h - \sigma/2, b_2 = \sigma/2 + h, c_2 = 1 + (2h - 2\alpha - \sigma)/(2r),$$

$$d_2 = 1/2 + h, e_2 = c_2, a_3 = (1 - \sigma)/2 + h, b_3 = (1 + \sigma)/2 + h,$$

$$c_3 = 1 + (2h - 2\alpha - \sigma + 1)/(2r), d_3 = 3/2 + h, e_3 = c_3, a_4 = h - \sigma/2, b_4 = (1 - \sigma)/2 + h,$$

$$c_4 = (\alpha + h)/r, d_4 = 1 - \sigma + h, e_4 = c_4, a_5 = r(1 + j) - \alpha - \sigma/2,$$

$$b_5 = r(1 + j) - \alpha + (1 - \sigma)/2, c_5 = 1 + j, d_5 = 1 - \alpha + r(1 + j),$$

$$e_5 = 1 - \sigma - \alpha + r(1 + j), m = 2p \text{ или } 2p + 1, z = (-y)^p \}$$

$$[y, \operatorname{Re} \alpha > 0, \operatorname{Re}(\alpha + \sigma/2) < r].$$

$$14. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\bar{y} + x)^\sigma}{(1 + V\bar{1} + x)^\gamma} dx = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ & [|\gamma| \leq 1], \\ V_8^- + V_9^- + V_{10}^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_8^+ = -2^{\sigma+2\alpha-\gamma} \sigma \Gamma \left[\begin{matrix} \alpha, -\sigma - 2\alpha \\ 1 - \sigma - \alpha \end{matrix} \right], E_9^+ = \gamma \Gamma \left[\begin{matrix} \alpha + \sigma/2, \gamma - \sigma - 2\alpha \\ 1 + \gamma - \alpha - \sigma/2 \end{matrix} \right] 2^{2\alpha+\sigma-\gamma},$$

$$E_{10}^+ = 2^{\sigma+2\alpha-\gamma-1} \gamma \sigma \Gamma \left[\begin{matrix} \alpha + (\sigma-1)/2, 1 + \gamma - \sigma - 2\alpha \\ 1 + \gamma - \alpha + (1-\sigma)/2 \end{matrix} \right],$$

$$E_8^- = 2^{\sigma-\gamma+2\alpha} \gamma \Gamma \left[\begin{matrix} \alpha, \gamma - 2\alpha \\ 1 + \gamma - \alpha \end{matrix} \right], E_9^- = -2^{\sigma+2\alpha-\gamma} \sigma \Gamma \left[\begin{matrix} \alpha - \gamma/2, \gamma - \sigma - 2\alpha \\ 1 - \alpha - \sigma + \gamma/2 \end{matrix} \right],$$

$$E_{10}^- = 2^{\sigma+2\alpha-\gamma-1} \gamma \sigma \Gamma \left[\begin{matrix} \alpha - (\gamma+1)/2, 1 + \gamma - \sigma - 2\alpha \\ 3/2 - \alpha - \sigma + \gamma/2 \end{matrix} \right],$$

$$s_1 = \alpha + \sigma/2, s_2 = 0, s_3 = 1/2, s_5 = \alpha + (\sigma - \gamma)/2, s_6 = \alpha + (\sigma - \gamma - 1)/2,$$

$$\begin{aligned}
s_7 &= \sigma/2, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = \gamma/2, \quad d_1 = (1 + \gamma)/2, \quad e_1 = 1 + \sigma/2 + \alpha, \\
f_1 &= (1 + \sigma)/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = -\sigma/2, \quad b_2 = \sigma/2, \quad c_2 = (\gamma - \sigma)/2 - \alpha, \\
d_2 &= (1 + \gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \quad f_2 = 1 + \gamma - \alpha - \sigma/2, \quad g_2 = 1 - \alpha - \sigma/2, \quad a_3 = (1 - \sigma)/2, \\
b_3 &= (1 + \sigma)/2, \quad c_3 = (1 + \gamma - \sigma)/2 - \alpha, \quad d_3 = 1 + (\gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \\
f_3 &= (3 - \sigma)/2 - \alpha + \gamma, \quad g_3 = (3 - \sigma)/2 - \alpha, \quad a_5 = (\gamma - \sigma)/2 - \alpha, \quad b_5 = (1 + \gamma - \sigma)/2 - \alpha, \\
c_5 &= \gamma/2, \quad d_5 = -\gamma/2, \quad e_5 = 1 - \alpha + \gamma/2, \quad f_5 = 1 - \sigma - \alpha + \gamma/2, \quad g_5 = 1/2, \\
a_6 &= (1 + \gamma - \sigma)/2 - \alpha, \quad b_6 = 1 - \alpha + (\gamma - \sigma)/2, \quad c_6 = (1 + \gamma)/2, \quad d_6 = (1 - \gamma)/2, \\
e_6 &= (3 + \gamma)/2 - \alpha, \quad f_6 = (3 + \gamma)/2 - \sigma - \alpha, \quad g_6 = 3/2, \quad a_7 = -\sigma/2, \quad b_7 = (1 - \sigma)/2, \\
c_7 &= \alpha, \quad d_7 = \alpha - \gamma, \quad e_7 = 1 - \sigma, \quad f_7 = 1 + \alpha - \gamma/2, \quad g_7 = (1 - \gamma)/2 + \alpha, \quad z = y\} \\
&\quad [|\arg y| < \pi, \operatorname{Re} \alpha > 0, \operatorname{Re} (2\alpha + \sigma - \gamma) < 0].
\end{aligned}$$

$$15. \quad \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\overline{y+x})^\sigma}{(V\bar{x} + V\overline{1+x})^\gamma} dx = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ + V_{11}^+ & [|\gamma| \leq 1], \\ V_8^- + V_9^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_8^+ = -2^{-2\alpha-\sigma} \sigma \Gamma \left[\begin{matrix} \alpha, & -\sigma-2\alpha \\ 1-\sigma-\alpha \end{matrix} \right], \quad E_9 = 2^{\sigma+2\alpha+1} \gamma \sigma \Gamma \left[\begin{matrix} -\sigma-2\alpha-1, & \alpha+1/2 \\ 1/2-\alpha-\sigma \end{matrix} \right],$$

$$E_{10}^+ = 2^{-2\sigma-\alpha} \gamma \Gamma \left[\begin{matrix} 2\alpha+\sigma, & (\gamma-\sigma)/2-\alpha \\ 1+\alpha+(\gamma+\sigma)/2 \end{matrix} \right],$$

$$E_{11}^+ = 2^{1-2\alpha-\sigma} \gamma \sigma \Gamma \left[\begin{matrix} 2\alpha+\sigma-1, & (\gamma-\sigma+1)/2-\alpha \\ (1+\gamma+\sigma)/2+\alpha \end{matrix} \right],$$

$$E_8^- = -2^{\sigma+2\alpha-2\gamma} \sigma \Gamma \left[\begin{matrix} \alpha-\gamma/2, & \gamma-\sigma-2\alpha \\ 1-\sigma-\alpha+\gamma/2 \end{matrix} \right],$$

$$E_9^- = 2^{\sigma-2\alpha} \gamma \Gamma \left[\begin{matrix} 2\alpha, & \gamma/2-\alpha \\ 1+\gamma/2+\alpha \end{matrix} \right],$$

$$\begin{aligned}
s_1 &= \alpha + \sigma/2, \quad s_2 = \alpha + (\sigma + 1)/2, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (-\gamma + \sigma)/2 + \alpha, \quad s_6 = \sigma/2, \\
a_1 &= \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = -\gamma/2, \quad d_1 = \gamma/2, \quad e_1 = 1 + \sigma/2 + \alpha, \quad f_1 = (1 + \sigma)/2 + \alpha, \\
g_1 &= 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \quad c_2 = (1 - \gamma)/2, \quad d_2 = (1 + \gamma)/2, \\
e_2 &= (3 + \sigma)/2 + \alpha, \quad f_2 = 1 + \sigma/2 + \alpha, \quad g_2 = 3/2, \quad a_3 = -\sigma/2, \quad b_3 = \sigma/2, \quad c_3 = (\gamma - \sigma)/2 - \alpha, \\
d_3 &= -(\gamma + \sigma)/2 - \alpha, \quad e_3 = 1/2, \quad f_3 = 1 - \alpha - \sigma/2, \quad g_3 = (1 - \sigma)/2 - \alpha, \quad a_4 = (1 - \sigma)/2, \\
b_4 &= (1 + \sigma)/2, \quad c_4 = (1 + \gamma - \sigma)/2 - \alpha, \quad d_4 = (1 - \gamma - \sigma)/2 - \alpha, \quad e_4 = 3/2, \\
f_4 &= (3 - \sigma)/2 - \alpha, \quad g_4 = 1 - \sigma/2 - \alpha, \quad a_5 = (\gamma - \sigma)/2 - \alpha, \quad b_5 = (1 + \gamma - \sigma)/2 - \alpha, \\
c_5 &= \gamma/2, \quad d_5 = (1 + \gamma)/2, \quad e_5 = 1 + \gamma/2 - \alpha, \quad f_5 = 1 - \sigma - \alpha + \gamma/2, \quad g_5 = 1 + \gamma, \\
a_6 &= -\sigma/2, \quad b_6 = (1 - \sigma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \quad e_6 = 1 - \sigma, \quad f_6 = 1 + \alpha - \gamma/2, \\
g_6 &= 1 + \alpha + \gamma/2, \quad z = -y\} \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma) > 0].
\end{aligned}$$

$$16. \quad \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \frac{(V\bar{y} + V\overline{y+x})^\sigma}{V\bar{y} + x} dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [|\gamma| \leq 1], \\ V_5^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_5^+ = 2^{\sigma+2\alpha+2r} \Gamma \left[\begin{matrix} \beta, & \alpha+jr, & 1-\sigma-2\alpha-2jr \\ \beta-j, & 1-\sigma-\alpha-jr \end{matrix} \right],$$

$$E_6^+ = \frac{1}{r \sqrt{\pi}} B \left(\beta, \frac{\sigma+2\alpha-2h-1}{2r} \right) \Gamma \left[\begin{matrix} (1-\sigma)/2+h, & 1/2-h \\ (1-\sigma)/2-h \end{matrix} \right],$$

$$E_7^+ = \frac{1}{r \sqrt{\pi}} B \left(\beta, \frac{\sigma+2\alpha-2h-2}{2r} \right) \Gamma \left[\begin{matrix} 1-\sigma/2+h, & -1/2-h \\ -\sigma/2-h \end{matrix} \right],$$

$$E_5^- = \frac{2^{\sigma-2h}}{r} B \left(\beta, \frac{\alpha+h}{r} \right) \Gamma \left[\begin{matrix} 1-\sigma+2h \\ 1-\sigma+h \end{matrix} \right], \quad s = (\sigma-1)/2 + \alpha + jr, \quad t = h,$$

$$t_1 = h + 1/2, \quad u = (\sigma-1)/2 - h, \quad a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 - \beta + j,$$

$$\begin{aligned}
 d_1 &= (1 + \sigma)/2 + \alpha + jr, \quad e_1 = \sigma/2 + \alpha + jr, \quad a_2 = (1 - \sigma)/2 + h, \quad b_2 = (1 + \sigma)/2 + h, \\
 c_2 &= 1 - \beta + (1 - \sigma - 2\alpha + 2h)/(2r), \quad d_2 = 1/2 + h, \quad e_2 = 1 + (1 - \sigma - 2\alpha + 2h)/(2r), \\
 a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = 1 - \beta + (2 - \sigma - 2\alpha + 2h)/(2r), \quad d_3 = 3/2 + h, \\
 e_3 &= 1 + (2 - \sigma - 2\alpha + 2h)/(2r), \quad a_4 = (1 - \sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \\
 c_4 &= (\sigma + h)/r, \quad d_4 = 1 - \sigma + h, \quad e_4 = \beta + (\sigma + h)/r, \quad z = (1 - \sigma)p
 \end{aligned}$$

$$\begin{aligned}
c_2 &= 1 - \rho + (1 - \sigma - 2\alpha + 2h)/(2r), \quad d_2 = 1/2 + h, \quad e_2 = 1 + (1 - \sigma - 2\alpha + 2h)/(2r), \\
a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = 1 - \rho + (2 - \sigma - 2\alpha + 2h)/(2r), \quad d_3 = 3/2 + h, \\
e_3 &= 1 + (2 - \sigma - 2\alpha + 2h)/(2r), \quad a_4 = (1 - \sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \quad c_4 = (\alpha + h)/r, \\
d_4 &= 1 - \sigma + h, \quad e_4 = \rho + (\alpha + h)/r, \quad a_6 = (1 - \sigma)/2 - \alpha + r(1 - \rho + j), \\
b_6 &= 1 - \sigma/2 - \alpha + r(1 - \rho + j), \quad c_6 = 1 - \rho + j, \quad d_6 = 1 - \alpha + r(1 - \rho + j), \\
e_6 &= 1 - \sigma - \alpha + r(1 - \rho + j), \quad z = (-1)^{q+p} y^p \} \\
&[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha + 2r\rho + \sigma - 1) < r].
\end{aligned}$$

$$19. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\overline{y+x})^\sigma}{(1-x^r) V\overline{y+x}} dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [|\arg y| \leq 1], \\ V_5^- + V_7^- & [|\arg y| \geq 1] \end{cases}$$

$$\begin{aligned}
\{E_5^+ &= (-1)^j j! 2^{\sigma+2\alpha+2rj} \Gamma \left[\begin{matrix} 1-\sigma-2\alpha-2rj, & \alpha+rj \\ 1-\sigma-\alpha-rj \end{matrix} \right], \\
E_6^+ &= -\frac{V\pi}{r} \operatorname{ctg} \left(\frac{1-\sigma-2\alpha+2h}{2r} \pi \right) \Gamma \left[\begin{matrix} (1-\sigma)/2+h, & 1/2-h \\ (1-\sigma)/2-h \end{matrix} \right], \\
E_7^+ &= -\frac{V\pi}{r} \operatorname{ctg} \left(\frac{2-\sigma-2\alpha+2h}{2r} \pi \right) \Gamma \left[\begin{matrix} 1-\sigma/2+h, & -1/2-h \\ -\sigma/2-h \end{matrix} \right], \\
E_5^- &= \frac{\pi}{r} 2^{\sigma-2h} \operatorname{ctg} \left(\frac{\alpha+h}{r} \pi \right) \Gamma \left[\begin{matrix} 1-\sigma+2h \\ 1-\sigma+h \end{matrix} \right], \\
E_7^- &= (-1)^{j+1} j! 2^{\sigma+2\alpha-2r(1+j)} \Gamma \left[\begin{matrix} 1-\sigma-2\alpha+2r(1+j), & \alpha-r(1+j) \\ 1-\sigma-\alpha+r(1+j) \end{matrix} \right], \\
s &= (\sigma-1)/2 + \alpha + jr, \quad t = h, \quad t_1 = h + 1/2, \quad u = (\sigma-1)/2 - h, \\
v &= (\sigma-1)/2 + \alpha - r(1+j), \quad a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 + j, \\
d_1 &= (1+\sigma)/2 + \alpha + rj, \quad e_1 = \sigma/2 + \alpha + rj, \quad a_2 = (1-\sigma)/2 + h, \\
b_2 &= (\sigma+1)/2 + h, \quad c_2 = e_2 = 1 + (1-\sigma-2\alpha+2h)/(2r), \quad d_2 = 1/2 + h, \\
a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = e_3 = 1 + (2-\sigma-2\alpha+2h)/(2r), \\
d_3 &= 3/2 + h, \quad a_4 = (1-\sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \quad c_4 = e_4 = (\alpha+h)/r, \\
d_4 &= 1 - \sigma + h, \quad a_6 = (1-\sigma)/2 - \alpha + r(1+j), \quad b_6 = 1 - \sigma/2 - \alpha + r(1+j), \\
c_6 &= 1 + j, \quad d_6 = 1 - \alpha + r(1+j), \quad e_6 = 1 - \sigma - \alpha + r(1+j), \quad z = (-y)^p \} \\
&[|\arg y| < \pi; 0 < \operatorname{Re} \alpha < r + (1 - \operatorname{Re} \sigma)/2].
\end{aligned}$$

$$20. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\overline{y+x})^\sigma}{(1+V\overline{1+x})^\gamma V\overline{y+x}} dx = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ & [|\arg y| \leq 1], \\ V_8^- + V_9^- + V_{10}^- & [|\arg y| \geq 1] \end{cases}$$

$$\begin{aligned}
\{E_8^+ &= 2^{\sigma+2\alpha-\gamma} B(\alpha, 1-\sigma-2\alpha), \\
E_9^+ &= 2^{2\alpha+\sigma-\gamma-1} \gamma \Gamma \left[\begin{matrix} \alpha + (\sigma-1)/2, & \gamma-2\alpha-\sigma+1 \\ 1+\gamma-\alpha + (1-\sigma)/2 \end{matrix} \right], \\
E_{10}^+ &= 2^{2\alpha+\sigma-\gamma-2} \gamma \sigma \Gamma \left[\begin{matrix} \alpha + \sigma/2 - 1, & \gamma+2-\sigma-2\alpha \\ 2+\gamma-\alpha-\sigma/2 \end{matrix} \right], \\
E_8^- &= 2^{\sigma+2\alpha-\gamma} \gamma \Gamma \left[\begin{matrix} \alpha, & \gamma-2\alpha \\ 1+\gamma-\alpha \end{matrix} \right], \quad E_9^- = 2^{\sigma+2\alpha-\gamma} B \left(\alpha - \frac{\gamma}{2}, 1+\gamma-\sigma-2\alpha \right), \\
E_{10}^- &= -2^{2\alpha+\sigma-\gamma-1} \gamma \Gamma \left[\begin{matrix} \alpha - (\gamma+1)/2, & 2-2\alpha+\gamma-\sigma \\ 1-\alpha-\sigma + (\gamma+1)/2 \end{matrix} \right], \\
s_1 &= (\sigma-1)/2 + \alpha, \quad s_2 = 0, \quad s_3 = 1/2, \quad s_5 = (\sigma-1)/2, \quad s_6 = (\sigma-\gamma-1)/2 + \alpha, \\
s_7 &= (\sigma-\gamma)/2 + \alpha - 1, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = \gamma/2, \quad d_1 = (1+\gamma)/2, \\
e_1 &= (1+\sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = (1-\sigma)/2,
\end{aligned}$$

$$\begin{aligned}
b_2 &= (1 + \sigma)/2, \quad c_2 = (1 - \sigma + \gamma)/2 - \alpha, \quad d_2 = 1 + (\gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \\
f_2 &= (3 - \sigma)/2 - \alpha + \gamma, \quad g_2 = (3 - \sigma)/2 - \alpha, \quad a_3 = 1 - \sigma/2, \quad b_3 = 1 + \sigma/2, \\
c_3 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad d_3 = (3 + \gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \quad f_3 = 2 + \gamma - \alpha - \sigma/2, \\
g_3 &= 2 - \alpha - \sigma/2, \quad a_5 = (1 - \sigma)/2, \quad b_5 = 1 - \sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha - \gamma, \quad e_5 = 1 - \sigma, \\
f_5 &= 1 + \alpha - \gamma/2, \quad g_5 = (1 - \gamma)/2 + \alpha, \quad a_6 = (1 - \sigma + \gamma)/2 - \alpha, \quad b_6 = 1 + (\gamma - \sigma)/2 - \alpha, \\
c_6 &= \gamma/2, \quad d_6 = -\gamma/2, \quad e_6 = 1 + \gamma/2 - \alpha, \quad f_6 = 1 - \alpha - \sigma + \gamma/2, \quad g_6 = 1/2, \\
a_7 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad b_7 = (3 + \gamma - \sigma)/2 - \alpha, \quad c_7 = (1 + \gamma)/2, \quad d_7 = (1 - \gamma)/2, \\
e_7 &= (3 + \gamma)/2 - \sigma - \alpha, \quad f_7 = (3 + \gamma)/2 - \alpha, \quad g_7 = 3/2, \quad z = y \} \\
&[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma - 1) < 0].
\end{aligned}$$

$$21. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\bar{y} + x)^\sigma}{(V\bar{x} + V\bar{1} + x)^\gamma V\bar{y} + x} dx = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ + V_{11}^+ & [|\gamma| \leq 1], \\ V_8^- + V_9^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_8^+ = 2^{\sigma+2\alpha} B(\alpha, 1 - \sigma - 2\alpha), \quad E_9^+ = -2^{\sigma+2\alpha+1} \gamma B\left(\alpha + \frac{1}{2}, -\sigma - 2\alpha\right),$$

$$E_{10}^+ = 2^{1-\sigma-2\alpha} \gamma \Gamma \left[\begin{matrix} 2\alpha + \sigma - 1, (1 + \gamma - \sigma)/2 - \alpha \\ \alpha + (\gamma + \sigma + 1)/2 \end{matrix} \right],$$

$$E_{11}^+ = 2^2 \sigma^{-2\alpha} \gamma \sigma \Gamma \left[\begin{matrix} 2\alpha + \sigma - 2, (\gamma - \sigma)/2 + 1 - \alpha \\ (\gamma + \sigma)/2 + \alpha \end{matrix} \right],$$

$$E_8^- = 2^{\sigma-2\alpha} \gamma \Gamma \left[\begin{matrix} 2\alpha, \gamma/2 - \alpha \\ 1 + \gamma/2 + \alpha \end{matrix} \right],$$

$$E_9^- = 2^{\sigma+2\alpha-2\gamma} B(\alpha - \gamma/2, 1 - \sigma - 2\alpha + \gamma),$$

$$\begin{aligned}
s_1 &= (\sigma - 1)/2 + \alpha, \quad s_2 = \sigma/2 + \alpha, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (\sigma - 1)/2, \\
s_6 &= (\sigma - 1 - \gamma)/2 + \alpha, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = \gamma/2, \quad d_1 = -\gamma/2, \quad e_1 = (1 + \sigma)/2 + \alpha, \\
f_1 &= \sigma/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \quad c_2 = (1 - \gamma)/2, \\
d_2 &= (\gamma + 1)/2, \quad e_2 = 1 + \sigma/2 + \alpha, \quad f_2 = (\sigma + 1)/2 + \alpha, \quad g_2 = 3/2, \quad a_3 = (1 - \sigma)/2, \\
b_3 &= (1 + \sigma)/2, \quad c_3 = (1 - \sigma + \gamma)/2 - \alpha, \quad d_3 = (1 - \sigma - \gamma)/2 - \alpha, \quad e_3 = 1/2, \\
f_3 &= (3 - \sigma)/2 - \alpha, \quad g_3 = 1 - \sigma/2 - \alpha, \quad a_4 = 1 - \sigma/2, \quad b_4 = 1 + \sigma/2, \\
c_4 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad d_4 = 1 - (\gamma + \sigma)/2 - \alpha, \quad e_4 = 3/2, \quad f_4 = 2 - \sigma/2 - \alpha, \\
g_4 &= (3 - \sigma)/2 - \alpha, \quad a_5 = (1 - \sigma)/2, \quad b_5 = 1 - \sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha + 1/2, \\
e_5 &= 1 + \gamma/2 + \alpha, \quad f_5 = 1 - \gamma/2 + \alpha, \quad g_5 = 1 - \sigma, \quad a_6 = (1 - \sigma + \gamma)/2 - \alpha, \\
b_6 &= 1 - (\sigma - \gamma)/2 - \alpha, \quad c_6 = \gamma/2, \quad d_6 = (\gamma + 1)/2, \quad e_6 = 1 + \gamma, \quad f_6 = 1 + \gamma/2 - \alpha, \\
g_6 &= 1 - \sigma - \alpha + \gamma/2, \quad z = y \} \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma - 1) < 0].
\end{aligned}$$

$$22. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\bar{y} + x)^\sigma dx}{(1 + V\bar{1} + x)^\gamma V\bar{(1+x)}(y+x)} = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ & [|\gamma| \leq 1], \\ V_8^- + V_9^- + V_{10}^- & [|\gamma| \geq 1] \end{cases}$$

$$\begin{aligned}
\{E_8^+ &= 2^{\sigma+2\alpha-\gamma} B(\alpha, 1 - \sigma - 2\alpha), \quad E_9^+ = 2^{2\alpha+\sigma-\gamma-1} B\left(\alpha + (\sigma - 1)/2, 2 + \gamma - 2\alpha - \sigma\right), \\
E_{10}^+ &= 2^{-2-\gamma+\sigma+2\alpha} \sigma B(\alpha + \sigma/2 - 1, 3 + \gamma - 2\alpha - \sigma), \quad E_8^- = 2^{\sigma+2\alpha-\gamma} B(\alpha, 1 + \gamma - 2\alpha), \\
E_9^- &= 2^{\sigma+2\alpha-\gamma-1} B(\alpha - (\gamma + 1)/2, 2 - \sigma - 2\alpha + \gamma), \\
E_{10}^- &= -2^{\sigma+2\alpha-\gamma-2} \gamma B(\alpha - \gamma/2 - 1, 3 + \gamma - \sigma - 2\alpha),
\end{aligned}$$

$$\begin{aligned}
s_1 &= (\sigma - 1)/2 + \alpha, \quad s_2 = 0, \quad s_3 = 1/2, \quad s_5 = (\sigma - 1)/2, \quad s_6 = \alpha + (\sigma - \gamma)/2 - 1, \\
s_7 &= \alpha + (\sigma - \gamma - 3)/2, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = (1 + \gamma)/2, \quad d_1 = 1 + \gamma/2, \\
e_1 &= (1 + \sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = (1 - \sigma)/2, \quad b_2 = (1 + \sigma)/2, \\
c_2 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad d_2 = (3 + \gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \quad f_2 = (3 - \sigma)/2 - \alpha, \\
g_2 &= (3 - \sigma)/2 + \gamma - \alpha, \quad a_3 = 1 - \sigma/2, \quad b_3 = 1 + \sigma/2, \quad c_3 = (3 + \gamma - \sigma)/2 - \alpha, \\
d_3 &= 2 + (\gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \quad f_3 = 2 - \sigma/2 - \alpha, \quad g_3 = 2 + \gamma - \sigma/2 - \alpha,
\end{aligned}$$

$$\begin{aligned}
 a_5 &= (1-\sigma)/2, \quad b_5 = 1-\sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha-\gamma, \quad e_5 = 1-\sigma, \quad f_5 = (1-\gamma)/2 + \alpha, \\
 g_5 &= \alpha-\gamma/2, \quad a_6 = 1 + (\gamma-\sigma)/2 - \alpha, \quad b_6 = (3+\gamma-\sigma)/2 - \alpha, \quad c_6 = (1+\gamma)/2, \\
 d_6 &= (1-\gamma)/2, \quad e_6 = (3+\gamma)/2 - \alpha - \sigma, \quad f_6 = 1/2, \quad g_6 = (3+\gamma)/2 - \alpha, \\
 a_7 &= (3-\sigma+\gamma)/2 - \alpha, \quad b_7 = 2 + (\gamma-\sigma)/2 - \alpha, \quad c_7 = 1 + \gamma/2, \quad d_7 = 1 - \gamma/2, \\
 e_7 &= 2 - \sigma - \alpha + \gamma/2, \quad f_7 = 3/2, \quad g_7 = 2 + \gamma/2 - \alpha, \quad z = y \} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma) < 1].
 \end{aligned}$$

$$23. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\overline{y+x})^\sigma dx}{(V\bar{x} + V\overline{1+x})^\gamma V(1+x)(y+x)} = \begin{cases} V_8^+ + V_9^+ + V_{10}^+ + V_{11}^+ & [|\gamma| \leq 1], \\ V_8^- + V_9^- & [|\gamma| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{ E_8^+ &= 2^{\sigma+2\alpha} B(\alpha, 1-\sigma-2\alpha), \quad E_9^+ = -2^{\sigma+2\alpha+1} \gamma B(\alpha+1/2, -\sigma-2\alpha), \\
 E_{10}^+ &= 2^{2-\sigma-2\alpha} B(\sigma+2\alpha-1, 1-\alpha+(\gamma-\sigma)/2), \\
 E_{11}^+ &= 2^{3-2\alpha-\sigma} \sigma B(2\alpha+\sigma-2, (3+\gamma-\sigma)/2-\alpha), \quad E_8^- = 2^{\sigma-2\alpha+1} B(2\alpha, (1+\gamma)/2-\alpha), \\
 E_9^- &= 2^{2\alpha+\sigma-2\gamma-1} B(\alpha-(\gamma+1)/2, 2+\gamma-\sigma-2\alpha), \\
 s_1 &= (\sigma-1)/2 + \alpha, \quad s_2 = \sigma/2 + \alpha, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (\sigma-1)/2, \\
 s_6 &= \alpha + (\sigma-\gamma)/2 - 1, \quad a_1 = \alpha, \quad b_1 = \alpha + \sigma, \quad c_1 = (1+\gamma)/2, \quad d_1 = (1-\gamma)/2, \\
 e_1 &= (1+\sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \\
 c_2 &= 1 + \gamma/2, \quad d_2 = 1 - \gamma/2, \quad e_2 = 1 + \sigma/2 + \alpha, \quad f_2 = (\sigma+1)/2 + \alpha, \quad g_2 = 3/2, \\
 a_3 &= (1-\sigma)/2, \quad b_3 = (1+\sigma)/2, \quad c_3 = 1 + (\gamma-\sigma)/2 - \alpha, \quad d_3 = 1 - (\gamma+\sigma)/2 - \alpha, \\
 e_3 &= 1/2, \quad f_3 = (3-\sigma)/2 - \alpha, \quad g_3 = 1 - \sigma/2 - \alpha, \quad a_4 = 1 - \sigma/2, \quad b_4 = 1 + \sigma/2, \\
 c_4 &= (3+\gamma-\sigma)/2 - \alpha, \quad d_4 = (3-\gamma-\sigma)/2 - \alpha, \quad e_4 = 3/2, \quad f_4 = 2 - \sigma/2 - \alpha, \\
 g_4 &= (3-\sigma)/2 - \alpha, \quad a_5 = (1-\sigma)/2, \quad b_5 = 1 - \sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha + 1/2, \\
 e_5 &= 1 - \sigma, \quad f_5 = (1-\gamma)/2 + \alpha, \quad g_5 = (1+\gamma)/2 + \alpha, \quad a_6 = 1 + (\gamma-\sigma)/2 - \alpha, \\
 b_6 &= (3+\gamma-\sigma)/2 - \alpha, \quad c_6 = (1+\gamma)/2, \quad d_6 = 1 + \gamma/2, \quad e_6 = (3+\gamma)/2 - \alpha, \\
 f_6 &= (3+\gamma)/2 - \alpha - \sigma, \quad g_6 = 1 + \gamma, \quad z = y \} \\
 &[|\arg y| < \pi; \operatorname{Re} (2\alpha + \sigma - \gamma) < 1; \operatorname{Re} \alpha > 0].
 \end{aligned}$$

$$24. \int_0^1 x^{\alpha-1} \frac{(1-x^r)^{\beta-1}}{y^2 + 2xy \cos \gamma + x^2} dx = \begin{cases} V_{12}^+ + V_{13}^+ & [|\gamma| \leq 1], \\ V_{12}^- & [|\gamma| \geq 1] \end{cases}$$

$$\begin{aligned}
 \left\{ E_{12}^+ &= (-1)^{l+1} \frac{\pi}{\sin \gamma} (1-\beta)_l, \quad E_{13}^+ = -\frac{1}{r \sin \gamma} \Gamma \left[\frac{\beta}{(\alpha-1-l)/r + \beta} \right], \right. \\
 E_{12}^- &= \frac{1}{r \sin \gamma} \Gamma \left[\frac{\beta}{(\alpha+l)/r + \beta} \right] \} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\gamma| < \pi; r > 0 - \text{произвольное}].
 \end{aligned}$$

$$25. \int_0^\infty x^{\alpha-1} \frac{(1+x^r)^{\rho-1}}{y^2 + 2xy \cos \gamma + x^2} dx = \begin{cases} V_{12}^+ + V_{13}^+ & [|\gamma| \leq 1], \\ V_{12}^- + V_{13}^- & [|\gamma| \geq 1] \end{cases}$$

$$\begin{aligned}
 \left\{ E_{12}^+ &= -\frac{\pi}{\sin \gamma} (1-\rho)_l, \quad E_{13}^+ = -\frac{1}{r \sin \gamma} \Gamma \left[\frac{1-\rho+(1+l-\alpha)/r}{1-\rho} \right], \right. \\
 E_{12}^- &= \frac{1}{r \sin \gamma} \Gamma \left[\frac{1-\rho-(l+\alpha)/r}{1-\rho} \right], \quad E_{13}^- = 1 \} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (\alpha + r\rho) < r+2; |\gamma| < \pi; r > 0 - \text{произвольное}].
 \end{aligned}$$

$$26. \int_0^\infty \frac{x^{\alpha-1}}{1-x^r} \frac{dx}{y^2 + 2xy \cos \gamma + x^2} = U,$$

$$U = -\frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \left\{ \frac{\sin [(\alpha + lr - 1) \gamma]}{\sin [(\alpha + lr) \pi]} y^{lr + \alpha - 2} + \right. \\ \left. + \frac{(-1)^l}{r} \operatorname{ctg} \left(\frac{\alpha - l - 1}{r} \pi \right) \sin (l\gamma) y^{l-1} \right\} \quad [0 < y \leq 1],$$

$$U = \frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \left\{ \frac{\sin [(\alpha - r - lr - 1) \gamma]}{\sin [(\alpha - r - lr) \pi]} y^{\alpha - r - lr - 2} + \right. \\ \left. + \frac{(-1)^l}{r} \operatorname{ctg} \left(\frac{\alpha + l}{r} \pi \right) \sin (l\gamma + \gamma) y^{l-2} \right\} \quad [y \geq 1];$$

[0 < Re α < r + 2; |γ| < π; r > 0 — произвольное].

$$27. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^{\sigma}}{y^2 + 2xy \cos \gamma + x^2} dx = \begin{cases} V_{14}^{+} + V_{15}^{+} & [|y| \leq 1], \\ V_{14}^{-} + V_{15}^{-} & [|y| \geq 1] \end{cases}$$

$$\{E_{14}^{+} = -2^{\sigma+2(\alpha-2)/r} \pi (-\sigma)_{2l}, \quad E_{15}^{+} = 2^{\sigma+2(\alpha-2)/r} \sigma r^{-1} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{14}^{-} = 2^{\sigma+2(\alpha-2)/r-1} \sigma \pi, \quad E_{15}^{-} = -2^{\sigma+2(\alpha-2)/r} \sigma r^{-1}, \quad z = 2^{-2/r} y, \quad h = -\sigma\}$$

[|arg y| < π; Re x > 0; Re(2α + σr) < 2; |γ| < π; r > 0 — произвольное].

$$28. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^{\sigma}}{\sqrt{1+x^r} (y^2 + 2xy \cos \gamma + x^2)} dx = \begin{cases} V_{14}^{+} + V_{15}^{+} & [|y| \leq 1], \\ V_{14}^{-} + V_{15}^{-} & [|y| \geq 1] \end{cases}$$

$$\{E_{14}^{+} = -2^{\sigma+2(\alpha-2)/r} \pi (1-\sigma)_{2l}, \quad E_{15}^{+} = -2^{\sigma+2(\alpha-2)/r} r^{-1} \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{14}^{-} = -2^{\sigma+2(\alpha-2)/r-1} \pi, \quad E_{15}^{-} = 2^{\sigma+2(\alpha-2)/r} r^{-1}, \quad z = 2^{-2/r} y, \quad h = 1-\sigma\}$$

[|arg y| < π; Re α > 0; Re(2α + σr) < 2 + r/2; |γ| < π; r > 0 — произвольное].

$$29. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1 + 2x^r \cos \lambda + x^{2r}) (y^2 + 2xy \cos \gamma + x^2)} = \begin{cases} V_{16}^{+} + V_{17}^{+} & [|y| \leq 1], \\ V_{16}^{-} + V_{17}^{-} & [|y| \geq 1] \end{cases}$$

[|arg y| < π; 0 < Re α < 2r + 2; |γ|, |λ| < π; r > 0 — произвольное].

$$30. \int_0^1 x^{\alpha-1} \frac{(1-x^r)^{\beta-1}}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \begin{cases} V_{12}^{+} + V_{13}^{+} + V_{18}^{+} + V_{19}^{+} & [|y| \leq 1], \\ V_{12}^{-} + V_{18}^{-} & [|y| \geq 1] \end{cases}$$

$$\{E_{12}^{+} = (-1)^{l+1} \frac{\pi}{2 \sin^3 \gamma} (1-\beta)_l y^{-2}, \quad E_{13}^{+} = -\frac{1}{2r \sin^3 \gamma} \Gamma\left[\beta + (\alpha - l - 1)/r\right] y^{-2}, \\ E_{18}^{+} = (-1)^l \frac{\pi}{2} (1-\beta)_l, \quad E_{19}^{+} = \frac{1}{2r} \Gamma\left[\beta + (\alpha - l - 1)/r\right], \\ E_{12}^{-} = \frac{1}{2r \sin^3 \gamma} \Gamma\left[\beta + (\alpha + l)/r\right] y^{-2}, \quad E_{18}^{-} = -\frac{1}{2r} \Gamma\left[\beta + (\alpha + l)/r\right]\}$$

[|arg y| < π; Re α, Re β > 0; |γ| < π; r > 0 — произвольное].

$$31. \int_0^{\infty} x^{\alpha-1} \frac{(1+x^r)^{\rho-1}}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \begin{cases} V_{12}^{+} + V_{13}^{+} + V_{18}^{+} + V_{19}^{+} & [|y| \leq 1], \\ V_{12}^{-} + V_{13}^{-} + V_{18}^{-} + V_{19}^{-} & [|y| \geq 1] \end{cases}$$

$$\{E_{12}^{+} = -\frac{\pi}{2 \sin^3 \gamma} (1-\rho)_l y^{-2}, \quad E_{13}^{+} = -\frac{1}{2r \sin^3 \gamma} \Gamma\left[\frac{1-\rho-(\alpha-l-1)/r}{1-\rho}\right] y^{-2}, \\ E_{18}^{+} = \frac{\pi}{2} (1-\rho)_l, \quad E_{19}^{+} = \frac{1}{2r} \Gamma\left[\frac{1-\rho-(\alpha-l-1)/r}{1-\rho}\right],$$

$$E_{12}^- = \frac{1}{2r \sin^3 \gamma} \Gamma \left[\frac{1-\rho-(\alpha+l)/r}{1-\rho} \right] y^{-2}, \quad E_{13}^- = \frac{y^{-2}}{2 \sin^2 \gamma},$$

$$E_{18}^- = -\frac{1}{2r} \Gamma \left[\frac{1-\rho-(\alpha+l)/r}{1-\rho} \right]$$

[$|\arg y| < \pi$; $\operatorname{Re} \alpha > 0$; $\operatorname{Re}(\alpha + \rho r) < r + 4$; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$32. \int_0^\infty \frac{x^{\alpha-1}}{1-x^r} \frac{dx}{(y^2 + 2xy \cos \gamma + x^2)^2} = \frac{1}{2} (y \sin \gamma)^{-2} U + U^* \quad [\text{см. 2.2.2.26}].$$

$$U^* = \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \left\{ \frac{(\alpha + lr - 1) \cos [\gamma (\alpha + lr - 2)]}{\sin [(\alpha + lr) \pi]} y^{\alpha + lr - 4} + \right.$$

$$\left. + \frac{(-1)^l}{r} l \cos (\gamma - l\gamma) \operatorname{ctg} \left(\frac{\alpha - l - 1}{r} \pi \right) y^{l-3} \right\} \quad [0 < y \leq 1].$$

$$U^* = \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \left\{ \frac{(1 - \alpha + r + rl) \cos [\gamma (\alpha - r - rl - 2)]}{\sin [(\alpha - r - rl) \pi]} y^{\alpha - r - rl - 4} + \right.$$

$$\left. + \frac{(-1)^{l+1}}{r} (l+1) \cos (\gamma l + 2\gamma) \operatorname{ctg} \left(\frac{\alpha + l}{r} \pi \right) y^{l-4} \right\} \quad [y \geq 1].$$

[$0 < \operatorname{Re} \alpha < r + 4$; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$33. \int_0^\infty x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^\sigma}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \begin{cases} V_{14}^+ + V_{15}^+ + V_{20}^+ + V_{21}^+ & [y \leq 1], \\ V_{14}^- + V_{15}^- + V_{20}^- + V_{21}^- & [y \geq 1] \end{cases}$$

$$\begin{cases} E_{14}^+ = -\frac{2^{\sigma+2}(\alpha-2)/r-1\pi}{\sin^2 \gamma} (-\sigma)_{2l} y^{-2}, \\ E_{15}^+ = \frac{2^{\sigma+2}(\alpha-2)/r-1\sigma}{ry^2 \sin^2 \gamma} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{20}^+ = 2^{\sigma+2}(\alpha-4)/r-1\pi (-\sigma)_{2l}, \\ E_{21}^+ = -2^{\sigma+2}(\alpha-4)/r-1\sigma r^{-1} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{14}^- = \frac{\pi\sigma}{\sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-2 y^{-2}, \quad E_{15}^- = -\frac{\sigma}{r \sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 y^{-2}, \\ E_{20}^- = -2^{\sigma+2}(\alpha-4)/r-2\pi\sigma, \quad E_{21}^- = 2^{\sigma+2}(\alpha-4)/r-1\sigma r^{-1}, \quad z = 2^{-2/r} y, \quad h = -\sigma \end{cases}$$

[$|\arg y| < \pi$; $\operatorname{Re} \alpha > 0$; $\operatorname{Re}(2\alpha + \sigma r) < 4$; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$34. \int_0^\infty x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^\sigma}{\sqrt{1+x^r} (y^2 + 2xy \cos \gamma + x^2)^2} dx =$$

$$= \begin{cases} V_{14}^+ + V_{15}^+ + V_{20}^+ + V_{21}^+ & [y' \leq 1], \\ V_{14}^- + V_{15}^- + V_{20}^- + V_{21}^- & [y \geq 1] \end{cases}$$

$$\begin{cases} E_{14}^+ = -\frac{\pi}{\sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 (1-\sigma)_{2l} y^{-2}, \\ E_{15}^+ = -\frac{1}{r \sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)] y^{-2}, \\ E_{20}^+ = 2^{\sigma+2}(\alpha-4)/r-1\pi (1-\sigma)_{2l}, \quad E_{21}^+ = r^{-1} 2^{\sigma+2}(\alpha-4)/r-1 \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)], \end{cases}$$

$$E_{14}^- = -\frac{\pi}{\sin^2 \gamma} 2^{\sigma+2(\alpha-2)/r-2} y^{-2}, \quad E_{15}^- = \frac{1}{r \sin^2 \gamma} 2^{\sigma+2(\alpha-2)/r-1} y^{-2},$$

$$E_{20}^- = 2^{\sigma+2(\alpha-4)/r-2} \pi, \quad E_{21}^- = -r^{-1} 2^{\sigma+2(\alpha-4)/r-1}, \quad z = 2^{-2/r} y, \quad h = 1 - \sigma \left\{ \begin{array}{l} [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha + \sigma r) < r + 8; |\gamma| < \pi; r > 0 - \text{произвольное}]. \end{array} \right.$$

$$35. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1 + 2x^r \cos \lambda + x^{2r}) (y^2 + 2xy \cos \gamma + x^2)^2} = U_1,$$

$$U_1 = 2^{-1} (y \sin \gamma)^{-2} [V_{16}^+ + V_{17}^+] + V_{22}^+ + V_{23}^+ \quad \left[\begin{array}{l} \{ |y| \leq 1 \} \\ \{ |y| \geq 1 \} \end{array} \right]$$

$$[|\arg y| < \pi; 0 < \operatorname{Re} \alpha < 2r + 4; |\gamma|, |\lambda| < \pi; r > 0 - \text{произвольное}].$$

$$36. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1 + 2x^r \cos \lambda + x^{2r})^2 (y^2 + 2xy \cos \gamma + x^2)^2} =$$

$$= (2 \sin^2 \lambda)^{-1} U_1 + U_1^* \quad [\text{см. 2.2.2.35}]$$

$$U_1^* = \frac{\pi}{4 \sin^2 \lambda \sin^2 \gamma} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l (l+1) \cos (\lambda l + 2\lambda)}{\sin [(\alpha + lr) \pi]} \times \right.$$

$$\times \left[\frac{\sin [\gamma (\alpha + lr - 1)]}{\sin \gamma} - (\alpha + lr - 1) \cos [\gamma (\alpha + lr - 2)] \right] y^{\alpha + lr - 4} +$$

$$+ \sum_{l=0}^{\infty} \frac{(-1)^l [(\alpha - l - 1)/r - 1]}{r \sin [(\alpha - l - 1) \pi / r]} \cos \left[\lambda \left(\frac{\alpha - l - 1}{r} - 2 \right) \right] \times$$

$$\times \left[l \cos (\gamma l - \gamma) - \frac{\sin l \gamma}{\sin \gamma} \right] y^{l-3} \left. \right\} \quad [|y| \leq 1].$$

$$U_1^* = \frac{\pi}{4 \sin^2 \lambda \sin^2 \gamma} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l l \cos (\lambda l - \lambda)}{\sin [(\alpha - lr - r) \pi]} \times \right.$$

$$\times \left[(\alpha - lr - r - 1) \cos [\gamma (\alpha - lr - r - 2)] - \frac{\sin [\gamma (\alpha - lr - r - 1)]}{\sin \gamma} \right] \times$$

$$\times y^{\alpha - r - lr - 4} + \sum_{l=0}^{\infty} \frac{(-1)^l [(\alpha + l)/r - 1] \cos [\lambda r^{-1} (\alpha + l - 2r)]}{r \sin [(\alpha + l) \pi / r]} \times$$

$$\times \left[(l - 1) \cos (\gamma l + 2\gamma) + \frac{\sin (\gamma l + \gamma)}{\sin \gamma} \right] y^{-l-4} \left. \right\} \quad [|y| \geq 1].$$

$$[|\arg y| < \pi; 0 < \operatorname{Re} \alpha < 4(r+1); |\gamma|, |\lambda| < \pi; r > 0 - \text{произвольное}].$$

$$37. \int_0^{\min(1, y)} x^{\alpha-1} (1 - x^r)^{\beta-1} \frac{(V_{\bar{y}} + V_{\bar{y}-x})^{\gamma} + (V_{\bar{y}} - V_{\bar{y}-x})^{\gamma}}{V_{\bar{y}-x}} dx =$$

$$= \begin{cases} V_5^+ & [0 < y \leq 1], \\ V_5^- + V_6^- & [y \geq 1] \end{cases}$$

$$\{E_5^+ = (-1)^j 2^{\gamma+2\alpha+2jr} (1 - \beta)_j B(\alpha + jr, \gamma + \alpha + jr),$$

$$E_5^- = 2^{\gamma-2h} r^{-1} (\gamma - 2h)_h B(\beta, r^{-1}(\alpha + h)),$$

$$E_6^- = 2^{\gamma-2h} r^{-1} (-\gamma - 2h)_h B(\beta, r^{-1}(\alpha + \gamma + h)),$$

$$s = \alpha + jr + (\gamma - 1)/2, \quad u = (\gamma - 1)/2 - h, \quad u_1 = -(1 + \gamma)/2 - h,$$

$$\begin{aligned} a_1 &= \alpha + jr, \quad b_1 = \alpha + \gamma + jr, \quad c_1 = 1 - \beta + j, \quad d_1 = \alpha + \gamma/2 + jr, \\ e_1 &= \alpha + (\gamma + 1)/2 + jr, \quad a_4 = 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h, \\ c_4 &= (\alpha + h)/r, \quad d_4 = 1 - \gamma + h, \quad e_4 = \beta + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h, \\ b_5 &= (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r, \quad d_5 = 1 + \gamma + h, \\ e_5 &= \beta + (\alpha + \gamma + h)/r, \quad z = y^\rho \end{aligned}$$

[$y, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0$ и $\operatorname{Re} \beta > 1/2$ (при $y=1$), или $\operatorname{Re} \beta > 0$ (при $y > 1$)].

$$38. \int_1^y x^{\alpha-1} (x^r - 1)^{\beta-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx = V_5^- + V_6^- + V_7^-$$

$$\begin{aligned} \{E_5^- &= r^{-12\gamma-2h} (\gamma - 2h)_h B(\beta, 1 - \beta - r^{-1}(\alpha + h)), \\ E_6^- &= r^{-12\gamma-2h} (-\gamma - 2h)_h B(\beta, 1 - \beta - r^{-1}(\alpha + \gamma + h)), \\ E_7^- &= (-1)^j 2^{\gamma+2\alpha-2r} (1 - \beta + j) (1 - \beta)_j B(\alpha - r(1 - \beta + j), \gamma + \alpha - r(1 - \beta + j)), \\ u &= (\gamma - 1)/2 - h, \quad u_1 = -(1 + \gamma)/2 - h, \quad v = \alpha + (\gamma - 1)/2 - r(1 - \beta + j), \\ a_4 &= 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h, \quad c_4 = (\alpha + h)/r, \quad d_4 = 1 - \gamma + h, \\ e_4 &= \beta + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h, \quad b_5 = (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r, \\ d_5 &= 1 + \gamma + h, \quad e_5 = \beta + (\alpha + \gamma + h)/r, \quad a_6 = 1 - \gamma/2 - \alpha + r(1 - \beta + j), \\ b_6 &= (1 - \gamma)/2 - \alpha + r(1 - \beta + j), \quad c_6 = 1 - \beta + j, \quad d_6 = 1 - \alpha + r(1 - \beta + j), \\ e_6 &= 1 - \gamma - \alpha + r(1 - \beta + j), \quad z = y^\rho \} \quad [y > 1, \operatorname{Re} \beta > 0]. \end{aligned}$$

$$39. \int_0^y x^{\alpha-1} (1 + x^r)^{\rho-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx =$$

$$= \begin{cases} V_5^+ & [0 < y \leq 1], \\ V_5^- + V_6^- + V_7^- & [y \geq 1] \end{cases}$$

$$\begin{aligned} \{E_5^+ &= 2^{\gamma+2\alpha+2jr} (1 - \rho)_j B(\alpha + jr, \gamma + \alpha + jr), \\ E_5^- &= \frac{2^{\gamma-2h}}{r} (\gamma - 2h)_h B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right), \\ E_6^- &= \frac{2^{\gamma-2h}}{r} (-\gamma - 2h)_h B\left(\frac{\alpha + \gamma + h}{r}, 1 - \rho - \frac{\alpha + \gamma + h}{r}\right), \\ E_7^- &= 2^{\gamma+2\alpha-2r} (1 - \rho + j) (1 - \rho)_j B(\alpha - r(1 - \rho + j), \gamma + \alpha - r(1 - \rho + j)), \\ s &= (\gamma - 1)/2 + \alpha + jr, \quad u = (\gamma - 1)/2 - h, \quad u_1 = -(1 + \gamma)/2 - h, \\ v &= \alpha + (\gamma - 1)/2 - r(1 - \rho + j), \quad a_1 = \alpha + jr, \quad b_1 = \gamma + \alpha + jr, \quad c_1 = 1 - \rho + j, \\ d_1 &= \gamma/2 + \alpha + jr, \quad e_1 = (\gamma + 1)/2 + \alpha + jr, \quad a_4 = 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h, \\ c_4 &= (\alpha + h)/r, \quad d_4 = 1 - \gamma + h, \quad e_4 = \rho + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h, \\ b_5 &= (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r, \quad d_5 = 1 + \gamma + h, \quad e_5 = \rho + (\alpha + \gamma + h)/r, \\ a_6 &= 1 - \gamma/2 - \alpha + r(1 - \rho + j), \quad b_6 = (1 - \gamma)/2 - \alpha + r(1 - \rho + j), \quad c_6 = 1 - \rho + j, \\ d_6 &= 1 - \alpha + r(1 - \rho + j), \quad e_6 = 1 - \gamma - \alpha + r(1 - \rho + j), \quad z = (-1)^q y^\rho \} \\ &\quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0]. \end{aligned}$$

$$40. \int_0^y \frac{x^{\alpha-1}}{1 - x^r} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx =$$

$$= \begin{cases} V_5^+ & [0 < y \leq 1], \\ V_5^- + V_6^- + V_7^- & [y \geq 1] \end{cases}$$

$$\begin{aligned} \{E_5^+ &= (-1)^j j! 2^{\gamma+2\alpha+2rj} B(\alpha + jr, \gamma + \alpha + jr), \\ E_5^- &= \frac{\pi}{r} \operatorname{ctg}\left(\frac{\alpha + h}{r} \pi\right) 2^{\gamma-2h} (\gamma - 2h)_h, \end{aligned}$$

$$E_6^- = \frac{\pi}{r} 2^{-\gamma-2h} (-\gamma-2h)_h \operatorname{ctg} \left(\frac{\alpha+\gamma+h}{r} \pi \right),$$

$$E_7^- = (-1)^{j+1} j! 2^{\gamma+2\alpha} 2r^{(j+1)} B(\alpha-r(j+1), \gamma+\alpha-r(j+1)),$$

$$s = \alpha + (\gamma-1)/2 + jr, u = (\gamma-1)/2 - h, u_1 = -(1+\gamma)/2 - h,$$

$$v = (\gamma-1)/2 + \alpha - r(j+1), a_1 = \alpha + jr, b_1 = \gamma + \alpha + jr, c_1 = 1 + j,$$

$$d_1 = \gamma/2 + \alpha + jr, e_1 = (\gamma+1)/2 + \alpha + jr, a_4 = 1 - \gamma/2 + h, b_4 = (1-\gamma)/2 + h,$$

$$c_4 = e_4 = 1/2, d_4 = 1 - \gamma + h, a_5 = 1 + \gamma/2 + h, b_5 = (1+\gamma)/2 + h,$$

$$c_5 = e_5 = 1/2, d_5 = 1 + \gamma + h, a_6 = 1 - \gamma/2 - \alpha + r(1+j),$$

$$b_6 = (1-\gamma)/2 - \alpha + r(1+j), c_6 = 1 + j, d_6 = 1 - \alpha + r(1+j),$$

$$e_6 = 1 - \gamma - \alpha + r(1+j), z = y^p \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha+\gamma) > 0].$$

$$41. \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\overline{y-x})^\gamma + (V\bar{y} - V\overline{y-x})^\gamma}{(1+V\sqrt{1+x})^{-\sigma} V\overline{y-x}} dx =$$

$$= \begin{cases} V_8^+ & [0 < y \leq 1], \\ V_8^- + V_9^- + V_{10}^- + V_{11}^- & [y \geq 1] \end{cases}$$

$$\{E_8^+ = 2^{2\alpha+\gamma+\sigma} B(\alpha, \gamma+\alpha), E_8^- = 2^{\gamma+\sigma+2\alpha} B(\alpha+\sigma/2, \alpha+\gamma+\sigma/2),$$

$$E_9^- = 2^{\gamma+\sigma+2\alpha-1} \sigma B(\alpha+(\sigma-1)/2, \gamma+\alpha+(\sigma-1)/2),$$

$$E_{10}^- = -2^{\sigma+2\alpha+\gamma} \sigma \Gamma \left[\begin{matrix} \alpha, -\sigma-2\alpha \\ 1-\alpha-\sigma \end{matrix} \right], E_{11}^- = -2^{\sigma+\gamma+2\alpha} \sigma \Gamma \left[\begin{matrix} \alpha+\gamma, -\sigma-2\alpha-2\gamma \\ 1-\sigma-\alpha-\gamma \end{matrix} \right],$$

$$s_1 = \alpha + (\gamma-1)/2, s_5 = \alpha + (\gamma+\sigma-1)/2, s_6 = \alpha + (\gamma+\sigma)/2 - 1,$$

$$s_7 = (\gamma-1)/2, s_8 = -(1+\gamma)/2, a_1 = \alpha, b_1 = \gamma + \alpha, c_1 = -\sigma/2,$$

$$d_1 = (1-\sigma)/2, e_1 = \gamma/2 + \alpha, f_1 = (\gamma+1)/2 + \alpha, g_1 = 1 - \sigma,$$

$$a_5 = 1 - \alpha - (\gamma+\sigma)/2, b_5 = (1-\gamma-\sigma)/2 - \alpha, c_5 = \sigma/2, d_5 = -\sigma/2,$$

$$e_5 = 1 - \alpha - \sigma/2, f_5 = 1 - \gamma - \alpha - \sigma/2, g_5 = 1/2, a_6 = (3-\sigma-\gamma)/2 - \alpha,$$

$$b_6 = 1 - (\gamma+\sigma)/2 - \alpha, c_6 = (1-\sigma)/2, d_6 = (1+\sigma)/2, e_6 = (3-\sigma)/2 - \alpha,$$

$$f_6 = (3-\sigma)/2 - \gamma - \alpha, g_6 = 3/2, a_7 = 1 - \gamma/2, b_7 = (1-\gamma)/2, c_7 = \sigma + \alpha, d_7 = \alpha,$$

$$e_7 = 1 + \sigma/2 + \alpha, f_7 = (1+\sigma)/2 + \alpha, g_7 = 1 - \gamma, a_8 = 1 + \gamma/2,$$

$$b_8 = (1+\gamma)/2, c_8 = \sigma + \alpha + \gamma, d_8 = \alpha + \gamma, e_8 = 1 + \sigma/2 + \alpha + \gamma,$$

$$f_8 = (1+\sigma)/2 + \alpha + \gamma, g_8 = 1 + \gamma, z = -y \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha+\gamma) > 0].$$

$$42. \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\overline{y-x})^\gamma + (V\bar{y} - V\overline{y-x})^\gamma}{(V\sqrt{x} + V\sqrt{1+x})^{-\sigma} V\overline{y-x}} dx =$$

$$= \begin{cases} V_8^+ + V_9^+ & [0 < y \leq 1], \\ V_8^- + V_9^- + V_{10}^- & [y \geq 1] \end{cases}$$

$$\{E_8^+ = 2^{\gamma+2\alpha} B(\alpha, \gamma+\alpha), E_9 = 2^{\gamma+2\alpha+1} \sigma B(\alpha+1/2, \gamma+\alpha+1/2),$$

$$E_8^- = 2^{\gamma+2\sigma+2\alpha} B(\alpha+\sigma/2, \gamma+\alpha+\sigma/2), E_9^- = -2^{\gamma-2\alpha} \sigma \Gamma \left[\begin{matrix} 2\alpha, -\alpha-\sigma/2 \\ 1+\alpha-\sigma/2 \end{matrix} \right],$$

$$E_{10}^- = -2^{2\alpha-3\gamma} \sigma \Gamma \left[\begin{matrix} 2\alpha+2\gamma, -\alpha-\gamma-\sigma/2 \\ 1+\alpha+\gamma-\sigma/2 \end{matrix} \right],$$

$$s_1 = \alpha + (\gamma-1)/2, s_2 = \alpha + \gamma/2, s_5 = \alpha + (\sigma+\gamma-1)/2, s_6 = (\gamma-1)/2,$$

$$s_7 = -(1+\gamma)/2, a_1 = \alpha, b_1 = \alpha + \gamma, c_1 = -\sigma/2, d_1 = \sigma/2,$$

$$e_1 = \gamma/2 + \alpha, f_1 = (\gamma+1)/2 + \alpha, g_1 = 1/2, a_2 = \alpha + 1/2,$$

$$b_2 = \alpha + \gamma + 1/2, c_2 = (1-\sigma)/2, d_2 = (\sigma+1)/2, e_2 = \alpha + (\gamma+1)/2,$$

$$f_2 = \alpha + 1 + \gamma/2, g_2 = 3/2, a_5 = 1 - (\gamma+\sigma)/2 - \alpha, b_5 = (1-\gamma-\sigma)/2 - \alpha,$$

$$c_5 = -\sigma/2, d_5 = (1-\sigma)/2, e_5 = 1 - \alpha - \sigma/2, f_5 = 1 - \gamma - \alpha - \sigma/2,$$

$$\begin{aligned}
 g_5 &= 1 - \sigma, \quad a_6 = 1 - \gamma/2, \quad b_6 = (1 - \gamma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \\
 e_6 &= 1 - \gamma, \quad f_6 = 1 + \sigma/2 + \alpha, \quad g_6 = 1 - \sigma/2 + \alpha, \quad a_7 = 1 + \gamma/2, \quad b_7 = (1 + \gamma)/2, \\
 c_7 &= \alpha + \gamma, \quad d_7 = \alpha + \gamma + 1/2, \quad e_7 = 1 + \sigma/2 + \alpha + \gamma, \quad f_7 = 1 - \sigma/2 + \alpha + \gamma, \\
 g_7 &= 1 + \gamma, \quad z = -y \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\bar{y-x})^\gamma + (V\bar{y} - V\bar{y-x})^\gamma}{(1 + V\sqrt{1+x})^{-\sigma} V(1+x)(y-x)} dx = \\
 = \begin{cases} V_8^+ & [0 < y \leq 1], \\ V_8^- + V_9^- + V_{10}^- + V_{11}^- & [y \geq 1] \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \{E_8^+ &= 2^{\gamma+\sigma+2\alpha} B(\alpha, \gamma + \alpha), \\
 E_8^- &= 2^{\gamma+2\alpha+\sigma-1} B(\alpha + (\sigma-1)/2, \gamma + \alpha + (\sigma-1)/2), \\
 E_9^- &= 2^{\gamma+2\alpha+\sigma-2} \sigma B(\alpha + \sigma/2 - 1, \gamma + \alpha + \sigma/2 - 1), \\
 E_{10}^- &= 2^{\sigma+2\alpha+\gamma+1} B(\alpha, 1 - \sigma - 2\alpha), \quad E_{11}^- = 2^{\sigma+2\alpha+\gamma+1} B(\alpha + \gamma, 1 - \sigma - 2\alpha - 2\gamma), \\
 s_1 &= \alpha + (\gamma-1)/2, \quad s_5 = \alpha + (\gamma + \sigma)/2 - 1, \quad s_6 = \alpha + (\gamma + \sigma - 3)/2, \\
 s_7 &= (\gamma-1)/2, \quad s_8 = -(1 + \gamma)/2, \quad a_1 = \alpha, \quad b_1 = \alpha + \gamma, \quad c_1 = 1 - \sigma/2, \quad d_1 = (1 - \sigma)/2, \\
 e_1 &= \gamma/2 + \alpha, \quad f_1 = (\gamma + 1)/2 + \alpha, \quad g_1 = 1 - \sigma, \quad a_5 = (3 - \gamma - \sigma)/2 - \alpha, \\
 b_5 &= 1 - (\gamma + \sigma)/2 - \alpha, \quad c_5 = (1 + \sigma)/2, \quad d_5 = (1 - \sigma)/2, \quad e_5 = (3 - \sigma)/2 - \alpha, \\
 f_5 &= (3 - \sigma)/2 - \gamma - \alpha, \quad g_5 = 1/2, \quad a_6 = 2 - (\gamma + \sigma)/2 - \alpha, \quad b_6 = (3 - \gamma - \sigma)/2 - \alpha, \\
 c_6 &= 1 + \sigma/2, \quad d_6 = 1 - \sigma/2, \quad e_6 = 2 - \sigma/2 - \alpha, \quad f_6 = 2 - \sigma/2 - \alpha - \gamma, \quad g_6 = 3/2, \\
 a_7 &= 1 - \gamma/2, \quad b_7 = (1 - \gamma)/2, \quad c_7 = \sigma + \alpha, \quad d_7 = \alpha, \quad e_7 = (1 + \sigma)/2 + \alpha, \\
 f_7 &= \sigma/2 + \alpha, \quad g_7 = 1 - \gamma, \quad a_8 = 1 + \gamma/2, \quad b_8 = (1 + \gamma)/2, \quad c_8 = \sigma + \alpha + \gamma, \\
 d_8 &= \alpha + \gamma, \quad e_8 = 1 + \gamma, \quad f_8 = (1 + \sigma)/2 + \alpha + \gamma, \quad g_8 = \sigma/2 + \alpha + \gamma, \quad z = -y \quad \} \\
 &[\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\bar{y-x})^\gamma + (V\bar{y} - V\bar{y-x})^\gamma}{(V\bar{x} + V\sqrt{1+x})^{-\sigma} V(1+x)(y-x)} dx = \\
 = \begin{cases} V_8^+ + V_9^+ & [0 < y \leq 1], \\ V_8^- + V_9^- + V_{10}^- & [y \geq 1] \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \{E_8^+ &= 2^{\gamma+2\alpha} B(\alpha, \gamma + \alpha), \quad E_9^+ = 2^{\gamma+2\alpha+1} \sigma B(\alpha + 1/2, \gamma + \alpha + 1/2), \\
 E_8^- &= 2^{\gamma+2\sigma+2\alpha-1} B(\alpha + (\sigma-1)/2, \alpha + \gamma + (\sigma-1)/2), \\
 E_9^- &= 2^{\gamma-2\alpha+1} B(2\alpha, (1 - \sigma)/2 - \alpha), \quad E_{10}^- = 2^{1-2\alpha-3\gamma} B(2\alpha + 2\gamma, (1 - \sigma)/2 - \alpha - \gamma), \\
 s_1 &= \alpha + (\gamma-1)/2, \quad s_2 = \alpha + \gamma/2, \quad s_5 = \alpha + (\gamma + \sigma)/2 - 1, \quad s_6 = (\gamma-1)/2, \\
 s_7 &= -(1 + \gamma)/2, \quad a_1 = \alpha, \quad b_1 = \alpha + \gamma, \quad c_1 = (1 - \sigma)/2, \quad d_1 = (1 + \sigma)/2, \\
 e_1 &= \alpha + \gamma/2, \quad f_1 = (\gamma + 1)/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \gamma + \alpha + 1/2, \\
 c_2 &= 1 - \sigma/2, \quad d_2 = 1 + \sigma/2, \quad e_2 = \alpha + (\gamma + 1)/2, \quad f_2 = \alpha + 1 + \gamma/2, \\
 g_2 &= 3/2, \quad a_5 = (3 - \gamma - \sigma)/2 - \alpha, \quad b_5 = 1 - (\gamma + \sigma)/2 - \alpha, \quad c_5 = (1 - \sigma)/2, \\
 d_5 &= 1 - \sigma/2, \quad e_5 = (3 - \sigma)/2 - \alpha, \quad f_5 = (3 - \sigma)/2 - \alpha - \gamma, \quad g_5 = 1 - \sigma, \quad a_6 = 1 - \gamma/2, \\
 b_6 &= (1 - \gamma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \quad e_6 = 1 - \gamma, \quad f_6 = (1 + \sigma)/2 + \alpha, \\
 g_6 &= (1 - \sigma)/2 + \alpha, \quad a_7 = 1 + \gamma/2, \quad b_7 = (1 + \gamma)/2, \quad c_7 = \alpha + \gamma, \\
 d_7 &= \alpha + \gamma + 1/2, \quad e_7 = 1 + \gamma, \quad f_7 = (1 + \sigma)/2 + \alpha + \gamma, \\
 g_7 &= (1 - \sigma)/2 + \alpha + \gamma, \quad z = -y \quad \} \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\bar{y-x})^\lambda + (V\bar{y} - V\bar{y-x})^\lambda}{(1 + 2x \cos \gamma + x^2) V\bar{y-x}} dx = U_2 = \\
 = \begin{cases} V_{12}^+ + V_{13}^+ + V_{24}^+ & [0 < y \leq 1], \\ V_{12}^- & [y \geq 1] \end{cases}
 \end{aligned}$$

$$\left\{ E_{12}^+ = -\frac{2^{\lambda-2l}}{\sin \gamma} \pi \Gamma \left[\begin{matrix} \lambda-l \\ \lambda-2l \end{matrix} \right], \right.$$

$$E_{13}^+ = -\frac{2^{\lambda+2(\alpha-l-1)/r}}{r \sin \gamma} \Gamma \left[\begin{matrix} \lambda+(\alpha-l-1)/r \\ \lambda+2(\alpha-l-1)/r \end{matrix} \right],$$

$$E_{24}^+ = -\frac{2^{-\lambda-2l} \pi}{\sin \gamma} \Gamma \left[\begin{matrix} -l-\lambda \\ -\lambda-2l \end{matrix} \right], \quad E_{12}^- = \frac{2^{\lambda+2(\alpha+l)/r}}{r \sin \gamma} \Gamma \left[\begin{matrix} \lambda+(\alpha+l)/r \\ \lambda+2(\alpha+l)/r \end{matrix} \right] \Bigg\}$$

[Re α , Re $(\alpha+\lambda)$ > 0; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$46. \int_0^y x^{\alpha-1} \frac{(V\bar{y} + V\bar{y-x})^\lambda + (V\bar{y} - V\bar{y-x})^\lambda}{(1+2x \cos \gamma + x^2)^2 V\bar{y-x}} dx =$$

$$= 2^{-1} (y \sin \gamma)^{-2} U_2 + \begin{cases} V_{18}^+ + V_{19}^+ + V_{25}^+, & \left\{ \begin{matrix} 0 < y \leq 1 \\ y \geq 1 \end{matrix} \right\}; \text{ см. 2.22.45} \\ V_{18}^- \end{cases}$$

$$\left\{ E_{18}^+ = 2^{\lambda-2l-1} \pi \Gamma \left[\begin{matrix} \lambda-l \\ \lambda-2l \end{matrix} \right], \quad E_{19}^+ = \frac{2^{\lambda+2(\alpha-l-1)/r-1}}{r} \Gamma \left[\begin{matrix} \lambda+(\alpha-l-1)/r \\ \lambda+2(\alpha-l-1)/r \end{matrix} \right], \right.$$

$$E_{25}^+ = 2^{-\lambda-2l-1} \pi \Gamma \left[\begin{matrix} -l-\lambda \\ \gamma-2(\lambda+l) \end{matrix} \right], \quad E_{18}^- = -\frac{2^{\lambda+2(\alpha+l)/r-1}}{r} \Gamma \left[\begin{matrix} \lambda+(l+\alpha)/r \\ \lambda+2(l+\alpha)/r \end{matrix} \right] \Bigg\}$$

[Re α , Re $(\alpha+\lambda)$ > 0; $|\gamma| < \pi$; $r > 0$ — произвольное].

2.2.3. Интегралы от $(x^\mu \mp a^\mu)^\rho$ и $(a^\mu - x^\mu)^\rho$.

$$1. \int_0^a (a^\mu - x^\mu)^{\beta-1} dx = \mu^{-1} a^\mu (\beta-1+1) B(\beta, 1/\mu) \quad [a, \mu, \operatorname{Re} \beta > 0].$$

$$2. \int_0^a (a^2 - x^2)^{n-1/2} dx = \pi a^{2n} \frac{(2n-1)!!}{2(2n)!!} \quad [a > 0].$$

$$3. \int_0^a (a^2 - x^2)^n dx = a^{2n+1} \frac{(2n)!!}{(2n+1)!!} \quad [a > 0].$$

$$4. \int_a^\infty (x^\mu - a^\mu)^{\beta-1} dx = \mu^{-1} a^\mu (\beta-1+1) B(1-\beta-1/\mu, \beta)$$

[$a, \mu > 0$; $0 < \operatorname{Re} \beta < 1 - 1/\mu$]

$$5. \int_0^\infty (x^\mu + z^\mu)^{-\rho} dx = \mu^{-1} z^{1-\mu\rho} B(1/\mu, \rho-1/\mu) \quad [\mu > 0; \mu |\arg z| < \pi; \mu \operatorname{Re} \rho > 1]$$

$$6. \int_0^\infty \frac{dx}{(x^2 + z^2)^n} = \frac{\pi}{2} z^{1-2n} \frac{(2n-3)!!}{(2n-2)!!} \quad [\operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

2.2.4. Интегралы от $x^\alpha (a^\mu \pm x^\mu)^\beta$.

$$1. \int_0^a \frac{x^{\alpha-1}}{x+a} dx = a^{\alpha-1} \beta(\alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$2. \int_0^a \frac{x^n}{x+a} dx = (-a)^n \left\{ \ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right\} \quad [a > 0].$$

$$3. \int_0^a \frac{dx}{\sqrt{x(x+a)}} = \frac{\pi}{2\sqrt{a}} \quad [a > 0].$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(x+a)^2} dx = \frac{a^{\alpha-2}}{2} - (\alpha-1) a^{\alpha-2} \beta(\alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$5. \int_0^a \frac{x^n}{(x+a)^2} dx = n (-a)^{n-1} \left\{ \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + \frac{(-1)^n}{2n} \right\} \quad [a > 0; n = 1, 2, 3, \dots].$$

$$6. \int_0^a \frac{x^{\alpha-1}}{(x+a)^{2\alpha}} dx = \frac{a^{-\alpha}}{2} B(\alpha, \alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$7. \int_0^a \frac{x^n dx}{\sqrt{x^2+a^2}} = I_n \quad [a > 0].$$

$$I_0 = \ln(1 + \sqrt{2}), \quad I_1 = a(\sqrt{2} - 1), \quad I_2 = \frac{a^2}{2} [\sqrt{2} - \ln(1 + \sqrt{2})],$$

$$I_n = \frac{a^n \sqrt{2}}{n} \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{\left(\frac{1-n}{2}\right)_k}{(1-n/2)_k} +$$

$$+ (-1)^{[(n+1)/2]} a^n \left\{ \frac{1 - (-1)^n \left(\frac{1-n}{2}\right)_{[(n+1)/2]}}{2 (-n/2)_{[(n+1)/2]}} \ln(1 + \sqrt{2}) - \right.$$

$$\left. - \frac{1 - (-1)^n}{2(n+1)} \frac{\left(\frac{1-n}{2}\right)_{[(n+1)/2]}}{(-n/2)_{[(n+1)/2]}} \right\}.$$

$$8. \int_0^a x^{\alpha-1} (a^\mu - x^\mu)^{\beta-1} dx = a^{\mu(\beta-1)+\alpha} \mu^{-1} B\left(\frac{\alpha}{\mu}, \beta\right) \quad [a, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0]$$

$$9. \int_0^a x^{2m} (a^2 - x^2)^{n-1/2} dx = \frac{\pi a^{2m+2n} (2m-1)!! (2n-1)!!}{2^{m+n+1} (m+n)!} \quad [a > 0]$$

$$10. \int_0^a x^{2m+1} (a^2 - x^2)^{n-1/2} dx = \frac{a^{2m+2n+1} 2^m m! (2n-1)!!}{(2m+2n+1)!!} \quad [a > 0].$$

$$11. \int_0^a \frac{x^{2m+\delta}}{\sqrt{a^2 - x^2}} dx = a^{2m+\delta} \frac{(2m+\delta-1)!!}{(2m+\delta)!!} \left\{ \frac{1}{\pi/2} \right\} \quad [a > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$12. \int_0^a x^{\alpha-1} (a-x)^{-\alpha} dx = \frac{\pi}{\sin \alpha \pi} \quad [a > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$13. \int_0^a x^{\alpha-1} (2a-x)^{\alpha-1} dx = \frac{1}{2} (2a)^{2\alpha-1} B(\alpha, \alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$14. \int_0^1 \frac{x^\alpha + x^\beta}{(x+1)^{\alpha+\beta+2}} dx = B(\alpha+1, \beta+1) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$15. \int_0^1 \frac{x^{\alpha-1} + x^{-\alpha}}{x+1} dx = \frac{\pi}{\sin \alpha \pi} \quad [0 < \operatorname{Re} \alpha < 1].$$

$$16. \int_0^1 \frac{x^\alpha - x^{-\alpha}}{x+1} dx = \frac{1}{\alpha} - \frac{\pi}{\sin \alpha \pi} \quad [0 < |\operatorname{Re} \alpha| < 1].$$

$$17. \int_0^1 \frac{x^{p-1} + x^{mn-p-1}}{(x^n+1)^m} dx = \binom{m-1-p/n}{m-1} \frac{\pi}{n \sin(p\pi/n)} \\ [m, n, p = 1, 2, 3, \dots; mn > p; p/n - \text{нецелое}].$$

$$18. = \frac{(p/n-1)! (m-1-p/n)!}{(m-1)! n} \quad [m, n, p = 1, 2, 3, \dots; mn > p; p/n - \text{целое}].$$

$$19. \int_0^1 \frac{x^\alpha - x^\beta}{(1-x)^{\alpha+\beta+2}} dx = \frac{\sin [(\alpha-\beta)\pi/2]}{\sin [(\alpha+\beta)\pi/2]} B(\alpha+1, \beta+1) \\ [\operatorname{Re}(\alpha+\beta) < 0; \operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$20. \int_0^1 \frac{x^\alpha - x^\beta}{1-x} dx = \psi(\beta+1) - \psi(\alpha+1) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$21. \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1-x} dx = \pi \operatorname{ctg} \alpha \pi \quad [0 < \operatorname{Re} \alpha < 1].$$

$$22. \int_0^1 \frac{x^\alpha - x^{-\alpha}}{1-x} dx = \pi \operatorname{ctg} \alpha \pi - \frac{1}{\alpha} \quad [|\operatorname{Re} \alpha| < 1].$$

$$23. \int_0^1 \frac{x^\alpha - x^\beta}{x^\mu - x^\nu} dx = \frac{1}{\mu - \nu} \left[\psi \left(\frac{\mu - \alpha - 1}{\mu - \nu} \right) - \psi \left(\frac{\mu - \beta - 1}{\mu - \nu} \right) \right]$$

[$\mu > \max(\nu, \alpha + 1, \beta + 1)$].

$$24. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} dx = z^{\alpha-\rho} B(\alpha, \rho - \alpha)$$

[$|\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho$].

$$25. \int_0^\infty \frac{x^{\alpha-1}}{x+z} dx = \frac{\pi z^{\alpha-1}}{\sin \alpha \pi}$$

[$|\arg z| < \pi; 0 < \operatorname{Re} \alpha < 1$].

$$26. \int_0^\infty \frac{x^{\alpha-1} dx}{x-a} = -\pi a^{\alpha-1} \operatorname{ctg} \alpha \pi$$

[$a > 0; 0 < \operatorname{Re} \alpha < 1$].

$$27. \int_0^\infty \frac{x^{\alpha-1}}{|x-a|^\rho} dx = \frac{a^{\alpha-\rho} \cos(\alpha - \rho/2) \pi}{\cos(\rho \pi/2)} B(\alpha, \rho - \alpha)$$

[$a > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho < 1$].

$$28. \int_0^\infty \frac{x^{\alpha-1}}{(x-a)^n} dx = \frac{(-1)^n \pi a^{\alpha-n}}{(n-1)! \sin \alpha \pi} \prod_{k=1}^{n-1} (\alpha - k)$$

[$a > 0; 0 < \operatorname{Re} \alpha < n; n = 1, 3, 5, \dots$].

$$29. \int_{-\infty}^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} dx = 0 \quad [0 < |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho].$$

$$30. \int_{-\infty}^\infty \frac{|x|^{\alpha-1}}{x-a} \begin{Bmatrix} 1 \\ \operatorname{sgn} x \end{Bmatrix} = \mp \pi \operatorname{tg} \frac{\pi}{2} \frac{\alpha \pi}{2} |a|^{\alpha-1} \begin{Bmatrix} \operatorname{sgn} a \\ 1 \end{Bmatrix}$$

[$\operatorname{Im} a = 0; 0 < \operatorname{Re} \alpha < 1$].

2.2.5. Интегралы от $(a^\mu \pm x^\mu)^\rho (b^\nu \mp x^\nu)^\sigma$.

Условия. a, b, c, d — действительные числа.

$$1. \int_a^b (x-a)^{\alpha-1} (b-x)^{\beta-1} dx = (b-a)^{\alpha+\beta-1} B(\alpha, \beta)$$

[$\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$].

$$2. \int_a^b \frac{(x-a)^{\alpha-1}}{(cx+d)^{\alpha+n+1}} dx = \frac{(b-a)^\alpha}{(ac+d)(bc+d)^\alpha} \sum_{k=0}^n \binom{n}{k} \frac{B(\alpha+k, n-k+1)}{(bc+d)^k (ac+d)^{n-k}}$$

[$(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0$].

$$3. \int_a^b \frac{(x-a)^{\alpha-1}}{(cx+d)^{\alpha+1}} dx = \frac{(b-a)^\alpha}{\alpha (ac+d)(bc+d)^\alpha}$$

[$(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0$].

$$4. \int_a^b \frac{(b-x)^{\beta-1}}{(cx+d)^{\beta+n+1}} dx = \frac{(b-a)^\beta}{(bc+d)(ac+d)^\beta} \sum_{k=0}^n \binom{n}{k} \frac{B(k+1, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}}$$

[$(ac+d)(bc+d) > 0; \operatorname{Re} \beta > 0$].

5. $\int_a^b \frac{(b-x)^{\beta-1}}{(cx+d)^{\beta+1}} dx = \frac{(b-a)^{\beta}}{\beta (bc+d)(ac+d)^{\beta}}$ [$(ac+d)(bc+d) > 0$; $\operatorname{Re} \beta > 0$].
6. $\int_{-a}^a \left(\frac{a-x}{a+x} \right)^{\beta-1} dx = \frac{2\pi(1-\beta)a}{\sin \beta\pi}$ [$0 < \operatorname{Re} \beta < 2$, $\beta \neq 1$].
7. $\int_{-a}^a \frac{(a^2-x^2)^m}{(cx+d)^{m+1/2}} dx = \frac{m!2^{m+1}}{(2m+1)!!} (\sqrt{ac+d} - \sqrt{d-ac})^{2m+1}$
[$d > |ac| > 0$].
8. $\int_{-a}^a \frac{(a^2-x^2)^m}{(y-x)^{m+1}} dx = 2(2a)^m Q_m\left(\frac{y}{a}\right)$ [$|y| < a$].
9. $\int_{-a}^a \frac{(d+icx)^m}{\sqrt{a^2-x^2}} dx = \pi (d^2+a^2c^2)^{m/2} P_m\left(\frac{d}{\sqrt{d^2+a^2c^2}}\right)$ [$m=0, \pm 1, \pm 2, \dots$].
10. $\int_{-a}^a \frac{(a^2-x^2)^{\pm 1/2}}{x-y} dx = \pm \pi \operatorname{sgn} y (y^2-a^2)^{\pm 1/2} - \begin{cases} \pi y \\ 0 \end{cases}$
[$-\infty < y < -a$ или $a < y < \infty$].
11. $= -\begin{cases} \pi y \\ 0 \end{cases}$ [$-a < y < a$].
12. $\int_0^a \frac{dx}{\sqrt{(a-x)(y-x)}} = \ln \frac{\sqrt{y} + \sqrt{a}}{\sqrt{y} - \sqrt{a}}$ [$0 < a < y$].
13. $\int_0^a \frac{dx}{\sqrt{(a-x)(x+y)}} = \frac{2}{\sqrt{a+y}} \ln \frac{\sqrt{a} + \sqrt{a+y}}{\sqrt{y}}$ [$a > 0$; $|\arg y| < \pi$].
14. $\int_0^a \frac{dx}{\sqrt{a-x}(x-y)} = -\frac{2}{\sqrt{y-a}} \operatorname{arctg} \sqrt{\frac{a}{y-a}}$ [$y > a > 0$].
15. $\int_0^a \frac{(a^2-x^2)^{\pm 1/2}}{x-y} dx = \pm \operatorname{sgn} y (y^2-a^2)^{\pm 1/2} \arccos\left(-\frac{a}{y}\right) - \begin{cases} a + \pi y/2 \\ 0 \end{cases}$
[$-\infty < y < -a$ или $a < y < \infty$]
16. $= (a^2-y^2)^{\pm 1/2} \ln \left| \frac{a + \sqrt{a^2-y^2}}{y} \right| - \begin{cases} a + \pi y/2 \\ 0 \end{cases}$ [$-a < y < a$].
17. $\int_0^a \frac{a^\mu - x^\mu}{a-x} dx = a^\mu [\zeta + \psi(\mu+1)]$ [$\operatorname{Re} \mu > -1$].
18. $\int_0^1 \sqrt{\frac{(1-k^2x^2)^{\pm 1}}{1-x^2}} dx = \begin{cases} E(k) \\ K(k) \end{cases},$ [$0 \leq k < 1$].

$$19. \int_0^{\infty} \frac{(ax+b)^{\beta-1}}{(cx+d)^{\beta+1}} dx = \frac{(ad)^{\beta} - (bc)^{\beta}}{\beta(ad-bc)(cd)^{\beta}} \quad [cd > 0; ad \neq bc; \operatorname{Re} \beta \geq 1].$$

$$20. \int_0^{\infty} \frac{dx}{\sqrt{x+z}(x-a)} = \frac{2}{\sqrt{z+a}} \ln \left| \frac{\sqrt{z} + \sqrt{z+a}}{\sqrt{z} - \sqrt{z+a}} \right|$$

[a > 0; |arg(1+a/z)| < π].

$$21. \int_0^{\infty} \frac{dx}{(x^2+z^2)^n(x-y)} = -\frac{1}{2(n-1)z^{2n-2}(z^2+y^2)} -$$

$$-\frac{(2n-3)!! y\pi}{(n-1)! 2^n z^{2n-1}(z^2+y^2)} + \frac{1}{z^2+y^2} \int_0^{\infty} \frac{dx}{(x^2+z^2)^{n-1}(x-y)}$$

[y, Re z > 0; n = 2, 3, 4, ...].

$$22. \int_0^{\infty} \frac{dx}{(x^2+z^2)(x \pm y)^n} = \frac{(-1)^{n-1}}{(n-1)!(z^2+y^2)} \frac{\partial^{n-1}}{\partial y^{n-1}} \left(\pm \frac{y\pi}{2z} - \ln \frac{y}{z} \right)$$

[Re z > 0; {arg y < π, n = 1, 2, 3, ...} ; {y > 0, n = 1, 3, 5, ...}]].

$$23. \int_0^{\infty} \frac{dx}{\sqrt{x^2+z^2}(x+y)} = \frac{1}{\sqrt{z^2+y^2}} \ln \frac{z(\sqrt{z^2+y^2}+z)}{y(\sqrt{z^2+y^2}-y)}$$

[Re z > 0; |arg y| < π].

$$24. \int_0^{\infty} \frac{dx}{\sqrt{x^2+z^2}(x^2+y^2)} = \frac{1}{y\sqrt{z^2-y^2}} \operatorname{arctg} \frac{\sqrt{z^2-y^2}}{y}$$

[Re z, Re y > 0; |arg(z^2-y^2)| < π; z ≠ y].

$$25. = y^{-2} \quad [z = y; \operatorname{Re} y > 0].$$

$$26. = \frac{1}{y\sqrt{y^2-z^2}} \ln \frac{y + \sqrt{y^2-z^2}}{z}$$

[Re z, Re y > 0; |arg(y^2-z^2)| < π; z ≠ y].

$$27. \int_0^{\infty} \frac{dx}{(a^2x^2+b^2)(c^2x^4+d^2)} = \frac{\pi}{2(a^4d^2+b^4c^2)} \left[\frac{a^3}{b} - \left(a^2 - \frac{b^2c}{d} \right) \sqrt{\frac{c}{2d}} \right]$$

[a, b, c, d > 0].

$$28. \int_{-\infty}^{\infty} \frac{dx}{(x^2+z^2)^{\rho}(x-y)} = -2\sqrt{\pi}\Gamma\left[\rho + \frac{1}{2}\right] \frac{yz^{1-2\rho}}{y^2+z^2} {}_2F_1\left(1, 1-\rho; \frac{3}{2}; \frac{y^2}{y^2+z^2}\right)$$

[Re z, Re ρ > 0; Im y = 0].

$$29. \int_{-\infty}^{\infty} \frac{(x-r_1e^{i\theta_1})^{\beta-1}}{(x+r_2e^{i\theta_2})^{\rho}} dx = 0 \quad [r_1, r_2 > 0; 0 < |\theta_1| < \pi, 0 < |\theta_2| < \pi; \theta_1\theta_2 < 0; \rho > \beta].$$

$$30. = \varepsilon \exp [\varepsilon i \pi \beta - i (\rho - \beta) (\theta_1 + \theta_3)] \frac{2\pi i}{\beta r_3^{\rho - \beta} B(\rho, -\beta)}$$

$$[r_1, r_2 > 0; \rho > \beta; r_3 = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}, r_3 \sin \theta_3 = r_2 \sin(\theta_2 - \theta_1),$$

$$r_3 \cos \theta_3 = r_1 + r_2 \cos(\theta_2 - \theta_1); |\theta_3| < \pi; \varepsilon = \begin{cases} -1, & \text{если } 0 < \theta_1 < \pi, 0 < \theta_2 < \pi, \\ 1, & \text{если } -\pi < \theta_1 < 0, -\pi < \theta_2 < 0 \end{cases}].$$

$$31. \int_{-\infty}^{\infty} \frac{dx}{(a+ix)^{\mu} (b-ix)^{\nu}} = 2\pi (a+b)^{1-\mu-\nu} \Gamma \left[\begin{matrix} \mu + \nu - 1 \\ \mu, \nu \end{matrix} \right] \quad [a, b > 0; \mu + \nu > 1].$$

$$32. = 0 \quad [ab < 0; \mu + \nu > 1].$$

$$33. \int_{-\infty}^{\infty} \frac{dx}{(a+ix)^{\mu} (b+ix)^{\nu}} = 0 \quad [\operatorname{Re} a, \operatorname{Re} b > 0; \mu + \nu > 1].$$

2.2.6. Интегралы от $\prod_{k=1}^3 (a_k \pm x)^{\beta_k}$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b (x-a)^{\alpha-1} (b-x)^{\beta-1} (cx+d)^{\gamma} dx =$$

$$= (b-a)^{\alpha+\beta-1} (ac+d)^{\gamma} B(\alpha, \beta) {}_2F_1 \left(\alpha, -\gamma; \alpha+\beta; \frac{c(a-b)}{ac+d} \right)$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(d+cb)/(d+ca)| < \pi].$$

$$2. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{|x-y|^{\alpha+\beta}} dx = \frac{(b-a)^{\alpha+\beta-1}}{|y-a|^{\beta} |y-b|^{\alpha}} B(\alpha, \beta)$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; y < a < b \text{ или } a < b < y].$$

$$3. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{x-y} dx = \frac{(b-a)^{\alpha+\beta-1}}{b-y} B(\alpha, \beta) {}_2F_1 \left(1, \beta; \alpha+\beta; \frac{b-a}{b-y} \right)$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; y < a < b \text{ или } a < b < y].$$

$$4. = (y-a)^{\alpha-1} (b-y)^{\beta-1} \pi \operatorname{ctg} \alpha \pi -$$

$$-(b-a)^{\alpha+\beta-2} B(\alpha, \beta-1) {}_2F_1 \left(1, 2-\alpha-\beta; 2-\beta; \frac{b-y}{b-a} \right)$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; a < y < b].$$

$$5. \int_a^b \left(\frac{x-a}{b-x} \right)^{\alpha-1} \frac{dx}{x-y} = \frac{\pi}{\sin \alpha \pi} \left(\left| \frac{a-y}{b-y} \right|^{\alpha-1} - 1 \right)$$

$$[0 < \operatorname{Re} \alpha < 2; y < a < b \text{ или } a < b < y].$$

$$6. = -\frac{\pi}{\sin \alpha \pi} \left[\cos \alpha \pi \left(\frac{y-a}{b-y} \right)^{\alpha-1} + 1 \right] \quad [0 < \operatorname{Re} \alpha < 2; a < y < b].$$

$$7. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{-\alpha}}{x-y} dx = \frac{\pi}{\sin \alpha \pi (b-y)} \left| \frac{a-y}{b-y} \right|^{\alpha-1}$$

$$[0 < \operatorname{Re} \alpha < 1; y < a < b \text{ или } a < b < y].$$

$$8. \quad = -\pi \operatorname{ctg} \alpha \pi \frac{(y-a)^{\alpha-1}}{(b-y)^{\alpha}} \quad [0 < \operatorname{Re} \alpha < 1; a < y < b].$$

$$9. \quad \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{(cx+d)^{\alpha+\beta+n}} dx =$$

$$= \frac{(b-a)^{\alpha+\beta-1}}{(bc+d)^{\alpha} (ac+d)^{\beta}} \sum_{k=0}^n \binom{n}{k} \frac{B(\alpha+k, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}}$$

$$[(ac+d)(bc+d) > 0; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$10. \quad \int_a^b \frac{\sqrt{(x-a)(b-x)}}{(cx+d)^n} dx = I_n \quad [c \neq 0; ac+d, bc+d > 0],$$

$$I_1 = \frac{\pi}{2c^2} (\sqrt{ac+d} - \sqrt{bc+d})^2,$$

$$I_2 = \frac{\pi}{2c^2} \frac{(\sqrt{ac+d} - \sqrt{bc+d})^2}{\sqrt{(ac+d)(bc+d)}}, \quad I_3 = \frac{\pi (b-a)^2}{8 [(ac+d)(bc+d)]^{3/2}}.$$

$$11. \quad \int_a^b x^m (x-a)^n (b-x)^p dx = \frac{m! (b-a)^{n+p+1}}{(m+n+p+1)!} \sum_{k=0}^m \frac{(n+k)! (m+p-k)!}{k! (m-k)!} a^{n-k} b^k$$

$$[m, n, p = 0, 1, 2, \dots].$$

$$12. \quad \int_a^b x^m (x-a)^{n/2} (b-x)^{n/2} dx =$$

$$= \frac{(-1)^m m!}{(m+n+1)!} \Gamma^2\left(\frac{n}{2} + 1\right) (b-a)^{n+1} \sum_{k=0}^m \binom{-n/2-1}{k} \binom{-n/2-1}{m-k} a^k b^{m-k}$$

$$[m = 0, 1, 2, \dots; n = -1, 0, 1, 2, \dots].$$

$$13. \quad \int_a^b \frac{dx}{x^{\varepsilon} \sqrt{(x-a)(b-x)}} = \frac{\pi}{\sqrt{ab}} \left\{ 2^{-1} (a^{-1} + b^{-1}) \right\} \quad \left[\varepsilon = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \right].$$

$$14. \quad \int_a^b \frac{x^m (x-a)^{\alpha-1}}{(cx+d)^{m+\alpha+1}} dx =$$

$$= \frac{(b-a)^{\alpha}}{(ac+d)(bc+d)^{\alpha}} \sum_{k=0}^m \binom{m}{k} b^k a^{m-k} \frac{B(\alpha+k, m-k+1)}{(bc+d)^k (ac+d)^{m-k}}$$

$$[(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0].$$

$$15. \quad \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (x+z)^{-\rho} dx = a^{\alpha+\beta-1} z^{-\rho} B(\alpha, \beta) {}_2F_1\left(\alpha, \rho; \alpha+\beta; -\frac{a}{z}\right)$$

$$[|\arg z| < \pi; a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$16. \int_0^a \frac{x^{\alpha-1}}{[(a-x)(y-x)]^{\alpha-1/2}} dx =$$

$$= \frac{\Gamma(\alpha-1) \Gamma(3/2-\alpha)}{2\sqrt{\pi}} \{(\sqrt{y}-\sqrt{a})^{2-2\alpha} - (\sqrt{y}+\sqrt{a})^{2-2\alpha}\}$$

[$y > a$; $0 < \operatorname{Re} \alpha < 3/2$].

$$17. \int_0^a \frac{dx}{\sqrt{x(a-x)(x \pm c)}} = \pm \frac{\pi}{\sqrt{c(c \pm a)}} \quad \left[\left\{ \begin{array}{l} a, c > 0 \\ c > a > 0 \end{array} \right\} \right].$$

$$18. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} \frac{dx}{x+a} = I_\alpha \quad [a, \operatorname{Re} \alpha > 0],$$

$$I_{1/4} = a^{-5/4} [\pi/2 + \sqrt{2} K(1/\sqrt{2})], \quad I_{3/4} = a^{-3/4} \pi/2,$$

$$I_{5/4} = -a^{-1/4} [\pi/2 - \sqrt{2} K(1/\sqrt{2})].$$

$$19. \int_0^a \sqrt{\frac{x}{a+x}} \frac{dx}{\sqrt{1+k^2x}} = \frac{1}{k^2} \left[\sqrt{2(1+k^2c)} - 2E\left(\frac{\pi}{4}, \sqrt{1-k^2a}\right) \right]$$

[$a > 0$; $0 < k \leq 1$].

$$20. \int_0^a \left(\frac{a-x}{a+x} \right)^{\pm 1/2} \frac{dx}{x-y} = \mp \frac{\pi}{2} \pm \left| \frac{y-a}{y+a} \right|^{\pm 1/2} \arccos \left(-\frac{a}{y} \right)$$

[$-\infty < y < -a$ или $a < y < \infty$].

$$21. = \mp \frac{\pi}{2} + \left(\frac{a-y}{a+y} \right)^{\pm 1/2} \ln \left| \frac{a + \sqrt{a^2 - y^2}}{y} \right| \quad [-a < y < a].$$

$$22. \int_a^b \frac{x^m (b-x)^{\beta-1}}{(cx+d)^{m+\beta+1}} dx = \frac{(b-a)^\beta}{(bc+d)(ac+d)^\beta} \sum_{k=0}^m \binom{m}{k} b^k a^{m-k} \frac{B(k+1, \beta+m-k)}{(bc+d)^k (ac+d)^{m-k}}$$

[$(ae+d)(bc+d)$, $\operatorname{Re} \beta > 0$].

$$23. \int_0^\infty \frac{x^{\alpha-1}}{[(x+z)(x+y)]^{\alpha-3/2}} dx = \frac{[y+z+2(\alpha-2)\sqrt{yz}]\sqrt{\pi}}{2(\sqrt{y}+\sqrt{z})^{2\alpha-4}} \Gamma\left[\frac{\alpha-3}{\alpha-3/2}\right]$$

[$|\arg z|, |\arg y| < \pi$; $\operatorname{Re} \alpha > 3$].

$$24. \int_0^\infty x^{\alpha-1} (x+y)^{-\rho} (x+z)^{-\lambda} dx =$$

$$= z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho+\lambda-\alpha) {}_2F_1\left(\alpha, \lambda; \rho+\lambda; 1-\frac{y}{z}\right)$$

[$|\arg y|, |\arg z| < \pi$; $0 < \operatorname{Re} \alpha < \operatorname{Re}(\rho+\lambda)$].

$$25. \int_0^\infty \frac{x^{\alpha-1} (x+y)^{\alpha-1}}{(2x+y)^{2\alpha}} dx = \frac{B(\alpha, \alpha)}{2y}$$

[$\operatorname{Re} \alpha > 0$; $|\arg y| < \pi$].

$$26. \int_0^{\infty} x^{\alpha-1} (x+z)^{-\lambda} \frac{dx}{x-y} = z^{-\lambda} (-y)^{\alpha-1} B(\alpha, 1-\alpha+\lambda) {}_2F_1\left(\alpha, \lambda; 1+\lambda; 1+\frac{y}{z}\right)$$

$$[|\arg(-y)| < \pi; 0 < \operatorname{Re} \alpha < 1 + \operatorname{Re} \lambda].$$

$$27. = -\pi \operatorname{ctg}(\alpha - \lambda) \pi y^{\alpha-1} (z+y)^{-\lambda} - \\ - \frac{z^{\alpha-\lambda}}{z+y} B(\alpha, \lambda-\alpha) {}_2F_1\left(1, 1-\lambda; 1+\alpha-\lambda; \frac{z}{z+y}\right)$$

$$[y > 0; 0 < \operatorname{Re} \alpha < 1 + \operatorname{Re} \lambda].$$

$$28. \int_0^{\infty} \frac{x^{\alpha-1} dx}{[(x+z)(x+y)]^{\alpha-1/2}} = \frac{\sqrt{\pi}}{(\sqrt{z} + \sqrt{y})^{2\alpha-2}} \Gamma\left[\frac{\alpha-1}{2}\right]$$

$$[|\arg z|, |\arg y| < \pi; \operatorname{Re} \alpha > 1].$$

2.2.7. Интегралы от $\prod_{k=1}^3 (a_k^{\mu_k} \pm x^{\mu_k})^{\rho_k}$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b \frac{x^{2m}}{\sqrt{1-x^2} \sqrt{b^2-x^2}} dx = \theta_m(b, a) \quad [0 < a < b < 1; m = 0, \pm 1, \pm 2, \dots].$$

$$\theta_0(b, a) = F(\varphi, b), \quad \theta_1(b, a) = F(\varphi, b) - E(\varphi, b) + a \sqrt{\frac{b^2-a^2}{1-a^2}}$$

$$\left[\varphi = \arcsin \frac{\sqrt{b^2-a^2}}{b \sqrt{1-a^2}} \right],$$

$$(2m+1) \theta_{m+1}(b, a) - 2m(b^2+1) \theta_m(b, a) - (2m-1) b^2 \theta_{m-1}(b, a) = \\ = -a^{2m+1} \sqrt{(1-a^2)(b^2-a^2)} \quad [m = 1, 2, 3, \dots].$$

$$\theta_{-1}(b, a) = b^{-2} [F(\varphi, b) - E(\varphi, b)] + b^{-2} a^{-1} \sqrt{(b^2-a^2)/(1-a^2)},$$

$$(2m+1) b^2 \theta_{-m-1}(b, a) - 2m(b^2+1) \theta_{-m}(b, a) + (2m-1) \theta_{1-m}(b, a) = \\ = a^{-2m-1} \sqrt{(1-a^2)(b^2-a^2)} \quad [m = 1, 2, 3, \dots].$$

$$2. \int_{-a}^a \frac{(a+x)^{\alpha-1} (a-x)^{\beta-1}}{(a^2+x^2)^{(\alpha+\beta)/2}} dx = \frac{1}{2} (2a)^{(\alpha+\beta)/2-1} B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

$$[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$3. \int_{-a}^a \left(\frac{a+x}{a-x}\right)^{\beta} \frac{dx}{x^2+z^2} = \frac{\pi}{2z \cos(\beta\pi/2)}$$

$$[|\operatorname{Re} \beta| < 1; a > 0].$$

$$4. \int_{-a}^a \frac{x(a^2-x^2)^m}{(cx+d)^{m+1/2}} dx = -\frac{m! 2^m (\sqrt{d+ac} - \sqrt{d-ac})^{2m+3}}{(2m+3)(2m-1)!!}$$

$$[d > |ac| > 0].$$

$$5. \int_0^a \frac{x^{\alpha-1} (a-x)^{n-1}}{1-zx^r} dx = (n-1)! a^{\alpha+n-1} \sum_{k=0}^{\infty} \frac{(za^r)^k}{(\alpha+kr)_n}$$

$$[r, \operatorname{Re} \alpha > 0; |za^r| < 1; n = 1, 2, 3, \dots].$$

$$6. \int_0^a \frac{a^n - x^n}{a - x} \frac{dx}{(a + x)^{n+1}} = \frac{2^{n-1}}{a} \sum_{k=1}^n \frac{2^k}{k} \quad [n=1, 2, 3, \dots].$$

$$7. \int_0^\infty \frac{x^{\alpha-1} dx}{(x^{2\mu} + 1)(x^{3\mu} + 1)} = \frac{\pi}{6\mu \sin(\alpha\pi/\mu)} \left[1 + 3\sqrt{2} \sin\left(\frac{\pi\alpha}{2\mu} + \frac{\pi}{4}\right) - 4 \sin\left(\frac{2\pi\alpha}{3\mu} - \frac{\pi}{6}\right) \right] \\ [0 < \operatorname{Re} \alpha < 5\mu].$$

$$8. \int_0^\infty \frac{x^{\alpha-1} dx}{(x+y)(x^{2/3} + y^{2/3})} = -\frac{3\pi}{8} y^{\alpha-5/3} \frac{1 - 4 \cos^2 \alpha\pi}{\sin \alpha\pi} \\ [|\arg y| < \pi; 0 < \operatorname{Re} \alpha < 5/3].$$

$$9. \int_0^\infty \left(\frac{x^p}{1+x^{2p}} \right)^q \frac{dx}{1-x^2} = 0 \quad [|p|q > -1].$$

$$10. \int_0^\infty \frac{x^m dx}{(x^2 + z^2)(x \pm y)^n} = \sum_{k=0}^m \binom{m}{k} y^{m-k} \frac{(\mp 1)^{m+n-1}}{(n-k-1)!} \frac{\partial^{n-k-1}}{\partial y^{n-k-1}} \left[\frac{1}{z^2 + y^2} \left(\pm \frac{y\pi}{2z} - \ln \frac{y}{z} \right) \right] \\ \left[\operatorname{Re} z > 0, \begin{cases} |\arg y| < \pi; n=0, 1, 2, \dots \\ y > 0; n=1, 3, 5, \dots \end{cases} \right].$$

$$11. \int_0^\infty x^{\alpha-1} \frac{x^\mu - y^\mu}{x-y} dx = y^{\alpha+\mu-1} \frac{\pi \sin \mu\pi}{\sin \alpha\pi \sin(\alpha+\mu)\pi} \\ [y > 0; 0 < \operatorname{Re} \alpha < 1; 0 < \operatorname{Re}(\alpha+\mu) < 1].$$

$$12. \int_0^\infty \left(\frac{x^\mu - x^{-\mu}}{x-1} \right)^2 dx = 2(1 - 2\pi\mu \operatorname{ctg} 2\mu\pi) \quad [|\operatorname{Re} \mu| < 1/2].$$

$$13. \int_0^\infty x^{\alpha-1} \left(\frac{x^\mu - x^{-\mu}}{x^\lambda - x^{-\lambda}} \right) dx = \frac{\pi \sin(\mu\pi/\lambda)}{\lambda [\cos(\mu\pi/\lambda) + \cos(\alpha\pi/\lambda)]} \\ [\mu - \lambda < \operatorname{Re} \alpha < \mu + \operatorname{Re} \alpha < \lambda].$$

$$14. \int_0^\infty \left(\frac{x^\mu - x^{-\mu}}{x^\lambda - x^{-\lambda}} \right)^n \frac{dx}{x} = I_n \quad [0 < \mu < \lambda],$$

$$I_1 = \frac{\pi}{\lambda} \operatorname{tg} \frac{\mu\pi}{2\lambda}, \quad I_2 = \frac{1}{\lambda} - \frac{\mu\pi}{\lambda^2} \operatorname{ctg} \frac{\mu\pi}{\lambda}, \\ I_3 = \frac{\pi}{8\lambda^3} \left[3(\lambda^2 - \mu^2) \operatorname{tg} \frac{\mu\pi}{2\lambda} - (\lambda^2 - 9\mu^2) \operatorname{tg} \frac{3\mu\pi}{2\lambda} \right].$$

$$15. \int_0^{\infty} \frac{x^{\mu}-1}{(x-1)^2} (x^{\lambda}-1) dx = \pi [\lambda \operatorname{ctg} \lambda \pi + \mu \operatorname{ctg} \mu \pi - (\mu + \lambda) \operatorname{ctg} (\mu + \lambda) \pi] - 1$$

$$[\lambda \neq -\mu; |\operatorname{Re}(\mu + \lambda)|, |\operatorname{Re} \lambda|, |\operatorname{Re} \mu| < 1].$$

$$16. = 2(\pi \mu \operatorname{ctg} \mu \pi - 1)$$

$$[\lambda = -\mu; |\operatorname{Re} \mu| < 1].$$

2.2.8. Интегралы от $\prod_{k=1}^n (a_k^{\mu_k} \pm x^{\mu_k})^{\rho_k}$, $n \geq 4$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1} (x+z)^m}{(cx+d)^{\alpha+\beta+n}} dx =$$

$$= \frac{(b-a)^{\alpha+\beta-1}}{(bc+d)^{\alpha} (ac+d)^{\beta}} \sum_{k=0}^n \frac{B(\alpha+k, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}} \sum_{l=0}^k \binom{m}{l} \binom{n-m}{k-l} (b+z)^l (a+z)^{m-l}$$

$$[(ac+d)(bc+d) > 0; m \leq n; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{-\alpha}}{(c_1 x + d_1)(c_2 x + d_2)} dx =$$

$$= \frac{\pi}{(c_1 d_2 - c_2 d_1) \sin \alpha \pi} \left[\frac{c_1 (ac_1 + d_1)^{\alpha-1}}{(bc_1 + d_1)^{\alpha}} - \frac{c_2 (ac_2 + d_2)^{\alpha-1}}{(bc_2 + d_2)^{\alpha}} \right]$$

$$[c_1 d_2 \neq c_2 d_1; (ac_j + d_j)(bc_j + d_j) > 0, j=1, 2; 0 < \operatorname{Re} \alpha < 1].$$

$$3. \int_a^b \frac{[(x-a)(b-x)]^{\alpha-1}}{(x-y)|x-c|^{2\alpha-r}} dx = \frac{\pi \operatorname{sgn}(y-c) \operatorname{ctg} \alpha \pi}{|y-c|^{2\alpha-r} [(y-a)(b-y)]^{1-\alpha}}$$

$$[r=0 \text{ или } 1; a < y < b; a < c < b; 0 < 2 \operatorname{Re} \alpha < 1+r].$$

$$4. \int_a^b \frac{[(x-a)(b-x)]^{\alpha-1} \operatorname{sgn}(x-c)}{(x-y)|x-c|^{2\alpha-r}} dx = \frac{(-1)^r \pi \operatorname{ctg} \alpha \pi}{|y-c|^{2\alpha-r} [(y-a)(b-y)]^{1-\alpha}}$$

$$[r=0 \text{ или } 1; a < y < b; a < c < b; 0 < \operatorname{Re} \alpha < (1+r)/2].$$

$$5. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (1-ux)^{-\rho} (1-vx)^{-\lambda} dx =$$

$$= a^{\alpha+\beta-1} B(\alpha, \beta) F_1(\alpha, \rho, \lambda, \alpha+\beta; ua, va)$$

$$[|\arg u| < \pi \text{ или } a|u| < 1; |\arg v| < \pi \text{ или } a|v| < 1; a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$6. \int_0^1 \frac{a_0 x^{\alpha_0-1} - a_1 x^{\alpha_1-1} - \dots - a_n x^{\alpha_n-1}}{1-x^{\lambda}} dx =$$

$$= \lambda^{-1} [a_1 \psi(\alpha_1/\lambda) + \dots + a_n \psi(\alpha_n/\lambda) - a_0 \psi(\alpha_0/\lambda)]$$

$$[a_0 = a_1 + a_2 + \dots + a_n; \lambda, \operatorname{Re} \alpha_i > 0; i=0, 1, 2, \dots, n].$$

$$7. \int_0^1 \frac{x^{\alpha-1} + x^{\alpha-(n-1)/2} + \dots + x^{\alpha-1/n} - nx^{\alpha n-1}}{1-x} dx = n \ln n$$

$$[\operatorname{Re} \alpha > 0; n=1, 2, 3, \dots].$$

$$8. \int_0^1 \frac{x^m}{1+x+x^2+\dots+x^{2m+2}} dx = \frac{\pi}{2m+3} \operatorname{tg} \frac{\pi}{2(2m+3)}.$$

$$9. \int_0^1 \frac{x^{2m} + x^{2m-2} + \dots + x^2 + 1}{x^{2m+2} + x^{2m} + \dots + x^2 + 1} dx = \frac{\pi}{2(m+2)} \operatorname{ctg} \frac{\pi}{2(m+2)}.$$

$$10. \int_0^1 \frac{x^{n-1} dx}{(x+1)(2x+1)\dots(nx+1)} = n! \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \ln(k+1)$$

[n = 1, 2, 3, ...].

$$11. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} (x-u)^{-\rho} (x-v)^{-\lambda} dx = a^{\alpha+\beta-\lambda-\rho-1} \times \\ \times B(1+\lambda+\rho-\alpha-\beta, \beta) F_1(1+\lambda+\rho-\alpha-\beta, \rho, \lambda, 1+\lambda+\rho-\alpha; u/a, v/a) \\ [a > 0; |\arg u| < \pi \text{ или } |u| < a; |\arg v| < \pi \text{ или } |v| < a, 0 < \operatorname{Re} \beta < \operatorname{Re}(\lambda+\rho-\alpha)+1].$$

$$12. \int_0^\infty \frac{(Ax+B) dx}{(c_1x+d_1)(c_2x+d_2)(c_3x+d_3)} = \frac{Bc_1-Ad_1}{pq} \ln \frac{c_1d_2}{c_2d_1} + \frac{Bc_3-Ad_3}{qr} \ln \frac{c_3d_2}{c_2d_3} \\ [p=c_1d_2-c_2d_1, q=c_1d_3-c_3d_1, r=c_2d_3-c_3d_2; c_1/d_1 > c_2/d_2 > c_3/d_3 > 0].$$

$$13. \int_0^\infty \frac{x^\mu - y^\mu}{x-y} \frac{x^\nu - 1}{x-1} dx = \frac{\pi}{1-y} [(y^{\mu+\nu} - 1) \operatorname{ctg}(\mu+\nu)\pi - \\ - y^\mu (y^\nu - 1) \operatorname{ctg} \nu\pi - (y^\mu - 1) \operatorname{ctg} \mu\pi + y^\mu \frac{\ln y}{\pi}] \\ [y > 0; y \neq 1; \nu \neq -\mu; |\operatorname{Re} \mu|, |\operatorname{Re} \nu|, |\operatorname{Re}(\mu+\nu)| < 1].$$

$$14. \int_0^\infty \frac{x^\mu - y^\mu}{x-y} \frac{x^{-\mu} - 1}{x-1} dx = \frac{\pi}{1-y} \left[\frac{y^\mu + 1}{\pi} \ln y - 2(y^\mu - 1) \operatorname{ctg} \mu\pi \right] \\ [y > 0; y \neq 1; |\operatorname{Re} \mu| < 1].$$

$$15. \int_0^\infty \frac{A_0x^n + A_1x^{n-1} + \dots + A_n}{(cx+d)^{n+2}} dx = \\ = \frac{1}{(n+1)cd} \left[\frac{A_0}{c^n} \binom{n}{0}^{-1} + \frac{A_1}{c^{n-1}d} \binom{n}{1}^{-1} + \dots + \frac{A_n}{d^n} \binom{n}{n}^{-1} \right] \quad [cd > 0].$$

$$16. \int_0^\infty \frac{A_0x^{2n} + A_1x^{2n-1} + \dots + A_{2n}}{(x^2+z^2)^{n+1}} dx = \\ = \frac{\pi}{n! 2^{n+1}z} \sum_{k=0}^n \frac{(2k-1)!! (2n-2k-1)!!}{z^{2k}} A_{2k} + \frac{1}{n! 2z^2} \sum_{k=0}^{n-1} k! (n-k-1)! \frac{A_{2k+1}}{z^{2k}} \\ [\operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} \frac{(A_1 x^2 + B_1)(A_2 x^2 + B_2) \dots (A_{n-1} x^2 + B_{n-1})}{(c_1 x^2 + d_1)(c_2 x^2 + d_2) \dots (c_n x^2 + d_n)} dx =$$

$$= \frac{\pi}{2} \left\{ \frac{(A_1 d_1 - B_1 c_1)(A_2 d_1 - B_2 c_1) \dots (A_{n-1} d_1 - B_{n-1} c_1)}{\sqrt{c_1 d_1} (c_2 d_1 - d_2 c_1)(c_3 d_1 - d_3 c_1) \dots (c_n d_1 - d_n c_1)} + \right.$$

$$+ \frac{(A_1 d_2 - B_1 c_2)(A_2 d_2 - B_2 c_2) \dots (A_{n-1} d_2 - B_{n-1} c_2)}{\sqrt{c_2 d_2} (c_1 d_2 - d_1 c_2)(c_3 d_2 - d_3 c_2) \dots (c_n d_2 - d_n c_2)} + \dots$$

$$\left. \dots + \frac{(A_1 d_n - B_1 c_n)(A_2 d_n - B_2 c_n) \dots (A_{n-1} d_n - B_{n-1} c_n)}{\sqrt{c_n d_n} (c_1 d_n - d_1 c_n)(c_2 d_n - d_2 c_n) \dots (c_{n-1} d_n - d_{n-1} c_n)} \right\}$$

$$[c_i, d_i > 0; c_i d_j \neq c_j d_i; i, j = 1, 2, 3, \dots, n; n = 2, 3, 4, \dots].$$

$$18. \int_{-\infty}^{\infty} \frac{P_n(x)}{Q_n(x) Q_n(-x)} dx = I_n$$

$[P_n(x) = A_0 x^{2n-2} + A_1 x^{2n-4} + \dots + A_{n-1}, Q_n(x) = B_0 x^n + B_1 x^{n-1} + \dots + B_n; \text{ все корни } Q_n(x) \text{ лежат в верхней полуплоскости}],$

$$I_n = \frac{\pi i}{B_0} \frac{M_n}{\Delta_n}, \quad M_n = \begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} \\ B_0 & B_2 & B_4 & \dots & 0 \\ 0 & B_1 & B_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & B_n \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} B_1 & B_3 & B_5 & \dots & 0 \\ B_0 & B_2 & B_4 & \dots & 0 \\ 0 & B_1 & B_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & B_n \end{vmatrix},$$

$$M_1 = A_0, \quad \Delta_1 = B_1; \quad M_2 = -A_0 + A_1 B_0 B_2^{-1}, \quad \Delta_2 = B_1;$$

$$M_3 = -A_0 B_2 + A_1 B_0 - B_0 B_1 A_2 B_3^{-1}, \quad \Delta_3 = B_0 B_3 - B_1 B_2;$$

$$M_4 = A_0 (-B_1 B_4 + B_2 B_3) - B_0 B_3 A_1 + B_0 B_1 A_2 + B_0 B_4^{-1} A_3 (B_0 B_3 - B_1 B_2),$$

$$\Delta_4 = B_0 B_3^2 + B_1^2 B_4 - B_1 B_2 B_3.$$

$$19. \int_0^{\infty} \frac{x^m}{1+x+x^2+\dots+x^n} dx = \frac{\pi}{n+1} \left[\operatorname{ctg} \frac{(m+1)\pi}{n+1} - \operatorname{ctg} \frac{(m+2)\pi}{n+1} \right]$$

$[n > m+1].$

2.2.9. Интегралы от $x^\alpha (ax^2 + bx + c)^\rho A(x)$.

Условия: a, b, c — действительные числа.

$$1. \int_0^1 \frac{x^{\alpha-1}}{x^2 + 2x \cos \gamma + 1} dx = \sum_{k=0}^{\infty} (-1)^k \frac{\sin \gamma k \operatorname{ctg} \gamma + \cos \gamma k}{\alpha + k} \quad [\operatorname{Re} \alpha > 0; |\gamma| < \pi].$$

$$2. \int_0^1 \frac{x^{\alpha-1} (1-x)^{\alpha-4+2\varepsilon}}{(ax^2 + 2bx + c)^{\alpha-3/2+\varepsilon}} dx =$$

$$= \frac{[b+c+(\alpha-2+\varepsilon)\sqrt{c}\sqrt{a+2b+c}] 2^{2-\alpha-\varepsilon} \sqrt{\pi} \Gamma(\alpha-3+\varepsilon)}{(b+c+\sqrt{c}\sqrt{a+2b+c})^{\alpha-2+\varepsilon} \Gamma(\alpha-3/2+\varepsilon)} \left\{ \frac{(a+2b+c)^{-3/2}}{c^{-3/2}} \right\}$$

$$[c, a+2b+c > 0; a < (\sqrt{c} + \sqrt{a+2b+c})^2; \operatorname{Re} \alpha > 3-\varepsilon, \varepsilon = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix}].$$

$$3. \int_0^1 \frac{x^{\alpha-1} (1-x)^{\beta}}{(ax^2+2bx+c)^{\alpha+1/2-\sigma}} dx =$$

$$= \frac{2^{-\alpha+\sigma} \sqrt{\pi} \Gamma(\alpha-\sigma)}{(b+c+\sqrt{c} \sqrt{a+2b+c})^{\alpha-\sigma} \Gamma(\alpha+1/2-\sigma)} \left\{ \frac{c^{-1/2}}{(a+2b+c)^{-1/2}} \right\}$$

$$\left[c, a+2b+c > 0; \quad a < (\sqrt{c} + \sqrt{a+2b+c})^2; \quad \operatorname{Re} \alpha > \sigma; \quad \beta = \left\{ \begin{matrix} \alpha \\ \alpha-2 \end{matrix} \right\}, \quad \sigma = \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \right].$$

$$4. \int_0^1 x^{\alpha-1} (1-x)^{-\alpha} \left(x^2 + 2xy \left\{ \begin{matrix} \cos \gamma \\ \operatorname{ch} \gamma \end{matrix} \right\} + y^2 \right)^{-1} dx =$$

$$= \frac{\pi y^{\alpha-2}}{\sin \alpha \pi} \left\{ \begin{matrix} \sin \gamma \\ \operatorname{sh} \gamma \end{matrix} \right\}^{-1} \left(y^2 + 2y \left\{ \begin{matrix} \cos \gamma \\ \operatorname{ch} \gamma \end{matrix} \right\} + 1 \right)^{-\alpha/2} \times$$

$$\times \left\{ \begin{matrix} \sin [\gamma - \alpha \operatorname{arctg} [\sin \gamma (y + \cos \gamma)^{-1}]] \\ \operatorname{sh} [\gamma - \alpha \operatorname{Arth} [\operatorname{sh} \gamma (y + \operatorname{ch} \gamma)^{-1}]] \end{matrix} \right\} \quad \left[\begin{matrix} |\arg y| < \pi; \quad 0 < \operatorname{Re} \alpha < 1; \quad |\gamma| < \pi \\ |\arg y| < \pi; \quad 0 < \operatorname{Re} \alpha < 1 \end{matrix} \right].$$

$$5. \int_0^1 \frac{x^{\alpha-1} (1+x \cos \gamma) dx}{x^2 + 2x \cos \gamma + 1} = \sum_{k=0}^{\infty} (-1)^k \frac{\cos k\gamma}{\alpha+k} \quad [|\gamma| < \pi; \quad \operatorname{Re} \alpha > 0].$$

$$6. \int_0^1 \frac{x^{\alpha} + x^{-\alpha}}{x^2 + 2x \cos \gamma + 1} dx = \frac{\pi \sin(\alpha\gamma)}{\sin \gamma \sin(\alpha\pi)} \quad [0 < |\gamma| < \pi; \quad |\operatorname{Re} \alpha| < 1].$$

$$7. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(ax^2+2bx+c)^{\rho}} = a^{-\alpha/2} c^{\alpha/2-\rho} B(\alpha, 2\rho-\alpha) {}_2F_1\left(\frac{\alpha}{2}, \rho-\frac{\alpha}{2}; \rho+\frac{1}{2}; 1-\frac{b^2}{ac}\right)$$

$$[a > 0; \quad b^2 < ac; \quad 0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho].$$

$$8. \int_0^{\infty} \frac{x^{\alpha-1}}{[(x+a)^2-b^2]^{\rho}} dx =$$

$$= B(\alpha, 2\rho-\alpha) (a^2-b^2)^{\alpha/2-\rho} {}_2F_1\left(\alpha, 2\rho-\alpha; \rho+\frac{1}{2}; \frac{1}{2} \left(1-\frac{a}{\sqrt{a^2-b^2}}\right)\right)$$

$$[b < a; \quad 0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho].$$

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(ax^2+2bx+c)^{\alpha+1/2-n}} = I_n \quad [a, c > 0; \quad \operatorname{Re} \alpha > 2n-1; \quad b > -\sqrt{ac}],$$

$$I_0 = \frac{\sqrt{\pi}}{\sqrt{c} (2\sqrt{ac}+2b)^{\alpha}} \Gamma\left[\frac{\alpha}{\alpha+1/2}\right], \quad I_1 = \frac{\sqrt{\pi}}{\sqrt{a} (2\sqrt{ac}+2b)^{\alpha-1}} \Gamma\left[\frac{\alpha-1}{\alpha-1/2}\right],$$

$$I_2 = \frac{[(\alpha-2)\sqrt{ac}+b]\sqrt{\pi}}{a^{3/2} (2\sqrt{ac}+2b)^{\alpha-2}} \Gamma\left[\frac{\alpha-3}{\alpha-3/2}\right],$$

$$I_3 = \frac{(\alpha-2)(\alpha-4)ac+3(\alpha-3)b\sqrt{ac}+3b^2}{a^{5/2} (2\sqrt{ac}+2b)^{\alpha-3}} \sqrt{\pi} \Gamma\left[\frac{\alpha-5}{\alpha-5/2}\right].$$

$$10. \int_0^{\infty} \frac{x^{m+2k+1/2}}{(ax^2+2bx+c)^n} dx =$$

$$= \frac{(-1)^{m+n-1} \pi (2m-1)!!}{\sqrt{a} 2^{2m+1/2} (n-1)!} \frac{\partial^{n-m-1}}{\partial a^k \partial c^{n-m-k-1}} [(b+\sqrt{ac})^{-m-1/2}]$$

$[a, c > 0; 0 \leq m+k \leq n-1; n=1, 2, 3, \dots \text{ при } b^2 < ac \text{ и } n=1, 3, 5, \dots \text{ при } b^2 > ac].$

$$11. \int_0^{\infty} \frac{x^{m+2k}}{(ax^2+2bx+c)^n} dx =$$

$$= \frac{(-1)^{n-1}}{(n-1)!} 2^{-m} \frac{\partial^{n-1}}{\partial a^k \partial b^m \partial c^{n-m-k-1}} \left(\frac{1}{\sqrt{ac-b^2}} \operatorname{arccctg} \frac{b}{\sqrt{ac-b^2}} \right) \quad [b^2 < ac].$$

$$12. = \frac{(-1)^{n-1}}{(n-1)!} 2^{-m-1} \frac{\partial^{n-1}}{\partial a^k \partial b^m \partial c^{n-m-k-1}} \left(\frac{1}{\sqrt{b^2-ac}} \ln \frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}} \right) \quad [ac < b^2].$$

$$13. = \left(\frac{b}{a} \right)^{m+2k-n+1} b^{-n} \frac{(m+2k)! (2n-2k-m-2)!}{(2n-1)!} \quad [ac = b^2],$$

$[a, b, c > 0, 0 \leq m+k \leq n-1; n=1, 2, 3, \dots].$

$$14. \int_0^{\infty} \frac{x dx}{(ax^2+2bx+c)^n} =$$

$$= \frac{(-1)^{n-1}}{2(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left(\frac{b}{(ac-b^2)^{3/2}} \operatorname{arccctg} \frac{b}{\sqrt{ac-b^2}} - \frac{1}{ac-b^2} \right) \quad [ac > b^2].$$

$$15. = \frac{(-1)^{n-1}}{2(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left(\frac{b}{(b^2-ac)^{3/2}} \ln \frac{\sqrt{ac}}{b+\sqrt{b^2-ac}} - \frac{1}{ac-b^2} \right) \quad [ac < b^2].$$

$$16. = \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad [ac = b^2],$$

$[a, b, c > 0; n=2, 3, 4, \dots].$

$$17. \int_0^{\infty} \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{\sqrt{ac-b^2}} \operatorname{arccctg} \frac{b}{\sqrt{ac-b^2}} \right)$$

$[ac > b^2].$

$$18. = \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{\sqrt{b^2-ac}} \ln \frac{\sqrt{ac}}{b+\sqrt{b^2-ac}} \right) \quad [ac < b^2].$$

$$19. = \frac{a^{n-1}}{(2n-1)b^{2n-1}} \quad [ac = b^2], \quad [a, b, c > 0; n=1, 2, 3, \dots].$$

$$20. \int_0^{\infty} \frac{x^{m+2k} dx}{(ax^2+2bx+c)^{n+1/2}} = \frac{(-1)^{n+m-1} 2^{n-k-1} m!}{(2n-1)!!} \frac{\partial^{n-m-1}}{\partial a^k \partial c^{n-m-k-1}} \left(\frac{1}{\sqrt{c} (\sqrt{ac}+b)^{m+1}} \right)$$

$[0 \leq m+k \leq n-1].$

$$21. = \frac{(-2)^{n-1}}{(2n-1)!!} \frac{\partial^{n-1}}{\partial a^{n-1}} \left(\frac{1}{\sqrt{a} (\sqrt{ac}+b)} \right)$$

$[m+2k=2n-1], [a, b, c > 0; n=1, 2, 3, \dots].$

$$22. \int_0^{\infty} \frac{dx}{(ax^2 + 2bx + c)^{n+1/2}} = \frac{(-2)^{n-1}}{(2n-1)!!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{\sqrt{c} \sqrt{ac} + b} \right)$$

[a, b, c > 0; n = 1, 2, 3, ...].

$$23. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2 + 2xy \cos \gamma + y^2)^{\rho}} dx =$$

$$= \frac{1}{2y^{2\rho-\alpha}} \left\{ B\left(\frac{\alpha}{2}, \rho - \frac{\alpha}{2}\right) {}_2F_1\left(\frac{\alpha}{2}, \rho - \frac{\alpha}{2}; \frac{1}{2}; \cos^2 \gamma\right) - \right.$$

$$\left. - (\alpha-1) \cos \gamma B\left(\frac{\alpha-1}{2}, \rho - \frac{\alpha-1}{2}\right) {}_2F_1\left(\frac{\alpha-1}{2}, \rho - \frac{\alpha-1}{2}; \frac{3}{2}; \cos^2 \gamma\right) \right\}$$

[|arg y| < π; 0 < |γ| < π; 0 < Re α < 2 Re ρ].

$$24. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2 + 2xy \cos \gamma + y^2} dx = \frac{\pi \sin[(1-\alpha)\gamma]}{y^{2-\alpha} \sin \gamma \sin \alpha \pi}$$

[|arg y| < π; 0 < |γ| < π; 0 < Re α < 2; α ≠ 1].

$$25. \int_0^{\infty} \frac{dx}{x^2 + 2xy \cos \gamma + y^2} = \frac{\gamma}{y \sin \gamma}$$

[0 < |γ| < π].

$$26. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2 + 2xy \cos \gamma + y^2)^2} dx = \frac{\pi \{ \sin[(1-\alpha)\gamma] - (1-\alpha) \sin \gamma \cos[(2-\alpha)\gamma] \}}{2y^{4-\alpha} \sin^3 \gamma \sin \alpha \pi}$$

[|arg y| < π; 0 < |γ| < π; 0 < Re α < 4; α ≠ 1, 2, 3].

$$27. \int_0^{\infty} \frac{dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{2\gamma - \sin 2\gamma}{4y^3 \sin^3 \gamma}$$

[0 < |γ| < π].

$$28. \int_0^{\infty} \frac{x dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{\sin \gamma - \gamma \cos \gamma}{2y^2 \sin^3 \gamma}$$

[0 < |γ| < π].

$$29. \int_0^{\infty} \frac{x^2 dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{2\gamma - \sin 2\gamma}{4y \sin^3 \gamma}$$

[0 < |γ| < π].

$$30. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(x^2 + 2xy \cos \gamma + y^2)^3} =$$

$$= \frac{\pi \{ 3 \sin[(1-\alpha)\gamma] - 3(1-\alpha) \sin \gamma \cos[(2-\alpha)\gamma] - (1-\alpha)(2-\alpha) \sin^2 \gamma \sin[(3-\alpha)\gamma] \}}{8y^{6-\alpha} \sin^5 \gamma \sin \alpha \pi}$$

[|arg y| < π, 0 < |γ| < π; 0 < Re α < 6; α ≠ 1, 2, 3, 4, 5].

$$31. \int_0^{\infty} \frac{x^n dx}{(x^2 + 2xy \cos \gamma + y^2)^3} = I_n$$

[0 < |γ| < π].

$$I_0 = \frac{12\gamma - 8 \sin 2\gamma + \sin 4\gamma}{32y^5 \sin^5 \gamma}, \quad I_1 = \frac{9 \sin \gamma + \sin 3\gamma - 12\gamma \cos \gamma}{32y^4 \sin^5 \gamma},$$

$$I_2 = \frac{2\gamma (2 + \cos 2\gamma) - 3 \sin 2\gamma}{16y^3 \sin^5 \gamma}, \quad I_3 = \frac{9 \sin \gamma + \sin 3\gamma - 12\gamma \cos \gamma}{32y^2 \sin^5 \gamma},$$

$$I_4 = \frac{12\gamma - 8 \sin 2\gamma + \sin 4\gamma}{32y \sin^5 \gamma}.$$

$$32. \int_0^\infty \frac{x^n dx}{(x^2 + 2xy \cos \gamma + y^2)^4} = J_n \quad [0 < |\gamma| < \pi],$$

$$J_0 = \frac{15\gamma - \sin \gamma \cos \gamma (15 + 10 \sin^2 \gamma + 8 \sin^4 \gamma)}{48y^7 \sin^7 \gamma},$$

$$J_1 = \frac{15 \sin \gamma - 15\gamma \cos \gamma - 5 \sin^3 \gamma - 2 \sin^5 \gamma}{48y^6 \sin^7 \gamma}.$$

$$33. \int_0^\infty \frac{x^{\alpha-1} |x-y|^{\beta-1}}{(x^2 + 2bx + y^2)^\rho} dx = y^{(\pm 1 - 1)/2} \frac{B\left(\rho - \alpha + \frac{1 \pm 1}{2}, \alpha - \frac{1 \pm 1}{2}\right)}{(2y + 2b)^{\alpha - (1 \pm 1)/2}}$$

$[\beta = 2\rho - 2\alpha + 1 \pm 1; \operatorname{Re} \alpha > (1 \pm 1)/2; \operatorname{Re}(\rho - \alpha) > (\mp 1 - 1)/2; y > 0; b > -y].$

$$34. = y^{(\pm 3 - 3)/2} \frac{\left[\left(2\rho - \alpha - \frac{1 \mp 3}{2} \right) y + (2\rho - 2\alpha + 1 \pm 3) b \right]}{(\rho - 1)(2y + 2b)^{\alpha - (1 \pm 3)/2}} \times$$

$$\times B\left(\rho - \alpha + \frac{1 \pm 3}{2}, \alpha - \frac{1 \pm 3}{2}\right)$$

$[\beta = 2\rho - 2\alpha + 1 \pm 3; \operatorname{Re} \alpha > (1 \pm 3)/2; \operatorname{Re}(\rho - \alpha) > (\mp 3 - 1)/2; y, b + y > 0].$

$$35. \int_0^\infty \frac{x^{\alpha-1}}{ax^2 + 2bx + c} dx = \frac{(y^{\alpha-1} - z^{\alpha-1})\pi}{a(z-y)\sin \alpha\pi} = \frac{\pi \operatorname{sh}[(1-\alpha)\varphi] c^{\alpha/2-1}}{a^{\alpha/2} \sin \alpha\pi \operatorname{sh} \varphi}$$

$[0 < \operatorname{Re} \alpha < 1; 0 < ac < b^2; ab > 0; ay = b - \sqrt{b^2 - ac}, az = b + \sqrt{b^2 - ac}, \varphi = \operatorname{Arch}(|b|/\sqrt{ac})].$

$$36. = \frac{\pi \sin[(1-\alpha)\psi] c^{\alpha/2-1}}{a^{\alpha/2} \sin \psi \sin \alpha\pi} \quad [0 < \operatorname{Re} \alpha < 1; ac > b^2; \psi = \arccos(b/\sqrt{ac})].$$

$$37. = -\frac{\pi(z^{\alpha-1} + y^{\alpha-1} \cos \alpha\pi)}{a(y+z)\sin \alpha\pi}$$

$[0 < \operatorname{Re} \alpha < 1; ac < 0; az = b + \sqrt{b^2 - ac}, ay = -b + \sqrt{b^2 - ac}].$

$$38. = \frac{\pi(y^{\alpha-1} - z^{\alpha-1})}{a(z-y)} \operatorname{ctg} \alpha\pi = \frac{\pi \operatorname{sh}[(1-\alpha)\varphi] c^{\alpha/2-1}}{a^{\alpha/2} \operatorname{sh} \varphi}$$

$[0 < \operatorname{Re} \alpha < 1; b^2 > ac > 0; ab < 0; ay = -b - \sqrt{b^2 - ac}, az = -b + \sqrt{b^2 - ac}, \varphi = \operatorname{Arch}(|b|/\sqrt{ac})].$

$$39. \int_0^\infty \frac{dx}{\sqrt{x(ax^2 + 2bx + c)}} = \frac{\pi}{\sqrt{2c} \sqrt{\sqrt{ac} + b}} \quad [b^2 < ac; a, b > 0].$$

$$40. \int_0^\infty \frac{Ax^3 + Bx^2 + Cx + D}{(ax^2 + 2bx + c)^{5/2}} dx = \frac{1}{3(\sqrt{ac} + b)^2} \times$$

$$\times \left[(2\sqrt{ac} + b) \left(\frac{A}{a^{3/2}} + \frac{D}{c^{3/2}} \right) + \frac{B}{\sqrt{a}} + \frac{C}{\sqrt{c}} \right] \quad [a, c > 0; b > -\sqrt{ac}].$$

$$41. \int_0^{\infty} \frac{dx}{(gx+h) \sqrt{ax^2+2bx+c}} = \frac{1}{r} \ln \frac{g(r\sqrt{c}+gc-hb)}{h(r\sqrt{a}+gb-ha)}$$

$$[g, h, a > 0; c \geq 0; b > -\sqrt{ac}; r = \sqrt{ah^2 - 2bgh + cg^2} > 0].$$

$$42. = \frac{1}{r} \arccos \left[1 - \frac{r^2}{gh(\sqrt{ac}+b)} \right]$$

$$[g, h, a > 0; c \geq 0; b > \sqrt{ac}; r = \sqrt{-ah^2 + 2bgh - cg^2} > 0].$$

$$43. = \frac{2}{h\sqrt{a}+g\sqrt{c}} \quad [ah^2 - 2bgh + cg^2 = 0; a, g, h > 0; c \geq 0; gb - ha \neq 0].$$

$$44. \int_{-\infty}^{\infty} \frac{x^m dx}{(ax^2+2bx+c)^n} =$$

$$= \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!! (ac-b^2)^{n-1/2}} \sum_{k=0}^{[m/2]} \binom{m}{2k} (2k-1)!! (2n-2k-3)!! (ac-b^2)^k b^{-2k}$$

$$[a > 0; ac > b^2; m=0, 1, 2, \dots, 2n-2; n=1, 2, 3, \dots].$$

$$45. \int_{-\infty}^{\infty} \frac{dx}{(ax^2+2bx+c)^n} = \frac{(2n-3)!! \pi a^{n-1}}{(2n-2)!! (ac-b^2)^{n-1/2}}$$

$$[a > 0; ac > b^2; n=1, 2, 3, \dots].$$

2.2.10. Интегралы от $x^a (ax^4+2bx^2+c)^p$.

Условия: a, b, c — действительные числа.

$$1. \int_0^1 \frac{x^2+1}{x^4 \pm x^2+1} dx = \frac{\pi}{2\sqrt{2 \pm 1}}. \quad 2. \int_1^{\infty} \frac{x^2+1}{x^4 \pm x^2+1} dx = \frac{\pi}{2\sqrt{2 \pm 1}}.$$

$$3. \int_0^{\infty} \frac{x^{a-1} dx}{(ax^4+2bx^2+c)^p} = \frac{1}{2} \int_0^{\infty} \frac{x^{a/2-1} dx}{(ax^2+2bx+c)^p} \quad [\text{см. 2.2.9.7}].$$

$$4. \int_0^{\infty} \frac{x^k dx}{ax^4+2bx^2+c} = I_k \quad [a, c > 0; b^2 > ac].$$

$$I_0 = \frac{\pi}{2\sqrt{2c}(b+\sqrt{ac})} \quad [c > 0; b > -\sqrt{ac}].$$

$$I_1 = \frac{1}{2\sqrt{b^2-ac}} \ln \frac{b+\sqrt{b^2-ac}}{\sqrt{ac}}, \quad I_2 = \frac{\pi}{2\sqrt{2a}(b+\sqrt{ac})}.$$

$$5. \int_0^{\infty} x^{2\rho-1} [(ax^2+b)^2+cx^2]^{-\rho} dx = \frac{\sqrt{\pi}}{2a^{(1+1)/2} b^{(1+1)/2} c^{\rho-1/2}} \Gamma \left[\begin{matrix} \rho-1/2 \\ \rho \end{matrix} \right]$$

$$[c > 0; \operatorname{Re} \rho > 1/2].$$

2.2.11. Интегралы, содержащие $A(\sqrt{ax^2+bx+c})$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(\sqrt{x^2+z^2}+z)^{\nu}} = (2z)^{\alpha-\nu} \Gamma \left[\begin{matrix} \alpha/2, \nu-\alpha \\ \nu+1-\alpha/2 \end{matrix} \right] \quad [\operatorname{Re} z > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu].$$

$$2. \int_0^{\infty} \frac{x^m dx}{(\sqrt{x^2 + z^2} + z)^n} = \left(\frac{z}{2}\right)^{m-n+1} \frac{n(n-m-2)!}{2(m+1)(m+3)\dots(2n-m-1)}$$

[Re z > 0; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$3. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(\sqrt{x^2 + z^2} \pm x)^v} = 2^{-\alpha} z^{\alpha-v} \Gamma \left[\alpha, (\pm v - \alpha)/2 \right] \Gamma \left[(\pm v + \alpha)/2 + 1 \right]$$

[Re z > 0; 0 < Re α < ± Re v].

$$4. \int_0^{\infty} \frac{x^m dx}{(\sqrt{x^2 + z^2} + x)^n} = \frac{2^m z^{m-n+1} n \cdot m!}{(n-m-1)(n-m+1)\dots(n+m+1)}$$

[Re z > 0; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$5. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left(x+z+\sqrt{\begin{Bmatrix} x^2 \\ z^2 \end{Bmatrix}+2xz}\right)^v} = 2^{1+\alpha} z^{\alpha-v} \left\{ \frac{\Gamma(2\alpha) \Gamma(v-\alpha) \Gamma^{-1}(1+\alpha+v)}{2^{-v} \Gamma(2v-2\alpha) \Gamma(\alpha) \Gamma^{-1}(1+2v-\alpha)} \right\}$$

[|arg z| < π; 0 < Re α < Re v].

$$6. \int_0^{\infty} \frac{x^m dx}{\left(x+z+\sqrt{\begin{Bmatrix} x^2 \\ z^2 \end{Bmatrix}+2xz}\right)^n} =$$

$$= \left\{ \begin{array}{l} n z^{m-n+1} 2^{-m} (2m+1)! (n-m-2)! [(m+n+1)!]^{-1} \\ 2n (2\alpha)^{m-n+1} m! (2n-2m-3)! [(2n-m-1)!]^{-1} \end{array} \right\}$$

[|arg z| < π; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$7. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left[x+z+\sqrt{(x+z)^2 - y^2 \begin{Bmatrix} x^2 \\ 1 \end{Bmatrix}}\right]^v} =$$

$$= 2^{-v} z^{\alpha-v} B(\alpha, v-\alpha) {}_2F_1\left(\frac{1}{2} \pm \frac{\alpha}{2} + \frac{v}{2} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \pm \frac{\alpha}{2} + \frac{v}{2} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}; 1+v; y^2 \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix}\right)$$

[|y| ≤ 1; |arg z| < π; 0 < Re α < Re v].

$$8. \int_a^{\infty} \frac{dx}{\sqrt{x^2 - b^2} (x + \sqrt{x^2 - b^2})^v} = \frac{1}{v} (a + \sqrt{a^2 - b^2})^{-v}$$

[|b| ≤ a; Re v > 0].

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left(\sqrt{x^2 + z^2} + \sqrt{\begin{Bmatrix} x^2 + y^2 \\ x^2 y^2 + z^2 \end{Bmatrix}}\right)^v} =$$

$$= 2^{-v-1} z^{\alpha-v} B\left(\frac{\alpha}{2}; \frac{v-\alpha}{2}\right) {}_2F_1\left(\frac{1}{2} \begin{Bmatrix} v-\alpha \\ \alpha \end{Bmatrix}, \frac{v+1}{2}; v+1; 1-y^2 \begin{Bmatrix} z^2 \\ 1 \end{Bmatrix}\right)$$

[Re y, Re z > 0; 0 < Re α < Re v].

$$10. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x^2 + z^2} (\sqrt{x^2 + z^2} + y)^v} =$$

$$= (2z)^{\alpha-v-1} B\left(\frac{\alpha}{2}, 1-\alpha+v\right) {}_2F_1\left(1-\alpha+v, v; 1+v-\frac{\alpha}{2}; \frac{z-y}{2z}\right)$$

[Re z > 0; |arg(y/z + 1)| < π; 0 < Re α < Re v + 1].

$$11. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x^2+z^2} (\sqrt{x^2+z^2}+z)^{\nu}} = (2z)^{\alpha-\nu-1} B\left(\frac{\alpha}{2}, 1+\nu-\alpha\right)$$

[Re $z > 0$; $0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1$].

$$12. \int_0^{\infty} \frac{x^m dx}{\sqrt{x^2+z^2} (\sqrt{x^2+z^2}+z)^n} = \frac{z^{m-n} (n-m-1)!}{(2n-m-1)(2n-m+1)\dots(m+1)}$$

[Re $z > 0$; $m=0, 1, 2, \dots, n-1$; $n=1, 2, 3, \dots$].

$$13. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} (\sqrt{x^2+z^2}+xy)^{-\nu} dx =$$

$$= 2^{-\alpha} z^{\alpha-\nu-1} B\left(\alpha, \frac{1-\alpha+\nu}{2}\right) {}_2F_1\left(\nu, \alpha; \frac{\alpha+\nu+1}{2}; \frac{1-y}{2}\right)$$

[Re $z > 0$; $|\arg(y+1)| < \pi$; $0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1$].

$$14. \int_0^{\infty} \frac{x^{\alpha-1} (\sqrt{x^2+z^2} \pm x)^{-\nu}}{\sqrt{x^2+z^2}} dx = z^{\alpha-\nu-1} 2^{-\alpha} B\left(\alpha, \frac{1-\alpha \pm \nu}{2}\right)$$

[Re $z > 0$; $0 < \operatorname{Re} \alpha < 1 \pm \operatorname{Re} \nu$]

$$15. \int_0^{\infty} \frac{x^m (\sqrt{x^2+z^2}+x)^{-n}}{\sqrt{x^2+z^2}} dx = \frac{z^{m-n} m!}{(n-m)(n-m+2)\dots(n+m)}$$

[Re $z > 0$; $m=0, 1, 2, \dots, n-1$; $n=1, 2, 3, \dots$].

$$16. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{2x+z} (x+z+\sqrt{z^2+2zx})^{\nu}} = 2^{\alpha-\nu} z^{\alpha-\nu-1/2} B(\alpha, 2\nu-2\alpha+1)$$

[$|\arg z| < \pi$; $0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2$].

$$17. \int_0^{\infty} \frac{x^m dx}{\sqrt{2x+z} (x+z+\sqrt{z^2+2zx})^n} = \frac{\sqrt{2} m! (2n-2m-2)!}{(2z)^{n-m-1/2} (2n-m-1)!}$$

[$|\arg z| < \pi$; $m=0, 1, 2, \dots, n-1$; $n=1, 2, 3, \dots$].

$$18. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x+z}} [x+z+y+2\sqrt{y(x+z)}]^{-\nu} dx =$$

$$= 2^{2\alpha-2\nu} z^{\alpha-\nu-1} \sqrt{y} B(2\nu-2\alpha+1, \alpha) {}_2F_1\left(\nu+1-\alpha, \nu+\frac{1}{2}; 2\nu+1-\alpha; 1-\frac{y}{z}\right)$$

[$|\arg z|, |\arg y|, |\arg(y/z)| < \pi$; $0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2$].

$$19. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x+2z} (x+z+\sqrt{x^2+2xz})^{\nu}} = 2^{1/2-\alpha} z^{\alpha-\nu-1/2} B\left(2\alpha, \nu-\alpha+\frac{1}{2}\right)$$

[$|\arg z| < \pi$; $0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2$].

$$20. \int_0^{\infty} \frac{x^m dx}{\sqrt{x+2z} (x+z+\sqrt{x^2+2xz})^n} = \frac{(2z)^{m+1/2} z^{-n} (2m+1)!}{(2n-2m-1)(2n-2m+1)\dots(2n+2m+1)}$$

[$|\arg z| < \pi$; $m=0, 1, 2, \dots, n-1$; $n=1, 2, 3, \dots$].

$$\begin{aligned}
 21. \quad & \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x+z}} [x+xy+z+2\sqrt{xy(x+z)}]^{-\nu} dx = \\
 & = 2^{1-2\alpha} z^{\alpha-\nu-1/2} \sqrt{y} B\left(2\alpha, \frac{1}{2} + \nu - \alpha\right) {}_2F_1\left(\alpha + \frac{1}{2}, \nu + \frac{1}{2}; \alpha + \nu + \frac{1}{2}; 1-y\right) \\
 & \quad [|\arg z|, |\arg y| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2].
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{(x+z)^2 - y^2} [x+z+\sqrt{(x+z)^2 - y^2}]^{\nu}} = \\
 & = 2^{-\nu} z^{\alpha-\nu-1} B(\alpha, 1+\nu-\alpha) {}_2F_1\left(\frac{\nu-\alpha+1}{2}, 1+\frac{\nu-\alpha}{2}; 1+\nu; \frac{y^2}{z^2}\right) \\
 & \quad [|y| < |z|; |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1].
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{(x+z)^2 - x^2 y^2} [x+z+\sqrt{(x+z)^2 - x^2 y^2}]^{\nu}} = \\
 & = 2^{-\nu} z^{\alpha-\nu-1} B(\alpha, \nu-\alpha+1) {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; 1+\nu; y^2\right) \\
 & \quad [|y| < 1; |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1].
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} [(z+\sqrt{a-x})^{\nu} + (z-\sqrt{a-x})^{\nu}] dx = \\
 & = 2\sqrt{\pi} \Gamma\left[\frac{\alpha}{2}\right] z^{\alpha-1/2} {}_2F_1\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2} + \alpha; \frac{a}{z^2}\right) \\
 & \quad [a, \operatorname{Re} \alpha > 0; z^2 \notin [0, a]].
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \int_0^a \frac{x^{\alpha-1}}{\sqrt{a^2 - x^2}} [(a+\sqrt{a^2 - x^2})^{\nu} + (a-\sqrt{a^2 - x^2})^{\nu}] dx = (2a)^{\alpha+\nu-1} B\left(\frac{\alpha}{2}, \nu + \frac{\alpha}{2}\right) \\
 & \quad [a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+2\nu) > 0].
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2+y^2)} (\sqrt{x^2+z^2} + \sqrt{x^2+y^2})^{\nu}} dx = \\
 & = 2^{-\nu-1} z^{\alpha-\nu-2} B\left(\frac{\alpha}{2}, 1+\frac{\nu-\alpha}{2}\right) {}_2F_1\left(1+\frac{\nu-\alpha}{2}, \frac{\nu+1}{2}; \nu+1; 1-\frac{y^2}{z^2}\right) \\
 & \quad [\operatorname{Re} z, \operatorname{Re} y > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 2].
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2 y^2 + z^2)} [\sqrt{x^2+z^2} + \sqrt{x^2 y^2 + z^2}]^{\nu}} dx = \\
 & = 2^{-\nu-1} z^{\alpha-\nu-2} B\left(\frac{\nu-\alpha}{2} + 1, \frac{\alpha}{2}\right) {}_2F_1\left(\frac{\alpha}{2}, \frac{\nu+1}{2}; \nu+1; 1-y^2\right) \\
 & \quad [\operatorname{Re} y, \operatorname{Re} z > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 2].
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & \int_0^{\infty} \frac{1}{\sqrt{x^2+z^2}} \left[\frac{1}{\sqrt{(x-c)^2+y^2}} + \frac{1}{\sqrt{(x+c)^2+y^2}} \right] dx = \frac{2\sqrt{u}}{\sqrt{yz}} K(\sqrt{1-u^2}) \\
 & \quad [u = v - \sqrt{v^2-1}, 2vyz = c^2 + y^2 + z^2; c, \operatorname{Re} z, \operatorname{Re} y > 0].
 \end{aligned}$$

2.2.12. Интегралы, содержащие разности алгебраических функций.

$$1. \int_0^1 \left(\frac{1}{1-x} - \frac{\lambda x^{\alpha-1}}{1-x^\lambda} \right) dx = C + \ln \lambda + \psi \left(\frac{\alpha}{\lambda} \right)$$

$$[|\arg \lambda| < \pi; \operatorname{Re} \alpha, \operatorname{Re}(\alpha - \lambda) > 0].$$

$$2. \int_0^\infty \left[\frac{x^{\alpha-1}}{(x+z)^\alpha} - \frac{x^{\sigma-1}}{(x+z)^\sigma} \right] dx = \psi(\sigma) - \psi(\alpha)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$3. \int_0^\infty \left[\frac{x^{\alpha-1}}{(x+z)^\alpha} - \frac{x^{-\alpha}}{(x+z)^{1-\alpha}} \right] dx = \pi \operatorname{ctg} \alpha \pi$$

$$[|\arg z| < \pi; 0 < \operatorname{Re} \alpha < 1].$$

$$4. \int_0^\infty \left[\frac{z^\alpha}{(x+z)^\alpha} - \frac{z^\sigma}{(x+z)^\sigma} \right] \frac{dx}{x} = \psi(\sigma) - \psi(\alpha)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$5. \int_0^\infty x^{\alpha-1} [(x+z)^{\rho-1} - x^{\rho-1}] dx = z^{\alpha+\rho-1} B(\alpha, 1-\alpha-\rho)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha > 0; 1 < \operatorname{Re}(\alpha + \rho) < 2].$$

$$6. \int_0^\infty \frac{(x+z)^\sigma - z^\sigma}{(x+z)^\rho x} dx = z^{\sigma-\rho} [\psi(\rho) - \psi(\rho-\sigma)]$$

$$[|\arg z| < \pi; \operatorname{Re} \rho > 0; \operatorname{Re} \rho > \operatorname{Re} \sigma].$$

$$7. \int_0^1 \left(\frac{x^{\alpha-1}}{1-x} - \frac{\lambda x^{\alpha\lambda-1}}{1-x^\lambda} \right) dx = \ln \lambda$$

$$[|\arg \lambda| < \pi; \operatorname{Re} \alpha, \operatorname{Re}(\lambda\alpha - \lambda) > 0].$$

$$8. \int_0^1 \left(\frac{x^{\alpha-1}}{1 \pm xy} \pm \frac{x^{-\alpha}}{y \pm x} \right) dx = \frac{\pi y^{-\alpha}}{\sin \alpha \pi} \begin{Bmatrix} 1 \\ \cos \alpha \pi \end{Bmatrix}$$

$$\left[0 < \operatorname{Re} \alpha < 1, \begin{Bmatrix} y > 0 \\ y < 0 \end{Bmatrix} \text{ или } |\arg y| < \pi \right].$$

$$9. \int_0^a [(x+a)^{\gamma-1} (a-x)^{\beta-1} + (x+a)^{\beta-1} (a-x)^{\gamma-1}] dx = (2a)^{\beta+\gamma-1} B(\beta, \gamma)$$

$$[a, \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$10. \int_0^1 [a^\alpha x^{\alpha-1} (a+b-ax)^{\beta-1} + b^\beta x^{\beta-1} (a+b-bx)^{\alpha-1}] dx = (a+b)^{\alpha+\beta-1} B(\alpha, \beta)$$

$$[a, b, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$11. \int_0^1 \left(\frac{1}{\sqrt{1-x^{2\mu}}} - \frac{1}{1+x^\mu} \right) \frac{dx}{x} = \frac{2}{\mu} \ln 2$$

$$[\mu > 0].$$

$$12. \int_0^\infty \frac{(a+bx)^\rho - (a+cx)^\rho}{x} dx = a^\rho \ln \frac{c}{b}$$

$$[a, b, c > 0; \operatorname{Re} \rho < 0]$$

$$13. \int_0^{\infty} \frac{a(x+a)^{\rho-1} x^{\alpha-1} + b(x+b)^{\alpha-1} x^{\rho-1}}{[(a+b)x+ab]^{\alpha+\rho}} dx = \frac{B(\alpha, \rho)}{a^{\alpha} b^{\rho}} \quad [a, b, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0].$$

$$14. \int_0^{\infty} \left[\left(1 + \frac{a}{px}\right)^{px} - \left(1 + \frac{a}{qx}\right)^{qx} \right] \frac{dx}{x} = (e^a - 1) \ln \frac{p}{q} \quad [a, p, q > 0].$$

$$15. \int_0^{\infty} \left(\frac{1}{1+x^{\alpha}} - \frac{1}{1+x^{\beta}} \right) \frac{dx}{x} = 0 \quad [\alpha, \beta - \text{произвольные}].$$

$$16. \int_0^{\infty} \left[\left(1 + \frac{x^{\alpha}}{y}\right)^{\mu} - \left(1 + \frac{x^{\beta}}{z}\right)^{\nu} \right] \frac{dx}{x} = \\ = \frac{\ln y}{\alpha} - \frac{\ln z}{\beta} - \zeta \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) - \frac{\psi(-\mu)}{\alpha} + \frac{\psi(-\nu)}{\beta} \\ [\alpha, \beta, \operatorname{Re} y, \operatorname{Re} z > 0; \operatorname{Re} \mu, \operatorname{Re} \nu < 0].$$

2.3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

2.3.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входит показательная функция. Одним из способов вычисления таких интегралов является сведение их с помощью замены переменной к интегралам, содержащим степенную и алгебраические функции. Кроме того, некоторые интегралы, содержащие показательную функцию, могут быть получены из других интегралов с помощью предельного перехода по параметру; например, если

$$\int_0^{\infty} (a+x)^{\beta} \varphi(x) dx = f(a, \beta),$$

то при соответствующих условиях

$$\int_0^{\infty} \varphi(x) e^{-px} dx = \lim_{a \rightarrow +\infty} \int_0^{\infty} \left(1 + \frac{x}{a}\right)^{-ap} \varphi(x) dx = \lim_{a \rightarrow +\infty} [a^{ap} f(a, -ap)].$$

2.3.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие степенную, алгебраические и показательную функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j = 1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. Кроме того, $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$1. \int_0^y x^{\alpha-1} (y^r - x^r)^{\beta-1} e^{-sx} dx = V_{26}$$

$$\{E_{26} = r^{-1} B(\beta, (\alpha+h)/r), u = \alpha + r(\beta-1) + h, \\ a = (\alpha+h)/r, b = \beta + (\alpha+h)/r, z = (-sy/p)^p\} \quad [y, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_y^{\infty} x^{\alpha-1} (x^r - y^r)^{\beta-1} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = r^{-1} B(\beta, 1 - \beta - (\alpha + h)/r), E_{27} = (-1)^j (1 - \beta)_j \Gamma[\alpha - r(1 - \beta + j)],$$

$$u = \alpha + r(\beta - 1) + h, u_1 = rj, v_1 = r(1 - \beta + j) - \alpha, a = (\alpha + h)/r,$$

$$b = \beta + (\alpha + h)/r, a_1 = 1 - \beta + j, b_1 = 1 - \alpha + r(1 - \beta + j), z = (-sy/p)^p\}$$

$$[y, \operatorname{Re} \beta > 0; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + r\beta) < r + 1)].$$

$$3. \int_0^{\infty} x^{\alpha-1} (y^r + x^r)^{\rho-1} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = \frac{1}{r} B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right), E_{27} = (1 - \rho)_j \Gamma[\alpha - r(1 - \rho + j)],$$

$$u = \alpha + r(\rho - 1) + h, u_1 = rj, v_1 = r(1 - \rho + j) - \alpha, a = (\alpha + h)/r,$$

$$b = \rho + (\alpha + h)/r, a_1 = 1 - \rho + j, b_1 = 1 - \alpha + r(1 - \rho + j),$$

$$z = (-1)^{p+q} (sy/p)^p\}$$

$$[\operatorname{Re} \alpha > 0; r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + r\rho) < r + 1)].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{y^r - x^r} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = \frac{\pi}{r} \operatorname{ctg}\left(\frac{\alpha + h}{r} \pi\right), E_{27} = (-1)^{j+1} j! \Gamma[\alpha - r(1 + j)],$$

$$u = \alpha - r + h, u_1 = rj, v_1 = r(1 + j) - \alpha, a = b = (\alpha + h)/r,$$

$$a_1 = 1 + j, b_1 = 1 - \alpha + r(1 + j), z = (-sy/p)^p\}$$

$$[y, \operatorname{Re} \alpha > 0; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} \alpha < r + 1)].$$

$$5. \int_0^{\infty} x^{\alpha-1} (y^{r/2} + \sqrt{y^r + x^r})^{\sigma} e^{-sx} dx = V_{28} + V_{29} + V_{30}$$

$$\{E_{28} = -\frac{\sigma}{2\sqrt{\pi}} \Gamma\left[\frac{j - \sigma/2, 1/2 - j, \alpha - r(j - \sigma/2)}{1 - \sigma/2 - j}\right],$$

$$E_{29} = -\frac{\sigma}{2\sqrt{\pi}} \Gamma\left[\frac{(1 - \sigma)/2 + j, -1/2 - j, \alpha - r(j + (1 - \sigma)/2)}{(1 - \sigma)/2 - j}\right],$$

$$E_{30} = -\frac{\sigma}{\alpha + h} 2^{\sigma+2} (\alpha + h)/r B\left(1 + \frac{\alpha + h}{r}, -\sigma - \frac{2}{r}(\alpha + h)\right),$$

$$u_1 = jr, u_2 = (j + 1/2)r, u = \alpha + r\sigma/2 + h, v_1 = (j - \sigma/2)r - \alpha,$$

$$v_2 = [j + (1 - \sigma)/2]r - \alpha, a_1 = j - \sigma/2, b_1 = j + \sigma/2, c_1 = 1 - \alpha + (j - \sigma/2)r,$$

$$d_1 = j + 1/2, a_2 = (1 - \sigma)/2 + j, b_2 = (1 + \sigma)/2 + j,$$

$$c_2 = 1 - \alpha + [j + (1 - \sigma)/2]r, d_2 = 3/2 + j, a_3 = (\alpha + h)/r,$$

$$b_3 = \sigma + (\alpha + h)/r, c_3 = 1 + \sigma/2 + (\alpha + h)/r, d_3 = (1 + \sigma)/2 + (\alpha + h)/r,$$

$$z = (-1)^{p+q} (sy/p)^p\}$$

$$[\operatorname{Re} \alpha > 0; r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + \sigma r/2) < 1)].$$

$$6. \int_0^{\infty} x^{\alpha-1} (x^{r/2} + \sqrt{y^r + x^r})^{\sigma} e^{-sx} dx = V_{28} + V_{30}$$

$$\{E_{28} = -\sigma 2^{\sigma-2} \Gamma\left[\frac{2j - \sigma, \alpha - r(j - \sigma/2)}{1 - \sigma + j}\right],$$

$$E_{30} = -\frac{\sigma}{\alpha+h} 2^{-2} (\alpha+h)/r - 1 B \left(1 + \frac{2}{r} (\alpha+h), -\frac{\sigma}{2} - \frac{\alpha+h}{r} \right),$$

$$u_1 = rj, u = \alpha + r\sigma/2 + h, v_1 = r(j - \sigma/2) - \alpha, a_1 = i - \sigma/2, \\ b_1 = (1 - \sigma)/2 + j, c_1 = 1 - \alpha + r(j - \sigma/2), d_1 = 1 - \sigma + j, a_3 = (\alpha + h)/r, \\ b_3 = (\alpha + h)/r + 1/2, c_3 = 1 + \sigma/2 + (\alpha + h)/r, d_3 = 1 - \sigma/2 + (\alpha + h)/r,$$

$$z = (-1)^{p+q} (sy/p)^p$$

$$[\operatorname{Re} \alpha > 0; r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} (\alpha + \sigma r/2) < 1)].$$

$$7. \int_0^\infty x^{\alpha-1} \frac{(y^{r/2} + \sqrt{y^r + x^r})^\sigma}{\sqrt{y^r + x^r}} e^{-sx} dx = V_{28} + V_{29} + V_{30}$$

$$\{E_{28} = \frac{1}{\sqrt{\pi}} \Gamma \left[\begin{matrix} (1-\sigma)/2 + j, 1/2 - j, \alpha - r(j + (1-\sigma)/2) \\ (1-\sigma)/2 - j \end{matrix} \right],$$

$$E_{29} = \frac{1}{\sqrt{\pi}} \Gamma \left[\begin{matrix} 1 - \sigma/2 + j, -1/2 - i, \alpha - r(j + 1 - \sigma/2) \\ -\sigma/2 - j \end{matrix} \right],$$

$$E_{30} = \frac{2^{\sigma+2} (\alpha+h)/r}{r} B \left(\frac{\alpha+h}{r}, 1 - \sigma - \frac{2}{r} (\alpha+h) \right),$$

$$u_1 = rj, u_2 = r(j + 1/2), u = r(\sigma - 1)/2 + \alpha + h, v_1 = [(1 - \sigma)/2 + j]r - \alpha, \\ v_2 = (1 - \sigma/2 + j)r - \alpha, a_1 = (1 - \sigma)/2 + j, b_1 = (1 + \sigma)/2 + j, \\ c_1 = 1 - \alpha + r[(1 - \sigma)/2 + j], d_1 = 1/2 + j, a_2 = 1 - \sigma/2 + j, b_2 = 1 + \sigma/2 + j, \\ c_2 = 1 - \alpha + r(1 - \sigma/2 + j), d_2 = 3/2 + j, a_3 = (\alpha + h)/r, \\ b_3 = \sigma + (\alpha + h)/r, c_3 = (1 + \sigma)/2 + (\alpha + h)/r, d_3 = \sigma/2 + (\alpha + h)/r,$$

$$z = (-1)^{p+q} (sy/p)^p$$

$$[\operatorname{Re} \alpha > 0, r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} (\alpha + \sigma r/2) < r/2 + 1)].$$

$$8. \int_0^\infty x^{\alpha-1} \frac{(x^{r/2} + \sqrt{y^r + x^r})^\sigma}{\sqrt{y^r + x^r}} e^{-sx} dx = V_{28} + V_{30}$$

$$\{E_{28} = 2^{\sigma-2} \Gamma \left[\begin{matrix} 1 - \sigma + 2j, \alpha - r(1 - \sigma + 2j)/2 \\ 1 - \sigma + j \end{matrix} \right],$$

$$E_{30} = \frac{2^{1-2} (\alpha+h)/r}{r} B \left(\frac{2}{r} (\alpha+h), \frac{1-\sigma}{2} - \frac{\alpha+h}{r} \right),$$

$$u_1 = rj, u = r(\sigma - 1)/2 + \alpha + h, v_1 = r[(1 - \sigma)/2 + j] - \alpha, a_1 = (1 - \sigma)/2 + j, \\ b_1 = 1 - \sigma/2 + j, c_1 = 1 - \alpha + r[(1 - \sigma)/2 + j], d_1 = 1 - \sigma + j, a_3 = (\alpha + h)/r, \\ b_3 = 1/2 + (\alpha + h)/r, c_3 = (1 + \sigma)/2 + (\alpha + h)/r, d_3 = (1 - \sigma)/2 + (\alpha + h)/r,$$

$$z = (-1)^{p+q} (sy/p)^p$$

$$[\operatorname{Re} \alpha > 0; r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} (\alpha + \sigma r/2) < r/2 + 1)].$$

$$9. \int_0^\infty \frac{x^{\alpha-1} e^{-sx}}{x^{2r} + 2(xy)^r \cos \gamma + y^{2r}} dx = U,$$

$$U = -\frac{\pi y^{\alpha-2r}}{r \sin \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l \sin [\gamma ((\alpha+l)/r - 1)]}{l! \sin [(\alpha+l) \pi/r]} (sy)^l -$$

$$- \sum_{l=0}^{\infty} (-1)^l \Gamma(\alpha - r - lr) \frac{\sin l\gamma}{\sin \gamma} s^{r(l+1)-\alpha} y^{r(l-1)}$$

$$[\operatorname{Re} \alpha > 0; |\gamma|, 2r |\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} \alpha < 2r + 1)].$$

$$\begin{aligned}
 10. \int_0^{\infty} \frac{x^{\alpha-1} e^{-sx} dx}{(x^{2r} + 2(xy)^r \cos \gamma + y^{2r})^2} = \\
 = \frac{U}{2 \sin^2 \gamma} + \frac{\pi y^{\alpha-2r}}{2r^2 \sin^2 \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l (\alpha+l-r) \cos [\gamma ((\alpha+l)/r - 2)]}{l! \sin [(\alpha+l) \pi/r]} (sy)^l + \\
 + \sum_{l=0}^{\infty} (-1)^l \Gamma(\alpha-r-lr) \frac{l \cos(\gamma l - \gamma)}{2 \sin^2 \gamma} s^{r(l+1)-\alpha} y^{r(l-1)}
 \end{aligned}$$

[$\operatorname{Re} \alpha > 0$, $|\gamma|$, $2r|\arg y| < \pi$ и $\operatorname{Re} s > 0$ (или $\operatorname{Re} s = 0$, $\operatorname{Re} \alpha < 4r + 1$); U см. 2.3.2.9].

$$11. \int_0^y \frac{x^{\alpha-1}}{\sqrt{y^r - x^r}} [(y^{r/2} + \sqrt{y^r - x^r})^{\sigma} + (y^{r/2} - \sqrt{y^r - x^r})^{\sigma}] e^{-sx} dx = V_{30}$$

$$\{E_{30} = \frac{2^{\sigma+2} (\alpha+h)/r}{r} B\left(\frac{\alpha+h}{r}, \sigma + \frac{\alpha+h}{r}\right),$$

$$u = r(\sigma-1)/2 + \alpha + h, \quad a_3 = (\alpha+h)/r, \quad b_3 = \sigma + (\alpha+h)/r,$$

$$c_3 = \sigma/2 + (\alpha+h)/r, \quad d_3 = (\sigma+1)/2 + (\alpha+h)/r, \quad z = (-sy/p)^p \} \\
 [y, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \sigma r) > 0].$$

$$12. \int_y^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^r - y^r}} [(x^{r/2} + \sqrt{x^r - y^r})^{\sigma} + (x^{r/2} - \sqrt{x^r - y^r})^{\sigma}] e^{-sx} dx = V_{28} + V_{29} + V_{30}$$

$$\{E_{28} = (\sigma-2j)_i 2^{\sigma-2} \Gamma[\alpha-r(1-\sigma+2j)/2],$$

$$E_{29} = (-\sigma-2j)_i 2^{-\sigma-2} \Gamma[\alpha-r(1+\sigma+2j)/2],$$

$$E_{30} = \frac{2^{1-\sigma-2} (\alpha+h)/r}{r} B\left(\frac{1-\sigma}{2} - \frac{\alpha+h}{r}, \frac{1+\sigma}{2} - \frac{\alpha+h}{r}\right),$$

$$u_1 = rj, \quad v_2 = r(\sigma+j), \quad u = r(\sigma-1)/2 + \alpha + h, \quad v_1 = r[(1-\sigma)/2 + j] - \alpha,$$

$$v_2 = r[(1+\sigma)/2 + j] - \alpha, \quad a_1 = (1-\sigma)/2 + j, \quad b_1 = 1 - \sigma/2 + j,$$

$$c_1 = 1 - \alpha + r[(1-\sigma)/2 + j], \quad d_1 = 1 - \sigma + j, \quad a_2 = (1+\sigma)/2 + j, \quad b_2 = 1 + \sigma/2 + j,$$

$$c_2 = 1 - \alpha + r[(1+\sigma)/2 + j], \quad d_2 = 1 + \sigma + j, \quad a_3 = 1/2 + (\alpha+h)/r,$$

$$b_3 = (\alpha+h)/r, \quad c_3 = (1+\sigma)/2 + (\alpha+h)/r, \quad d_3 = (1-\sigma)/2 + (\alpha+h)/r, \quad z = (-sy/p)^p \} \\
 [y > 0; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha \pm \sigma r/2) < 1 + r/2)].$$

$$13. \int_0^{\infty} x^{\alpha-1} e^{-ax^r - sx} dx = U_1 \quad \left[z = \left(\frac{p}{s}\right)^p \left(\frac{a}{q}\right)^q; \operatorname{Re} \alpha, \operatorname{Re} a, \operatorname{Re} s > 0 \right],$$

$$U_1 = \sum_{j=0}^{q-1} \frac{(-a)^j}{j!} \Gamma(\alpha+rj) s^{-\alpha-rj} {}_{p+1}F_q \left(\begin{matrix} 1, \Delta(p, \alpha+rj); (-1)^q z \end{matrix} \right) \\
 \Delta(q, 1+j)$$

[$0 < r < 1$].

$$U_1 = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!r} \Gamma\left(\frac{\alpha+h}{r}\right) a^{-(\alpha+h)/r} s^h {}_{q+1}F_p \left(\begin{matrix} 1, \Delta(q, (\alpha+h)/r); (-1)^p z^{-1} \end{matrix} \right) \\
 \Delta(p, 1+h)$$

[$r > 1$].

$$\begin{aligned}
 14. \int_0^{\infty} x^{\alpha-1} e^{-ax-r} - sx \, dx = \\
 = \sum_{j=0}^{q-1} \frac{(-a)^j}{j!} \Gamma(\alpha-rj) s^{rj-\alpha} {}_1F_{p+q} \left(\begin{matrix} 1; \\ \Delta(p, 1-\alpha+rj), \Delta(q, 1+j) \end{matrix} \middle| \frac{z}{s} \right) + \\
 + \sum_{h=0}^{p-1} \frac{(-1)^h}{h!r} \Gamma\left(-\frac{\alpha+h}{r}\right) a^{(\alpha+h)/r} s^h {}_1F_{p+q} \left(\begin{matrix} 1; \\ \Delta(q, 1+(\alpha+h)/r), \Delta(p, 1+h) \end{matrix} \middle| \frac{z}{s} \right) \\
 \left[z = (-1)^{p+q} \left(\frac{s}{p}\right)^p \left(\frac{a}{q}\right)^q; \operatorname{Re} a, \operatorname{Re} s > 0 \right].
 \end{aligned}$$

2.3.3. Интегралы от $x^\alpha e^{-px}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px} \, dx = \Gamma(\alpha) p^{-\alpha} \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0] \text{ или } [0 < \operatorname{Re} \alpha < 1, \operatorname{Re} p = 0].$$

$$2. \int_0^{\infty} x^n e^{-px} \, dx = n! p^{-n-1} \quad [\operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} x^{n-1/2} e^{-px} \, dx = \frac{(2n-1)!! \sqrt{\pi}}{2^n p^{n+1/2}} \quad [\operatorname{Re} p > 0].$$

$$4. \int_{\gamma-i\infty}^{\gamma+i\infty} z^{\alpha-1} e^{-pz} \, dz = \frac{2\pi i}{(-p)^\alpha \Gamma(1-\alpha)} \begin{cases} 0 \\ 1 \end{cases} \quad \left[\begin{cases} p > 0 \\ p < 0 \end{cases}; \operatorname{Re} \alpha < 1; \gamma > 0 \right].$$

2.3.4. Интегралы от $(x \pm a)^\beta e^{-px}$.

$$1. \int_a^{\infty} (x-a)^{\beta-1} e^{-px} \, dx = p^{-\beta} e^{-pa} \Gamma(\beta) \quad [a > 0, \operatorname{Re} \beta, \operatorname{Re} p > 0 \text{ (или } 0 < \operatorname{Re} \beta < 1, \operatorname{Re} p = 0)].$$

$$2. \int_0^{\infty} \frac{e^{-px}}{(x+z)^\rho} \, dx = p^{\rho-1} e^{pz} \Gamma(1-\rho, pz) \quad [\operatorname{Re} p > 0, |\arg z| < \pi].$$

$$3. \int_0^{\infty} \frac{e^{-px}}{x+z} \, dx = -e^{pz} \operatorname{Ei}(-pz) \quad [\operatorname{Re} p > 0].$$

$$4. \int_0^{\infty} \frac{e^{-px}}{(x+z)^2} \, dx = \frac{1}{z} + p e^{pz} \operatorname{Ei}(-pz) \quad [\operatorname{Re} p \geq 0; |\arg z| < \pi].$$

$$5. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x+z}} \, dx = \sqrt{\frac{\pi}{p}} e^{pz} \operatorname{erfc}(\sqrt{pz}) \quad [|\arg z| < \pi; \operatorname{Re} p > 0].$$

$$6. \int_0^{\infty} \frac{e^{-px}}{(x+z)^{3/2}} \, dx = \frac{2}{\sqrt{z}} - 2\sqrt{\pi p} e^{pz} \operatorname{erfc}(\sqrt{pz}) \quad [|\arg z| < \pi; \operatorname{Re} p \geq 0].$$

$$7. \int_0^{\infty} \frac{e^{-px}}{x-y} dx = -e^{-py} \operatorname{Ei}(py) \quad [y, \operatorname{Re} p > 0].$$

$$8. \int_0^{\infty} \frac{e^{-p|x|}}{x-y} \left\{ \frac{1}{\operatorname{sgn} x} \right\} dx = [-e^{-p|y|} \operatorname{Ei}(p|y|) \pm e^{p|y|} \operatorname{Ei}(-p|y|)] \left\{ \frac{\operatorname{sgn} y}{1} \right\}$$

[Re $p \geq 0$, $y \neq 0$, Im $y = 0$].

$$9. \int_{-\infty}^{\infty} \frac{e^{\pm i\lambda x}}{(y \pm ix)^{\rho}} dx = \frac{2\pi}{\Gamma(\rho)} \lambda^{\rho-1} e^{-\lambda y} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

[$\begin{Bmatrix} \lambda > 0 \\ \lambda < 0 \end{Bmatrix}$; Re y , Re $\rho > 0$].

$$10. \int_{-i\infty}^{i\infty} \frac{e^{-pz}}{(z \pm y)^{\rho}} dz = 2\pi i \frac{\rho^{\rho-1}}{\Gamma(\rho)} e^{-\pi i \rho - py} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

[ρ , y , Re $\rho > 0$].

2.3.5. Интегралы от $(x^n \pm a^n)^{\rho} e^{-px}$.

Условие: $a > 0$.

$$1. \int_{-a}^a (a^2 - x^2)^{\beta-1} e^{-px} dx = 2^{\beta-1/2} \sqrt{\pi} \Gamma(\beta) (\rho/a)^{1/2-\beta} I_{\beta-1/2}(ap)$$

[Re $\beta > 0$].

$$2. \int_0^a (a-x)^{\beta-1} e^{-px} dx = \rho^{-\beta} e^{-ap} \gamma(\beta, -ap)$$

[Re $\beta > 0$].

$$3. \int_{-a}^a (a^2 - x^2)^{\beta-1} e^{i\lambda x} dx = \sqrt{\pi} \Gamma(\beta) (2a/\lambda)^{\beta-1/2} J_{\beta-1/2}(a\lambda)$$

[Re $\beta > 0$].

$$4. \int_a^{\infty} (x^2 - a^2)^{\beta-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \Gamma(\beta) \left(\frac{2a}{\rho}\right)^{\beta-1/2} K_{\beta-1/2}(ap)$$

[Re β , Re $\rho > 0$].

$$5. \int_a^{\infty} \frac{e^{\varepsilon i \lambda x}}{(x^2 - a^2)^{1-\beta}} dx = \varepsilon i \sqrt{\pi} \Gamma(\beta) 2^{\beta-3/2} \left(\frac{\lambda}{a}\right)^{1/2-\beta} H_{1/2-\beta}^{(3/2-\varepsilon/2)}(\lambda a)$$

[$\varepsilon = \pm 1$; Re $\beta > 0$, $\pm \operatorname{Re}(i\lambda) < 0$ (или $0 < \operatorname{Re} \beta < 1$, Re $(i\lambda) = 0$)].

$$6. \int_0^{\infty} \frac{e^{-px}}{(x^2 + z^2)^{\rho}} dx =$$

$$= 2^{-\rho-1/2} \sqrt{\pi} \Gamma(1-\rho) \left(\frac{z}{\rho}\right)^{1/2-\rho} [\mathbf{H}_{1/2-\rho}(\rho z) - Y_{1/2-\rho}(\rho z)]$$

[Re z , Re $\rho > 0$].

$$7. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x^2 + z^2)^{\rho}} dx = 2^{3/2-\rho} \sqrt{\pi} \Gamma(\rho) \left(\frac{\lambda}{z}\right)^{\rho-1/2} K_{\rho-1/2}(\lambda z)$$

[λ , Re $z > 0$; $0 < \operatorname{Re} \rho < 1$].

$$8. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + z^2} dx = \frac{\pi}{z} e^{-|\lambda|z}$$

[Re $z > 0$; Im $\lambda = 0$].

$$9. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^4 + z^4} dx = \frac{\pi}{z^3} e^{-z|\lambda|/2} \sin\left(\frac{z|\lambda|}{\sqrt{2}} + \frac{\pi}{4}\right) \quad [\operatorname{Re} z^2 > 0; \operatorname{Im} \lambda = 0].$$

$$10. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^5 + z^5} dx = \frac{\pi}{3z^4} e^{-z|\lambda|} + \frac{2\pi}{3z^4} e^{-z|\lambda|/2} \sin\left(z|\lambda|\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) \quad [\operatorname{Re} z^3 > 0; \operatorname{Im} \lambda = 0].$$

2.3.6. Интегралы от $x^\alpha (x \pm a)^\beta e^{-px}$.

Условие: $a > 0$.

$$1. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} e^{-px} dx = B(\alpha, \beta) a^{\alpha+\beta-1} {}_1F_1(\alpha; \alpha+\beta; -ap) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_0^a x^{\alpha-1} (a-x)^{\alpha-1} e^{-px} dx = \sqrt{\pi} \Gamma(\alpha) (a/p)^{\alpha-1/2} e^{-ap/2} I_{\alpha-1/2}(ap/2) \quad [\operatorname{Re} \alpha > 0].$$

$$3. \int_0^a \sqrt{\frac{x}{a-x}} e^{-px} dx = \frac{\pi a}{2} e^{-ap/2} \left[I_0\left(\frac{ap}{2}\right) - I_1\left(\frac{ap}{2}\right) \right].$$

$$4. \int_0^a \frac{e^{-px}}{\sqrt{x(x+a)}} dx = \frac{\pi}{2\sqrt{a}} e^{ap} [1 - \operatorname{erfc}^2(\sqrt{ap})].$$

$$5. \int_0^1 \frac{xe^x}{(x+1)^2} dx = \frac{e}{2} - 1.$$

$$6. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} e^{-px} dx = \Gamma(\beta) a^{\alpha+\beta-1} e^{-ap} \Psi(\beta, \alpha+\beta; ap) \quad [\operatorname{Re} \beta, \operatorname{Re} p > 0].$$

$$7. \int_a^\infty x^{\alpha-1} (x-a)^{\alpha-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{p}\right)^{\alpha-1/2} \Gamma(\alpha) e^{-ap/2} K_{\alpha-1/2}\left(\frac{ap}{2}\right) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$8. \int_{2a}^\infty \frac{e^{-px}}{\sqrt{x-a} x} dx = \frac{\pi}{2\sqrt{a}} \operatorname{erfc}^2(\sqrt{ap}) \quad [\operatorname{Re} p > 0].$$

$$9. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} e^{-px} dx = \Gamma(\alpha) z^{\alpha-\rho} \Psi(\alpha, \alpha+1-\rho; pz) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$10. \int_0^\infty x^{\alpha-1} (x+z)^{\alpha-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \Gamma(\alpha) (p/z)^{1/2-\alpha} e^{pz/2} K_{\alpha-1/2}(pz/2) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$11. \int_0^{\infty} \sqrt{\frac{x}{x+z}} e^{-px} dx = \frac{z}{2} e^{pz/2} \left[K_1\left(\frac{pz}{2}\right) - K_0\left(\frac{pz}{2}\right) \right] \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$12. \int_0^{\infty} x^{\alpha-1} (x+z)^{\pm 1/2 - \alpha} e^{-px} dx = 2^{\alpha} \left\{ \frac{(2\pi)^{-1/2}}{z^{-1/2}} \right\} \Gamma(\alpha) e^{pz/2} D_{\delta-2\alpha}(\sqrt{2pz})$$

$$\left[\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$13. \int_0^{\infty} \frac{x^{\alpha-1}}{x+z} e^{-px} dx = \Gamma(\alpha) z^{\alpha-1} e^{pz} \Gamma(1-\alpha, pz)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} p > 0 \text{ (или } \operatorname{Re} p = 0, 0 < \operatorname{Re} \alpha < 2)].$$

$$14. \int_0^{\infty} \frac{x^n}{x+z} e^{-px} dx = (-1)^{n+1} z^n e^{pz} \operatorname{Ei}'(-pz) + \sum_{k=1}^n (k-1)! (-z)^{n-k} p^{-k}$$

$$[\operatorname{Re} p > 0].$$

$$15. \int_0^{\infty} \frac{x^{n-1/2}}{x+z} e^{-px} dx = (-1)^n \pi z^{n-1/2} e^{pz} \operatorname{erfc}(\sqrt{pz}) +$$

$$+ 2^{1-n} \sqrt{\pi} p^{1/2-n} \sum_{k=0}^{n-1} (2n-2k-3)!! (-2pz)^k \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1}}{x-y} e^{-px} dx = \Gamma(\alpha-1) p^{1-\alpha} {}_1F_1(1; \alpha; -py) - \pi y^{\alpha-1} \operatorname{ctg} \alpha \pi e^{-py}$$

$$[y, \operatorname{Re} \alpha, \operatorname{Re} p > 0; \alpha \neq 1].$$

$$17. \int_0^{\infty} x^{\alpha-1} (x \mp iy)^{\alpha-1} e^{-px} dx = \pm \frac{i\sqrt{\pi}}{2} \Gamma(\alpha) \left(\frac{p}{y}\right)^{1/2-\alpha} e^{\mp ipy/2} \times$$

$$\times H_{\alpha-1/2}^{(3/2 \mp 1/2)}\left(\frac{py}{2}\right) \quad [y, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$18. \int_{-\infty}^{\infty} (z-ix)^{-\rho} (y-ix)^{-\sigma} e^{i\lambda x} dx =$$

$$= 2\pi \frac{(-\lambda)^{\rho+\sigma-1}}{\Gamma(\rho+\sigma)} e^{\lambda z} {}_1F_1(\sigma; \rho+\sigma; -\lambda(z-y)) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left[\begin{Bmatrix} \lambda > 0 \\ \lambda < 0 \end{Bmatrix}; \operatorname{Re} y, \operatorname{Re} z > 0; \operatorname{Re}(\rho+\sigma) > 1 \right].$$

$$19. \int_{-\infty}^{\infty} (z+ix)^{-\rho} (y-ix)^{-\sigma} e^{i\lambda x} dx =$$

$$= \operatorname{sgn} \lambda \frac{2\pi}{\Gamma(\delta)} (y+z)^{1-\rho-\sigma} e^{-|\lambda|z} \Psi(1-\delta, 2-\rho-\sigma; |\lambda|(y+z))$$

$$[\operatorname{Re} y, \operatorname{Re} z > 0, \operatorname{Re}(\rho+\sigma) > 1; \delta = \sigma \text{ при } \lambda < 0, \delta = \rho \text{ при } \lambda > 0].$$

2.3.7. Интегралы от $x^\alpha (x^2 \pm a^2)^\beta e^{-px}$ и $x^\alpha (a^2 - x^2) e^{-px}$.
Условие: $a > 0$.

1. $\int_0^a x^{\alpha-1} (a^2 - x^2)^{\beta-1} e^{-px} dx =$

$$= \frac{1}{2} a^{\alpha+2\beta} {}_2B\left(\frac{\alpha}{2}, \beta\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, \frac{\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) -$$

$$- \frac{1}{2} p a^{2\beta+\alpha-1} {}_2B\left(\beta, \frac{\alpha+1}{2}\right) {}_1F_2\left(\frac{\alpha+1}{2}; \frac{3}{2}, \beta + \frac{\alpha+1}{2}; \frac{a^2 p^2}{4}\right) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$
2. $\int_0^a \frac{x^{n-1/2}}{\sqrt{a^2 - x^2}} e^{-px} dx = \frac{(-1)^n \sqrt{\pi}}{2} \frac{\partial^n}{\partial p^n} \left\{ \sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) \left[I_{1/4}\left(\frac{ap}{2}\right) + I_{-1/4}\left(\frac{ap}{2}\right) \right] \right\}.$
3. $\int_a^\infty x^{\alpha-1} (x^2 - a^2)^{\beta-1} e^{-px} dx =$

$$= \frac{1}{2} a^{\alpha+2\beta-2} {}_2B\left(\beta, 1-\beta-\frac{\alpha}{2}\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, \frac{\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) -$$

$$- \frac{1}{2} p a^{\alpha+2\beta-1} {}_2B\left(\beta, \frac{1-\alpha}{2}-\beta\right) {}_1F_2\left(\frac{1+\alpha}{2}; \frac{3}{2}, \frac{1+\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) +$$

$$+ (2p)^2 a^{\alpha+2\beta-2} \Gamma(\alpha+2\beta-2) {}_1F_2\left(1-\beta; 2-\beta-\frac{\alpha}{2}, \frac{3-\alpha}{2}-\beta; \frac{a^2 p^2}{4}\right) \quad [\operatorname{Re} \beta, \operatorname{Re} p > 0].$$
4. $\int_a^\infty x^{n+1} (x^2 - a^2)^{\beta-1} e^{-px} dx =$

$$= (-1)^n \frac{(2a)^\beta + 1/2}{2\sqrt{\pi}} \Gamma(\beta) \frac{\partial^n}{\partial p^n} \left[p^{1/2-\beta} K_{\beta+1/2}(ap) \right] \quad [\operatorname{Re} \beta, \operatorname{Re} p > 0].$$
5. $\int_a^\infty \frac{x^{n-3/2}}{\sqrt{x^2 - a^2}} e^{-px} dx = \frac{(-1)^n}{\sqrt{2\pi}} \frac{\partial^n}{\partial p^n} \left\{ p^{3/2} \left[K_{3/4}\left(\frac{ap}{2}\right) - K_{1/4}\left(\frac{ap}{2}\right) \right] \right\}$

$$[\operatorname{Re} p > 0].$$
6.
$$= \frac{(-1)^{n-1}}{\sqrt{2\pi}} \frac{\partial^{n-1}}{\partial p^{n-1}} \left[\sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) \right] \quad [\operatorname{Re} p > 0; n = 1, 2, 3, \dots].$$
7.
$$= \frac{(-1)^n a}{\sqrt{2\pi}} \frac{\partial^{n-2}}{\partial p^{n-2}} \left[\sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) K_{3/4}\left(\frac{ap}{2}\right) \right] \quad [\operatorname{Re} p > 0; n = 2, 3, 4, \dots].$$
8. $\int_0^\infty \frac{x^{\alpha-1}}{(x^2 + z^2)^\rho} e^{-px} dx =$

$$= p^{2\rho-\alpha} \Gamma(\alpha-2\rho) {}_1F_2\left(\rho; 1+\rho-\frac{\alpha}{2}, \frac{1-\alpha}{2} + \rho; -\frac{p^2 z^2}{4}\right) +$$

$$+ \frac{z^{\alpha-2\rho}}{2} {}_2B\left(\rho-\frac{\alpha}{2}, \frac{\alpha}{2}\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, 1+\frac{\alpha}{2}-\rho; -\frac{p^2 z^2}{4}\right) -$$

$$- \frac{p}{2} z^{1+\alpha-2\rho} {}_2B\left(\rho-\frac{\alpha+1}{2}, \frac{\alpha+1}{2}\right) {}_1F_2\left(\frac{\alpha+1}{2}; \frac{3}{2}, \frac{3+\alpha}{2}-\rho; -\frac{p^2 z^2}{4}\right) \quad [\operatorname{Re} z, \operatorname{Re} \alpha > 0, \operatorname{Re} p \geq 0; \text{ если } \operatorname{Re} p = 0, \text{ то добавляется условие } \operatorname{Re}(\alpha-2\rho) < 1].$$

$$9. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(x^2+z^2)^n} dx = \frac{(-2)^n}{(n-1)!} \Gamma(\alpha-1) \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{\alpha-2} [e^{ipz} + \pi \alpha i/2 \Gamma(2-\alpha, ipz) + e^{-ipz} - \pi \alpha i/2 \Gamma(2-\alpha, -ipz)] \} \\ [\operatorname{Re} \alpha, \operatorname{Re} p, \operatorname{Re} z > 0; n=1, 2, 3, \dots].$$

$$10. \int_0^{\infty} \frac{x^{k-1/2} e^{-px}}{(x^2+z^2)^n} dx = \frac{(-1)^{k+n-1} 2^{3/2-n} \pi}{(n-1)!} \frac{\partial^k}{\partial p^k} \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{-3/2} [\cos(pz) (1/2 - S(pz)) - \sin(pz) (1/2 - C(pz))] \} \\ [\operatorname{Re} p, \operatorname{Re} z > 0; k=0, 1, 2, \dots; n=1, 2, 3, \dots].$$

$$11. \int_0^{\infty} \frac{x^m e^{-px}}{(x^2+z^2)^n} dx = \frac{(-1)^{m+n-1} 2^{1-n}}{(n-1)!} \frac{\partial^m}{\partial p^m} \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{-1} [\sin(pz) \operatorname{ci}(pz) - \cos(pz) \operatorname{si}(pz)] \} \quad [\operatorname{Re} p, \operatorname{Re} z > 0; n=1, 2, 3, \dots].$$

$$12. \int_0^{\infty} \frac{x^{m-1/2} e^{-px}}{(x^2+z^2)^{n+1/2}} dx = \frac{(-1)^{m+n} \pi^{3/2}}{4 (2n-1)!!} \frac{\partial^m}{\partial p^m} \left(\frac{\partial}{z \partial z} \right)^n \times \\ \times \left\{ V_{\rho} \left[J_{1/4}^2 \left(\frac{\rho z}{2} \right) + Y_{1/4}^2 \left(\frac{\rho z}{2} \right) \right] \right\} \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$13. \int_0^{\infty} \frac{x^m e^{-px}}{x^2+z^2} dx = z^{m-1} \left[\sin \left(pz - \frac{\pi m}{2} \right) \operatorname{ci}(pz) - \right. \\ \left. - \cos \left(pz - \frac{\pi m}{2} \right) \operatorname{si}(pz) \right] + p^{1-m} \sum_{k=1}^{[m/2]} (m-2k)! (-p^2 z^2)^{k-1} \\ [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{x^2-y^2} dx = \Gamma(\alpha-2) p^{2-\alpha} {}_1F_2 \left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}, \frac{p^2 y^2}{4} \right) - \\ - \frac{y^{\alpha-2} \pi}{2 \sin \alpha \pi} (e^{py} + \cos \alpha \pi e^{-py}) \quad [y, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$15. \int_0^{\infty} \frac{x^m e^{-px}}{x^2-y^2} dx = \frac{y^{m-1}}{2} [(-1)^m e^{py} \operatorname{Ei}(-py) - e^{-py} \operatorname{Ei}(py)] + \\ + p^{1-m} \sum_{k=1}^{[m/2]} (m-2k)! (py)^{2k-2} \quad [y, \operatorname{Re} p > 0].$$

$$16. \int_0^{\infty} \frac{x^m e^{-px}}{x^4-y^4} dx = -\frac{y^{m-3}}{4} \left\{ e^{-py} \operatorname{Ei}(py) + (-1)^{m+1} e^{py} \operatorname{Ei}(-py) + \right. \\ \left. + 2 \operatorname{ci}(py) \sin \left(py - \frac{\pi m}{2} \right) - 2 \operatorname{si}(py) \cos \left(py - \frac{\pi m}{2} \right) \right\} + \\ + p^{3-m} \sum_{k=1}^{[m/4]} (m-4k)! (py)^{4k-4} \quad [y, \operatorname{Re} p > 0].$$

$$17. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{x^{\alpha-1} e^{i\lambda x}}{x^2+z^2} dx = \pi i z^{\alpha-2} e^{\lambda z} - \pi \alpha i / 2 \quad [\gamma, \operatorname{Re} z > 0; \lambda < 0, 0 < \operatorname{Re} \alpha < 2].$$

2.3.8. Интегралы от $x^\alpha \prod_k (a^{\mu_k} \pm x^{\mu_k})^{\rho_k} e^{-px}$.

$$1. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (x+z)^{-\rho} e^{-px} dx = B(\alpha, \beta) z^{-\rho} a^{\alpha+\beta-1} \times \\ \times \Phi_1(\alpha, \rho, \alpha+\beta, -a/z, ap) \quad [a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0, |\arg(1+a/z)| < \pi].$$

$$2. \int_0^\infty x^{\alpha-1} (x+z)^{-\alpha} (2x+z)^{-1/2} e^{-px} dx = z^{-1/2} e^{pz/2} \Gamma(\alpha) [D_{-\alpha}(\sqrt{pz})]^2 \\ [a, \operatorname{Re} \alpha, \operatorname{Re} p > 0, |\arg z| < \pi].$$

$$3. \int_0^\infty \frac{x^m e^{-px}}{y^3+y^2x+yx^2+x^3} dx = \frac{y^{m-2}}{2} \left\{ \operatorname{ci}(p|y|) \left[\sin\left(py - \frac{\pi m}{2}\right) + \cos\left(py - \frac{\pi m}{2}\right) \right] + \right. \\ \left. + \operatorname{sgn} y \operatorname{si}(p|y|) \left[\sin\left(py - \frac{\pi m}{2}\right) - \cos\left(py - \frac{\pi m}{2}\right) \right] - \right. \\ \left. - (-1)^m e^{py} \operatorname{Ei}(-py) \right\} \quad [\operatorname{Re} p > 0; \operatorname{Im} y = 0].$$

$$4. \int_{-\infty}^\infty \frac{e^{i\lambda x}}{(z^2+x^2)(y \pm ix)^\sigma} dx = \frac{\pi}{z} e^{\pm \lambda z} (y \pm z)^{-\sigma} \\ \left[\operatorname{Re} y, \operatorname{Re} z > 0; \operatorname{Re} \sigma > -1; \left\{ \begin{array}{l} \lambda > 0 \\ \lambda < 0, y \neq z \end{array} \right\} \right].$$

$$5. \int_{-\infty}^\infty \left(\frac{iy-x}{iy+x} \right)^n \frac{e^{-i\lambda x}}{x^2+y^2} dx = (-1)^{n-1} 2\pi \lambda e^{-\lambda y} L_{n-1}^1(2\lambda y) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \left[\left\{ \begin{array}{l} \lambda > 0 \\ \lambda < 0 \end{array} \right\}; y > 0; n = 1, 2, 3, \dots \right].$$

$$6. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{(ix)^\alpha}{ix-z} \prod_{k=1}^n (z_k+ix)^{\alpha_k} e^{-i\lambda x} dx = -2\pi e^{-\lambda z} z^\alpha \prod_{k=1}^n (z+z_k)^{\alpha_k} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \left[\left\{ \begin{array}{l} \operatorname{Re} z > 0 \\ \operatorname{Re} z < 0 \end{array} \right\}; \lambda, \gamma, \operatorname{Re} z_k > 0, k=1, 2, \dots, n, \operatorname{Re} \alpha > -1; \operatorname{Re} \left(\alpha + \sum_{k=1}^n \alpha_k \right) < 1 \right].$$

$$7. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{(ix)^\alpha}{x^2+z^2} \prod_{k=1}^n (z_k+ix)^{\alpha_k} e^{-i\lambda x} dx = \pi e^{-\lambda z} z^{\alpha-1} \prod_{k=1}^n (z+z_k)^{\alpha_k} \\ \left[\lambda, \gamma, \operatorname{Re} z, \operatorname{Re} z_k > 0; k=1, 2, \dots, n; \operatorname{Re} \left(\alpha + \sum_{k=1}^n \alpha_k \right) < 2; \operatorname{Re} \alpha > -1 \right]$$

2.3.9. Интегралы от $x^\alpha (\sqrt{x \pm z} + ax^\mu + bz^\mu)^\rho e^{-px}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x+z} \pm \sqrt{x})^\nu}{(x+z)^r} e^{-px} dx = 2^{2r-\alpha} z^{\alpha+\nu/2-r} (\mp \nu)^{1-2r} \times \\ \times \Gamma \left[\begin{matrix} \alpha, r-\alpha \mp \nu/2 \\ 1+\alpha-r \mp \nu/2 \end{matrix} \right] {}_2F_2 \left(\alpha, \alpha+\frac{1}{2}; 1+\alpha \pm \frac{\nu}{2}-r, 1+\alpha \mp \frac{\nu}{2}-r; pz \right) + \\ + 2^\nu z^{(\nu \mp \nu)/2} p^{r-\alpha \mp \nu/2} {}_2F_2 \left(r \mp \frac{\nu}{2}, \frac{1 \mp \nu}{2}+r; 1 \mp \nu, 1+r-\alpha \mp \frac{\nu}{2}; pz \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$2. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x+z} \pm \sqrt{z})^\nu}{(x+z)^r} e^{-px} dx = (\pm \nu)^{1-2r} p^{(1-\nu)/2-\alpha} z^{1/2-r} \Gamma \left(\alpha + \frac{\nu-1}{2} \right) \times \\ \times {}_2F_2 \left(\frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{3}{2}-2r, \frac{3-\nu}{2}-\alpha; pz \right) + (\pm \nu)^{2r} p^{2r-\alpha-\nu/2} z^r \Gamma \left(\alpha + \frac{\nu}{2}-2r \right) \times \\ \times {}_2F_2 \left(2r+\frac{\nu}{2}, 2r-\frac{\nu}{2}; \frac{1}{2}+2r, 1+2r-\frac{\nu}{2}-\alpha; pz \right) + \\ + 2^{2\alpha+\nu} z^{\nu/2+\alpha-r} (\mp \nu)^{1-2r} \Gamma \left[\begin{matrix} \alpha, 2r-\nu-2\alpha \\ 1-\alpha-(\nu \pm \nu)/2 \end{matrix} \right] \times \\ \times {}_2F_2 \left(\frac{\nu \pm \nu}{2} + \alpha, \alpha; \frac{1+\nu}{2} + \alpha, 1+\alpha+\frac{\nu}{2}-2r; pz \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} p, \operatorname{Re}(2\alpha+\nu \mp \nu) > 0; |\arg z| < \pi].$$

$$3. \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x+z}} (\sqrt{x+z} + \sqrt{z})^{1-\alpha} e^{-px} dx = 2^{1-\alpha/2} p^{-\alpha/2} \Gamma(\alpha) e^{pz/2} D_{-\alpha}(\sqrt{2pz}) \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$4. \int_0^\infty \sqrt{\frac{\sqrt{x+z} + \sqrt{z}}{x(x+z)}} e^{-px} dx = \sqrt{2} z^{1/4} e^{pz/2} K_{1/4} \left(\frac{pz}{2} \right) \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

2.3.10. Интегралы от $x^\alpha (\sqrt{x^2+z^2} + ax + bz)^\rho e^{-px}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2+z^2} \pm x)^\nu}{(x^2+z^2)^r} e^{-px} dx = \\ = 2^{2r-\alpha-1} (\mp \nu)^{1-2r} z^{\alpha+\nu-2r} \Gamma \left[\begin{matrix} \alpha, r-(\alpha \pm \nu)/2 \\ 1+(\alpha \mp \nu)/2-r \end{matrix} \right] \times \\ \times {}_2F_3 \left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}, 1+\frac{\alpha \pm \nu}{2}-r, 1+\frac{\alpha \mp \nu}{2}-r; -\frac{p^2 z^2}{4} \right) - \\ - 2^{2r-\alpha-2} p z^{1+\alpha+\nu-2r} (\mp \nu)^{1-2r} \Gamma \left[\begin{matrix} \alpha+1, r-(\alpha \pm \nu+1)/2 \\ (3+\alpha \mp \nu)/2-r \end{matrix} \right] \times \\ \times {}_2F_3 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2}+1; \frac{3}{2}, \frac{3+\alpha \pm \nu}{2}-r, \frac{3+\alpha \mp \nu}{2}-r; -\frac{p^2 z^2}{4} \right) + \\ + 2^{\pm \nu} p^{2r \mp \nu-\alpha} z^{\nu \mp \nu} \Gamma(\alpha \pm \nu-2r) \times \\ \times {}_2F_3 \left(r \mp \frac{\nu}{2}, r+\frac{1 \mp \nu}{2}; 1 \mp \nu, 1+r-\frac{\alpha \pm \nu}{2}, \frac{1 \mp \nu-\alpha}{2}+r; -\frac{p^2 z^2}{4} \right) \\ [r=0 \text{ или } 1/2, \operatorname{Re} \alpha, \operatorname{Re} p, \operatorname{Re} z > 0]$$

$$2. \int_0^{\infty} \frac{(Vx^2+z^2+x)^v}{Vx^2+z^2} e^{-px} dx = \frac{\pi z^v}{\operatorname{sh} v\pi} [J_{-v}(pz) - J_v(pz)] \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$3. \int_0^{\infty} \frac{(Vx^2+z^2+x)^v}{Vx Vx^2+z^2} e^{-px} dx = \\ = \left(\frac{\pi}{2}\right)^{3/2} Vp z^v \left[J_{1/4-v/2}\left(\frac{pz}{2}\right) Y_{-1/4-v/2}\left(\frac{pz}{2}\right) - J_{-1/4-v/2}\left(\frac{pz}{2}\right) Y_{1/4-v/2}\left(\frac{pz}{2}\right) \right] \\ [\operatorname{Re} p, \operatorname{Re} z > 0]$$

$$4. \int_0^{\infty} x^{\alpha-1} \frac{(Vx^2+z^2 \pm z)^v}{(x^2+z^2)^r} e^{-px} dx = \\ = (\mp v)^{1-2r} 2^{\alpha+v-1} z^{\alpha+v-2r} \Gamma \left[\frac{\alpha}{2}, \frac{2r-v-\alpha}{1-(v+\alpha \pm v)/2} \right] \times \\ \times {}_2F_3 \left(\frac{v \pm v + \alpha}{2}, \frac{\alpha}{2}; \frac{1}{2}, \frac{1+v+\alpha}{2}, 1 + \frac{v+\alpha}{2} - 2r; -\frac{p^2 z^2}{4} \right) - \\ - 2^{v+\alpha} (\mp v)^{1-2r} p z^{\alpha+v-2r+1} \Gamma \left[\frac{(\alpha+1)/2, 2r-v-\alpha-1}{(1-\alpha-v \mp v)/2} \right] \times \\ \times {}_2F_3 \left(\frac{\alpha+1}{2}, \frac{1+\alpha+v \pm v}{2}; \frac{3}{2}, 1 + \frac{v+\alpha}{2}, \frac{3+v+\alpha}{2} - 2r; -\frac{p^2 z^2}{4} \right) + \\ + p^{1-\alpha-v} (\pm z v)^{1-2r} \Gamma(v+\alpha-1) \times \\ \times {}_2F_3 \left(\frac{1+v}{2}, \frac{1-v}{2}; \frac{3}{2} - 2r, 1 - \frac{v+\alpha}{2}, \frac{3-\alpha-v}{2}; -\frac{p^2 z^2}{4} \right) + \\ + p^{4r-v-\alpha} (\pm z v)^{2r} \Gamma(v+\alpha-4r) \times \\ \times {}_2F_3 \left(\frac{v}{2} + 2r, 2r - \frac{v}{2}; 1 - \frac{v+\alpha}{2} + 2r, \frac{1-v-\alpha}{2} + 2r, \frac{1}{2} + 2r; -\frac{p^2 z^2}{4} \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} p, \operatorname{Re} z, \operatorname{Re}(\alpha+v \mp v) > 0].$$

$$5. \int_{-\infty}^{\infty} \frac{(Vx^2+z^2+x)^v}{Vx^2+z^2} e^{\pm i\lambda x} dx = 4z^v e^{\pm i v \pi/2} K_v(\lambda z) \quad [\lambda, \operatorname{Re} z > 0; |\operatorname{Re} v| < 1].$$

2.3.11. Интегралы от $x^\alpha [(Vax^u+b \pm Vcx^v+d)^0 \pm \pm (Vax^u+b \mp Vcx^v+d)^0] e^{-px}$.

$$1. \int_0^a x^{\alpha-1} \frac{(V\bar{a}+V\bar{a}-x)^v - (-1)^{2r} (V\bar{a}-V\bar{a}-x)^v}{(a-x)^r} e^{-px} dx = \\ = 2^{2\alpha+v} a^{v/2+\alpha-r} v^{1-2r} \Gamma \left[\frac{\alpha, v+\alpha}{v+2\alpha-2r+1} \right] \times \\ \times {}_2F_2 \left(\alpha, v+\alpha; \frac{v+1}{2} + \alpha, \frac{v}{2} + \alpha + 1 - 2r; -ap \right) \\ [r=0 \text{ или } 1/2, a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+v) > 0].$$

$$\begin{aligned}
 2. \int_0^a x^{\alpha-1} \frac{(V\overline{a+x} + V\overline{a-x})^v - (-1)^{2r} (V\overline{a+x} - V\overline{a-x})^v}{(a^2 - x^2)^r} e^{-px} dx = \\
 = 2^{v/2} \int_0^a x^{\alpha-1} \frac{(a + V\overline{a^2 - x^2})^{v/2} - (-1)^{2r} (a - V\overline{a^2 - x^2})^{v/2}}{(a^2 - x^2)^r} e^{-px} dx \\
 [r=0 \text{ или } 1/2; \text{ см. 2.3.11.3}].
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^a x^{\alpha-1} \frac{(a + V\overline{a^2 - x^2})^v - (-1)^{2r} (a - V\overline{a^2 - x^2})^v}{(a^2 - x^2)^r} e^{-px} dx = \\
 = 2^{\alpha+v-1} v^{1-2r} a^{\alpha+v-2r} \Gamma \left[\frac{\alpha/2}{v+\alpha+1-2r} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha}{2}, v+\frac{\alpha}{2}; \frac{1}{2}, \frac{v+\alpha+1}{2}, \frac{v+\alpha}{2}+1-2r; \frac{a^2 p^2}{4} \right) - \\
 - 2^{v+\alpha} p v^{1-2r} a^{1+\alpha+v-2r} \Gamma \left[\frac{(\alpha+1)/2}{\alpha+v+2-2r} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}+v, \frac{3}{2}, \frac{v+\alpha}{2}+1, \frac{v+\alpha+3}{2}-2r; \frac{a^2 p^2}{4} \right) \\
 [r=0 \text{ или } 1/2, a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+v) > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_a^\infty x^{\alpha-1} \frac{(V\overline{x} + V\overline{x-a})^v - (-1)^{2r} (V\overline{x} - V\overline{x-a})^v}{(x-a)^r} e^{-px} dx = \\
 = v^{1-2r} 2^{2r-2\alpha} a^{v/2+\alpha-r} \Gamma \left[\frac{r-\alpha+v/2}{1-2\alpha} \right] \times \\
 \times {}_2F_2 \left(\alpha, \frac{1}{2}+\alpha; 1+\alpha-r-\frac{v}{2}, 1+\alpha+\frac{v}{2}-r; -ap \right) + \\
 + (-1)^{1-2r} a^v 2^{-v} p^{r+v/2-\alpha} \Gamma \left(\alpha-r-\frac{v}{2} \right) \times \\
 \times {}_2F_2 \left(r+\frac{v}{2}, \frac{1+v}{2}+r; 1+v, 1+r+\frac{v}{2}-\alpha; -ap \right) + \\
 + (-1)^{1-2r} a^{-v} 2^v p^{r-v/2-\alpha} \Gamma \left(\alpha+\frac{v}{2}-r \right) \times \\
 \times {}_2F_2 \left(r-\frac{v}{2}, \frac{1-v}{2}+r; 1-v, 1+r-\frac{v}{2}-\alpha; -ap \right) \\
 [r=0 \text{ или } 1/2, a, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_a^\infty x^{\alpha-1} \frac{(V\overline{x+a} + V\overline{x-a})^v - (-1)^{2r} (V\overline{x+a} - V\overline{x-a})^v}{(x^2 - a^2)^r} e^{-px} dx = \\
 = 2^{v/2} \int_a^\infty x^{\alpha-1} \frac{(x + V\overline{x^2 - a^2})^{v/2} - (-1)^{2r} (x - V\overline{x^2 - a^2})^{v/2}}{(x^2 - a^2)^r} e^{-px} dx \\
 [r=0 \text{ или } 1/2, \text{ см. 2.3.11.7}].
 \end{aligned}$$

$$6. \int_a^\infty [(\sqrt{x+a} + \sqrt{x-a})^v - (\sqrt{x+a} - \sqrt{x-a})^v] e^{-px} dx =$$

$$= \left(\frac{v}{p}\right) (2a)^{v/2} K_{v/2}(ap) \quad [a, \operatorname{Re} p > 0].$$

$$7. \int_a^\infty x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^v - (-1)^{2r} (x - \sqrt{x^2 - a^2})^v}{(x^2 - a^2)^r} e^{-px} dx =$$

$$= 2^{2r-1-\alpha} a^{\alpha+v-2r} v^{1-2r} \Gamma \left[\frac{r + (v-\alpha)/2, r - (v+\alpha)/2}{1-\alpha} \right] \times$$

$$\times {}_2F_3 \left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; \frac{1}{2}, 1 + \frac{\alpha-v}{2} - r, 1 + \frac{v+\alpha}{2} - r; \frac{a^2 p^2}{4} \right) -$$

$$- 2^{2r-\alpha} v^{1-2r} p a^{\alpha+v+1-2r} \Gamma \left[\frac{r + (v-\alpha-1)/2, r - (v+\alpha+1)/2}{-\alpha} \right] \times$$

$$\times {}_2F_3 \left(\frac{1+\alpha}{2}, 1 + \frac{\alpha}{2}; \frac{3}{2}, \frac{3+\alpha-v}{2} - r, \frac{3+\alpha+v}{2} - r; \frac{a^2 p^2}{4} \right) +$$

$$+ (-1)^{1-2r} 2^{-v} a^{2v} p^{2r+v-\alpha} \Gamma(\alpha - v - 2r) \times$$

$$\times {}_2F_3 \left(r + \frac{v}{2}, \frac{1+v}{2} + r; 1+v, 1+r + \frac{v-\alpha}{2}, \frac{1+v-\alpha}{2} + r; \frac{a^2 p^2}{4} \right) +$$

$$+ (-1)^{1-2r} 2^v a^{-2v} p^{2r-v-\alpha} \Gamma(\alpha + v - 2r) \times$$

$$\times {}_2F_3 \left(r - \frac{v}{2}, \frac{1-v}{2} + r; 1-v, 1+r - \frac{v+\alpha}{2}, \frac{1-v-\alpha}{2} + r; \frac{a^2 p^2}{4} \right)$$

$$[r=0 \text{ или } 1/2; a, \operatorname{Re} p > 0].$$

$$8. \int_a^\infty \frac{(x + \sqrt{x^2 - a^2})^v + (x - \sqrt{x^2 - a^2})^v}{\sqrt{x} \sqrt{x^2 - a^2}} e^{-px} dx =$$

$$= \sqrt{\frac{2p}{\pi}} a^v K_{v/2+1/4} \left(\frac{ap}{2} \right) K_{v/2-1/4} \left(\frac{ap}{2} \right) \quad [a, \operatorname{Re} p > 0].$$

$$9. \int_a^\infty \frac{(x + \sqrt{x^2 - a^2})^v + (x - \sqrt{x^2 - a^2})^v}{\sqrt{x^2 - a^2}} e^{-px} dx = 2a^v K_v(ap) \quad [a, \operatorname{Re} p > 0].$$

$$10. \int_a^\infty [(x + \sqrt{x^2 - a^2})^v - (x - \sqrt{x^2 - a^2})^v] e^{-px} dx = 2p^{-1} v a^v K_v(ap)$$

$$[a, \operatorname{Re} p > 0].$$

$$11. \int_0^\infty x^{-r} \frac{(\sqrt{x+z} + \sqrt{x})^v - (-1)^{2r} (\sqrt{x+z} - \sqrt{x})^v}{(x+z)^{1-\alpha}} e^{-px} dx =$$

$$= e^{pz} \int_a^\infty x^{\alpha-1} \frac{(\sqrt{x} + \sqrt{x-a})^v - (-1)^{2r} (\sqrt{x} - \sqrt{x-a})^v}{(x-a)^r} e^{-px} dx \Big|_{a-z}$$

$$[r=0 \text{ или } 1/2, \operatorname{Re} p > 0; |\arg z| < \pi; \text{ см. 2.3.11.4}].$$

$$12. \int_0^{\infty} [(\sqrt{x+z} + \sqrt{x})^{\nu} - (\sqrt{x+z} - \sqrt{x})^{\nu}] e^{-px} dx = \\ = \nu p^{-1} z^{\nu/2} e^{pz/2} K_{\nu/2}(pz/2) \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$13. \int_0^{\infty} x^{-r} \frac{(\sqrt{x+2z} + \sqrt{x})^{\nu} - (-1)^{2r} (\sqrt{x+2z} - \sqrt{x})^{\nu}}{(x+z)^{1-\alpha} (x+2z)^r} e^{-px} dx = \\ = 2^{\nu/2} e^{pz} \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu/2} - (-1)^{2r} (x - \sqrt{x^2 - a^2})^{\nu/2}}{(x^2 - a^2)^r} e^{-px} dx \Big|_{a=z} \\ [r=0 \text{ или } 1/2; \operatorname{Re} p > 0; |\arg z| < \pi; \text{ см. 2.3.12.7}].$$

$$14. \int_0^{\infty} x^{-r} \frac{(x+z + \sqrt{x^2+2xz})^{\nu} - (-1)^{2r} (x+z - \sqrt{x^2+2xz})^{\nu}}{(x+z)^{1-\alpha} (x+2z)^{-r}} e^{-px} dx = \\ = e^{pz} \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu} - (-1)^{2r} (x - \sqrt{x^2 - a^2})^{\nu}}{(x^2 - a^2)^r} e^{-px} dx \Big|_{a=z} \\ [r=0 \text{ или } 1/2; \operatorname{Re} p > 0; |\arg z| < \pi; \text{ см. 2.3.12.7}].$$

$$15. \int_0^{\infty} [(\sqrt{x^2+z^2} + x)^{\nu} \pm (\sqrt{x^2+z^2} - x)^{\nu}] e^{-px} dx = \\ = \frac{(2\nu)^{(1\pm 1)/2}}{p} z^{\nu} S_{(1\pm 1)/2, \nu}(pz) \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$16. \int_0^{\infty} \frac{(\sqrt{x^2+z^2} + x)^{\nu} \pm (\sqrt{x^2+z^2} - x)^{\nu}}{\sqrt{x^2+z^2}} e^{-px} dx = \\ = 2z^{\nu} \nu^{(1\pm 1)/2} S_{(\pm 1-1)/2, \nu}(pz) \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} x^{\alpha-1} \frac{(2z^2 - c^2 + xz + c \sqrt{x^2 + 2zx + c^2})^{1/2} (c + \sqrt{x^2 + 2zx + c^2})^{1-\alpha}}{(x+2z) \sqrt{x^2 + 2zx + c^2}} e^{-px} dx = \\ = \sqrt{2} \Gamma(\alpha) e^{pz} D_{-\alpha}(\sqrt{2p(z + \sqrt{z^2 - c^2})}) D_{-\alpha}(\sqrt{2p(z - \sqrt{z^2 - c^2})}) \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

2.3.12. Интегралы от $x^{\alpha} (e^{qx} + z)^{\lambda} e^{-px}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(e^{qx} + z)^n} dx = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(k+1)_{n-1} (-z)^k}{(p+qk+qn)^{\alpha}} \\ [\operatorname{Re} q, \operatorname{Re}(p+qn) > 0; |z| \leq 1; z \neq -1, \operatorname{Re} \alpha > 0, \text{ или } z = -1, \operatorname{Re} \alpha > n; n = 1, 2, 3, \dots].$$

$$2. \int_0^{\infty} \frac{x e^{-px}}{(e^{-px} + 1)^2} dx = \frac{\ln 2}{p^2} \quad [p \neq 0].$$

$$3. \int_0^{\infty} \frac{x^{2n} e^{-px}}{(e^{-px} + 1)^2} dx = (-1)^{n-1} p^{-2n-1} (2^{2n-1} - 1) \pi^{2n} B_{2n} \quad [p \neq 0].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{qx}+1} e^{-px} dx = (2q)^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{p+q}{2q}\right) - \zeta\left(\alpha, \frac{p}{2q} + 1\right) \right]$$

[Re α , Re q , Re($p+q$) > 0].

$$5. \int_0^{\infty} \frac{x^{\alpha-1} e^{qx/2}}{e^{qx}+1} dx = \left(\frac{2}{q}\right)^{\alpha} \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha}}$$

[Re α > 0, $q \neq 0$].

$$6. \int_0^{\infty} \frac{x^n e^{-px}}{e^{qx}+1} dx = q^{-n-1} (-1)^n \beta^{(n)}\left(1 + \frac{p}{q}\right)$$

[Re q , Re($p+q$) > 0].

$$7. \int_0^{\infty} \frac{x^{2n} e^{qx/2}}{e^{qx}+1} dx = \frac{(-1)^n}{2} \left(\frac{\pi}{q}\right)^{2n+1} E_{2n}$$

[$q \neq 0$].

$$8. \int_0^{\infty} \frac{x e^{-mx}}{e^x+1} dx = (-1)^m \left[\frac{\pi^2}{12} + \sum_{k=1}^m \frac{(-1)^k}{k^2} \right].$$

$$9. \int_0^{\infty} \frac{x e^{qx/2}}{e^{qx}+1} dx = \frac{4}{q^2} G$$

[$q \neq 0$].

$$10. \int_{-\infty}^{\infty} \frac{x^n e^{-px}}{(e^x+z)^{\rho}} dx = (-1)^n \frac{\partial^n}{\partial \rho^n} [z^{-p-\rho} B(-\rho, p+\rho)] =$$

$$= (-1)^{n-1} \frac{\partial^{n-1}}{\partial \rho^{n-1}} \{z^{-p-\rho} B(-\rho, p+\rho) [\ln z + \psi(-\rho) - \psi(p+\rho)]\}$$

[|arg z | < π , $-\text{Re } \rho < \text{Re } p < 0$, $n = 1, 2, 3, \dots$].

$$11. \int_{-\infty}^{\infty} \frac{x^n e^{-px}}{(e^x+z)^m} dx = \frac{(-1)^{n-1} \pi (p+1)_{m-1}}{(m-1)!} \frac{\partial^n}{\partial p^n} \frac{z^{-p-m}}{\sin p\pi}$$

[|arg z | < π ; $-m < \text{Re } p < 0$; $m = 1, 2, 3, \dots$].

$$12. \int_{-\infty}^{\infty} \frac{x e^{mx}}{(e^x+z)^n} dx = z^{m-n} \frac{(m-1)! (n-m-1)!}{(n-1)!} \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right)$$

[|arg z | < π , $m = 1, 2, \dots, n-1$; $n = 2, 3, 4, \dots$].

$$13. \int_{-\infty}^{\infty} \frac{x e^{mx}}{(e^x+z)^{n+1/2}} dx =$$

$$= 2^m z^{m-n-1/2} \frac{(2n-2m-1)!! (m-1)!}{(2n-1)!!} \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right)$$

[|arg z | < π , $m = 1, 2, \dots, n$; $n = 1, 2, 3, \dots$].

$$14. \int_{-\infty}^{\infty} \frac{x e^{(m+1/2)x}}{(e^x + z)^{n+1/2}} dx =$$

$$= \left(\frac{z}{2}\right)^{m-n} \frac{(2m-1)!!}{(2n-1)!!} (n-m-1)! \left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right)$$

$$[|\arg z| < \pi, m=0, 1, 2, \dots, n-1, n=1, 2, 3, \dots].$$

$$15. \int_{-\infty}^{\infty} \frac{x e^{mx}}{(e^x + z)^{n+\varepsilon/3}} dx =$$

$$= (-1)^m z^{m-n-\varepsilon/3} \frac{(m-1)!}{\left(\frac{3-\varepsilon}{3}-n\right)_m} \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 3 \sum_{k=1}^{n-m} \frac{1}{3k+\varepsilon-3} + \frac{3}{2} \ln 3 \pm \frac{\pi}{2\sqrt{3}} \right)$$

$$\left[|\arg z| < \pi, m=1, 2, \dots, n, n=1, 2, 3, \dots; \varepsilon = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \right].$$

$$16. \int_{-\infty}^{\infty} \frac{x e^{(m+\varepsilon/3)x}}{(e^x + z)^{n+\varepsilon/3}} dx = z^{m-n} \left(n-m-\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right)! \Gamma \left[\begin{matrix} m+\varepsilon/3 \\ n+\varepsilon/3 \end{matrix} \right] \times$$

$$\times \left(\ln z + 3 \sum_{k=1}^m \frac{1}{3k+\varepsilon-3} - \sum_{k=1}^{n-m-1} \frac{1}{k} - \frac{3}{2} \ln 3 \mp \frac{\pi}{2\sqrt{3}} \right)$$

$$\left[|\arg z| < \pi, m=0, 1, 2, \dots, n-1, n=1, 2, 3, \dots, \varepsilon = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \right].$$

$$17. \int_{-\infty}^{\infty} \frac{x e^{(m+\varepsilon/3)x}}{(e^x + z)^{n+1-\varepsilon/3}} dx = z^{m-n+1/3} B \left(m + \frac{\varepsilon}{3}, n-m \pm \frac{1}{3} \right) \times$$

$$\times \left(\ln z + 3 \sum_{k=1}^m \frac{1}{3k+\varepsilon-3} - 3 \sum_{k=1}^{n-m+1-\varepsilon} \frac{1}{3k+\varepsilon-3} \right)$$

$$\left[|\arg z| < \pi, m=0, 1, 2, \dots, n+1-\varepsilon, n=\varepsilon-1, 1, 2, \dots, \varepsilon = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \right].$$

$$18. \int_{-\infty}^{\infty} \frac{x^2 e^{mx}}{(e^x + z)^n} dx = z^{m-n} \frac{(m-1)! (n-m-1)!}{(n-1)!} \times$$

$$\times \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right)^2 - \sum_{k=1}^{m-1} \frac{1}{k^2} - \sum_{k=1}^{n-m-1} \frac{1}{k^2} + \frac{\pi^2}{3} \right]$$

$$[|\arg z| < \pi, m=1, 2, 3, \dots, n-1, n=2, 3, 4, \dots].$$

$$19. \int_{-\infty}^{\infty} \frac{x^2 e^{mx}}{(e^x + z)^{n+1/2}} dx = 2^m z^{m-n-1/2} \frac{(2n-2m-1)!! (m-1)!}{(2n-1)!!} \times$$

$$\times \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right)^2 - \sum_{k=1}^{m-1} \frac{1}{k^2} - 4 \sum_{k=1}^{n-m} \frac{1}{(2k-1)^2} + \frac{2}{3} \pi^2 \right]$$

$$[|\arg z| < \pi, m=1, 2, \dots, n, n=1, 2, 3, \dots].$$

$$\begin{aligned}
 20. \quad \int_{-\infty}^{\infty} \frac{x^2 e^{(m+1/2)x}}{(e^x + z)^{n+1/2}} dx &= \left(\frac{z}{2}\right)^{m-n} \frac{(2m-1)!! (n-m-1)!}{(2n-1)!!} \times \\
 &\times \left[\left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right)^2 - \right. \\
 &\quad \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{n-m-1} \frac{1}{k^2} + \frac{2}{3} \pi^2 \right] \\
 &[|\arg z| < \pi; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int_{-\infty}^{\infty} \frac{x^3 e^{mx}}{(e^x + z)^n} dx &= z^{m-n} \frac{(m-1)! (n-m-1)!}{(n-1)!} \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right)^3 + \right. \\
 &+ 3 \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right) \left(\frac{\pi^2}{3} - \sum_{k=1}^{m-1} \frac{1}{k^2} - \sum_{k=1}^{n-m-1} \frac{1}{k^2} \right) + \\
 &+ 2 \left(\sum_{k=1}^{m-1} \frac{1}{k^3} - \sum_{k=1}^{n-m-1} \frac{1}{k^3} \right) \Big] \\
 &[|\arg z| < \pi; m=1, 2, \dots, n-1; n=2, 3, 4, \dots].
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \int_{-\infty}^{\infty} \frac{x^3 e^{mx}}{(e^x + z)^{n+1/2}} dx &= 2^m z^{m-n-1/2} \frac{(2n-2m-1)!! (m-1)!}{(2n-1)!!} \times \\
 &\times \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right)^3 + \right. \\
 &+ 3 \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right) \left(\frac{2}{3} \pi^2 - \sum_{k=1}^{m-1} \frac{1}{k^2} - 4 \sum_{k=1}^{n-m} \frac{1}{(2k-1)^2} \right) + \\
 &+ 2 \left(\sum_{k=1}^{m-1} \frac{1}{k^3} - 8 \sum_{k=1}^{n-m} \frac{1}{(2k-1)^3} + 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} \right) \Big] \\
 &[|\arg z| < \pi; m=1, 2, \dots, n; n=1, 2, 3, \dots].
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \int_{-\infty}^{\infty} \frac{x^3 e^{(m+1/2)x}}{(e^x + z)^{n+1/2}} dx &= \\
 &= \left(\frac{z}{2}\right)^{m-n} \frac{(2m-1)!! (n-m-1)!}{(2n-1)!!} \left[\left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right)^3 + \right. \\
 &\quad \left. + 3 \left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right) \times \right.
 \end{aligned}$$

$$\times \left(\frac{2}{3} \pi^2 - 4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{n-m-1} \frac{1}{k^2} \right) +$$

$$+ 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{n-m-1} \frac{1}{k^3} - 16 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \right) \Bigg]$$

[$|\arg z| < \pi$; $m=0, 1, 2, \dots, n-1$; $n=1, 2, 3, \dots$].

2.3.13. Интегралы от $x^\alpha (e^x - z)^\lambda e^{-\rho x}$.

$$1. \int_0^{\ln 2} \frac{x e^x}{(e^x - 1)^\rho} dx = \frac{1}{1-\rho} \left[\ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2-\rho+k} \right] \quad [\operatorname{Re} \rho < 2, \rho \neq 1].$$

$$2. \int_0^a \frac{x e^{-\rho x}}{e^x - 1} dx = \sum_{k=1}^{\infty} \frac{(1+\rho)_k}{k k!} (1-e^a)^k [\psi(1+\rho) - \psi(1+\rho+k)]$$

[$0 < a < \ln 2$].

$$3. = - (e^a - 1)^{-\rho} \sum_{k=0}^{\infty} \frac{(1+\rho)_k}{1+\rho+k} \frac{(1-e^a)^{-k}}{k!} [\psi(1+\rho) +$$

$$+ \frac{1}{1+\rho+k} - \psi(1+\rho+k) + \ln(e^a - 1)] + \psi'(1+\rho) \quad [a > \ln 2].$$

$$4. \int_0^{\ln 2} \frac{x e^x}{e^x - 1} dx = \frac{\pi^2}{12}.$$

$$5. \int_a^{\infty} \frac{x e^x}{e^{2x} - 1} dx = I(a),$$

$$I[\ln(\sqrt{2}-1)] = \frac{\pi^2}{16} - \frac{1}{4} \ln^2(\sqrt{2}-1),$$

$$I\left(\ln \frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{24} - \frac{1}{12} \ln^2(\sqrt{5}-2).$$

$$6. \int_0^{\infty} \frac{x^n e^{-\rho x}}{(e^x - 1)^\rho} dx = (-1)^n \frac{\partial^n}{\partial \rho^n} B(1-\rho, \rho+n) =$$

$$= (-1)^{n-1} \frac{\partial^{n-1}}{\partial \rho^{n-1}} \{B(1-\rho, \rho+n) [\psi(1+\rho) - \psi(\rho+n)]\}$$

[$\operatorname{Re}(\rho+n) > 0$; $\operatorname{Re} \rho < n+1$; $n=1, 2, 3, \dots$].

$$7. \int_0^{\infty} \frac{x e^{-mx}}{(e^x - 1)^{1/2-n}} dx = \frac{\pi}{2^{m-1} m!} (2n-1)!! (2m-2n-1)!! \times$$

$$\times \left(\ln 2 - \sum_{k=1}^{m-n} \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^m \frac{1}{k} \right) \quad [n=0, 1, 2, \dots, m].$$

$$8. \int_0^{\infty} \frac{x e^{-(m-1/2)x}}{(e^x - 1)^{1/2-n}} dx = - \frac{(2n-1)!! (m-n-1)!}{(2m-1)!! 2^{n-m-1}} \times$$

$$\times \left(\ln 2 - \sum_{k=1}^m \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^{m-n-1} \frac{1}{k} \right)$$

[$n=0, 1, 2, \dots, m-1$; $m=1, 2, 3, \dots$].

$$9. \int_0^{\infty} \frac{x e^x}{(e^x - 1)^{3/2}} dx = 2\pi.$$

$$10. \int_0^{\infty} \frac{x e^x}{(e^x - 1)^{(4+k)/3}} dx = 2^{1-k} \sqrt{3} \pi$$

[$k=0$ или 1].

$$11. \int_0^{\infty} \frac{x e^{-mx}}{(e^x - 1)^{(4+\varepsilon)/3-n}} dx = \frac{3}{2(m!)} \Gamma\left(n - \frac{1+\varepsilon}{3}\right) \Gamma\left(m-n + \frac{4+\varepsilon}{3}\right) \times$$

$$\times \left[\ln 3 + \frac{(-1)^\varepsilon \pi}{3\sqrt{3}} - 2 \sum_{k=1}^{m-n+1} \frac{1}{3k-2+\varepsilon} + \frac{2}{3} \sum_{k=1}^m \frac{1}{k} \right]$$

[$n=0, 1, 2, \dots, m+1$; $\varepsilon=0$ или 1].

$$12. \int_0^{\infty} \frac{x e^{-(m-1/3-\varepsilon/3)x}}{(e^x - 1)^{(4+\varepsilon)/3-n}} dx = - \frac{3}{2} \Gamma\left[n - 1/3 - \varepsilon/3\right] (m-n)! \times$$

$$\times \left[\ln 3 - \frac{(-1)^\varepsilon \pi}{3\sqrt{3}} + \frac{2}{3} \sum_{k=1}^{m-n} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{3k-1-\varepsilon} \right]$$

[$n=0, 1, 2, \dots, m$; $\varepsilon=0$ или 1].

$$13. \int_0^{\infty} \frac{x e^{-(m-2/3+\varepsilon/3)x}}{(e^x - 1)^{(4+\varepsilon)/3-n}} dx = - B\left(n - \frac{1+\varepsilon}{3}, m-n + \frac{2+2\varepsilon}{3}\right) \times$$

$$\times \left[\frac{(-1)^\varepsilon \pi}{\sqrt{3}} + 3(-1)^\varepsilon \sum_{k=1}^{m-n+\varepsilon} \frac{1}{3k-1-\varepsilon} - 3 \sum_{k=1}^m \frac{1}{3k-2+\varepsilon} \right]$$

[$n=0, 1, 2, \dots, m+\varepsilon$; $\varepsilon=0$ или 1].

$$14. \int_0^{\infty} \frac{x^2 e^{-mx}}{(e^x - 1)^{1/2-n}} dx = \frac{\pi (2n-1)!! (2m-2n-1)!!}{2^m m!} \times$$

$$\times \left[\frac{\pi^2}{3} + \sum_{k=1}^m \frac{1}{k^2} - 4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^2} + \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right]$$

[$n=0, 1, 2, \dots, m$].

$$15. \int_0^{\infty} \frac{x^2 e^{-(m-1/2)x}}{(e^x - 1)^{1/2-n}} dx = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[\frac{\pi^2}{3} + \sum_{k=1}^{m-n-1} \frac{1}{k^2} - \right. \\ \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} \frac{x^2 e^{(m+1)x}}{(e^x - 1)^{1/2+n}} dx = \\ = (-1)^{m+n-1} 2^{2+m} \pi m! \frac{(2n-2m-3)!!}{(2n-1)!!} \left[2 \ln 2 + \sum_{k=1}^m \frac{1}{k} - 2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} \right] \\ [m=0, n-1; n=1, 2].$$

$$17. \int_0^{\infty} \frac{x^2 e^{(m+1)x}}{(e^x - 1)^{(1+\varepsilon)/3+n}} dx = 2 (-1)^{m-1} m! \Gamma\left(\frac{2-\varepsilon}{3} - n\right) \Gamma\left(n-m-\frac{2-\varepsilon}{3}\right) \times \\ \times \left[\frac{3}{2} \ln 3 + \frac{(-1)^\varepsilon \pi}{2\sqrt{3}} - 3 \sum_{k=1}^{n-m-1} \frac{1}{3k-2+\varepsilon} + \sum_{k=1}^m \frac{1}{k} \right] \\ [m=0, n-1; n=1, 2; \varepsilon=0 \text{ или } 1].$$

$$18. \int_0^{\infty} \frac{x^3 e^{-mx}}{(e^x - 1)^{1/2-n}} dx = \frac{\pi (2n-1)!! (2m-2n-1)!!}{m! 2^m} \times \\ \times \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^3} - \sum_{k=1}^m \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^2} - \sum_{k=1}^m \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right) - \right. \\ \left. - \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^3 \right] [n=0, 1, 2, \dots, m].$$

$$19. \int_0^{\infty} \frac{x^3 e^{-(m-1/2)x}}{(e^x - 1)^{1/2-n}} dx = \\ = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{m-n-1} \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{m-n-1} \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \ln 2 \right) + \right. \\ \left. + \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^3 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$20. \int_0^{\infty} \frac{x^3 e^{(m+1)x}}{(e^x - 1)^{1/2+n}} dx = \frac{(-1)^{m+n+1} 3\pi m! 2^{m+1}}{(2n-1)!!} (2n-2m-3)!! \times$$

$$\times \left[\frac{\pi^2}{3} - 4 \sum_{k=1}^{n-m-1} \frac{1}{(2k-1)^2} + \sum_{k=1}^m \frac{1}{k^2} + \left(2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right]$$

[$m=0, 1, 2, n-1; n=1, 2, 3$].

$$21. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{e^{qx} - z} dx = \Gamma(\alpha) q^{-\alpha} \Phi\left(z, \alpha, \frac{p}{q} + 1\right)$$

[$\operatorname{Re}(p/q) > -1; |\arg(1-z)| < \pi, \operatorname{Re} \alpha > 0$] или
[$|z| \leq 1, z \neq 1, \operatorname{Re} \alpha > 0$] или [$z=1, \operatorname{Re} \alpha > 1$].

$$22. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{e^{qx} - 1} dx = \Gamma(\alpha) q^{-\alpha} \zeta\left(\alpha, \frac{p}{q} + 1\right)$$

[$\operatorname{Re} \alpha > 1; \operatorname{Re}(p/q) > -1$].

$$23. \int_0^{\infty} \frac{x^n e^{-px}}{e^{qx} - 1} dx = (-q)^{-n-1} \psi^{(n)}\left(\frac{p}{q} + 1\right)$$

[$\operatorname{Re}(p/q) > -1; n=1, 2, 3, \dots$].

$$24. \int_0^{\infty} \frac{x^{2n-1} e^{qx/2}}{e^{qx} - 1} dx = \frac{2^{2n}-1}{4n} \left(\frac{2\pi}{q}\right)^{2n} |B_{2n}|$$

[$q \neq 0; n=1, 2, 3, \dots$].

$$25. \int_0^{\infty} \frac{x e^{-mx}}{e^x - 1} dx = \frac{\pi^2}{6} - \sum_{k=1}^m \frac{1}{k^2}.$$

$$26. \int_0^{\infty} \frac{x^3 e^{-mx}}{e^x - 1} dx = \frac{\pi^4}{15} - 6 \sum_{k=1}^m \frac{1}{k^4}.$$

$$27. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(e^x - z)^2} dx = \Gamma(\alpha) [\Phi(z, \alpha-1, p+1) - (p+1) \Phi(z, \alpha, p+1)]$$

[$\operatorname{Re} \alpha > 0; \operatorname{Re} p > -2; |\arg(1-z)| < \pi$].

$$28. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(e^x - 1)^2} dx = \Gamma(\alpha) [\zeta(\alpha-1, p+1) - (p+1) \zeta(\alpha, p+1)]$$

[$\operatorname{Re} \alpha > 2; \operatorname{Re} p > -2$].

$$29. \int_0^{\infty} \frac{x^{2n} e^x}{(e^x - 1)^2} dx = 2^{2n-1} \pi^{2n} |B_{2n}|$$

[$n=1, 2, 3, \dots$].

$$30. \int_{-\infty}^{\infty} \frac{x^n e^{-px}}{(e^x - a)^m} dx = \frac{(-1)^{n+m-1} \pi}{(m-1)!} \frac{\partial^n}{\partial p^n} [(p+1)_{m-1} a^{-p-m} \operatorname{ctg} p\pi]$$

[$a > 0; -1 < \operatorname{Re} p < 0; m=1, 3, 5, \dots, n=0, 1, 2, \dots, a \neq 1$ (или $m=1, 3, 5, \dots, m < n+1, n=1, 2, 3, \dots, a=1$)].

2.3.14. Интегралы от $(ae^{qx} + be^{-qx} + c)^{-1} f(x)$.

Условие: γ — действительное число.

1. $\int_0^{\ln 2} \frac{x dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2.$
2. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^x + 1} dx = (1 - 2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha)$ [Re $\alpha > 0$].
3. $\int_0^{\infty} \frac{dx}{e^x + 1} = \ln 2.$
4. $\int_0^{\infty} \frac{x^{2n-1}}{e^x + 1} dx = (-1)^{n+1} \frac{\pi^{2n}}{2n} (2^{2n-1} - 1) B_{2n}$ [$n = 1, 2, 3, \dots$].
5. $\int_0^{\infty} \frac{x^{\alpha-1} dx}{e^x - z} = \frac{\Gamma(\alpha)}{z} \text{Li}_{\alpha}(z)$ [Re $\alpha > 1$; $|\arg(1 - z)| < \pi$].
6. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^x - 1} dx = \Gamma(\alpha) \zeta(\alpha)$ [Re $\alpha > 1$].
7. $\int_0^{\infty} \frac{x^{2n-1}}{e^x - 1} dx = (-1)^{n+1} \frac{(2\pi)^{2n}}{4n} B_{2n}$ [$n = 1, 2, 3, \dots$].
8. $\int_0^{\infty} \frac{(1+ix)^n - (1-ix)^n}{e^{\pi x} + 1} dx = \frac{i}{n+1} - \frac{2^{n+1}i}{n+1} B_{n+1}$ [$n = 1, 2, 3, \dots$].
9. $\int_0^{\infty} \frac{(1+ix)^{2n} - (1-ix)^{2n}}{e^{2\pi x} - 1} dx = \frac{i(2n-1)}{2(2n+1)}$ [$n = 1, 2, 3, \dots$].
10. $\int_0^{\infty} \frac{x dx}{(x^2 + z^2)^n (e^{qx} - 1)} =$

$$= \frac{2^n}{(n-1)!} \left\{ \left(-\frac{\partial}{z \partial z} \right)^{n-1} \left[\ln \left(\frac{qz}{2\pi} \right) - \psi \left(\frac{qz}{2\pi} \right) \right] - \frac{\pi(2n-3)!!}{qz^{2n-1}} \right\}$$
[Re $q, \text{Re } z > 0$; $n = 1, 2, 3, \dots$].
11. $\int_0^{\infty} \frac{x dx}{(x^2 + z^2)^2 (e^{qx} - 1)} = \frac{\pi^2}{q^2 z^4} \sum_{k=0}^{\infty} |B_{2k+2}| \left(\frac{2\pi}{qz} \right)^{2k}$ [Re $q, \text{Re } z > 0$].
12. $\int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{e^x + 1} \frac{dx}{x} = \ln \Gamma \left[\begin{matrix} (\mu+1)/2, & 1+\nu/2 \\ (\nu+1)/2, & 1+\mu/2 \end{matrix} \right]$ [Re $\mu, \text{Re } \nu > -1$]

$$13. \int_0^{\infty} \frac{e^{-\mu x} - e^{(\mu+1)x}}{e^x + 1} \frac{dx}{x} = \ln \left(-\operatorname{tg} \frac{\mu}{2} \right) \quad [-1 < \operatorname{Re} \mu < 0].$$

$$14. \int_0^{\infty} x \frac{e^{2x/3} - e^{x/3}}{e^x + 1} dx = \frac{2}{3} \pi^2.$$

$$15. \int_{-\infty}^{\infty} \frac{e^{\mu x} - e^{\nu x}}{e^q x + 1} \frac{dx}{x} = \ln \left(\operatorname{tg} \frac{\mu \pi}{2q} \operatorname{ctg} \frac{\nu \pi}{2q} \right) \quad [0 < \operatorname{Re} (\mu/q) < 1; 0 < \operatorname{Re} (\nu/q) < 1].$$

$$16. \int_0^{\infty} \frac{e^{px}}{e^x - 1} \prod_{i=1}^n (e^{-v_i x} - 1) \frac{dx}{x} = \sum_{k=0}^n (-1)^k \ln \Gamma (1 + p + v_{i_1} + v_{i_2} + \dots + v_{i_k})$$

$[n=1, 2, 3, \dots; \operatorname{Re} p > -1; \operatorname{Re} p > -1 - \operatorname{Re} (v_{i_1} + v_{i_2} + \dots + v_{i_k})]$; символ $\sum^*$ означает, что k -й член содержит $\binom{n}{k}$ слагаемых, которые отличаются наборами индексов $i_1, i_2, \dots, i_k$ взятых из множества $i=1, 2, \dots, n$.

$$17. \int_0^{\infty} \frac{e^{\mu x} + e^{(1-\mu)x} - 2e^{x/2}}{e^x - 1} \frac{dx}{x} = -\ln (\sin \mu \pi) \quad [0 < \operatorname{Re} \mu < 1].$$

$$18. \int_0^{\infty} x^n \frac{e^{-mx} - 1}{e^x - 1} dx = -n! \sum_{k=1}^m \frac{1}{k^{n+1}}.$$

$$19. \int_0^{\infty} x \frac{e^{\mu x} + e^{(1-\mu)x}}{e^x - 1} dx = \frac{\pi^2}{\sin^2 \mu \pi} \quad [0 < \operatorname{Re} \mu < 1].$$

$$20. \int_0^{\infty} x^{2n} \frac{e^{3x/2} + e^{x/2}}{(e^x - 1)^2} dx = (2^{2n} - 1)(2\pi)^{2n} |B_{2n}| \quad [n=1, 2, 3, \dots].$$

$$21. \int_0^{\infty} x^2 \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} dx = \frac{2\pi^2}{3} - 2.$$

$$22. \int_0^{\infty} \frac{x e^x}{a^2 e^x - (a^2 \pm b^2) \sqrt{e^x - 1}} dx = \frac{2\pi}{ab} \left\{ \operatorname{arctg} (b/a) \right\} \quad [a, b > 0].$$

$$23. \int_0^{\infty} \frac{a^x - b^x}{c^x - d^x} dx = \frac{1}{\ln (c/d)} \left\{ \Psi \left[\frac{\ln (c/b)}{\ln (c/d)} \right] - \Psi \left[\frac{\ln (c/a)}{\ln (c/d)} \right] \right\} \quad [a, b, c, d > 0; \max (a, b) < \max (c, d)].$$

$$24. \int_0^{\infty} \frac{x dx}{e^x + e^{-x} - 1} = 1,1719536194 \dots$$

$$25. \int_0^{\infty} \frac{x e^{-x} dx}{e^x + e^{-x} - 1} = 0,3118211319 \dots$$

$$26. \int_{-\infty}^{\infty} \frac{e^{\mu x} - e^{\nu x}}{e^{qx} - 1} \frac{dx}{x} = \ln \frac{\sin [\nu \pi / (2q)]}{\sin [\mu \pi / (2q)]} \quad [0 < \operatorname{Re} (\nu/q) < 1; 0 < \operatorname{Re} (\mu/q) < 1].$$

$$27. \int_{-\infty}^{\infty} \frac{x^{2n} dx}{e^x + e^{-x} + 2 \cos \gamma} = \frac{(-1)^n (2n)! (2\pi)^{2n+1}}{(2n+1)! \sin \gamma} B_{2n+1} \left(\frac{\gamma + \pi}{2\pi} \right) \quad [|\gamma| < \pi]$$

$$28. \int_{-\infty}^{\infty} \frac{x^2 dx}{e^x + e^{-x} + 2 \cos \gamma} = \frac{\gamma (\pi^2 - \gamma^2)}{3 \sin \gamma} \quad [|\gamma| < \pi].$$

$$29. \int_{-\infty}^{\infty} \frac{dx}{a^2 e^x + e^{-x} + 2a \cos \gamma} = \frac{\gamma}{a \sin \gamma} \quad [|\gamma| < \pi; a > 0].$$

$$30. \int_{-\infty}^{\infty} \frac{x dx}{a^2 e^x + e^{-x} + 2a \cos \gamma} = -\frac{\gamma \ln a}{a \sin \gamma} \quad [|\gamma| < \pi; a > 0].$$

2.3.15. Интегралы от $A(x) e^{-px^2 - qx}$.

$$1. \int_a^{\infty} (x-a)^{\beta-1} e^{-px^2 - qx} dx = \Gamma(\beta) (2p)^{-\beta/2} \exp \left[\frac{q^2}{8p} - \frac{a}{2} (q + ap) \right] D_{-\beta} \left(\frac{2ap + q}{\sqrt{2p}} \right)$$

$[\operatorname{Re} \beta, \operatorname{Re} p > 0]$ или $[\operatorname{Re} \beta, \operatorname{Re} q > 0, \operatorname{Re} p = 0]$ или $[\operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0, 0 < \operatorname{Re} \beta < 2]$.

$$2. \int_0^{\infty} e^{-i\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} e^{-\pi i/4} \quad [\lambda > 0].$$

$$3. \int_0^{\infty} x^{\alpha-1} e^{-px^2 - qx} dx = \Gamma(\alpha) (2p)^{-\alpha/2} \exp [q^2 / (8p)] D_{-\alpha}(q/\sqrt{2p})$$

$[\operatorname{Re} \alpha, \operatorname{Re} p > 0]$ или $[\operatorname{Re} \alpha, \operatorname{Re} q > 0, \operatorname{Re} p = 0]$
или $[0 < \operatorname{Re} \alpha < 2, \operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0]$.

$$4. \int_0^{\infty} e^{-px^2 - qx} dx = \frac{1}{2} \sqrt{\frac{\pi}{p}} \exp \left(\frac{q^2}{4p} \right) \operatorname{erfc} \left(\frac{q}{2\sqrt{p}} \right)$$

$[\operatorname{Re} p > 0]$ или $[\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$

$$5. \int_0^{\infty} \frac{e^{-px^2 - qx}}{\sqrt{x}} dx = \frac{1}{2} \sqrt{\frac{q}{p}} \exp \left(\frac{q^2}{8p} \right) K_{1/4} \left(\frac{q^2}{8p} \right)$$

$[\operatorname{Re} p > 0]$ или $[\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$.

$$6. \int_0^{\infty} \sqrt{x} e^{-px^2 - qx} dx = \frac{1}{8} \left(\frac{q}{p} \right)^{3/2} \exp \left(\frac{q^2}{8p} \right) \left\{ K_{3/4} \left(\frac{q^2}{8p} \right) - K_{1/4} \left(\frac{q^2}{8p} \right) \right\}$$

$[\operatorname{Re} p > 0]$ или $[\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$.

$$7. \int_0^{\infty} x^n e^{-px^2 - qx} dx = \frac{(-1)^n}{2} \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \left[\exp\left(\frac{q^2}{4p}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right) \right]$$

[Re $p > 0$] или [Re $p = 0$, Re $q > 0$].

$$8. \int_{-\infty}^{\infty} x^n e^{-px^2 - qx} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \exp\left(\frac{q^2}{4p}\right) =$$

$$9. = n! \sqrt{\frac{\pi}{p}} \left(-\frac{q}{2p}\right)^n \exp\left(\frac{q^2}{4p}\right) \sum_{k=0}^{[n/2]} \frac{p^k q^{2k}}{k! (n-2k)!} =$$

$$10. = \left(\frac{i}{2}\right)^n \sqrt{\pi} p^{-(n+1)/2} \exp\left(\frac{q^2}{4p}\right) H_n\left(\frac{iq}{2\sqrt{p}}\right) \quad [\operatorname{Re} p > 0].$$

$$11. \int_{-\infty}^{\infty} e^{-px^2 - qx} dx = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right) \quad [\operatorname{Re} p > 0].$$

$$12. \int_{-\infty}^{\infty} (ax^2 + 2bx + c) e^{-px^2 - qx} dx =$$

$$= \frac{\sqrt{\pi}}{4} p^{-5/2} [4cp^2 - 4pqb + a(q^2 + 2p)] \exp\left(\frac{q^2}{4p}\right) \quad [\operatorname{Re} p > 0].$$

$$13. \int_{-\infty}^{\infty} e^{i(\lambda x^2 + \mu x)} dx = \sqrt{\frac{\pi}{\lambda}} \exp\left[\frac{i(\lambda\pi - \mu^2)}{4\lambda}\right] \quad [\lambda > 0].$$

2.3.16. Интегралы от $A(x) e^{-px - q/x}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px - q/x} dx = 2 \left(\frac{q}{p}\right)^{\alpha/2} K_{\alpha}(2\sqrt{pq}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$2. \int_0^{\infty} x^{n-1/2} e^{-px - q/x} dx = (-1)^n \sqrt{\pi} \frac{\partial^n}{\partial p^n} (p^{-1/2} e^{-2\sqrt{pq}}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$3. \int_0^{\infty} x^{-n-1/2} e^{-px - q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}} \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$4. \int_0^{\infty} \frac{1}{\sqrt{x(x+z)}} e^{-px - q/x} dx = \frac{\pi}{\sqrt{z}} e^{pz + q/z} \operatorname{erfc}\left(\sqrt{\frac{q}{z}} + \sqrt{pz}\right)$$

[Re $p, \operatorname{Re} q \geq 0$; $|\arg z| < \pi$].

$$5. \int_{-\infty}^{\infty} e^{-p|x - q/x|^v} dx = \frac{2}{v} p^{-1/v} \Gamma\left(\frac{1}{v}\right) \quad [v, q, \operatorname{Re} p > 0].$$

$$6. \int_0^{\infty} e^{i(\lambda x - \mu/x)} \frac{dx}{x} = 2K_0(2\sqrt{\lambda\mu}) \quad [\mu, \lambda > 0].$$

$$7. \int_0^{\infty} e^{i(\lambda x - \mu/x)} \frac{dx}{\sqrt{x}} = (i+1) \sqrt{\frac{\pi}{2\lambda}} e^{-2\sqrt{\lambda\mu}} \quad [\lambda > 0, \mu \geq 0].$$

$$8. \int_{\gamma-i\infty}^{\gamma+i\infty} x^{\alpha-1} e^{px-q/x} dx = 2\pi i \left(\frac{q}{p}\right)^{\alpha/2} J_{\alpha}(2\sqrt{pq}) \quad [p, q, \gamma > 0; \operatorname{Re} \alpha < 1].$$

2.3.17. Интегралы от $A(x) e^{-b\sqrt{x^2 \pm a^2} - px}$.

$$1. \int_a^{\infty} \frac{1}{\sqrt{x-a}} \exp(-b\sqrt{x^2-a^2} - px) dx = \\ = b \sqrt{\frac{2a}{p^2-b^2}} \left[\frac{1}{z_-} \exp(-a\sqrt{p^2-b^2}) \operatorname{erfc}(z_-) - \frac{1}{z_+} \exp(a\sqrt{p^2-b^2}) \operatorname{erfc}(z_+) \right] \\ [z_{\pm} = \sqrt{a(p \pm \sqrt{p^2-b^2})}; \operatorname{Re} b, \operatorname{Re}(p+b) > 0].$$

$$2. \int_a^{\infty} \frac{1}{\sqrt[4]{(x^2-a^2)^3}} \exp(-b\sqrt{x^2-a^2} - px) dx = \sqrt{\frac{b}{\pi}} K_{1/4}\left(\frac{z_+^2}{2}\right) K_{1/4}\left(\frac{z_-^2}{2}\right) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0].$$

$$3. \int_a^{\infty} \frac{1}{\sqrt{x-a}(x+a)} \exp(-b\sqrt{x^2-a^2} - px) dx = 2 \sqrt{\frac{2}{a}} e^{pa} \operatorname{erfc}(z_-) \operatorname{erfc}(z_+) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0].$$

$$4. \int_a^{\infty} \frac{(x+a)^{2\beta}}{(x^2-a^2)^{\beta+3/4}} \exp(-b\sqrt{x^2-a^2} - px) dx = \\ = \sqrt{\frac{2}{a}} \Gamma\left(\frac{1}{2} - 2\beta\right) D_{2\beta-1/2}(\sqrt{2} z_+) D_{2\beta-1/2}(\sqrt{2} z_-) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}), \operatorname{Re} b, \operatorname{Re}(p+b) > 0; \operatorname{Re} \beta < 1/4].$$

$$5. \int_a^{\infty} \frac{[(x+\sqrt{x^2-a^2})^{\nu} \exp(b\sqrt{x^2-a^2}) + (x-\sqrt{x^2-a^2})^{\nu} \exp(-b\sqrt{x^2-a^2})]}{\sqrt{x^2-a^2}} \times \\ \times e^{-px} dx = 2a^{\nu} (p+b)^{\nu/2} (p-b)^{-\nu/2} K_{\nu}(a\sqrt{p^2-b^2}) \quad [\operatorname{Re}(p \pm b) > 0].$$

$$6. \int_0^{\infty} \frac{1}{\sqrt{x}\sqrt{x^2+a^2}} \exp(-b\sqrt{x^2+a^2} - px) dx = \sqrt{\frac{p}{\pi}} K_{1/4}(z_+^2/2) K_{1/4}(z_-^2/2) \\ [z_{\pm}^2 = a(b \pm \sqrt{b^2-p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p+b) > 0].$$

$$7. \int_0^{\infty} \frac{(a + \sqrt{x^2 + a^2})^{1/2}}{\sqrt{x^2 + a^2}} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \frac{p \sqrt{2a}}{\sqrt{b^2 - p^2}} \left\{ \frac{1}{z_-} \exp(-a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_-) - \frac{1}{z_+} \exp(a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_+) \right\}$$

$$[z_{\pm}^2 = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p + b) > 0].$$

$$8. \int_0^{\infty} \frac{\exp(-b \sqrt{x^2 + a^2} - px)}{\sqrt{x^2 + a^2} (a + \sqrt{x^2 + a^2})^{1/2}} dx = 2 \sqrt{\frac{2}{a}} e^{ab} \operatorname{erfc}(z_+) \operatorname{erfc}(z_-)$$

$$[z_{\pm}^2 = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p + b) > 0].$$

$$9. \int_0^{\infty} \frac{x^{-2\beta-1/2}}{\sqrt{x^2 + a^2}} (a + \sqrt{x^2 + a^2})^{2\beta} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \sqrt{\frac{2}{a}} \Gamma\left(\frac{1}{2} - 2\beta\right) D_{2\beta-1/2}(\sqrt{2} z_+) D_{2\beta-1/2}(\sqrt{2} z_-)$$

$$[z_{\pm}^2 = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p + b) > 0; \operatorname{Re} \beta < 1/4].$$

$$10. \int_0^{\infty} [(\sqrt{x^2 + a^2} + x)^{\nu} e^{bx} + (\sqrt{x^2 + a^2} - x)^{\nu} e^{-bx}] e^{-c \sqrt{x^2 + a^2}} \frac{dx}{\sqrt{x^2 + a^2}} =$$

$$= 2a^{\nu} (c + b)^{\nu/2} (c - b)^{-\nu/2} K_{\nu}(a \sqrt{c^2 - b^2}) \quad [\operatorname{Re} a, \operatorname{Re}(c \pm b) > 0]$$

2.3.18. Интегралы от $e^{g(x)} f(x)$.

$$1. \int_0^1 x^{-px} dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k}.$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px^{\mu}} dx = \frac{1}{\mu} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} e^{-p x^2} dx = -E_1(-p) \quad [\operatorname{Re} p > 0].$$

$$4. \int_{-\infty}^{\infty} (a + b|x|) e^{-|x| + i\lambda x} dx = 2 \frac{\lambda^2(a - b) + a + b}{(\lambda^2 + 1)^2} \quad [\lambda > 0].$$

$$5. \int_{-\infty}^{\infty} x e^{x - p e^{\mu x}} dx = \mu^{-2} \Gamma\left(\frac{1}{\mu}\right) p^{-1/\mu} \left[\psi\left(\frac{1}{\mu}\right) - \ln p \right] \quad [\mu, \operatorname{Re} p > 0].$$

$$6. \int_{-\infty}^{\infty} x e^{x - e^x} dx = C.$$

2.3.19. Интегралы, содержащие разности алгебраических и показательной функций.

$$1. \int_0^1 \left(\frac{re^{1-x-r}}{1-x^r} - \frac{x^p}{1-x} \right) \frac{dx}{x} = \psi(p) - \ln r \quad [r, \operatorname{Re} p > 0].$$

$$2. \int_0^1 (e^{1-1/x} - x^p) \frac{dx}{x(1-x)} = \psi(p) \quad [\operatorname{Re} p > 0].$$

$$3. \int_0^1 \left(\frac{re^{1-x-r}}{1-x^r} - \frac{e^{1-x^{-1}}}{1-x} \right) \frac{dx}{x} = -\ln r \quad [r > 0]$$

$$4. \int_0^1 \frac{1 - e^{-x} - e^{-1/x}}{x} dx = C.$$

$$5. \int_0^\infty x^{\alpha-1} \left[e^{-px} - \sum_{k=0}^{n-1} \frac{(-px)^k}{k!} \right] dx = \Gamma(\alpha) p^{-\alpha} \quad [\operatorname{Re} p \geq 0, -n < \operatorname{Re} \alpha < 1-n].$$

$$6. \int_0^\infty x^{-1} [e^{-x} - (x+1)^{-p}] dx = \psi(p) \quad [\operatorname{Re} p > 0].$$

$$7. \int_0^\infty \left(e^{-px} - \frac{1}{1+qx} \right) \frac{dx}{x} = \ln \frac{q}{p} - C \quad [\operatorname{Re} p > 0, |\arg q| < \pi].$$

$$8. \int_0^\infty \frac{e^{-px} - 1}{x(x+z)} dx = \frac{1}{z} [e^{pz} E_1(-pz) - \ln(pz)] \quad [\operatorname{Re} p > 0, |\arg z| < \pi].$$

$$9. \int_0^\infty \left(e^{-px} - \frac{1}{1+q^\mu x^\mu} \right) \frac{dx}{x} = \ln \frac{q}{p} - C \quad [\mu, \operatorname{Re} p > 0, \mu |\arg q| < \pi].$$

$$10. \int_0^\infty \left[\frac{e^{-px} - 1}{px} + \frac{1}{(1+q^\mu x^\mu)^\sigma} \right] \frac{dx}{x} = \ln \frac{p}{q} + C \left(1 - \frac{1}{\mu} \right) - \frac{\psi(\sigma)}{\mu} - 1, \\ [\mu, \operatorname{Re} \sigma, \operatorname{Re} p > 0, \mu |\arg q| < \pi].$$

$$11. \int_0^\infty \left(\frac{q}{e^{qx} - 1} - \frac{1}{xe^{qx}} \right) dx = C.$$

$$12. \int_0^\infty \left(\frac{e^{-2x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi.$$

$$13. \int_0^\infty \left(\frac{e^{-x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = \frac{1}{2} \ln \frac{\pi}{4}.$$

$$14. \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-px}}{x} dx = \ln \Gamma(p) - \left(p - \frac{1}{2} \right) \ln p + p - \frac{1}{2} \ln(2\pi)$$

[Re $p > 0$].

$$15. \int_0^{\infty} \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{e^{-x} - 1} \right) \frac{e^{-px}}{x} dx = \left(p - \frac{1}{2} \right) \ln p + \frac{1}{2} \ln(2\pi) - p - \ln \Gamma(p)$$

[Re $p > 0$].

$$16. \int_0^{\infty} \left(\frac{p^2 e^{-x}}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right) \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3p^2}{4}$$

[Re $p > 0$].

$$17. \int_0^{\infty} \frac{(e^{-px} - e^{-qx})^2}{x^2} dx = 2p \ln \frac{2p}{p+q} + 2q \ln \frac{2q}{p+q}$$

[Re $p, \text{Re } q > 0$].

$$18. \int_0^{\infty} \left(\frac{e^{-x}}{x} - \frac{e^{-px}}{1 - e^{-x}} \right) dx = \psi(p)$$

[Re $p > 0$].

$$19. \int_0^{\infty} [(p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx}] \frac{dx}{x^2} =$$

$$= (r-q)p \ln p + (p-r)q \ln q + (q-p)r \ln r$$

[Re $p, \text{Re } q, \text{Re } r > 0$].

$$20. \int_0^{\infty} \left[(p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right] \frac{dx}{x} =$$

$$= p \ln p - q \ln q - (p-q) \left(1 + \ln \frac{r}{m} \right)$$

[$m, \text{Re } p, \text{Re } q, \text{Re } r > 0$].

$$21. \int_0^{\infty} \left(\frac{e^{-qx}}{e^{-x} - 1} + \frac{e^{-rx}}{x} \right) e^{-px} dx = \psi(q+p) - \ln(r+p)$$

[Re $(q+p), \text{Re } (r+p) > 0$].

$$22. \int_0^{\infty} \left(qe^{-x} - \frac{1 - e^{-qx}}{e^x - 1} e^{-rx} \right) \frac{dx}{x} = \ln \Gamma \left[\frac{q+r+1}{r+1} \right]$$

[Re $r, \text{Re } (q+r) > -1$].

$$23. \int_0^{\infty} \left[1 - e^{-x} - \frac{(1 - e^{-qx})(1 - e^{-px})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(p, q)$$

[Re $p, \text{Re } q > 0$].

$$24. \int_0^{\infty} \left(\frac{a + be^{-qx}}{ce^{lx} + g + he^{-qx}} - \frac{a + be^{-px}}{ce^{px} + g + he^{-px}} \right) \frac{dx}{x} = \frac{a+b}{c+g+h} \ln \frac{q}{p}$$

[Re $p, \text{Re } q > 0$].

$$25. \int_0^{\infty} \left(\frac{e^{-pnx} - e^{-qnx}}{n} - \frac{e^{-pmx} - e^{-qmx}}{m} \right) \frac{dx}{x^2} = (q-p) \ln \frac{m}{n}$$

[$m, n, \operatorname{Re} p, \operatorname{Re} q > 0$].

$$26. \int_0^{\infty} e^{-px} \prod_{i=1}^n (e^{-v_i x} - 1) \frac{dx}{x^m} = \frac{1}{(n-1)!} \frac{\partial^{n-m}}{\partial p^{n-m}} \times$$

$$\times \sum_{k=0}^n (-1)^{k+m} (p + v_{i_1} + v_{i_2} + \dots + v_{i_k})^{n-1} \ln (p + v_{i_1} + v_{i_2} + \dots + v_{i_k})$$

[$0 \leq m \leq n$; $\operatorname{Re} p > 0$; $\operatorname{Re} p > -\operatorname{Re}(v_{i_1} + v_{i_2} + \dots + v_{i_k})$; символ $\sum^*$ означает, что k -й член суммы содержит $\binom{n}{k}$ слагаемых, которые отличаются наборами индексов $i_1, i_2, \dots, i_k$, взятых из множества $i = 1, 2, \dots, n$].

$$27. \int_0^{\infty} \left[e^{-pxq} - \frac{y^{\sigma}}{(x^r + y^r)^{\sigma}} \right] \frac{dx}{x} = \left(\frac{1}{r} - \frac{1}{q} \right) C - \ln y - \frac{\ln p}{q} + \frac{\psi(\sigma)}{r}$$

[$r, q, \operatorname{Re} p, \operatorname{Re} \sigma > 0, r | \arg y| < \pi$].

$$28. \int_0^{\infty} (e^{-pxq} - e^{-nxr}) \frac{dx}{x} = \frac{\ln h + C}{r} - \frac{\ln p + C}{q}$$

[$r, q, \operatorname{Re} h, \operatorname{Re} p > 0$].

$$29. \int_{-\infty}^{\infty} \frac{\alpha^x}{(a\beta^x + b\gamma^x)^{\rho}} dx = \frac{a^{-\lambda} b^{-\mu}}{\ln(\gamma/\beta)} \Gamma \left[\begin{matrix} \mu, \lambda \\ \rho \end{matrix} \right]$$

$$\left[\mu = \frac{\ln(\alpha/\beta^{\rho})}{\ln(\gamma/\beta)}, \lambda = \frac{\ln(\gamma^{\rho}/\alpha)}{\ln(\gamma/\beta)}; a, b, \rho > 0, 0 < \beta < \alpha^{1/\rho} < \gamma \right].$$

2.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

2.4.1. Введение.

Этот раздел содержит интегралы, в подынтегральные выражения которых входят гиперболические функции. Для их вычисления можно использовать следующие свойства. Интегралы, содержащие гиперболические функции, могут быть преобразованы к интегралам от показательной или тригонометрических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\operatorname{sh}^2 bx$ и $\operatorname{ch}^2 bx$, сводятся к интегралам от функций $\operatorname{sh} b k x$ и $\operatorname{ch} b k x$, если использовать формулы для степеней гиперболических функций (см. приложение I). В ряде случаев интегралы от гиперболических функций с помощью замены переменной сводятся к интегралам от алгебраических и других функций. Например,

$$\int_0^a \varphi(x) \operatorname{ch}^2 bx dx = \frac{1}{2} \int_0^a \varphi(x) dx + \frac{1}{4} \int_0^a \varphi(x) e^{2bx} dx + \frac{1}{4} \int_0^a \varphi(x) e^{-2bx} dx =$$

$$= \int_0^a \varphi(x) \cos^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 + \operatorname{ch} 2ix) dx =$$

$$= \frac{1}{b} \int_1^{\operatorname{ch} ab} x^2 \varphi \left(\frac{\ln(x + \sqrt{x^2 - 1})}{b} \right) \frac{dx}{\sqrt{x^2 - 1}} \quad [a, b > 0].$$

В разделе «Гиперболические функции» a, b, c — комплексные числа.

2.4.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие гиперболические, а также степенную, алгебраические и показательную функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций $V_r^\pm$, V_j , $j = 1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

В п. 2 4 2 приняты следующие обозначения: $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \operatorname{sh} bx, \\ 0, & \text{если взят } \operatorname{ch} bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2] + k}, & \text{если взят } \operatorname{sh} bx, \\ 1, & \text{если взят } \operatorname{ch} bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \operatorname{sh}^n bx, \quad n = 1, 3, 5, \dots, \\ 0, & \text{если взят } \operatorname{sh}^n bx, \quad n = 2, 4, 6, \dots, \\ 0, & \text{если взят } \operatorname{ch}^n bx, \quad n = 1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{-1} \left(\frac{\alpha}{2} + \gamma + h \right), \quad \tilde{b} = \frac{by}{2} (n - 2k).$$

$$1. \int_0^y x^{\alpha-1} (y^{2r} - x^{2r})^{\beta-1} \begin{cases} \operatorname{sh} bx \\ \operatorname{ch} bx \end{cases}^n dx = \\ = V_{31} + \frac{1 + (-1)^n}{2} (\mp 1)^{n/2} y^{2r(\beta-1)+\alpha} \frac{2^{-n-1} n!}{r [(n/2)!]^2} B \left(\frac{\alpha}{2r}, \beta \right)$$

$$\left\{ E_{31} = (-1)^h (\mp 1)^{[n/2]} \frac{\sqrt{\pi}}{2^n} \frac{B(\beta, \xi)}{r \Gamma(1/2 + 2\gamma + h)}, \right.$$

$$u = \alpha + 2r(\beta - 1), \quad a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{b}/p)^{2p} \left. \right\}$$

$$[y, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta n; |\arg b| < \pi].$$

$$2. \int_0^y x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} - x^{2r}})^v + (y^r - \sqrt{y^{2r} - x^{2r}})^v}{\sqrt{y^{2r} - x^{2r}}} \begin{cases} \operatorname{sh} bx \\ \operatorname{ch} bx \end{cases}^n dx = \\ = V_{35} + \frac{1 + (-1)^n}{2} (\mp 1)^{n/2} 2^{v-n+\alpha/r-1} y^{\alpha+rv} \frac{n! \sqrt{\pi}}{r [(n/2)!]^2} \Gamma \left[\frac{\alpha/(2r), v + \alpha/(2r)}{v + \alpha/r} \right]$$

$$\left\{ E_{35} = (-1)^h (\mp 1)^{[n/2]} \sqrt{\pi} \frac{2^{v+2\xi-n}}{r} \Gamma \left[\frac{\xi, v + \xi}{v + 2\xi, 1/2 + 2\gamma + h} \right], \right.$$

$$u = \alpha + r(v - 1), \quad a_1 = \xi, \quad b_1 = v + \xi, \quad c_1 = v/2 + \xi, \quad d_1 = (v + 1)/2 + \xi,$$

$$e_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{b}/p)^{2p} \left. \right\}$$

$$[y > 0; \operatorname{Re} \alpha > -n\delta, \operatorname{Re} \alpha > -n\delta - 2r \operatorname{Re} v; |\arg b| < \pi].$$

$$3. \int_0^\infty x^{\alpha-1} e^{-sx^{2r}} \begin{cases} \operatorname{sh} bx \\ \operatorname{ch} bx \end{cases}^n dx =$$

$$= U + \frac{1 + (-1)^n}{2} (\mp 1)^{n/2} 2^{-n-1} s^{-\alpha/(2r)} \frac{n!}{r [(n/2)!]^2} \Gamma \left(\frac{\alpha}{2r} \right),$$

$$U = V_{40}$$

$$[r > 1/2].$$

$$U = 2^{1-n} \sqrt{\pi} \sum_{k=0}^{[(n-1)/2]} (\mp 1)^k \binom{n}{k} \times$$

$$\times s^{-\alpha} \varphi^{2\gamma} \Gamma \left[\frac{\alpha + 2\gamma}{1/2 + 2\gamma} \right] {}_2F_1 \left(\begin{matrix} \Delta(2, \alpha + 2\lambda); \\ 1/2 + 2\gamma \end{matrix}; \frac{b^2 (n - 2k)^2}{s^2} \right) \quad [r = 1/2]$$

$$\left\{ E_{40} = (-1)^h (\mp 1)^{[n/2]} \frac{\sqrt{\pi}}{2^n} \Gamma \left[\frac{\xi}{1/2 + 2\gamma + h} \right], \right.$$

$$u = -\alpha/(2r), \quad v_1 = 2(h + \gamma), \quad \varphi = 2^{-1} (n - 2k) b s^{-1/(2r)},$$

$$a_1 = \xi, \quad b_1 = 1/2 + 2\gamma + h, \quad z = \left[\frac{b(n - 2k)}{2\rho} \right]^{2p} q^q s^{-q} \left. \right\}$$

[Re s > 0; Re α > -δn; arg b произволен при r > 1/2 (n | Re b| < Re s при r = 1/2)].

2.4.3. Интегралы от $A(x) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^\sigma$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^\infty \frac{x^{\alpha-1}}{\operatorname{sh} cx} dx = I_\alpha \quad [\operatorname{Re} c > 0; \operatorname{Re} \alpha > 1],$$

$$I_\alpha = \frac{2^\alpha - 1}{2^\alpha - 1} \Gamma(\alpha) \zeta(\alpha) = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^\infty \frac{1}{(2k+1)^\alpha},$$

$$I_{2n} = \frac{2^{2n} - 1}{2n} \left(\frac{\pi}{c} \right)^{2n} |B_{2n}|, \quad I_2 = \frac{\pi^2}{4c^2}, \quad I_4 = \frac{\pi^4}{8c^4} \quad [\text{см. 5.1.4}].$$

$$2. \int_0^\infty \frac{x^{\alpha-1}}{\operatorname{ch} cx} dx = J_\alpha \quad [\operatorname{Re} \alpha, \operatorname{Re} c > 0],$$

$$J_\alpha = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^\alpha} = c^{-\alpha} 2^{1-2\alpha} \Gamma(\alpha) [\zeta(\alpha, 1/4) - \zeta(\alpha, 3/4)],$$

$$J_{2n+1} = \left(\frac{\pi}{2c} \right)^{2n+1} |E_{2n}|, \quad J_1 = \frac{\pi}{2c}, \quad J_3 = \left(\frac{\pi}{2c} \right)^3 \quad [\text{см. 5.1.4}].$$

$$3. \int_0^\infty x^{\alpha-1} \left\{ \frac{\operatorname{sh} cx}{\operatorname{ch} cx} \right\}^{-2} dx = I_\alpha \quad [\operatorname{Re} c > 0, \operatorname{Re} \alpha > 2\delta],$$

$$I_\alpha = 2^{2-\alpha} c^{-\alpha} (1 - 2^{(2-\alpha)(1-\delta)}) \Gamma(\alpha) \zeta(\alpha - 1), \quad I_{2n+1} = \pi^{2n} c^{-2n-1} (1 - 2^{(2-\alpha)(1-\delta)}) |B_{2n}|.$$

$$4. \int_0^\infty \frac{x dx}{\operatorname{ch} cx} = \frac{\ln 2}{c^2} \quad [\operatorname{Re} c > 0].$$

$$5. \int_0^\infty \frac{dx}{\operatorname{ch}^n cx} = \frac{(2r-2)!! (1-s)}{(2r-1)!! c} + \frac{(2r-1)!! s\pi}{(2r)!! 2c}$$

$$[r = [n/2], \quad s = n - 2r; \quad \operatorname{Re} c > 0; \quad n = 2, 3, 4, \dots].$$

$$6. \int_0^a (a-x)^{\beta-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = 2a^{\beta} \beta^{-1} [{}_1F_1(1; \beta+1; ab) \mp {}_1F_1(1; \beta+1; -ab)]$$

[a, Re $\beta > 0$; |arg b | < π].

$$7. \int_0^a (a^2-x^2)^{\beta-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ \begin{matrix} L_{\beta-1/2}(ab) \\ I_{\beta-1/2}(ab) \end{matrix} \right\}$$

[a, Re $\beta > 0$; |arg b | < π].

$$8. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx =$$

$$= \frac{1}{2} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_1F_1(\alpha; \alpha+\beta; ab) \mp {}_1F_1(\alpha; \alpha+\beta; -ab)]$$

[a, Re $\beta > 0$; Re $\alpha > -\delta$; |arg b | < π].

$$9. \int_0^a x^{\alpha-1} (a-x)^{\alpha-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = \sqrt{\pi} \left(\frac{a}{b} \right)^{\alpha-1/2} \left\{ \begin{matrix} \text{sh } (ab/2) \\ \text{ch } (ab/2) \end{matrix} \right\} \Gamma(\alpha) I_{\alpha-1/2} \left(\frac{ab}{2} \right)$$

[a, Re $\alpha > 0$; |arg b | < π].

$$10. \int_0^a x^{\alpha-1} (a^2-x^2)^{\beta-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx =$$

$$= \frac{b^{\delta} a^{2\beta+\alpha+\delta-2}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; \frac{a^2 b^2}{4}\right)$$

[a, Re $\beta > 0$; Re $\alpha > -\delta$; |arg b | < π].

$$11. \int_0^a x (a^2-x^2)^{\beta-1} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ \begin{matrix} I_{\beta+1/2}(ab) \\ L_{\beta+1/2}(ab) \end{matrix} \right\} + \left\{ \begin{matrix} 0 \\ 2^{-1} \beta a^{2\beta} \end{matrix} \right\}$$

[a, Re $\beta > 0$; |arg b | < π].

$$12. \int_0^a \frac{1}{\sqrt{x(a-x)}} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = \frac{\pi}{2} \sqrt{b'} I_{\pm 1/4}^2 \left(\frac{ab}{2} \right)$$

[a > 0; |arg b | < π].

$$13. \int_0^{\infty} \frac{x}{(x^2+z^2)} \frac{dx}{\text{sh } bx} = \frac{\pi}{2bz} - \beta \left(\frac{bz}{\pi} + 1 \right)$$

[Re $z > 0$].

$$14. = \ln 2 - \frac{1}{2}$$

[bz = π ; Re $z > 0$].

$$15. = \frac{\pi}{2} - 1$$

[2bz = π ; Re $z > 0$].

$$16. = 2^{-1/2} [\pi + 2 \ln(1 + \sqrt{2})] - 2$$

[4bz = π ; Re $z > 0$].

$$17. \int_0^{\infty} \frac{dx}{(x^2+z^2) \text{ch } bx} = \frac{1}{z} \beta \left(\frac{bz}{\pi} + \frac{1}{2} \right)$$

[Re $z > 0$].

$$18. = \frac{b}{2\pi} (2 - \pi)$$

[bz = π ; Re $z > 0$].

$$19. \quad = \frac{2b}{\pi} \ln 2 \quad [2bz = \pi; \operatorname{Re} z > 0].$$

$$20. \quad = 2^{3/2} b [1 - 2\pi^{-1} \ln(1 + \sqrt{2})] \quad [4bz = \pi; \operatorname{Re} z > 0].$$

$$21. \quad \int_0^\infty \frac{x^5 dx}{(x^6 + z^6) \operatorname{sh} bx} = \frac{1}{3} \left(\ln 2 - 1 + \frac{\pi}{\operatorname{ch}(\pi \sqrt{3}/2)} \right) \quad [bz = \pi; 6|\arg z| < \pi].$$

$$22. \quad \int_0^\infty \frac{(1+ix)^n - (1-ix)^n}{\operatorname{sh}(x\pi/2)} dx = 2i - 2i(-1)^{n/2} \frac{1+(-1)^n}{2} |E_n|.$$

2.4.4. Интегралы от $\prod_k \left\{ \frac{\operatorname{sh} b_k x}{\operatorname{ch} b_k x} \right\}$.

Обозначение: $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$.

$$1. \quad \int_0^\infty \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \operatorname{tg} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$2. \quad \int_0^\infty \frac{\operatorname{sh}^2 bx}{\operatorname{sh}^2 cx} dx = \frac{c - \pi b \operatorname{ctg}(b\pi/c)}{2c^2} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$3. \quad \int_0^\infty \frac{1}{\operatorname{ch} cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\pi}{2c \cos[b\pi/(2c)]} - \frac{1}{c} \beta \left(\frac{b+c}{2c} \right) \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$4. \quad \int_0^\infty \frac{\operatorname{ch} bx}{\operatorname{ch}^v cx} dx = \frac{2^{v-1}}{c \Gamma(v)} \Gamma \left(\frac{vc+b}{2c} \right) \Gamma \left(\frac{vc-b}{2c} \right) \quad [\operatorname{Re}(vc) > |\operatorname{Re} b|].$$

$$5. \quad \int_0^\infty \frac{\operatorname{ch} bx}{\operatorname{ch}^n cx} dx = \frac{2^{n+2\lambda-3}\pi}{(n-1)!} b^{1-2\lambda} c^{2\lambda-2} \left\{ \frac{\sec[b\pi/(2c)]}{\operatorname{cosec}[b\pi/(2c)]} \right\} \prod_{k=1}^{[(n-1)/2]} \left[(k-\lambda)^2 - \frac{b^2}{4c^2} \right] \\ \left[n \operatorname{Re} c > |\operatorname{Re} b|; \lambda = \frac{1-(-1)^n}{4}, n \geq 1 \right]$$

$$6. \quad \int_0^\infty \operatorname{sh}^{\mu-1} bx \operatorname{ch}^{\nu-1} bx dx = \frac{1}{2b} B \left(\frac{\mu}{2}, 1 - \frac{\mu+\nu}{2} \right) \quad [b > 0; 0 < \operatorname{Re} \mu < 2 - \operatorname{Re} \nu].$$

$$7. \quad \int_0^\infty \operatorname{sh}^{\nu-1} ax \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \\ = \frac{1}{2a \sqrt{\pi}} \left\{ \frac{\sin[b\pi/(2a)]}{\cos[b\pi/(2a)]} \right\} \Gamma \left[\frac{(\nu+\delta)/2}{1-\nu/2} \right] \Gamma \left(\frac{1-\nu}{2} + \frac{b}{2a} \right) \Gamma \left(\frac{1-\nu}{2} - \frac{b}{2a} \right) \\ [a > 0; -\delta < \operatorname{Re} \nu < 1 - |\operatorname{Re} b|/a].$$

$$8. \int_0^{\infty} \text{sh}^{\nu-1} ax \text{ch} ax \begin{Bmatrix} \text{sh} bx \\ \text{ch} bx \end{Bmatrix} dx =$$

$$= \pm \frac{b}{4a^2 \sqrt{\pi}} \begin{Bmatrix} \cos [b\pi/(2a)] \\ \sin [b\pi/(2a)] \end{Bmatrix} \Gamma \left[\frac{(\nu + \delta)/2}{(1 - \nu)/2} \right] \Gamma \left(\frac{b - \nu a}{2a} \right) \Gamma \left(-\frac{b + \nu a}{2a} \right)$$

[$a > 0$; $-\delta < \text{Re } \nu < -|\text{Re } b|/a$].

$$9. \int_0^{\infty} \frac{\text{sh} ax}{\text{sh} cx} \begin{Bmatrix} \text{sh} bx \\ \text{ch} bx \end{Bmatrix} dx =$$

$$= \frac{\pi}{2c} \left(\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c} \right)^{-1} \begin{Bmatrix} \sin (b\pi/c) \\ \sin (a\pi/c) \end{Bmatrix} + \left\{ \frac{(2c)^{-1}}{0} \right\} \left[\psi \left(\frac{a+c-b}{2c} \right) - \psi \left(\frac{a+c+b}{2c} \right) \right]$$

[$\text{Re } c > |\text{Re } a| + |\text{Re } b|$].

$$10. \int_0^{\infty} \begin{Bmatrix} \text{sh} ax \text{ sh} bx \\ \text{ch} ax \text{ ch} bx \end{Bmatrix} \frac{dx}{\text{ch} cx} =$$

$$= \frac{\pi}{c} \begin{Bmatrix} \sin [a\pi/(2c)] \sin [b\pi/(2c)] \\ \cos [a\pi/(2c)] \cos [b\pi/(2c)] \end{Bmatrix} [\cos (a\pi/c) + \cos (b\pi/c)]^{-1}$$

[$\text{Re } c > |\text{Re } a| + |\text{Re } b|$].

$$11. \int_0^{\infty} \frac{\text{sh} ax \text{ ch} bx}{\text{ch} cx} dx = \frac{1}{4c} \left[\psi \left(\frac{3c-a+b}{4c} \right) - \psi \left(\frac{3c+a-b}{4c} \right) + \right.$$

$$\left. + \psi \left(\frac{3c-a-b}{4c} \right) - \psi \left(\frac{3c+a+b}{4c} \right) + \frac{2\pi \sin (a\pi/c)}{\cos (a\pi/c) + \cos (b\pi/c)} \right]$$

[$\text{Re } c > |\text{Re } a| + |\text{Re } b|$].

$$12. \int_0^{\infty} \frac{\text{sh} ax \text{ sh} cx}{\text{ch}^2 cx} dx = \frac{\pi a}{2c^2 \cos [a\pi/(2c)]}$$

[$\text{Re } c > |\text{Re } a|$].

2.4.5. Интегралы от $A(x) \prod_k \begin{Bmatrix} \text{sh} b_k x \\ \text{ch} b_k x \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\text{sh} cx} \begin{Bmatrix} \text{sh} bx \\ \text{ch} bx \end{Bmatrix} dx = \frac{\Gamma(\alpha)}{(2c)^\alpha} \left[\zeta \left(\alpha, \frac{1}{2} - \frac{b}{2c} \right) \mp \zeta \left(\alpha, \frac{1}{2} + \frac{b}{2c} \right) \right]$$

[$\text{Re } x > 1 - \delta$; $\text{Re } c > |\text{Re } b|$].

$$2. \int_0^{\infty} \frac{x^{2n+\sigma}}{\text{sh} cx} \begin{Bmatrix} \text{sh} bx \\ \text{ch} bx \end{Bmatrix} dx = \frac{\pi}{2c} \frac{\partial^{2n+\sigma}}{\partial b^{2n+\sigma}} \text{tg} \frac{b\pi}{2c}$$

[$\text{Re } c > |\text{Re } b|$; $\sigma = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$].

$$3. \int_0^{\infty} \frac{\text{sh}^2 bx}{x \text{ sh} cx} dx = -\frac{1}{2} \ln \cos \frac{b\pi}{c}$$

[$\text{Re } c > 2$, $\text{Re } b$].

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{ch} cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx =$$

$$= 2^{-2\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{c-b}{4c}\right) - \zeta\left(\alpha, \frac{c+b}{4c}\right) + \zeta\left(\alpha, \frac{3c+b}{4c}\right) - \zeta\left(\alpha, \frac{3c-b}{4c}\right) \right]$$

[Re $\alpha > -\delta$; Re $c > |\operatorname{Re} b|$].

$$5. \int_0^{\infty} \frac{\operatorname{sh} bx}{x \operatorname{ch} cx} dx = \ln \operatorname{tg} \left(\frac{b+c}{4c} \pi \right)$$

[Re $c > |\operatorname{Re} b|$].

$$6. \int_0^{\infty} \frac{x^{2n+\delta}}{\operatorname{ch} cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\pi}{2c} \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} \sec \frac{b\pi}{2c}$$

[Re $c > |\operatorname{Re} b|$; $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$].

$$7. \int_0^{\infty} x^{\alpha-1} \frac{\operatorname{ch} bx}{\operatorname{ch} cx} dx = \Gamma(\alpha) \sum_{k=0}^{\infty} (-1)^k \{ [c(2k+1)-b]^{-\alpha} + [c(2k+1)+b]^{-\alpha} \}$$

[Re $\alpha > 0$; Re $c > |\operatorname{Re} b|$].

$$8. \int_0^{\infty} x \frac{\operatorname{sh} bx}{\operatorname{ch}^2 cx} dx = \frac{\pi}{4c^3} \sin^{-2} \frac{b\pi}{2c} \left(2c \sin \frac{b\pi}{2c} - b\pi \cos \frac{b\pi}{2c} \right)$$

[2 Re $c > |\operatorname{Re} b|$].

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\operatorname{sh} ax \operatorname{ch} bx} = \frac{\Gamma(\alpha)}{(2b)^{\alpha}} \left[{}_1F_1 \left(-1; \alpha; \frac{1}{2} + \frac{a}{2b} \right) + {}_1F_1 \left(-1; \alpha; \frac{1}{2} - \frac{a}{2b} \right) \right]$$

[Re $a, \operatorname{Re} b > 0$; Re $\alpha > 1$].

$$10. \int_0^{\infty} x^{\alpha-1} \frac{\operatorname{sh} cx}{\operatorname{ch}^2 cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha-1}}$$

[Re $c > 0$; Re $\alpha > -1$; см. 5.1.4].

$$11. \int_0^{\infty} x \operatorname{ch}^{\mu-1} ax \operatorname{sh} ax dx = -\frac{\sqrt{\pi}}{2\mu a^2} \Gamma \left[\frac{-\mu/2}{(1-\mu)/2} \right]$$

[$\mu > 0$; Re $\mu < 0$].

$$12. \int_0^{\infty} x^{\alpha-1} \frac{\operatorname{ch} cx}{\operatorname{sh}^2 cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\alpha-1}}$$

[Re $\alpha > 2$; см. 5.1.4].

$$13. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{sh} bx}{x \operatorname{sh} cx} dx = \frac{1}{2} \ln \frac{\cos [(a-b)\pi/(2c)]}{\cos [(a+b)\pi/(2c)]}$$

[Re $c > |\operatorname{Re} a| + |\operatorname{Re} b|$].

$$14. \int_0^{\infty} \frac{\operatorname{sh}^2 ax \operatorname{ch} bx}{x \operatorname{sh} cx} dx = \frac{1}{4} \ln \frac{1 + \cos (b\pi/c)}{\cos (2a\pi/c) + \cos (b\pi/c)}$$

[Re $c > |\operatorname{Re} a| + |\operatorname{Re} b|$].

$$15. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{ch} bx}{x \operatorname{ch} cx} dx = \frac{1}{2} \ln \frac{\cos [b\pi/(2c)] + \sin [a\pi/(2c)]}{\cos [b\pi/(2c)] - \sin [a\pi/(2c)]}$$

[Re $c > |\operatorname{Re} a| + |\operatorname{Re} b|$].

$$16. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{\pi}{z} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(bk\pi/c)}{cz + k\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$17. = (-1)^n b \frac{e^{ibn\pi/c}}{2n} + \frac{c}{2\pi n} \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} e^{ibk\pi/c} +$$

$$+ (-1)^n \frac{c}{2\pi n} e^{ibn\pi/c} \ln(1 + e^{ib\pi/c}) + \frac{c}{2\pi n!} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{(1+z)^{n-1}}{z} \ln(1+z) \right] \Big|_{z=e^{ib\pi/c}}$$

$$[cz = \pi n; \operatorname{Re} c > |\operatorname{Re} b|; \operatorname{Re} z > 0; n = 1, 2, 3, \dots].$$

$$18. = -\frac{b}{2} \cos \frac{b\pi}{c} + \frac{c}{\pi} \sin \frac{b\pi}{c} \ln \left(2 \cos \frac{b\pi}{2c} \right) \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$19. = c \sin \frac{b\pi}{2c} - \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \operatorname{ctg} \left(\frac{c-b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$20. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\operatorname{sh} bx}{\operatorname{ch}(\pi x/z)} dx = -2 \sin \frac{bz}{2} + \frac{\pi}{2} \sin bz - \operatorname{ch} bz \ln \operatorname{tg} \frac{bz + \pi}{4}$$

$$[z > 0; z \neq |\operatorname{Re} b| < \pi].$$

$$21. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\operatorname{ch} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2cz} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k \cos(bk\pi/c)}{cz + k\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$22. = \frac{b\pi}{2c} \sin \frac{b\pi}{c} - \frac{1}{2} + \frac{1}{2} \cos \frac{b\pi}{c} \ln \left[2 \left(1 + \cos \frac{b\pi}{c} \right) \right]$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$23. = \frac{\pi}{2} \cos \frac{b\pi}{2c} - 1 + \sin \frac{b\pi}{2c} \ln \operatorname{tg} \left(\frac{c+b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$24. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\operatorname{ch} bx}{\operatorname{ch} cx} dx = \frac{(-1)^n}{2n+1} \exp \left[\frac{ib}{c} \left(n + \frac{1}{2} \right) \pi \right] \left[-ib + \right.$$

$$\left. + \frac{c}{\pi} \ln(1 + e^{ib\pi/c}) \right] + \frac{c \exp[ib\pi/(2c)]}{\pi(2n+1)} \sum_{k=0}^{n-1} \frac{(-1)^k}{k-n} \exp(ibk\pi/c) +$$

$$+ \frac{c \exp[ib\pi/(2c)]}{\pi(2n+1)(n+1)} {}_2F_1(n+1, 1; n+2; -e^{ib\pi/c})$$

$$[cz = \pi(n+1/2); \operatorname{Re} c > |\operatorname{Re} b|].$$

$$25. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} - \frac{c}{2} \cos \frac{b\pi}{c} + \frac{2c}{\pi} \sin \frac{b\pi}{c} \ln \operatorname{tg} \left(\frac{b+c}{4c} \pi \right)$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$26. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \left(2 \cos \frac{b\pi}{2c} \right) + b \sin \frac{b\pi}{2c} \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|].$$

2.4.6. Интегралы от $(a + b \operatorname{ch}^n bx + c \operatorname{sh}^n bx)^p \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.

$$1. \int_0^a \frac{\operatorname{ch} bx dx}{(\operatorname{ch} a - \operatorname{ch} x)^v} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-v)}{\operatorname{sh}^{v-1/2} a} P_{b-1/2}^{v-1/2}(\operatorname{ch} a)$$

$$[a > 0; \operatorname{Re} v < 1; |\arg b| < \pi].$$

$$2. \int_0^{\infty} \frac{dx}{a + b \operatorname{ch} x} = \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctg} \frac{\sqrt{b^2 - a^2}}{a + b} \quad [|b| > |a|].$$

$$3. = \frac{1}{\sqrt{a^2 - b^2}} \ln \left| \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \right| \quad [0 < |b| < |a|].$$

$$4. = a^{-1} \quad [b = a].$$

$$5. \int_0^{\infty} \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \left| \frac{a + b + \sqrt{a^2 + b^2}}{a + b - \sqrt{a^2 + b^2}} \right| \quad [ab \neq 0].$$

$$6. \int_0^{\infty} \frac{\operatorname{ch} bx}{a + \operatorname{ch} cx} dx = \frac{\pi}{c \sqrt{a^2 - 1}} \operatorname{cosec} \frac{b\pi}{c} \operatorname{sh} \left(\frac{b}{c} \operatorname{Arch} a \right) \quad [a > 1; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$7. = \frac{\pi}{c \sqrt{1 - a^2}} \operatorname{cosec} \frac{b\pi}{c} \sin \left(\frac{b}{c} \arccos a \right) \quad [-1 < a < 1; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{\operatorname{ch} bx}{(a + \operatorname{ch} cx)^v} dx = \frac{1}{c} \sqrt{\frac{\pi}{2}} (a^2 - 1)^{1/4 - v/2} \Gamma \left[\frac{v + b/c}{v}, \frac{v - b/c}{v} \right] P_{b/c - 1/2}^{1/2 - v}(a),$$

$$[a \notin [-1, 1]; \operatorname{Re}(v \pm b/c) > 0].$$

$$10. \int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{a + \operatorname{ch} cx}} dx = \frac{\pi}{c \sqrt{2}} \sec \frac{b\pi}{c} P_{b/c - 1/2}(a)$$

$$[-1 < a < 1; \operatorname{Re} c > 2|\operatorname{Re} b|].$$

$$11. \int_0^{\infty} \frac{\operatorname{ch} bx dx}{(\cos t + \operatorname{ch} cx)^2} = \frac{\pi [b \sin t \cos(bt/c) - c \cos t \sin(bt/c)]}{c^2 \sin^3 t \sin(b\pi/c)}$$

$$[2\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi].$$

$$12. \int_0^{\infty} \frac{\operatorname{ch} b^v \operatorname{ch} cx}{(1 + \operatorname{ch} cx)^v} dx = \frac{2^{v-1}}{c} B \left(v + \frac{b}{c}, v - \frac{b}{c} \right)$$

$$[\operatorname{Re} v > 0; \operatorname{Re}(vc) > \operatorname{Re} c + |\operatorname{Re} b|].$$

$$13. \int_0^{\infty} \frac{\cos \tau - \operatorname{ch} bx}{\cos t - \operatorname{ch} cx} d\tau = \frac{\pi}{c} \frac{\sin[b(\pi - t)/c]}{\sin t \sin(b\pi/c)} - \frac{\pi - t}{c \sin t} \cos v$$

$$[\operatorname{Re} c > |\operatorname{Re} b|; 0 < \tau < \pi; 0 < t < \pi].$$

$$14. \int_0^{\infty} \left\{ \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} ax \operatorname{ch} bx} \right\} \frac{dx}{\cos t + \operatorname{ch} cx} = \frac{\pi}{c \sin t} \left(\cos \frac{2b\pi}{c} - \cos \frac{2a\pi}{c} \right)^{-1} \times$$

$$\times \left\{ \frac{\sin[a(\pi - t)/c] \sin[b(\pi + t)/c] - \sin[a(\pi + t)/c] \sin[b(\pi - t)/c]}{\cos[a(\pi - t)/c] \cos[b(\pi + t)/c] - \cos[a(\pi + t)/c] \cos[b(\pi - t)/c]} \right\}$$

$$[0 < |t| < \pi, \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|; \pi \operatorname{Re} c > |\operatorname{Re}(ct)|].$$

$$15. \int_0^{\infty} \left\{ \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} ax \operatorname{ch} bx} \right\} \frac{dx}{1 + \operatorname{ch} cx} = \frac{2\pi}{c^2 [\cos(2b\pi/c) - \cos(2a\pi/c)]} \times$$

$$\times \left[b \left\{ \begin{matrix} \sin(a\pi/c) \cos(b\pi/c) \\ \cos(a\pi/c) \sin(b\pi/c) \end{matrix} \right\} - a \left\{ \begin{matrix} \cos(a\pi/c) \sin(b\pi/c) \\ \sin(a\pi/c) \cos(b\pi/c) \end{matrix} \right\} \right] \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|]$$

$$16. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi \sin(bt/c)}{2c \operatorname{ch}(b\pi/c) \sin(t/2)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|; 0 < |t| < \pi]$$

$$17. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \cos(b\pi/c)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|]$$

$$18. \int_0^{\infty} \frac{\operatorname{sh} cx \operatorname{sh} bx}{(\cos t + \operatorname{ch} cx)^2} dx = \frac{b\pi \sin(bt/c)}{c^2 \sin t \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi]$$

$$19. \int_0^{\infty} \frac{dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{\sqrt{b^2 - c^2}}{b + c} \quad [|b| > |c|]$$

$$20. = \frac{1}{\sqrt{c^2 - b^2}} \ln \left| \frac{b + c + \sqrt{c^2 - b^2}}{b + c - \sqrt{c^2 - b^2}} \right| \quad [0 < |b| < |c|]$$

$$21. = c^{-1} \quad [b = c]$$

$$22. \int_0^{\infty} \left\{ \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} ax \operatorname{ch} bx} \right\} \frac{dx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} = \frac{\pi}{4(a^2 - b^2)} \left\{ \frac{b}{a} \right\} \quad [\operatorname{Re} a, \operatorname{Re} b > 0]$$

$$23. \int_0^{\infty} \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \left| \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} \right|$$

$$[b^2 < a^2 + c^2; |a| \neq |b|]$$

$$24. = \frac{2(a - b)}{c(a - b - c)} \quad [b^2 = a^2 + c^2; c(a - b - c) < 0]$$

$$25. = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left(\operatorname{arctg} \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + \varepsilon \pi \right)$$

$[b^2 > a^2 + c^2, \varepsilon = 0 \text{ при } (b - a)(a + b + c) > 0, \varepsilon = 1 \text{ при } (b - a)(a + b + c) < 0, a < b + c; \varepsilon = -1 \text{ при } (b - a)(a + b + c) < 0, a > b + c]$

$$26. \int_0^{\infty} \frac{\operatorname{ch} cx \operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \sec \frac{b\pi}{2c} \operatorname{ch} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$27. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{sh}^2 t + \operatorname{ch}^2 cx} dx = \frac{\pi}{c} \left(\operatorname{sh} 2t \sin \frac{b\pi}{2c} \right)^{-1} \operatorname{sh} \frac{bt}{c} \quad [2\operatorname{Re} c > |\operatorname{Re} b|]$$

$$28. \int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{\operatorname{sh}^2 t + \operatorname{ch}^2 cx}} dx = \frac{\pi}{2c} \sec \frac{b\pi}{2c} P_{b/(2a) - 1/2}(\operatorname{ch} 2t) \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$29. \int_0^{\infty} \frac{\operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{sh}^{-1} \frac{b\pi}{2c} \operatorname{sh} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [2\operatorname{Re} c > |\operatorname{Re} b|].$$

$$30. \int_0^{\infty} \frac{dx}{b \operatorname{ch}^2 x + c \operatorname{sh}^2 x} = \frac{1}{\sqrt{bc}} \operatorname{arctg} \frac{c}{\sqrt{bo}} \quad [bc > 0].$$

$$31. = \frac{1}{2\sqrt{-bc}} \ln \left| \frac{c - \sqrt{-bc}}{b + \sqrt{-bc}} \right| \quad [bc < 0].$$

2.4.7. Интегралы от $A(x) (a + b \operatorname{ch}^n bx + c \operatorname{sh}^n bx)^0 \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.

$$1. \int_0^a \frac{x dx}{\operatorname{sh}(x/2) \sqrt{\operatorname{ch} a - \operatorname{ch} x}} = \frac{\pi \sqrt{2}}{\operatorname{sh}(a/2)} \arcsin \operatorname{th} \frac{a}{2} \quad [a > 0].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\cos t + \operatorname{ch} x} = J_{\alpha} \quad [\operatorname{Re} \alpha > 0; |t| < \pi].$$

$$J_{\alpha} = \frac{\pi^{\alpha} 2^{\alpha-1}}{\sin t \sin(\alpha\pi/2)} \left[\zeta \left(1-\alpha, \frac{1}{2} - \frac{t}{2\pi} \right) - \zeta \left(1-\alpha, \frac{1}{2} + \frac{t}{2\pi} \right) \right] = \\ = \frac{2\Gamma(\alpha)}{\sin t} \sum_{k=1}^{\infty} \frac{\sin kt}{k^{\alpha}},$$

$$J_2 = \frac{4}{\sin t} \left[t \ln 2 - 2L \left(\frac{t}{2} \right) \right], \quad J_3 = \frac{t}{3 \sin t} (\pi^2 - t^2),$$

$$J_5 = \frac{t}{15 \sin t} (\pi^2 - t^2) (7\pi^2 - 3t^2).$$

$$3. \int_0^{\infty} \frac{x^{\alpha-1}}{\cos t + \operatorname{ch} x} \left\{ \frac{\operatorname{sh}(x/2)}{\operatorname{ch}(x/2)} \right\} dx = \pi^{\alpha} 2^{2\alpha-3} \left\{ \frac{\sin(t/2) \cos(\alpha\pi/2)}{\cos(t/2) \sin(\alpha\pi/2)} \right\}^{-1} \times \\ \times \left[\zeta \left(1-\alpha, \frac{1}{4} - \frac{t}{4\pi} \right) \mp \zeta \left(1-\alpha, \frac{1}{4} + \frac{t}{4\pi} \right) \pm \zeta \left(1-\alpha, \frac{3}{4} + \frac{t}{4\pi} \right) - \right. \\ \left. - \zeta \left(1-\alpha, \frac{3}{4} - \frac{t}{4\pi} \right) \right] \quad [\operatorname{Re} \alpha > 0; |t| < \pi].$$

$$4. \int_0^{\infty} \frac{x \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi}{c^3} \operatorname{sh}^{-2} \frac{b\pi}{c} \left(c \sin \frac{b\pi}{c} - b\pi \cos \frac{b\pi}{c} \right) \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$5. \int_0^{\infty} \frac{x^{2n+1} \operatorname{sh} bx}{(1 + \operatorname{ch} cx)^v} dx = -\frac{\pi}{c^2} \Gamma \left[\frac{2-v}{v-2}, \frac{2v}{2v} \right] \frac{\partial^{2n+1}}{\partial b^{2n+1}} \frac{b}{\sin(b\pi/c)} \quad [\operatorname{Re}(v) > |\operatorname{Re} b|].$$

$$6. \int_0^{\infty} \frac{x \operatorname{ch}(x/2) dx}{\cos t + \operatorname{ch} x} = \frac{4}{\cos(t/2)} \left[\frac{\pi}{2} \ln 2 - L \left(\frac{\pi+t}{4} \right) - L \left(\frac{\pi-t}{4} \right) \right] \\ [t \neq \pi(2k+1); k=0, \pm 1, \pm 2, \dots].$$

$$7. \int_0^{\infty} \frac{x^{2n+1} \operatorname{sh} cx \, dx}{(\cos t + \operatorname{ch} cx)^2} = I_n \quad [\operatorname{Re} c > 0; |t| < \pi],$$

$$I_n = \frac{2(2n+1)!}{c^{2n+2} \sin t} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos kt}{k^{2n+1}},$$

$$I_0 = -\frac{t + \pi}{c^2 \sin t}, \quad I_1 = \frac{t(\pi^2 - t^2)}{c^4 \sin t}.$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{(x^2 + z^2)(\cos t + \operatorname{ch} cx)} \, dx = \frac{\pi e^{-ibx}}{2z(\cos cz + \cos t)} +$$

$$+ \frac{\pi}{\operatorname{sh} t} \sum_{k=0}^{\infty} \left\{ \frac{\exp[-ib\pi[(2k+1)\pi - t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) - t]^2} - \frac{\exp[-ib\pi[(2k+1)\pi + t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) + t]^2} \right\}$$

$$[\operatorname{Re} c \geqslant 1, \operatorname{Re} b \geqslant 1; |t| < \pi; \operatorname{Re} z > 0].$$

$$9. \int_0^{\infty} \frac{x^{-1/2}}{1 + 2\operatorname{ch}^2 cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} \, dx = \sqrt{\frac{2\pi}{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{kc - \sqrt{k^2 c^2 - b^2}}{k^2 c^2 - b^2} \right)^{1/2} \sin \frac{k\pi}{3}$$

$$[2\operatorname{Re} c > |\operatorname{Re} b|].$$

$$10. \int_0^{\infty} \frac{x^{2n+1} \operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \operatorname{sh} bx \, dx =$$

$$= \frac{\pi}{2c\sqrt{2}} \frac{\partial^{2n+1}}{\partial b^{2n+1}} \left[\sec \frac{b\pi}{2c} \operatorname{ch} \left(\frac{b}{2c} \ln(1 + \sqrt{2}) \right) \right] \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$11. \int_0^{\infty} \frac{x^{\alpha-1} \operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} \, dx = \frac{\Gamma(\alpha)}{2(a^2 - b^2)^{\alpha/2}} \left[\left(\frac{a+b}{a-b} \right)^{\alpha/2} - \left(\frac{a-b}{a+b} \right)^{\alpha/2} \right] \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha}}$$

$$[\operatorname{Re} a, \operatorname{Re} b > 0, \operatorname{Re} \alpha > -2; \text{см. 5.1.4}].$$

$$12. \int_0^{\infty} \frac{x^n \, dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{(2n)!}{(b+c)^{2n+1}} \Phi \left(\frac{c-b}{c+b}, n+1, \frac{1}{2} \right) \quad [b, c > 0].$$

2.4.8. Интегралы от $A(x) \operatorname{ch} bx \begin{Bmatrix} \operatorname{sh} \sqrt{a^2 - x^2} \\ \operatorname{ch} \sqrt{a^2 - x^2} \end{Bmatrix}$.

Условие: $a > 0$.

$$1. \int_0^a (a^2 - x^2)^{(\delta-1)/2} \operatorname{ch} bx \begin{Bmatrix} \operatorname{sh}(c\sqrt{a^2 - x^2}) \\ \operatorname{ch}(c\sqrt{a^2 - x^2}) \end{Bmatrix} \, dx = \frac{\pi a^{\delta} c^{\delta}}{2(b^2 + c^2)^{\delta/2}} I_{\delta}(a\sqrt{b^2 + c^2})$$

$$[\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$2. \int_0^a \frac{\operatorname{ch} bx}{(a^2 - x^2)^{3/4}} \begin{Bmatrix} \operatorname{sh}(c\sqrt{a^2 - x^2}) \\ \operatorname{ch}(c\sqrt{a^2 - x^2}) \end{Bmatrix} \, dx =$$

$$= \left(\frac{\pi}{2} \right)^{3/2} \sqrt{c} I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} - b) \right] I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} + b) \right].$$

$$3. \int_0^a \sqrt{a^2 - x^2} \operatorname{ch} (c \sqrt{a^2 - x^2}) \operatorname{ch} bx \, dx = \\ = \frac{\pi a^2 c^2}{2(b^2 + c^2)} \left[I_0(a \sqrt{b^2 + c^2}) + \frac{b^2 - c^2}{ac^2 \sqrt{b^2 + c^2}} I_1(a \sqrt{b^2 + c^2}) \right].$$

$$4. \int_0^a \frac{\operatorname{ch} (c \sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \operatorname{ch} bx \, dx = \\ = \left(\frac{\pi}{2} \right)^{3/2} \sqrt{b} I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right].$$

2.4.9. Интегралы, содержащие $\operatorname{th} ax$, $\operatorname{cth} ax$ и разности гиперболических и алгебраических функций.

$$1. \int_0^\infty x^{\alpha-1} \operatorname{th} ax \, dx = 2^{1-\alpha} a^{-\alpha} (2^{1-\alpha} - 1) \Gamma(\alpha) \zeta(\alpha) \quad [\operatorname{Re} a > 0; -1 < \operatorname{Re} \alpha < 0].$$

$$2. \int_0^\infty x^{\alpha-1} \left\{ \frac{\operatorname{th} ax - 1}{\operatorname{cth} ax - 1} \right\} dx = 2^{1-\alpha} a^{-\alpha} \Gamma(\alpha) \zeta(\alpha) \left\{ \begin{matrix} 2^{1-\alpha} - 1 \\ 1 \end{matrix} \right\} \quad [a > 0; \operatorname{Re} \alpha > \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}].$$

$$3. \int_0^\infty \frac{x}{(x^2 + 1)^\rho} \operatorname{th} \frac{\pi x}{2} \, dx = 2^{-2\rho-2} \zeta(2\rho-3) \quad [\operatorname{Re} \rho > 1].$$

$$4. \int_0^\infty \frac{\operatorname{th} (cx/2)}{x \operatorname{ch} cx} \, dx = \ln 2 \quad [c > 0].$$

$$5. \int_0^\infty \frac{\operatorname{ch} bx - 1}{x \operatorname{sh} cx} \, dx = -\ln \cos \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$6. \int_0^\infty \left(\frac{a}{\operatorname{sh} ax} - \frac{b}{\operatorname{sh} bx} \right) \frac{dx}{x} = (b - a) \ln 2 \quad [\operatorname{Re} a, \operatorname{Re} b > 0].$$

$$7. \int_0^\infty x^{\alpha-1} \left(\frac{1}{\operatorname{sh} x} - \frac{1}{x} \right) dx = 2(1 - 2^{-\alpha}) \Gamma(\alpha) \zeta(\alpha) \quad [|\operatorname{Re} \alpha| < 1].$$

2.4.10. Интегралы от $x^\alpha e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^\sigma$.

$$1. \int_0^\infty e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{p^2 - b^2} \left\{ \begin{matrix} b \\ p \end{matrix} \right\} \quad [\operatorname{Re} p > |\operatorname{Re} b|].$$

$$2. \int_0^\infty e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^2 dx = \frac{1}{p(p^2 - 4b^2)} \left\{ \begin{matrix} 2b^2 \\ p^2 - 2b^2 \end{matrix} \right\} \quad [\operatorname{Re} p > 2|\operatorname{Re} b|].$$

$$3. \int_0^{\infty} e^{-px} \operatorname{sh}^n bx dx = b^n p^{1-\sigma} n! \prod_{k=0}^{[n/2]} [\rho^2 - b^2 (2k + \sigma)^2]^{-1}$$

$$[2\sigma = 1 - (-1)^n; \operatorname{Re} p > n |\operatorname{Re} b|].$$

$$4. \int_0^{\infty} e^{-px} \operatorname{sh}^{v-1} bx dx = 2^{-v} b^{-1} B\left(v, \frac{p}{2b} + \frac{1-v}{2}\right)$$

$$[\operatorname{Re} v, \operatorname{Re} b > 0; \operatorname{Re} p > \operatorname{Re}(vb - b)].$$

$$5. \int_0^{\infty} e^{-px} \operatorname{ch}^n bx dx = 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \frac{p}{p^2 - (n-2k)^2 b^2} + \frac{1 + (-1)^n}{2} \binom{n}{\frac{n+1}{2}} \frac{1}{2^n p}$$

$$[\operatorname{Re} p > n |\operatorname{Re} b|; n = 1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch}^n cx} dx = I_n$$

$$[\operatorname{Re} p > -n |\operatorname{Re} c|],$$

$$I_1 = \frac{1}{c} \beta\left(\frac{p+c}{2c}\right), \quad I_2 = \frac{p}{c^2} \beta\left(\frac{p}{2c}\right) - \frac{1}{c},$$

$$I_{2m+1} = \frac{2^{2m-2}}{c} B(m, m) - \frac{1}{2mc}.$$

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{cases} \operatorname{sh} bx \\ \operatorname{ch} bx \end{cases} dx = \frac{\Gamma(\alpha)}{2} [(\rho - b)^{-\alpha} \mp (\rho + b)^{-\alpha}]$$

$$\left[\operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} \alpha > -\begin{cases} 1 \\ 0 \end{cases} \right].$$

$$8. \int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh} bx dx = \frac{1}{2} \ln \frac{p+b}{p-b}$$

$$[\operatorname{Re} p > |\operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh}^2 bx dx = \frac{1}{2} \ln \sqrt{1 - \frac{4b^2}{p^2}}$$

$$[\operatorname{Re} p > 2 |\operatorname{Re} b|]$$

$$10. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{cases} \operatorname{sh} bx \\ \operatorname{ch} bx \end{cases}^m dx =$$

$$= 2^{1-m} \Gamma(\alpha) \sum_{k=0}^{[(m-1)/2]} (-1)^{[m/2]-k} \binom{m}{k} \{[\rho - b(m-2k)]^{-\alpha} \mp [\rho + b(m-2k)]^{-\alpha}\} +$$

$$+ \frac{1 + (-1)^m}{2} 2^{-m} p^{-\alpha} \Gamma(\alpha) \binom{n}{n/2} \left[\operatorname{Re} \alpha > -\begin{cases} 1 \\ 0 \end{cases}; \operatorname{Re} p > m |\operatorname{Re} b| \right].$$

$$11. \int_0^{\infty} x^m e^{-px} \operatorname{sh}^n bx dx = 2^{-n} m! \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(p - bn + 2kb)^{m+1}}$$

$$[\operatorname{Re} p > n |\operatorname{Re} b|].$$

$$12. \int_0^{\infty} \frac{e^{-px}}{x^m} \operatorname{sh}^n bx dx = I_m^n$$

$$[\operatorname{Re} p > n |\operatorname{Re} b|; m \leq n; m, n = 1, 2, 3, \dots].$$

$$I_1^{2n} = 2^{-2n} \sum_{k=0}^{n-1} (-1)^{k-1} \binom{2n}{k} \ln [p^2 - 4(n-k)^2 b^2] - (-1)^n 2^{-2n} \binom{2n}{n} \ln p,$$

$$I_1^{2n+1} = 2^{-2n-1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \ln \frac{p+b(2n-2k+1)}{p-b(2n-2k+1)}, \quad I_1^1 = \frac{1}{2} \ln \frac{p+b}{p-b},$$

$$I_2^2 = \frac{b}{2} \ln \frac{p+2b}{p-2b} + \frac{p}{4} \ln \left(1 - \frac{4b^2}{p^2} \right),$$

$$I_2^3 = \frac{p}{8} \ln \frac{p-3b}{p+3b} - \frac{3p}{4} \ln \frac{p-b}{p+b} + \frac{3b}{8} \ln \frac{p^2-b^2}{p^2-3b^2}.$$

$$13. \int_0^\infty x^{\alpha-1} \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{2^{1-\alpha}}{c^\alpha} \Gamma(\alpha) \zeta\left(\alpha, \frac{p+c}{2c}\right) \quad [\operatorname{Re} \alpha > 1, \operatorname{Re} p > -|\operatorname{Re} c|].$$

$$14. \int_0^\infty x^{2n+1} \frac{e^{-cx}}{\operatorname{sh} cx} dx = \frac{\pi^{2n+2}}{2(n+1)c^{2n+2}} |B_{2n+2}| \quad [\operatorname{Re} c > 0].$$

$$15. \int_0^\infty x^3 \frac{e^{-2ncx}}{\operatorname{sh} cx} dx = \frac{1}{8} \left(\frac{\pi}{c}\right)^4 - \frac{12}{c^4} \sum_{k=1}^n \frac{1}{(2k-1)^4} \quad [\operatorname{Re} c > 0; n = 1, 2, 3, \dots].$$

$$16. \int_0^\infty x^{\alpha-1} \frac{e^{-px}}{\operatorname{ch} cx} dx = 2^{1-2\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{1}{4} + \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{3}{4} + \frac{p}{4c}\right) \right] \quad [\operatorname{Re} \alpha > 0, \operatorname{Re} p > -|\operatorname{Re} c|].$$

$$17. \int_0^\infty x^{\alpha-1} \frac{e^{-cx}}{\operatorname{ch} cx} dx = \frac{2^{1-\alpha}}{c^\alpha} (1 - 2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha) \quad [\operatorname{Re} \alpha, \operatorname{Re} c > 0].$$

$$18. \int_0^\infty x^{2n+1} \frac{e^{-cx}}{\operatorname{ch} cx} dx = \frac{1 - 2^{1-2n}}{2(n+1)c^{2n+2}} \pi^{2n+2} |B_{2n+2}| \quad [\operatorname{Re} c > 0].$$

$$19. \int_{-\infty}^\infty \frac{e^{iax}}{\operatorname{ch} cx} dx = \frac{\pi}{c \operatorname{ch} [a\pi/(2c)]} \quad [a, \operatorname{Re} c > 0].$$

2.4.11. Интегралы от e^{-px} и гиперболических функций.

$$1. \int_0^\infty e^{-px} |\operatorname{sh} bx| dx = \frac{b}{p^2 - b^2} \operatorname{ctg} \frac{p\pi}{2b} \quad [\operatorname{Re} p > |\operatorname{Re} b|].$$

$$2. \int_0^\infty e^{-px} |\operatorname{ch} bx| dx = \frac{1}{p^2 - b^2} \left(p - b \operatorname{cosech} \frac{p\pi}{2b} \right) \quad [\operatorname{Re} p > |\operatorname{Re} b|].$$

$$3. \int_0^\infty e^{-px} |\operatorname{sh} bx|^v dx = - \frac{\pi}{(v+1) 2^{v+1} b} \left[\operatorname{sh} \frac{p\pi}{2b} \operatorname{B} \left(1 + \frac{v}{2} + \frac{p}{2b}, 1 + \frac{v}{2} - \frac{p}{2b} \right) \right]^{-1} \quad [\operatorname{Re} p > \operatorname{Re} v | \operatorname{Re} b|].$$

$$4. \int_0^{\infty} e^{-px} \operatorname{th} cx \, dx = \frac{1}{c} \beta\left(\frac{p}{2c}\right) - \frac{1}{p} \quad [c, \operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} \, dx = \frac{1}{2c} \left[\psi\left(\frac{c+p+b}{2c}\right) - \psi\left(\frac{c+p-b}{2c}\right) \right] \\ [\operatorname{Re} p > |\operatorname{Re} b| - |\operatorname{Re} c|].$$

$$6. \int_0^{\infty} e^{-cx} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} \, dx = \frac{1}{b} - \frac{\pi}{2c} \operatorname{ctg} \frac{b\pi}{2c} \quad [2 \operatorname{Re} c > |\operatorname{Re} b|].$$

$$7. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} px}{\operatorname{ch}^2 px} \, dx = \frac{1 - \ln 2}{p} \quad [\operatorname{Re} p > 0].$$

2.4.12. Интегралы от $A(x)$, e^{-px} и гиперболических функций.

$$1. \int_{-a}^a x^k \sqrt{a^2 - x^2} e^{-px} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} \, dx = \frac{a^{2+k}}{2} [I_k(ap+ab) \mp I_k(ap-ab)] \\ [k=0 \text{ или } 1; a, \operatorname{Re} p > 0; |\arg b| < \pi].$$

$$2. \int_0^{\infty} \frac{(x+z)^n}{x} e^{-px} \operatorname{sh} bx \, dx = (-1)^n b p^{-n-1} \sum_{k=0}^{\infty} \left(\frac{b}{2p}\right)^{2k} L_n^{-n-2k-1}(pz) \\ [\operatorname{Re} p > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$3. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{th} cx \, dx = 2^{1-2\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{2c+p}{4c}\right) \right] - p^{-\alpha} \Gamma(\alpha) \\ [\operatorname{Re} p > 0; \operatorname{Re} \alpha > -1].$$

$$4. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{th} cx \, dx = \ln \frac{p}{4c} + 2 \ln \Gamma\left(\frac{p}{4c}\right) - 2 \ln \Gamma\left(\frac{p+2c}{4c}\right) \quad [c, \operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{cth} cx \, dx = \Gamma(\alpha) \left[2^{1-\alpha} c^{-\alpha} \zeta\left(\alpha, \frac{p}{2c}\right) - p^{-\alpha} \right] \\ [c, \operatorname{Re} p > 0; \operatorname{Re} \alpha > 1].$$

$$6. \int_0^{\infty} x e^{-px} \operatorname{cth} px \, dx = \frac{1}{p^2} \left(\frac{\pi^2}{3} - 1 \right) \quad [\operatorname{Re} p > 0].$$

$$7. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{sh} bx \, dx = \frac{1}{4} \ln \frac{p^2 - (a-b)^2}{p^2 - (a+b)^2} \quad [\operatorname{Re} p > |\operatorname{Re} a| + |\operatorname{Re} b|].$$

$$8. \int_0^{\infty} \frac{1}{x^2} e^{-px} \operatorname{sh} ax \operatorname{sh} bx dx =$$

$$= \frac{a}{4} \ln \frac{(p+b)^2 - a^2}{(p-b)^2 - a^2} + \frac{b}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2} + \frac{p}{4} \ln \frac{p^2 - (a+b)^2}{p^2 - (a-b)^2}$$

[Re $p \geq |\operatorname{Re} a| + |\operatorname{Re} b|$].

$$9. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{ch} bx dx = \frac{1}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2}$$

[Re $p > |\operatorname{Re} a| + |\operatorname{Re} b|$; $p^2 + a^2 \geq b^2$].

$$10. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{sh} px} dx = \frac{1}{2} \ln \frac{b\pi}{p \sin(b\pi/p)}$$

[Re $p > |\operatorname{Re} b|$].

$$11. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{ch} 2bx} dx = \ln \Gamma\left(\frac{p+6b}{8b}\right) - \ln \Gamma\left(\frac{p+2b}{4b}\right) - \frac{1}{2} \ln \frac{p}{8b}$$

[Re $p > 0$].

$$12. \int_0^{\infty} \frac{1}{x} \frac{e^{-2px}}{\operatorname{ch} px} \operatorname{th} \frac{px}{2} dx = 2 \ln \frac{\pi}{2\sqrt{2}}$$

[Re $p > 0$].

2.4.13. Интегралы от $x^\alpha (e^x + a)^p e^{-px} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} + 1} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \Gamma(\alpha) \begin{Bmatrix} b^{-\alpha} i^{-\alpha-\delta} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} +$$

$$+ \frac{(2a)^{-\alpha}}{2} \left[\zeta\left(\alpha, \frac{a-b}{2a}\right) \mp \zeta\left(\alpha, \frac{a+b}{2a}\right) - \zeta\left(\alpha, -\frac{b}{2a}\right) \pm \zeta\left(\alpha, \frac{b}{2a}\right) \right]$$

[Re $\alpha > -\delta$; Re $a > |\operatorname{Re} b|$; $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$].

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} - 1} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \frac{e^{-\alpha}}{2} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{a-b}{a}\right) \mp \zeta\left(\alpha, \frac{a+b}{a}\right) \right]$$

[Re $\alpha > \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$; Re $a > |\operatorname{Re} b|$].

$$3. \int_0^{\infty} \frac{\operatorname{sh} bx}{e^{ax} \pm 1} dx = \mp \frac{1}{2b} \pm \frac{\pi}{2a \sin(b\pi/a)} \begin{Bmatrix} 1 \\ \cos(b\pi/2) \end{Bmatrix}$$

[Re $a > |\operatorname{Re} b|$].

$$4. \int_0^{\infty} \frac{x \operatorname{ch} bx}{e^{ax} - 1} dx = \frac{\pi^2}{2a^2 \sin^2(b\pi/a)} - \frac{1}{2b^2}$$

[Re $a > |\operatorname{Re} b|$].

$$5. \int_0^{\infty} \frac{\operatorname{sh}^2 bx}{e^{ax} - 1} dx = -\frac{1}{2a} \left[\zeta + \operatorname{Re} \psi\left(-\frac{2b}{a}\right) \right]$$

[Re $a > 2|\operatorname{Re} b|$].

$$6. \int_0^{\infty} \frac{x^{-1} \operatorname{sh}^2 bx}{e^{ax} + 1} dx = \frac{1}{4} \ln \left(\frac{a}{b\pi} \operatorname{tg} \frac{b\pi}{a} \right)$$

[Re $a > 2|\operatorname{Re} b|$].

$$7. \int_0^{\infty} \frac{x^{-1} \operatorname{sh}^2 bx}{e^{ax} - 1} dx = \frac{1}{4} \ln \left(\frac{a}{2b\pi} \sin \frac{2b\pi}{a} \right) \quad [\operatorname{Re} a > 2; \operatorname{Re} b \neq 0].$$

$$8. \int_0^{\infty} \frac{e^{-ax}}{(e^{-ax} + 1)^2} \operatorname{ch} bx dx = \frac{\pi b}{2a^2 \sin(b\pi/a)} \quad [\operatorname{Re} a > |\operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{\operatorname{sh} bx}{e^{ax} - e^{cx}} dx = \frac{1}{2(c-a)} \left[\psi \left(\frac{a-b}{a-c} \right) - \psi \left(\frac{a+b}{a-c} \right) \right] \quad [\max(\operatorname{Re} a, \operatorname{Re} c) > |\operatorname{Re} b|].$$

$$10. \int_0^{\infty} \frac{x^{2n} \operatorname{sh} bx}{e^{mcx} - e^{(m-2)c}} dx = \frac{\partial^{2n}}{\partial b^{2n}} \left[\pm \frac{\pi}{4c} \left\{ \operatorname{tg} [b\pi/(2c)] \right\} + \right. \\ \left. + \frac{1}{2b} \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} - \sum_{k=1}^{[(m-1)/2]} \frac{b}{c^2 (2k-2\lambda)^2 - b^2} \right] \\ \left[\operatorname{Re}(mc-b) > 0; \operatorname{Re}[(m-2)c-b] > 0; \left\{ \begin{matrix} \lambda = 1/2, m = 1, 3, 5, \dots \\ \lambda = 0, m = 2, 4, 6, \dots \end{matrix} \right\} \right].$$

$$11. \int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x + 1)^{\rho}} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = I_{\rho}^{\pm} \quad [0 < \operatorname{Re} a + |\operatorname{Re} b| < \rho].$$

$$I_{\rho}^{\pm} = \frac{1}{\Gamma(\rho)} \left\{ \begin{matrix} -i \operatorname{Im} z \\ \operatorname{Re} z \end{matrix} \right\}, \quad z = \Gamma(a-b) \Gamma(\rho-a+b),$$

$$I_1^{\pm} = \mp \frac{\pi}{\sin(a-b)\pi \sin(a+b)\pi} \left\{ \begin{matrix} \cos a\pi \sin b\pi \\ \sin a\pi \cos b\pi \end{matrix} \right\},$$

$$I_2^{\pm} = \mp \frac{\pi}{\sin(a-b)\pi \sin(a+b)\pi} \left[(1-a) \left\{ \begin{matrix} \cos a\pi \sin b\pi \\ \sin a\pi \cos b\pi \end{matrix} \right\} + b \left\{ \begin{matrix} \sin a\pi \cos b\pi \\ \cos a\pi \sin b\pi \end{matrix} \right\} \right],$$

$$I_n^{\pm} = \frac{1}{n-a} [(n-1-a) I_{n-1}^{\pm} - b I_{n-1}^{\mp}] \quad [n = 2, 3, 4, \dots].$$

2.4.14. Интегралы от $x^{\alpha} e^{-px} \left\{ \frac{b + \operatorname{sh} ax}{b + \operatorname{ch} ax} \right\}^{\sigma}$.

$$1. \int_a^{\infty} e^{-px} (\operatorname{ch} x - \operatorname{ch} a)^{\nu-1} dx = -i \sqrt{\frac{2}{\pi}} e^{\nu\pi i} \Gamma(\nu) \operatorname{sh}^{\nu-1/2} a Q_{\rho-1/2}^{\nu-1/2}(\operatorname{ch} a) \\ [a, \operatorname{Re} \nu > 0; \operatorname{Re} \rho > |\operatorname{Re} \nu - 1|].$$

$$2. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} x + \operatorname{ch} a} dx = \frac{1}{\operatorname{sh} a} \left[\frac{1}{\rho} + 2\rho \sum_{k=1}^{\infty} \frac{(-1)^k e^{-ak}}{p^2 - k^2} \right] \\ [a > 0; \operatorname{Re} \rho > -1; \text{см. 5.2.5}].$$

$$3. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} x - \cos a} dx = -\frac{2}{\sin a} \sum_{k=1}^{\infty} \frac{\sin ak}{\rho + ak} \quad [\operatorname{Re} \rho > -1; a \neq \pm 2\pi n; \text{см. 5.4.3}].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{\operatorname{ch} px + \cos a} dx = \left(\frac{2\pi}{p}\right)^{\alpha} (\sin a \sin \alpha\pi)^{-1} \left[\cos\left(\frac{2a+\alpha\pi}{2}\right) \times \right. \\ \left. \times \zeta\left(1-\alpha, \frac{\pi+a}{2\pi}\right) - \cos\left(\frac{2a-\alpha\pi}{2}\right) \zeta\left(1-\alpha, \frac{\pi-a}{2\pi}\right) \right] \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |a| < \pi].$$

$$5. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{\operatorname{ch} px - \cos a} dx = \frac{2\Gamma(\alpha)}{p^{\alpha} \sin a} \sum_{k=0}^{\infty} \frac{\sin ak}{(k+1)^{\alpha}} \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; a \neq \pm 2\pi n; \text{ см. 5.4.3}]$$

$$6. \int_0^{\infty} x^{2n+1} \frac{e^{-px} - \cos a}{\operatorname{ch} px - \cos a} dx = I_n \quad [\operatorname{Re} p > 0],$$

$$I_0 = p^{-2} \left(a\pi - \frac{a^2}{2} - \frac{\pi^2}{3} \right), \quad I_n = 2(2n+1)! p^{-2n-2} \sum_{k=1}^{\infty} \frac{\cos ak}{k^{2n+2}} \quad [\text{см. 5.4.2}].$$

$$7. \int_0^{\infty} \frac{e^{px} + \cos a}{(\operatorname{ch} px + \cos a)^2} dx = \frac{1}{p} \left(\frac{a}{\sin a} + \frac{1}{1 + \cos a} \right) \quad [\operatorname{Re} p > 0; |a| < \pi].$$

$$8. \int_0^{\infty} e^{-px} (\operatorname{ch} cx - 1)^{\nu} dx = 2^{-\nu} c^{-1} B\left(\frac{p}{c} - \nu, 2\nu + 1\right) \\ [\operatorname{Re} c > 0, 2 \operatorname{Re} \nu > -1; \operatorname{Re} p > \operatorname{Re}(c\nu)].$$

$$9. \int_{-\infty}^{\infty} \frac{e^{-ibx}}{\operatorname{sh} x + \operatorname{sh} a} dx = -\frac{i\pi e^{iab}}{\operatorname{sh} b\pi \operatorname{ch} a} (\operatorname{ch} b\pi - e^{2iab}) \quad [a, b > 0].$$

2.4.15. Интегралы от $x^{\alpha} e^{-ax^2-cx} \left\{ \begin{matrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{matrix} \right\}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-ax^2} \left\{ \begin{matrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{matrix} \right\} dx = \\ = \frac{(2a)^{-\alpha/2}}{2} \Gamma(\alpha) \exp\left(\frac{b^2}{8a}\right) \left[D_{-\alpha}\left(-\frac{b}{\sqrt{2a}}\right) \mp D_{-\alpha}\left(\frac{b}{\sqrt{2a}}\right) \right] \quad [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}].$$

$$2. \int_0^{\infty} e^{-ax^2} \left\{ \begin{matrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{matrix} \right\} dx = \mp \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \left\{ \begin{matrix} \operatorname{erf}(-b/(2\sqrt{a})) \\ 1 \end{matrix} \right\} \quad [\operatorname{Re} a > 0].$$

$$3. \int_0^{\infty} x^{2n+\delta} e^{-ax^2} \left\{ \begin{matrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{matrix} \right\} dx = (-1)^{n+\delta} i^{\delta} \sqrt{\pi} 2^{-2n-1-\delta} a^{-n-1/2-\delta/2} \times \\ \times \exp\left(\frac{b^2}{4a}\right) H_{2n+\delta}\left(\frac{i\delta b}{2\sqrt{a}}\right) \quad [\operatorname{Re} a > 0; \delta = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}].$$

4. $\int_0^{\infty} x^2 e^{-ax^2} \operatorname{sh} bx \, dx = \frac{\sqrt{\pi}}{8a^{5/2}} (2a + b^2) \exp\left(\frac{b^2}{4a}\right) \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) - \frac{b}{4a^2}$ [Re $a > 0$].
5. $\int_0^{\infty} \frac{1}{x} e^{-ax^2} \operatorname{sh} bx \, dx = \frac{\pi}{2} \operatorname{erf}\left(\frac{ib}{2\sqrt{a}}\right)$ [Re $a > 0$].
6. $\int_0^{\infty} x^{1/2} e^{-ax^2} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \frac{\pi}{4} \left(\frac{b}{2a}\right)^{3/2} \exp\left(\frac{b^2}{8a}\right) \left[I_{\mp 1/4}\left(\frac{b^2}{8a}\right) + I_{\pm 3/4}\left(\frac{b^2}{8a}\right) \right]$ [Re $a > 0$].
7. $\int_0^{\infty} x^{-1/2} e^{-ax^2} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \frac{\pi}{2} \sqrt{\frac{b}{2a}} \exp\left(\frac{b^2}{8a}\right) I_{\pm 1/4}\left(\frac{b^2}{8a}\right)$ [Re $a > 0$].
8. $\int_0^{\infty} e^{-ax^2-cx} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \frac{1}{4} \sqrt{\frac{\pi}{a}} \left\{ \exp\left[\frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp \exp\left[\frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\}$ [Re $a > 0$].
9. $\int_0^{\infty} x^{\alpha-1} e^{-ax^2-cx} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx =$
 $= \frac{\Gamma(\alpha)}{2(2a)^{\alpha/2}} \exp\left(\frac{c^2+b^2}{8a}\right) \left[\exp\left(-\frac{bc}{4a}\right) D_{-\alpha}\left(\frac{c-b}{\sqrt{2a}}\right) \mp \exp\left(\frac{bc}{4a}\right) D_{-\alpha}\left(\frac{c+b}{\sqrt{2a}}\right) \right]$
[Re $a > 0$; Re $\alpha > -\left\{\frac{1}{0}\right\}$].
10. $\int_0^{\infty} x e^{-ax^2-cx} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = -\frac{1}{8a} \sqrt{\frac{\pi}{a}} \left\{ (c-b) \exp\left[\mp \frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp (c+b) \exp\left[\mp \frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\} + \left\{ \begin{matrix} 0 \\ (2a)^{-1} \end{matrix} \right\}$ [Re $a > 0$].
11. $\int_0^{\infty} x^{\delta} \frac{e^{-ax^2}}{x^2+z^2} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx =$
 $= \frac{\pi}{4} \left\{ \begin{matrix} i \\ z^{-1} \end{matrix} \right\} e^{az^2} \left[2 \left\{ \begin{matrix} i \sin bz \\ \cos bz \end{matrix} \right\} - \operatorname{erf}\left(z\sqrt{a} + \frac{ib}{2\sqrt{a}}\right) \pm \operatorname{erf}\left(z\sqrt{a} - \frac{ib}{2\sqrt{a}}\right) \right]$
[Re a , Re $z > 0$; $\delta = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$].
12. $\int_0^{\infty} e^{-ax^2} \operatorname{sh} [(2n+1)x] \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-1} dx = \left\{ \begin{matrix} 1 \\ -i \end{matrix} \right\} \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^n (\pm 1)^k \exp\left(\pm \frac{k^2}{a}\right) \right]$ [Re $a > 0$].
13. $\int_{-\infty}^{\infty} e^{-ax^2-cx} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2+c^2}{4a}\right) \begin{Bmatrix} \operatorname{sh} [bc/(2a)] \\ \operatorname{ch} [bc/(2a)] \end{Bmatrix}$ [Re $a > 0$].

2.4.16. Интегралы от $A(x) \exp(-p \sqrt{x^2+z^2}) \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix}$.

$$1. \int_0^\infty \exp(-p \sqrt{x^2+z^2}) \operatorname{ch} bx \, dx = \frac{pz}{\sqrt{p^2-b^2}} K_1(z \sqrt{p^2-b^2})$$

[$\operatorname{Re} p > |\operatorname{Re} b|$; $\operatorname{Re} z > 0$].

$$2. \int_0^\infty x^{\alpha-1} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{sh} bx \, dx = J_\alpha$$

[$\operatorname{Re} \alpha > -1$; $\operatorname{Re} z > 0$; $\operatorname{Re} p > |\operatorname{Re} b|$].

$$J_{2n} = z \frac{\partial^{2n-2}}{\partial b^{2n-2}} \left[\frac{b}{\sqrt{p^2-b^2}} K_1(z \sqrt{p^2-b^2}) \right],$$

$$J_4 = \frac{bz^2}{(p^2-b^2)^2} (3p^2+b^2) K_2(z \sqrt{p^2-b^2}) + \frac{b^3 z^3}{(p^2-b^2)^{3/2}} K_1(z \sqrt{p^2-b^2}).$$

$$3. \int_0^\infty x^{\alpha-1} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{ch} bx \, dx = I_\alpha$$

[$\operatorname{Re} \alpha, \operatorname{Re} z > 0$; $\operatorname{Re} p > |\operatorname{Re} b|$].

$$I_{2n+1} = \frac{\partial^{2n}}{\partial b^{2n}} K_0(z \sqrt{p^2-b^2}),$$

$$I_3 = \frac{z(p^2+b^2)}{(p^2-b^2)^{3/2}} K_1(z \sqrt{p^2-b^2}) + \frac{b^2 z^2}{p^2-b^2} K_0(z \sqrt{p^2-b^2}).$$

$$4. \int_0^\infty x^{1-\delta} (\sqrt{x^2+z^2}-z)^{\pm 1/2} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx =$$

$$= b^\delta \sqrt{\frac{\pi}{2}} \frac{(\sqrt{p^2-b^2}+p)^{\mp 1/2}}{\sqrt{p^2-b^2}} \exp(-z \sqrt{p^2-b^2})$$

[$\operatorname{Re} z > 0$; $\operatorname{Re} p > |\operatorname{Re} b|$; $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$].

2.4.17. Интегралы от $x^\alpha e^{f(x)} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix}$.

$$1. \int_0^\infty e^{-ax} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx = \frac{z}{2b} [e^{\pi i/2} S_{0, 1/3}(ze^{3\pi i/2}) \mp S_{0, 1/3}(z)]$$

[$z = 2(b/3)^{3/2} a^{-1/2}$; $\operatorname{Re} a > 0$].

$$2. \int_0^\infty x^{\alpha-1} e^{-px-q/x} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx =$$

$$= q^{\alpha/2} [(p-b)^{-\alpha/2} K_\alpha(2\sqrt{q}\sqrt{p-b}) \mp (p+b)^{-\alpha/2} K_\alpha(2\sqrt{q}\sqrt{p+b})]$$

[$\operatorname{Re} p > |\operatorname{Re} b|$; $\operatorname{Re} q > 0$].

$$3. \int_0^\infty x^{-k/2} e^{-px-q/x} \begin{Bmatrix} \operatorname{sh} bx \\ \operatorname{ch} bx \end{Bmatrix} dx =$$

$$= \frac{\sqrt{\pi}}{2q^{(k-1)/4}} [(p-b)^{(k-3)/4} \exp(-2\sqrt{q(p-b)}) \mp$$

$$\mp (p+b)^{(k-3)/4} \exp(-2\sqrt{q(p+b)})]$$

[$k=1$ или 3 ; $\operatorname{Re} p > |\operatorname{Re} b|$; $\operatorname{Re} q > 0$].

$$4. \int_{-\infty}^{\infty} \exp(ax - e^x) \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx = \mp |\Gamma(a-b)| \left\{ \begin{matrix} \sin t \\ \cos t \end{matrix} \right\}$$

$$\left[t = \ln \frac{\Gamma(a-b)}{|\Gamma(a-b)|}; \operatorname{Re} a > |\operatorname{Re} b| \right].$$

2.4.18. Интегралы от $f(x) \exp \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} dx =$$

$$= \pm \frac{\pi}{2} \left(\operatorname{tg} \frac{b\pi}{2} \right)^{\mp 1} [J_b(a) - J_b(a)] - \frac{\pi}{2} [E_b(a) + Y_b(a)] = b^\delta S_{-\delta, b}(a)$$

[$\operatorname{Re} a > 0$; b — нецелое].

$$2. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} \operatorname{ch} x dx = \frac{b^\delta}{a} S_{1-\delta, b}(a)$$

[$\operatorname{Re} a > 0$].

$$3. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\} \operatorname{sh}^{-1/2} x dx =$$

$$= \frac{\pi \sqrt{\pi a}}{8\delta} \left[e^{\pi b i/2} H_{(1+b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1-b)/2}^{(2)} \left(\frac{a}{2} \right) - e^{-\pi b i/2} H_{(1-b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1+b)/2}^{(2)} \left(\frac{a}{2} \right) \right]$$

[$\operatorname{Re} a > 0$].

$$4. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{ch} bx dx = K_b(a)$$

[$\operatorname{Re} a > 0$].

$$5. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} bx \operatorname{sh} x dx = \frac{b}{a} K_b(a)$$

[$\operatorname{Re} a > 0$].

$$6. \int_0^{\infty} e^{-a \operatorname{ch} x} \left\{ \begin{matrix} \operatorname{sh}(x/2) \operatorname{sh} bx \\ \operatorname{ch}(x/2) \operatorname{ch} bx \end{matrix} \right\} dx = \mp \left\{ \begin{matrix} i \operatorname{Im} K_{1/2-b}(a) \\ \operatorname{Re} K_{1/2-b}(a) \end{matrix} \right\}$$

[$a, b > 0$].

$$7. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx}{\sqrt{\operatorname{ch} x}} dx = \sqrt{\frac{a}{2\pi}} K_{b/2+1/4} \left(\frac{a}{2} \right) K_{b/2-1/4} \left(\frac{a}{2} \right)$$

[$\operatorname{Re} a > 0$].

$$8. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx \operatorname{sh} x}{\sqrt{\operatorname{ch} x} \sqrt{\operatorname{ch} x - 1}} dx = K_b(a)$$

[$\operatorname{Re} a > 0$].

$$9. \int_0^{\infty} e^{-a \operatorname{sh} x - px} dx = \frac{\pi}{\operatorname{sh} p\pi} [J_p(a) - J_p(a)]$$

[$\operatorname{Re} a > 0$].

$$10. \int_0^{\infty} e^{-a \operatorname{sh} x - px} \frac{dx}{V \operatorname{sh} x} =$$

$$= \frac{\pi}{2} V \frac{\pi a}{2} \left[J_{1/4-p/2} \left(\frac{a}{2} \right) J_{-1/4-p/2} \left(\frac{a}{2} \right) + Y_{1/4-p/2} \left(\frac{a}{2} \right) Y_{-1/4-p/2} \left(\frac{a}{2} \right) \right]$$

[Re $a > 0$].

$$11. \int_0^{\infty} e^{-a \operatorname{ch} x - px} dx = \frac{1}{\operatorname{sh} p\pi} \left[\int_0^{\pi} e^{a \cos x} \cos px dx - \pi I_p(a) \right]$$

[Re $a > 0$].

$$12. \int_{-\infty}^{\infty} e^{-a \operatorname{ch} x - px} dx = 2K_p(a)$$

[Re $a > 0$].

$$13. \int_{-\infty}^{\infty} e^{ia \operatorname{sh} x - px} dx = 2e^{-p\pi i/2} K_p(a)$$

[|Re p | < 1; $a > 0$].

$$14. \int_{-\infty}^{\infty} e^{\pm ia \operatorname{ch} x - px} dx = \pm \pi i e^{\pm p\pi i/2} H_p^{(3/2 \mp 1/2)}(a)$$

[$\pm \operatorname{Re}(ia) < 0$].

2.4.19. Интегралы от $A(x) e^{-px} \left\{ \frac{\operatorname{sh} V \overline{ax^2 + bx + c}}{\operatorname{ch} V \overline{ax^2 + bx + c}} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{e^{-px}}{V x V \overline{a^2 - x^2}} \operatorname{ch}(c V \overline{a^2 - x^2}) dx =$$

$$= -\frac{\pi}{2} V \pi p \left[I_{1/4}(z_+) I_{1/4}(z_-) - I_{-1/4}(z_+) I_{-1/4}(z_-) \right]$$

[$2z_{\pm} = \sqrt{c^2 + p^2} \pm c$; $a, c > 0$].

$$2. \int_0^a \frac{e^{-px}}{V \overline{ax - x^2}} \operatorname{ch}(c V \overline{ax - x^2}) dx = \pi e^{-ap/2} I_0 \left(\frac{a}{2} V \overline{p^2 + c^2} \right)$$

[$a, c > 0$].

$$3. \int_a^{\infty} x^{\varepsilon} e^{-px} \operatorname{sh}(c V \overline{x^2 - a^2}) dx = \frac{a^{1+\varepsilon} c p^{\varepsilon}}{(p^2 - c^2)^{(1+\varepsilon)/2}} K_{1+\varepsilon}(a V \overline{p^2 - c^2})$$

[$\varepsilon = 0$ или 1; $a > 0$; Re $p > |\operatorname{Re} c|$].

$$4. \int_a^{\infty} \frac{e^{-px}}{V x + a} \left\{ \frac{\operatorname{sh}(c V \overline{x^2 - a^2})}{\operatorname{ch}(c V \overline{x^2 - a^2})} \right\} dx =$$

$$= V \frac{\pi}{2} \frac{c^{\delta}}{V \overline{p^2 - c^2}} (V \overline{p^2 - c^2} + p)^{-1/2} \exp(-a V \overline{p^2 - c^2})$$

[$a > 0$; Re $p > |\operatorname{Re} c|$].

$$5. \int_a^{\infty} \frac{e^{-px}}{V x - a} \operatorname{ch}(c V \overline{x^2 - a^2}) dx = V \frac{\pi}{2} V \frac{p + V \overline{p^2 - c^2}}{p^2 - c^2} \exp(-a V \overline{p^2 - c^2})$$

[$a > 0$; Re $p > |\operatorname{Re} c|$].

6.
$$\int_a^\infty (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \begin{matrix} \text{sh} (c \sqrt{x^2-a^2}) \\ \text{ch} (c \sqrt{x^2-a^2}) \end{matrix} \right\} dx =$$

$$= \frac{2^{\pm 1/2}}{a \sqrt{c}} \Gamma \left(\alpha + \frac{\delta}{2} \right) W_{1/4-\alpha, 1/4} (z_+) M_{1/4-\alpha, \pm 1/4} (z_-)$$

$$[z_{\pm} = a(p \pm \sqrt{p^2 - c^2}); a > 0; \text{Re } p > |\text{Re } c|].$$
7.
$$\int_a^\infty \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \begin{matrix} \text{sh} (c \sqrt{x^2-a^2}) \\ \text{ch} (c \sqrt{x^2-a^2}) \end{matrix} \right\} dx = \frac{\pi}{\sqrt{2a}} e^{ap} \text{erfc} (z_+) \left\{ \begin{matrix} \text{erf} (z_-) \\ 1 \end{matrix} \right\}$$

$$[z_{\pm}^2 = a(p \pm \sqrt{p^2 - c^2}); a > 0; \text{Re } p > |\text{Re } c|].$$
8.
$$\int_a^\infty \frac{e^{-px}}{(x^2-a^2)^{3/4}} \left\{ \begin{matrix} \text{sh} (c \sqrt{x^2-a^2}) \\ \text{ch} (c \sqrt{x^2-a^2}) \end{matrix} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4} (z_+) I_{\pm 1/4} (z_-)$$

$$[2z_{\pm} = a(p \pm \sqrt{p^2 - c^2}); a > 0; \text{Re } p > |\text{Re } c|].$$
9.
$$\int_a^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2-a^2}} \text{ch} (c \sqrt{x^2-a^2}) dx = \frac{1}{\sqrt{2\pi p}} K_{1/4} (z_+) K_{1/4} (z_-)$$

$$[2z_{\pm} = a(p \pm \sqrt{p^2 - c^2}); a > 0; \text{Re } p > |\text{Re } c|].$$
10.
$$\int_a^\infty \frac{x^\varepsilon e^{-px}}{\sqrt{x^2-a^2}} \text{ch} (c \sqrt{x^2-a^2}) dx = \frac{a^\varepsilon p^\varepsilon}{(p^2-c^2)^{\varepsilon/2}} K_\varepsilon (a \sqrt{p^2-c^2})$$

$$[\varepsilon = 0 \text{ или } 1; a > 0; \text{Re } p > |\text{Re } c|].$$
11.
$$\int_a^\infty \frac{(x + \sqrt{x^2-a^2})^\nu \mp (x - \sqrt{x^2-a^2})^\nu}{\sqrt{x^2-a^2}} e^{-px} \left\{ \begin{matrix} \text{sh} (c \sqrt{x^2-a^2}) \\ \text{ch} (c \sqrt{x^2-a^2}) \end{matrix} \right\} dx =$$

$$= a^\nu \left[\left(\frac{p+c}{p-c} \right)^{\nu/2} \mp \left(\frac{p+c}{p-c} \right)^{-\nu/2} \right] K_\nu (a \sqrt{p^2-c^2}) \quad [a > 0; \text{Re } p > |\text{Re } c|].$$
12.
$$\int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2+z^2}} \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+z^2}) \\ \text{sh} (b \sqrt{x^2+z^2}) \end{matrix} \right\} dx =$$

$$= \frac{\pi i^\delta}{4} \sqrt{\pi p} \left[\left\{ \begin{matrix} J_{1/4} (z_+) \\ Y_{1/4} (z_+) \end{matrix} \right\} Y_{1/4} (z_-) \mp \left\{ \begin{matrix} Y_{1/4} (z_+) \\ J_{1/4} (z_+) \end{matrix} \right\} J_{1/4} (z_-) \right]$$

$$[2z_{\pm} = z(\sqrt{p^2-b^2} \pm ib); \text{Re } z > 0; \text{Re } p > |\text{Re } b|].$$
13.
$$\int_0^\infty e^{-px} \text{sh} (b \sqrt{x^2+xz}) dx = \frac{bz}{2 \sqrt{p^2-b^2}} e^{pz/2} K_1 \left(\frac{z}{2} \sqrt{p^2-b^2} \right)$$

$$[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$$
14.
$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-px} \text{ch} (b \sqrt{x^2+xz}) dx =$$

$$= \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2-b^2}}{p^2-b^2}} \exp \left[\frac{z}{2} (p - \sqrt{p^2-b^2}) \right]$$

$$[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$$

$$15. \int_0^\infty \frac{e^{-px}}{\sqrt{x+z}} \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+xz}) \\ \text{ch} (b \sqrt{x^2+xz}) \end{matrix} \right\} dx =$$

$$= \sqrt{\frac{\pi}{2}} b^\delta (p + \sqrt{p^2 - b^2})^{-1/2} (p^2 - b^2)^{-1/2} \exp \left[\frac{z}{2} (p - \sqrt{p^2 - b^2}) \right]$$

[Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

$$16. \int_0^\infty x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+xz}) \\ \text{ch} (b \sqrt{x^2+xz}) \end{matrix} \right\} dx =$$

$$= \frac{2^{\delta+1/2}}{z \sqrt{b}} \Gamma \left(\alpha + \frac{\delta}{2} \right) e^{pz/2} W_{1/4-\alpha, 1/4} (z_+) M_{1/4-\alpha, \pm 1/4} (z_-)$$

[$2z_\pm = z(p \pm \sqrt{p^2 - b^2})$; $2\text{Re } \alpha > -\delta$; Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

$$17. \int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x+z}} \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+xz}) \\ \text{ch} (b \sqrt{x^2+xz}) \end{matrix} \right\} dx = \frac{\pi}{\sqrt{z}} e^{pz} \text{erfc} (z_+) \left\{ \begin{matrix} \text{erf} (z_-) \\ 1 \end{matrix} \right\}$$

[$2z_\pm = z(p \pm \sqrt{p^2 - b^2})$; Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

$$18. \int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x+z}} \text{ch} (b \sqrt{x^2+xz}) dx = e^{pz/2} K_0 \left(\frac{z}{2} \sqrt{p^2 - b^2} \right)$$

[Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

$$19. \int_0^\infty (x^2+xz)^{-3/4} e^{-px} \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+xz}) \\ \text{ch} (b \sqrt{x^2+xz}) \end{matrix} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{pz/2} K_{1/4} (z_+) I_{\pm 1/4} (z_-)$$

[$4z_\pm = z(p \pm \sqrt{p^2 - b^2})$; Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

$$20. \int_0^\infty \frac{1}{\sqrt{x^2+xz}} \left[\left(\frac{z}{2} + x + \sqrt{x^2+xz} \right)^\nu \mp \left(\frac{z}{2} + x - \sqrt{x^2+xz} \right)^\nu \right] e^{-px} \times$$

$$\times \left\{ \begin{matrix} \text{sh} (b \sqrt{x^2+xz}) \\ \text{ch} (b \sqrt{x^2+xz}) \end{matrix} \right\} dx = \left(\frac{z}{2} \right)^\nu \left[\left(\frac{p+b}{p-b} \right)^{\nu/2} \mp \left(\frac{p+b}{p-b} \right)^{-\nu/2} \right] e^{pz/2} K_\nu \left(\frac{z}{2} \sqrt{p^2 - b^2} \right)$$

[Re $p > |\text{Re } b|$; $|\arg z| < \pi$].

2.4.20. Интегралы от $A(x) e^{f(x)} \left\{ \begin{matrix} \text{sh} [cx/(a^2-x^2)] \\ \text{ch} [cx/(a^2-x^2)] \end{matrix} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a^2-x^2}} \exp \left(-b \frac{a^2+x^2}{a^2-x^2} \right) \left\{ \begin{matrix} \text{sh} [cx/(a^2-x^2)] \\ \text{ch} [cx/(a^2-x^2)] \end{matrix} \right\} dx =$$

$$= 2^{\delta+(2\alpha-3)/4} \sqrt{\frac{a}{c^2}} \Gamma \left(\frac{\alpha+\delta}{2} \right) (\sqrt{4a^2b^2-c^2} + 2ab)^{1/4} \times$$

$$\times D_{-\alpha} \left(\sqrt{\frac{\sqrt{4a^2b^2-c^2} + 2ab}{a}} \right) M_{1/4-\alpha/2, \pm 1/4} \left(\frac{\sqrt{4a^2b^2-c^2} - 2ab}{2a} \right)$$

[Re $\alpha > -\delta$; $2ab > c > 0$; $a > 0$].

$$\begin{aligned}
 2. \int_a^\infty \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-\frac{b}{x^2-a^2}\right) \left\{ \begin{array}{l} \text{sh } [cx/(x^2-a^2)] \\ \text{ch } [cx/(x^2-a^2)] \end{array} \right\} dx = \\
 = \frac{a^{\alpha-1}}{\sqrt{c} 2^{\alpha/2+1/4-\delta}} \exp\left(\frac{b}{2a^2}\right) \Gamma\left(\frac{1+\delta-\alpha}{2}\right) [Vb^2-a^2c^2+b]^{1/4} \times \\
 \times D_{\alpha-1}\left(\frac{Vb^2-a^2c^2+b}{a}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{b-Vb^2-a^2c^2}{2a^2}\right) \\
 [\text{Re } \alpha < 1+\delta, b > ac > 0, a > 0].
 \end{aligned}$$

$$\begin{aligned}
 3. \int_a^\infty \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-b \frac{x^2+a^2}{x^2-a^2}\right) \left\{ \begin{array}{l} \text{sh } [cx/(x^2-a^2)] \\ \text{ch } [cx/(x^2-a^2)] \end{array} \right\} dx = \\
 = 2^{\delta-(2\alpha+1)/4} a^{\alpha-3/4} c^{-1/2} \Gamma\left(\frac{1+\delta-\alpha}{2}\right) (2ab + V4a^2b^2-c^2)^{1/4} \times \\
 \times D_{\alpha-1}\left(V \frac{2ab + V4a^2b^2-c^2}{a}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{2ab - V4a^2b^2-c^2}{2a}\right) \\
 [\text{Re } \alpha < 1+\delta, 2ab > c > 0, a > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-\frac{b}{x^2+z^2}\right) \left\{ \begin{array}{l} \text{sh } [cx/(x^2+z^2)] \\ \text{ch } [cx/(x^2+z^2)] \end{array} \right\} dx = \\
 = \frac{2^{\delta-1/2}}{\sqrt{\pi c}} z^{\alpha-1/2} \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \Gamma\left(\frac{\alpha+\delta}{2}\right) \exp\left(-\frac{b}{2z^2}\right) \times \\
 \times M_{1/4-\alpha/2, \pm 1/4}\left(\frac{b+Vb^2+c^2z^2}{2z^2}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{Vb^2+c^2z^2-b}{2z^2}\right) \\
 [-\delta < \text{Re } \alpha < 1+\delta; b, c, \text{Re } z > 0].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-b \frac{z^2-x^2}{z^2+x^2}\right) \left\{ \begin{array}{l} \text{sh } [cx/(x^2+z^2)] \\ \text{ch } [cx/(x^2+z^2)] \end{array} \right\} dx = \\
 = \frac{2^{\delta-1/2} z^{\alpha-1/2}}{\sqrt{\pi c}} \Gamma\left(\frac{\alpha+\delta}{2}\right) \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \times \\
 \times M_{1/4-\alpha/2, \pm 1/4}\left(\frac{V4b^2z^2+c^2+2bz}{2z}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{V4b^2z^2+c^2-2bz}{2z}\right) \\
 [-\delta < \text{Re } \alpha < 1+\delta; b, c, \text{Re } z > 0].
 \end{aligned}$$

2.4.21. Интегралы, содержащие $A(x)$, $e^{f(x)}$, $\text{sh } g(x)$, $\text{ch } g(x)$.

$$1. \int_0^\infty e^{-px} \left\{ \begin{array}{l} \text{sh } a[x] \\ \text{ch } a[x] \end{array} \right\} dx = \frac{e^p - 1}{p(e^{2p} - 2e^p \text{ch } a + 1)} \left\{ \begin{array}{l} \text{sh } a \\ e^p - \text{ch } a \end{array} \right\} \quad [\text{Re } p > |\text{Re } a|].$$

$$2. \int_0^\infty e^{-px} b[x] \left\{ \begin{array}{l} \text{sh } a[x] \\ \text{ch } a[x] \end{array} \right\} dx = \frac{e^p - 1}{p(e^{2p} - 2be^p \text{ch } a + b^2)} \left\{ \begin{array}{l} b \text{ sh } a \\ e^p - b \text{ ch } a \end{array} \right\} \\
 [\text{Re } p > \ln b + |\text{Re } a|; b \geq 1].$$

$$3. \int_0^{\infty} e^{-ax^2} \left\{ \begin{matrix} \text{sh } cx^2 \\ \text{ch } cx^2 \end{matrix} \right\} \text{ch } bx \, dx =$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{a^2 - c^2}} e^{-aA} \left(B_+ \left\{ \begin{matrix} \text{sh } cA \\ \text{ch } cA \end{matrix} \right\} - B_- \left\{ \begin{matrix} \text{ch } cA \\ \text{sh } cA \end{matrix} \right\} \right) = \frac{\sqrt{\pi}}{2 \sqrt[4]{a^2 - c^2}} e^{-aA} \left\{ \begin{matrix} \text{sh } C \\ \text{ch } C \end{matrix} \right\}$$

$$\left[A = \frac{b^2}{4(c^2 - a^2)}, B_{\pm} = \left\{ \frac{1}{2} (a \pm \sqrt{a^2 - c^2}) \right\}^{1/2}, 2C = \frac{1}{2} \ln \frac{a+c}{a-c} - Ac; \text{Re } a > |\text{Re } c| \right].$$

$$4. \int_0^{\infty} x^{-1/2} e^{-px-q/x} \left\{ \begin{matrix} \text{sh } (bx+c/x) \\ \text{ch } (bx+c/x) \end{matrix} \right\} dx =$$

$$= \frac{\sqrt{\pi}}{r} \exp \left(-rs \sqrt[4]{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - rs \sqrt[4]{\frac{(p-b)(q-c)}{(p+b)(q+c)}} \right) \left\{ \begin{matrix} \text{sh } d \\ \text{ch } d \end{matrix} \right\}$$

$$\left[d = \frac{1}{4} \ln \frac{p+b}{p-b} + rs \left(\sqrt[4]{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - \sqrt[4]{\frac{(p-b)(q-c)}{(p+b)(q+c)}} \right), \right.$$

$$\left. r = \sqrt[4]{p^2 - b^2}, s = \sqrt[4]{q^2 - c^2}; \text{Re } p > |\text{Re } b|; \text{Re } q > |\text{Re } c| \right].$$

2.4.22. Интегралы, содержащие разности $A(x)$, e^{-px} , $\left\{ \begin{matrix} \text{sh } bx \\ \text{ch } bx \end{matrix} \right\}$.

$$1. \int_0^{\infty} e^{-px} \frac{1 - \text{ch } ax}{x} dx = \frac{1}{2} \ln \left(1 - \frac{a^2}{p^2} \right) \quad [\text{Re } p > |\text{Re } a|].$$

$$2. \int_0^{\infty} e^{-px} \frac{1 - \text{ch } ax}{x^2} dx = -p \ln \frac{\sqrt{p^2 - a^2}}{p} - \frac{a}{2} \ln \frac{p+a}{p-a} \quad [\text{Re } p > |\text{Re } a|].$$

$$3. \int_0^{\infty} e^{-px} \frac{x - \text{sh } x}{x^n} dx = I_n \quad [\text{Re } p > 1]$$

$$I_1 = \frac{1}{p} - \frac{1}{2} \ln \frac{p+1}{p-1}, \quad I_2 = \frac{1}{2} \ln \left(1 - \frac{1}{p^2} \right) + \frac{p}{2} \ln \frac{p+1}{p-1} - 1,$$

$$I_3 = \frac{1}{2} \left[p \ln \left(1 - \frac{1}{p^2} \right) + \frac{p^2 - 1}{2} \ln \frac{p+1}{p-1} - p \right].$$

$$4. \int_0^{\infty} \frac{e^{-px}}{x} \left(1 - \frac{1}{\text{ch } x} \right) dx = 2 \ln \Gamma \left(\frac{p+3}{4} \right) - 2 \ln \Gamma \left(\frac{p+1}{4} \right) - \ln \frac{p}{4} \quad [\text{Re } p > 0].$$

$$5. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \frac{1}{\text{sh } x} \right) dx = \psi \left(\frac{p+1}{2} \right) - \ln \frac{p}{2} \quad [\text{Re } p > 0].$$

$$6. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \text{cth } x \right) dx = \psi \left(\frac{p}{2} \right) - \ln \frac{p}{2} + \frac{1}{p} \quad [\text{Re } p > 0].$$

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px} (\text{cth } cx - 1) dx = 2^{1-\alpha} c^{-\alpha} \Gamma(\alpha) \zeta \left(\alpha, 1 + \frac{p}{2c} \right)$$

$$[\text{Re } \alpha > 1; \text{Re } p > -2].$$

$$8. \int_0^{\infty} x^{-1} \frac{(1-e^{-x})^2}{\operatorname{ch} x + \cos(m\pi/n)} dx = \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \times$$

$$\times \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(n+(-1)^r k+1)/(rn), (n+(-1)^r k+1)/(rn), (k+2)/(rn), k/(rn)}{(n+(-1)^r k)/(rn), (n+(-1)^r k+2)/(rn), (k+1)/(rn), (k+1)/(rn)} \right]$$

$$\left[\left\{ \begin{array}{l} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{array} \right\} \right].$$

$$9. \int_0^{\infty} x^n \frac{1-e^{-2mx}}{\operatorname{sh}^2 x} dx = J_n,$$

$$J_1 = \frac{m\pi^2}{6} - \sum_{k=1}^{m-1} \frac{m-k}{k^2}, \quad J_2 = m\zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3},$$

$$J_3 = \frac{1}{60} m\pi^4 - \frac{3}{2} \sum_{k=1}^{m-1} \frac{m-k}{k^4}.$$

$$10. \int_0^{\infty} x \frac{1-(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = \frac{m}{12} \pi^2 + \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^2}.$$

$$11. \int_0^{\infty} x^n \frac{1+(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = I_n,$$

$$I_2 = \frac{3m}{4} \zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3}, \quad I_3 = \frac{7m\pi^4}{480} + \frac{3}{2} \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^4}.$$

$$12. \int_{-\infty}^{\infty} x \frac{1-e^{ax}}{\operatorname{sh} x} dx = -\frac{\pi^2}{2} \operatorname{tg}^2 \frac{a\pi}{2} \quad [|\operatorname{Re} a| < 1].$$

$$13. \int_0^{\infty} \frac{e^{-ax} - e^{-cx} \operatorname{ch} bx}{x} dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2} \quad [\operatorname{Re} a > 0; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$14. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x} \operatorname{ch} bx dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Re} b|].$$

$$15. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x^2} \operatorname{sh} bx dx = \frac{b}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} + \frac{c}{2} \ln \frac{c+b}{c-b} - \frac{a}{2} \ln \frac{a+b}{a-b}$$

$$[\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Re} b|].$$

$$16. \int_0^{\infty} \frac{1+x-e^x}{x \operatorname{sh} x} e^{-px} dx = 2 \ln \Gamma\left(\frac{p+1}{2}\right) - 2 \ln \Gamma\left(\frac{p}{2}\right) - \psi\left(\frac{p+1}{2}\right)$$

$$[\operatorname{Re} p > 0].$$

$$17. \int_0^{\infty} \frac{(1 - e^{-ax})(1 - e^{-(a+1)x})}{x \operatorname{sh} x} dx = 2a \ln 2 \quad [\operatorname{Re} a > -1].$$

$$18. \int_0^{\infty} \frac{1}{x} \left(ae^{-bx} - e^{-cx} \frac{\operatorname{sh} ax}{x} \right) dx = \frac{a}{2} \ln \frac{c^2 - a^2}{b^2} + \frac{c}{2} \ln \frac{c+a}{c-a} - a$$

[$\operatorname{Re} b > 0$; $\operatorname{Re} c > |\operatorname{Re} a|$]

$$19. \int_0^{\infty} \frac{1}{x} \frac{1 - \operatorname{ch} ax}{e^{2ncx} - e^{2(n+1)cx}} dx = -\frac{1}{2} \ln \frac{2c \sin [a\pi/(2c)]}{a\pi} + \frac{1}{2} \sum_{k=1}^n \ln \left(1 - \frac{a^2}{4k^2 c^2} \right)$$

[$a < 2(n+1)c$; $a < 2nc$]

$$20. \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{\operatorname{ch} x - \cos(m\pi/n)} \frac{dx}{x} =$$

$$= \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(n+b+(-1)^r k)/(rn), (a+k)/(rn)}{(n+a+(-1)^r k)/(rn), (b+k)/(rn)} \right]$$

[$\operatorname{Re} a, \operatorname{Re} b > -1$; $\left\{ \begin{array}{l} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{array} \right\}$].

$$21. \int_0^{\infty} \left[e^{-x} \operatorname{tg} \frac{m\pi}{2n} - \frac{e^{-ax} \sin(m\pi/n)}{\operatorname{ch} x + \cos(m\pi/n)} \right] \frac{dx}{x} =$$

$$= \operatorname{tg} \frac{m\pi}{2n} \ln(rn) + 2 \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{km}{n} \pi \ln \Gamma \left[\frac{(a+n+(-1)^r k)/(rn)}{(a+k)/(rn)} \right]$$

[$\operatorname{Re} a > -1$; $\left\{ \begin{array}{l} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{array} \right\}$].

$$22. \int_0^{\infty} e^{-px} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{x^n} dx = I_n \quad [\operatorname{Re} p > |\operatorname{Re} a|, |\operatorname{Re} b|],$$

$$I_1 = \frac{1}{2} \ln \frac{p^2 - b^2}{p^2 - a^2}, \quad I_2 = \frac{p}{2} \ln \frac{p^2 - a^2}{p^2 - b^2} + \frac{b}{2} \ln \frac{p-b}{p+b} + \frac{a}{2} \ln \frac{p+a}{p-a}.$$

$$23. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - x \operatorname{ch} x}{x^n} dx = J_n \quad [\operatorname{Re} p > 1],$$

$$J_1 = \frac{1}{2} \ln \frac{p+1}{p-1} - \frac{p}{p^2-1}, \quad J_2 = 1 - \frac{p}{2} \ln \frac{p+1}{p-1},$$

$$J_3 = \frac{1}{4} \left[(p^2-1) \ln \frac{p+1}{p-1} - 2p \right].$$

$$24. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - 2x \operatorname{ch} x}{x^2} \operatorname{sh} x dx = \frac{p}{4} \ln \frac{p^2-4}{p^2} \quad [\operatorname{Re} p > 2].$$

$$25. \int_0^{\infty} \frac{1}{x} \left(\frac{\operatorname{sh} ax}{\operatorname{sh} x} - ae^{-2x} \right) dx = \ln \left(\pi \cos \frac{a\pi}{2} \right) - 2 \ln \Gamma \left(\frac{a+1}{2} \right) \quad [|\operatorname{Re} a| < 1].$$

$$26. \int_0^{\infty} \frac{ae^{-bx} \operatorname{sh} cx - ce^{-hx} \operatorname{sh} ax}{x^2} dx = \frac{ac}{2} \ln \frac{c^2 - a^2}{b^2 - c^2} + \frac{ch}{2} \ln \frac{h+a}{h-a} - \frac{ab}{2} \ln \frac{b+a}{b-a}$$

[Re $b > | \operatorname{Re} c |$; Re $h > | \operatorname{Re} a |$].

$$27. \int_0^{\infty} \frac{1}{x} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{e^{mcx} - e^{(m-2)c x}} dx = \frac{1}{2} \left[\left\{ \ln \cos \{b\pi/(2c)\} - \ln \cos \{a\pi/(2c)\} \right\} - \right. \\ \left. - \sum_{k=1}^{[(m-1)/2]} \ln \frac{c^2 (2k-2\lambda)^2 - b^2}{c^2 (2k-2\lambda)^2 - a^2} + \left\{ \ln (a/b) \right\} \right] \\ \left[a < (m-2)c; b < (m-2)c; \begin{cases} \lambda = 1/2, & m = 1, 3, 5, \dots \\ \lambda = 0, & m = 2, 4, 6, \dots \end{cases} \right].$$

$$28. \int_0^{\infty} (e^{-ax} - e^{-bx}) \operatorname{cth} cx dx = \frac{1}{b} - \frac{1}{a} + \frac{1}{c} \left[\psi \left(\frac{b}{2c} \right) - \psi \left(\frac{a}{2c} \right) \right]$$

[Re $a, \operatorname{Re} b > 0$].

2.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

2.5.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входят тригонометрические функции. Для их вычисления можно использовать следующие свойства.

Интегралы, содержащие тригонометрические функции, могут быть преобразованы к интегралам от показательной или гиперболических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\sin^n bx$ и $\cos^n bx$, сводятся к интегралам от функций $\sin kbx$ и $\cos kbx$, если использовать формулы для степеней тригонометрических функций (см. приложение I). В ряде случаев интегралы от тригонометрических функций с помощью замены переменной сводятся к интегралам от алгебраических и других функций. Например,

$$\int_0^a \varphi(x) \sin^2 bx dx = \frac{1}{2} \int_0^a \varphi(x) dx - \frac{1}{4} \int_0^a \varphi(x) e^{2ibx} dx - \frac{1}{4} \int_0^a \varphi(x) e^{-2ibx} dx = \\ = - \int_0^a \varphi(x) \operatorname{sh}^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 - \cos 2bx) dx = \\ = \frac{1}{b} \int_0^{\sin ab} x^2 \varphi \left(\frac{\arcsin x}{b} \right) \frac{dx}{\sqrt{1-x^2}} \quad [ab \leq \pi/2].$$

2.5.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие тригонометрические, а также степенную, алгебраические, показательную и гиперболические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_l , $l=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

В п. 2.5.2 приняты следующие обозначения: $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \sin bx, \\ 0, & \text{если взят } \cos bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2] + k}, & \text{если взят } \sin bx, \\ 1, & \text{если взят } \cos bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \sin^n bx, \quad n = 1, 3, 5, \dots, \\ 0, & \text{если взят } \sin^n bx, \quad n = 2, 4, 6, \dots, \\ 0, & \text{если взят } \cos^n bx, \quad n = 1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{-1}(\alpha/2 + \gamma + h), \quad \tilde{b} = by(n - 2k)/2.$$

$$1. \int_0^y x^{\alpha-1} (y^{2r} - x^{2r})^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = V_{31} + \frac{1 + (-1)^n}{2} 2^{-n-1} \times \\ \times y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, \beta\right)$$

$$\left\{ E_{31} = V\pi 2^{-n} \frac{B(\beta, \xi)}{r\Gamma(1/2 + 2\gamma + h)}, \quad u = \alpha + 2r(\beta - 1), \right.$$

$$a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (-\tilde{b}^2/p^2)^p \left\{ \right. \\ \left. [y, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta n; |\arg b| < \pi]. \right.$$

$$2. \int_y^\infty x^{\alpha-1} (x^{2r} - y^{2r})^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{31} + V_{32} + \frac{1 + (-1)^n}{2} 2^{-n-1} y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\beta, 1 - \frac{\alpha}{2r} - \beta\right)$$

$$\left\{ E_{31} = V\pi 2^{-n} \frac{B(\beta, 1 - \beta - \xi)}{r\Gamma(1/2 + 2\gamma + h)}, \quad E_{32} = V\pi 2^{-n} \Gamma\left[\begin{matrix} \beta, & 1 - b_2 \\ \beta - j, & c_2 \end{matrix}\right], \right.$$

$$u = \alpha + 2r(\beta - 1), \quad v_2 = 2r(1 - \beta + j) - \alpha, \quad a_1 = \xi, \quad b_1 = \beta + \xi, \\ c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \beta + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \beta + j),$$

$$c_2 = (1 - \alpha)/2 + \gamma + r(1 - \beta + j), \quad z = (-\tilde{b}^2/p^2)^p \left\{ \right. \\ \left. [y, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha + 2r\beta) < (1 - (-1)^n)/2 + 2r] \right.$$

$$3. \int_0^\infty x^{\alpha-1} (x^{2r} + y^{2r})^{\rho-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{31} + V_{32} + \frac{1 + (-1)^n}{2} 2^{-n-1} y^{2r(\rho-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, 1 - \rho - \frac{\alpha}{2r}\right)$$

$$\left\{ E_{31} = V\pi 2^{-n} \frac{B(\xi, 1 - \rho - \xi)}{r\Gamma(1/2 + 2\gamma + h)}, \quad E_{32} = V\pi 2^{-n} \Gamma\left[\begin{matrix} 1 - \rho + h, & 1 - b_1 \\ 1 - \rho, & c_1 \end{matrix}\right], \right.$$

$$u = 2r(\rho - 1) + \alpha, \quad v_2 = 2r(1 - \rho + j) - \alpha, \quad a_1 = \xi, \quad b_1 = \rho + \xi, \\ c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \rho + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \rho + j),$$

$$c_2 = (1 - \alpha)/2 + \gamma + r(1 - \rho + j), \quad z = (-1)^{p+q} \tilde{b}^{2p} p^{-2p} \left\{ \right.$$

$$\left. [b > 0; 2r|\arg y| < \pi; \operatorname{Re} \alpha > -\delta n; \operatorname{Re}(\alpha + 2r\rho) < (1 - (-1)^n)/2 + 2r]. \right.$$

$$4. \int_0^\infty \frac{x^{\alpha-1}}{x^{2r} - y^{2r}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = V_{33} + V_{34} - \frac{1 + (-1)^n}{2} 2^{-n-1} y^{\alpha-2r} \frac{n!\pi}{r[(n/2)!]^2} \operatorname{ctg} \frac{\alpha\pi}{2r}$$

$$\{E_{33} = -\frac{\pi^{3/2}}{2^n r} \frac{\operatorname{ctg} \xi \pi}{\Gamma(1/2 + 2\gamma + h)}, \quad E_{34} = \frac{\sqrt{\pi}}{2^n} \Gamma\left[\begin{matrix} 1-a_2 \\ b_2 \end{matrix}\right],$$

$$u = \alpha - 2r, \quad v_2 = 2r(1+j) - \alpha, \quad a_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \gamma + r(1+j) - \alpha/2, \\ b_2 = (1 - \alpha)/2 + \gamma + r(1+j), \quad z = (-\tilde{b}^2/\rho^2)^{\rho} \\ [y, b > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re} \alpha < (1 - (-1)^n)/2 + 2r].$$

$$5. \int_0^{\infty} x^{\alpha-1} |x^{2r} - y^{2r}|^{\sigma-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{31} + V_{32} + \frac{1 + (-1)^n}{2} 2^{-n-1} y^{2r(\sigma-1)+\alpha} \frac{n!}{r[(n/2)!]^2} \frac{\sin[(\alpha + r\sigma)/(2r)]}{\sin(\pi\sigma/2)} \times \\ \times B\left(\frac{\alpha}{2r}, 1 - \sigma - \frac{\alpha}{2r}\right)$$

$$\{E_{31} = \frac{\sqrt{\pi} 2^{-n} \sin[(\sigma/2 + \xi)\pi] B(\xi, 1 - \sigma - \xi)}{r \sin(\pi\sigma/2) \Gamma(1/2 + 2\gamma + h)}, \quad E_{32} = (-1)^j (1 - \sigma)_j \sqrt{\pi} 2^{-n},$$

$$u = \alpha + 2r(\sigma - 1), \quad v_2 = 2r(1 - \sigma + j) - \alpha, \quad a_1 = \xi, \quad b_1 = \sigma + \xi, \\ c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \sigma + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \sigma + j), \\ c_2 = (1 - \alpha)/2 + \gamma + (1 - \sigma + j)r, \quad z = (-\tilde{b}^2/\rho^2)^{\rho} \\ [y, b, \operatorname{Re} \sigma > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re}(\alpha + 2r\sigma) < (1 - (-1)^n)/2 + 2r].$$

$$6. \int_0^{\infty} x^{\alpha-1} \frac{(y^r + \sqrt{x^{2r} + y^{2r}})^v}{(x^{2r} + y^{2r})^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I_{\rho}$$

$$[\rho = 0, 1/2].$$

$$I_{\rho} = V_{35} + V_{36} + V_{37} +$$

$$+ \frac{1 + (-1)^n}{2} 2^{v-n-1+\alpha/r} (-v)^{1-2\rho} \frac{n!}{r[(n/2)!]^2} y^{rv-2\rho+\alpha} \Gamma\left[\begin{matrix} \alpha/(2r), 2\rho - v - \alpha/r \\ 1 - v - \alpha/(2r) \end{matrix}\right]$$

$$\{E_{35} = \frac{(-v)^{1-2\rho}}{r} \sqrt{\pi} 2^{v-n+2\xi} \Gamma\left[\begin{matrix} \xi, 2\rho - v - 2\xi \\ 1 - v - \xi, 1/2 + 2\gamma + h \end{matrix}\right],$$

$$E_{36} = (-v)^{1-2\rho} 2^{2\rho-1-n} \Gamma\left[\begin{matrix} \rho + j - v/2, 1/2 - j, 1 - d_2 \\ 1 - v/2 - \rho - j, e_2 \end{matrix}\right],$$

$$E_{37} = (-v)^{-2\rho+1} 2^{2\rho-1-n} \Gamma\left[\begin{matrix} \rho + j + (1-v)/2, -1/2 - j, 1 - d_3 \\ (1-v)/2 - \rho - j, e_3 \end{matrix}\right],$$

$$u = \alpha + r(v - 2\rho), \quad v_2 = r(2\rho - v + 2j) - \alpha, \quad v_3 = r(2\rho + 1 - v + 2j) - \alpha,$$

$$a_1 = v + \xi, \quad b_1 = \xi, \quad c_1 = 1 - \rho + v/2 + \xi, \quad d_1 = (1 + v)/2 - \rho + \xi,$$

$$e_1 = 1/2 + 2\gamma + h, \quad a_2 = \rho - v/2 + j, \quad b_2 = v/2 + \rho + j, \quad c_2 = 1/2 + j,$$

$$d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, \quad e_2 = (1 - \alpha)/2 + \gamma + r(\rho - v/2 + j),$$

$$a_3 = \rho + (1 - v)/2 + j, \quad b_3 = (1 + v)/2 + \rho + j, \quad c_3 = 3/2 + j, \quad d_3 = d_2 + r/2,$$

$$e_3 = e_2 + r/2, \quad z = (-1)^{\rho+q} (\tilde{b}/\rho)^{2\rho}$$

$$[b > 0; 2r |\arg y| < \pi; -\delta n < \operatorname{Re} \alpha < 2r\rho - r \operatorname{Re} v + (1 - (-1)^n)/2].$$

$$7. \int_0^{\infty} x^{\alpha-1} \frac{(x^r + \sqrt{x^{2r} + y^{2r}})^v}{(x^{2r} + y^{2r})^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J_{\rho}$$

$$[\rho = 0, 1/2].$$

$$J_{\rho} = V_{35} + V_{36} +$$

$$+ \frac{1 + (-1)^n}{2} 2^{2\rho-n-1-\alpha/r} (-v)^{1-2\rho} \frac{n!}{r[(n/2)!]^2} y^{rv-2\rho+\alpha} \Gamma\left[\begin{matrix} \rho - v/2 - \alpha/(2r), \alpha/r \\ 1 - \rho - v/2 + \alpha/(2r) \end{matrix}\right]$$

$$\{E_{35} = (-v)^{1-2\rho} \frac{\sqrt{\pi}}{r} 2^{2\rho-2\xi-n} \Gamma \left[\begin{matrix} 2\xi, \rho-v/2-\xi \\ 1-\rho-v/2+\xi, 1/2+2\gamma+h \end{matrix} \right],$$

$$E_{36} = (-v)^{1-2\rho} \sqrt{\pi} 2^{v-2j-n} \Gamma \left[\begin{matrix} 2\rho-v+2j, 1-d_2 \\ 1-v+j, e_2 \end{matrix} \right],$$

$$u = \alpha + r(v-2\rho), v_2 = r(2\rho-v+2j) - \alpha, a_1 = \xi, b_1 = \xi + 1/2,$$

$$c_1 = 1 - \rho - v/2 + \xi, d_1 = 1 - \rho + v/2 - \xi, e_1 = 1/2 + 2\gamma + h,$$

$$a_2 = \rho - v/2 + j, b_2 = \rho + (1-v)/2 + j, c_2 = 1 - v + j,$$

$$d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, e_2 = (1 - \alpha)/2 + \gamma + r(\rho - v/2 + j).$$

$$z = (-1)^{p+q} (\tilde{b}/\rho)^{2p}$$

$$[b > 0; 2r|\arg y| < \pi; -\delta n < \operatorname{Re} \alpha < 2r\rho - r\operatorname{Re} v + (1 - (-1)^n)/2].$$

$$8. \int_0^y x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} - x^{2r}})^v + (y^r - \sqrt{y^{2r} - x^{2r}})^v}{\sqrt{y^{2r} - x^{2r}}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{35} + \frac{1 + (-1)^n}{2} 2^{v-n-1+\alpha/r} \pi y^{rv+\alpha} \frac{n!}{r[(n/2)!]^2} \Gamma \left[\begin{matrix} \alpha/(2r), v+\alpha/(2r) \\ v+\alpha/r \end{matrix} \right]$$

$$\{E_{35} = \sqrt{\pi} \frac{2^{v+2\xi-n}}{r} \Gamma \left[\begin{matrix} \xi, v+\xi \\ v+2\xi, 1/2+2\gamma+h \end{matrix} \right],$$

$$u = \alpha + r(v-1), a_1 = \xi, b_1 = v + \xi, c_1 = v/2 + \xi, d_1 = (v+1)/2 + \xi,$$

$$e_1 = 1/2 + 2\gamma + h, z = (-\tilde{b}^2/p^2)^p$$

$$[y > 0; \operatorname{Re} \alpha > -n\delta; \operatorname{Re} \alpha > -n\delta - 2r\operatorname{Re} v; |\arg b| < \pi].$$

$$9. \int_y^\infty x^{\alpha-1} \frac{(x^r + \sqrt{x^{2r} - y^{2r}})^v + (x^r - \sqrt{x^{2r} - y^{2r}})^v}{\sqrt{x^{2r} - y^{2r}}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{35} + V_{36} + V_{37} + \frac{1 + (-1)^n}{2} 2^{-n-\alpha/r} y^{r(v-1)+\alpha} \times$$

$$\times \frac{n!}{r[(n/2)!]^2} \Gamma \left[\begin{matrix} (1+v)/2 - \alpha/(2r), (1-v)/2 - \alpha/(2r) \\ 1 - \alpha/r \end{matrix} \right]$$

$$\{E_{35} = \frac{2^{-n-2\xi+1}}{r} \sqrt{\pi} \Gamma \left[\begin{matrix} (1+v)/2 - \xi, (1-v)/2 - \xi \\ 1-2\xi, 1/2+2\gamma+h \end{matrix} \right],$$

$$E_{36} = 2^{-n-v-2j} \sqrt{\pi} \Gamma \left[\begin{matrix} -v-j, 1-d_2 \\ -v-2j, e_2 \end{matrix} \right], E_{37} = 2^{-n+v-2j} \sqrt{\pi} \Gamma \left[\begin{matrix} v-j, 1-d_3 \\ v-2j, e_3 \end{matrix} \right],$$

$$u = \alpha + r(v-1), v_2 = r(1+v+2j) - \alpha, v_3 = r(1-v+2j) - \alpha, a_1 = \xi,$$

$$b_1 = \xi + 1/2, c_1 = (1-v)/2 + \xi, d_1 = (1+v)/2 + \xi, e_1 = 1/2 + 2\gamma + h,$$

$$a_2 = (1+v)/2 + j, b_2 = 1 + v/2 + j, c_2 = 1 + v + j,$$

$$d_2 = 1 - \gamma + r[(1+v)/2 + j] - \alpha/2, e_2 = (1 - \alpha)/2 + \gamma + r[(1+v)/2 + j],$$

$$a_3 = (1-v)/2 + j, b_3 = 1 - v/2 + j, c_3 = 1 - v + j,$$

$$d_3 = 1 - \gamma + r[(1-v)/2 + j] - \alpha/2, e_3 = (1 - \alpha)/2 + \gamma + r[(1-v)/2 + j],$$

$$z = (-1)^p \tilde{b}^{2p} p^{-2p} \quad [y, b > 0; \operatorname{Re} \alpha < r(1 - |\operatorname{Re} v|) + (1 - (-1)^n)/2].$$

$$10. \int_0^\infty \frac{x^{\alpha-1}}{x^{2r} + 2(xy)^r \cos \beta + y^{2r}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{38} + V_{39} - \frac{1 + (-1)^n}{2} 2^{-n-1} y^{-2r+\alpha} \frac{\pi n!}{r[(n/2)!]^2} \frac{\sin [\beta(\alpha-r)/r]}{\sin \beta \sin (\pi\alpha/r)}$$

$$\{E_{38} = -\frac{2^{1-n}}{r} \pi^{3/2} \frac{1}{\sin \beta}, E_{39} = -2^{-n} \frac{\sqrt{\pi}}{\sin \beta}, u = \alpha - 2r,$$

$$\begin{aligned}
 v_1 &= 2\gamma + 2l, \quad v_2 = r(1+l) - \alpha, \quad a_l = \beta(2\gamma + 2l + \alpha - r)r^{-1}, \\
 b_l &= (2\gamma + 2l + \alpha)\pi/r, \quad c_l = 1/2 + 2\gamma + l, \quad d_l = \beta l, \quad e_l = 1 + l - \alpha/r, \\
 f_l &= \gamma + \alpha/2 - r(1+l)/2, \quad g_l = (1-\alpha)/2 + \gamma + r(1+l)/2 \\
 &[y, b > 0; |\beta| < \pi; -n\delta < \operatorname{Re} \alpha < 2r + (1 - (-1)^n)/2; r > 0 - \text{произвольное}].
 \end{aligned}$$

$$11. \int_0^\infty x^{\alpha-1} e^{-sx} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = U + \frac{1 + (-1)^n}{2} 2^{-n-1} s^{-\alpha/(2r)} \frac{n!}{r [(n/2)!]^2} \Gamma\left(\frac{\alpha}{2r}\right),$$

$$U = V_{40} \quad [r > 1/2],$$

$$U = V_{41} \quad [r < 1/2],$$

$$U = V_{\pi} 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} s^{-\alpha} \varphi^{2\gamma} \Gamma\left[\frac{\alpha + 2\gamma}{1/2 + 2\gamma}\right] {}_2F_1\left(\Delta(2, \alpha + 2\gamma); -\frac{b^2(n-2k)^2}{s^2}; 1/2 + 2\gamma\right)$$

$$[r = 1/2]$$

$$\left\{ E_{40} = \frac{V_{\pi}}{2^n r} \Gamma\left[\frac{\xi}{1/2 + 2\gamma + h}\right], \quad u = -\frac{\alpha}{2r}, \quad v_1 = 2(h + \gamma), \right.$$

$$\varphi = 2^{-1}(n-2k)bs^{-1/(2r)}, \quad a_1 = \xi, \quad b_1 = 1/2 + 2\gamma + h,$$

$$z = (-1)^p \left[\frac{(n-2k)b}{2p}\right]^{2p} q^q s^{-q}; \quad E_{41} = \frac{V_{\pi}}{2^n} \Gamma\left[\frac{\gamma + rj + \alpha/2}{1/2 + \gamma - rj - \alpha/2}\right],$$

$$v_2 = -2rj - \alpha, \quad a_2 = \gamma + rj + \alpha/2, \quad b_2 = (1 + \alpha)/2 - \gamma + rj,$$

$$z_2 = (-1)^{p+q} \left[\frac{(n-2k)b}{2p}\right]^{-2p} q^{-q} s^q \}$$

$$[\operatorname{Re} s > 0; \operatorname{Re} \alpha > -\delta n; \arg b \text{ произволен (при } r > 1/2), b > 0 \text{ (при } r < 1/2); n | \operatorname{Im} b | < \operatorname{Re} s \text{ (при } r = 1/2)].$$

$$12. \int_0^\infty x^{\alpha-1} e^{-sx} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{42} + V_{43} + \frac{1 + (-1)^n}{2} 2^{-n-1} s^{\alpha/(2r)} \frac{n!}{r [(n/2)!]^2} \Gamma\left(-\frac{\alpha}{2r}\right)$$

$$\left\{ E_{42} = \frac{V_{\pi}}{2^n r} \Gamma\left[\frac{-\xi}{1/2 + 2\gamma + h}\right], \quad E_{43} = \frac{V_{\pi}}{2^n} \Gamma\left[\frac{\gamma - rj + \alpha/2}{(1-\alpha)/2 + \gamma + rj}\right], \right.$$

$$u = \alpha/(2r), \quad v_1 = 2(\gamma + h), \quad v_2 = 2rj - \alpha, \quad \varphi = 2^{-1}b(n-2k)s^{1/(2r)},$$

$$a_1 = 1/2 + 2\gamma + h, \quad b_1 = 1 + \xi, \quad a_2 = 1 - \gamma + rj - \alpha/2, \quad b_2 = (1 - \alpha)/2 + \gamma + rj,$$

$$z = (-1)^{p+q} \left(\frac{n-2k}{2p}b\right)^{2p} s^q q^{-q} \} \quad [b, \operatorname{Re} s > 0; \operatorname{Re} \alpha < (1 - (-1)^n)/2].$$

$$13. \int_0^\infty x^{\alpha-1} \sin^m ax \left\{ \frac{\sin^nbx}{\cos^nbx - c} \right\} dx = I \left[\frac{1 - (-1)^m}{4}, 1, (-1)^{[m/2] + l} \right],$$

$$I(p, \sigma, \tau) = U + \left[\frac{1 + (-1)^n}{2} \frac{n!}{2^n [(n/2)!]^2} - \left\{ \frac{\theta}{c} \right\} \right] \frac{a^{-\alpha/r}}{r} 2^{-m} V_{\pi} \times$$

$$\times \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \left(\frac{2}{m-2l}\right)^{\alpha/r} \Gamma\left[\frac{p + \alpha/(2r)}{p + 1/2 - \alpha/(2r)}\right] +$$

$$+ \frac{1 + (-1)^m}{2} 2^{-m-n} \frac{b^{\alpha m} V_{\pi}}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \left(\frac{2}{n-2k}\right)^{\alpha} \Gamma\left[\frac{\gamma + \alpha/2}{\gamma + (1-\alpha)/2}\right]$$

$$[r \neq 1 \text{ (} r = 1 \text{ см. 2 5 13 20-21), } c = 1 \text{ при } n = 2, 4, 6, \dots; c = 0 \text{ при } n = 1, 3, 5, \dots].$$

$$U = 2^{1-m-n} \pi \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} A_{kl} \left[\frac{n-2k}{2} b \right]^{-\alpha},$$

$$A_{kl} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(j+\rho)} \Gamma \left[\begin{matrix} a_1 \\ c_1, 1-b_1 \end{matrix} \right] {}_{2p+1}F_{2q} \left(\begin{matrix} \Delta(\rho, a_1), \Delta(\rho, b_1), 1; z \\ \Delta(q, c_1), \Delta(q, 1+j) \end{matrix} \right)$$

[r < 1],

$$A_{kl} = \sum_{h=0}^{p-1} \frac{(-1)^h}{rh!} \varphi^{-2\xi} \Gamma \left[\begin{matrix} \rho + \xi \\ 1/2 + \rho - \xi, 1/2 + 2\gamma + h \end{matrix} \right] \times \\ \times {}_{2q+1}F_{2p} \left(\begin{matrix} \Delta(q, \xi + \rho), \Delta(q, 1/2 - \rho + \xi), 1; z \\ \Delta(\rho, 1/2 + 2\gamma + h), \Delta(\rho, 1+h) \end{matrix} \right) \quad [r > 1]$$

$$\{a_1 = \gamma + \alpha/2 + r(j + \rho), b_1 = (1 + \alpha)/2 - \gamma + r(j + \rho),$$

$$c_1 = 1/2 + 2\rho + j, \varphi = \frac{a(m-2l)}{2} \left(\frac{2}{(n-2k)b} \right)^r, z = (-1)^{p+q} \rho^{2p} q^{2q} \varphi^{2q}\}$$

$$[a, b > 0; -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < 0 \text{ (при } m+n=2, 4, 6, \dots), \text{ или } \\ -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < \max(1, r) \text{ (при } m+n=1, 3, 5, \dots)]$$

$$14. \int_0^\infty x^{\alpha-1} \cos^m ax^r \left\{ \frac{\sin^n bx}{\cos^n bx - c} \right\} dx = I(0, 0, 1) \quad [\text{см } 2.5.2 \text{ 13}].$$

$$15. \int_0^\infty x^{\alpha-1} \sin^m ax^r \left\{ \frac{\sin^n bx}{\cos^n bx - c} \right\} dx = J \left[\frac{1 - (-1)^m}{4}, 1, (-1)^{[m/2] + l} \right],$$

$$J(\rho, \sigma, \tau) = 2^{1-m-n} \pi \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \times \\ \times \left[\frac{b(n-2k)}{2} \right]^{-\alpha} \left\{ \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(\rho+j)} \Gamma \left[\begin{matrix} 1-b_1 \\ a_1, c_1 \end{matrix} \right] \times \right. \\ \times {}_1F_{2p+2q} \left(\begin{matrix} 1; \\ \Delta(q, a_1), \Delta(\rho, b_1), \Delta(\rho, c_1), \Delta(q, 1+j) \end{matrix} \right) + \\ \left. + \sum_{h=0}^{p-1} \frac{(-1)^h}{rh!} \varphi^{2\xi} \Gamma \left[\begin{matrix} \rho - \xi \\ 1/2 + \rho + \xi, 1/2 + 2\gamma + h \end{matrix} \right] \times \right. \\ \times {}_1F_{2p+2q} \left(\begin{matrix} 1; \\ \Delta(q, 1 - \rho + \xi), \Delta(q, 1/2 + \rho + \xi), \Delta(\rho, 1+h), \Delta(\rho, 1/2 + 2\gamma + h) \end{matrix} \right) \left. \right\} + \\ + \left[\frac{1 + (-1)^n}{2} \frac{n}{2^n [(n/2)!]^2} - \left\{ \begin{matrix} 0 \\ c \end{matrix} \right\} \right] \frac{a^{\alpha/r}}{r} \frac{\sqrt{\pi}}{2^m} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \times \\ \times \left(\frac{m-2l}{2} \right)^{\alpha/r} \Gamma \left[\begin{matrix} \rho - \alpha/(2r) \\ 1/2 + \rho + \alpha/(2r) \end{matrix} \right] + \frac{1 + (-1)^m}{2} 2^{-m-n} b^{-\alpha} \sqrt{\pi} \times \\ \times \frac{m!}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \left(\frac{2}{n-2k} \right)^\alpha \Gamma \left[\begin{matrix} \gamma + \alpha/2 \\ \gamma + (1-\alpha)/2 \end{matrix} \right]$$

$$\{a_1 = 1/2 + 2\rho + j, \quad b_1 = 1 - \gamma - \alpha/2 + r(\rho + j), \quad c_1 = (1 - \alpha)/2 + \gamma + r(\rho + j),$$

$$\psi = a \frac{m-2l}{2} \left(b \frac{n-2k}{2} \right)^r, \quad z = (-1)^{p+q} \psi^{2q} \rho^{-2p} q^{-2q} \}$$

$$\left[a, b > 0; \quad -2c - \delta n - \sigma r m < \operatorname{Re} \alpha < \left(\frac{1 - (-1)^n}{2} + \frac{1 - (-1)^m}{2} r \right); \right. \\ \left. c = 1 \text{ при } n = 2, 4, 6, \dots; \quad c = 0 \text{ при } n = 1, 3, 5, \dots \right].$$

$$16. \int_0^\infty x^{\alpha-1} \cos^m ax^r \left\{ \frac{\sin^n bx}{\cos^n bx - c} \right\} dx = J(0, 0, 1) \quad [\text{см. 2.5.2.15}].$$

$$17. \int_0^\infty x^{\alpha-1} e^{-sx^r} \sin ax^r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(\sigma) |_{\sigma=1} \quad [r > 0].$$

$$I(\sigma) = U + \frac{1 + (-1)^n}{2} \frac{2^{-n}}{r} (s^2 + a^2)^{-\alpha/(2r)} \frac{n}{[(n/2)!]^2} \Gamma\left(\frac{\alpha}{r}\right) \cos\left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right),$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\sigma\pi}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (rl + \alpha)/2}{\gamma + (1 - rl - \alpha)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{\alpha-lr} \quad [r < 1].$$

$$U = \frac{\sqrt{\pi}}{2^{n-1}r} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left(\frac{\pi\sigma}{2} - 2\xi \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{2\xi}{1/2 + h + 2\gamma}\right] (s^2 + a^2)^{-\xi} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad [r > 1].$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} A_k(z) \quad [r = 1],$$

$$z = \left(b \frac{n-2k}{\sqrt{s^2 + a^2}} \right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s}\right] \times \\ \times \Gamma\left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad \text{при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\sigma\pi}{2} + l \operatorname{arctg} \frac{a}{s}\right) \Gamma\left[\frac{\gamma + (\alpha + l)/2}{\gamma + (1 - \alpha - l)/2}\right] \times \\ \times (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{-\alpha-l} \quad \text{при } |z| > 1$$

$$[a, \operatorname{Re} s > 0, \operatorname{Re} \alpha > -\sigma r - \delta n; \quad b - \text{произвольное (при } r > 1), \quad |\operatorname{Im} b| n < \operatorname{Re} s \text{ (при } r = 1) \\ \text{и } b > 0 \text{ (при } r < 1)] \text{ или } [\operatorname{Re} s = 0; \quad a, b > 0, \quad -\sigma r - \delta n < \operatorname{Re} \alpha < \max(r, 1) \text{ (при } r \neq 1) \\ \text{или } -\sigma r - \delta n < \operatorname{Re} \alpha < 0 \text{ (при } r = 1)].$$

$$18. \int_0^\infty x^{\alpha-1} e^{-sx^r} \cos ax^r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(\sigma) |_{\sigma=0} \quad [I(\sigma) \text{ см. 2.5.2.17}].$$

$$19. \int_0^{\infty} x^{\alpha-1} e^{-sx-r} \sin ax^{-r} \left\{ \frac{\sin bx}{\cos lx} \right\}^n dx = J(\sigma) \Big|_{\sigma=1},$$

$$J(\sigma) = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \cos \left(\frac{\sigma\pi}{2} + l \operatorname{arctg} \frac{a}{s} \right) \times \\ \times \Gamma \left[\frac{\gamma + (\alpha - lr)/2}{\gamma + (1 - \alpha + lr)/2} \right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2} \right)^{rl-\alpha} + \\ + \frac{\sqrt{\pi}}{r 2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} \cos \left(\frac{\sigma\pi}{2} + 2\xi \operatorname{arctg} \frac{a}{s} \right) \times \\ \times \Gamma \left[\frac{-2\xi}{2\gamma + 1/2 + h} \right] (s^2 + a^2)^{\xi} \left(b \frac{n-2k}{2} \right)^{2\gamma+2h} + \\ + \frac{1 + (-1)^n (s^2 + a^2)^{\alpha/(2r)} n!}{2 \cdot 2^n [(n/2)!]^2} \cos \left(\frac{\pi\sigma}{2} + \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s} \right) \Gamma \left(-\frac{\alpha}{r} \right) \\ [a, b > 0; \operatorname{Re} \alpha < \sigma + (1 - (-1)^n)/2 \text{ и } \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0 \text{ и } \operatorname{Re} \alpha > -r - n\delta)].$$

$$20. \int_0^{\infty} x^{\alpha-1} e^{-sx-r} \cos ax^{-r} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J(\sigma) \Big|_{\sigma=0} \quad [\text{см. 25.2 19}].$$

$$21. \int_0^{\infty} x^{\alpha-1} e^{-sx-r} \sin ax^r \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^n dx = K(\sigma) \Big|_{\sigma=1} \quad [r > 0].$$

$$K(\sigma) = U + \frac{1 + (-1)^n}{2} (\mp 1)^{n/2} \frac{2^n n}{r [(n/2)!]^2} \Gamma \left(\frac{\alpha}{r} \right) (s^2 + a^2)^{-\alpha/(2r)} \times \\ \times \cos \left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s} \right),$$

$$U = \frac{\sqrt{\pi}}{2^{n-1} r} \sum_{k=0}^{[(n-1)/2]} (\mp 1)^k \binom{n}{k} \sum_{h=0}^{\infty} \frac{1}{h!} \cos \left(\frac{\pi\sigma}{2} - 2\xi \operatorname{arctg} \frac{a}{s} \right) \times \\ \times \Gamma \left[\frac{2\xi}{1/2 + 2\gamma + h} \right] (s^2 + a^2)^{-\xi} \left(b \frac{n-2k}{2} \right)^{2h+2\gamma} \quad [r > 1],$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} (\mp 1)^k \binom{n}{k} A_k(z) \quad [r = 1],$$

$$z = - \left(b \frac{n-2k}{\sqrt{s^2 + a^2}} \right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^{\infty} \frac{1}{h!} \cos \left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s} \right] \times \\ \times \Gamma \left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h} \right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2} \right)^{2h+2\gamma} \text{ при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^{\infty} \frac{(-1)^l \sin [c - \gamma + (\alpha + l)/2] \pi}{l! \sin c \pi} \cos \left(\frac{\sigma \pi}{2} + l \operatorname{arctg} \frac{a}{s} \right) \times \\ \times \Gamma \left[\frac{\gamma + (\alpha + l)/2}{\gamma + (1 - \alpha - l)/2} \right] (s^2 + a^2)^{l/2} \left(b \frac{n - 2k}{2} \right)^{-\alpha - l} \quad \text{при } |z| > 1$$

[c — произвольное вещественное, a , $\operatorname{Re} s > 0$, $\operatorname{Re} \alpha > -\sigma r - \delta n$; b — произвольное (при $r > 1$, или $n | \operatorname{Re} b | < \operatorname{Re} s$ (при $r = 1$)).]

$$22. \int_0^{\infty} x^{\alpha-1} e^{-sx} \cos ax \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^n dx = K(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2 5 2 21}]$$

2.5.3. Интегралы от $x^\alpha \left\{ \frac{\sin bx}{\cos bx} \right\}^\sigma$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^\alpha dx = \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{(\alpha + 1)/2}{\alpha/2 + 1} \right] \quad [\operatorname{Re} \alpha > -1].$$

$$2. \int_0^{\pi/2} x \left\{ \frac{\sin x}{\cos x} \right\}^n dx = \\ = \left(1 + 2 \left[\frac{n+1}{2} \right] \pm n \right) A \frac{(n-1)!! \pi}{4n!!} \pm \sum_{k=0}^{[(n-1)/2]} \frac{((1-n)/2)_k}{(-n/2)_k (n-2k)^2} \\ [A=1 \text{ при } n=1, 3, 5, \dots; A=\pi/2 \text{ при } n=2, 4, 6, \dots].$$

$$3. \int_0^{\pi/2} x^m \left\{ \frac{\sin x}{\cos x} \right\} dx = \\ = \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k-\delta)!} \left(\frac{\pi}{2} \right)^{m-2k-\delta} + (-1)^{[m/2]} m! \left(2 \left[\frac{m}{2} \right] + \delta - m \right).$$

$$4. \int_0^{\pi/2} \frac{\sin^2 x}{x} dx = \frac{1}{2} [C + \ln \pi - \operatorname{ci}(\pi)].$$

$$5. \int_0^{\pi} x^m \left\{ \frac{\sin nx}{\cos nx} \right\} dx = \frac{(-1)^{n-\delta+2}}{n^{m+1}} \sum_{k=0}^{[(m+\delta-1)/2]} (-1)^k \frac{m!}{(m-2k+\delta-1)!} \times \\ \times (\pi n)^{m-2k+\delta-1} + (-1)^{[(m-\delta+1)/2]} m! n^{-m-1} \left(2 \left[\frac{m-\delta+1}{2} \right] - m + \begin{cases} 1 \\ 0 \end{cases} \right).$$

$$6. \int_0^{\pi} x^{\alpha-1} \sin^n x dx = -\frac{(\alpha-1)(\alpha-2)}{n^2} \int_0^{\pi} x^{\alpha-3} \sin^n x dx + \frac{n-1}{n} \int_0^{\pi} x^{\alpha-1} \sin^{n-2} x dx \\ [\operatorname{Re} \alpha > 2-n; n=2, 3, 4, \dots].$$

$$7. \int_0^{\pi} x \cos^{2n} x \, dx = \frac{(2n-1)!! \pi^2}{2(2n)!!}.$$

$$8. \int_0^{\pi k} x \sin^n x \, dx = \frac{\pi(n-1)!!}{2(n!!)} \begin{cases} \pi k^2, & n=0, 2, 4, \dots \\ 2k, & n=1, 3, 5, \dots \end{cases}$$

$$9. \int_0^{2\pi l} x^m \cos nx \, dx = - \sum_{k=0}^{m-1} \frac{k!}{n^{k+1}} \binom{m}{k} (2\pi l)^{m-k} \cos \frac{k+1}{2} \pi$$

[$n, l = 1, 2, 3, \dots$].

$$10. \int_0^{\infty} x^{\alpha-1} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \frac{\Gamma(\alpha)}{b^{\alpha}} \begin{cases} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{cases} \quad [b > 0, -\delta < \operatorname{Re} \alpha < 1].$$

$$11. \int_0^{\infty} x^{-1/2} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \sqrt{\frac{\pi}{2b}} \quad [b > 0].$$

$$12. \int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2} \operatorname{sgn} b \quad [\operatorname{Im} b = 0].$$

$$13. \int_0^{\infty} x^{\alpha-1} \sin^n bx \, dx = I_{\alpha, n}$$

[$\operatorname{Im} b = 0, -n < \operatorname{Re} \alpha < 2\lambda, n = 1, 2, 3, \dots; \lambda = (1 - (-1)^n)/4$],

$$I_{\alpha, n} = 2^{\alpha-n} \sqrt{\pi} |b|^{-\alpha} (\operatorname{sgn} b)^n \Gamma \left[\begin{matrix} \lambda + \alpha/2 \\ \lambda + (1-\alpha)/2 \end{matrix} \right] \sum_{j=0}^{[(n-1)/2]} (-1)^{[n/2]+j} \frac{n! (n-2j)^{-\alpha}}{j! (n-j)!}$$

при $\alpha \neq -2(\lambda + k), k = 0, 1, 2, \dots$,

$$I_{-2\lambda-2k, n} = \lim_{\alpha \rightarrow -2\lambda-2k} I_{\alpha, n} \quad \text{при } \alpha \rightarrow -2\lambda-2k,$$

$$I_{1/2-m, 2l-1} = (-1)^{[m/2]} \sqrt{\pi} 2^{m-2l+3/2} \frac{|b|^{m-1/2}}{(2m-1)!!} \operatorname{sgn} b \times$$

$$\times \sum_{j=1}^l (-1)^{j-1} \binom{2l-1}{l+j-1} (2j-1)^{m-1/2} \quad [0 \leq m \leq 2l-1],$$

$$I_{1/2-m, 2l} = (-1)^{[(m+1)/2]} \sqrt{\pi} 2^{2m-2l} \frac{|b|^{m-1/2}}{(2m-1)!!} \sum_{j=1}^l (-1)^j \binom{2l}{l+j} j^{m-1/2}$$

[$m = 1, 2, \dots, 2l$].

$$I_{-1/2, 1} = \sqrt{2\pi |b|} \operatorname{sgn} b, \quad I_{-3/2, 2} = \frac{4}{3} b \sqrt{\pi |b|},$$

$$I_{-1/2, 3} = \frac{3-\sqrt{3}}{4} \sqrt{\pi |b|} \operatorname{sgn} b,$$

$$I_{0, 2l-1} = 2^{1-2l} \pi \sum_{j=0}^{l-1} (-1)^{l+j+1} \frac{(2l-1)!}{j! (2l-j-1)!} \operatorname{sgn} b,$$

$$I_{0, (2m+1)/(2n+1)} = 2^{(2m-4n-1)/(2n+1)} B((2m+1)/(4n+2), (2m+1)/(4n+2)) \quad [b > 0].$$

$$I_{-m, n} = \frac{2^{-n} b^m \pi}{m! T(m\pi/2)} \sum_{j=0}^{[(n-1)/2]} (-1)^{[(n-1)/2]+j} \frac{n! (n-2j)^m}{j! (n-j)!}$$

$$[b > 0, T(x) = \cos x \text{ при } n = 2l - 1; T(x) = \sin x \text{ при } n = 2l, l = 1, 2, 3, \dots].$$

$$I_{-1, 2l} = \frac{\pi |b| (2l-3)!!}{2 (2l-2)!!}, \quad I_{-2, 2l-1} = \frac{\pi b |b| (2l-1) (2l-5)!!}{4 (2l-2)!!},$$

$$I_{\alpha, 2} = -2^{-\alpha-1} \Gamma(\alpha) |b|^{-\alpha} \cos(\alpha\pi/2) \quad [-2 < \operatorname{Re} \alpha < 0],$$

$$I_{\alpha, 3} = 2^{-2} |b|^{-\alpha} (3 - 3^{-\alpha}) \sin(\alpha\pi/2) \Gamma(\alpha) \operatorname{sgn} b \quad [-3 < \operatorname{Re} \alpha < 1].$$

$$I_{0,3} = \frac{\pi}{4} \operatorname{sgn} b, \quad I_{-1,3} = \frac{3}{4} b \ln 3, \quad I_{-2,3} = \frac{3}{8} \pi b |b|.$$

$$14. \int_0^{\infty} x^{\alpha-1} \cos^{2n+1} bx \, dx =$$

$$= 2^{\alpha-2n-1} \sqrt{\pi} (2n+1)! b^{-\alpha} \Gamma\left[\frac{\alpha/2}{(1-\alpha)/2}\right] \sum_{k=0}^n \frac{(2n-2k+1)^{-\alpha}}{k! (2n-k+1)!} \quad [b > 0, 0 < \operatorname{Re} \alpha < 1].$$

2.5.4. Интегралы от $x^m \left\{ \frac{\sin x}{\cos x} \right\}^{-n}$.

$$1. \int_0^{\pi/4} \frac{x^m}{\sin^2 x} \, dx = -\left(\frac{\pi}{4}\right)^m \left[1 - \frac{4m}{\pi(m-1)} + \frac{4m}{\pi} \sum_{k=1}^{\infty} \frac{\pi^{2k} |B_{2k}|}{2^{2k} (2k)! (m+2k-1)} \right] \quad [m=2, 3, 4, \dots].$$

$$2. \int_0^{\pi/4} \frac{x^2}{\left\{ \frac{\sin x}{\cos x} \right\}^2} \, dx = \frac{\pi}{4} \ln 2 + \frac{\pi^2}{16} \pm G.$$

$$3. \int_0^{\pi/4} \frac{dx}{\cos^n x} = -\sum_{k=1}^{[n/2]} \frac{(1-n/2)_{k-1}}{((1-n)/2)_k} 2^{n/2-k} + \left(n - 2 \left[\frac{n}{2} \right] \right) \frac{(2[n/2]-1)!!}{(2[n/2])!!} \ln(1+\sqrt{2}) \quad [n=1, 2, 3, \dots].$$

$$4. \int_0^{\pi/2} \frac{x^m}{\sin x} \, dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1} - 1}{4^{2k-1} (m+2k)} \zeta(2k) \right] \quad [m=1, 2, 3, \dots].$$

$$5. \int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2G.$$

$$6. \int_0^{\pi/2} \frac{x^m}{\sin^2 x} \, dx = m \left(\frac{\pi}{2}\right)^{m-1} \left[\frac{1}{m-1} + \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k} \pi^{2k}}{(m+2k-1) (2k)!} \right] \quad [m=2, 3, 4, \dots].$$

$$7. \int_0^{\pi/2} \frac{x^{2+2\sigma}}{\sin^2 x} dx = \frac{\pi^{2\sigma+1}}{2\sigma} \ln 2 - \frac{9\pi\sigma}{4} \zeta(3)$$

[$\sigma=0$ или 1].

$$8. \int_0^{\pi} \frac{x}{\cos x} dx = -4G.$$

2.5.5. Интегралы от $(x+z)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a (a-x)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{b^\delta a^{\beta+\delta}}{\beta(\beta+1)^\delta} {}_1F_2 \left(1; 1+\frac{\beta}{2}, \frac{1+\beta}{2}+\delta; -\frac{a^2 b^2}{4} \right)$$

[$a, \operatorname{Re} \beta > 0; |\arg b| < \pi$].

$$2. \int_a^\infty (x-a)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\Gamma(\beta)}{b^\beta} \begin{Bmatrix} \sin(ab + \beta\pi/2) \\ \cos(ab + \beta\pi/2) \end{Bmatrix}$$

[$a, b > 0; 0 < \operatorname{Re} \beta < 1$].

$$3. \int_0^\infty \frac{1}{(x+z)^\rho} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{b^\delta z^{1+\delta-\rho}}{(\rho-1)(\rho-2)^\delta} {}_1F_2 \left(1; \frac{3-\rho}{2}, 1+\delta-\frac{\rho}{2}; -\frac{b^2 z^2}{4} \right) +$$

$$+ \Gamma(1-\rho) b^{\rho-1} \begin{Bmatrix} \cos(\rho\pi/2 + bz) \\ \sin(\rho\pi/2 + bz) \end{Bmatrix}$$

[$b, \operatorname{Re} \rho > 0; |\arg z| < \pi$].

$$4. \int_0^\infty \frac{1}{\sqrt{x+z}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \sqrt{\frac{\pi}{2b}} \left[\cos bz \mp \sin bz \pm 2C(bz) \begin{Bmatrix} \sin bz \\ \cos bz \end{Bmatrix} - 2S(bz) \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix} \right]$$

[$b > 0; |\arg z| < \pi$].

$$5. \int_0^\infty \frac{1}{x+z} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \pm \operatorname{ci}(bz) \begin{Bmatrix} \sin bz \\ \cos bz \end{Bmatrix} - \operatorname{si}(bz) \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix}$$

[$b > 0; |\arg z| < \pi$].

$$6. \int_0^\infty \frac{1}{x-y} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \pm \begin{Bmatrix} \cos by \\ \sin by \end{Bmatrix} [\operatorname{si}(by) + \pi] - \begin{Bmatrix} \sin by \\ \cos by \end{Bmatrix} \operatorname{ci}(by)$$

[$b, y > 0$].

$$7. \int_{-\infty}^\infty \frac{1}{x+z} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \pi \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix}$$

[$b > 0, \operatorname{Im} z \neq 0$].

2.5.6. Интегралы от $(x^2 \pm a^2)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}^n, (a^2 - x^2)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}^n$.

$$1. \int_0^a (a^2 - x^2)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \begin{Bmatrix} H_{\beta-1/2}(ab) \\ J_{\beta-1/2}(ab) \end{Bmatrix}$$

[$a, \operatorname{Re} \beta > 0; |\arg b| < \pi$].

$$2. \int_a^{\infty} (x^2 - a^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pm \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ \begin{matrix} J_{1/2-\beta}(ab) \\ Y_{1/2-\beta}(ab) \end{matrix} \right\}$$

[a, b > 0, 0 < Re β < 1].

$$3. \int_0^{\infty} \frac{\sin bx}{(x^2 + z^2)^{\rho}} dx = \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} b^{\rho-1/2} [I_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)]$$

[b, Re ρ, Re z > 0].

$$4. \int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)^{\rho}} dx = \left(\frac{2z}{b} \right)^{1/2-\rho} \frac{\sqrt{\pi}}{\Gamma(\rho)} K_{1/2-\rho}(bz)$$

[b, Re ρ, Re z > 0].

$$5. \int_0^{\infty} \frac{\sin bx}{x^2 + z^2} dx = \frac{1}{2z} [e^{-bz} \operatorname{Ei}(bz) - e^{bz} \operatorname{Ei}(-bz)]$$

[b, Re z > 0].

$$6. \int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)^{m+1}} dx = \frac{\pi e^{-bz}}{(2z)^{2m+1} m!} \sum_{k=0}^m \frac{(2m-k)! (2bz)^k}{k! (m-k)!}$$

[b, Re z > 0].

$$7. \int_0^{\infty} \frac{\sin bx}{x^2 - y^2} dx = \frac{1}{y} [\cos by \operatorname{Si}(by) - \sin by \operatorname{ci}(by)]$$

[b, y > 0].

$$8. \int_0^{\infty} \frac{\cos by}{x^2 - y^2} dx = -\frac{\pi}{2y} \sin by$$

[b, y > 0].

$$9. \int_0^{\infty} \frac{1}{x^2 - y^2} \left\{ \frac{\sin bx}{\cos bx} \right\}^2 dx = \pm \frac{\pi}{4y} \sin 2by$$

[b, y > 0].

$$10. \int_0^{\infty} \frac{\sin^{2n} bx}{x^2 + z^2} dx = \frac{(-1)^n \pi}{2^{2n+1} z} \left\{ 2^{2n} \operatorname{sh}^{2n} bz - 2 \sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{sh} [2(n-k)bz] \right\}$$

[b, Re z > 0].

$$11. \int_0^{\infty} \frac{\sin^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 - e^{-2bz})$$

[b, Re z > 0].

$$12. \int_0^{\infty} \frac{\sin^{2n+1} bx}{x^2 + z^2} dx =$$

$$= \frac{(-1)^{n-1}}{2^{2n+2} z} \left\{ e^{(2n+1)bz} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{-2kbz} \operatorname{Ei}[(2k-2n-1)bz] + \right.$$

$$\left. + e^{-(2n+1)bz} \sum_{k=1}^{2n+1} (-1)^{k-1} \binom{2n+1}{k} e^{2kbz} \operatorname{Ei}[(2n+1-2k)bz] \right\}$$

[b, Re z > 0].

$$13. \int_0^{\infty} \frac{\cos^{2n} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1}z} \binom{2n}{n} + \frac{\pi}{2^{2n}} \sum_{k=1}^n \binom{2n}{n+k} e^{-2kbz} \quad [b, \operatorname{Re} z > 0].$$

$$14. \int_0^{\infty} \frac{\cos^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 + e^{-2bz}) \quad [b, \operatorname{Re} z > 0].$$

$$15. \int_0^{\infty} \frac{\cos^{2n+1} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1}z} \sum_{k=0}^n \binom{2n+1}{n+k+1} e^{-(2k+1)bz} \quad [b, \operatorname{Re} z > 0].$$

2.5.7. Интегралы от $x^\alpha (x \pm a)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$, $x^\alpha (a-x)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = (-i)^\delta 2^{-1} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_1F_1(\alpha; \alpha+\beta; iab) \mp {}_1F_1(\alpha; \alpha+\beta; -iab)]$$

[$a, \operatorname{Re} \beta > 0, \operatorname{Re} \alpha > -\delta, |\arg b| < \pi$].

$$2. \int_0^a x^{\alpha-1} (a-x)^{\alpha-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\pi} \left(\frac{a}{b}\right)^{\alpha-1/2} \begin{Bmatrix} \sin(ab/2) \\ \cos(ab/2) \end{Bmatrix} \Gamma(\alpha) J_{\alpha-1/2}\left(\frac{ab}{2}\right)$$

[$a > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi$].

$$3. \int_0^a \frac{1}{\sqrt{x(a-x)}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\pi}{2} \sqrt{b} J_{\pm 1/4}\left(\frac{ab}{2}\right) \quad [a > 0; |\arg b| < \pi].$$

$$4. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = B(\beta, 1-\alpha-\beta-\delta) b^\delta a^{\alpha+\beta+\delta-1} \times$$

$$\times {}_2F_3\left(\frac{\alpha+1}{2}, \frac{1+2\delta}{2}; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2\delta}{2}, \frac{1}{2}+\delta; -\frac{a^2b^2}{4}\right) +$$

$$+ (\beta-1) \Gamma(\alpha+\beta-2) \begin{Bmatrix} \sin[(\alpha+\beta)\pi/2] \\ \cos[(\alpha+\beta)\pi/2] \end{Bmatrix} ab^{2-\alpha-\beta} \times$$

$$\times {}_2F_3\left(1-\frac{\beta}{2}, \frac{3-\beta}{2}; \frac{3}{2}, 2-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}; -\frac{a^2b^2}{4}\right) \mp$$

$$\mp \Gamma(\alpha+\beta-1) \begin{Bmatrix} \cos[(\alpha+\beta)\pi/2] \\ \sin[(\alpha+\beta)\pi/2] \end{Bmatrix} b^{1-\alpha-\beta} \times$$

$$\times {}_2F_3\left(1-\frac{\beta}{2}, \frac{1-\beta}{2}; \frac{1}{2}, 1-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}; -\frac{a^2b^2}{4}\right)$$

[$a, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 2$].

$$5. \int_a^\infty x^{\alpha-1} (x-a)^{\alpha-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \pm \frac{\sqrt{\pi}}{2} \left(\frac{a}{b}\right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha}\left(\frac{ab}{2}\right) \begin{Bmatrix} \cos(ab/2) \\ \sin(ab/2) \end{Bmatrix} \mp Y_{1/2-\alpha}\left(\frac{ab}{2}\right) \begin{Bmatrix} \sin(ab/2) \\ \cos(ab/2) \end{Bmatrix} \right]$$

[$a, b > 0; 0 < \operatorname{Re} \alpha < 1$].

$$\begin{aligned}
 6. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= B(\alpha+\delta, \rho-\alpha-\delta) b^\delta z^{\alpha+\delta-\rho} \times \\
 &\times {}_2F_3\left(\frac{\alpha+2\delta}{2}, \frac{\alpha+1}{2}; \delta+\frac{1}{2}, \delta+\frac{\alpha+1-\rho}{2}, \frac{2+\alpha-\rho}{2}; -\frac{b^2 z^2}{4}\right) + \\
 &+ \Gamma(\alpha-\rho) \left\{ \frac{\sin[(\alpha-\rho)\pi/2]}{\cos[(\alpha-\rho)\pi/2]} \right\} b^{\rho-\alpha} \times \\
 &\times {}_2F_3\left(\frac{\rho}{2}, \frac{\rho+1}{2}; \frac{1}{2}, 1+\frac{\rho-\alpha}{2}, \frac{1-\alpha+\rho}{2}; -\frac{b^2 z^2}{4}\right) + \\
 &+ \rho \Gamma(\alpha-\rho-1) \left\{ \frac{\cos[(\rho-\alpha)\pi/2]}{\sin[(\rho-\alpha)\pi/2]} \right\} z b^{\rho+1-\alpha} \times \\
 &\times {}_2F_3\left(\frac{\rho+1}{2}, 1+\frac{\rho}{2}; \frac{3}{2}, \frac{3+\rho-\alpha}{2}, 1+\frac{\rho-\alpha}{2}; -\frac{b^2 z^2}{4}\right) \\
 &[b > 0; \operatorname{Re} \alpha > -\delta; \operatorname{Re}(\alpha-\rho) < 1; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^\infty \frac{x^{\alpha-1}}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= \\
 &= i^\delta 2^{-1} \Gamma(\alpha) z^{\alpha-1} [e^{-ibz} \Gamma(1-\alpha, -ibz) \mp e^{ibz} \Gamma(1-\alpha, ibz)] \\
 &[b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 2].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^{1-\alpha}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= \\
 &= \frac{\sqrt{\pi}}{2} \left(\frac{z}{b}\right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha}\left(\frac{bz}{2}\right) \left\{ \frac{\cos(bz/2)}{\sin(bz/2)} \right\} \pm Y_{1/2-\alpha}\left(\frac{bz}{2}\right) \left\{ \frac{\sin(bz/2)}{\cos(bz/2)} \right\} \right] \\
 &[b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 1].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_0^\infty \frac{x^{\sigma-1/2}}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= 2\pi \left(\sigma - \frac{1}{2}\right) z^{\sigma-1/2} \left[\sqrt{2} S(bz) \left\{ \frac{\cos(bz+\pi/4)}{\sin(bz+\pi/4)} \right\} \mp \right. \\
 &\mp \sqrt{2} C(bz) \left\{ \frac{\sin(bz+\pi/4)}{\cos(bz+\pi/4)} \right\} \pm \left. \frac{\sin bz}{\cos bz} \right] \mp \frac{1-(-1)^\sigma}{2} \sqrt{\frac{\pi}{2b}} \\
 &[\sigma = 0 \text{ или } 1; b > 0; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_0^\infty \frac{x^{\alpha-1}}{x-y} \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= -\pi y^{\alpha-1} \operatorname{ctg} \alpha\pi \left\{ \frac{\sin by}{\cos by} \right\} - \\
 &- b^{2-\alpha} y \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} {}_1F_2\left(1; 2, \frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4}\right) \mp \\
 &\mp b^{1-\alpha} \Gamma(\alpha-1) \left\{ \frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi/2)} \right\} {}_1F_2\left(1; \frac{3-\alpha}{2}, 1+\frac{\alpha}{2}; -\frac{b^2 y^2}{4}\right) \\
 &[b, y > 0; -\delta < \operatorname{Re} \alpha < 2].
 \end{aligned}$$

$$\begin{aligned}
 11. \int_0^\infty x^{2n} [(\beta+ix)^{-\nu} \mp (\beta-ix)^{-\nu}] \left\{ \frac{\sin bx}{\cos bx} \right\} dx &= \\
 &= \frac{(-1)^n}{\Gamma(\nu)} i^\delta (2n)! \pi b^{\nu-2n-1} e^{-b\beta} L_{2n}^{\nu-2n-1}(b\beta) \\
 &[b, \operatorname{Re} \beta > 0; 0 \leq 2n < \operatorname{Re} \nu].
 \end{aligned}$$

$$12. \int_0^{\infty} x^{2n+1} [(\beta + ix)^{-\nu} \pm (\beta - ix)^{-\nu}] \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{(-1)^{n+1}}{\Gamma(\nu)} i^{1-\delta} (2n+1)! \pi b^{\nu-2n-2} e^{-b\beta} L_{2n+1}^{\nu-2n-2}(b\beta)$$

$$[b, \operatorname{Re} \beta > 0; -2-\delta < 2n < \operatorname{Re} \nu - 1].$$

2.5.8. Интегралы от $x^\alpha (x^2 - a^2)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$, $x^\alpha (a^2 - x^2)^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a x^{\alpha-1} (a^2 - x^2)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{b^\delta a^{2\beta+\alpha+\delta-2}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; -\frac{a^2 b^2}{4}\right)$$

$$[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi].$$

$$2. \int_0^a x (a^2 - x^2)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \pm \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \begin{Bmatrix} J_{\beta+1/2}(ab) \\ H_{\beta+1/2}(ab) \end{Bmatrix} + (1-\delta) \frac{a^{2\beta}\beta}{2}$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi].$$

$$3. \int_a^\infty x^{\alpha-1} (x^2 - a^2)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{b^\delta}{2} a^{\delta+2\beta+\alpha-2} B\left(\beta, 1-\beta-\frac{\alpha+\delta}{2}\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \beta+\frac{\alpha+\delta}{2}, \delta+\frac{1}{2}; -\frac{a^2 b^2}{4}\right) +$$

$$+ \frac{\sqrt{\pi}}{2} \left(\frac{b}{2}\right)^{2-2\beta-\alpha} \Gamma\left[\frac{(\alpha+\delta)/2+\beta-1}{(3+\delta-\alpha)/2-\beta}\right] {}_1F_2\left(1-\beta; 2-\beta-\frac{\alpha}{2}, \frac{3-\alpha}{2}-\beta; -\frac{a^2 b^2}{4}\right)$$

$$[a, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+2\beta) < 3].$$

$$4. \int_a^\infty x (x^2 - a^2)^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b}\right)^{\beta-1/2} a \Gamma(\beta) \begin{Bmatrix} Y_{-\beta-1/2}(ab) \\ J_{-\beta-1/2}(ab) \end{Bmatrix}$$

$$[a, b > 0; 0 < \operatorname{Re} \beta < 1/2].$$

$$5. \int_a^\infty \frac{1}{\sqrt{x(x^2 - a^2)}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = -\frac{\pi}{2} \sqrt{b} J_{\pm 1/4}\left(\frac{ab}{2}\right) Y_{\pm 1/4}\left(\frac{ab}{2}\right) \quad [a, b > 0].$$

$$6. \int_a^\infty \frac{(x^2 - a^2)^{\beta-1}}{x} \cos bx dx =$$

$$= \frac{\pi a^{2\beta}}{2 \sin \beta \pi} \left\{ 1 - \frac{\pi ab}{2} [J_{1/2-\beta}(ab) H_{-1/2-\beta}(ab) - J_{-1/2-\beta}(ab) H_{1/2-\beta}(ab)] \right\}$$

$$[a, b > 0; 0 < \operatorname{Re} \beta < 3/2].$$

$$7. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+y^2} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \pm \frac{\pi}{2} y^{\alpha-2} \left\{ \begin{matrix} \operatorname{tg}(\alpha\pi/2) \sin by \\ \operatorname{ctg}(\alpha\pi/2) \cos by \end{matrix} \right\} - \\ - \Gamma(\alpha-2) \left\{ \begin{matrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{matrix} \right\} b^2 {}_2F_2\left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4}\right) \\ [b, y > 0; -\delta < \operatorname{Re} \alpha < 3].$$

$$8. \int_0^{\infty} \frac{x}{x^2+y^2} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \left\{ \begin{matrix} (\pi/2) \cos by \\ \sin by \operatorname{Si}(by) - \cos by \operatorname{ci}(by) \end{matrix} \right\} \quad [b, y > 0].$$

2.5.9. Интегралы от $\frac{x^{\alpha}}{(x^2+z^2)^{\rho}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\}^n$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2+z^2)^{\rho}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \frac{b^{\delta}}{2} z^{\delta+\alpha-2\rho} B\left(\frac{\alpha+\delta}{2}, \rho-\frac{\alpha+\delta}{2}\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \delta+\frac{1}{2}, 1-\rho+\frac{\alpha+\delta}{2}; \frac{b^2 z^2}{4}\right) + \\ + b^{2\rho-\alpha} \Gamma(\alpha-2\rho) \left\{ \begin{matrix} \sin[(\alpha/2-\rho)\pi] \\ \cos[(\alpha/2-\rho)\pi] \end{matrix} \right\} {}_1F_2\left(\rho; \frac{1-\alpha}{2}+\rho, 1-\frac{\alpha}{2}+\rho; \frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1+2\operatorname{Re} \rho].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+z^2} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \frac{\pi z^{\alpha-2}}{2} \left\{ \begin{matrix} \cos(\alpha\pi/2) \\ \sin(\alpha\pi/2) \end{matrix} \right\}^{-1} \left\{ \begin{matrix} \operatorname{sh} bz \\ \operatorname{ch} bz \end{matrix} \right\} - \\ - \Gamma(\alpha-2) \left\{ \begin{matrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{matrix} \right\} b^{2-\alpha} {}_1F_2\left(1; \frac{3-\alpha}{2}, 2-\frac{\alpha}{2}; \frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 3].$$

$$3. \int_0^{\infty} \frac{x^{2n} \sin bx dx}{(x^2+z^2)^{\rho}} = \\ = (-1)^n \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} \frac{\partial^{2n}}{\partial b^{2n}} \left\{ b^{\rho-1/2} [I_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)] \right\} \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n].$$

$$4. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^{\rho}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = (-1)^{n+\delta} \frac{\sqrt{\pi}}{\Gamma(\rho)} (2z)^{1/2-\rho} \frac{\partial^{2n+1}}{\partial b^{2n+1}} [b^{\rho-1/2} K_{\rho-1/2}(bz)] \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n + \delta/2; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$5. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^m} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \frac{(-1)^{m+n-1}}{(m-1)!} 2^{-m} \pi \left(\frac{\partial}{z \partial z} \right)^{m-1} (z^{2n-1+\delta} e^{-bz}) \\ [b, \operatorname{Re} z > 0; n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{x^{2n\delta}}{(x^2+z^2)^2} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \frac{1+\delta-bz}{4z^{1-\delta}} \pi e^{-bz} \quad [b, \operatorname{Re} z > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$7. \int_0^{\infty} \frac{1}{\sqrt{x(x^2+z^2)}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \sqrt{\frac{\pi b}{2}} I_{\pm 1/4} \left(\frac{bz}{2} \right) K_{1/4} \left(\frac{bz}{2} \right) \quad [b, \operatorname{Re} z > 0].$$

$$8. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^m} dx = \frac{\pi b e^{-bz}}{2^{2m-2} (m-1)!} z^{3-2m} \sum_{k=0}^{m-2} \frac{(2m-k-4)! (2bz)^k}{k! (m-2-k)!} \quad [b, \operatorname{Re} z > 0; m = 1, 2, 3, \dots].$$

$$9. \int_0^{\infty} \frac{x \sin bx}{x^2+z^2} dx = \frac{\pi}{2} e^{-bz} \quad [b, \operatorname{Re} z > 0].$$

$$10. \int_0^{\infty} \frac{\sin bx}{x(x^2+z^2)^2} dx = \frac{\pi}{2z^4} \left[1 - \frac{e^{-bz}}{2} (2+bz) \right] \quad [b, \operatorname{Re} z > 0].$$

$$11. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^{\rho}} dx = \frac{\sqrt{\pi}}{\Gamma(\rho)} \left(\frac{2}{b} \right)^{1/2-\rho} z^{3/2-\rho} K_{3/2-\rho}(bz) \quad [b, \operatorname{Re} z > 0, \operatorname{Re} \rho > 1/2].$$

$$12. \int_0^{\infty} \frac{x \cos bx}{x^2+z^2} dx = -\frac{1}{2} [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] \quad [b, \operatorname{Re} z > 0].$$

$$13. \int_0^{\infty} \frac{x \cos bx}{(x^2+z^2)^{\rho}} dx = \frac{z^{3/2-\rho} b^{\rho-1/2}}{\sqrt{2}} \left[2^{-\rho} \Gamma(1-\rho) I_{\rho-3/2}(bz) + \sqrt{2} e^{-\pi i(1-2\rho)/4} S_{-\rho-1/2, 3/2-\rho}(ibz) \right] \quad [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > 1/2].$$

$$14. \int_0^{\infty} \frac{x \sin^{2n+1} bx}{x^2+z^2} dx = \frac{(-1)^{n-1}}{2^{2n+2}} e^{-(2n+1)bz} \times \\ \times \left[(1 - e^{2(2n+1)bz}) (1 - e^{-2bz})^{2n+1} - 2 \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{2kbz} \right] \quad [b, \operatorname{Re} z > 0].$$

$$15. \int_0^{\infty} \frac{\sin^2 bx}{x(x^2+z^2)} dx = \frac{1}{4z^2} \{ 2[C + \ln(bz)] - [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] \} \quad [b, \operatorname{Re} z > 0].$$

$$16. \int_0^{\infty} \frac{\sin^2 bx}{x^2(x^2+z^2)} dx = \frac{\pi}{4z^3} (e^{-2bz} + 2bz - 1) \quad [b, \operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} \frac{x dx}{(x^2+z^2) \sin bx} = \frac{\pi}{2 \operatorname{sh} bz} \quad [\operatorname{Re} z > 0].$$

$$18. \int_0^{\infty} \frac{x \cos^{2n+1} bx}{x^2 + z^2} dx = -\frac{e^{(2n+1)bz}}{2^{2n+2}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{2kbz} \operatorname{Ei}[(2n-2k+1)bz] -$$

$$-\frac{e^{(2n+1)bz}}{2^{2n+2}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{-2kbz} \operatorname{Ei}[(2k-2n-1)bz] \quad [b, \operatorname{Re} z > 0].$$

2.5.10. Интегралы от $\frac{x^m}{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\}^l$.

$$1. \int_0^{\infty} \frac{x^{2k+1+\delta}}{x^{2n} + z^{2n}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = 0$$

$$\left[b > 0; -\delta < k < n + \frac{1-\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$2. \int_0^{\infty} \frac{x^{2k+\delta}}{x^{2n} + z^{2n}} \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \mp \frac{\pi}{2n} z^{2k+1-2n-\delta} \sum_{l=0}^{n-1} \exp\left(-bz \sin \frac{2l+1}{2n} \pi\right) \left\{ \begin{matrix} \cos t \\ \sin t \end{matrix} \right\}$$

$$\left[t = \frac{2l+1}{2n} (2k+1-\delta)\pi + bz \cos \frac{2l+1}{2n} \pi; b > 0; 0 \leq k < n + \frac{\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$3. \int_0^{\infty} \frac{x^k \sin bx}{x^4 + z^4} dx = I_k$$

$$\left[b > 0; |\arg z| < \frac{\pi}{4} \right],$$

$$I_{-1} = \frac{\pi}{2z^4} \left[1 - \exp\left(-\frac{bz}{\sqrt{2}}\right) \cos\left(\frac{bz}{\sqrt{2}}\right) \right],$$

$$I_{2 \pm 1} = \frac{\pi}{2z^{1 \mp 1}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left\{ \begin{matrix} \sin(bz/\sqrt{2}) \\ \cos(bz/\sqrt{2}) \end{matrix} \right\}.$$

$$4. \int_0^{\infty} \frac{x^k \cos bx}{x^4 + z^4} dx = J_k$$

$$\left[b > 0; |\arg z| < \frac{\pi}{4} \right],$$

$$J_{1 \pm 1} = \frac{\pi \sqrt{2}}{4z^{2 \mp 1}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left(\cos \frac{bz}{\sqrt{2}} \mp \sin \frac{bz}{\sqrt{2}} \right).$$

$$5. \int_0^{\infty} \frac{x^{2k} \cos bx}{x^{2n} - y^{2n}} dx = -y^{2k+1-2n} \sum_{l=0}^{n-1} \exp\left(-by \sin \frac{l\pi}{n}\right) \sin\left(\frac{2k+1}{n} l\pi + by \cos \frac{l\pi}{n}\right)$$

$$[b, y > 0; 0 \leq k < n].$$

$$6. \int_0^{\infty} \frac{x^k \sin by}{x^4 - y^4} dx = I_k$$

$$[b, y > 0; k = -1, 0, 1, 2, 3].$$

$$I_{-1} = \frac{\pi}{4y^4} (e^{-by} + \cos by - 2),$$

$$I_{1 \pm 1} = \frac{1}{4y^{2 \mp 1}} [2 \cos by \operatorname{Si}(by) - 2 \sin by \operatorname{ci}(by) \mp e^{by} \operatorname{Ei}(-by) \pm e^{-by} \operatorname{Ei}(by)],$$

$$I_{2 \pm 1} = \frac{\pi}{4y^{1 \mp 1}} (\cos by \pm e^{-by}).$$

$$7. \int_0^{\infty} \frac{x^k \cos bx}{x^4 - y^4} dx = J_k \quad [b, y > 0; k = 0, 1, 2, 3],$$

$$J_{1 \pm 1} = -\frac{\pi}{4y^2 \mp 1} (\sin by \mp e^{-by}),$$

$$J_{2 \pm 1} = -\frac{1}{4y^1 \mp 1} [2 \cos by \operatorname{ci}(by) + 2 \sin by \operatorname{Si}(by) \pm e^{-by} \operatorname{Ei}(by) \pm e^{by} \operatorname{Ei}(-by)].$$

$$8. \int_{-\infty}^{\infty} \frac{\sin x}{x(x-z)} dx = \frac{\pi}{z} (e^{\pm iz} - 1) \quad [\pm \operatorname{Im} z > 0].$$

$$9. = \frac{\pi}{y} (\cos y - 1) \quad [z = y; \operatorname{Im} y = 0].$$

$$10. \int_0^{\infty} \frac{x^{2 \mp 1} \sin bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_{\pm} b} \left(\pm z^{1 \mp 1} \sin A_{\pm} b + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y \cos A_{\pm} b \right) \\ [2A_{\pm}^2 = \sqrt{z^4 + y^2} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$11. \int_0^{\infty} \frac{x^{1 \mp 1} \cos bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_{\pm} b} (z^4 + y^2)^{-1/2} \times \\ \times \left[\pm z^{1 \mp 1} (A_{\pm} \sin A_{\pm} b + A_{\mp} \cos A_{\pm} b) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y (A_{\pm} \cos A_{\pm} b - A_{\mp} \sin A_{\pm} b) \right] \\ [2A_{\pm}^2 = \sqrt{z^4 + y^2} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$12. \int_{-\infty}^{\infty} \frac{cx + d}{x^2 + 2qx + p} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \pi \exp(-b \sqrt{p - q^2}) \left[c \begin{Bmatrix} \cos bq \\ \sin bq \end{Bmatrix} \pm \frac{cq - d}{\sqrt{p - q^2}} \right] \\ [b > 0; p > q^2].$$

$$13. \int_0^{\infty} \frac{\sin^{2l} mx\pi}{(1 - x^2/1^2)(1 - x^2/2^2) \dots (1 - x^2/n^2)} dx = 0 \quad [l, m, n = 1, 2, 3, \dots].$$

$$14. \int_0^{\infty} \frac{\sin^{2l+1} mx\pi}{x(1 - x^2/1^2)(1 - x^2/2^2) \dots (1 - x^2/n^2)} dx = \frac{1 - (-1)^m}{2} 2^{2(n-l)-1} \frac{(2l)!}{(2n)!} \left(\frac{n!}{l!} \right)^2 \pi \\ [l, m, n = 1, 2, 3, \dots].$$

$$15. \int_0^{\infty} \frac{x^m \sin bx dx}{c^3 \pm c^2 x + cx^2 \pm x^3} = I_m \quad [b, c > 0],$$

$$I_1 = (4c)^{-1} \{ \pi (e^{-bc} - \cos bc) \pm [e^{-bc} \operatorname{Ei}(bc) - e^{bc} \operatorname{Ei}(-bc) - \\ - 2 \sin bc \operatorname{ci}(bc) + 2 \cos bc \operatorname{Si}(bc)] \},$$

$$I_2 = 2^{-2} [e^{bc} \operatorname{Ei}(-bc) - e^{-bc} \operatorname{Ei}(bc) + 2 \operatorname{ci}(bc) \sin bc - \\ - 2 \operatorname{Si}(bc) \cos bc] \pm \pi (e^{-bc} + \cos bc).$$

$$16. \int_0^{\infty} \frac{x^m \sin bx}{x^4 + 2a^2 x^2 + c^4} dx = I_m \quad [a, b, c > 0],$$

$$I_{-1} = \frac{\pi}{2c^4} \left\{ 1 - \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \times \right. \\ \left. \times \left[\cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) + \frac{a^2}{\sqrt{c^4 - a^4}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right] \right\},$$

$$I_1 = \frac{\pi}{2\sqrt{c^4 - a^4}} \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right),$$

$$I_3 = \frac{\pi}{2} \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \left[\cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) - \frac{a^2}{c\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right].$$

$$17. \int_0^\infty \frac{x^m \cos bx}{x^4 + 2ax^2 + c^4} dx = J_m \quad [a, b, c > 0]$$

$$J_0 = \frac{\pi}{2\sqrt{2}c^2} \exp \left(-b \sqrt{\frac{c^2 + a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2 + a^2}} \cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) + \frac{1}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right],$$

$$J_2 = \frac{\pi}{2\sqrt{2}} \exp \left(-b \sqrt{\frac{c^2 + a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2 + a^2}} \cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) - \frac{1}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right].$$

$$18. \int_0^\infty \left[\frac{1}{(x-a)^2 + z^2} \mp \frac{1}{(x+a)^2 + z^2} \right] \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\pi}{z} e^{-bz} \begin{Bmatrix} \sin ab \\ \cos ab \end{Bmatrix} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|].$$

$$19. \int_0^\infty \left[\frac{x-a}{(x-a)^2 + z^2} \mp \frac{x+a}{(x+a)^2 + z^2} \right] \sin bx dx = \mp \pi e^{-bz} \begin{Bmatrix} \sin ab \\ \cos ab \end{Bmatrix} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|].$$

$$20. \int_0^\infty \frac{\sin bx dx}{x(x^2 + 2^2) \dots (x^2 + 4n^2)} = \\ = \frac{\pi}{(2n)!} \frac{(-1)^n}{2^{2n+1}} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2(k-n)b} + (-1)^n \binom{2n}{n} \right] \\ [b > 0; n = 1, 2, 3, \dots].$$

$$21. \int_0^\infty \frac{\cos bx dx}{(x^2 + 1^2)(x^2 + 3^2) \dots [x^2 + (2n+1)^2]} = \\ = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{(2k-2n-1)b} \quad [b > 0].$$

2.5.11. Интегралы от $x^\alpha (\sqrt{x^2 \pm z^2} + ax + bz)^\nu A(x) \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$\begin{aligned}
 1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + z)^\nu}{(x^2 + z^2)^r} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\
 = \frac{1}{2} (vz)^{1-2r} b^{1-\alpha-\nu} \Gamma(\alpha + \nu - 1) \begin{Bmatrix} \cos [(\alpha + \nu) \pi/2] \\ \sin [(\alpha + \nu) \pi/2] \end{Bmatrix} \times \\
 \times {}_2F_3 \left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{3}{2} - 2r, 1 - \frac{\alpha + \nu}{2}, \frac{3 - \alpha - \nu}{2}; \frac{b^2 z^2}{4} \right) + \\
 + (vz)^{2r} b^{4r-\alpha-\nu} \Gamma(\alpha + \nu - 4r) \begin{Bmatrix} \sin [(\alpha + \nu)/2 - 2r] \pi \\ \cos [(\alpha + \nu)/2 - 2r] \pi \end{Bmatrix} \times \\
 \times {}_2F_3 \left(2r - \frac{\nu}{2}, 2r + \frac{\nu}{2}; \frac{1}{2} + 2r, 2r + \frac{1 - \alpha - \nu}{2}, 1 + 2r - \frac{\alpha + \nu}{2}; \frac{b^2 z^2}{4} \right) + \\
 + b^\delta z^{\alpha+\nu+\delta-2r} 2^{\alpha+\nu+\delta-1} (-v)^{1-2r} \Gamma \left[\frac{(\alpha + \delta)/2, 2r - \nu - \alpha - \delta}{1 - \nu - (\alpha + \delta)/2} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha + \delta}{2}, \nu + \frac{\alpha + \delta}{2}; \delta + \frac{1}{2}, \frac{1 + \nu + \alpha + \delta}{2}, 1 + \frac{\nu + \alpha + \delta}{2} - 2r; \frac{b^2 z^2}{4} \right) \\
 [r = 0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} \nu + 2r + 1].
 \end{aligned}$$

$$\begin{aligned}
 2. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + z)^{1/2-\alpha}}{\sqrt{x^2 + z^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\
 = \frac{1}{z} \sqrt{\frac{2}{b}} \Gamma \left(\frac{\alpha + \delta}{2} \right) W_{1/4 - \alpha/2, \pm 1/4} (bz) M_{\alpha/2 - 1/4, \pm 1/4} (bz) \\
 [b, \operatorname{Re} z > 0; \operatorname{Re} \alpha > -\delta].
 \end{aligned}$$

$$3. \int_0^\infty x^\delta \frac{(\sqrt{x^2 + z^2} + z)^{1/2-\delta}}{\sqrt{x^2 + z^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\frac{\pi}{2b}} e^{-bz} \quad [b, \operatorname{Re} z > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$\begin{aligned}
 4. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + x)^\nu}{(x^2 + z^2)^r} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\
 = \frac{b^\delta}{2} \left(-\frac{\nu}{2} \right)^{1-2r} z^{\alpha+\nu-2r+\delta} \Gamma \left[\frac{\alpha + \delta, r - (\alpha + \nu + \delta)/2}{1 + (\alpha + \delta - \nu)/2 - r} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha + 1}{2}, \frac{\alpha}{2} + \delta; \delta + \frac{1}{2}, 1 + \frac{\alpha + \delta - \nu}{2} - r, 1 + \frac{\alpha + \nu + \delta}{2} - r; \frac{b^2 z^2}{4} \right) + \\
 + 2^\nu b^{2r-\alpha-\nu} \Gamma(\alpha + \nu - 2r) \begin{Bmatrix} \sin [(\alpha + \nu)/2 - r] \pi \\ \cos [(\alpha + \nu)/2 - r] \pi \end{Bmatrix} \times \\
 \times {}_2F_3 \left(r - \frac{\nu}{2}, \frac{1-\nu}{2} + r; 1 - \nu, r + \frac{1-\nu-\alpha}{2}, r + 1 - \frac{\nu + \alpha}{2}; \frac{b^2 z^2}{4} \right) \\
 [r = 0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} \nu + 2r + 1].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_0^\infty \frac{(\sqrt{x^2 + z^2} + x)^\nu}{\sqrt{x(x^2 + z^2)}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\frac{\pi b}{2}} z^{\nu/2} I_{\pm 1/4 - \nu/2} \left(\frac{bz}{2} \right) K_{\pm 1/4 + \nu/2} \left(\frac{bz}{2} \right) \\
 [b, \operatorname{Re} z > 0; \operatorname{Re} \nu < 3/2].
 \end{aligned}$$

$$6. \int_0^{\infty} \frac{(\sqrt{x^2+z^2}+x)^{\nu} + (\sqrt{x^2+z^2}-x)^{\nu}}{\sqrt{x^2+z^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = 2z^{\nu} \begin{Bmatrix} \sin(\nu\pi/2) \\ \cos(\nu\pi/2) \end{Bmatrix} K_{\nu}(bz)$$

[$b, \operatorname{Re} z > 0; |\operatorname{Re} \nu| < 1$].

$$7. \int_0^{\infty} \frac{(x+z+\sqrt{x^2+2xz})^{\nu} + (x+z-\sqrt{x^2+2xz})^{\nu}}{\sqrt{x^2+2xz}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \pi z^{\nu} \left[\begin{Bmatrix} \cos(bz - \nu\pi/2) \\ \sin(bz - \nu\pi/2) \end{Bmatrix} J_{\nu}(bz) \pm \begin{Bmatrix} \sin(bz - \nu\pi/2) \\ \cos(bz - \nu\pi/2) \end{Bmatrix} Y_{\nu}(bz) \right]$$

[$b > 0; |\operatorname{Re} \nu| < 1; |\arg z| < \pi$].

$$8. \int_0^a \frac{(x+i\sqrt{a^2-x^2})^{\nu} + (x-i\sqrt{a^2-x^2})^{\nu}}{\sqrt{a^2-x^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{\pi a^{\nu}}{2} \begin{Bmatrix} \operatorname{cosec}(\nu\pi/2) \\ \sec(\nu\pi/2) \end{Bmatrix} [J_{\nu}(ab) \mp J_{-\nu}(ab)]$$

[$a > 0; |\arg b| < \pi$].

$$9. \int_a^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} + (x-\sqrt{x^2-a^2})^{\nu}}{\sqrt{x^2-a^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \pm \pi a^{\nu} \left[\begin{Bmatrix} \cos(\nu\pi/2) \\ \sin(\nu\pi/2) \end{Bmatrix} J_{\nu}(ab) \mp \begin{Bmatrix} \sin(\nu\pi/2) \\ \cos(\nu\pi/2) \end{Bmatrix} Y_{\nu}(ab) \right]$$

[$a, b > 0; |\operatorname{Re} \nu| < 1$].

$$10. \int_a^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} + (x-\sqrt{x^2-a^2})^{\nu}}{\sqrt{x(x^2-a^2)}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = -\frac{\pi}{2} \sqrt{b} a^{\nu} \times$$

$$\times \left[J_{\pm 1/4 + \nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4 - \nu/2} \left(\frac{ab}{2} \right) + J_{\pm 1/4 - \nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4 + \nu/2} \left(\frac{ab}{2} \right) \right]$$

[$a, b > 0; |\operatorname{Re} \nu| < 3/2$].

2.5.12. Интегралы от $\prod_k \begin{Bmatrix} \sin a_k x \\ \cos a_k x \end{Bmatrix}^{\mu_k} \begin{Bmatrix} \sin b_k x \\ \cos b_k x \end{Bmatrix}^{\nu_k}$,

$$\prod_k \sin^{\mu_k} a_k x \cos^{\nu_k} b_k x.$$

$$1. \int_0^{\pi/4} \frac{\sin^{2\mu} x}{\cos^{\mu+1/2} x} dx = 2^{-2\mu-1} B(2\mu+1, -\mu)$$

[$-1/2 < \operatorname{Re} \mu < 0$].

$$2. \int_0^{\pi/4} \frac{\cos^{2\mu} 2x}{\cos^{2\mu+2} x} dx = 2^{2\mu} B(\mu+1, \mu+1)$$

[$\operatorname{Re} \mu > -1$].

$$3. \int_0^{\pi/4} \frac{\cos^{n-1/2} 2x}{\cos^{2n+1} x} dx = \frac{(2n-1)!!}{2(2n)!!} \pi.$$

$$4. \int_0^{\pi/4} \frac{\sin^m x \cos^{\nu} 2x}{\cos^{2\nu+m+2} x} dx = \frac{1}{2} B\left(\frac{m+1}{2}, \nu+1\right)$$

[$\operatorname{Re} \nu > -1$].

$$5. \int_0^{\pi/4} \frac{\sin^m x \cos^{n-1/2} 2x}{\cos^{2n+m+1} x} dx = \frac{(m-1)!! (2n-1)!!}{(m+2n)!!} \begin{cases} \pi/2, & m=0, 2, 4, \dots \\ 1, & m=1, 3, 5, \dots \end{cases} \\ [n=1, 2, 3, \dots].$$

$$6. \int_0^{\pi/2} \sin^n x dx = \frac{(n+1)!!}{n!!} \begin{cases} 1, & n=1, 3, 5, \dots \\ \pi/2, & n=0, 2, 4, \dots \end{cases}$$

$$7. \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx = 2 \sum_{k=1}^{n/2} \frac{(-1)^{k-1}}{2k-1} \quad [n=2, 4, \dots].$$

$$8. = \pi/2 \quad [n=1, 3, 5, \dots].$$

$$9. \int_0^{\pi/2} \frac{\sin 2nx}{\cos x} dx = (-1)^n 2 \sum_{k=1}^n \frac{(-1)^k}{2k-1}.$$

$$10. \int_0^{\pi/2} \frac{\cos (2n+1)x}{\cos x} dx = (-1)^n \frac{\pi}{2}.$$

$$11. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2^{-\mu} \pi \left\{ \Gamma \left[\frac{\mu}{(1+\mu-b)/2}, \frac{(1+\mu+b)/2} \right] \times \right. \\ \times \left\{ \frac{\sin (b\pi/2)}{\cos (b\pi/2)} \right\} \mp \frac{b}{2} \Gamma \left[\frac{\mu}{1+(\mu+b)/2}, \frac{1+(\mu-b)/2} \right] \left\{ \frac{\cos (b\pi/2)}{\sin (b\pi/2)} \right\} \times \\ \times {}_3F_2 \left(\frac{\mu}{2}, \frac{\mu+1}{2}, \frac{1}{2}; 1+\frac{\mu+b}{2}, 1+\frac{\mu-b}{2}; 1 \right) \Bigg\} \quad [\operatorname{Re} \mu > -\left\{ \frac{1}{0} \right\}; | \arg b | < \pi].$$

$$12. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin (\mu+1)x}{\cos (\mu+1)x} \right\} dx = \frac{1}{\mu} \left\{ \frac{\sin (\mu\pi/2)}{\cos (\mu\pi/2)} \right\} \quad \left[\operatorname{Re} \mu > -\left\{ \frac{1}{0} \right\} \right].$$

$$13. \int_0^{\pi/2} \sin^{2n+\delta} x \left\{ \frac{\sin (2m+1)x}{\cos 2mx} \right\} dx = \frac{(-1)^m}{2^{2n+1+\delta}} \binom{2n+\delta}{n-m} \begin{cases} \pi \\ 1 \end{cases} \quad \left[m \leq n; \delta = \left\{ \frac{1}{0} \right\} \right].$$

$$14. = 0 \quad \left[m > n; \delta = \left\{ \frac{1}{0} \right\} \right].$$

$$15. \int_0^{\pi/2} \sin^{2n+\delta-1} x \left\{ \frac{\sin (2m+1)x}{\cos 2mx} \right\} dx = \\ = \frac{(-1)^m (2n-1)!! (n-1)! 2^{n-1}}{(2n-2m-1)!! (2m+2n+1)!!} \begin{cases} 2n \\ 2m+2n+1 \end{cases} \quad \left[m \leq n; \delta = \left\{ \frac{1}{0} \right\} \right].$$

$$16. = \frac{(-1)^n (n-1)! 2^{n-1} (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \begin{cases} 2n \\ 2m+2n+1 \end{cases} \quad \left[m > n; \delta = \left\{ \frac{1}{0} \right\} \right].$$

$$17. \int_0^{\pi/2} \cos^{v-1} x \sin bx \, dx = \pi b 2^{-1-v} \left[1 + (v+b)/2, 1 + (v-b)/2 \right]^v \times \\ \times {}_3F_2 \left(\frac{v}{2}, \frac{1+v}{2}, \frac{1}{2}; 1 + \frac{v+b}{2}, 1 + \frac{v-b}{2}; 1 \right) \quad [\operatorname{Re} v > 0; |\arg b| < \pi].$$

$$18. \int_0^{\pi/2} \cos^{v-1} x \sin (v+1) x \, dx = \frac{1}{v} \quad [\operatorname{Re} v > 0].$$

$$19. \int_0^{\pi/2} \cos^n x \sin mx \, dx = \sum_{j=0}^{k-1} \frac{n! (m+n-2j-2)!!}{(n-j)! (m+n)!!} + \theta \frac{n! (m-n-2)!!}{(m+n)!!} \\ [k = \min(m, n), \theta = 2 \text{ при } m-n=4l+2 > 0; \theta = 1 \text{ при } m-n=2l+1 > 0; \theta = 0 \text{ при } m-n=4l \geq 0 \text{ или } m-n < 0].$$

$$20. \int_0^{\pi/2} \cos^n x \sin nx \, dx = \frac{1}{2^{n+1}} \sum_{j=1}^n \frac{2^j}{j}.$$

$$21. \int_0^{\pi/2} \cos^{v-1} x \sin (2m+1) x \, dx = (-1)^m \sum_{j=0}^m \frac{(-1)^j 2^j}{v+j} \binom{m}{j} \quad [\operatorname{Re} v > 0].$$

$$22. \int_0^{\pi/2} \cos^{v-1} x \cos bx \, dx = 2^{-v} \pi \Gamma \left[(1+v+b)/2, (1+v-b)/2 \right]^v \\ [\operatorname{Re} v > 0; |\arg b| < \pi].$$

$$23. \int_0^{\pi/2} \cos^{v-1} x \cos (v+1) x \, dx = 0 \quad [\operatorname{Re} v > 0].$$

$$24. \int_0^{\pi/2} \cos^n x \cos mx \, dx = \frac{n!}{(n-m)!! (n+m)!!} \begin{cases} 1, & n-m=1, 3, 5, \dots \\ \pi/2, & n-m=0, 2, 4, \dots \end{cases}$$

$$25. = B \frac{n!}{(m-n)(m-n+2) \dots (m+n)} \\ [n < m; B = 0, \text{ если } m-n \text{ четно; } B = \pm 1, \text{ если } m-n=2l \pm 1, l=0, 1, 2, \dots].$$

$$26. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{v-1} x \, dx = \frac{1}{2} B \left(\frac{\mu}{2}, \frac{v}{2} \right) \quad [\operatorname{Re} \mu, \operatorname{Re} v > 0].$$

$$27. \int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{\pi a}{2} - \frac{\sin a\pi}{2} [2a\beta(a) - 1] \quad [a > 0].$$

$$28. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{v-1} x \begin{cases} \sin (\mu+v) x \\ \cos (\mu+v) x \end{cases} dx = \\ = \frac{\pi}{2} \left\{ \sec (\mu\pi/2) \right\} \Gamma \left[\mu+v, 1-\mu \right]^v \quad [\operatorname{Re} v > 0; \operatorname{Re} \mu > -\{ \frac{1}{0} \}].$$

29. $\int_0^\pi \sin^{\mu-1} x \cos^{\nu-1} x \begin{cases} \sin (\mu + \nu) x \\ \cos (\mu + \nu) x \end{cases} dx = \begin{cases} \sin (\mu\pi/2) \\ \cos (\mu\pi/2) \end{cases} B(\mu, \nu)$
 $\left[\operatorname{Re} \nu > 0; \operatorname{Re} \mu > -\left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\} \right].$
30. $\int_0^\pi \frac{\sin nx \cos mx}{\sin x} dx = 0$ $[n \leq m \text{ или } n > m; m + n = 2k].$
31. $= \pi$ $[n > m; m + n = 2k - 1, k = 1, 2, 3, \dots].$
32. $\int_a^{a+\pi} \sin^m x \cos^n x dx = \frac{1 + (-1)^n}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$ $[m + n = 0, 2, 4, \dots].$
33. $\int_0^\pi \frac{\sin nx}{\sin x} dx = \frac{1 - (-1)^n}{2} \pi.$
34. $\int_0^\pi \frac{\sin 2nx}{\cos x} dx = (-1)^n 4 \sum_{k=1}^n \frac{(-1)^k}{2k-1}.$
35. $\int_0^\pi \frac{\cos (2n+1)x}{\cos x} dx = (-1)^n \pi.$
36. $\int_0^\pi \sin^{\mu-1} x \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \frac{2^{1-\mu} \pi}{\mu B((\mu+b+1)/2, (\mu-b+1)/2)} \begin{cases} \sin (b\pi/2) \\ \cos (b\pi/2) \end{cases}$
 $[\operatorname{Re} \mu > 0; |\arg b| < \pi].$
37. $\int_0^\pi \sin^n x \begin{cases} \sin (2m+1)x \\ \cos 2mx \end{cases} dx = 2 \int_0^{\pi/2} \sin^n x \begin{cases} \sin (2m+1)x \\ \cos 2mx \end{cases} dx$
 $[\text{см. 2.5.12.13—16}].$
38. $\int_0^\pi \sin^{2n} x \cos 2mx dx = \frac{(-1)^m \pi}{2^{2n}} \binom{2n}{n-m}$ $[n \geq m].$
39. $= 0$ $[n < m].$
40. $\int_0^\pi \cos^n x \begin{cases} \sin mx \\ \cos mx \end{cases} dx = [1 \mp (-1)^{m+n}] \int_0^{\pi/2} \cos^n x \begin{cases} \sin mx \\ \cos mx \end{cases} dx$
 $[\text{см. 2.5.12.19—25}].$
41. $\int_0^\pi \cos^n x \cos bx dx = \frac{(-1)^n \sin b\pi}{2^n (n+b)} {}_2F_1\left(-n, \frac{b+n}{2}; 1 - \frac{b+n}{2}; -1\right)$
 $[b \neq 0, \pm 1, \pm 2, \dots].$
42. $\int_0^\pi \cos^n x \cos mx dx = 0$ $[m + n = 1, 3, 5, \dots \text{ или } m > n, m + n = 2, 4, 6, \dots].$

$$43. = \frac{(-1)^{m+n-1} 2^{1-n} \pi (-n)_l (l)_l}{(m+n)(l-1)!} \quad [m \leq n; m+n=2l, l=1, 2, 3, \dots]$$

$$44. \int_a^{a+2\pi} \sin^m x \cos^n x dx = \frac{[1+(-1)^m][1+(-1)^n]}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

2.5.13. Интегралы от $x^\alpha \sin^\mu ax \cos^\nu bx$.

$$1. \int_0^{\pi/4} x \frac{\sin^{\mu-1} x}{\cos^{\mu+1} x} dx = \frac{\pi}{4\mu} - \frac{1}{2\mu} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1].$$

$$2. \int_0^{\pi/4} x \frac{\sin^n x}{\cos^{n+2} x} dx =$$

$$= \frac{1}{n+1} \left\{ \frac{\pi}{4} - (-1)^{[(n+1)/2]} \left(\frac{1-(-1)^n}{2} \frac{\pi}{4} + \frac{1+(-1)^n}{2} \ln 2 \right) - \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{n-2k} \right\}.$$

$$3. \int_0^{\pi/4} x^2 \frac{\sin x}{\cos^3 x} dx = \frac{\ln 2}{2} - \frac{\pi}{4} + \frac{\pi^2}{16}.$$

$$4. \int_0^{\pi/4} x^2 \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + 6G \right).$$

$$5. \int_0^{\pi/4} \frac{x dx}{\sin x \sqrt{\cos 2x}} = \frac{\pi}{2} \ln(1 + \sqrt{2}).$$

$$6. \int_{\pi/4}^{\pi/2} x \frac{\cos^{\mu-1} x}{\sin^{\mu+1} x} dx = \frac{1}{2\mu} \left[\frac{\pi}{2} + \beta\left(\frac{\mu+1}{2}\right) \right] \quad [\operatorname{Re} \mu > 0].$$

$$7. \int_0^{\pi/2} x \left\{ \frac{\sin^{\mu-1} x \cos x}{\cos^{\mu-1} x \sin x} \right\} dx = \frac{\pi}{2\mu} \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} \mp \frac{\sqrt{\pi}}{2\mu} \Gamma \left[\frac{(\mu+1)/2}{\mu/2+1} \right] \quad [\operatorname{Re} \mu > -\left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}].$$

$$8. \int_0^{\pi/2} x \sin^{2n} x \cos x dx = \frac{1}{2(2n+1)} \left[\pi - \frac{2(2n)!!}{(2n+1)!!} \right].$$

$$9. \int_0^{\pi/2} x \sin^{2n+1} x \cos x dx = \frac{1}{4(n+1)} \left[1 - \frac{(2n+1)!!}{(2n+2)!!} \right].$$

$$10. \int_0^{\pi/2} x^2 \frac{\cos x}{\sin^2 x} dx = 4G - \frac{\pi^2}{4}.$$

$$11. \int_0^{\pi/2} x^3 \frac{\cos x}{\sin^3 x} dx = \frac{3\pi}{2} \ln 2 - \frac{\pi^3}{16}.$$

$$12. \int_0^{\pi/2} x \cos^n x \sin x dx = \frac{n!!}{(n+1)!! (n+1)} \begin{cases} 1, & n=0, 2, 4, \dots \\ \pi, & n=1, 3, 5, \dots \end{cases}$$

$$13. \int_0^{\pi/2} x \left\{ \frac{\cos x}{\sin x} \right\}^{\mu-1} \left\{ \frac{\sin x}{\cos x} \right\}^{-\mu-1} dx = \pm \frac{\pi}{2\mu \cos(\mu\pi/2)} \quad [0 < \pm \operatorname{Re} \mu < 1].$$

$$14. \int_0^{\pi/2} x \sin^n x \cos^{2m-1} x dx = \frac{B(n/2, m+1/2)}{2(2m+n-1)} + \frac{n-1}{2m+n-1} \int_0^{\pi/2} x \sin^{n-2} x \cos^{2m-1} x dx \quad [m, n=1, 2, 3, \dots].$$

$$15. \int_0^{\pi/2} x \cos^{v-1} x \sin bx dx = \frac{\pi}{2^{v+1}} \Gamma(v) \frac{\psi((v+b+1)/2) - \psi((v-b+1)/2)}{\Gamma((v+b+1)/2) \Gamma((v-b+1)/2)} \\ [\operatorname{Re} v > 0; -1 - \operatorname{Re} v < \operatorname{Re} b < 1 + \operatorname{Re} v].$$

$$16. \int_0^{\pi} x \sin^{\mu-1} x \cos x dx = -\frac{\sqrt{\pi}}{\mu} \Gamma\left[\frac{(\mu+1)/2}{\mu/2+1}\right] \quad [\operatorname{Re} \mu > 0].$$

$$17. \int_0^{\pi} x \cos^n x \sin x dx = \frac{\pi}{n+1} \left\{ (-1)^n + \frac{1 - (-1)^n}{2} \frac{(2[(n+1)/2] - 1)!!}{(2[(n+1)/2])!!} \right\}.$$

$$18. \int_0^{\pi} x \frac{\sin x}{\cos x} dx = -\pi \ln 2.$$

$$19. \int_0^{\pi} x \sin^{2n-1} x \cos^m x dx = \frac{(-1)^{m-1}}{2(2n+m-1)} B\left(\frac{m}{2}, n+\frac{1}{2}\right) + \frac{m-1}{2n+m-1} \int_0^{\pi} x \sin^{2n-1} x \cos^{m-2} x dx \\ [m, n=1, 2, 3, \dots].$$

$$20. \int_0^{\infty} x^{\alpha-1} \sin^n ax \sin^m bx dx = I_{\alpha}^{n, m}.$$

$$I_{\alpha}^{n, m} = \sqrt{\pi} 2^{1-m-n+\alpha} \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} \times \\ \times \sum_{l=0}^{[(m-1)/2]} (-1)^{[m/2]+l} \frac{m!}{l! (m-l)!} A_{kl}(t), \quad t = \frac{(n-2k)^2 a^2}{(m-2l)^2 b^2},$$

$$A_{kl}(t) = (4t)^\lambda b^{-\alpha} (m-2l)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] \times \\ \times \{ {}_2F_1(\lambda + \gamma + \alpha/2, \lambda - \gamma + (1+\alpha)/2; 2\lambda + 1/2; t) + 2\lambda - 1 \} + \\ + (2\gamma - 1) 4\gamma t^{-\gamma} a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \text{ при } 0 < t < 1,$$

$$A_{kl}(t) = 4\gamma t^{-\gamma} a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \times \\ \times \{ {}_2F_1(\lambda + \gamma + \alpha/2, \gamma - \lambda + (1+\alpha)/2; 2\gamma + 1/2; t^{-1}) + 2\gamma - 1 \} + \\ + (2\lambda - 1) (4t)^\lambda b^{-\alpha} (m-2l)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] \text{ при } t > 1,$$

$$A_{kl}(1) = a^{-\alpha} (n-2k)^{-\alpha} \left\{ \sqrt{\pi} \Gamma \left[\frac{\lambda + \gamma + \alpha/2, -\alpha}{\lambda - \gamma + (1-\alpha)/2, \gamma - \lambda + (1-\alpha)/2, \lambda + \gamma - \alpha/2} \right] + \right. \\ \left. + 4^\lambda (2\lambda - 1) \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] + 4^\gamma (2\gamma - 1) \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \right\}$$

$$\left[\lambda = \frac{1 - (-1)^n}{4}, \quad \gamma = \frac{1 - (-1)^m}{4}; \quad a, b > 0; \quad -m - n < \operatorname{Re} \alpha < 0 \quad \leftarrow m - n < \operatorname{Re} \alpha < 1, \text{ если} \right. \\ \left. \text{одно из чисел } m, n \text{ нечетно или если } m, n \text{ нечетны и } a \neq b; \quad m, n = 1, 2, 3, \dots \right];$$

$$I_{0,1}^{1,1} = \frac{1}{2} \ln \frac{a+b}{|a-b|} \quad [a \neq b]; \quad I_{1,1}^{1,1} = \frac{\pi}{2} \min(a, b);$$

$$I_{0,1}^{2,1} = \frac{\pi}{4} \quad [b < 2a], \quad I_{0,1}^{3,1} = \frac{\pi}{8} \quad [b = 2a], \quad I_{0,1}^{2,1} = 0 \quad [b > 2a];$$

$$I_{-1}^{2,1} = \frac{a}{2} \ln \frac{2a+b}{|2a-b|} + \frac{b}{4} \ln \frac{|4a^2 - b^2|}{b^2}, \quad [b \neq 2a],$$

$$I_{-1}^{2,1} = a \ln 2 \quad [b = 2a]; \quad I_{-2}^{2,1} = \frac{\pi}{8} b (4a - b), \quad [b \leq 2a],$$

$$I_{-2}^{2,1} = \frac{\pi}{2} a^2 \quad [b \geq 2a]; \quad I_{-1}^{2,2} = \frac{\pi}{4} \min(a, b);$$

$$I_{-2}^{2,2} = \frac{a^2}{4} \ln \frac{|a^2 - b^2|}{a^2} + \frac{b^2}{4} \ln \frac{|a^2 - b^2|}{b^2} - \frac{ab}{2} \ln \frac{a+b}{|a-b|} \quad [a \neq b],$$

$$I_{-2}^{2,2} = a^2 \ln 2 \quad [b = a]; \quad I_{-3}^{2,2} = \frac{1}{6} \pi a^2 (3b - a) \quad [a \leq b],$$

$$I_{-3}^{2,2} = \frac{1}{6} \pi b^2 (3a - b) \quad [b \leq a]; \quad I_{0,1}^{3,1} = \frac{1}{8} \ln \left| \frac{(a+b)^3 (3a-b)}{(a-b)^3 (3a+b)} \right| \quad [b \neq a, b \neq 3a];$$

$$I_{-1}^{3,1} = \frac{\pi}{4} b \quad [b \leq a], \quad I_{-1}^{3,1} = \frac{\pi}{8} (3a - b) \quad [a \leq b \leq 2a], \quad I_{-1}^{3,1} = 0 \quad [b \geq 3a];$$

$$I_{-n}^{n,1} = \frac{\pi}{2} \left\{ a^n - \frac{2^{1-n}}{n!} \sum_{k=0}^{[(na-b)/(2a)]} (-1)^k \binom{n}{k} [a(n-2k) - b]^n \right\} \quad [b \leq na],$$

$$I_{-n}^{n,1} = \frac{\pi}{2} a^n \quad [b = am, m \geq n, m = 1, 2, 3, \dots].$$

$$21. \int_0^\infty x^{\alpha-1} \sin^n ax \cos^n bx \, dx = J_{\alpha}^{n,m},$$

$$\begin{aligned}
 J_{\alpha}^{n, m} &= \sqrt{\pi} 2^{1-m-n+\alpha} \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} \times \\
 &\times \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} B_{kl}(t) + \frac{1+(-1)^m}{2} \sqrt{\pi} 2^{\alpha-m-n} a^{-\alpha} \frac{m!}{[(m/2)!]^2} \times \\
 &\times \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2} \right], \quad t = \frac{(n-2k)^2 a^2}{(m-2l)^2 b^2}, \\
 B_{kl}(t) &= (4t)^{\lambda} b^{-\alpha} (m-2l)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{(1-\alpha)/2 - \lambda} \right] {}_2F_1 \left(\lambda + \alpha/2, \right. \\
 &\quad \left. \lambda + (1+\alpha)/2; 2\lambda + 1/2; t \right) - 1 + 2\lambda \} \text{ при } 0 < t < 1, \\
 B_{kl}(t) &= a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2} \right] {}_2F_1 \left(\lambda + \alpha/2, \right. \\
 &\quad \left. (1+\alpha)/2 - \lambda; 1/2; t^{-1} \right) + (2\lambda - 1) (4t)^{\lambda} [b(m-2l)]^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{(1-\alpha)/2 - \lambda} \right] \text{ при } t > 1, \\
 B_{kl}(1) &= -4^{\lambda} a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{(1-\alpha)/2 - \lambda} \right] (1 - 2\lambda - \alpha^{-2\lambda} 2^{-\alpha-1}) \\
 &\quad \left[\lambda = \frac{1-(-1)^n}{4}; a, b > 0; -n < \operatorname{Re} \alpha < 0 \text{ } (-n < \operatorname{Re} \alpha < 1, \text{ если } n \text{ или } m \text{ нечетно); } n, \right. \\
 &\quad \left. m = 1, 2, 3, \dots \right];
 \end{aligned}$$

$$J_{1/2}^{1,1} = \frac{\sqrt{\pi}}{2} \sqrt{\frac{a + \sqrt{a^2 - b^2}}{a^2 - b^2}} \quad [a > b], \quad J_{1/2}^{1,1} = \frac{1}{4} \sqrt{\frac{\pi}{a}} \quad [b = a],$$

$$J_{1/2}^{1,2} = -\frac{\sqrt{\pi}}{2} \sqrt{\frac{b - \sqrt{b^2 - a^2}}{b^2 - a^2}} \quad [b > a], \quad J_0^{1,2} = \frac{\pi}{2} \quad [a > b],$$

$$J_0^{1,1} = \frac{\pi}{4} \quad [a = b], \quad J_{\frac{1}{2}}^{1,1} = 0 \quad [b > a],$$

$$J_0^{2,1} = \frac{1}{4} \ln \frac{|4a^2 - b^2|}{b^2} \quad [b \neq 2a], \quad J_{\frac{1}{2}}^{2,1} = \frac{\pi}{4} (2a - b) \quad [b \leq 2a],$$

$$J_{\frac{1}{2}}^{2,1} = 0 \quad [b \geq 2a], \quad J_0^{1,2} = \frac{\pi}{4} \quad [a < 2b], \quad J_0^{1,2} = \frac{3}{8} \pi \quad [a = 2b],$$

$$J_0^{1,2} = \frac{\pi}{2} \quad [a > 2b], \quad J_{\frac{1}{2}}^{2,2} = \frac{2a-b}{4} \pi \quad [a \geq b], \quad J_{\frac{1}{2}}^{2,2} = \frac{\pi a}{4} \quad [a \leq b];$$

$$J_0^{3,1} = \frac{\pi}{4} \quad [b < a], \quad J_{\frac{1}{2}}^{3,1} = \frac{\pi}{16} \quad [b = a],$$

$$J_0^{3,1} = -\frac{\pi}{8} \quad [a < b < 3a], \quad J_{\frac{1}{2}}^{3,1} = -\frac{\pi}{16} \quad [b = 3a], \quad J_{\frac{1}{2}}^{3,1} = 0 \quad [b > 3a];$$

$$J_{\frac{1}{2}}^{3,1} = \frac{3a}{8} \ln \left| \frac{9a^2 - b^2}{a^2 - b^2} \right| + \frac{b}{8} \ln \left| \frac{(a-b)^2 (3a+b)}{(a+b)^3 (3a-b)} \right| \quad [b \neq a, b \neq 3a],$$

$$J_{\frac{1}{2}}^{3,1} = \frac{\ln 2}{2} a \quad [b = a], \quad J_{\frac{1}{2}}^{3,1} = \frac{3}{4} (\ln 3 - 2 \ln 2) a \quad [b = 3a];$$

$$J_{\frac{1}{2}}^{3,1} = \frac{\pi}{8} (3a^2 - b^2) \quad [b \leq a], \quad J_{\frac{1}{2}}^{3,1} = \frac{\pi}{16} (3a - b)^2 \quad [a \leq b \leq 3a],$$

$$J_{-2}^{1,1} = 0 \quad [b \geq 3a]; \quad J_0^{3,2} = \frac{\pi}{4} \quad [2b < a],$$

$$J_0^{3,2} = \frac{5}{32} \pi \quad [2b = a], \quad J_0^{3,2} = \frac{\pi}{16} \quad [a < 2b < 3a],$$

$$J_0^{3,2} = \frac{3}{32} \pi, \quad [2b = 3a], \quad J_0^{3,2} = \frac{\pi}{8} \quad [2b > 3a];$$

$$J_{-1}^{3,2} = \frac{3 \ln 3}{8} a + \frac{3}{16} a \ln \left| \frac{9a^2 - 4b^2}{a^2 - 4b^2} \right| + \frac{b}{8} \ln \left| \frac{(a-2b)^3 (3a+2b)}{(a+2b)^3 (3a-2b)} \right| \quad [2b \neq a, 2b \neq 3a],$$

$$J_{-1}^{3,2} = \frac{3 \ln 3 + 2 \ln 2}{8} a \quad [2b = a], \quad J_{-1}^{3,2} = \frac{3}{4} (\ln 3 - \ln 2) a \quad [2b = 3a];$$

$$J_{-2}^{3,2} = \frac{\pi}{8} (3a^2 - 2b^2) \quad [2b \leq a], \quad J_{-2}^{3,2} = \frac{\pi}{32} [6a^2 + (3a - 2b)^2] \quad [a \leq 2b \leq 3a],$$

$$J_{-2}^{3,2} = \frac{3}{16} \pi a^2 \quad [2b \geq 3a].$$

$$J_0^{2j+1, m} = \frac{1}{2} B \left(j + \frac{1}{2}, \left[\frac{m+1}{2} \right] + \frac{1}{2} \right) =$$

$$= \frac{(2j-1)!! (2[(m+1)/2] - 1)!!}{2^{j+1} + [(m+1)/2]! (j + [(m+1)/2])!} \pi \quad [b = a];$$

$$J_0^{2j+1, -1} = \frac{\pi}{2} \frac{(2j-1)!!}{(2j)!!} \quad [b = a]; \quad J_0^{2j+1, 2k} = \frac{1}{2} B \left(j + \frac{1}{2}, k + \frac{1}{2} \right) \quad [b = a];$$

$$J_{-1}^{2j, 1} = \frac{(2j-3)!!}{(2j)!!} \frac{\pi a}{2} \quad [b = a];$$

$$J_{1-n}^{n, 1} = \frac{\pi n}{2^n} \sum_{k=0}^{[(an-b)/(2a)]} \frac{(-1)^k}{k! (n-k)!} [a(n-2k) - b]^{n-1} \quad [b < na],$$

$$J_{1-n}^{n, 1} = 0 \quad [b \geq na]; \quad J_{1-k}^{2j+1, 1} = \frac{(2j-k)!!}{(2j-k+3)!!} \frac{\pi}{2} a^{k-1} \quad [b = a; k = 1, 2, 3].$$

$$22. \int_0^\infty \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{x} dx = \frac{1}{2} B \left(\frac{\mu-1}{2}, \frac{\nu}{2} \right)$$

$[\mu > 1, \nu > 0; \mu = 2k/(2l+1), \nu = 2m/(2n+1)].$

$$23. \int_0^\infty x^{\alpha-1} \cos^n ax \cos^m bx dx = \sqrt{\pi} 2^{1-m} n^{\alpha} \Gamma \left[\frac{\alpha/2}{(1-\alpha)/2} \right] \times$$

$$\times \left\{ \sum_{k=0}^{[(n-1)/2]} \frac{n!}{k! (n-k)!} \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} C_{kl}(t) + \right.$$

$$+ \frac{1+(-1)^n}{4} \frac{n! b^{-\alpha}}{[(n/2)!]^2} \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} (m-2l)^{-\alpha} +$$

$$\left. + \frac{1+(-1)^m}{4} \frac{m! a^{-\alpha}}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} \frac{n!}{k! (n-k)!} (n-2k)^{-\alpha} \right\},$$

$$t = \frac{(n-2k)^2 a^2}{(m-2l)^2 b^2},$$

$$C_{kl}(t) = (m-2l)^{-\alpha} b^{-\alpha} {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}; t\right) \text{ при } 0 < t < 1,$$

$$C_{kl}(t) = (n-2k)^{-\alpha} a^{-\alpha} {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}; t^{-1}\right) \text{ при } t > 1$$

$[a, b > 0; 0 < \operatorname{Re} \alpha < 1; m, n = 1, 2, 3, \dots, \text{ кроме случая, когда } m \text{ и } n \text{ четны одновременно}].$

2.5.14. Интегралы от $\frac{x^\alpha}{x^2 \pm z^2} \sin^\mu ax \cos^\nu bx$.

$$1. \int_0^\infty \frac{x^{\alpha-1}}{x^2 + z^2} \sin^n ax \sin bx \, dx = I_{\alpha, b}^n \quad [\alpha = 1, 2; a, b, \operatorname{Re} z > 0].$$

$$I_{1, (2k-2)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k-1} z} [(1 - e^{-2az})^{2k-2} - 1] \operatorname{sh} az,$$

$$I_{1, (2k-1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k} z} (1 - e^{-2az})^{2k-1},$$

$$I_{1, (2k+1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k} z} (1 - e^{-2az})^{2k-1} e^{-2az},$$

$$I_{1, 3(2k-1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2z} \operatorname{sh}^{2k-1} az e^{-3(2k-1)az},$$

$$I_{2, (2k-1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} e^{az} [(1 - e^{-2az})^{2k} - 1],$$

$$I_{2, 2ka}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} [(1 - e^{-2az})^{2k} - 1],$$

$$I_{2, 2(k+1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} e^{-2az} (1 - e^{-2az})^{2k},$$

$$I_{2, 4ka}^{2k} = \frac{(-1)^k \pi}{2} e^{-4kaz} \operatorname{sh}^{2k} az.$$

$$2. \int_0^\infty \frac{x^{\alpha-1}}{x^2 + z^2} \sin^n ax \cos bx \, dx = J_{\alpha, b}^n \quad [\alpha = 1, 2; a, b, \operatorname{Re} z > 0].$$

$$J_{2, (2k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} [(1 - e^{-2az})^{2k-1} - 1],$$

$$J_{2, 2(k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} [(1 - e^{-2az})^{2k-1} - 1] e^{-az},$$

$$J_{2, (2k+1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} (1 - e^{-2az})^{2k-1} e^{-2az},$$

$$J_{2, 2(2k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2} \operatorname{sh}^{2k-1} az e^{-2(2k-1)az},$$

$$J_{1, (2k-1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k} z} [(1 - e^{-2az})^{2k-1} - 1] \operatorname{sh} az,$$

$$J_{1, 2ka}^{2k} = \frac{(-1)^k \pi}{2^{2k+1} z} (1 - e^{-2az})^{2k},$$

$$J_{1, 2(k+1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1} z} (1 - e^{-2az})^{2k} e^{-2az},$$

$$J_{1, 4ka}^{2k} = \frac{(-1)^k \pi}{2z} e^{-4kaz} \operatorname{sh}^{2k} az.$$

$$3. \int_0^\infty \frac{x^{2-1}}{x^2 + z^2} \cos^n ax \sin bx \, dx = I_{a,b} \quad [a, b, \operatorname{Re} z > 0],$$

$$I_{1, na} = \frac{1}{2^{n+1} z} \sum_{l=1}^n \binom{n}{l} [e^{-2laz} \operatorname{Ei}(2laz) - e^{2laz} \operatorname{Ei}(-2laz)],$$

$$I_{2, (n+2)a} = \frac{\pi}{2^{n+1}} e^{-2az} (1 + e^{-2az})^n,$$

$$I_{2, (n+1)a} = \frac{\pi}{2^{n+1}} e^{-az} (1 + e^{-2az})^n, \quad I_{2, na} = \frac{\pi}{2^{n+1}} [(1 + e^{-2az})^n - 1],$$

$$I_{2, (n-1)a} = \frac{\pi}{2^{n+1}} (1 + e^{-2az})^n e^{az}, \quad I_{2, 3a} = \frac{\pi}{2} e^{-3az} \operatorname{ch}^n az.$$

$$4. \int_0^\infty \frac{1}{x^2 + z^2} \cos^n ax \cos bx \, dx = I_b \quad [a, b, \operatorname{Re} z > 0],$$

$$I_{(n+2)a} = \frac{\pi}{2^{n+1} z} e^{-2az} (1 + e^{-2az})^n, \quad I_{(n+1)a} = \frac{\pi}{2^{n+1} z} e^{-az} (1 + e^{-2az})^n,$$

$$I_{na} = \frac{\pi}{2^{n+1} z} (1 + e^{-2az})^n, \quad I_{(n-1)a} = \frac{\pi}{2^{n+1} z} e^{az} (1 + e^{-2az})^n.$$

$$5. \int_0^\infty \frac{x^\delta \cos^n ax}{x^2 - y^2} \begin{Bmatrix} \sin any \\ \cos any \end{Bmatrix} dx = \frac{\pi}{2} (-y)^{\delta-1} \cos^n ay \begin{Bmatrix} \cos any \\ \sin any \end{Bmatrix} - 2^{n-1} \delta \pi$$

$$[n=1, 2, 3, \dots; a>0; \delta=\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$6. \int_0^\infty \frac{1}{x^2 + z^2} \frac{\sin bx}{\sin ax} dx = \frac{\pi}{2z} \frac{\operatorname{sh} bz}{\operatorname{sh} az} \quad [0 < b < a; \operatorname{Re} z > 0],$$

$$7. \int_0^\infty \frac{x^{\pm 1}}{x^2 + z^2} \frac{\sin bx}{\cos ax} dx = \mp \frac{\pi}{2} z^{-1 \pm 1} \frac{\operatorname{sh} bz}{\operatorname{ch} az} \quad [0 < b < a; \operatorname{Re} z > 0].$$

$$8. \int_0^\infty \frac{x}{x^2 + z^2} \frac{\cos bx}{\sin ax} dx = \frac{\pi}{2} \frac{\operatorname{ch} bz}{\operatorname{sh} az} \quad [0 < b < a; \operatorname{Re} z > 0].$$

$$9. \int_0^\infty \frac{1}{x^2 + z^2} \frac{\cos bx}{\cos ax} dx = \frac{\pi}{2z} \frac{\operatorname{ch} bz}{\operatorname{ch} az} \quad [0 < b < a; \operatorname{Re} z > 0].$$

$$10. \int_0^{\infty} \frac{1}{x^2 - y^2} \frac{\sin bx}{\sin ax} dx = -\frac{\pi}{y} \frac{\sin^2(by/2)}{\sin ay} \quad [0 < b < a; y > 0].$$

$$11. \int_0^{\infty} \frac{x^{y-1}}{x^2 - y^2} \frac{\sin bx}{\cos ax} dx = 0 \quad [0 < b < a; y > 0].$$

2.5.15. Интегралы от $x^{\alpha} \prod_k \sin^{\delta_k} a_k x \cos^{\nu} b_k x$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} x^{(1-n)/2} \prod_{k=0}^n \sin b_k x dx = 0 \quad \left[b_k > 0; b_0 > \sum_{k=1}^n b_k \right].$$

$$2. \int_0^{\infty} x^{-n-1} \prod_{k=0}^n \sin b_k x dx = -\frac{\pi}{2} \prod_{k=0}^n b_k \quad \left[b_k > 0; b_0 > \sum_{k=1}^n b_k \right].$$

$$3. \int_0^{\infty} x^{\alpha-1} \sin ax \sin bx \sin cx dx = I_{\alpha} \quad [-3 < \operatorname{Re} x < 1; a \geq b \geq c \geq 0, \text{ исключая } a=b \geq c=0].$$

$$I_{\alpha} = \frac{\Gamma(\alpha)}{4} \sin\left(\frac{\alpha\pi}{4}\right) [-(a+b+c)^{-\alpha} + (a-b+c)^{-\alpha} + \\ + (a+b-c)^{-\alpha} - \varepsilon |a-b-c|^{-\alpha}] \\ [\alpha \neq 0, -1, -2; \varepsilon = \operatorname{sgn}(a-b-c), \varepsilon = 0 \text{ при } a=b+c].$$

$$I_0 = \frac{\pi}{8} (1 - \varepsilon),$$

$$I_{-1} = \frac{a+b+c}{4} \ln(a+b+c) - \frac{a-b+c}{4} \ln(a-b+c) - \\ - \frac{a+b-c}{4} \ln(a+b-c) + \frac{a-b-c}{4} \ln|a-b-c|,$$

$$I_{-2} = \frac{\pi}{16} [(a+b+c)^2 - (a+b-c)^2 - (a-b+c)^2 + \varepsilon (a-b-c)^2].$$

$$4. \int_0^{\infty} x^{\alpha-1} \prod_{j=1}^m \sin a_j x \prod_{k=1}^n \cos b_k x \begin{cases} \sin \xi x \\ \cos \xi x \end{cases} dx = \\ = 2^{\alpha+m-1} \sqrt{\pi} a_1 \dots a_m \xi^{-\alpha-m-1/2} \Gamma\left[\frac{(\alpha+m+\delta)/2}{(1+\delta-\alpha-m)/2}\right] \times \\ \times F_c^{(m+n)}\left[\frac{\alpha+m}{2}, \frac{\alpha+m+1}{2}, \underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_m, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_n; \right. \\ \left. \frac{a_1^2}{\xi^2}, \dots, \frac{a_m^2}{\xi^2}, \frac{b_1^2}{\xi^2}, \dots, \frac{b_n^2}{\xi^2}\right] \\ [a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0; \xi > a_1 + \dots + a_m + b_1 + \dots + b_n; -m-\delta < \operatorname{Re} x < 1].$$

$$5. \int_0^{\infty} x^{\delta-m+2l} \prod_{j=1}^m \sin a_j x \prod_{k=1}^n \cos b_k x \begin{Bmatrix} \sin \xi x \\ \cos \xi x \end{Bmatrix} dx = 0$$

$$[a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0, \xi > a_1 + \dots + a_m + b_1 + \dots + b_n;$$

$$l=0, 1, 2, \dots, \left[\frac{m-\delta}{2}\right]-1; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$6. \int_0^{\infty} x^{-m-1} \prod_{j=1}^m \sin a_j x \prod_{k=1}^n \cos b_k x \sin \xi x dx = \frac{1}{2} \pi a_1 \dots a_m \xi^{-1/2}$$

$$[a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0; \xi > a_1 + \dots + a_m + b_1 + \dots + b_n].$$

$$7. \int_0^{\infty} \frac{1}{x^2} \prod_{k=1}^n \cos^{p_k} a_k x \sin b x \sin x dx = \frac{\pi}{2} \quad \left[b > \sum_{k=1}^n a_k p_k; a_k, p_k > 0 \right].$$

2.5.16. Интегралы, содержащие $\left(a + b \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix}\right)^0$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a (\cos x - \cos a)^{\nu-1} \cos bx dx = \sqrt{\pi/2} \sin^{\nu-1/2} a \Gamma(\nu) P_{b-1/2}^{1/2-\nu}(\cos a)$$

$[b, \operatorname{Re} \nu > 0; 0 < a < \pi].$

$$2. \int_0^a \frac{\cos(n+1/2)x}{\sqrt{\cos x - \cos a}} dx = \frac{\pi}{\sqrt{2}} P_n(\cos a) \quad [a > 0].$$

$$3. \int_0^{\pi/2} \frac{dx}{a + b \sin x} = \frac{1}{\sqrt{a^2 - b^2}} \arccos \frac{b}{a} \quad [|b| < a].$$

$$4. = \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \frac{b + \sqrt{b^2 - a^2}}{a} \right| \quad [0 < |a| < |b|].$$

$$5. = 1/a \quad [b = a].$$

$$6. \int_0^{\pi/2} \frac{\sin x dx}{1 + \sin x} = \int_0^{\pi/2} \frac{\cos x dx}{1 + \cos x} = \frac{\pi}{2} - 1.$$

$$7. \int_0^{\pi/2} \frac{1 - a \cos 2nx}{1 - 2a \cos 2nx + a^2} \cos^m x \cos mx dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + 2^{-m-1} \pi$$

$$[|a| < 1; n=1, 2, 3, \dots].$$

$$8. \int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a \cos 2x + a^2} dx = \frac{\pi}{4a} \left[\left(\frac{1+a^{\pm 1}}{2} \right)^n - 2^{-n} \right]$$

$$\left[\begin{Bmatrix} |a| < 1 \\ |a| > 1 \end{Bmatrix}; n=1, 2, 3, \dots \right].$$

9.
$$\int_0^{\pi/2} \frac{\sin^{2n} x \cos^{-2n-\beta} x \cos \beta x}{1-2a \cos 2x + a^2} dx = \pm \frac{(-1)^n \pi (1-a)^{2n-1}}{2^{1-\beta} (1+a)^{2n+\beta+1}} \left\{ \frac{1}{a^\beta} \right\}$$

$$\left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; \beta < 1-2n \right].$$
10.
$$\int_0^{\pi/2} \frac{dx}{(a+b \cos x)^2} = \frac{1}{a^2-b^2} \left(\frac{a}{\sqrt{a^2-b^2}} \arccos \frac{b}{a} - \frac{b}{a} \right) \quad [a > |b|].$$
11.
$$= \frac{1}{b^2-a^2} \left(\frac{b}{a} - \frac{a}{\sqrt{b^2-a^2}} \ln \frac{b+\sqrt{b^2-a^2}}{a} \right) \quad [b > a > 0].$$
12.
$$\int_0^{\pi/2} \frac{\cos^{2n-\beta} x \cos \beta x}{(1-2a \cos 2x + a^2)^{n+1}} dx = \frac{\pi (1-a)^{-2n-1}}{2^{2n-\beta+1} (a+1)^{\beta+1}} \sum_{k=0}^n \binom{\beta}{k} \binom{2n-k}{n} (a-1)^k$$

$$[|a| < 1; \beta < 2n+1].$$
13.
$$\int_0^{\pi/2} \frac{\sin^{2m} x \cos^\mu x \cos \beta x}{(1-2a \cos 2x + a^2)^n} dx =$$

$$= \frac{(-1)^m \pi (1-a)^{2m-2n+1}}{2^{2n-\beta-1} (1+a)^{2n+\beta+1}} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} \binom{\beta}{k} \binom{2m}{l} \binom{2n-k-l-2}{n-1} (-2)^l (a-1)^k$$

$$[\beta = 2n-2m-\mu-2; |a| < 1; \mu > -1].$$
14.
$$\int_0^{\pi/2} \frac{x^{n+2}}{1-\cos x} dx = -\left(\frac{\pi}{2}\right)^{n+2} + \left(\frac{\pi}{2}\right)^{n+1} (n+2) \left[\frac{2}{n+1} - \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (n+2k+1)} \right].$$
15.
$$\int_0^{\pi/2} \frac{x - \sin x}{1 - \cos x} dx = 2 - \frac{\pi}{2}.$$
16.
$$\int_0^{\pi/2} \frac{x \cos x dx}{1+2a \sin x + a^2} = \frac{\pi}{2a} \ln(1+a^{\pm 1}) - a^{-1 \pm 1} \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k+1)^2}$$

$$\left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\} \right].$$
17.
$$\int_0^{\pi/2} \frac{x dx}{1+\sin x} = \ln 2. \quad 18. \int_0^{\pi/2} \frac{x dx}{1+\cos x} = \frac{\pi}{2} - \ln 2.$$
19.
$$\int_0^{\pi/2} \frac{x \cos x dx}{1+\sin x} = \pi \ln 2 - 2G. \quad 20. \int_0^{\pi/2} \frac{x \sin x dx}{1 \pm \cos x} = \mp \frac{\pi}{2} \ln 2 + 2G.$$
21.
$$\int_0^{\pi/2} \frac{x^2 dx}{1-\cos x} = 4G + \pi \ln 2 - \frac{\pi^3}{4}.$$

$$22. \int_0^{\pi} \frac{\cos nx}{a+b \cos x} dx = \frac{\pi}{\sqrt{a^2-b^2}} \left(\frac{\sqrt{a^2-b^2}-a}{b} \right)^n \quad [a > b > 0].$$

$$23. \int_0^{\pi} \frac{\cos nx}{\cos a - \cos x} dx = -\frac{\pi \sin an}{\sin a} \quad [a \neq 0].$$

$$= -\pi n \quad [a = 0].$$

$$24. \int_0^{\pi} \frac{\sin^n x}{1 \pm \cos x} dx = 2^{n-1} B\left(\frac{n-1}{2}, \frac{n+1}{2}\right) \quad [n=2, 3, 4, \dots].$$

$$25. \int_0^{\pi} \frac{\sin^n x}{a+b \cos x} dx =$$

$$= 2^{n-2} \frac{a}{b^2} \sum_{l=1}^k \left(\frac{b^2-a^2}{4b^2} \right)^{l-1} B\left(\frac{n+1-2l}{2}, \frac{n+1-2l}{2}\right) + A \left(\frac{b^2-a^2}{b^2} \right)^k$$

$$\left[A = \pi b^{n-2} (a - \sqrt{a^2-b^2}) \text{ при } n=2k+2, A = \frac{1}{b} \ln \frac{a+b}{a-b} \text{ при } n=2k+1, 0 < |b| \leq a \right].$$

$$26. \int_0^{\pi} \frac{\sin nx \sin x}{1-2a \cos x + a^2} dx = \frac{\pi}{2} a^{\pm n-1} \quad \left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases}; n=1, 2, 3, \dots \right].$$

$$27. \int_0^{\pi} \frac{\cos nx \cos x}{1-2a \cos x + a^2} dx = \frac{\pi}{2a} \left(1 - \frac{1+a^2}{|a^2-1|} a^{-n} \right) \quad \left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases}; n=1, 2, 3, \dots \right].$$

$$28. \int_0^{\pi} \frac{\cos nx}{(a+b \cos x)^\mu} dx = \frac{\pi (a^2-b^2)^{-\mu/2}}{(1-\mu)_n} P_n^{-\mu} \left(\frac{a}{\sqrt{a^2-b^2}} \right) \quad [a > |b|].$$

$$29. \int_0^{\pi} \frac{\cos mx}{(1-2a \cos x + a^2)^n} dx = \frac{\pi a^l}{|1-a^2|^{2n-1}} \sum_{k=0}^{n-1} \binom{m+n-1}{k} \times$$

$$\times \binom{2n-k-2}{n-1} (a^{\pm 2} - 1)^k \quad \left[\begin{cases} l=-m, |a| > 1 \\ l=2n+m-2, |a| < 1 \end{cases} \right].$$

$$30. \int_0^{\pi} \frac{\cos nx}{(1-2a \cos x + a^2)^{n+1}} dx = \binom{2n}{n} \frac{a^n \pi}{|1-a^2|^{2n+1}} \quad [|a| \neq 1].$$

$$31. \int_0^{\pi} (1-2a \cos x + a^2)^{-n} \cos mx dx = 0 \quad [n < m].$$

$$32. = \pi (-a)^m (1+a^2)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k} \binom{n-k}{m+k} \left(\frac{a}{1+a^2} \right)^{2k} \quad [n \geq m].$$

$$33. \int_0^\pi \frac{\cos nx \, dx}{1 - 2a \cos x + a^2} = \frac{\pi a^{\pm n}}{|1 - a^2|} \quad \left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\} \right].$$

$$34. \int_0^\pi \frac{\sin nx - a \sin (n+1)x}{1 - 2a \cos x + a^2} \, dx = 0 \quad [|a| < 1].$$

$$35. \int_0^\pi \frac{dx}{(a + b \cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}} \quad [|b| < a].$$

$$36. \int_0^\pi (1 - 2a \cos x + a^2)^n \, dx = \pi \sum_{k=0}^n \binom{n}{k}^2 a^{2k}.$$

$$37. \int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^n} =$$

$$= \frac{\pi}{|1 - a^2|^{2n-1}} \left\{ \begin{array}{c} a^{2n-2} \\ 1 \end{array} \right\} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{2n-k-2}{n-1} |1 - a^{\mp 2}|^k$$

$$\left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; n=1, 2, 3, \dots \right].$$

$$38. \int_0^\pi \frac{dx}{(a + b \cos x)^{\pm m}} = \pi (a^2 - b^2)^{\mp m/2} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right) \quad \left[\left\{ \begin{array}{l} n=m-1 \\ n=m \end{array} \right\}; a > |b| \right].$$

$$39. \int_0^\pi \frac{\cos x}{1 + (a + b \sin x)^2} \left\{ \begin{array}{c} \sin 2nx \\ \cos (2n+1)x \end{array} \right\} dx =$$

$$= \mp \frac{\pi}{b} \sin \left[\left(2n + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \right) \operatorname{arctg} \sqrt{\frac{s}{2}} \right] \left[\operatorname{tg} \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \right]^{2n+1-\delta}$$

$$[s = a^2 - b^2 - 1 + \sqrt{(a^2 - b^2 - 1)^2 + 4a^2}; a > 0, n=1, 2, 3, \dots].$$

$$40. \int_0^\pi \frac{\sin x \sin px \, dx}{(a^2 - 2ab \cos x + b^2)(a^{2p} - 2a^p c^p \cos px + c^{2p})} = \frac{\pi b^{p-1} c^{-p}}{2a^{p+1}(c^p - b^p)}$$

$$[0 < a < c; 0 < a < b; p > 0]$$

$$41. \int_0^\pi \frac{\sin^{2\nu-1} x}{(a + b \cos x)^\mu} \, dx =$$

$$= 2^{\nu-1/2} \sqrt{\pi} (-ib)^{1/2-\nu} \Gamma(\nu) (\sqrt{a^2 - b^2})^{\nu-\mu-1/2} P_{\nu-\mu-1/2}^{1/2-\nu} \left(\frac{a}{\sqrt{a^2 - b^2}} \right)$$

$$[a > |b|; \operatorname{Re} \nu > 0].$$

$$42. \int_0^{\pi} \frac{\cos^n x dx}{(a+b \cos x)^{n+1}} =$$

$$= \frac{\pi}{2^n (a+b)^n \sqrt{a^2-b^2}} \sum_{k=0}^n (-1)^k \frac{(2n-2k-1)!! (2k-1)!!}{(n-k)! k!} \left(\frac{a+b}{a-b} \right)^k$$

[$a > |b| > 0$].

$$43. \int_0^{\pi} \frac{\sin^{\mu-1} x}{(1-2a \cos x + a^2)^{\nu}} dx = B\left(\frac{\mu}{2}, \frac{1}{2}\right) {}_2F_1\left(\nu, \nu + \frac{1-\mu}{2}; \frac{1+\mu}{2}; a^2\right)$$

[$|a| < 1$; $\operatorname{Re} \mu > 0$].

$$44. \int_a^{\pi} \frac{\sin(n+1/2)x}{\sqrt{\cos a - \cos x}} dx = \frac{\pi}{\sqrt{2}} P_n(\cos a)$$

[$a > 0$].

$$45. \int_0^{\pi} \frac{dx}{\sqrt{a \pm b \cos x}} = \frac{2}{\sqrt{a+b}} K\left(\sqrt{\frac{2b}{a+b}}\right)$$

[$a > b > 0$].

$$46. \int_0^{\pi} \frac{dx}{\sqrt{3-\cos x}} = \frac{1}{4\sqrt{\pi}} \Gamma^2\left(\frac{1}{4}\right).$$

$$47. \int_0^{\pi} \frac{dx}{\sqrt{1 \pm 2a \cos x + a^2}} = 2K(a)$$

[$|a| < 1$].

$$48. \int_0^{\pi} \frac{\sin x dx}{\sqrt{1 \pm 2a \cos x + a^2}} = \begin{cases} 2|a|^{-1} \\ 2 \end{cases}$$

[$\begin{cases} |a| > 1 \\ |a| \leq 1 \end{cases}$].

$$49. \int_0^{\pi} \frac{\cos x dx}{\sqrt{1-2a \cos x + a^2}} = \frac{2}{a} [K(a) - E(a)]$$

[$|a| < 1$].

$$50. \int_0^{\pi} \frac{\cos(n+1/2)x dx}{\sqrt{\cos x + 1}} = \frac{(-1)^n \pi}{2}.$$

$$51. \int_0^{\pi} \frac{x dx}{a \pm b \cos x} = \frac{\pi^2}{2\sqrt{a^2-b^2}} \pm \frac{4}{\sqrt{a^2-b^2}} \sum_{k=1}^{\infty} \frac{(a-\sqrt{a^2-b^2})^{2k+1}}{(2k+1)^2 b^{2k+2}}$$

[$a > b > 0$].

$$52. \int_0^{\pi} \frac{x \sin x dx}{a+b \cos x} = \frac{\pi}{b} \ln \frac{a+\sqrt{a^2-b^2}}{2(a-b)}$$

[$a > |b| > 0$].

$$53. \int_0^{\pi} \frac{x - \sin x}{1 - \cos x} dx = 2. \quad 54. \int_0^{\pi} \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 4G.$$

$$55. \int_0^{\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{\pi}{a} \ln(1 + a^{\pm 1}) \quad \left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases} \right].$$

$$56. \int_0^{\pi} \frac{x \sin x}{(a + b \cos x)^2} dx = \frac{\pi}{b} \left(\frac{1}{a-b} - \frac{1}{\sqrt{a^2 - b^2}} \right) \quad [0 < |b| < a].$$

$$57. \int_0^{\pi} \frac{x^2 dx}{1 - \cos x} = 4\pi \ln 2.$$

$$58. \int_0^{\pi} x^2 \frac{a \cos x + b}{(a + b \cos x)^2} dx = \frac{2\pi}{b} \ln \frac{2(a-b)}{a + \sqrt{a^2 - b^2}} \quad [0 < |b| < a].$$

$$59. \int_0^{2\pi} (1 - \cos x)^n \begin{cases} \sin nx \\ \cos nx \end{cases} dx = (-1)^n \frac{\pi}{2^{n-1}} \begin{cases} 0 \\ 1 \end{cases}.$$

$$60. \int_0^{2\pi} (a + b \cos x)^m \sin nx dx = 0 \quad [a > |b|].$$

$$61. \int_0^{2\pi} \frac{dx}{(1 - 2a \cos x + a^2)^{n+1}} = \frac{2\pi}{|1 - a^2|^{n+1}} \sum_{k=0}^n \frac{(n+k)!}{(k!)^2 (n-k)!} |a^{\mp 2} - 1|^{-k}, \quad \left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases} \right].$$

$$62. \int_0^{2\pi} \frac{\sin(x/2) dx}{(1 - 2a \cos x + a^2)^{n+1}} = \pm \frac{1}{2na} [(1-a)^{-2n} - (1+a)^{-2n}] \quad \left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases} \right].$$

$$63. \int_0^{2\pi} \frac{\cos(n-1/2)x}{1 - 2a \cos x + a^2} dx = 0 \quad [|a| \neq 1; n=0, \pm 1, \pm 2, \dots].$$

$$64. \int_0^{2\pi} \begin{cases} \sin 2nx \\ \cos(2n+1)x \end{cases} \frac{dx}{a + b \sin x} = 0 \quad [|a| > |b| > 0].$$

$$65. \int_0^{2\pi} \begin{cases} \sin(2n+1)x \\ \cos 2nx \end{cases} \frac{dx}{a + b \sin x} = \frac{(-1)^{n+\delta} 2\pi (a - \sqrt{a^2 - b^2})^{2n+\delta}}{b^{2n} \delta \sqrt{a^2 - b^2}} \quad [|a| > |b| > 0].$$

$$66. \int_0^{2\pi} \begin{cases} \sin(nx/2) \sin(x/2) \\ \cos(nx/2) \cos(x/2) \end{cases} \frac{dx}{1 - 2a \cos x + a^2} = 0 \quad [|a| \neq 1].$$

$$67. \int_0^{2\pi} \frac{\sin(n-1/2)x \sin(x/2)}{1 - 2a \cos x + a^2} dx = \frac{\pi a^m}{1+a}$$

[$m=n-1$ при $|a|<1$; $m=-n$ при $|a|>1$; $n=1, 2, 3, \dots$].

$$68. \int_0^{2\pi} \frac{\cos(n-1/2)x \cos(x/2)}{1-2a \cos x + a^2} dx = \frac{\pi a^m}{|1-a|}$$

[$m=n-1$ при $|a|<1$; $m=-n$ при $|a|>1$; $n=1, 2, 3, \dots$].

$$69. \int_0^{2\pi} \frac{c + \cos x}{(a - b \cos x)^{3/2}} dx = -\frac{4}{b \sqrt{a+b}} \left[K\left(\sqrt{\frac{2b}{a+b}}\right) + \frac{a+bc}{a-b} E\left(\sqrt{\frac{2b}{a+b}}\right) \right]$$

[$a>b>0$].

$$70. \int_0^{2\pi} \frac{x \sin x}{a \pm b \cos x} dx = \pm \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a \pm b)}$$

[$a>|b|>0$].

$$71. \int_0^{2\pi} \frac{x \sin nx}{a \pm b \cos x} dx =$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}} \left\{ (\mp b)^{-n} [(a + \sqrt{a^2 - b^2})^n - (a - \sqrt{a^2 - b^2})^n] \ln \frac{2\sqrt{a \pm b}}{\sqrt{a+b} + \sqrt{a-b}} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\mp b)^{-k}}{(n-k)} [(a + \sqrt{a^2 - b^2})^k - (a - \sqrt{a^2 - b^2})^k] \right\}$$

[$a>|b|>0$].

$$72. \int_0^{2\pi} \frac{x \cos nx}{a \pm b \cos x} dx = \frac{2\pi^2}{\sqrt{a^2 - b^2}} \left(\frac{a - \sqrt{a^2 - b^2}}{\pm b} \right)^n$$

[$a>|b|>0$].

$$73. \int_0^{2\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{a} \ln(1 - a^{\pm 1})$$

[$\left\{ \begin{array}{l} |a|<1 \\ |a|>1 \end{array} \right\}$].

$$74. \int_0^{2\pi} \frac{x \sin nx dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1-a^2} \left[(a^{-n} - a^n) \ln(1 - a^{\pm 1}) + \sum_{k=1}^{n-1} \frac{a^{-k} - a^k}{n-k} \right]$$

[$\left\{ \begin{array}{l} |a|<1 \\ |a|>1 \end{array} \right\}$; $n=1, 2, 3, \dots$].

$$75. \int_0^{\infty} x^{-1} \frac{\sin x}{a \pm b \cos 2x} dx = \frac{\pi}{2\sqrt{a^2 - b^2}}$$

[$|a|>|b|$].

$$76. = 0$$

[$|a|<|b|$].

$$77. \int_0^{\infty} x^{-1} \frac{\sin nx}{1 - 2a \cos x + a^2} dx = \frac{\pi}{2} \frac{1 - a^{\pm n}}{(1-a)^2}$$

[$\left\{ \begin{array}{l} 0<a<1 \\ a>1 \end{array} \right\}$; $n=1, 2, 3, \dots$].

$$78. \int_0^{\infty} x^{-1} \frac{\sin bx}{1 - 2a \cos x + a^2} dx = \frac{\pi}{2} \frac{1 + a - 2a^{[b]+1}}{(1-a^2)(1-a)}$$

[$0<a<1$; $b \neq 0, 1, 2, \dots$].

$$79. = \frac{\pi}{2} \frac{a^{[b]} + a^{[b]+1} - 2}{(1-a^2)(1-a)} a^{-[b]}$$

[$a>1$; $b \neq 0, 1, 2, \dots$].

$$80. \int_0^{4\pi} \frac{\sin(mx/2) dx}{(1-2a \cos x + a^2)^n} = 0.$$

$$81. \int_0^\infty \left[\operatorname{tg}^{\pm \mu} x \left\{ \frac{\sin 2x}{\cos 2x} \right\} + (\mp a)^{1-\delta} \right] \frac{dx}{1 \mp 2a \cos 2x + a^2} =$$

$$= (\mp 1)^{1-\delta} \frac{\pi}{4a} \left\{ \frac{\sin(\mu\pi/2)}{\cos(\mu\pi/2)} \right\} \left(1 + \operatorname{sgn}(a^2 - 1) \left| \frac{1-a}{1+a} \right|^\mu \right)$$

$$\left[|a| \neq 1; -1-\delta < \operatorname{Re} \mu < 1; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$82. \int_0^\infty \frac{x^{-1}}{(1-2a \cos 2x + a^2)^{n+1}} \left\{ \frac{\operatorname{tg} x}{\sin x} \right\} dx = \frac{\pi}{2 |1-a^2|^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 a^{2k}$$

$$[|a| \neq 1].$$

$$83. \int_0^\infty x^{-1} \frac{\operatorname{tg} x}{a + b \cos kx} dx = \frac{\pi}{2 \sqrt{b^2 - a^2}} \quad [k=2, 4; |a| > |b|].$$

$$84. \quad = 0 \quad [k=2, 4; |a| < |b|].$$

$$85. \int_0^\infty \frac{x^{-1} \sin^{2m+1} x}{1-2a \cos x + a^2} dx =$$

$$= \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left(\left| \frac{1-a}{1+a} \right|^{2m-1} - \sum_{k=0}^{2m} \frac{(-1)^k}{k!} \Gamma \left[\frac{m+1/2}{1/2+m-k} \right] \frac{4^k a^k}{(1+a)^{2k}} \right)$$

$$[|a| \neq 1].$$

$$86. \int_0^\infty \frac{x^{-1} \sin^{2m+1} x \cos^n x}{(1-2a \cos x + a^2)^v} dx = \frac{\pi n!}{2^{n+1} (2m+n+1)! (1+a)^{2v}} \times$$

$$\times \sum_{k=0}^n \frac{(-1)^k (2m+2n-2k+1)!!}{k! (n-k)!} \times$$

$$\times (2m+2k-1)!! {}_2F_1 \left(m+n-k+3/2, v; 2m+n+2; \frac{4a}{(1+a)^2} \right) \quad [|a| \neq 1].$$

$$87. \int_0^\infty \frac{1}{1-2a \cos cx + a^2} \frac{dx}{x^2 + z^2} = \frac{\pi}{2z(1-a^2)} \frac{1 + ae^{\mp cz}}{1 - ae^{\mp cz}}$$

$$[c, \operatorname{Re} z > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$88. \int_0^\infty \frac{1}{1-2a \cos cx + a^2} \frac{dx}{x^2 - y^2} = - \frac{\pi a \sin cy}{y |1-a^2| (1-2a \cos cy + a^2)}$$

$$[c, y > 0; |a| \neq 1].$$

$$89. \int_0^{\infty} \frac{\cos bx}{1-2a \cos cx + a^2} \frac{dx}{x^2 + z^2} = \frac{\pi}{2z(1-a^2)} \frac{e^{-bz} + ae^{\pm z(b-c)}}{1 - ae^{\mp cz}}$$

$$[b, c, \operatorname{Re} z > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$90. \int_0^{\infty} \frac{\cos bx}{1-2a \cos cx + a^2} \frac{dx}{x^2 - y^2} =$$

$$= -\frac{\pi}{2y|1-a^2|} \frac{|1-a^2| \sin by + 2a^{\pm(b+c)/c} \sin cy}{1-2a \cos cy + a^2} [b, c, y > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$91. \int_0^{\infty} \frac{x \sin bx}{1-2a \cos cx + a^2} \frac{dx}{x^2 + z^2} = \frac{\pi}{2} \frac{e^{-bz} - ae^{\pm b/c}}{(1 - ae^{\mp cz})(1 - ae^{\pm cz})}$$

$$[c, b, \operatorname{Re} z > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$92. \int_0^{\infty} \frac{x \sin bx}{1-2a \cos cx + a^2} \frac{dx}{x^2 - y^2} = -\frac{\pi}{2|1-a^2|} \frac{a^{\pm b/c} - \cos by}{1-2a \cos cy + a^2} \left\{ \frac{1}{a^2} \right\}$$

$$[b, c, y > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$93. \int_{-\infty}^{\infty} \frac{1 - \cos ax}{x(x-y)} dx = \frac{\sin ay}{y} [a > 0; \operatorname{Im} y = 0].$$

2.5.17. Интегралы, содержащие $(a \cos x + b \sin x + c)^{-1}$.

$$1. \int_0^a \frac{x}{\cos x + \operatorname{tg} a \sin x} \frac{dx}{\cos x} = -\frac{a}{\operatorname{tg} a} \ln |\cos a| [a > 0].$$

$$2. \int_a^{\pi/2} \frac{x}{\cos x + \operatorname{tg} a \sin x} \frac{dx}{\sin x} = \left(\frac{\pi}{2} + a \right) \ln |\sin a| [a > 0].$$

$$3. \int_0^{\pi/4} \frac{A \cos x + B \sin x}{a \cos x + b \sin x} dx = \frac{Ab - Ba}{2(a^2 + b^2)} \ln \frac{(a+b)^2}{2a^2} + \frac{Aa + Bb}{a^2 + b^2} \frac{\pi}{4} [b \neq -a].$$

$$4. \int_0^{\pi/4} x \frac{\cos x - \sin x}{\cos x + \sin x} dx = \frac{\pi}{4} \ln 2 - \frac{G}{2}.$$

$$5. \int_0^{\pi/4} \frac{x}{\cos x + \sin x} \left\{ \frac{\sin x}{\cos x} \right\}^{-1} dx = \mp \frac{\pi}{8} \ln 2 + G \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right\}.$$

$$6. \int_0^{\pi/4} \frac{(\pi/4 - x)}{\cos x - \sin x} \frac{dx}{\cos x} = G - \frac{\pi}{8} \ln 2.$$

$$7. \int_0^{\pi/4} \frac{x \operatorname{tg} x}{\cos x + \sin x} \frac{dx}{\cos x} = \frac{\pi}{4} - \frac{\pi}{8} \ln 2 - \frac{\ln 2}{2}.$$

$$8. \int_0^{\pi/4} \frac{x dx}{(a \cos x + b \sin x)^2} = \frac{1}{a^2 + b^2} \ln \frac{a+b}{a\sqrt{2}} + \frac{\pi}{4} \frac{a-b}{(a^2 + b^2)(a+b)} \quad [a, b > 0].$$

$$9. \int_0^{\pi/2} x \frac{\cos x \mp \sin x}{\cos x \pm \sin x} dx = \pm \frac{\pi}{4} \ln 2 - G.$$

$$10. \int_0^{\pi/2} \frac{x dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 \pm G.$$

$$11. \int_0^{\pi/2} \frac{x dx}{(a \cos x + b \sin x) \sqrt{\sin 2x}} =$$

$$= \frac{\pi}{\sqrt{2ab}} \left[\operatorname{arctg} \left(\sqrt{\frac{2a}{b}} + 1 \right) + \operatorname{arctg} \left(\sqrt{\frac{2a}{b}} - 1 \right) - \operatorname{arctg} \frac{a}{b} \right] \quad [a, b > 0].$$

$$12. \int_0^{\pi/2} \frac{x dx}{(\sin x + \cos x) \sqrt{\sin 2x}} = \frac{\pi^2}{4\sqrt{2}}.$$

$$13. \int_0^{\pi/2} \frac{x dx}{(a \cos x + b \sin x)^2} = \frac{1}{a^2 + b^2} \left(\ln \frac{b}{a} + \frac{\pi a}{2b} \right) \quad [a, b > 0].$$

$$14. \int_0^{\pi/2} \frac{\sin^{u-1} x \cos^{v-1} x}{(a \cos x + b \sin x)^{u+v}} dx = \frac{B(u, v)}{a^v b^u} \quad [a, b, \operatorname{Re} u, \operatorname{Re} v > 0].$$

$$15. \int_0^{2\pi} (a \cos x + b \sin x)^n dx = \frac{1 + (-1)^n}{2} \frac{(n-1)!!}{(n)!!} 2\pi (a^2 + b^2)^{n/2}.$$

$$16. \int_0^{2\pi} \frac{dx}{a + ib \cos x + ic \sin x} = \frac{2\pi \operatorname{sgn} a}{\sqrt{a^2 + b^2 + c^2}} \quad [a \neq 0].$$

$$17. = 0$$

$$18. \int_0^{\infty} x^{-n-1/2} \sin^n x (\cos ax + \sin ax) dx =$$

$$= \frac{\pi \sqrt{2}}{(2n-1)!!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k+a)^{n-1/2} \quad [a > n].$$

2.5.18. Интегралы, содержащие $(a \cos^2 x + b \sin^2 x + c)^{-n}$.

$$1. \int_0^{\pi/4} \frac{x \sin x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x) \sqrt{\cos 2x}} = \\ = \frac{\pi}{b} \ln \cos \left(\frac{\pi}{8} - \frac{\arccos b}{2} \right) - \frac{\pi}{b} \ln \sin \left(\frac{\pi}{8} + \frac{\arccos b}{2} \right) \quad [a^2 + b^2 = 1].$$

$$2. \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = I_n \quad [a, b > 0],$$

$$I_1 = \frac{\pi}{2ab}, \quad I_2 = \frac{\pi}{4ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \quad I_3 = \frac{\pi}{16ab} \left(\frac{3}{a^4} + \frac{2}{a^2 b^2} + \frac{3}{b^4} \right),$$

$$I_4 = \frac{\pi}{32ab} \left(\frac{5}{a^6} + \frac{3}{a^4 b^2} + \frac{3}{a^2 b^4} + \frac{5}{b^6} \right).$$

$$3. \int_0^{\pi/2} \frac{\cos 2nx \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \binom{2n}{n} \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi \quad [a, b > 0].$$

$$4. \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^{\mu+\nu}} dx = \frac{a^{-2\nu} b^{-2\mu}}{2} B(\mu, \nu) \quad [a, b, \operatorname{Re} \mu, \operatorname{Re} \nu > 0],$$

$$5. \int_0^{\pi/2} \frac{\cos^{p+2n} x \cos px \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{p+k-1}{k} \frac{b^{p-1}}{(2a)^{2n-k+1} (a+b)^{p+k}} \\ [a, b > 0; p > -2n-1].$$

$$6. \int_0^{\pi/2} \frac{x \sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b} \quad [a, b > 0].$$

$$7. \int_0^{\pi/2} \frac{\cos \mu x \cos^\mu x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2a(a+b)^\mu} \quad [a, b > 0; \mu > -1].$$

$$8. \int_0^{\pi/2} \frac{\sin \mu x \cos^{\mu-1} x \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2b(a+b)^\mu} \quad [a, b, \mu > 0].$$

$$9. \int_0^{\pi/2} \frac{x \operatorname{ctg} x}{a^2 \cos^2 x - (-1)^k b^2 \sin^2 x} dx = \frac{\pi}{2^k a^2} \ln \frac{a^k + b^k}{b^k} \quad [a, b > 0; k=1 \text{ или } 2].$$

$$10. \int_0^{\pi/2} \frac{x \sin x \, dx}{a^2 \cos^2 x - b^2 \sin^2 x} = \\ = -\frac{2}{b \sqrt{a^2 + b^2}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin \left[(2k+1) \arcsin \frac{b}{\sqrt{a^2 + b^2}} \right] \quad [a, b > 0].$$

$$11. \int_0^{\pi/2} \frac{\cos 2nx}{1-b^2 \sin^2 x} dx = \frac{(-1)^n \pi}{2\sqrt{1-b^2}} \left(\frac{1-\sqrt{1-b^2}}{b} \right)^{2n} \quad [|b| < 1].$$

$$12. \int_0^{\pi/2} \frac{A \cos^2 x + B \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{4ab} \left(\frac{A}{a^2} + \frac{B}{b^2} \right) \quad [a, b > 0].$$

$$13. \int_0^{\pi/2} x \frac{\sin 2x}{(1+b \sin^2 x)^2} dx = \frac{\pi}{2b} \frac{\sqrt{1+b}-1}{1+b} \quad [b > -1].$$

$$14. \int_0^{\pi/2} \frac{x \sin 2x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{2ab^2(a+b)} \quad [a, b > 0].$$

$$15. \int_0^{\pi/2} \frac{x \cos^3 x dx}{\sin x (a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{2a^4} \ln \left(1 + \frac{a}{b} \right) - \frac{\pi}{4a^2(a+b)} \quad [a, b > 0].$$

$$16. \int_0^{\pi/2} x^2 \frac{\cos x (1-a^2 + \sin^2 x)}{(a^2 - \cos^2 x)^2} dx =$$

$$= -\frac{\pi^2}{4a^2} + \frac{4}{a} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arcsin a]}{(2k+1)^2} \quad [|a| < 1].$$

$$17. \int_0^{\pi} \frac{x \sin x dx}{a+b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{b}{a}} \quad [a, b > 0].$$

$$18. = \frac{\pi}{2\sqrt{|ab|}} \ln \frac{\sqrt{a} + \sqrt{|b|}}{\sqrt{a} - \sqrt{|b|}} \quad [a > -b > 0].$$

$$19. \int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a, b > 0].$$

$$20. \int_0^{\pi} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a} \quad [a, b > 0].$$

$$21. \int_0^{\pi} \frac{x dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}} \quad [|a| > 1].$$

$$22. = 0 \quad [|a| < 1].$$

$$23. \int_0^{\pi} \frac{x \cos x dx}{a^2 - \cos^2 x} = \frac{4}{\sqrt{1-a^2}} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arccos a]}{(2k+1)^2} \quad [|a| < 1].$$

$$24. \int_0^{\pi} \frac{x \sin 2x}{a^2 - \cos^2 x} dx = \pi \ln [4(1-a^2)] \quad [|a| < 1].$$

$$25. \quad = 2\pi \ln [2(1 - a^2 + a\sqrt{a^2 - 1})] \quad [|a| > 1]$$

$$26. \quad \int_0^\pi \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} dx = \frac{\pi^2}{2} \frac{\sqrt{a^2 - 1} - a}{a(a^2 - 1)} \quad [a > 1].$$

$$27. \quad \int_0^\pi x^2 \frac{1 - a^2 + (2a^2 - 1) \sin^2 x}{(a^2 - \cos^2 x)^2} dx = \pi \ln [4(1 - a^2)] \quad [|a| < 1].$$

$$28. \quad = 2\pi \ln [2(1 - a^2 + |a|\sqrt{a^2 - 1})] \quad [|a| > 1].$$

$$29. \quad \int_0^\pi x^2 \frac{\cos x (a^2 - 1 - \sin^2 x)}{(a^2 - \cos^2 x)^2} dx = \frac{\pi}{a} \ln \frac{1 - a}{1 + a} \quad [|a| \neq 0 \text{ и } 1].$$

$$30. \quad \int_0^\infty \frac{1}{x} \frac{\sin x \cos 2mx \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2a} \frac{b^{2m-1}}{(a+b)^{2m}} \quad [a, b > 0; n = 2m \text{ или } 2m-1].$$

$$31. \quad \int_0^\infty \frac{1}{x} \frac{\sin^k x \cos^l x}{a^2 \cos^2 x + b^2 \sin^2 x} \frac{dx}{\cos 2x} = I^{k,l} \quad [a, b > 0].$$

$$I^{1,0} = \frac{\pi}{2ab} \frac{b^2 - a^2}{b^2 + a^2}, \quad I^{1,1} = I^{1,2} = \frac{\pi}{2a} \frac{b}{a^2 + b^2}, \quad I^{3,0} = -\frac{\pi}{2b} \frac{a}{a^2 + b^2}.$$

$$32. \quad \int_0^\infty \frac{1}{x^2 + z^2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi \operatorname{sh} 2z}{4z(b^2 \operatorname{sh}^2 z - a^2 \operatorname{ch}^2 z)} \left(\frac{b}{a} - \frac{a}{b} - \frac{2}{\operatorname{sh} 2z} \right) \quad [a, b, \operatorname{Re} z > 0; \operatorname{th} z \neq a/b].$$

$$33. \quad \int_0^\infty \frac{x \sin 2x dx}{(x^2 + z^2)(a^2 \cos^2 x + b^2 \sin^2 x)} = \frac{\pi}{2(b^2 \operatorname{sh}^2 z - a^2 \operatorname{ch}^2 z)} \left(\frac{b-a}{b+a} - e^{-2z} \right) \quad [a, b, \operatorname{Re} z > 0].$$

$$34. \quad \int_0^\infty \frac{\sin^k x \cos^l x}{(a^2 \cos^2 x + b^2 \sin^2 x)^m} \frac{dx}{x} = I_m^{k,l} \quad [a, b > 0].$$

$$I_1^{1,0} = I_1^{1,-1} = \frac{\pi}{2ab}, \quad I_1^{3,0} = I_1^{3,-1} = \frac{\pi}{2b(a+b)}, \quad I_1^{1,1} = I_1^{1,2} = \frac{\pi}{2a(a+b)},$$

$$I_1^{3,-1} = \frac{\pi}{2b(a+b)}, \quad I_2^{1,0} = I_2^{1,-1} = \frac{\pi}{4} \frac{a^2 + b^2}{a^3 b^3}, \quad I_2^{3,0} = I_2^{3,-1} = \frac{\pi}{4ab^3},$$

$$I_2^{1,1} = I_2^{1,2} = \frac{\pi}{4a^3 b}, \quad I_3^{1,0} = I_3^{1,-1} = \frac{\pi}{16} \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5},$$

$$I_3^{3,0} = I_3^{3,-1} = \frac{\pi}{16} \frac{3a^2 + b^2}{a^3 b^5}, \quad I_3^{1,1} = I_3^{1,2} = \frac{\pi}{16} \frac{a^2 + 3b^2}{a^5 b^3},$$

$$I_4^{1,0} = \frac{\pi}{32} \frac{5a^6 + 3a^4 b^2 + 3a^2 b^4 + 5b^6}{a^7 b^7}, \quad I_4^{3,0} = I_4^{3,-1} = \frac{\pi}{32} \frac{5a^4 + 2a^2 b^2 + b^4}{a^5 b^7},$$

$$I_{4,0}^{5,0} = I_{4,-1}^{5,-1} = \frac{\pi}{32} \frac{5a^2 + b^2}{a^3 b^7}, \quad I_{4,1}^{1,1} = I_{4,2}^{1,2} = \frac{\pi}{32} \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5},$$

$$I_{4,3}^{1,3} = I_{4,4}^{1,4} = \frac{\pi}{32} \frac{a^2 + 5b^2}{a^7 b^3}, \quad I_{4,1}^{3,1} = I_{4,2}^{3,2} = \frac{\pi}{32} \frac{a^2 + b^2}{a^5 b^5}.$$

$$35. \int_0^\infty \frac{1}{x} \frac{(1 - \cos x) \cos^l x}{(a^2 \cos^2 x + b^2 \sin^2 x)^m} \frac{dx}{\sin x} = I_m^l \quad [a, b > 0].$$

$$I_1^0 = \frac{\pi}{2ab}, \quad I_1^2 = \frac{\pi}{2a} \frac{1}{a+b}, \quad I_2^0 = \frac{\pi}{4} \frac{a^2 + b^2}{a^3 b^3},$$

$$I_2^2 = \frac{\pi}{4a^3 b}, \quad I_3^0 = \frac{\pi}{16} \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5}, \quad I_3^2 = \frac{\pi}{16} \frac{a^2 + 3b^2}{a^5 b^3},$$

$$I_4^2 = \frac{\pi}{32} \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5}, \quad I_4^4 = \frac{\pi}{32} \frac{a^2 + 5b^2}{a^7 b^3}.$$

2.5.19. Интегралы, содержащие $(a \cos^2 x + b \sin^2 x + c)^{1/2}$.

$$1. \int_0^a \frac{x}{\sin x \sqrt{\operatorname{tg}^2 a \cos^2 x - \sin^2 x}} dx = -\frac{\pi}{2 \operatorname{tg} a} \ln \left| \operatorname{tg} \frac{\pi - 2a}{4} \right| \quad [a > 0].$$

$$2. \int_0^a \frac{x \sin x dx}{\cos^2 x \sqrt{\operatorname{tg}^2 a \cos^2 x - \sin^2 x}} = \frac{\pi}{4 \cos a} (1 - \cos a) \quad [a > 0].$$

$$3. \int_0^a \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 a - \sin^2 x}} = \frac{\pi}{\cos^2 a} \sin^2 \frac{a}{2} \quad [a > 0].$$

$$4. \int_0^a \frac{x \sin x dx}{(1 - \sin^2 b \sin^2 x) \sqrt{\sin^2 a - \sin^2 x}} = \\ = (2 \cos b \sqrt{1 - \sin^2 b \sin^2 a})^{-1} \pi \ln \frac{\cos b + \sqrt{1 - \sin^2 b \sin^2 a}}{2 \cos a \cos^2 (b/2)} \quad [a > 0].$$

$$5. \int_a^{\pi/2} \frac{x \cos x dx}{\sqrt{\sin^2 x - \sin^2 a}} = \frac{\pi}{2} \ln (1 + \cos a) \quad [|a| < \pi/2].$$

$$6. \int_0^{\pi/4} \frac{x \sin x dx}{\cos x \sqrt{\cos^2 x + k^2 \sin^2 x}} = \frac{1}{1 - k^2} \left[F \left(\frac{\pi}{4}, \sqrt{1 - k^2} \right) - \frac{\pi}{2 \sqrt{2(1 + k^2)}} \right] \\ [0 < k < 1].$$

$$7. \int_0^{\pi/2} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = (ik)^{\delta-1} K \left(\sqrt{1 + k^2} \right) \quad [0 < k < 1; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$8. \int_0^{\pi/2} \frac{\left\{ \frac{\sin x}{\cos x} \right\}^2 dx}{\sqrt{1 + \sin^2 x}} = \pm \sqrt{2} E \left(\frac{1}{\sqrt{2}} \right) \mp 2^{1/2} K \left(\frac{1}{\sqrt{2}} \right).$$

$$9. \int_0^{\pi/2} \operatorname{tg}^2 \frac{x}{2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = 2\sqrt{1-k^2} - 2E(k) + K(k) \quad [0 < k < 1].$$

$$10. \int_0^{\pi/2} x \sin 2x \sqrt{1-k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2} dx = \\ = -\frac{1}{9k^2} \left[\frac{3\pi}{2} \left\{ \frac{1}{k^3} \right\} + k'^2 K(k) - 2(1+k'^2) E(k) \right] \quad [0 < k < 1; k' = \sqrt{1-k^2}].$$

$$11. \int_0^{\pi/2} \cos 2nx \sqrt{1-k^2 \sin^2 x} dx = \\ = (-1)^{n+1} \frac{\pi}{2} \sum_{j=0}^{\infty} \frac{(2n+2j-3)!! (2n+2j-1)!!}{j! (2n+j)!} \left(\frac{k}{2} \right)^{2n+2j} \quad [0 \leq k < 1].$$

$$12. \int_0^{\pi/2} \frac{x \sin 2x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{1}{k^2} [2E(k) - \pi \sqrt{1-k^2}] \quad [0 < k < 1].$$

$$13. \int_0^{\pi/2} \frac{x \sin 2x}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{k^2} [\pi - 2E(k)] \quad [0 < k < 1].$$

$$14. \int_0^{\pi/2} \frac{\cos 2nx}{\sqrt{1-k^2 \sin^2 x}} dx = (-1)^n \frac{\pi}{2} \sum_{j=0}^{\infty} \frac{[(2n+2j-1)!!]^2}{j! (2n+j)!} \left(\frac{k}{2} \right)^{2n+2j} \\ [0 \leq k < 1].$$

$$15. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^2 \left(1 - a^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = \\ = 2(k')^{-2\delta} \left(\sin 2\beta \sqrt{1-k'^2 \sin^2 \beta} \right)^{-1/2} \times \\ \times \left[\frac{\pi}{2} - K(k) E(\beta, k') - E(k) F(\beta, k') + K(k) F(\beta, k') \right] \\ \left[\begin{cases} a^2 = 1 - k'^2 \sin^2 \beta \\ a^2 = \cos^2 \beta \end{cases}; 0 < k < 1; k' = \sqrt{1-k^2}; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$16. \int_0^{\pi} \sin nx (\sqrt{1-k^2 \cos^2 x})^{\pm 1} dx = 0 \quad [0 < k \leq 1; n=2, 4, 6, \dots].$$

$$17. = \frac{2}{n} (1-k^2)^{\pm 1/2} {}_3F_2 \left(\mp \frac{1}{2}, \frac{1}{2}, 1; 1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{k^2}{k^2-1} \right) \\ [0 < k \leq 1; n=1, 3, 5, \dots].$$

$$18. \int_0^{2\pi} \cos nx (\sqrt{1-k^2 \cos^2 x})^{\pm 1} dx = 0 \quad [0 < k < 1; n=1, 3, 5, \dots].$$

$$19. = \frac{\pi (n-3)!! k^n}{2^{3n/2-1} (n/2)!} \left\{ \frac{-1}{n-1} \right\} {}_2F_1 \left(\frac{n-1}{2}, \frac{n+1}{2}; n+1; k^2 \right) \\ [0 < k < 1; n=2, 4, 6, \dots].$$

$$20. \int_0^{n\pi} \frac{x dx}{\sqrt{1-k^2 \sin^2 x}} = \pi n^2 K(k) \quad [0 < k < 1].$$

$$21. \int_0^{n\pi} \frac{x \sin x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{(-1)^n \pi n}{2k} \ln \frac{1-k}{1+k} \quad [0 < k < 1].$$

$$22. \int_0^\infty \frac{\sin x}{x} \sqrt{1-k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2} dx = E(k) \quad [0 < k < 1].$$

$$23. \int_0^\infty \frac{\operatorname{tg} x}{x} \sqrt{1-k^2 \left\{ \frac{\sin rx}{\cos rx} \right\}^2} dx = E(k) \quad [0 < k < 1; r=1, 2].$$

$$24. \int_0^\infty \frac{\sin x}{x} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = K(k) \quad [0 < k < 1].$$

$$25. \int_0^\infty \frac{\sin x}{x} \left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right).$$

$$26. \int_0^\infty \frac{\operatorname{tg} x}{x} \left(1 - k^2 \left\{ \frac{\sin rx}{\cos rx} \right\}^2 \right)^{-1/2} dx = K(k) \quad [r=1, 2; 0 < k < 1].$$

$$27. \int_0^\infty \frac{1}{x} \operatorname{tg} x \left(1 + \left\{ \frac{\sin rx}{\cos rx} \right\}^2 \right)^{-1/2} dx = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right) \quad [r=1, 2].$$

$$28. \int_0^\infty \frac{\sin x \cos^n x}{x \sqrt{1-k^2 \sin^2 x}} dx = I_n \quad [0 < k < 1; k' = \sqrt{1-k^2}],$$

$$I_1 = I_2 = k^{-2} [E(k) - k'^2 K(k)],$$

$$I_3 = I_4 = 3^{-1} k^{-4} [(2 - 3k^2) k'^2 K(k) - 2(k'^2 - k^2) E(k)].$$

$$29. \int_0^\infty \frac{\sin x \cos^n x}{x \sqrt{1-k^2 \cos^2 x}} dx = J_n \quad [0 < k < 1].$$

$$J_1 = J_2 = k^{-2} [K(k) - E(k)],$$

$$J_3 = J_4 = 3^{-1} k^{-4} [(2 + k^2) K(k) - 2(1 + k^2) E(k)].$$

$$30. \int_0^\infty \frac{1}{x} \sin x \cos^r x \left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \mp \sqrt{2} \left[E\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \left\{ \frac{2}{1} \right\} K\left(\frac{1}{\sqrt{2}}\right) \right] \quad [r=1, 2].$$

$$31. \int_0^\infty \frac{1}{x} \sin^3 x \cos^r x \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \frac{1}{3k^4} [(1 + k'^2) E(k) - 2k'^2 K(k)] \quad [r=1, 2; 0 < k < 1; k' = \sqrt{1-k^2}].$$

$$32. \int_0^{\infty} \frac{\sin^3 x \cos x}{x \sqrt{1 - k^2 \sin^2 2x}} dx = \frac{1}{4k^2} [K(k) - E(k)] \quad [0 < k < 1].$$

$$33. \int_0^{\infty} \frac{\sin^3 x \cos x}{x \sqrt{1 - k^2 \cos^2 2x}} dx = \frac{1}{4k^2} [E(k) - (1 - k^2) K(k)] \quad [0 < k < 1]$$

$$34. \int_0^{\infty} \frac{1}{x} \sin^3 x \cos x \left(1 + \left\{ \frac{\sin 2x}{\cos 2x} \right\}^2\right)^{-1/2} dx = \mp \frac{\sqrt{2}}{4} \left[\frac{1}{2} \left\{ \frac{1}{2} \right\} K\left(\frac{1}{\sqrt{2}}\right) - E\left(\frac{1}{\sqrt{2}}\right) \right].$$

$$35. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{x \sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{k^2} [K(k) - E(k)] \quad [0 < k < 1].$$

$$36. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{x \sqrt{1 - k^2 \cos^2 x}} dx = \frac{1}{k^2} [E(k) - (1 - k^2) K(k)] \quad [0 < k < 1].$$

$$37. \int_0^{\infty} \frac{1}{x} \cos^2 2x \operatorname{tg} x \left(1 - k^2 \left\{ \frac{\sin 2x}{\cos 2x} \right\}^2\right)^{-1/2} dx = \pm \frac{1}{k^2} \left\{ \begin{array}{l} E(k) - (1 - k^2) K(k) \\ E(k) - K(k) \end{array} \right\} \quad [0 < k < 1].$$

$$38. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{x \sqrt{1 + \sin^2 2x}} dx = \sqrt{2} \left[K\left(\frac{1}{\sqrt{2}}\right) - E\left(\frac{1}{\sqrt{2}}\right) \right].$$

$$39. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\sin^4 x \operatorname{tg} x}{\cos^4 x \operatorname{tg}(x/2)} \right\} \left(1 - k^2 \left\{ \frac{\cos x}{\sin x} \right\}^2\right)^{-1/2} dx = \\ = \frac{1}{3k^4} [(2 \pm 3k^2) k'^2 K(k) - 2(k'^2 - k^2) E(k)] \quad [0 < k < 1; k' = \sqrt{1 - k^2}].$$

$$40. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\sin^4 x \operatorname{tg} x}{\cos^4 x \operatorname{tg}(x/2)} \right\} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2\right)^{-1/2} dx = \frac{1}{3k^4} [(2 + k^2) K(k) - 2(1 + k^2) E(k)] \quad [0 < k < 1].$$

$$41. \int_0^{\infty} \frac{\sin^2 4x \operatorname{tg} x}{x \sqrt{1 - k^2 \sin^2 2x}} dx = \frac{4}{3k^4} [(2 - k^2) E(k) - 2(1 - k^2) K(k)] \quad [0 < k < 1].$$

$$42. \int_0^{\infty} \frac{\sqrt{1 - k^2 \sin^2 x}}{x^2 + z^2} dx = \frac{1}{2z} \operatorname{sh} 2z \left[\left(\frac{1}{\operatorname{sh}^2 z} + k^2 \right) \Pi\left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k\right) - k^2 K(k) \right] \quad [\operatorname{Re} z > 0; 0 < k < 1].$$

$$43. \int_0^{\infty} \frac{dx}{(x^2 + z^2) \sqrt{1 - k^2 \sin^2 x}} = \frac{1}{z} \operatorname{cth} z \Pi\left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k\right) \quad [0 < k < 1; \operatorname{Re} z > 0].$$

$$44. \int_0^{\infty} \frac{x \sin x \, dx}{(x^2 + z^2) \sqrt{1 - k^2 \sin^2 x}} = \operatorname{ch} z \left[K(k) - \Pi \left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k \right) \right] \quad [0 < k < 1; \operatorname{Re} z > 0].$$

2.5.20. Интегралы от $A(x) \left\{ \frac{\sin ax}{\cos ax} \right\}^{\mu} \left\{ \frac{\sin (bx+c)}{\cos (bx+c)} \right\}^{\nu}$,

$$A(x) \left\{ \frac{\sin ax}{\cos ax} \right\}^{\mu} \left\{ \frac{\cos (bx+c)}{\sin (bx+c)} \right\}^{\nu}.$$

$$1. \int_0^a \frac{\sin [c(a-x)]}{(x+z)^2} dx = c \{ \sin t [\Phi_1(t-ac) - \Phi_1(t)] - \cos t [\Phi_2(t-ac) - \Phi_2(t)] \},$$

$$\Phi_1(t) = t^{-1} \cos t + \operatorname{si}(t), \quad \Phi_2(t) = t^{-1} \sin t - \operatorname{ci}(t)$$

$$[t = c(z+a); \quad a, c > 0; \quad |\arg z| < \pi].$$

$$2. \int_0^a \frac{x \, dx}{\cos x \cos (a-x)} = -\frac{a}{\sin a} \ln \cos a \quad [0 < a < \pi/2].$$

$$3. \int_0^{\pi} \sin^{\mu-1} x \cos \left[a \left(\frac{\pi}{2} - x \right) \right] dx = 2^{\mu-1} \Gamma \left[\begin{matrix} (\mu-a)/2, (\mu+a)/2, \mu \\ \mu-a, \mu+a \end{matrix} \right] \quad [\operatorname{Re} \mu > 0].$$

$$4. \int_0^{\pi} \sin^{n-1} x \cos [m(a-x)] dx = \frac{\pi \cos [m(a-\pi/2)]}{2^{n-1} n B \left(\frac{n+m+1}{2}, \frac{n-m+1}{2} \right)} \quad [n \geq m; \quad m, n = 1, 2, 3, \dots].$$

$$5. \int_0^{2\pi} [a + ib \cos (a-x)]^{\nu} \left\{ \frac{\sin mx}{\cos mx} \right\} dx = \frac{2\pi i^{3m}}{(1+\nu)_m} (a^2 + b^2)^{\nu/2} \left\{ \frac{\sin cm}{\cos cm} \right\} P_{\nu}^m(c). \quad [c = a/\sqrt{a^2 + b^2}].$$

$$6. \int_0^{2\pi} [a + b \cos (c-x)]^{\nu} (d + h \cos x)^{\nu-1} dx = 2\pi \frac{g^{\nu}}{\sqrt{d^2 - h^2}} P_{\nu} \left(\frac{ad - bh \cos c}{g} \right) \quad [g = \sqrt{(a^2 - b^2)(d^2 - h^2)}; \quad |b| < a; \quad |h| < d].$$

$$7. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\sin (bx+c)}{\cos (bx+c)} \right\} dx = b^{-\alpha} \Gamma(\alpha) \left\{ \frac{\sin (c + \alpha\pi/2)}{\cos (c + \alpha\pi/2)} \right\} \quad [b > 0; \quad 0 < \operatorname{Re} \alpha < 1].$$

$$8. \int_0^{\infty} x^{\alpha-1} \frac{\sin (\pi\alpha/2 - bx)}{x^2 - y^2} dx = -\frac{\pi}{2} y^{\alpha-2} \cos \left(\frac{\pi\alpha}{2} - by \right) \quad [y, b > 0; \quad 0 < \alpha < 2].$$

2.5.21. Интегралы от $\left\{ \frac{\sin (ax^p + bx^q + c)}{\cos (ax^p + bx^q + c)} \right\}$.

$$1. \int_0^{1+\sigma} x^{\sigma} \sin (ax^2 - 2ax) dx = \frac{(-1)^{\sigma+1}}{2} \{ \operatorname{Si}(a) - \pi [C^2(a) + S^2(a)] \}$$

$$[\sigma = 0 \text{ или } 1; \quad a > 0].$$

$$2. \int_1^{\infty} \frac{1}{x} \sin(ax^2 - 2ax) dx = \frac{1}{2} \operatorname{si}(a) + \frac{\pi}{2} [1 - C(a) - S(a)] \quad [a > 0].$$

$$3. \int_0^{\infty} \left\{ \begin{matrix} \sin(ax^2 + 2bx) \\ \cos(ax^2 + 2bx) \end{matrix} \right\} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left(\cos \frac{b^2}{a} \mp \sin \frac{b^2}{a} \right) \quad [a > 0].$$

$$4. \int_0^{\infty} \frac{1}{x} \sin(ax^2 \mp 2ax) dx = \frac{\pi}{2} \left[\frac{1}{2} - C^2(a) - S^2(a) \mp C(a) \mp S(a) \right] \quad [a > 0].$$

$$5. \int_0^{\infty} \cos(x^3 \mp ax) dx = \frac{\pi \sqrt{a}}{3 \sqrt{3}} \left\{ \begin{matrix} J_{-1/3}(c) + J_{1/3}(c) \\ I_{-1/3}(c) - I_{1/3}(c) \end{matrix} \right\} \quad [a > 0; c = 2(a/3)^{3/2}].$$

$$6. \int_0^{\infty} \left\{ \begin{matrix} \sin(ax^p + bx^q) \\ \cos(ax^p + bx^q) \end{matrix} \right\} dx = \frac{1}{p} \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \left\{ \begin{matrix} \sin c \\ \cos c \end{matrix} \right\} \\ \left[c = \frac{k(q-p)+1}{2p} \pi, \quad a, b, p, q > 0 \right].$$

$$7. \int_{-\infty}^{\infty} \left\{ \begin{matrix} \sin(ax^2 + 2bx + c) \\ \cos(ax^2 + 2bx + c) \end{matrix} \right\} dx = \sqrt{\frac{\pi}{a}} \left\{ \begin{matrix} \sin d \\ \cos d \end{matrix} \right\} \quad \left[d = \frac{\pi}{4} + \frac{ac - b^2}{a}; \quad a > 0 \right].$$

$$8. \int_{-\infty}^{\infty} \frac{\sin a \sqrt{|x|} \operatorname{sgn} x}{x - y} dx = \cos a \sqrt{|y|} + e^{-a \sqrt{|y|}} \quad [a > 0; \operatorname{Im} y = 0].$$

2.5.22. Интегралы от $x^\alpha \left\{ \begin{matrix} \sin ax^p \sin bx \\ \cos ax^p \cos bx \end{matrix} \right\}$, $x^\alpha \left\{ \begin{matrix} \sin ax^p \cos bx \\ \cos ax^p \sin bx \end{matrix} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \left\{ \begin{matrix} \sin ax^2 \\ \cos ax^2 \end{matrix} \right\} \sin bx dx = \\ = \sqrt{\frac{\pi}{2a}} \left[\left\{ \begin{matrix} \cos [b^2/(4a)] \\ \sin [b^2/(4a)] \end{matrix} \right\} C\left(\frac{b^2}{4a}\right) \pm \left\{ \begin{matrix} \sin [b^2/(4a)] \\ \cos [b^2/(4a)] \end{matrix} \right\} S\left(\frac{b^2}{4a}\right) \right] \quad [a, b > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} \left\{ \begin{matrix} \sin ax^2 \\ \cos ax^2 \end{matrix} \right\} \sin bx dx =$$

$$\frac{b}{2} a^{-(\alpha+1)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \left\{ \begin{matrix} \cos [(1-\alpha)\pi/4] \\ \sin [(1-\alpha)\pi/4] \end{matrix} \right\} {}_2F_3\left(\frac{\alpha+3}{4}, \frac{\alpha+1}{4}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^4}{64a^2}\right) \mp \\ \frac{b^3 a^{-(\alpha+3)/2}}{12} \Gamma\left(\frac{\alpha+3}{2}\right) \left\{ \begin{matrix} \cos [(1+\alpha)\pi/4] \\ \sin [(1+\alpha)\pi/4] \end{matrix} \right\} {}_2F_3\left(\frac{5+\alpha}{4}, \frac{3+\alpha}{4}; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; -\frac{b^4}{64a^2}\right) \\ [a, b > 0; -1-2\delta < \operatorname{Re} \alpha < 2].$$

$$3. \int_0^{\infty} x \left\{ \begin{matrix} \sin ax^2 \\ \cos ax^2 \end{matrix} \right\} \sin bx dx = \frac{b}{4a} \sqrt{\frac{\pi}{2a}} \left(\sin \frac{b^2}{4a} \pm \cos \frac{b^2}{4a} \right) \quad [a, b > 0].$$

$$4. \int_0^{\infty} x^{-1/2} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} \sin bx \, dx = \mp \frac{\pi}{4} \sqrt{\frac{b}{a}} \left(\sin \frac{b^2}{8a} \pm \cos \frac{b^2}{8a} \right) J_{1/4} \left(\frac{b^2}{8a} \right)$$

[a, b > 0]

$$5. \int_0^{\infty} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} \cos bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left(\cos \frac{b^2}{4a} \mp \sin \frac{b^2}{4a} \right)$$

[a, b > 0].

$$6. \int_0^{\infty} x^{\alpha-1} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} \cos bx \, dx =$$

$$= \frac{a^{-\alpha/2}}{2} \Gamma \left(\frac{\alpha}{2} \right) \begin{Bmatrix} \sin (\alpha\pi/4) \\ \cos (\alpha\pi/4) \end{Bmatrix} {}_2F_3 \left(\frac{2+\alpha}{4}, \frac{\alpha}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; -\frac{b^4}{64a^2} \right) \mp$$

$$\mp \frac{b^2}{4} a^{-\alpha/2-1} \Gamma \left(\frac{\alpha}{2} + 1 \right) \begin{Bmatrix} \cos (\alpha\pi/4) \\ \sin (\alpha\pi/4) \end{Bmatrix} {}_2F_3 \left(1 + \frac{\alpha}{4}, \frac{2+\alpha}{4}; \frac{3}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^4}{64a^2} \right)$$

[a, b > 0; -2\delta < \operatorname{Re} \alpha < 2].

$$7. \int_0^{\infty} x \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} \cos bx \, dx =$$

$$= \frac{\delta}{2a} \mp \frac{b}{2a} \sqrt{\frac{\pi}{2a}} \left[\begin{Bmatrix} \sin [b^2/(4a)] \\ \cos [b^2/(4a)] \end{Bmatrix} C \left(\frac{b^2}{4a} \right) \mp \begin{Bmatrix} \cos [b^2/(4a)] \\ \sin [b^2/(4a)] \end{Bmatrix} S \left(\frac{b^2}{4a} \right) \right] \quad [a, b > 0].$$

$$8. \int_0^{\infty} x^{-1/2} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} \cos bx \, dx = \mp \frac{\pi}{2} \sqrt{\frac{b}{2a}} \begin{Bmatrix} \sin c \\ \cos c \end{Bmatrix} J_{-1/4} \left(\frac{b^2}{8a} \right)$$

[c = (b^2 - \pi a)/(8a); a, b > 0].

$$9. \int_0^{\infty} x^{1/2} \cos ax^2 \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = -\frac{\pi}{8\sqrt{2}} \left(\frac{b}{a} \right)^{3/2} \left[\begin{Bmatrix} \sin [(b^2 - \pi a)/(8a)] \\ \cos [(b^2 + \pi a)/(8a)] \end{Bmatrix} J_{\pm 1/4} \left(\frac{b^2}{8a} \right) \mp \right. \\ \left. + \begin{Bmatrix} \cos [(b^2 - \pi a)/(8a)] \\ \sin [(b^2 + \pi a)/(8a)] \end{Bmatrix} J_{\pm 3/4} \left(\frac{b^2}{8a} \right) \right] \quad [a, b > 0].$$

$$10. \int_0^{\infty} \frac{1}{x^2} \sin ax^2 \cos bx \, dx = \frac{b\pi}{2} \left[S \left(\frac{b^2}{4a} \right) - C \left(\frac{b^2}{4a} \right) + \sqrt{\pi a} \sin \frac{b^2 + \pi a}{4a} \right]$$

[a, b > 0].

$$11. \int_0^{\infty} \begin{Bmatrix} \sin a^3 x^3 \sin bx \\ \cos a^3 x^3 \cos bx \end{Bmatrix} dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left[J_{1/3}(c) + J_{-1/3}(c) \mp \frac{\sqrt{3}}{\pi} K_{1/3}(c) \right]$$

[c = 2b^{3/2}/(3a)^{3/2}; a, b > 0].

$$12. \int_0^{\infty} x^{-1/2} \sin(2a\sqrt{x}) \sin cx \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \sqrt{\frac{\pi}{2(c-b)}} \left[\begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix} C(d) \mp \begin{Bmatrix} \cos d \\ \sin d \end{Bmatrix} S(d) \right] \mp$$

$$\mp \sqrt{\frac{\pi}{2(c+b)}} \left[\begin{Bmatrix} \sin h \\ \cos h \end{Bmatrix} C(h) \mp \begin{Bmatrix} \cos h \\ \sin h \end{Bmatrix} S(h) \right]$$

[c > b > 0; a > 0; d = \frac{a^2}{c-b}, h = \frac{a^2}{c+b}].

$$13. \quad = \mp \frac{1}{2} \sqrt{\frac{\pi}{c}} \left[\cos \frac{a^2}{2c} C \left(\frac{a^2}{2c} \right) \mp \left\{ \begin{matrix} \cos [a^2/(2c)] \\ \sin [a^2/(2c)] \end{matrix} \right\} S \left(\frac{a^2}{2c} \right) \right] \\ [c=b>0, \quad a>0].$$

$$14. \quad \int_0^\infty x^{-1/2} \sin (2a \sqrt{x}) \cos cx \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \sqrt{\frac{\pi}{2(c+b)}} \left[\left\{ \begin{matrix} \cos h \\ \sin h \end{matrix} \right\} C(h) \pm \left\{ \begin{matrix} \sin h \\ \cos h \end{matrix} \right\} S(h) \right] - \\ - \sqrt{\frac{\pi}{2(c-b)}} \left[\left\{ \begin{matrix} \sin d \\ \cos d \end{matrix} \right\} S(d) \mp \left\{ \begin{matrix} \cos d \\ \sin d \end{matrix} \right\} C(d) \right] \\ \left[c>b>0, \quad a>0, \quad d=\frac{a^2}{c-b}, \quad h=\frac{a^2}{c+b} \right].$$

$$15. \quad = \frac{1}{2} \sqrt{\frac{\pi}{c}} \left[\left\{ \begin{matrix} \cos [a^2/(2c)] \\ \sin [a^2/(2c)] \end{matrix} \right\} C \left(\frac{a^2}{2c} \right) \pm \sin \frac{a^2}{2c} S \left(\frac{a^2}{2c} \right) \right] \\ [c=b>0, \quad a>0].$$

$$16. \quad \int_0^\infty x^{-1/2} \cos (2a \sqrt{x}) \sin cx \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2(c-b)}} (\cos d \pm \sin d) \mp \frac{1}{2} \sqrt{\frac{\pi}{2(c+b)}} (\cos h \pm \sin h) \\ \left[c>b>0, \quad a>0, \quad d=\frac{a^2}{c-b}, \quad h=\frac{a^2}{c+b} \right].$$

$$17. \quad = \mp \frac{1}{4} \sqrt{\frac{\pi}{c}} \left(\cos \frac{a^2}{2c} \pm \sin \frac{a^2}{2c} \right) \\ [b=c>0, \quad a>0].$$

$$18. \quad \int_0^\infty x^{-1/2} \cos (2a \sqrt{x}) \cos cx \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2(c+b)}} (\cos h \mp \sin h) + \frac{1}{2} \sqrt{\frac{\pi}{2(c-b)}} (\sin d \mp \cos d) \\ \left[c>b>0, \quad a>0, \quad d=\frac{a^2}{c-b}, \quad h=\frac{a^2}{c+b} \right].$$

$$19. \quad = \frac{1}{4} \sqrt{\frac{\pi}{c}} \left(\cos \frac{a^2}{2c} \mp \sin \frac{a^2}{2c} \right) \\ [b=c>0, \quad a>0].$$

2.5.23. Интегралы от $A(x) \left\{ \begin{matrix} \sin (ax + b/x) \\ \cos (ax + b/x) \end{matrix} \right\}$.

Обозначение $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \quad \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x^2+1}} \left\{ \begin{matrix} \sin (ax + a/x) \\ \cos (ax + a/x) \end{matrix} \right\} dx = \pm \left(\frac{\pi}{2} \right)^{3/2} \sqrt{a} [J_\mu(a) J_\nu(a) - Y_\mu(a) Y_\nu(a)] \\ [4\mu = -1 - 2\alpha - 2\delta, \quad 4\nu = 3 + 2\alpha - 2\delta, \quad a>0, \quad -1 < \operatorname{Re} \alpha < 2].$$

$$2. \quad \int_0^1 \frac{x^{\alpha-1}}{\sqrt{1-x^2}} \left\{ \begin{matrix} \sin (ax - a/x) \\ \cos (ax - a/x) \end{matrix} \right\} dx = \mp \sqrt{\frac{\pi a}{2}} I_{(\alpha+\delta-1)/2}(a) K_{(\alpha-\delta)/2}(a) \\ [\operatorname{Re} \alpha > -1; \quad a>0].$$

$$3. \int_0^1 \left\{ \frac{\sin(ax + a/x) \sin(ax - a/x)}{\cos(ax + a/x) \cos(ax - a/x)} \right\} \frac{dx}{1+x^2} = \frac{\pi}{4} \left\{ \frac{\sin(-2a)}{\exp(-2a)} \right\} \quad [a > 0].$$

$$4. \int_0^\infty x^{\alpha-1} \left\{ \frac{\sin(ax + b/x)}{\cos(ax + b/x)} \right\} dx = \frac{\pi}{2} \left(\frac{b}{a} \right)^{\alpha/2} \left\{ \frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi/2)} \right\} [J_{-\alpha}(2\sqrt{ab}) \pm J_\alpha(2\sqrt{ab})] \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$5. \int_0^\infty x^{-1/2} \left\{ \frac{\sin(ax + b/x)}{\cos(ax + b/x)} \right\} dx = \sqrt{\frac{\pi}{a}} \left\{ \frac{\sin c}{\cos c} \right\} \quad [c = 2\sqrt{ab} + \pi/4, a, b > 0].$$

$$6. \int_0^\infty x^{-3/2} \left\{ \frac{\sin(ax + b/x)}{\cos(ax + b/x)} \right\} dx = \sqrt{\frac{\pi}{b}} \left\{ \frac{\sin c}{\cos c} \right\} \quad [c = 2\sqrt{ab} + \pi/4, a, b > 0].$$

$$7. \int_0^\infty x^{\alpha-1} \left\{ \frac{\sin(ax - b/x)}{\cos(ax - b/x)} \right\} dx = 2 \left(\frac{b}{a} \right)^{\alpha/2} \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} K_\alpha(2\sqrt{ab}) \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$8. \int_0^\infty x^{-1/2} \left\{ \frac{\sin(ax - b/x)}{\cos(ax - b/x)} \right\} dx = \sqrt{\frac{\pi}{2a}} e^{-2\sqrt{ab}} \quad [a, b > 0].$$

$$9. \int_0^\infty x^{-3/2} \left\{ \frac{\sin(ax - b/x)}{\cos(ax - b/x)} \right\} dx = -\sqrt{\frac{\pi}{2b}} e^{-2\sqrt{ab}} \quad [a, b > 0].$$

$$10. \int_0^\infty x^{-2} \left\{ \frac{\sin(ax - b/x^2)}{\cos(ax - b/x^2)} \right\} dx = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \quad [a, b > 0].$$

$$11. \int_0^\infty \frac{x^\delta}{x^2 + z^2} \left\{ \frac{\sin(ax - b/x)}{\cos(ax - b/x)} \right\} dx = \frac{\pi}{2} z^{\delta-1} \exp\left(-az - \frac{b}{z}\right) \\ [a, b, \operatorname{Re} z > 0, \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}]$$

2.5.24. Интегралы от $x^\alpha \left\{ \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} \right\}$, $x^\alpha \left\{ \frac{\sin(a/x) \cos bx}{\cos(a/x) \sin bx} \right\}$.

$$1. \int_0^\infty x^{\alpha-1} \left\{ \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} \right\} dx = \\ = \frac{\pi}{4} \left(\frac{a}{b} \right)^{\alpha/2} \frac{1}{\sin(\alpha\pi/2)} [J_{\pm\alpha}(2\sqrt{ab}) - J_{\mp\alpha}(2\sqrt{ab}) + 2\pi^{-1} \sin \alpha\pi K_\alpha(2\sqrt{ab})] \\ [ab > 0, |\operatorname{Re} \alpha| < 1 + \delta, \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$2. \int_0^\infty \frac{1}{x} \left\{ \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} \right\} dx = \pm \frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a, b > 0].$$

$$3. \int_0^{\infty} x^{-1/2} \begin{Bmatrix} \sin (a/x) \sin bx \\ \cos (a/x) \cos bx \end{Bmatrix} dx = \\ = \pm \frac{1}{2} \sqrt{\frac{\pi}{2b}} [\sin (2 \sqrt{ab}) - \cos (2 \sqrt{ab}) \pm e^{-2 \sqrt{ab}}] \quad [a, b > 0].$$

$$4. \int_0^{\infty} x^{-3/2} \begin{Bmatrix} \sin (a/x) \sin bx \\ \cos (a/x) \cos bx \end{Bmatrix} dx = \\ = \pm \frac{1}{2} \sqrt{\frac{\pi}{2a}} [\sin (2 \sqrt{ab}) - \cos (2 \sqrt{ab}) \pm e^{-2 \sqrt{ab}}] \quad [a, b > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} \begin{Bmatrix} \cos (a/x) \sin bx \\ \sin (a/x) \cos bx \end{Bmatrix} dx = \\ = \frac{\pi}{4} \left(\frac{a}{b}\right)^{\alpha/2} \frac{1}{\cos (\alpha \pi / 2)} [J_{\alpha} (2 \sqrt{ab}) + J_{-\alpha} (2 \sqrt{ab}) \pm 2 \pi^{-1} \sin \alpha \pi K_{\alpha} (2 \sqrt{ab})] \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$6. \int_0^{\infty} x^{-1/2} \begin{Bmatrix} \cos (a/x) \sin bx \\ \sin (a/x) \cos bx \end{Bmatrix} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2b}} [\sin (2 \sqrt{ab}) + \cos (2 \sqrt{ab}) \pm e^{-2 \sqrt{ab}}] \quad [a, b > 0].$$

$$7. \int_0^{\infty} x^{-3/2} \begin{Bmatrix} \cos (a/x) \sin bx \\ \sin (a/x) \cos bx \end{Bmatrix} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} [\sin (2 \sqrt{ab}) + \cos (2 \sqrt{ab}) \mp e^{-2 \sqrt{ab}}] \\ [a, b > 0].$$

2.5.25. Интегралы от $A(x) \begin{Bmatrix} \sin (c \sqrt{a^2 \pm x^2}) \sin bx \\ \cos (c \sqrt{a^2 \pm x^2}) \cos bx \end{Bmatrix}$.

$A(x) \begin{Bmatrix} \sin (c \sqrt{a^2 \pm x^2}) \cos bx \\ \cos (c \sqrt{a^2 \pm x^2}) \sin bx \end{Bmatrix}$.

$$1. \int_0^a \sin (c \sqrt{a^2 - x^2}) \cos bx \, dx = \frac{\pi}{2} \frac{ac}{\sqrt{b^2 + c^2}} J_1 (a \sqrt{b^2 + c^2}) \quad [a > 0].$$

$$2. \int_0^a \frac{\sin (c \sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx \, dx = \\ = \frac{\pi}{2} \sqrt{\frac{\pi c}{2}} J_{1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} - b) \right] J_{1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} + b) \right] \quad [a > 0].$$

$$3. \int_0^a \frac{\cos (c \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} \cos bx \, dx = \frac{\pi}{2} J_0 (a \sqrt{b^2 + c^2}) \quad [a > 0].$$

$$4. \int_0^a \frac{\cos(c \sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx \, dx =$$

$$= \left(\frac{\pi}{2}\right)^{3/2} \sqrt{c} J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - b) \right] J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + b) \right] \quad [a > 0].$$

$$5. \int_0^a \sqrt{a^2 - x^2} \cos(c \sqrt{a^2 - x^2}) \cos bx \, dx =$$

$$= \frac{\pi}{2} \frac{a^2 c^2}{b^2 + c^2} \left[J_0(a \sqrt{b^2 + c^2}) + \frac{b^2 - c^2}{ac^2 \sqrt{b^2 + c^2}} J_1(a \sqrt{b^2 + c^2}) \right] \quad [a > 0].$$

$$6. \int_0^a \frac{\cos(c \sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \cos bx \, dx =$$

$$= \left(\frac{\pi}{2}\right)^{3/2} \sqrt{b} J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] J_{1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right] \quad [a > 0].$$

$$7. \int_0^\infty \frac{\sin(c \sqrt{x^2 + y^2})}{x \sqrt{x^2 + y^2}} \sin cx \, dx = \frac{\pi}{2y} \sin cy \quad [c, y > 0].$$

$$8. \int_0^\infty \frac{\sin(c \sqrt{x^2 + y^2})}{(x^2 + y^2)^2} \cos bx \, dx = \frac{\pi c}{2y} e^{-by} \quad [b > y > 0].$$

$$9. \int_0^\infty \frac{\sin(c \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \cos bx \, dx = \frac{\pi}{2} J_0(y \sqrt{c^2 - b^2}) \begin{cases} 1 \\ 0 \end{cases} \quad \left[\begin{cases} c > b > 0 \\ b > c > 0 \end{cases}; y > 0 \right].$$

$$10. \int_0^\infty \frac{\sin(c \sqrt{x^2 + y^2})}{\sqrt{x(x^2 + y^2)}} \cos bx \, dx =$$

$$= \left(\frac{\pi}{2}\right)^{3/2} \sqrt{b} J_{-1/4} \left[\frac{y}{2} (c - \sqrt{c^2 - b^2}) \right] J_{1/4} \left[\frac{y}{2} (c + \sqrt{c^2 - b^2}) \right] \quad [c > b > 0; y > 0].$$

$$11. \int_0^\infty \frac{\sin(c \sqrt{x^2 + y^2}) \cos bx}{(x^2 + a^2) \sqrt{x^2 + y^2}} \, dx = \frac{\pi}{2a} \frac{e^{-ab} \sin(c \sqrt{y^2 - a^2})}{\sqrt{y^2 - a^2}}$$

$$[b > c > 0; a, y > 0; a \neq y].$$

$$12. = \pi c (2a)^{-1} e^{-ab} \quad [b > c > 0; a = y > 0].$$

$$13. \int_0^\infty x^\delta \frac{\cos(c \sqrt{x^2 + y^2})}{x^2 + a^2} \begin{cases} \sin bx \\ \cos bx \end{cases} \, dx = \frac{\pi}{2a^{1-\delta}} e^{-ab} \cos(c \sqrt{y^2 - a^2})$$

$$[b > c > 0; a, y > 0; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$14. \int_0^{\infty} x^{-1/2} \frac{\cos(c \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= -\frac{\pi}{2} \sqrt{\frac{\pi b}{2}} J_{\pm 1/4} \left[\frac{y}{2} (c - \sqrt{c^2 - b^2}) \right] Y_{1/4} \left[\frac{y}{2} (c + \sqrt{c^2 - b^2}) \right]$$

[c > b > 0; y > 0].

$$15. \int_0^{\infty} \frac{\cos(c \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \cos bx dx = K_0(y \sqrt{b^2 - c^2})$$

[b > c > 0; y > 0].

$$= -\frac{\pi}{2} Y_0(y \sqrt{c^2 - b^2})$$

[c > b > 0, y > 0].

2.5.26. Интегралы, содержащие $\begin{Bmatrix} \operatorname{tg} ax \\ \operatorname{ctg} ax \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/4} \operatorname{tg}^{\mu} x dx = \frac{1}{2} \beta \left(\frac{\mu + 1}{2} \right)$$

[Re $\mu > -1$].

$$2. \int_0^{\pi/4} \operatorname{tg}^m x dx = \frac{(-1)^{[(m+1)/2]}}{4} \begin{Bmatrix} \pi \\ \ln 4 \end{Bmatrix} + \sum_{k=0}^{[m/2]-1} \frac{(-1)^k}{m-2k-1}$$

[m = {0, 2, 4, ...}].

$$3. \int_0^{\pi/4} x \operatorname{tg}^m x dx = I_m,$$

$$I_1 = \frac{G}{2} - \frac{\pi}{8} \ln 2, \quad I_2 = -\left(\frac{\ln 2}{2} + \frac{\pi^2}{32} - \frac{\pi}{4} \right), \quad I_3 = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} \ln 2 - \frac{G}{2}.$$

$$4. \int_0^{\pi/4} x^n \operatorname{tg} x dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^n \sum_{k=1}^{\infty} \frac{(4^k - 1) \zeta(2k)}{4^{2k-1} (n + 2k)}$$

[n = 1, 2, 3, ...].

$$5. \int_0^{\pi/4} x^n \operatorname{ctg} x dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^n \left[\frac{2}{n} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (n + 2k)} \right]$$

[n = 1, 2, 3, ...].

$$6. \int_0^{\pi/4} x \operatorname{ctg} x dx = \frac{\pi}{8} \ln 2 + \frac{G}{2}.$$

$$7. \int_0^{\pi/2} \operatorname{tg}^{\mu} x dx = \frac{\pi}{2 \cos(\mu\pi/2)}$$

[|Re μ | < 1].

$$8. \int_0^{\pi/2} x^n \operatorname{ctg} x dx = \left(\frac{\pi}{2} \right)^n \left[\frac{1}{n} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k (n + 2k)} \right]$$

[n = 1, 2, 3, ...].

$$9. \int_0^{\pi/2} x \operatorname{ctg} x \, dx = \frac{\pi}{2} \ln 2. \quad 10. \int_0^{\pi} x \operatorname{tg} x \, dx = -\pi \ln 2.$$

$$11. \int_0^{\infty} \frac{\operatorname{tg}^{(2 \pm 1)/3} ax}{x} \, dx = 2^{-(1 \pm 1)/2} \pi \operatorname{sgn} a.$$

$$12. \int_0^{\pi/4} \operatorname{tg}^{\mu} x \left\{ \frac{\sin x}{\cos x} \right\}^2 \, dx = \frac{1 \pm \mu}{2} \beta \left(\frac{\mu + 1}{2} \right) \quad [\operatorname{Re} \mu > -1].$$

$$13. \int_0^{\pi/4} \frac{(\pi/4 - x) \operatorname{tg} x}{\cos 2x} \, dx = \frac{\pi}{8} - \frac{\pi}{8} \ln 2.$$

$$14. \int_0^{\pi/2} \operatorname{tg}^{\mu} x \left\{ \frac{\sin 2x}{\cos 2x} \right\} \, dx = \pm \frac{\pi \mu}{2} \left\{ \frac{\sin (\mu \pi / 2)}{\cos (\mu \pi / 2)} \right\}^{-1} \quad [|\operatorname{Re} \mu| < 1 + \delta].$$

$$15. \int_0^{\pi/2} \operatorname{tg}^{\mu} x \sin^{v-2} x \left\{ \frac{\sin vx}{\cos vx} \right\} \, dx = \mp \left\{ \frac{\cos c}{\sin c} \right\} B(\mu + v - 1, 1 - \mu) \quad [2c = (\mu + v) \pi; \operatorname{Re} \mu < 1, \operatorname{Re} (\mu + v) > 1 - \delta].$$

$$16. \int_0^{\pi/2} \frac{\operatorname{tg}^{\mu} (x/2)}{2 \pm \sin x} \, dx = \frac{1}{3} \frac{\psi[(\mu + 2)/3] - \psi[(\mu + 1)/3]}{\beta[(\mu + 2)/3] + \beta[(\mu + 1)/3]} \quad [\operatorname{Re} \mu > -1].$$

$$17. \int_0^{\pi/2} \frac{\operatorname{tg}^m (x/2)}{1 + \cos (l\pi/n) \sin x} \, dx = \frac{1}{2n \sin (l\pi/n)} \sum_{k=0}^{n-1} (-1)^{k-1} \sin \frac{kl}{n} \pi \left[\psi \left(\frac{n+m+k}{2n} \right) - \psi \left(\frac{m+k}{2n} \right) \right] \quad [l+n=1, 3, 5, \dots].$$

$$18. = \frac{1}{n \sin (l\pi/n)} \sum_{k=0}^{[(n-1)/2]} (-1)^{k-1} \sin \frac{kl}{n} \pi \left[\psi \left(\frac{n+m-k}{n} \right) - \psi \left(\frac{m+k}{n} \right) \right] \quad [l+n=0, 2, 4, \dots; m=1, 2, 3, \dots].$$

$$19. \int_0^{\pi/2} \operatorname{ctg}^{\mu} x \cos^{v-2} x \left\{ \frac{\sin vx}{\cos vx} \right\} \, dx = \left\{ \frac{\cos (\mu \pi / 2)}{\sin (\mu \pi / 2)} \right\} B(\mu + v - 1, 1 - \mu) \quad [\operatorname{Re} \mu < 1 + \delta; \operatorname{Re} (\mu + v) > 1].$$

$$20. \int_0^{\pi/2} \operatorname{ctg} x \cos^{v-2} x \sin vx \, dx = \pi/2 \quad [\operatorname{Re} v > 0].$$

$$21. \int_0^{\pi/2} \frac{x \operatorname{ctg} x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2.$$

$$22. \int_0^{\pi/2} x \frac{\operatorname{ctg}^{\mu-1} x}{\sin^2 x} dx = \frac{\pi}{2\mu \cos(\mu\pi/2)} \quad [0 < \operatorname{Re} \mu < 1].$$

$$23. \int_0^{\pi/2} \frac{x dx}{(\operatorname{tg}^{2/3} x + 1) (\sin x \cos^2 x)^{2/3}} = \frac{3 \cdot 2^{1/3} \pi^2}{4}.$$

$$24. \int_0^{\pi/2} \frac{x (\operatorname{tg} x)^{2pq-1}}{(\operatorname{tg}^{2p} x + 1)^{2q} \cos^2 x} dx = 2^{-2q-2} \frac{\pi^{3/2}}{p} \Gamma \left[q + 1/2 \right] \quad [q > 0].$$

$$25. \int_0^a \frac{x \operatorname{tg} x}{\sqrt{\operatorname{tg}^2 a - \operatorname{tg}^2 x} (\operatorname{tg}^2 b - \operatorname{tg}^2 x) \cos^2 x} dx =$$

$$= \frac{\pi}{2 \sqrt{\operatorname{tg}^2 b - \operatorname{tg}^2 a}} \left[a + \operatorname{arctg} \frac{\cos b}{\sqrt{\sin(b-a) \sin(b+a)}} - \frac{\pi}{2} \right] \quad [0 < a < b < \pi/2].$$

$$26. \int_0^{\infty} \frac{\operatorname{tg}^{\mu} ax}{x^2 + z^2} dx = \frac{\pi}{2z \cos(\mu\pi/2)} \operatorname{th}^{\mu} az \quad [a, \operatorname{Re} z > 0; |\operatorname{Re} \mu| < 1].$$

$$27. \int_0^{\infty} \frac{x}{x^2 + z^2} \left\{ \begin{matrix} \operatorname{tg} ax \\ \operatorname{ctg} ax \end{matrix} \right\} dx = \mp \frac{\pi}{2} \left\{ \begin{matrix} \operatorname{th} az \\ \operatorname{cth} az \end{matrix} \right\} \quad [a, \operatorname{Re} z > 0].$$

$$28. \int_0^{\infty} x^{-2n} \sin^{2n-1} x \cos(2n-1)x \operatorname{tg} x dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} 2^{2n-1} \pi |B_{2n}|$$

$$[n=1, 2, 3, \dots].$$

$$29. \int_0^{\infty} x^{-n} \sin^{2m} x \cos 2mx \operatorname{tg} x dx = 0 \quad [n=1, 2, 3, \dots, 2m+1].$$

$$30. \int_0^{\infty} x^{-n} \sin^{2m-1} x \cos 2mx \operatorname{tg} x dx = 0 \quad [n=1, 2, 3, \dots, 2m].$$

2.5.27. Интегралы, содержащие тригонометрические функции от тригонометрических функций.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} \left\{ \begin{matrix} \sin(a \sin x) \sin(2n+1)x \\ \cos(a \sin x) \cos 2nx \end{matrix} \right\} dx = \frac{\pi}{2} J_{2n+\delta}(a) \quad [|\arg a| < \pi].$$

$$2. \int_0^{\pi/2} \left\{ \begin{matrix} \sin(a \cos x) \\ \cos(a \cos x) \end{matrix} \right\} \cos bx dx =$$

$$= \frac{1}{b} \left(\frac{ab}{1-b^2} \right)^{\delta} \left\{ \begin{matrix} \cos(b\pi/2) \\ \sin(b\pi/2) \end{matrix} \right\} {}_1F_2 \left(\frac{\delta+1}{2}; 1+\frac{b+\delta}{2}, 1+\frac{\delta-b}{2}; -\frac{a^2}{4} \right) =$$

$$= (-b)^{1-\delta} \left\{ \begin{matrix} \cos(b\pi/2) \\ \sin(b\pi/2) \end{matrix} \right\} s_{\delta-1, \delta}(a) \quad [b > 0].$$

3. $\int_0^{\pi/2} \left\{ \begin{matrix} \sin(a \cos x) \\ \cos(a \cos x) \end{matrix} \right\} \cos nx \, dx = \frac{\pi}{2} \left[\left\{ \begin{matrix} \sin(n\pi/2) \\ \cos(n\pi/2) \end{matrix} \right\} J_n(a) \mp \left\{ \begin{matrix} \cos(n\pi/2) \\ \sin(n\pi/2) \end{matrix} \right\} E_n(a) \right].$
4. $\int_0^{\pi/2} x \left\{ \begin{matrix} \sin(n\pi \sin x) \cos x \\ \sin(n\pi \cos x) \sin x \end{matrix} \right\} dx = \pm \frac{1}{2n} J_e(n\pi) + \frac{(\mp 1)^{n+1}}{2}, \quad [n=1, 2, 3, \dots].$
5. $\int_0^{\pi/2} \sin(z \operatorname{tg} x - vx) \sin^{v-2} x \, dx = 0 \quad [\operatorname{Re} v > 0; |\arg z| < \pi].$
6. $\int_0^{\pi/2} \cos(z \operatorname{tg} x - vx) \cos^v x \, dx = \pi z^{v/2} 2^{-1-v/2} \frac{1}{\Gamma(1+(v+\gamma)/2)} W_{v/2, -(v+1)/2}(2z) \quad [\operatorname{Re} v > -1; |\arg z| < \pi, \gamma+v \neq -2, -4, -6, \dots]$
7. $\int_0^{\pi/2} \cos(z \operatorname{tg} x + vx) \cos^v x \, dx = 2^{-v-1} \pi e^{-z} \quad [\operatorname{Re} v > -1; |\arg z| < \pi].$
8. $\int_0^{\pi/2} \cos(z \operatorname{tg} x - vx) \cos^{v-2} x \, dx = \frac{\pi}{\Gamma(v)} e^{-z} z^{v-1} \quad [\operatorname{Re} v > 1; |\arg z| < \pi].$
9. $\int_0^{\pi/2} \sin(n \operatorname{tg} x + vx) \frac{\cos^{v-1} x}{\sin x} \, dx = \frac{\pi}{2} \quad [\operatorname{Re} v > 0; n=1, 2, 3, \dots].$
10. $\int_0^{\pi/2} \frac{\sin(z \operatorname{tg} x - nx) + \sin nx}{\sin x} \cos^{n-1} x \, dx = \pi \quad [|\arg z| < \pi; n=1, 2, 3, \dots].$
11. $\int_0^{\pi} \sin(a \sin x) \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = 2 \cos \frac{b\pi}{2} \left\{ \begin{matrix} \sin(b\pi/2) \\ \cos(b\pi/2) \end{matrix} \right\} s_{0, b}(a) \quad [b > 0].$
12. $\int_0^{\pi} \cos(a \sin x) \left\{ \begin{matrix} \sin bx \\ \cos bx \end{matrix} \right\} dx = -2b \sin \frac{b\pi}{2} \left\{ \begin{matrix} \sin(b\pi/2) \\ \cos(b\pi/2) \end{matrix} \right\} s_{-1, b}(a) \quad [b > 0].$
13. $\int_0^{\pi} \left\{ \begin{matrix} \sin(a \sin x) \sin nx \\ \cos(a \sin x) \cos nx \end{matrix} \right\} dx = [1 \mp (-1)^n] \frac{\pi}{2} J_n(a).$
14. $\int_0^{\pi} \left\{ \begin{matrix} \sin(a \sin x) \cos nx \\ \cos(a \sin x) \sin nx \end{matrix} \right\} dx = [(-1)^{n+1} \mp 1] \frac{\pi}{2} E_n(a).$
15. $\int_0^{\pi} \left\{ \begin{matrix} \sin(a \cos x) \\ \cos(a \cos x) \end{matrix} \right\} \cos nx \, dx = \pi \left\{ \begin{matrix} \sin(n\pi/2) \\ \cos(n\pi/2) \end{matrix} \right\} J_n(a).$
16. $\int_0^{\pi} \left\{ \begin{matrix} \sin(vx - a \sin x) \\ \cos(vx - a \sin x) \end{matrix} \right\} dx = \pi \left\{ \begin{matrix} E_v(a) \\ J_v(a) \end{matrix} \right\}.$

$$17. \int_0^{\pi} \cos (nx - a \sin x) dx = \pi J_n(a).$$

$$18. \int_{-\pi}^{\pi} \left\{ \begin{matrix} \sin (a \cos x) \\ \cos (a \cos x) \end{matrix} \right\} \sin nx dx = 0,$$

$$19. \int_{-\pi}^{\pi} \left\{ \begin{matrix} \sin (a \cos x) \\ \cos (a \cos x) \end{matrix} \right\} \cos nx dx = 2\pi \left\{ \begin{matrix} \sin (n\pi/2) \\ \cos (n\pi/2) \end{matrix} \right\} J_n(a).$$

2.5.28. Интегралы, содержащие x^a и тригонометрические функции от тригонометрических функций.

$$1. \int_0^{\pi/2} x \left\{ \begin{matrix} \sin (a \operatorname{tg} x) \\ \cos (a \operatorname{tg} x) \operatorname{tg} x \end{matrix} \right\} dx = \pm \frac{\pi}{4} e^{-a} [C + \ln(2a) \mp e^{2a} \operatorname{Ei}(-2a)] \quad [a > 0].$$

$$2. \int_0^{\pi/2} x \cos(a \operatorname{tg} x) \frac{dx}{\sin 2x} = -\frac{\pi}{4} \operatorname{Ei}(-a) \quad [a > 0].$$

$$3. \int_0^{\pi/2} x \sin(a \operatorname{ctg} x) \frac{dx}{\sin^2 x} = \frac{1 - e^{-a}}{2a} \pi \quad [a > 0].$$

$$4. \int_0^{\pi} x \cos(n\pi \cos x) \sin x dx = \delta_0 n\pi.$$

$$5. \int_0^{\pi} x \sin(n\pi \cos x) \sin x dx = \frac{1}{n} [(-1)^n - J_0(n\pi)] \quad [n = 1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{1}{x} \sin(a \operatorname{tg} x) \left\{ \begin{matrix} 1 \\ \cos x \end{matrix} \right\} dx = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0].$$

$$7. \int_0^{\infty} \frac{1}{x} \sin(a \operatorname{tg} x) \left\{ \begin{matrix} \sin 2x \\ \cos^2 x \operatorname{tg}(x/2) \end{matrix} \right\} dx = \frac{\pi}{4} (1 + a) e^{-a} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \quad [a > 0].$$

$$8. \int_0^{\infty} \frac{1}{x} \sin(a \operatorname{tg} x) \operatorname{tg} x \left\{ \begin{matrix} \operatorname{tg} x \\ \operatorname{tg}(x/2) \end{matrix} \right\} dx = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

$$9. \int_0^{\infty} \frac{1}{x} \sin(a \operatorname{tg} x) \operatorname{ctg} x \left\{ \begin{matrix} \sin x \\ \operatorname{tg}(x/2) \end{matrix} \right\} dx = \frac{\pi}{2} (1 - e^a) \quad [a > 0].$$

$$10. \int_0^{\infty} \frac{1}{x} \cos(a \operatorname{tg} x) \left\{ \begin{matrix} \operatorname{tg} x \\ \operatorname{tg}(x/2) \end{matrix} \right\} dx = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

$$11. \int_0^{\infty} \frac{1}{x} \cos(a \operatorname{tg} x) \left\{ \frac{\sin^3 x}{\sin^2 x \operatorname{tg} x} \right\} dx = \frac{\pi}{16} (1-a) e^{-a} \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} \quad [a > 0].$$

$$12. \int_0^{\infty} x \sin(a \operatorname{tg}^2 x) \frac{dx}{x^2 + z^2} = \frac{\pi}{2} [\exp(-a \operatorname{th} z) - e^{-a}] \quad [a, \operatorname{Re} z > 0].$$

$$13. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \cos x \frac{dx}{x^2 + z^2} = \frac{\pi}{2z} [\operatorname{ch} z \exp(-a \operatorname{th} z) - e^{-a} \operatorname{sh} z] \quad [a, \operatorname{Re} z > 0].$$

$$14. \int_0^{\infty} x \frac{\cos(a \operatorname{tg}^2 x)}{\sin 2x} \frac{dx}{x^2 + z^2} = \frac{\pi}{2 \operatorname{sh} 2z} \exp(-a \operatorname{th} z) \quad [a, \operatorname{Re} z > 0].$$

$$15. \int_0^{\infty} x \cos(a \operatorname{tg}^2 x) \operatorname{tg} x \frac{dx}{x^2 + z^2} = \frac{\pi}{2 \operatorname{ch} z} [e^{-a} \operatorname{ch} z - \exp(-a \operatorname{th} z) \operatorname{sh} z] \quad [a, \operatorname{Re} z > 0].$$

$$16. \int_0^{\infty} x \cos(a \operatorname{tg}^2 x) \operatorname{ctg} 2^k x \frac{dx}{x^2 + z^2} = \frac{\pi}{2} [\operatorname{cth} 2^k z \exp(-a \operatorname{th} z) - e^{-a}] \quad [k=1 \text{ или } 2; a, \operatorname{Re} z > 0].$$

2.5.29. Интегралы, содержащие разности тригонометрических и алгебраических функций.

$$1. \int_0^a \frac{1 - \cos bx}{x} dx = C + \ln(ab) - \operatorname{ci}(ab) \quad [a, b > 0].$$

$$2. \int_0^a \left(\frac{1}{x} - \operatorname{ctg} x \right) dx = \ln \frac{a}{\sin a} \quad [a > 0].$$

$$3. \int_0^{\infty} \frac{1 - \cos^n x}{x^2} dx = \frac{\pi m}{4^m} \binom{2m}{m} \quad \left[m = \left[\frac{n+1}{2} \right]; n=1, 2, 3, \dots \right].$$

$$4. \int_0^{\infty} x^{\alpha-1} (\cos^{2n} bx - 1) dx = 2^{-2n} \sqrt{\pi} b^{-\alpha} (2n)! \Gamma \left[\frac{\alpha/2}{(1-\alpha)/2} \right] \sum_{k=0}^{n-1} \frac{(n-k)^{-\alpha}}{k! (2n-k)!} \quad [b > 0; -2 < \operatorname{Re} \alpha < 0; n=1, 2, 3, \dots].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left(\cos^{2n} bx - \frac{(2n)!}{2^{2n} (n!)^2} \right) dx = 2^{-2n} \sqrt{\pi} b^{-\alpha} (2n)! \Gamma \left[\frac{\alpha/2}{(1-\alpha)/2} \right] \sum_{k=0}^{n-1} \frac{(n-k)^{-\alpha}}{k! (2n-k)!} \quad [b > 0; 0 < \operatorname{Re} \alpha < 1; n=1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{x^n - \sin^n x}{x^{n+2}} dx = J_n \quad [n=1, 2, 3, \dots].$$

$$J_n = \frac{\pi}{2^n (n-1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n-1},$$

$$J_1 = \frac{\pi}{4}, \quad J_2 = \frac{\pi}{3}, \quad J_3 = \frac{13}{32}\pi, \quad J_4 = \frac{7}{15}\pi.$$

$$7. \int_0^{\infty} \left(\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)} \right) \frac{dx}{x} = \frac{C}{2} - \frac{3}{4}.$$

$$8. \int_0^{\infty} \left(\frac{\sin x}{x} - \frac{1}{1+x} \right) \frac{dx}{x} = 1 - C.$$

$$9. \int_0^{\infty} \left(\cos x - \frac{1}{1+x^k} \right) \frac{dx}{x} = -C \quad [k=1, 2].$$

$$10. \int_0^{\infty} \left(\frac{1}{x} - \operatorname{ctg} x \right) \frac{dx}{x} = \frac{\pi}{2}.$$

$$11. \int_0^1 \left(\cos ax \pm \cos \frac{a}{x} \right) \frac{dx}{1 \pm x^2} = \frac{\pi}{2} \begin{cases} \exp(-a) \\ \sin a \end{cases} \quad [a > 0].$$

$$12. \int_0^{\pi/4} \frac{\pi/4 - x \operatorname{tg} x}{\cos 2x} dx = \frac{\pi}{8} \ln 2 + \frac{G}{2}.$$

$$13. \int_0^{\pi/2} \frac{(1 - x \operatorname{ctg} x)}{\sin^2 x} dx = \frac{\pi}{4}. \quad 14. \int_0^{\pi/2} \frac{4x^2 \cos x + (\pi - x)x}{\sin x} dx = \pi^2 \ln 2.$$

$$15. \int_0^a \frac{1 - \cos bx}{x} dx - \int_a^{\infty} \frac{\cos bx}{x} dx = C + \ln(ab) \quad [a, b > 0].$$

$$16. \int_0^{\infty} \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a} \quad [a, b > 0].$$

$$17. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} |b - a| \quad [a, b > 0].$$

$$18. \int_0^{\infty} \frac{\cos^n ax - \cos^n bx}{x} dx = \left[1 - \frac{1 + (-1)^n (n-1)!!}{2} \right] \ln \frac{b}{a} \quad [a, b > 0; n=1, 2, 3, \dots].$$

$$19. \int_0^{\infty} \frac{\sin^{2n} ax - \sin^{2n} bx}{x} dx = -\frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} \quad [a, b > 0; n=1, 2, 3, \dots].$$

$$20. \int_0^{\infty} \frac{a \sin bx - b \sin ax}{x^2} dx = ab \ln \frac{a}{b} \quad [a, b > 0].$$

$$21. \int_0^{\infty} \frac{\cos ax - \cos bz}{x(x+z)} dx = \\ = \frac{1}{z} \left[\text{ci}(az) \cos az + \text{si}(az) \sin az - \text{ci}(bz) \cos bz - \text{si}(bz) \sin bz + \ln \frac{b}{a} \right] \\ [a, b > 0; |\arg z| < \pi].$$

$$22. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2(x^2+z^2)} dx = \frac{\pi}{2z^3} [(b-a)z + e^{-bz} - e^{-az}] \quad [a, b, \text{Re} z > 0].$$

$$23. \int_0^{\infty} \frac{\cos ax \pm \sin ax}{x^{1/2}(x^2+z^2)} dx = \frac{\pi e^{-az}}{z \sqrt{2}} z^{\pm 1/2} \quad [\text{Re} z > 0; a \geq 0].$$

$$24. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^m} dx = J_{mn}^n$$

$$J_{5/2}^1 = \frac{\sqrt{2\pi}}{3}, \quad J_4^2 = \frac{\pi}{6}, \quad J_5^2 = \frac{1}{4}, \quad J_6^2 = \frac{\pi}{15},$$

$$J_4^3 = \frac{1}{4} \ln 3 + \frac{1}{3}, \quad J_5^3 = \frac{3\pi}{32}, \quad J_6^3 = \frac{9}{40} \ln 3 - \frac{1}{10},$$

$$J_7^3 = \frac{5}{192} \pi, \quad J_8^3 = \frac{23}{280} - \frac{27}{1120} \ln 3, \quad J_9^3 = \frac{47}{2560} \pi.$$

$$25. \int_0^{\infty} \frac{3 - 4 \sin^2 ax}{x} \sin^2 ax dx = \frac{\ln 2}{2} \quad [a > 0].$$

$$26. \int_0^{\infty} \left(\cos ax \pm \cos \frac{a}{x} \right) \frac{dx}{1 \pm x^2} = \pi \begin{cases} \exp(-a) \\ \sin a \end{cases} \quad [a > 0].$$

$$27. \int_0^{\pi/4} \frac{\sin x - x \cos x}{(\sin 2x)^{3/2}} dx = \frac{\pi}{2\sqrt{2}} (\sqrt{2} - 1).$$

$$28. \int_0^{\pi/4} \left[\left(\frac{\pi}{2} - x \right) \text{tg} x - \text{ctg} x \right] \frac{dx}{\cos 2x} = -\frac{\pi}{4} \ln 2.$$

$$29. \int_0^{\infty} \frac{1}{x} \sum_{k=1}^n a_k \cos b_k x dx = - \sum_{k=1}^n a_k \ln b_k \quad \left[b_k > 0, \sum_{k=1}^n a_k = 0 \right].$$

$$30. \int_0^{\infty} \frac{\cos^n ax \cos (anx) - \cos^n bx \cos (bnx)}{x} dx = (1 - 2^{-n}) \ln \frac{b}{a} \\ [a, b > 0; n = 1, 2, 3, \dots].$$

$$31. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^m} \sin bx \, dx = I_m^n \quad [b > 0].$$

$$I_1^1 = \frac{|1-b^2|}{4} \ln \frac{1+b}{|1-b|} + \frac{b}{2} \quad [b \neq 1], \quad I_1^1 = \frac{1}{2} \quad [b = 1];$$

$$I_4^1 = \frac{\pi}{4} b \left(1 - \frac{b^2}{3}\right) \quad [b \leq 1], \quad I_4^1 = \frac{\pi}{6} \quad [b \geq 1];$$

$$I_2^2 = \frac{b}{4} \ln \frac{|4-b^2|}{b^2} + \frac{2-b^2}{b(4-b^2)} \quad [b \neq 2];$$

$$I_3^2 = \frac{\pi}{8} (2-b^2) \quad [b < 2], \quad I_3^2 = -\frac{\pi}{8} \quad [b = 2], \quad I_3^2 = 0 \quad [b > 2];$$

$$I_4^2 = \frac{b(b^2-6)}{24} \ln \frac{b^2}{|4-b^2|} + \frac{1}{6} \ln \frac{b+2}{|b-2|} - \frac{b}{6} \quad [b \neq 2];$$

$$I_4^2 = \frac{1}{3} (\ln 2 - 1) \quad [b = 2]; \quad I_5^2 = \frac{\pi}{96} b (b^3 - 12b + 16) \quad [b \leq 2];$$

$$I_5^2 = 0 \quad [b \geq 2];$$

$$I_6^2 = \frac{b^3}{480} (20-b^2) \ln \frac{b^2}{|4-b^2|} + \frac{4-5b^2}{60} \ln \frac{2+b}{|2-b|} + b \frac{22+b^2}{120} \quad [b \neq 2];$$

$$I_6^2 = \frac{13}{30} - \frac{8}{15} \ln 2 \quad [b = 2]; \quad I_7^2 = \frac{\pi}{2880} (192b - 80b^3 + 30b^4 - b^6) \quad [b \leq 2];$$

$$I_7^2 = \frac{\pi}{18} \quad [b \geq 2].$$

$$32. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^m} \cos bx \, dx = J_m^n \quad [b > 0].$$

$$J_2^1 = \frac{b}{2} \ln \frac{|1-b|}{1+b} + 1 \quad [b \neq 1]; \quad J_2^1 = \frac{\pi}{4} (1-b^2) \quad [b \leq 1];$$

$$J_2^1 = 0 \quad [b \geq 1]; \quad J_2^2 = -\frac{\pi}{4} b \quad [b < 2], \quad J_2^2 = 0 \quad [b > 2];$$

$$J_3^2 = \frac{b^2-2}{8} \ln \frac{b^2}{|4-b^2|} - \frac{1}{2} \quad [b \neq 2]; \quad J_4^2 = \frac{\pi}{24} (b^3 - 6b + 4) \quad [b \leq 2];$$

$$J_4^2 = 0 \quad [b \geq 2]; \quad J_5^2 = \frac{b^2}{96} (12-b^2) \ln \frac{b^2}{|4-b^2|} - \frac{b}{6} \ln \frac{2+b}{|2-b|} + \frac{b^2+6}{24} \quad [b \neq 2];$$

$$J_5^2 = \frac{5}{12} - \frac{2}{3} \ln 2 \quad [b = 2].$$

2.5.30. Интегралы от $e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}^n$.

$$1. \int_0^{\pi/2} e^{-px} \sin^n x \, dx = p^{1-2\lambda} n! \prod_{k=0}^{[n/2]} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ 1 - p^{2\lambda} e^{\pm p\pi/2} \left[1 + \sum_{k=1}^{[n/2]} \frac{(p^2 + 4\lambda^2)(p^2 + 4(\lambda+1)^2) \dots (p^2 + 4(\lambda+k-1)^2)}{(2k+2\lambda)!} \right] \right\}$$

$$\left[\begin{Bmatrix} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{Bmatrix}; \quad |\arg p| < \pi \right].$$

$$2. \int_0^{\pi/2} e^{-px} \cos^n x dx = p^{1-2\lambda} n! \prod_{k=0}^{[n/2]} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ (-1)^{n-1} e^{-p\pi/2} + p^{2\lambda} \left[1 + \sum_{k=1}^{[n/2]} \frac{(p^2 + 4\lambda^2)(p^2 + 4(\lambda+1)^2) \dots (p^2 + 4(\lambda+k-1)^2)}{(2k+2\lambda)!} \right] \right\}$$

$$\left[\begin{cases} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{cases}; |\arg p| < \pi \right].$$

$$3. \int_0^{\pi} e^{px} \sin^v x dx = \frac{\pi e^{-\pi p/2}}{2^v (v+1) B\left(\frac{v+ip}{2} + 1, \frac{v-ip}{2} + 1\right)} \quad [\operatorname{Re} v > -1].$$

$$4. \int_0^{2\pi} e^{imx} \begin{Bmatrix} \sin nx \\ \cos nx \end{Bmatrix} dx = \pi \begin{Bmatrix} i \\ 1 \end{Bmatrix} \delta_{m,n}.$$

$$5. \int_0^{\pi m} e^{-px} \sin^n x dx = n! p^{1-2\lambda} [1 - (-1)^{2\lambda m} e^{-mp\pi}] \prod_{k=0}^{[n/2]} [p^2 + (2k+2\lambda)^2]^{-1}$$

$$\left[\begin{cases} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{cases}; |\arg p| < \pi \right].$$

$$6. \int_{\pi/2}^{\pi(m+1/2)} e^{-px} \cos^n x dx =$$

$$= n! e^{-p\pi/2} (-p)^{1-2\lambda} \{ \exp[-m\pi(p+2i\lambda)] - 1 \} \prod_{j=0}^{[n/2]} [p^2 + (2j+2\lambda)^2]^{-1}$$

$$\left[\begin{cases} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{cases}; |\arg p| < \pi \right].$$

$$7. \int_{\pi/2}^{\infty} e^{-px} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^n dx = A n! e^{-p\pi/2} \prod_{k=0}^{[n/2]} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ 1 + \delta \left[\frac{p^2 + 4\lambda^2}{(2+2\lambda)!} + \frac{[p^2 + 4\lambda^2][p^2 + (2\lambda+2)^2]}{(4+2\lambda)!} + \dots \right. \right.$$

$$\left. \dots + \frac{[p^2 + 4\lambda^2][p^2 + (2\lambda+2)^2] \dots [p^2 + (n-2)^2]}{n!} \right\}$$

$$\left[\begin{cases} A = -(-p)^{\delta}, & \lambda = 1/2, & n = 1, 3, 5, \dots \\ A = p, & \lambda = 0, & n = 2, 4, 6, \dots \end{cases}; \operatorname{Re} p > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$8. \int_0^{\infty} e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{1}{b^2 + p^2} \begin{Bmatrix} b \\ p \end{Bmatrix} \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$9. \int_0^{\infty} e^{-px} \sin^2 bx dx = 2b^2 p^{-1} (p^2 + 4b^2)^{-1} \quad [\operatorname{Re} p > 2|\operatorname{Im} b|].$$

$$10. \int_0^{\infty} e^{-px} \sin^n bx dx = b^n p^{1-2\lambda} n! \prod_{k=0}^{[n/2]} [p^2 + b^2 (2k+2\lambda)^2]^{-1}$$

$$\left[\begin{cases} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{cases}; \operatorname{Re} p > n|\operatorname{Im} b| \right].$$

$$11. \int_0^{\infty} e^{-px} \cos^n bx \, dx =$$

$$= 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \frac{p}{p^2 + (n-2k)^2 b^2} + \frac{1 + (-1)^n}{2} \cdot \frac{1}{2^n p} \left(\left[\frac{n+k}{2} \right] \right)$$

[Re $p > n | \operatorname{Im} b |$; $n = 1, 2, 3, \dots$].

2.5.31. Интегралы от $x^\alpha e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/2} x^n e^{-px} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = (-1)^n \frac{d^n}{dp^n} \left[\frac{p^{1-\delta} + (-p)^\delta e^{-p\pi/2}}{1+p^2} \right].$$

$$2. \int_0^{\pi} x^n e^{-px} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = (-1)^n \frac{d^n}{dp^n} \left(p^{1-\delta} \frac{1 + e^{-p\pi}}{1+p^2} \right).$$

$$3. \int_{\pi/2}^{\infty} x e^{-px} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \frac{e^{-p\pi/2}}{(p^2+1)^2} \left[\frac{\pi p^\delta}{2} (p^2+1) + \left\{ \frac{p^2-1}{2p} \right\} \right] \quad [\operatorname{Re} p > 0].$$

$$4. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{b^\delta}{p^{\alpha+\delta}} \Gamma(\alpha+\delta) {}_2F_1 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2} + \delta; \delta + \frac{1}{2}; -\frac{b^2}{p^2} \right) = \frac{\Gamma(\alpha)}{(b^2+p^2)^{\alpha/2}} \begin{Bmatrix} \sin c \\ \cos c \end{Bmatrix}$$

[$c = \alpha \operatorname{arctg}(b/p)$; $\operatorname{Re} \alpha > -\delta$; $\operatorname{Re} p > | \operatorname{Im} b |$].

$$5. \int_0^{\infty} x^n e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = (-1)^n \frac{\partial^n}{\partial p^n} \frac{b^\delta p^{1-\delta}}{p^2 + b^2} =$$

$$= n! \left(\frac{p}{p^2 + b^2} \right)^{n+1} \sum_{k=0}^{[(n+1-\delta)/2]} (-1)^k \binom{n+1}{2k+\delta} \left(\frac{b}{p} \right)^{2k+\delta} \quad [\operatorname{Re} p > | \operatorname{Im} b |].$$

$$6. \int_0^{\infty} x^n e^{-px} \sin bx \, dx = I_n \quad [\operatorname{Re} p > | \operatorname{Im} b |; n = -1, 0, 1, 2, 3, 4].$$

$$I_{-1} = \operatorname{arctg} \frac{b}{p}, \quad I_0 = \frac{b}{b^2 + p^2}, \quad I_1 = \frac{2pb}{(p^2 + b^2)^2}, \quad I_2 = \frac{2b(3p^2 - b^2)}{(b^2 + p^2)^3},$$

$$I_3 = \frac{24bp(p^2 - b^2)}{(b^2 + p^2)^4}, \quad I_4 = \frac{24b}{(b^2 + p^2)^5} (5p^4 - 10p^2b^2 + b^4).$$

$$7. \int_0^{\infty} x^n e^{-px} \cos bx \, dx = J_n \quad [\operatorname{Re} p > | \operatorname{Im} b |].$$

$$J_0 = \frac{p}{p^2 + b^2}, \quad J_1 = \frac{p^2 - b^2}{(b^2 + p^2)^2}, \quad J_2 = \frac{2p(p^2 - 3b^2)}{(b^2 + p^2)^3},$$

$$J_3 = \frac{6}{(b^2 + p^2)^3} (p^4 - 6p^2b^2 + b^4), \quad J_4 = \frac{24p}{(b^2 + p^2)^5} (p^4 - 10p^2b^2 + 5b^4).$$

$$8. \int_0^\infty x^{-3/2} e^{-px} \sin bx \, dx = \sqrt{2\pi} (\sqrt{p^2 + b^2} - p)^{1/2} \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$9. \int_0^\infty x^{n-1/2} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{\partial^n}{\partial p^n} \left(\frac{\sqrt{b^2 + p^2} - p}{b^2 + p^2} \right)^{1/2} \\ [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$10. \int_0^\infty x^{\alpha-1} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = 2^{1-n} \Gamma(\alpha) \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]-k} \binom{n}{k} [b^2(n-2k)^2 + p^2]^{-\alpha/2} \left\{ \frac{\sin c}{\cos c} \right\} + \\ + \frac{1 + (-1)^n}{2} 2^{-n} p^{-\alpha} \Gamma(\alpha) \binom{n}{n/2} \\ [c = \alpha \operatorname{arctg}[b(n-2k)/p]; \operatorname{Re} \alpha > -\delta; \operatorname{Re} p > n |\operatorname{Im} b|].$$

$$11. \int_0^\infty \frac{e^{-px}}{x^m} \sin^n bx \, dx = I_m^n \quad [\operatorname{Re} p > n |\operatorname{Im} b|; m \leq n; m, n = 1, 2, 3, \dots],$$

$$I_1^{2n} = \frac{(-1)^{n+1}}{4^n} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \ln [p^2 + 4(n-k)^2 b^2] - 2^{-2n} \binom{2n}{n} \ln p,$$

$$I_1^{2n+1} = \frac{(-1)^n}{4^n} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \operatorname{arctg} \left[\frac{b}{p} (2n-2k+1) \right],$$

$$I_1^1 = \operatorname{arctg} \frac{b}{p}, \quad I_1^2 = \frac{1}{4} \ln \left(1 + \frac{4b^2}{p^2} \right),$$

$$I_2^2 = b \operatorname{arctg} \frac{2b}{p} - \frac{p}{4} \ln \left(1 + \frac{4b^2}{p^2} \right).$$

2.5.32. Интегралы от $A(x) e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_{-a}^a \sqrt{a^2 - x^2} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= \frac{i^\delta a^2}{2} \{ I_0[a(p-ib)] \mp I_0[a(p+ib)] \}$$

$$[a, \operatorname{Re} p > 0; |\arg b| < \pi].$$

$$2. \int_{-a}^a x \sqrt{a^2 - x^2} e^{-px} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{i\delta}{2} a^3 \{I_1[a(p - ib)] \mp I_1[a(p + ib)]\}$$

[a, Re p > 0; |arg b| < π].

$$3. \int_0^\infty \frac{(x+z)^n}{x} e^{-px} \sin bx dx = (-1)^n b p^{-n-1} \sum_{k=0}^\infty (-1)^k \left(\frac{b}{2p}\right)^{2k} L_n^{-n-2k-1}(pz)$$

[Re p > |Im b|; |arg z| < π].

$$4. \int_0^\infty \frac{x e^{-px}}{x^4 + z^4} \cos px dx = \frac{\pi}{4z^2} e^{-pz} \sqrt{2}$$

[p > 0; |arg z| < π/4].

$$5. \int_0^\infty \frac{x e^{-px}}{x^4 - y^4} \cos px dx = -\frac{\pi}{4y^2} e^{-py} \sin py$$

[y, p > 0].

$$6. \int_0^{\pi/2} e^{ipx} \sin^\mu x \cos^\nu x dx = 2^{-\mu-\nu-1} \times$$

$$\times \left\{ e^{\pi i(p-\nu-1)/2} B\left(\frac{p-\mu-\nu}{2}, \nu+1\right) {}_2F_1\left(-\mu, \frac{p-\mu-\nu}{2}; 1+\frac{p-\mu+\nu}{2}; -1\right) + \right.$$

$$\left. + e^{\pi i(\mu+1)/2} B\left(\frac{p-\mu-\nu}{2}, \mu+1\right) {}_2F_1\left(-\nu, \frac{p-\mu-\nu}{2}; 1+\frac{p+\mu-\nu}{2}; -1\right) \right\}$$

[Re μ, Re ν > -1].

2.5.33. Интегралы от $x^\alpha e^{px} \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix}^\mu \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}^\nu$, $x^\alpha e^{px} \sin^\mu ax \cos^\nu bx$.

$$1. \int_0^{\pi/2} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x dx = e^{\pi i \mu/2} B(\mu, \nu) =$$

$$= 2^{1-\mu-\nu} e^{\pi i \mu/2} \left\{ \frac{1}{\mu} {}_2F_1(1-\nu, 1; 1+\mu; -1) + \frac{1}{\nu} {}_2F_1(1-\mu, 1; 1+\nu; -1) \right\}$$

[Re μ, Re ν > 0].

$$2. \int_0^\pi e^{ipx} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{\pi i e^{\pi i(p-\nu)/2}}{2^{\mu+\nu-2} \mu B\left(\frac{\mu+\nu-p}{2}, \frac{\mu-\nu+p}{2}+1\right)} \times$$

$$\times {}_2F_1\left(1-\nu, \frac{p-\mu-\nu}{2}+1; 1+\frac{\mu-\nu+p}{2}; -1\right)$$

[Re μ, Re ν > 0].

$$3. \int_0^\pi e^{-px} \sin^{\mu-1} x \cos x dx = \frac{\pi p 2^{2-\mu} e^{-p\pi/2}}{(\mu^2 + p^2) B\left(\frac{\mu+ip}{2}, \frac{\mu-ip}{2}\right)}$$

[Re μ > 0].

$$4. \int_0^\infty e^{-px} \sin^n ax \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx =$$

$$= \frac{2^{-n-2}}{a(n+1)} \exp\left(\frac{1-\delta \mp n}{2} \pi i\right) \left[\left(\frac{b+ip+na}{2}\right)^{-1} \pm (-1)^n \left(\frac{b-ip+na}{2}\right)^{-1} \right]$$

[a, b, Re p > 0; δ = {1 \choose 0}].

$$5. \int_0^{\infty} e^{-px} \frac{\sin bx}{\sin cx} dx = \frac{1}{2c} \left[\psi \left(\frac{c+b-ip}{2c} \right) - \psi \left(\frac{c-b-ip}{2c} \right) \right]$$

[b, Re p > 0; c ≠ 0].

$$6. \int_0^{\infty} e^{-px} \frac{\sin nx}{\sin x} dx = 2p \sum_{k=1}^{[n/2]} \left[p^2 + \left(2k - \frac{1+(-1)^n}{2} \right)^2 \right]^{-1} + \frac{1-(-1)^n}{2} \frac{1}{p}$$

[Re p > 0].

$$7. \int_0^{\infty} e^{-px} \cos(2n+1)x \operatorname{tg} x dx = \frac{2n+1}{p^2 + (2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k (2k+1)}{p^2 + (2k+1)^2}$$

[Re p > 0].

$$8. \int_0^{\infty} \frac{1}{x} e^{-px} \sin ax \sin bx dx = \frac{1}{4} \ln \frac{p^2 + (a+b)^2}{p^2 + (a-b)^2} \quad [\operatorname{Re} p > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$9. \int_0^{\infty} \frac{1}{x^2} e^{-px} \sin ax \sin bx dx =$$

$$= \frac{a}{2} \operatorname{arctg} \frac{2pb}{p^2 + a^2 - b^2} + \frac{b}{2} \operatorname{arctg} \frac{2pa}{p^2 + b^2 - a^2} + \frac{p}{4} \ln \frac{p^2 + (a-b)^2}{p^2 + (a+b)^2}$$

[Re p ≥ |Im a| + |Im b|].

$$10. \int_0^{\infty} \frac{1}{x} e^{-px} \sin ax \cos bx dx = \frac{1}{2} \operatorname{arctg} \frac{2ap}{p^2 + b^2 - a^2} + \left\{ \begin{matrix} 0 \\ \pi/2 \end{matrix} \right\}$$

[Re p > |Im a| + |Im b|; {p^2 + b^2 ≥ a^2} / {p^2 + b^2 < a^2}].

2.5.34. Интегралы от $x^\alpha R(e^{ax}) \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} + 1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \Gamma(\alpha) \left\{ b^{-\alpha} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} + \right.$$

$$\left. + \frac{i^\delta}{2} (2a)^{-\alpha} \left[\zeta \left(\alpha, \frac{a+ib}{2a} \right) \mp \zeta \left(\alpha, \frac{a-ib}{2a} \right) - \zeta \left(\alpha, \frac{ib}{2a} \right) \pm \zeta \left(\alpha, -\frac{ib}{2a} \right) \right] \right\}$$

[Re α > -δ; Re a > |Im b|].

$$2. \int_0^{\infty} \frac{\sin bx}{e^{ax} + 1} dx = \frac{1}{2b} - \frac{\pi}{2a \operatorname{sh}(b\pi/a)} \quad [\operatorname{Re} a > |\operatorname{Im} b|].$$

$$3. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} - 1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{i^\delta}{2} e^{-\alpha} \Gamma(\alpha) \left[\zeta \left(\alpha, 1 + i \frac{b}{a} \right) \mp \zeta \left(\alpha, 1 - i \frac{b}{a} \right) \right]$$

[Re α > 1 - δ; Re a > |Im b|].

$$4. \int_0^{\infty} \frac{\sin bx}{e^{ax} - 1} dx = \frac{\pi}{2a} \operatorname{cth} \frac{b\pi}{a} - \frac{1}{2b} \quad [\operatorname{Re} a > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{x \cos bx}{e^{ax} - 1} dx = \frac{1}{2b^2} - \frac{\pi^2}{2a^2 \operatorname{sh}^2(b\pi/a)} \quad [\operatorname{Re} a > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{\sin^2 bx}{e^{ax} - 1} dx = \frac{1}{2a} \left[\zeta + \operatorname{Re} \psi \left(2i \frac{b}{a} \right) \right] \quad [\operatorname{Re} a > 2|\operatorname{Im} b|].$$

$$7. \int_0^{\infty} \frac{e^{-ax}}{(e^{-ax} + 1)^2} \cos bx dx = \frac{\pi b}{2a^2 \operatorname{sh}(b\pi/2)} \quad [\operatorname{Re} a > |\operatorname{Im} b|].$$

$$8. \int_0^{\infty} e^{-px} (1 - e^{-ax})^{\beta-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{(-i)^{\delta}}{2a} \left[B\left(\beta, \frac{p-ib}{a}\right) \mp B\left(\beta, \frac{p+ib}{a}\right) \right] \\ [\operatorname{Re} a > 0; \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} \beta > -\delta].$$

$$9. \int_0^{\infty} \frac{\sin bx}{e^{ax} - e^{cx}} dx = \frac{1}{2i(a-c)} \left[\psi\left(\frac{a+ib}{a-c}\right) - \psi\left(\frac{a-ib}{a-c}\right) \right] \\ [\max(\operatorname{Re} a, \operatorname{Re} c) > |\operatorname{Im} b|].$$

$$10. \int_0^{\infty} \frac{x^{2n} \begin{Bmatrix} \sin bx \\ x \cos bx \end{Bmatrix}}{e^{mcx} - e^{(m-2)c x}} dx = \\ = (-1)^n \frac{\partial^{2n+1-\delta}}{\partial b^{2n+1-\delta}} \left[\frac{\pi}{4c} \begin{Bmatrix} \operatorname{th}[b\pi/(2c)] \\ \operatorname{cth}[b\pi/(2c)] \end{Bmatrix} - \frac{1}{2b} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \sum_{k=1}^{[(m-1)/2]} \frac{b}{b^2 + c^2 (2k-2\lambda)^2} \right] \\ [b, c > 0; \begin{cases} \lambda=1/2, & m=1, 3, 5, \dots \\ \lambda=0, & m=2, 4, 6, \dots \end{cases}].$$

$$11. \int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x + 1)^{\rho}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = I_{\rho}^{\pm} \quad [\operatorname{Im} \rho = 0; 0 < \operatorname{Re} a + |\operatorname{Im} b| < \rho].$$

$$I_{\rho}^{\pm} = \frac{1}{\Gamma(\rho)} \begin{Bmatrix} \operatorname{Im} z \\ \operatorname{Re} z \end{Bmatrix}, \quad z = \Gamma(a+ib) \Gamma(\rho-a-ib),$$

$$I_1^{\pm} = \mp \frac{\pi}{\sin^2 a\pi \operatorname{ch}^2 b\pi + \cos^2 a\pi \operatorname{sh}^2 b\pi} \begin{Bmatrix} \cos a\pi \operatorname{sh} b\pi \\ \sin a\pi \operatorname{ch} b\pi \end{Bmatrix},$$

$$I_2^{\pm} = - \frac{\pi}{\sin^2 a\pi \operatorname{ch}^2 b\pi + \cos^2 a\pi \operatorname{sh}^2 b\pi} \left[(1-a) \begin{Bmatrix} \cos a\pi \operatorname{sh} b\pi \\ \sin(-a\pi) \operatorname{ch} b\pi \end{Bmatrix} + \right. \\ \left. + b \begin{Bmatrix} \sin a\pi \operatorname{ch} b\pi \\ \cos a\pi \operatorname{sh} b\pi \end{Bmatrix} \right],$$

$$I_n^{\pm} = \frac{1}{n-1} [(n-1-a) I_{n-1}^{\pm} \mp b I_{n-1}^{\mp}] \quad [n=2, 3, 4, \dots].$$

2.5.35. Интегралы от $f(x, e^{ax}, \sin bx, \cos bx)$.

$$1. \int_0^{\infty} e^{-x} \operatorname{sgn}(\sin x) dx = -\operatorname{th} \frac{\pi}{2}.$$

$$2. \int_0^{\infty} e^{-px} |\sin bx| dx = \frac{b}{p^2 + b^2} \operatorname{cth} \frac{p\pi}{2b} \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$3. \int_0^{\infty} e^{-px} |\cos bx| dx = \frac{1}{p^2 + b^2} \left\{ p + \frac{b}{\operatorname{sh} [p\pi/(2b)]} \right\} \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$4. \int_0^{\infty} e^{-px} |\sin bx|^v dx = \frac{\pi}{(v+1) 2^{v+1} b} \left[\operatorname{sh} \frac{p\pi}{2b} B \left(1 + \frac{v}{2} + \frac{ip}{2b}, 1 + \frac{v}{2} - \frac{ip}{2b} \right) \right]^{-1} \\ [2 \operatorname{Re} p > \operatorname{Re} v, |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} a \pm \cos x} dx = \frac{p}{\operatorname{sh} a} \left[\frac{1}{p^2 + 1} + 2 \sum_{k=1}^{\infty} \frac{(\mp 1)^k e^{-ka}}{p^2 + k^2} \right] \quad [a, \operatorname{Re} p > 0].$$

$$6. \int_0^{\pi/2} \frac{(e^{ix} \cos x)^n + (e^{-ix} \sin x)^n}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{ab} \left(\frac{b}{a+b} \right)^n \quad [a, b > 0].$$

$$7. \int_{-\pi/2}^{\pi/2} e^{ipx} \cos^{\mu-1} x (a^2 e^{ix} + b^2 e^{-ix})^v dx = \\ = \frac{\pi c^{2v} F(-v, (\mu+v \mp p-1)/2; (\mu-v \pm p+1)/2; (a^2/b^2)^{\pm 1})}{2^{\mu-1} \mu B((\pm p + \mu - v + 1)/2, (v + \mu \mp p + 1)/2)} \\ [c = \max(a, b); \operatorname{Re} \mu > 0; \left\{ \begin{array}{l} |a| < |b| \\ |a| > |b| \end{array} \right\}].$$

$$8. \int_{-\pi}^{\pi} (a + \cos x)^v e^{inx} dx = \frac{2\pi}{(v+1)_n} (a^2 - 1)^{v/2} P_v^n \left(\frac{a}{\sqrt{a^2 - 1}} \right) \quad [a > 1].$$

2.5.36. Интегралы, содержащие $e^{-ax^2 - cx} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-ax^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\frac{\pi}{4a}} e^{-b^2/(4a)} \begin{Bmatrix} -i \operatorname{erf} [ib/(2\sqrt{a})] \\ 1 \end{Bmatrix} \quad [\operatorname{Re} a > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-ax^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{b^{\delta}}{2} a^{-(\alpha+\delta)/2} \Gamma \left(\frac{\alpha+\delta}{2} \right) {}_1F_1 \left(\frac{\alpha+\delta}{2}; \delta + \frac{1}{2}; -\frac{b^2}{4a} \right) = \\ = 2^{-(\alpha+3)/2} \sqrt{\pi} \begin{Bmatrix} \sec(\alpha\pi/2) \\ \operatorname{cosec}(\alpha\pi/2) \end{Bmatrix} a^{-\alpha/2} \exp \left(-\frac{b^2}{8a} \right) \left[D_{\alpha-1} \left(-\frac{b}{\sqrt{2a}} \right) \mp D_{\alpha-1} \left(\frac{b}{\sqrt{2a}} \right) \right] \\ [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\delta].$$

$$3. \int_0^{\infty} x^{-1/2} e^{-ax^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\pi}{2} \sqrt{\frac{b}{2a}} \exp\left(-\frac{b^2}{8a}\right) I_{\pm 1/4}\left(\frac{b^2}{8a}\right) \quad [\operatorname{Re} a > 0].$$

$$4. \int_0^{\infty} x^{1/2} e^{-ax^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \pm \frac{\pi}{4} \left(\frac{b}{2a}\right)^{3/2} \exp\left(-\frac{b^2}{8a}\right) \left[I_{\mp 1/4}\left(\frac{b^2}{8a}\right) - I_{\pm 3/4}\left(\frac{b^2}{8a}\right) \right] \quad [\operatorname{Re} a > 0].$$

$$5. \int_0^{\infty} x^{2n+\delta} e^{-ax^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = (-1)^n \sqrt{\pi} 2^{-2n-1-\delta} a^{-(2n+1+\delta)/2} \exp\left(-\frac{b^2}{4a}\right) H_{2n+\delta}\left(\frac{b}{2\sqrt{a}}\right) \\ \quad [\operatorname{Re} a > 0, \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$6. \int_0^{\infty} \frac{1}{x} e^{-ax^2} \sin bx \, dx = \frac{\pi}{2} \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) \quad [\operatorname{Re} a > 0].$$

$$7. \int_0^{\infty} x e^{-ax^2} \cos bx \, dx = \frac{1}{2a} + \frac{ib}{2} \sqrt{\frac{\pi}{a^3}} \exp\left(-\frac{b^2}{4a}\right) \operatorname{erf}\left(\frac{ib}{2\sqrt{a}}\right) \quad [\operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} e^{-ax^2-cx} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \frac{i^{\delta}}{4} \sqrt{\frac{\pi}{a}} \left\{ \exp\left[\frac{(c+ib)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+ib}{2\sqrt{a}}\right) \mp \exp\left[\frac{(c-ib)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-ib}{2\sqrt{a}}\right) \right\} \\ \quad [\operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} x^{\alpha-1} e^{-ax^2-cx} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \frac{i^{\delta} \Gamma(\alpha)}{2(2a)^{\alpha/2}} \exp\left(\frac{c^2-b^2}{8a}\right) \left[\exp\left(\frac{ibc}{4a}\right) D_{-\alpha}\left(\frac{c+ib}{\sqrt{2a}}\right) \mp \exp\left(-\frac{ibc}{4a}\right) D_{-\alpha}\left(\frac{c-ib}{\sqrt{2a}}\right) \right] \\ \quad [\operatorname{Re} a > 0, \operatorname{Re} \alpha > -\delta].$$

$$10. \int_0^{\infty} x e^{-ax^2-cx} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = -\frac{i^{\delta}}{8a} \sqrt{\frac{\pi}{a}} \left\{ (c+ib) \exp\left[\mp \frac{(c+ib)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+ib}{2\sqrt{a}}\right) \mp \right. \\ \left. \mp (c-ib) \exp\left[\mp \frac{(c-ib)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-ib}{2\sqrt{a}}\right) \right\} + \frac{1-\delta}{2a} \quad [\operatorname{Re} a > 0].$$

$$11. \int_0^{\infty} x^{(1 \pm 1)/2} \frac{e^{-ax^2}}{x^2 + z^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \frac{\pi}{4} z^{\delta-1} e^{az^2} \left[\pm e^{-bz} \operatorname{erf} \left(z \sqrt{a} - \frac{b}{2\sqrt{a}} \right) - e^{bz} \operatorname{erf} \left(z \sqrt{a} + \frac{b}{2\sqrt{a}} \right) + 2 \frac{\{\operatorname{sh} bz\}}{\{\operatorname{ch} bz\}} \right] \\ [\operatorname{Re} a, \operatorname{Re} z > 0].$$

$$12. \int_0^{\infty} e^{-ax^2} \sin(2n+1)x \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{-1} dx = \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^n (\pm 1)^k \exp \left(-\frac{k^2}{a} \right) \right] \\ [\operatorname{Re} a > 0].$$

$$13. \int_0^{\infty} x e^{-ax^2} \operatorname{tg} cx dx = \frac{c}{a} \sqrt{\frac{\pi}{a}} \sum_{k=1}^{\infty} (-1)^k k \exp \left(-\frac{c^2 k^2}{a} \right) \quad [\operatorname{Re}(c^2/a) > 0].$$

$$14. \int_0^{\infty} \frac{e^{-ax^2} dx}{1 - 2c \cos x + c^2} = \frac{1}{|1 - c^2|} \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \exp \left(-\frac{k^2}{2a} \right) c^{\pm k} \right] \\ [\operatorname{Re} a > 0, \{ \frac{c}{c_1} < 1 \}].$$

2.5.37. Интегралы от $x^{\alpha} e^{-px - q/x} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-q/x} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = i^{\delta} \left(\frac{q}{b} \right)^{\alpha/2} \left[\exp \left(-\frac{\alpha \pi i}{4} \right) K_{\alpha} (2e^{\pi i/4} \sqrt{qb}) \mp \exp \left(\frac{\alpha \pi i}{4} \right) K_{\alpha} (2e^{-\pi i/4} \sqrt{qb}) \right] \\ [b, \operatorname{Re} q > 0, \operatorname{Re} \alpha < 1].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px - q/x} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = i^{\delta} q^{\alpha/2} [(p+ib)^{-\alpha/2} K_{\alpha} (2\sqrt{q} \sqrt{p+ib}) \mp (p-ib)^{-\alpha/2} K_{\alpha} (2\sqrt{q} \sqrt{p-ib})] \\ [\operatorname{Re} p > \operatorname{Im} b, \operatorname{Re} q > 0]$$

$$3. \int_0^{\infty} x^{-3/2} e^{-px - q/x} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{q} z_{\pm}} \begin{Bmatrix} \sin (2z_{\pm} \sqrt{q}) \\ \cos (2z_{\pm} \sqrt{q}) \end{Bmatrix} \\ [2z_{\pm}^2 = \sqrt{p^2 + b^2} \pm p, \operatorname{Re} p > \operatorname{Im} b; \operatorname{Re} q > 0].$$

$$4. \int_0^{\infty} x^{-1/2} e^{-px - q/x} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \sqrt{\frac{\pi}{p^2 + b^2}} e^{-2\sqrt{q} z_{\pm}} \left[z_{\pm} \begin{Bmatrix} \sin (2z_{\pm} \sqrt{q}) \\ \cos (2z_{\pm} \sqrt{q}) \end{Bmatrix} \pm z \begin{Bmatrix} \cos (2z_{\pm} \sqrt{q}) \\ \sin (2z_{\pm} \sqrt{q}) \end{Bmatrix} \right] \\ [2z_{\pm}^2 = \sqrt{p^2 + b^2} \pm p; \operatorname{Re} p > \operatorname{Im} b; \operatorname{Re} q > 0].$$

2.5.38. Интегралы, содержащие e^{-a/x^2} , $e^a \sqrt{x}$, $\begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$\begin{aligned} 1. \int_0^\infty x^{\alpha-1} e^{-ax^{-2}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \frac{\Gamma(\alpha)}{b^\alpha} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} {}_0F_2\left(1 - \frac{\alpha}{2}, \frac{1-\alpha}{2}; \frac{ab^2}{4}\right) + \\ + \frac{b^\delta}{2} a^{(\alpha+\delta)/2} \Gamma\left(-\frac{\alpha+\delta}{2}\right) {}_0F_2\left(1 + \frac{\alpha+\delta}{2}, \delta + \frac{1}{2}; \frac{ab^2}{4}\right) \\ [b, \operatorname{Re} a > 0, \operatorname{Re} \alpha < 1]. \end{aligned}$$

$$\begin{aligned} 2. \int_0^\infty x^{-2} e^{-ax^{-2}} \sin bx \, dx = \\ = b \sum_{k=0}^\infty \frac{(b^2 a)^k}{k! (2k+1)!} [\psi(2k+2) + 2^{-1} \psi(k+1) - \ln(b \sqrt{a})] \quad [b, \operatorname{Re} a > 0]. \end{aligned}$$

$$3. \int_0^\infty x^{-2} e^{-ax^{-2}} \cos bx \, dx = \frac{\pi}{2 \sqrt{a}} \sum_{k=0}^\infty \frac{(-b \sqrt{a})^k}{k! \Gamma[(k+1)/2]} \quad [b, \operatorname{Re} a > 0].$$

$$\begin{aligned} 4. \int_0^\infty e^{-p \sqrt{x}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \begin{Bmatrix} b^{-1} \\ 0 \end{Bmatrix} + p \sqrt{\frac{\pi}{2}} b^{-3/2} \left\{ \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \left[\frac{1}{2} - C(z) \right] \mp \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix} \left[\frac{1}{2} - S(z) \right] \right\} \\ [z = p^2/(4b); b, \operatorname{Re} p > 0]. \end{aligned}$$

$$\begin{aligned} 5. \int_0^\infty x^{\alpha-1} e^{-p \sqrt{x}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = i^\delta \Gamma(2\alpha) (2b)^{-\alpha} \left\{ \exp\left[-i\left(\frac{\alpha\pi}{2} + \frac{p^2}{8b}\right)\right] D_{2\alpha}\left(\frac{p}{\sqrt{2b}} e^{-\pi i/4}\right) \mp \right. \\ \left. \mp \exp\left[i\left(\frac{\alpha\pi}{2} + \frac{p^2}{8b}\right)\right] D_{-2\alpha}\left(\frac{p}{\sqrt{2b}} e^{\pi i/4}\right) \right\} \quad [b, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -\delta]. \end{aligned}$$

$$\begin{aligned} 6. \int_0^\infty x^{-3/4} e^{-p \sqrt{x}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \mp \frac{\pi}{2} \sqrt{\frac{p}{b}} \left[J_{1/4}\left(\frac{p^2}{8b}\right) \begin{Bmatrix} \cos c \\ \sin c \end{Bmatrix} \pm Y_{1/4}\left(\frac{p^2}{8b}\right) \begin{Bmatrix} \sin c \\ \cos c \end{Bmatrix} \right] \\ [c = p^2/(8b) + \pi/8; b, \operatorname{Re} p > 0]. \end{aligned}$$

$$\begin{aligned} 7. \int_0^\infty \frac{e^{-p \sqrt{x}}}{x} \sin bx \, dx = \pi \left[\frac{1}{2} - C(z) \right]^2 + \pi \left[\frac{1}{2} - S(z) \right]^2 \\ [z = p^2/(4b); b, \operatorname{Re} p > 0]. \end{aligned}$$

$$8. \int_0^{\infty} \frac{\sin bx}{e^{2\pi\sqrt{x}} - 1} dx = I(b) - \frac{1}{2b} + I\left(\frac{\pi^2}{b}\right) \sqrt{\frac{2\pi^3}{b^3}} \quad [\text{см. 2.5.38.9}].$$

$$9. \int_0^{\infty} \frac{\cos bx}{e^{2\pi\sqrt{x}} - 1} dx = I(b),$$

$$I(0) = \frac{1}{12}, \quad I\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}, \quad I(\pi) = \frac{2 - \sqrt{2}}{8}, \quad I(2\pi) = \frac{1}{16}.$$

$$10. \int_0^{\infty} \frac{x \cos(\pi x/2)}{e^{2\pi\sqrt{x}} - 1} dx = \frac{13 - 4\pi}{8\pi^2}.$$

$$11. \int_0^{\infty} \frac{x \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx = \frac{1}{64} \left(\frac{1}{2} - \frac{3}{\pi} + \frac{5}{\pi^2} \right).$$

$$12. \int_0^{\infty} \frac{x^2 \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx = \frac{1}{256} \left(1 - \frac{5}{\pi} - \frac{5}{\pi^2} \right).$$

$$13. \int_0^{\infty} \frac{x^3 \sin(\pi x/2)}{e^{2\pi\sqrt{x}} - 1} dx = \frac{17}{16} - \frac{8}{3\pi} - \frac{7}{\pi^2} + \frac{35}{2\pi^3} - \frac{105}{16\pi^4}.$$

$$14. \int_0^{\infty} \frac{x^3 \cos(\pi x/2)}{e^{2\pi\sqrt{x}} - 1} dx = -\frac{8}{3\pi} + \frac{7}{\pi^2} + \frac{35}{2\pi^3} + \frac{873}{16\pi^4}.$$

$$15. \int_0^{\infty} e^{-ax^3} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{i^{\delta} z}{2b} \left[e^{\pi i/4} S_{0, 1/3}(ze^{3\pi i/4}) \mp e^{-\pi i/4} S_{0, 1/3}(ze^{-3\pi i/4}) \right]$$

$$[z = 2(b/3)^{3/2} a^{-1/2}; \operatorname{Re} a > 0].$$

2.5.39. Интегралы, содержащие $e^{A(x)}$ и $\begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2 + z^2}} \exp\left(b \frac{z^2 - x^2}{z^2 + x^2}\right) \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix} dx =$$

$$= \frac{2^{\pm 1} z^{\alpha-3/2}}{\sqrt{\pi c}} \Gamma\left(\frac{\alpha + \delta}{2}\right) \Gamma\left(\frac{1 + \delta - \alpha}{2}\right) M_{\alpha/2-1/4, \pm 1/4}(b + \sqrt{b^2 - c^2 z^2}) \times$$

$$\times M_{\alpha/2-1/4, \pm 1/4}(b - \sqrt{b^2 - c^2 z^2}) \quad [b, c, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1 + \delta].$$

$$2. \int_0^{\infty} \exp(-p\sqrt{x^2 + z^2}) \cos bx dx = \frac{pz}{\sqrt{p^2 + b^2}} K_1(z\sqrt{p^2 + b^2})$$

$$[b, \operatorname{Re} p, \operatorname{Re} z > 0].$$

$$3. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \sin bx \, dx = J_{\alpha} \quad [b, \operatorname{Re} p, \operatorname{Re} z > 0, \operatorname{Re} \alpha > -1].$$

$$J_{2n} = (-1)^{n-1} z \frac{\partial^{2n-2}}{\partial b^{2n-2}} \left[\frac{b}{\sqrt{p^2+b^2}} K_1(z \sqrt{p^2+b^2}) \right].$$

$$4. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \cos bx \, dx = I_{\alpha} \quad [b, \operatorname{Re} \alpha, \operatorname{Re} p, \operatorname{Re} z > 0].$$

$$I_{2n+1} = (-1)^n \frac{\partial^{2n}}{\partial b^{2n}} [K_0(z \sqrt{p^2+b^2})].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left(x^2 + \frac{z^2}{2} \right) (x^2+z^2)^{\alpha/2-1} \exp(-b \sqrt{x^2+z^2}) \begin{cases} \sin bx \\ \cos bx \end{cases} dx =$$

$$= \left(\frac{2}{b} \right)^{\alpha-1/2} z^{\alpha+1/2} \Gamma \left[\frac{\alpha+\delta/2}{(\delta+1)/2-\alpha} \right] K_{\alpha+1/2}(bz) \quad [\operatorname{Re} \alpha > -\delta; b, \operatorname{Re} z > 0].$$

$$6. \int_0^{\infty} x^{1-\delta} (\sqrt{x^2+z^2}-z)^{\pm 1/2} \frac{\exp(-p \sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \begin{cases} \sin bx \\ \cos bx \end{cases} dx =$$

$$= b^{\delta} \sqrt{\frac{\pi}{2}} \frac{(\sqrt{p^2+b^2}+p)^{\mp 1/2}}{\sqrt{p^2+b^2}} \exp(-z \sqrt{p^2+b^2})$$

$$\left[b, \operatorname{Re} p, \operatorname{Re} z > 0; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

2.5.40. Интегралы, содержащие показательную функцию от показательной и тригонометрических функций и $\begin{cases} \sin bx \\ \cos bx \end{cases}$.

$$1. \int_{-\infty}^{\infty} \exp(ax - e^x) \begin{cases} \sin bx \\ \cos bx \end{cases} dx = |\Gamma(a+ib)| \begin{cases} \sin \varphi \\ \cos \varphi \end{cases}$$

$$[\varphi = \arg \Gamma(a+ib), \arg \Gamma(z) = -i \ln \frac{\Gamma(z)}{|\Gamma(z)|}; \operatorname{Re} a > |\operatorname{Im} b|].$$

$$2. \int_{-\pi/2}^{\pi/2} x e^{z \sin x} \cos x \, dx = \frac{\pi}{z} [\operatorname{ch} z - I_0(z)] \quad [|\arg z| < \pi].$$

$$3. \int_0^{\pi} e^{z \cos x} \cos nx \, dx = \pi I_n(z) \quad [|\arg z| < \pi].$$

$$4. \int_0^{\pi} x e^{z \cos x} \sin x \, dx = \frac{\pi}{z} [I_0(z) - e^{-z}] \quad [|\arg z| < \pi].$$

$$5. \int_0^{\pi} e^{iz \cos x} \cos nx \, dx = i^n \pi J_n(z) \quad [|\arg z| < \pi].$$

$$6. \int_0^{\pi} e^{iz \cos x} \sin^{v-1} x \, dx = \sqrt{\pi} \left(\frac{2}{z}\right)^{(v-1)/2} \Gamma\left(\frac{v}{2}\right) J_{(v-1)/2}(z)$$

[Re $v > 0$; $|\arg z| < \pi$].

$$7. \int_0^{\pi} e^{\pm i(vx - z \sin x)} \, dx = \pi [J_v(z) \pm iE_v(z)]$$

[$|\arg z| < \pi$].

$$8. \int_0^{\pi} \{e^{i[(v-1)x - z \sin x]} - e^{i[(v+1)x - z \sin x]}\} \, dx =$$

$$= 2\pi [J'_v(z) + iE'_v(z)] \quad [|\arg z| < \pi].$$

$$9. \int_{-\pi}^{\pi} \frac{dx}{1 - \exp(-a - ib \cos x - ic \sin x)} = 2\pi \sum_{k=0}^{\infty} e^{-ka} J_0(k \sqrt{b^2 + c^2})$$

[$a, b, c > 0$].

$$10. \int_a^{a+2\pi} e^{i(nx - z \sin x)} \, dx = 2\pi J_n(z)$$

[$|\arg z| < \pi$; $n = 0, \pm 1, \pm 2, \dots$].

$$11. \int_0^{\infty} e^{-(px - z \sin x)} \, dx = \frac{1}{p} + 2 \sum_{k=1}^{\infty} \frac{p}{p^2 + k^2} J_k\left(\frac{kz}{p}\right)$$

[Re $p > 0$; $|\arg z| < \pi$].

$$12. \int_0^{\pi/2} \frac{\cos^{\mu-1} x}{\sin^{2\mu} x} e^{\pm i(1-\mu)x - \gamma \operatorname{ctg} x} \, dx =$$

$$= \pm \frac{i}{2} e^{\mp i\gamma/2} \sqrt{\frac{\pi}{\gamma}} \gamma^{1-\mu} \Gamma(\mu) H_{\mu-1/2}^{(3/2 \mp 1/2)}\left(\frac{\gamma}{2}\right) \quad [\operatorname{Re} \gamma, \operatorname{Re} \mu > 0].$$

$$13. \int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{e^{k\pi \operatorname{ctg} x} - 1} = \frac{(-1)^{n-1}}{2n+1} \left\{ \frac{n}{2^{-2}(2n-1)} \right\}$$

[$k = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$; $n = 1, 2, 3, \dots$].

$$14. \int_0^{\infty} e^{-\operatorname{tg}^2 x} \frac{\sin x \, dx}{x \cos^2 x} = \frac{\sqrt{\pi}}{2}.$$

2.5.41. Интегралы, содержащие $A(x)$, $e^{f(x)}$, $\left\{ \begin{matrix} \sin(ax^2 + bx^{\pm 1} + c) \\ \cos(ax^2 + bx^{\pm 1} + c) \end{matrix} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-px} \left\{ \begin{matrix} \sin(bx + c) \\ \cos(bx + c) \end{matrix} \right\} \, dx = \frac{1}{p^2 + b^2} \left(p \left\{ \begin{matrix} \sin c \\ \cos c \end{matrix} \right\} \pm b \left\{ \begin{matrix} \cos c \\ \sin c \end{matrix} \right\} \right).$$

$$2. \int_0^{\infty} e^{-px} \left\{ \begin{matrix} \sin a[x] \\ \cos a[x] \end{matrix} \right\} \, dx = \frac{e^p - 1}{p(e^{2p} - 2e^p \cos a + 1)} \left\{ \begin{matrix} \sin a \\ e^p - \cos a \end{matrix} \right\} \quad [a, \operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} e^{-px} b^{[x]} \begin{Bmatrix} \sin a [x] \\ \cos a [x] \end{Bmatrix} dx = \frac{e^p - 1}{p (e^{2p} - 2be^p \cos a + b^2)} \begin{Bmatrix} b \sin a \\ e^p - b \cos a \end{Bmatrix}$$

[$a > 0; b \geq 1; \operatorname{Re} p > \ln b$].

$$4. \int_0^{\infty} e^{-px} \begin{Bmatrix} \sin bx^2 \\ \cos bx^2 \end{Bmatrix} dx = \sqrt{\frac{\pi}{2b}} \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \left[\frac{1}{2} - S(z) \right] \pm \sqrt{\frac{\pi}{2b}} \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix} \left[\frac{1}{2} - C(z) \right]$$

[$z = p^2/(4b); b, \operatorname{Re} p > 0$].

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{Bmatrix} \sin bx^2 \\ \cos bx^2 \end{Bmatrix} dx =$$

$$= \frac{i\delta}{2} \Gamma(\alpha) (2b)^{-\alpha/2} \left\{ \exp \left[-i \left(\frac{\alpha\pi}{4} + \frac{p^2}{8b} \right) \right] D_{-\alpha} \left(\frac{p}{\sqrt{2b}} e^{-\pi i/4} \right) \mp \right.$$

$$\left. \mp \exp \left[i \left(\frac{\alpha\pi}{4} + \frac{p^2}{8b} \right) \right] D_{-\alpha} \left(\frac{p}{\sqrt{2b}} e^{\pi i/4} \right) \right\}$$

[$b, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -2\delta$].

$$6. \int_0^{\infty} x^{-1/2} e^{-px} \begin{Bmatrix} \sin bx^2 \\ \cos bx^2 \end{Bmatrix} dx = \mp \frac{\pi}{4} \sqrt{\frac{p}{b}} \left[J_{1/4} \left(\frac{p^2}{8b} \right) \begin{Bmatrix} \cos c \\ \sin c \end{Bmatrix} \pm Y_{1/4} \left(\frac{p^2}{8b} \right) \begin{Bmatrix} \sin c \\ \cos c \end{Bmatrix} \right]$$

[$c = p^2/(8b) + \pi/8; b, \operatorname{Re} p > 0$].

$$7. \int_0^{\infty} x e^{-px} \begin{Bmatrix} \sin bx^2 \\ \cos bx^2 \end{Bmatrix} dx = \frac{\delta}{2b} + \frac{p}{2} \sqrt{\frac{\pi}{2}} b^{-3/2} \left\{ \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \left[\frac{1}{2} - C(z) \right] \mp \right.$$

$$\left. \mp \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix} \left[\frac{1}{2} - S(z) \right] \right\}$$

[$z = p^2/(4b); b, \operatorname{Re} p > 0$].

$$8. \int_0^{\infty} \frac{e^{-px}}{x} \sin bx^2 dx = \frac{\pi}{2} \left[\frac{1}{2} - C(z) \right]^2 + \frac{\pi}{2} \left[\frac{1}{2} - S(z) \right]^2$$

[$z = p^2/(4b); b, \operatorname{Re} p > 0$].

$$9. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{Bmatrix} \sin (bx^2 + px) \\ \cos (bx^2 + px) \end{Bmatrix} dx = \frac{\Gamma(\alpha)}{(2b)^{\alpha/2}} \exp \left(-\frac{2p^2}{b} \right) \begin{Bmatrix} \sin (\alpha\pi/2) \\ \cos (\alpha\pi/2) \end{Bmatrix} D_{-\alpha} \left(\frac{p}{\sqrt{b}} \right)$$

[$b > 0; \operatorname{Re} \alpha > -\delta; |\arg p| < \pi/4$].

$$10. \int_0^{\infty} x e^{-px} \begin{Bmatrix} \sin bx^2 \sin px \\ \cos bx^2 \cos px \end{Bmatrix} dx = \frac{p}{4} \sqrt{\frac{\pi}{2b^3}} \exp \left(-\frac{p^2}{2b} \right)$$

[$b > 0; |\arg p| < \pi/4$].

$$11. \int_0^{\infty} x^2 e^{-px} [\sin (bx^2 \pm px) \pm \cos (bx^2 \pm px)] dx =$$

$$= 2^{-3/2} b^{-5/2} \sqrt{\pi} (b - p^2) \exp \left(-\frac{p^2}{2b} \right) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

[$b > 0; |\arg p| < \pi/4$].

$$12. \int_0^{\infty} e^{-ax^2} \begin{Bmatrix} \sin cx^2 \\ \cos cx^2 \end{Bmatrix} \cos bx dx = \mp \frac{1}{2} \sqrt{\frac{\pi}{a^2 + c^2}} e^{-aA} \left[B_{\pm} \begin{Bmatrix} \sin cA \\ \cos cA \end{Bmatrix} \mp \right.$$

$$\left. \mp B_{\pm} \begin{Bmatrix} \cos cA \\ \sin cA \end{Bmatrix} \right] = \frac{\sqrt{\pi}}{2 \sqrt[4]{a^2 + c^2}} e^{-aA} \begin{Bmatrix} \sin D \\ \cos D \end{Bmatrix}$$

$$\left[A = \frac{b^2}{4(a^2 + c^2)}, B_{\pm} = \frac{1}{\sqrt{2}} (\sqrt{a^2 + c^2} \pm a)^{1/2}, 2D = \operatorname{arctg}(c/a) - Ac; b > 0; \operatorname{Re} a > |\operatorname{Im} c| \right].$$

$$13. \int_0^{\infty} x^{-1/2} e^{-px - q/x} \begin{Bmatrix} \sin (bx + c/x) \\ \cos (bx + c/x) \end{Bmatrix} dx = \frac{\sqrt{\pi}}{r} e^{-2rs \cos (A+B)} \begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix}$$

[$d = A + 2rs \sin (A+B)$, $r = \sqrt[4]{p^2 + b^2}$, $s = \sqrt[4]{q^2 + c^2}$,
 $2A = \operatorname{arctg} (b/p)$, $2B = \operatorname{arctg} (c/q)$; $\operatorname{Re} p > |\operatorname{Im} b|$; $\operatorname{Re} q > |\operatorname{Im} c|$].

$$14. \int_0^{\infty} x^{-3/2} e^{-px - q/x} \begin{Bmatrix} \sin (bx + c/x) \\ \cos (bx + c/x) \end{Bmatrix} dx = \frac{\sqrt{\pi}}{s} e^{-2rs \cos (A+B)} \begin{Bmatrix} \sin f \\ \cos f \end{Bmatrix}$$

[$f = B + 2rs \sin (A+B)$, r, s, A, B см. 2.5.41.13; $\operatorname{Re} p > |\operatorname{Im} b|$; $\operatorname{Re} q > |\operatorname{Im} c|$].

$$15. \int_{-\infty}^{\infty} x^n e^{-(ax^2 + bx + c)} \begin{Bmatrix} \sin (qx + r) \\ \cos (qx + r) \end{Bmatrix} dx =$$

$$= \left(-\frac{1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - 4ac - q^2}{4a}\right) \sum_{k=0}^{[n/2]} \frac{n! a^k}{(n-2k)! k!} \times$$

$$\times \sum_{j=0}^{n-2k} \binom{n-2k}{j} b^{n-2k-j} q^j \begin{Bmatrix} \sin h \\ \cos h \end{Bmatrix} \quad [h = r - bq/(2a) - \pi j/2; \operatorname{Re} a > 0].$$

$$16. \int_{-\infty}^{\infty} e^{-(ax^2 + bx + c)} \begin{Bmatrix} \sin (px^2 + qx + r) \\ \cos (px^2 + qx + r) \end{Bmatrix} dx =$$

$$= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp\left[\frac{a(b^2 - 4ac) - (aq^2 - 2bpq + 4cp^2)}{4(a^2 + p^2)}\right] \times$$

$$\times \begin{Bmatrix} \sin \left[\frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - 4pr) - (b^2p - 2abq + 4a^2r)}{4(a^2 + p^2)}\right] \\ \cos \left[\frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - 4pr) - (b^2p - 2abq + 4a^2r)}{4(a^2 + p^2)}\right] \end{Bmatrix} \quad [\operatorname{Re} a > 0].$$

$$17. \int_0^{\infty} e^{-ax^4 + bx^2} \left[2x \sqrt{a} \begin{Bmatrix} \cos cx^3 \\ \sin cx^3 \end{Bmatrix} \mp \sqrt{b} \begin{Bmatrix} \sin cx^3 \\ \cos cx^3 \end{Bmatrix} \right] dx = \frac{\pi}{2} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

[$c = 2\sqrt{ab}$; $a, b > 0$].

2.5.42. Интегралы от $A(x) e^{-px} \begin{Bmatrix} \sin \sqrt{ax^2 + bx + c} \\ \cos \sqrt{ax^2 + bx + c} \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{e^{-px}}{\sqrt{x} \sqrt{a^2 - x^2}} \cos (c \sqrt{a^2 - x^2}) dx =$$

$$= -\frac{\pi}{2} \sqrt{\pi p} [J_{1/4}(z_+) J_{1/4}(z_-) - J_{-1/4}(z_+) J_{-1/4}(z_-)]$$

[$2z_{\pm} = c \pm \sqrt{c^2 - p^2}$; $a, c > 0$].

$$2. \int_0^a \frac{e^{-px}}{\sqrt{ax - x^2}} \cos (c \sqrt{ax - x^2}) dx = \pi e^{-ap/2} J_0\left(\frac{a}{2} \sqrt{c^2 - p^2}\right) \quad [a, c > 0].$$

$$3. \int_a^\infty x^k e^{-px} \sin(c \sqrt{x^2 - a^2}) dx = \frac{a^{1+k} c p^k}{(p^2 + c^2)^{(1+k)/2}} K_{1+k}(a \sqrt{p^2 + c^2})$$

$$[k=0 \text{ или } 1; a>0; \operatorname{Re} p > |\operatorname{Im} c|]$$

$$4. \int_a^\infty \frac{e^{-px}}{\sqrt{x+a}} \left\{ \begin{array}{l} \sin(c \sqrt{x^2 - a^2}) \\ \cos(c \sqrt{x^2 - a^2}) \end{array} \right\} dx =$$

$$= \sqrt{\frac{\pi}{2}} \frac{c^\delta}{\sqrt{p^2 + c^2}} (\sqrt{p^2 + c^2} + p)^{1/2 - \delta} \exp(-a \sqrt{p^2 + c^2})$$

$$[a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$5. \int_a^\infty \frac{e^{-px}}{\sqrt{x-a}} \cos(c \sqrt{x^2 - a^2}) dx = \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2 + c^2}}{p^2 + c^2}} \exp(-a \sqrt{p^2 + c^2})$$

$$[a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$6. \int_a^\infty (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \begin{array}{l} \sin(c \sqrt{x^2 - a^2}) \\ \cos(c \sqrt{x^2 - a^2}) \end{array} \right\} dx =$$

$$= \frac{2^{\pm 1/2}}{a \sqrt{c}} \Gamma\left(\alpha + \frac{\delta}{2}\right) W_{1/4-\alpha, 1/4}(z_+) M_{\alpha-1/4, \pm 1/4}(z_-)$$

$$[z_\pm = a(\sqrt{p^2 + c^2} \pm p); a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$7. \int_a^\infty \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \begin{array}{l} \sin(c \sqrt{x^2 - a^2}) \\ \cos(c \sqrt{x^2 - a^2}) \end{array} \right\} dx = \frac{(-i)^\delta \pi}{\sqrt{2a}} e^{ap} \operatorname{erfc}(z_+) \left\{ \begin{array}{l} \operatorname{erf}(iz_-) \\ 1 \end{array} \right\}$$

$$[z_\pm = a(\sqrt{p^2 + c^2} \pm p); a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$8. \int_a^\infty \frac{e^{-px}}{(x^2 - a^2)^{3/4}} \left\{ \begin{array}{l} \sin(c \sqrt{x^2 - a^2}) \\ \cos(c \sqrt{x^2 - a^2}) \end{array} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$$

$$[z_\pm = \frac{a}{2} (\sqrt{p^2 + c^2} \pm p); a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$9. \int_a^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2 - a^2}} \cos(c \sqrt{x^2 - a^2}) dx = \frac{1}{\sqrt{2\pi p}} K_{1/4}(z_+) K_{1/4}(z_-)$$

$$[z_\pm = \frac{a}{2} (\sqrt{p^2 + c^2} \pm p); a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$10. \int_a^\infty \frac{x^m}{\sqrt{x^2 - a^2}} e^{-px} \cos(c \sqrt{x^2 - a^2}) dx = \frac{a^m p^m}{(p^2 + c^2)^{m/2}} K_m(a \sqrt{p^2 + c^2})$$

$$[m=0 \text{ или } 1; a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$11. \int_a^\infty \frac{(x + \sqrt{x^2 - a^2})^\nu \mp (x - \sqrt{x^2 - a^2})^\nu}{\sqrt{x^2 - a^2}} e^{-px} \left\{ \begin{array}{l} \sin(c \sqrt{x^2 - a^2}) \\ \cos(c \sqrt{x^2 - a^2}) \end{array} \right\} dx =$$

$$= 2a^\nu \left\{ \begin{array}{l} \sin d \\ \cos d \end{array} \right\} K_\nu(a \sqrt{p^2 + c^2}) \quad [d = \nu \operatorname{arctg}(c/p); a>0; \operatorname{Re} p > |\operatorname{Im} c|].$$

$$\begin{aligned}
 12. \int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2+z^2}} \left\{ \begin{array}{l} \sin(b \sqrt{x^2+z^2}) \\ \cos(b \sqrt{x^2+z^2}) \end{array} \right\} dx = \\
 = \frac{\pi}{4} \sqrt{\pi p} \left[\left\{ \begin{array}{l} Y_{1/4}(z_+) \\ J_{1/4}(z_+) \end{array} \right\} J_{1/4}(z_-) \mp \left\{ \begin{array}{l} J_{1/4}(z_+) \\ Y_{1/4}(z_+) \end{array} \right\} Y_{1/4}(z_-) \right] \\
 [2z_\pm = z(\sqrt{p^2+b^2} \pm b); \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Im} b|].
 \end{aligned}$$

$$\begin{aligned}
 13. \int_0^\infty e^{-px} \sin(b \sqrt{x^2+xz}) dx = \frac{bz}{2 \sqrt{p^2+b^2}} e^{pz/2} K_1\left(\frac{z}{2} \sqrt{p^2+b^2}\right) \\
 [\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 14. \int_0^\infty \frac{1}{\sqrt{x}} e^{-px} \cos(b \sqrt{x^2+xz}) dx = \\
 = \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2+b^2}}{p^2+b^2}} \exp\left[-\frac{z}{2} (\sqrt{p^2+b^2} - p)\right] \\
 [\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 15. \int_0^\infty \frac{e^{-px}}{\sqrt{x+z}} \left\{ \begin{array}{l} \sin(b \sqrt{x^2+xz}) \\ \cos(b \sqrt{x^2+xz}) \end{array} \right\} dx = \\
 = \sqrt{\frac{\pi}{2}} b^\delta (\sqrt{p^2+b^2} + p)^{1/2-\delta} (p^2+b^2)^{-1/2} \exp\left[-\frac{z}{2} (\sqrt{p^2+b^2} - p)\right] \\
 [\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 16. \int_0^\infty x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \begin{array}{l} \sin(b \sqrt{x^2+xz}) \\ \cos(b \sqrt{x^2+xz}) \end{array} \right\} dx = \\
 = \frac{2^{\delta+1/2}}{z \sqrt{b}} \Gamma\left(\alpha + \frac{\delta}{2}\right) e^{pz/2} W_{1/4-\alpha, 1/4}(z_+) M_{\alpha-1/4, \pm 1/4}(z_-) \\
 [z_\pm = \frac{z}{2} (\sqrt{p^2+b^2} \pm p); 2 \operatorname{Re} \alpha > -\delta; \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 17. \int_0^\infty \frac{e^{-px}}{\sqrt{x} (x+z)} \left\{ \begin{array}{l} \sin(b \sqrt{x^2+xz}) \\ \cos(b \sqrt{x^2+xz}) \end{array} \right\} dx = \frac{\pi}{\sqrt{z}} e^{pz} \operatorname{erfc}(z_+) \left\{ \begin{array}{l} -i \operatorname{erf}(iz_-) \\ 1 \end{array} \right\} \\
 [z_\pm^2 = \frac{z}{2} (\sqrt{p^2+b^2} \pm p); \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 18. \int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x+z}} \cos(b \sqrt{x^2+xz}) dx = e^{pz/2} K_0\left(\frac{z}{2} \sqrt{p^2+b^2}\right) \\
 [\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 19. \int_0^\infty (x^2+xz)^{-3/4} e^{-px} \left\{ \begin{array}{l} \sin(b \sqrt{x^2+xz}) \\ \cos(b \sqrt{x^2+xz}) \end{array} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{pz/2} K_{1/4}(z_+) I_{\pm 1/4}(z_-) \\
 [z_\pm = \frac{z}{4} (\sqrt{p^2+b^2} \pm p); \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].
 \end{aligned}$$

$$20. \int_0^{\infty} \frac{1}{V x^2 + xz} \left[\left(\frac{z}{2} + x + V x^2 + xz \right)^v + \left(\frac{z}{2} + x - V x^2 + xz \right)^v \right] \times \\ \times e^{-px} \begin{Bmatrix} \sin(b V x^2 + xz) \\ \cos(b V x^2 + xz) \end{Bmatrix} dx = 2 \left(\frac{z}{2} \right)^v \begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix} e^{pz/2} K_v \left(\frac{z}{2} V p^2 + b^2 \right) \\ [d = v \operatorname{arctg}(b/p); \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi].$$

2.5.43. Интегралы от $A(x) e^{f(x)} \begin{Bmatrix} \sin [cx/(x^2 \pm a^2)] \\ \cos [cx/(x^2 \pm a^2)] \end{Bmatrix}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{x^{\alpha-1}}{V a^2 - x^2} \exp \left(-b \frac{a^2 + x^2}{a^2 - x^2} \right) \begin{Bmatrix} \sin [cx/(a^2 - x^2)] \\ \cos [cx/(a^2 - x^2)] \end{Bmatrix} dx = \\ = 2^{\delta + (2\alpha-3)/4} \sqrt[4]{\frac{a}{c^3}} \Gamma \left(\frac{\alpha + \delta}{2} \right) (V 4a^2b^2 + c^2 + 2ab)^{1/4} \times \\ \times D_{-\alpha} \left(\sqrt{\frac{V 4a^2b^2 + c^2 + 2ab}{a}} \right) M_{\alpha/2 - 1/4, \pm 1/4} \left(\frac{V 4a^2b^2 + c^2 - 2ab}{2a} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha > -\delta]$$

$$2. \int_a^{\infty} \frac{x^{\alpha-1}}{V x^2 - a^2} \exp \left(-\frac{b}{x^2 - a^2} \right) \begin{Bmatrix} \sin [cx/(x^2 - a^2)] \\ \cos [cx/(x^2 - a^2)] \end{Bmatrix} dx = \\ = \frac{a^{\alpha-1}}{V c 2^{\alpha/2 + 1/4 - \delta}} \exp \left(\frac{b}{2a^2} \right) \Gamma \left(\frac{1 + \delta - \alpha}{2} \right) (V b^2 + a^2c^2 + b)^{1/4} \times \\ \times D_{\alpha-1} \left(\frac{V V b^2 + a^2c^2 + b}{a} \right) M_{1/4 - \alpha/2, \pm 1/4} \left(\frac{V b^2 + a^2c^2 - b}{2a^2} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha < 1 + \delta].$$

$$3. \int_a^{\infty} \frac{x^{\alpha-1}}{V x^2 - a^2} \exp \left(-b \frac{x^2 + a^2}{x^2 - a^2} \right) \begin{Bmatrix} \sin [cx/(x^2 - a^2)] \\ \cos [cx/(x^2 - a^2)] \end{Bmatrix} dx = \\ = 2^{\delta - (2\alpha+1)/4} a^{\alpha-3/4} c^{-1/2} \Gamma \left(\frac{1 + \delta - \alpha}{2} \right) (V 4a^2b^2 + c^2 + 2ab)^{1/4} \times \\ \times D_{\alpha-1} \left(\sqrt{\frac{V 4a^2b^2 + c^2 + 2ab}{a}} \right) M_{1/4 - \alpha/2, \pm 1/4} \left(\frac{V 4a^2b^2 + c^2 - 2ab}{2a} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha < 1 + \delta].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{V x^2 + z^2} \exp \left(-\frac{b}{x^2 + z^2} \right) \begin{Bmatrix} \sin [cx/(x^2 + z^2)] \\ \cos [cx/(x^2 + z^2)] \end{Bmatrix} dx = \\ = \frac{2^{2\delta}}{V 2\pi c} \Gamma \left(\frac{1 - \alpha + \delta}{2} \right) \Gamma \left(\frac{\delta + \alpha}{2} \right) z^{\alpha-1/2} \exp \left(-\frac{b}{2z^2} \right) \times \\ \times M_{1/4 - \alpha/2, \pm 1/4} \left(\frac{b + V b^2 - c^2z^2}{2z^2} \right) M_{1/4 - \alpha/2, \pm 1/4} \left(\frac{b - V b^2 - c^2z^2}{2z^2} \right) \\ [b, c, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1 + \delta].$$

$$\begin{aligned}
 5. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-b \frac{z^2-x^2}{z^2+x^2}\right) \left\{ \begin{array}{l} \sin [cx/(x^2+z^2)] \\ \cos [cx/(x^2+z^2)] \end{array} \right\} dx = \\
 = \frac{2^{1/2} \pm 1 z^{\alpha-1/2}}{\sqrt{\pi c}} \Gamma\left(\frac{\alpha+\delta}{2}\right) \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \times \\
 \times M_{1/4-\alpha/2, \pm 1/4}\left(\frac{2zb + \sqrt{4b^2z^2 - c^2}}{2z}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{2zb - \sqrt{4b^2z^2 - c^2}}{2z}\right) \\
 [b, c, \operatorname{Re} z > 0, -\delta < \operatorname{Re} \alpha < 1 + \delta].
 \end{aligned}$$

2.5.44. Интегралы, содержащие показательную и тригонометрические функции от тригонометрических функций.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/2} e^{z \cos x} \left\{ \begin{array}{l} \sin (z \sin x) \\ \cos (z \sin x) \end{array} \right\} dx = \left\{ \begin{array}{l} -\operatorname{Ei}(z) + \operatorname{ci}(-z) \\ -\operatorname{si}(-z) \end{array} \right\} \quad [|\arg(-z)| < \pi].$$

$$2. \int_0^{\pi/2} e^{z \cos x} \left\{ \begin{array}{l} \sin (z \sin x) \\ \cos (z \sin x) \end{array} \right\} dx = \left\{ \begin{array}{l} 2 \operatorname{shi}(z) \\ \pi \end{array} \right\} \quad [|\arg z| < \pi].$$

$$3. \int_0^{\pi} e^{z \cos x} \left\{ \begin{array}{l} \sin (z \sin x) \sin nx \\ \cos (z \sin x) \cos nx \end{array} \right\} dx = \frac{\pi z^n}{2n!} \quad [|\arg z| < \pi; n=1, 2, 3, \dots].$$

$$4. \int_0^{\pi} e^{z \cos x} \cos (z \sin x + bx) dx = z^{-b} \sin(b\pi) \gamma(b, z) \quad [|\arg z|, |\arg b| < \pi].$$

$$5. \int_0^{\pi} e^{z \cos x} \cos (z \sin x + x/2) dx = \sqrt{\frac{\pi}{z}} \operatorname{erf}(\sqrt{z}) \quad [|\arg z| < \pi].$$

$$\begin{aligned}
 6. \int_0^{\pi} e^{z(\cos bx + \cos cx)} \left\{ \begin{array}{l} \sin (z \sin bx) \sin (z \sin cx) \\ \cos (z \sin bx) \cos (z \sin cx) \end{array} \right\} dx = \\
 = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{z^{(b+ck)}}{\Gamma(1+bk) \Gamma(1+ck)} + \left\{ \begin{array}{l} 0 \\ \pi \end{array} \right\}.
 \end{aligned}$$

$$7. \int_0^{\pi} e^{z \cos x} \frac{\sin (z \sin x)}{\sin x} dx = \pi \operatorname{sh} z \quad [|\arg z| < \pi].$$

$$8. \int_0^{\pi} e^{z \cos x} \cos (z \sin x) \frac{\sin 2nx}{\sin x} dx = \pi \sum_{k=0}^{n-1} \frac{z^{2k+1}}{(2k+1)!} \quad [|\arg z| < \pi].$$

$$9. \int_0^{\pi} e^{z \cos x} \sin (z \sin x) \operatorname{tg}^{\pm 1} \frac{x}{2} dx = \pm \pi (1 - e^z) \quad [|\arg z| < \pi].$$

$$10. \int_0^{\pi} e^{z \cos x} \frac{\cos(z \sin x)}{\cos x + c} dx = \frac{\pi}{\sqrt{c^2 - 1}} e^{z(\sqrt{c^2 - 1} - c)} \quad [|\arg z| < \pi; |c| > 1].$$

$$11. \int_0^{\pi} e^{z \cos x} \sin(z \sin x) \frac{\sin bx}{1 - 2a \cos bx + a^2} dx = \frac{\pi}{2a} \sum_{k=1}^{\infty} \frac{a^k z^{kb}}{\Gamma(kb + 1)} \quad [|\arg z| < \pi; |a| < 1].$$

$$12. \int_0^{\pi} e^{z \cos x} \cos(z \sin x) \frac{1 - a \cos bx}{1 - 2a \cos bx + a^2} dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{a^k z^{kb}}{\Gamma(kb + 1)} + \pi \quad [|\arg z| < \pi; |a| < 1].$$

$$13. \int_0^{2\pi} e^{a \cos x + b \sin x} \left\{ \begin{array}{l} \sin(p \cos x + q \sin x + nx) \\ \cos(p \cos x + q \sin x + nx) \end{array} \right\} dx =$$

$$= (-i)^{\delta} \pi [(p-b)^2 + (q+a)^2]^{-n/2} [(A+iB)^{n/2} I_n(\sqrt{C-iD}) + (A-iB)^{n/2} I_n(\sqrt{C+iD})]$$

$$[(p-b)^2 + (q+a)^2 > 0; A=a^2-b^2+p^2-q^2, B=2(ab+pq), C=a^2+b^2-p^2-q^2, D=2(ap+bq)].$$

$$14. \int_0^{2\pi} e^{a \cos x + b \sin x} \left\{ \begin{array}{l} \sin(b \cos x - a \sin x + nx) \\ \cos(b \cos x - a \sin x + nx) \end{array} \right\} dx = \frac{2\pi}{n!} (a^2 + b^2)^{n/2} \left\{ \begin{array}{l} \sin \varphi \\ \cos \varphi \end{array} \right\}$$

$$[\varphi = n \operatorname{arctg}(b/a)].$$

$$15. \int_0^{\infty} \frac{1}{x} e^{a \cos x} \sin(a \sin x) \cos nx \, dx = \frac{\pi}{2} \left[e^a - \sum_{k=0}^{n-1} \frac{a^k}{k!} - \frac{a^n}{2n!} \right] \quad [a > 0].$$

$$16. \int_0^{\infty} \frac{1}{x} e^{a \cos x} \cos(a \sin x) \sin nx \, dx = \frac{\pi}{2} \left[\sum_{k=0}^{n-1} \frac{a^k}{k!} + \frac{a^n}{2n!} \right] \quad [a > 0; n=1, 2, 3, \dots].$$

$$17. \int_0^{\infty} \frac{1}{x} e^{a \cos x} \sin(a \sin x + nx) \, dx = \frac{\pi}{2} e^a \quad [a > 0].$$

$$18. \int_0^{\infty} \frac{x^{\delta} e^{a \cos cx}}{x^2 + y^2} \left\{ \begin{array}{l} \sin(a \sin cx) \\ \cos(a \sin cx) \end{array} \right\} dx = \pm \frac{\pi}{2} y^{\delta-1} e^{a \cos ac} \left\{ \begin{array}{l} \cos(a \sin ac) \\ \sin(a \sin ac) \end{array} \right\} - \frac{\pi \delta}{2}$$

$$[a, c, y > 0; \delta = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right\}].$$

$$19. \int_0^{\infty} \frac{x^{\delta} e^{a \cos cx}}{x^2 + z^2} \left\{ \begin{array}{l} \sin(a \sin cx + bx) \\ \cos(a \sin cx + bx) \end{array} \right\} dx = \frac{\pi}{2} z^{\delta-1} \exp(-bx + ae^{-cz})$$

$$[a, b, c, \operatorname{Re} z > 0; \delta = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right\}].$$

$$20. \int_0^{\infty} \frac{x^{1-\delta} e^{a \cos cx}}{(x^2 + z^2) \sin cx} \left\{ \begin{array}{l} \sin(a \sin cx + cx) \\ \cos(a \sin cx + cx) \end{array} \right\} dx = \frac{\pi z^{-\delta}}{2 \operatorname{sh} cz} [e^a - \exp(ae^{-cz} - cz)]$$

$$[a, c, \operatorname{Re} z > 0, \delta = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right\}].$$

$$21. \int_0^{\infty} \frac{e^{a \cos cx}}{x^2 + z^2} \frac{\sin(a \sin cx)}{\sin cx} dx = \frac{\pi}{2z \operatorname{sh} cz} [e^a - \exp(ae^{-cz})] \quad [a, c, \operatorname{Re} z > 0].$$

$$22. \int_0^{\infty} \frac{e^{a \cos cx}}{x^2 - y^2} \frac{\sin(a \sin cx)}{\sin cx} dx = \frac{\pi}{2y \sin cy} [-e^a + e^{a \cos cy} \cos(a \sin cy)] \quad [a, c, y > 0].$$

$$23. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{1 - e^{a \cos cx} \cos(a \sin cx)}{\sin cx} dx = \frac{\pi}{2 \operatorname{sh} cz} [e^a - \exp(ae^{-cz})] \quad [a, c, \operatorname{Re} z > 0].$$

$$24. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{1 - e^{a \cos cx} \cos(a \sin cx)}{\sin cx} dx = \frac{\pi}{2 \sin cy} e^{a \cos cy} \sin(a \sin cy) \quad [a, c, y > 0].$$

$$25. \int_0^{\infty} \frac{e^{a \cos cx}}{x^2 + z^2} \sin(a \sin cx) \operatorname{tg}^{\pm 1} cx \, dx = \pm \frac{\pi}{2z} \operatorname{th}^{\pm 1} cz [\exp(ae^{-cz}) - e^a] \quad [a, c, \operatorname{Re} z > 0].$$

$$26. \int_0^{\pi/2} e^{-p \operatorname{ctg} x} \sin^{\nu-1} x \cos^{\mu-1} x \begin{cases} \sin [p/2 - (\mu-1)x] \\ \cos [p/2 - (\mu-1)x] \end{cases} dx = \\ = \frac{\sqrt{\pi}}{2} p^{1/2-\mu} \Gamma(\mu) \begin{cases} J_{\mu-1/2}(p/2) \\ Y_{\mu-1/2}(p/2) \end{cases} \quad [\nu=1/2-\mu \text{ или } \nu=1-2\mu; p, \operatorname{Re} \mu > 0].$$

$$27. \int_0^{\pi/2} e^{-p \operatorname{tg}^2 x} \cos^{\nu-1} x \cos [2p \operatorname{tg} x - (\nu+1)x] dx = -\frac{1}{2} \Gamma\left(-\frac{\nu}{2}\right) p^{\nu/2} e^{-p} \sin \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu < 0; p > 0].$$

2.5.45. Интегралы, содержащие разности функций $A(x)$, e^{-px} , $\begin{cases} \sin bx \\ \cos bx \end{cases}$.

$$1. \int_0^{\infty} \frac{1 - e^{-ax}}{x^n} \sin bx \, dx = I_n \quad [\operatorname{Re} a, b > 0],$$

$$I_1 = \operatorname{arctg} \frac{a}{b}, \quad I_2 = a \operatorname{arctg} \frac{b}{a} + \frac{b}{2} \ln \left(1 + \frac{a^2}{b^2} \right).$$

$$2. \int_0^{\infty} \frac{1 - e^{-ax}}{x} \cos bx \, dx = \frac{1}{2} \ln \left(1 + \frac{a^2}{b^2} \right) \quad [\operatorname{Re} a, b > 0].$$

$$3. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx \, dx = \ln b - \frac{1}{2} [\psi(ib) + \psi(-ib)] \quad [b > 0].$$

$$4. \int_0^{\infty} e^{-px} \frac{1 - \cos ax}{x} dx = \frac{1}{2} \ln \left(1 + \frac{a^2}{p^2} \right) \quad [\operatorname{Re} p > |\operatorname{Im} a|].$$

$$5. \int_0^{\infty} e^{-px} \frac{1 - \cos ax}{x^2} dx = a \operatorname{arctg} \frac{a}{p} - \frac{p}{2} \ln \left(1 + \frac{a^2}{p^2} \right) \quad [\operatorname{Re} p > |\operatorname{Im} a|].$$

$$6. \int_0^{\infty} e^{-px} \frac{x - \sin x}{x^n} dx = I_n \quad [\operatorname{Re} p > 0],$$

$$I_1 = p^{-1} - \operatorname{arctg} p, \quad I_2 = 2^{-1} \ln(1 + p^{-2}) + p \operatorname{arctg} p - 1,$$

$$I_3 = 2^{-1} [p \ln(1 + p^{-2}) + (p^2 + 1) \operatorname{arctg} p - p].$$

$$7. \int_0^{\infty} \frac{1 - \cos ax}{x(e^{2\pi x} - 1)} dx = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a} \quad [a > 0].$$

$$8. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C.$$

$$9. \int_0^{\infty} \frac{\cos ax - e^{-ax}}{x(x^4 + z^4)} dx = \frac{\pi}{2z^4} \exp \left(-\frac{az}{\sqrt{2}} \right) \sin \left(\frac{az}{\sqrt{2}} \right) \quad [a > 0; |\arg z| < \pi/4].$$

$$10. \int_0^{\infty} \frac{e^{-ax} - e^{-cx} \cos bx}{x} dx = \frac{1}{2} \ln \frac{c^2 + b^2}{a^2} \quad [\operatorname{Re} a > 0; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$11. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x} \cos bx \, dx = \frac{1}{2} \ln \frac{b^2 + c^2}{b^2 + a^2} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Im} b|].$$

$$12. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x^2} \sin bx \, dx = \frac{b}{2} \ln \frac{b^2 + c^2}{b^2 + a^2} + c \operatorname{arctg} \frac{b}{c} - a \operatorname{arctg} \frac{b}{a} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{1}{x} \left(ae^{-bx} - e^{-cx} \frac{\sin ax}{x} \right) dx = \frac{a}{2} \ln \frac{a^2 + c^2}{b^2} + c \operatorname{arctg} \frac{a}{c} - a \quad [\operatorname{Re} b > 0; \operatorname{Re} c > |\operatorname{Im} a|].$$

$$14. \int_0^{\infty} \frac{1 - \cos ax}{x [e^{2ncx} - e^{2(n-1)cx}]} dx = \frac{1}{2} \ln \frac{2c \operatorname{sh} [a\pi/(2c)]}{a\pi} - \frac{1}{2} \sum_{k=1}^{n-1} \ln \left(1 + \frac{a^2}{4k^2 c^2} \right) \quad [a, c > 0; n=1, 2, 3, \dots].$$

$$15. \int_0^{\infty} e^{-px} \frac{\cos ax - \cos bx}{x^n} dx = I_n \quad [\operatorname{Re} p > |\operatorname{Im} a|, |\operatorname{Im} b|].$$

$$I_1 = \frac{1}{2} \ln \frac{p^2 + b^2}{p^2 + a^2}, \quad I_2 = \frac{p}{2} \ln \frac{p^2 + a^2}{p^2 + b^2} + b \operatorname{arctg} \frac{b}{p} - a \operatorname{arctg} \frac{a}{p}.$$

$$16. \int_0^{\infty} e^{-px} \frac{\sin x - x \cos x}{x^n} dx = J_n \quad [\operatorname{Re} p > 0],$$

$$J_1 = \operatorname{arccotg} p - \frac{p}{p^2 + 1}, \quad J_2 = 1 - p \operatorname{arccotg} p, \quad J_3 = \frac{1}{2} [(p^2 + 1) \operatorname{arccotg} p - p].$$

$$17. \int_0^{\infty} \frac{ae^{-bx} \sin cx - ce^{-hx} \sin ax}{x^2} dx = \frac{ac}{2} \ln \frac{a^2 + h^2}{c^2 + b^2} + ch \operatorname{arccotg} \frac{h}{a} - ab \operatorname{arccotg} \frac{b}{c}$$

$$[\operatorname{Re} b > |\operatorname{Im} c|; \operatorname{Re} h > |\operatorname{Im} a|].$$

$$18. \int_0^{\infty} \frac{\cos ax - \cos bx}{x [e^{mcx} - e^{(m-2)c x}]} dx =$$

$$= \frac{1}{2} \left[\left\{ \ln \operatorname{ch} [b\pi/(2c)] - \ln \operatorname{ch} [a\pi/(2c)] \right\} - \sum_{k=1}^{[(m-1)/2]} \ln \frac{b^2 + c^2 (2k - 2\lambda)^2}{a^2 + c^2 (2k - 2\lambda)^2} + \right.$$

$$\left. + \left\{ \ln \frac{0}{(a/b)} \right\} \right] \quad \left[a, b, \operatorname{Re} c > 0, \left\{ \begin{array}{l} \lambda = 1/2, \quad m = 1, 3, 5, \dots \\ \lambda = 0, \quad m = 2, 4, 6, \dots \end{array} \right\} \right].$$

$$19. \int_0^{\infty} e^{-px} \frac{2x \cos x - \sin x}{x^2} \sin x dx = \frac{p}{4} \ln \left(1 + \frac{4}{p^2} \right) \quad [\operatorname{Re} p > 0].$$

$$20. \int_0^{\infty} e^{-x^2} \frac{2x \cos x - \sin x}{x^2} \sin x dx = \frac{\sqrt{\pi}}{2} (1 - e^{-1}).$$

2.5.46. Интегралы от $\prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^{\nu_k},$

$$\prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\cos b_k x}{\sin b_k x} \right\}^{\nu_k}.$$

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$

$$1. \int_0^{\pi/2} \operatorname{ch} ax \cos^n bx dx = n! b^{n+2\lambda} a^{1-2\lambda} \left\{ \frac{\operatorname{ch} [a\pi/(2b)]}{\operatorname{sh} [a\pi/(2b)]} \right\} \prod_{k=0}^{[(n+1)/2]} [a^2 + (2k + 2\lambda)^2 b^2]^{-1}$$

$$\left[|\arg b|, |\arg a| < \pi; \left\{ \begin{array}{l} \lambda = 1/2, \quad n = 1, 3, 5, \dots \\ \lambda = 0, \quad n = 2, 4, 6, \dots \end{array} \right\} \right].$$

$$2. \int_0^{\infty} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$3. \int_0^{\infty} \frac{\sin^2 bx}{\operatorname{sh}^2 cx} dx = \frac{\pi b \operatorname{cth} (b\pi/c) - c}{2c^2} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$4. \int_0^{\infty} \frac{\sin bx}{\operatorname{ch} cx} dx = \frac{i}{2c} \left[\psi \left(\frac{c - ib}{4c} \right) - \psi \left(\frac{c + ib}{4c} \right) \right] - \frac{\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch} cx} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch}^{\nu} cx} dx = \frac{2^{\nu-2}}{c \Gamma(\nu)} \Gamma\left(\frac{\nu}{2} + \frac{ib}{2c}\right) \Gamma\left(\frac{\nu}{2} - \frac{ib}{2c}\right) \quad [\operatorname{Re}(\nu c) > |\operatorname{Im} b|].$$

$$7. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch}^n cx} dx = \frac{2^{n-2} \pi}{(n-1)!} b^{1-2\lambda} c^{2\lambda-2} \left\{ \begin{array}{l} \operatorname{sech} [b\pi/(2c)] \\ \operatorname{cosech} [b\pi/(2c)] \end{array} \right\} \prod_{k=1}^{[(n-1)/2]} \left[\frac{b^2}{4c^2} + (k-\lambda)^2 \right] \\ \left[n \operatorname{Re} c > |\operatorname{Im} b|; \left\{ \begin{array}{l} \lambda = 1/2, \quad n = 1, 3, 5, \dots \\ \lambda = 0, \quad n = 2, 4, 6, \dots \end{array} \right\} \right].$$

$$8. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} cx} \left\{ \begin{array}{l} \sin bx \\ \cos bx \end{array} \right\} dx = \\ = \frac{\pi}{2c} [\operatorname{ch}(b\pi/c) + \cos(a\pi/c)]^{-1} \left\{ \begin{array}{l} \operatorname{sh}(b\pi/c) \\ \sin(a\pi/c) \end{array} \right\} + \frac{\delta i}{2c} \left[\psi\left(\frac{a+c+ib}{2c}\right) - \psi\left(\frac{a+c-ib}{2c}\right) \right] \\ [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$9. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{sh} cx} \sin bx dx = \frac{\pi}{2c} \frac{\operatorname{sh}(b\pi/c)}{\operatorname{ch}(b\pi/c) + \cos(a\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$10. \int_0^{\infty} \left\{ \begin{array}{l} \operatorname{sh} ax \sin bx \\ \operatorname{ch} ax \cos bx \end{array} \right\} \frac{dx}{\operatorname{ch} cx} = \\ = \frac{\pi}{c} \left\{ \begin{array}{l} \sin[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)] \\ \cos[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)] \end{array} \right\} \left(\operatorname{ch} \frac{b\pi}{c} + \cos \frac{a\pi}{c} \right)^{-1} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$11. \int_0^{\infty} \left\{ \begin{array}{l} \operatorname{sh} ax \cos bx \\ \operatorname{ch} ax \sin bx \end{array} \right\} \frac{dx}{\operatorname{ch} cx} = \\ = \frac{1}{4c} \left\{ \begin{array}{l} 1 \\ -i \end{array} \right\} \left[\psi\left(\frac{3c-a-ib}{4c}\right) - \psi\left(\frac{3c+a+ib}{4c}\right) \pm \psi\left(\frac{3c-a+ib}{4c}\right) \mp \psi\left(\frac{3c+a-ib}{4c}\right) \right] + \\ + 2\pi [\cos(a\pi/c) + \operatorname{ch}(b\pi/c)]^{-1} \left\{ \begin{array}{l} \sin(a\pi/c) \\ i \operatorname{sh}(b\pi/c) \end{array} \right\} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$12. \int_0^{\infty} \left\{ \begin{array}{l} \sin ax \sin bx \\ \cos ax \cos bx \end{array} \right\} \frac{dx}{\operatorname{ch} cx} = \\ = \frac{\pi}{c} \left\{ \begin{array}{l} \operatorname{sh}[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)] \\ \operatorname{ch}[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)] \end{array} \right\} \left(\operatorname{ch} \frac{a\pi}{c} + \operatorname{ch} \frac{b\pi}{c} \right)^{-1} \quad [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{\sin ax \cos bx}{\operatorname{sh} cx} dx = \frac{\pi \operatorname{sh}(a\pi/c)}{2c [\operatorname{ch}(a\pi/c) + \operatorname{ch}(b\pi/c)]} \quad [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$14. \int_0^{\infty} \frac{\operatorname{sh} cx \sin bx}{\operatorname{ch}^2 cx} dx = \frac{\pi b}{2c^2 \operatorname{ch}[b\pi/(2c)]} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$15. \int_0^{\infty} \operatorname{sh}^{\nu-1} ax \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= \frac{1}{2a\sqrt{\pi}} \left\{ \frac{\operatorname{sh} [b\pi/(2a)]}{\operatorname{ch} [b\pi/(2a)]} \right\} \Gamma \left[\frac{(\nu+\delta)/2}{1-\nu/2} \right] \Gamma \left(\frac{1-\nu}{2} + \frac{ib}{2a} \right) \Gamma \left(\frac{1-\nu}{2} - \frac{ib}{2a} \right)$$

[Re $\nu > -\delta$; Re $[(\nu-1)a] < -|\operatorname{Im} b|$].

$$16. \int_0^{\infty} \operatorname{sh}^{\nu-1} ax \operatorname{ch} ax \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= \frac{b}{4a^2\sqrt{\pi}} \left\{ \frac{\operatorname{ch} [b\pi/(2a)]}{\operatorname{sh} [b\pi/(2a)]} \right\} \Gamma \left[\frac{(\nu+\delta)/2}{(1-\nu)/2} \right] \Gamma \left(\frac{ib-\nu a}{2a} \right) \Gamma \left(-\frac{ib+\nu a}{2a} \right)$$

[Re $\nu > -\delta$; Re $(\nu a) < -|\operatorname{Im} b|$].

$$17. \int_0^{\infty} \left[1 - \left\{ \frac{\operatorname{th} ax}{\operatorname{cth} ax} \right\} \right] \sin bx \, dx = \frac{1}{b} - \frac{\pi}{2a} \left\{ \frac{1}{\operatorname{cth} [b\pi/(2a)]} \right\}$$

[$b, \operatorname{Re} a > 0$].

2.5.47. Интегралы от $x^{\alpha} \prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^{\nu_k}$,

$$x^{\alpha} \prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\cos b_k x}{\sin b_k x} \right\}^{\nu_k}.$$

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{sh} cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\Gamma(\alpha) i^{\delta}}{(2c)^{\alpha}} \left\{ \zeta \left[\alpha, \frac{1}{2} \left(1 + \frac{ib}{c} \right) \right] \mp \zeta \left[\alpha, \frac{1}{2} \left(1 - \frac{ib}{c} \right) \right] \right\}$$

[Re $\alpha > 1-\delta$; Re $c > |\operatorname{Im} b|$].

$$2. \int_0^{\infty} \frac{x^{2n+1-\delta}}{\operatorname{sh} cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^n \frac{\pi}{2c} \frac{\partial^{2n+1-\delta}}{\partial b^{2n+1-\delta}} \operatorname{th} \frac{b\pi}{2c}$$

[Re $c > |\operatorname{Im} b|$; $\delta = \begin{cases} 1 \\ 0 \end{cases}$].

$$3. \int_0^{\infty} \frac{\sin^2 bx}{x \operatorname{sh} cx} dx = \frac{1}{2} \ln \operatorname{ch} \frac{b\pi}{c}$$

[Re $c > 2|\operatorname{Im} b|$].

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{ch} cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= \frac{i^{\delta} \Gamma(\alpha)}{(4c)^{\alpha}} \left[\zeta \left(\alpha, \frac{c+ib}{4c} \right) \mp \zeta \left(\alpha, \frac{c-ib}{4c} \right) \pm \zeta \left(\alpha, \frac{3c-ib}{4c} \right) - \zeta \left(\alpha, \frac{3c+ib}{4c} \right) \right]$$

[Re $\alpha > -\delta$; Re $c > |\operatorname{Im} b|$].

$$5. \int_0^{\infty} \frac{x^{2n+\delta}}{\operatorname{ch} cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^{n+\delta} \frac{\pi}{2c} \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} \operatorname{sech} \frac{b\pi}{2c}$$

[Re $c > |\operatorname{Im} b|$; $\delta = \begin{cases} 1 \\ 0 \end{cases}$].

$$6. \int_0^{\infty} \frac{\sin bx}{x \operatorname{ch} cx} dx = 2 \operatorname{arctg} \left[\exp \left(\frac{b\pi}{2c} \right) \right] - \frac{\pi}{2}$$

[Re $c > |\operatorname{Im} b|$].

$$7. \int_0^{\infty} x \frac{\sin bx}{\operatorname{ch}^2 cx} dx = \frac{\pi}{4c^3} \operatorname{cosech}^2 \frac{b\pi}{2c} \left(b\pi \operatorname{ch} \frac{b\pi}{2c} - 2c \operatorname{sh} \frac{b\pi}{2c} \right) \quad [2 \operatorname{Re} c > |\operatorname{Im} b|].$$

$$8. \int_0^{\infty} \frac{\cos bx}{x} \left\{ \frac{\operatorname{th} ax}{\operatorname{cth} ax} \right\} dx = \pm \ln \left\{ \frac{\operatorname{cth} [b\pi/(4a)]}{2 \operatorname{sh} [b\pi/(2a)]} \right\} \quad [b, \operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} \frac{\operatorname{sh} ax \sin bx}{x \operatorname{sh} cx} dx = \operatorname{arctg} \left(\operatorname{tg} \frac{a\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \right) \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$10. \int_0^{\infty} \frac{\sin ax \sin bx}{x \operatorname{sh} cx} dx = \frac{1}{2} \left[\ln \operatorname{ch} \left(\frac{a+b}{2c} \pi \right) - \ln \operatorname{ch} \left(\frac{a-b}{2c} \pi \right) \right] \quad [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$11. \int_0^{\infty} \frac{\operatorname{ch} bx \sin^2 ax}{x \operatorname{sh} cx} dx = \frac{1}{4} \ln \frac{\operatorname{ch} (2a\pi/c) + \cos (b\pi/c)}{1 + \cos (b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b| + |\operatorname{Im} a|].$$

$$12. \int_0^{\infty} \frac{\operatorname{sh} ax \cos bx}{x \operatorname{ch} cx} dx = \frac{1}{2} \ln \frac{\operatorname{ch} [b\pi/(2c)] + \sin [a\pi/(2c)]}{\operatorname{ch} [b\pi/(2c)] - \sin [a\pi/(2c)]} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{\cos ax \sin bx}{x \operatorname{ch} cx} dx = \operatorname{arctg} \frac{\operatorname{sh} [b\pi/(2c)]}{\operatorname{ch} [a\pi/(2c)]} \quad [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

2.5.49. Интегралы, содержащие $A(x) (a + b \operatorname{ch} x)^{\nu} \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix}$.

$$1. \int_0^a \frac{\cos bx}{(\operatorname{ch} a - \operatorname{ch} x)^{\nu}} dx = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-\nu)}{\operatorname{sh}^{\nu-1/2} a} P_{ib-1/2}^{\nu-1/2}(\operatorname{ch} a) \quad [a > 0, \operatorname{Re} \nu < 1].$$

$$2. \int_0^{\infty} \frac{\cos bx}{a + \operatorname{ch} cx} dx = \frac{\pi}{c \sqrt{a^2 - 1}} \operatorname{cosech} \frac{b\pi}{c} \sin \left(\frac{b}{c} \operatorname{Arch} a \right) \quad [a > 1; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$3. = -\frac{\pi}{c \sqrt{a^2 - 1}} \operatorname{sech} \frac{b\pi}{c} \sin \left[\frac{b}{c} \operatorname{Arch} (-a) \right] \quad [a < -1; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$4. = \frac{\pi}{c \sqrt{1 - a^2}} \operatorname{cosech} \frac{b\pi}{c} \operatorname{sh} \left(\frac{b}{c} \arccos a \right) \quad [-1 < a < 1; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{\cos bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \operatorname{sh} (b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{\cos bx}{(a + \operatorname{ch} cx)^{\nu}} dx = \frac{e^{b\pi/c}}{c} (a^2 - 1)^{-\nu/2} \Gamma \left[\begin{matrix} \nu - ib/c \\ \nu \end{matrix} \right] Q_{\nu-1}^{bi/c} \left(\frac{a}{\sqrt{a^2 - 1}} \right) \quad [b, \operatorname{Re}(c\nu) > 0; a \notin [-1, 1]].$$

$$7. \int_0^{\infty} \frac{\cos bx}{\sqrt{a + \operatorname{ch} cx}} dx = \frac{\pi}{c\sqrt{2}} \operatorname{sech} \frac{b\pi}{c} P_{ib/c-1/2}(a) \quad [-1 < a < 1, \operatorname{Re} c > 2 + |\operatorname{Im} b|].$$

$$8. \int_0^{\infty} \frac{\cos bx}{(1 + \operatorname{ch} cx)^v} \operatorname{ch} cx dx = \frac{2^{v-1}}{c} B\left(v + \frac{ib}{c}, v - \frac{ib}{c}\right) \\ [\operatorname{Re} v > 0; \operatorname{Re}(vc) > \operatorname{Re} c + |\operatorname{Im} b|].$$

$$9. \int_0^{\infty} \frac{x \sin bx}{1 + \operatorname{ch} cx} dx = \frac{\pi}{c^2} \operatorname{cosech}^2 \frac{b\pi}{c} \left(b\pi \operatorname{ch} \frac{b\pi}{c} - c \operatorname{sh} \frac{b\pi}{c} \right) \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$10. \int_0^{\infty} \frac{x^{2n+\delta}}{(1 + \operatorname{ch} cx)^v} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{(-1)^{n+\delta-1} \pi}{c^2} \Gamma\left[\begin{matrix} 2-v \\ v-2, 2v \end{matrix}\right] \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} \frac{b}{\operatorname{sh}(b\pi/c)} \\ \left[\operatorname{Re}(vc) > |\operatorname{Im} b|; \delta = \left\{ \frac{1}{0} \right\} \right].$$

$$11. \int_0^{\infty} \begin{Bmatrix} \operatorname{sh} ax \sin bx \\ \operatorname{ch} ax \cos bx \end{Bmatrix} \frac{dx}{\cos t + \operatorname{ch} cx} = \frac{\pi}{c \sin t} \left(\operatorname{ch} \frac{2b\pi}{c} - \cos \frac{2a\pi}{c} \right)^{-1} \times \\ \times \begin{Bmatrix} \sin[a(\pi-t)/c] \operatorname{sh}[b(\pi+t)/c] - \sin[a(\pi+t)/c] \operatorname{sh}[b(\pi-t)/c] \\ \cos[a(\pi-t)/c] \operatorname{ch}[b(\pi+t)/c] - \cos[a(\pi+t)/c] \operatorname{ch}[b(\pi-t)/c] \end{Bmatrix} \\ [0 < |t| < \pi; \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|; \pi \operatorname{Re} c > |\operatorname{Re}(ct)|].$$

$$12. \int_0^{\infty} \begin{Bmatrix} \operatorname{sh} ax \sin bx \\ \operatorname{ch} ax \cos bx \end{Bmatrix} \frac{dx}{1 + \operatorname{ch} cx} = \frac{2\pi}{c^2 [\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)]} \times \\ \times \left[b \begin{Bmatrix} \sin(a\pi/c) \operatorname{ch}(b\pi/c) \\ \cos(a\pi/c) \operatorname{sh}(b\pi/c) \end{Bmatrix} \mp a \begin{Bmatrix} \cos(a\pi/c) \operatorname{sh}(b\pi/c) \\ \sin(a\pi/c) \operatorname{ch}(b\pi/c) \end{Bmatrix} \right] \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{\operatorname{sh} cx \sin bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi}{c} \operatorname{ch} \frac{bt}{c} \operatorname{cosech} \frac{b\pi}{c} \quad [b, c > 0, 0 < |t| < \pi].$$

$$14. \int_0^{\infty} \frac{\operatorname{ch} cx \cos bx}{\cos t + \operatorname{ch} cx} dx = -\frac{\pi}{c} \operatorname{sh} \frac{bt}{c} \operatorname{ctg} t \operatorname{cosech} \frac{b\pi}{c} \quad [b, c > 0; 0 < |t| < \pi].$$

$$15. \int_0^{\infty} \frac{\operatorname{ch} cx \cos bx}{1 + \operatorname{ch} cx} dx = -\frac{\pi b}{c^2} \operatorname{cosech} \frac{b\pi}{c} \quad [b, c > 0].$$

$$16. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \sin bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi \operatorname{sh}(bt/c)}{2c \operatorname{ch}(b\pi/c) \sin(t/2)} \quad [\operatorname{Re} c > 2 + |\operatorname{Im} b|; 0 < |t| < \pi].$$

$$17. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \sin bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \operatorname{ch}(b\pi/c)} \quad [\operatorname{Re} c > 2 + |\operatorname{Im} b|].$$

$$18. \int_0^{\infty} \frac{\operatorname{sh} cx \sin bx}{(\cos t + \operatorname{ch} cx)^2} dx = \frac{b\pi \operatorname{sh}(bt/c)}{c^2 \sin t \operatorname{sh}(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Im} b|; |t| < \pi].$$

$$19. \int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)(\cos t + \operatorname{ch} cx)} dx = \frac{\pi e^{bz}}{2z(\cos cz + \cos t)} +$$

$$+ \frac{c\pi}{\operatorname{sh} t} \sum_{k=0}^{\infty} \left\{ \frac{\exp[-b\pi[(2k+1)\pi - t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) - t]^2} - \frac{\exp[-b\pi[(2k+1)\pi + t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) + t]^2} \right\}$$

$$[\operatorname{Re} c \geq |\operatorname{Im} b|; |t| < \pi, \operatorname{Re} z > 0].$$

2.5.49. Интегралы, содержащие $A(x)(a + b \operatorname{ch}^2 x)^{\nu} \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{\cos bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{cosec} \frac{b\pi}{2c} \sin \frac{b \ln(1 + \sqrt{2})}{2c} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|].$$

$$2. \int_0^{\infty} \frac{\operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \cos bx dx = \frac{\pi}{2c\sqrt{2}} \operatorname{sech} \frac{b\pi}{2c} \cos \frac{b \ln(1 + \sqrt{2})}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$3. \int_0^{\infty} \frac{\cos bx}{\operatorname{sh}^2 t + \operatorname{ch}^2 cx} dx = \frac{\pi}{c} \left(\operatorname{sh} 2t \operatorname{sh} \frac{b\pi}{2c} \right)^{-1} \sin \frac{bt}{c} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|].$$

$$4. \int_0^{\infty} \frac{\cos bx}{\sqrt{\operatorname{sh}^2 t + \operatorname{ch}^2 cx}} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} P_{ib/(2a)-1/2}(\operatorname{ch} 2t) \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{x^{-1/2}}{1 + 2 \operatorname{ch}^2 cx} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \sqrt{\frac{2\pi}{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\sqrt{c^2 k^2 + b^2} + kc}{c^2 k^2 + b^2} \right)^{1/2} \sin \frac{k\pi}{3}$$

$$[2 \operatorname{Re} c > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{x^{2n+1} \operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \sin bx dx =$$

$$= (-1)^{n+1} \frac{\pi}{2c\sqrt{2}} \frac{\partial^{2n+1}}{\partial b^{2n+1}} \left[\operatorname{sech} \frac{b\pi}{2a} \cos \frac{b \ln(1 + \sqrt{2})}{2a} \right] \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

2.5.50. Интегралы, содержащие $A(x)(\operatorname{ch} ax \pm \cos bx)^{-1} \begin{Bmatrix} \sin cx \\ \cos cx \end{Bmatrix}$.

$$1. \int_0^{\infty} \begin{Bmatrix} \operatorname{sh} ax \sin bx \\ \operatorname{ch} ax \cos bx \end{Bmatrix} \frac{dx}{\operatorname{ch} 2ax + \cos 2bx} = \frac{\pi}{4(a^2 + b^2)} \begin{Bmatrix} b \\ a \end{Bmatrix} \quad [b, \operatorname{Re} a > 0].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1} \operatorname{sh} ax \sin bx}{\operatorname{ch} 2ax + \cos 2bx} dx = \frac{\Gamma(\alpha)}{(a^2 + b^2)^{\alpha/2}} \sin \left(\alpha \operatorname{arctg} \frac{b}{a} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha}}$$

$$[b, \operatorname{Re} a > 0; \operatorname{Re} \alpha > -2].$$

$$3. \int_0^{\infty} \frac{\sin nx}{x (\operatorname{ch} x + \cos x)} dx = \frac{\pi}{4} \quad [n = 1, 2, 3, \dots].$$

$$4. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\sin bx}{\operatorname{ch} cx - \cos bx} dx = \frac{b}{2c} \frac{b^2 + 2c^2}{b^2 + c^2} - \operatorname{arctg} \frac{b}{c} \quad [by = 2\pi; b, \operatorname{Re} c > 0].$$

$$5. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\sin bx}{\operatorname{ch} cx \pm \cos bx} dx = \pm \operatorname{arctg} \frac{b}{c} + \left\{ \frac{-bc^{-1}}{bc(c^2 + b^2)^{-1}} \right\} \\ [by = \pi; y, \operatorname{Re} c > 0].$$

$$6. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\operatorname{ch} ax \sin bx}{\operatorname{ch} 2ax - \cos 2bx} dx = -\frac{b^3}{2a(a^2 + b^2)} \quad [yb = \pi; b, \operatorname{Re} a > 0].$$

2.5.51. Интегралы, содержащие $(x^2 + z^2)^{-1} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}^{-1} \left\{ \frac{\sin bx}{\cos bx} \right\}$,

$(x^2 + z^2)^{-1} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}^{-1} \left\{ \frac{\cos bx}{\sin bx} \right\}$.

$$1. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{\pi}{2cz^2} - \frac{\pi e^{-bz}}{2z \sin cz} - c\pi \sum_{k=1}^{\infty} \frac{(-1)^k e^{-bk\pi/c}}{\pi^2 k^2 - c^2 z^2} \\ [\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi n].$$

$$2. = (-1)^n b \frac{e^{-bn\pi/c}}{2n} + \frac{c}{2\pi n} \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} e^{-bk\pi/c} + \\ + (-1)^n \frac{c}{2\pi n} e^{-bn\pi/c} \ln(1 + e^{-b\pi/c}) + \frac{c}{2\pi n!} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{(1+z)^{n-1}}{z} \ln(1+z) \right) \Big|_{z=e^{-b\pi/c}} \\ [cz = \pi n; \operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; n = 1, 2, 3, \dots].$$

$$3. = -\frac{b}{2} \operatorname{ch} \frac{b\pi}{c} + \frac{c}{\pi} \operatorname{sh} \frac{b\pi}{c} \ln \left(2 \operatorname{ch} \frac{b\pi}{2c} \right) \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$4. = c \operatorname{sh} \frac{b\pi}{2c} - \frac{2c}{\pi} \operatorname{ch} \frac{b\pi}{2c} \operatorname{arctg} \left(\operatorname{sh} \frac{b\pi}{2c} \right) = \\ = \frac{2c}{\pi} [e^{b\pi/(2c)} \operatorname{arctg} e^{-b\pi/(2c)} - e^{-b\pi/(2c)} \operatorname{arctg} e^{b\pi/(2c)}] \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. = -2\sqrt{2} c e^{-b\pi/(4c)} + \frac{2\sqrt{2} c}{\pi} \operatorname{sh} \frac{b\pi}{4c} \ln \frac{\operatorname{ch} [b\pi/(4c)] + 1/\sqrt{2}}{\operatorname{ch} [b\pi/(4c)] - 1/\sqrt{2}} + \\ + \frac{4\sqrt{2} c}{\pi} \operatorname{ch} \frac{b\pi}{4c} \operatorname{arctg} \frac{1}{\sqrt{2} \operatorname{sh} [b\pi/(4c)]} \quad [4cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\sin bx}{\operatorname{ch} cx} dx = \pi^2 \sum_{k=0}^{\infty} (-1)^k \frac{(k+1/2)^2 e^{-bz} - (k+1/2) e^{-b\pi(k+1/2)/c}}{\pi^2 (k+1/2)^2 - c^2 z^2} \\ [\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi(n+1/2)].$$

$$7. = -2 \operatorname{sh} \frac{b\pi}{2c} + \frac{\pi}{2} \operatorname{sh} \frac{b\pi}{c} - 2 \operatorname{ch} \frac{b\pi}{c} \operatorname{arctg} \operatorname{th} \frac{b\pi}{4c} \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$

8.
$$= \frac{\pi}{\sqrt{2}} e^{-b\pi/(4c)} + \frac{\operatorname{sh} [b\pi/(4c)]}{\sqrt{2}} \ln \frac{\operatorname{ch} [b\pi/(4c)] + 1/\sqrt{2}}{\operatorname{ch} [b\pi/(4c)] - 1/\sqrt{2}} -$$

$$- \sqrt{2} \operatorname{ch} \frac{b\pi}{4c} \operatorname{arctg} \frac{1}{\sqrt{2} \operatorname{sh} [b\pi/(4c)]} \quad [4cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
9.
$$\int_0^\infty \frac{x}{x^2 + z^2} \frac{\cos bx}{\operatorname{sh} cx} dx = \frac{\pi e^{-bz}}{2 \sin cz} + \pi^2 \sum_{k=1}^\infty \frac{(-1)^k k e^{-bk\pi/c}}{\pi^2 k^2 - c^2 z^2}$$

$$[\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi n].$$
10.
$$= -\frac{1}{2} + \frac{\pi b}{2c} e^{-b\pi/c} + \operatorname{ch} \frac{b\pi}{c} \ln (1 + e^{-b\pi/c}) =$$

$$= -\frac{b\pi}{2c} \operatorname{sh} \frac{b\pi}{c} - \frac{1}{2} + \frac{1}{2} \operatorname{ch} \frac{b\pi}{c} \ln \left[2 \left(1 + \operatorname{ch} \frac{b\pi}{c} \right) \right] \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
11.
$$= 2 \operatorname{sh} \frac{b\pi}{2c} \operatorname{arctg} (e^{-b\pi/(2c)}) + \frac{\pi}{2} e^{-b\pi/(2c)} - 1 =$$

$$= \frac{\pi}{2} \operatorname{ch} \frac{b\pi}{2c} - 1 + \operatorname{sh} \frac{b\pi}{2c} \operatorname{arctg} \operatorname{sh} \frac{b\pi}{2c} \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
12.
$$\int_0^\infty \frac{1}{x^2 + z^2} \frac{\cos bx}{\operatorname{ch} cx} dx = \frac{\pi}{z} \sum_{k=0}^\infty (-1)^k \frac{\pi (k + 1/2)^2 e^{-bz} - cze^{-b\pi(2k+1)/(2c)}}{\pi^2 (k + 1/2)^2 - c^2 z^2}$$

$$[\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi(n + 1/2)].$$
13.
$$= \frac{(-1)^n}{2n+1} \exp \left[-\frac{b}{c} (n + 1/2) \pi \right] \left[b + \frac{c}{\pi} \ln (1 + e^{-b\pi/c}) \right] +$$

$$+ \frac{ce^{-b\pi/(2c)}}{\pi(2n+1)} \sum_{k=0}^{n-1} \frac{(-1)^k}{k-n} e^{-bk\pi/c} + \frac{ce^{-b\pi/(2c)}}{\pi(2n+1)(n+1)} F(n+1, 1; n+2; -e^{-b\pi/c})$$

$$[sz = \pi(n + 1/2); \operatorname{Re} c > |\operatorname{Im} b|].$$
14.
$$= \frac{2c}{\pi} \operatorname{ch} \frac{b\pi}{2c} - e^{b\pi/c} \operatorname{arctg} (e^{-b\pi/(2c)}) - e^{-b\pi/c} \operatorname{arctg} (e^{b\pi/(2c)})$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
15.
$$= be^{-b\pi/(2c)} + \frac{2c}{\pi} \operatorname{ch} \frac{b\pi}{c} \ln (1 + e^{-2b\pi/c}) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
16.
$$= 2\sqrt{2} ce^{-b\pi/(4c)} + \frac{4\sqrt{2}c}{\pi} \operatorname{sh} \frac{b\pi}{4c} \operatorname{arctg} \frac{1}{\sqrt{2} \operatorname{sh} [b\pi/(4c)]} -$$

$$- \frac{2\sqrt{2}c}{\pi} \operatorname{ch} \frac{b\pi}{4c} \ln \frac{\operatorname{ch} [b\pi/(4c)] + 1/\sqrt{2}}{\operatorname{ch} [b\pi/(4c)] - 1/\sqrt{2}} \quad [4cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|].$$
17.
$$\int_0^\infty \frac{\operatorname{th} ax \sin bx}{x^2 + z^2} dx = b \operatorname{ch} \frac{b\pi}{2a} - \frac{2a}{\pi} \operatorname{sh} \frac{b\pi}{2a} \ln \left(2 \operatorname{sh} \frac{b\pi}{2a} \right) \quad [2az = \pi; b, \operatorname{Re} z > 0].$$
18.
$$= -\frac{\pi}{2z} e^{bz} + \frac{\operatorname{sh} bz}{z} \ln \operatorname{cth} \frac{bz}{2} + \frac{2}{z} \operatorname{ch} bz \operatorname{arctg} e^{bz} \quad [4az = \pi; b, \operatorname{Re} z > 0].$$
19.
$$\int_0^\infty \frac{x \operatorname{th} ax \cos bx}{x^2 + z^2} dx = -bze^{-bz} + \operatorname{sh} bz \ln (1 - e^{-2bz}) \quad [2az = \pi; b, \operatorname{Re} z > 0].$$

$$20. \quad = -\frac{\pi}{2} e^{bz} \operatorname{ch} bz \ln \operatorname{cth} \frac{bz}{2} + 2 \operatorname{sh} bz \operatorname{arctg} e^{bz} \quad [4az = \pi; b, \operatorname{Re} z > 0].$$

$$21. \quad \int_0^\infty \frac{\operatorname{cth} ax \sin bx}{x^2 + z^2} dx = \frac{2a}{\pi} \operatorname{sh} \frac{b\pi}{2a} \ln \operatorname{cth} \frac{b\pi}{4a} \quad [2az = \pi; b, \operatorname{Re} z > 0].$$

$$22. \quad \int_0^\infty \frac{x \operatorname{cth} ax \cos bx}{x^2 + z^2} dx = -2 + \frac{\pi}{2} e^{-bz} + \operatorname{ch} bz \ln \operatorname{cth} \frac{bz}{2} + 2 \operatorname{sh} bz \operatorname{arctg} e^{-bz} \quad [4az = \pi, b, \operatorname{Re} z > 0].$$

$$23. \quad \int_0^\infty \frac{1}{x^2 + z^2} \left\{ \frac{\operatorname{sh} ax \cos bx}{\operatorname{ch} ax \sin bx} \right\} \frac{dx}{\operatorname{sh} cx} = \pm \frac{\pi}{2z} \frac{e^{-bz}}{\sin cz} \left\{ \frac{\sin az}{\cos az} \right\} \pm$$

$$\pm c\pi \sum_{k=1}^\infty (-1)^k \frac{\exp(-kb\pi/c)}{\pi^2 k^2 - c^2 z^2} \left\{ \frac{\sin(ak\pi/c)}{\cos(ak\pi/c)} \right\} + \left\{ \frac{0}{\pi(2cz^2)^{-1}} \right\}$$

$$[\operatorname{Re} c > | \operatorname{Re} a | + | \operatorname{Im} b |; \operatorname{Re} z > 0; cz \neq \pi n].$$

$$24. \quad = \frac{1}{2} \exp\left(-\frac{b\pi}{c}\right) \left[\pm b \left\{ \frac{\sin(a\pi/c)}{\cos(a\pi/c)} \right\} - a \left\{ \frac{\cos(a\pi/c)}{\sin(a\pi/c)} \right\} \right] +$$

$$+ \frac{c}{2\pi} \left\{ \frac{\operatorname{ch}(b\pi/c) \sin(a\pi/c)}{\operatorname{sh}(b\pi/c) \cos(a\pi/c)} \right\} \ln \left[1 + 2 \exp\left(-\frac{b\pi}{c}\right) \cos \frac{a\pi}{c} + \exp\left(-\frac{2b\pi}{c}\right) \right] \mp$$

$$\mp \frac{c}{\pi} \left\{ \frac{\operatorname{sh}(b\pi/c) \cos(a\pi/c)}{\operatorname{ch}(b\pi/c) \sin(a\pi/c)} \right\} \operatorname{arctg} \frac{\sin(a\pi/c)}{\exp(b\pi/c) + \cos(a\pi/c)}$$

$$[cz = \pi, \operatorname{Re} c > | \operatorname{Re} a | + | \operatorname{Im} b |].$$

$$25. \quad = \pm c \exp\left(-\frac{b\pi}{2c}\right) \left\{ \frac{\sin[a\pi/(2c)]}{\cos[a\pi/(2c)]} \right\} \mp$$

$$\mp \frac{c}{\pi} \left\{ \frac{\operatorname{ch}[b\pi/(2c)] \cos[a\pi/(2c)]}{\operatorname{sh}[b\pi/(2c)] \sin[a\pi/(2c)]} \right\} \ln \frac{\operatorname{ch}[b\pi/(2c)] + \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] - \sin[a\pi/(2c)]} +$$

$$+ \frac{2c}{\pi} \left\{ \frac{\operatorname{sh}[b\pi/(2c)] \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] \cos[a\pi/(2c)]} \right\} \operatorname{arctg} \frac{\cos[a\pi/(2c)]}{\operatorname{sh}[b\pi/(2c)]}$$

$$[2cz = \pi; \operatorname{Re} c > | \operatorname{Re} a | + | \operatorname{Im} b |].$$

$$26. \quad \int_0^\infty \frac{1}{x^2 + z^2} \left\{ \frac{\operatorname{sh} ax \sin bx}{\operatorname{ch} ax \cos bx} \right\} \frac{dx}{\operatorname{ch} cx} =$$

$$= \exp\left(-\frac{b\pi}{2c}\right) \left[b \left\{ \frac{\sin[a\pi/(2c)]}{\cos[a\pi/(2c)]} \right\} \mp a \left\{ \frac{\cos[a\pi/(2c)]}{\sin[a\pi/(2c)]} \right\} \right] \mp$$

$$\mp \frac{c}{\pi} \left\{ \frac{\operatorname{sh}[b\pi/(2c)] \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] \cos[a\pi/(2c)]} \right\} \ln \left[1 + 2 \exp\left(-\frac{b\pi}{c}\right) \cos \frac{a\pi}{c} + \exp\left(-\frac{2b\pi}{c}\right) \right] +$$

$$+ \frac{c}{\pi} \left\{ \frac{\operatorname{ch}[b\pi/(2c)] \cos[a\pi/(2c)]}{2 \operatorname{sh}[b\pi/(2c)] \sin[a\pi/(2c)]} \right\} \operatorname{arctg} \left[\frac{\sin(a\pi/c)}{1 + \exp(-b\pi/c) \cos(a\pi/c)} \left\{ \frac{1}{\exp(-b\pi/c)} \right\} \right]$$

$$[2cz = \pi, \operatorname{Re} c > | \operatorname{Re} a | + | \operatorname{Im} b |].$$

$$27. \quad \int_0^\infty \frac{x}{x^2 + z^2} \left\{ \frac{\operatorname{sh} ax \sin bx}{\operatorname{ch} ax \cos bx} \right\} \frac{dx}{\operatorname{sh} cx} =$$

$$= \frac{\pi e^{-bz}}{2 \sin cz} \left\{ \frac{\sin az}{\cos az} \right\} + \pi^2 \sum_{k=1}^\infty (-1)^k \frac{k \exp(-kb\pi/c)}{\pi^2 k^2 - c^2 z^2} \left\{ \frac{\sin(ak\pi/c)}{\cos(ak\pi/c)} \right\}$$

$$[\operatorname{Re} c > | \operatorname{Re} a | + | \operatorname{Im} b |; \operatorname{Re} z > 0; cz \neq \pi n].$$

$$\begin{aligned}
 28. \quad &= \frac{\pi}{2c} \exp\left(-\frac{b\pi}{c}\right) \left[b \begin{Bmatrix} \sin(a\pi/c) \\ \cos(a\pi/c) \end{Bmatrix} \mp a \begin{Bmatrix} \cos(a\pi/c) \\ \sin(a\pi/c) \end{Bmatrix} \right] - \frac{1}{2} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mp \\
 &\mp \frac{1}{2} \begin{Bmatrix} \operatorname{sh}(b\pi/c) \sin(a\pi/c) \\ \operatorname{ch}(b\pi/c) \cos(a\pi/c) \end{Bmatrix} \ln \left[1 + 2 \exp\left(-\frac{b\pi}{c}\right) \cos \frac{a\pi}{c} + \exp\left(-\frac{2b\pi}{c}\right) \right] + \\
 &+ \begin{Bmatrix} \operatorname{ch}(b\pi/c) \cos(a\pi/c) \\ \operatorname{sh}(b\pi/c) \sin(a\pi/c) \end{Bmatrix} \operatorname{arctg} \frac{\sin(a\pi/c)}{\exp(b\pi/c) + \cos(a\pi/c)} \\
 &[cz = \pi; \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].
 \end{aligned}$$

$$\begin{aligned}
 29. \quad &= \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + \frac{\pi}{2} \exp\left(-\frac{b\pi}{2c}\right) \begin{Bmatrix} \sin[a\pi/(2c)] \\ \cos[a\pi/(2c)] \end{Bmatrix} + \\
 &+ \frac{1}{2} \begin{Bmatrix} \cos[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)] \\ \sin[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)] \end{Bmatrix} \ln \frac{\operatorname{ch}[b\pi/(2c)] + \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] - \sin[a\pi/(2c)]} \mp \\
 &\mp \begin{Bmatrix} \sin[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)] \\ \cos[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)] \end{Bmatrix} \operatorname{arctg} \frac{\cos[a\pi/(2c)]}{\operatorname{sh}[b\pi/(2c)]} \\
 &[2cz = \pi, \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].
 \end{aligned}$$

$$\begin{aligned}
 30. \quad &\int_0^\infty \frac{\operatorname{ch} x}{\sqrt{\operatorname{ch}^2 x - 1}} [(\operatorname{ch} x + \sqrt{\operatorname{ch}^2 x - 1})^\nu - (\operatorname{ch} x - \sqrt{\operatorname{ch}^2 x - 1})^\nu] \cos bx \, dx = \\
 &= \frac{\pi \sin \nu\pi}{\operatorname{ch} b\pi + \cos \nu\pi} \quad [b, \operatorname{Re} \nu > 0].
 \end{aligned}$$

2.5.52. Интегралы, содержащие $A(x)$, $\operatorname{sh} bx$, $\operatorname{ch} bx$, $\sin ax^2$, $\cos bx^2$.

$$1. \int_0^\infty \frac{x^n}{\operatorname{sh} \pi x} \begin{Bmatrix} \sin \pi x^2 \\ \cos \pi x^2 \end{Bmatrix} dx = I_n^\pm,$$

$$I_1^+ = \frac{1}{4\pi}, \quad I_3^+ = \frac{1}{16\pi}, \quad I_1^- = \frac{1}{8}, \quad I_3^- = \frac{1}{16} \left(\frac{1}{4} - \frac{3}{\pi^2} \right).$$

$$2. \int_0^\infty \frac{x^2}{\operatorname{ch} \pi x} \begin{Bmatrix} \sin \pi x^2 \\ \cos \pi x^2 \end{Bmatrix} dx = \pm \frac{1}{8} \begin{Bmatrix} 1 \\ 2\pi \end{Bmatrix} - \frac{1}{8\sqrt{2}}.$$

$$\begin{aligned}
 3. \quad &\int_0^\infty \frac{\sin bx}{\operatorname{sh} cx} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} dx = \pm \frac{\pi}{2c} \operatorname{cosech} \frac{b\pi}{2c} \left[\begin{Bmatrix} \sin[b^2\pi/(4c^2)] \\ \cos[b^2\pi/(4c^2)] \end{Bmatrix} - \begin{Bmatrix} 0 \\ \operatorname{ch}(b\pi/c) \end{Bmatrix} \right] \\
 &[a\pi = c^2; a, b, \operatorname{Re} c > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \quad &\int_0^\infty \frac{\cos bx}{\operatorname{ch} cx} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} dx = -\frac{\pi}{c} \sum_{k=0}^\infty (\pm 1)^{k-1} \exp \left[-\left(k + \frac{1}{2}\right) \frac{b\pi}{c} \right] \begin{Bmatrix} \sin d_k \\ \cos d_k \end{Bmatrix} + \\
 &+ \sqrt{\frac{\pi}{a}} \sum_{k=0}^\infty \exp \left[-\frac{bc}{a} \left(k + \frac{1}{2}\right) \right] \begin{Bmatrix} \sin h_k \\ \cos h_k \end{Bmatrix} \\
 &\left[d_k = \frac{a\pi^2}{c^2} \left(k + \frac{1}{2}\right)^2, \quad h_k = \frac{\pi}{4} - \frac{b^2}{4a} + \frac{c^2}{a} \left(k + \frac{1}{2}\right)^2; a, b, \operatorname{Re} c > 0 \right].
 \end{aligned}$$

$$\begin{aligned}
 5. \quad &\int_0^\infty \frac{\cos bx}{\operatorname{ch} cx} \begin{Bmatrix} \sin ax^2 \\ \cos ax^2 \end{Bmatrix} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} \left[\begin{Bmatrix} \cos d \\ \sin d \end{Bmatrix} \mp \frac{1}{\sqrt{2}} \right] \\
 &[d = b^2\pi/(4c^2), a\pi = c^2, a, b > 0].
 \end{aligned}$$

$$6. \int_0^{\infty} \frac{\sin bx}{\operatorname{th} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \pm \frac{1}{2} \operatorname{th} \frac{b}{2} \left\{ \frac{\sin [\pi/4 + b^2/(4\pi)]}{\cos [\pi/4 + b^2/(4\pi)] - 1} \right\} \quad [b > 0].$$

$$7. \int_0^{\infty} \frac{\operatorname{sh} bx}{\operatorname{sh} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \frac{1}{2} \operatorname{cosec} \frac{b}{2} \left\{ \frac{\sin [b^2/(4\pi)]}{\cos (b/2) - \cos [b^2/(4\pi)]} \right\} \quad [| \operatorname{Re} b | < \pi].$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{ch} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \pm 2^{-3/2} \sec \frac{b}{2} \left[\sqrt{2} \left\{ \frac{\cos [b^2/(4\pi)]}{\sin [b^2/(4\pi)]} \right\} - 1 \right] \quad [| \operatorname{Re} b | < \pi].$$

$$9. \int_0^{\infty} \frac{\cos bx}{1 + 2 \operatorname{ch} (2x\pi/\sqrt{3})} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \\ = \pm \left(4 \operatorname{ch} \frac{b}{\sqrt{3}} - 2 \right)^{-1} \left\{ \frac{\cos d}{\sin d} \right\} + \left\{ \frac{-\sqrt{3}}{1} \right\} \quad [d = \frac{\pi}{12} - \frac{b^2}{4\pi}, b > 0].$$

$$10. \int_0^{\infty} \frac{x \operatorname{sh} (x\pi/2) \sin (x\pi/2) \sin ax^2}{\operatorname{ch} x\pi + \cos x\pi} dx = \frac{1}{4} \frac{\partial \theta_1(z, e^{-2\alpha})}{\partial z} \Big|_{z=0} \quad [a > 0].$$

2.5.53. Интегралы от $A(x) \left\{ \frac{\operatorname{sh} (\sqrt{a^2 - x^2}) \sin bx}{\operatorname{ch} (\sqrt{a^2 - x^2}) \cos bx} \right\}$,

$$A(x) \left\{ \frac{\operatorname{sh} (\sqrt{a^2 - x^2}) \cos bx}{\operatorname{ch} (\sqrt{a^2 - x^2}) \sin bx} \right\}.$$

$$1. \int_0^a \operatorname{sh} (c \sqrt{a^2 - x^2}) \cos bx dx = \frac{\pi ac}{2 \sqrt{c^2 - b^2}} I_1 (a \sqrt{c^2 - b^2}) \quad [0 < b < c; a > 0].$$

$$2. = \frac{\pi a^2 c}{4} \quad [b = c > 0; a > 0].$$

$$3. = \frac{\pi ac}{2 \sqrt{b^2 - c^2}} J_1 (a \sqrt{b^2 - c^2}) \quad [0 < c < b; a > 0].$$

$$4. \int_0^a \frac{\operatorname{sh} (c \sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx dx = \\ = \frac{\pi}{2} \sqrt{\frac{\pi c}{2}} J_{3/4} \left[\frac{a}{2} (b - \sqrt{b^2 - c^2}) \right] J_{1/4} \left[\frac{a}{2} (b + \sqrt{b^2 - c^2}) \right] \quad [a > 0].$$

$$5. \int_0^a \frac{\operatorname{ch} (c \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} \cos bx dx = \frac{\pi}{2} I_0 (a \sqrt{c^2 - b^2}) \quad [0 < b < c; a > 0].$$

$$6. = \frac{\pi}{2} \quad [b = c > 0; a > 0].$$

$$7. = \frac{\pi}{2} J_0 (a \sqrt{b^2 - c^2}) \quad [0 < c < b; a > 0].$$

$$8. \int_0^a \frac{\operatorname{ch}(c \sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx \, dx =$$

$$= \left(\frac{\pi}{2}\right)^{3/2} \sqrt{c} J_{-1/4} \left[\frac{a}{2} (b - \sqrt{b^2 - c^2}) \right] J_{-1/4} \left[\frac{a}{2} (b + \sqrt{b^2 - c^2}) \right] \quad [a > 0].$$

$$9. \int_0^a \sqrt{a^2 - x^2} \operatorname{ch}(c \sqrt{a^2 - x^2}) \cos bx \, dx =$$

$$= \frac{\pi a^2 c^2}{2(c^2 - b^2)} \left[I_0(a \sqrt{c^2 - b^2}) - \frac{b^2 + c^2}{ac^2 \sqrt{c^2 - b^2}} I_1(a \sqrt{c^2 - b^2}) \right]$$

$$[0 < b < c; a > 0].$$

$$10. = \frac{\pi a^2}{4} (4 + a^2 b^2) \quad [c = b > 0; a > 0].$$

$$11. = \frac{\pi a^2 c^2}{2(c^2 - b^2)} \left[J_0(a \sqrt{b^2 - c^2}) - \frac{b^2 + c^2}{ac^2 \sqrt{b^2 - c^2}} J_1(a \sqrt{b^2 - c^2}) \right]$$

$$[0 < c < b, a > 0].$$

$$12. \int_0^a \frac{\operatorname{ch}(c \sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \cos bx \, dx =$$

$$= \left(\frac{\pi}{2}\right)^{3/2} \sqrt{b} J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 - c^2} - ic) \right] J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 - c^2} + ic) \right] \quad [a > 0].$$

2.5.54. Интегралы, содержащие тригонометрические функции от гиперболических функций.

$$1. \int_0^\infty \cos(a \operatorname{ch} x) \, dx = -\frac{\pi}{2} Y_0(a) \quad [\operatorname{Im} a = 0; |\operatorname{Re} a| < 1/2].$$

$$2. \int_0^\infty \operatorname{sh}^\mu x \cos(a \operatorname{ch} x) \, dx = -\frac{\sqrt{\pi}}{2} \Gamma\left(\frac{\mu+1}{2}\right) \left(\frac{a}{2}\right)^{-\mu/2} Y_{-\mu/2}(a)$$

$$[\operatorname{Im} a = 0; |\operatorname{Re} \mu| < 1].$$

$$3. \int_0^\infty \frac{\cos bx}{\sqrt{\operatorname{ch} x}} \left\{ \begin{matrix} \sin(a \operatorname{ch} x) \\ \cos(a \operatorname{ch} x) \end{matrix} \right\} dx = -\frac{\pi^{3/2}}{4} \sqrt{\frac{a}{2}} \times$$

$$\times \left[J_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) Y_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) + J_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) Y_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) \right]$$

$$[a, b > 0].$$

$$4. \int_0^\infty \frac{\sin bx}{\sqrt{\operatorname{sh} x}} \left\{ \begin{matrix} \sin(a \operatorname{sh} x) \\ \cos(a \operatorname{sh} x) \end{matrix} \right\} dx =$$

$$= -\frac{i}{2} \sqrt{\frac{\pi a}{2}} \left[I_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) K_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) - I_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) K_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) \right]$$

$$[a, b > 0].$$

$$5. \int_0^{\infty} \frac{\cos bx}{\sqrt{\operatorname{sh} x}} \left\{ \begin{matrix} \sin (a \operatorname{sh} x) \\ \cos (a \operatorname{sh} x) \end{matrix} \right\} dx =$$

$$= \frac{1}{2} \sqrt{\frac{\pi a}{2}} \left[I_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) K_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) + I_{\pm 1/4 + ib/2} \left(\frac{a}{2} \right) K_{\pm 1/4 - ib/2} \left(\frac{a}{2} \right) \right]$$

[a, b > 0].

$$6. \int_0^{\infty} \left\{ \begin{matrix} \operatorname{sh} cx \sin (a \operatorname{sh} x) \\ \operatorname{ch} cx \cos (a \operatorname{sh} x) \end{matrix} \right\} dx = \left\{ \begin{matrix} \sin (c\pi/2) \\ \cos (c\pi/2) \end{matrix} \right\} K_c(a) \quad [a > 0, |\operatorname{Re} c| < 1].$$

$$7. \int_0^{\infty} \operatorname{ch} cx \left\{ \begin{matrix} \sin (a \operatorname{ch} x - c\pi/2) \\ \cos (a \operatorname{ch} x - c\pi/2) \end{matrix} \right\} dx = \pm \frac{\pi}{2} \left\{ \begin{matrix} J_c(a) \\ Y_c(a) \end{matrix} \right\} \quad [a > 0; |\operatorname{Re} c| < 1].$$

2.5.55. Интегралы, содержащие гиперболические функции от тригонометрических функций.

$$1. \int_0^{\pi/2} \operatorname{ch} \left(a \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right) \cos 2nx \, dx = (-1)^n \frac{\pi}{2} I_{2n}(a).$$

$$2. \int_0^{\pi/2} \operatorname{sh} \left(a \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right) \left\{ \begin{matrix} \sin (2n+1)x \\ \cos (2n+1)x \end{matrix} \right\} dx = (-1)^n \frac{\pi}{2} I_{2n+1}(a).$$

$$3. \int_0^{\pi/2} \operatorname{ch} (a \cos x) \sin^{\mu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{a} \right)^{\mu/2} \Gamma \left(\frac{\mu+1}{2} \right) I_{\mu/2}(a)$$

[Re a > 0, Re μ > -1].

$$4. \int_0^{\pi/2} \left\{ \begin{matrix} \operatorname{sh} (a \cos x) \\ \operatorname{ch} (a \cos x) \end{matrix} \right\} \frac{\cos bx}{\sqrt{\cos x}} dx = \left(\frac{\pi}{2} \right)^{3/2} \sqrt{a} I_{\pm 1/4 + b/2} \left(\frac{a}{2} \right) I_{\pm 1/4 - b/2} \left(\frac{a}{2} \right)$$

[a, b > 0].

$$5. \int_0^{\pi/2} \operatorname{ch} (a \cos x) \cos (c \sin x) \, dx = \frac{\pi}{2} J_0(\sqrt{c^2 - a^2})$$

[a, c > 0].

$$6. \int_0^{\pi} \operatorname{ch} (a \sin x) \cos nx \, dx = (-1)^{n/2} \frac{1 + (-1)^n}{2} \pi I_n(a).$$

$$7. \int_0^{\pi} \operatorname{ch} (a \cos x) \sin^{\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{a} \right)^{\mu/2} \Gamma \left(\frac{\mu+1}{2} \right) I_{\mu/2}(a)$$

[Re a > 0; Re μ > -1].

$$8. \int_0^{\pi} x \operatorname{ch} (a \cos x) \sin x \, dx = \frac{\pi}{a} \operatorname{sh} a$$

[a ≠ 0].

$$9. \int_0^{\infty} \frac{\operatorname{sh}(a \sin x)}{x} \sin x \sin 2nx \cos(a \cos x) dx =$$

$$= (-1)^{n-1} \frac{\pi a^{2n-1}}{8(2n-1)!} \left[1 + \frac{a^2}{2n(2n+1)} \right] \quad [a > 0].$$

$$10. \int_0^{\infty} \frac{\operatorname{ch}(a \sin x)}{x} \sin x \cos(2n-1)x \cos(a \cos x) dx =$$

$$= (-1)^{n-1} \frac{\pi a^{2n-2}}{8(2n-2)!} \left[1 - \frac{a^2}{2n(2n-1)} \right] \quad [a > 0].$$

$$11. \int_0^{\infty} \frac{\operatorname{sh}(a \sin x)}{x} \cos x \cos 2nx \cos(a \cos x) dx =$$

$$= \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^{n-1} \pi a^{2n-1}}{8(2n-1)!} \left[1 - \frac{3a^2}{2n(n+1)} \right] \quad [a > 0].$$

$$12. \int_0^{\infty} \frac{\operatorname{ch}(a \sin bx)}{x^2 - y^2} \begin{Bmatrix} \sin(a \cos bx) \\ \cos(a \cos bx) \end{Bmatrix} dx =$$

$$= \mp \frac{\pi}{2y} \begin{Bmatrix} \cos(a \cos by) \\ \sin(a \cos by) \end{Bmatrix} \operatorname{sh}(a \sin by) \quad [a, b, y > 0].$$

$$13. \int_0^{\infty} \frac{x \operatorname{sh}(a \sin bx)}{x^2 - y^2} \begin{Bmatrix} \sin(a \cos bx) \\ \cos(a \cos bx) \end{Bmatrix} dx =$$

$$= \mp \frac{\pi}{2} \left[\begin{Bmatrix} \cos(a \cos by) \\ \sin(a \cos by) \end{Bmatrix} \operatorname{ch}(a \sin by) - \begin{Bmatrix} 1 \\ a \cos by \end{Bmatrix} \right] \quad [a, b, y > 0].$$

$$14. \int_0^{\infty} \frac{\operatorname{ch}(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} \begin{Bmatrix} \sin(2a \cos^2 x) \\ \cos(2a \cos^2 x) \end{Bmatrix} dx = \frac{\pi}{2bc} \begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix}$$

$$[d = 2ac(b+c)^{-1}; a, b, c > 0].$$

$$15. \int_0^{\infty} \frac{\sin(a \operatorname{sh} x)}{\operatorname{sh} x} \sin(a \operatorname{ch} x) dx = \frac{\pi}{2} \sin a \quad [a > 0]$$

2.5.56. Интегралы, содержащие e^{-px} , $\operatorname{sh} ax$, $\operatorname{ch} ax$, $\sin bx$, $\cos bx$.

$$1. \int_0^{\infty} e^{-px} \begin{Bmatrix} \operatorname{sh} cx \sin cx \\ \operatorname{ch} cx \cos cx \end{Bmatrix} dx = \frac{p}{p^4 + 4c^4} \begin{Bmatrix} 2c^2 \\ p^2 \end{Bmatrix} \quad [\operatorname{Re} p > |\operatorname{Re} c| + |\operatorname{Im} c|].$$

$$2. \int_0^{\infty} e^{-px} \begin{Bmatrix} \operatorname{sh} cx \cos cx \\ \operatorname{ch} cx \sin cx \end{Bmatrix} dx = \frac{c(p^2 + 2c^2)}{p^4 + 4c^4} \quad [\operatorname{Re} p > |\operatorname{Re} c| + |\operatorname{Im} c|].$$

$$3. \int_0^{\infty} e^{-px} \operatorname{sh}^{\mu} cx \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \\ = \frac{(-i)^{\delta} \Gamma(\mu+1)}{2^{\mu+2} c} \left\{ \Gamma \left[\frac{(p-\mu c-ib)/(2c)}{(p+\mu c-ib)/(2c)+1} \right] \mp \Gamma \left[\frac{(p-\mu c+ib)/(2c)}{(p+\mu c+ib)/(2c)+1} \right] \right\} \\ \left[\operatorname{Re} p > |\operatorname{Re}(\mu c)| + |\operatorname{Im} b|; \operatorname{Re} \mu > -1-\delta, \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$4. \int_0^{\infty} e^{-px} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{1}{2ci} \left[\psi \left(\frac{p+c+ib}{2c} \right) - \psi \left(\frac{p+c-ib}{2c} \right) \right] \\ [\operatorname{Re} p > |\operatorname{Re} c| + |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} e^{-cx} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \operatorname{cth} \frac{b\pi}{2c} - \frac{1}{b} \quad [b, \operatorname{Re} c > 0].$$

$$6. \int_0^{\infty} \frac{\operatorname{sh} ax \sin bx}{e^{cx} - 1} dx = -\frac{b}{2(a^2+b^2)} \pm \frac{\pi}{2c} \frac{\operatorname{sh}(2b\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} + \\ + \frac{i}{2c} \left[\psi \left(\frac{a+ib}{c} + 1 \right) - \psi \left(\frac{a-ib}{c} + 1 \right) \right] \quad [b > 0; \operatorname{Re} c > |\operatorname{Re} a|].$$

$$7. \int_0^{\infty} \frac{\operatorname{sh} ax \cos bx}{e^{cx} \pm 1} dx = \mp \frac{a}{2(a^2+b^2)} \pm \frac{\pi}{\sigma c} \frac{\sin(a\sigma\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} \begin{Bmatrix} \operatorname{ch}(b\pi/c) \\ 1 \end{Bmatrix} \\ \left[\sigma = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}; b > 0; \operatorname{Re} c > |\operatorname{Re} a| \right].$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} ax \sin bx}{e^{cx} \pm 1} dx = \pm \frac{b}{2(a^2+b^2)} \mp \frac{\pi}{\sigma c} \frac{\operatorname{sh}(b\sigma\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} \begin{Bmatrix} \cos(a\pi/c) \\ 1 \end{Bmatrix} \\ \left[\sigma = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}; b > 0; \operatorname{Re} c > |\operatorname{Re} a| \right].$$

2.5.57. Интегралы, содержащие $e^{j(x)}$, $\operatorname{sh} ax$, $\operatorname{ch} ax$, $\sin bx$, $\cos bx$.

$$1. \int_0^{\infty} e^{-px^2} \begin{Bmatrix} \operatorname{sh} cx \sin bx \\ \operatorname{ch} cx \cos bx \end{Bmatrix} dx = \frac{1}{2} \sqrt{\frac{\pi}{p}} \exp \left(-\frac{c^2-b^2}{4p} \right) \begin{Bmatrix} \sin [bc/(2p)] \\ \cos [bc/(2p)] \end{Bmatrix} \\ [\operatorname{Re} p > 0].$$

$$2. \int_0^{\infty} x^2 e^{-px^2} \begin{Bmatrix} \operatorname{sh} cx \sin cx \\ \operatorname{ch} cx \cos cx \end{Bmatrix} dx = \frac{1}{4} \sqrt{\frac{\pi}{p^3}} \left[\begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix} \pm \begin{Bmatrix} \cos d \\ \sin d \end{Bmatrix} \right] \\ [d = c^2/(2p); \operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} x^{-1/2} e^{-px^2} \begin{Bmatrix} \operatorname{sh} cx \sin cx \\ \operatorname{ch} cx \cos cx \end{Bmatrix} dx = \frac{\pi}{2^{5/4}} \sqrt{\frac{c}{p}} J_{-1/4} \left(\frac{c^2}{4p} \right) \begin{Bmatrix} \cos [c^2/(4p) + \pi/4] \\ \sin [c^2/(4p) + \pi/4] \end{Bmatrix} \\ [\operatorname{Re} p > 0].$$

4. $\int_0^{\infty} x e^{-px^2} \begin{Bmatrix} \operatorname{ch} cx \sin bx \\ \operatorname{sh} cx \cos bx \end{Bmatrix} dx =$
 $= \frac{\sqrt{\pi}}{8p^2} \sqrt{\frac{b^2 + c^2}{p}} \exp\left(-\frac{b^4 - c^4}{16p^3}\right) \left[c \begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix} \pm b \begin{Bmatrix} \cos d \\ \sin d \end{Bmatrix} \right]$
 $[d = bc(3p)^{-3}(b^2 + c^2); \operatorname{Re} p > 0].$
5. $\int_0^{\infty} x^2 e^{-px^2} \begin{Bmatrix} \operatorname{ch} cx \sin cx \\ \operatorname{sh} cx \cos cx \end{Bmatrix} dx = \frac{c}{4p} \sqrt{\frac{\pi}{p}} \left(\cos \frac{c^2}{2p} \pm \sin \frac{c^2}{2p} \right) \quad [\operatorname{Re} p > 0].$
6. $\int_0^{\infty} x^{-1/2} e^{-px^2} \begin{Bmatrix} \operatorname{ch} cx \sin cx \\ \operatorname{sh} cx \cos cx \end{Bmatrix} dx = \pm \frac{\pi}{2^{5/4}} \sqrt{\frac{c}{p}} J_{1/4}\left(\frac{c^2}{4p}\right) \begin{Bmatrix} \cos [c^2/(4p) - \pi/4] \\ \sin [c^2/(4p) - \pi/4] \end{Bmatrix}$
 $[\operatorname{Re} p > 0].$
7. $\int_0^{\infty} x^2 e^{-px^2} \operatorname{sh} px^2 \cos bx dx = \sqrt{\frac{\pi c}{2}} \exp\left(-\frac{b^2}{8c}\right) - \frac{\pi b}{4} \operatorname{erfc}\left(\frac{b}{\sqrt{8c}}\right)$
 $[b, \operatorname{Re} c > 0].$
8. $\int_0^{\infty} e^{-px^2} \operatorname{ch} cx \begin{Bmatrix} \sin bx^2 \\ \cos bx^2 \end{Bmatrix} dx = \frac{\sqrt{\pi}}{2(p^2 + b^2)^{1/4}} \exp\left[-\frac{p^2}{4(p^2 + b^2)}\right] \begin{Bmatrix} \sin d \\ \cos d \end{Bmatrix}$
 $\left[d = \frac{bc^2}{4(p^2 + b^2)} + \frac{1}{2} \operatorname{arctg} \frac{b}{p}; b, \operatorname{Re} p > 0\right].$
9. $\int_0^{\infty} x^{-1/2} e^{-px^2} (\operatorname{sh} cx \pm \sin cx) dx = \frac{\pi}{4} \sqrt{\frac{c}{2p}} I_{1/4}\left(\frac{c^2}{8p}\right) \begin{Bmatrix} \operatorname{ch} [c^2/(8p)] \\ \operatorname{sh} [c^2/(8p)] \end{Bmatrix}$
 $[\operatorname{Re} p > 0].$
10. $\int_0^{\infty} x^{-1/2} e^{-px^2} (\operatorname{ch} cx \pm \cos cx) dx = \frac{\pi}{4} \sqrt{\frac{c}{2p}} I_{-1/4}\left(\frac{c^2}{8p}\right) \begin{Bmatrix} \operatorname{ch} [c^2/(8p)] \\ \operatorname{sh} [c^2/(8p)] \end{Bmatrix}$
 $[\operatorname{Re} p > 0].$
11. $\int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} (x/2) \sin bx dx = \operatorname{Im} K_{1/2+ib}(a) \quad [a, b > 0].$
12. $\int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{ch} (x/2) \cos bx dx = \operatorname{Re} K_{1/2+ib}(a) \quad [a, b > 0].$
13. $\int_0^{\infty} e^{-px^2} \operatorname{sh}\left(\frac{c}{\operatorname{ch} c - \cos x}\right) dx = \pi e^{c^2 p} \left[\theta_3\left(2cp, \frac{4p}{i\pi}\right) + \hat{\theta}_3\left(2cp, \frac{4p}{i\pi}\right) \right] - \frac{1}{2} \sqrt{\frac{\pi}{p}}$
 $[\operatorname{Re} c, \operatorname{Re} p > 0].$

$$\hat{\theta}_3(v, x) = \frac{1}{\sqrt{\pi x}} \left[\sum_{k=0}^{\infty} e^{-(v+k)^2/x} - \sum_{k=-1}^{\infty} e^{-(v+k)^2/x} \right].$$

2.6. ЛОГАРИФМИЧЕСКАЯ ФУНКЦИЯ

2.6.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входит логарифмическая функция. С помощью замены переменной они могут быть преобразованы к интегралам от показательной и степенной функций. Некоторые интегралы от логарифмов можно получить путем дифференцирования по параметру. Например,

$$\begin{aligned} \int_0^{\infty} x^{\alpha-1} \ln^n (1+x) dx &= \int_0^{\infty} x^n e^x (e^x - 1)^{\alpha-1} dx = \\ &= \frac{\partial^n}{\partial \rho^n} \int_0^{\infty} x^{\alpha-1} (1+x)^{\rho} dx \Big|_{\rho=0} = \frac{\partial^n}{\partial \rho^n} B(\alpha, -\rho-\alpha) \Big|_{\rho=0}. \quad [-n < \operatorname{Re} \alpha < 0]. \end{aligned}$$

2.6.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие логарифмическую, а также степенную, алгебраические, показательную и тригонометрические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Значения параметров, входящих в интегралы общего вида, приводятся в фигурных скобках после каждого интеграла; $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$\begin{aligned} 1. \int_0^y x^{\alpha-1} (y^r - x^r)^{\beta-1} \left\{ \begin{array}{l} \ln(cx+d) \\ \ln|cx-d| \end{array} \right\} dx &= \\ &= \frac{c}{rd} y^{\alpha+r\beta-r+1} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha+h+1}{r}, \beta\right) \left(\mp \frac{cy}{d}\right)^h \chi_1 + \frac{y^{\alpha+r\beta-r}}{r} B\left(\frac{\alpha}{r}, \beta\right) \ln d \end{aligned}$$

[|cy| ≤ |d|].

$$\begin{aligned} 2. &= \frac{dy^{\alpha+\beta r-r-1}}{cr} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha-h-1}{r}, \beta\right) \left(\mp \frac{d}{cy}\right)^h \chi_2 + \\ &+ y^{r(\beta-1)} \left(\frac{d}{c}\right)^{\alpha} \pi \sum_{h=0}^{q-1} \frac{(1+\beta)_h}{h! (\alpha+hr)} \left\{ \begin{array}{l} \operatorname{cosec} [(\alpha+hr)\pi] \\ \operatorname{ctg} [(\alpha+hr)\pi] \end{array} \right\} \left(\frac{d}{cy}\right)^{hr} \chi_3 + \\ &+ y^{\alpha+\beta r-r} r^{-2} B\left(\frac{\alpha}{r}, \beta\right) \left[r \ln(cy) + \psi\left(\frac{\alpha}{r}\right) - \psi\left(\frac{\alpha}{r} + \beta\right) \right] \end{aligned}$$

[|cy| > |d|].

{Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_j = 1$, $j=1, 2, 3$; если $r > 0$ — рациональное ($r=p/q$) то

$$\begin{aligned} \chi_1 &= {}_{q+2}F_{q+1} \left(1, (h+1)/p, \Delta(q, (\alpha+h+1)r^{-1}); z \right), \\ \chi_2 &= {}_{q+2}F_{q+1} \left(1, (h+1)/p, \Delta(q, 1-\beta-r^{-1}(\alpha-h-1)); z^{-1} \right), \\ \chi_3 &= {}_{q+2}F_{q+1} \left(1, \Delta(q, 1+h-\beta), (\alpha+hr)/p; z^{-1} \right), \quad z = \left(\mp \frac{cy}{d} \right)^p. \end{aligned}$$

[$\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } 0 < x < y \\ c, d > 0 \end{array} \right\}$].

$$\begin{aligned}
 3. \int_y^\infty x^{\alpha-1} (x^r - y^r)^{\beta-1} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\
 = y^{r(\beta-1)} \left\{ \int_0^{y^{-1}} x^{r(1-\beta)-\alpha-1} (y^{-r} - x^r)^{\beta-1} \ln(dx+c) dx + \right. \\
 \left. + \frac{y^\alpha}{r^2} B\left(1-\beta-\frac{\alpha}{r}, \beta\right) \left[r \ln y - \psi\left(1-\beta-\frac{\alpha}{r}\right) + \psi\left(1-\frac{\alpha}{r}\right) \right] \right\} \\
 \left[0 < \operatorname{Re} \beta < 1 - \operatorname{Re} \alpha/r; \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } x > y \\ c, d > 0 \end{array} \right\} \right].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^\infty x^{\alpha-1} (x^r + y^r)^{\rho-1} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\
 = \frac{c}{rd} y^{\alpha+r\beta} r^{-1} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha+h+1}{r}, 1-\rho-\frac{\alpha+h+1}{r}\right) \left(\mp \frac{cy}{d}\right)^h \chi_1 + \\
 + \left(\frac{c}{d}\right)^{r-r\rho-\alpha} \pi \sum_{h=0}^{q-1} \frac{(-1)^h (1-\rho)_h}{h! (r-r\rho-\alpha+rh)} \left\{ \begin{array}{l} \cos^2 c [(r-r\rho-\alpha+rh)\pi] \\ \operatorname{ctg} [(r-r\rho-\alpha+rh)\pi] \end{array} \right\} \left(\frac{cy}{d}\right)^{rh} \chi_2 + \\
 + \frac{y^{\alpha+r\rho}}{r} B\left(\frac{\alpha}{2}, 1-\rho-\frac{\alpha}{r}\right) \ln d \quad [|cy| \leq d].
 \end{aligned}$$

$$\begin{aligned}
 5. = y^{r(\rho-1)} \left(\frac{d}{c}\right)^\alpha \pi \sum_{h=0}^{q-1} \frac{(1-\rho)_h (-1)^h}{h! (\alpha+rh)} \left\{ \begin{array}{l} \operatorname{cosec} [(\alpha+rh)\pi] \\ \operatorname{ctg} [(\alpha+rh)\pi] \end{array} \right\} \left(\frac{d}{cy}\right)^{rh} \chi_3 + \\
 + y^{r(\rho-1)+\alpha} \frac{d}{cr} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha-h-1}{r}, 1-\rho+\frac{h+1-\alpha}{r}\right) \left(\mp \frac{d}{cy}\right)^h \chi_4 + \\
 + r^{-2} y^{r(\rho-1)+\alpha} B\left(1-\rho-\frac{\alpha}{r}, \frac{\alpha}{r}\right) \left[r \ln(cy) + \psi\left(\frac{\alpha}{r}\right) - \psi\left(1-\rho-\frac{\alpha}{r}\right) \right] \\
 [|cy| > d].
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_j = 1$, $j=1, 2, 3, 4$; если $r > 0$ — рациональное ($r=p/q$), то

$$\begin{aligned}
 \chi_1 &= {}_{q+2}F_{q+1} \left(\begin{array}{c} 1, (h+1)/p, \Delta(q, (\alpha+h+1)r^{-1}); z \end{array} \right), \\
 \chi_2 &= {}_{q+2}F_{q+1} \left(\begin{array}{c} 1, \Delta(q, 1-\rho+h), (1-\rho+h)q^{-1}-\alpha p^{-1}; z \end{array} \right), \\
 \chi_3 &= {}_{q+2}F_{q+1} \left(\begin{array}{c} 1, \Delta(q, 1-\rho+h), (\alpha+rh)p^{-1}; z^{-1} \end{array} \right), \\
 \chi_4 &= {}_{q+2}F_{q+1} \left(\begin{array}{c} 1, \Delta(q, 1-\rho+(1+h-\alpha)r^{-1}), (h+1)/p; z^{-1} \end{array} \right), \\
 z &= (-1)^q \left(\mp \frac{cy}{d}\right)^p. \\
 &\left[\operatorname{Re} \alpha > 0, \operatorname{Re}(\alpha+r\rho) < r; r|\arg y| < \pi, \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } 0 < x < \infty \\ c, d > 0 \end{array} \right\} \right].
 \end{aligned}$$

$$6. \int_0^{\infty} \frac{x^{\alpha-1}}{x^r - y^r} \left\{ \ln(cx+d) \right\} dx = -\frac{\pi y^{\alpha-r+1} c}{rd} \sum_{h=0}^{p-1} \frac{1}{h+1} \operatorname{ctg} \left(\frac{\alpha+h+1}{r} \pi \right) \times$$

$$\times \left(\mp \frac{cy}{d} \right)^h \chi_1 - \pi^2 \left(\frac{c}{d} \right)^{r-\alpha} \sum_{h=0}^{q-1} \frac{1}{\alpha-r-rh} \left\{ \operatorname{cosec} [(r+rh-\alpha)\pi] \right\} \left(\frac{cy}{d} \right)^{rh} \chi_2 -$$

$$- \frac{\pi}{r} y^{\alpha-r} \operatorname{ctg} \frac{\alpha\pi}{r} \ln d \quad [|cy| \leq |d|].$$

$$7. = -\pi^2 y^{-r} \left(\frac{d}{c} \right)^{\alpha} \sum_{h=0}^{q-1} \frac{1}{\alpha+rh} \left\{ \operatorname{cosec} [(\alpha+rh)\pi] \right\} \left(\frac{d}{cy} \right)^{rh} \chi_3 +$$

$$+ \frac{\pi dy^{\alpha-r-1}}{rc} \sum_{h=0}^{p-1} \frac{1}{h+1} \operatorname{ctg} \left(\frac{h-\alpha}{r} \pi \right) \left(\mp \frac{d}{cy} \right)^h \chi_4 -$$

$$- \pi r^{-2} y^{\alpha-r} \operatorname{ctg} \frac{\alpha\pi}{r} \left[r \ln(cy) - 2\pi \operatorname{cosec} \frac{2\alpha}{r} \pi \right] \quad [|cy| > |d|].$$

{Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_j=1$, $j=1, 2, 3, 4$, если $r > 0$ — рациональное ($r=p/q$), то

$$\chi_1 = {}_2F_1(1, (h+1)p^{-1}; 1+(h+1)p^{-1}; z),$$

$$\chi_2 = {}_2F_1(1, (r+rh-\alpha)p^{-1}; 1+(r+rh-\alpha)p^{-1}; z),$$

$$\chi_3 = {}_2F_1(1, (\alpha+rh)p^{-1}; 1+(\alpha+rh)p^{-1}; z^{-1}),$$

$$\chi_4 = {}_2F_1(1, (h+1)p^{-1}; 1+(h+1)p^{-1}; z^{-1}), \quad z = (\mp cy/d)^p \}$$

$$\left[0 < \operatorname{Re} \alpha < r; y > 0, \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } x > 0 \\ c, d > 0 \end{array} \right\} \right].$$

$$8. \int_0^{\infty} x^{\alpha-1} \frac{(y^{r/2} + \sqrt{x^r + y^r})^v}{(x^r + y^r)^{\rho}} \left\{ \ln(cx+d) \right\} dx = U_{\rho} \quad [\rho = 0, 1/2]$$

$$U_{\rho} = \frac{cy^{\alpha+r\nu/2-\rho r+1}}{dr} \sum_{h=0}^{p-1} \frac{2^{\nu+2(h+\alpha+1)/r}}{h+1} \left(\frac{\nu r}{\nu r + \alpha + h + 1} \right)^{1-2\rho} \times$$

$$\times B \left(\frac{\alpha+h+1}{r}, 2\rho - \nu - \frac{2}{r}(\alpha+h+1) \right) \left(\mp \frac{cy}{d} \right)^h \chi_1 +$$

$$+ \nu^{1-2\rho} \pi \left(\frac{c}{d} \right)^{r(1-\nu)/2-\alpha} y^{r/2-\rho r} \sum_{h=0}^{q-1} \frac{(-1)^h ((1-\nu)/2)_h ((1+\nu)/2)_h}{h! (3/2-2\rho)_h [r(1-\nu)/2 + hr - \alpha]} \times$$

$$\times \left\{ \operatorname{cosec} [r(1-\nu)/2 + hr - \alpha] \pi \right\} \left(\frac{cy}{d} \right)^{rh} \chi_2 +$$

$$+ \nu^{2\rho} \pi \left(\frac{c}{d} \right)^{2r\rho-\alpha-r\nu/2} y^{\rho r} \sum_{h=0}^{q-1} \frac{(-1)^h (2\rho-\nu/2)_h (2\rho+\nu/2)_h}{h! (1/2+2\rho)_h [(2\rho-\nu/2+n)r-\alpha]} \times$$

$$\times \left\{ \operatorname{cosec} [(2\rho-\nu/2+h)r-\alpha] \pi \right\} \left(\frac{cy}{d} \right)^{rh} \chi_3 +$$

$$+ \frac{\ln d}{r} 2^{\nu+2\alpha/r} \left(\frac{\nu r}{\nu r + \alpha} \right)^{1-2\rho} y^{\alpha-2\rho+r\nu/2} B \left(\frac{\alpha}{r}, 2\rho - \nu - \frac{2\alpha}{r} \right) \quad [|cy| \leq |d|].$$

$$\begin{aligned}
 U_p = & 2^v \left(\frac{d}{c} \right)^\alpha y^{rv/2 - \rho r} \pi \sum_{h=0}^{q-1} \frac{(-1)^h ((1-v)/2)_h (2\rho - v/2)_h}{h! (1-v)_h (\alpha + rh)} \left\{ \operatorname{cosec}(\alpha + rh)\pi \right. \\
 & \times \left. \operatorname{ctg}(\alpha + rh)\pi \right\} \times \\
 & \times \left(\frac{d}{cy} \right)^{rh} \chi_4 + \frac{d}{c} y^{\alpha + rv/2 - \rho r - 1} \sum_{h=0}^{p-1} \frac{2^{v+2(\alpha-h)/r}}{h+1} \left(\frac{vr}{vr+h-\alpha} \right)^{1-2\rho} \times \\
 & \times B\left(\frac{\alpha-h}{r}, 2\rho - v - 2\frac{\alpha-h}{r} \right) \left(\mp \frac{d}{cy} \right)^h \chi_5 + \\
 & + r^{-2} y^{\alpha + rv/2 - \rho r} 2^{v+2\alpha/r} \left(\frac{vr}{vr+\alpha} \right)^{1-2\rho} B\left(\frac{\alpha}{r}, 2\rho - v - \frac{2\alpha}{r} \right) \times \\
 & \times \left[r \ln(cy) + 2\ln 2 - 2\psi\left(2\rho - v - \frac{2\alpha}{r} \right) + \psi\left(\frac{\alpha}{r} \right) + \psi\left(1 - v - \frac{\alpha}{r} \right) \right] \quad [|cy| > |d|].
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $p = q = \infty$ и $\chi_j = 1$, $j = 1, 2, 3, 4, 5$; если $r > 0$ — рациональное ($r = p/q$), то

$$\chi_1 = {}_{2q+2}F_{2q+1} \left(1, \Delta\left(q, \frac{1+h+\alpha}{r} \right), \Delta\left(q, v + \frac{1+h+\alpha}{r} \right), \frac{h+1}{p}; z \right. \\
 \left. \Delta\left(q, \frac{1+v}{2} + \frac{h+\alpha}{r} \right), \Delta\left(q, 1-2\rho + \frac{v}{2} + \frac{h+\alpha}{r}, 1 + \frac{h+1}{p} \right) \right),$$

$$\chi_2 = {}_{2q+2}F_{2q+1} \left(1, \Delta\left(q, \frac{1-v}{2} + h \right), \Delta\left(q, \frac{1+v}{r} + h \right), \left(r \frac{1-v}{2} + hr - \alpha \right) \frac{1}{p}; z \right. \\
 \left. \Delta\left(q, \frac{3}{2} - 2\rho + h \right), \Delta(q, h+1), 1 + \left(\frac{r}{2} (1-v) + hr - \alpha \right) \frac{1}{p} \right),$$

$$\begin{aligned}
 \chi_3 = & \\
 = {}_{2q+2}F_{2q+1} & \left(1, \Delta\left(q, 2\rho - \frac{v}{2} + h \right), \Delta\left(q, \frac{v}{2} + 2\rho + h \right), \left(2\rho - \frac{v}{2} + h \right) \frac{r}{p} - \frac{\alpha}{p}; z \right. \\
 & \left. \Delta\left(q, \frac{1}{2} + 2\rho + h \right), \Delta(q, h+1), 1 + \left(\left(2\rho - \frac{v}{2} + h \right) r - \alpha \right) \frac{1}{p} \right),
 \end{aligned}$$

$$\chi_4 = {}_{2q+2}F_{2q+1} \left(1, \Delta\left(q, \frac{1-v}{2} + h \right), \Delta\left(q, 2\rho - \frac{v}{2} + h \right), (\alpha + rh) \frac{1}{p}; \frac{1}{z} \right. \\
 \left. \Delta(q, 1-v+h), \Delta(q, 1+h), 1 + (\alpha + rh) \frac{1}{p} \right),$$

$$\chi_5 = {}_{2q+2}F_{2q+1} \left(1, \frac{h+1}{p}, \Delta\left(q, \frac{1-v}{2} + \frac{h-\alpha}{r} \right), \Delta\left(q, 2\rho - \frac{v}{2} + \frac{h-\alpha}{r} \right); \frac{1}{z} \right. \\
 \left. 1 + \frac{h+1}{p}, \Delta\left(q, 1-v + \frac{h-\alpha}{r} \right), \Delta\left(q, 1 + \frac{h-\alpha}{r} \right) \right),$$

$$z = (-1)^q \left(\mp \frac{cy}{d} \right)^p.$$

$$\left[\operatorname{Re} \alpha > 0, \operatorname{Re}(2\alpha + vr) < 2\rho; r |\arg y| < \pi, \left\{ |\arg(cx + d)| < \pi \text{ при } x > 0 \right\} \right].$$

$$\begin{aligned}
 9. \int_0^{\infty} x^{\alpha-1} \frac{(x^{r/2} + \sqrt{x^r + y^r})^v}{(x^r + y^r)^{\rho}} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\
 = y^{rv/2-\rho} \left\{ \int_0^{\infty} x^{or-rv/2-\alpha-1} \frac{(y^{-r/2} + \sqrt{x^r + y^{-r}})^v}{(x^r + y^{-r})^{\rho}} \ln(xd+c) dx + \right. \\
 \left. + \frac{2^{2\rho-2\alpha/r}}{r^2} \left(\frac{2rv}{rv+2\rho r-2\alpha} \right)^{1-2\rho} y^{\alpha} B\left(\rho - \frac{\alpha}{r} - \frac{v}{2}, 2\frac{\alpha}{r}\right) \times \right. \\
 \left. \times \left[r \ln y - 2 \ln 2 - \psi\left(\rho - \frac{\alpha}{r} - \frac{v}{2}\right) + 2\psi\left(2\frac{\alpha}{r}\right) - \psi\left(\rho + \frac{\alpha}{r} - \frac{v}{2}\right) + \frac{2r(1-2\rho)}{rv+2\rho r-2\alpha} \right] \right\} \\
 \left[\rho=0 \text{ или } 1/2, \operatorname{Re} \alpha > 0, \operatorname{Re}(2\alpha + vr) < 2\rho r; r|\arg y| < \pi; \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } x > 0 \\ c, d > 0 \end{array} \right\} \right].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_0^{\infty} x^{\alpha-1} e^{-sx^r} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\
 = \frac{d}{cr} s^{(1-\alpha)/r} \sum_{h=0}^{p-1} \frac{1}{h+1} \Gamma\left(\frac{\alpha-h-1}{r}\right) \left(\mp \frac{ds^{1/r}}{c}\right)^h \chi_1 + \left(\frac{d}{c}\right)^{\alpha} \frac{\pi}{r} \sum_{h=0}^{q-1} \frac{(-1)^h}{h! (\alpha+hr)} \times \\
 \times \left\{ \begin{array}{l} \operatorname{cosec}(\alpha+hr)\pi \\ \operatorname{ctg}(\alpha+hr)\pi \end{array} \right\} \left(\frac{d^r s}{c^r}\right)^h \chi_2 + r^{-2} s^{-\alpha/r} \Gamma\left(\frac{\alpha}{r}\right) \left[r \ln c - \ln s + \psi\left(\frac{\alpha}{r}\right) \right].
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_1=\chi_2=1$; если $r > 0$ — рациональное ($r=p/q$), то

$$\begin{aligned}
 \chi_1 = {}_2F_{p+1} \left(\begin{array}{c} 1, (h+1)p^{-1}; \\ 1+(h+1)p^{-1}, \Delta(q, 1+(1+h-\alpha)r^{-1}) \end{array} \middle| z \right), \\
 \chi_2 = {}_2F_{q+1} \left(\begin{array}{c} 1, (\alpha+hr)p^{-1}; \\ 1+(\alpha+hr)p^{-1}, \Delta(q, 1+h) \end{array} \middle| z \right), \quad z = (-1)^q \left(\frac{s}{q}\right)^q \left(\mp \frac{d}{c}\right)^p \cdot \left. \right\} \\
 \left[\operatorname{Re} \alpha, \operatorname{Re} s > 0; \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } x > 0 \\ c, d > 0 \end{array} \right\} \right].
 \end{aligned}$$

$$11. \int_0^{\infty} x^{\alpha-1} \sin bx^r \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = U(\delta) \Big|_{\delta=1},$$

$$\begin{aligned}
 U(\delta) = \frac{c}{rd} 2^{1/r} b^{-(\alpha+1)/r} \sum_{h=0}^{p-1} \frac{1}{h+1} \Gamma\left(\frac{\alpha-h}{r}\right) \cos \frac{r\delta+h-\alpha}{2r} \pi \left(\mp \frac{d}{cb^{1/r}}\right)^h \chi_1 + \\
 + \pi b^{\delta} \left(\frac{d}{c}\right)^{\alpha+r\delta} \sum_{h=0}^{q-1} \frac{(-1)^h}{h! (\delta+1/2)_h (\alpha+2hr+\delta r)} \left\{ \begin{array}{l} \operatorname{cosec}(\alpha+2hr+\delta r)\pi \\ \operatorname{ctg}(\alpha+2hr+\delta r)\pi \end{array} \right\} \left(\frac{bd^r}{2c^r}\right)^{2h} \chi_2 + \\
 + \frac{b^{-\alpha/r}}{2r^2} \Gamma\left(\frac{\alpha}{r}\right) \cos \frac{r\delta-\alpha}{2r} \pi \left[2r \ln c - 2 \ln b + \psi\left(\frac{\alpha}{r}\right) + \left(\operatorname{tg} \frac{\alpha\pi}{2r}\right)^{1-2\delta} \right].
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_1=\chi_2=1$; если $r > 0$ — рациональное ($r=p/(2q)$), то

$$\chi_1 = {}_2F_{2q+1} \left(\begin{array}{c} 1, (h+1)p-1; \\ 1+\frac{h+1}{p}, \Delta\left(q, \frac{1}{2} + \frac{1}{2r}(h+1-\alpha)\right), \Delta\left(q, 1+\frac{h+1-\alpha}{2r}\right) \end{array} \middle| z \right),$$

$$\chi_2 = {}_2F_{2\gamma+1} \left(1, (\alpha + h + r\delta) p^{-1}; \right. \\ \left. 1 + (\alpha + h + r\delta) p^{-1}, \Delta(q, h+1), \Delta(q, h+\delta+1/2) \right), \\ z = (-1)^q \left(-\frac{d}{c} \right)^p \left(\frac{b}{2q^2} \right)^q \left\{ \begin{array}{l} -r\delta < \operatorname{Re} \alpha < r; b > 0, \\ c, d > 0 \end{array} \right\} \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } x > 0 \end{array} \right\}.$$

$$12. \int_0^\infty x^{\alpha-1} \cos bx \left\{ \begin{array}{l} \ln(cx+d) \\ \ln cx-d \end{array} \right\} dx = U(\delta) \Big|_{\delta=0} \quad [U(\delta) \text{ см в 2.6.2 11].}$$

2.6.3. Интегралы от $x^\alpha \ln^\sigma x$.

$$1. \int_0^a x^{\alpha-1} \ln^\sigma \frac{a}{x} dx = a^\alpha \alpha^{-\sigma-1} \Gamma(\sigma+1) \quad [a, \operatorname{Re} \alpha > 0, \operatorname{Re} \sigma > -1].$$

$$2. \int_0^1 x^{\alpha-1} \ln^n x dx = (-1)^n \alpha^{-n-1} n! \quad [\operatorname{Re} \alpha > 0].$$

$$3. \int_0^\infty \frac{a \ln(1+bx) - b \ln(1+ax)}{x^2} dx = ab \ln \frac{a}{b} \quad [a, b > 0].$$

2.6.4. Интегралы от $\frac{x^\alpha}{(x^\mu + a^\mu)^\rho} \ln^\sigma x$.

$$1. \int_0^a \frac{x^{\alpha-1}}{(x^\mu + a^\mu)^\rho} \ln^\sigma \frac{a}{x} dx = a^{\alpha-\mu\rho} \Gamma(\sigma+1) \sum_{k=0}^\infty \frac{(1-\rho-k)_k}{k! (\alpha + \mu k)^{\sigma+1}} \\ [a, \mu, \operatorname{Re} \alpha > 0, \operatorname{Re} \sigma > -1].$$

$$2. \int_0^1 \frac{x^{\mu\rho/2-1} \ln x}{(x^\mu + z^\mu)^\rho} dx = \frac{\ln z}{\mu z^{\mu\rho/2}} B\left(\frac{\rho}{2}, \frac{\rho}{2}\right) \quad [\mu, \operatorname{Re} \rho > 0; \mu |\arg z| < \pi].$$

$$3. \int_0^1 \frac{x^{\alpha-1} \ln x}{1+x^\mu} dx = I_\mu^\alpha \quad [\mu, \operatorname{Re} \alpha > 0].$$

$$I_\mu^\alpha = \frac{1}{\mu^2} \beta'\left(\frac{\alpha}{\mu}\right), \quad I_\mu^{\mu n} = \frac{(-1)^n}{\mu^2} \left[\frac{\pi^2}{12} + \sum_{k=1}^{n-1} \frac{(-1)^k}{k^2} \right],$$

$$I_\mu^{\mu/2} = -\frac{4}{\mu^2} G, \quad I_\mu^{\mu/2-1} - I_\mu^{\mu/2+1} = -\frac{\pi^2}{\mu^2} \frac{\sin(\pi/\mu)}{\cos^2(\pi/\mu)}.$$

$$4. \int_0^1 \frac{x^{\alpha-1} \ln^n x}{1+x^\mu} dx = J_\mu^{\alpha, n} \quad [\mu, \operatorname{Re} \alpha > 0; n = 1, 2, 3, \dots].$$

$$J_\mu^{\alpha, n} = \frac{1}{\mu^{n+1}} \beta^{(n)}\left(\frac{\alpha}{\mu}\right), \quad J_\mu^{\mu, 2l} = \frac{2^{2l}-1}{2^{2l} \mu^{2l+1}} (2l)! \zeta(2l+1),$$

$$J_\mu^{\mu, 2l-1} = \frac{1-2^{2l-1}}{2l \mu^{2l}} \pi^{2l} |B_{2l}|, \quad J_\mu^{\mu/2, n} = (-1)^n \left(\frac{2}{\mu}\right)^{n+1} n! \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{n+1}},$$

$$J_{\mu}^{\mu/2, 2l} = \left(\frac{\pi}{\mu}\right)^{2l+1} \frac{1}{2} |E_{2l}|, \quad J_{\mu}^{n\mu, 2} = \frac{2}{\mu^3} \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3}, \quad J_{\mu}^{\mu/2, 2} = \frac{\pi^3}{2\mu^3},$$

$$J_{\mu}^{\mu/4, 2} + J_{\mu}^{3\mu/4, 2} = \frac{\sqrt{2}}{\mu^3} \pi^3, \quad J_{\mu}^{n\mu, 3} = (-1)^n \mu^{-4} \left[\frac{7\pi^4}{120} + 6 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^4} \right],$$

$$J_{\mu}^{\mu/2, 4} = \frac{5}{2\mu^5} \pi^5, \quad J_{\mu}^{\mu, 5} = -\frac{31}{252\mu^6} \pi^6, \quad J_{\mu}^{\mu/2, 6} = \frac{61}{2} \left(\frac{\pi}{\mu}\right)^7, \quad J_{\mu}^{\mu, 7} = -\frac{127}{240} \left(\frac{\pi}{\mu}\right)^8.$$

$$5. \int_0^1 \frac{x^{\alpha-1} \ln^n x}{(1+x^{\mu})^2} dx = J_{\mu}^{\alpha, n} \quad [\mu, \operatorname{Re} \alpha > 0, n = 1, 2, 3, \dots]$$

$$J_{\mu}^{\mu, 1} = -\frac{\ln 2}{\mu^2}, \quad J_{\mu}^{3\mu/2, 1} = -\frac{4 \ln 2}{\mu^2},$$

$$J_{\mu}^{n\mu, 1} = \frac{(-1)^n}{\mu^2} \left[\ln 2 - \frac{(n-1)\pi^2}{12} - \sum_{k=1}^{n-2} \frac{(-1)^k}{k^2} (n-k-1) \right], \quad J_{\mu}^{\mu, 2} = \frac{\pi^2}{6\mu^3},$$

$$J_{\mu}^{\mu/2, 2} = \frac{\pi^3}{2\mu^3}, \quad J_{\mu}^{n\mu, 2} = \frac{(-1)^n}{\mu^3} \left[\frac{3}{2} (n-1) \zeta(3) + 2 \sum_{k=1}^{n-2} \frac{(-1)^k}{k^3} (n-k-1) - \frac{\pi^2}{6} \right],$$

$$J_{\mu}^{\mu, 3} + (-1)^n J_{\mu}^{n\mu, 3} = \mu^{-4} \left[6 \sum_{k=1}^{n-2} (-1)^{k-1} \frac{n-k-1}{k^4} - \frac{7\pi^4}{120} (n-1) \right], \quad J_{\mu}^{\mu, 4} = \frac{7\pi^4}{30\mu^5}.$$

$$6. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{(x^{\mu} + z^{\mu})^{\rho}} dx = \frac{1}{\mu} \left(\frac{\partial}{\partial \alpha} \right)^n \left[z^{\alpha-\rho\mu} B \left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu} \right) \right]$$

$$[\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$7. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{(x^{\mu} + z^{\mu})^{\rho}} dx = \frac{z^{\alpha-\rho\mu}}{\mu^2} B \left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu} \right) \left[\mu \ln z + \psi \left(\frac{\alpha}{\mu} \right) - \psi \left(\rho - \frac{\alpha}{\mu} \right) \right]$$

$$[\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$8. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{(x^{\mu} + z^{\mu})^m} dx = \frac{z^{\alpha-\mu m} (1-\sigma/\mu)_{m-1} \pi}{(m-1)! \mu^2 \sin(\alpha\pi/\mu)} \left(\mu \ln z - \sum_{k=1}^{m-1} \frac{\mu}{\mu - \alpha} - \pi \operatorname{ctg} \frac{\alpha\pi}{\mu} \right)$$

$$[\mu > 0, \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu m; m = 1, 2, 3, \dots]$$

$$9. \int_0^{\infty} \frac{\ln x}{(x+z)^{m+1/2}} dx = \frac{2z^{1/2-m}}{2m-1} \left(\ln z + 2 \ln 2 - \sum_{k=1}^{m-2} \frac{1}{k} - 2 \sum_{k=m-1}^{2m-3} \frac{1}{k} \right)$$

$$[|\arg z| < \pi; m = 1, 2, 3, \dots]$$

$$10. \int_0^{\infty} \frac{x^{\alpha-1} \ln^2 x}{(x^{\mu} + z^{\mu})^{\rho}} dx =$$

$$= \frac{z^{\alpha-\rho\mu}}{\mu^3} B \left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu} \right) \{ [\mu \ln z + \psi(\alpha/\mu) - \psi(\rho - \alpha/\mu)]^2 + \psi'(\alpha/\mu) + \psi'(\rho - \alpha/\mu) \}$$

$$[\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$11. \int_0^{\infty} \frac{x^{\mu\rho/2-1} \ln^{2n} x}{(x^{\mu}+1)^{\rho}} dx = 2^{2n+2} \mu^{-2n-1} (2n)! \sum_{k=0}^{\infty} \frac{(1-\rho-k)_k}{k! (\rho+2k)^{2n+1}} \quad [\mu, \operatorname{Re} \rho > 0].$$

$$12. \int_0^{\infty} \frac{x^{\mu/2-1} \ln^n x}{x^{\mu}+1} dx = 2^{n+2} \frac{1+(-1)^n}{2} n! \mu^{-n-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \quad [\mu > 0].$$

$$13. \int_0^{\infty} \frac{x^{\mu-1} \ln^{2n+1} x}{(x^{\mu}+1)^2} dx = 0 \quad [\mu > 0].$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{x^{\mu}+z^{\mu}} dx = \frac{\pi}{\mu} \frac{\partial^n}{\partial \alpha^n} \frac{z^{\alpha/\mu}}{\sin(\alpha\pi/\mu)} \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu].$$

$$15. \int_0^{\infty} \frac{x^{\alpha-1} \ln^2 x}{x^{\mu}+z^{\mu}} dx = \frac{\pi z^{\alpha/\mu}}{\mu^3 \sin(\alpha\pi/\mu)} \left[\mu^2 \ln^2 z - 2\pi\mu \ln z \operatorname{ctg} \frac{\alpha\pi}{\mu} + \frac{\pi^2}{\sin^2(\alpha\pi/\mu)} \left(1 + \cos^2 \frac{\alpha\pi}{\mu} \right) \right] \\ [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^{\mu}+z^{\mu})(x^2+z^2)} \ln \frac{x}{a} dx = 0 \quad [z > 0; \alpha - \mu < 2].$$

2.6.5. Интегралы от $\frac{x^{\alpha}}{(a^{\mu}-x^{\mu})^{\rho}} \ln^{\sigma} x$.

$$1. \int_0^a \frac{x^{\alpha-1} \ln^n x}{(a^{\mu}-x^{\mu})^{1-\beta}} dx = \frac{a^{\mu(\beta-1)}}{\mu} \frac{\partial^n}{\partial \alpha^n} \left[a^{\alpha} B\left(\beta, \frac{\alpha}{\mu}\right) \right] \\ [a, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -n \text{ при } a=1)].$$

$$2. \int_0^a x^{\alpha-1} (a^{\mu}-x^{\mu})^m \ln^n x dx = (-1)^n n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(\alpha+\mu k)^{n+1}} \quad [a, \mu > 0].$$

$$3. \int_0^a x^{\alpha-1} (a^{\mu}-x^{\mu})^m \ln^{\sigma} \frac{a}{x} dx = a^{\alpha+\mu m} \Gamma(\sigma+1) \sum_{k=0}^m \frac{(-1)^k (1+m-k)_k}{k! (\alpha+\mu k)^{\sigma+1}} \\ [a, \mu, \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -m-1].$$

$$4. \int_0^a \frac{x^{\alpha-1} \ln x}{(a^{\mu}-x^{\mu})^{1-\beta}} dx = \frac{a^{\alpha+\mu(\beta-1)}}{\mu^2} B\left(\beta, \frac{\alpha}{\mu}\right) \left[\mu \ln a + \psi\left(\frac{\alpha}{\mu}\right) - \psi\left(\beta + \frac{\alpha}{\mu}\right) \right] \\ [a, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -1 \text{ при } a=1)].$$

$$5. \int_0^a \frac{x^{\alpha-1} \ln^2 x}{(a^{\mu}-x^{\mu})^{1-\beta}} dx = \frac{a^{\alpha+\mu\beta-\mu}}{\mu^3} B\left(\beta, \frac{\alpha}{\mu}\right) \left\{ \left[\mu \ln a + \psi\left(\frac{\alpha}{\mu}\right) - \psi\left(\beta + \frac{\alpha}{\mu}\right) \right]^2 + \psi'\left(\frac{\alpha}{\mu}\right) - \psi'\left(\beta + \frac{\alpha}{\mu}\right) \right\} \\ [a, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -2 \text{ при } a=1)].$$

$$6. \int_0^1 \frac{x^{\mu-1}}{z^{\mu}-x^{\mu}} \ln^{\sigma} \frac{1}{x} dx = \frac{\Gamma(\sigma+1)}{\mu^{\sigma+1}} \sum_{k=1}^{\infty} \frac{z^{-\mu k}}{k^{\sigma+1}} \quad [|z| > 1; \mu > 0; \operatorname{Re} \sigma > -1].$$

$$7. \int_0^1 \frac{x^{\alpha-1}}{1-x^{\mu}} \ln^{\sigma} \frac{1}{x} dx = \Gamma(\sigma+1) \mu^{-\sigma-1} \zeta\left(\sigma+1, \frac{\alpha}{\mu}\right) \quad [\operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$8. \int_0^1 \frac{x^{\mu-1}}{1-x^{\mu}} \ln^{\sigma} \frac{1}{x} dx = \Gamma(\sigma+1) \mu^{-\sigma-1} \zeta(\sigma+1) \quad [\mu, \operatorname{Re} \sigma > 0].$$

$$9. \int_0^1 \frac{x^{\alpha-1}}{1-x^{\mu}} \ln^n x dx = I_{\mu}^{\alpha, n}, \quad I_{\mu}^{\alpha, n} = -\mu^{-n-1} \psi^{(n)}\left(\frac{\alpha}{\mu}\right) \quad [\mu, \operatorname{Re} \alpha > 0; n = 1, 2, 3, \dots].$$

$$I_{\mu}^{n, 1} = \frac{1}{\mu^2} \left(\sum_{k=1}^{n-1} \frac{1}{k^2} - \frac{\pi^2}{6} \right) \quad [n = 1, 2, 3, \dots],$$

$$I_k^{m, 1} + I_k^{k-m, 1} = -\frac{\pi^2}{k^2 \sin^2(m\pi/k)} \quad [k > m].$$

$$I_{\mu}^{\mu/2, 1} = -\frac{\pi^2}{2\mu^2}, \quad I_{\mu}^{\mu/2-1, 1} + I_{\mu}^{\mu/2+1, 1} = -\frac{\pi^2}{\mu^2 \cos^2(\pi/\mu)},$$

$$I_{\mu}^{3\mu/2, 1} = \frac{4}{\mu^2} \left(1 - \frac{\pi^2}{8} \right), \quad I_{\mu}^{\mu/6, 2} - I_{\mu}^{\mu/3, 2} = \frac{8\sqrt{3}}{\mu^3} \pi^3,$$

$$I_{\mu}^{\mu(n+p), 2} = \frac{2}{\mu^3} \sum_{k=n}^{\infty} \frac{1}{(k+p)^3} \quad [p = 1 \text{ или } 1/2],$$

$$I_{\mu}^{(n+1)\mu, 3} = \frac{1}{\mu^4} \left[6 \sum_{k=0}^{n-1} \frac{1}{(k+1)^4} - \frac{\pi^4}{15} \right], \quad I_{\mu}^{(n+1/2)\mu, 3} = \frac{1}{\mu^4} \left[96 \sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} - \pi^4 \right],$$

$$I_{\mu}^{\mu, 5} = -\frac{8}{63} \left(\frac{\pi}{\mu} \right)^6, \quad I_{\mu}^{\mu, 7} = -\frac{8}{15} \left(\frac{\pi}{\mu} \right)^8.$$

$$10. \int_0^1 \frac{x^{\mu-1}}{1-x^{\mu}} \ln^{2n-1} x dx = (-1)^n \frac{(2\pi)^{2n}}{4n\mu^{2n}} B_{2n} \quad [\mu > 0; n = 1, 2, 3, \dots].$$

$$11. \int_0^1 \frac{x^{\mu/2-1}}{1-x^{\mu}} \ln^{2n-1} x dx = \left(\frac{2\pi}{\mu} \right)^{2n} \frac{1-2^{2n}}{4n} B_{2n} \quad [\mu > 0; n = 1, 2, 3, \dots].$$

$$12. \int_0^1 \frac{x^{\mu/2-1}}{1-x^{\mu}} \ln^{2n} x dx = (2n)! \mu^{-2n-1} (2^{2n} - 1) \zeta(2n+1) \quad [\mu > 0; n = 1, 2, 3, \dots].$$

$$13. \int_0^1 \frac{x^{\alpha-1}}{(1-x^\mu)^2} \ln^n x dx = J_{\mu}^{\alpha, n} \quad [\mu, \operatorname{Re} \alpha > 0; m = 1, 2, 3, \dots].$$

$$J_{\mu}^{\mu, 1} - J_{\mu}^{n\mu, 1} = \mu^{-2} \left[\sum_{k=1}^{n-2} \frac{n-k-1}{k^2} - \frac{\pi^2}{6} (n-1) \right],$$

$$J_{\mu}^{\mu/2, 1} - J_{\mu}^{(n+1/2)\mu, 1} = \mu^{-2} \left[4 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^2} - \frac{n\pi^2}{2} \right],$$

$$J_{\mu}^{\mu, 2} = \frac{2\pi^2}{3\mu^3}, \quad J_{\mu}^{(n+1)\mu, 2} = \frac{2}{\mu^3} \left(\frac{\pi^2}{3} - n\zeta(3) + \sum_{k=1}^{n-1} \frac{n-k}{k^3} \right),$$

$$J_{\mu}^{\mu, 3} - J_{\mu}^{(n+1)\mu, 3} = \frac{1}{\mu^4} \left(6 \sum_{k=1}^{n-1} \frac{n-k}{k^4} - \frac{n\pi^4}{15} \right),$$

$$J_{\mu}^{\mu/2, 3} - J_{\mu}^{(n+1/2)\mu, 3} = \frac{1}{\mu^4} \left(96 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^4} - n\pi^4 \right),$$

$$J_{\mu}^{\mu, 4} = \frac{4\pi^4}{15\mu^5}, \quad J_{\mu}^{\mu/2, 2m} + J_{\mu}^{3\mu/2, 2m} = \left(\frac{2\pi}{\mu} \right)^{2m} \frac{2^{2m}-1}{\mu} |B_{2m}|.$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{(x^\mu - 1)^m} dx = \frac{(-1)^m \pi}{\mu (m-1)!} \frac{\partial^n}{\partial \alpha^n} \left[\left(1 - \frac{\alpha}{\mu} \right)_{m-1} \operatorname{ctg} \frac{\alpha\pi}{\mu} \right]$$

[0 < Re α < μm; n ≥ m; m, n = 1, 2, 3, ...].

$$15. \int_0^{\infty} \frac{x^{\mu/2-1} \ln^n x}{x^\mu - 1} dx = \frac{2^{n+1}-1}{(n+1)\mu^{n+1}} \pi^{n+1} |B_{n+1}| (1 - (-1)^n) \quad [\mu > 0].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{x^\mu - y^\mu} dx = -\frac{\pi}{\mu} \frac{\partial^n}{\partial \alpha^n} \left(y^{\alpha-\mu} \operatorname{ctg} \frac{\alpha\pi}{\mu} \right) \quad [\mu, y > 0; 0 < \operatorname{Re} \alpha < \mu].$$

$$17. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{x^\mu - y^\mu} dx = \frac{\pi y^{\alpha-\mu}}{\mu^2} \left[\frac{\pi}{\sin^2(\alpha\pi/\mu)} - \mu \ln y \operatorname{ctg} \frac{\alpha\pi}{\mu} \right]$$

[μ, y > 0; 0 < Re α < μ].

$$18. \int_0^{\infty} \frac{x^{\alpha-1} \ln^2 x}{x^\mu - y^\mu} dx = \frac{\pi y^{\alpha-\mu}}{\mu^3} \left[\frac{2\pi}{\sin^2(\alpha\pi/\mu)} \left(\mu \ln y - \pi \operatorname{ctg} \frac{\alpha\pi}{\mu} \right) - \mu^2 \ln^2 y \operatorname{ctg} \frac{\alpha\pi}{\mu} \right]$$

[μ, y > 0; 0 < Re α < μ].

2.6.6. Интегралы от $\frac{x^\alpha \ln^\sigma x}{ax^2 + bx + c}$.

$$1. \int_a^b \frac{\ln x dx}{(x+a)(x+b)} = \frac{\ln(ab)}{2(b-a)} \ln \frac{(a+b)^2}{4ab} \quad [0 < a < b].$$

$$2. \int_a^b \frac{\ln^{2n} x}{(x+1)^2} dx = \frac{1}{1-2n} \sum_{k=0}^{n-1} \binom{2n+1}{2k} (\pi i)^{2n-2k} \int_a^b \frac{\ln^{2k} x}{(x+1)^2} dx$$

[$a=0, b=\infty$, или $a=0, b=1$, или $a=1, b=\infty$].

$$3. \int_0^a \frac{\ln x}{x^2+a^2} dx = \frac{\pi}{2a} \ln a - \frac{G}{a} \quad [a > 0].$$

$$4. \int_0^1 \frac{x^{\alpha-1}}{x^2-2ax \cos \gamma + a^2} \ln^\sigma \frac{1}{x} dx = \frac{\Gamma(\sigma+1)}{\sin \gamma} \sum_{k=1}^{\infty} \frac{\sin(k\gamma) a^{-k-1}}{(\alpha+k-1)^{\sigma+1}}$$

[$a > 1; 0 < \gamma < 2\pi; \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1$].

$$5. \int_0^1 \frac{x^{\alpha-1} \ln^n x}{x^2-2ax \cos \gamma + a^2} dx = \pi \cos \gamma \frac{\partial^n}{\partial \alpha^n} \left\{ a^{\alpha-2} \frac{\sin[(\alpha-1)(\gamma-\pi)]}{\sin \alpha \pi} \right\}$$

[$a > 0; 0 < \gamma < 2\pi; \operatorname{Re} \alpha > 0$].

$$6. \int_0^1 \frac{\ln^2 x}{x^2-2x \cos \gamma + 1} dx = \frac{\gamma(\gamma-2\pi)(\gamma-\pi)}{6 \sin \gamma} \quad [0 < \gamma < 2\pi].$$

$$7. \int_0^1 \frac{\ln^4 x}{x^2-2x \cos \gamma + 1} dx = \frac{\gamma(\gamma-2\pi)}{30 \sin \gamma} (4\pi^2 - 3\gamma^2 + 6\gamma\pi) \quad [0 < \gamma < 2\pi].$$

$$8. \int_0^1 \frac{x^k \ln^n x}{x^2 \pm x - 1} dx = I_{\pm}^{k,n},$$

$$I_{+}^{0,1} = -0,7813024129 \dots, \quad I_{-}^{0,1} = -1,17195361935 \dots,$$

$$I_{+}^{1,1} = -0,15766014915 \dots, \quad I_{-}^{1,1} = -0,3118211319 \dots,$$

$$I_{+}^{0,2} = \frac{8\pi^2}{81\sqrt{3}}, \quad I_{-}^{0,2} = \frac{10\pi^3}{81\sqrt{3}}.$$

$$9. \int_0^1 \frac{(x^\alpha - x^{-\alpha}) \ln x}{x^2-2x \cos \gamma + 1} dx = \frac{\pi \sin[(\gamma-\pi)\alpha]}{\sin \gamma \sin \alpha \pi} \{ \pi \operatorname{ctg} \alpha \pi - (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] \}$$

[$0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1$].

$$10. \int_0^1 \frac{(x^\alpha + x^{-\alpha}) \ln^2 x}{x^2-2x \cos \gamma + 1} dx =$$

$$= \frac{\pi \sin[(\gamma-\pi)\alpha]}{\sin \gamma \sin \alpha \pi} \{ \gamma^2 - 2\gamma\pi + 2\pi \operatorname{ctg} \alpha \pi \{ (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] - \pi \operatorname{ctg} \alpha \pi \} \}$$

[$0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1$].

$$11. \int_0^1 x^{\alpha-1} \left(1 + \frac{x \ln x}{1-x} \right) \frac{dx}{1-x} = \alpha \psi'(\alpha) - 1 \quad [\operatorname{Re} \alpha > 0].$$

$$12. \int_0^1 \left(1 + \frac{x \ln x}{1-x} \right) \frac{dx}{1-x} = \frac{\pi^2}{6} - 1.$$

$$13. \int_0^1 x^{\alpha-1} \left[\frac{1 + (\alpha-1) \ln x}{1-x} + \frac{x \ln x}{(1-x)^2} \right] dx = \psi'(\alpha) - 1 \quad [\operatorname{Re} \alpha > 0].$$

$$14. \int_1^\infty \frac{\ln^\sigma x}{x(x-z)} dx = \frac{\Gamma(\sigma+1)}{z} \operatorname{Li}_{\sigma+1}(z) \\ [|\arg(1-z)| < \pi, \operatorname{Re} \sigma > -1] \text{ или } [z=1, \operatorname{Re} \sigma > 0].$$

$$15. \int_1^\infty \frac{\ln x}{x(x-y)} dx = \frac{1}{y} \left[\frac{\pi^2}{3} - \operatorname{Li}_2\left(\frac{1}{y}\right) - \frac{1}{2} \ln^2 y \right] \quad [y \geq 1].$$

$$16. \int_1^\infty \frac{\ln^2 x}{x(x-y)} dx = \frac{1}{y} \left[\operatorname{Li}_3\left(\frac{1}{y}\right) - \frac{1}{6} \ln^3 y + \frac{\pi^2}{3} \ln y \right] \quad [y \geq 1].$$

$$17. \int_0^\infty \frac{\ln^n x dx}{(x+a)(x+b)} = I_n \quad [\operatorname{Im} a = \operatorname{Im} b = 0].$$

$$I_0 = \frac{1}{a-b} \ln \left| \frac{a}{b} \right|,$$

$$I_1 = \frac{1}{2(a-b)} (\ln^2 |a| - \ln^2 |b| + A \operatorname{sgn} a) \quad [A=0 \text{ при } ab > 0; A=\pi^2 \text{ при } ab < 0].$$

$$I_2 = \frac{1}{3(a-b)} (\ln^3 |a| - \ln^3 |b| + B\pi^2)$$

$$[B = \ln |a/b| \text{ при } a, b > 0; B = \ln(ab^2) \text{ при } a > 0, b < 0; B = -\ln(a^2b) \text{ при } a < 0, b > 0; \\ B = 2 \ln |b/a| \text{ при } a, b < 0].$$

$$18. \int_0^\infty \frac{x^{\alpha-1} \ln^n x}{x^2 - 2ax \cos \gamma + a^2} d\gamma = \frac{\pi}{\sin \gamma} \frac{\partial^n}{\partial \alpha^n} \left\{ a^{\alpha-2} \frac{\sin [(\alpha-1)(\gamma-\pi)]}{\sin \alpha\pi} \right\} \\ [a > 0; 0 < \gamma < 2\pi, 0 < \operatorname{Re} \alpha < 2].$$

$$19. \int_0^\infty \frac{x^{\alpha-1} \ln x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{\pi a^{\alpha-2}}{\sin \gamma \sin^2 \alpha\pi} \{ \ln a \sin \alpha\pi \sin [(\alpha-1)(\gamma-\pi)] + \\ + \gamma \sin \alpha\pi \cos [(\alpha-1)(\gamma-\pi)] + \pi \sin [\gamma(\alpha-1)] \} \\ [a > 0; 0 < \gamma < 2\pi; 0 < \operatorname{Re} \alpha < 2, \alpha \neq 1].$$

$$20. \int_0^\infty \frac{\ln x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{(\pi-\gamma) \ln a}{a \sin \gamma} \quad [a > 0; 0 < \gamma < 2\pi].$$

$$21. \int_0^\infty \frac{(x^\alpha - x^{-\alpha}) \ln x}{x^2 - 2x \cos \gamma + 1} dx = \frac{2\pi \sin [\alpha(\gamma-\pi)]}{\sin \gamma \sin \alpha\pi} \{ \pi \operatorname{ctg} \alpha\pi - (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] \} \\ [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1].$$

$$22. \int_0^\infty \frac{(x^\alpha + x^{-\alpha}) \ln x}{x^2 - 2x \cos \gamma + 1} dx = \\ = \frac{2\pi \sin [\alpha(\gamma-\pi)]}{\sin \gamma \sin \alpha\pi} \{ \gamma^2 - 2\gamma\pi + 2\pi \operatorname{ctg} \alpha\pi \{ (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] - \pi \operatorname{ctg} \alpha\pi \} \} \\ [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1].$$

$$23. \int_0^{\infty} \frac{\ln^2 x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{\pi - \gamma}{a \sin \gamma} \left(\ln^2 a + \frac{4}{3} \pi^2 \right) \quad [a > 0; 0 < \gamma < 2\pi].$$

$$24. \int_0^{\infty} \frac{\ln^{2n-1} x}{x^2 + bx + 1} dx = 0 \quad [|b| < 2; n = 1, 2, 3, \dots].$$

2.6.7. Интегралы от $x^\alpha (a^2 - x^2)^\rho \ln^n x$.

$$1. \int_0^a x^{1/n-1} (a^2 - x^2)^{-1/n} \ln x dx = \frac{2 \ln a - \pi \operatorname{ctg} [\pi/(2n)]}{8a^{1/n} \cos [\pi/(2n)]} B\left(\frac{1}{2n}, \frac{1}{2n}\right) \quad [a > 0; n = 2, 3, 4, \dots]$$

$$2. \int_0^a x^2 (a^2 - x^2)^{n/2-1} \ln x dx =$$

$$= Ba^{n+1} \frac{(n-2)!!}{(n+1)!!} \left[\ln a + \frac{n}{n+1} - \sum_{k=1}^{[n/2]} \frac{1}{2k-r} - (1-r) \ln 2 \right]$$

[$a > 0$; $B = \pi/2$, $r = 0$ при $n = 1, 3, 5, \dots$; $B = 1$, $r = 1$ при $n = 2, 4, 6, \dots$].

$$3. \int_0^a \frac{x^m \ln x}{\sqrt{a^2 - x^2}} dx = Aa^m \frac{(m-1)!!}{m!!} \left[(-1)^{m+1} \ln 2 + \ln a + (-1)^{m+1} \sum_{k=1}^m \frac{(-1)^k}{k} \right]$$

[$a > 0$; $A = \pi/2$ при $m = 0, 2, 4, \dots$; $A = 1$ при $m = 1, 3, 5, \dots$].

$$4. \int_0^a x^m \sqrt{a^2 - x^2} \ln x dx =$$

$$= Aa^{m+2} \frac{(m-1)!!}{(m+2)!!} \left[\ln a + (-1)^{m+1} \ln 2 + (-1)^{m+1} \sum_{k=1}^m \frac{(-1)^k}{k} - \frac{1}{m+2} \right]$$

[$a > 0$; $A = \pi/2$ при $m = 0, 2, 4, \dots$; $A = 1$ при $m = 1, 3, 5, \dots$].

$$5. \int_0^a (a^2 - x^2)^{n/2-1} \ln x dx = Ba^{n-1} \frac{(n-2)!!}{(n-1)!!} \left[\ln a - \sum_{k=1}^{[n/2]} \frac{1}{2k-r} - (1-r) \ln 2 \right]$$

[$a > 0$; $B = \pi/2$, $r = 0$ при $n = 1, 3, 5, \dots$; $B = 1$, $r = 1$ при $n = 2, 4, 6, \dots$].

$$6. \int_0^a \frac{x^m \ln^2 x}{\sqrt{a^2 - x^2}} dx =$$

$$= Aa^m \frac{m!!}{(m+1)!!} \left\{ \left[\ln(2a) + \sum_{k=1}^m \frac{(-1)^k}{k} \right]^2 + (-1)^m \left[\frac{\pi^2}{12} + \sum_{k=1}^m \frac{(-1)^k}{k^2} \right] \right\}$$

[$a > 0$; $A = \pi/2$ при $m = 0, 2, 4, \dots$; $A = 1$ при $m = 1, 3, 5, \dots$].

$$7. \int_0^1 \frac{\ln x}{\sqrt{2-x^2}} dx = -\frac{\pi}{8} \ln 2 - \frac{G}{2}.$$

2.6.8. Интегралы от $A(x) \ln^n x$.

$$1. \int_a^b \frac{\ln x dx}{V(x^2 - a^2)(b^2 - x^2)} = \frac{1}{2b} K\left(\frac{V(b^2 - a^2)}{b}\right) \ln(ab) \quad [0 < a < b].$$

$$2. \int_0^a \frac{\ln x dx}{V(a^2 - x^2)(x^2 + z^2)} = \frac{1}{2V(a^2 + z^2)} \left[K\left(\frac{a}{V(a^2 + z^2)}\right) \ln(az) - \frac{\pi}{2} K\left(\frac{z}{V(a^2 + z^2)}\right) \right] \\ [a, \operatorname{Re} z > 0].$$

$$3. \int_0^a \frac{\ln x dx}{V(a^2 - x^2)(b^2 - x^2)} = \frac{1}{2b} \left[K\left(\frac{a}{b}\right) \ln(ab) - \frac{\pi}{2} K\left(\frac{V(b^2 - a^2)}{b}\right) \right] \\ [0 < a < b].$$

$$4. \int_0^1 \frac{1+x}{1-x} \ln x dx = 1 - \frac{\pi^2}{3}. \quad 5. \int_0^1 \frac{1+x}{1+x^3} \ln x dx = -\frac{3+\pi}{27} \pi^2.$$

$$6. \int_0^1 \frac{\ln x}{(x^2 + z^2) V(1 - x^2)} dx = \frac{\pi}{2z V(z^2 + 1)} \ln \frac{z}{z + V(z^2 + 1)} \quad [\operatorname{Re} z > 0].$$

$$7. \int_0^1 \frac{x^2 \ln x dx}{(1 - x^2)(1 + x^4)} = -\frac{\pi^2}{16(2 + \sqrt{2})}.$$

$$8. \int_0^1 \left(\frac{1+x}{1-x}\right)^\rho \frac{\ln x}{1-x^2} dx = \frac{1}{2\rho} \left[\zeta + 2\psi(-\rho) - \psi\left(-\frac{\rho}{2}\right) \right] \quad [0 < \operatorname{Re} \rho < 1].$$

$$9. \int_1^\infty x^{\mu/2-1} \frac{(x^{\mu/2} \pm V(x^{\mu/2}-1))^v}{V(x^{\mu/2}-1)} \ln x dx = \pm \frac{2}{v\mu^2} \left[\psi\left(\frac{v}{4}\right) - \psi\left(\frac{2+v}{4}\right) \right] - \frac{4}{\mu^2 v^2} \\ [\pm \operatorname{Re} v < 0; \mu > 0].$$

$$10. \int_0^\infty \frac{\ln x dx}{V(x^2 + z^2)(x^2 + y^2)} = \frac{\ln(zy)}{2z} K\left(\frac{V(z^2 - y^2)}{z}\right) \quad [z > y > 0].$$

$$11. \int_0^\infty x^{\alpha-1} \frac{(V(x^{\mu/2}+1) \pm 1)^v}{(x^{\mu/2}+1)^r} \ln^n x dx = \\ = \frac{(\mp v)^{1-2r}}{\mu} 2^v \frac{\partial^n}{\partial x^n} \left(2^{2x/\mu} \Gamma \left[\frac{(v \mp v)/2 + \alpha/\mu, 2r-v-2\alpha/\mu}{1-\alpha/\mu-(v \pm v)/2} \right] \right) \\ [r=0 \text{ или } 1/2; \mu > 0; \mu \operatorname{Re}(\pm v - v)/2 < \operatorname{Re} \alpha < \mu r - \mu \operatorname{Re} v/2].$$

$$12. \int_0^\infty x^{\alpha-1} \frac{(V(x^{\mu/2}+1) \pm 1)^v}{(x^{\mu/2}+1)^r} \ln x dx = \\ = \frac{(\mp v)^{1-2r}}{\mu^2} 2^{v+2\alpha/\mu} \Gamma \left[\frac{\alpha/\mu + (v \mp v)/2, 2r-v-2\alpha/\mu}{1-\alpha/\mu-(v \pm v)/2} \right] \times \\ \times \left[2 \ln 2 + \psi\left(\frac{\alpha}{\mu} + \frac{v \mp v}{2}\right) + \psi\left(1 - \frac{\alpha}{\mu} - \frac{v \pm v}{2}\right) - 2\psi\left(2r-v-\frac{2\alpha}{\mu}\right) \right] \\ [r=0 \text{ или } 1/2; \mu > 0; \mu \operatorname{Re}(\pm v - v)/2 < \operatorname{Re} \alpha < \mu r - \mu \operatorname{Re} v/2].$$

$$13. \int_0^{\infty} x^{\alpha-1} \frac{(V \overline{x^{\mu}+1} \pm x^{\mu/2})^{\nu}}{(x^{\mu}+1)^r} \ln^n x dx =$$

$$= \frac{(\mp \nu)^{1-2r}}{\mu} 2^{2r} \frac{\partial^n}{\partial \alpha^n} \left(2^{-2\alpha/\mu} \Gamma \left[\begin{matrix} 2\alpha/\mu, r \mp \nu/2 - \alpha/\mu \\ 1-r+\alpha/\mu \mp \nu/2 \end{matrix} \right] \right)$$

[$r=0$ или $1/2$, $\mu > 0$; $0 < \operatorname{Re} \alpha < \mp \mu \operatorname{Re} \nu/2 + \mu r$].

$$14. \int_0^{\infty} x^{\alpha-1} \frac{(V \overline{x^{\mu}+1} \pm x^{\mu/2})^{\nu}}{(x^{\mu}+1)^r} \ln x dx =$$

$$= \frac{(\mp \nu)^{1-2r}}{\mu^2} 2^{2r-2\alpha/\mu} \Gamma \left[\begin{matrix} 2\alpha/\mu, r \mp \nu/2 - \alpha/\mu \\ 1-r+\alpha/\mu \mp \nu/2 \end{matrix} \right] \left[-2 \ln 2 + 2\psi \left(\frac{2\alpha}{\mu} \right) - \right.$$

$$\left. - \psi \left(1-r+\frac{\alpha}{\mu} \mp \frac{\nu}{2} \right) + \psi \left(r \mp \frac{\nu}{2} - \frac{\alpha}{\mu} \right) \right]$$

[$r=0$ или $1/2$; $\mu > 0$; $0 < \operatorname{Re} \alpha < \mu r \mp \mu \operatorname{Re} \nu/2$].

$$15. \int_0^1 x^{\alpha-1} \frac{(1+V \overline{1-x^{\mu}})^{\nu} + (-1)^{2r-1} (1-V \overline{1-x^{\mu}})^{\nu}}{(1-x^{\mu})^r} \ln^n x dx =$$

$$= \frac{2^{\nu}}{\mu} \nu^{1-2r} \frac{\partial^n}{\partial \alpha^n} \left(2^{\alpha/\mu} \Gamma \left[\begin{matrix} \alpha/\mu, \nu + \alpha/\mu \\ 2\alpha/\mu + \nu + 1 - 2r \end{matrix} \right] \right)$$

[$r=0$ или $1/2$; $\mu, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \mu \nu) > 0$].

$$16. \int_0^1 x^{\alpha-1} \frac{(1+V \overline{1-x^{\mu}})^{\nu} + (-1)^{2r-1} (1-V \overline{1-x^{\mu}})^{\nu}}{(1-x^{\mu})^r} \ln x dx =$$

$$= \frac{2^{\nu+\alpha/\mu}}{\mu^2} \nu^{1-2r} \Gamma \left[\begin{matrix} \alpha/\mu, \nu + \alpha/\mu \\ 2\alpha/\mu + \nu + 1 - 2r \end{matrix} \right] \left\{ \ln 2 + \psi \left(\frac{\alpha}{\mu} \right) + \right.$$

$$\left. + \psi \left(\nu + \frac{\alpha}{\mu} \right) - 2\psi \left(\frac{2\alpha}{\mu} + \nu + 1 - 2r \right) \right\}$$

[$r=0$ или $1/2$; $\mu, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \mu \nu) > 0$].

$$17. \int_0^1 x^{\alpha-1} \frac{(V \overline{1+x^{\mu}} + V \overline{1-x^{\mu}})^{\nu} + (-1)^{2r-1} (V \overline{1+x^{\mu}} - V \overline{1-x^{\mu}})^{\nu}}{(1-x^{2\mu})^r} \ln^n x dx =$$

$$= \frac{\nu^{1-2r}}{\mu} 2^{\nu+2r-2} \frac{\partial^n}{\partial \alpha^n} \left(2^{\alpha/\mu} \Gamma \left[\begin{matrix} \alpha/(2\mu), \alpha/(2\mu) + \nu/2 \\ \alpha/\mu + \nu/2 + 1 - 2r \end{matrix} \right] \right)$$

[$r=0$ или $1/2$; $\mu, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \mu \nu) > 0$].

$$18. \int_0^1 x^{\alpha-1} \frac{(V \overline{1+x^{\mu}} + V \overline{1-x^{\mu}})^{\nu} + (-1)^{2r-1} (V \overline{1+x^{\mu}} - V \overline{1-x^{\mu}})^{\nu}}{(1-x^{2\mu})^r} \ln x dx =$$

$$= \frac{\nu^{1-2r}}{\mu^2} 2^{\nu+2r+\alpha/\mu-3} \Gamma \left[\begin{matrix} \alpha/(2\mu), \alpha/(2\mu) + \nu/2 \\ \alpha/\mu + \nu/2 + 1 - 2r \end{matrix} \right] \left\{ 2 \ln 2 + \right.$$

$$\left. + \psi \left(\frac{\alpha}{2\mu} \right) + \psi \left(\frac{\alpha}{2\mu} + \frac{\nu}{2} \right) - 2\psi \left(\frac{\alpha}{\mu} + \frac{\nu}{2} + 1 - 2r \right) \right\}$$

[$r=0$ или $1/2$; $\mu, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \mu \nu) > 0$].

2.6.9. Интегралы от $x^\alpha \ln^n (ax+b)$.

$$1. \int_0^a x^{\alpha-1} \ln^n (a+x) dx = a^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k) k!} \frac{\partial^n}{\partial \rho^n} \left(a^\rho \Gamma \left[\begin{matrix} k-\rho \\ -\rho \end{matrix} \right] \right)_{\rho=0}$$

[$a > 0$; $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -n$ при $a=1$)].

$$2. \int_0^a x^{\alpha-1} \ln (a+x) dx = a^\alpha \alpha^{-1} [\ln (2a) - \beta (\alpha+1)]$$

[$a > 0$; $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -1$ при $a=1$)].

$$3. \int_0^a x^{n-1} \ln (a+x) dx = \frac{a^n}{n} \left[(-1)^{n-1} \sum_{k=1}^n \frac{(-1)^k}{k} + \ln a + \frac{1 - (-1)^n}{2} \ln 4 \right]$$

[$a > 0$].

$$4. \int_0^a x^{n-1/2} \ln (a+x) dx = \frac{a^{n+1/2}}{2n+1} \left[2 \ln (2a) + (-1)^n \left(\pi - 4 \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right) \right]$$

[$a > 0$].

$$5. \int_0^a x^{\alpha-1} \ln^n (a-x) dx = a^\alpha \frac{\partial^n}{\partial \beta^n} [a^\beta B(\alpha, \beta+1)]|_{\beta=0}$$

[a , $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -n$ при $a=1$)].

$$6. \int_0^1 \frac{1}{x} \ln^n (1-x) dx = (-1)^n n! \zeta (n+1).$$

$$7. \int_0^a x^{\alpha-1} \ln (a-x) dx = \alpha^{-1} a^\alpha [\ln a - C - \psi (\alpha+1)]$$

[a , $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -1$ при $a=1$)].

$$8. \int_0^1 x^n \ln (1 \pm x) dx = \frac{(-1)^{n+1}}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} + \frac{1 - (-1)^{n+1}}{n+1} \ln 2.$$

$$9. \int_0^1 \frac{\ln (1 \pm x)}{x} dx = \frac{\pi^2}{24} (-1 \pm 3). \quad 10. \int_0^{1/2} \frac{\ln (1-x)}{x} dx = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.$$

$$11. \int_0^1 x^{-3/2} \ln (1 \pm x) dx = \pi \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 4 \end{matrix} \right\} \ln 2.$$

$$12. \int_0^a x^{\alpha-1} \ln^2 (a+x) dx =$$

$$= a^\alpha \alpha^{-1} \left\{ 2 (\ln a - C) [\ln 2 - \beta (\alpha+1)] + \ln^2 a + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k(\alpha+k)} \right\}$$

[$a > 0$; $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -2$ при $a=1$)].

$$13. \int_0^a x^{\alpha-1} \ln^2(a-x) dx = a^{\alpha} \alpha^{-1} \{ [\ln a - C - \psi(1+\alpha)]^2 + \pi^2/6 - \psi'(1+\alpha) \}$$

[$a > 0$; $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -2$ при $a=1$)].

$$14. \int_0^1 \frac{\ln^2(1+x)}{x} dx = 2 \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} - \frac{\pi^2 C}{6}.$$

$$15. \int_0^1 \frac{\ln^2(1-x)}{x} dx = 2\zeta(3). \quad 16. \int_0^1 \frac{\ln^2(1-x)}{x^2} dx = \frac{\pi^2}{3}.$$

$$17. \int_1^{n+1} \ln[x] dx = \ln(n!).$$

$$18. \int_a^{\infty} x^{\alpha-1} \ln^n(x+a) dx = a^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^n}{\partial \rho^n} \left(\Gamma \left[\begin{matrix} k-\rho \\ -\rho \end{matrix} \right] \frac{a^{\rho}}{k-\alpha-\rho} \right)_{\rho=0}$$

[$a > 0$; $\operatorname{Re} \alpha < 0$].

$$19. \int_a^{\infty} x^{\alpha-1} \ln(x \pm a) dx = \mp \alpha^{-1} a^{\alpha} \left\{ \begin{matrix} \beta(-\alpha) + \ln(2a) \\ \psi(-\alpha) - \ln a + C \end{matrix} \right\}$$

[$a > 0$; $\operatorname{Re} \alpha < 0$].

$$20. \int_a^{\infty} x^{-n-1} \ln(x+a) dx = \frac{a^{-n}}{n} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k} + \ln a + \frac{1-(-1)^n}{2} \ln 4 \right]$$

[$a > 0$].

$$21. \int_0^{\infty} x^{\alpha-1} \ln|1 \pm x| dx = \frac{\pi}{\alpha \sin \alpha \pi} \left\{ \begin{matrix} 1 \\ \cos \alpha \pi \end{matrix} \right\}$$

[$-1 < \operatorname{Re} \alpha < 0$].

$$22. \int_a^{\infty} x^{-(n+3/2)} \ln(x+a) dx = \frac{a^{-n-1/2}}{2n+1} \left[2 \ln(2a) + (-1)^n \left(\pi - 4 \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \right) \right]$$

[$a > 0$].

$$23. \int_a^{\infty} x^{\alpha-1} \ln^2(x+a) dx = \alpha^{-1} a^{\alpha} \left\{ 2 \left(C + \frac{1}{\alpha} - \ln a \right) [\ln 2 + \beta(-\alpha)] + \right.$$

$$\left. + \frac{2}{\alpha} C - \ln^2 a - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-\alpha)^2} + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k(k-\alpha)} \right\}$$

[$a > 0$, $\operatorname{Re} \alpha < 0$].

$$24. \int_a^{\infty} x^{\alpha-1} \ln^2(x-a) dx = -\alpha^{-1} a^{\alpha} \left\{ [C + \psi(-\alpha) - \ln a]^2 + \frac{\pi^2}{6} + \psi'(-\alpha) \right\}$$

[$a > 0$, $\operatorname{Re} \alpha < 0$].

2.6.10. Интегралы от $x^{\alpha} (ax+b)^{\beta} \ln^n(cx+d)$.

$$1. \int_0^a \frac{x^{\alpha-1}}{(a+x)^{\rho}} \ln^n(a+x) dx = (-1)^n a^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k) k!} \frac{\partial^n}{\partial \rho^n} \left(a^{-\rho} \Gamma \left[\begin{matrix} \rho+k \\ \rho \end{matrix} \right] \right)$$

[$a > 0$, $\operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -n$ при $a=1$)].

$$2. \int_0^a \frac{1}{(a+x)^\rho} \ln^n (a+x) dx = (-1)^n \frac{\partial^n}{\partial \rho^n} \left(a^{1-\rho} \frac{2^{1-\rho} - 1}{1-\rho} \right) \quad [a > 0].$$

$$3. \int_0^1 (1 \pm x)^{\beta-1} \ln^n (1 \pm x) dx = \mp \frac{(-1)^n n!}{\beta^{n+1}} + (1 \pm 1) n! 2^{\beta-1} \sum_{k=0}^n \frac{(-1)^k (\ln 2)^{n-k}}{(n-k)! \beta^{k+1}} \\ [\operatorname{Re} \beta > 0 \text{ для } \ln^n (1-x)].$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln (a+x) dx = \\ = a^{\alpha-\rho} \beta(\alpha) [\ln a + \psi(\rho)] - a^{\alpha-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k (\rho)_k}{(\alpha+k) k!} \psi(\rho+k) \\ [a > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -1 \text{ при } a=1)].$$

$$5. \int_0^a \frac{\ln (a+x)}{(a+x)^\rho} dx = \\ = -\frac{a^{1-\rho}}{(1-\rho)^2} \{ (2^{1-\rho} - 1) [1 + (\rho-1) \ln a] + (\rho-1) 2^{1-\rho} \ln 2 \} \quad [a > 0; \rho \neq 1].$$

$$6. \int_0^a \frac{\ln (a+x)}{a+x} dx = \ln 2 \ln (2a) \quad [a > 0].$$

$$7. \int_0^a x^{-1/2} \frac{\ln (a+x)}{a+x} dx = 2a^{-1/2} \left(\frac{\pi}{2} \ln 2 + \frac{\pi}{4} \ln a - G \right) \quad [a > 0].$$

$$8. \int_0^1 \frac{\ln (1+x)}{1+x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}.$$

$$9. \int_0^1 \frac{\ln (1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi(\rho) - \psi(\rho+k)].$$

$$10. \int_0^1 \frac{\ln (1+x)}{(1+x)^4} dx = -\frac{1}{24} \left(\ln 2 + \frac{2}{9} \right).$$

$$11. \int_0^1 x^2 \frac{\ln (1+x)}{(1+x)^4} dx = \frac{1}{24} \left(\frac{25}{9} - 7 \ln 2 \right).$$

$$12. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln^2 (a+x) dx = a^{\alpha-\rho} \beta(\alpha) [(\ln a + \psi(\rho))^2 - \psi'(\rho)] + \\ + a^{\alpha-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k) k!} [\psi^2(\rho+k) + \psi'(\rho+k) - 2\psi(\rho+k) (\ln a + \psi(\rho))] \\ [a > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -2 \text{ при } a=1)].$$

$$13. \int_0^1 \frac{\ln^2(1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi^2(\rho+k) - \psi'(\rho+k) - 2\psi(\rho+k)\psi(\rho)].$$

$$14. \int_0^1 \frac{\ln^2(1+x)}{x^2(1+x)^\rho} dx = \left\{ \psi^2(\rho) - \psi'(\rho) + \right. \\ \left. + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)!} [\psi^2(\rho+k+2) + \psi'(\rho+k+2) - 2\psi(\rho+k+2)\psi(\rho)] \right\}.$$

$$15. \int_0^1 \frac{\ln(1-x)}{x(1+x)^\rho} dx = \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} (\mathbf{C} + \psi(2+k)) - \frac{\pi^2}{6}.$$

$$16. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln(1+x) dx = \frac{1}{\beta} {}_3F_2(1, 1, 1; 2, \beta+1; -1) \quad [\operatorname{Re} \beta > 0].$$

$$17. \int_0^1 \frac{\ln^2(1-x)}{x(1+x)^\rho} dx = \\ = \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} [\psi'(k+2) - \psi(k+2)(\psi(k+2) + 2\mathbf{C}) - \mathbf{C}^2 - \pi^2/6].$$

$$18. \int_0^1 \frac{\ln^2(1-x)}{x^2(1+x)} dx = a^{-1-\rho} \left\{ \rho \psi''(1) + \frac{\pi^2}{3} + \right. \\ \left. + \rho(\rho+1) \sum_{k=0}^{\infty} \frac{(\rho+2)_k (-1)^k}{(k+2)(k+1)^2 k!} \left[\frac{\pi^2}{6} - \psi'(2+k) + (\mathbf{C} + \psi(2+k))^2 \right] \right\}.$$

$$19. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln^2(1+x) dx = \\ = \frac{2}{\beta} \left[\sum_{k=0}^{\infty} \frac{(-1)^{k+1} k! \psi(k+1)}{(1+\beta)_k (k+1)} - \mathbf{C} {}_3F_2(1, 1, 1; 2, 1+\beta; -1) \right] \quad [\operatorname{Re} \beta > 0],$$

$$20. \int_0^1 \frac{(1-x)^{\beta-1}}{x^2} \ln^2(1+x) dx = \frac{2}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(\beta+1)_k (k+2)} [\mathbf{C} + \psi(k+2)] \\ [\operatorname{Re} \beta > 0].$$

$$21. \int_0^1 \frac{\ln(1 \pm x)}{\sqrt{x}} \frac{dx}{1+x} = \frac{3 \pm 1}{4} \pi \ln 2 - 2\mathbf{G}.$$

$$22. \int_0^a x^{\alpha-1} \frac{\ln^n (a-x)}{(cx+d)^\rho} dx =$$

$$= d^{-\rho} a^\alpha \frac{\partial^n}{\partial \beta^n} \left[a^\beta B(\alpha, \beta+1) {}_2F_1 \left(\rho, \alpha; \alpha+\beta+1; -\frac{ac}{d} \right) \right]_{\beta=0}$$

[a, Re $\alpha > 0$, $|\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$23. \int_0^a x^{\alpha-1} \frac{\ln (a-x)}{(cx+d)^\rho} dx = \frac{d^{-\rho} a^\alpha}{\alpha} {}_2F_1 \left(\rho, \alpha; \alpha+1; -\frac{ac}{d} \right) (\ln a - C) -$$

$$- d^{-\rho} a^\alpha \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\alpha+k) k!} \left(-\frac{ac}{d} \right)^k \psi(\alpha+1+k)$$

[a, Re $\alpha > 0$, $|\arg(cx+d)| < \pi$ при $0 \leq x \leq a$, $|ac| < |d|$].

$$24. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln^n (a-x) dx = a^{\alpha-1} \frac{\partial^n}{\partial \beta^n} [a^\beta B(\alpha, \beta)]$$

[a, Re $\beta > 0$; Re $\alpha > 0$ (Re $\alpha > -n$ при $a=1$)]

$$25. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln (a-x) dx = a^{\alpha+\beta-1} B(\alpha, \beta) [\ln a + \psi(\beta) - \psi(\alpha+\beta)]$$

[a, Re $\beta > 0$; Re $\alpha > 0$ (Re $\alpha > -1$ при $a=1$)].

$$26. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln (1-x) dx = -\psi'(\beta)$$

[Re $\beta > 0$].

$$27. \int_0^a \frac{x^{\alpha-1}}{(a-x)^{1-\beta}} \ln^2 (a-x) dx =$$

$$= \frac{a^{\alpha+\beta-1}}{\mu} B(\beta, \alpha) \{[\psi(\beta) - \psi(\beta+\alpha) + \ln a]^2 + \psi'(\beta) - \psi'(\beta+\alpha)\}$$

[a, Re $\beta > 0$; Re $\alpha > 0$ (Re $\alpha > -2$ при $a=1$)].

$$28. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln^n (cx+d) dx =$$

$$= a^{\alpha+\beta-1} B(\alpha, \beta) \frac{\partial^n}{\partial \rho^n} \left[{}_2F_1 \left(-\rho, \alpha; \alpha+\beta; -\frac{ac}{d} \right) \right]_{\rho=0}$$

[a, Re α , Re $\beta > 0$; $|\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$29. \int_0^a (a-x)^{\beta-1} \ln (cx+d) dx = \int_0^a x^{\beta-1} \ln (ac+d-cx) dx.$$

$$30. \int_0^1 \ln \frac{1+x}{2} \frac{dx}{1-x} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.$$

$$31. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln (cx+d) dx = a^{\alpha+\beta-1} B(\alpha, \beta) \ln d +$$

$$+ cd^{-1} a^{\alpha+\beta} B(\alpha+1, \beta) {}_3F_2(\alpha+1, 1, 1; 2, \alpha+\beta+1; -ac/d)$$

[a, Re α , Re $\beta > 0$; $|\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$32. \int_0^a \frac{\ln [(1-b)x/a+b]}{\sqrt{x}} \frac{dx}{a-x} = -\frac{2}{\sqrt{a}} \arccos^2 \sqrt{b} \quad [a > 0; 0 < b \leq 1].$$

$$33. \int_0^a \frac{\ln(a-cx)}{\sqrt{x}\sqrt{a-x}} dx = 2\pi \ln \frac{1+\sqrt{1-c}}{2} + \pi \ln a \quad [0 < c \leq 1]$$

$$34. \int_0^a \sqrt{a-x} \frac{\ln(d+cx)}{\sqrt{x}} dx = \pi a \ln \frac{d+\sqrt{d(d+ac)}}{2} + \frac{\pi a}{2} \frac{\sqrt{d}-\sqrt{d+ac}}{\sqrt{d}+\sqrt{d+ac}} \quad [a, b, c > 0].$$

$$35. \int_0^1 \frac{\ln(1-cx)}{\sqrt{x}\sqrt{1-x}} dx = 2\pi (\sqrt{1-c}-1) \quad [0 < c < 1].$$

$$36. \int_0^1 \frac{\ln(1-cx)}{x\sqrt{1-x}} dx = \frac{\pi^2}{2} - 2 \arccos^2 \sqrt{c} \quad [0 < c \leq 1].$$

$$37. = \pi^2/2 + 2 \ln(\sqrt{c} + \sqrt{c-1}) \quad [c \geq 1].$$

$$38. \int_a^\infty x^{\alpha-1} \frac{\ln^n(cx+d)}{(cx+d)^\rho} dx = (-1)^n \frac{\partial^n}{\partial \rho^n} \left[\frac{a^{\alpha-\rho} c^{-\rho}}{\rho-\alpha} {}_2F_1\left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac}\right) \right] \\ [Re(\alpha-\rho) < 0; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty].$$

$$39. \int_a^\infty x^{\alpha-1} \frac{\ln(cx+d)}{(cx+d)^\rho} dx = \\ = \frac{a^{\alpha-\rho} c^{-\rho}}{\rho-\alpha} \left\{ [\ln(ac) + \psi(\rho)] {}_2F_1\left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac}\right) - \right. \\ \left. - (\rho-\alpha) \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\rho-\alpha+k)k!} \left(-\frac{d}{ac}\right)^k \left[\psi(\rho+k) - \frac{1}{\rho-\alpha+k} \right] \right\} \\ [Re(\alpha-\rho) < 0; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty, |d| < |ac|].$$

$$40. \int_1^\infty \frac{\ln(x \pm 1)}{\sqrt{x}(x+1)} dx = \frac{3 \pm 1}{4} \pi \ln 2 + 2G.$$

$$41. \int_a^\infty x^{\alpha-1} \frac{\ln^n(x-a)}{(cx+d)^\rho} dx = \\ = a^{\alpha-\rho} c^{-\rho} \frac{\partial^n}{\partial \beta^n} \left[a^\beta B(\beta+1, \rho-\alpha-\beta) {}_2F_1\left(\rho, \rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \right]_{\beta=0} \\ [Re(\alpha-\rho) < 0, a > 0, |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty]$$

$$42. \int_a^\infty x^{\alpha-1} \frac{\ln(x-a)}{(cx+d)^\rho} dx = \frac{a^{\alpha-\rho} c^{-\rho}}{\rho-\alpha} \left[(\ln a - C) {}_2F_1\left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac}\right) - \right. \\ \left. - (\rho-\alpha) \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\rho-\alpha+k)k!} \left(-\frac{d}{ac}\right)^k \psi(\rho-\alpha+k) \right] \\ [Re(\alpha-\rho) < 0, a > 0, |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty, |d| < |ac|].$$

$$43. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln^n (x-a) dx = a^{\alpha-1} \frac{\partial^n}{\partial \beta^n} [a^{\beta} B(\beta, 1-\alpha-\beta)]$$

[a, Re $\beta > 0$, Re ($\alpha + \beta$) < 1].

$$44. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln (x-a) dx =$$

$$= a^{\alpha+\beta-1} B(\beta, 1-\alpha-\beta) [\ln a + \psi(\beta) - \psi(1-\alpha-\beta)]$$

[a, Re $\beta > 0$; Re ($\alpha + \beta$) < 1].

$$45. \int_a^{\infty} x^{\alpha-1} \frac{\ln^2 (x-a)}{(x-a)^{1-\beta}} dx =$$

$$= a^{\alpha+\beta-1} B(1-\alpha-\beta, \beta) \{[\psi(\beta) - \psi(1-\alpha-\beta) + \ln a]^2 + \psi'(\beta) + \psi'(1-\alpha-\beta)\}$$

[a, Re $\beta > 0$, Re ($\alpha + \beta$) < 1].

$$46. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln^n (cx+d) dx = a^{\alpha+\beta-1} \frac{\partial^n}{\partial \rho^n} \times$$

$$\times [(ac)^{\rho} B(\beta, 1-\alpha-\beta-\rho) {}_2F_1(-\rho, 1-\alpha-\beta-\rho; 1-\alpha-\rho; -d/(ac))]_{\rho=0}$$

[a, Re $\beta > 0$, Re ($\alpha + \beta$) < 1, $|\arg (cx+d)| < \pi$ при $a \leq x < \infty$].

$$47. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln (cx+d) dx =$$

$$= a^{\alpha+\beta-1} B(\beta, 1-\alpha-\beta) [\ln (ac) + \psi(1-\alpha) - \psi(1-\alpha-\beta)] +$$

$$+ dc^{-1} a^{\alpha+\beta-2} B(\beta, 2-\alpha-\beta) {}_3F_2(2-\alpha-\beta, 1, 1; 2, 2-\alpha; -d/(ac))$$

[a, Re $\beta > 0$, Re ($\alpha + \beta$) < 1; $|\arg (cx+d)| < \pi$ при $a \leq x < \infty$].

$$48. \int_0^{\infty} \frac{x^{\alpha-1}}{(cx+d)^{\rho}} \ln^n (cx+d) dx = (-1)^n \left(\frac{d}{c}\right)^{\alpha} \frac{\partial^n}{\partial \rho^n} [d^{-\rho} B(\alpha, \rho-\alpha)]$$

[Re α , Re ($\rho - \alpha$) > 0; $|\arg (cx+d)| < \pi$ при $0 \leq x < \infty$].

$$49. \int_0^{\infty} \frac{x^{\alpha-1}}{(cx+d)^{\rho}} \ln (cx+d) dx = \frac{d^{\alpha-\rho}}{c^{\alpha}} B(\alpha, \rho-\alpha) [\ln d + \psi(\rho) - \psi(\rho-\alpha)]$$

[Re α , Re ($\rho - \alpha$) > 0; $|\arg (cx+d)| < \pi$ при $0 \leq x < \infty$].

$$50. \int_0^{\infty} \frac{x^{\alpha-1}}{cx+d} \ln (cx+d) dx = \frac{d^{\alpha-1} \pi}{c^{\alpha} \sin \alpha \pi} [\ln d - C - \psi(1-\alpha)]$$

[0 < Re α < 1; $|\arg (cx+d)| < \pi$ при $0 \leq x < \infty$].

$$51. \int_0^{\infty} \frac{x^{\alpha-1}}{cx+d} \ln^2 (cx+d) dx = \frac{d^{\alpha-1} \pi}{c^{\alpha} \sin \alpha \pi} \{[\ln d - C - \psi(1-\alpha)]^2 - \pi^2/6 + \psi'(1-\alpha)\}$$

[0 < Re α < 1; $|\arg (cx+d)| < \pi$ при $0 \leq x < \infty$]

$$52. \int_0^{\infty} \frac{\ln (x+1)}{(x+1)^{\rho}} \frac{dx}{x} = \psi'(\rho)$$

[Re $\rho > 0$].

$$53. \int_0^{\infty} x^{\alpha-1} \frac{\ln^2(x+a)}{(x+a)^{\rho}} dx =$$

$$= a^{\alpha-\rho} B(\alpha, \rho-\alpha) \{ [\psi(\rho) - \psi(\rho-\alpha) + \ln a]^2 - \psi'(\rho) + \psi'(\rho-\alpha) \}$$

$$[|\arg a| < \pi; \operatorname{Re}(\alpha-\rho) < 0; \operatorname{Re} \alpha > 0 \text{ (} \operatorname{Re} \alpha > -2 \text{ при } a=1 \text{)}].$$

$$54. \int_0^{\infty} \frac{\ln^2(1+x)}{x(1+x)^{\rho}} dx = -\psi''(\rho) \quad [\operatorname{Re} \rho > 0].$$

$$55. \int_0^{\infty} \frac{\ln^2(1+x)}{x^2(1+x)^{\rho}} dx = 2\psi'(\rho+1) + \rho\psi''(\rho+1) \quad [\operatorname{Re} \rho > 1].$$

$$56. \int_0^{\infty} x^{\alpha-1} \frac{\ln(x+b)}{(x+a)^{\rho}} dx = a^{\alpha-\rho} B(\alpha, \rho-\alpha) \left[\ln b + \psi(\rho) - \psi(\rho-\alpha) - \right.$$

$$\left. - \frac{\alpha}{\rho} (1-ab^{-1}) {}_3F_2(\alpha+1, 1, 1; 2, \rho+1; 1-ab^{-1}) \right]$$

$$[\operatorname{Re}(\alpha-\rho) < 0; |\arg a|, |\arg b| < \pi; \operatorname{Re} \alpha > 0 \text{ (} \operatorname{Re} \alpha > -1 \text{ при } b=1 \text{)}].$$

$$57. \int_0^{\infty} \frac{\ln(x+1)}{x(x+a)^{\rho}} dx = \frac{a^{-\rho}}{\rho} [\rho\psi'(\rho) - (1-a) {}_3F_2(1, 1, 1; 2, \rho+1; 1-a)]$$

$$[\operatorname{Re} \rho > 0; |\arg a| < \pi].$$

$$58. \int_0^{\infty} \frac{x^{\alpha-1}}{(ax+b)^{\sigma}} \ln^n(cx+d) dx =$$

$$= (-1)^n \left(\frac{d}{c} \right)^{\alpha} b^{-\sigma} \frac{\partial^n}{\partial \rho^n} [d^{-\rho} B(\alpha, \rho+\sigma-\alpha) {}_2F_1(\sigma, \alpha; \sigma+\rho; 1-ad/(bc))]_{\rho=0}$$

$$[\operatorname{Re}(\alpha-\sigma) < 0, \operatorname{Re} \alpha > 0; |\arg(cx+d)|, |\arg(ax+b)| < \pi \text{ при } 0 \leq x < \infty].$$

$$59. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln(x+a) dx =$$

$$= a^{\alpha+\beta-1} \{ B(\beta, 1-\alpha-\beta) [\ln a + \psi(1-\alpha) - \psi(1-\alpha-\beta)] +$$

$$+ B(\beta, 2-\alpha-\beta) {}_3F_2(2-\alpha-\beta, 1, 1; 2-\alpha, 2; -1) \}$$

$$[a, \operatorname{Re} \beta > 0, \operatorname{Re}(\alpha+\beta) < 1].$$

$$60. \int_0^{\infty} \frac{x^{\alpha-1}}{(ax+b)^{\sigma}} \ln(cx+d) dx = \left(\frac{b}{a} \right)^{\alpha} b^{-\sigma} B(\alpha, \sigma-\alpha) \left\{ [\ln d - \psi(\sigma-\alpha)] + \right.$$

$$\left. + \left(\frac{ad}{bc} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \left(1 - \frac{ad}{bc} \right)^k \psi(\sigma+k) \right\}$$

$$[\operatorname{Re}(\alpha-\sigma) < 0, \operatorname{Re} \alpha > 0; |1-ad/(bc)| < 1; |\arg(cx+d)|, |\arg(ax+b)| < \pi \text{ при } 0 \leq x < \infty].$$

$$61. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{x(ax+b)}} dx = \frac{2\pi}{\sqrt{ab}} \ln \frac{\sqrt{ad} + \sqrt{bc}}{\sqrt{ad} - \sqrt{bc}}$$

$$[|\arg(cx+d)|, |\arg(ax+b)| < \pi \text{ при } 0 \leq x < \infty].$$

$$62. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{x(ax+b)}} dx = \frac{2\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{bc}{ad}}$$

[$a, b > 0$; $|\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$63. \int_0^{\infty} x^{\pm 1/2} \frac{\ln(cx+d)}{(ax+b)^2} dx = \frac{\pi}{2} a^{-1 \pm 1/2} b^{-1 \mp 1/2} \left(\ln \frac{\sqrt{bc} + \sqrt{ad}}{\sqrt{a}} \mp \frac{\sqrt{bc}}{\sqrt{bc} + \sqrt{ad}} \right)$$

[$|\arg(cx+d)|, |\arg(ax+b)| < \pi$ при $0 \leq x < \infty$].

$$64. \int_0^{\infty} \frac{\ln|a-x|}{\sqrt{x(b+x)}} dx = \frac{\pi}{\sqrt{b}} \ln(a+b)$$

[$a, b > 0$].

2.6.11. Интегралы от $x^\alpha (ax^2+b)^\beta \ln(cx+d)$.

$$1. \int_0^a \frac{\ln(c^2+ax)}{c^2+x^2} dx = \frac{1}{c} \operatorname{arctg} \frac{a}{c} \left[\ln c + \frac{1}{2} \ln(a^2+c^2) \right]$$

[$a, c > 0$].

$$2. \int_0^1 \frac{\ln(1 \pm x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - \left\{ \begin{matrix} 0 \\ G \end{matrix} \right\}.$$

$$3. \int_0^1 x^{\pm 1} \frac{\ln(1+x)}{(1+x^2)^2} dx = \frac{\pi}{16} (\ln 2 \mp 1) \pm \frac{3}{8} \ln 2.$$

$$4. \int_0^a \frac{\ln(cx+d)}{\sqrt{a^2-x^2}} dx = \frac{\pi}{2} \ln \frac{d + \sqrt{d^2 - a^2 c^2}}{2}$$

[$a > 0$; $d > a|c|$].

$$5. = (\pi/2) \ln(2ac/d^2)$$

[$0 < d < a|c|$].

$$6. \int_0^a \frac{\ln(a \pm x)}{\sqrt{a^2-x^2}} dx = \frac{\pi}{2} \ln \frac{a}{2} \pm 2G$$

[$a > 0$].

$$7. \int_0^a \frac{\ln(cx+1)}{x\sqrt{a^2-x^2}} dx = \frac{\pi^2}{8a} - \frac{1}{2a} \arccos^2(ac)$$

[$a > 0$; $|ac| \leq 1$].

$$8. \int_0^a x \frac{\ln(cx+d)}{\sqrt{a^2-x^2}} dx =$$

$$= \frac{1}{2c} [\pi(d - \sqrt{d^2 - a^2 c^2}) + 2\sqrt{d^2 - a^2 c^2} \arcsin(ac/d) + 2a(\ln d - 1)]$$

[$d \geq a|c|$, $a > 0$].

$$9. = \frac{1}{2c} \left[\pi + 2\sqrt{a^2 c^2 - d^2} \ln \frac{ac + \sqrt{a^2 c^2 - d^2}}{d} + 2ac(\ln d - 1) \right]$$

[$0 < d \leq a|c|$; $a > 0$].

$$10. \int_1^{\infty} \frac{\ln(x \pm 1)}{x^2+1} dx = \frac{\pi}{8} \ln 2 + \left\{ \begin{matrix} G \\ 0 \end{matrix} \right\}. \quad 11. \int_1^{\infty} \frac{\ln(x-1)}{x^2+x+1} dx = \frac{\pi \ln 3}{6\sqrt{3}}.$$

$$12. \int_0^{\infty} \frac{\ln(cx+d)}{(ax+b)^2} dx = \frac{1}{bc-ad} \left(\frac{c}{a} \ln \frac{bc}{a} - \frac{d}{b} \ln d \right)$$

[$bc \neq ad$, $|\arg(cx+d)| < \pi$ при $0 \leq x < \infty$]

$$13. = \frac{c}{a^2 d} (\ln d + 1)$$

[$bc = ad$, $|\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$14. \int_0^{\infty} \frac{\ln |1 \pm x|}{1+x^2} dx = \frac{\pi}{4} \ln 2 \pm G.$$

$$15. \int_a^{\infty} \frac{\ln(ax+b)}{x^2+c^2} dx = \frac{a}{b} {}_3F_2 \left(1, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{a^2 c^2}{b^2} \right) +$$

$$+ \frac{1}{2c} \left[\ln \frac{b}{ac} \cdot \ln \frac{b+ac}{b-ac} - \frac{\pi}{2} \ln \left(1 - \frac{a^2 c^2}{b^2} \right) + \pi \ln b \right]$$

[$\operatorname{Re} c > 0$, $|\arg(ax+b)| < \pi$ при $x > 0$].

$$16. \int_0^{\infty} x^{\pm 1/2} \frac{\ln |a-x|}{(b+x)^2} dx = \frac{\pi b^{\pm 1/2}}{2b} \ln(a+b) \pm \frac{\pi b^{\pm 1/2}}{a+b}$$

[$a, b > 0$].

$$17. \int_0^{\infty} \frac{x \ln |cx+d|}{(z^2+x^2)^2} dx = \frac{c^2}{2(d^2+c^2 z^2)} \left(\ln |cz| + \frac{\pi d}{2cz} + \frac{d^2}{c^2 z^2} \ln d \right)$$

[$d, \operatorname{Re} z > 0, \operatorname{Im} c = 0$].

$$18. \int_0^{\infty} \frac{z^2-x^2}{(z^2+x^2)^2} \ln(cx+d) dx = \frac{c}{d^2+c^2 z^2} \left(d \ln \left| \frac{cz}{d} \right| - \frac{cz\pi}{2} \right)$$

[$d, \operatorname{Re} z > 0, \operatorname{Im} c = 0$]

2.6.12. Интегралы от $x^\alpha (a_1 x + b_1)^\beta (a_2 x + b_2)^\gamma \ln^n (cx + d)$.

$$1. \int_a^b \frac{\ln |x-y|}{(x-a)^{1-\alpha} (b-x)^\alpha} dx = \pi \frac{\ln(b-a) + \psi(\alpha) + C}{\sin \alpha \pi} +$$

$$+ \operatorname{ctg} \alpha \pi \left(\frac{y-a}{b-y} \right)^\alpha {}_2F_1 \left(1, \alpha; 1+\alpha; -\frac{y-a}{b-y} \right)$$

[$a < y < b$, $0 < \operatorname{Re} \alpha < 1$].

$$2. \int_a^b \left(\frac{x-a}{b-x} \right)^\alpha \ln |x-y| dx = \pi \alpha (b-a) \frac{\ln(b-a) + \psi(\alpha+1) + C - 1}{\sin \alpha \pi} -$$

$$- \pi \frac{x-a}{\sin \alpha \pi} + \pi \frac{\operatorname{ctg} \alpha \pi}{\alpha+1} (y-a)^{\alpha-1} (b-y)^{-\alpha} {}_2F_1 \left(1, \alpha; 2+\alpha; -\frac{y-a}{b-y} \right)$$

[$\operatorname{Re} \alpha = 0$, $\alpha \neq 0$, $a < y < b$].

$$3. \int_a^b \frac{\ln |x-y|}{V(x-a)(b-x)} dx = \pi \ln \frac{b-a}{4}$$

[$a < y < b$].

$$4. \int_0^1 \ln(1-x) \frac{1-(\pm x)^{n+1}}{1-x} dx = \mp \sum_{j=1}^{n+1} \frac{(\pm 1)^j}{j} \sum_{k=1}^j \frac{1}{k}.$$

$$5. \int_0^1 \ln(1+x) \frac{1 - (\pm x)^{n+1}}{1 \mp x} dx = 2 \ln 2 \sum_{k=0}^{[n/2]} \frac{(\mp 1)^k}{2k+1} \pm \sum_{j=1}^{n+1} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}.$$

$$6. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln^n(a-x)}{(cx+d)^\rho} dx =$$

$$= d^{-\rho} a^{\alpha-1} \frac{\partial^n}{\partial \beta^n} [a^\beta B(\alpha, \beta) {}_2F_1(\rho, \alpha; \alpha+\beta; -ac/d)]$$

[$a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$7. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln(a-x)}{(cx+d)^\rho} dx =$$

$$= d^{-\rho} a^{\alpha+\beta-1} B(\alpha, \beta) \left\{ {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) [\ln a + \psi(\beta)] - \right.$$

$$\left. - \sum_{k=0}^{\infty} \frac{(\rho)_k (\alpha)_k}{(\alpha+\beta)_k k!} \left(-\frac{ac}{d}\right)^k \psi(\alpha+\beta+k) \right\}$$

[$a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |ac| < d; |\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$8. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln^n(cx+d)}{(cx+d)^\rho} dx =$$

$$= (-1)^n a^{\alpha+\beta-1} B(\alpha, \beta) \left(\frac{\partial}{\partial \rho}\right)^n \left[d^{-\rho} {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) \right]$$

[$a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$9. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln(cx+d)}{(cx+d)^\rho} dx =$$

$$= a^{\alpha+\beta-1} B(\alpha, \beta) d^{-\rho} \left\{ [\ln d + \psi(\rho)] {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) - \right.$$

$$\left. - \sum_{k=0}^{\infty} \frac{(\rho)_k (\alpha)_k}{(\alpha+\beta)_k k!} \left(-\frac{ac}{d}\right)^k \psi(\rho+k) \right\}$$

[$a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |ac| < |d|; |\arg(cx+d)| < \pi$ при $0 \leq x \leq a$].

$$10. \int_0^1 \frac{(1-x^2) \ln(1+x)}{(ax+b)^2 (bx+a)^2} dx =$$

$$= \frac{1}{a^2-b^2} \left\{ \frac{1}{a-b} \left[\frac{a+b}{ab} \ln(a+b) - \frac{\ln b}{a} - \frac{\ln a}{b} \right] + \frac{4 \ln 2}{b^2-a^2} \right\} \quad [a, b > 0; a \neq b].$$

$$11. \int_0^1 \frac{(1+x^2) \ln(1+x)}{(a^2+x^2)(a^2x^2+1)} dx = \frac{1}{2a(a^2+1)} \left[\frac{\pi}{2} \ln(a^2+1) - 2 \operatorname{arctg} a \ln a \right]$$

[$a > 0$].

$$12. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln^n (x-a)}{(cx+d)^\rho} dx =$$

$$= \frac{\partial^n}{\partial \beta^n} \left[a^{\alpha+\beta-\rho-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \right]$$

[Re ($\rho-\alpha-\beta$) > -1, Re β > 0, |arg ($cx+d$)| < π при $a \leq x < \infty$].

$$13. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln (x-a)}{(cx+d)^\rho} dx =$$

$$= a^{\alpha+\beta-\rho-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \left\{ {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \times \right.$$

$$\times [\ln a + \psi(\beta)] - \sum_{k=0}^\infty \frac{(\rho)_k (1+\rho-\alpha-\beta)_k}{(1+\rho-\alpha)_k k!} \left(-\frac{ac}{d}\right)^{-k} \psi(1+\rho-\alpha-\beta+k) \Big\}$$

[Re ($\alpha+\beta-\rho$) < 1, a, Re β > 0, |ac| > |d|; |arg ($cx+d$)| < π при $a \leq x < \infty$].

$$14. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln^n (cx+d)}{(cx+d)^\rho} dx =$$

$$= (-1)^n \frac{\partial^n}{\partial \rho^n} \left[a^{\alpha+\beta-\rho-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \times \right.$$

$$\times {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \Big]$$

[Re ($\alpha+\beta-\rho$) < 1, Re β > 0; |arg ($cx+d$)| < π при $a \leq x < \infty$].

$$15. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln (cx+d)}{(cx+d)^\rho} dx = a^{\alpha+\beta-\rho-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \times$$

$$\times \left\{ {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) [\ln(ac) + \psi(\rho)] - \right.$$

$$- \sum_{k=0}^\infty \frac{(\rho)_k (1+\rho-\alpha-\beta)_k}{(1+\rho-\alpha)_k k!} \left(-\frac{d}{ac}\right)^k [\psi(\rho+k) + \psi(1+\rho-\alpha-\beta+k) -$$

$$\left. - \psi(1+\rho-\alpha+k)] \Big\}$$

[Re ($\alpha+\beta-\rho$) < 1; a, Re β > 0; |d| < |ac|; |arg ($cx+d$)| < π при $a \leq x < \infty$].

$$16. \int_0^\infty \frac{x^{\alpha-1} \ln^n (cx+d)}{(ax+b)^\sigma (cx+d)^\rho} dx =$$

$$= (-1)^n \left(\frac{d}{c}\right)^\alpha b^{-\sigma} \frac{\partial^n}{\partial \rho^n} \left[d^{-\rho} B(\alpha, \rho+\sigma-\alpha) {}_2F_1\left(\sigma, \alpha; \sigma+\rho; 1-\frac{ad}{bc}\right) \right]$$

[Re α > 0; Re ($\alpha-\sigma-\rho$) < 0; |arg ($ax+b$)|, |arg ($cx+d$)| < π при $0 \leq x < \infty$].

$$17. \int_0^\infty \frac{x^{\alpha-1} \ln (cx+d)}{(ax+b)^\sigma (cx+d)^\rho} dx = \frac{d^{\alpha-\rho}}{c^\alpha b^\sigma} B(\alpha, \rho+\sigma-\alpha) \left\{ {}_2F_1\left(\sigma, \alpha; \sigma+\rho; 1-\frac{ad}{bc}\right) \times \right.$$

$$\times [\ln d - \psi(\rho+\sigma-\alpha)] + \sum_{k=0}^\infty \frac{(\sigma)_k (\alpha)_k}{(\sigma+\rho)_k k!} \left(1-\frac{ad}{bc}\right)^k \psi(\sigma+\rho+k) \Big\}$$

[Re ($\alpha-\sigma-\rho$) < 0; Re α > 0; $\left|1-\frac{ad}{bc}\right| < 1$; |arg ($cx+d$)|, |arg ($ax+b$)| < π при $0 \leq x < \infty$].

$$18. \int_0^{\infty} x^{\alpha-1} (x+a)^k (x+2a)^{-\rho} \ln(x+2a) dx =$$

$$= a^{k+\alpha-\rho} 2^{\alpha-\rho} (\alpha+\rho-1)^k \Gamma \left[\begin{matrix} \alpha, \rho-\alpha-k \\ \rho \end{matrix} \right] [\ln(2a) + \psi(\rho) -$$

$$-\psi(\rho-\alpha-k) - k(\alpha+\rho-1)^{-1}]$$

[$k=0$ или 1 ; $\operatorname{Re} \alpha > 0$, $\operatorname{Re}(\alpha-\rho) < -k$; $|\arg a| < \pi$].

$$19. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{x}(ax+b)(gx+h)} dx =$$

$$= \frac{2\pi}{\sqrt{d}(bg-ah)} \left[\sqrt{\frac{dg}{h}} \ln \left(1 + \sqrt{\frac{ch}{dg}} \right) - \sqrt{\frac{da}{b}} \ln \left(1 + \sqrt{\frac{bc}{da}} \right) \right] +$$

$$+ \frac{\pi \ln d}{\sqrt{bh}(\sqrt{ah} + \sqrt{bg})} \quad [bg \neq ah, a, b, c, d, g, h > 0].$$

$$20. \int_0^{\infty} \frac{\sqrt{x} \ln(cx+d)}{(ax+b)(gx+h)} dx =$$

$$= \frac{2\pi d}{bg-ah} \left[\sqrt{\frac{b}{ad}} \ln \left(1 + \sqrt{\frac{bc}{ad}} \right) - \sqrt{\frac{h}{gd}} \ln \left(1 + \sqrt{\frac{hc}{gd}} \right) \right] + \frac{\pi \ln d}{\sqrt{ag}(\sqrt{bg} + \sqrt{ah})}$$

[$bg \neq ah$; $a, b, c, d, g, h > 0$].

$$21. \int_0^{\infty} \frac{1-x^2}{(ax+b)^2} \frac{\ln(1+x)}{(bx+a)^2} dx = \frac{1}{ab(a^2-b^2)} \ln \frac{b}{a}$$

[$a \neq b, a, b > 0$].

2.6.13. Интегралы от $x^\alpha \ln^n \frac{ax+b}{cx+d}$.

$$1. \int_a^b \ln \frac{x+b}{x+a} \frac{dx}{x} = \frac{1}{2} \ln^2 \frac{b}{a}$$

[$0 < a < b$].

$$2. \int_a^b \ln \frac{c+x}{c-x} \frac{dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \frac{\pi}{b} F \left(\arcsin \frac{b}{c}, \frac{a}{b} \right)$$

[$|b| < |c|, a < b$].

$$3. \int_a^b \ln^n \frac{b-x}{x-a} \frac{dx}{x-y} = \sum_{k=0}^{[(n+1)/2]} (\pi i)^{2k} a_{2k}^n \ln^{n+1-2k} \frac{b-y}{y-a}$$

[$a < y < b$; a_{2k}^n находятся по рекуррентным формулам

$$\sum_{j=0}^k \binom{m}{2j+1} a_{2k-2j}^{m-2j-1} = \binom{m}{2k}, \quad k=0, 1, 2, \dots, \left[\frac{n+1}{2} \right], \quad \binom{k}{k+1} \equiv 0].$$

$$4. \int_0^a \ln^n \frac{a-x}{x} dx = -\frac{1+(-1)^n}{2(n+1)} \sum_{k=0}^{n/2-1} \binom{n+1}{2k} (\pi i)^{n-2k} \int_0^a \ln^{2k} \frac{a-x}{x} dx \quad [a > 0].$$

$$5. \int_0^a x^n \ln \frac{a+x}{a-x} dx = \frac{2}{n+1} a^{n+1} \left\{ \sum_{k=0}^{[(n-1)/2]} \frac{1}{n-2k} + \frac{1+(-1)^n}{2} \ln 2 \right\} \quad [a > 0].$$

$$6. \int_0^a \ln \frac{a+x}{a-x} \frac{dx}{x} = \frac{\pi^2}{4} \quad [a \neq 0].$$

$$7. \int_0^{\sqrt{2}-1} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{8} - \frac{1}{2} \ln^2 (\sqrt{2}-1).$$

$$8. \int_0^{\sqrt{5}-2} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{1}{6} \ln^2 (\sqrt{5}-2).$$

$$9. \int_0^{(\sqrt{5}-1)/2} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{6} - \frac{3}{2} \ln^2 \frac{\sqrt{5}-1}{2}.$$

$$10. \int_0^a x^{-1/2} \ln \frac{a+bx}{a-bx} \frac{dx}{\sqrt{a-x}} = 2\pi \ln \operatorname{ctg} \frac{\pi - \arcsin b}{4} \quad [-1 \leq b \leq 1; a > 0].$$

$$11. \int_0^a \ln \frac{b+x}{b-x} \frac{dx}{x \sqrt{a^2-x^2}} = \frac{\pi}{a} \arcsin \frac{a}{b} \quad [0 < a \leq b].$$

$$12. \int_0^a \ln^\sigma \frac{a+x}{a-x} dx = 2a \Gamma(\sigma+1) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^\sigma} \quad [\operatorname{Re} \sigma > 0].$$

$$13. \int_0^a \ln^\sigma \frac{a+x}{a-x} \frac{dx}{(a^2-x^2)^\rho} = (2a)^{1-2\rho} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(2\rho-k-1)_k (k+1-\rho)^{-1-\sigma}}{k!} \quad [\operatorname{Re} \sigma > -1, \operatorname{Re} \rho < 1; a > 0].$$

$$14. \int_0^a \ln \frac{a+x}{a-x} \frac{dx}{\sqrt{a^2-x^2}} = 4G.$$

$$15. \int_a^\infty x^{\alpha-1} \ln \frac{x+a}{x-a} dx = -\frac{a^\alpha}{\alpha} \left[\psi\left(\frac{1-\alpha}{2}\right) + \mathbf{C} + \ln 4 \right] \quad [\operatorname{Re} \alpha < 1, \alpha \neq 0; a > 0].$$

$$16. \int_a^\infty \ln \frac{x+a}{x-a} \frac{dx}{x} = \frac{\pi^2}{4} \quad [a > 0].$$

$$17. \int_a^\infty x^{-n} \ln \frac{x+a}{x-a} dx = \frac{a^{1-n}}{n-1} \left\{ \frac{1+(-1)^n}{2} \left[\ln 4 + \sum_{k=1}^{[(n-1)/2]} \frac{1}{k} \right] + \right. \\ \left. + [1-(-1)^n] \sum_{k=1}^{[(n-1)/2]} \frac{1}{2k-1} \right\} \quad [a > 0; n=2, 3, 4, \dots].$$

$$18. \int_a^{\infty} \ln^{\sigma} \frac{x+a}{x-a} dx = 2a\Gamma(\sigma+1)\zeta(\sigma) \quad [\operatorname{Re} \sigma > 1; a > 0].$$

$$19. \int_a^{\infty} \ln^{\sigma} \frac{x+a}{x-a} \frac{dx}{(x^2-a^2)^{\rho}} = (2a)^{1-2\rho} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(2\rho-k-1)_k (-1)^k}{k! (k-\rho+1)^{\sigma+1}} \\ [a > 0; \operatorname{Re} \rho < 1; \operatorname{Re}(\sigma+2\rho) > 1].$$

$$20. \int_a^{\infty} \ln \frac{x+a}{x-a} \frac{dx}{\sqrt{x^2-a^2}} = \frac{\pi^2}{2} \quad [a > 0].$$

$$21. \int_0^{\infty} x^{\alpha-1} \ln \frac{cx+d}{cx+b} dx = \frac{\pi c^{-\alpha}}{\alpha \sin \alpha\pi} (d^{\alpha} - b^{\alpha}) \quad [b, c, d > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$22. \int_0^{\infty} x^{\alpha-1} \ln \left| \frac{cx+d}{cx-b} \right| dx = \frac{\pi (d^2 - b^2 \cos \alpha\pi)}{\alpha c^2 \sin \alpha\pi} \quad [b, c, d > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$23. \int_0^{\infty} x^{\alpha-1} \ln \left| \frac{x+a}{x-a} \right| dx = \frac{\pi a^{\alpha}}{\alpha} \operatorname{tg} \frac{\alpha\pi}{2} \quad [a > 0; |\operatorname{Re} \alpha| < 1, \alpha \neq 0].$$

$$24. \int_0^{\infty} \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x} = \frac{\pi^2}{2} \quad [a > 0].$$

$$25. \int_0^{\infty} \ln \frac{x+a}{x} \frac{dx}{x+a} = \frac{\pi^2}{4} \quad [a > 0].$$

$$26. \int_0^{\infty} \ln \left| \frac{x-a}{x-b} \right| \frac{dx}{x-y} = 0 \quad [y < a < b \text{ или } y > b > a].$$

$$27. = \pi \quad [a < y < b].$$

$$28. \int_0^{\infty} \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x(x^2+z^2)} = \frac{\pi}{z^2} \operatorname{arctg} \frac{z}{a} \quad [a, \operatorname{Re} z > 0].$$

2.6.14. Интегралы от $A(x) \ln(ax^2+bx+c)$.

Обозначение: $k' = \sqrt{1-k^2}$, $0 < k < 1$.

$$1. \int_0^1 x^n \ln(1+x^2) dx = \frac{1}{n+1} \left\{ \frac{\pi}{2} (-1)^{[n/2]} + \ln 2 - \frac{2}{n+1} + 2 \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{n-2k-1} \right\}.$$

$$2. \int_0^1 \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \\ = \frac{\ln a}{c(c+d)} + \frac{b}{ad^2+bc^2} \left[2 \sqrt{\frac{a}{b}} \operatorname{arctg} \sqrt{\frac{b}{a}} + \frac{bc-ad}{b(c+d)} \ln \frac{a+b}{a} - 2 \frac{c}{d} \ln \frac{c+d}{c} \right] \\ [a, b, c, d > 0].$$

$$3. \int_0^1 \frac{\ln(1-x^2)}{(a+bx)\sqrt{1-x^2}} dx = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} \quad [a, b > 0; a \neq b].$$

$$4. \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = (2-k^2) K(k) - (2-\ln k') E(k).$$

$$5. \int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}} \ln(1-k^2x^2) dx = k^{-2} (1+k'^2 - k'^2 \ln k') K(k) - (2-\ln k') E(k).$$

$$6. \int_0^1 \frac{\ln(1-x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} K(k) - \frac{\pi}{2} K(k').$$

$$7. \int_0^1 \frac{\ln(1 \pm kx^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2 \pm 2k}{\sqrt{k}} K(k) - \frac{\pi}{8} K(k').$$

$$8. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln k' K(k).$$

$$9. \int_0^1 x^2 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = \\ = \frac{1}{9k^2} \{(-2+11k^2-6k^4+3k'^2 \ln k') K(k) + [2-10k^2-3(1-2k^2) \ln k'] E(k)\}.$$

$$10. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{k'^2} [(k^2-2) K(k) + (2+\ln k') E(k)].$$

$$11. \int_0^1 \frac{x^2 \ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{(k'k)^2} [(2+\ln k') E(k) - (1+k'^2+k'^2 \ln k') K(k)].$$

$$12. \int_0^1 x^2 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{k^2} [(k^2-2+\ln k') K(k) + (2-\ln k') E(k)].$$

$$13. \int_0^\infty \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \\ = \frac{\ln b}{cd} + \frac{b}{ad^2+bc^2} \left(\pi \sqrt{\frac{a}{b}} + 2 \frac{c}{d} \ln \frac{c}{d} + \frac{ad}{bc} \ln \frac{a}{b} \right) \quad [a, b, c, d > 0].$$

$$14. \int_0^\infty \frac{\ln(a+bx^2)}{c+dx^2} dx = \frac{\pi}{\sqrt{bd}} \ln \left(\sqrt{a} + \sqrt{\frac{bc}{d}} \right) \quad [a, b, c, d > 0].$$

$$15. \int_0^{\infty} \frac{x^{\alpha-1}}{x+z} \ln(x^2+z^2) dx =$$

$$= \frac{\pi z^{\alpha-1}}{\sin \alpha \pi} \left[\ln(2z^2) - (1-\alpha) \sin \frac{\alpha \pi}{2} \beta\left(\frac{1-\alpha}{2}\right) - (2-\alpha) \cos \frac{\alpha \pi}{2} \beta\left(1-\frac{\alpha}{2}\right) \right]$$

[$0 < \operatorname{Re} \alpha < 1$ при $z \neq i$; $-2 < \operatorname{Re} \alpha < 1$ при $z = i$, $\operatorname{Re} z > 0$].

$$16. \int_0^1 \ln(1+2bx+x^2) \frac{dx}{x} = \frac{\pi^2}{6} - \frac{\arccos^2 b}{2} \quad [|b| \leq 1].$$

$$17. \int_0^1 \frac{\ln(1+2bx-ax^2)}{\sqrt{x}\sqrt{1-x}} dx = 2\pi \ln \frac{\sqrt{c}+1+\sqrt{(\sqrt{c}+1)^2+a}}{4}$$

[$0 < 2b < a \leq 1$; $c = 1+2b-a$].

$$18. \int_0^{\infty} x^{\alpha-1} \ln(1+2bx+x^2) dx = \frac{2\pi \cos(\alpha \arccos b)}{\alpha \sin \alpha \pi} \quad [-1 < b \leq 1; -1 < \operatorname{Re} \alpha < 0].$$

$$19. \int_{-\infty}^{\infty} \ln(a^2-2abx+x^2) \frac{dx}{x^2+z^2} = \frac{\pi}{z} \ln(z^2+2az\sqrt{1-b^2}+a^2)$$

[$a, \operatorname{Re} z > 0$; $-1 < b \leq 1$].

$$20. \int_0^{1/2} \ln^n[x(1-x)] \frac{dx}{1-x} = I_n \quad [n = 1, 2, 3, \dots].$$

$$I_n = -\frac{2^n}{n+1} \ln^{n+1} \frac{1}{2} + \frac{1}{2} \frac{\partial^n}{\partial v^n} \left[B(v, v+1) - \frac{1}{v} \right]_{v=0},$$

$$I_4 = \frac{16}{5} \ln^5 2 + 8 [18\zeta(5) - \pi^2 \zeta(3)].$$

2.6.15. Интегралы от $A(x) \ln \frac{P(x)}{Q(x)}$.

$$1. \int_0^1 x^{\alpha-1} \ln(1+x+x^2+\dots+x^{n+1}) dx = \alpha^{-1} \left[\psi(\alpha+1) - \psi\left(\frac{\alpha}{n+2}+1\right) \right]$$

[$\operatorname{Re} \alpha > -1$, $\alpha \neq 0$].

$$2. \quad = \pi^2(n+1)/(6n+12) \quad [\alpha = 0].$$

$$3. \int_0^{\infty} x^{\alpha-1} \ln(1+x+x^2+\dots+x^{n+1}) dx = \pi \alpha^{-1} \left(\operatorname{ctg} \frac{\alpha \pi}{n+2} - \operatorname{ctg} \alpha \pi \right)$$

[$-1 < \operatorname{Re} \alpha < 0$].

$$4. \int_0^1 \frac{\ln(x^{-1} \pm x)}{1+x^2} dx = \frac{\pi}{8} (3 \pm 1) \ln 2.$$

$$5. \int_0^1 \frac{x \ln(x^{-1} \pm x)}{1+x^4} dx = \frac{\pi}{16} \ln 2 \pm \frac{\pi}{4}.$$

$$6. \int_1^{\infty} \frac{\ln(x \pm x^{-1})}{1+x^2} dx = \frac{\pi}{8} (3 \pm 1) \ln 2.$$

$$7. \int_1^{\infty} \frac{x \ln(x \pm x^{-1})}{1+x^4} dx = \frac{\pi}{16} \ln 2 \pm \frac{G}{4}.$$

$$8. \int_0^1 x^{\alpha-1} \ln \frac{(1+x)(a^2+x)}{(a+x)^2} dx = \frac{\pi(a^{\alpha}-1)^2}{\alpha \sin \alpha \pi} \quad [\operatorname{Re} \alpha > -1; a > 0].$$

$$= \ln^2 a \quad [\alpha = 0; a > 0].$$

$$9. \int_0^1 \ln \frac{(1-ax)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln(1+a) \quad [a > 0]$$

$$10. \int_0^{\infty} \ln \frac{x^2+a^2}{x^2+b^2} dx = (a-b)\pi \quad [a, b \geq 0].$$

$$11. \int_0^{\infty} \ln \left| \frac{x^2-a^2}{x^2-b^2} \right| \frac{dx}{x-y} = \pm \pi \quad [0 < a < |y| < b].$$

$$12. = 0 \quad [\pm y \notin [a, b], 0 < a < b].$$

$$13. \int_0^{\infty} x^{\alpha-1} \ln \frac{(x+a)^2+c^2}{(x+b)^2+c^2} dx =$$

$$= -\frac{2}{\alpha} \left[(b^2+c^2)^{\alpha/2} \cos \left(\alpha \operatorname{arctg} \frac{c}{b} \right) - (a^2+c^2)^{\alpha/2} \cos \left(\alpha \operatorname{arctg} \frac{c}{a} \right) \right]$$

$$[0 < \operatorname{Re} \alpha < 1, a, b, c > 0].$$

$$14. \int_0^{\infty} x^{\alpha-1} \ln \frac{a^2+2abx+x^2}{(a+x)^2} dx = \frac{2\pi a^{\alpha} [\cos(\pi \arccos b) - 1]}{\alpha \sin \alpha \pi}$$

$$[|\operatorname{Re} \alpha| < 1; -1 < b \leq 1, a > 0]$$

$$15. \int_0^{\infty} \ln \frac{(1+x)(x+a^2)}{(x+a)^2} \frac{dx}{x} = \ln^2 a \quad [a > 0]$$

2.6.16. Интегралы от $A(x) \ln^{\sigma} (\sqrt{ax^{\mu}+bx^{\mu}}+cx^{\nu})$.

$$1. \int_a^1 \frac{\ln(\sqrt{1-x^2}+x)}{\sqrt{1-x^2}} dx =$$

$$= \left(\arccos c + \varphi - \frac{\pi}{2} \right) \ln c + \frac{1}{2} L(\arccos a + \varphi) - \frac{1}{2} L(\arccos a - \varphi) - L(\varphi),$$

$$\cos \varphi = \sqrt{\frac{1-a^2}{1-c^2}} \quad [c \leq a < 1].$$

$$2. = (\arccos c - \pi/2) \ln c \quad [a = c < 1].$$

$$3. \int_0^a x^{\alpha-1} \ln (\sqrt{a^2-x^2}+a) dx = \frac{a^\alpha}{\alpha} \left\{ \ln a + \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{\alpha/2}{(\alpha+1)/2} \right] - \frac{1}{\alpha} \right\}$$

[a, Re $\alpha > 0$].

$$4. \int_0^a \frac{x^{\alpha-1}}{(x^\mu+z^\mu)^r} \ln^n (\sqrt{x^\mu+z^\mu} \pm z^{\mu/2}) dx =$$

$$= \frac{a^{\alpha-\mu r} z^{-\mu r}}{\mu} \frac{\partial^n}{\partial v^n} \left[2^{\pm v} z^{\pm \mu v/2} a^{\mu(v \mp v)/2} \left(\frac{\alpha}{\mu} + \frac{v \mp v}{2} \right)^{-1} \times \right.$$

$$\left. \times {}_3F_2 \left(\frac{\alpha}{\mu} + \frac{v \mp v}{2}, \frac{1 \mp v}{2}, 2r \mp \frac{v}{2}; 1 + \frac{\alpha}{\mu} + \frac{v \mp v}{2}, 1 \mp v; -\frac{a^\mu}{z^\mu} \right) \right]_{v=0}$$

[$r=0$ или $1/2$; a, μ , Re $\alpha > 0$; $\mu | \arg z | < \pi$].

$$5. \int_0^a \frac{x^{\alpha-1}}{\sqrt{x^\mu+z^\mu}} \ln (\sqrt{x^\mu+z^\mu} \pm z^{\mu/2}) dx =$$

$$= \pm a^\alpha (az)^{-\mu/2} \sum_{k=0}^{\infty} \frac{(1/2)_k}{\alpha + \mu k} \left(-\frac{a^\mu}{z^\mu} \right)^k \left[\ln (2z^{\mu/2}) + \right.$$

$$\left. + \frac{\mu}{2} (1 \mp 1) \left(\frac{1}{\alpha + k\mu} - \ln a \right) - \psi(2k+1) + \psi(k+1) \right]$$

[a, μ , Re $\alpha > 0$; $\mu | \arg z | < \pi$].

$$6. \int_0^a x^{\alpha-1} \ln (\sqrt{x^\mu+z^\mu} \pm z^{\mu/2}) dx =$$

$$= \mp \frac{a^\alpha}{2} \sum_{k=1}^{\infty} \frac{(1/2)_k}{(\alpha + \mu k) k k!} \left(-\frac{a^\mu}{z^\mu} \right)^k \pm \frac{a^\alpha}{\alpha} \left[\ln (2z^{\mu/2}) - \mu \frac{1 \mp 1}{2} (\alpha \ln a - 1) \right]$$

[a, μ , Re $\alpha > 0$; $\mu | \arg z | < \pi$].

$$7. \int_0^a \ln (\sqrt{x^2+a^2}+x) dx = a [1 - \sqrt{2} + \ln (1 + \sqrt{2}) + \ln a] \quad [a > 0].$$

$$8. \int_0^1 \ln (\sqrt{x^2+1}+x) \frac{dx}{x} = \sum_{k=0}^{\infty} \frac{(1/2-k)_k}{k! (2k+1)^2}.$$

$$9. \int_0^a \frac{\ln (\sqrt{x^2+a^2}+x)}{\sqrt{x^2+a^2}} dx = \frac{1}{2} \ln (1 + \sqrt{2}) [\ln (1 + \sqrt{2}) + 2 \ln a] \quad [a > 0].$$

$$10. \int_0^a x \frac{\ln (\sqrt{x^2+a^2}+x)}{\sqrt{x^2+a^2}} dx = a [(\sqrt{2}-1) \ln a + \sqrt{2} \ln (\sqrt{2}+1) - 1] \quad [a > 0].$$

$$11. \int_0^1 \frac{\ln (ax + \sqrt{1-b^2x^2})}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln 2 - 2L \left(\frac{\arcsin a}{2} \right) - 2L \left(\frac{\pi - \arcsin a}{2} \right)$$

[$a^2 + b^2 = 1$].

$$12. \int_0^1 \frac{\ln(1 \pm \sqrt{a^2 - c^2 x^2})}{1 - a^2 + c^2 x^2} dx = \frac{1}{2c \sqrt{1 - a^2}} \left\{ \left(\arcsin b \pm \gamma \mp \frac{\pi}{2} \right) \ln(1 - a^2) + \right. \\ \left. + 2L(\arcsin b) \mp [2L(\gamma) - L(\arcsin b + \gamma) + L(\arcsin b - \gamma)] \right\} \\ \left[b = \frac{c}{\sqrt{1 + c^2 - a^2}}; c^2 < a^2; \cos \gamma = \frac{b}{a}; 0 < b < a < 1 \right].$$

$$13. \int_0^1 \frac{\ln(1 \pm \sqrt{1 - k^2 x^2})}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx = \frac{\ln k}{2} K(k) + \pi \left(-\frac{1}{4} \pm \frac{1}{2} \right) K(k') \\ [0 < k < 1; k' = \sqrt{1 - k^2}].$$

$$14. \int_0^1 \frac{\ln(\sqrt{1 + kx} \pm \sqrt{1 - kx})}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx = \frac{\ln 4k}{4} K(k) + \frac{\pi}{8} (2 \mp 1) K(k') \\ [0 < k < 1; k' = \sqrt{1 - k^2}].$$

$$15. \int_0^1 \ln \frac{1 + a \sqrt{1 - x^2}}{1 - a \sqrt{1 - x^2}} \frac{dx}{1 - x} = \pi \arcsin a \quad [|a| \leq 1].$$

$$16. \int_0^1 \ln \frac{1 + a \sqrt{1 - x^2}}{1 - a \sqrt{1 - x^2}} \frac{dx}{x^2 + z^2} = \frac{\pi a}{\sqrt{1 - a^2}} \left(\frac{\sqrt{1 + z^2} + a + z \sqrt{1 - a^2}}{\sqrt{1 + z^2} - a z - \sqrt{1 - a^2}} \right)^{1/2} \\ [|a| < 1; \operatorname{Re} z > 0].$$

$$17. \int_0^1 x \ln \left| \frac{x + \sqrt{1 - x^2}}{x - \sqrt{1 - x^2}} \right| \frac{dx}{1 - x^2} = \frac{\pi^2}{4}.$$

$$18. \int_0^1 \ln \frac{1 + a \sqrt{1 - k^2 x^2}}{1 - a \sqrt{1 - k^2 x^2}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = \pi F(\arcsin a, \sqrt{1 - k^2}) \\ [|a| < 1; 0 < k < 1].$$

$$19. \int_0^a \ln \frac{a + b \sqrt{(1 - x^2)(a^2 - x^2)}}{a - b \sqrt{(1 - x^2)(a^2 - x^2)}} \frac{dx}{\sqrt{1 - x^2}} = \pi \ln \frac{ab + \sqrt{a^2 b^2 + (1 + \sqrt{1 - b^2})^3}}{1 + \sqrt{1 - b^2}} \\ [a > 0; |b| \leq 1].$$

$$20. \int_0^a \ln \frac{\sqrt{a^2 - b^2 x^2} - x \sqrt{1 - b^2}}{a - x} \frac{dx}{x} = \frac{1}{2} \arcsin^2 b \quad [|b| \leq 1].$$

$$21. \int_a^\infty x^{\alpha-1} \ln(x + \sqrt{x^2 - a^2}) dx = -\frac{a^\alpha}{\alpha} \left\{ \ln a + \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{-\alpha/2}{(1-\alpha)/2} \right] \right\} \\ [a > 0; \operatorname{Re} \alpha < 0].$$

$$22. \int_a^\infty x^{-n-1} \ln(x + \sqrt{x^2 - a^2}) dx = \frac{a^{-n}}{n} \left(\ln a + B \frac{(n-2)!!}{(n-1)!!} \right) \\ [a > 0; B = \pi/2 \text{ при } n = 1, 3, 5, \dots; B = 1 \text{ при } n = 2, 4, 6, \dots].$$

$$23. \int_a^{\infty} \ln(a + \sqrt{x^2 + a^2}) \frac{dx}{x^2} = \frac{1}{a} [2 - \sqrt{2} + \ln(1 + \sqrt{2}) + \ln a] \quad [a > 0];$$

$$24. \int_1^{\infty} \ln^{\sigma}(x + \sqrt{x^2 - 1}) \frac{dx}{(x^2 - 1)^{1-\beta}} = 2^{1-2\beta} \Gamma(\sigma + 1) \sum_{k=0}^{\infty} \frac{(-1)^k (2\beta - k)_k}{k! (2k + 1 - 2\beta)^{\sigma+1}} \\ [\operatorname{Re} \beta < 1/2; \operatorname{Re}(\sigma + \beta) > 0].$$

$$25. \int_1^{\infty} \ln(x + \sqrt{x^2 - 1}) \frac{dx}{x^2 - 1} = \frac{\pi^2}{4}.$$

$$26. \int_a^{\infty} \frac{\ln(a + \sqrt{x^2 + a^2})}{x \sqrt{x^2 + a^2}} dx = \\ = \frac{1}{2a} \ln(1 + \sqrt{2}) [\ln(1 + \sqrt{2}) - 2 \ln a] + a^{-1} \sum_{k=0}^{\infty} \frac{(1/2 - k)_k}{k! (2k + 1)^2} \quad [a > 0].$$

$$27. \int_a^{\infty} x^k (x^2 - a^2)^{\alpha-1} (x \pm \sqrt{x^2 - a^2})^{\nu} \ln(x \pm \sqrt{x^2 - a^2}) dx = \\ = \left(\frac{1}{2} \mp \nu\right) k 2^{-k-2\alpha-1} a^{\nu+2\alpha+k-1} \Gamma \left[\frac{2\alpha, (1-k \mp \nu)/2 - \alpha}{(1 \mp \nu + k)/2 + \alpha} \right] \times \\ \times \left[2 \ln a \mp \psi \left(\frac{1-k \mp \nu}{2} - \alpha \right) \pm \psi \left(\frac{1 \mp \nu + k}{2} + \alpha \right) + \frac{2k}{\nu} \right] \\ [k=0 \text{ или } 1; a, \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha \pm \nu) < 1-k].$$

$$28. \int_0^{\infty} x^{\alpha-1} \frac{\ln(\sqrt{x^{\mu} + z^{\mu}} \pm z^{\mu/2})}{\sqrt{x^{\mu} + z^{\mu}}} dx = \\ = \frac{z^{\alpha-\mu/2}}{\mu} 2^{2\alpha/\mu-1} B \left(\frac{\alpha}{\mu}, 1 - \frac{2\alpha}{\mu} \right) \left[\mu \ln z + 2\beta \left(1 - 2 \frac{\alpha}{\mu} \right) - (1 \mp 1) \pi \operatorname{ctg} \left(\frac{\alpha\pi}{\mu} \right) \right] \\ [\mu > 0, 0 < \operatorname{Re} \alpha < \mu/2, \mu |\arg z| < \pi].$$

$$29. \int_0^{\infty} x^{\alpha-1} \frac{\ln(\sqrt{x^{\mu} + z^{\mu}} \pm x^{\mu/2})}{\sqrt{x^{\mu} + z^{\mu}}} dx = \\ = \frac{z^{\alpha-\mu/2}}{\mu} 2^{-2\alpha/\mu} B \left(\frac{2\alpha}{\mu}, \frac{1}{2} - \frac{\alpha}{\mu} \right) \left[\mu \ln z \pm \pi \operatorname{tg} \left(\frac{\alpha\pi}{\mu} \right) \right] \\ [\mu > 0, \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu/2, z \neq 1 \text{ (} |2 \operatorname{Re} \alpha| < \mu \text{ при } z=1)].$$

$$30. \int_0^{\infty} \frac{\ln(\sqrt{x^{\mu} + 1} \pm x^{\mu/2})}{\sqrt{x^{\mu} + 1}} \frac{dx}{x} = \pm \frac{\pi^2}{2\mu} \quad [\mu > 0].$$

$$31. \int_0^{\infty} x^{\alpha-1} \ln(\sqrt{x^{\mu} + 1} \pm x^{\mu/2}) dx = \mp 2^{-2\alpha/\mu} \alpha^{-1} B(2\alpha/\mu, -\alpha/\mu) \\ [\mu > 0; -\mu < 2 \operatorname{Re} \alpha < 0].$$

$$32. \int_0^{\infty} \frac{\ln(\sqrt{x^2+z^2}+x)}{x^2+z^2} dx = \frac{\pi}{2z} \ln z + \frac{2G}{z} \quad [\operatorname{Re} z > 0].$$

$$33. \int_0^{\infty} x \frac{\ln(\sqrt{x^2+z^2}+x)}{(x^2+z^2)^{\rho}} dx = -\frac{z^{2-2\rho}}{2(1-\rho)} \left\{ \ln z + \frac{\sqrt{\pi}}{2} \Gamma\left[\frac{\rho-1}{2}\right] \right\} \quad [\operatorname{Re} \rho > 1; \operatorname{Re} z > 0].$$

$$34. \int_0^{\infty} x^{\alpha-1} \frac{\ln^{\sigma}(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}} dx = 2^{1-\alpha} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha-k)_k}{k! (2k+1-\alpha)^{\sigma+1}} \quad [\operatorname{Re} \alpha < 1; \operatorname{Re}(\alpha+2\sigma) > 0].$$

$$35. \int_0^{\infty} \frac{x}{(x^2+z^2)^{\rho}} \ln \frac{\sqrt{x^2+z^2}+x}{\sqrt{x^2+z^2}-x} dx = \frac{z^{2-2\rho} 2^{2\rho-4}}{\rho-1} B(\rho-1, \rho-1) \quad [\operatorname{Re} z > 0; \operatorname{Re} \rho > 1].$$

$$36. \int_0^{\infty} \frac{x}{(x^2+z^2)^{\rho}} \ln \frac{\sqrt{x^2+2z^2}+x}{\sqrt{x^2+2z^2}-x} dx = \frac{2^{\rho-4} z^{2-2\rho}}{\rho-1} B\left(\frac{\rho-1}{2}, \frac{\rho-1}{2}\right) \quad [\operatorname{Re} z > 0; \operatorname{Re} \rho > 1].$$

$$37. \int_0^{\infty} \ln \frac{\sqrt{x^2+z^2}+a^2}{\sqrt{x^2+z^2}-a^2} \frac{dx}{\sqrt{x^2+z^2}} = \pi \arcsin \frac{a}{z} \quad [0 < a < |z|; \operatorname{Re} z > 0].$$

$$38. \int_0^{\infty} \frac{x^{\alpha-1} \{ \ln \{ a^2+b^2+x^2 + \sqrt{[x^2+(a+b)^2][x^2+(a-b)^2]} \} - \ln(2ab) \}}{\sqrt{[x^2+(a+b)^2][x^2+(a-b)^2]}} dx =$$

$$= \frac{ab}{|b^2-a^2|^{\alpha/2}} \Gamma\left[\frac{\alpha}{2}, \frac{(3-\alpha)/2, (3-\alpha)/2}{1-\alpha/2}\right] Q_{-\alpha/2-1}\left(\frac{b^2+a^2}{|b^2-a^2|}\right) \quad [0 < \operatorname{Re} \alpha < 1].$$

$$39. \int_0^{\infty} x^{\alpha-1} \frac{(\sqrt{x^{\mu}+z^{\mu}} \pm z^{\mu/2})^{\nu}}{(x^{\mu}+z^{\mu})^r} \ln(\sqrt{x^{\mu}+z^{\mu}} \pm z^{\mu/2}) dx =$$

$$= \frac{z^{\alpha+(\nu-2r)\mu/2}}{\mu} (\pm \nu)^{1-2r} \left(\frac{\alpha}{\mu} + \frac{\nu \pm \nu}{2}\right)^{2r-1} 2^{\nu+2\alpha/\mu-1} \times$$

$$\times B\left(\frac{\nu \mp \nu}{2} + \frac{\alpha}{\mu}, 2r - \nu - \frac{2\alpha}{\mu}\right) \times$$

$$\times \left[\ln(4z^{\mu}) - 2\psi\left(\nu - 2\frac{\alpha}{\mu} + 2r\right) + 2\psi\left(\frac{1 \pm 1}{2} \mp \nu \mp \frac{\alpha}{\mu}\right) - 2\nu^{2r-1}(1-2r) \right]$$

$$[r=0 \text{ или } 1/2; \mu > 0; 2\operatorname{Re} \alpha > \mu \operatorname{Re}(\pm \nu - \nu); \operatorname{Re}(\mu\nu + 2\alpha) < 2\mu r; \mu |\arg z| < \pi].$$

$$40. \int_0^{\infty} x^{\alpha-1} \frac{(\sqrt{x^{\mu}+z^{\mu}} \pm x^{\mu/2})^{\nu}}{(x^{\mu}+z^{\mu})^r} \ln(\sqrt{x^{\mu}+z^{\mu}} \pm x^{\mu/2}) dx =$$

$$= \frac{z^{\alpha+(\nu-2r)\mu/2}}{\mu} (\mp \nu)^{1-2r} 2^{2r-2\alpha/\mu-1} \left(\frac{\alpha}{\mu} \mp \frac{\nu}{2}\right)^{2r-1} B\left(2\frac{\alpha}{\mu}, r \mp \frac{\nu}{2} - \frac{\alpha}{\mu}\right) \times$$

$$\times \left[\mu \ln z \mp \psi\left(r \mp \frac{\nu}{2} - \frac{\alpha}{\mu}\right) \pm \psi\left(1 \mp \frac{\nu}{2} \mp \frac{\alpha}{\mu} - r\right) + 2\nu^{2r-1}(1-2r) \right]$$

$$[r=0 \text{ или } 1/2; \mu > 0; \operatorname{Re}(2\alpha \pm \mu\nu) < 2r\mu; \mu |\arg z| < \pi; \operatorname{Re} \alpha > 0 (2\operatorname{Re} \alpha > -\mu \text{ при } z=1)].$$

$$\begin{aligned}
 41. \int_0^{\infty} \frac{x^k}{(x^2+z^2)^{\rho}} & \left[(\sqrt{x^2+z^2}+x)^{\nu} \ln (\sqrt{x^2+z^2}+x) + \right. \\
 & \left. + (-1)^k (\sqrt{x^2+z^2}-x)^{\nu} \ln (\sqrt{x^2+z^2}-x) \right] dx = \\
 & = z^{1+\nu+k} 2^{\rho} 2^{2\rho} 2^{k-3} \left(\frac{\nu}{\rho-1} \right)^k B \left(\frac{\nu-k-1}{2} + \rho, \rho - \frac{\nu+k+1}{2} \right) \times \\
 & \times \left[2 \ln z + \psi \left(\frac{\nu-k-1}{2} + \rho \right) - \psi \left(\rho - \frac{\nu+k+1}{2} \right) + 2k\nu^{-k} \right] \\
 & [k=0 \text{ или } 1; \operatorname{Re} z > 0; 2 \operatorname{Re} \rho > k+1+|\operatorname{Re} \nu|].
 \end{aligned}$$

$$\begin{aligned}
 42. \int_0^{\infty} x^k \frac{(2z^2+x^2)^{(k-1)/2}}{(x^2+z^2)^{\rho}} & \left[(\sqrt{x^2+2z^2}+x)^{\nu} \ln (\sqrt{x^2+2z^2}+x) + \right. \\
 & \left. + (-1)^k (\sqrt{x^2+2z^2}-x)^{\nu} \ln (\sqrt{x^2+2z^2}-x) \right] dx = \\
 & = 2^{\nu/2+\rho-2k-4} z^{2k+\nu-2\rho} \left(\frac{\nu}{\rho-1} \right)^k B \left(\frac{\nu-2k+2\rho}{4}, \frac{2\rho-\nu-2k}{4} \right) \times \\
 & \times \left[2 \ln (2z^2) + \psi \left(\frac{\nu-2k+2\rho}{4} \right) - \psi \left(\frac{2\rho-\nu-2k}{4} \right) + 4k\nu^{-k} \right] \\
 & [k=0 \text{ или } 1, \operatorname{Re} z > 0, 2 \operatorname{Re} \rho > 2k+|\operatorname{Re} \nu|].
 \end{aligned}$$

2.6.17. Интегралы, содержащие $A(x)$ и $\frac{1}{\ln^n x}$.

$$1. \int_0^1 (x^{\beta} - x^{\gamma}) \frac{dx}{\ln x} = \ln \frac{\beta+1}{\gamma+1} \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > -1].$$

$$2. \int_0^1 (x^{\alpha} - 1)^n \frac{dx}{\ln x} = \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} \ln (1 + \alpha k)$$

[$n \operatorname{Re} \alpha > -1; n=1, 2, 3, \dots$].

$$3. \int_0^1 \frac{(x-1)^2}{x^2+1} \frac{dx}{\ln x} = \ln \frac{\pi}{4}.$$

$$4. \int_0^1 \frac{(x^{\beta} - x^{-\beta})^2}{x^2-1} \frac{dx}{\ln x} = -\ln (\cos \beta \pi) \quad [|\operatorname{Re} \beta| < 1/2].$$

$$5. \int_0^1 \frac{x^{\beta} - x^{\gamma}}{(1-cx)^n} \frac{dx}{\ln x} = \ln \frac{\beta+1}{\gamma+1} + \sum_{k=1}^{\infty} \binom{n+k-1}{k} c^k \ln \frac{\beta+k+1}{\gamma+k+1}$$

[$|c| < 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > -1$].

$$6. \int_0^1 \frac{x^{\beta} - x^{\gamma}}{x^{\mu} + 1} \frac{dx}{\ln x} = \ln \Gamma \left[\frac{(\beta + \mu + 1)/(2\mu), (\gamma + 1)/(2\mu)}{(\gamma + \mu + 1)/(2\mu), (\beta + 1)/(2\mu)} \right]$$

[$\mu > 0; \operatorname{Re} \beta; \operatorname{Re} \gamma > -1$].

$$7. \int_0^1 \frac{x^{\alpha-1} - x^{\alpha}}{x^{\mu} + 1} \frac{dx}{\ln x} = \ln \left(\operatorname{tg} \frac{\alpha \pi}{2\mu} \right) \quad [\mu > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$8. \int_0^1 x^{\mu-1} \frac{x^{\mu}-1}{x^{\mu}+1} \frac{dx}{\ln x} = \ln \frac{\pi}{2} \quad [\mu > 0].$$

$$9. \int_0^1 x^{\alpha-1} (x^{\beta}-1)^n (x^{\gamma}-1)^m \frac{dx}{\ln x} =$$

$$= \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{j=0}^m (-1)^j \binom{m}{j} \ln [\alpha + \gamma(m-j) + \beta(n-k)]$$

$$[\operatorname{Re} \alpha > \max(0, -n \operatorname{Re} \beta, -m \operatorname{Re} \gamma, -n \operatorname{Re} \beta - m \operatorname{Re} \gamma)].$$

$$10. \int_0^1 \frac{x^{\alpha-1}}{x^{\gamma}-1} (x^{\beta}-1)^n \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^{k+n} \ln \Gamma \left(\frac{\beta(n-k) + \alpha}{\gamma} \right)$$

$$[\operatorname{Re} \alpha > \max(0, \operatorname{Re} \gamma, -n \operatorname{Re} \beta, \operatorname{Re} \gamma - n \operatorname{Re} \beta)].$$

$$11. \int_0^1 x^{\alpha-1} (x^{\beta}-1) (x^{\gamma}-1) \frac{dx}{\ln x} = \ln \frac{\alpha(\beta+\gamma+\alpha)}{(\beta+\alpha)(\gamma+\alpha)}$$

$$[\operatorname{Re} \alpha > \max(0, -\operatorname{Re} \beta, -\operatorname{Re} \gamma, -\operatorname{Re} \beta - \operatorname{Re} \gamma)]$$

$$12. \int_0^1 \frac{x-1}{x+1} \frac{dx}{(x^2+1) \ln x} = \frac{\ln 2}{2}. \quad 13. \int_0^1 x^2 \frac{x-1}{x+1} \frac{dx}{(x^2+1) \ln x} = \ln \frac{\pi}{2\sqrt{2}}.$$

$$14. \int_0^1 x^{\alpha-1} \frac{(x^{\beta}-1)(x^{\gamma}-1)}{x^{\mu}-1} \frac{dx}{\ln x} = \ln \Gamma \left[\frac{\alpha/\mu, (\alpha+\beta+\gamma)/\mu}{(\alpha+\beta)/\mu, (\alpha+\gamma)/\mu} \right]$$

$$[\mu > 0; \operatorname{Re} \alpha > \max(0, -\operatorname{Re} \beta, -\operatorname{Re} \gamma, -\operatorname{Re} \beta - \operatorname{Re} \gamma)].$$

$$15. \int_0^1 \frac{(x-1)^2}{1+2x \cos(m\pi/n) + x^2} \frac{dx}{\ln x} = \frac{1}{\sin(m\pi/n)} \sum_{k=1}^{(n-1)/r} (-1)^k \sin \frac{km\pi}{n} \times$$

$$\times \ln \Gamma \left[r \frac{n - (-1)^r k + 1}{2n}, r \frac{n - (-1)^r k + 1}{2n}, r \frac{k+2}{2n}, r \frac{rk}{2n} \right]$$

$$\left[r \frac{k+1}{2n}, r \frac{k+1}{2n}, r \frac{n - (-1)^r k}{2n}, r \frac{n - (-1)^r k + 2}{2n} \right]$$

$$[m < n; r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l, l=1, 2, 3, \dots].$$

$$16. \int_0^1 \left(\frac{x^{\beta}-x^{\gamma}}{x-1} - \beta + \gamma \right) \frac{dx}{\ln x} = \ln \Gamma \left[\frac{\beta+1}{\gamma+1} \right] \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > -1].$$

$$17. \int_0^1 \left[x-1 - \frac{(x^{\beta}-1)(x^{\gamma}-1)}{x-1} \right] \frac{dx}{x \ln x} = \ln B(\beta, \gamma) \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$18. \int_0^1 \left(\frac{x^{\beta\mu}-1}{x^{\mu}-1} - \frac{x^{\beta}-1}{x-1} \right) \frac{dx}{x \ln x} = \mu \ln \beta \quad [\mu, \operatorname{Re} \beta > 0].$$

$$19. \int_0^1 \left(\frac{1}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = -\psi(q) \quad [\operatorname{Re} q > 0].$$

$$20. \int_1^\infty \left[\frac{(\mu-1)(x-1)}{x^2} - \frac{1}{x} + \frac{1}{x^\mu} \right] \frac{dx}{(x-1) \ln x} = \ln \Gamma(\mu) \quad [\mu > 0].$$

$$21. \int_0^\infty \frac{x^\beta - x^\gamma}{x^\mu \pm 1} \frac{dx}{\ln x} = \\ = \pm \ln \left(\sin \frac{\beta+1}{2\mu} \pi \right) - \ln \left(\cos \frac{\beta+1}{2\mu} \pi \right) \mp \ln \left(\sin \frac{\gamma+1}{2\mu} \pi \right) + \ln \left(\cos \frac{\gamma+1}{2\mu} \pi \right) \\ [\mu > 0; \operatorname{Re} \beta, \operatorname{Re} \gamma > -1].$$

$$22. \int_0^\infty \frac{(x^\beta - x^{-\beta})^2}{x^2 - 1} \frac{dx}{\ln x} = -2 \ln (\cos \beta \pi) \quad [|\operatorname{Re} \beta| < 1/2].$$

$$23. \int_0^1 \ln [a + (1-a)x] \frac{dx}{\ln x} = - \sum_{k=1}^\infty \frac{\ln(k+1)}{k} \left(\frac{a-1}{a} \right)^k \quad [a > 1/2].$$

$$24. \int_0^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\ln x} \ln(1+cx) dx = \sum_{k=1}^\infty \frac{c^k}{k} \ln \frac{\alpha+k}{\beta+k} + \ln \frac{\alpha}{\beta} \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |c| < 1].$$

$$25. \int_0^1 x^{\alpha-1} \frac{(x^\beta - 1)^n (x^\gamma - 1)^m}{\ln^2 x} dx = (-1)^{m+n} \sum_{k=0}^n (-1)^k \binom{n}{k} \times \\ \times \sum_{j=0}^m (-1)^j \binom{m}{j} [\alpha + (n-k)\beta + (m-j)\gamma] \ln [\alpha + (n-k)\beta + (m-j)\gamma] \\ [\operatorname{Re} \alpha > \max(0, -n \operatorname{Re} \beta, -m \operatorname{Re} \gamma, -n \operatorname{Re} \beta - m \operatorname{Re} \gamma)].$$

$$26. \int_0^1 x^{\alpha-1} \frac{(x^\beta - 1)(x^\gamma - 1)}{\ln^2 x} dx = \\ = \alpha \ln \alpha - (\alpha + \beta) \ln(\alpha + \beta) - (\alpha + \gamma) \ln(\alpha + \gamma) + (\alpha + \beta + \gamma) \ln(\alpha + \beta + \gamma) \\ [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \beta), \operatorname{Re}(\alpha + \gamma), \operatorname{Re}(\alpha + \beta + \gamma) > 0].$$

$$27. \int_0^1 \frac{(x^\beta - x^\gamma)^2}{\ln^2 x} dx = \\ = (2\gamma + 1) \ln(2\gamma + 1) - 2(\gamma + \beta + 1) \ln(\gamma + \beta + 1) + (2\beta + 1) \ln(2\beta + 1) \\ [\operatorname{Re} \gamma, \operatorname{Re} \beta > -1/2]$$

$$28. \int_0^1 \left[\frac{1}{\ln^2 x} - \frac{x}{(1-x)^2} \right] dx = \mathbf{C} - \frac{1}{2}.$$

$$29. \int_0^1 \left[\frac{1}{\ln^2 x} + \frac{(\rho-2)x - \rho + 1}{(x-1)^2} x^{\rho-1} \right] dx = \rho - \psi(\rho) - \frac{3}{2} \quad [\operatorname{Re} \rho > 0].$$

$$30. \int_0^1 [x^{p-1}(q-r) + x^{q-1}(r-p) + x^{r-1}(p-q)] \frac{dx}{\ln^2 x} = \\ = p(q-r) \ln p + q(r-p) \ln q + r(p-q) \ln r \quad [p, q, r > 0].$$

$$31. \int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right. \\ \left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)} \right] \frac{dx}{\ln^2 x} = \frac{1}{2} \left[\frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \right. \\ \left. + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)} + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)} \right] \\ [p, q, r, s > 0].$$

$$32. \int_0^1 \left(\frac{x-1}{\ln x} - x \right) \frac{dx}{\ln x} = \ln 2 - 1.$$

$$33. \int_0^1 \left(\frac{x^q - 1}{x \ln x} - q \right) \frac{dx}{\ln x} = q(\ln q - 1) \quad [q > 0].$$

$$34. \int_0^1 x^{\alpha-1} \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)} \right] \frac{dx}{\ln x} = -\ln \Gamma(\alpha) + \left(\alpha - \frac{1}{2} \right) \ln \alpha - \alpha + \ln \sqrt{2\pi} \\ [\operatorname{Re} \alpha > 0].$$

$$35. \int_0^1 \left(\frac{1}{x \ln x} + \frac{2}{1-x^2} \right) \frac{dx}{\ln x} = \ln 2.$$

$$36. \int_0^1 \left[\frac{1}{\ln x} + \frac{x(3-x)}{2(1-x)} \right] \frac{dx}{x \ln x} = \frac{\ln(2\pi)}{2}.$$

$$37. \int_0^1 \left(\frac{1}{\ln x} + \frac{1+x^2}{1-x^2} \right) \frac{dx}{\ln x} = \ln 2 - 1.$$

$$38. \int_0^1 \left[p-1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x} \right) x^{p-1} \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) \ln p + p - \frac{\ln(2\pi)}{2} \\ [p > 0]$$

$$39. \int_0^1 \ln(1+x) \frac{x \ln x + 1 - x}{x \ln^2 x} dx = \ln \frac{4}{\pi}.$$

$$40. \int_0^\infty x^{\alpha-1} \ln(1+x^{-\alpha}) \left[(p+\alpha)x^p - (q+\alpha)x^q + \frac{x^q - x^p}{\ln x} \right] \frac{dx}{\ln x} = \\ = -\alpha \ln \left(\operatorname{tg} \frac{q\pi}{2\alpha} \operatorname{ctg} \frac{p\pi}{2\alpha} \right) \quad [p+\alpha, q+\alpha < 0; p, q > 0].$$

$$41. \int_0^1 \left(\frac{x^q - 1}{x \ln^2 x} - \frac{q}{x \ln x} - \frac{q^2}{2} \right) \frac{dx}{\ln x} = \frac{q^2}{2} \ln q - \frac{3}{4} q^2 \quad [q > 0].$$

$$42. \int_0^1 \left[\left(p - \frac{1}{2} \right) x^3 + \frac{1}{2} (x^{2p-1} - 1) \left(1 - \frac{1}{\ln x} \right) \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) (\ln p - 1) \quad [p > 0].$$

$$43. \int_0^1 \left[\left(q - \frac{1}{2} \right) \frac{x^{p-1} - x^{r-1}}{\ln x} + p \frac{x^{pq-1}}{1-x^p} - r \frac{x^{rq-1}}{1-x^r} \right] \frac{dx}{\ln x} = \\ = (p-r) \left[1/2 - q - \ln \Gamma(q) + \ln \sqrt{2\pi} \right] \quad [q > 0].$$

$$44. \int_0^1 \left(\frac{x^q - 1}{x \ln^3 x} - \frac{q}{x \ln^2 x} - \frac{q^2}{2x \ln x} - \frac{q^3}{6} \right) \frac{dx}{\ln x} = \frac{q^3}{6} \ln q - \frac{11}{36} q^3 \quad [q > 0].$$

$$45. \int_0^1 [x - (1 - \ln x)^q] \frac{dx}{x \ln x} = -\psi(-q) \quad [\operatorname{Re} q < 0].$$

2.6.18. Интегралы от $\frac{A(x) \ln^n x}{\ln^m x + a}$.

$$1. \int_0^1 \frac{dx}{\ln x + a} = e^{-a} \operatorname{Ei}(a) \quad [a \neq 0].$$

$$2. \int_0^1 \frac{x^{\alpha-1} dx}{(\ln x \pm a)^n} = \\ = \frac{1}{(n-1)!} \left[\alpha^{n-1} e^{\mp \alpha a} \operatorname{Ei}(\pm \alpha a) - \sum_{k=1}^{n-1} (n-k-1)! \alpha^{k-1} (\pm a)^{k-n} \right] \\ \left[\begin{array}{l} \{a, \operatorname{Re} \alpha > 0, n=1, 3, 5, \dots\} \\ \{a, \operatorname{Re} \alpha > 0, n=1, 2, 3, \dots\} \end{array} \right].$$

$$3. \int_1^\infty \frac{dx}{x^2 (\ln x + a)} = -e^a \operatorname{Ei}(-a) \quad [a \neq 0].$$

$$4. \int_0^1 \frac{x^{\alpha-1} \ln^n x}{\ln^2 x + a^2} dx = \frac{1}{a} \frac{\partial^n}{\partial \alpha^n} [\sin \alpha a \operatorname{ci}(\alpha a) - \cos \alpha a \operatorname{si}(\alpha a)] \quad [a, \operatorname{Re} \alpha > 0].$$

$$5. \int_0^1 \frac{x^{\alpha-1} \ln x}{\ln^2 x + a^2} dx = \sin \alpha a \operatorname{si}(\alpha a) + \cos \alpha a \operatorname{ci}(\alpha a) \quad [a, \operatorname{Re} \alpha > 0].$$

$$6. \int_0^1 \frac{x^{\mu/2-1}}{\ln^2 x + a^2} \frac{dx}{1+x^\mu} = \frac{1}{2a} \beta\left(\frac{a\mu + \pi}{4\pi}\right) \quad [a\mu > -\pi; \mu > 0].$$

7. $= \frac{\mu}{8\pi} (4 - \pi)$ [$a\mu = 2\pi$].
8. $= \frac{\mu \ln 2}{2\pi}$ [$a\mu = \pi$].
9. $= \frac{\mu \sqrt{2}}{2\pi} [\pi + 2 \ln (\sqrt{2} - 1)]$ [$2a\mu = \pi$].
10. $\int_0^1 \frac{x^\alpha - x^{-\alpha}}{\ln^2 x + a^2} \frac{dx}{1-x^2} = \frac{2\pi}{a} \sum_{k=1}^{\infty} (-1)^k \frac{\sin k\alpha\pi}{2a + k\pi}$ [$a > 0$, $|\operatorname{Re} \alpha| < 1$].
11. $\int_0^1 \frac{\ln x}{\ln^2 x + a^2} \frac{dx}{1-x^2} = -\frac{1}{4} (1 - 2 \ln 2)$ [$a = \pi$].
12. $= \frac{1}{4} (2 - \pi)$ [$2a = \pi$].
13. $= \frac{1}{2\sqrt{2}} [2\sqrt{2} - \pi + 2 \ln (\sqrt{2} - 1)]$ [$4a = \pi$].
14. $\int_0^1 \frac{x \ln x}{\ln^2 x + a^2} \frac{dx}{1-x^2} = \frac{1}{2} \left[\frac{\pi}{2a} + \ln \frac{\pi}{a} + \psi \left(\frac{a}{\pi} \right) \right]$ [$\operatorname{Re} a > 0$].
15. $\int_0^1 \frac{\ln x}{(\ln^2 x + a^2)^2} \frac{dx}{1-x} = -\frac{\pi^2}{a^4} \sum_{k=1}^{\infty} \left(\frac{2\pi}{a} \right)^{2k-2} |B_{2k}|$ [$\operatorname{Re} a > 0$].
16. $\int_0^1 \frac{\ln (1-x^\mu) dx}{x (\ln^2 x + a^2)} = -\frac{\pi}{a} \ln \Gamma \left(\frac{a\mu + 2\pi}{2\pi} \right) + \frac{\pi}{2a} \ln (a\mu) - \frac{\mu}{2} \left(\ln \frac{2\pi}{a\mu} + 1 \right)$ [$a, \mu > 0$].

2.6.19. Интегралы от произведений логарифмов.

1. $\int_0^1 \ln x \ln (1 \pm x) dx = 2 - \frac{3 \mp 1}{24} \pi^2 - (1 \pm 1) \ln 2.$
2. $\int_0^1 x^n \ln x \ln (1 \pm x) dx =$
 $= \frac{(-1)^n}{n+1} \left[\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n+1} + \frac{1}{k} \right) - \frac{3 \mp 1}{24} \pi^2 \right] + \frac{(1 \pm 1) \{(-1)^{n+1} - 1\}}{2(n+1)^2} \ln 2.$
3. $\int_0^1 \ln^m x \ln^n (1-x) \frac{dx}{x} = I_{m,n}$ [$m, n = 1, 2, 3, \dots$].
- $I_{m,n} = \frac{\partial^{m+n}}{\partial \nu^m \partial \mu^n} B(\nu, \mu + 1) \Big|_{\mu=\nu=0}, \quad I_{m,1} = (-1)^{m+1} m! \zeta(m+2),$
 $I_{1,2} = -\frac{\pi^4}{180}, \quad I_{2,2} = 8\zeta(5) - \frac{2}{3} \pi^2 \zeta(3), \quad I_{3,2} = 6\zeta^2(3) - \frac{\pi^6}{105}.$

4. $\int_0^1 \ln^\sigma \frac{1}{x} \ln(1 \pm x) \frac{dx}{x} = -\Gamma(\sigma+1) \zeta(\sigma+2) \left\{ \begin{matrix} 2^{-\sigma-1} & -1 \\ & 1 \end{matrix} \right\} \quad [\operatorname{Re} \sigma > -1].$
5. $\int_0^1 \ln^{2n} x \ln(1 \pm x) \frac{dx}{x} = -\frac{\pi^{2n+2} 2^{2n+1}}{(2n+1)(2n+2)} |B_{2n+2}| \left\{ \begin{matrix} 2^{-2n-1} & -1 \\ & 1 \end{matrix} \right\}.$
6. $\int_0^1 \ln^\sigma \frac{1}{x} \ln(1 - bx^\mu) \frac{dx}{x} = -\frac{\Gamma(\sigma+1)}{\mu^{\sigma+1}} \operatorname{Li}_{\sigma+2}(b) \quad [\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1].$
7. $\int_0^1 \ln^\sigma \frac{1}{x} \ln(1 - 2bx^\mu \cos \gamma + b^2 x^{2\mu}) \frac{dx}{x} = -2\Gamma(\sigma+1) \mu^{-\sigma-1} \sum_{k=1}^{\infty} \frac{b^k \cos(k\gamma)}{k^{\sigma+2}} \quad [\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1].$
8.
$$\int_0^\infty x^{\alpha-1} \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} =$$

$$= \frac{\pi^2}{(a-1) \sin^2 \alpha \pi} [(a^{\alpha-1} + 1) \ln a - 2\pi (a^{\alpha-1} - 1) \operatorname{ctg} \alpha \pi] \quad [a > 0; a \neq 1; 0 < \operatorname{Re} \alpha < 2, \alpha \neq 1].$$
9. $\int_0^\infty \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \frac{\ln a}{6(a-1)} (4\pi^2 + \ln^2 a) \quad [a > 0, a \neq 1].$
10. $\int_0^\infty \frac{\ln |1 \pm x|}{x} \frac{x \ln x - x - a}{(x+a)^2} dx = \frac{1}{2(a \mp 1)} \left(\ln^2 a + \begin{Bmatrix} 0 \\ \pi^2 \end{Bmatrix} \right) \quad [a > 0].$
11. $\int_0^\infty \frac{\ln^2 x dx}{(x-1)^2} = \frac{2}{3} \pi^2.$
12. $\int_0^\infty \frac{\ln x}{x^2} \ln(1 + ax^2) dx = \pi \sqrt{a} \left(1 - \frac{\ln a}{2} \right) \quad [\operatorname{Re} a > 0].$
13. $\int_0^\infty \ln x \ln \frac{x^2 + a^2}{x^2 + b^2} dx = \pi \left(b - a + \ln \frac{a^a}{b^b} \right) \quad [a, b > 0].$
14. $\int_0^\infty \ln x \ln \frac{a^2 + 2bx + x^2}{a^2 - 2bx + x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad [|b| \leq a].$
15.
$$\int_0^\infty \ln(1 + a^2 x^2) \ln(b^2 + c^2 x^2) \frac{dx}{x^2} =$$

$$= \frac{2\pi}{b} [(ab + c) \ln(ab + c) - ab \ln a - c \ln c] \quad [a, b, c > 0].$$

$$16. \int_0^{\infty} \ln(1+a^2x^2) \ln(1+c^2x^2) \frac{dx}{x^4} = \\ = \frac{2\pi}{3} [ac(a+c) + a^3 \ln a + c^3 \ln c - (a^3 + c^3) \ln(a+c)] \quad [a, c > 0].$$

$$17. \int_0^{\infty} \ln(1+a^2x^{1+2}) \ln(b^2+c^2x^{1+2}) \frac{dx}{x^{1+1}} = \frac{2\pi}{c} [(b+ac) \ln(b+ac) - b \ln b - ac] \\ [a, b, c > 0].$$

2.6.20. Интегралы, содержащие $\ln \ln x$.

$$1. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{x + e^{i\lambda}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-ik\lambda} (C + \ln k) \quad [|\operatorname{Im} \lambda| = 0].$$

$$2. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k}.$$

$$3. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x)^2} = \frac{1}{2} \left(\ln \frac{\pi}{2} - C \right).$$

$$4. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+2x \cos \gamma + x^2} = \frac{\pi}{2 \sin \gamma} \ln \left\{ (2\pi)^{\gamma/\pi} \Gamma \left[\frac{(\pi+\gamma)/(2\pi)}{(\pi-\gamma)/(2\pi)} \right] \right\} \quad [|\gamma| < \pi].$$

$$5. \int_0^1 \ln \ln \frac{1}{x} \frac{x^{n-1}}{1+x^2+x^4+\dots+x^{2n}} dx = \frac{\pi}{2n+2} \operatorname{tg} \frac{\pi}{2n+2} \ln \frac{2\pi}{r} + \\ + \frac{\pi}{n+1} \sum_{k=1}^{n/r} (-1)^{k-1} \sin \frac{k\pi}{n+1} \ln \Gamma \left[\frac{r(n+1-(-1)^r k)/(2n+2)}{rk/(2n+2)} \right] \\ [r=2 \text{ при } n=2l, r=1 \text{ при } n=2l-1, l=1, 2, 3, \dots].$$

$$6. \int_0^1 x^{\alpha-1} \ln \ln \frac{1}{x} \ln^{\sigma} \frac{1}{x} dx = \alpha^{-\sigma-1} \Gamma(\sigma+1) [\psi(\sigma+1) - \ln \alpha] \\ [\operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1].$$

$$7. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x^2) \sqrt{\ln(1/x)}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + 2 \ln 2 + C].$$

2.6.21. Интегралы от $x^{\alpha} e^{-px^{\mu}} \ln^n x$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px^{\mu}} \ln^n x dx = \frac{1}{\mu} \left(\frac{\partial}{\partial \alpha} \right)^n \left[p^{-\alpha/\mu} \Gamma \left(\frac{\alpha}{\mu} \right) \right] \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px^{\mu}} \ln x dx = \mu^{-2} p^{-\alpha/\mu} \Gamma(\alpha/\mu) [\psi(\alpha/\mu) - \ln p] \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} x^n e^{-px} \ln x \, dx = \frac{n!}{p^{n+1}} \left(\sum_{k=1}^n \frac{1}{k} - \mathbf{C} - \ln p \right) \quad [\operatorname{Re} p > 0].$$

$$4. \int_0^{\infty} e^{-px} \ln x \, dx = -\frac{1}{p} (\mathbf{C} + \ln p) \quad [\operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} x^{n-1/2} e^{-px} \ln x \, dx = \frac{\sqrt{\pi} (2n-1)!!}{2^n p^{n+1/2}} \left[2 \sum_{k=1}^n \frac{1}{2k-1} - \mathbf{C} - \ln(4p) \right] \quad [\operatorname{Re} p > 0].$$

$$6. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}} \ln x \, dx = -\sqrt{\frac{\pi}{p}} [\mathbf{C} + \ln(4p)] \quad [\operatorname{Re} p > 0].$$

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px^{\mu}} \ln^2 x \, dx = \mu^{-3} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \left\{ \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right]^2 + \Psi'\left(\frac{\alpha}{\mu}\right) \right\} \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$8. \int_0^{\infty} x^{\alpha-1} e^{-px^{\mu}} \ln^3 x \, dx = \mu^{-4} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \left\{ \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right]^3 + 3 \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right] \Psi'\left(\frac{\alpha}{\mu}\right) + \Psi''\left(\frac{\alpha}{\mu}\right) \right\} \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

2.6.22. Интегралы от $x^{\alpha} e^{ax^2+bx^{\pm 1}} \ln^n x$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px^2-2qx} \ln^n x \, dx = \left(\frac{\partial}{\partial \alpha} \right)^n \left[(4p)^{-\alpha/2} \Gamma(\alpha) \Psi(\alpha/2, 1/2; q^2/p) \right] \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0] \text{ или } [\operatorname{Re} \alpha > 0, \operatorname{Re} p = 0, \operatorname{Re} q > 0] \text{ или } [0 < \operatorname{Re} \alpha < 2, \operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px^2-2qx} \ln x \, dx = -2^{-\alpha-1} p^{-\alpha/2} \Gamma(\alpha) \ln p \Psi(\alpha/2, 1/2; q^2/p) + \\ + \frac{p^{-\alpha/2}}{4} \Gamma\left(\frac{\alpha}{2}\right) \sum_{k=0}^{\infty} \frac{(\alpha/2)_k \Psi(\alpha/2+k)}{(1/2)_k k!} \left(\frac{q^2}{p}\right)^k - \\ - \frac{q}{2} p^{-(\alpha+1)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \sum_{k=0}^{\infty} \frac{((\alpha+1)/2)_k \Psi((\alpha+1)/2+k)}{(3/2)_k k!} \left(\frac{q^2}{p}\right)^k$$

$[\operatorname{Re} \alpha, \operatorname{Re} p > 0]$ или $[\operatorname{Re} \alpha > 0, \operatorname{Re} p = 0, \operatorname{Re} q > 0]$ или $[0 < \operatorname{Re} \alpha < 2, \operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0]$.

$$3. \int_0^{\infty} x(px^2 - qx - 1) e^{-px^2+2qx} \ln x \, dx = \\ = \frac{1}{4p} + \frac{q}{4p} \sqrt{\frac{\pi}{p}} \exp(q^2/p) [1 + \operatorname{erf}(q/\sqrt{p})] \quad [\operatorname{Re} p > 0].$$

$$4. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \ln^n x dx = 2 \left(\frac{\partial}{\partial \alpha} \right)^n \left[\left(\frac{q}{p} \right)^{\alpha/2} K_{\alpha} (2 \sqrt{pq}) \right] \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \ln x dx = -2\pi \operatorname{ctg} \alpha\pi (q/p)^{\alpha/2} K_{\alpha} (2 \sqrt{pq}) - \\ - \frac{\pi}{\sin \alpha\pi} [\ln p I_{-\alpha} (2 \sqrt{pq}) + \ln q I_{\alpha} (2 \sqrt{pq}) - \\ - p^{-\alpha} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! \Gamma(k-\alpha+1)} \psi(k-\alpha+1) - q^{\alpha} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! \Gamma(k+\alpha+1)} \psi(k+\alpha+1)] \\ [\operatorname{Re} p, \operatorname{Re} q > 0; \alpha \neq 0, \pm 1, \pm 2, \dots].$$

$$6. \int_0^{\infty} x^n e^{-px-q/x} \ln x dx = \frac{(-1)^n q^{n+1}}{2} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! (k+n+1)!} \times \\ \times [\ln^2 q - \ln^2 p - 2 \ln q \psi(k+n+2) + \psi^2(k+n+2) - \psi^2(k+1) + \\ + \psi'(k+1) - \psi'(k+n+2)] + p^{-n-1} \sum_{k=0}^n \frac{(-pq)^k}{k!} (n-k)! [\psi(n-k+1) - \ln p] \\ [\operatorname{Re} p, \operatorname{Re} q > 0; n = -1, 0, 1, 2, \dots].$$

$$7. \int_0^{\infty} e^{-px-q/x} \ln x \frac{dx}{x} = \ln \frac{q}{p} K_0 (2 \sqrt{pq}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$8. \int_0^{\infty} x^{-1 \pm 1/2} e^{-px-q/x} \ln x dx = \\ = \sqrt{\frac{\pi}{p}} \left(\frac{p}{q} \right)^{(1 \mp 1)/4} \left[\frac{1}{2} e^{-2\sqrt{pq}} \ln \frac{q}{p} \mp e^{2\sqrt{pq}} \operatorname{Ei}(-4\sqrt{pq}) \right] \\ [\operatorname{Re} p, \operatorname{Re} q > 0].$$

$$9. \int_0^{\infty} \left(p - \frac{q}{x^2} \right) e^{-px-q/x} \ln x dx = 2K_0 (2 \sqrt{pq}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0].$$

2.6.23. Интегралы от $x^{\alpha} e^{-px} \ln(a+bx)$.

$$1. \int_0^1 x e^x \ln(1-x) dx = 1 - e.$$

$$2. \int_{2a}^{\infty} \frac{e^{-px}}{x} \ln(bx-ab) dx = \frac{1}{2} \operatorname{Ei}^2(-ap) - \ln(ab) \operatorname{Ei}(-2ap) \quad [a, b, \operatorname{Re} p > 0].$$

$$3. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(a+bx)^{\rho}} \ln^n(a+bx) dx = \\ = (-1)^n \left(\frac{a}{b} \right)^{\alpha} \Gamma(\alpha) \left(\frac{\partial}{\partial \rho} \right)^n \left[a^{-\rho} \Psi \left(\alpha, 1-\rho+\alpha; \frac{ap}{b} \right) \right] \\ [\operatorname{Re} p > 0; |\arg(a+bx)| < \pi \text{ при } x \geq 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -n \text{ при } a=1)].$$

$$\begin{aligned}
 4. \int_0^{\infty} x^{\alpha-1} e^{-px} \ln(a+bx) dx = \\
 = -\left(\frac{a}{b}\right)^{\alpha} \frac{\pi}{\alpha \sin \alpha \pi} {}_1F_1\left(\alpha; \alpha+1; \frac{ap}{b}\right) + \Gamma(\alpha) p^{-\alpha} \left\{ \left[\ln \frac{ap}{b} - \psi(\alpha) \right] + \right. \\
 \left. + \frac{ap}{b(1-\alpha)} {}_2F_2\left(1, 1; 2, 2-\alpha; \frac{ap}{b}\right) \right\} \\
 [\operatorname{Re} p > 0; |\arg(a+bx)| < \pi \text{ при } x \geq 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -1 \text{ при } a=1)].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_0^{\infty} e^{-px} \ln \left\{ \frac{(a+bx)}{|a-bx|} \right\} dx = \frac{1}{p} \left[\ln a - e^{\pm ap/b} \operatorname{Ei} \left(\mp \frac{ap}{b} \right) \right] \\
 [\operatorname{Re} p > 0; \{ |\arg(a/b)| < \pi \}].
 \end{aligned}$$

$$\begin{aligned}
 6. \int_0^{\infty} \frac{e^{-px}}{x} \ln(1+bx) dx = \frac{1}{2} \left(\ln^2 \frac{p}{b} + 2\mathfrak{C} \ln \frac{p}{b} + \mathfrak{C}^2 + \frac{\pi^2}{2} \right) - \\
 - \sum_{k=1}^{\infty} \frac{2k^{-k} + \psi(k) - \ln p + \ln b}{k!k} \left(\frac{p}{b} \right)^k \quad [\operatorname{Re} p, \operatorname{Re} b > 0].
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^{\infty} e^{-px} \ln(x+a+b) \left(\frac{1}{x+a} + \frac{1}{x+b} \right) dx = (1 + \ln a \ln b) \ln(a+b) + \\
 + e^{-(a+b)p} \{ \operatorname{Ei}(-ap) \operatorname{Ei}(-bp) + [1 - \ln(ab)] \operatorname{Ei}(-ap-bp) \} \quad [a, b, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_0^{\infty} e^{-px} \ln \left| \frac{x+a}{x-a} \right| dx = \frac{2}{p} [\operatorname{shi}(ap) \operatorname{ch}(ap) - \operatorname{chi}(ap) \operatorname{sh}(ap)] \\
 [a, \operatorname{Re} p > 0].
 \end{aligned}$$

2.6.24. Интегралы от $A(x) e^{-px} \ln(ax^2+bx+c)$.

$$\begin{aligned}
 1. \int_0^a \frac{e^{-px}}{\sqrt{x(a-x)}} \ln[x(a-x)] dx = \\
 = \pi e^{-ap/2} \left[\frac{\pi}{2} Y_0\left(\frac{iap}{2}\right) + \left(\ln \frac{a^2}{8} - \mathfrak{C} \right) I_0\left(\frac{ap}{2}\right) \right] \quad [a, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 2. \int_a^{\infty} \frac{e^{-px}}{\sqrt{x^2-a^2}} \ln(x^2-a^2) dx = -\left(\mathfrak{C} + \ln \frac{2p}{a} \right) K_0(ap) \quad [a, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 3. \int_{a+b}^{\infty} \frac{e^{-px}}{x} \ln[(x-a)(x-b)] dx = \\
 = \operatorname{Ei}(-ap) \operatorname{Ei}(-bp) - \ln(ab) \operatorname{Ei}(-ap-bp) \quad [a, b, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^{\infty} e^{-px} \ln[(a+x)(b+x)] \frac{dx}{x+a+b} = \\
 = e^{p(a+b)} \{ \operatorname{Ei}(-ap) \operatorname{Ei}(-bp) - \ln(ab) \operatorname{Ei}(-ap-bp) \} \\
 [\operatorname{Re} p > 0; |\arg(a+b)| < \pi].
 \end{aligned}$$

$$5. \int_0^{\infty} e^{-px} \ln(x^2 + a^2) dx = \frac{2}{p} [\ln a - \operatorname{ci}(ap) \cos ap - \operatorname{si}(ap) \sin ap]$$

[Re a , Re $p > 0$].

$$6. \int_0^{\infty} e^{-px} \left\{ \frac{\ln |x^2 - a^2|}{\ln(x^2 - a^2)} \right\} dx = \frac{1}{p} [2 \ln a - e^{ap} \operatorname{Ei}(-ap) - e^{-ap} \operatorname{Ei}(ap)]$$

[Re $p > 0$; $\left\{ \begin{array}{l} a > 0 \\ |\operatorname{Im} a| > 0 \end{array} \right\}$].

$$7. \int_0^{\infty} \frac{e^{-px}}{x} \ln(1 + a^2 x^2) dx = \operatorname{ci}^2\left(\frac{p}{a}\right) + \operatorname{si}^2\left(\frac{p}{a}\right)$$

[Re a , Re $p > 0$].

$$8. \int_0^{\infty} \frac{e^{-px}}{x} \ln |1 - a^2 x^2| dx = \operatorname{Ei}\left(\frac{p}{a}\right) \operatorname{Ei}\left(-\frac{p}{a}\right)$$

[a , Re $p > 0$].2.6.25. Интегралы от $e^{-px} \ln A(x)$.

$$1. \int_a^{\infty} e^{-px} \ln(x + \sqrt{x^2 - a^2}) dx = \frac{1}{p} [K_0(ap) + \ln ae^{-ap}]$$

[a , Re $p > 0$].

$$2. \int_a^{\infty} e^{-px} \ln(\sqrt{x+a} + \sqrt{x-a}) dx = \frac{1}{2p} [K_0(ap) + \ln(2a) e^{-ap}]$$

[a , Re $p > 0$].

$$3. \int_0^{\infty} e^{-px} \ln(\sqrt{x+z} + \sqrt{x}) dx = \frac{1}{2p} [e^{pz/2} K_0(pz/2) + \ln z]$$

[Re $p > 0$; $|\arg z| < \pi$].

$$4. \int_0^{\infty} e^{-px} \ln(\sqrt{x+ib} + \sqrt{x-ib}) dx = \frac{\pi}{4p} [H_0(bp) - Y_0(bp) + \ln \sqrt{2b}]$$

[b , Re $p > 0$].

$$5. \int_0^{\infty} e^{-px} \ln(\sqrt{x^2 + z^2} + x) dx = \frac{\pi}{2p} [H_0(pz) - Y_0(pz)] + \frac{\ln z}{p}.$$

$$6. \int_0^{\infty} x^n e^{-px} \ln(\sqrt{x^2 + z^2} + x) dx =$$

$$= \frac{(-1)^n \pi}{2} \frac{\partial^n}{\partial p^n} \left\{ \frac{1}{p} \left[H_0(pz) - Y_0(pz) + \frac{2}{\pi} \ln z \right] \right\} \quad [\operatorname{Re} p, \operatorname{Re} z > 0]$$

$$7. \int_0^{\infty} x e^{-px} \ln(\sqrt{x^2 + z^2} + x) dx =$$

$$= \frac{\pi}{2p^2} [H_0(pz) - Y_0(pz)] + \frac{\pi z}{2p} [H_1(pz) - Y_1(pz)] - \frac{z}{p} + \frac{1}{p^2} \ln z$$

[Re p , Re $z > 0$].

$$8. \int_0^{\infty} e^{-px} \ln (\sqrt{x^2 + 2zx} + x + z) dx = \frac{1}{p} [e^{pz} K_0(pz) + \ln z]$$

[Re $p > 0$; $|\arg z| < \pi$].

$$9. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2 + z^2}} \ln (\sqrt{x^2 + z^2} + x) dx =$$

$$= \frac{\pi}{2} \ln z [H_0(pz) - Y_0(pz)] - S_{-1,0}(pz) \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

2.6.26. Интегралы от $R(e^x) \ln x$.

$$1. \int_0^{\infty} \frac{\ln x}{e^x + a} dx = a^{-1} \left[\sum_{k=1}^{\infty} \frac{(-a)^k}{k} \ln' k - C \ln(1+a) \right] \quad [-1 < a \leq 1].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{(e^x + 1)^n} dx = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k (k+n-1)!}{k! (k+n)^{\alpha}} [\psi(\alpha) - \ln(k+n)]$$

[Re $\alpha > 0$; $n = 1, 2, 3, \dots$].

$$3. \int_0^{\infty} \frac{e^x \ln x}{e^{2x} + 2e^x \cos \gamma + 1} dx =$$

$$= \frac{1}{2 \sin \gamma} \left\{ \gamma \ln(2\pi) - \pi \ln \pi + \pi \ln \cos \frac{\gamma}{2\pi} + 2\pi \ln \Gamma \left(\frac{\pi + \gamma}{2\pi} \right) \right\} \quad [0 < \gamma < \pi].$$

$$4. \int_0^{\infty} \frac{e^x \ln x}{(e^x + 1)^2} dx = -\frac{C}{2} + \frac{1}{2} \ln \frac{\pi}{2}.$$

2.6.27. Интегралы, содержащие x^{α} , e^{-px} и $\frac{1}{\ln^n(x+a)}$.

$$1. \int_0^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\ln x} e^{-px} dx = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \ln \frac{\alpha+k}{\beta+k} \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg p| < \pi].$$

$$2. \int_0^{\infty} e^{-x} \frac{dx}{\ln x} = 0.$$

$$3. \int_0^{\infty} (x^{\alpha-1} - x^{\beta-1}) e^{-px} \frac{dx}{\ln x} = \frac{1}{p} [\lambda(p, \alpha-1) - \lambda(p, \beta-1)]$$

[Re $\alpha, \operatorname{Re} \beta, \operatorname{Re} p > 0$].

$$4. \int_0^{\infty} \left[e^{-x} - \frac{x(1+x)^{-q-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln q$$

[Re $q > 0$].

$$5. \int_0^{\infty} \left[(q-1)e^{-x} + \frac{(1+x)^{-q} - (1+x)^{-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln \Gamma(q)$$

[Re $q > 0$].

$$6. \int_0^{\infty} \frac{x^{n-1} e^{-px}}{\ln^2 x + \pi^2} dx = (-1)^{n-1} [\psi^{(n)}(p) - e^p] \quad [\operatorname{Re} p > 0].$$

2.6.28. Интегралы от $x^a \ln(a + be^{-px})$

$$1. \int_0^{\infty} x^n \ln(1 - e^{-ax}) dx = - \left(\frac{2}{a}\right)^{n+1} \frac{\pi^{n+2} B_{n,2}}{(n+2)(n+1)} \quad [a > 0].$$

$$2. \int_0^{\infty} \ln(1 \pm e^{-ax}) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln \Gamma\left(\frac{az}{2\pi} + \frac{3 \mp 1}{4} \pi\right) - \\ - \frac{\pi}{4z} \left[\ln(2\pi az) \pm \ln\left(\frac{2\pi}{az}\right) \right] + \frac{a}{2} \left(1 + \ln \frac{2\pi}{az}\right) \quad [a, \operatorname{Re} z > 0].$$

$$3. \int_0^{\infty} \ln \frac{a + be^{-px}}{a + be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q} \quad [a+b, pq > 0].$$

2.6.29. Интегралы от $x^a R(\operatorname{sh} x, \operatorname{ch} x) \ln x$.

$$1. \int_1^{\infty} \frac{\operatorname{sh} mx}{\operatorname{sh} nx} \ln x dx = \frac{\pi}{2n} \operatorname{tg} \frac{m\pi}{2n} \ln \frac{2\pi}{r} + \\ + \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(n - (-1)^r k) r / (2n)}{kr / (2n)} \right] \\ [r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l; l=1, 2, 3, \dots]$$

$$2. \int_1^{\infty} \frac{\operatorname{ch} mx}{\operatorname{ch} nx} \ln x dx = \frac{\pi}{2n} \ln \left(\frac{2\pi}{r}\right) \sec \frac{m\pi}{2n} + \\ + \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \cos \frac{(2-1)m\pi}{2n} \ln \Gamma \left[\frac{r [2n - (-1)^r (2k-1)] / (4n)}{r (2k-1) / (4n)} \right] \\ [r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l, l=1, 2, 3, \dots].$$

$$3. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} ax} dx = \frac{\pi}{2a} \left[4 \ln \Gamma\left(\frac{3}{4}\right) - \ln(\pi a) \right] \quad [\operatorname{Re} a > 0].$$

$$4. \int_0^{\infty} \frac{\ln x}{V x \operatorname{ch} x} dx = 2 \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + \ln 4 + C].$$

$$5. \int_0^{\infty} \frac{\ln x}{\operatorname{ch}^2 x} dx = \ln \frac{\pi}{4} - C.$$

$$6. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} x + c} dx = \frac{\pi}{V 1 - c^2} \left[2 \ln \Gamma\left(\frac{1+t}{t}\right) + \frac{1}{2} \ln(1+c) + \right. \\ \left. + (t-1) \ln \pi + (t-1/2) \ln 2 \right] \quad [-1 < c \leq 1; \pi t = \arccos c]$$

$$7. \int_0^{\infty} \frac{\ln x}{2 \operatorname{ch}^2 x + 1} dx = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^{\infty} (-1)^k \frac{C + \ln(4k)}{\sqrt{k}} \sin \frac{\pi k}{3}.$$

$$8. \int_0^{\infty} x^{\alpha-1} \frac{\alpha \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = 2\Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^{\alpha}} \quad [\operatorname{Re} \alpha > 0].$$

$$9. \int_0^{\infty} x^n \frac{(n+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = \frac{(-1)^n}{2^n} \beta^{(n)} \left(\frac{1}{2} \right).$$

$$10. \int_0^{\infty} x^{2n} \frac{(2n+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = - \left(\frac{\pi}{2} \right)^{2n+1} |E_{2n}|.$$

$$11. \int_0^{\infty} x^{2n-1} \frac{n \operatorname{sh} 2ax - ax}{\operatorname{sh}^2 ax} \ln x dx = - \frac{1}{n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0; n = 1, 2, 3, \dots].$$

$$12. \int_0^{\infty} x^{2n-1} \frac{2ne^{-ax} \operatorname{sh} ax - ax}{\operatorname{sh}^2 ax} \ln x dx = - \frac{1}{2n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}|$$

[a > 0; n = 1, 2, 3, ...].

$$13. \int_0^{\infty} x^{2n} \frac{2ax \operatorname{ch} ax - (2n+1) \operatorname{sh} ax}{\operatorname{sh}^3 ax} \ln x dx = \frac{1}{a} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}|$$

[a > 0; n = 1, 2, 3, ...].

$$14. \int_0^{\infty} x^{2n} \frac{2ax \operatorname{sh} ax - (2n+1) \operatorname{ch} ax}{\operatorname{ch}^3 ax} \ln x dx = \frac{2}{a} (2^{2n-1} - 1) \left(\frac{\pi}{2a} \right)^{2n} |B_{2n}|$$

[a > 0; n = 1, 2, 3, ...].

$$15. \int_0^{\infty} x^{n-1} \frac{ax \operatorname{ch} ax - n \operatorname{sh} ax}{\operatorname{sh}^2 ax} \ln x dx = \frac{2^{n-1} - 1}{n} \left(\frac{\pi}{a} \right)^n |B_n|$$

[a > 0; n = 2, 4, 6, ...].

$$16. = 2(n-1)! \frac{2^n - 1}{(2a)^n} \zeta(n) \quad [a > 0; n = 1, 3, 5, \dots].$$

2.6.30. Интегралы от $A(x) R(\operatorname{sh} x, \operatorname{ch} x) \ln A(x)$.

$$1. \int_0^{\infty} \frac{\ln(a + bx^2)}{\operatorname{ch} x} dx = \frac{2\pi}{c} \ln \Gamma \left[\frac{(2\sqrt{ac} + 3\pi\sqrt{b})/(4\pi\sqrt{b})}{(2c\sqrt{a} + \pi\sqrt{b})/(4\pi\sqrt{b})} \right] - \frac{\pi}{c} \ln \frac{c}{2\pi\sqrt{b}}$$

[a, b, c > 0].

$$2. \int_0^{\infty} \frac{\ln(1 + bx^2)}{\operatorname{sh}^2 cx} dx = \frac{2}{c} \left[\ln \frac{c}{\pi\sqrt{b}} + \frac{\pi\sqrt{b}}{2c} - \psi \left(1 + \frac{c}{\pi\sqrt{b}} \right) \right] \quad [b, c > 0].$$

$$3. \int_0^{\infty} \frac{\operatorname{ch} cx}{\operatorname{sh}^2 cx} \ln \left(1 + \frac{4c^2}{\pi^2} x^2 \right) dx = \frac{\pi - 2}{c} \quad [c > 0].$$

$$4. \int_0^{\infty} \frac{\operatorname{ch} cx}{\operatorname{sh}^2 cx} \ln \left(1 + \frac{16c^2}{\pi^2} x^2 \right) dx = \frac{1}{c} [\pi \sqrt{2} - 4 + 2 \sqrt{2} \ln (1 + \sqrt{2})] \quad [c > 0].$$

$$5. \int_0^{\infty} \frac{\operatorname{sh} (2\pi x \sqrt{b/3})}{\operatorname{sh} (\pi x \sqrt{b})} \ln (a + bx^2) dx = \\ = \sqrt{\frac{3}{b}} \ln \Gamma \left[\frac{(\sqrt{a} + 4)/6, (\sqrt{a} + 5)/6}{(\sqrt{a} + 1)/6, (\sqrt{a} + 2)/6} \right] + \sqrt{\frac{3}{b}} \ln 6 \quad [a, b > 0].$$

$$6. \int_0^{\infty} \frac{\operatorname{ch} cx + cx \operatorname{sh} cx}{x^2 \operatorname{ch}^2 cx} \ln \left(1 + \frac{4c^2}{\pi^2} x^2 \right) dx = \frac{4c}{\pi} \ln 2 \quad [c > 0].$$

$$7. \int_0^{\infty} \frac{\operatorname{ch} cx + cx \operatorname{sh} cx}{x^2 \operatorname{ch}^2 cx} \ln \left(1 + \frac{c^2}{\pi^2} x^2 \right) dx = \frac{c}{\pi} (4 - \pi) \quad [c > 0].$$

$$8. \int_a^{\infty} \frac{e^{-px}}{\sqrt{x^2 - a^2}} \operatorname{sh} (b \sqrt{x^2 - a^2}) \ln \frac{x + \sqrt{x^2 - a^2}}{a} dx = \\ = \frac{1}{2} \ln \frac{p+b}{p-b} K_0 (a \sqrt{p^2 - b^2}) \quad [a > 0; \operatorname{Re} p > |\operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2 + 2zx}} \operatorname{sh} (b \sqrt{x^2 + 2zx}) \ln \frac{x+z+\sqrt{x^2+2zx}}{z} dx = \\ = e^{zp} \ln \frac{p+b}{p-b} K_0 (z \sqrt{p^2 - b^2}) \quad [\operatorname{Re} p > |\operatorname{Re} b|; |\arg z| < \pi].$$

2.6.31. Интегралы, содержащие логарифмическую функцию от гиперболических функций.

$$1. \int_0^{\infty} e^{-px} \ln (\operatorname{sh} ax) dx = -\frac{1}{p} \left[\ln 2 + \frac{a}{2p} + \psi \left(\frac{p}{2a} \right) - C \right] \quad [a, \operatorname{Re} p > 0].$$

$$2. \int_0^{\infty} e^{-2nax} \ln (\operatorname{sh} ax) dx = \frac{1}{2an} \left[\frac{1}{n} + \ln 2 - 2\beta (2n+1) \right] \\ [a > 0; n = 1, 2, 3, \dots].$$

$$3. \int_0^{\infty} e^{-px} \ln (\operatorname{ch} ax) dx = \frac{1}{p} \left[\beta \left(\frac{p}{2a} \right) - \frac{a}{p} \right] \quad [a, \operatorname{Re} p > 0].$$

$$4. \int_{-\infty}^{\infty} x \ln \operatorname{ch} x dx = 0. \quad 5. \int_0^{\infty} \ln \operatorname{ch} x \frac{dx}{1-x^2} = 0.$$

$$6. \int_0^{\infty} \ln \operatorname{ch} x \frac{dx}{\operatorname{ch} 2x} = \frac{G}{2} - \frac{\pi}{8} \ln 2. \quad 7. \int_0^{\infty} \ln \operatorname{th} x dx = -\pi^2/8.$$

$$8. \int_0^{\infty} x^{\alpha-1} \ln \operatorname{th} x \, dx = \frac{\pi^{\alpha+1} 2^{-\alpha}}{\alpha \sin(\alpha\pi/2)} (2^{\alpha+1} - 1) \zeta(-\alpha) \quad [\operatorname{Re} \alpha > 0].$$

$$9. \int_0^{\infty} \ln \operatorname{th} x \frac{dx}{\operatorname{ch} x} = -\frac{\pi}{2} \ln 2.$$

$$10. \int_0^{\infty} x^{\alpha-1} \ln \frac{\operatorname{ch} ax + \operatorname{sh} b}{\operatorname{ch} ax - \operatorname{sh} b} dx = -\left(\frac{2}{a}\right)^{\alpha} \frac{\pi^{\alpha+1}}{\alpha \sin(\alpha\pi/2)} \times \\ \times \left[\zeta\left(-\alpha, \frac{\pi-2b}{4\pi}\right) - \zeta\left(-\alpha, \frac{\pi+2b}{4\pi}\right) + \zeta\left(-\alpha, \frac{3\pi+2b}{4\pi}\right) - \zeta\left(-\alpha, \frac{3\pi-2b}{4\pi}\right) \right] \\ [\operatorname{Re} \alpha > 0; 2b > \pi].$$

2.6.32. Интегралы от $A(x) \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln^n x$.

$$1. \int_0^{\infty} x^{\alpha-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln^n x \, dx = \frac{\partial^n}{\partial \alpha^n} \left[b^{-\alpha} \Gamma(\alpha) \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} \right] \\ [b > 0; \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} < \operatorname{Re} \alpha < 1].$$

$$2. \int_0^{\infty} x^{\alpha-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln x \, dx = \frac{\Gamma(\alpha)}{b^{\alpha}} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} \left[\psi(\alpha) - \ln b \pm \frac{\pi}{2} \operatorname{tg}^{\mp 1} \frac{\alpha\pi}{2} \right] \\ [b > 0; \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} < \operatorname{Re} \alpha < 1].$$

$$3. \int_0^{\infty} \frac{\sin bx}{x} \ln x \, dx = -\frac{\pi}{2} (\mathbf{C} + \ln b) \quad [b > 0].$$

$$4. \int_0^{\infty} \frac{1}{\sqrt{x}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln x \, dx = -\sqrt{\frac{\pi}{2b}} \left[\ln(4b) + \mathbf{C} \mp \frac{\pi}{2} \right] \quad [b > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln^2 x \, dx = \\ = \frac{\Gamma(\alpha)}{b^{\alpha}} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix} \left\{ \left[\psi(\alpha) - \ln b \pm \frac{\pi}{2} \operatorname{tg}^{\mp 1} \frac{\alpha\pi}{2} \right]^2 + \psi'(\alpha) - \frac{\pi^2}{4} \begin{Bmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{Bmatrix}^{-2} \right\} \\ [b > 0; \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} < \operatorname{Re} \alpha < 1].$$

$$6. \int_0^{\infty} \frac{\sin bx}{x} \ln^2 x \, dx = \frac{\pi}{2} \left(\mathbf{C}^2 + \frac{\pi^2}{12} + 2\mathbf{C} \ln b + \ln^2 b \right) \quad [b > 0].$$

$$7. \int_0^{\infty} \frac{\sin^2 bx}{x^2} \ln x \, dx = -\frac{b\pi}{2} [\mathbf{C} + \ln(2b) - 1] \quad [b > 0].$$

$$8. \int_0^{\infty} \frac{1}{x^2 + z^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln(ax) dx =$$

$$= \frac{\pi}{2} z^{(\pm 1 - 1)/2} [e^{-bz} \ln(az) \mp e^{bz} \operatorname{Ei}(-bz) - e^{-bz} \operatorname{Ei}(bz)] \quad [b, \operatorname{Re} z > 0; |\arg a| < \pi].$$

$$9. \int_0^{\infty} \frac{1}{x^2 - y^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln(ax) dx =$$

$$= \frac{\pi}{2} y^{(\pm 1 - 1)/2} \left[\pm \left\{ \frac{\cos by}{\sin by} \right\} [\ln(ay) - \operatorname{ci}(by)] - \operatorname{si}(by) \left\{ \frac{\sin by}{\cos by} \right\} \right]$$

$$[b, y > 0; |\arg a| < \pi].$$

$$10. \int_0^{\infty} \frac{\cos ax - \cos bx}{x} \ln x dx = \ln \frac{a}{b} \left[C + \frac{1}{2} \ln(ab) \right] \quad [a, b > 0].$$

$$11. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} \ln x dx = \frac{\pi}{2} [(a - b)(C - 1) + a \ln a - b \ln b] \quad [a, b > 0].$$

2.6.33. Интегралы от $A(x) \left\{ \frac{\sin bx}{\cos bx} \right\} \ln A(x)$.

$$1. \int_0^a \sin bx \ln(a - x) dx = \frac{1}{b} \{ \ln a - \sin ab \operatorname{Si}(ab) - \cos ab [\operatorname{ci}(ab) - C - \ln b] \}$$

$$[a, b > 0]$$

$$2. \int_0^{\infty} \frac{\cos bx}{x} \ln(1 + ax) dx = \frac{1}{2} \left[\operatorname{si}^2\left(\frac{b}{a}\right) + \operatorname{ci}^2\left(\frac{b}{a}\right) \right] \quad [a, b > 0].$$

$$3. \int_0^{\infty} \sin bx \ln \left| \frac{x+a}{x-a} \right| dx = \frac{\pi}{b} \sin ab \quad [a, b > 0].$$

$$4. \int_0^{\infty} \cos bx \ln \left| \frac{x+a}{x-c} \right| dx = \frac{1}{b} \left[\frac{\pi}{2} (\cos bc - \cos ab) + \right.$$

$$\left. + \cos bc \operatorname{Si}(bc) + \cos ab \operatorname{Si}(ab) - \sin bc \operatorname{ci}(bc) - \sin ab \operatorname{ci}(ab) \right] \quad [a, b, c > 0].$$

$$5. \int_0^{\infty} \frac{\cos bx}{x} \ln \left| \frac{x+a}{x-a} \right| dx = -\pi \operatorname{si}(ab) \quad [a > 0, b \geq 0].$$

$$6. \int_0^a \cos bx \ln(a^2 - x^2) dx = \frac{\sin ab}{b} \left[\operatorname{ci}(2ab) - C - \ln \frac{b}{a} \right] - \frac{\cos ab}{b} \operatorname{Si}(2ab)$$

$$[a, b > 0].$$

$$7. \int_0^a \frac{\cos bx}{\sqrt{a^2 - x^2}} \ln(a^2 - x^2) dx = \frac{\pi^2}{4} Y_0(ab) - \frac{\pi}{2} J_0(ab) \left(C + \ln \frac{2b}{a} \right) \quad [a, b > 0].$$

$$8. \int_0^{\infty} \frac{\sin bx}{x} \ln(1+ax^2) dx = -\pi \operatorname{Ei}\left(-\frac{b}{\sqrt{a}}\right) \quad [a, b > 0].$$

$$9. \int_0^{\infty} \frac{\cos bx}{x^2} \ln(1+ax^2) dx = \pi \left[b \operatorname{Ei}\left(-\frac{b}{\sqrt{a}}\right) + \sqrt{b} \exp\left(-\frac{b}{\sqrt{a}}\right) \right] \quad [a > 0, b \geq 0].$$

$$10. \int_0^{\infty} \frac{\cos bx}{x^2+z^2} \ln(x^2+z^2) dx = -\left(C + \ln \frac{2b}{z}\right) K_0(bz) \quad [b, \operatorname{Re} z > 0].$$

$$11. \int_0^{\infty} \cos bx \ln\left(1+\frac{z^2}{x^2}\right) dx = \frac{\pi}{b} (1-e^{-bz}) \quad [b, \operatorname{Re} z > 0].$$

$$12. \int_0^{\infty} \cos bx \ln\left|1-\frac{y^2}{x^2}\right| dx = \frac{\pi}{b} [1-\cos(by)] \quad [b, y > 0].$$

$$13. \int_0^{\infty} \cos bx \ln \frac{a^2+x^2}{c^2+x^2} dx = \frac{\pi}{b} (e^{-bc} - e^{-ac}) \quad [b, \operatorname{Re} a, \operatorname{Re} c > 0].$$

$$14. \int_0^{\infty} x \sin bx \ln \frac{a^2+x^2}{c^2+x^2} dx = \frac{\pi}{b^2} [(1+bc)e^{-bc} - (1+ab)e^{-ab}] \quad [b, \operatorname{Re} a, \operatorname{Re} c > 0].$$

$$15. \int_0^{\infty} \frac{\sin bx}{x} \ln \frac{a^2+x^2g^2}{a^2+x^2h^2} dx = \pi \left[\operatorname{Ei}\left(-\frac{ab}{h}\right) - \operatorname{Ei}\left(-\frac{ab}{g}\right) \right] \quad [b, g, h, \operatorname{Re} a > 0].$$

$$16. \int_0^{\infty} \sin bx \ln \frac{a^2+(x+c)^2}{a^2+(x-c)^2} dx = \frac{2\pi}{b} e^{-ab} \sin(bc) \quad [b, \operatorname{Re} a > 0; |\operatorname{Im} c| \leq \operatorname{Re} a].$$

$$17. \int_0^{\infty} \frac{\sin bx}{\sqrt{x^2+z^2}} \ln(x+\sqrt{x^2+z^2}) dx = \frac{\pi}{2} \{K_0(bz) + \ln z [I_0(bz) - L_0(bz)]\} \quad [b, \operatorname{Re} z > 0].$$

$$18. \int_0^{\infty} \frac{\cos bx}{\sqrt{x^2+z^2}} \ln(x+\sqrt{x^2+z^2}) dx = \frac{1}{2} [S_{1,0}(ibz) + S_{-1,0}(-ibz)] + \ln z K_0(bz) \quad [b, \operatorname{Re} z > 0].$$

$$19. \int_1^{\infty} \frac{\cos bx}{x} \ln(x+\sqrt{x^2-1}) dx = -\frac{1}{4} \sum_{k=1}^{\infty} \frac{2\psi(k) + 3k^{-1} - 2 \ln(b/2)}{k(k!)^2} \left(-\frac{b^2}{4}\right)^k - \frac{1}{4} \left(C^2 + \frac{\pi^2}{3}\right) \quad [b > 0].$$

2.6.34. Интегралы, содержащие $\ln \left\{ \frac{\sin bx}{\cos bx} \right\}$.

$$1. \int_0^{\pi/4} x^{\alpha-1} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx =$$

$$= -\frac{1}{2\alpha} \left(\frac{\pi}{4} \right)^{\alpha} \left[\ln 2 + \frac{2}{\alpha} \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (\alpha+2k)} \left\{ \begin{matrix} 1 \\ 2^{2k-1} - 1 \end{matrix} \right\} \right] \quad [\operatorname{Re} \alpha > \left\{ \begin{matrix} 0 \\ -2 \end{matrix} \right\}].$$

$$2. \int_0^{\pi/4} x \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{7 \pm 28}{128} \zeta(3) \mp \frac{\pi}{8} G - \frac{\pi^2}{32} \ln 2.$$

$$3. \int_0^{\pi/4} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\pi}{4} \ln 2 \mp \frac{G}{2}.$$

$$4. \int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+2} x} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx =$$

$$= -\frac{1}{n+1} \left[\frac{\ln 2}{2} + \frac{(-1)^{[n/2]}}{4} \left\{ \begin{matrix} \pi \\ \ln 4 \end{matrix} \right\} + \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{n-2k-1} \right] + \frac{1 \pm 1 \pm 2(-1)^n}{2(n+1)^2}.$$

$$5. \int_0^{\pi/4} \cos^{v-1} 2x \sin 2x \ln \sin x dx = -(4v)^{-1} [C + \psi(v+1) + \ln 2] \quad [\operatorname{Re} v > 0].$$

$$6. \int_0^{\pi/4} \cos^{v-1} 2x \operatorname{tg} 2x \ln \cos x dx = \frac{1}{4(1-v)} \beta(v) \quad [\operatorname{Re} v > 0].$$

$$7. \int_0^{\pi/2} x^{\alpha-1} \ln \sin x dx = -\frac{1}{\alpha} \left(\frac{\pi}{2} \right)^{\alpha} \left[\frac{1}{\alpha} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k (\alpha+2k)} \right] \quad [\operatorname{Re} \alpha > 0].$$

$$8. \int_0^{\pi/2} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\pi}{2} \ln 2.$$

$$9. \int_0^{\pi/2} x \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \pm \frac{7}{16} \zeta(3) - \frac{\pi^2}{8} \ln 2.$$

$$10. \int_0^{\pi/2} x^2 \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \pm \frac{7-1}{32} \pi \zeta(3) - \frac{\pi^3}{24} \ln 2.$$

$$11. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^{\mu-1} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\sqrt{\pi}}{2} \Gamma \left[\begin{matrix} \mu/2 \\ (\mu+1)/2 \end{matrix} \right] \beta(\mu) \quad [\operatorname{Re} \mu > 0].$$

$$12. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^n \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = A (-1)^{n+1} \frac{(n-1)!!}{n!!} \left[\ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right]$$

[$A=1$, если $n=1, 3, 5, \dots$; $A=\pi/2$, если $n=0, 2, 4, \dots$].

$$13. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \ln 2 - 1.$$

$$14. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^{\mu-1} \ln \left\{ \frac{\cos x}{\sin x} \right\} dx =$$

$$= -\frac{\sqrt{\pi}}{4} \Gamma \left[\frac{\mu/2}{(\mu+1)/2} \right] \left[C + 2 \ln 2 + \psi \left(\frac{\mu+1}{2} \right) \right] \quad [\operatorname{Re} \mu > -2].$$

$$15. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^n \ln \left\{ \frac{\cos x}{\sin x} \right\} dx = I_n \quad [n=-2, -1, 0, 1, 2, \dots].$$

$$I_{-2} = -\frac{\pi}{2}, \quad I_{-1} = -\frac{\pi^2}{8}, \quad I_0 = -\frac{\pi}{2} \ln 2, \quad I_1 = -1,$$

$$I_2 = -\frac{\pi}{8} (1 + 2 \ln 2), \quad I_n = -\frac{A (n-1)!!}{2 n!!} \left[\sum_{k=1}^{[(n+1)/2]} \frac{1}{k-r} + B \right].$$

[$A=\pi/2$, $r=0$, $B=2 \ln 2$, если $n=2, 4, 6, \dots$; $A=1$, $r=1/2$, $B=0$, если $n=1, 3, 5, \dots$].

$$16. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{4} B \left(\frac{\mu}{2}, \frac{\nu}{2} \right) \left[\left\{ \psi \left(\frac{\mu}{2} \right) \right\} - \psi \left(\frac{\mu+\nu}{2} \right) \right]$$

$$\left[\left\{ \operatorname{Re} \mu > 0, \operatorname{Re} \nu > -2 \right\} \right].$$

$$17. \int_0^{\pi/2} \frac{\cos^{\nu-1} x}{\sin x} \ln \cos x dx = -\frac{1}{4} \psi' \left(\frac{\nu}{2} \right) \quad [\operatorname{Re} \nu > 0].$$

$$18. \int_0^{\pi/2} \cos 2nx \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\pi}{4n} \left\{ \frac{1}{(-1)^n} \right\} \quad [n=1, 2, 3, \dots].$$

$$19. \int_0^{\pi/2} x \left\{ \frac{\sin x}{\cos x} \right\} \ln \left\{ \frac{\cos x}{\sin x} \right\} dx = \pm (\ln 2 - 2) - \frac{\pi}{4} (1 \mp 1).$$

$$20. \int_0^{\pi/2} \cos^{\nu-1} x \sin \nu x \sin x \ln \cos x dx = 2^{-\nu-2} \pi [C + \psi(\nu) - 1/\nu - \ln 4] \quad [\nu > 0].$$

$$21. \int_0^{\pi/2} \frac{1}{(a \sin x + b \cos x)^2} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{a^2 + b^2} \left[\mp \left| \frac{a}{b} \right|^{\pm 1} \ln \left| \frac{a}{b} \right| - \frac{\pi}{2} \right]$$

[$a > 0$; $b \neq 0$, $\operatorname{Im} b = 0$]

$$22. \int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{\pi}{2ab} \ln \left[\frac{1}{a+b} \left\{ \frac{b}{a} \right\} \right] \quad [a, b > 0].$$

$$23. \int_0^{\pi/2} \frac{\sin 2x}{(a \sin^2 x + b \cos^2 x)^2} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{2(b-a)} \left\{ \frac{b^{-1}}{a^{-1}} \right\} \ln \frac{a}{b} \quad [a, b > 0].$$

$$24. \int_0^{\pi/2} \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \pm \frac{\pi}{2(a+b)} \left\{ \frac{b^{-1}}{a^{-1}} \right\} \quad [a, b > 0].$$

$$25. \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2 x}} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{1}{2} K(k) \ln k - \\ - \frac{\pi}{4} K(\sqrt{1-k^2}) + \frac{1}{8} (1+k) \ln(1-k^2) K(k) \quad [0 < k < 1].$$

$$26. \int_0^{\pi} \sin^{\mu-1} x \ln \sin x dx = -\sqrt{\pi} \Gamma \left[\frac{\mu/2}{(\mu+1)/2} \right] \beta(\mu) \quad [\operatorname{Re} \mu > 0].$$

$$27. \int_0^{\pi} x \sin^{\mu-1} x \ln \sin x dx = -\frac{\pi}{2} B\left(\frac{1}{2}, \frac{\mu}{2}\right) \beta(\mu) \quad [\operatorname{Re} \mu > 0].$$

$$28. \int_0^{\pi} x \ln \left\{ \frac{\sin x}{|\cos x|} \right\} dx = -\frac{\pi^2}{2} \ln 2.$$

$$29. \int_0^{\pi} x^2 \ln \sin x dx = -\frac{\pi^3}{3} \ln 2 - \frac{\pi}{2} \zeta(3).$$

$$30. \int_0^{\pi n} x \ln |\sin x| dx = -\frac{\pi^2 n^2}{2} \ln 2. \quad 31. \int_0^{\pi} \sin 2nx \ln \sin x dx = 0.$$

$$32. \int_0^{\pi} \sin[(2n+1)x] \ln \sin x dx = \frac{2}{2n+1} \left(\ln 2 - 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} - \frac{1}{2n+1} \right).$$

$$33. \int_0^{\pi} \cos[(2n+1)x] \ln \sin x dx = 0.$$

$$34. \int_0^{\pi} \cos 2nx \ln \sin x dx = -\frac{\pi}{2n} \quad [n=1, 2, 3, \dots].$$

$$35. \int_0^{\pi} \sin^{\mu} x \cos \left[\mu \left(\frac{\pi}{2} - x \right) \right] \ln \sin x dx = -2^{-\mu} \pi \ln 2 \quad [\mu > -1].$$

$$36. \int_0^{\pi} \sin^{2n} 2x \cos 2x \ln \sin x \, dx = -\frac{\pi}{4n+2} \frac{(2n-1)!!}{(2n)!!}.$$

$$37. \int_0^{\pi} \frac{\ln \sin x \, dx}{a+b \cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \quad [a > 0, a > b].$$

$$38. \int_0^{\pi n} \frac{1}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{n\pi}{|1-a^2|} \ln \frac{1+a^{2b\varphi}}{2}$$

[$n=1, 2$; $\varphi = \operatorname{sgn}(1-|a|)$, $|a| \neq 1$].

$$39. \int_0^{\pi} \frac{\cos x \ln \sin x \, dx}{1-2a \cos x + a^2} = \frac{\pi}{2a} \left| \frac{1+a^2}{1-a^2} \right| \ln(1-a^{2\varphi}) - \frac{\pi a^{\varphi} \ln 2}{|1-a^2|}$$

[$|a| \neq 1$; $\varphi = \operatorname{sgn}(1-|a|)$].

$$40. \int_0^{\pi} \frac{\cos x}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} dx = \frac{\pi}{2a|1-a^2|} [(1+a^2) \ln(1+a^{\varphi}) - 2a^{1+\varphi} \ln 2]$$

[$|a| \neq 1$; $\varphi = \operatorname{sgn}(1-|a|)$].

$$41. \int_0^{2\pi} \frac{\cos(x/2)}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 0 \quad [|a| \neq 1].$$

$$42. \int_0^{2\pi} \frac{\cos^2(x/2) \ln \sin(x/2)}{1-2a \cos x + a^2} dx = \frac{\pi}{2a} \left| \frac{a+1}{a-1} \right| \ln(1-a^{\varphi}) - \frac{\pi a^{(1-\varphi)/2}}{|a-1|} \ln 2$$

[$|a| \neq 1$; $\varphi = \operatorname{sgn}(1-|a|)$].

$$43. \int_0^{\pi} \frac{\sin x}{\sqrt{1+\sin^2 x}} \ln \sin x \, dx = -\frac{\pi}{8} \ln 2.$$

$$44. \int_0^{\pi} \frac{\sin^3 x}{\sqrt{1+\sin^2 x}} \ln \sin x \, dx = \frac{\ln 2 - 1}{4}.$$

$$45. \int_0^{\infty} \frac{\ln |\cos x|}{x^2} dx = -\frac{\pi}{2}.$$

$$46. \int_0^{\infty} \ln \left\{ \frac{|\sin ax|}{|\cos ax|} \right\} \frac{dx}{x^2+z^2} = \frac{\pi}{2z} \ln \frac{1-e^{-2az}}{2} \quad [a, \operatorname{Re} z > 0].$$

$$47. \int_0^{\infty} \ln \left\{ \frac{|\sin ax|}{|\cos ax|} \right\} \frac{dx}{x^2-y^2} = \frac{(1+i)\pi^2}{8y} - \frac{\pi a}{2} \quad [a, y > 0].$$

$$48. \int_0^{\infty} \frac{\cos bx}{x^2} \ln |\cos ax| \, dx = \frac{\pi}{2} (b \ln 2 - \pi) \quad [a, b > 0].$$

$$49. \int_0^{\infty} \frac{\cos bx}{x^2 + z^2} \ln |\cos ax| dx = \frac{\pi}{2z} [\operatorname{ch} bz \ln (1 + e^{-2az}) - e^{-bz} \ln 2]$$

[$0 < b < 2a$; $\operatorname{Re} z > 0$].

$$50. \int_0^{\infty} \frac{\cos bx}{x^n (x^2 + z^2)} \ln |\cos ax| dx = I_n$$

[$a, b, \operatorname{Re} z > 0$].

$$I_1 = \frac{\pi}{2} z^{-2} [\operatorname{sh} bz \ln (1 + e^{-2az}) - (1 - e^{-bz}) \ln 2],$$

$$I_2 = \frac{\pi}{2} z^{-3} [(bz + e^{-bz}) \ln 2 - az - \operatorname{ch} bz \ln (1 + e^{-2az})].$$

$$51. \int_0^{\infty} \frac{1}{x^2 + z^2} x^{-1} \sin bx \ln \cos^2 ax dx = \pi \ln (1 + e^{-2az}) z^{-4} \operatorname{sh} bz - z^{-2} + \pi z^{-2} e^{-bz} \ln 2$$

[$0 \leq b < 2a$; $\operatorname{Re} z > 0$].

$$52. \int_0^{\infty} \frac{\operatorname{tg} x}{x} \ln |\cos x| dx = -\frac{\pi}{2} \ln 2.$$

$$53. \int_0^{\infty} \frac{1}{x^2 + z^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln (2 + 2 \cos x) dx = \mp \pi \left\{ \frac{\operatorname{sh} bz}{z^{-1} \operatorname{ch} bz} \right\} \ln (1 + e^{-z})$$

[$\operatorname{Re} z > 0$; $0 < b < 1$].

$$54. \int_0^{\infty} \frac{1}{x^2 + z^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln (2 - 2 \cos x) dx = \mp \pi \left\{ \frac{\operatorname{sh} bz}{z^{-1} \operatorname{ch} bz} \right\} \ln (1 - e^{-z})$$

[$\operatorname{Re} z > 0$; $0 < b < 1$].

2.6.35. Интегралы, содержащие $\ln^n \left\{ \frac{\sin bx}{\cos bx} \right\}$.

$$1. \int_0^{\pi/4} \operatorname{tg} x \ln^{2n} \cos 2x dx = \frac{2^{2n} - 1}{2^{2n+1}} (2n)! \zeta(2n+1)$$

[$n = 1, 2, 3, \dots$].

$$2. \int_0^{\pi/4} \operatorname{tg} x \ln^{2n+1} \cos 2x dx = \frac{1 - 2^{2n+1}}{4(n+1)} \pi^{2n+2} |B_{2n+2}|.$$

$$3. \int_0^{\pi/4} \cos^{v-1} 2x \operatorname{tg} x \ln^n \cos 2x dx = \frac{1}{2} \beta^{(n)}(v)$$

[$\operatorname{Re} v > 0$].

$$4. \int_0^{\pi/4} \sin^{\mu-1} 2x \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \ln^n \sin 2x dx = \left\{ \frac{(-1)^n 2^{-1} n! \zeta(n+1, \mu)}{2^{-1} \beta^{(n)}(\mu)} \right\}$$

[$\operatorname{Re} \mu > 0$].

$$5. \int_0^{\pi/6} \ln^2 (2 \sin x) dx = \frac{7\pi^3}{216}.$$

$$6. \int_0^{\pi/6} x \ln^2 (2 \sin x) dx = \frac{17\pi^4}{25920}.$$

$$7. \int_0^{\pi/2} \ln^2 \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{\pi}{2} \left(\ln^2 2 + \frac{\pi^2}{12} \right).$$

$$8. \int_0^{\pi/2} \ln^n (2 \sin x) dx = I_n,$$

$$I_1 = 0, \quad I_2 = \frac{\pi^3}{24}, \quad I_3 = -\frac{3\pi}{4} \zeta(3), \quad I_4 = \frac{19}{480} \pi^5.$$

$$9. \int_0^{\pi/2} x \ln^2 (2 \sin x) dx = \text{Li}_4 \left(\frac{1}{2} \right) + \frac{1}{24} \ln^4 2 - \frac{\pi^2}{24} \ln^2 2 + \frac{7}{8} \ln 2 \zeta(3) - \frac{19}{2880} \pi^4.$$

$$10. \int_0^{\pi/2} x^2 \ln^2 (2 \sin x) dx = \pi \text{Li}_4 \left(\frac{1}{2} \right) + \frac{\pi}{24} \ln^4 2 + \frac{7}{8} \pi \ln 2 \zeta(3) - \frac{37\pi^5}{2880}.$$

$$11. \int_0^{\pi/2} \ln^2 (2 \sin x) \ln (2 \cos x) dx = \frac{1}{8} \pi \zeta(3).$$

$$12. \int_0^{\pi} x^2 \ln^2 (2 \sin x) dx = -\frac{7\pi^5}{240}.$$

2.6.36. Интегралы от $A(x, \sin x, \cos x) \ln(a \cos x + b \sin x + c)$.

$$1. \int_0^{\pi/2} \ln(1 \pm \sin x) dx = -\frac{\pi}{2} \ln 2 \pm 2G.$$

$$2. \int_0^{\pi/2} \ln(1 \pm \cos x) dx = -\frac{\pi}{2} \ln 2 \pm 2G.$$

$$3. \int_0^{\pi/2} \ln \left(1 + a \left\{ \frac{\sin x}{\cos x} \right\} \right) \left\{ \frac{\sin x}{\cos x} \right\}^{-1} dx = \frac{\pi^2}{8} - \frac{1}{2} \arccos^2 a \quad [|a| < 1].$$

$$4. \int_0^{\pi/2} \frac{\cos x \ln(1 \pm a \cos x)}{1 - a^2 \cos^2 x} dx =$$

$$= \frac{1}{a \sqrt{1 - a^2}} \left[L \left(\frac{\pi}{2} - \arccos a \right) - \frac{\arccos a}{2} \ln(1 - a^2) + \frac{1 \mp 1}{4} \pi \ln(1 - a^2) \right] \quad [0 < a < 1].$$

$$5. \int_0^{\pi} \ln(a \pm b \cos x) dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \quad [a, b > 0].$$

$$6. \int_0^{\pi} \ln(1 + b \cos x) \frac{dx}{\cos x} = \pi \arcsin b \quad [|b| < 1].$$

$$7. \int_0^{\infty} \frac{x \sin bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = -\pi \operatorname{sh} bz \ln(1 \pm e^{-z}) - \frac{\pi}{2} e^{-bz} \ln 2$$

[$b, \operatorname{Re} z > 0$].

$$8. \int_0^{\infty} \frac{\cos bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = \frac{\pi}{z} \left[\operatorname{ch} bz \ln(1 \pm e^{-z}) - \frac{1}{2} e^{-bz} \ln 2 \right]$$

[$b, \operatorname{Re} z > 0$].

$$9. \int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi \ln |a|$$

[$|b| \leq |a|$].

$$10. \int_0^{\pi} \frac{\{\sin(x/2)\}^2}{\{\cos(x/2)\}^2} \ln(1 - 2a \cos x + a^2) dx = \pm \frac{\pi a}{2}$$

[$|a| < 1$].

$$11. = \pi \ln |a| \pm \pi/(2a)$$

[$|a| > 1$].

$$12. \int_0^{\pi} \frac{\ln(1 - 2a \cos x + a^2) dx}{1 - 2b \cos x + b^2} = \frac{2\pi}{|1 - b^2|} \ln(1 - ab^{\pm 1})$$

[$\left\{ \begin{array}{l} |b| < 1 \\ |b| > 1 \end{array} \right\}, |a| \leq 1$].

$$13. = \frac{2\pi}{|1 - b^2|} \ln |a - b^{\pm 1}|$$

[$\left\{ \begin{array}{l} |b| < 1 \\ |b| > 1 \end{array} \right\}, |a| > 1$].

$$14. \int_0^{\pi n} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi n \ln |a|$$

[$|b| \leq |a|$].

$$15. \int_0^{\pi n} \cos mx \ln(1 - 2a \cos x + a^2) dx = -\frac{\pi n}{m} a^{\pm m}$$

[$\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; n=1, 2; m=1, 2, 3, \dots$].

$$16. \int_0^{\pi n} \frac{\{\sin x \sin mx\}}{\{\cos x \cos mx\}} \ln(1 - 2a \cos x + a^2) dx = \frac{\pi n}{2} \left(\pm \frac{b^{m+1}}{m+1} - \frac{b^{m-1}}{m-1} \right)$$

[$n=1, 2; b = \left\{ \begin{array}{l} a, |a| < 1 \\ a^{-1}, |a| > 1 \end{array} \right\}$].

$$17. \int_0^{2\pi} \cos[(m-1/2)x] \ln(1 - 2a \cos x + a^2) dx = 0$$

[$|a| \neq 1$].

$$18. \int_0^{2\pi} \frac{\{\sin(x/2) \sin nx\}}{\{\cos(x/2) \cos nx\}} \ln(1 - 2a \cos x + a^2) dx = 0$$

[$|a| \neq 1$].

$$19. \int_0^{2\pi} \frac{\{\sin(x/2) \sin[(n-1/2)x]\}}{\{\cos(x/2) \cos[(n-1/2)x]\}} \ln(1 - 2a \cos x + a^2) dx = \pi \left[\pm \frac{b^n}{n} - \frac{b^{n-1}}{n-1} \right]$$

[$b = \left\{ \begin{array}{l} a, |a| < 1 \\ a^{-1}, |a| > 1 \end{array} \right\}; n=2, 3, 4, \dots$].

$$20. \int_0^{\infty} \ln(1 - 2a \cos x + a^2) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln(1 - ae^{-z})$$

[$|a| \leq 1; \operatorname{Re} z > 0$].

$$21. \quad = \frac{\pi}{z} \ln(|a| - \operatorname{sgn} a e^{-z}) \quad [|a| \geq 1; \operatorname{Re} z > 0].$$

$$22. \quad \int_0^\infty \ln(1 - 2a \cos x + a^2) \frac{dx}{(x^2 + z^2)^2} = \frac{\pi}{2z^3} \ln(1 - a e^{-z}) - \frac{\pi a}{2z^2(e^z - a)} \quad [|a| \leq 1; \operatorname{Re} z > 0].$$

$$23. \quad = \frac{\pi}{2z^3} \ln(|a| - \operatorname{sgn} a e^{-z}) - \frac{\pi e^{-z}}{2z^2 a - e^{-z}} \quad [|a| \geq 1, \operatorname{Re} z > 0].$$

$$24. \quad \int_0^\infty \frac{\sin bx}{x} \ln(1 - 2a \cos x + a^2) dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{a^k}{k} [1 + \operatorname{sgn}(b-k)] \quad [b > 0; |a| < 1].$$

$$25. \quad = -\frac{\pi}{2} \left\{ \sum_{k=1}^{[b]} \frac{a^k}{k} [1 + \operatorname{sgn}(b-k)] - 2 \ln |a| \right\} \quad [b > 0, |a| > 1].$$

$$26. \quad \int_0^\infty \frac{\cos bx}{x^2 + z^2} \ln(1 - 2a \cos x + a^2) dx = \frac{\pi}{z} \left\{ \operatorname{ch} bz \ln(1 - a e^{-z}) + \sum_{k=1}^{[b]} \frac{a^k}{k} \operatorname{sh}[z(b-k)] \right\} \quad [|a| < 1; b, \operatorname{Re} z > 0].$$

$$27. \quad = \frac{\pi}{z} \left\{ \operatorname{ch} bz \ln(|a| - \operatorname{sgn} a e^{-z}) + \sum_{k=1}^{[b]} \frac{a^k}{k} \operatorname{sh}[z(b-k)] - \operatorname{sh} bz \ln |a| \right\}, \quad [|a| > 1, b, \operatorname{Re} z > 0].$$

$$28. \quad \int_0^{\pi/4} \ln(\cos x \pm \sin x) dx = -\frac{\pi}{8} \ln 2 \pm \frac{1}{2} G.$$

$$29. \quad \int_0^{2\pi} \ln(a + b \cos x + c \sin x) dx = 2\pi \ln \left[(a + \sqrt{a^2 - b^2 - c^2})/2 \right] \quad [a^2 > b^2 + c^2].$$

2.6.37. Интегралы, содержащие $\ln \frac{a_1 \cos x + b_1 \sin x + c_1}{a_2 \cos x + b_2 \sin x + c_2}$.

$$1. \quad \int_0^{\pi/2} \ln \frac{a + b \sin x}{a - b \sin x} \frac{dx}{\sin x} = \pi \operatorname{arcsin} \frac{b}{a} \quad [|b| \leq |a|].$$

$$2. \quad \int_0^{\pi/2} \ln \frac{a + b \cos x}{a - b \cos x} \frac{dx}{\cos x} = \pi \operatorname{arcsin} \frac{b}{a} \quad [|b| \leq |a|].$$

$$3. \quad \int_0^{\pi/2} \frac{\cos x}{1 - a^2 \cos^2 x} \ln \frac{1 + b \cos x}{1 - b \cos x} dx = \frac{\pi}{2a \sqrt{1 - a^2}} \ln \frac{1 + ab + \sqrt{(1 - a^2)(1 - b^2)}}{1 - ab + \sqrt{(1 - a^2)(1 - b^2)}} \quad [0 < b < a < 1].$$

4. $\int_0^\pi \ln \frac{1 + \sin x}{1 + a \sin x} \frac{dx}{\sin x} = \arccos^2 a$ [|a| < 1]
5. $\int_0^\pi \sin(2n+1)x \ln \frac{1 + 2a \cos x + a^2}{1 - 2a \cos x + a^2} dx = (-1)^n \frac{2\pi a^{-(2n+1)}}{2n+1}$ [{|a| < 1}, {|a| > 1}]
6. $\int_0^{2\pi} \cos mx \ln \frac{1 - 2a \cos x + a^2}{1 - 2a \cos nx + a^2} dx = \frac{2\pi}{m} (na^{\pm m/n} - a^{\pm m})$ [{|a| < 1}, {|a| > 1}]
7. $\int_0^{2\pi} \operatorname{ctg} \frac{x}{2} \ln \frac{1 + 2a \cos x + a^2}{1 + 2a \cos nx + a^2} dx = 0$ [ln a = 0]
8. $\int_0^\infty \ln \frac{1 + \sin x}{1 - \sin x} \frac{dx}{x} = \frac{\pi^2}{2}$ 9. $\int_0^\infty \ln \left| \frac{\cos x + \sin x}{\cos x - \sin x} \right| \frac{dx}{x} = \frac{\pi^2}{4}$
10. $\int_0^\infty \ln \frac{1 - 2a \cos bx + a^2}{1 - 2a \cos cx + a^2} \frac{dx}{x} = 2 \ln(1 - a^{\pm 1}) \ln \left| \frac{c}{b} \right|$ [{|a| > 1 или a = -1}, -1 ≤ a < 1]

2.6.38. Интегралы, содержащие $\ln(a \cos^2 x + b \sin^2 x + c)$.

Обозначение $k' = \sqrt{1 - k^2}$, $0 < k < 1$.

1. $\int_0^{\pi/2} \ln(1 + b \sin^2 x) dx = \pi \ln \frac{1 + \sqrt{1+b}}{2}$ [b ≥ -1]
2. $\int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^2 \ln(1 + b \sin^2 x) dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+b}}{2} \mp \frac{1}{2} \cdot \frac{1 - \sqrt{1+b}}{1 + \sqrt{1+b}} \right)$ [b ≥ -1]
3. $\int_0^{\pi/2} \frac{\ln(1 - a^2 \cos^2 x)}{1 - b^2 \cos^2 x} dx = -\frac{\pi}{\sqrt{1-b^2}} \ln \frac{1 + \sqrt{1-b^2}}{\sqrt{1-b^2} + \sqrt{1-a^2}}$ [|a|, |b| < 1]
4. $\int_0^{\pi/2} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \frac{|a| + |b|}{2}$ [ab ≠ 0]
5. $\int_0^{\pi/2} x^n \sin x \cos x \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-m/2} \ln \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right) dx = J_m^n$ •
- $J_{-1}^1 = \frac{1}{27k^2} \left[3\pi \left\{ \frac{k'^3(1-3 \ln k')}{-1} \right\} \pm (22k'^2 + 6k^4 - 3k'^2 \ln k') K(k) \mp (2 - k^2)(14 - 6 \ln k') E(k) \right],$

$$J_1^1 = \pm k^{-2} [(2 - k^2) K(k) - (4 - \ln k') E(k)] + \pi k^{-2} \left\{ \frac{k'(1 - \ln k')}{-1} \right\},$$

$$J_1^{-1} = \pm k^{-2} \left[\left(2 - k^2 - \left\{ \frac{k'^2}{1} \right\} \ln k' \right) K(k) - (2 - \ln k') E(k) \right],$$

$$J_3^1 = \mp k^{-2} (2 + \ln k') K(k) + \pi k^{-2} \left\{ \frac{(1 + \ln k')/k'}{-1} \right\}.$$

$$6. \int_0^{\pi/2} \ln \left(\frac{1 + \sin \varphi \cos^2 x}{1 - \sin \varphi \cos^2 x} \right) dx = \pi \ln \frac{1 + \sin(\varphi/2)}{\cos(\varphi/2)} \quad [|\varphi| < \pi/2].$$

$$7. \int_0^{\infty} \frac{\sin x}{x} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} \ln \left(1 \pm k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right) dx = \\ = \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} K(k) - \frac{\pi}{8} K(\sqrt{1 - k^2}).$$

$$8. \int_0^{\infty} \frac{\sin^n x \cos^l x}{x} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-m/2} \ln \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right) dx = I_m^{n, l},$$

$$I_{-1}^{1, -1} = I_{-1}^{1, 0} = (2 - k^2) K(k) - (2 - \ln k') E(k), \quad I_1^{1, 0} = I_1^{1, -1} = \ln k' K(k),$$

$$I_1^{1, 2} = \pm k^{-2} \left[\left(2 - k^2 - \left\{ \frac{k'^2}{1} \right\} \ln k' \right) K(k) - (2 - \ln k') E(k) \right],$$

$$I_1^{3, -1} = I_1^{3, 0} = \pm k^{-2} \left[\left(k^2 - 2 + \left\{ \frac{1}{k'^2} \right\} \ln k' \right) K(k) + (2 - \ln k') E(k) \right],$$

$$I_3^{1, 0} = I_3^{1, -1} = k'^{-2} [(k^2 - 2) K(k) + (2 + \ln k') E(k)],$$

$$I_3^{1, 1} = I_3^{1, 2} = \pm k^{-2} \left\{ \frac{1}{k'^{-2}} \right\} \left[\left(2 - k^2 + \left\{ \frac{1}{k'^2} \right\} \ln k' \right) K(k) - (2 + \ln k') E(k) \right],$$

$$I_3^{3, 0} = I_3^{3, -1} = \mp k^{-2} \left\{ \frac{k'^2}{1} \right\} \left[\left(2 - k^2 + \left\{ \frac{k'}{1} \right\} \ln k' \right) K(k) - (2 + \ln k') E(k) \right].$$

$$9. \int_0^{\infty} \ln(a^2 \cos^2 cx + b^2 \sin^2 cx) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} [\ln(a \operatorname{ch} cz + b \operatorname{sh} cz) - cz]$$

$$[a, b, c, \operatorname{Re} z > 0].$$

$$10. \int_0^{\infty} \frac{\operatorname{tg}(x/2) \ln(\sin^2 x + k' \cos^2 x)}{\sqrt{1 - k^2 \cos^2 x}} dx = \frac{1}{2} \ln \frac{2(k')^{3/2}}{1 + k'} K(k).$$

$$11. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\operatorname{tg} x}{\sin x} \right\} \frac{\ln(\sin^2 x + k' \cos^2 x)}{\sqrt{1 - k^2 \cos^2 x}} dx = \frac{1}{2} \ln \frac{2(\sqrt{k'})^3}{1 + k'} K(k).$$

$$12. \int_{\varphi}^{\pi/2} \frac{\ln(\sin x + \sqrt{\sin^2 x - \sin^2 \varphi})}{1 - a^2 \cos^2 x} dx =$$

$$= -\frac{1}{\sqrt{1 - a^2}} \left[\operatorname{arctg} \left(\frac{\operatorname{tg} \varphi}{\sqrt{1 - a^2}} \right) \ln \sin \varphi + \frac{\pi}{2} \ln \frac{1 + \sqrt{1 - a^2}}{\sqrt{1 - a^2} + \sqrt{1 - a^2 \cos^2 \varphi}} \right]$$

$$[a < 1; 0 < \varphi < \pi/2].$$

2.6.39. Интегралы, содержащие $\ln A(\operatorname{tg} x)$.

$$1. \int_0^{\pi/4} \ln \operatorname{tg} x \, dx = -G. \quad 2. \int_0^{\pi/4} \sin 2x \ln \operatorname{tg} x \, dx = -\frac{1}{2} \ln 2.$$

$$3. \int_0^{\pi/4} \frac{\ln \operatorname{tg} x}{\cos 2x} \, dx = -\frac{\pi^2}{8}.$$

$$4. \int_0^{\pi/4} \frac{\cos 2x}{1 \pm a \sin 2x} \ln \operatorname{tg} x \, dx = -\frac{\arcsin a}{4a} (\pi \mp \arcsin a) \quad [|a| < 1].$$

$$5. \int_0^{\pi/4} \frac{\cos 2x \ln \operatorname{tg} x}{1 \pm a \sin^2 2x} \, dx = -\frac{\pi}{4\sqrt{|a|}} \left\{ \operatorname{Arsh} \sqrt{|a|} \right\} \arcsin \sqrt{|a|} \quad [|a| < 1].$$

$$6. \int_0^{\pi/4} \ln (1 \pm \operatorname{tg} x) \, dx = \frac{\pi}{8} \ln 2 - \left\{ \begin{matrix} 0 \\ G \end{matrix} \right\}.$$

$$7. \int_0^{\pi/4} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8}.$$

$$8. \int_0^{\pi/4} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\operatorname{tg} 2x} = \pm \frac{\pi^2}{16}.$$

$$9. \int_0^{\pi/4} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2} - 1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|.$$

$$10. \int_0^{\pi/4} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1 - 2^{2n+1}}{2^{2n+2}n} (2n)! \zeta(2n+1)$$

[$n = 1, 2, 3, \dots$].

$$11. \int_0^{\pi/4} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \sin 2x)^n \frac{dx}{\operatorname{tg} 2x} = \frac{(-1)^n}{2} n! \zeta(n+2).$$

$$12. \int_0^{\pi/4} \frac{\sin 2x \ln \operatorname{tg} x}{1 - a^2 \sin^2 2x} \, dx = \frac{1}{2a\sqrt{1-a^2}} [L(\arcsin a) - \arcsin a \ln 2] \quad [|a| < 1].$$

$$13. \int_{\varphi}^{\pi/4} \frac{\sin^2 x \ln \operatorname{tg} x \, dx}{(\sin^2 \varphi + \operatorname{tg}^2 v \sin^2 2x) \sqrt{\sin^2 2x - \sin^2 \varphi}} =$$

$$= -\frac{\pi}{2} \frac{\cos^2 v}{\sin \varphi \sin v} \ln \frac{\sin v + \sqrt{1 - \cos^2 \varphi \cos^2 v}}{\sin \varphi (1 + \sin v)} \quad [0 < \varphi < \pi/2; \quad 0 < v < \pi/2].$$

$$14. \int_0^{\pi/2} \ln \operatorname{tg} x \, dx = 0.$$

$$15. \int_0^{\pi/2} \cos^{a-1} x \cos (a+1) x \ln \operatorname{tg} x dx = \frac{\pi}{2a} \quad [a > 0].$$

$$16. \int_0^{\pi/2} \cos^{a-1} x \operatorname{ctg} x \sin (a+1) x \ln \operatorname{tg} x dx = \frac{\pi}{2} [C + \psi (a+1)] \quad [a > -1].$$

$$17. \int_0^{\pi/2} \operatorname{tg} x \ln \operatorname{tg} (x/2) (\ln \cos x)^n dx = (-1)^n 2^{-n-3} n! \zeta (n+2, 1/2).$$

$$18. \int_0^{\pi/2} \frac{\ln \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} dx = -\ln \sqrt{1-k^2} K(k) \quad [0 < k < 1].$$

$$19. \int_0^{\pi/2} \ln (1 + \operatorname{tg} x) dx = \frac{\pi}{4} \ln 2 + G. \quad 20. \int_0^{\pi} x \ln |\operatorname{tg} x| dx = 0.$$

$$21. \int_0^{\infty} \frac{\ln |\operatorname{tg} x|}{x^2 + z^2} dx = \frac{\pi}{2z} \ln \operatorname{th} z \quad [\operatorname{Re} z > 0].$$

$$22. \int_0^{\infty} \ln \left| \frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x} \right| \frac{dx}{x} = \frac{\pi^2}{4}.$$

$$23. \int_0^{\infty} \ln \left(1 + b^2 \left\{ \frac{\operatorname{tg} ax}{\operatorname{ctg} ax} \right\}^2 \right) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln \left(1 + b \left\{ \frac{\operatorname{th} az}{\operatorname{cth} az} \right\} \right) \quad [a, b, \operatorname{Re} z > 0].$$

2.6.40. Интегралы, содержащие показательную, логарифмическую, гиперболыческие и тригонометрические функции.

$$1. \int_0^{\infty} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} \ln x dx = \\ = \frac{1}{b^2 + p^2} \left[\left\{ -\frac{p}{b} \right\} \operatorname{arctg} \frac{b}{p} - C \left\{ \frac{b}{p} \right\} - \frac{1}{2} \left\{ \frac{b}{p} \right\} \ln (b^2 + p^2) \right] \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} \ln x dx = \\ = \Gamma(\alpha) (p^2 + b^2)^{-\alpha/2} \left\{ \frac{\sin c}{\cos c} \right\} \left[\psi(\alpha) - \frac{1}{2} \ln (p^2 + b^2) \pm \operatorname{arctg} \frac{b}{p} \left\{ \frac{\operatorname{ctg} c}{\operatorname{tg} c} \right\} \right] \\ \left[c = \alpha \operatorname{arctg} \frac{b}{p}; \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} \alpha > \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \right].$$

$$3. \int_0^{\infty} e^{-px} \frac{\sin bx}{x} \ln x dx = -\operatorname{arctg} \frac{b}{p} \left[C + \frac{1}{2} \ln (p^2 + b^2) \right] \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$\begin{aligned}
 4. \int_0^{\infty} e^{-q x} \frac{\sin bx}{x} \ln x \, dx = \\
 = \frac{i}{2} \ln \left(\frac{b}{q} \right) [K_0(2e^{-\pi i/4} \sqrt{bq}) - K_0(2e^{\pi i/4} \sqrt{bq})] + \\
 + \frac{\pi}{4} [K_0(2e^{\pi i/4} \sqrt{bq}) + K_0(2e^{-\pi i/4} \sqrt{bq})] \quad [b, \operatorname{Re} q > 0].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_0^{\infty} \sin bx \ln(1 + e^{-ax}) \, dx = \\
 = \frac{1}{4b} \left[4 \ln 2 - \psi \left(1 + \frac{ib}{2a} \right) - \psi \left(1 - \frac{ib}{2a} \right) + \psi \left(\frac{a+ib}{2a} \right) + \psi \left(\frac{a-ib}{2a} \right) \right] \\
 [b, \operatorname{Re} a > 0]
 \end{aligned}$$

$$\begin{aligned}
 6. \int_0^{\infty} \sin bx \ln(1 - e^{-ax}) \, dx = -\frac{1}{2b} \left[2C + \psi \left(1 + \frac{ib}{a} \right) + \psi \left(1 - \frac{ib}{a} \right) \right] \\
 [b, \operatorname{Re} a > 0].
 \end{aligned}$$

$$7. \int_0^{\infty} \cos bx \ln(1 \pm e^{-ax}) \, dx = \frac{a}{2b^2} - \frac{\pi}{2b \operatorname{sh}(b\pi/a)} \left\{ \frac{1}{\operatorname{ch}(b\pi/a)} \right\} \quad [b, \operatorname{Re} a > 0].$$

$$\begin{aligned}
 8. \int_0^{\infty} \operatorname{sh} ax \sin bx \ln(1 - e^{-2ax}) \, dx = \frac{a}{a^2 + b^2} \left(\frac{2ab}{a^2 + b^2} - \frac{\pi}{2} \operatorname{th} \frac{b\pi}{2a} \right) \\
 [b, \operatorname{Re} a > 0].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_0^{\infty} \operatorname{ch} ax \cos bx \ln(1 - e^{-2ax}) \, dx = \frac{a(b^2 - a^2)}{(a^2 + b^2)^2} - \frac{\pi b}{2(a^2 + b^2)} \operatorname{th} \frac{b\pi}{2a} \\
 [b, \operatorname{Re} a > 0].
 \end{aligned}$$

$$10. \int_0^{\infty} \cos bx \ln \left(1 + \frac{\cos a}{\operatorname{ch} x} \right) dx = \frac{\pi}{b \operatorname{sh} b\pi} \left(\operatorname{ch} \frac{b\pi}{2} - \operatorname{ch} ab \right) \quad [b > 0; |\operatorname{Re} a| < \pi].$$

$$11. \int_0^{\infty} \cos bx \ln \frac{\operatorname{ch} x + \sin a}{\operatorname{ch} x - \sin a} \, dx = \frac{\pi \operatorname{sh} ab}{b \operatorname{ch}(b\pi/2)} \quad [b > 0; 2|\operatorname{Re} a| < \pi].$$

$$12. \int_0^{\infty} \cos bx \ln \frac{\operatorname{ch} x + \cos a}{\operatorname{ch} x + \cos c} \, dx = \frac{\pi}{b \operatorname{sh} ab} (\operatorname{ch} cb - \operatorname{ch} ab) \quad [b > 0; |\operatorname{Re} a|, |\operatorname{Re} c| < \pi].$$

$$13. \int_0^{\infty} e^{-px} \ln \left\{ \frac{2|\sin ax|}{2|\cos ax|} \right\} dx = -p \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(p^2 + 4a^2 k^2)} \quad [\operatorname{Re} p > 0].$$

$$14. \int_0^{\infty} e^{-px} \ln |\operatorname{ctg} ax| \, dx = 2p \sum_{k=1}^{\infty} \frac{1}{(2k-1)(p^2 + 4(2k-1)^2 a^2)} \quad [\operatorname{Re} p > 0].$$

$$15. \int_0^{\pi} e^{a \cos x} [\ln (2a \sin^2 x) + C] dx = -\pi K_0(a) \quad [a > 0]$$

$$16. \int_1^{\infty} \frac{e^{-px}}{\sqrt{x^2-1}} \sin(b \sqrt{x^2-1}) \ln(x + \sqrt{x^2-1}) dx = \operatorname{arctg} \left(\frac{b}{p} \right) K_0(\sqrt{p^2+b^2})$$

[Re $p > |\operatorname{Im} b|$].

$$17. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2+2ax}} \sin(b \sqrt{x^2+2ax}) \ln \frac{a+x+\sqrt{x^2+2ax}}{a} dx =$$

$$= e^{ap} \operatorname{arctg} \left(\frac{b}{p} \right) K_0(a \sqrt{b^2+p^2}) \quad [\operatorname{Re} p > |\operatorname{Im} b|].$$

$$18. \int_0^{\infty} x^{\alpha-1} e^{-x} \begin{cases} \sin(b \ln x) \\ \cos(b \ln x) \end{cases} dx = |\Gamma(\alpha + ib)| \begin{cases} \sin \arg \Gamma(\alpha + ib) \\ \cos \arg \Gamma(\alpha + ib) \end{cases}$$

[Re $\alpha > |\operatorname{Im} b|$; $\arg \Gamma(x) = -i \ln \frac{\Gamma(x)}{|\Gamma(x)|}$].

2.7. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

2.7.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входят обратные тригонометрические функции. Они могут быть преобразованы к интегралам от логарифмической функции и от степенных функций, если воспользоваться соответствующими формулами, приведенными в приложении I. В ряде случаев интегралы от обратных тригонометрических функций с помощью замены переменной сводятся к интегралам от тригонометрических функций. Например,

$$\int_0^1 \arcsin x dx = -i \int_0^1 \ln(ix + \sqrt{1-x^2}) dx = \int_0^{\pi/2} x \cos x dx.$$

2.7.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие обратные тригонометрические, а также степенную, алгебраические и тригонометрические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. В п. 2.7.2 приняты следующие обозначения: $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \sin bx, \\ 0, & \text{если взят } \cos bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2]+k}, & \text{если взят } \sin bx, \\ 1, & \text{если взят } \cos bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \sin^n bx, \quad n=1, 3, 5, \dots, \\ 0, & \text{если взят } \sin^n bx, \quad n=2, 4, 6, \dots, \\ 0, & \text{если взят } \cos^n bx, \quad n=1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{-1} \left(\frac{\alpha}{2} + \gamma + h \right), \quad \tilde{b} = \frac{by}{2} (n - 2k).$$

$$1. \int_0^y x^{\alpha-1} \begin{cases} \sin bx \\ \cos bx \end{cases}^n \arcsin \left(\frac{x}{y} \right)^r dx = I(\sigma) \Big|_{\sigma=1},$$

$$I(\sigma) = U + \frac{1 + (-1)^n}{2} 2^{n-1} \frac{n! y^\alpha \sqrt{\pi}}{\alpha [(n/2)!]^2} \left\{ (-1)^\sigma \Gamma \left[\frac{1/2 + \alpha/(2r)}{1 + \alpha/(2r)} \right] + \sigma \sqrt{\pi} \right\},$$

$$U = \frac{(-1)^\sigma 2^{-n-1} \pi}{r y^{-\alpha}} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \tilde{b}^{2\gamma+2h} \frac{1}{\xi \Gamma(2\gamma+1/2+h)} \left\{ \Gamma \left[\frac{1/2 + \xi}{1 + \xi} \right] \times \right.$$

$$\times {}_q F_{2p+q+1} \left(\begin{matrix} 1, \xi/q, \Delta(q, 1/2 + \xi); \\ 1 + \xi/q, \Delta(q, 1 + \xi), \Delta(p, 2\gamma+1/2+h), \Delta(p, 1+h) \end{matrix} \right) +$$

$$\left. + (-1)^\sigma \sigma \sqrt{\pi} {}_2 F_{2p+1} \left(\begin{matrix} 1, \xi/q; \\ \Delta(p, 2\gamma+1/2+h), \Delta(p, 1+h), 1 + \xi/q \end{matrix} \right) \right\},$$

$$z = (-1)^p \tilde{b}^{2p} p^{-2p} \quad [y, b > 0; \operatorname{Re} \alpha > -r\sigma - \delta n].$$

$$2. \int_0^y x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \arccos \left(\frac{x}{y} \right)^r dx = I(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2.7.2.1}].$$

$$3. \int_y^\infty x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \arcsin \left(\frac{y}{x} \right)^r dx = J(\sigma) \Big|_{\sigma=1}.$$

$$J(\sigma) = U - \frac{1 + (-1)^n}{2} 2^{n-1} \frac{n! y^\alpha \sqrt{\pi}}{\alpha [(n/2)!]^2} \left\{ (-1)^\sigma \Gamma \left[\frac{1/2 - \alpha/(2r)}{1 - \alpha/(2r)} \right] + \sigma \sqrt{\pi} \right\},$$

$$U = \frac{(-1)^{\sigma-1} 2^{n-1} \pi y^\alpha}{r} \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{\xi h!} \tilde{b}^{2\gamma+2h} \Gamma \left[\frac{1/2 - \xi}{1 - \xi, 1/2 + 2\gamma + h} \right] \times$$

$$\times {}_q F_{2p+q+1} \left(\begin{matrix} 1, \frac{\xi}{q}, \Delta(q, \xi); \\ 1 + \frac{\xi}{q}, \Delta(p, 1+h), \Delta\left(q, \frac{1}{2} + \xi\right), \Delta\left(p, \frac{1}{2} + 2\gamma + h\right) \end{matrix} \right) -$$

$$- \frac{(-1)^\sigma \sqrt{\pi}}{2^n} y^\alpha \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(1/2)_j}{j! (1+2j)} \times$$

$$\times \tilde{b}^{r(1+2j)-\alpha} \Gamma \left[\frac{\gamma + (\alpha-r)/2 - jr}{\gamma + (1+r-\alpha)/2 + jr} \right] \times$$

$$\times {}_q F_{2p+q+1} \left(\begin{matrix} \frac{1+2j}{2j}, 1, \Delta\left(q, \frac{1}{2} + j\right); \\ 1 + \frac{1+2j}{2q}, \Delta(q, 1+j), \Delta\left(p, \gamma + \frac{1+r-\alpha}{2} + jr\right), \Delta(p, \beta) \end{matrix} \right) +$$

$$+ (1-\sigma) \pi^{3/2} 2^{-n-1} y^\alpha \Gamma \left[\frac{\gamma + \alpha}{1/2 + \gamma - \alpha} \right] \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \tilde{b}^{-\alpha} -$$

$$- \sigma \pi^{3/2} 2^{-n-1} \frac{y^\alpha}{r} \sum_{h=0}^{p-1} \frac{(-1)^h}{\xi h!} \tilde{b}^{2\gamma+2h} \frac{1}{\Gamma(1/2 + 2\gamma + h)} \times$$

$$\times {}_2 F_{2p+1} \left(\begin{matrix} 1, \xi/q; \\ 1 + \xi/q, \Delta(p, 1/2 + 2\gamma + h), \Delta(p, 1+h) \end{matrix} \right),$$

$$z = (-1)^p \tilde{b}^{2p} p^{-2p} \quad [\beta = 1 - \gamma + jr + (r - \alpha)/2; y, b > 0; \operatorname{Re} \alpha < 0].$$

$$4. \int_y^\infty x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \arccos \left(\frac{y}{x} \right)^r dx = J(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2.7.2.3}].$$

$$5. \int_0^y x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \sin \left[v \arccos \left(\frac{x}{y} \right)^r \right] dx = K(\sigma) \Big|_{\sigma=1},$$

$$K(\sigma) = V_{35} + \frac{1 + (-1)^n}{2} 2^{-n-\alpha/r} \frac{\pi n!}{r [(n/2)!]^2} \left(\frac{v}{2} \right)^\sigma y^{\alpha+r(\sigma-1)} \times \\ \times \Gamma \left[\frac{\alpha/r}{\alpha/(2r) + (v+1+\sigma)/2, \alpha/(2r) + (1+\sigma-v)/2} \right]$$

$$\left\{ E_{35} = \left(\frac{v}{2} \right)^\sigma \frac{\pi^{3/2}}{2^{n-1} r} 2^{-2\xi} \Gamma \left[\frac{2\xi}{\xi + (1+\sigma+v)/2, \xi + (1+\sigma-v)/2, 2\gamma + 1/2 + h} \right], \right. \\ u = \alpha + r(\sigma-1), \quad a_1 = \xi, \quad b_1 = \xi + 1/2, \quad c_1 = \xi + (1+\sigma+v)/2, \\ d_1 = \xi + (1+\sigma-v)/2, \quad e_1 = 2\gamma + 1/2 + h, \quad z = (-\tilde{b}^2/p^2)p \Big\} \\ [y, b > 0; \operatorname{Re} \alpha > -\delta n].$$

$$6. \int_0^y x^{\alpha-1} (y^{2r} - x^{2r})^{-1/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \cos \left[v \arccos \left(\frac{x}{y} \right)^r \right] dx = K(\sigma) \Big|_{\sigma=0}$$

[см. 2.7.2.5].

$$7. \int_y^\infty x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \sin \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = L(\sigma) \Big|_{\sigma=1},$$

$$L(\sigma) = V_{35} + V_{36} + V_{37} + \frac{1 + (-1)^n}{2} 2^{-n+\alpha/r-1} \frac{\pi n! v^\sigma}{r [(n/2)!]^2} \times \\ \times y^{\alpha+r(\sigma-1)} \Gamma \left[\frac{1-\sigma-\alpha/r}{1+v/2-\alpha/(2r), 1-v/2-\alpha/(2r)} \right]$$

$$\left\{ E_{35} = \pi^{3/2} v^\sigma \frac{2^{2\xi-n}}{r} \Gamma \left[\frac{1-\sigma-2\xi}{1+v/2-\xi, 1-v/2-\xi, 2\gamma + 1/2 + h} \right], \right.$$

$$E_{36} = \pi^{3/2} v^\sigma 2^{-n-\sigma} \frac{(-1)^j}{(1/2)_j} \Gamma \left[\frac{1-d_2}{1-a_2, 1-b_2, e_2} \right],$$

$$E_{37} = -\pi^{3/2} v^\sigma 2^{1-n-\sigma} \frac{(-1)^j}{(3/2)_j} \Gamma \left[\frac{1-d_3}{1-a_3, 1-b_3, e_3} \right],$$

$$u = \alpha + r(\sigma-1), \quad v_2 = r(1-\sigma+2j) - \alpha, \quad v_3 = r(2-\sigma+2j) - \alpha,$$

$$a_1 = \xi - v/2, \quad b_1 = \xi + v/2, \quad c_1 = (1+\sigma)/2 + \xi, \quad d_1 = \sigma/2 + \xi,$$

$$e_1 = 2\gamma + 1/2 + h, \quad a_2 = (1-v-\sigma)/2 + j, \quad b_2 = (1+v-\sigma)/2 + j,$$

$$c_2 = 1/2 + j, \quad d_2 = 1 - \gamma - \alpha/2 + r[(1-\sigma)/2 + j],$$

$$e_2 = \gamma + (1-\alpha)/2 + r[(1-\sigma)/2 + j], \quad a_3 = 1 - (\sigma+v)/2 + j,$$

$$b_3 = 1 - (\sigma-v)/2 + j, \quad c_3 = 3/2 + j,$$

$$d_3 = 1 - \gamma - \alpha/2 + r(1-\sigma/2 + j), \quad e_3 = \gamma + (1-\alpha)/2 + r(1-\sigma/2 + j),$$

$$z = (-\tilde{b}^2/p^2)p \Big\} \quad [y, b > 0; \operatorname{Re} \alpha < r(1-\sigma)].$$

$$8. \int_y^\infty x^{\alpha-1} (x^{2r} - y^{2r})^{-1/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \cos \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = L(\sigma) \Big|_{\sigma=0}$$

[см. 2.7.2.7].

$$9. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \sin \left[v \operatorname{arctg} \left(\frac{x}{y} \right)^r \right] dx = M(\sigma) |_{\sigma=1},$$

$$M(\sigma) = V_{35} + V_{36} + V_{37} + \\ + \frac{1 + (-1)^n}{2} \frac{(-1)^{\sigma} \sqrt{\pi} 2^{\alpha/r-n} n!}{\Gamma(-v) [(n/2)!]^2} y^{\alpha+rv} \Gamma \left[\frac{\sigma/2 + \alpha/(2r), -v - \alpha/r}{(1+\sigma)/2 - \alpha/(2r)} \right]$$

$$\left\{ E_{35} = (-1)^{\sigma} \frac{2^{2\xi-n\pi}}{r\Gamma(-v)} \Gamma \left[\frac{-v-2\xi, \sigma/2+\xi}{(1+\sigma)/2-\xi, 2\gamma+1/2+h} \right], \right.$$

$$E_{36} = (-1)^{\sigma+j} \frac{2^{v-n-1\pi}}{\Gamma(-v) (1/2)_j} \Gamma \left[\frac{a_2, 1-d_2}{1-b_2, e_2} \right],$$

$$E_{37} = (-1)^{\sigma+j-1} \frac{2^{v-n\pi}}{\Gamma(-v) (3/2)_j} \Gamma \left[\frac{a_3, 1-d_3}{1-b_3, e_3} \right],$$

$$u = \alpha + rv, \quad v_2 = r(2j-v) - \alpha, \quad v_3 = r(2j+1-v) - \alpha,$$

$$a_1 = \sigma/2 + \xi, \quad b_1 = (1-\sigma)/2 + \xi, \quad c_1 = 1 + v/2 + \xi,$$

$$d_1 = (1+v)/2 + \xi, \quad e_1 = 2\gamma + 1/2 + h, \quad a_2 = (\sigma-v)/2 + j,$$

$$b_2 = (1-\sigma-v)/2 + j, \quad c_2 = 1/2 + j, \quad d_2 = 1 - \gamma - \alpha/2 + r(j-v/2),$$

$$e_2 = (1-\alpha)/2 + \gamma + r(j-v/2), \quad a_3 = (1-v+\sigma)/2 + j,$$

$$b_3 = 1 - (\sigma+v)/2 + j, \quad c_3 = 3/2 + j, \quad d_3 = 1 - \gamma - \alpha/2 + r[j + (1-v)/2],$$

$$e_3 = (1-\alpha)/2 + \gamma + r[j + (1-v)/2], \quad z = (-1)^{p+q} \tilde{b}^{2p} p^{-2p}$$

$$\left[b, y > 0, -n\delta - r\sigma < \operatorname{Re} \alpha < \frac{1 - (-1)^n}{2} - r \operatorname{Re} v \right].$$

$$10. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \cos \left[v \operatorname{arctg} \left(\frac{x}{y} \right)^r \right] dx = M(\sigma) |_{\sigma=0}$$

[см. 2.7 2.9].

$$11. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \sin \left[v \operatorname{arctg} \left(\frac{y}{x} \right)^r \right] dx = N(\sigma) |_{\sigma=1},$$

$$N(\sigma) = V_{35} + V_{36} + \\ + \frac{1 + (-1)^n}{2} \frac{(-1)^{\sigma} \sqrt{\pi} 2^{-v-n-\alpha/r} n!}{\Gamma(-v) [(n/2)!]^2} y^{\alpha+vr} \Gamma \left[\frac{\alpha/r, (\sigma-v)/2 - \alpha/(2r)}{(1+\sigma+v)/2 + \alpha/(2r)} \right]$$

$$\left\{ E_{35} = \frac{(-1)^{\sigma} \pi}{r\Gamma(-v)} 2^{-n-v-2\xi} \Gamma \left[\frac{2\xi, (\sigma-v)/2 - \xi}{(1+\alpha+v)/2 + \xi, 2\gamma+1/2+h} \right], \right.$$

$$E_{36} = \frac{(-1)^{\sigma} \pi}{\Gamma(-v)} 2^{-n-2j-\sigma} \Gamma \left[\frac{\sigma-v+2j, 1-d_2}{c_2, e_2} \right],$$

$$u = \alpha + vr, \quad v_2 = r(2j+\sigma-v) - \alpha, \quad a_1 = \xi, \quad b_1 = 1/2 + \xi,$$

$$c_1 = (1+\sigma+v)/2 + \xi, \quad d_1 = 1 + (v-\sigma)/2 + \xi, \quad e_1 = 1/2 + 2\gamma + h,$$

$$a_2 = (\sigma-v)/2 + j, \quad b_2 = (1+\sigma-v)/2 + j, \quad c_2 = 1/2 + \sigma + j,$$

$$d_2 = 1 - \gamma - \alpha + r[j + (\sigma-v)/2], \quad e_2 = 1/2 + \gamma - \alpha/2 + r[j + (\sigma-v)/2],$$

$$z = (-1)^{p+q} \tilde{b}^{2p} p^{-2p} \quad [b, y > 0; -n\delta < \operatorname{Re} \alpha < (1 - (-1)^n)/2 + r\sigma - r \operatorname{Re} v].$$

$$12. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \cos \left[v \operatorname{arctg} \left(\frac{y}{x} \right)^r \right] dx = N(\sigma) |_{\sigma=0}$$

[см. 2.7.2.11].

2.7.3. Интегралы, содержащие $\begin{Bmatrix} \arcsin bx \\ \arccos bx \end{Bmatrix}$.

$$1. \int_0^1 \arcsin(e^{i\varphi} x) dx = \operatorname{sgn}(\cos \varphi) \arcsin(\sqrt{1 - \sin \varphi}) - \\ - \cos \varphi + \sqrt{2 \sin \varphi} \cos[(2\varphi + \pi)/4] + i \{ \ln(\sqrt{1 + \sin \varphi} + \sqrt{\sin \varphi}) + \\ + \sin \varphi - \sqrt{2 \sin \varphi} \sin[(2\varphi + \pi)/4] \} \quad [0 \leq \varphi \leq \pi].$$

$$2. \int_0^1 x^{\alpha-1} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \frac{\pi}{2\alpha} \left[\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mp \frac{1}{\sqrt{\pi}} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \right] \quad \left[\operatorname{Re} \alpha > -\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$3. \int_0^1 \frac{x}{\sqrt{(1-x^2)(1-k^2x^2)}} \begin{Bmatrix} \arcsin kx \\ \arccos kx \end{Bmatrix} dx = \mp \frac{\pi}{2k} \begin{Bmatrix} \ln \sqrt{1-k^2} \\ \ln(1+k) \end{Bmatrix} \quad [0 < k < 1].$$

$$4. \int_0^1 e^{-px} \arcsin x dx = \frac{\pi}{2p} [I_0(p) - L_0(p) - e^{-p}] \quad [|\arg p| < \pi].$$

$$5. \int_0^1 x e^{-px} \arcsin x dx = \frac{\pi}{2p^2} [L_0(p) - I_0(p) + pL_1(p) - pI_1(p)] + \frac{1}{p} \quad [|\arg p| < \pi].$$

$$6. \int_0^1 \sin(n\pi x) \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \mp \frac{1}{2n} [(-1)^n - J_0(\pi n)] \quad [n = 1, 2, 3, \dots].$$

$$7. \int_0^1 \ln x \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \pm (2 - \ln 2) - \begin{Bmatrix} \pi/2 \\ 0 \end{Bmatrix}.$$

$$8. \int_0^1 \frac{\arccos x}{\ln x} dx = - \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1}.$$

$$9. \int_{-1}^1 e^{ax} \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = \pm \frac{\pi}{a} \left[\begin{Bmatrix} \operatorname{ch} a \\ e^{-a} \end{Bmatrix} - I_0(a) \right] \quad [a \neq 0].$$

$$10. \int_{-1}^1 \operatorname{ch} ax \arccos x dx = \frac{\pi}{a} \operatorname{sh} a \quad [a \neq 0].$$

$$11. \int_{-1}^1 \begin{Bmatrix} \sin(n\pi x) \\ \cos(n\pi x) \end{Bmatrix} \arccos x dx = \frac{1}{n} \begin{Bmatrix} (-1)^n - J_0(\pi n) \\ 0 \end{Bmatrix} \quad [n = 1, 2, 3, \dots].$$

2.7.4. Интегралы от $A(x)$, $\left\{ \begin{array}{l} \operatorname{arctg} x \\ \operatorname{arccctg} x \end{array} \right\}^n$.

$$1. \int_0^1 \left\{ \begin{array}{l} \operatorname{arctg}(e^{i\varphi}x) \\ \operatorname{arccctg}(e^{i\varphi}x) \end{array} \right\} dx = \frac{\pi}{4} \mp \frac{\cos \varphi}{2} \ln(2 \cos \varphi) \mp \frac{\varphi}{2} \sin \varphi \pm \\ \pm \frac{i}{2} \left[\frac{1}{2} \ln \frac{1 + \sin \varphi}{1 - \sin \varphi} + \sin \varphi \ln(2 \cos \varphi) - \varphi \cos \varphi \right] \quad [|\varphi| < \pi/2].$$

$$2. \int_0^1 \operatorname{arctg}^2 x \, dx = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 - G.$$

$$3. \int_0^a x^{\alpha-1} \left\{ \begin{array}{l} \operatorname{arctg}(x/a) \\ \operatorname{arccctg}(x/a) \end{array} \right\} dx = \frac{a^\alpha}{4\alpha} \left[\pi \mp 2\beta \left(\frac{\alpha+1}{2} \right) \right] \quad [a > 0; \operatorname{Re} \alpha > -\{0\}].$$

$$4. \int_0^a \frac{1}{x} \operatorname{arctg}(x/a) \, dx = G.$$

$$5. \int_0^a x^n \left\{ \begin{array}{l} \operatorname{arctg}(x/a) \\ \operatorname{arccctg}(x/a) \end{array} \right\} dx = \\ = \frac{a^{n+1}}{n+1} \left\{ \frac{\pi}{4} \mp (-1)^{[(n+1)/2]} \left[\frac{\pi(1 - (-1)^n)}{8} + \frac{1 \mp (-1)^n}{4} \ln 2 \right] \mp \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{n-2k} \right\} \\ [a > 0].$$

$$6. \int_0^\infty \operatorname{arccctg}^n x \, dx = I_n \quad [n = 2, 3, 4, \dots]$$

$$I_n = n \left(\frac{\pi}{2} \right)^{n-1} \left[\frac{1}{n-1} + \sum_{k=1}^\infty \frac{(-1)^k \pi^{2k} B_{2k}}{(n+2k-1)(2k)!} \right], \quad I_2 = \pi \ln 2.$$

$$7. \int_0^\infty x^{\alpha-1} \left\{ \begin{array}{l} \operatorname{arctg} x \\ \operatorname{arccctg} x \end{array} \right\} dx = \mp \frac{\pi}{2\alpha \cos(\alpha\pi/2)} \quad [0 < \mp \operatorname{Re} \alpha < 1].$$

$$8. \int_0^a \frac{\operatorname{arctg} bx}{x \sqrt{a^2 - x^2}} dx = \frac{\pi}{2a} \ln(ab + \sqrt{1 + a^2 b^2}) \quad [a, \operatorname{Re} b > 0].$$

$$9. \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \left\{ \begin{array}{l} \operatorname{arctg} bx \\ \operatorname{arccctg} bx \end{array} \right\} dx = \frac{\pi}{4b} (ab \pm 2\sqrt{1 + a^2 b^2} \mp ab \mp 2) \quad [a, \operatorname{Re} b > 0].$$

$$10. \int_0^a \frac{x \operatorname{arctg} bx}{\sqrt{a^2 - x^2} (x^2 - y^2)} dx = \frac{\pi}{2\sqrt{y^2 - a^2}} \left[\operatorname{arccctg} by - \operatorname{arctg} \sqrt{\frac{1 + a^2 b^2}{b^2 (y^2 - a^2)}} \right] \\ [0 < a < y; \operatorname{Re} b > 0].$$

$$11. \int_0^a \frac{x \operatorname{arctg} x}{\sqrt{c^2 - x^2} (x^2 - y^2)} dx = I(a, c, y) \quad [0 < a \leq c < y].$$

$$I\left(\frac{1}{\sqrt{3}}, 1, \sqrt{3}\right) = -\frac{\sqrt{2}}{576} \pi^2, \quad I\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 1\right) = -\frac{\sqrt{2}}{96} \pi^2,$$

$$I\left(\sqrt{\frac{5}{3}}, \sqrt{2}, \sqrt{3}\right) = -\frac{\pi^2}{30}, \quad I\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right) = \sqrt{10} \left(\frac{2\pi}{15} \operatorname{arctg} \sqrt{\frac{5}{3}} - \frac{\pi^2}{25} \right),$$

$$I\left(\frac{1}{\sqrt{3}}, \sqrt{2}, \sqrt{3}\right) = \frac{\pi}{6} \operatorname{arctg} \sqrt{\frac{5}{3}} - \frac{\pi^2}{20}, \quad I(\sqrt{2}, \sqrt{2}, \sqrt{3}) = -\frac{\pi^2}{12},$$

$$I(1, \sqrt{2}, \sqrt{3}) = -\frac{\pi^2}{96}.$$

$$12. \int_0^\infty \frac{\operatorname{arctg}^n bx}{x^2 + z^2} dx = I_n \quad [b, \operatorname{Re} z > 0],$$

$$I_1 = \frac{1}{z} \left[\frac{\pi^2}{6} - \frac{1}{4} \ln^2(1 + bz) - \frac{1}{2} \operatorname{Li}_2\left(\frac{1}{1 + bz}\right) - \frac{1}{2} \operatorname{Li}_2(1 - bz) \right],$$

$$I_2 = \frac{\pi}{4z} \left[\frac{\pi^2}{6} - \ln^2 \frac{1 + bz}{2} - 2 \operatorname{Li}_2\left(\frac{1 - bz}{2}\right) \right].$$

$$13. \int_0^\infty \frac{\operatorname{arctg} bx}{\sqrt{x} (x + z)} dx = \frac{\pi}{\sqrt{z}} [\operatorname{arctg}(\sqrt{2bz} + 1) + \operatorname{arctg}(\sqrt{2bz} - 1) - \operatorname{arctg} bz]$$

$$[b > 0; |\arg z| < \pi].$$

$$14. \int_0^\infty \frac{\operatorname{arctg} bx}{x (x^2 + z^2)} dx = \frac{\pi}{2z^2} \ln(1 + bz) \quad [b, \operatorname{Re} z > 0].$$

$$15. \int_0^\infty \frac{\operatorname{arctg} ax - \operatorname{arctg} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b} \quad [a, b > 0].$$

$$16. \int_0^\infty \operatorname{arctg} ax \operatorname{arctg} bx dx = \frac{\pi}{2} \ln \frac{(a + b)^{1/a + 1/b}}{a^{1/b} b^{1/a}} \quad [a, b > 0].$$

$$17. \int_{-\infty}^\infty \frac{\operatorname{arctg} ax - \operatorname{arctg} bx}{x} \operatorname{arctg} cx dx = \frac{1}{2} \pi^2 \ln \frac{a}{b} \quad [a, b, c > 0].$$

$$18. \int_0^\infty \frac{\arcsin ax \operatorname{arctg} bx}{x^2} dx = \frac{b\pi}{2} \ln \frac{a + \sqrt{a^2 + b^2}}{a \sqrt{a^2 + b^2}} +$$

$$+ \frac{\pi a}{2} \ln \frac{b + \sqrt{a^2 + b^2}}{a} - \frac{\pi a}{2} \operatorname{arctg} \frac{b}{a} \quad [a, b > 0].$$

$$19. \int_0^\infty \frac{\operatorname{arctg} ax \operatorname{arctg} bx}{x^2} dx = \frac{\pi}{2} \ln \frac{(a + b)^{a+b}}{a^a b^b} \quad [a, b > 0].$$

$$20. \int_0^{\infty} \frac{1}{x^2} \left(1 - \frac{1}{x} \operatorname{arctg} x\right) dx = \frac{\pi}{4}.$$

2.7.5. Интегралы от $A(x)$, e^{-px} , $\begin{Bmatrix} \operatorname{arctg} bx \\ \operatorname{arctg} bx \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-px} \begin{Bmatrix} \operatorname{arctg} bx \\ \operatorname{arctg} bx \end{Bmatrix} dx = \frac{\pi}{2p} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mp \frac{1}{p} \left[\operatorname{ci} \left(\frac{p}{b} \right) \sin \frac{p}{b} - \operatorname{si} \left(\frac{p}{b} \right) \cos \frac{p}{b} \right]$$

[$b, \operatorname{Re} p > 0$].

$$\begin{aligned} 2. \int_0^{\infty} x^{\alpha-1} e^{-px} \begin{Bmatrix} \operatorname{arctg} bx \\ \operatorname{arctg} bx \end{Bmatrix} dx = \\ = \frac{\pi}{2p^{\alpha}} \Gamma(\alpha) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mp \frac{p^{1-\alpha}}{b} \Gamma(\alpha-1) {}_2F_3 \left(\frac{1}{2}, 1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}, \frac{3}{2}; -\frac{p^2}{4b^2} \right) \mp \\ \mp \frac{p\pi b^{-\alpha-1}}{2(\alpha+1) \sin(\alpha\pi/2)} {}_1F_2 \left(\frac{1+\alpha}{2}; \frac{3}{2}, \frac{3+\alpha}{2}; -\frac{p^2}{4b^2} \right) \mp \\ \mp \frac{\pi b^{-\alpha}}{2\alpha \cos(\alpha\pi/2)} {}_1F_2 \left(\frac{\alpha}{2}; \frac{1}{2}, 1+\frac{\alpha}{2}; -\frac{p^2}{4b^2} \right) \end{aligned} \quad [b, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -1].$$

$$\begin{aligned} 3. \int_0^{\infty} x e^{-px} \begin{Bmatrix} \operatorname{arctg} bx \\ \operatorname{arctg} bx \end{Bmatrix} dx = \frac{1}{p^2} \left[\begin{Bmatrix} 0 \\ \pi/2 \end{Bmatrix} \mp \operatorname{ci} \left(\frac{p}{b} \right) \sin \frac{p}{b} \mp \operatorname{si} \left(\frac{p}{b} \right) \cos \frac{p}{b} \right] \mp \\ \mp \frac{1}{pb} \left[\operatorname{si} \left(\frac{p}{b} \right) \sin \frac{p}{b} - \operatorname{ci} \left(\frac{p}{b} \right) \cos \frac{p}{b} \right] \end{aligned} \quad [b, \operatorname{Re} p > 0].$$

$$\begin{aligned} 4. \int_0^{\infty} x^{\alpha-1} e^{-px^2} \operatorname{arctg} bx dx = \\ = \frac{\pi p^{-\alpha/2}}{4} \Gamma \left(\frac{\alpha}{2} \right) - \frac{\pi b^{-\alpha}}{2\alpha \cos(\alpha\pi/2)} {}_1F_1 \left(\frac{\alpha}{2}; 1+\frac{\alpha}{2}; \frac{p}{b^2} \right) - \\ - \frac{p^{(1-\alpha)/2}}{2b} \Gamma \left(\frac{\alpha+1}{2} \right) {}_2F_2 \left(\frac{1}{2}, 1; \frac{3}{2}, \frac{3-\alpha}{2}; \frac{p}{b^2} \right) \end{aligned} \quad [a, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -1].$$

$$5. \int_0^{\infty} \frac{1}{(x^2+1)^2} \begin{Bmatrix} x^2 \\ x^3-x \end{Bmatrix} e^{-x^2} \operatorname{arctg} x dx = \frac{\sqrt{\pi}}{8} \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}.$$

$$6. \int_0^{\infty} \frac{\operatorname{arctg} bx}{e^{cx}-1} dx = \frac{\pi}{c} \left[\ln \Gamma \left(\frac{c}{2\pi b} \right) - \frac{c-\pi b}{2\pi b} \ln \frac{c}{2\pi b} + \frac{c}{2\pi b} - \frac{1}{2} \ln(2\pi) \right]$$

[$b, \operatorname{Re} c > 0$].

$$7. \int_0^{\infty} \frac{\operatorname{arctg} x^3}{e^{2\pi x}-1} dx = \frac{1}{4} \ln(2\pi) - \frac{\pi}{4\sqrt[3]{3}} - \frac{1}{2} \ln(1+e^{-\pi\sqrt[3]{3}}).$$

$$8. \int_0^{\infty} \frac{\operatorname{arctg} x^3}{e^{4\pi x}-1} dx = \frac{1}{8} \ln(12\pi) - \frac{\pi}{4\sqrt[3]{3}} + \frac{1}{4} \ln(1-e^{-2\pi\sqrt[3]{3}}).$$

$$9. \int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arctg} \frac{1}{x} - e^{-bx} \right) \frac{dx}{x} = C + \ln b \quad [b > 0].$$

$$10. \int_0^{\infty} \frac{\operatorname{arctg} bx}{\operatorname{sh} cx} dx = \frac{\pi}{c} \left[\ln \Gamma \left(\frac{c}{2\pi b} \right) - \ln \Gamma \left(\frac{c + \pi b}{2\pi b} \right) - \ln \left(2 \sqrt{\frac{b\pi}{c}} \right) \right] \\ [b, \operatorname{Re} c > 0].$$

2.7.6. Интегралы от $A(x)$, $\left\{ \begin{smallmatrix} \sin ax \\ \cos ax \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} \operatorname{arctg} bx^{\pm 1} \\ \operatorname{arctg} bx^{\pm 1} \end{smallmatrix} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} x^{\alpha-1} \left\{ \begin{smallmatrix} \sin bx \\ \cos bx \end{smallmatrix} \right\} \operatorname{arctg} ax dx = \\ = + \frac{\pi b^{\delta} a^{-\alpha-\delta}}{2(\alpha+\delta)} \left\{ \begin{smallmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{smallmatrix} \right\}^{-1} {}_1F_2 \left(\frac{\alpha+\delta}{2}; 1 + \frac{\alpha+\delta}{2}, \delta + \frac{1}{2}; \frac{b^2}{4a^2} \right) \mp \\ \mp \frac{b^{1-\alpha}}{a} \left\{ \begin{smallmatrix} \cos(\alpha\pi/2) \\ \sin(\alpha\pi/2) \end{smallmatrix} \right\} \Gamma(\alpha-1) {}_2F_3 \left(\frac{1}{2}, 1, \frac{3}{2}, \frac{3-\alpha}{2}, 1 - \frac{\alpha}{2}; \frac{b^2}{4a^2} \right) + \\ + \frac{b^{\alpha\pi}}{2} \Gamma(\alpha) \left\{ \begin{smallmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{smallmatrix} \right\} \quad [-\delta < \operatorname{Re} \alpha < 1, a, b > 0].$$

$$2. \int_0^{\infty} \frac{\cos bx}{x} \operatorname{arctg} ax dx = -\frac{\pi}{2} E_1 \left(-\frac{b}{a} \right) \quad [a, b > 0].$$

$$3. \int_0^{\infty} \sin bx \operatorname{arctg} ax dx = \frac{\pi}{2b} (1 - e^{-b/a}) \quad [a, b > 0].$$

$$4. \int_0^{\infty} \cos bx \operatorname{arctg} ax dx = \frac{1}{2b} \left[e^{-b/a} E_1 \left(\frac{b}{a} \right) - e^{b/a} E_1 \left(-\frac{b}{a} \right) \right] \quad [a, b > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left\{ \begin{smallmatrix} \sin bx \\ \cos bx \end{smallmatrix} \right\} \operatorname{arctg} \frac{a}{x} dx = \\ = \mp \frac{b^{\delta} a^{1+\delta}}{2(\alpha+\delta)} \left\{ \begin{smallmatrix} \sin(\alpha\pi/2) \\ \cos(\alpha\pi/2) \end{smallmatrix} \right\}^{-1} {}_1F_2 \left(\frac{\alpha+\delta}{2}; \delta + \frac{1}{2}, 1 + \frac{\alpha+\delta}{2}; \frac{a^2 b^2}{4} \right) \mp \\ \mp ab^{1-\alpha} \Gamma(\alpha-1) \left\{ \begin{smallmatrix} \cos(\alpha\pi/2) \\ \sin(\alpha\pi/2) \end{smallmatrix} \right\} {}_2F_3 \left(1, \frac{1}{2}; \frac{3}{2}, 1 - \frac{\alpha}{2}, \frac{3-\alpha}{2}; \frac{a^2 b^2}{4} \right) \\ [-\delta < \operatorname{Re} \alpha < 2; a, b > 0].$$

$$6. \int_0^{\infty} \cos bx \operatorname{arctg} \frac{a}{x^2} dx = \frac{\pi}{b} e^{-b\sqrt{a/2}} \sin \left(b \sqrt{\frac{a}{2}} \right) \quad [a, b > 0].$$

$$7. \int_0^{\infty} \frac{\sin cx}{\cos cx + d} \operatorname{arctg} \frac{a}{x} dx = \frac{\pi}{c} \ln \frac{d+1+\sqrt{d^2-1}}{e^{-ac}+d+\sqrt{d^2-1}} \quad [a, c > 0; d > 1].$$

$$8. \int_0^{\infty} \frac{\operatorname{tg} x}{b^2 \sin^2 x + c^2 \cos^2 x} \operatorname{arctg} \frac{a}{x} dx = \frac{\pi}{2b^2} \ln \left(1 + \frac{b}{c} \operatorname{th} a \right) \quad [a, b, c > 0].$$

$$9. \int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arctg} \frac{1}{x} - \cos bx \right) \frac{dx}{x} = C + \ln b \quad [b > 0].$$

2.7.7. Интегралы, содержащие $\operatorname{arctg} x$ и логарифмическую функцию.

$$1. \int_0^1 \ln x \operatorname{arctg} x dx = \frac{\ln 2}{2} + \frac{\pi^2}{48} - \frac{\pi}{4}.$$

$$2. \int_0^1 \frac{1}{x} \ln^{v-1} \frac{1}{x} \operatorname{arctg} bx dx = \Gamma(v) b \sum_{k=0}^{\infty} \frac{(-b^2)^k}{(2k+1)^{v+1}} \quad [|b| < 1; \operatorname{Re} v > 0].$$

$$3. \int_0^1 \ln^{n-1} x (\ln x + n) \operatorname{arctg} x dx = (-2)^{-n-1} n! (2^{-n} - 1) \zeta(n+1).$$

$$4. \int_0^{\infty} \frac{1}{x^3} \ln(1+a^2 x^2) \operatorname{arctg} bx dx = \frac{\pi}{2} [ab + (a^2 - b^2) \ln(a+b) + b^2 \ln b - a^2 \ln a] \quad [a, b > 0].$$

$$5. \int_0^{\infty} \frac{\ln^{v-1} (x + \sqrt{x^2 + 1})}{\sqrt{x^2 + 1}} \operatorname{arctg} x dx = \\ = -\frac{\pi(2\pi)^v}{v \cos(v\pi/2)} [\zeta(-v, 3/4) - \zeta(-v, 1/4)] \quad [-1 < \operatorname{Re} v < 0].$$

2.7.8. Интегралы, содержащие $\begin{Bmatrix} \operatorname{arctg} f(x) \\ \operatorname{arccotg} f(x) \end{Bmatrix}$.

$$1. \int_0^{\infty} \sin bx \operatorname{arctg} \frac{2ax}{x^2 + c^2} dx = \frac{\pi}{b} e^{-b\sqrt{a^2 + c^2}} \operatorname{sh} ab \quad [a, b, c > 0]$$

$$2. \int_0^{\infty} x^{\alpha-1} \operatorname{arctg}(ae^{-x}) dx = \pi^{-\alpha-1} a \Gamma(\alpha) \Phi(-a^2, \alpha+1, 1/2) \quad [a, \operatorname{Re} \alpha > 0].$$

$$3. \int_0^{\infty} \frac{x}{x^2 + z^2} \operatorname{arctg} \frac{a \sin cx}{1 + a \cos cx} dx = \frac{\pi}{2} \ln(1 + ae^{-cz}) \quad [a > -e^{cz}].$$

$$4. \int_0^1 \left(\frac{1-x^2}{k^2 - \lambda^2} \right)^{\pm 1/2} \operatorname{arctg} (\operatorname{tg} \lambda \sqrt{1-k^2 x^2}) dx = \\ = \frac{\pi}{2k} [E(\lambda, k) - \operatorname{ctg} \lambda (1 - \sqrt{1-k^2 \sin^2 \lambda})] - \frac{\pi(1-k^2)}{2k} F(\lambda, k) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad [0 < k < 1].$$

$$5. \int_0^{\pi} \left\{ \frac{\sin nx \cos x}{\cos nx \sin x} \right\} \operatorname{arctg} \frac{a \sin x}{1 - a \cos x} dx = \frac{\pi}{4} \left(\frac{a^{n+1}}{n+1} \pm \frac{a^{n-1}}{n-1} \right) \quad [|a| < 1]$$

$$6. \int_0^{\pi} \frac{\cos^n x}{\sin x} \operatorname{arctg} \frac{a \sin x}{1 - a \cos x} dx = -\frac{\pi}{2} \ln [(1-a)(1+a)^{2n-1}] \quad [n=0, 1; a > 0].$$

$$7. \int_0^{\pi/2} \left\{ \frac{\operatorname{arctg} (a \sqrt{1-k^2 \sin^2 x})}{\operatorname{arctg} (a \sqrt{1-k^2 \sin^2 x})} \right\} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{\pi}{2} \left\{ F(\operatorname{arctg} a, k) \right. \\ \left. F(\operatorname{arctg} (a \sqrt{1-k^2}), k) \right\} \quad [0 < k < 1].$$

2.7.9. Интегралы, содержащие $f \left(\left\{ \begin{matrix} \operatorname{arctg} x \\ \operatorname{arctg} x \end{matrix} \right\} \right)$.

$$1. \int_0^{\infty} x^{\nu} (x^2+1)^{\nu/2} e^{-px} \left\{ \frac{\sin (\nu \operatorname{arctg} x + c)}{\cos (\nu \operatorname{arctg} x + c)} \right\} dx = \\ = -\frac{\sqrt{\pi}}{2} p^{-\nu-1/2} \Gamma(\nu+1) \left[\left\{ \frac{\sin (p/2 + c)}{\cos (p/2 + c)} \right\} Y_{\nu+1/2} \left(\frac{p}{2} \right) \pm \left\{ \frac{\cos (p/2 + c)}{\sin (p/2 + c)} \right\} J_{\nu+1/2} \left(\frac{p}{2} \right) \right] \\ [\operatorname{Re} p > 0; \operatorname{Re} \nu > -1].$$

$$2. \int_0^{\infty} \frac{\sin (\nu \operatorname{arctg} x)}{(x^2+1)^{\nu/2} (e^{2\pi x} + 1)} dx = \frac{1}{2(\nu-1)} - \frac{\zeta(\nu)}{2^{\nu}} \quad [\operatorname{Re} \nu > 1].$$

$$3. \int_0^{\infty} \frac{\sin (\nu \operatorname{arctg} ax)}{(a^2 x^2 + 1)^{\nu/2} (e^{2\pi x} - 1)} dx = \frac{\zeta(\nu, a^{-1})}{2a^{\nu}} - \frac{1}{2a(\nu-1)} - \frac{1}{4} \\ [\operatorname{Re} a > 0; \operatorname{Re} \nu > 1].$$

$$4. \int_0^{\infty} \frac{(x+1)^{\nu/2}}{\sqrt{x}} e^{-px} \cos bx \cos (\nu \operatorname{arctg} \sqrt{x}) dx = \\ = \sqrt{\frac{\pi}{2}} (2p)^{-(\nu+1)/2} \exp \left(\frac{4p^2 - b^2}{8p} \right) \left[e^{b/2} D_{\nu} \left(\frac{2p+a}{\sqrt{2p}} \right) + e^{-b/2} D_{\nu} \left(\frac{2p-a}{\sqrt{2p}} \right) \right] \\ [b, \operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} (x^2+y^2)^{-\mu/2} (x^2+z^2)^{-\nu/2} \left\{ \frac{\sin [\mu \operatorname{arctg} (x/y)] \sin [\nu \operatorname{arctg} (x/z)]}{\cos [\mu \operatorname{arctg} (x/y)] \cos [\nu \operatorname{arctg} (x/z)]} \right\} dx = \\ = \frac{\pi}{2(y+z)^{\mu+\nu-1}} \Gamma \left[\begin{matrix} \mu+\nu-1 \\ \mu, \nu \end{matrix} \right] \quad [y, z > 0; \operatorname{Re} (\mu+\nu) > 1].$$

$$6. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2+2ax \cos \lambda + a^2)^{\nu}} \left\{ \frac{\sin (2\rho \operatorname{arctg} [a \sin \lambda / (x+a \cos \lambda)])}{\cos (2\rho \operatorname{arctg} [a \sin \lambda / (x+a \cos \lambda)])} \right\} dx = \\ = a^{\alpha-2\rho} B(\alpha, 2\rho-\alpha) \left\{ \frac{\sin [(2\rho-\alpha)\lambda]}{\cos [(2\rho-\alpha)\lambda]} \right\} \quad [0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho; a > 0; 0 \leq \lambda < \pi].$$

2.8. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

Интегралы, содержащие обратные гиперболические функции, сводятся к интегралам, содержащим гиперболические функции, с помощью подстановок вида $y = \operatorname{Arsh} x$, $y = \operatorname{Arch} x$, $y = \operatorname{Arth} x$, $y = \operatorname{Arcth} x$.

Кроме того, они могут быть сведены к интегралам от логарифмической, степенной и других функций, если воспользоваться формулами, приведенными в приложении I. Полученные таким образом интегралы отнесены в соответствующие разделы.

Например, с помощью формулы

$$\operatorname{ch} (v \operatorname{Arsh} x) = \frac{1}{2} [(\sqrt{x^2 + 1} + x)^v + (\sqrt{x^2 + 1} - x)^v]$$

получаем

$$\begin{aligned} \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{ch} (v \operatorname{Arsh} x) dx &= \\ &= \frac{1}{2} \int_0^{\infty} x^{\alpha-1} e^{-px} (\sqrt{x^2 + 1} + x)^v dx + \frac{1}{2} \int_0^{\infty} x^{\alpha-1} e^{-px} (\sqrt{x^2 + 1} - x)^v dx \\ &\quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0]. \end{aligned}$$

Г л а в а 3. КРАТНЫЕ ИНТЕГРАЛЫ

3.1. ДВОЙНЫЕ ИНТЕГРАЛЫ

3.1.1. Введение.

В этом разделе помещены двойные интегралы и указаны некоторые способы их приведения к одномерным интегралам. Кроме того, указаны различные системы координат на плоскости, которые могут быть использованы при вычислении двойных интегралов.

Отметим, что основным способом вычисления двойных интегралов является сведение их к повторным [20]. Например, для вычисления двойного интеграла $\iint_G f(x, y) dx dy$ его можно свести к одному или нескольким повторным

$$\int_a^b \int_{x_1(y)}^{x_2(y)} f(x, y) dx dy \equiv \int_a^b \left(\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right) dy \equiv \int_a^b dy \int_{x_1(y)}^{x_2(y)} f(x, y) dx,$$

или

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y) dy dx \equiv \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right) dx \equiv \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy,$$

при этом сначала вычисляются внутренние интегралы.

В частности, если G — треугольник, ограниченный прямыми $y=0$, $y=x$ и $x=a$ ($a > 0$), то

$$\iint_G f(x, y) dx dy = \int_0^a \left(\int_0^x f(x, y) dy \right) dx = \int_0^a \left(\int_y^a f(x, y) dx \right) dy.$$

Кроме того, часто используется метод замены переменных интегрирования [20].

Пусть задано взаимно-однозначное отображение $x=x(u, v)$, $y=y(u, v)$, области Γ плоскости переменных u, v на область G плоскости переменных x, y , причем

а) x, y как функции u, v непрерывны в Γ вместе с первыми частными производными;

б) якобиан

$$\Delta = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} \neq 0 \quad \text{в } \Gamma.$$

Тогда справедлива следующая формула:

$$\iint_G f(x, y) dx dy = \iint_{\Gamma} f[x(u, v), y(u, v)] |\Delta| du dv.$$

Приведем некоторые системы координат на плоскости (dl — элемент длины, dS — элемент площади) [16].

1) Полярные координаты:

$$x = u \cos v, \quad y = u \sin v,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi \quad (\text{обычно используются обозначения } u = \rho, \quad v = \varphi),$$

$$dl^2 = du^2 + u^2 dv^2, \quad dS = u du dv.$$

2) Обобщенные полярные координаты:

$$x = au \cos v, \quad y = bu \sin v,$$

$$a, b > 0, \quad 0 \leq u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = (a^2 \cos^2 v + b^2 \sin^2 v) du^2 + (b^2 - a^2) u \sin 2v du dv + (a^2 \sin^2 v + b^2 \cos^2 v) u^2 dv^2,$$

$$dS = abu du dv.$$

3) Эллиптические координаты ($x, y > 0$):

$$x = \left[\frac{(u + a^2)(v + a^2)}{a^2 - b^2} \right]^{1/2}, \quad y = \left[\frac{(u + b^2)(v + b^2)}{b^2 - a^2} \right]^{1/2},$$

$$0 \leq b^2 < a^2, \quad -b^2 < u < \infty, \quad -a^2 < v < -b^2,$$

$$dl^2 = \frac{1}{4} \frac{u - v}{(u + a^2)(u + b^2)} du^2 - \frac{u - v}{(v + a^2)(v + b^2)} dv^2,$$

$$dS = \frac{u - v}{4 \sqrt{-(u + a^2)(u + b^2)(v + a^2)(v + b^2)}} du dv.$$

4) Вырожденные эллиптические координаты:

$$x = \operatorname{ch} u \cos v, \quad y = \operatorname{sh} u \sin v,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = (\operatorname{ch}^2 u - \cos^2 v) (du^2 + dv^2), \quad dS = (\operatorname{ch}^2 u - \cos^2 v) du dv.$$

5) Параболические координаты:

$$x = u^2 - v^2, \quad y = 2uv,$$

$$-\infty < u < \infty, \quad 0 \leq v < \infty,$$

$$dl^2 = 4(u^2 + v^2)(du^2 + dv^2), \quad dS = 4(u^2 + v^2) du dv.$$

6) Биполярные координаты:

$$x = \frac{\operatorname{ch} u}{\operatorname{ch} u + \cos v}, \quad y = \frac{\sin v}{\operatorname{ch} u + \cos v},$$

$$-\infty < u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = \frac{du^2 + dv^2}{(\operatorname{ch} u + \cos v)^2}, \quad dS = \frac{1}{(\operatorname{ch} u + \cos v)^2} du dv.$$

3.1.2. Общие формулы; интегралы от алгебраических функций.

В этом пункте предполагается, что все рассматриваемые интегралы существуют; $a, b > 0$.

$$\begin{aligned} 1. \quad \int \int_{\substack{x \geq 0, y \geq 0 \\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \leq 1}} x^{\alpha-1} y^{\beta-1} f \left[\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \right] dx dy = \\ = \frac{a^\alpha b^\beta}{pq} \Gamma \left[\frac{\alpha}{p}, \frac{\beta}{q} \right] \int_0^1 f(t) t^{\alpha/p + \beta/q - 1} dt \quad [\alpha, \beta, p, q > 0]. \end{aligned}$$

$$2. \iint_{x^2+y^2 \leq 1} f(\alpha x + \beta y + \gamma) dx dy = 2 \int_{-1}^1 \sqrt{1-t^2} f(t \sqrt{\alpha^2 + \beta^2} + \gamma) dt$$

[$\alpha^2 + \beta^2 \neq 0$; α, β, γ — действительные числа].

$$3. \iint_{|x|+|y| \leq 1} f(x+y) dx dy = \int_{-1}^1 f(t) dt.$$

$$4. \int_0^1 \int_0^1 f(xy) (1-x)^{\alpha-1} y^{\alpha} (1-y)^{\beta-1} dx dy =$$

$$= B(\alpha, \beta) \int_0^1 f(t) (1-t)^{\alpha+\beta-1} dt \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$5. \int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} (1-x)^{\gamma-\alpha-1} (1-y)^{\gamma-\beta-1} (1-xyz)^{-\gamma} dx dy =$$

$$= \Gamma \left[\begin{matrix} \alpha, \beta, \gamma-\alpha, \gamma-\beta \\ \gamma, \gamma \end{matrix} \right] {}_2F_1(\alpha, \beta; \gamma; z)$$

[$\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$; $\operatorname{Re} \gamma > \operatorname{Re} \beta > 0$; $|\arg(1-z)| < \pi$].

$$6. \iint_{\substack{x \geq 0, y \geq 0 \\ x+y \leq 1}} x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta-\beta' \\ \gamma \end{matrix} \right] F_1(\alpha, \beta, \beta', \gamma; u, v) \quad [\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta-\beta') > 0].$$

$$7. \iint_{\substack{x \geq 0, y \geq 0 \\ x+y \leq 1}} x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta-\beta' \\ \gamma \end{matrix} \right] F_3(\alpha, \alpha', \beta, \beta', \gamma; u, v)$$

[$\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta-\beta') > 0$].

$$8. \int_0^1 \int_0^1 x^{\beta-1} y^{\beta'-1} (1-x)^{\gamma-\beta-1} (1-y)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta, \gamma'-\beta' \\ \gamma, \gamma' \end{matrix} \right] F_2(\alpha, \beta, \beta', \gamma, \gamma'; u, v)$$

[$\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta), \operatorname{Re}(\gamma'-\beta') > 0$].

$$9. \int_0^1 \int_0^1 \frac{x^{\alpha-1} y^{\beta-1} (1-x)^{\gamma-\alpha-1} (1-y)^{\gamma'-\beta-1} dx dy}{(1-ux)^{\gamma+\gamma'-\alpha-1} (1-vy)^{\gamma+\gamma'-\beta-1} (1-ux-vy)^{\alpha+\beta-\gamma-\gamma'+1}} =$$

$$= \Gamma \left[\begin{matrix} \alpha, \beta, \gamma-\alpha, \gamma'-\beta \\ \gamma, \gamma' \end{matrix} \right] F_4(\alpha, \beta, \gamma, \gamma'; u(1-v), v(1-u))$$

[$\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re}(\gamma-\alpha), \operatorname{Re}(\gamma'-\beta) > 0$].

$$10. \iint_{\substack{x \geq 0, y \geq 0 \\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \leq 1}} x^{\alpha-1} y^{\beta-1} dx dy = \frac{a^{\alpha} b^{\beta}}{\alpha q + \beta p} \Gamma \left[\begin{matrix} \alpha/p, \beta/q \\ \alpha/p + \beta/q \end{matrix} \right]$$

[$p, q, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0$].

$$11. \int_0^a \int_0^b \frac{xy}{\sqrt{x^2+y^2}} dx dy = \frac{ab}{3(a+b)}.$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$$

$$12. \int_0^r \int_0^{\sqrt{r^2-x^2}} \frac{dx dy}{(a^2+x^2+y^2)^a} = \begin{cases} \frac{\pi}{1-a} [(a^2+r^2)^{1-a} - a^{2(1-a)}] & [a \neq 1], \\ \pi \ln(1+r^2) & [a = 1]. \end{cases}$$

3.1.3. Интегралы, содержащие показательную функцию.
Условие. $a, b, p, q > 0$.

$$1. \int_0^\infty \int_0^\infty f(ax+by) e^{-px-xy} dx dy =$$

$$= -\frac{1}{pq(bp-aq)} \left[aq \int_0^\infty f(x) e^{-px/a} dx - bp \int_0^\infty f(y) e^{-qy/b} dy \right].$$

$$2. \int_0^\infty \int_0^\infty f(|x-y|) e^{-px-xy} dx dy = \frac{1}{pq(p+q)} \left[p \int_0^\infty e^{-qy} f(y) dy + q \int_0^\infty e^{-px} f(x) dx \right].$$

$$3. \int_0^\infty \int_0^\infty f(xy) e^{-px-xy} dx dy = 2 \int_0^\infty K_0(2\sqrt{pqt}) f(t) dt.$$

$$4. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} f\left(\frac{xy}{x+y}\right) e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq(p+q)}} \int_0^\infty e^{-(\sqrt{p}+\sqrt{q})^2 t} f(t) dt.$$

$$5. \int_0^\infty \int_0^\infty x^\mu y^\nu (x+y)^\lambda e^{-px-xy} dx dy =$$

$$= \frac{\Gamma(\mu+1) \Gamma(\nu+1) \Gamma(\mu+\nu+\lambda+2)}{\Gamma(\mu+\nu+2) p^{\mu+1} q^{\nu+1}} {}_2F_1\left(-\lambda, \mu+1; \mu+\nu+2; \frac{p-q}{p}\right)$$

[Re μ , Re ν , Re $(\mu+\nu+\lambda+2) > 0$].

$$6. \int_0^\infty \int_0^\infty x^{\mu-1} y^{\nu-1} e^{-px-xy} dx dy = \frac{\Gamma(\mu) \Gamma(\nu)}{p^\mu q^\nu}$$

[Re μ , Re $\nu > 0$].

$$7. \int_0^\infty \int_0^\infty (x+y)^{\nu-1} e^{-px-xy} dx dy = \Gamma(\nu) \frac{p^\nu - q^\nu}{(pq)^\nu (p-q)}$$

[Re $\nu > 0$].

$$8. \int_0^\infty \int_0^\infty \frac{1}{x+y} e^{-px-xy} dx dy = \frac{1}{p-q} \ln \frac{q}{p}$$

[$p \neq q$].

$$9. \int_0^\infty \int_0^\infty (xy)^{\nu/2} (x+y)^{-(\nu+1)/2} e^{-px-xy} dx dy = \Gamma\left(\frac{\nu}{2}+1\right) \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})^{\nu+1}}.$$

$$10. \int_0^\infty \int_0^\infty \sqrt{x+y} e^{-px-xy} dx dy = \frac{\sqrt{\pi} (p+q+\sqrt{pq})}{2(pq)^{3/2} (\sqrt{p}+\sqrt{q})}.$$

$$11. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq} (\sqrt{p} + \sqrt{q})}.$$

$$12. \int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{x+y} e^{-px-xy} dx dy = \frac{\pi}{2 \sqrt{pq} (\sqrt{p} + \sqrt{q})^2}.$$

$$13. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy(x+y)}} e^{-px-xy} dx dy = 2 \sqrt{\frac{\pi}{p}} K \left(\sqrt{1 - \frac{q}{p}} \right).$$

$$14. \int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y(x+y)}} e^{-px-xy} dx dy =$$

$$= \frac{\sqrt{\pi}}{\sqrt{p(p-q)}} \left[K \left(\sqrt{1 - \frac{q}{p}} \right) - E \left(\sqrt{1 - \frac{q}{p}} \right) \right].$$

$$15. \int_0^\infty \int_0^\infty \frac{\sqrt{x+y}}{\sqrt{xy}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{q \sqrt{p}} E \left(\sqrt{1 - \frac{q}{p}} \right).$$

$$16. \int_0^\infty \int_0^\infty \frac{\sqrt{y}}{\sqrt{x(x+y)^3}} e^{-px-xy} dx dy = \frac{2 \sqrt{\pi p}}{p-q} \left[K \left(\sqrt{1 - \frac{q}{p}} \right) - E \left(\sqrt{1 - \frac{q}{p}} \right) \right].$$

$$17. \int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{\sqrt{(x+y)^3}} e^{-px-xy} dx dy =$$

$$= \frac{\sqrt{\pi} (p+q)}{\sqrt{p(p-q)^2}} K \left(\sqrt{1 - \frac{q}{p}} \right) - \frac{2 \sqrt{\pi p}}{(p-q)^2} E \left(\sqrt{1 - \frac{q}{p}} \right).$$

$$18. \int_0^\infty \int_0^\infty \frac{1}{[x^2 y (x+y)]^{1/4}} e^{-px-xy} dx dy = \frac{2 \sqrt{2} \Gamma(3/4)}{p^{1/4} q^{1/2}} K \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right].$$

$$19. \int_0^\infty \int_0^\infty \frac{1}{[xy^3 (x+y)^3]^{1/4}} e^{-px-xy} dx dy = \frac{2 \sqrt{2} \Gamma(1/4)}{p^{1/4}} K \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right].$$

$$20. \int_0^\infty \int_0^\infty \left[\frac{y}{x(x+y)^3} \right]^{1/4} e^{-px-xy} dx dy =$$

$$= \frac{4 \sqrt{2} \Gamma(5/4) p^{1/4}}{\sqrt{q} (p-q)} E \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right] -$$

$$- \frac{2 \sqrt{2} \Gamma(5/4) p^{-1/4}}{\sqrt{q} (\sqrt{p} - \sqrt{q})} K \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right].$$

$$\begin{aligned}
 21. \int_0^\infty \int_0^\infty \left[\frac{x}{y(x+y)^5} \right]^{1/4} e^{-px-xy} dx dy = \\
 = \frac{4 \sqrt{2} \Gamma(3/4) p^{1/4}}{p-q} E \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right] - \\
 - \frac{2 \sqrt{2} \Gamma(3/4) p^{-1/4}}{\sqrt{q} (\sqrt{p} - \sqrt{q})} K \left[\sqrt{\frac{1}{2} \left(1 - \sqrt{\frac{q}{p}} \right)} \right].
 \end{aligned}$$

$$\begin{aligned}
 22. \int_0^\infty \int_0^\infty \frac{(xy)^{(v-3)/2}}{(x+y)^{v/2} [xy/(x+y) + 2a]^{(v+1)/2}} e^{-px-xy} dx dy = \\
 = \frac{2^{v/2} \sqrt{\pi} \Gamma\left(\frac{v+1}{2}\right)}{a(v-1)(v-2)} e^{a(\sqrt{p} + \sqrt{q})^2} D_{2-v} [2 \sqrt{a} (\sqrt{p} + \sqrt{q})] \quad [\operatorname{Re} v > -2].
 \end{aligned}$$

$$24. \int_0^\infty \int_0^\infty \frac{xy}{(x+y)^{3/2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq} (\sqrt{p} + \sqrt{q})^3}.$$

$$25. \int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y}(x+y)} e^{-px-xy} dx dy = \frac{\pi}{\sqrt{p} (\sqrt{p} + \sqrt{q})}.$$

$$\begin{aligned}
 26. \int_0^\infty \int_0^\infty (xy+1)^{-3/2} e^{-px-xy} dx dy = \\
 = \frac{8}{\pi} [\cos(2\sqrt{pq}) \operatorname{ci}(2\sqrt{pq}) - \sin(2\sqrt{pq}) \operatorname{si}(2\sqrt{pq})].
 \end{aligned}$$

$$27. \int_0^\infty \int_0^\infty \frac{(xy)^{-1/2}}{4xy+1} e^{-px-xy} dx dy = \frac{\pi^2}{2} [H_0(\sqrt{pq}) - Y_0(\sqrt{pq})].$$

$$28. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{1}{\sqrt{p^2+q^2}} \ln \frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}.$$

$$29. \int_0^\infty \int_0^\infty (\sqrt{x^2+y^2} + x)^{1/2} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{2(pq)^{3/2}} \left(\sqrt{2q} + \frac{p^{3/2}}{p+\sqrt{2pq}+q} \right).$$

$$30. \int_0^\infty \int_0^\infty (\sqrt{x^2+y^2} - x)^{1/2} e^{-px-xy} dx dy = \frac{1}{2} \sqrt{\frac{\pi}{pq^3}} \left(\frac{\sqrt{p} + \sqrt{2q}}{p+\sqrt{2pq}+q} \right).$$

$$31. \int_0^\infty \int_0^\infty \frac{(\sqrt{x^2+y^2} + x)^{1/2}}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi} (\sqrt{p} + \sqrt{2q})}{\sqrt{pq} (p+\sqrt{2pq}+q)}.$$

$$32. \int_0^\infty \int_0^\infty \frac{(\sqrt{x^2+y^2} - x)^{1/2}}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{q} (p+\sqrt{2pq}+q)}.$$

$$33. \int_0^\infty \int_0^\infty \frac{(V x^2 + 2axy + y^2 - ax - y)^{1/2}}{V x^2 + 2axy + y^2} e^{-px - qy} dx dy = \frac{V(1-a)\pi}{Vp(p + V2(a+1)pq + q)} \\ [a \geq -1].$$

$$34. \int_0^\infty \int_0^\infty e^{-px - qy + xy} dx dy = e^{-pq} E_1(pq).$$

$$35. \int_0^\infty \int_0^\infty e^{-px - qy - xy} dx dy = -e^{pq} E_1(-pq).$$

$$36. \int_0^\infty \int_0^\infty x^{\nu-1} e^{-px - qy - xy} dx dy = \Gamma(\nu) p^{-\nu/2} q^{\nu/2-1} e^{pq/2} W_{-\nu/2, (1-\nu)/2}(pq).$$

$$37. \int_0^\infty \int_0^\infty y^\nu e^{-px - qy - x^2y} dx dy = \\ = \frac{\pi^{3/2}}{\sin \pi \nu} 2^{-\nu-3/2} p^{(2\nu+1)/2} q^{-(2\nu+1)/4} [H_{-\nu-1/2}(p\sqrt{q}) - Y_{-\nu-1/2}(p\sqrt{q})] \\ [-1/2 < \nu < 0].$$

$$38. \int_0^\infty \int_0^\infty e^{-px - qy - x^2y} dx dy = \frac{1}{Vq} [\sin(p\sqrt{q}) \operatorname{ci}(p\sqrt{q}) - \cos(p\sqrt{q}) \operatorname{si}(p\sqrt{q})].$$

$$39. \int_0^\infty \int_0^\infty ye^{-px - qy - x^2y} dx dy = -\cos(p\sqrt{q}) \operatorname{ci}(p\sqrt{q}) - \sin(p\sqrt{q}) \operatorname{si}(p\sqrt{q}).$$

$$40. \int_{-\infty}^\infty \int_{-\infty}^\infty (\alpha x^2 + 2\beta xy + \gamma y^2) e^{-(ax^2 + 2bxy + cy^2)} dx dy = \\ = \frac{\pi}{2} (c\alpha - 2b\beta + a\gamma) (ac - b^2)^{-3/2} \quad [a > 0, ac > b^2].$$

$$41. \int_{-\infty}^\infty \int_{-\infty}^\infty x^n y^n e^{-(x^2 - 2axy + y^2)/[2(1-a^2)]} dx dy = \\ = 2\pi n! (1-a^2)^{(n+1)/2} P_n\left(\frac{-ia}{\sqrt{1-a^2}}\right) \quad [|a| < 1].$$

$$42. \int_0^\infty \int_0^\infty e^{-px - qy - (x/y)} dx dy = \frac{1}{p^2} e^{q/p} E_1\left(-\frac{q}{p}\right) + \frac{1}{pq}.$$

$$43. \int_0^\infty \int_0^\infty \frac{1}{y} e^{-px - qy - (x/y)} dx dy = -\frac{1}{p} e^{q/p} E\left(-\frac{q}{p}\right).$$

$$44. \int_0^\infty \int_0^\infty \frac{1}{Vxy} e^{-px - qy - (x/y)} dx dy = \frac{\pi}{Vpq} e^{q/p} \operatorname{erfc}\left(\sqrt{\frac{q}{p}}\right).$$

$$45. \int_0^\infty \int_0^\infty \frac{1}{V y} e^{-px - qy - \alpha^2 x^2 / (4y)} dx dy = \frac{V \pi}{V q (p + \alpha V q)} \quad [|\arg \alpha| < \pi/4]$$

$$46. \int_0^\infty \int_0^\infty \frac{1}{V xy} e^{-px - qy - x^2 / (4y)} dx dy = \frac{\pi}{V q (p + V q)^{1/2}}$$

$$47. \int_0^\infty \int_0^\infty xy^{-3/2} e^{-px - qy - \alpha^2 x^2 / (4y)} dx dy = \frac{2 V \pi}{\alpha} \cdot \frac{1}{p + \alpha V q}$$

[|\arg \alpha| \leq \pi/4, \alpha \neq 0].

$$48. \int_0^\infty \int_0^\infty x^v y^{-v-1} e^{-px - qy - x^2 / (4y)} dx dy = \frac{2^{v+1} \pi q^{v/2}}{\sin \pi v V p^2 - q} \operatorname{sh} \left(v \operatorname{Arch} \frac{p}{V q} \right) \quad [|\operatorname{Re} v| < 1].$$

$$49. \int_0^\infty \int_0^\infty \frac{1}{x V y} (1 - e^{av}) e^{-px - qy - x^2 / (4y)} dx dy = V \frac{\pi}{q} \ln \frac{p + V q - a}{p + V q} \quad [p + V q > a].$$

$$50. \int_0^\infty \int_0^\infty \frac{1}{V xy} e^{-px - qy - xy / (x+y)} dx dy = \frac{\pi (V p + V q)}{pq [1 + (V p + V q)^2]^{1/2}}.$$

$$51. \int_0^\infty \int_0^\infty \frac{V xy}{x+y} e^{-px - qy + xy / (x+y)} dx dy = \frac{\pi}{2 pq (p + q + 2 V pq + 1)^{3/2}}.$$

$$52. \int_0^\infty \int_0^\infty \frac{V xy}{x+y} e^{-px - qy - xy / (x+y)} dx dy = \frac{\pi (V p + V q)}{2 V pq [1 + (V p + V q)^2]^{3/2}}.$$

$$53. \int_0^\infty \int_0^\infty \frac{(xy)^{(2v-1)/2}}{(x+y)^{v+1}} e^{-px - qy - a^2 xy / (x+y)} dx dy =$$

$$= \frac{\pi \Gamma(2v)}{2^{v-1} a^v pq [(V p + V q)^2 + a^2]^{-v/2}} P_{v-1}^{-v} \left[\frac{V p + V q}{(V p + V q)^2 + a^2} \right] \quad [\operatorname{Re} v > -1/2].$$

$$54. \int_0^\infty \int_0^\infty (xy)^{v-1} e^{-px - qy - 1/(x^2 y)} dx dy =$$

$$= 4q^{-v/2} K_v [(2p V q)^{1/2} e^{\pi i/4}] K_v [(2p V q)^{1/2} e^{-\pi i/4}].$$

$$55. \int_0^\infty \int_0^\infty \frac{1}{V xy} e^{-px - qy - 1/(x^2 y)} dx dy = \pi V \frac{2}{pq} e^{-2p^{1/2} q^{1/4}}.$$

$$\begin{aligned}
 56. \int_0^\infty \int_0^\infty (xy)^{-1/4} e^{-px - qy \pm 2a\sqrt{xy}} dx dy = \\
 = \frac{2\sqrt{\pi} (pq)^{1/4}}{pq - a^2} E \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] + \\
 + \frac{2\sqrt{\pi} (pq)^{-1/4} (-\sqrt{pq} \pm a)}{pq - a^2} K \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] \quad [pq > a^2].
 \end{aligned}$$

$$\begin{aligned}
 57. \int_0^\infty \int_0^\infty (xy)^{-3/4} e^{-px - qy \pm 2a\sqrt{xy}} dx dy = \\
 = 4\sqrt{\pi} (pq)^{-1/4} K \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] \quad [pq > a^2].
 \end{aligned}$$

$$\begin{aligned}
 58. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x^2 + y^2}} e^{-px - qy - a\sqrt{x^2 + y^2}} dx dy = \\
 = \frac{1}{\sqrt{p^2 + q^2 - a^2}} \ln \frac{p + q + a + \sqrt{p^2 + q^2 - a^2}}{p + q + a - \sqrt{p^2 + q^2 - a^2}}.
 \end{aligned}$$

3.1.4. Интегралы, содержащие гиперболические функции.

Условие: $a, b, c, p, q > 0$.

$$\begin{aligned}
 1. \int_{-\infty}^\infty \int_{-\infty}^\infty f'(a \operatorname{sh} x + b \operatorname{ch} x \operatorname{sh} y + c \operatorname{ch} x \operatorname{ch} y) \operatorname{ch} x dx dy = \\
 = -\frac{2\pi \operatorname{sgn} c}{\sqrt{c^2 - a^2 - b^2}} f(\operatorname{sgn} c \sqrt{c^2 - a^2 - b^2}) \quad [c^2 > a^2 + b^2; \lim_{x \rightarrow +\infty} f(x) = 0].
 \end{aligned}$$

$$2. \int_0^{\ln(1+\sqrt{2})} \int_0^{\ln(1+\sqrt{2})} \frac{\operatorname{ch} x \operatorname{ch} y dx dy}{(1 + k^2 \operatorname{sh} x \operatorname{sh} y) \sqrt{\operatorname{sh} x \operatorname{sh} y (1 - \operatorname{sh} x) (1 - \operatorname{sh} y)}} = 2\pi K(k).$$

$$3. \int_0^{\ln(1+\sqrt{2})} \int_0^{\ln(1+\sqrt{2})} \frac{\operatorname{ch} x \operatorname{ch} y dx dy}{\sqrt{(1 - \operatorname{sh}^2 x \operatorname{sh}^2 y) (1 - \operatorname{sh}^2 x) (1 - \operatorname{sh}^2 y)}} = K^2 \left(\frac{\sqrt{2}}{2} \right).$$

$$4. \int_0^k \int_0^\infty \frac{y dx dy}{\sqrt{\operatorname{ch}^2 x - y^2}} = E(k) - (1 - k^2) K(k).$$

$$5. \int_0^k \int_0^\infty \frac{y^3 dx dy}{\operatorname{ch}^2 x \sqrt{\operatorname{ch}^2 x - y^2}} = \frac{1}{3} [(2 - k^2) E(k) - 2(1 - k^2) K(k)].$$

$$6. \int_0^\infty \int_0^\infty \operatorname{sh}(x + y) e^{-px - qy} dx dy = \frac{p + q}{(p^2 - 1)(q^2 - 1)}.$$

$$7. \int_0^{\infty} \int_0^{\infty} \operatorname{ch}(x+y) e^{-px-xy} dx dy = \frac{pq+1}{(p^2-1)(q^2-1)}.$$

$$8. \int_0^{\infty} \int_0^{\infty} \operatorname{ch} \sqrt{xy} e^{-px-xy} dx dy = \frac{4}{4pq-1} + 4(4pq-1)^{-3/2} \arcsin \frac{1}{2\sqrt{pq}} \quad [pq > 1/4].$$

$$9. \int_0^{\infty} \int_0^{\infty} \frac{1}{xy} (\operatorname{ch} \sqrt{xy} - 1) e^{-px-xy} dx dy = -2 \arcsin^2 \frac{1}{2\sqrt{pq}} \quad [pq \geq 1/4].$$

$$10. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{xy}} \operatorname{sh} \sqrt{xy} e^{-px-xy} dx dy = \frac{4}{(4pq-1)^{1/2}} \arcsin \frac{1}{2\sqrt{pq}} \quad [pq > 1/4].$$

$$11. \int_0^{\infty} \int_0^{\infty} \frac{1}{(xy^3)^{1/2}} (\operatorname{sh} \sqrt{xy} - \sqrt{xy}) e^{-px-xy} dx dy = \\ = \frac{2}{pq} \left[1 - (4pq-1)^{1/2} \arcsin \frac{1}{2\sqrt{pq}} \right] \quad [pq \geq 1/4].$$

$$12. \int_0^{\infty} \int_0^{\infty} \frac{\operatorname{sh}(a\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{1}{2\sqrt{p^2+q^2-a^2}} \ln \frac{pq-a\sqrt{p^2+q^2-a^2}}{pq+a\sqrt{p^2+q^2-a^2}}.$$

$$13. \int_0^{\infty} \int_0^{\infty} \frac{\operatorname{ch}(a\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \\ = \frac{1}{2\sqrt{p^2+q^2-a^2}} \ln \frac{p^2+pq+q^2+(p+q)\sqrt{p^2+q^2-a^2}-a^2}{p^2+pq+q^2-(p+q)\sqrt{p^2+q^2-a^2}-a^2}.$$

3.1.5. Интегралы, содержащие тригонометрические функции.

Условия: $a, b, c, p, q > 0$; $k^2 < 1$.

$$1. \int_0^{\pi} \int_0^{2\pi} f(a \sin x \sin y + b \cos x \sin y + c \cos y) \sin y dx dy = \\ = 2\pi \int_{-1}^1 f(t\sqrt{a^2+b^2+c^2}) dt \quad [\text{Функция } f(t) \text{ непрерывна при } |t| < \sqrt{a^2+b^2+c^2}].$$

$$2. \int_0^{\pi} \int_0^{\pi} f' \left(a \operatorname{ctg} x - \frac{b \cos y - c}{\sin x \sin y} \right) \frac{dx dy}{\sin^2 x \sin y} = \\ = -\frac{2\pi \operatorname{sgn} c}{\sqrt{c^2-a^2-b^2}} f(\operatorname{sgn} c \sqrt{c^2-a^2-b^2}) \quad [c^2 > a^2+b^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$3. \int_0^{\infty} \int_0^{\pi} f'(a \cos x \operatorname{sh} y + b \operatorname{ch} y) \operatorname{sh} y dx dy = -\frac{\pi \operatorname{sgn} b}{\sqrt{b^2-a^2}} f(\operatorname{sgn} b \sqrt{b^2-a^2}) \\ [b^2 > a^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$4. \int_0^{\infty} \int_0^{2\pi} f' [(a \sin x + b \cos x) \operatorname{sh} y + c \operatorname{ch} y] \operatorname{sh} y \, dx \, dy = \\ = - \frac{2\pi \operatorname{sgn} c}{\sqrt{c^2 - a^2 - b^2}} f(\operatorname{sgn} c \sqrt{c^2 - a^2 - b^2}) \quad [c^2 > a^2 + b^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$5. \int_0^{\infty} \int_0^{\pi} f(a \cos x \operatorname{sh} y + b \operatorname{ch} y) \sin x \operatorname{sh}^2 y \, dx \, dy = \\ = 2 \int_0^{\infty} f(\operatorname{sgn} b \sqrt{b^2 - a^2} \operatorname{ch} y) \operatorname{sh}^2 y \, dy \quad [b^2 > a^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$6. \int_0^{\pi/2} \int_0^{\pi/2} \cos(2a \sin x \sin y) \, dx \, dy = \frac{\pi^2}{4} J_0^2(a).$$

$$7. \int_0^{\pi} \int_0^{\pi} \frac{dx \, dy}{4 - \alpha^2 (1 - \cos x)(1 - \cos y)} = \frac{\pi}{2\alpha} K\left(\frac{1}{\alpha}\right) \quad [\alpha > 1].$$

$$8. \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{3} K^2\left(\frac{\sqrt{2}}{2}\right).$$

$$9. \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \cos^2 y \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{72} \left[20K^2\left(\frac{\sqrt{2}}{2}\right) - 9\pi^2 K^{-2}\left(\frac{\sqrt{2}}{2}\right) \right].$$

$$10. \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{8} \left[4K^2\left(\frac{\sqrt{2}}{2}\right) + \pi^2 K^{-2}\left(\frac{\sqrt{2}}{2}\right) \right].$$

$$11. \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = \frac{\pi}{2\sqrt{1 - k^2}}.$$

$$12. \int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = K(k).$$

$$13. \int_0^{\pi/2} \int_0^{\pi/2} \frac{dx \, dy}{\sqrt{1 - \sin^2 x \cos^2 y}} = K^2\left(\frac{\sqrt{2}}{2}\right).$$

$$14. \int_0^{\pi/2} \int_0^y \frac{\operatorname{ctg} y \, dx \, dy}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{4} K(\sqrt{1 - k^2}) + \frac{1}{2} \ln k K(k).$$

$$15. \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \, dx \, dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2} \operatorname{cosec} \alpha.$$

$$16. \int_0^{\pi/2} \int_0^{\pi/2} \frac{dx dy}{\sqrt{1-k_1^2 \sin^2 x - k_2^2 \sin^2 y}} = \frac{2}{1+\sqrt{1-k_2^2}} K \left(\frac{\sqrt{1-k_1^2} - \sqrt{1-k_1^2-k_2^2}}{1+\sqrt{1-k_2^2}} \right) K' \left(\frac{\sqrt{1-k_1^2} + \sqrt{1-k_1^2-k_2^2}}{1+\sqrt{1-k_2^2}} \right) [k_1^2 + k_2^2 < 1].$$

$$17. \int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos^2 x dx dy}{\sqrt{1-\sin^2 x \sin^2 y}} = \frac{1}{8} \left[4K^2 \left(\frac{\sqrt{2}}{2} \right) - \pi^2 K^{-2} \left(\frac{\sqrt{2}}{2} \right) \right].$$

$$18. \int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos^2 x \cos^2 y dx dy}{\sqrt{1-\sin^2 x \cos^2 y}} = \frac{1}{8} \left[4K^2 \left(\frac{\sqrt{2}}{2} \right) - 3\pi^2 K^{-2} \left(\frac{\sqrt{2}}{2} \right) \right].$$

$$19. \int_0^{\pi/2} \int_0^y \frac{\sin 2y dx dy}{(1-k^2 \sin^2 y) \sqrt{1-k^2 \sin^2 x}} = -\frac{K(k)}{4k^2} \ln(1-k^2).$$

$$20. \int_0^{\varphi} \int_0^y \frac{dx dy}{\sqrt{(1-k^2 \sin^2 x)(1-k^2 \sin^2 y)}} = \frac{F^2(\varphi, k)}{2}.$$

$$21. \int_0^{\pi/2} \int_0^y \frac{dx dy}{\sqrt{(1-k^2 \sin^2 x)(1-k^2 \sin^2 y)}} = \frac{K^2(k)}{2}.$$

$$22. \int_0^{\pi/2} \int_0^{\pi/2} \frac{[k^2 \cos^2 x + (1-k^2) \cos^2 y] dx dy}{\sqrt{(1-k^2 \sin^2 x)[1-(1-k^2) \sin^2 y]}} = \frac{\pi}{2}.$$

$$23. \int_0^{\pi/2} \int_0^y \sqrt{\frac{1-k^2 \sin^2 x}{1-k^2 \sin^2 y}} dx dy = \frac{E(k) K(k)}{2} - \frac{\ln(1-k^2)}{4}.$$

$$24. \int_0^{\pi/2} \int_0^y \sqrt{\frac{1-k^2 \sin^2 x}{1-k^2 \sin^2 y}} \sin 2y dx dy = \frac{1}{k^2} \left\{ (2-k^2) K(k) - \left[2 + \frac{1}{4} \ln(1-k^2) \right] E(k) \right\}.$$

$$25. \int_0^{\pi/2} \int_0^y \sqrt{\frac{1-k^2 \sin^2 x}{1-k^2 \sin^2 y}} \sin y dx dy = \frac{\pi}{2\sqrt{1-k^2}}.$$

$$26. \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{(1+\sin x)(1+\sin y)}{\sin x \sin y}} \frac{dx dy}{1-k^2 \sin x \sin y} = 2\pi K(k).$$

$$27. \int_0^k \int_0^{\pi/2} y \sqrt{1-y^2 \sin^2 x} dx dy = \frac{1}{3} [(1+k^2) E(k) - (1-k^2) K(k)].$$

$$28. \int_0^1 \int_0^{\pi/2} \sqrt{1-y^2 \sin^2 x} dx dy = G + \frac{1}{2}.$$

$$29. \int_0^1 \int_0^{\pi/2} \sqrt{\cos^2 x + y^2 \sin^2 x} dx dy = \frac{\pi^2}{8}.$$

$$30. \int_0^k \int_0^{\varphi} y \sqrt{1-y^2 \sin^2 x} dx dy = \\ = \frac{1}{3} [(1+k^2) E(\varphi, k) - (1-k^2) F(\varphi, k) - (1-\sqrt{1-k^2 \sin^2 \varphi}) \operatorname{ctg} \varphi].$$

$$31. \int_0^1 \int_0^{\varphi} y \sqrt{1-y^2 \sin^2 x} dx dy = \frac{\sin^2 \varphi - \cos \varphi + 1}{3 \sin \varphi}.$$

$$32. \int_0^k \int_0^{\pi/2} \frac{\sqrt{1-y^2 \sin^2 x}}{1-y^2} dx dy = kK(k).$$

$$33. \int_0^1 \int_0^{\pi/2} \frac{dx dy}{\sqrt{\cos^2 x + y^2 \sin^2 x}} = \frac{\pi^2}{4}. \quad 34. \int_0^1 \int_0^{\pi/2} \frac{dx dy}{\sqrt{1-y^2 \sin^2 x}} = 2G.$$

$$35. \int_0^k \int_0^{\pi/2} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = E(k) - (1-k^2) K(k).$$

$$36. \int_0^1 \int_0^{\varphi} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = \frac{1 - \cos \varphi}{\sin \varphi}.$$

$$37. \int_0^k \int_0^{\varphi} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = E(\varphi, k) - (1-k^2) F(\varphi, k) + (\sqrt{1-k^2 \sin^2 \varphi} - 1) \operatorname{ctg} \varphi.$$

$$38. \int_0^k \int_0^{\varphi} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} = \\ = (k^2 - \alpha^2) \Pi(\varphi, \alpha^2, k) + E(\varphi, k) - F(\varphi, k) + (\sqrt{1-k^2 \sin^2 \varphi} - 1) \operatorname{ctg} \varphi.$$

$$39. \int_0^k \int_0^{\pi/2} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} = (k^2 - \alpha^2) \Pi(\pi/2, \alpha^2, k) - K(k) + E(k).$$

$$40. \int_0^1 \int_0^{\varphi} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} = \\ = (1-\alpha^2) \Pi(\varphi, \alpha^2, 1) - \alpha^2 \Pi(\varphi, \alpha^2, 0) + (1 - \cos \varphi) / \sin \varphi - \ln(\operatorname{tg} \varphi + \sec \varphi).$$

$$41. \int_0^1 \int_0^{\pi/2} \frac{dx dy}{(1+y) \sqrt{1-y^2 \sin^2 x}} = \frac{\pi^2}{8}.$$

$$42. \int_0^\infty \int_0^\infty e^{-p\sqrt{x^2+y^2}} \cos ax \cos by dx dy = \frac{\pi}{2} \frac{p}{(a^2+b^2+p^2)^{3/2}}.$$

$$43. \int_0^\infty \int_0^\infty e^{-px-xy} \sin(x+y) dx dy = \frac{p+q}{(p^2+1)(q^2+1)}.$$

$$44. \int_0^\infty \int_0^\infty e^{-px-xy} \cos(x+y) dx dy = \frac{pq-1}{(p^2+1)(q^2+1)}.$$

$$45. \int_0^\infty \int_0^\infty e^{-px-xy} \cos(xy) dx dy = \sin(pq) \operatorname{ci}(pq) - \cos(pq) \operatorname{si}(pq).$$

$$46. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} e^{-px-xy} \sin \frac{xy}{x+y} dx dy = \sqrt{\frac{\pi}{pq}} \frac{\sqrt{p} + \sqrt{q}}{(\sqrt{p} + \sqrt{q})^4 + 1}.$$

$$47. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} e^{-px-xy} \cos \frac{xy}{x+y} dx dy = \sqrt{\frac{\pi}{pq}} \frac{(\sqrt{p} + \sqrt{q})^3}{(\sqrt{p} + \sqrt{q})^4 + 1}.$$

$$48. \int_0^\infty \int_0^\infty e^{-px-xy} \sin(x\sqrt{y}) dx dy = \sqrt{\frac{\pi}{q}} - \pi p e^{p^2 q} \operatorname{erfc}(p\sqrt{q}).$$

$$49. \int_0^\infty \int_0^\infty e^{-px-xy} \cos(x\sqrt{y}) dx dy = -p e^{p^2 q} \operatorname{Ei}(-p^2 q).$$

$$50. \int_0^\infty \int_0^\infty e^{-px-xy} \sin(y\sqrt{x}) dx dy = \sqrt{\frac{\pi}{2p}} e^{pq^2/2} D_{-2}(q\sqrt{2p}).$$

$$51. \int_0^\infty \int_0^\infty y^{n-1} e^{-px-xy} \sin(y\sqrt{2x}) dx dy = \sqrt{\frac{\pi}{2}} n! p^{(n-2)/2} e^{pq^2/4} D_{-(n+1)}(q\sqrt{p}).$$

$$52. \int_0^\infty \int_0^\infty e^{-px-xy} \sin(2\sqrt{axy}) dx dy = \frac{\pi \sqrt{a}}{2} (pq+a)^{-3/2}.$$

$$53. \int_0^\infty \int_0^\infty y^n e^{-px-xy} \sin \sqrt{xy} dx dy = \frac{\pi (2n+1)! p^n}{n! (4pq+1)^{n+3/2}}.$$

$$54. \int_0^\infty \int_0^\infty \frac{1}{y} e^{-px-xy} \sin(x\sqrt{y}) dx dy = \frac{\pi}{p} e^{p^2 q} \operatorname{erfc}(p\sqrt{q}).$$

$$55. \int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-px-xy} \sin(2\sqrt{axy}) dx dy = \frac{\pi}{p} \sqrt{\frac{a}{pq+a}}.$$

$$56. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-px-xy} \sin(x\sqrt{y}) dx dy = -e^{p^2q} E_1(-p^2q).$$

$$57. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-px-xy} \cos(2\sqrt{axy}) dx dy = \frac{\sqrt{\pi q}}{pq+a}.$$

$$58. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-px-xy} \sin(2\sqrt{axy}) dx dy = \frac{1}{pq+a} \sqrt{\frac{\pi a}{p}}.$$

$$59. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-px-xy} \cos(x\sqrt{y}) dx dy = \pi e^{p^2q} \operatorname{erfc}(p\sqrt{q}).$$

$$60. \int_0^{\infty} \int_0^{\infty} \frac{x^{n-1/2}}{\sqrt{y}} e^{-px-xy} \cos\sqrt{xy} dx dy = \frac{2\pi (2n)! q^n}{n! (4pq+1)^{n+1/2}}.$$

$$61. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{xy}} e^{-px-xy} \cos(2\sqrt{axy}) dx dy = \frac{\pi}{\sqrt{pq+a}}.$$

$$62. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{xy}} e^{-px-xy} \cos 2\sqrt{xy} \cos 2\sqrt{2ay} dx dy = \pi (pq+1)^{-1/2} e^{-ap/(pq+1)}.$$

$$63. \int_0^{\infty} \int_0^{\infty} (xy)^{-1/4} e^{-px-xy} [\cos(2a\sqrt{xy}) \pm \sin(2a\sqrt{xy})] dx dy =$$

$$= \sqrt{\pi} (pq+a^2)^{-3/4} \left\{ 2E \left[\sqrt{\frac{1}{2} \left(1 \mp \frac{a}{\sqrt{pq+a^2}} \right)} \right] - \right.$$

$$\left. - K \left[\sqrt{\frac{1}{2} \left(1 \mp \frac{a}{\sqrt{pq+a^2}} \right)} \right] \right\}.$$

$$64. \int_0^{\infty} \int_0^{\infty} (xy)^{-3/4} e^{-px-xy} [\cos(2a\sqrt{xy}) \pm \sin(2a\sqrt{xy})] dx dy =$$

$$= 4 \sqrt{\pi} (pq+a^2)^{-1/4} K \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq+a^2}} \right)} \right].$$

$$65. \int_0^{\infty} \int_0^{\infty} \frac{1}{xy} e^{-px-xy} \sin\sqrt{xy} dx dy = 2\pi \ln \left(\sqrt{1 + \frac{1}{4pq}} + \frac{1}{2\sqrt{pq}} \right).$$

$$66. \int_0^{\infty} \int_{-\pi}^{\pi} e^{-y(a+ib \cos x + ic \sin x)} dx dy = \frac{2\pi}{\sqrt{a^2+b^2+c^2}}.$$

$$67. \int_0^{\infty} \int_0^{\infty} \frac{1}{xy} (\cos \sqrt{xy} - 1) e^{-px-xy} dx dy = -2 \operatorname{Arsh}^2 \frac{1}{2\sqrt{pq}}.$$

$$68. \int_0^{\infty} \int_0^{\infty} \cos \sqrt{xy} e^{-px-xy} dx dy = \frac{4}{4pq+1} - 4(4pq+1)^{-3/2} \operatorname{Arsh} \frac{1}{2\sqrt{pq}}.$$

$$69. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{xy}} e^{-px-xy} \sin \sqrt{xy} dx dy = \frac{4}{(4pq+1)} \operatorname{Arsh} \frac{1}{2\sqrt{pq}}.$$

$$70. \int_0^{\infty} \int_0^{\infty} (xy^3)^{-1/2} e^{-px-xy} (\sqrt{xy} - \sin \sqrt{xy}) dx dy = \\ = -\frac{2}{pq} \left[1 - (1+4pq)^{1/2} \operatorname{Arsh} \frac{1}{2\sqrt{pq}} \right].$$

3.1.6. Интегралы, содержащие логарифмическую функцию.

Условие: $a, p, q > 0$.

$$1. \int_0^{\infty} \int_0^{\infty} (\ln xy + 2C) e^{-px-xy} dx dy = -\frac{1}{pq} \ln pq$$

$$2. \int_0^{\infty} \int_0^{\infty} [\ln(x+y) + C] e^{-px-xy} dx dy = \frac{q \ln p - p \ln q}{pq(p-q)}.$$

$$3. \int_0^{\infty} \int_0^{\infty} \ln(x^2+y^2) e^{-px-xy} dx dy = -\frac{2}{pq} \left[C + \ln p - \frac{2p^2(\ln p - \ln q) + \pi pq}{2(p^2+q^2)} \right].$$

$$4. \int_0^{\infty} \int_0^{\infty} \ln \left(1 + \frac{y^2}{x^2} \right) e^{-px-xy} dx dy = \frac{2p(\ln p - \ln q) + \pi q}{q(p^2+q^2)}.$$

$$5. \int_0^{\infty} \int_a^{\infty} \frac{y^x - y^a}{\ln y} e^{-px-xy} dx dy = \lambda(qe^p, a).$$

$$6. \int_0^1 \int_0^1 e^{xy \ln(xy)} dx dy = \int_0^1 x^x dx.$$

3.1.7. Интегралы, содержащие обратные тригонометрические и обратные гиперболические функции.

Условие: $p, q > 0$.

$$1. \int_0^{\infty} \int_0^{\infty} \operatorname{arctg} \frac{y}{x} e^{-px-xy} dx dy = \frac{2q (\ln q - \ln p) + \pi p}{2q (p^2 + q^2)}.$$

$$2. \int_0^{\infty} \int_0^{\infty} \operatorname{arctg} \sqrt{\frac{y}{x}} e^{-px-xy} dx dy = \frac{\pi}{2q (p + \sqrt{pq})}.$$

$$3. \int_0^{\infty} \int_0^{\infty} \operatorname{Arsh} \frac{x}{y} e^{-px-xy} dx dy = \frac{1}{p \sqrt{p^2 + q^2}} \ln \frac{p + q + \sqrt{p^2 + q^2}}{p + q - \sqrt{p^2 + q^2}}.$$

3.2. ТРОЙНЫЕ ИНТЕГРАЛЫ

3.2.1. Введение.

В этом разделе помещены тройные интегралы и указаны некоторые способы их приведения к одномерным интегралам. Кроме того, указаны различные системы координат в пространстве, которые могут быть использованы при вычислении тройных интегралов.

Отметим, что основным способом вычисления тройных интегралов является сведение их к повторным [20], которые находят путем последовательного интегрирования по каждой из переменных в отдельности.

Кроме того, часто используется метод замены переменных интегрирования. Пусть задано взаимно однозначное отображение $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ области Ω пространства переменных u, v, w на область V пространства переменных x, y, z , причем

- а) x, y, z как функции u, v, w непрерывны в Ω вместе со всеми первыми частными производными;
- б) якобиан

$$\Delta = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v & \partial x / \partial w \\ \partial y / \partial u & \partial y / \partial v & \partial y / \partial w \\ \partial z / \partial u & \partial z / \partial v & \partial z / \partial w \end{vmatrix} \neq 0.$$

Тогда справедлива следующая формула:

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{\Omega} f[x(u, v, w), y(u, v, w), z(u, v, w)] |\Delta| du dv dw.$$

3. Некоторые системы координат в пространстве (dl — элемент длины, dS — элемент площади, dV — элемент объема) [16].

1) Цилиндрические координаты:

$$\begin{aligned} x &= u \cos v, & y &= u \sin v, & z &= w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & -\infty &< w < \infty, \\ dl^2 &= du^2 + u^2 dv^2 + dw^2, \\ dS^2 &= (u du dv)^2 + (du dw)^2 + (u dv dw)^2, \\ dV &= u du dv dw, \end{aligned}$$

(обычно используются обозначения $u = \rho$, $v = \varphi$, $w = z$).

2) Сферические координаты:

$$x = u \cos v \sin w, \quad y = u \sin v \sin w, \quad z = u \cos w,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w \leq \pi,$$

$$dl^2 = du^2 + u^2 \sin^2 w dv^2 + u^2 dw^2,$$

$$dS^2 = (u \sin w du dv)^2 + (u du dw)^2 + (u^2 \sin w dv dw)^2,$$

$$dV = u^2 \sin w du dv dw$$

(обычно используются обозначения $u = \rho$, $v = \varphi$, $w = \psi$).

3) Обобщенные сферические координаты:

$$x = au \cos v \sin w, \quad y = bu \sin v \sin w, \quad z = cu \cos w,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w \leq \pi,$$

$$dV = abc u^2 \sin w du dv dw.$$

$$3') \quad x = au \cos^n v \sin^n w, \quad y = bu \sin^m v \sin^n w, \quad z = cu \cos^r w,$$

$$dV = abcmnu^2 \cos^{m-1} v \sin^{m-1} v \cos^{n-1} w \sin^{2n-1} w du dv dw.$$

4) Эллипсоидальные координаты:

$$x^2 = \frac{(u+a^2)(v+a^2)(w+a^2)}{(b^2-a^2)(c^2-a^2)}, \quad y^2 = \frac{(u+b^2)(v+b^2)(w+b^2)}{(a^2-b^2)(c^2-b^2)},$$

$$z^2 = \frac{(u+c^2)(v+c^2)(w+c^2)}{(a^2-c^2)(b^2-c^2)},$$

$$0 \leq c^2 < b^2 < a^2, \quad -a^2 < w < -b^2 < v < -c^2 < u < \infty,$$

$$dl^2 = \frac{1}{4} [\mu(u)(u-v)(u-w) du^2 - \mu(v)(u-v)(v-w) dv^2 + \\ + \mu(w)(u-w)(v-w) dw^2],$$

$$dS^2 = \frac{1}{16} [-\mu(u)\mu(v)(u-v)^2(u-w)(v-w)(du dv)^2 + \\ + \mu(u)\mu(w)(u-v)(u-w)^2(v-w)(du dw)^2 - \\ - \mu(v)\mu(w)(u-v)(u-w)(v-w)^2(dv dw)^2],$$

$$dV = \frac{1}{8} \sqrt{-\mu(u)\mu(v)\mu(w)(u-v)(u-w)(v-w)} du dv dw,$$

$$\mu(t) = [(t+a^2)(t+b^2)(t+c^2)]^{-1}.$$

5) Вырожденные эллипсоидальные «вытянутые» координаты:

$$x = \operatorname{sh} u \cos v \sin w, \quad y = \operatorname{sh} u \sin v \sin w, \quad z = \operatorname{ch} u \cos w,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w \leq \pi,$$

$$dl^2 = (\operatorname{sh}^2 u + \sin^2 w) (du^2 + dw^2) + \operatorname{sh}^2 u \sin^2 w dv^2,$$

$$dS^2 = (\operatorname{sh}^2 u + \sin^2 w) \operatorname{sh}^2 u \sin^2 w (du^2 + dw^2) dv^2 + (\operatorname{sh}^2 u + \sin^2 w)^2 (du dw)^2,$$

$$dV = (\operatorname{sh}^2 u + \sin^2 w) \operatorname{sh} u \sin w du dv dw.$$

6) Вырожденные эллипсоидальные «сплюснутые» координаты:

$$x = \operatorname{ch} u \cos v \sin w, \quad y = \operatorname{ch} u \sin v \sin w, \quad z = \operatorname{sh} u \cos w,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w \leq \pi,$$

$$dl^2 = (\operatorname{sh}^2 u + \cos^2 w) (du^2 + dw^2) + \operatorname{sh}^2 u \sin^2 w dv^2,$$

$$dS^2 = (\operatorname{sh}^2 u + \cos^2 w) \operatorname{ch}^2 u \sin^2 w (du^2 + dw^2) dv^2 + (\operatorname{sh}^2 u + \cos^2 w) (du dw)^2,$$

$$dV = (\operatorname{sh}^2 u + \cos^2 w) \operatorname{ch} u \sin w du dv dw.$$

7) Сферо-конические координаты:

$$x^2 = \frac{u(v+a^2)(w+a^2)}{(a^2-b^2)a^2}, \quad y^2 = \frac{u(v+b^2)(w+b^2)}{(b^2-a^2)b^2}, \quad z^2 = \frac{uvw}{a^2b^2},$$

$$-a^2 < w < -b^2 < v < 0 < u < \infty,$$

$$dl^2 = \frac{1}{4} \left[\frac{du^2}{u} - \lambda(v) u(v-w) dv^2 + \lambda(w) u(v-w) dw^2 \right],$$

$$dS^2 = \frac{1}{16} [-\lambda(v)(v-w)(du dv)^2 + \lambda(w)(v-w)(du dw)^2 - \\ - \lambda(v)\lambda(w)u^2(v-w)^2(dv dw)^2],$$

$$dV = \frac{1}{8} \sqrt{-u\lambda(v)\lambda(w)(v-w)} du dv dw,$$

$$\lambda(t) = [t(t+a^2)(t+b^2)]^{-1}.$$

8) Параболоидальные координаты:

$$x = 2uw \cos v, \quad y = 2uw \sin v, \quad z = u^2 - w^2,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w < \infty,$$

$$dl^2 = 4[(u^2 + w^2)(du^2 + dw^2) + u^2w^2 dv^2],$$

$$dS^2 = 16[(u^2 + w^2)u^2w^2(du^2 + dw^2)dv^2 + (u^2 + w^2)^2 du^2 dw^2],$$

$$dV = 8(u^2 + w^2)uw du dv dw.$$

9) Тороидальные координаты:

$$x = \frac{\operatorname{sh} u \cos v}{\operatorname{ch} u - \cos w}, \quad y = \frac{\operatorname{sh} u \sin v}{\operatorname{ch} u - \cos w}, \quad z = \frac{\sin w}{\operatorname{ch} u - \cos w},$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad -\pi \leq w \leq \pi,$$

$$dl^2 = \frac{1}{(\operatorname{ch} u - \cos w)^2} (du^2 + \operatorname{sh}^2 u dv^2 + dw^2),$$

$$dS^2 = \frac{1}{(\operatorname{ch} u - \cos w)^4} [\operatorname{sh}^2 u (du^2 + dw^2) dv^2 + du^2 dw^2],$$

$$dV = \frac{\operatorname{sh} u}{(\operatorname{ch} u - \cos w)^3} du dv dw.$$

10) Биполярные координаты:

$$x = \frac{\sin u \cos v}{\operatorname{ch} w - \cos u}, \quad y = \frac{\sin u \sin v}{\operatorname{ch} w - \cos u}, \quad z = \frac{\operatorname{sh} w}{\operatorname{ch} w - \cos u},$$

$$0 \leq u < \pi, \quad 0 \leq v < 2\pi, \quad -\infty < w < \infty,$$

$$dl^2 = \frac{1}{(\operatorname{ch} w - \cos u)^2} (du^2 + \sin^2 u dv^2 + dw^2),$$

$$dS^2 = \frac{1}{(\operatorname{ch} w - \cos u)^4} [\sin^2 u (du^2 + dw^2) dv^2 + du^2 dv^2],$$

$$dV = \frac{\sin u}{(\operatorname{ch} w - \cos u)^3} du dv dw.$$

3.2.2. Общие формулы; интегралы от алгебраических функций.

Условие: $a, b, c, p, q, r, \alpha, \beta, \gamma > 0$.

$$1. \int\limits_{\substack{x \geq 0, y \geq 0, z \geq 0 \\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1}} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} f\left[\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r\right] dx dy dz =$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \Gamma\left[\frac{\alpha/p, \beta/q, \gamma/r}{\alpha/p + \beta/q + \gamma/r}\right] \int_0^1 f(t) t^{\alpha/p + \beta/q + \gamma/r - 1} dt$$

[при условии, что интеграл справа сходится абсолютно].

$$2. \int\limits_{\substack{x \geq 0, y \geq 0, z \geq 0 \\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1}} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx dy dz = \frac{a^\alpha b^\beta c^\gamma}{pqr} \times$$

$$\times \Gamma\left[\frac{\alpha/p, \beta/q, \gamma/r}{\alpha/p + \beta/q + \gamma/r + 1}\right].$$

$$3. \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} f(x+y+z) dx dy dz =$$

$$= \Gamma\left[\frac{\alpha, \beta, \gamma}{\alpha + \beta + \gamma}\right] \int_0^a f(t) t^{\alpha + \beta + \gamma - 1} dt$$

[при условии, что интеграл справа сходится абсолютно].

$$4. \int\limits_{\substack{x^2 + y^2 + z^2 \leq 1}} \frac{xyz dx dy dz}{x^2 + y^2 + z^2} =$$

$$= \frac{a^2 b^2 c^2}{8(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)} \left[b^2 c^2 \ln \frac{c}{b} + c^2 a^2 \ln \frac{a}{c} + a^2 b^2 \ln \frac{b}{a} \right].$$

$$5. \int\limits_{\substack{x \geq 0, y \geq 0, z \geq 0 \\ x^4 + y^4 + z^4 \leq a^4}} dx dy dz = \frac{a^3 \Gamma^4(1/4)}{48\pi \sqrt{2}}.$$

$$6. \int\limits_{x^2 + y^2 + z^2 \leq r^2} \frac{dx dy dz}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} = \frac{4}{3} \pi r^3 (a^2 + b^2 + c^2)^{-1/2}$$

[$a^2 + b^2 + c^2 > r^2$].

$$7. \int\limits_{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1} \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}} =$$

$$= \frac{2\pi abc}{\sqrt{a^2 - c^2}} F\left(\arcsin \frac{\sqrt{a^2 - c^2}}{a}, \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\right) \quad [a > b > c].$$

$$8. \int \int \int_{\substack{x \geq 0, y \geq 0, z \geq 0, \\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1}} \frac{xyz}{\sqrt{x^2 + y^2 + z^2}} dx dy dz =$$

$$= \frac{1}{15} a^2 b^2 c^2 \frac{ab + bc + ca}{(a+b)(b+c)(c+a)}.$$

3.2.3. Интегралы, содержащие показательную, гиперболические и тригонометрические функции.

Условие: $a, b, c, p, q, r > 0$.

$$1. \int \int \int_{x^2 + y^2 + z^2 \leq r^2} e^{-x^2 - y^2 - (z-a)^2} dx dy dz =$$

$$= \frac{\pi}{a\sqrt{2}} [e^{-(r+a)^2/2} - e^{-(r-a)^2/2}] + \frac{\pi^{3/2}}{2} \left[\operatorname{erf} \left(\frac{r+a}{\sqrt{2}} \right) + \operatorname{erf} \left(\frac{r-a}{\sqrt{2}} \right) \right].$$

$$2. \int_0^\infty \int_0^\infty \int_0^\infty \frac{(xyz)^{(v-2)/2}}{(xy + yz + xz)^{(v+1)/2}} e^{-px - qy - rz} dx dy dz =$$

$$= \frac{8\pi\Gamma\left(\frac{v+2}{2}\right)}{v(v-1)(v-2)} \cdot \frac{1}{(\sqrt{p} + \sqrt{q} + \sqrt{r})^{v-2}} \quad [\operatorname{Re} v > -2].$$

$$3. \int_a^\beta \int_a^\beta \int_a^\beta \frac{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z dx dy dz}{(1 - \operatorname{sh} x \operatorname{sh} y \operatorname{sh} z) \sqrt{(1 - \operatorname{sh}^2 x)(1 - \operatorname{sh}^2 y)(1 - \operatorname{sh}^2 z)}} =$$

$$= 4\pi K^2(k) [\alpha = \ln(\sqrt{2} - 1), \beta = \ln(\sqrt{2} + 1)].$$

$$4. \int_0^\infty \int_0^{2\pi} \int_0^\pi f \{ [(a \cos y + b \sin y) \sin x + c \operatorname{ch} x] \operatorname{sh} z + d \operatorname{ch} z \} \operatorname{sh}^2 z \sin x dx dy dz =$$

$$= 4\pi \int_0^\infty f(\operatorname{sgn} d \sqrt{d^2 - a^2 - b^2 - c^2} \operatorname{ch} t) \operatorname{sh}^2 t dt \quad [d^2 > a^2 + b^2 + c^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$5. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} =$$

$$= 4\pi (18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}) K^2 [(2 - \sqrt{3})(\sqrt{3} - \sqrt{2})].$$

$$6. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x \cos y - \cos x \cos z - \cos y \cos z} = \sqrt{3} \pi K^2 \left(\sin \frac{\pi}{12} \right).$$

$$7. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi K^2 \left(\frac{\sqrt{2}}{2} \right).$$

3.3. МНОГОМЕРНЫЕ ИНТЕГРАЛЫ

3.3.1. Введение.

Обозначения: $x = (x_1, x_2, \dots, x_n)$, $a = (a_1, \dots, a_n)$ $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$,
 $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, $ax = a_1x_1 + a_2x_2 + \dots + a_nx_n$, $dx = dx_1 dx_2 \dots dx_n$.

Замена переменных (сферические координаты):

$$x_1 = \rho \cos \varphi_1,$$

$$x_2 = \rho \sin \varphi_1 \cos \varphi_2,$$

$$x_3 = \rho \sin \varphi_1 \sin \varphi_2 \cos \varphi_3,$$

$$\dots \dots \dots$$

$$x_{n-1} = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \cos \varphi_{n-1},$$

$$x_n = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \sin \varphi_{n-1},$$

$$0 \leq \rho < \infty, \quad 0 \leq \varphi_k \leq \pi, \quad k = 1, 2, \dots, n-2, \quad 0 \leq \varphi_{n-1} < 2\pi,$$

$$dV = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \dots \sin^2 \varphi_{n-3} \sin \varphi_{n-2} d\rho d\varphi_1 d\varphi_2 \dots d\varphi_{n-1} -$$

— элемент объема.

При $x^2 \leq 1$ можно использовать формулу

$$\int_{x^2 \leq 1} \dots \int f(x) dx = (-1)^n \int_0^\pi \dots \int_0^\pi f(\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots$$

$$\dots, \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} \cos \theta_n) \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^2 \theta_{n-1} \sin \theta_n d\theta_1 d\theta_2 \dots d\theta_n.$$

$$1. \int_a^x \varphi(x_1) dx_1 \int_a^{x_1} \varphi(x_2) dx_2 \dots \int_a^{x_{n-1}} \varphi(x_n) dx_n \int_a^{x_n} f(t) dt =$$

$$= \frac{1}{n!} \int_a^x f(t) \left(\int_t^x \varphi(y) dy \right)^n dt.$$

$$2. \int_a^x dx_1 \int_0^{x_1} dx_2 \dots \int_a^{x_{n-1}} dx_n \int_0^{x_n} f(t) dt = \frac{1}{n!} \int_a^x (x-t)^n f(t) dt.$$

$$3. \int_0^\infty e^{-ax_1} \int_0^{x_1} e^{ax_2} \int_{x_2}^\infty e^{-ax_3} \dots \int_0^{x_{2n-1}} e^{ax_{2n}} \int_{x_{2n}}^\infty e^{-ax_{2n+1}} \times$$

$$\times x_{2n+1}^m dx_1 dx_2 \dots dx_{2n+1} = \frac{1}{a^{m+2n+1}} \frac{(m+1)!(m+2n)!}{n!(m+n+1)!} \quad [a > 0].$$

3.3.2. Интегралы по области $x_1^2 + x_2^2 + \dots + x_n^2 \leq r^2$.

$$1. \int_{x^2 \leq r^2} \dots \int f(x^2) dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r t^{n-1} f(t^2) dt.$$

$$2. \int_{x^2 \leq r^2} \dots \int f(ax) dx = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{-r}^r (r^2 - t^2)^{(n-1)/2} f(|a|t) dt.$$

$$3. \int_{x^2 \leq r^2} \dots \int f(x^2, ax) dx = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r t^{n-1} dt \int_0^\pi f(t^2, |a| t \cos u) \sin^{n-2} u du$$

[$\operatorname{Im} a_k = 0; k = 1, 2, \dots, n$].

$$4. \int_{x^2 \leq r^2} \dots \int f(ax+b) (r^2 - x^2)^m dx =$$

$$= \frac{\Gamma(m+1) \pi^{(n-1)/2}}{\Gamma\left(m + \frac{n+1}{2}\right)} \int_{-r}^r f(|a|t+b) (r^2 - t^2)^{(2m+n-1)/2} dt.$$

$$5. \int_{x^2 \leq r^2} \dots \int f(x^2, ax, bx) dx =$$

$$= \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} \int_{-r}^r dt \int_{-\sqrt{r^2-t^2}}^{\sqrt{r^2-t^2}} du \int_0^{\sqrt{r^2-t^2-u^2}} dv v^{n-3} f(t^2+u^2+v^2, |a|t, \beta t + \gamma u) dv$$

[$\beta = \frac{1}{|a|} ab, \gamma = \frac{a^2 b^2 - (a^*)^2}{a^2}$].

$$6. \int_{x^2 \leq r^2} \dots \int f(x^2) g\left(\frac{ax}{x^2}\right) dx =$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r f(y^2) y^{n-1} dy \int_{-1}^1 g(|a|t) (1-t^2)^{(n-1)/2} dt.$$

$$7. \int_{x^2 \leq r^2} \dots \int F_1(x_1) F_2(x_2) \dots F_n(x_n) dx =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{r^2(t+iu)} du}{t+iu} \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-(t+iu)x_k^2} F_k(x_k) dx_k$$

[$t > 0$ — произвольный параметр].

$$8. \int_{x^2 \leq r^2} \dots \int F(x^2) \prod_{k=1}^n \cos a_k x_k dx =$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r F(t^2) t^{n-1} dt \int_{-1}^1 e^{i|a|tu} (1-u^2)^{(n-3)/2} du.$$

$$9. \int_{x^2 \leq r^2} \dots \int dx = \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)}.$$

$$10. \int_{x^2 \leq 1} \dots \int (ax)^{2m} dx = \frac{(2m-1)!!}{2^m} \frac{\pi^{n/2}}{\Gamma(m+1+n/2)} |a|^{2m}.$$

$$11. \int_{x^2 \leq 1} \dots \int (ax)^{2m+1} dx = 0.$$

$$12. \int \dots \int_{x^2 \leq 1} (a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2)^m (bx)^{2l} dx =$$

$$= \frac{\pi^{n/2} (2l)!}{2^{2l} l! \Gamma(l+m+1+n/2)} D_y^m \left[\left(\sum_{k=1}^n \frac{b_k^2}{1-a_k y} \right)^l \prod_{j=1}^n (1-a_j)^{-1/2} \right]_{y=0}.$$

$$13. \int \dots \int_{x^2 \leq 1} |x|^{4l} (ax)^{2m} dx = \frac{\pi^{n/2} (2m)! |a|^{2m}}{2^{2m-1} (n+2l+2m) \Gamma(m+n/2)}.$$

$$14. \int \dots \int_{x^2 \leq 1} \frac{(ax)^{2m}}{|x|^{2m}} dx = \frac{\pi^{n/2}}{2^{m-1} n} \Gamma \left[\frac{2m+1}{m+1, n/2+m} \right] |a|^{2m}.$$

$$15. \int \dots \int_{x^2 \leq 1} \frac{(ax)^{2m} dx}{(b_1 x_1^2 + b_2 x_2^2 + \dots + b_n x_n^2)^v} =$$

$$= \frac{\pi^{n/2}}{2^{2m}} \Gamma \left[\frac{2m+1}{v, n/2+m-v+1, m+1} \right] \int_0^\infty t^{v-1} \prod_{j=1}^n (1+b_j t)^{-1/2} \left(\sum_{k=1}^n \frac{a_k^2}{1+b_k t} \right)^m dt$$

[$m > 1, 0 < v < m + n/2; b_k > 0, k = 1, 2, \dots, n$].

$$16. \int \dots \int_{x^2 \leq 1} e^{ax} dx = (2\pi)^{n/2} |a|^{-n/2} I_{n/2}(|a|).$$

$$17. \int \dots \int_{x^2 \leq 1} (1-x^2)^p e^{ax} dx = (2\pi)^{n/2} 2^p \Gamma(p+1) |a|^{-p-n/2} I_{p+n/2}(|a|).$$

$$18. \int \dots \int_{x^2 \leq 1} e^{ax/|x|} dx = \frac{(2\pi)^{n/2}}{n} |a|^{-n/2+1} I_{n/2-1}(|a|).$$

$$19. \int \dots \int_{x^2 \leq 1} e^{-a/(1-x^2)} dx = J(a, n) \quad [a > 0],$$

$$J(a, 2m+1) = \frac{\pi^{(2m+1)/2}}{m! \Gamma\left(\frac{2m+1}{2}\right)} e^{-a} \left(m + \frac{1}{2}\right)^{-1} \times$$

$$\times \sum_{k=0}^m \binom{m}{k} \left(m - \frac{1}{2}\right)_k k! a^{m-k} \frac{d^{m-k}}{da^{m-k}} \left\{ a e^{a/2} \left[K_1\left(\frac{a}{2}\right) - K_0\left(\frac{a}{2}\right) \right] \right\} =$$

$$= (-1)^m \frac{\pi^{(2m+3)/2}}{\Gamma\left(\frac{2m+1}{2}\right)} e^{-a/2} k_{-2m-1}\left(\frac{a}{2}\right),$$

$$J(a, 2m) = \frac{\pi^m}{(m-1)! m!} e^{-a} \frac{d^{m-1}}{da^{m-1}} \{ a^{m-1} [1 + a e^{-a} \text{Ei}(-a)] \},$$

$$J(a, 2) = \pi e^{-a} [1 + a e^{-a} \text{Ei}(-a)],$$

$$J(a, 3) = \frac{2\pi a}{3} e^{-a/2} \left[(2a+1) K_1\left(\frac{a}{2}\right) - (2a+3) K_0\left(\frac{a}{2}\right) \right].$$

$$20. \int_{x^2 \leq r^2} \dots \int \prod_{k=1}^n \cos a_k x_k dx = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{-r}^r (r^2 - t^2)^{(n-1)/2} e^{i|a|t} dt$$

[Im $a_k = 0$, $k=1, 2, \dots, n$].

3.3.3. Интегралы по области $\frac{x_1^{\alpha_1}}{a_1} + \frac{x_2^{\alpha_2}}{a_2} + \dots + \frac{x_n^{\alpha_n}}{a_n} \leq 1$.

Условие. $r, a_k, b_k, \alpha_k, \beta_k, v_k > 0$, $k=1, 2, \dots, n$.

$$1. \int_{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1} \dots \int \frac{F\left(\sum_{k=1}^n a_k^{-2} x_k^2\right)}{|x-b|^{2\rho}} dx = \frac{\rho \pi^{n/2}}{\Gamma(\rho+1) \Gamma(n/2-\rho)} \times$$

$$\times \int_{y_0}^{\infty} y^{-\rho-1} \prod_{i=1}^n \left(\frac{1}{a_i^2} + \frac{1}{y}\right)^{-1/2} dy \int_{t_0}^1 F(t) \left(t - \sum_{k=1}^n \frac{b_k^2}{y + a_k^2}\right)^{n/2-\rho-1} dt$$

$$\left[t_0 = \sum_{k=1}^n \frac{b_k^2}{y + a_k^2}; y_0 = 0, \text{ если } b_1, b_2, \dots, b_n \text{ принадлежат области интегрирования;} \right.$$

в противном случае y_0 — положительный корень уравнения $\frac{b_1^2}{y_0 + a_1^2} + \frac{b_2^2}{y_0 + a_2^2} + \dots + \frac{b_n^2}{y_0 + a_n^2} = 1$]

$$2. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1} + \left(\frac{x_2}{a_2}\right)^{\beta_2} + \dots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1}} \prod_{k=1}^n x_k^{v_k-1} dx =$$

$$= \Gamma\left[\frac{v_1}{\beta_1}, \frac{v_2}{\beta_2}, \dots, \frac{v_n}{\beta_n}\right] \prod_{k=1}^n \frac{a_k^{v_k}}{\beta_k}.$$

$$3. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \Omega}} f\left[\left(\frac{x_1}{a_1}\right)^{\beta_1} + \left(\frac{x_2}{a_2}\right)^{\beta_2} + \dots + \left(\frac{x_n}{a_n}\right)^{\beta_n}\right] \prod_{k=1}^n x_k^{v_k-1} dx =$$

$$= \Gamma\left[\frac{v_1}{\beta_1}, \frac{v_2}{\beta_2}, \dots, \frac{v_n}{\beta_n}\right] \prod_{k=1}^n \frac{a_k^{v_k}}{\beta_k} \int_{t_1}^{t_2} f(t) t^{v_1/\beta_1 + v_2/\beta_2 + \dots + v_n/\beta_n - 1} dt$$

[$t_1=1, t_2=+\infty, -\infty < v_k < \infty$ при Ω $(x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \geq 1, t_1=0,$
 $t_2=1, v_k > 0$ при Ω $(x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \leq 1, k=1, 2, \dots, n$].

$$\begin{aligned}
 4. \quad & \int \dots \int_{\substack{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1 \\ a_1^2 + a_2^2 + \dots + a_n^2 \leq 1}} \frac{dx}{|x-b|^{2p}} = \\
 & = \frac{p\pi^{n/2}}{\Gamma(p+1)\Gamma(n/2-p-1)} \int_{y_0}^{\infty} t^{-p-1} \left(1 - \sum_{i=1}^n \frac{b_i^2}{t+a_i^2}\right)^{n/2-p} \prod_{k=1}^n \left(\frac{1}{a_k^2} + \frac{1}{t}\right)^{-1/2} dt \\
 & \quad [0 < p < n/2; y_0 \text{ см. в 3.3.3.1}].
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} \frac{1}{\left(1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \dots - x_n^{\alpha_n}\right)^\mu} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \Gamma \left[\frac{v_1/\alpha_1, v_2/\alpha_2, \dots, v_n/\alpha_n, 1-\mu}{1-\mu + v_1/\alpha_1 + v_2/\alpha_2 + \dots + v_n/\alpha_n} \right] \prod_{k=1}^n \alpha_k^{-1} \quad [\mu < 1].
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \Omega}} \left(\prod_{k=1}^n x_k^{v_k-1} \right) \frac{dx}{\left(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n}\right)^\mu} = \\
 & = \varepsilon \Gamma \left[\frac{v_1/\alpha_1, v_2/\alpha_2, \dots, v_n/\alpha_n}{v_1/\alpha_1 + v_2/\alpha_2 + \dots + v_n/\alpha_n} \right] \frac{1}{\mu - v_1/\alpha_1 - v_2/\alpha_2 - \dots - v_n/\alpha_n} \prod_{k=1}^n \alpha_k^{-1} \\
 & \quad [\varepsilon = 1, -\infty < v_1, \dots, v_n < \infty, \mu > v_1/\alpha_1 + v_2/\alpha_2 + \dots + v_n/\alpha_n \text{ при } \Omega \quad x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + \\
 & \quad + x_n^{\alpha_n} \geq 1] \text{ или } [\varepsilon = -1, \mu < v_1/\alpha_1 + v_2/\alpha_2 + \dots + v_n/\alpha_n \text{ и } v_1, \dots, v_n > 0 \text{ при } \Omega. \\
 & \quad x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} \left(\frac{1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \dots - x_n^{\alpha_n}}{1 + x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n}} \right)^{1/2} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{v_1/\alpha_1, v_2/\alpha_2, \dots, v_n/\alpha_n, \beta/2}{\beta, (\beta+1)/2} \right] - \\
 & \quad - \Gamma \left[\frac{v_1/\alpha_1, v_2/\alpha_2, \dots, v_n/\alpha_n, (\beta+1)/2}{\beta, (\beta+2)/2} \right] \prod_{k=1}^n \alpha_k^{-1} \\
 & \quad [\beta = v_1/\alpha_1 + v_2/\alpha_2 + \dots + v_n/\alpha_n].
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \int \dots \int_{\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \dots + \left(\frac{x_n}{a_n}\right)^2 \leq 1} \left(1 - \sum_{k=1}^n a_k^{-2} x_k^2\right)^\lambda f(bx) dx = \\
 & = \left(\prod_{k=1}^n a_k\right) \frac{\pi^{(n-1)/2} \Gamma(\lambda+1)}{\Gamma\left(\lambda + \frac{n+1}{2}\right)} \int_{-1}^1 (1-t^2)^{\lambda+(n-1)/2} f(ct) dt \\
 & \quad \left[c = \sqrt{\sum_{k=1}^n a_k^2 b_k^2}\right].
 \end{aligned}$$

3.3.4. Интегралы по области $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, x_1 + x_2 + \dots + x_n \leq a$.

$$\begin{aligned}
 1. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} F_1(x_1) F_2(x_2) \dots F_n(x_n) dx = \\
 & = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{a(t+iy)} dy}{t+iy} \prod_{k=1}^n \int_0^{\infty} e^{-(t+iy)x_k} F_k(x_k) dx_k \\
 & \quad [a > 0; t > 0 - \text{произвольный параметр}].
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} F(x_1 + x_2 + \dots + x_n) e^{ax} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \frac{1}{2\pi} \prod_{k=1}^n \Gamma(v_k) \int_0^a F(y) dy \int_{-\infty}^{\infty} \frac{e^{y(t+im)}}{\prod_{k=1}^n (t - a_k + it)^{v_k}} du \\
 & \quad [a > 0; t > 0 - \text{произвольный параметр}].
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \frac{f(x_1 + x_2 + \dots + x_n)}{(ax+b)^{v_1+v_2+\dots+v_n}} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \Gamma\left[\frac{v_1, v_2, \dots, v_n}{v_1+v_2+\dots+v_n}\right] \int_0^1 f(t) t^{v_1+v_2+\dots+v_n-1} \prod_{k=1}^n (b+a_k t)^{-v_k} dt \\
 & \quad [a_k \geq 0, k=1, 2, \dots, n; b > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} (1-x_1)^{\mu_1-1} (1-x_1-x_2)^{\mu_2-1} \dots \\
 & \dots (1-x_1-x_2-\dots-x_n)^{\mu_n-1} F_1(1-x_1) \prod_{k=2}^n x_k^{v_k-1} F_k\left(\frac{1-x_1-x_2-\dots-x_n}{1-x_1-x_2-\dots-x_{k-1}}\right) dx = \\
 & = \prod_{k=1}^n \int_0^1 t_k^{v_k-1} (1-t_k)^{s_k} F_k(1-t_k) dt_k \\
 & \quad [s_k = \mu_k + \mu_{k+1} + \dots + \mu_n + v_{k+1} + v_{k+2} + \dots + v_n, k=1, 2, \dots, n-1; s_n = \mu_n].
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} \left(\prod_{k=1}^n x_k^{v_k-1} \right) F(x_1 + x_2 + \dots + x_n) \times \\
 & \times F_1 \left(\frac{x_2 + x_3 + \dots + x_n}{x_1 + x_2 + \dots + x_n} \right) F_2 \left(\frac{x_3 + x_4 + \dots + x_n}{x_2 + x_3 + \dots + x_n} \right) \dots F_{n-1} \left(\frac{x_n}{x_{n-1} + x_n} \right) dx = \\
 & = \int_0^a t_1^{s_1-1} F(t_1) dt_1 \prod_{k=2}^n \int_0^1 t_k^{s_k-1} (1-t_k)^{v_{k-1}-1} F_{k-1}(t_k) dt_k
 \end{aligned}$$

[при условии, что интеграл в правой части равенства сходится; $s_k = v_k + v_{k+1} + \dots + v_n$, $\text{Im } v_k = 0, k = 1, 2, \dots, n$].

$$\begin{aligned}
 6. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1}} dx = \frac{a_1 a_2 \dots a_n}{n!} \quad [a_i > 0, i = 1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} (ax)^m dx = \frac{m!}{(m+n)!} \sum_{k=1}^n \frac{a_k^{n+m-1}}{\Phi'(a_k)} \\
 & [\Phi(x) = (x-a_1)(x-a_2)\dots(x-a_n)].
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \prod_{k=1}^n x_k^{\alpha_k} X_k^{\lambda_k} dx = \\
 & = \prod_{k=1}^n \Gamma \left[\begin{matrix} \sum_{m=1}^k \lambda_m + \sum_{m=1}^k \alpha_m + k, \alpha_{k+1} + 1 \\ \sum_{m=1}^k \lambda_m + \sum_{m=1}^k \alpha_k + k + 1 \end{matrix} \right] \\
 & \left[X_k = \sum_{m=1}^k x_m; \alpha_k, \lambda_k + \alpha_k > -1, k = 1, 2, \dots, n \right]
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \prod_{k=1}^l (a_1^{(k)} x_1 + a_2^{(k)} x_2 + \dots + a_n^{(k)} x_n)^{m_k} x_k^{v_k-1} dx = \\
 & = \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ \sum_{k=1}^n m_k + \sum_{k=1}^n v_k + 1 \end{matrix} \right] D_{y_1}^{m_1} D_{y_2}^{m_2} \dots D_{y_l}^{m_l} \prod_{s=1}^l (1 - Y_s)^{-v_s} \Big|_{y_1 = y_2 = \dots = y_l = 0} \\
 & [Y_i = a_i^{(1)} y_1 + a_i^{(2)} y_2 + \dots + a_i^{(l)} y_l, i = 1, 2, \dots, l; v_j > 0, j = 1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \dots \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{1}{(ax)^\mu} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ v_1 + v_2 + \dots + v_n - \mu + 1, \mu \end{matrix} \right] \int_0^\infty t^{\mu-1} \prod_{k=1}^n (1 + a_k t)^{-v_k} dt \\
 & \quad [a_k, v_k, \mu, v_1 + v_2 + \dots + v_n - \mu + 1 > 0].
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \dots \int_{x_1 + x_2 + \dots + x_n \leq a} \frac{(bx)^m}{(c_0 + cx)^{v_1 + v_2 + \dots + v_n + m + 1}} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \frac{1}{c_0} \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ v_1 + v_2 + \dots + v_n + m + 1 \end{matrix} \right] \frac{d^m}{dt^m} \prod_{k=1}^n \left(\frac{c_0}{a} + c_k - b_k t \right)^{-v_k} \Big|_{t=0}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \dots \int_{ax \leq 1} \frac{dx}{\sqrt{x_1 x_2 \dots x_n (1 - ax)}} = \frac{\tau^{(n+1)/2}}{\Gamma \left(\frac{n+1}{2} \right)} \prod_{k=1}^n a_k^{1/2} \\
 & \quad [a_k > 0, k = 1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \dots \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{(1 - x_1 - x_2 - \dots - x_n)^{v_0-1}}{(ax + b)^\mu} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 & = \Gamma \left[\begin{matrix} v_0, v_1, \dots, v_n \\ v_0 + v_1 + \dots + v_n - \mu, \mu \end{matrix} \right] \int_0^\infty t^{\mu-1} (1 + bt)^{-v_0} \prod_{k=1}^n [1 + (b + a_k t)]^{-v_k} dt \\
 & \quad [a_k, b, v_0, v_k, \mu > 0; v_0 + v_1 + \dots + v_n > \mu; k = 1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & = \Gamma \left[\begin{matrix} v_0, v_1, \dots, v_n \\ v_0 + v_1 + \dots + v_n \end{matrix} \right] b^{-v_0} \prod_{k=1}^n (a_k + b)^{-v_k} \\
 & \quad [a_k, b, v_0, v_k, \mu > 0; v_0 + v_1 + \dots + v_n = \mu; k = 1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & = (-1)^m \Gamma \left[\begin{matrix} v_0, v_1, \dots, v_n \\ v_0 + v_1 + \dots + v_n - \mu + m, \mu \end{matrix} \right] \times \\
 & \quad \times \int_0^\infty \frac{d^m}{db^m} \left\{ t^{\mu-m-1} (1 + bt)^{-v_0} \prod_{k=1}^n [1 + (a_k + b)t]^{-v_k} \right\} dt \\
 & \quad [a_k, b, v_0, v_k, \mu > 0; v_0 + v_1 + \dots + v_n < \mu; k = 1, 2, \dots, n].
 \end{aligned}$$

$$16. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \dots \int \left(\prod_{k=1}^n x_k^{v_k-1} \right) e^{-\rho(x_1 + x_2 + \dots + x_n)} dx =$$

$$= \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ v_1 + v_2 + \dots + v_n \end{matrix} \right] \int_0^1 t^{v_1 + v_2 + \dots + v_n - 1} e^{-\rho t} dt \quad [v_k > 0, k = 1, 2, \dots, n].$$

$$17. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} e^{ax} dx = \sum_{k=1}^n \frac{1}{a_k \varphi'(a_k)} (e^{a_k} - 1)$$

$$[\varphi(x) = (x - a_1)(x - a_2) \dots (x - a_n)].$$

3.3.5. Интегралы вида $\int_a^b \dots \int_a^b f(x) dx$.

$$1. \int_0^\infty \dots \int_0^\infty f(x_1 + x_2 + \dots + x_n) g(x_1 x_2 \dots x_n) x_1^{1/n} x_2^{2/n} \dots x_{n-1}^{(n-1)/n} dx =$$

$$= \frac{V_n (2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty t^{n-1} g(t^n) dt \int_0^\infty y^{(n-3)/2} f(y + nt) dy.$$

$$2. \int_0^\infty \dots \int_0^\infty e^{-(x_1 + x_2 + \dots + x_n)} x_1^{1/n} x_2^{2/n} \dots x_{n-1}^{(n-1)/n} f(x_1 x_2 \dots x_n) dx =$$

$$= V_n (2\pi)^{(n-1)/2} \int_0^\infty t^{n-1} e^{-nt} f(t^n) dt.$$

$$3. \int_{-\infty}^\infty \dots \int_{-\infty}^\infty F(a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2) e^{ibx} dx =$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right) V_{a_1 a_2 \dots a_n}} \int_0^\infty F(t^2) t^{n-1} dt \int_{-1}^1 e^{irv t} (1 - y^2)^{(n-3)/2} dy$$

$$[a_k > 0, k = 1, 2, \dots, n; r = b_1^2/a_1 + b_2^2/a_2 + \dots + b_n^2/a_n].$$

$$4. \int_0^\infty \dots \int_0^\infty \frac{1}{(a_0 + ax)^\mu} e^{-bx} \prod_{k=1}^n x_k^{v_k-1} dx =$$

$$= \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ \mu \end{matrix} \right] \int_0^\infty e^{-a_0 t} t^{\mu-1} \prod_{k=1}^n (b_k + a_k t)^{-v_k} dt$$

$$[a_k, b_k, v, \mu > 0 \text{ и либо } a_0 > 0, \text{ либо } a_0 = 0 \text{ и } v_1 + v_2 + \dots + v_n > \mu].$$

$$\begin{aligned}
 5. \int_0^\infty \dots \int_0^\infty \frac{1}{(a_0 + ax)^\mu} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 = \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n, \mu - v_1 - v_2 - \dots - v_n \\ \mu \end{matrix} \right] a_0^{-\mu} \prod_{k=1}^n \left(\frac{a_0}{a_k} \right)^{v_k} \\
 [a_k, \mu, v_k > 0, k=1, 2, \dots, n].
 \end{aligned}$$

$$\begin{aligned}
 6. \int_0^\infty \dots \int_0^\infty \frac{1}{[1 + (a_1 x_1)^{\mu_1} + (a_2 x_2)^{\mu_2} + \dots + (a_n x_n)^{\mu_n}]^s} \prod_{k=1}^n x_k^{v_k-1} dx = \\
 = \Gamma \left[\begin{matrix} v_1/\mu_1, v_2/\mu_2, \dots, v_n/\mu_n, s - v_1/\mu_1 - v_2/\mu_2 - \dots - v_n/\mu_n \\ s \end{matrix} \right] \prod_{k=1}^n \mu_k^{-1} a_k^{-\mu_k v_k} \\
 [s, a_k, \mu_k, v_k > 0, k=1, 2, \dots, n]
 \end{aligned}$$

$$\begin{aligned}
 7. \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \frac{dx}{[p^2 + (x-a)^2]^\mu [q^2 + (x-b)^2]^\nu} = \\
 = \frac{\pi^{n/2} \Gamma(\mu + \nu - n/2)}{\Gamma(\mu) \Gamma(\nu)} \int_0^1 \frac{t^{\mu-1} (1-t)^{\nu-1} dt}{[p^2 t + q^2 (1-t) + t(1-t)(a-b)^2]^{\mu+\nu-n/2}} \\
 [\operatorname{Re} \mu, \operatorname{Re} \nu > 0].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \frac{dx}{[p^2 + (x-a)^2]^{(n+1)/2} [q^2 + (x-b)^2]^{(n-1)/2}} = \\
 = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot \frac{1}{p[(p+q)^2 + (a-b)^2]^{(n-1)/2}}.
 \end{aligned}$$

$$\begin{aligned}
 9. \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-Q(x_1, x_2, \dots, x_n) + 2bx + c} dx = \left(\frac{\pi^n}{|A|} \right)^{1/2} e^{-B/A} \\
 \left[Q = \sum_{i,j=1}^n a_{ij} x_i x_j - \text{положительно определенная квадратичная форма, } a_{ij} = a_{ji}, A = \det Q = \det(a_{ij}), B = \det \left(\frac{a_{ij}}{b_i} \frac{b_i}{c} \right) \right].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_0^\infty \dots \int_0^\infty e^{-\left(x_1 + x_2 + \dots + x_n + \frac{a^{n+1}}{x_1 x_2 \dots x_n}\right)} x_1^{\frac{1}{n+1}-1} x_2^{\frac{2}{n+1}-1} \dots x_n^{\frac{n}{n+1}-1} dx = \\
 = \frac{1}{\sqrt{n+1}} (2\pi)^{n/2} e^{-(n+1)a} \quad [a > 0].
 \end{aligned}$$

$$11. \int_0^1 \dots \int_0^1 f(x_1 x_2 \dots x_n) (1-x_1)^{v_1-1} \prod_{k=2}^n x_k^{v_1+v_2+\dots+v_{k-1}} (1-x_k)^{v_k-1} dx = \\ = \Gamma \left[\begin{matrix} v_1, v_2, \dots, v_n \\ v_1+v_2+\dots+v_n \end{matrix} \right] \int_0^1 f(t) (1-t)^{v_1+v_2+\dots+v_n-1} dt$$

[при условии, что интеграл в правой части сходится абсолютно].

$$12. \int_0^1 \dots \int_0^1 (x_1 x_2 \dots x_n)^{\alpha-1} \prod_{k=1}^n (1-x_k)^{\beta-1} \left| \prod_{i < l} (x_i - x_l) \right|^{2z} dx = \\ = \prod_{k=1}^n \Gamma \left[\begin{matrix} 1+kz, \alpha+(k-1)z, \beta+(k-1)z \\ 1+z, \alpha+\beta+(n+k-2)z \end{matrix} \right]$$

[$\operatorname{Re} \alpha, \operatorname{Re} \beta > 0, \operatorname{Re} z > \max(-1/n, -\operatorname{Re} \alpha/(n-1), -\operatorname{Re} \beta/(n-1))$].

$$13. \int_0^1 \dots \int_0^1 (x_1 + x_2 + \dots + x_n)^2 dx = \frac{n(3n+1)}{12}.$$

$$14. \int_0^1 \dots \int_0^1 \prod_{k=1}^n x_k^{\alpha_k-1} (1-x_k)^{\beta_k-\alpha_k-1} (1-x_1 x_2 \dots x_n z)^{-\nu} dx = \\ = \Gamma \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1-\alpha_1, \beta_2-\alpha_2, \dots, \beta_n-\alpha_n \\ \beta_1, \beta_2, \dots, \beta_n \end{matrix} \right] \times \\ \times {}_{n+1}F_n(\nu, \alpha_1, \alpha_2, \dots, \alpha_n; \beta_1, \beta_2, \dots, \beta_n; z) \\ [\operatorname{Re} \beta_k > \operatorname{Re} \alpha_k > 0, |\arg(1-z)| < \pi].$$

$$15. \int_0^1 \dots \int_0^1 \frac{x_1^{n-1} x_2^{n-2} \dots x_{n-1} dx}{\left(1 + \sum_{k=1}^n x_1^2 x_2^2 \dots x_k^2 \right)^{(n+1)/2}} = \frac{\pi^{(n+1)/2}}{n! 2^{n+1} \Gamma(n/2 + 3/2)}.$$

$$16. \int_0^1 \dots \int_0^1 \frac{x_1^{n-1} x_2^{n-2} \dots x_{n-1} dx}{\left[\frac{1}{(n+1)(n+2)} + \sum_{k=1}^n \frac{x_1^2 x_2^2 \dots x_k^2}{(n-k+1)(n-k+2)} \right]^{(n+1)/2}} = \\ = \frac{(n+1)\pi^{(n+1)/2}}{\sqrt{n+2} \Gamma(n/2 + 3/2)}.$$

Г л а в а 4. КОНЕЧНЫЕ СУММЫ

4.1. ВВЕДЕНИЕ

В этой главе помещены некоторые конечные суммы, содержащие различные элементарные функции. Отметим, что для вычисления конечной суммы $\sum a_k$ можно ввести вспомогательную сумму $\sum a_k x^k$ (или привести данную сумму к такой форме), записать ее в виде обобщенного гипергеометрического ряда и получить ответ в виде обобщенной гипергеометрической функции. Иногда может оказаться полезной следующая формула замены индекса суммирования:

$$\sum_{k=n_1}^{n_2} a_k = \sum_{k=n-n_2}^{n-n_1} a_{n-k},$$

в частности,

$$\sum_{k=0}^n a_k = \sum_{k=0}^n a_{n-k}.$$

Всюду в этой главе $l, m, n = 0, 1, 2, 3, \dots$, если не указаны другие условия; кроме того, по определению $\sum_{k=n_1}^{n_2} a_k = 0$ при $n_2 < n_1$.

4.1.1. Суммы вида $\sum (\pm 1)^k P(k)$.

$$1. \quad \sum_{k=1}^n k^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k (n+1)^{m-k+1}.$$

$$2. \quad = \sum_{k=1}^m (-1)^k \binom{n+1}{k+1} \sum_{l=1}^k (-1)^l \binom{k}{l} l^m.$$

$$3. \quad = \frac{1}{m+1} [B_{m+1}(n+1) - B_{m+1}] \quad [m=1, 2, 3, \dots].$$

$$4. \quad \sum_{k=1}^n (-1)^{k+1} k^m = \sum_{k=1}^m (-1)^{k+1} \left[\binom{n+1}{k+1} - 2^{m+1} \binom{[n/2]+1}{k+1} \right] \sum_{l=0}^k (-1)^l \binom{k}{l} l^m.$$

$$5. \quad = \frac{1}{2} (-1)^{n+1} [E_m(n+1) + (-1)^n E_m(0)] \quad [m=1, 2, 3, \dots].$$

$$6. \sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

$$7. \sum_{k=1}^n (-1)^k k = (-1)^n \left[\frac{n+1}{2} \right] \quad \left[\text{здесь } \left[\frac{n+1}{2} \right] - \text{целая часть } (n+1)/2 \right].$$

$$8. \sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1). \quad 9. \sum_{k=1}^n (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}.$$

$$10. \sum_{k=1}^n k^3 = \frac{1}{4} n^2 (n+1)^2.$$

$$11. \sum_{k=1}^n (-1)^k k^3 = \frac{1}{8} [1 - (-1)^n (1 - 6n^2 - 4n^3)].$$

$$12. \sum_{k=1}^n k^4 = \frac{n}{30} (n+1)(2n+1)(3n^2 + 2n - 1).$$

$$13. \sum_{k=1}^n (-1)^k k^4 = (-1)^n \frac{n^4 + 2n^3 - n}{2}.$$

$$14. \sum_{k=1}^n k^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1).$$

$$15. \sum_{k=1}^n (-1)^{k-1} k^5 = \frac{1}{4} [1 + (-1)^n (5n^2 - 5n^4 - 2n^5 - 1)].$$

$$16. \sum_{k=0}^n (k+a)^m = \frac{1}{m+1} [\bar{B}_{m+1}(n+1+a) - B_{m+1}(a)].$$

$$17. \sum_{k=0}^n (-1)^k (k+a)^m = \frac{1}{2} [(-1)^n E_m(a+n+1) + E_m(a)] - (-1)^n a^m.$$

$$18. \sum_{k=0}^n (\pm 1)^k (2k+1)^m = \\ = \sum_{k=1}^m \frac{(-1)^k}{k!} \sum_{l=1}^k (-1)^l \binom{k}{l} l^m \left[a^k \frac{d^k}{da^k} \sum_{j=0}^n (\pm 1)^j a^{2j+1} \right]_{a=1}.$$

$$19. \sum_{k=0}^n (2k+1) = (n+1)^2. \quad 20. \sum_{k=0}^n (-1)^k (2k+1) = (-1)^n (n+1).$$

$$21. \sum_{k=0}^n (2k+1)^2 = \frac{1}{3} (n+1) (2n+1) (2n+3).$$

$$22. \sum_{k=0}^n (-1)^k (2k+1)^2 = (-1)^n 2 (n+1)^2 - \frac{1+(-1)^n}{2}.$$

$$23. \sum_{k=0}^n (2k+1)^3 = (n+1)^2 (2n^2 + 4n + 1).$$

$$24. \sum_{k=0}^n (k+a)^l (k+b)^m = \\ = (a-b)^l \sum_{k=0}^l \frac{(-1)^k}{k! (k+m+1)} (b-a)^{-k} [B_{k+m+1}(n+b+1) - B_{k+m+1}(b)].$$

$$25. \sum_{k=1}^n (k+a)(k+b) = \frac{1}{6} n(n+1) [2n+1+3(a+b)] + nab.$$

$$26. \sum_{k=1}^n k(k+1) \dots (k+m) = \frac{1}{m+2} n(n+1) \dots (n+m+1).$$

$$27. \sum_{k=1}^n k(k+1) = \frac{1}{3} n(n+1)(n+2).$$

$$28. \sum_{k=1}^n k(k+1)(k+2) = \frac{1}{4} n(n+1)(n+2)(n+3).$$

$$29. \sum_{k=1}^n k(k+3)(k+6) = \frac{1}{4} n(n+1)(n+6)(n+7).$$

$$30. \sum_{k=1}^n k(k+4)(k+8) = \frac{1}{4} n(n+1)(n+8)(n+9).$$

$$31. \sum_{k=0}^n (2k+1)(2k+3)(2k+5) = \frac{15}{8} + \frac{1}{8} (2n+1)(2n+3)(2n+5)(2n+7).$$

$$32. \sum_{k=0}^n (3k+1)(3k+4)(3k+7) = \frac{14}{3} + \frac{1}{12} (3n+1)(3n+4)(3n+7)(3n+10).$$

$$33. \sum_{k=1}^n k! = (n+1)! - 1.$$

$$34. \sum_{k=0}^n k! (n-k)! = S_n, \quad S_1 = 2, \quad S_n = \frac{n+1}{2} S_{n-1} + n!.$$

4.1.2. Суммы вида $\sum \frac{(\pm 1)^k}{k^s}$.

$$1. \sum_{k=1}^n \frac{1}{k^s} = \zeta(s) - \zeta(s, n+1).$$

$$2. \sum_{k=1}^n \frac{1}{k^m} = \frac{(-1)^m}{(m-1)!} [\psi^{(m-1)}(1) - \psi^{(m-1)}(n+1)] \quad [m = 1, 2, 3, \dots]$$

$$3. \sum_{k=1}^n \frac{1}{k} = C + \psi(n+1) = S_n,$$

$$S_1 = 1, \quad S_2 = \frac{3}{2}, \quad S_3 = \frac{11}{6}, \quad S_4 = \frac{25}{12}, \quad S_5 = \frac{137}{60}, \quad S_6 = \frac{147}{60},$$

$$S_7 = \frac{1089}{420}, \quad S_8 = \frac{2283}{840}, \quad S_9 = \frac{7129}{2520}, \quad S_{10} = \frac{7381}{2520}.$$

$$4. \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - \psi'(n+1) = T_n,$$

$$T_1 = 1, \quad T_2 = \frac{5}{4}, \quad T_3 = \frac{49}{36}, \quad T_4 = \frac{205}{144}, \quad T_5 = \frac{5269}{3600}.$$

$$5. \sum_{k=1}^n \frac{1}{k^3} = \zeta(3) + \frac{1}{2} \psi''(n+1).$$

$$6. \sum_{k=1}^n \frac{(-1)^k}{k^s} = (2^{1-s} - 1) \zeta(s) - 2^{1-s} \zeta\left(s, \frac{n}{2} + 1\right) + \zeta(s, n+1).$$

$$7. \sum_{k=1}^n \frac{(-1)^k}{k^m} = \frac{1}{(m-1)!} (-1)^m [\beta^{(m-1)}(1) - (-1)^n \beta^{(m-1)}(n+1)].$$

$$8. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} = \ln 2 - (-1)^n \beta(n+1) = U_n,$$

$$U_1 = 1, \quad U_2 = \frac{1}{2}, \quad U_3 = \frac{5}{6}, \quad U_4 = \frac{7}{12}, \quad U_5 = \frac{47}{60}, \quad U_6 = \frac{37}{60},$$

$$U_7 = \frac{319}{420}, \quad U_8 = \frac{533}{840}, \quad U_9 = \frac{1879}{2520}, \quad U_{10} = \frac{1627}{2520}.$$

$$9. \sum_{k=1}^n \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12} - (-1)^n \beta'(n+1) = V_n,$$

$$V_1 = 1, \quad V_2 = \frac{3}{4}, \quad V_3 = \frac{31}{36}, \quad V_4 = \frac{115}{144}, \quad V_5 = \frac{3019}{3600}.$$

$$10. \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=1}^n \frac{1}{k+n}.$$

4.1.3. Суммы вида $\sum \frac{(\pm 1)^k}{(k+a)^s}$.

$$1. \sum_{k=0}^n \frac{1}{(k+a)^s} = \zeta(s, a) - \zeta(s, n+a+1).$$

$$2. \sum_{k=0}^n \frac{(\pm 1)^k}{k+a} = \mp \left\{ \begin{matrix} \psi(a) \\ \beta(a) \end{matrix} \right\} + (\pm 1)^n \left\{ \begin{matrix} \psi(n+a+1) \\ \beta(n+a+1) \end{matrix} \right\}.$$

$$3. \sum_{k=0}^n \frac{1}{2k+1} = \frac{1}{2} \left[\psi\left(n + \frac{3}{2}\right) + C \right] + \ln 2.$$

$$4. \sum_{k=0}^n \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + \frac{1}{2} (-1)^n \beta\left(n + \frac{3}{2}\right).$$

$$5. \sum_{k=0}^n \frac{(\pm 1)^k}{(k+a)^2} = \pm \left\{ \begin{matrix} \psi'(a) \\ \beta'(a) \end{matrix} \right\} - (\pm 1)^n \left\{ \begin{matrix} \psi'(n+a+1) \\ \beta'(n+a+1) \end{matrix} \right\}.$$

$$6. \sum_{k=0}^{n-1} \frac{1}{(2k+1)^2} = \frac{1}{8} \left[\pi^2 - 2\psi'\left(\frac{1}{2} \pm n\right) \right].$$

4.1.4. Суммы вида $\sum \frac{(\pm 1)^k}{(k+a)(k+b)}$.

$$1. \sum_{k=1}^n \frac{(\pm 1)^k}{k(k+a)} = \frac{1}{a} \left[\left\{ \begin{matrix} C \\ \ln 2 \end{matrix} \right\} + \left\{ \begin{matrix} \psi(a+1) \\ \beta(a+1) \end{matrix} \right\} - (\pm 1)^n \left\{ \begin{matrix} \psi(n+a+1) - \psi(n+1) \\ \beta(n+a+1) - \beta(n+1) \end{matrix} \right\} \right].$$

$$2. \sum_{k=1}^n \frac{1}{k(k+m)} = \frac{n}{m} \sum_{k=1}^m \frac{1}{k(k+n)}.$$

$$3. \sum_{k=1}^n \frac{(-1)^k}{k(k+2m)} = \frac{1}{2m} \sum_{k=1}^{2m} (-1)^k \left[\frac{1}{k} - \frac{(-1)^n}{k+n} \right].$$

$$4. \sum_{k=1}^n \frac{1}{(k+m)(k+l)} = \frac{n}{m-l} \sum_{k=1}^{m-l} \frac{1}{(k+l)(k+l+n)}$$

$l < m$.

$$5. \sum_{k=0}^n \frac{1}{(ka+b)(ka+a+b)} = \frac{1}{ab} - \frac{1}{a(na+a+b)}.$$

$$6. \sum_{k=0}^n \frac{1}{(ka+b)(ka+ma+b)} = \frac{1}{ma} \left(\sum_{k=0}^n \frac{1}{ka+b} - \sum_{k=m}^{m+n} \frac{1}{ka+b} \right).$$

$$7. \sum_{k=0}^n \frac{1}{(k+a)(k+b)} = \frac{1}{b-a} [\psi(n+a+1) - \psi(a) - \psi(n+b+1) + \psi(b)].$$

$$8. \sum_{k=0}^n \frac{(-1)^k}{(k+a)(k+b)} = \\ = \frac{1}{b-a} \{ \beta(a) - \beta(b) + (-1)^n [\beta(n+a+1) - \beta(n+b+1)] \}.$$

$$9. \sum_{k=0}^n \frac{1}{k^2-a^2} = \frac{1}{2a} [\psi(n-a+1) - \psi(n+a+1) + \psi(a) - \psi(-a)].$$

$$10. \sum_{k=1}^n \frac{1}{4k^2-1} = \frac{n}{2n+1}.$$

$$11. \sum_{k=0}^n \frac{(-1)^k}{k^2-a^2} = \frac{(-1)^n}{4a} [\beta(n-a+1) - \beta(n+a+1)] - \frac{\pi}{2 \sin \pi a} - \frac{1}{2a^2}.$$

$$12. \sum_{k=1}^n \frac{(-1)^k}{(k+2m)(k+2l)} = \frac{1}{2(m-l)} \sum_{k=1}^{2(m-l)} (-1)^k \left(\frac{1}{k+2l} - \frac{(-1)^n}{k+2l+n} \right).$$

4.1.5. Суммы вида $\sum \frac{(\pm 1)^k}{(k+a)(k+b)(k+c)}.$

$$1. \sum_{k=0}^n \frac{(\pm 1)^{k+1}}{(k+a)(k+b)(k+c)} = \frac{1}{(b-a)(c-a)} \left[(\pm 1)^n \left\{ \psi(n+a+1) \right\} - \right. \\ \left. - \left\{ \psi(a) \right\} \right] + \frac{1}{(a-b)(c-b)} \left[(\pm 1)^{n+1} \left\{ \psi(n+b+1) \right\} - \left\{ \psi(b) \right\} \right] + \\ + \frac{1}{(c-a)(c-b)} \left[(\pm 1)^{n+1} \left\{ \psi(n+c) \right\} - \left\{ \psi(c) \right\} \right].$$

$$2. \sum_{k=1}^n \frac{1}{k(k+m)(k+l)} = \frac{n}{m-l} \left[\frac{1}{l} \sum_{k=1}^l \frac{1}{k(k+n)} - \frac{1}{m} \sum_{k=1}^m \frac{1}{k(k+n)} \right].$$

$$3. \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

$$4. \sum_{k=1}^n \frac{(-1)^k}{k(k+2m)(k+2l)} = \frac{1}{4(m-l)} \left\{ \frac{1}{l} \sum_{k=1}^{2l} (-1)^k \left[\frac{1}{k} - \frac{(-1)^n}{k+n} \right] - \right. \\ \left. - \frac{1}{m} \sum_{k=1}^{2m} (-1)^k \left[\frac{1}{k} - \frac{(-1)^k}{k+n} \right] \right\} \quad [m \neq l].$$

$$5. \sum_{k=0}^n \frac{1}{(mk+l)(mk+m+l)(mk+2m+l)} = \\ = \frac{1}{2m} \left[\frac{1}{l(m+l)} - \frac{1}{(mn+m+l)(mn+2m+l)} \right].$$

$$6. \sum_{k=0}^n \frac{1}{(k+a)(k+a+1)(k+a+2)} = \frac{1}{2a(a+1)} - \frac{1}{2(n+a+1)(n+a+2)}.$$

$$7. \sum_{k=0}^n \frac{1}{(k+1)(2k+1)(2k+3)} = 2 \sum_{k=1}^n \frac{1}{k+n+1} - \frac{2n+2}{2n+3}.$$

$$8. \sum_{k=0}^n \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{12} - \frac{1}{4(2n+3)(2n+5)}.$$

$$9. \sum_{k=0}^n \frac{1}{(2k+1)(2k+3)(2k+7)} = \frac{11}{180} - \frac{6n+17}{12(2n+3)(2n+5)(2n+7)}.$$

4.1.6. Суммы вида $\sum \frac{P(k)}{Q(k)}$ и другие.

$$1. \sum_{k=1}^n \frac{k}{(k+m)!} m^k = \frac{1}{(m-1)!} - \frac{m^{n+1}}{(m+n)!}.$$

$$2. \sum_{k=1}^n \frac{1}{k(k+1) \dots (k+m)} = \frac{1}{m} \left[\frac{1}{m!} - \frac{1}{(n+1)(n+2) \dots (n+m)} \right].$$

$$3. \sum_{k=0}^n \frac{1}{(k+a)(k+a+1) \dots (k+a+m)} = \\ = \frac{1}{m} \left[\frac{1}{a(a+1)(a+2) \dots (a+m-1)} - \frac{1}{(n+a+1)(n+a+2) \dots (n+a+m)} \right].$$

$$4. \sum_{k=0}^n \frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{a-b+1} \left[\frac{\Gamma(n+a+1)}{\Gamma(n+b)} - \frac{\Gamma(a)}{\Gamma(b-1)} \right].$$

$$5. \sum_{k=0}^n \frac{\Gamma(a-k)}{\Gamma(b-k)} = \frac{1}{b-a-1} \left[\frac{\Gamma(a-n)}{\Gamma(b-n-1)} - \frac{\Gamma(a+1)}{\Gamma(b)} \right].$$

$$6. \sum_{k=0}^n \frac{1}{k! (k+2)} = 1 - \frac{1}{(n+2)!}.$$

$$7. \sum_{k=1}^n \frac{(\pm 1)^{k+1}}{k} \sum_{m=1}^k \frac{1}{m} = - \int_0^{\infty} \frac{1 - (\pm t)^n}{1 \mp t} \ln(1-t) dt.$$

$$8. \sum_{k=1}^n \frac{kk!}{(2k+1)!} 2^k = \frac{1}{2} - \frac{n!}{(2n+1)!} 2^{n-1}.$$

$$9. \sum_{k=0}^n (-1)^{n+k} \frac{(n+k-1)!}{(n-k)! (k!)^2} \sum_{m=1}^{n+k-1} \frac{1}{m} = \frac{1}{n^2}.$$

$$10. \sum_{k=0}^{n-1} \frac{2^k}{1+a^{2k}} \left\{ \frac{1}{a^{2k}} \right\} = \frac{1}{a-1} \left\{ -\frac{1}{a} \right\} - \frac{2^n}{a^{2n}-1} \left\{ \frac{1}{a^{2n}} \right\}.$$

$$11. \sum_{k=0}^{n-1} \frac{1}{1-a^{2k+1}} a^{2k} = \frac{1}{1-a} - \frac{1}{1-a^{2n}}.$$

4.1.7. Суммы вида $\sum a_k x^k$.

$$1. \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}. \quad 2. \sum_{k=1}^n kx^k = (1-x)^{-2} [x + (nx - n - 1)x^{n+1}].$$

$$3. \sum_{k=1}^n k^2 x^k = (1-x)^{-3} \{x(1+x) - x^{n+1} [(n+1)^2 - (2n^2 + 2n - 1)x + n^2 x^2]\}.$$

$$4. \sum_{k=1}^n k^3 x^k = x(1-x)^{-4} [x^2 + 4x + 1 - (n+1)^3 x^n + \\ + (3n^3 + 6n^2 - 4)x^{n+1} - (3n^3 + 3n^2 - 3n + 1)x^{n+2} + n^3 x^{n+3}].$$

$$5. \sum_{k=1}^n k(k+1)x^k = \\ = (1-x)^{-3} [2x - (n+1)(n+2)x^{n+1} + 2n(n+2)x^{n+2} - n(n+1)x^{n+3}].$$

$$6. \sum_{k=0}^n k! x^k = \int_0^{\infty} \frac{1 - (xt)^{n+1}}{1 - xt} e^{-t} dt.$$

$$7. \sum_{k=0}^n \frac{x^k}{(k+a)^s} = \Phi(x, s, a) - x^{n+1} \Phi(x, s, n+a+1).$$

$$8. \sum_{k=1}^n \frac{1}{k} x^k = -\ln(1-x) - x^{n+1} \int_0^1 t^n (1-x+xt)^{-n-1} dt.$$

9.
$$\sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1} = \operatorname{arctg} x + (-1)^n \int_0^x \frac{t^{2n+2} dt}{t^2+1}.$$
10.
$$\sum_{k=0}^n \frac{x^k}{k!} = \frac{1}{n!} e^x \Gamma(n+1, x).$$
11.
$$\sum_{k=0}^n \frac{(-1)^k}{\Gamma(k+v)} x^k = \frac{n!}{\Gamma(n+v)} L_n^{v-1}(x).$$
12.
$$\sum_{k=0}^{[n/2]} \frac{(-1)^k}{k! (n-2k)!} x^{2k} = \frac{x^n}{n!} H_n\left(\frac{1}{2x}\right).$$
13.
$$\sum_{k=0}^n \frac{(-1)^k}{(2k+\delta)! (n-k)!} x^{2k+\delta} = \frac{(-1)^n}{(2n+\delta)!} H_{2n+\delta}\left(\frac{x}{2}\right) \quad [\delta=0 \text{ или } 1].$$
14.
$$\sum_{k=0}^n \frac{\Gamma(n-k+1/2)}{k!} x^{2k} = \pi x^{n+1/2} [H_{n+1/2}(2x) - Y_{n+1/2}(2x)].$$
15.
$$\sum_{k=0}^{[n/2]} \frac{(n-k-1)!}{k!} x^{2k} = \frac{4}{n} x^{n+1} O_n(2x).$$
16.
$$\sum_{k=0}^n \frac{k!}{(2k)!} x^k = 1 + \frac{\sqrt{\pi x}}{2} e^{x/4} \left[\frac{2^n}{\sqrt{\pi} (2n-1)!!} \Gamma\left(n+\frac{1}{2}, \frac{x}{2}\right) - \operatorname{erfc}\left(\frac{\sqrt{x}}{2}\right) \right] \quad [x \geq 0].$$
17.
$$\sum_{k=0}^{[n/2]} \frac{(2k-1)!!}{(2n-2k+1)!!} x^{2k} = \frac{1}{2} \pi x^n \left[E_{n+1}\left(\frac{1}{x}\right) + H_{n+1}\left(\frac{1}{x}\right) \right].$$
18.
$$\sum_{k=0}^{[n/2]} \frac{(2n-2k-1)!!}{(2k+1)!!} x^{2k} = (-1)^n \frac{\pi}{2} x^n [E_{-n-1}(x) + H_{-n-1}(x)].$$
19.
$$\sum_{k=0}^n \frac{(2n-k)!}{k! (n-k)!} x^k = \frac{1}{\sqrt{\pi}} x^{n+1/2} e^{x/2} K_{n+1/2}\left(\frac{x}{2}\right).$$
20.
$$\sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(v+n-k)}{k! (n-2k)!} x^{2k} = \Gamma(v) x^n C_n^v\left(\frac{1}{2x}\right).$$

$$21. \sum_{k=1}^n (\pm 1)^k \frac{(2n-k)!}{(n-k)! \Gamma\left(\frac{1}{2} - n + k\right)} x^{2k} =$$

$$= 2^{2n-1} \pi x^{2n+1} \frac{n!}{(2n)! \Gamma\left(\frac{1}{2} - n\right)} \left\{ I_{-n-1/2}^2(x) - I_{n+1/2}^2(x) \right\}.$$

$$22. \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k! (n-2k)!} x^{2k} = \frac{2x^n}{n} T_n\left(\frac{1}{2x}\right).$$

$$23. \sum_{k=0}^n (-1)^k \frac{\Gamma\left(n+k+\frac{1}{2}+\delta\right)}{(2k+\delta)! (n-k)!} x^{2k+\delta} =$$

$$= (-1)^n 2^{2n+\delta} \frac{n!}{(2n+\delta)!} \Gamma\left(n+\frac{1}{2}+\delta\right) P_{2n+\delta}\left(\frac{x}{2}\right) \quad [\delta=0 \text{ или } 1].$$

$$24. \sum_{k=0}^n (-1)^k \frac{\Gamma(k+\nu)}{(2k+\delta)! (n-k)!} x^{2k+\delta} =$$

$$= (-1)^\delta \frac{\pi}{\sin \pi \nu \Gamma(n+1+\delta-\nu)} C_{2n+\delta}^{\nu-n-\delta}\left(\frac{x}{2}\right) \quad [\delta=0 \text{ или } 1].$$

$$25. \sum_{k=0}^n \frac{x^k}{(2k)! [(n-k)!]^2} = \frac{(x-4)^n}{(2n)!} P_{2n}\left(\sqrt{\frac{x}{x-4}}\right).$$

$$26. \sum_{k=0}^n (-1)^k \frac{(n+k-1)!}{(2k)! (n-k)!} x^k = \frac{1}{n} T_n\left(1 - \frac{x}{2}\right).$$

$$27. \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(2k+1)! (n-k)!} x^k = \frac{1}{n+1} U_n\left(1 - \frac{x}{2}\right).$$

4.1.8. Суммы Гида $\sum (\pm 1)^k \left[\frac{k}{n}\right]$.

Всюду в этом пункте $[a]$ означает целую часть a .

$$1. \sum_{k=1}^{[n/2]} \left[\frac{k}{2}\right] = \frac{1}{4} \left[\frac{n}{2}\right]^2 + \frac{1}{8} \{1 - (-1)^{[n/2]}\} \{1 + 2(-1)^{[n/2]}\}.$$

$$2. \sum_{k=1}^{[n/2]} (-1)^k \left[\frac{k}{2}\right] = \frac{1}{4} \left[\frac{n}{2}\right] \{1 + (-1)^{[n/2]}\}.$$

$$3. \sum_{k=1}^n (-1)^{[k/2]} \left[\frac{k}{2}\right] = \frac{(-1)^{[n/2]}}{4} \{1 + (-1)^n\} \{1 - (-1)^{[n/2]}\} +$$

$$+ \frac{(-1)^{[n/2]}}{4} \{1 - (-1)^n\} \{n + (-1)^{[(n+1)/2]}\}.$$

$$4. \sum_{k=1}^{[n/2]} (-1)^{[k/2]} \left[\frac{k}{2} \right] = \frac{1}{2} (-1)^{[(n-2)/4]} \{1 - (-1)^{[n/2]}\} \left[\frac{n+2}{4} \right] + \\ + \frac{1}{2} (-1)^{[(n-2)/4]} \{(-1)^{[(n+2)/4]} - 1\}.$$

$$5. \sum_{k=1}^n \left[\frac{k}{3} \right] = \frac{1}{6} (n-1)(n+4) - \left[\frac{2n-1}{3} \right] - \frac{1}{6} \{(-1)^{[2n/3]} + (-1)^{n+[n/3]}\}.$$

$$6. \sum_{k=1}^n (-1)^k \left[\frac{k}{3} \right] = \frac{1}{4} \left\{ (-1)^{[n/3]} + 2(-1)^{[n/3]} \left[\frac{n}{3} \right] - 1 \right\} - \\ - (-1)^{[(n-1)/3]} \frac{1 - (-1)^{n+[n/3]}}{2} \left[\frac{n-1}{3} \right].$$

$$7. \sum_{k=0}^{n-1} \left[x + \frac{mk}{n} \right] = [nx] + \frac{1}{2} (m-1)(n-1) \quad [m, n - \text{взаимно простые числа}].$$

4.2. БИНОМИАЛЬНЫЕ КОЭФФИЦИЕНТЫ

Обозначения: a, b — любые комплексные числа, при которых биномиальные коэффициенты существуют (относительно определения и свойств биномиальных коэффициентов см. приложение 1); $!, m, n = 0, 1, 2, \dots$

4.2.1. Суммы вида $\sum (\pm 1)^k \binom{a_k}{b_k}$.

$$1. \sum_{k=0}^n \binom{n}{k} = 2^n. \quad 2. \sum_{k=0}^n (-1)^k \binom{a}{k} = \binom{n-a}{n} = \prod_{k=1}^n \left(1 - \frac{a}{k}\right).$$

$$3. \sum_{k=0}^n (-1)^k \binom{n}{k} = \delta_{0,n}. \quad 4. \sum_{k=0}^n (-1)^k \binom{m}{k} = (-1)^n \binom{m-1}{n}.$$

$$5. \sum_{k=m}^n (-1)^k \binom{a}{k} = (-1)^m \binom{a-1}{m-1} + (-1)^n \binom{a-1}{n}.$$

$$6. \sum_{k=0}^{[n/2]} \binom{n}{k} = 2^{n-1} + \frac{1 + (-1)^n}{4} \binom{n}{n/2}.$$

$$7. \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} = 2^{n-1} - \frac{1 + (-1)^n}{4} \binom{n}{n/2}.$$

$$8. \sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^{n-1}. \quad 9. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} = 2^{n/2} \cos \frac{n\pi}{4}.$$

$$10. \sum_{k=0}^{[n/2]} \binom{2n}{2k} = 2^{2n-2} + \frac{1+(-1)^n}{2} \binom{2n-1}{n}.$$

$$11. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} = 2^{n-1} \quad [n \geq 1].$$

$$12. \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} = 2^{n/2} \sin \frac{n\pi}{4}.$$

$$13. \sum_{k=0}^{[(n-1)/2]} \binom{2n}{2k+1} = 2^{2n-2} + \frac{1-(-1)^n}{2} \binom{2n-1}{n} \quad [n \geq 1].$$

$$14. \sum_{k=0}^{[n/3]} \binom{n}{3k} = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right).$$

$$15. \sum_{k=0}^{[(n-1)/3]} \binom{n}{3k+1} = \frac{1}{3} \left[2^n + 2 \cos \frac{(n-2)\pi}{3} \right].$$

$$16. \sum_{k=0}^{[n/3]} \binom{n}{3k+2} = \frac{1}{3} \left[2^n + 2 \cos \frac{(n+2)\pi}{3} \right].$$

$$17. \sum_{k=0}^{[n/4]} \binom{n}{4k} = \frac{1}{4} \left[2^n + 2^{(n+2)/2} \cos \frac{n\pi}{4} \right].$$

$$18. \sum_{k=0}^{[n/m]} \binom{n}{mk} = \frac{2^n}{m} \sum_{k=1}^m \cos^n \frac{k\pi}{m} \cos \frac{nk\pi}{m}.$$

$$19. \sum_{k=0}^{[(n-m)/l]} \binom{n}{m+kl} = \frac{1}{l} \sum_{k=1}^l 2 \cos^n \frac{k\pi}{l} \cos \frac{(n-2m)k\pi}{l} \quad [n \geq m \geq 0, l-1 \geq m].$$

$$20. \sum_{k=0}^{m-n} (-1)^k \binom{m}{n+k} = \binom{m-1}{n-1}. \quad 21. \sum_{k=0}^n (-1)^k \binom{a}{n-k} = \binom{a-1}{n}.$$

$$22. \sum_{k=0}^{[n/2]} \binom{2n}{n+2k} = 2^{2n-2} + \binom{2n-1}{n}.$$

$$23. \sum_{k=0}^{[n/2]} \binom{n-k}{k} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.$$

$$24. \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} = \frac{(-1)^{[n/3]} + (-1)^{[(n+1)/3]}}{2}.$$

$$25. \sum_{k=m}^n \binom{k-a}{k} = (-1)^m \binom{a-2}{m-1} + (-1)^n \binom{a-2}{n}.$$

$$26. \sum_{k=0}^n \binom{a-k}{b} = \binom{a+1}{b+1} - \binom{a-n}{b+1}.$$

$$27. \sum_{k=0}^n \binom{a+k}{m} = \binom{a+n+1}{m+1} - \binom{a}{m+1}.$$

$$28. \sum_{k=0}^n \binom{a+k}{a} = \binom{a+n+1}{n}.$$

$$29. \sum_{k=0}^n (-1)^k \binom{m+k}{l} = (-1)^n \binom{m}{l-n}.$$

$$30. \sum_{k=m}^n \binom{k}{a} = \binom{n+1}{a+1} - \binom{m}{a+1}.$$

$$31. \sum_{k=0}^n \binom{2k+1}{m} = \frac{(-1)^{m+1}}{2^{m+2}} \left[\sum_{k=0}^{m-1} (-1)^k \binom{2n+3}{k} 2^{k+1} \right].$$

4.2.2. Суммы вида $\sum a_k \binom{b_k}{c_k}$.

$$1. \sum_{k=1}^n k^m \binom{n}{k} = \left(x \frac{d}{dx} \right)^m [(1+x)^n - 1] \Big|_{x=1}.$$

$$2. = 2^{n-m} \binom{n}{m} m! + 2 \sum_{k=1}^{m-1} (-1)^k \binom{n}{k} \frac{1}{2^k} \sum_{j=1}^k (-1)^j \binom{k}{j} j^m.$$

$$3. \sum_{k=0}^n (-1)^k k^m \binom{n}{k} = 0$$

[$m < n$].

$$4. \sum_{k=0}^n (-1)^k k^n \binom{n}{k} = (-1)^n n!$$

$$5. \sum_{k=1}^n (-1)^k k^{n+1} \binom{n}{k} = \frac{(-1)^n n}{2} (n+1)!$$

$$6. \sum_{k=1}^n \binom{n}{k} = 2^n - 1. \quad 7. \sum_{k=1}^n k \binom{n}{k} = 2^{n-1} n.$$

$$8. \sum_{k=1}^n k^2 \binom{n}{k} = 2^{n-2} n (n+1). \quad 9. \sum_{k=1}^n k^3 \binom{n}{k} = 2^{n-3} n^2 (n+3).$$

$$10. \sum_{k=1}^n k^4 \binom{n}{k} = 2^{n-4} n (n+1) (n^2 + 5n - 2).$$

$$11. \sum_{k=0}^n (-1)^k (k)_m \binom{n}{k} = 0 \quad [m < n].$$

$$12. \sum_{k=1}^n k \binom{2n}{k} = n 2^{2n-1}.$$

$$13. \sum_{k=1}^n k \binom{2n+1}{k} = (2n+1) 2^{2n-1} - \frac{2n+1}{2} \binom{2n}{n}.$$

$$14. \sum_{k=1}^n k \binom{2n+1}{2k+1} = (2n-1) 2^{2n-2} \quad [n \geq 1].$$

$$15. \sum_{k=0}^n (-1)^k (k+a)^m \binom{n}{k} = 0 \quad [m=0, 1, \dots, n-1].$$

$$16. \sum_{k=0}^n (-1)^k (k+a)^n \binom{n}{k} = (-1)^n n!$$

$$17. \sum_{k=0}^n (-1)^k (k+a)^{n-m} \binom{n}{k} = 0 \quad [m=1, 2, \dots, n].$$

$$18. \sum_{k=0}^n (-1)^k [(k+a)^{n+1} - (k+b)^{n+1}] \binom{n}{k} = (-1)^n (n+1)! (a-b).$$

$$19. \sum_{k=0}^n (-1)^k (m+k)^{n+1} \binom{n}{k} = (-1)^n \left(m + \frac{n}{2}\right) (n+1)!$$

$$20. \sum_{k=0}^n (-1)^k (m+k)^{n+2} \binom{n}{k} = (-1)^n \frac{(n+2)!}{24} [3n^2 + (12m+1)n + 12m^2].$$

$$21. \sum_{k=0}^n (-1)^k (n-k)^{n+3} \binom{n}{k} = \frac{n^2 (n+1)^2 (n+2) (n+3)}{48} n!$$

$$\begin{aligned}
 22. \quad & \sum_{k=0}^n (-1)^k (k+a)^{n-1/2} \binom{n}{k} = \\
 & = \frac{1}{2\pi} (-1)^{n-1/2} \Gamma\left(n+\frac{1}{2}\right) \int_0^\infty \{\cos[(n+2a)t] - \sin[(n+2a)t]\} t^{-n-1/2} \sin^n t \, dt \\
 & \quad [a < -n < 0].
 \end{aligned}$$

$$23. \quad \sum_{\substack{k=0 \\ k \neq n/2}}^n (n-2k)^{\pm 2m+1} \binom{n}{k} = 0 \quad [m=0, 1, 2, \dots].$$

$$24. \quad \sum_{\substack{k=0 \\ k \neq n/2}}^n (-1)^k (n-2k)^{n \pm 2m+1} \binom{n}{k} = 0 \quad [m=1, 2, 3, \dots].$$

$$25. \quad \sum_{k=0}^{[(n-1)/2]} (-1)^k (n-2k)^s \binom{n}{k} = \frac{1}{\pi} 2^n \Gamma(s+1) \sin \frac{(n-s)\pi}{2} \int_0^\infty t^{-s-1} \sin^n t \, dt \\
 [n=1, 2, 3, \dots; -1 \leq s < n].$$

$$26. \quad \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} (n-2k)^n = 2^{n-1} n!$$

$$27. \quad \sum_{k=0}^{[n/2]} (-1)^k (n-2k)^{n+2} \binom{n}{k} = \frac{1}{3} 2^{n-2} n (n+2)!$$

$$28. \quad \sum_{k=0}^{[(n-1)/2]} (n-2k)^2 \binom{n}{k} = \frac{n(2n-2)!}{[(n-1)!]^2}.$$

$$29. \quad \sum_{k=0}^n (n-2k)^2 \binom{n}{k} = 2^n n. \quad 30. \quad \sum_{k=0}^n (-1)^k (n-2k)^2 \binom{n}{k} = 8\delta_{2,n}.$$

$$31. \quad \sum_{\substack{k=0 \\ k \neq n/2}}^n (-1)^k (n-2k)^m \binom{2n}{2k} = 0 \\
 [m+n - \text{нечетное число}, m=0, \pm 1, \pm 2, \dots].$$

$$32. \quad \sum_{k=1}^{n-1} (-1)^k (n-k)^s \binom{2n}{k} = (-1)^{n+1} \frac{1}{\pi} \Gamma(s+1) 2^{2n-2} \sin \frac{\pi s}{2} \int_0^\infty t^{-s-1} \sin^{2n} t \, dt \\
 [0 < s < 2n].$$

$$33. \quad \sum_{k=0}^n (-1)^k (n-k)^{2n} \binom{2n}{k} = \frac{1}{2} (2n)!.$$

$$34. \sum_{k=0}^n (-1)^k (n-k)^{2m} \binom{2n}{k} = 0 \quad [m=1, 2, \dots, n-1].$$

$$35. \sum_{k=1}^n k! k n^{-k} \binom{n}{k} = n.$$

$$36. \sum_{k=1}^n k^{k-1} (n-k+1)^{n-k} \binom{n}{k} = n (n+1)^{n-1}.$$

$$37. \sum_{k=1}^n (\pm 1)^k k n^{n-1-k} \binom{n}{k} = \pm (n \pm 1)^{n-1} \quad [n \geq 1].$$

$$38. \sum_{k=1}^{n-1} k^{n-k-1} (n-k)^{k-1} \binom{n}{k} = 2n^{n-2} \quad [n \geq 2].$$

$$39. \sum_{k=0}^n (-1)^k k^{n-k} (k \mp 1)^{k-1} \binom{n}{k} = \mp (\pm 1)^n.$$

$$40. \sum_{k=0}^n (n-k)^{n-k} (k-1)^{k-1} \binom{n}{k} = -(n-1)^n.$$

$$41. \sum_{k=0}^n (-1)^k (a+kc)^{k-1} (b+kc)^{n-k} \binom{n}{k} = \frac{1}{a} (b-a)^n.$$

$$42. \sum_{k=0}^n \frac{1}{m+k} \binom{n}{k} = (m-1)! \left[\frac{n! (-1)^m}{(m+n)!} - \frac{2^n}{m!} \sum_{k=1}^m (-2)^k \frac{m! n!}{(m-k)! (n+k)!} \right].$$

$$43. \sum_{k=0}^n \frac{(-1)^k k^m}{a+k} \binom{n}{k} = (-1)^m a^{m-1} \binom{a+n}{n}^{-1} \quad [m \leq n].$$

$$44. \sum_{k=0}^n \frac{(-1)^k}{m+k} \binom{n}{k} = \frac{1}{m} \binom{m+n}{m}^{-1}.$$

$$45. \sum_{k=0}^n \frac{(-1)^k}{k+a} \binom{n}{k} = \frac{n!}{a(a+1) \dots (a+n)}.$$

$$46. \sum_{\substack{k=0 \\ k \neq m}}^n \frac{(-1)^k}{m-k} \binom{n}{k} = (-1)^n \binom{n}{m} \left(\sum_{k=1}^{n-m} \frac{1}{k} - \sum_{k=1}^m \frac{1}{k} \right) \quad [m=0, 1, \dots, n].$$

$$47. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^n \frac{1}{k}. \quad 48. \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{1}{n+1} (2^{n+1} - 1).$$

$$49. \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1}. \quad 50. \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{(2n)!!}{(2n+1)!!}.$$

$$51. \sum_{k=1}^n \frac{(-1)^{k+1}}{k(k+1)} \binom{n}{k} = \sum_{k=2}^{n+1} \frac{1}{k}. \quad 52. \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{m=1}^k \frac{1}{m} = \frac{1}{n}.$$

$$53. \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{m=1}^{2k} \frac{1}{m} = \frac{1}{2n} + \frac{(2n-2)!!}{(2n-1)!!}.$$

$$54. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n-k} \binom{n-k}{k} = \begin{cases} (-1)^n 2/n & [n=3m], \\ (-1)^{n-1}/n & [n \neq 3m]. \end{cases}$$

$$55. \sum_{k=0}^n \frac{1}{a-k} \binom{a-k}{n-k} = \frac{1}{a-n} \binom{a}{n}.$$

$$56. \sum_{k=1}^n \frac{1}{a+k} \binom{a+k}{k} = \frac{1}{a} \left[\binom{a+n}{n} - 1 \right].$$

$$57. \sum_{k=0}^n (-1)^k \frac{1}{(k+a)^2} \binom{n}{k} = n! \left(\prod_{k=0}^n \frac{1}{k+a} \right) \left(\sum_{k=0}^n \frac{1}{k+a} \right) \quad [a \neq 0, -1, -2, \dots].$$

$$58. \sum_{k=0}^n (-1)^k \frac{1}{(k+1)^2} \binom{n}{k} = \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{1}{k}. \quad 59. \sum_{k=0}^n \frac{n-2k}{(n-2k)^2 - a^2} \binom{n}{k} = 0.$$

4.2.3. Суммы вида $\sum a_k \binom{b_k}{c_k} x^k$.

$$1. \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n. \quad 2. \sum_{k=0}^{[n/2]} \binom{n}{2k} x^k = \frac{(1+\sqrt{x})^n + (1-\sqrt{x})^n}{2}.$$

$$3. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^k = \frac{(1+\sqrt{x})^n - (1-\sqrt{x})^n}{2\sqrt{x}}.$$

$$4. \sum_{k=0}^{[n/2]} (-3)^k \binom{n}{2k} = \begin{cases} (-1)^n 2^n & [n=3m], \\ (-1)^{n-1} 2^{n-1} & [n \neq 3m]. \end{cases}$$

$$5. \sum_{k=0}^{[(n-1)/2]} (-3)^k \binom{n}{2k+1} = \begin{cases} 0 & [n=3m], \\ 2^{n-1} & [n=6m+1, 6m+2], \\ -2^{n-1} & [n=6m+4, 6m+5]. \end{cases}$$

$$6. \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k = 2^{-n-1} (1+4x)^{-1/2} [(1+\sqrt{1+4x})^{n+1} - (1-\sqrt{1+4x})^{n+1}].$$

$$7. \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{2k} = x^n U_n \left(\frac{1}{2x} \right).$$

$$8. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^{2k}} \binom{n-k}{k} = (n+1) 2^{-n}.$$

$$9. \sum_{k=0}^n \binom{2n-k}{k} x^k = \frac{1}{2^n} \frac{(2x+1 \pm \sqrt{1+4x})^{2n+1} + (2x)^{2n+1}}{(2x+1 \pm \sqrt{1+4x})^n (4x+1 \pm \sqrt{1+4x})}.$$

$$10. \sum_{k=0}^n 2^k \binom{2n-k}{n} = 2^{2n}. \quad 11. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{2n-k}{k} = \frac{2n+1}{2^{2n}}.$$

$$12. \sum_{k=0}^n \frac{1}{2^k} \binom{n+k}{k} = 2^n. \quad 13. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n+k}{2k} = (-1)^n (2n+1).$$

$$14. \sum_{k=0}^n \frac{1}{2^k} \binom{n+k}{2k} = \frac{2^{2n+1} + 1}{3 \cdot 2^n}.$$

$$15. \sum_{k=0}^n (-1)^k \binom{n+k}{2k} x^{2k} = \sec \left(\arcsin \frac{x}{2} \right) \cos \left[(2n+1) \arcsin \frac{x}{2} \right].$$

$$16. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n+k+1}{2k+1} = (-1)^n (n+1).$$

$$17. \sum_{k=0}^n (-1)^k \binom{n+k+1}{2k+1} x^{2k+1} = \sec \left(\arcsin \frac{x}{2} \right) \sin \left[2(n+1) \arcsin \frac{x}{2} \right].$$

$$18. \sum_{k=0}^n k^m \binom{n}{k} x^k = (1+x)^n \sum_{k=0}^m (-1)^k \binom{n}{k} \frac{x^k}{(1+x)^k} \sum_{l=0}^k (-1)^l \binom{k}{l} l^m.$$

$$19. \sum_{k=0}^n \frac{k^m}{2^{2k}} \binom{2k}{k} = S_m,$$

$$S_0 = \frac{2n+1}{2^{2n}} \binom{2n}{n}, \quad S_1 = \frac{n}{3} S_0, \quad S_2 = \frac{n(3n+2)}{15} S_0, \quad S_3 = \frac{n(15n^2+18n+2)}{15 \cdot 7} S_0.$$

$$20. \sum_{k=0}^n \frac{(-1)^k}{k+a} \binom{n}{k} x^k = x^{-a} \int_0^x t^{a-1} (1 \pm t)^n dt \quad [a \neq 0, -1, \dots, -n].$$

$$21. \sum_{k=1}^n \frac{1}{k} \binom{n}{k} x^k = [(1+x)^n - 1] \ln x - n \int_0^x (1+t)^{n-1} \ln t \, dt.$$

$$22. \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} x^k = \frac{(1+x)^{n+1} - 1}{(n+1)x}.$$

$$23. \sum_{k=0}^{[n/2]} \frac{1}{2k+1} \binom{n}{2k} x^{2k} = \frac{(1+x)^{n+1} - (1-x)^{n+1}}{2(n+1)x}.$$

$$24. \sum_{k=0}^{[(n-1)/2]} \frac{1}{k+1} \binom{n}{2k+1} x^{2k} = \frac{(1+x)^{n+1} + (1-x)^{n+1} - 2}{(n+1)x^2}.$$

$$25. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n-k} \binom{n-k}{k} x^{2k} = \frac{2}{n} x^n T_n \left(\frac{1}{2x} \right).$$

$$26. = \frac{1}{2^n n} [(1 + \sqrt{1-4x^2})^n + (1 - \sqrt{1-4x^2})^n].$$

$$27. \sum_{k=0}^{[n/2]} \frac{6^k}{n-k} \binom{n-k}{k} = \frac{3^n + (-1)^n 2^n}{n}.$$

$$28. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n-2k} \binom{n-k-1}{k} x^{2k} = \frac{2}{n} x^n T_n \left(\frac{1}{2x} \right).$$

$$29. \sum_{k=0}^n (-1)^k \frac{(k+a)^{n+1} - 1}{k+a-1} \binom{n}{k} = (-1)^n n!$$

$$30. \sum_{k=0}^n \frac{k!}{a^{k+1}} \binom{a}{k} = 1 - \frac{(n+1)!}{a^{n+1}} \left(\frac{a}{n+1} \right).$$

$$31. \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{x^k}{k!} = L_n^\alpha(-x).$$

$$32. \sum_{k=0}^m \binom{m+1}{k} (1^k + 2^k + \dots + n^k) = (n+1)^{m+1} - 1.$$

$$33. \sum_{k=0}^n \binom{2n-k}{n} \frac{x^k}{k!} = \frac{1}{n! \sqrt{\pi}} x^{n+1/2} e^{x/2} K_{n+1/2} \left(\frac{x}{2} \right).$$

4.2.4. Суммы вида $\sum a_k \binom{b_k}{c_k}^{-1}$.

$$1. \sum_{k=0}^n \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

$$2. \sum_{k=0}^n (-1)^k \binom{a}{k}^{-1} = \frac{a+1}{a+2} \left[1 + (-1)^n \binom{a+1}{n+1}^{-1} \right].$$

$$3. \sum_{k=0}^n \binom{a+k}{k}^{-1} = \frac{a}{a-1} \left[1 - \binom{a+n}{n+1}^{-1} \right].$$

$$4. \sum_{k=0}^n \binom{a+k}{m}^{-1} = \frac{m}{m-1} \left[\binom{a-1}{m-1}^{-1} - \binom{n+a}{m-1}^{-1} \right].$$

$$5. \sum_{k=1}^n (-1)^{k+1} \binom{2n}{k}^{-1} = \frac{1}{2(n+1)} + \frac{(-1)^{n+1}}{2} \binom{2n}{n}^{-1}.$$

$$6. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^{-1} = \frac{2n+1}{n+1}. \quad 7. \sum_{k=1}^{2n-1} (-1)^{k+1} k \binom{2n}{k}^{-1} = \frac{n}{n+1}.$$

$$8. \sum_{k=1}^m (n-2k) \binom{n}{k}^{-1} = 1 - (n+1) \binom{n+1}{m+1}^{-1}.$$

$$9. \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1} \binom{2n}{k}^{-1} = (2n+1) \sum_{k=1}^n \frac{1}{(k+n)^2}.$$

$$10. \sum_{k=1}^{2n-1} (-1)^{k+1} \binom{2n}{k}^{-1} \sum_{m=1}^k \frac{1}{m} = \frac{n}{2(n+1)^2} + \frac{1}{2n+2} \sum_{k=1}^{2n} \frac{1}{k}.$$

$$11. \sum_{k=1}^{2n+1} (-1)^{k+1} \binom{2n+1}{k}^{-1} \sum_{m=1}^k \frac{1}{m} = \frac{2n+2}{2n+3} \sum_{k=1}^{2n+2} \frac{1}{k}.$$

$$12. \sum_{k=0}^n \binom{n}{k}^{-1} x^k = (n+1) \left(\frac{x}{1+x} \right)^{n+1} \sum_{k=1}^{n+1} \frac{1}{k} \frac{1+x^k}{1+x} \left(\frac{1+x}{x} \right)^k.$$

$$13. \sum_{k=1}^n \frac{2^{2k-1}}{k} \binom{2k}{k}^{-1} = 2^{2n} \binom{2n}{n}^{-1} - 1.$$

4.2.5. Суммы вида $\sum (\pm 1)^k \binom{a_k}{b_k} \binom{c_k}{d_k}$.

$$1. \sum_{k=0}^n \binom{n^2}{k} = \binom{2n}{n}. \quad 2. \sum_{k=0}^{[(n-1)/2]} \binom{n^2}{k} = \frac{1}{2} \binom{2n}{n} - \frac{1+(-1)^n}{4} \binom{n}{n/2}^2.$$

$$3. \sum_{k=0}^n (-1)^k \binom{n^2}{k} = \frac{(-1)^{n/2} [1+(-1)^n]}{2} \binom{n}{n/2}.$$

$$4. \sum_{k=0}^n \binom{2n^2}{k} = \frac{1}{2} \binom{4n}{2n} + \frac{1}{2} \binom{2n^2}{n}.$$

$$5. \sum_{k=0}^n (-1)^k \binom{2n^2}{k} = \frac{(-1)^n}{2} \left[\binom{2n}{n} + \binom{2n^2}{n} \right].$$

$$6. \sum_{k=0}^{2n} (-1)^k \binom{2n^2}{k} = (-1)^n \binom{2n}{n}.$$

$$7. \sum_{k=0}^{[n/2]} \binom{n^2}{2k} = \frac{1}{2} \binom{2n}{n} + \frac{(-1)^{n/2} [1+(-1)^n]}{4} \binom{n}{n/2}.$$

$$8. \sum_{k=0}^{[(n-1)/2]} \binom{n^2}{2k+1} = \frac{1}{2} \binom{2n}{n} - \frac{(-1)^{n/2} [1+(-1)^n]}{4} \binom{n}{n/2}.$$

$$9. \sum_{k=0}^{[(n-1)/2]} \binom{2n^2}{2k+1} = \frac{1}{4} \binom{4n}{2n} - \frac{(-1)^n}{4} \binom{2n}{n} + \frac{1-(-1)^n}{4} \binom{2n^2}{n}.$$

$$10. \sum_{k=0}^n \binom{a}{k} \binom{b}{k} = \sum_{k=0}^n (-1)^k \binom{n-a}{k} \binom{b+n-k}{n}.$$

$$11. \sum_{k=0}^n \binom{n}{k} \binom{m-n}{k} = \binom{m}{n}. \quad 12. \sum_{k=0}^n \binom{-1/2}{k} \binom{1/2}{k} = \frac{2n+1}{2^{2n}} \binom{2n}{n}^2.$$

$$13. \sum_{k=0}^n \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}.$$

$$14. \sum_{k=0}^n \binom{-a}{k} \binom{a}{m+k} = \frac{a}{a-m} \binom{-a-1}{n} \binom{a-1}{m+n}.$$

$$15. \sum_{k=0}^m \binom{a}{k} \binom{-a}{n-k} = - \sum_{k=m+1}^n \binom{a}{k} \binom{-a}{n-k}.$$

$$16. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{b}{n-k} = \sum_{k=0}^{[n/2]} (-1)^k \binom{a}{k} \binom{b-a}{n-2k}.$$

$$17. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a}{n-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{a}{n/2}.$$

$$18. \sum_{k=0}^m \binom{a}{k} \binom{1-a}{n-k} = \frac{(n-1)(1-a) - m}{n(n-1)} \binom{a-1}{m} \binom{-a}{n-m-1}$$

$$[n \geq 2, \quad 1 \leq m \leq n-1].$$

$$19. \sum_{k=0}^n \binom{n}{k} \binom{a+b}{a-k} = \binom{a+b+n}{a}.$$

$$20. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2a-2n}{a-k} = (-1)^n \binom{a}{n} \binom{2a}{a} \binom{2a}{2n}^{-1}.$$

$$21. \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2a-2n}{a-k} = \frac{(-1)^n}{2} \binom{2a}{a} \binom{2a}{2n}^{-1} \left[\binom{a}{n} + \binom{a}{n}^2 \right].$$

$$22. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{2n-a}{n-k} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-a).$$

$$23. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2a-n}{a-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{a}{n/2} \binom{2a}{a} \binom{2a}{n}^{-1}.$$

$$24. \sum_{k=0}^n \binom{n}{k} \binom{n+2a}{k+a} = \binom{2a+2n}{a+n}. \quad 25. \sum_{k=0}^n \binom{n}{k} \binom{a}{m+k} = \binom{n+a}{n+m}.$$

$$26. \sum_{k=0}^n \binom{n}{k} \binom{a}{k-m} = \binom{n+a}{n-m}. \quad 27. \sum_{k=0}^n \binom{a}{k} \binom{a}{2n-k} = \frac{1}{2} \binom{2a}{2n} + \frac{1}{2} \binom{a}{n}^2.$$

$$28. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a}{2n-k} = \frac{(-1)^n}{2} \left[\binom{a}{n} + \binom{a}{n}^2 \right].$$

$$29. \sum_{k=0}^{2n} (-1)^k \binom{a}{k} \binom{a}{2n-k} = (-1)^n \binom{a}{n}.$$

$$30. \sum_{k=0}^{[n/2]} (-1)^k \binom{a}{k} \binom{-a}{n-2k} = (-1)^n \binom{a}{n}.$$

$$31. \sum_{k=0}^n \binom{a}{k} \binom{a}{2n-k+1} = \sum_{k=0}^n \binom{a}{n-k} \binom{a}{n+k+1} = \frac{1}{2} \binom{2a}{2n+1}.$$

$$32. \sum_{k=0}^{[n/2]} (-1)^k \binom{a}{k} \binom{2n-2a}{n-2k} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-a).$$

$$33. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2a}{k+a} = (-1)^n \binom{2n}{n} \binom{2n+2a}{a} \binom{n+a}{n}^{-1}.$$

$$34. \sum_{k=0}^{2m-n+1} (-1)^k \binom{n}{k} \binom{n}{2m+1-k} = 0.$$

$$35. \sum_{k=-n}^n (-1)^k \binom{2n}{k} \binom{2m}{m-k} = (-1)^n \frac{(2m)! (2n)!}{m! n! (m+n)!}.$$

$$36. \sum_{k=0}^n \binom{a+k}{a} \binom{b+n-k}{b} = \binom{a+b+n+1}{n}.$$

$$37. \sum_{k=0}^n \binom{a+k}{a} \binom{b-k}{b-n} = \binom{a+b+1}{n}.$$

$$38. \sum_{k=0}^n (-1)^k \binom{a+k}{a} \binom{a+n-k}{a} = \binom{a+n/2}{n/2} \frac{1+(-1)^n}{2}.$$

$$39. \sum_{k=l}^{n-m} \binom{k}{l} \binom{n-k}{m} = \binom{n+l}{l+m+1}.$$

$$40. \sum_{k=m}^{n-m} (-1)^k \binom{k}{m} \binom{n-k}{m} = (-1)^m \frac{1+(-1)^n}{2} \binom{n/2}{m}.$$

$$41. \sum_{k=0}^{n-1} \binom{k}{m} \binom{k+l}{l} = \binom{n}{m} \binom{n+l}{l} \frac{n-m}{l+m+1}.$$

$$42. \sum_{k=0}^n \binom{n+2k+1}{n} \binom{n+2m-2k+1}{n} = \frac{1}{2} \binom{2m+2n+3}{2n+1} - \frac{1}{2} \binom{m+n+1}{n+1}.$$

$$43. \sum_{k=0}^n \binom{a+kc}{k} \binom{b-kc}{n-k} = \sum_{k=0}^n \binom{a+b-k}{n-k} c^k.$$

$$44. \sum_{k=0}^n \binom{a+kb}{k} \binom{m-a-kb}{n-k} = \begin{cases} b^{m+1} (b-1)^{n-m-1} & [0 \leq m \leq n-1], \\ \frac{b^{n+1}-1}{b-1} & [m=n] \end{cases}$$

$$45. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^{n+1} \delta_{n,m}. \quad 46. \sum_{k=m}^n \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m}.$$

$$47. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+kb}{m} = (-1)^n b^n \delta_{n,m} \quad [0 \leq m \leq n].$$

$$48. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{b-k}{b-n} = \binom{b-a}{n}. \quad 49. \sum_{k=m}^n (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^n \delta_{n,m}.$$

$$50. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a+n-k-1}{a-1} = 0.$$

$$51. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+m+n-k}{m} = 0 \quad [m < n].$$

$$52. \sum_{k=0}^n \binom{n}{k} \binom{k/2}{m} = \frac{n}{m} 2^{n-2m} \binom{n-m-1}{m-1}.$$

$$53. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a+n-k}{a-1} = (-1)^n \binom{a}{n+1}.$$

$$54. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k/2}{m} = (-1)^m 2^{n-2m} \left[\binom{2m-n-1}{m-1} - \binom{2m-n-1}{m} \right].$$

$$55. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+k}{m} = (-1)^n \binom{a}{m-n}.$$

$$56. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-k}{m} = \binom{a-n}{m-n}.$$

$$57. \sum_{k=0}^n \binom{4n}{2k} \binom{2n-k}{n} = \frac{2^{2n+1}}{3} \binom{3n}{n}.$$

$$58. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{2n-k}{n} = \binom{2n-a}{n}.$$

$$59. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a-k}{a-n} = \delta_{0,n}. \quad 60. \sum_{k=0}^n (-1)^k \binom{m}{k} \binom{m+n-k-1}{m-1} = 0.$$

$$61. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{m} = (-1)^n 2^{2n-m} \binom{n}{m-n}.$$

$$62. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n+m} = 2^{n-m} \binom{n}{m}.$$

$$63. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-2k}{2n-m} = 2^m \binom{n}{m}.$$

$$64. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{a}{k} \binom{2a-2k}{2a-n} = 2^n \binom{a}{n}.$$

$$65. \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \binom{2n-2k+a}{n-1} = \binom{n-a-2}{n-1}.$$

$$66. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m}{k} \binom{m+n-2k-1}{m-1} = \binom{m}{n}.$$

$$67. \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{4n-2k+1}{2n+1} = (n+1) 2^{2n}.$$

$$68. \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k-3}{n} = \frac{(n-1)n}{2}.$$

$$69. \sum_{k=0}^{\lfloor (m-1)(n-1)/m \rfloor} (-1)^k \binom{n}{k} \binom{mn-mk-m}{n} = \binom{m+n-2}{n-1}.$$

$$70. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m-2k}{n-1} = 0 \quad \{m \geq 2n\}.$$

$$71. \sum_{k=0}^n (-1)^k \binom{2k}{k} \binom{n+k}{2k} = (-1)^n. \quad 72. \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} = 2^{2n}.$$

$$73. \sum_{k=0}^n (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = [1 + (-1)^n] 2^{n-1} \binom{n}{n/2}.$$

$$74. \sum_{k=0}^n \binom{4k}{2k} \binom{4n-4k}{2n-2k} = 2^{2n-1} \binom{2n}{n} + 2^{4n-1}.$$

$$75. \sum_{k=0}^{n-1} \binom{4k+2}{2k+1} \binom{4n-4k-2}{2n-2k-1} = 2^{4n-1} - 2^{2n-1} \binom{2n}{n}.$$

$$76. \sum_{k=0}^{[n/2]} \binom{a}{2k} \binom{a}{n-2k} = \frac{1}{2} \binom{2a}{n} + \frac{(-1)^{n/2} [1 + (-1)^n]}{4} \binom{a}{n/2}.$$

$$77. \sum_{k=0}^n \binom{a}{2k} \binom{a}{2n-2k} = \frac{1}{2} \binom{2a}{2n} + \frac{(-1)^n}{2} \binom{a}{n}.$$

$$78. \sum_{k=0}^{n-1} \binom{n}{2k} \binom{m+k}{n-1} = \binom{2m+1}{n-1}.$$

$$79. \sum_{k=m}^{[n/2]} \binom{n}{2k} \binom{k}{m} = 2^{n-2m-1} \frac{n}{n-2m} \binom{n-m-1}{m}.$$

$$80. \sum_{k=0}^n \binom{2a}{2k} \binom{a-k}{a-n} = \frac{a}{a+n} 2^{2n} \binom{a+n}{2n}.$$

$$81. \sum_{k=0}^n \binom{2a+1}{2k} \binom{a-k}{a-n} = 2^{2n} \binom{a+n}{2n}.$$

$$82. \sum_{k=0}^n \binom{a}{2k} \binom{a+n-k-1}{a-1} = \binom{a+2n-1}{2n}.$$

$$83. \sum_{k=0}^{[n/2]} \binom{a}{2k} \binom{2n-a}{n-2k} = \frac{1}{2} \left[\binom{2n}{n} + (-1)^n 2^{2n} \binom{(a-1)/2}{n} \right].$$

$$84. \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{a+k}{n-1} = \binom{2a+1}{n-1}. \quad 85. \sum_{k=0}^n \binom{4n+1}{2k} \binom{2n-k}{n} = 2^{2n} \binom{3n}{n}.$$

$$86. \sum_{k=0}^n \binom{2a+1}{2k+1} \binom{a-k}{a-n} = \frac{2a+1}{2n+1} 2^{2n} \binom{a+n}{2n}.$$

$$87. \sum_{k=0}^n \binom{a}{2k+1} \binom{a+n-k-1}{a-1} = \binom{a+2n}{2n+1}.$$

$$88. \sum_{k=0}^{n-1} \binom{2a}{2k+1} \binom{a-k-1}{a-n} = \frac{n}{a+n} 2^{2n} \binom{a+n}{2n}.$$

$$89. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \binom{a+k}{n-1} = \binom{2a}{n-1}.$$

$$90. \sum_{k=m}^{[(n-1)/2]} \binom{n}{2k+1} \binom{k}{m} = 2^{n-2m-1} \binom{n-m-1}{m}.$$

$$91. \sum_{k=0}^n \binom{2a}{2k+1} \binom{a-k}{a-n} = \frac{2a+1}{2n+1} 2^{2n} \binom{a+n}{2n}.$$

$$92. \sum_{k=0}^{[(n-2m)/2]} \binom{n}{2k+1} \binom{n-1}{2m-2k} = \frac{1}{2} \binom{2n-1}{2m+1} + \frac{(-1)^m}{2} \binom{n-1}{m}.$$

$$93. \sum_{k=0}^{[(n-1)/2]} \binom{a}{2k+1} \binom{2n-a}{n-2k-1} = \frac{1}{2} \binom{2n}{n} - (-1)^n 2^{2n-1} \binom{(a-1)/2}{n}.$$

$$94. \sum_{k=0}^{[(n-1)/2]} \binom{a}{2k+1} \binom{a}{n-2k-1} = \frac{1}{2} \binom{2a}{n} - \frac{(-1)^{n/2} [1 + (-1)^n]}{4} \binom{a}{n/2}.$$

4.2.6. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k}.$

$$1. \sum_{k=1}^n k \binom{n}{k} \binom{a}{k} = n \binom{a+n-1}{n}.$$

$$2. \sum_{k=0}^n k^m \binom{n}{k}^2 = S_m,$$

$$S_0 = \binom{2n}{n}, \quad S_1 = \frac{n}{2} \binom{2n}{n}, \quad S_2 = n^2 \binom{2n-2}{n-1}, \quad S_3 = \frac{n^2(n+1)}{2} \binom{2n-2}{n-1}.$$

$$3. \sum_{k=0}^{[(n-1)/2]} (n-2k)^2 \binom{n}{k}^2 = n \binom{2n-2}{n-1}.$$

$$4. \sum_{k=1}^n \binom{n}{k}^2 \sum_{m=1}^k \frac{1}{m} = \binom{2n}{n} \left(\sum_{k=1}^n \frac{1}{k} - \sum_{k=n+1}^{2n} \frac{1}{k} \right).$$

$$5. \sum_{k=1}^n k \binom{a+k}{m} \binom{a-k}{m} = \frac{m+1}{2} \left[\binom{a+1}{m+1} \binom{a}{m+1} - \binom{a+n+1}{m+1} \binom{a-n}{m+1} \right].$$

$$6. \sum_{k=0}^n [ma - (a+b)k] \binom{a}{k} \binom{b}{m-k} = (n+1)(m-n) \binom{a}{n+1} \binom{b}{m-n}.$$

$$7. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} \binom{n+k}{n} = 2 \sum_{k=1}^n \frac{1}{k}.$$

$$8. \sum_{k=0}^n \frac{(-1)^k}{m-k} \binom{m-k}{n} \binom{n}{k} = 0 \quad [2n < m]$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k-1}{n-1} \sum_{m=1}^{n+k-1} \frac{1}{m} = \frac{(-1)^n}{n}.$$

$$10. \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k-1}{n-1} \sum_{m=1}^k \frac{1}{m} = \frac{(-1)^n}{n}.$$

$$11. \sum_{k=0}^n \frac{(-1)^k}{n+k} \binom{n}{k} \binom{n+k}{n} \sum_{m=1}^k \frac{1}{m} = \frac{(-1)^n}{n^2}.$$

$$12. \sum_{k=0}^n \frac{1}{a+k} \binom{2k}{k} \binom{2n-2k}{n-k} = 2^n \frac{(2a+1)(2a+3) \dots (2a+2n-1)}{a(a+1) \dots (a+n)}.$$

$$13. \sum_{k=0}^n \frac{(-1)^k}{a+k} \binom{2k}{k} \binom{n+k}{2k} = \frac{(-1)^n}{a} \binom{a-1}{n} \binom{a+n}{n}^{-1}.$$

$$14. \sum_{k=m}^n \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{n-m+1}{2(n+1)} \binom{2m}{m} \binom{2n-2m+2}{n-m+1}.$$

$$15. \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k-1} = \binom{2n+1}{n-1}.$$

$$16. \sum_{k=0}^n \frac{1}{2k-1} \binom{2k}{k} \binom{2n-2k}{n-k} = -\delta_{0,n}.$$

$$17. \sum_{k=0}^n \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1}.$$

$$18. \sum_{k=0}^n \frac{1}{(k+1)(n-k+1)} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{(2n+2)!}{(n+1)!(n+2)!}.$$

$$19. \sum_{k=0}^n \frac{1}{(2k-1)(2n-2k-1)} \binom{2k}{k} \binom{2n-2k}{n-k} = (-1)^n 2^{2n} \binom{1}{n}.$$

$$20. \sum_{k=0}^n \frac{1}{(2k-1)(2n-2k+1)} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1}{n(2n+1)} 2^{2n-1} \binom{2n}{n}^{-1}.$$

$$21. \sum_{k=0}^n \binom{a+kc}{k} \binom{b-kc}{n-k} \frac{p+qk}{(a+kc)(b-kc)} = \frac{p(a+b-nc) + naq}{a(a+b)(b-nc)} \binom{a+b}{n}.$$

$$22. \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \binom{ka}{k} \binom{na-ka}{n-k} = \frac{2a}{n} \binom{na}{n} \sum_{k=1}^{n-1} \frac{1}{na-n+k}.$$

$$23. \sum_{k=0}^n \frac{1}{(mk+1)(mn-mk+1)} \binom{mk+1}{k} \binom{mn-mk+1}{n-k} = \frac{2}{2+mn} \binom{2+mn}{n}.$$

$$24. \sum_{k=0}^n \frac{2k+1}{n+k+1} \binom{a-k-1}{a-n-1} \binom{a+k}{a-n} = \binom{a}{n}^2.$$

$$25. \sum_{k=1}^n \binom{m}{k} k^n \sum_{l=0}^{n-k} (-1)^l \binom{m-k}{l} = m^n \quad [n \geq 1].$$

$$26. \sum_{k=0}^{[2n/3]} (-1)^k \sum_{l=0}^{[k/2]} \binom{n-k}{k-l} \binom{k-l}{l} = \frac{1}{2} [1 + (-1)^{[n/2]}].$$

4.2.7. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k} x^k$.

$$1. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} x^k = \sum_{k=0}^n \binom{n}{k} \binom{a+n-k}{n} (x-1)^k.$$

$$2. \sum_{k=0}^n \binom{n}{k} \binom{a+k}{k} (b-c)^{n-k} c^k = \sum_{k=0}^n \binom{n}{k} \binom{a}{k} b^{n-k} c^k.$$

$$3. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} 2^k = \sum_{k=0}^n \binom{n}{k} \binom{a+k}{n}.$$

$$4. \sum_{k=0}^n \binom{n}{k}^2 x^k = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (x-1)^k.$$

$$5. \sum_{k=0}^n \binom{n}{k}^2 x^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} (x-1)^{n-k}.$$

$$6. \sum_{k=0}^n \binom{n}{k}^2 x^k = (1-x)^n P_n \left(\frac{1+x}{1-x} \right).$$

$$7. \sum_{k=0}^n 2^k \binom{n}{k}^2 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{3n-2k}{2n}.$$

$$8. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1/2}{k} x^{2k} = P_{2n} (\sqrt{1-x^2}).$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+3/2}{k} x^k = (1-x)^{-1/2} P_{2n+1} (\sqrt{1-x}).$$

$$10. \sum_{k=0}^n \binom{n}{k} \binom{k}{m} x^k = \binom{n}{m} x^m (1+x)^{n-m}.$$

$$11. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{2k} = (2x)^n P_n \left(\frac{1}{x} \right).$$

$$12. \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} x^k = P_n (1+2x).$$

$$13. \sum_{k=m}^n \frac{(-1)^k}{2^k} \binom{n}{k} \binom{n+k}{m+n} = \frac{(-1)^{(m+n)/2}}{2^n} \binom{n}{(n-m)/2}$$

[$n-m$ — четное число]

$$14. \sum_{k=0}^n (-2)^k \binom{a}{k} \binom{2n-k}{n} = \frac{2^n}{n!} \prod_{k=1}^n (2k-a-1).$$

$$15. \sum_{k=0}^n (-1)^k 2^{n-k} \binom{n}{k} \binom{n+2m+k}{n+m} =$$

$$= \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{n+m}{n/2} \binom{n+2m}{m} \binom{n+m}{n}^{-1}.$$

$$16. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2m-n+k}{m} \frac{1}{2^k} =$$

$$= (-1)^{n/2} [1 + (-1)^n] \frac{1}{2^{n+1}} \binom{m}{n/2} \binom{2m-n}{m} \binom{m}{m-n}^{-1}.$$

$$17. \sum_{k=n-m}^n \frac{(-1)^k}{2^k} \binom{n}{k} \binom{n+k}{2n-m} = \frac{(-1)^{m/2+n}}{2^{n+1}} [1 + (-1)^m] \binom{n}{m/2}.$$

$$18. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} = \frac{1}{2^{2n}} \binom{2n}{n}.$$

$$19. \sum_{k=0}^{2n} \frac{(-1)^k}{3^k} \binom{2n}{k} \binom{2n+2k}{n+k} = \frac{1}{3^{2n}} \binom{2n}{n}.$$

$$20. \sum_{k=0}^{[(2m-n)/2]} \frac{1}{2^{2k}} \binom{m}{k} \binom{m-k}{n-m+k} = 2^{n-2m} \binom{2m}{n}.$$

$$21. \sum_{k=m}^n (-1)^k 2^{n-k} \binom{n}{k} \binom{2k}{k-m} = \frac{(-1)^m + (-1)^n}{2} \binom{n}{(n-m)/2}.$$

$$22. \sum_{k=0}^{[n/2]} \frac{1}{2^{2k}} \binom{a}{k} \binom{a-k}{n-2k} = \frac{1}{2^n} \binom{2a}{n}.$$

$$23. \sum_{k=0}^{2m} (-2)^k \binom{n}{k} \binom{2n-k}{2m-k} = (-1)^m \binom{n}{m}.$$

$$24. \sum_{k=0}^n (-2)^k \binom{n}{k} \binom{2n-2k}{2m-k} = \binom{n}{m}.$$

$$25. \sum_{k=0}^n (-1)^k 2^{2k} \binom{a}{k} \binom{2a-2k}{n-k} = (-1)^n \binom{2a}{n}.$$

$$26. \sum_{k=0}^n (-1)^k 2^{2k} \binom{a}{k} \binom{4a-2k}{n-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{2a}{n/2}.$$

$$27. \sum_{k=0}^n (-2)^k \binom{a}{k} \binom{b-2k}{n-k} = \sum_{k=0}^{[n/2]} \binom{a}{k} \binom{b-2a}{n-2k}.$$

$$28. \sum_{k=0}^n (-2)^k \binom{a}{k} \binom{2a-k}{n-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{a}{n/2}.$$

$$29. \sum_{k=0}^{2m+1} (-2)^k \binom{n+k-1}{n-1} \binom{2m+2n+k}{2m+1-k} = 0.$$

$$30. \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} \left(\frac{x+1}{x-1} \right)^k = \frac{2^n}{(x-1)^n} P_n^{(\alpha, \beta)}(x).$$

$$31. \sum_{k=m}^{[n/2]} \frac{(-1)^k}{2^{2k}} \binom{n-k}{k} \binom{k}{m} = \frac{(-1)^m}{2^n} \binom{n+1}{2m+1}.$$

$$32. \sum_{k=0}^n (-1)^k 2^{2k} \binom{k}{m} \binom{n+k}{2k} = (-1)^n \binom{n+m}{2m} 2^{2m} \frac{2n+1}{2m+1}.$$

$$33. \sum_{k=0}^n (-1)^k 2^{2k} \binom{k}{m} \binom{n+k}{2k+1} = (-1)^{n-1} \binom{n+m}{2m+1} 2^{2m}.$$

$$34. \sum_{k=0}^n \frac{(-1)^k}{n+k} 2^{2k} \binom{k}{m} \binom{n+k}{2k} = (-1)^n \frac{1}{n+m} 2^{2m} \binom{n+m}{2m}.$$

$$35. \sum_{k=0}^n \frac{(-1)^k}{2k+1} 2^{2k} \binom{k}{m} \binom{n+k}{2k} = \frac{(-1)^n}{2n+1} 2^{2m} \binom{n+m}{2m}.$$

$$36. \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2k}{k} \binom{2n-k}{n} = \frac{1}{2^{2n}} \binom{4n+1}{2n}.$$

$$37. \sum_{k=0}^{[n/2]} \frac{1}{2^{2k}} \binom{n}{2k} \binom{2k}{k} = \frac{1}{2^n} \binom{2n}{n}.$$

$$38. \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} = 2^{2n} x^n P_n \left[\frac{1}{2} \left(x + \frac{1}{x} \right) \right].$$

$$39. \sum_{k=0}^n 2^{2k} \binom{a}{2k} \binom{a-2k}{n-k} = \binom{2a}{2n}.$$

$$40. \sum_{k=0}^n 2^{2k+1} \binom{a+1}{2k+1} \binom{a-2k}{n-k} = \binom{2a+2}{2n+1}.$$

$$41. \sum_{k=1}^n \frac{(-2)^k k}{2n-k} \binom{a}{k} \binom{2n-k}{n} = 2^{2n} \binom{n-1-a/2}{n}.$$

$$42. \sum_{k=0}^n \frac{(-1)^k}{2^k} \frac{n-k}{n+k} \binom{2a}{n-k} \binom{n+k}{n} = 2^n \binom{a}{n}.$$

$$43. \sum_{k=m}^{[n/2]} \frac{(-1)^k}{(n-k) 2^{2k}} \binom{n-k}{k} \binom{k}{m} = \frac{(-1)^m 2^{1-n}}{n} \binom{n}{2m}.$$

$$44. \sum_{k=0}^n k! \binom{n}{k} \binom{n+k}{n} x^k = y_n(2x).$$

$$45. \sum_{k=0}^{[n/2]} (-1)^k k! \binom{n}{k} \binom{n-k}{k} x^{2k} = x^n H_n \left(\frac{1}{2x} \right).$$

4.2.8. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k}^{-1}$.

$$1. \sum_{k=m}^n \binom{a}{k} \binom{b}{k}^{-1} = \frac{b+1}{b-a+1} \left[\binom{a}{m} \binom{b+1}{m}^{-1} - \binom{a}{n+1} \binom{b+1}{n+1}^{-1} \right].$$

$$2. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+k}{a}^{-1} = \frac{a}{a+n}.$$

$$3. \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{k+a-1}{a}^{-1} = -a \sum_{k=0}^{n-1} \frac{1}{k+a}.$$

$$4. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{l}^{-1} = \frac{l}{n+l} \binom{n+m}{m-l}^{-1}.$$

$$5. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k}^{-1} = \frac{[1+(-1)^n](2n+1)}{2(n+1)}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n+1}{2k}^{-1} = 1 + \frac{(-1)^n - 1}{2(n+2)}.$$

$$7. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n+1}{2k+1}^{-1} = \frac{[1-(-1)^n]}{2(n+2)} + (-1)^n.$$

$$8. \sum_{k=0}^n (-1)^k \binom{2n}{2k} \binom{n}{k}^{-1} = \frac{1+(-1)^n}{2(1-n)}.$$

$$9. \sum_{k=0}^n \binom{n}{k} \binom{n+2a}{k+a}^{-1} = \frac{2a+n+1}{2a+1} \binom{2a}{a}^{-1}.$$

$$10. \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2n+2a+1}{k+a}^{-1} = 0.$$

$$11. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2a}{k+a}^{-1} = \binom{a+n}{n} \binom{2a}{a}^{-1} \binom{2a+2n}{2n}^{-1}.$$

$$12. \sum_{k=0}^n (-1)^k \binom{2n+1}{2k} \binom{n}{k}^{-1} = (-1)^n \left[1 + \frac{1 - (-1)^n}{2n} \right].$$

$$13. \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \binom{n}{k}^{-1} = \frac{n}{n-1} + \frac{(-1)^n - 1}{2n(n-1)}.$$

$$14. \sum_{k=0}^n (-1)^k \binom{2n+1}{2k+1} \binom{n}{k}^{-1} = 1 + \frac{1 - (-1)^n}{2n}.$$

$$15. \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{a+k}{k}^{-1} = S_m,$$

$$S_0 = \frac{a}{a+n}, \quad S_1 = \frac{-n}{2(2n-1)}.$$

$$16. \sum_{k=1}^n (-1)^k k^{2m} \binom{n}{k} \binom{n+k}{n}^{-1} = 0$$

$$[1 < 2m < 2n-1].$$

$$17. \sum_{k=1}^n k \binom{n}{k} \binom{a}{k}^{-1} = \frac{n(1+a)}{(n-a-1)(n-a-2)}.$$

$$18. \sum_{k=1}^n k^2 \binom{n}{k} \binom{a}{k}^{-1} = -\frac{n(1+a)(a+n+1)}{(n-a-1)(n-a-2)(n-a-3)}.$$

$$19. \sum_{k=1}^n k \binom{n}{k} \binom{2n-1}{k}^{-1} = \frac{2n}{n+1}.$$

$$20. \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} \binom{k+n+1}{n+1}^{-1} = (n+1) \binom{n+1/2}{n}^{-2}.$$

$$21. \sum_{k=0}^n \frac{4k+1}{2n+2k+1} 2^{2k} \binom{2n}{n+k} \binom{2n+2k}{n+k}^{-1} = 1.$$

$$22. \sum_{k=0}^n \frac{a+n+1-2k}{a+n+1-k} \binom{a}{k} \binom{a+n-k}{n}^{-1} = 1.$$

$$23. \sum_{k=0}^n \binom{n}{k} \binom{k+m}{k}^{-1} x^k = \left[m + (x+1)^{n+m} - \sum_{k=0}^m \binom{n+m}{k} x^k \right] x^{-m} \binom{n+m}{n}^{-1}.$$

$$24. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n}{k} \binom{2k}{k}^{-1} = \frac{1}{1-2n}.$$

$$25. \sum_{k=1}^n \frac{(-1)^{k-1} 2^{2k}}{k} \binom{n}{k} \binom{2k}{k}^{-1} = 2 \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k}.$$

$$26. \sum_{k=0}^{2n+1} (-1)^k 2^{2k} \binom{3n+2}{n+k+1} \binom{2n+2k+1}{n+k}^{-1} = \frac{1}{3} \binom{3n+2}{n+1} \binom{2n+1}{n}^{-1}.$$

$$27. \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{k}{n} \binom{m}{k}^{-1} = \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-1} \frac{1}{k} \quad [0 \leq m \leq n].$$

$$28. \sum_{k=0}^n \binom{n}{k}^{-2} = \frac{3(n+1)^2}{2n+3} \binom{2n+2}{n+1}^{-1} \sum_{k=1}^{n+1} \frac{1}{k} \binom{2k}{k}.$$

$$29. \sum_{k=1}^n k 2^k \binom{n}{k} \binom{2n}{k}^{-1} = 2n+1 - 2^{2n} \binom{2n}{n}^{-1}.$$

$$30. \sum_{k=1}^n k 2^k \binom{n}{k} \binom{2n+1}{k}^{-1} = 2n+4 - 2^{2n+2} \binom{2n+1}{n}^{-1}.$$

4.2.9. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k} \binom{f_k}{g_k}$.

$$1. \sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{n}{n/2} \binom{3n/2}{n}.$$

$$2. \sum_{k=0}^n \binom{n}{k}^2 \binom{a+k}{2n} = \binom{a}{n}^2. \quad 3. \sum_{k=0}^n \binom{n}{k}^2 \binom{a+2n-k}{2n} = \binom{a+n}{n}^2.$$

$$4. \sum_{k=0}^n \binom{a}{k}^2 \binom{2a+n-k}{2a} = \binom{a+n}{n}^2.$$

$$5. \sum_{k=0}^n \binom{n}{k}^2 \binom{k}{n-m} = \binom{n}{m} \binom{n+m}{m}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{2a-n}{a-k} = (-1)^{n/2} \frac{1 + (-1)^n}{2} \binom{n}{n/2} \binom{a+n/2}{n} \binom{2a}{a} \binom{2a}{n}^{-1}.$$

$$7. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{n+k}{n}^2 = \binom{2n}{n}.$$

$$8. \sum_{k=0}^n (-1)^k \binom{2m}{k} \binom{2n-2m}{n-k}^2 = (-1)^m \binom{2m}{m} \binom{2n-m}{n}^2 \binom{2n-m}{m}^{-1}.$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{a}{n-k} = \frac{1+(-1)^n}{2} \binom{a}{n/2} \binom{-a-1}{n/2}.$$

$$10. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m-a-1}{k+l} \binom{a}{m-l-k} = (-1)^l \binom{m+n}{m-l} \binom{a}{m}.$$

$$11. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{-a+m-1}{m+k} \binom{a-m-n}{l-m-k} = (-1)^m \binom{l+n}{l-m} \binom{a}{l}.$$

$$12. \sum_{k=-m}^m (-1)^k \binom{m+n}{m+k} \binom{n+l}{n+k} \binom{l+m}{l+k} = \frac{(l+m+n)!}{l! m! n!}.$$

$$13. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2m}{m-n+k} \binom{2l}{l-n+k} =$$

$$= (-1)^n \frac{(l+m+n)! (2l)! (2m)! (2n)!}{l! m! n! (l+m)! (l+n)! (m+n)!}.$$

$$14. \sum_{k=0}^n \binom{m}{k} \binom{n}{k} \binom{m+n+k}{m+n} = \binom{m+n}{n}^2.$$

$$15. \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{a+m+n+k}{m+n} = \binom{a+m+n}{n} \binom{a+m+n}{m}.$$

$$16. \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{m+n+l-k}{m+n} = \binom{m+l}{m} \binom{n+l}{n}.$$

$$17. \sum_{k=0}^n \binom{a}{k} \binom{b}{k} \binom{a+b+n-k}{n-k} = \binom{a+n}{n} \binom{b+n}{n}.$$

$$18. \sum_{k=0}^n \binom{a}{k} \binom{n}{n-k} \binom{k}{m} = \binom{a}{m} \binom{a+n-m}{n-m}.$$

$$19. \sum_{k=0}^n \binom{n}{k} \binom{a}{m-k} \binom{m-k}{n} = \binom{a}{m-n} \binom{a}{n}.$$

$$20. \sum_{k=0}^n \binom{n}{k} \binom{m}{n-k} \binom{a+n-k}{m+n} = \binom{a}{m} \binom{a}{n}.$$

$$21. \sum_{k=0}^{\min(m, n)} \binom{a}{k} \binom{b}{m-n+k} \binom{a+b+n-k}{a+b} = \binom{a+m}{n} \binom{b+n}{m}.$$

$$22. \sum_{k=0}^n \sum_{l=0}^m (-1)^{k+l} \binom{n}{n-k} \binom{m}{m-l} \binom{k+l}{k} = \delta_{n,m}.$$

$$23. \sum_{k=0}^n \binom{m-a+b}{k} \binom{n+a-b}{n-k} \binom{a+k}{m+n} = \binom{a}{m} \binom{b}{n}.$$

$$24. \sum_{k=0}^n \binom{a}{k} \binom{a+m-2}{k+m-1} \binom{a-k-1}{a-n-1} = \binom{a+m-2}{n+m-1} \binom{a+n+m-1}{n}.$$

$$25. \sum_{k=\max(l,m)}^n (-1)^k \binom{n}{k} \binom{k}{l} \binom{k}{m} = (-1)^n \binom{n}{m} \binom{m}{n-l}.$$

$$26. \sum_{k=m}^n (-1)^k \binom{n}{k} \binom{k}{m} \binom{2n-k}{m} = (-1)^m \binom{n}{m}^2.$$

$$27. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1+(-1)^n}{2} \binom{n}{n/2}^2.$$

$$28. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{m} = 2^{n-m} \binom{n}{m} \binom{n+m}{m}.$$

$$29. \sum_{k=0}^{n-n} \binom{2n}{k} \binom{n-k}{m} \binom{m+n-k-1}{m-1} = \binom{2n-1}{n} \binom{n}{m}.$$

$$30. \sum_{k=0}^n (-1)^k \binom{a+b+1}{k} \binom{a+n-k}{a} \binom{b+n-k}{b} = \binom{a}{n} \binom{b}{n}.$$

$$31. \sum_{k=0}^n \frac{(-1)^k}{k+a} \binom{2n}{k} \binom{2n-k}{n}^2 = \frac{1}{a} \binom{2n}{n} \binom{2n+a}{n} \binom{a+n}{n}^{-1}.$$

$$32. \sum_{k=0}^n \frac{(-1)^k}{2n-l+1} \binom{2n}{k} \binom{2n-k}{n}^2 = \frac{1}{2n+1}.$$

$$33. \sum_{k=m}^{[n/2]} \frac{1}{2^{2k}} \binom{n}{2k} \binom{2k}{k} \binom{k}{m} = \frac{1}{2^n} \binom{n}{m} \binom{2n-2m}{n}.$$

$$34. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k+m} = \frac{(-1)^n + (-1)^m}{2^{2n+1}} \binom{n+m}{(n+m)/2} \binom{n-m}{(n-m)/2}$$

$$35. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} \binom{n+k}{n} = \frac{1+(-1)^n}{2^{2n+1}} \binom{n}{n/2}^2.$$

$$4.2.10. \text{ Суммы вида } \sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k} \binom{f_k}{g_k}^{-1}.$$

$$1. \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{mn+n}{k}^{-1} = \frac{[(mn)!]^2}{(mn-n)! (mn+n)!}.$$

$$2. \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k}^2 \binom{m}{k}^{-1} = 2 \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^m \frac{1}{k} \quad [m \geq n \geq 1]$$

$$3. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{b}{k}^{-1} = \binom{b-a}{n} \binom{b}{n}^{-1}.$$

$$4. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} \binom{b+k}{k}^{-1} = \binom{a+b+n}{n} \binom{b+n}{n}^{-1}.$$

$$5. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{b}{k} \binom{a+b-1}{k}^{-1} = (-1)^n \binom{a-1}{n} \binom{b-1}{n} \binom{a+b-1}{n}^{-1}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-m}{k} \binom{a}{k}^{-1} = 0 \quad [m < n].$$

$$7. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+a}{a} \binom{k+b}{b}^{-1} = \prod_{k=0}^{n-1} \left(1 - \frac{a+1}{b-k+1}\right).$$

$$8. \sum_{k=0}^{2n} (-1)^k \frac{2k+1}{n+k+1} \binom{2n}{k} \binom{2k}{k} \binom{n+k}{k}^{-1} = 1.$$

$$9. \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \binom{n+1}{k} \binom{m}{k}^{-1} = \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^n \frac{1}{k} \quad [1 \leq m \leq n+1].$$

$$10. \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{2n}{k}^{-1} x^k = (-1)^n \binom{2n}{n}^{-1} x^n P_n \left(1 - \frac{2}{x}\right).$$

$$11. \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}^{-1} x^k = \binom{2n}{n}^{-1} (x-4)^n P_{2n} \left(\sqrt{\frac{x}{x-4}}\right).$$

$$12. \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{2k}{k}^{-1} x^{2k} = T_n \left(1 - \frac{x^2}{2}\right).$$

$$13. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{2a}{k}^{-1} x^k = \binom{a-1/2}{n}^{-1} \left(-\frac{x}{4}\right)^n C_n^{1/2+a-n} \left(1 - \frac{2}{x}\right).$$

$$14. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-n+1/2}{k} \binom{a}{k}^{-1} x^k = \binom{a}{2n}^{-1} \left(-\frac{x}{4}\right)^n C_{2n}^{1+a-2n} \left(\sqrt{1 - \frac{1}{x}}\right).$$

$$15. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2a-n+1}{k} \binom{a}{k}^{-1} x^k = (-1)^n \binom{2a+1}{n}^{-1} C_n^{-a-1/2} (1-2x).$$

$$16. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{-1/2}{k} \binom{n-1/2}{k}^{-1} x^{2k} = \binom{2n}{n}^{-1} (4x)^n P_n \left(\frac{x^2+1}{2x}\right).$$

$$17. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{2k}{k}^{-1} x^{2k} = \binom{1-a}{n}^{-1} C_{2n}^{a-n} \left(\frac{x}{2}\right).$$

$$18. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1/2}{k} \binom{2k}{k}^{-1} x^{2k} = (-1)^n \binom{2n}{n}^{-1} 2^{2n} P_{2n} \left(\frac{x}{2}\right).$$

$$19. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{a}{k}^{-1} x^{2k} = \binom{a}{n}^{-1} x^n C_n^{1+a-n} \left(\frac{1}{2x}\right) \quad [a > -1/2 + n].$$

$$20. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{n-1/2}{k}^{-1} x^{2k} = \binom{2n}{n}^{-1} (4x)^n P_n \left(\frac{1}{2x}\right).$$

$$21. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{n}{k}^{-1} x^{2k} = x^n U_n \left(\frac{1}{2x}\right).$$

$$22. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} \binom{m+1/2}{k}^{-1} = \frac{1}{2^{2n}} \binom{2m+2n}{m+n} \binom{2m}{m}^{-1}.$$

$$23. \sum_{k=0}^n \frac{(-1)^k}{2k+1} 2^{2k} \binom{n}{k} \binom{a+k}{k} \binom{2k}{k}^{-1} = \frac{2^{2n}}{2n+1} \binom{n-a-1/2}{n} \binom{2n}{n}^{-1}.$$

$$24. \sum_{k=0}^n \frac{(-1)^k 2^{2k}}{a+k} \binom{n}{k} \binom{a+k}{k} \binom{2k}{k}^{-1} = \frac{(-1)^n}{a} \binom{2a}{2n} \binom{a}{n}^{-1}.$$

$$25. \sum_{k=1}^n \frac{(-1)^{k+1} 2^{2k}}{k} \binom{n}{k} \binom{2k}{k}^{-1} = 2 \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k}.$$

$$26. \sum_{k=0}^n \frac{(-1)^k}{2k+1} 2^{2k} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k}^{-1} = \frac{(-1)^n}{2n+1}.$$

$$27. \sum_{k=0}^1 \binom{n}{k} \binom{m}{k} \binom{l}{k} \binom{m+n+l}{k}^{-1} = \binom{m+n}{n} \binom{n+l}{n} \binom{n+m+l}{n}^{-1}.$$

$$28. \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \binom{2m+n-1}{k} \binom{m}{k}^{-2} = 2\psi(m) - 2\psi(m+n).$$

4.3. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

4.3.1. Суммы вида $\sum \alpha_k \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\}_k$.

$$1. \sum_{k=0}^n \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = \text{cosech} \frac{x}{2} \text{sh} \frac{n+1}{2} x \left\{ \frac{\text{sh}[(nx/2)+a]}{\text{ch}[(nx/2)+a]} \right\}.$$

$$2. \sum_{k=0}^n (-1)^k \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = \\ = \frac{\text{sech}(x/2)}{2} \left[\left\{ \frac{\text{sh}[a-(x/2)]}{\text{ch}[a-(x/2)]} \right\} + (-1)^n \left\{ \frac{\text{sh}[(2nx+x+2a)/2]}{\text{ch}[(2nx+x+2a)/2]} \right\} \right].$$

$$3. \sum_{k=1}^n k \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = \\ = -\text{cosech}^2 \frac{x}{2} \left[(n+1) \left\{ \frac{\text{sh}(nx+a)}{\text{ch}(nx+a)} \right\} - n \left\{ \frac{\text{sh}(nx+x+a)}{\text{ch}(nx+x+a)} \right\} - \left\{ \frac{\text{sh} a}{\text{ch} a} \right\} \right].$$

$$4. \sum_{k=1}^n (-1)^k k \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = \\ = \text{sech}^2 \frac{x}{2} \left[(-1)^n (n+1) \left\{ \frac{\text{sh}(nx+a)}{\text{ch}(nx+a)} \right\} + (-1)^n n \left\{ \frac{\text{sh}(nx+x+a)}{\text{ch}(nx+x+a)} \right\} - \left\{ \frac{\text{sh} a}{\text{ch} a} \right\} \right].$$

$$5. \sum_{k=0}^n \binom{n}{k} \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = 2^n \text{ch}^n \frac{x}{2} \left\{ \frac{\text{sh}[(nx/2)+a]}{\text{ch}[(nx/2)+a]} \right\}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k} \left\{ \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} \right\} = \\ = 2^{n-1} \text{sh}^n \frac{x}{2} [1 + (-1)^n] \left\{ \frac{\text{sh}[(nx/2)+a]}{\text{ch}[(nx/2)+a]} \right\} - \\ - 2^{n-1} \text{sh}^n \frac{x}{2} [1 - (-1)^n] \left\{ \frac{\text{ch}[(nx/2)+a]}{\text{sh}[(nx/2)+a]} \right\}.$$

$$7. \sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{sh} (n-k) x = \\ = \frac{(-1)^{n-1}}{\pi} 2^{2n+1} x \int_0^{\infty} \frac{1}{x^2 + 4t^2} \sin^{2n} t \, dt + 2^{2n-1} \operatorname{sh}^{2n} \frac{x}{2}.$$

$$8. \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \operatorname{sh} (2n-2k+1) x = 2^{2n} \operatorname{sh}^{2n+1} x.$$

$$9. \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{n-k} \binom{2n}{k} \operatorname{sh} (n-k) x = 2^{2n-1} \int_0^x \frac{\{\operatorname{ch} (x/2)\}^{2n}}{\{\operatorname{sh} (x/2)\}} dx - \frac{(\pm 1)^n}{2} \binom{2n}{n} x.$$

$$10. \sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{ch} (n-k) x = 2^{2n-1} \operatorname{sh}^{2n} \frac{x}{2} + \frac{(-1)^n}{2} \binom{2n}{n}.$$

$$11. \sum_{k=0}^n \frac{1}{2n-2k+1} \binom{2n+1}{k} \operatorname{sh} (2n-2k+1) x = 2^{2n} \int_0^x \operatorname{ch}^{2n+1} x \, dx.$$

$$12. \sum_{k=0}^{[n/2]} \binom{n}{k} \operatorname{ch} (n-2k) x = 2^{n-1} \operatorname{ch}^n x + \frac{1}{4} [1 + (-1)^n] \binom{n}{n/2}.$$

$$13. \sum_{k=0}^{n-1} r^k \begin{Bmatrix} \operatorname{sh} kx \\ \operatorname{ch} kx \end{Bmatrix} = \\ = \frac{1}{1 - 2r \operatorname{ch} x + r^2} \left[r^{n+1} \begin{Bmatrix} \operatorname{sh} (n-1)x \\ \operatorname{ch} (n-1)x \end{Bmatrix} - r^n \begin{Bmatrix} \operatorname{sh} nx \\ \operatorname{ch} nx \end{Bmatrix} \pm r \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right].$$

4.3.2. Суммы, содержащие $\operatorname{sech} x$ и $\operatorname{th} x$.

$$1. \sum_{k=0}^{n-1} 2^{2k} \operatorname{sech}^2 2^k x = \operatorname{cosech}^2 x - 2^{2n} \operatorname{cosech}^2 2^n x.$$

$$2. \sum_{k=1}^n \frac{1}{2^{2k}} \operatorname{sech}^2 \frac{x}{2^k} = \frac{1}{2^{2n}} \operatorname{cosech}^2 \frac{x}{2^n} - \operatorname{cosech}^2 x.$$

$$3. \sum_{k=1}^n \frac{1}{2^k} \operatorname{th} \frac{x}{2^k} = \operatorname{cth} x - \frac{1}{2^n} \operatorname{cth} \frac{x}{2^n}.$$

$$4. \sum_{k=0}^{n-1} 2^k \operatorname{th} 2^k x = 2^n \operatorname{cth} 2^n x - \operatorname{cth} x.$$

$$5. \sum_{k=1}^n \frac{1}{2^{2k}} \operatorname{th}^2 \frac{x}{2^k} = \operatorname{cth}^2 x - \frac{2}{3} \left(1 - \frac{1}{2^{2n}} \right) - \frac{1}{2^{2n}} \operatorname{cth}^2 \frac{x}{2^n}.$$

4.4. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

4.4.1. Суммы вида $\sum \alpha_k \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\}$.

$$1. \sum_{k=0}^n k^{2m} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = (-1)^m \frac{d^{2m}}{dx^{2m}} \sum_{k=0}^n \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\}.$$

$$2. \sum_{k=1}^n k^{2m+1} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = \varepsilon \frac{d^{2m+1}}{dx^{2m+1}} \sum_{k=1}^n \left\{ \begin{matrix} \cos (kx + a) \\ \sin (kx + a) \end{matrix} \right\}$$

[$\varepsilon = \pm 1$, если m — нечетное; $\varepsilon = \mp 1$, если m — четное].

$$3. \sum_{k=0}^n (-1)^k k^{2m} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = (-1)^m \frac{d^{2m}}{dx^{2m}} \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\}.$$

$$4. \sum_{k=1}^n (-1)^k k^{2m+1} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = \varepsilon \frac{d^{2m+1}}{dx^{2m+1}} \sum_{k=1}^n (-1)^k \left\{ \begin{matrix} \cos (kx + a) \\ \sin (kx + a) \end{matrix} \right\}$$

[$\varepsilon = \pm 1$, если m — нечетное; $\varepsilon = \mp 1$, если m — четное].

$$5. \sum_{k=0}^n \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = \operatorname{cosec} \frac{x}{2} \sin \frac{(n+1)x}{2} \left\{ \begin{matrix} \sin [(nx/2) + a] \\ \cos [(nx/2) + a] \end{matrix} \right\}.$$

$$6. \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = \sec \frac{x}{2} \sin \frac{(n+1)(x+\pi)}{2} \left\{ \begin{matrix} \sin [n(x+\pi)/2 + a] \\ \cos [n(x+\pi)/2 + a] \end{matrix} \right\}.$$

$$7. \sum_{k=1}^n k \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} =$$

$$= \operatorname{cosec}^2 \frac{x}{2} \left[(n+1) \left\{ \begin{matrix} \sin (nx + a) \\ \cos (nx + a) \end{matrix} \right\} - n \left\{ \begin{matrix} \sin (nx + x + a) \\ \cos (nx + x + a) \end{matrix} \right\} - \left\{ \begin{matrix} \sin a \\ \cos a \end{matrix} \right\} \right].$$

$$8. \sum_{k=1}^n (-1)^k k \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} =$$

$$= \sec^2 \frac{x}{2} \left[(-1)^n (n+1) \left\{ \begin{matrix} \sin (nx + a) \\ \cos (nx + a) \end{matrix} \right\} + (-1)^n n \left\{ \begin{matrix} \sin (nx + x + a) \\ \cos (nx + x + a) \end{matrix} \right\} - \left\{ \begin{matrix} \sin a \\ \cos a \end{matrix} \right\} \right].$$

$$9. \sum_{k=0}^n \binom{n}{k} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = 2^n \cos^n \frac{x}{2} \left\{ \begin{matrix} \sin [(nx/2) + a] \\ \cos [(nx/2) + a] \end{matrix} \right\}.$$

$$10. \sum_{k=0}^n (-1)^k \binom{n}{k} \left\{ \begin{matrix} \sin (kx + a) \\ \cos (kx + a) \end{matrix} \right\} = (-1)^n 2^n \sin^n \frac{x}{2} \left\{ \begin{matrix} \sin [n(x+\pi)/2 + a] \\ \cos [n(x+\pi)/2 + a] \end{matrix} \right\}.$$

- $$\begin{aligned}
 11. \quad & \sum_{k=0}^{[n/2]} \binom{n}{2k} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \\
 & = 2^{n-1} \left[\cos^n \frac{x}{4} \begin{Bmatrix} \sin (nx/4) \\ \cos (nx/4) \end{Bmatrix} + (-1)^n \sin^n \frac{x}{4} \begin{Bmatrix} \sin [n(2\pi + x)/4] \\ \cos [n(2\pi + x)/4] \end{Bmatrix} \right]. \\
 12. \quad & \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \begin{Bmatrix} \sin (2k+1)x \\ \cos (2k+1)x \end{Bmatrix} = \\
 & = 2^{n-1} \left[\cos^n \frac{x}{2} \begin{Bmatrix} \sin (nx/2) \\ \cos (nx/2) \end{Bmatrix} - (-1)^n \sin^n \frac{x}{2} \begin{Bmatrix} \sin [n(\pi + x)/2] \\ \cos [n(\pi + x)/2] \end{Bmatrix} \right]. \\
 13. \quad & \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \\
 & = 2^{n-1} \left[\cos^n \frac{x}{4} \begin{Bmatrix} \sin [(n-2)x/4] \\ \cos [(n-2)x/4] \end{Bmatrix} - (-1)^n \sin^n \frac{x}{4} \begin{Bmatrix} \sin [(2n\pi + nx - 2x)/4] \\ \cos [(2n\pi + nx - 2x)/4] \end{Bmatrix} \right]. \\
 14. \quad & \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \\
 & = 2^{n-1} \left[\cos^n \frac{\pi+x}{4} \begin{Bmatrix} \sin [n(\pi+x)/4] \\ \cos [n(\pi+x)/4] \end{Bmatrix} + (-1)^n \sin^n \frac{\pi+x}{4} \begin{Bmatrix} \sin [(3n\pi + nx)/4] \\ \cos [(3n\pi + nx)/4] \end{Bmatrix} \right]. \\
 15. \quad & \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \\
 & = 2^{n-1} \left[(-1)^n \sin^n \frac{\pi+x}{4} \begin{Bmatrix} \cos [(3n\pi + nx - 2x)/4] \\ \sin [(3n\pi + nx - 2x)/4] \end{Bmatrix} \right. \\
 & \quad \left. + \cos^n \frac{\pi+x}{4} \begin{Bmatrix} \cos [(n\pi + nx - 2x)/4] \\ \sin [(n\pi + nx - 2x)/4] \end{Bmatrix} \right]. \\
 16. \quad & \sum_{k=0}^{n-1} (\pm 1)^k \binom{2n}{k} \cos (2n-2k)x = -\frac{(\pm 1)^n}{2} \binom{2n}{n} + (\pm 1)^n 2^{2n-1} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix}^{2n}. \\
 17. \quad & \sum_{k=0}^n (\pm 1)^k \binom{2n+1}{k} \cos (2n-2k+1)x = (\pm 1)^n 2^{2n} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix}^{2n+1}. \\
 18. \quad & \sum_{k=0}^{[n/2]} \binom{n}{k} \cos (n-2k)x = 2^{n-1} \cos^n x + \frac{1+(-1)^n}{4} \binom{n}{n/2}. \\
 19. \quad & \sum_{k=1}^n (-1)^k \binom{2n}{n-k} k^{2m-1} \sin 2kx = (-1)^m 2^{2n-2m} \frac{d^{2m-1}}{dx^{2m-1}} \sin^{2n} x. \\
 20. \quad & \sum_{k=1}^n (-1)^k \binom{2n}{n-k} k^{2m} \cos 2kx = (-1)^m 2^{2n-2m-1} \frac{d^{2m}}{dx^{2m}} \sin^{2n} x.
 \end{aligned}$$

$$21. \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} (2k+1)^{2m-1} \cos(2k+1)x = (-1)^{m+1} 2^{2n} \frac{d^{2m-1}}{dx^{2m-1}} \sin^{2n+1} x.$$

$$22. \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} (2k+1)^{2m} \sin(2k+1)x = (-1)^m 2^{2n} \frac{d^{2m}}{dx^{2m}} \sin^{2n+1} x.$$

$$23. \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{n-k} \binom{2n}{k} \sin(n-k)x =$$

$$= (\pm 1)^n 2^{2n-1} \int_0^x \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\}^{2n} dx - \frac{(\pm 1)^n}{2} x \binom{2n}{n}.$$

$$24. \sum_{k=0}^n \frac{(\pm 1)^k}{2n-2k+1} \binom{2n+1}{k} \left\{ \frac{\sin(2n-2k+1)x}{\cos(2n-2k+1)x} \right\} = (\pm 1)^{n-1} 2^{2n} \int_0^x \left\{ \frac{\cos x}{\sin x} \right\}^{2n+1} dx.$$

$$25. \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \cos(n-2k)x = 4^n P_n(\cos x).$$

$$26. \sum_{k=0}^n \frac{\Gamma(k+\alpha) \Gamma(n-k+\alpha)}{k! (n-k)!} \cos(n-2k)x = \Gamma^2(\alpha) C_n^\alpha(\cos x).$$

$$27. \sum_{k=1}^n r^k \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} = (1-2r\cos x+r^2)^{-1} \left[-r^2 \left\{ \frac{\sin x}{\cos x} \right\} + r \left\{ \frac{\sin(x+a)}{\cos(x+a)} \right\} + \right.$$

$$\left. + r^{n+2} \left\{ \frac{\sin(nx+a)}{\cos(nx+a)} \right\} - r^{n+1} \left\{ \frac{\sin[(n+1)x+a]}{\cos[(n+1)x+a]} \right\} \right].$$

4.4.2. Суммы вида $\sum a_k \left\{ \frac{\sin kx}{\cos kx} \right\}^m$.

$$1. \sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m} = \frac{1}{2^{2m-1}} \sum_{k=1}^m (\mp 1)^k \binom{2m}{m-k} \times$$

$$\times \frac{\cos(n+1)kx \sin nkx}{\sin kx} + \frac{n}{2^{2n}} \binom{2m}{m}.$$

$$2. \sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m+1} =$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^m (\mp 1)^k \binom{2m+1}{m-k} \left\{ \frac{\sin[(2k+1)(n+1)x/2]}{\cos[(2k+1)(n+1)x/2]} \right\} \frac{\sin[(2k+1)nx/2]}{\sin[(2k+1)x/2]}.$$

$$3. \sum_{k=1}^n (-1)^{k-1} \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m} =$$

$$= \frac{1}{2^m} \sum_{k=1}^m (\mp 1)^k \binom{2m}{m-k} \frac{\cos kx + (-1)^{n-1} \cos (2n+1) kx}{\cos kx} + \frac{1 - (-1)^n}{2^{2m+1}} \binom{2m}{m}.$$

$$4. \sum_{k=1}^n (-1)^{k-1} \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m+1} = \frac{1}{2^{2m+1}} \sum_{k=0}^m (\mp 1)^k \binom{2m+1}{m-k} \times$$

$$\times \frac{\left\{ \frac{\sin [(2k+1)x/2]}{\cos [(2k+1)x/2]} \right\} + (-1)^{n-1} \left\{ \frac{\sin [(2n+1)(2k+1)x/2]}{\cos [(2n+1)(2k+1)x/2]} \right\}}{\cos [(2k+1)x/2]}.$$

$$5. \sum_{k=1}^n (\pm 1)^k a^k k^p \left\{ \frac{\sin kx}{\cos kx} \right\}^m = \left(a \frac{d}{da} \right)^p \sum_{k=1}^n (\pm 1)^k a^k \left\{ \frac{\sin kx}{\cos kx} \right\}^m.$$

$$6. \sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^2 = \frac{n}{2} \mp \frac{1}{2} \sin nx \cos (n+1)x \operatorname{cosec} x.$$

$$7. \sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^3 = \frac{3}{4} \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} \left\{ \frac{\sin [(n+1)x/2]}{\cos [(n+1)x/2]} \right\} \mp$$

$$\mp \frac{1}{4} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2} \left\{ \frac{\sin [3(n+1)x/2]}{\cos [3(n+1)x/2]} \right\}.$$

$$8. \sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^4 =$$

$$= \frac{1}{8} [3n \mp 4 \sin nx \cos (n+1)x \operatorname{cosec} x + \sin 2nx \cos 2(n+1)x \operatorname{cosec} 2x].$$

$$9. \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{\pi k}{n} = \frac{1}{2} [1 - (-1)^{m+n}] \quad [m=0, 1, 2, \dots, n-1].$$

$$10. \sum_{k=0}^{n-1} (-1)^k \cos^n \frac{\pi k}{n} = \frac{n}{2^{n-1}}.$$

$$11. \sum_{k=1}^{[n/2]} \cos^{2m} \frac{\pi k}{n} = \frac{n}{2^{2m+1}} \binom{2m}{m} - \frac{1}{2} \quad [m < n].$$

$$12. \sum_{k=1}^n \sin^{2m} \frac{\pi k}{2n} = \frac{1}{2} + \frac{n}{2^{2m}} \binom{2m}{m} \quad [m < 2n].$$

$$13. \sum_{k=1}^n \left\{ \frac{\sin(2\pi kl/n)}{\cos(2\pi kl/n)} \right\}^m = [1 + (-1)^m] n (m!) \left[\left(\frac{m}{2} \right)! \right]^{-2} \frac{1}{2^{m+1}}$$

[$m, l = 1, 2, 3, \dots; ml < n$].

$$14. \sum_{k=0}^{n-1} \left\{ \frac{\sin(2\pi k^2/n)}{\cos(2\pi k^2/n)} \right\} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} \mp \sin \frac{n\pi}{2} \right).$$

4.4.3. Суммы вида $\sum a_k \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^m$.

$$1. \sum_{k=1}^n 2^{2k} \sin^4 \frac{x}{2^k} = 2^{2n} \sin^2 \frac{x}{2^n} - \sin^2 x.$$

$$2. \sum_{k=0}^n \frac{(-1)^k}{3^k} \cos^3 3^k x = \frac{3}{4} \left[\cos x + \frac{(-1)^n}{3^{n+1}} \cos 3^{n+1} x \right].$$

$$3. \sum_{k=1}^n \frac{(-1)^k}{2^{2k}} \cos^4 2^k x = \frac{(-1)^n}{2^{2n+2}} \cos^2 2^{n+1} x - \frac{1}{4} \cos^2 2x + \frac{1}{20} [1 - (-4)^{-n}].$$

4.4.4. Суммы вида $\sum a_k \left\{ \frac{\sin kx}{\cos kx} \right\}^l \left\{ \frac{\sin ky}{\cos ky} \right\}^m$.

$$1. \sum_{k=0}^{n-1} \sin \frac{lk\pi}{n} \sin \frac{mk\pi}{n} = 0$$

[либо числа $l-m, l+m$ не делятся на $2n$, либо оба делятся на $2n$].

$$2. \quad \quad \quad = \frac{n}{2} \quad \quad \quad [l-m \text{ делится, } l+m \text{ не делится на } 2n].$$

$$3. \quad \quad \quad = -\frac{n}{2} \quad \quad \quad [l-m \text{ не делится, } l+m \text{ делится на } 2n].$$

$$4. \sum_{k=0}^{n-1} \sin \frac{km\pi}{n} \cos \frac{kl\pi}{n} = 0.$$

$$5. \sum_{k=0}^{n-1} \cos \frac{mk\pi}{n} \cos \frac{kl\pi}{n} = 0$$

[m, l — целые; $m+l, m-l$ не делятся на n ; $m \neq l$].

$$6. \sum_{k=0}^{n-1} \cos \frac{2lk\pi}{n} \cos \frac{2m'k\pi}{n} = 0$$

[$l-m, l+m$ не делятся на n].

$$7. \quad \quad \quad = n/2 \quad \quad \quad [\text{лишь одно из чисел } l-m, l+m \text{ делится на } n].$$

$$8. \quad \quad \quad = n \quad \quad \quad [l-m, l+m \text{ делятся на } n].$$

$$9. \sum_{k=0}^{2n-1} \sin \frac{km\pi}{n} \sin \frac{kl\pi}{n} = \begin{cases} 0, & m \neq l, \\ n, & m = l \neq 0 \end{cases} \quad [m, l < n].$$

$$10. \sum_{k=0}^{2n-1} \sin \frac{km\pi}{n} \cos \frac{kl\pi}{n} = 0 \quad [m, l < n].$$

$$11. \sum_{k=0}^{2n-1} \cos \frac{mk\pi}{n} \cos \frac{kl\pi}{n} = 0 \quad [m \neq l].$$

$$12. \quad \quad \quad = n \quad [m=l \text{ не делится на } n].$$

$$13. \quad \quad \quad = 2n \quad [m=l \text{ делится на } n].$$

$$14. \sum_{k=0}^{n-1} (-1)^k \cos \frac{2km\pi}{n} \cos^n \frac{k\pi}{n} = \frac{n}{2^n} \binom{n}{m} \quad [1 \leq m \leq n-1].$$

$$15. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \sin^{2k} x \cos^{n-2k} x = \cos nx.$$

$$16. \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \sin^{2k+1} x \cos^{n-2k-1} x = \sin nx.$$

$$17. \sum_{k=0}^n 2^k \sin \frac{x}{2^k} \sin^2 \frac{x}{2^{k+1}} = 2^{n-1} \sin \frac{x}{2^n} - \frac{1}{4} \sin 2x.$$

$$18. \sum_{k=0}^n (-2)^k \sin \frac{x}{2^k} \cos^3 \frac{x}{2^k} = \frac{1}{8} \sin 4x + (-2)^{n-2} \sin \frac{x}{2^{n-1}}.$$

4.4.5. Разные суммы, содержащие $\sin x$, $\cos x$.

$$1. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n} \pi} \sin^{2m+1} \frac{2k+1}{2n} \pi = n \sec nx \sin^{2m} x \quad [m < n].$$

$$2. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n} \pi} \sin \frac{2k+1}{2n} \pi \cos^m \frac{2k+1}{2n} \pi = n \sec nx \cos^m x \quad [m < n].$$

$$3. \sum_{k=1}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{k\pi}{n}} \cos^{2m} \frac{k\pi}{2n} = \frac{1}{4} \operatorname{cosec} \frac{x}{2} \left(\operatorname{cosec} \frac{x}{2} - 2n \operatorname{cosec} nx \cos^{2m-1} \frac{x}{2} \right) \quad [1 \leq m \leq n].$$

$$4. \sum_{k=1}^n \frac{(-1)^k}{\cos x - \cos \frac{2k\pi}{2n+1}} \cos^{2m+1} \frac{k\pi}{2n+1} = \frac{1}{4} \operatorname{cosec} \frac{x}{2} \left[\operatorname{cosec} \frac{x}{2} - (2n+1) \operatorname{cosec} (2n+1) \frac{x}{2} \cos^{2m} \frac{x}{2} \right] \quad [m < n].$$

$$5. \sum_{k=1}^n \frac{(-1)^k}{\cos x - \cos \frac{2k\pi}{2n+1}} \cos \frac{k\pi}{2n+1} \sin^{2m} \frac{k\pi}{2n+1} =$$

$$= -\frac{1}{4} (2n+1) \operatorname{cosec} \frac{(2n+1)x}{2} \sin^{2m-1} \frac{x}{2} \quad [m < 2n+2].$$

$$6. \sum_{k=0}^{n-1} \frac{(-1)^k}{1 - 2x \cos \frac{2k+1}{2n} \pi + x^2} \sin \frac{2k+1}{2n} \pi = \frac{nx^{n-1}}{1+x^{2n}}.$$

$$7. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n} \pi} \cos \frac{2k+1}{2n} m\pi \sin \frac{2k+1}{2n} \pi = n \cos mx \sec nx$$

$$[m < n].$$

$$8. \sum_{k=0}^{[(n-2)/2]} \frac{1}{\cos x - \cos [(2k+1)\pi/n]} = \frac{n}{2} \operatorname{cosec} x \operatorname{tg} \frac{nx}{2} - \frac{1}{8} [1 - (-1)^n] \sec^2 \frac{x}{2}.$$

$$9. \sum_{k=0}^{n-1} \frac{2^k}{2 \cos (2^k x) - 1} \sin (2^k x) = \frac{2^n}{2^n \cos (2^n x) + 1} \sin (2^n x) - \frac{\sin x}{2 \cos x + 1}.$$

$$10. \sum_{k=2}^n \prod_{m=1}^{k-1} \sin \frac{m\pi}{k} = 3 - \frac{n+2}{2^{n-1}}.$$

$$11. \sum_{k=2}^n \prod_{m=1}^{k-1} \cos \frac{m\pi}{k} = -\frac{1}{5 \cdot 2^n} \{2^n + (-1)^{[n/2]} + 3(-1)^{[(n+1)/2]}\}.$$

$$12. \sum_{k=1}^n \prod_{m=1}^{2k} \sin \frac{m\pi}{2k+1} = \frac{1}{9} \left[11 - \frac{1}{2^{2n}} (6n+11) \right].$$

$$13. \sum_{k=1}^n (-1)^{k+1} \prod_{m=1}^{2k} \sin \frac{m\pi}{2k+1} = \frac{1}{25} \left[13 + \frac{(-1)^{n+1}}{2^{2n}} (10n+13) \right].$$

$$14. \sum_{k=1}^n \prod_{m=1}^{2k} \cos \frac{m\pi}{2k+1} = \frac{(-1)^n}{5 \cdot 4^n} - \frac{1}{5}.$$

$$15. \sum_{k=1}^n (-1)^{k+1} \prod_{m=0}^{2k} \cos \frac{m\pi}{2k+1} = \frac{1}{3 \cdot 4^n} - \frac{1}{3}.$$

$$16. \sum_{k=1}^n \prod_{m=0}^{k-1} \sin \frac{2m+1}{2k} \pi = 2 - \frac{1}{2^{n-1}}.$$

$$17. \sum_{k=1}^n (-1)^{k-1} \prod_{m=0}^{k-1} \sin \frac{2m+1}{2k} \pi = \frac{2}{3} + \frac{(-1)^{n-1}}{3 \cdot 2^{n-1}}.$$

$$18. \sum_{k=2}^n \prod_{m=0}^{k-1} \cos \frac{2m+1}{2k} \pi = \frac{1}{5 \cdot 2^n} \{(-1)^{[(n-1)/2]} + 3(-1)^{[n/2]} - 2^n + 1\}.$$

$$19. \sum_{k=1}^n \prod_{m=0}^{k-1} (-1)^m \sin \frac{2m+1}{2k} \pi = \frac{2}{5} + \frac{(-1)^{[n/2]}}{5 \cdot 2^{n-1}} [1 - 2(-1)^n].$$

$$20. \sum_{k=1}^n \prod_{m=0}^{k-1} \sin^l \frac{2m+1}{2k} \pi = \frac{2^{nl} - 1}{2^{(n-1)l} (2^l - 1)}.$$

$$21. \sum_{k=1}^n \prod_{m=1}^{2k} \cos^l \frac{m\pi}{2k+1} = (-1)^l \frac{2^{2nl} - (-1)^{nl}}{2^{2nl} [2^{2l} - (-1)^l]}.$$

$$22. \sum_{k=1}^n \prod_{m=0}^{2k-1} \cos^l \frac{2m+1}{4k} \pi = (-1)^l \frac{2^{2nl} - (-1)^{nl}}{2^{(2n-1)l} [2^{2l} - (-1)^l]}.$$

4.4.6. Суммы, содержащие $\sec x$ и $\operatorname{cosec} x$.

$$1. \sum_{\substack{k=0 \\ k \neq n/2}}^n \sec \frac{k\pi}{n} = 0.$$

$$2. \sum_{k=0}^n \sec \frac{2k\pi}{2n+1} = (-1)^n \frac{2n+1}{2} + \frac{1}{2}.$$

$$3. \sum_{k=0}^{n-1} \sec^2 \left(\frac{k\pi}{n} + x \right) = n^2 \operatorname{cosec}^2 \left(\frac{n\pi}{2} + nx \right).$$

$$4. \sum_{k=1}^{[(n-1)/2]} \sec^2 \frac{k\pi}{n} = \frac{1}{12} (4n^2 - 7) + \frac{(-1)^{n+1}}{12} (2n^2 + 1).$$

$$5. \sum_{k=1}^{n-1} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{n^2 - 1}{3}.$$

$$6. \sum_{k=1}^{[(n-1)/2]} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{1}{6} (n^2 - 1) - \frac{1}{4} [1 + (-1)^n].$$

$$7. \sum_{k=0}^{[(n-2)/2]} \operatorname{cosec}^2 \frac{2k+1}{2n} \pi = \frac{n^2}{2} + \frac{1}{4} [(-1)^n - 1].$$

$$8. \sum_{k=1}^n \operatorname{cosec}^2 \frac{\pi k}{2n+1} = \frac{n(2n+2)}{3}.$$

$$9. \sum_{k=0}^{n-1} \left\{ \frac{\sec [x + (k\pi/n)]}{\operatorname{cosec} [x + (k\pi/n)]} \right\}^2 = n^2 \left\{ \frac{\operatorname{cosec} [n(2x + \pi)/2]}{\operatorname{cosec} nx} \right\}^2.$$

$$10. \sum_{k=0}^{n-1} \left\{ \frac{\sec [x + (2k\pi/n)]}{\operatorname{cosec} [x + (2k\pi/n)]} \right\}^2 = \frac{1}{2} [1 - (-1)^n] n^2 \left\{ \frac{\operatorname{cosec} [n(2x + \pi)/2]}{\operatorname{cosec} nx} \right\}^2 + \\ + \frac{1}{4} [1 + (-1)^n] n^2 \left\{ \frac{\operatorname{cosec} [(2nx + \pi)/4]}{\operatorname{cosec} (nx/2)} \right\}^2.$$

$$11. \sum_{k=1}^{2n} (-1)^k \sin^2 \frac{k\pi}{2n+1} \sec \frac{k\pi}{2n+1} = (-1)^n (2n+1) \quad [n \geq 1].$$

$$12. \sum_{k=0}^{n-1} \cos \left[(2m+1) \left(x + \frac{k\pi l}{n} \right) \right] \operatorname{cosec} \left(x + \frac{k\pi l}{n} \right) = n \operatorname{ctg} nx \\ [m=0, 1, \dots, n-1; l \text{ и } n \text{ — взаимно простые числа}].$$

$$13. \sum_{k=1}^n \cos \left(x + \frac{k\pi}{n} \right) \operatorname{cosec}^3 \left(x + \frac{k\pi}{n} \right) = n^3 \cos nx \operatorname{cosec}^3 nx.$$

$$14. \sum_{k=1}^n \left\{ \frac{\sec (kx+a)}{\operatorname{cosec} (kx+a)} \frac{\sec [(k+1)x+a]}{\operatorname{cosec} [(k+1)x+a]} \right\} = \\ = \operatorname{cosec} x \left[\pm \left\{ \frac{\operatorname{tg} [(n+1)x+a]}{\operatorname{ctg} [(n+1)x+a]} \right\} \mp \left\{ \frac{\operatorname{tg} (x+a)}{\operatorname{ctg} (x+a)} \right\} \right].$$

$$15. \sum_{k=0}^{n-1} (-1)^k \left\{ \frac{\sec (kx+a)}{\operatorname{cosec} (kx+a)} \frac{\sec [(k+2)x+a]}{\operatorname{cosec} [(k+2)x+a]} \right\} = \\ = \frac{1}{2} \sec x \left[\left\{ \frac{\sec (x+a)}{\operatorname{cosec} (x+a)} \frac{\sec a}{\operatorname{cosec} a} \right\} - (-1)^n \left\{ \frac{\sec (nx+a)}{\operatorname{cosec} (nx+a)} \frac{\sec [(n+1)x+a]}{\operatorname{cosec} [(n+1)x+a]} \right\} \right].$$

$$16. \sum_{k=0}^n \operatorname{cosec} \frac{x}{2^k} = \operatorname{ctg} \frac{x}{2^{n+1}} - \operatorname{ctg} x.$$

$$17. \sum_{k=1}^n \frac{1}{2^{2k}} \sec^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{2^{2n}} \operatorname{cosec}^2 \frac{x}{2^n}.$$

$$18. \sum_{k=0}^{n-1} \frac{1}{2^k} \sin^2 \frac{x}{2^{k+2}} \operatorname{cosec} \frac{x}{2^k} = \frac{1}{2} \operatorname{cosec} x - \frac{1}{2^{n+1}} \operatorname{cosec} \frac{x}{2^n}.$$

$$19. \sum_{k=0}^n \frac{1}{2^{3k}} \sin \frac{x}{2^k} \sec^3 \frac{x}{2^k} = \frac{1}{2^{3n}} \cos \frac{x}{2^n} \operatorname{cosec}^3 \frac{x}{2^n} - 8 \cos 2x \operatorname{cosec}^3 2x.$$

4.4.7. Суммы, содержащие $\operatorname{tg} a_k$ и $\operatorname{ctg} a_k$.

$$1. \sum_{k=0}^{n-1} \left\{ \begin{matrix} \operatorname{tg} [x + (km\pi/n)] \\ \operatorname{ctg} [x + (km\pi/n)] \end{matrix} \right\} = \mp n \left\{ \begin{matrix} \operatorname{ctg} [(2nx + \pi)/2] \\ \operatorname{ctg} nx \end{matrix} \right\}$$

$\{m, n - \text{взаимно простые числа}\}.$

$$2. \sum_{k=0}^{n-1} (-1)^k \left\{ \begin{matrix} \operatorname{tg} [(2k+1)\pi/(4n)] \\ \operatorname{ctg} [(2k+1)\pi/(4n)] \end{matrix} \right\} = (-1)^{n-1} n.$$

$$3. \sum_{k=0}^{2n-1} (-1)^k \operatorname{ctg} \frac{2k+1}{8n} \pi = 2n.$$

$$4. \sum_{k=0}^{n-1} \left\{ \begin{matrix} \operatorname{tg} [x + (k\pi/n)] \\ \operatorname{ctg} [x + (k\pi/n)] \end{matrix} \right\}^2 = n^2 - n + n^2 \left\{ \begin{matrix} \operatorname{ctg} [(2nx + \pi)/2] \\ \operatorname{ctg} nx \end{matrix} \right\}^2.$$

$$5. \sum_{k=1}^{[(n-1)/2]} \operatorname{tg}^2 \frac{k\pi}{n} = \frac{1}{6} (n-1) [2n-1 - (-1)^n (n+1)].$$

$$6. \sum_{k=1}^{n-1} \operatorname{ctg}^2 \frac{k\pi}{n} = \frac{1}{3} (n-1) (n-2).$$

$$7. \sum_{k=1}^{[(n-1)/2]} \operatorname{ctg}^2 \frac{k\pi}{n} = \frac{1}{6} (n-1) (n-2).$$

$$8. \sum_{k=1}^n \operatorname{ctg}^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}.$$

$$9. \sum_{k=1}^n \operatorname{tg}^4 \frac{k\pi}{2n+1} = \frac{n}{3} (2n+1) (4n^2+6n-1).$$

$$10. \sum_{k=1}^n \operatorname{ctg}^4 \frac{k\pi}{2n+1} = \frac{n}{45} (2n-1) (4n^2+10n-9).$$

$$11. \sum_{k=0}^n \frac{1}{2^k} \operatorname{tg} \frac{x}{2^k} = \frac{1}{2^n} \operatorname{ctg} \frac{x}{2^n} - 2 \operatorname{ctg} 2x.$$

$$12. \sum_{k=0}^n \frac{1}{2^{2k}} \operatorname{tg}^2 \frac{x}{2^k} = \frac{1}{3 \cdot 2^{2n-1}} (2^{2n+2} - 1) + 4 \operatorname{ctg}^2 2x - 2^{-2n} \operatorname{ctg}^2 \frac{x}{2^n}.$$

$$13. \sum_{k=0}^{n-1} \left\{ \frac{\operatorname{tg}(kx+a) \operatorname{tg}[(k+1)x+a]}{\operatorname{ctg}(kx+a) \operatorname{ctg}[(k+1)x+a]} \right\} = \operatorname{ctg} x \left[\pm \frac{\operatorname{tg}(nx+a)}{\operatorname{ctg}(nx+a)} \mp \operatorname{tg} a \right] - n.$$

$$14. \sum_{k=0}^{n-1} \operatorname{tg} \frac{x}{2^{k+1}} \sec \frac{x}{2^k} = \operatorname{tg} x - \operatorname{tg} \frac{x}{2^n}.$$

$$15. \sum_{k=0}^{n-1} 2^k \sin \frac{x}{2^k} \operatorname{tg}^2 \frac{x}{2^{k+1}} = \operatorname{tg} x - 2^n \operatorname{tg} \frac{x}{2^n}.$$

$$16. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \operatorname{tg}^{2k} x = \frac{\cos nx}{\cos^n x}.$$

$$17. \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \operatorname{tg}^{2k+1} x = \frac{\sin nx}{\cos^n x}.$$

$$18. \sum_{k=1}^n \prod_{m=1}^{2k} \operatorname{tg} \frac{m\pi}{2k+1} = (-1)^n (n+1) - 1.$$

$$19. \sum_{k=1}^n \prod_{m=1}^{2k-1} \operatorname{tg} \frac{2m+1}{4k} \pi = -\frac{1 - (-1)^n}{2}.$$

$$20. \sum_{k=1}^n \prod_{\substack{m=1 \\ m \neq k}}^n \operatorname{ctg}(x_k - x_m) = \sin \frac{n\pi}{2}.$$

$$21. \sum_{k=1}^n \operatorname{ctg} x_k \prod_{\substack{m=1 \\ m \neq k}}^n \operatorname{ctg}(x_k - x_m) = \sin \frac{(n+1)\pi}{2} + (-1)^{n+1} \prod_{k=1}^n \operatorname{ctg} x_k.$$

4.4.8. Суммы, содержащие логарифмы.

$$1. \sum_{k=1}^n \ln(ka+b) = n \ln a + \ln \Gamma\left(\frac{b}{a} + n + 1\right) - \ln \Gamma\left(\frac{b}{a} + 1\right).$$

$$2. \sum_{k=0}^n (a+k) \ln(a+k) = \int_0^{n+1} \ln \Gamma(a+t) dt + (n+1)a + \frac{1}{2}(n+1)(n - \ln 2\pi)$$

[$a \geq 0$].

$$3. \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(ka+b) = \int_0^1 (1-t^a)^n t^{b-1} \frac{dt}{\ln t}$$

[$a, b > 0$].

$$4. \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(b-ka) = \int_0^1 (t^a - 1)^n t^{b-na-1} \frac{dt}{\ln t}$$

[$a \geq 0$; $b > nx$].

$$5. \sum_{k=1}^n \frac{(-1)^k k^{2m}}{(n-k)! (n+k)!} \ln 2k = (-1)^{m-1} 2^{2n-2m-1} \frac{(2m)!}{(2n)!} \int_0^{\infty} t^{-2m-1} \sin^{2n} t \, dt$$

[n > m].

$$6. \sum_{k=0}^n \frac{(-1)^k (2k+1)^{2n+1}}{(n-k)! (n+k+1)!} \ln (2k+1) =$$

$$= (-1)^{m-1} 2^{2n} \frac{(2m+1)!}{(2n+1)!} \int_0^{\infty} t^{-2m-2} \sin^{2n+1} t \, dt \quad [m, n = 1, 2, 3, \dots].$$

$$7. \sum_{k=2}^n k^m \ln (k!) = \frac{1}{m+1} \left[B_{m+1}(n+1) \ln (n!) - \sum_{k=1}^n B_{m+1}(k) \ln k \right].$$

$$8. \sum_{k=2}^n \ln (k!) = (n+1) \ln (n+1)! - \sum_{k=2}^{n+1} k \ln k.$$

$$9. \sum_{k=2}^n k \ln (k!) = \frac{n(n+1)}{2} \ln (n!) - \frac{1}{2} \sum_{k=2}^n k(k-1) \ln k.$$

$$10. \sum_{k=2}^n k^2 \ln (k!) = \frac{n}{6} (n+1)(2n+1) \ln (n!) - \frac{1}{6} \sum_{k=2}^n k(k-1)(2k-1) \ln k.$$

$$11. \sum_{k=0}^{n-1} \ln \Gamma (ka+b) =$$

$$= \int_0^{\infty} \left[\left(b-1 + \frac{n-1}{2} a \right) n e^{-t} - \frac{n}{e^t - 1} + \frac{1 - e^{-nat}}{(1 - e^{-at})(e^t - 1)} e^{(1-b)t} \right] \frac{dt}{t}.$$

$$12. \sum_{k=0}^n (-1)^{k-1} \ln \Gamma (ka+b) = \int_0^1 (1-t^{-a}) t^{na+b-1} \frac{dt}{(1-t) \ln t} \quad [a < 0; b > -na].$$

$$13. \sum_{k=0}^{n-1} \left[1 - (k+1) \ln \frac{2k+3}{2k+1} \right] = n - n \ln (2n+1) + \ln \frac{(2n)!}{n! 2^n}.$$

$$14. \sum_{k=1}^{n-1} \cos kx \ln \sin \frac{k\pi}{2n} = -\frac{\pi}{2x} \int_0^{1/2} \sin 2nxt \operatorname{ctg} \pi t \, dt.$$

$$15. \sum_{k=1}^{[(n-1)/2]} \cos \frac{2mk\pi}{n} \ln \sin \frac{k\pi}{n} = \frac{1}{2} \left[\ln 2n + \psi \left(\frac{m}{n} \right) + C \right] + \frac{\pi}{4} \operatorname{ctg} \frac{m\pi}{n}$$

[m = 1, 2, ..., n-1; n = 2, 3, 4, ...].

$$16. \sum_{k=0}^{[(n-1)/2]} \cos \frac{m(2k+1)}{n} \pi \ln \sin \frac{2k+1}{2n} \pi = -\frac{1}{2} \beta\left(\frac{m}{n}\right) + \frac{\pi}{4} \operatorname{cosec} \frac{m\pi}{n}$$

[$m=1, 2, \dots, n-1; n=2, 3, 4, \dots$].

4.4.9. Суммы, содержащие арктангенсы.

$$1. \sum_{k=1}^n \operatorname{arctg} \frac{2}{k^2} = -\frac{\pi}{4} - \operatorname{arctg} n - \operatorname{arctg} (n+1).$$

$$2. \sum_{k=1}^n \operatorname{arctg} \frac{a}{1+k(k+1)a^2} = \operatorname{arctg} \frac{na}{1+(n+1)a^2}.$$

$$3. \sum_{k=0}^{n-1} \operatorname{arctg} \frac{a}{1+a(ka+b)+(ka+b)^2} = \operatorname{arctg} (na+b) - \operatorname{arctg} b.$$

$$4. \sum_{k=1}^n \operatorname{arctg} \frac{2k}{2+k^2+k^4} = \operatorname{arctg} \frac{n+n^2}{2+n+n^2}.$$

$$5. \sum_{k=1}^{2n} \operatorname{arctg} \left(x \sec \frac{2k\pi}{2n+1} \right) = \operatorname{arctg} \{ \operatorname{sh} (2n \operatorname{Arsh} x) \operatorname{sech} [(2n+1) \operatorname{Arsh} x] \}.$$

$$6. \sum_{k=0}^{n-1} (-1)^k \operatorname{arctg} \frac{x - \cos \frac{2k+1}{2n} \pi}{\sin \frac{2k+1}{2n} \pi} = \operatorname{arctg} x^n - \frac{\pi}{4} [1 + (-1)^n].$$

$$7. \sum_{k=0}^{n-1} (-1)^k \operatorname{arctg} \frac{x \sin \frac{2k+1}{2n} \pi}{1 - x \cos \frac{2k+1}{2n} \pi} = \operatorname{arctg} x^n.$$

Глава 5. РЯДЫ

5.1. ЧИСЛОВЫЕ РЯДЫ

5.1.1. Введение.

В этом разделе приведены суммы рядов вида $\sum_k a_k$, где a_k — комплексные числа или функции некоторых параметров, причем зависимость от параметров такова, что они не записываются в виде степенных рядов (см. 5.2) или тригонометрических рядов (см. 5.4).

В некоторых случаях для вычисления суммы ряда $\sum_k a_k$ можно ввести вспомогательную сумму $\sum_k a_k x^k$, а затем воспользоваться формулами из (5.2) и теоремой Абеля:

Пусть ряд $\sum_{k=0}^{\infty} a_k x^k$ сходится при $|x| < 1$. Если сходится также ряд $\sum_{k=0}^{\infty} a_k$ и $0 < x < 1$, то

$$\lim_{x \rightarrow 1-0} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

При суммировании рядов могут оказаться полезными следующие формулы, в которых функции $f(x)$, $F(x)$, $G(x)$, $M(x)$ и $\varphi(x)$ связаны соотношениями

$$F(x) = \int_0^{\infty} e^{-xt} f(t) dt,$$

$$M(s) = \int_0^{\infty} t^{s-1} G(t) dt,$$

$$F(s) \Gamma(-s) = \int_0^{\infty} e^{-st} \varphi(t) dt.$$

$$1. \sum_{k=1}^{\infty} F[g(k)] = \int_0^{\infty} \int_0^{\infty} \frac{h(x, y) f(y)}{e^x - 1} dx dy,$$

где

$$\int h(x, y) e^{-px} dx = e^{-yg(p)}.$$

$$2. \sum_{k=1}^{\infty} F(k) = \int_0^{\infty} \frac{f(x)}{e^x - 1} dx. \quad 3. \sum_{k=1}^{\infty} (-1)^k F(k) = - \int_0^{\infty} \frac{f(x)}{e^x + 1} dx.$$

$$4. \sum_{k=0}^{\infty} G(k+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(s) \zeta(s, a) ds \quad [c > 1].$$

$$5. \sum_{k=1}^{\infty} F(k^{1/2}) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} dx \int_0^{\infty} y \frac{e^{-y^2/4x} f(y)}{e^x - 1} dy.$$

$$6. \sum_{k=1}^{\infty} F(\ln k) = \int_0^{\infty} dx \int_0^1 \frac{x^{y-1} f(y)}{\Gamma(y)(e^x - 1)} dy + \int_1^{\infty} f(y) \zeta(y) dy.$$

$$7. \sum_{k=0}^{\infty} (-1)^k \frac{a^k}{k!} F(k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) a^s \Gamma(-s) ds = \varphi(\ln a).$$

$$8. \sum_{k=0}^{\infty} F(ka) x^k = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t) \frac{1 - x \cos at}{1 - 2x \cos at + x^2} dt.$$

$$9. \sum_{k=1}^{\infty} F(ka) x^k = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(t) \frac{1 - x \sin at}{1 - 2x \cos at + x^2} dt.$$

где

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t) \cos xt dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(t) \sin xt dt.$$

5.1.2. Ряды вида $\sum \frac{(\pm 1)^k}{k^s}$.

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \quad [\operatorname{Re} s > 1].$$

$$\sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(1) = S_n \quad [n = 2, 3, 4, \dots].$$

$$S_{2n} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}| \quad [n = 1, 2, 3, \dots].$$

$$S_2 = \frac{\pi^2}{6} = 1,64493407, \quad S_3 = 1,20205690,$$

$$S_4 = \frac{\pi^4}{90} = 1,08232323, \quad S_5 = 1,03692776,$$

$$S_6 = \frac{\pi^6}{945} = 1,01734206, \quad S_7 = 1,008349277,$$

$$S_8 = \frac{\pi^8}{9450} = 1,00407736, \quad S_9 = 1,00200839,$$

$$S_{10} = \frac{5 \cdot 2^8 \pi^{10}}{33 \cdot 10!} = 1,00099458.$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = (2^{1-s} - 1) \zeta(s) = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + 1} dt = T_s \quad [\operatorname{Re} s > 0].$$

$$T_{2n} = \frac{1}{(2n)!} (1 - 2^{2n-1}) \pi^{2n} |B_{2n}| \quad [n = 1, 2, 3, \dots].$$

$$T_1 = -\ln 2 = -0,699314718, \quad T_2 = -\frac{\pi^2}{12} = -0,82246703,$$

$$T_3 = -0,90154268, \quad T_4 = -\frac{7\pi^4}{720} = -0,94703283,$$

$$T_5 = -0,97211977, \quad T_6 = -\frac{31\pi^6}{32 \cdot 945} = -0,98555109,$$

$$T_7 = -0,99259382, \quad T_8 = -\frac{127\pi^8}{128 \cdot 9450} = -0,99623300,$$

$$T_9 = -0,99809430, \quad T_{10} = -\frac{73\pi^{10}}{5 \cdot 33 \cdot 81 \cdot 512} = -0,99903951.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{[k/2]}}{k} = \frac{\pi}{4} - \frac{1}{2} \ln 2. \quad 5. \sum_{k=0}^{\infty} \frac{(-1)^{[k/2]}}{k+1} = \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^{[k\sqrt{2}]}}{k} = -0,5154.$$

5.1.3. Ряды вида $\sum \frac{(\pm 1)^k}{(k+a)^s}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} = \zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - e^{-t}} dt \quad [\operatorname{Re} s > 1].$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k+a)^n} = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(a) \quad [n = 2, 3, 4, \dots].$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+a)^2} = -\int_0^1 \frac{t^{a-1}}{1-t} \ln t dt.$$

$$4. \sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^n} = (-1)^{n-1} \frac{\pi}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} \operatorname{ctg} a\pi.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s} = 2^{-s} \left[\zeta \left(s, \frac{a}{2} \right) - \zeta \left(s, \frac{a+1}{2} \right) \right] = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1+e^{-t}} dt$$

[Re s > 0].

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^n} = \frac{(-1)^{n-1}}{(n-1)!} \beta^{(n-1)}(a)$$

[n = 1, 2, ...].

$$7. \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^n} = (-1)^{n-1} \pi \frac{d^{n-1}}{da^{n-1}} \operatorname{cosec} a\pi$$

[n = 1, 2, 3, ...].

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^2} = - \int_0^1 \frac{t^{a-1}}{1+t} \ln t dt.$$

5.1.4. Ряды вида $\sum \frac{(\pm 1)^k}{(2k+1)^s}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = (1-2^{-s}) \zeta(s) = A_s$$

[Re s > 1].

$$A_{2n} = \frac{1}{2(2n)!} (2^{2n} - 1) \pi^{2n} |B_{2n}|.$$

$$A_2 = \frac{\pi^2}{8} = 1,23370055,$$

$$A_3 = 1,05179974,$$

$$A_4 = \frac{\pi^4}{96} = 1,01467803,$$

$$A_5 = 1,00452376,$$

$$A_6 = \frac{\pi^6}{960} = 1,00144708,$$

$$A_7 = 1,00047155,$$

$$A_8 = \frac{17\pi^8}{256 \cdot 60} = 1,00015518,$$

$$A_9 = 1,00005135,$$

$$A_{10} = \frac{31\pi^{10}}{35 \cdot 81 \cdot 1024} = 1,00001704.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = 2^{-s} \left[\zeta \left(s, \frac{1}{4} \right) - \zeta \left(s, \frac{3}{4} \right) \right] = \frac{1}{2\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\operatorname{ch} t} dt = \hat{A}_s$$

[Re s > 0].

$$\hat{A}_{2n+1} = \frac{1}{2(2n)!} \left(\frac{\pi}{2} \right)^{2n+1} |E_{2n}|,$$

$$\hat{A}_1 = \frac{\pi}{4} = 0,78539816,$$

$$\hat{A}_2 = G = 0,91596559,$$

$$\hat{A}_3 = \frac{\pi^3}{32} = 1,96894615,$$

$$\hat{A}_4 = 0,9889455,$$

$$\hat{A}_5 = \frac{5\pi^5}{64 \cdot 4!} = 0,99615783,$$

$$\hat{A}_6 = 0,99868522,$$

$$\hat{A}_7 = \frac{61\pi^7}{256 \cdot 6!} = 0,99955451, \quad \hat{A}_8 = 0,99984999,$$

$$\hat{A}_9 = \frac{1385\pi^9}{1024 \cdot 8!} = 0,99994968, \quad \hat{A}_{10} = 0,99998316.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^{[k/2]}}{(2k+1)^s} = 8^{-s} \left[2\zeta\left(s, \frac{1}{8}\right) + 2\zeta\left(s, \frac{3}{8}\right) - 2^{2s} (2^s - 1) \zeta(s) \right] = C_s$$

[Re s > 0]

$$C_{2n+1} = \frac{(-1)^n \pi^{2n+1}}{(2n+1)! \sqrt{2}} \left[B_{2n+1}\left(\frac{1}{4}\right) - 2^{2n+1} B_{2n+1}\left(\frac{1}{8}\right) \right],$$

$$C_1 = \frac{\pi \sqrt{2}}{4}, \quad C_3 = \frac{3\pi^3 \sqrt{2}}{2^7}, \quad C_5 = \frac{19\pi^5 \sqrt{2}}{2^{13}}, \quad C_7 = \frac{307\pi^7 \sqrt{2}}{5 \cdot 2^{18}}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^{[(k+1)/2]}}{(2k+1)^n} = \hat{C}_n,$$

$$\hat{C}_1 = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1), \quad \hat{C}_2 = \frac{\pi^2 \sqrt{2}}{16}, \quad \hat{C}_4 = \frac{11\pi^4 \sqrt{2}}{3 \cdot 2^9},$$

$$\hat{C}_6 = \frac{361\pi^6 \sqrt{2}}{15 \cdot 2^{15}}, \quad \hat{C}_8 = \frac{24611\pi^8 \sqrt{2}}{9 \cdot 35 \cdot 2^{20}}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^{[k/3]}}{2k+1} = \frac{5\pi}{12}.$$

5.1.5. Ряды вида $\sum \frac{(-1)^k}{kn+m}$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{ka+b} = \int_0^1 \frac{t^{b-1}}{t^a+1} dt = \frac{1}{a} \beta\left(\frac{b}{a}\right).$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} = \frac{\pi}{2n} \operatorname{cosec} \frac{m\pi}{n} - \frac{2}{n} \sum_{k=0}^{[(n-2)/2]} \cos\left[(2k+1) \frac{m\pi}{n}\right] \ln\left[\sin(2k+1) \frac{\pi}{2n}\right]$$

[n=2, 3, 4, ...; m=1, 2, ..., n-1; см. приложение I.M]

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+nl+m} = (-1)^l \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} - \sum_{k=0}^{l-1} \frac{(-1)^k}{kn+m} \right].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+m} = (-1)^{m+1} \ln 2 + (-1)^{m+1} \sum_{k=1}^{m-1} \frac{(-1)^k}{k}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2. \quad 6. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3k+2} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} - (-1)^2 \ln 2 \right)$$

[$2 = 1$ или 2].

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{4k+2 \pm 1} = \frac{1}{4\sqrt{2}} [\pi \mp 2 \ln(1 + \sqrt{2})].$$

5.1.6. Ряды вида $\sum \frac{(\pm 1)^k}{(k+a)(k+b)}$.

$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(ka+b)(ka+c)} = - \int_0^1 dy \int_0^1 \frac{x^{b(1-y)+cy-1}}{1+x^a} \ln x dx.$$

$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)(k+b)} = \pm \frac{1}{b-a} \left\{ \begin{matrix} \psi(b) - \psi(a) \\ \beta(b) - \beta(a) \end{matrix} \right\}.$$

[$a=b$ см. в 5.1.3].

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+b)} = \frac{1}{b-a} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k+b} \right]$$

[см. 5.1.5].

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)(k+b)} = \frac{\pi}{b-a} (\operatorname{ctg} \pi a - \operatorname{ctg} \pi b)$$

[$a=b$ см. в 5.1.3.5].

$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)(k+a+1)} = \pm \frac{1}{a} + 2\beta(a) \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}.$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(kn+m)(kn+nl+m)} = \frac{1}{nl} \sum_{k=0}^{l-1} \frac{1}{kn+m}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kn+m)(kn+m+n)} = \frac{1}{n} \left[2 \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} - \frac{1}{m} \right].$$

$$8. \sum_{k=1}^{\infty} \frac{1}{(k+l)(k+m)} = \frac{1}{m-l} \sum_{k=1}^{m-l} \frac{1}{k+l}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+l)(k+m)} = \frac{1}{m-l} [(-1)^m - (-1)^l] \ln 2 +$$

$$+ \frac{(-1)^m}{m-l} \sum_{k=1}^{m-l} \frac{(-1)^k}{k} - (-1)^l \sum_{k=1}^{l-1} \frac{(-1)^k}{k(k+m-l)}$$

[$l < m$].

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(k+a)} = \frac{1}{a} \left\{ \begin{matrix} \psi(a+1) + C \\ \beta(a+1) + \ln 2 \end{matrix} \right\}.$$

$$11. \sum_{k=-\infty}^{\infty} \frac{(\pm 1)^k}{k(k+a)} = \frac{1}{a^2} - \frac{\pi}{a \sin \pi a} \begin{Bmatrix} \cos \pi a \\ 1 \end{Bmatrix}.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k(kn+m)} = \frac{1}{m} \left[\psi\left(\frac{m}{n} + 1\right) + C \right].$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k(kn+nl+m)} = \frac{n}{nl+m} \left[\sum_{k=1}^l \frac{1}{kn+m} + \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{k(kn+m)} \right].$$

$$14. \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{1}{n} \sum_{k=1}^n \frac{1}{k}.$$

$$15. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+n)} = \frac{1}{n} [(-1)^n - 1] \ln 2 + \frac{(-1)^n}{n} \sum_{k=1}^n \frac{(-1)^k}{k}.$$

$$16. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(kn+m)^2} = \frac{1}{n^2} \left\{ \psi'(m/n) \right\}.$$

5.1.7. Ряды вида $\sum \frac{(\pm 1)^k}{k(kn+m)}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1. \quad 2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} = 1 - 2 \ln 2.$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}. \quad 4. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)} = -\frac{1}{4}.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \frac{11}{18}. \quad 6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+3)} = \frac{5}{18} - \frac{2}{3} \ln 2.$$

$$7. \sum_{k=1}^{\infty} \frac{1}{k(k+4)} = \frac{25}{48}. \quad 8. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+4)} = \frac{7}{48}.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2 \ln 2. \quad 10. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)} = 2 - \frac{\pi}{2} - \ln 2.$$

$$11. \sum_{k=1}^{\infty} \frac{1}{k(2k+3)} = \frac{8}{9} - \frac{2}{3} \ln 2.$$

$$12. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+3)} = -\frac{4}{9} + \frac{\pi}{6} - \frac{1}{3} \ln 2.$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k(3k+1)} = 3 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+1)} = 3 - \frac{\pi}{\sqrt{3}} - 2 \ln 2.$$

$$15. \sum_{k=1}^{\infty} \frac{1}{k(3k+2)} = \frac{3}{4} - \frac{\pi}{4\sqrt{3}} - \frac{3}{4} \ln 3.$$

$$16. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+2)} = \frac{3}{4} - \frac{\pi}{2\sqrt{3}}.$$

$$17. \sum_{k=1}^{\infty} \frac{1}{k(4k+1)} = 4 - \frac{\pi}{2} - 3 \ln 2.$$

$$18. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+1)} = 4 - \frac{\pi}{\sqrt{2}} - \ln 2 - \sqrt{2} \ln(1 + \sqrt{2}).$$

$$19. \sum_{k=1}^{\infty} \frac{1}{k(4k+3)} = \frac{4}{9} + \frac{\pi}{6} - \ln 2.$$

$$20. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+3)} = \frac{4}{9} - \frac{\pi}{3\sqrt{2}} - \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1 + \sqrt{2}).$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k(8k+1)} = 8 - \frac{\pi}{2} (\sqrt{2} + 1) - 4 \ln 2 - \sqrt{2} \ln(1 + \sqrt{2}).$$

$$22. \sum_{k=1}^{\infty} \frac{1}{k(8k+3)} = \frac{8}{9} - \frac{\pi}{6} (\sqrt{2} - 1) - \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1 + \sqrt{2}).$$

$$23. \sum_{k=1}^{\infty} \frac{1}{k(8k+5)} = \frac{8}{25} - \frac{\pi}{10} (\sqrt{2} - 1) - \frac{4}{5} \ln 2 + \frac{\sqrt{2}}{5} \ln(1 + \sqrt{2}).$$

$$24. \sum_{k=1}^{\infty} \frac{1}{k(8k+7)} = \frac{8}{49} + \frac{\pi}{28} (\sqrt{2} + 1) - \frac{4}{7} \ln 2 - \frac{\sqrt{2}}{7} \ln(1 + \sqrt{2}).$$

5.1.8. Ряды вида $\sum \frac{(\pm 1)^k}{(k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+m)} = \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{1}{k}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)} = 2 \ln 2. \quad 3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)} = \frac{\pi}{2} - \ln 2.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)} = \frac{3}{4} \ln 3 + \frac{\pi}{4\sqrt{3}}. \quad 5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+1)} = \frac{\pi}{2\sqrt{3}}.$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)} = \frac{3}{2} \ln 3 - \frac{\pi}{2\sqrt{3}}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+2)} = \frac{\pi}{\sqrt{3}} - 2 \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)} = \frac{\pi}{6} + \ln 2.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)} = -\frac{1}{3} \ln 2 + \frac{1}{3\sqrt{2}} [\ln(3+\sqrt{2}) + \pi].$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)} = 3 \ln 2 - \frac{\pi}{2}.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)} = -\ln 2 + \frac{\pi}{\sqrt{2}} - \frac{1}{\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+1)} = \frac{\pi}{14}(\sqrt{2}+1) + \frac{4}{7} \ln 2 + \frac{\sqrt{2}}{7} \ln(\sqrt{2}+1).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+3)} = \frac{\pi}{10}(\sqrt{2}-1) + \frac{4}{5} \ln 2 - \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1).$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+5)} = \frac{4}{3} \ln 2 - \frac{\pi}{6}(\sqrt{2}-1) - \frac{\sqrt{2}}{3} \ln(\sqrt{2}+1).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+7)} = 4 \ln 2 - \frac{\pi}{2}(\sqrt{2}+1) + \sqrt{2} \ln(\sqrt{2}+1).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(k+2)(2k+1)} = \frac{1}{3} + \frac{2}{3} \ln 2.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)(2k+1)} = \frac{\pi}{6} - \frac{1}{3} + \frac{1}{3} \ln 2.$$

5.1.9. Ряды вида $\sum \frac{(\pm 1)^k}{(2k+1)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)} = \frac{1}{2}. \quad 2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} = \frac{\pi}{4} - \frac{1}{2}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+1)} = \frac{\pi}{2\sqrt{3}} - 2 \ln 2 + \frac{3}{2} \ln 3.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(3k+1)} = \pi \left(\frac{\sqrt{3}}{3} - \frac{1}{2} \right) + \ln 2.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+2)} = \frac{\pi}{2\sqrt{3}} + 2 \ln 2 - \frac{3}{2} \ln 3.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(3k+2)} = \pi \left(\frac{1}{2} - \frac{\sqrt{3}}{3} \right) + \ln 2.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)} = \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+1)} = \frac{\pi}{4} (\sqrt{2}-1) - \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)} = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+3)} = \frac{\pi}{4} (1-\sqrt{2}) + \frac{\sqrt{2}}{2} \ln(\sqrt{2}-1).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+5)} = \frac{2}{3} - \frac{\pi}{12} - \frac{1}{6} \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+5)} = -\frac{2}{15} + \frac{\pi}{20} (1+\sqrt{2}) + \frac{1}{5\sqrt{2}} (\sqrt{2}-1).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+7)} = \frac{2}{15} + \frac{\pi}{20} - \frac{1}{10} \ln 2.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+1)} = \frac{\pi}{12} (\sqrt{2}+1) + \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{6} \ln(\sqrt{2}+1).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+3)} = \frac{\pi}{4} (\sqrt{2}-1) + \ln 2 - \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+5)} = \frac{\pi}{4} (\sqrt{2}-1) - \ln 2 + \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+7)} = \frac{\pi}{12} (\sqrt{2}+1) - \frac{1}{3} \ln 2 - \frac{\sqrt{2}}{6} \ln(\sqrt{2}+1).$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)} = \frac{1}{5} + \frac{\pi}{20} + \frac{1}{10} \ln 2.$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)} = -\frac{1}{5} + \frac{\pi}{20} (1+\sqrt{2}) + \frac{1}{5\sqrt{2}} \ln(1+\sqrt{2}).$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)} = \frac{1}{3} - \frac{\pi}{12} + \frac{1}{6} \ln 2.$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)} = -\frac{1}{3} + \frac{\pi}{12} (1+\sqrt{2}) - \frac{1}{3\sqrt{2}} \ln(1+\sqrt{2}).$$

5.1.10. Ряды вида $\sum \frac{(\pm 1)^k}{(3k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)} = \frac{\pi}{3\sqrt{3}}. \quad 2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)} = \frac{2}{3} \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)} = \frac{1}{3}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+4)} = \frac{1}{3} - \frac{2}{9} \ln 2 - \frac{\pi\sqrt{3}}{3}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+5)} = \frac{1}{8} + \frac{\pi}{12\sqrt{3}}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+5)} = \frac{\pi}{6\sqrt{3}} - \frac{1}{8}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(6k+1)} = \frac{\pi}{3\sqrt{3}} + \frac{2}{3} \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(6k+1)} = \frac{\pi}{3} \left(1 - \frac{\sqrt{3}}{3}\right) - \frac{1}{3} \ln 2 + \frac{1}{3\sqrt{3}} \ln(2 + \sqrt{3}).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(6k+5)} = \frac{2}{9} \left(\frac{\pi}{\sqrt{3}} - \ln 2\right).$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(6k+5)} = \frac{\pi}{9} \left(\frac{\sqrt{3}}{3} - 1\right) + \frac{1}{18\sqrt{3}} \ln(7 + 4\sqrt{3}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+4)} = \frac{1}{2} - \frac{\pi}{6\sqrt{3}}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+4)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{2}. \quad 13. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+5)} = \frac{1}{6}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+5)} = \frac{2}{9} \left(\frac{\pi}{\sqrt{3}} - \ln 2\right) - \frac{1}{6}.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(6k+1)} = \frac{2\pi}{9\sqrt{3}} + \frac{2}{9} \ln 2.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(6k+1)} = \frac{\pi}{9} \left(1 - \frac{\sqrt{3}}{3}\right) + \frac{1}{9} \ln 2 + \frac{1}{9\sqrt{3}} \ln(2 + \sqrt{3}).$$

5.1.11. Ряды вида $\sum \frac{(\pm 1)^k}{(4k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(4k+m)(4k+4l+m)} = \frac{1}{4l} \sum_{k=0}^{l-1} \frac{1}{4k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} = \frac{\pi}{8}.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)} = \frac{1}{4\sqrt{2}} \ln(3 + 2\sqrt{2}).$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)} = \frac{1}{4}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)} = -\frac{1}{4} + \frac{1}{8\sqrt{2}} [\pi + \ln(3+2\sqrt{2})].$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+7)} = \frac{1}{18} + \frac{\pi}{24}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+7)} = \frac{\pi}{12\sqrt{2}} - \frac{1}{18}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+1)} = \frac{\pi\sqrt{2}}{8} + \frac{1}{4} \ln 2 + \frac{\sqrt{2}}{4} \ln(\sqrt{2}+1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+3)} = \frac{\pi}{8} (2-\sqrt{2}) - \frac{1}{4} \ln 2 + \frac{\sqrt{2}}{4} \ln(1+\sqrt{2}).$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+5)} = \frac{\pi\sqrt{2}}{24} - \frac{1}{12} \ln 2 + \frac{\sqrt{2}}{12} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+7)} = \frac{\pi}{40} (2+\sqrt{2}) - \frac{1}{20} \ln 2 - \frac{\sqrt{2}}{20} \ln(1+\sqrt{2}).$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)} = \frac{1}{2} - \frac{\pi}{8}.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+5)} = \frac{\pi}{4\sqrt{2}} - \frac{1}{2}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)} = \frac{1}{12}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+7)} = \frac{\pi}{8\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln(1+\sqrt{2}) - \frac{1}{12}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+1)} = \frac{\pi}{40} (2+\sqrt{2}) + \frac{1}{20} \ln 2 + \frac{\sqrt{2}}{20} \ln(1+\sqrt{2}).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+3)} = \frac{\pi\sqrt{2}}{24} + \frac{1}{12} \ln 2 - \frac{\sqrt{2}}{12} \ln(1+\sqrt{2}).$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+5)} = \frac{\pi}{8} (2 - \sqrt{2}) + \frac{1}{4} \ln 2 - \frac{\sqrt{2}}{4} \ln(1 + \sqrt{2}).$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+7)} = \frac{\pi}{8} \sqrt{2} - \frac{1}{4} \ln 2 - \frac{\sqrt{2}}{4} \ln(1 + \sqrt{2}).$$

5.1.12. Ряды вида $\sum \frac{(\pm 1)^k}{(5k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(5k+m)(5k+5l+m)} = \frac{1}{5l} \sum_{k=0}^{l-1} \frac{1}{5k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(5k+1)(5k+6)} = \frac{1}{5}. \quad 3. \sum_{k=0}^{\infty} \frac{1}{(5k+2)(5k+7)} = \frac{1}{10}.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(5k+3)(5k+8)} = \frac{1}{15}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(5k+4)(5k+6)} = \frac{1}{2} - \sqrt{2+\sqrt{5}} \frac{\pi}{10}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(5k+4)(5k+6)} = -\frac{1}{2} + \frac{2\pi}{5\sqrt{10-2\sqrt{5}}}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(5k+4)(5k+9)} = \frac{1}{20}.$$

5.1.13. Ряды вида $\sum \frac{(\pm 1)^k}{(6k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(6k+m)(6k+6l+m)} = \frac{1}{6l} \sum_{k=0}^{l-1} \frac{1}{6k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(6k+1)(6k+5)} = \frac{\pi\sqrt{3}}{24}.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+1)(6k+5)} = \frac{1}{12\sqrt{3}} \ln(2 + \sqrt{3}).$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(6k+1)(6k+7)} = \frac{1}{6}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+1)(6k+7)} = -\frac{1}{6} + \frac{\pi}{18} + \frac{1}{36\sqrt{3}} \ln(7+4\sqrt{3}).$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(6k+5)(6k+7)} = \frac{1}{2} - \frac{\pi\sqrt{3}}{12}. \quad 7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+5)(6k+7)} = \frac{\pi}{6} - \frac{1}{2}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(6k+5)(6k+11)} = \frac{1}{30}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+5)(6k+11)} = -\frac{1}{30} + \frac{1}{18} [\pi - \sqrt{3} \ln(2+\sqrt{3})].$$

5.1.14. Ряды вида $\sum \frac{(\pm 1)^k}{(8k+l)(8k+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(8k+m)(8k+8l+m)} = \frac{1}{8l} \sum_{k=0}^{l-1} \frac{1}{8k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(8k+1)(8k+5)} = \frac{\pi}{16} + \frac{\sqrt{2}}{16} \ln(3+2\sqrt{2}).$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(8k+1)(8k+9)} = \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{16} \ln(1+\sqrt{2}).$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(8k+3)(8k+5)} = \frac{\pi}{16} (\sqrt{2}-1).$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(8k+1)(8k+7)} = \frac{\pi}{48} (\sqrt{2}+1).$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(8k+3)(8k+7)} = \frac{\pi}{32} \sqrt{2} - \frac{\sqrt{2}}{16} \ln(1+\sqrt{2}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(8k+5)(8k+7)} = \frac{\pi}{16} - \frac{\sqrt{2}}{8} \ln(1+\sqrt{2}).$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(8k+7)(8k+9)} = \frac{1}{2} - (\sqrt{2}+1) \frac{\pi}{16}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(8k+7)(8k+9)} = -\frac{1}{2} + (\sqrt{2}+1)^{1/2} \frac{\pi}{8}.$$

5.1.15. Ряды вида $\sum \frac{(\pm 1)^k}{(ka+b)(ka+c)(ka+d)}$.

$$1. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(ka+b)(ka+c)(ka+d)} = \int_0^1 dy \int_0^y dz \int_0^1 \frac{x^{b(1-y)+c(y-z)+dz-1}}{1+x^a} \ln^2 x dx.$$

$$2. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)(k+b)(k+c)} =$$

$$= -\frac{1}{(b-a)(c-a)} \left\{ \frac{\psi(a)}{\beta(a)} \right\} - \frac{1}{(a-b)(c-b)} \left\{ \frac{\psi(b)}{\beta(b)} \right\} - \frac{1}{(a-c)(b-c)} \left\{ \frac{\psi(c)}{\beta(c)} \right\}.$$

$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)(k+b)^2} = \frac{\pm 1}{(b-a)^2} \left\{ \frac{\psi(b) - \psi(a)}{\beta(b) - \beta(a)} \right\} + \frac{1}{b-a} \left\{ \frac{\psi'(b)}{\beta'(b)} \right\}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+a)(k+a+1)(k+a+2)} = \frac{1}{2a(a+1)} \quad [a \neq 0, -1, -2, \dots].$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+a+1)(k+a+2)} = 2\beta(a) - \frac{3}{2a} + \frac{1}{2(a+1)}$$

[a \neq 0, -1, -2, \dots].

$$6. \sum_{k=0}^{\infty} \frac{1}{(kn+m)(kn+m+n)(kn+m+2n)} = \frac{1}{2mn(m+n)}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kn+m)(kn+m+n)(kn+m+2n)} =$$

$$= \frac{2}{n^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} - \frac{3}{2mn^2} + \frac{1}{2(m+n)n^2} \quad [\text{см. 5.1.5.2}].$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(kn+m)} = \frac{n}{m^2} \ln 2 - \frac{\pi^2}{12m} + \frac{n^2}{m^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{kn+m} \quad [\text{см. 5.1.5.2}].$$

$$9. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(k+a)^2} = \frac{1}{a^2} \left\{ \frac{\psi(a+1)+c}{\beta(a+1) - \ln 2} \right\} - \frac{1}{a} \left\{ \frac{\psi'(a+1)}{\beta'(a+1)} \right\}.$$

$$10. \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k(k+a)^2} = \frac{2}{a^3} - \frac{\pi}{a^2} \left(\operatorname{ctg} \pi a + \frac{\pi a}{\sin^2 \pi a} \right).$$

$$11. \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k}{k(k+a)^2} = \frac{2}{a^3} - \frac{\pi}{\sin \pi a} \left(\frac{1}{a^2} + \frac{\pi}{a} \operatorname{ctg} \pi a \right).$$

5.1.16. Ряды вида $\sum \frac{(\pm 1)^k}{k(kn_1+m_1)(kn_2+m_2)}$.

1. $\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = \frac{\pi^2}{6} - 1.$
2. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)} = 2 \ln 2 - \frac{\pi^2}{12} - 1.$
3. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = 2 - \frac{\pi^2}{6}.$
4. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)^2} = 2 - \frac{\pi^2}{12} - 2 \ln 2.$
5. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}.$
6. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)} = \frac{5}{4} - 2 \ln 2.$
7. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+3)} = \frac{7}{36}.$
8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+3)} = \frac{1}{12} - \frac{1}{3} \ln 2.$
9. $\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+3)} = \frac{5}{36}.$
10. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)(k+3)} = -\frac{19}{36} - \frac{1}{3} \ln 2.$
11. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} = 3 - 4 \ln 2.$
12. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+1)} = 3 - \pi.$
13. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+3)} = \frac{4}{3} \ln 2 - \frac{7}{9}.$
14. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+3)} = \frac{17}{9} - \frac{\pi}{3} - \frac{4}{3} \ln 2.$
15. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+1)} = 4 - \frac{\pi\sqrt{3}}{4} - \frac{9}{4} \ln 3.$
16. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+1)} = 4 - 2 \ln 2 - \frac{\pi\sqrt{3}}{2}.$
17. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+2)} = 1 + \frac{\pi\sqrt{3}}{4} - \frac{9}{4} \ln 3.$
18. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+2)} = \frac{9}{4} - \frac{\pi\sqrt{3}}{2} + 2 \ln 2.$

$$19. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(4k+1)} = 5 - \frac{2\pi}{3} - 4 \ln 2.$$

$$20. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(4k+1)} = 5 - \frac{2\sqrt{2}}{3} \pi - \frac{2}{3} \ln 2 - \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(4k+3)} = \frac{7}{9} + \frac{2\pi}{3} - 4 \ln 2.$$

$$22. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(4k+3)} = \frac{13}{3} + \frac{2}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$23. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)(2k+3)} = \frac{5}{18} - \frac{1}{3} \ln 2.$$

$$24. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(2k+3)} = \frac{1}{3} + \ln 2 - \frac{\pi}{3}.$$

$$25. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)(3k+1)} = 2 - \frac{\pi\sqrt{3}}{2} + 4 \ln 2 - \frac{9}{2} \ln 3.$$

$$26. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(3k+1)} = 5 + \pi(1-\sqrt{3}) - 4 \ln 2.$$

$$27. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)(3k+2)} = \frac{7}{4} + \frac{\pi\sqrt{3}}{4} - 4 \ln 2 + \frac{9}{4} \ln 3.$$

$$28. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(3k+2)} = \frac{7}{4} - \pi\left(1 - \frac{\sqrt{3}}{2}\right) - 2 \ln 2.$$

$$29. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)(4k+1)} = 6 - \pi - 4 \ln 2.$$

$$30. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(4k+1)} = 6 + \frac{\pi}{2}(1-2\sqrt{2}) - 2\sqrt{2} \ln(1+\sqrt{2}).$$

$$31. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)(4k+3)} = \frac{10}{9} - \frac{\pi}{3}.$$

$$32. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(4k+3)} = \frac{2}{3} + \frac{\pi}{2}(\sqrt{2}-1) - \frac{1}{3} \ln 2 - \sqrt{2} \ln(1+\sqrt{2}).$$

$$33. \sum_{k=1}^{\infty} \frac{1}{k(4k+1)(4k+3)} = \frac{4}{9} - \frac{\pi}{3} - \ln 2.$$

$$34. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+1)(4k+3)} = \frac{16}{9} - \frac{\pi}{3\sqrt{2}} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

5.1.17. Ряды вида $\sum \frac{(\pm 1)^k}{(k+1)(kn_1+m_1)(kn_2+m_2)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} = 2 \ln 2 - 1.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(2k+3)} = 1 - \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(3k+1)} = \frac{\pi\sqrt{3}}{6} - 4 \ln 2 + \frac{9}{4} \ln 3.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(3k+1)} = 2 \ln 2 - \pi \left(1 - \frac{\sqrt{3}}{2}\right).$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(3k+1)} = \frac{\pi\sqrt{3}}{2} + 4 \ln 2 - \frac{9}{2} \ln 3.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(3k+2)} = 4 \ln 2 + \pi(1 - \sqrt{3}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)} = \frac{\pi}{3}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+1)} = \frac{\pi}{6}(2\sqrt{2}-3) + \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+3)} = \pi - 4 \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+3)} = \frac{\pi}{2}(1-2\sqrt{2}) + \ln 2 + 2\sqrt{2} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+5)} = \frac{\pi}{3} + \frac{8}{3} \ln 2 - \frac{8}{3}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+5)} = \frac{8}{3} + \pi \left(\frac{1}{6} - \sqrt{2} \right) - \ln 2 - 2\sqrt{2} \ln(1+\sqrt{2}).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+7)} = \frac{4}{5} \ln 2 - \frac{\pi}{15} - \frac{8}{45}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+7)} = \\ = \frac{8}{45} + \frac{\pi}{5} \left(1 - \frac{\sqrt{2}}{3} \right) - \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{15} \ln(1+\sqrt{2}).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)} = \frac{\pi}{15} + \frac{4}{5} \ln 2 - \frac{2}{5}.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+1)} = \\ = \frac{2}{5} - \frac{\pi}{30} (3 - 2\sqrt{2}) - \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{15} \ln(1+\sqrt{2}).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)} = \frac{8}{3} \ln 2 - \frac{\pi}{3} - \frac{2}{3}.$$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+3)} = \frac{2}{3} + \frac{\pi}{3} (\sqrt{2} - 1) - \ln 2 + \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)(3k+2)} = \frac{\pi\sqrt{3}}{4} - \frac{3}{4} \ln 3.$$

$$20. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+1)(3k+2)} = 2 \ln 2 - \frac{\pi}{2\sqrt{3}}.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)(3k+4)} = \frac{3}{2} (\ln 3 - 1).$$

$$22. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+2)(3k+4)} = \frac{3}{2} - 2 \ln 2.$$

$$23. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)} = \frac{\pi}{3} - \ln 2.$$

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)(4k+3)} = -\frac{\pi}{6\sqrt{2}} + \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{2} \ln(1+\sqrt{2}).$$

$$25. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+5)} = \frac{\pi}{6} + \ln 2 - 1.$$

$$26. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)(4k+5)} = \frac{\pi}{3\sqrt{2}} - \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$27. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+7)} = \frac{1}{3} \ln 2 - \frac{2}{27}.$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)(4k+5)} = 3 \ln 2 - 2.$$

$$29. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)(4k+5)} = 2 - \ln 2 - \frac{1}{\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$30. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)(4k+7)} = \ln 2 - \frac{\pi}{6} - \frac{1}{9}.$$

$$31. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)(4k+7)} = \frac{1}{9} + \frac{\pi}{6\sqrt{2}} - \frac{1}{3} \ln 2 - \frac{\pi}{3\sqrt{2}} \ln(1+\sqrt{2}).$$

$$32. \sum_{k=0}^{\infty} \frac{1}{(k+1)(6k+5)(6k+7)} = \frac{3}{2} \ln 3 + 2 \ln 2 - 3.$$

$$33. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(6k+5)(6k+7)} = 3 - \ln 2 - \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}).$$

5.1.18. Ряды вида $\sum \frac{(\pm 1)^k}{(2k+m)(kn_1+m_1)(kn_2+m_2)}.$

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{12}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(2k+5)} = \frac{\pi}{8} - \frac{1}{3}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(2k+7)} = \frac{11}{180}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(2k+7)} = \frac{\pi}{24} - \frac{4}{45}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)} = \frac{\pi}{10} + \frac{1}{5} \ln 2 - \frac{1}{10}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(4k+1)} = \left(\frac{1}{3\sqrt{2}} - \frac{1}{10} \right) \pi - \frac{1}{10} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+3)} = \frac{\pi}{6} - \frac{1}{3} \ln 2 - \frac{1}{6}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(4k+3)} = \frac{1}{6} + \frac{\pi}{12} (1-2\sqrt{2}) - \frac{\sqrt{2}}{3} \ln(\sqrt{2}-1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+5)} = \frac{5}{6} - \frac{\pi}{6} - \frac{1}{3} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(4k+5)} =$$

$$= -\frac{5}{6} + \pi \left(\frac{1}{3\sqrt{2}} - \frac{1}{8} \right) + \frac{1}{6} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+7)} = \frac{7}{30} - \frac{\pi}{10} + \frac{1}{5} \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+3)} = \frac{1}{2} \ln 2.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+1)(4k+3)} = (\sqrt{2}-1) \frac{\pi}{4}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+5)} = \frac{\pi}{12} + \frac{1}{6} \ln 2 - \frac{1}{6}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+1)(4k+5)} = \frac{1}{6} + \frac{\pi}{12} \left(\frac{\sqrt{2}}{2} - 1 \right) + \frac{1}{6\sqrt{2}} \ln(1+\sqrt{2}).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+7)} = \frac{\pi}{30} + \frac{1}{10} \ln 2 - \frac{1}{45}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)(4k+5)} = \frac{\pi}{6} - \frac{1}{6} \ln 2 - \frac{1}{3}.$$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+3)(4k+5)} = \frac{1}{3} - \frac{\pi}{12} (1-2\sqrt{2}) + \frac{1}{3\sqrt{2}} \ln(1+\sqrt{2}).$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)(4k+7)} = \frac{\pi}{20} - \frac{1}{10} \ln 2 - \frac{1}{30}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+5)(4k+7)} = \frac{4}{15} - \frac{\pi}{15} - \frac{1}{30} \ln 2.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+3)} = \frac{\pi}{15} - \frac{1}{30} \ln 2 - \frac{1}{15}.$$

$$22. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+3)} = \frac{1}{15} - \frac{\pi}{60} (1+\sqrt{2}) + \frac{4}{15} \ln(1+\sqrt{2}).$$

$$23. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+5)} = \frac{3}{10} - \frac{\pi}{20} - \frac{1}{10} \ln 2.$$

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+5)} = \frac{\pi}{20} \left(\frac{3\sqrt{2}}{2} - 1 \right) - \frac{3}{10} + \frac{3\sqrt{2}}{20} \ln(1+\sqrt{2}).$$

$$25. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+7)} = \frac{4}{45} + \frac{1}{10} \ln 2 - \frac{\pi}{30}.$$

$$26. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+7)} = \frac{\pi}{20} \left(1 - \frac{2\sqrt{2}}{3} \right) - \frac{4}{45} + \frac{\sqrt{2}}{10} \ln(1+\sqrt{2}).$$

$$27. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)(4k+5)} = \frac{2}{3} - \frac{\pi}{6} - \frac{1}{6} \ln 2.$$

$$28. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)(4k+5)} = \frac{\pi}{12} (2\sqrt{2}-1) - \frac{2}{3} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$29. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)(4k+7)} = \frac{1}{6} - \frac{\pi}{12} + \frac{1}{6} \ln 2.$$

$$30. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)(4k+7)} = \frac{\pi}{12} (1 - \sqrt{2}) - \frac{1}{6} + \frac{\sqrt{2}}{6} \ln(1 + \sqrt{2}).$$

5.1.19. Ряды вида $\sum \frac{(\pm 1)^k}{(3k+m)(3k+m_1)(3k+m_2)}.$

$$1. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+4)} = \frac{\pi\sqrt{3}}{18} - \frac{1}{6}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)(3k+4)} = \frac{1}{6} - \frac{\pi}{9\sqrt{3}} + \frac{2}{9} \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+5)} = \frac{\pi}{12\sqrt{3}} - \frac{1}{24}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)(3k+5)} = \frac{1}{4} - \frac{7\pi}{36\sqrt{3}} + \frac{13}{36} \ln 2.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+2)(3k+7)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{60}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)(3k+7)} = \frac{2}{15} \ln 2 - \frac{1}{40}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)(3k+7)} = \frac{1}{24}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+4)(3k+7)} = \frac{2\pi}{27\sqrt{3}} + \frac{2}{27} \ln 2 - \frac{11}{72}.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+5)(3k+8)} = \frac{1}{60}.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+5)(3k+8)} = \frac{2\pi}{27\sqrt{3}} - \frac{2}{27} \ln 2 - \frac{13}{180}.$$

5.1.20. Ряды вида $\sum \frac{(\pm 1)^k}{(4k+m)(4k+m_1)(4k+m_2)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+5)} = \frac{\pi}{16} - \frac{1}{8}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+5)} = \frac{1}{8} - \frac{\pi}{16\sqrt{2}} + \frac{1}{16\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+7)} = \frac{\pi}{48} - \frac{1}{72}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+7)} = \frac{1}{72} - \frac{\pi}{48\sqrt{2}} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)(4k+7)} = \frac{7}{72} - \frac{\pi}{48}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)(4k+7)} = \frac{\pi}{48\sqrt{2}} - \frac{7}{72} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)(4k+7)} = \frac{5}{24} - \frac{\pi}{16}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+5)(4k+7)} = \frac{\pi}{16\sqrt{2}} - \frac{5}{24} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$$

5.1.21. Ряды вида $\sum \frac{(\pm 1)^k}{(kn_1+m_1)(kn_2+m_2)(kn_3+m_3)(kn_4+m_4)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)(k+a+1)(k+a+2)(k+a+3)} = \frac{1}{3a(a+1)(a+2)} \quad [a \neq 0, -1, -2, \dots]$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+a+1)(k+a+2)(k+a+3)} = \\ = \frac{4}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} - \frac{7}{6a} + \frac{2}{3(a+1)} - \frac{1}{6(a+2)},$$

$$3. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)^2(k+b)^2} = \\ = \pm \frac{2}{(b-a)^3} \left\{ \psi(a) - \psi(b) \right\} \pm \frac{1}{(b-a)^2} \left\{ \psi'(a) + \psi'(b) \right\}$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+n)^2} = \frac{\pi^2}{3n^2} - \frac{1}{n^2} \sum_{k=1}^n \frac{1}{k^2} - \frac{2}{n^3} \sum_{k=1}^n \frac{1}{k}.$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 (k+n)^2} = \frac{2}{n^3} [1 - (-1)^n] \ln 2 - \frac{\pi^2}{12n^2} [1 + (-1)^n] + \\ + \frac{2(-1)^{n+1}}{n^3} \sum_{k=1}^n \frac{(-1)^k}{k} + (-1)^{n+1} \sum_{k=1}^n \frac{(-1)^k}{k^2}.$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)^2} = \frac{\pi^2 - 9}{3}. \quad 7. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 (k+1)^2} = 4 \ln 2 - 3.$$

$$8. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+2)^2} = \frac{\pi^2}{12} - \frac{11}{16}. \quad 9. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 (k+2)^2} = \frac{5}{16} - \frac{\pi^2}{24}.$$

$$10. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{18}.$$

$$11. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)(k+3)} = \frac{8}{9} - \frac{4}{3} \ln 2.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)(2k+3)} = \frac{17}{9} - \frac{8}{3} \ln 2.$$

$$13. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+1)(2k+3)} = \frac{5}{9} - \frac{\pi}{3} + \frac{2}{3} \ln 2.$$

$$14. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+1)(3k+2)} = \frac{1}{18} - \frac{\pi}{6\sqrt{3}}.$$

$$15. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+1)(3k+2)} = \frac{11}{4} - 4 \ln 2.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (2k+1)(2k+3)} = \frac{1}{2} - \frac{\pi^2}{24}.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2 (2k+1)(2k+3)} = \frac{\pi}{4} - \frac{1}{2} - \frac{\pi^2}{24}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (3k+2) (3k+4)} = \frac{1}{2} - \frac{\pi}{6\sqrt{3}} - \frac{\pi^2}{54}.$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2 (3k+2) (3k+4)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{2} - \frac{\pi^2}{108}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(k+1) (k+2) (2k+1) (2k+3)} = \frac{8}{3} \ln 2 - \frac{5}{3}.$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (k+2) (2k+1) (2k+3)} = \frac{5}{3} - \frac{\pi}{3} - \frac{2}{3} \ln 2.$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(k+1) (k+2) (2k+1) (2k+5)} = \frac{1}{9}.$$

$$23. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (k+2) (2k+1) (2k+5)} = \frac{5}{9} - \frac{2}{3} \ln 2.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1)} = \frac{1}{5} + \frac{2\pi}{15} - \frac{2}{5} \ln 2.$$

$$25. \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (2k+3) (4k+1)} = \\ = \frac{1}{3} \ln 2 + \frac{\pi}{5} \left(\frac{2\sqrt{2}}{3} - 1 \right) + \frac{4\sqrt{2}}{15} \ln(1+\sqrt{2}) - \frac{1}{5}.$$

$$26. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3)} = \frac{2\pi}{3} - \frac{10}{3} \ln 2 + \frac{1}{3}.$$

$$27. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (2k+3) (4k+3)} = \\ = \frac{\pi}{3} (1-2\sqrt{2}) + \frac{4\sqrt{2}}{3} \ln(1+\sqrt{2}) - \frac{1}{3}.$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+5)} = \frac{2\pi}{3} + \frac{10}{3} \ln 2 - \frac{13}{3}.$$

$$29. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+3) (4k+5)} = \frac{\pi}{3} - \frac{10}{3} \ln 2 + \frac{4}{3}.$$

$$30. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)(4k+3)} = 2 \ln 2 - \frac{\pi}{3}.$$

$$31. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+1)(4k+3)} = \\ = \frac{\pi}{2} \left(\frac{4\sqrt{2}}{3} - 1 \right) - \frac{1}{3} \ln 2 - \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$32. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)(4k+5)} = \frac{2}{3} - \frac{2}{3} \ln 2.$$

$$33. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)} = \frac{\pi}{5} - \frac{14}{15} \ln 2 + \frac{2}{15}.$$

$$34. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+5)} = \frac{4\pi}{15} + \frac{6}{5} \ln 2 - \frac{8}{5}.$$

$$35. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)(4k+5)} = \frac{\pi}{3} + \frac{10}{3} \ln 2 - \frac{10}{3}.$$

$$36. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)(3k+2)(3k+4)} = \frac{1}{2} + \frac{\pi}{4\sqrt{3}} - \frac{3}{4} \ln 3.$$

$$37. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+1)(3k+2)(3k+4)} = \frac{1}{2} - \frac{\pi}{2\sqrt{3}} + \frac{1}{3} \ln 2.$$

$$38. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)(4k+5)} = \frac{\pi}{12} - \ln 2 + \frac{1}{2}.$$

$$39. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2(2k+3)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

$$40. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2(2k+3)^2} = \frac{1}{2} - \frac{\pi}{8}.$$

$$41. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2(2k+5)^2} = \frac{\pi^2}{64} - \frac{1}{9}.$$

$$42. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2(2k+5)^2} = \frac{1}{8} G - \frac{11}{144}.$$

$$43. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 (4k+1) (4k+3)} = \frac{\pi}{2} - \frac{\pi^2}{8}.$$

$$44. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 (4k+1) (4k+3)} = \frac{1}{4\sqrt{2}} \ln(3+2\sqrt{2}) - \frac{1}{4} G.$$

$$45. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+1) (4k+3)} = \frac{1}{30} + \frac{4}{15} \ln 2 - \frac{\pi}{30}.$$

$$46. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) (2k+3) (4k+1) (4k+3)} =$$

$$= -\frac{1}{30} + \frac{\pi}{60} (8\sqrt{2}-7) - \frac{\sqrt{2}}{2} \ln(1+\sqrt{2}).$$

$$47. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+1) (4k+5)} = \frac{\pi}{15} + \frac{2}{15} \ln 2 - \frac{7}{30}.$$

$$48. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+3) (4k+5)} = \frac{\pi}{6} - \frac{1}{2}.$$

$$49. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (4k+1) (4k+3)^2} = 0,112742.$$

$$50. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (4k+1) (4k+3) (4k+5)} = \frac{1}{6} \ln 2 - \frac{\pi}{24} + \frac{1}{12}.$$

$$51. \sum_{k=0}^{\infty} \frac{1}{(2k+3) (4k+1) (4k+3) (4k+5)} = \frac{7\pi}{120} + \frac{1}{30} \ln 2 - \frac{11}{60}.$$

$$52. \sum_{k=0}^{\infty} \frac{1}{(8k+1) (8k+3) (8k+5) (8k+7)} = \frac{\pi}{96} (2+\sqrt{2})^{-1}.$$

$$53. \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2 (4k+3)^2} = \frac{\pi(\pi-2)}{32}.$$

$$54. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)^2 (4k+3)^2} = 0,11037723.$$

$$55. \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2 (4k+7)^2} = \frac{\pi}{16} + \frac{\pi^2}{32} - \frac{1}{2}.$$

5.1.22. Ряды вида $\sum \frac{(\pm 1)^k}{(kn_1 + m_1)(kn_2 + m_2) \dots (kn_s + m_s)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (2k+1) (2k+3)^2} = 7 - \frac{5\pi^2}{12} - 4 \ln 2.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1)^2 (2k+3)^2} = \frac{3}{2} - 2 \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+3)} = \frac{22}{15} \ln 2 - \frac{4\pi}{15} - \frac{1}{15}.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+5)} = \frac{17}{15} - \frac{2\pi}{15} - \frac{14}{15} \ln 2.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+7)} = \frac{19}{135} - \frac{2}{15} \ln 2.$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3) (4k+5)} = \frac{7}{3} - \frac{10}{3} \ln 2.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3) (4k+7)} = \frac{2}{15} - \frac{14}{15} \ln 2 - \frac{11}{45}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+5) (4k+7)} = \frac{4\pi}{15} + \frac{22}{15} \ln 2 - \frac{83}{45}.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+3) (4k+5)} = \frac{4}{3} \ln 2 - \frac{\pi}{6} - \frac{1}{3}.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (4k+1) (4k+3) (4k+5)} =$$

$$= \frac{1}{12\sqrt{2}} \left[\frac{35}{6} \pi - 6 \ln (3 + 2\sqrt{2}) + \frac{1}{3} (1 - \ln 2) \right].$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+3) (4k+7)} = \frac{8}{15} \ln 2 - \frac{\pi}{10} - \frac{1}{135}.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+5) (4k+7)} = \frac{43}{135} - \frac{4}{15} \ln 2 - \frac{\pi}{30}.$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{30} - \frac{16}{15} \ln 2 + \frac{29}{45}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{13}{15} - \frac{\pi}{30} - \frac{16}{15} \ln 2.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+7)} = \frac{\pi}{30} - \frac{4}{15} \ln 2 + \frac{13}{135}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+5)(4k+7)} = \frac{\pi}{10} + \frac{8}{15} \ln 2 - \frac{91}{135}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{6} - \frac{4}{3} \ln 2 - \frac{13}{9}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{13}{54} - \frac{1}{3} \ln 2.$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{1}{15} \ln 2 - \frac{\pi}{20} + \frac{2}{15}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+3)(4k+7)} = \frac{1}{45} + \frac{1}{15} \ln 2 - \frac{\pi}{60}.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{20} + \frac{1}{15} \ln 2 - \frac{1}{5}.$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+5)(4k+7)} = \frac{\pi}{60} + \frac{1}{15} \ln 2 - \frac{4}{45}.$$

$$23. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{7}{180} + \frac{1}{30} \ln 2 - \frac{\pi}{60}.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{60} + \frac{1}{30} \ln 2 - \frac{13}{180}.$$

5.1.23. Ряды вида $\sum \frac{(\pm 1)^k}{(kn_1+m_1)(kn_2+m_2) \dots (kn_i+m_i)}$, $i=6, 7, 8, 9$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)^3} = 10 - \pi^2.$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 (k+1)^3} = 10 - 12 \ln 2 - \frac{3}{2} \zeta(3).$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)^2 (k+2)^2} = \frac{\pi^4}{4} - \frac{39}{16}.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 (k+1)^2 (k+2)^2} = \frac{\pi^2}{24} - \frac{7}{16}.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)^2 (2k+1)^2} = 2\pi^4 + \frac{\pi^2}{3} - 23.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 (k+1)^2 (2k+1)^2} = 12 \ln 2 - 23 + 16G.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{6}{5} \ln 2 - \frac{\pi}{15} - \frac{3}{5}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+3)(4k+7)} = \frac{2}{5} - \frac{\pi}{15} - \frac{7}{135}.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+5)(4k+7)} = \frac{67}{135} - \frac{\pi}{15} - \frac{2}{5} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{47}{45} - \frac{\pi}{15} - \frac{6}{5} \ln 2.$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{2}{5} \ln 2 - \frac{\pi}{30} - \frac{22}{135}.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{52}{135} - \frac{\pi}{30} - \frac{2}{5} \ln 2.$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3 (2k+3)^3} = \frac{1}{2} - \frac{3\pi^2}{64}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3 (2k+3)^3} = -\frac{1}{2} + \frac{3\pi}{32} - \frac{\pi^3}{128}.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 (2k+3)^2 (2k+5)^2} = \frac{3\pi^2}{256} - \frac{1}{9}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+1) (4k+3) (4k+5) (4k+7)} = \frac{1}{18} - \frac{\pi}{60}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(4k+1)^3 (4k+3)^3} = \frac{3\pi}{64} - \frac{3\pi^2}{128} - \frac{\pi^3}{256}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(4k+3)^3 (4k+7)^3} = \frac{1}{2} - \frac{3\pi}{64} - \frac{3\pi^2}{128} + \frac{\pi^3}{256}.$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+3) (4k+5) (4k+7)} = \\ = \frac{2}{5} \ln 2 - \frac{37}{135}.$$

$$20. \sum_{k=1}^{\infty} \frac{1}{k^4 (k+1)^4} = \frac{\pi^4}{45} + \frac{10\pi^2}{3} - 35.$$

$$21. \sum_{k=1}^{\infty} \frac{1}{[k (k+1) (k+2) (k+3)]^2} = \frac{5\pi^2}{54} - \frac{197}{216}.$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4 (2k+3)^4} = \frac{\pi^2}{768} (\pi^2 + 30) - \frac{1}{2}.$$

$$23. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1) (2k+3) (2k+5) (2k+7)]^2} = \frac{\pi}{1728} - \frac{7}{4050}.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(4k+1)^4 (4k+3)^4} = \frac{1}{1536} \pi (\pi^3 - 6\pi^2 + 30\pi - 60).$$

$$25. \sum_{k=1}^{\infty} \frac{1}{[k (k+1) (k+2) (k+3)]^3} = \frac{35\pi^2}{648} - \frac{6217}{11\,664}.$$

5.1.24. Ряды вида $\sum \frac{(\pm 1)^k}{(k+a_1) (k+a_2) \dots (k+a_n)}.$

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a_1) (k+a_2) \dots (k+a_n)} = (-1)^n \sum_{k=1}^n \prod_{\substack{l=1 \\ l \neq k}}^n \frac{1}{a_k - a_l} \psi(a_k) \\ [a_i \neq 0, -1, -2, \dots; a_i \neq a_l \text{ при } i \neq l].$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(ka+b)(ka+b+1) \dots (ka+b+n)} = \frac{1}{n!} \int_0^1 \frac{t^{b-1} (1-t)^n}{1-t^a} dt$$

[$a, b > 0$].

$$3. \sum_{k=1}^{\infty} \frac{1}{(a+1)(2a+1) \dots (ka+1)} = \frac{1}{a} \quad \left[a \neq 0, -1, -\frac{1}{2}, -\frac{1}{3}, \dots \right].$$

$$4. \sum_{k=0}^{\infty} \frac{1}{a(a+1) \dots (a+k)} = \int_0^1 t^{a-1} e^{1-t} dt \quad [\operatorname{Re} a > 0].$$

$$5. \sum_{k=0}^{\infty} \frac{k!}{a(a+2) \dots (a+2k)} = \int_0^1 \frac{t^{a/2-1}}{1+t} dt \quad [\operatorname{Re} a > 0].$$

$$6. \sum_{k=1}^{\infty} \frac{p(k)}{q_1(k) q_2(k) q_3(k)} =$$

$$= - \sum_{i=1}^m a_i \psi(1+\alpha_i) - \sum_{i=1}^r [b_{1i} \psi(1+\beta_i) - b_{2i} \psi'(1+\beta_i)] -$$

$$- \sum_{i=1}^s [c_{1i} \psi(1+\gamma_i) - c_{2i} \psi'(1+\gamma_i) + \frac{c_{3i}}{2} \psi''(1+\gamma_i)],$$

где

$$q_1(k) = (k+\alpha_1)(k+\alpha_2) \dots (k+\alpha_m),$$

$$q_2(k) = (k+\beta_1)^2 (k+\beta_2)^2 \dots (k+\beta_r)^2,$$

$$q_3(k) = (k+\gamma_1)^3 (k+\gamma_2)^3 \dots (k+\gamma_s)^3,$$

$p(k)$ — многочлен степени не выше $m+2r+3s-2$, все постоянные $\alpha_i, \beta_i, \gamma_i$ различны и

$$\frac{p(k)}{q_1(k) q_2(k) q_3(k)} = \sum_{i=1}^m \frac{a_i}{k+\alpha_i} + \sum_{i=1}^r \left[\frac{b_{1i}}{k+\beta_i} + \frac{b_{2i}}{(k+\beta_i)^2} \right] +$$

$$+ \sum_{i=1}^s \left[\frac{c_{1i}}{k+\gamma_i} + \frac{c_{2i}}{(k+\gamma_i)^2} + \frac{c_{3i}}{(k+\gamma_i)^3} \right],$$

$$\sum_{i=1}^m a_i + \sum_{i=1}^r b_{1i} + \sum_{i=1}^s c_{1i} = 0.$$

$$7. \sum_{k=1}^{\infty} \frac{1}{[k(k+1) \dots (k+n)]^m} = S_{m,n},$$

$$S_{1,n} = \frac{1}{n!n}, \quad S_{2,n} = \frac{(2n-1)!! 2^n}{(n!)^3} \left[\frac{\pi^2}{6} - 3 \sum_{k=1}^n \frac{(k-1)!}{k 2^k (2k-1)!!} \right].$$

$$8. \sum_{k=1}^{\infty} \frac{1}{k^m (k+1)^n} = (-1)^m \sum_{k=0}^{n-2} \binom{m+k-1}{1} \zeta(n-k) + \\ + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} \zeta(m-k) + (-1)^{m-1} \binom{m+n-1}{n-1}.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = \\ = \frac{(-1)^n}{(n-1)!} \sum_{k=1}^{[n/2]} \frac{(2n-2k-1)! (2\pi)^{2k}}{(n-2k)! (2k)!} |B_{2k}| + (-1)^{n-1} \binom{2n-1}{n-1}.$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^m (k+1)^n} = (-1)^{m-1} \sum_{k=0}^{n-2} \binom{m+k-1}{k} (1-2^{1-n+k}) \zeta(n-k) + \\ + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} (1-2^{1-m+k}) \zeta(m-k) + \\ + (-1)^n \binom{m+n-1}{n-1} + (-1)^{m-1} \binom{m+n-2}{m-1} 2 \ln 2.$$

$$11. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n (k+1)^n} = 2 (-1)^{n-1} \sum_{k=1}^{[(n-1)/2]} \binom{2n-2k-2}{n-1} (1-2^{-2k}) \zeta(2k+1) + \\ + (-1)^n \binom{2n-1}{n-1} + (-1)^{n-1} \binom{2n-2}{n-1} 2 \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^m (2k+3)^n} = \\ = \frac{1}{2^{m+n}} \left[(-1)^m \sum_{k=0}^{n-2} \binom{m+k-1}{k} (2^{n-k}-1) \zeta(n-k) + \right. \\ \left. + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} (2^{m-k}-1) \zeta(m-k) + \right. \\ \left. + (-1)^{m-1} \sum_{k=0}^{m-1} \binom{m+k-1}{k} 2^{n-k} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n (2k+3)^n} = \\ = \frac{(-1)^n}{2^{2n} (n-1)!} \sum_{k=1}^{[n/2]} \frac{(2n-2k-1)! (2\pi)^{2k} (2^{2k}-1)}{(n-2k)! (2k)!} B_{2k} + \frac{(-1)^{n-1}}{2}.$$

$$\begin{aligned}
 14. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m (2k+3)^n} = \\
 = \frac{(-1)^{m-1}}{2^{m+2n}} \sum_{k=0}^{n-2} \binom{m+k-1}{k} 2^k \left[\zeta \left(n-k, \frac{1}{4} \right) - \zeta \left(n-k, \frac{3}{4} \right) \right] + \\
 + \frac{1}{2^{2m+n}} \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} 2^k \left[\zeta \left(m-k, \frac{1}{4} \right) - \zeta \left(m-k, \frac{3}{4} \right) \right] + \\
 + \frac{(-1)^m}{2^m} \sum_{k=0}^{n-1} \binom{m+k-1}{k} \left(\frac{1}{2} \right)^k + \frac{(-1)^{m-1} \pi}{2^{m+n}} \binom{m+n-2}{m-1}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n (2k+3)^n} = \frac{(-1)^n}{2} + \\
 + \frac{(-1)^{n+1} \pi}{2^{2n} (n-1)!} \sum_{k=0}^{[(n-1)/2]} \frac{(2n-k-2)!}{(n-k-1)! (2k)!} \pi^{2k} E_{2k}.
 \end{aligned}$$

5.1.25. Ряды вида $\sum \frac{(\pm 1)^k k^{\pm m}}{(k^2 \pm a^2)^s}$.

$$1. \quad \sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2)^s} = a^{-2s} + \frac{2\sqrt{\pi}}{\Gamma(s)} (2a)^{1/2-s} \int_0^{\infty} \frac{1}{e^t - 1} t^{s-1} J_{s-1/2}(at) dt$$

[Re s > 1/2; |a| < 1].

$$2. \quad \sum_{k=0}^{\infty} \frac{1}{(k+a)^2 - b^2} = \frac{1}{2b} [\psi(a+b) - \psi(a-b)].$$

$$3. \quad \sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2 \pm b^2} = \frac{\pi}{b} \left[\frac{\{\operatorname{ch} 2\pi b\}}{\{\cos 2\pi b\}} - \cos 2\pi a \right]^{-1} \left\{ \frac{\operatorname{sh} 2\pi b}{\sin 2\pi b} \right\}.$$

$$4. \quad \sum_{k=0}^{\infty} \frac{1}{k^2 \pm a^2} = \pm \frac{1}{2a^2} \pm \frac{\pi}{2a} \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\}.$$

$$5. \quad \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{k^2 - n^2} = \frac{3}{4n^2}$$

[n = 1, 2, 3, ...].

$$6. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2 \pm a^2} = \pm \frac{1}{2a^2} \pm \frac{\pi}{2a} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}.$$

$$7. \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(-1)^k}{k^2 - n^2} = \frac{1}{4n^2} [2 + (-1)^n] \quad [n=1, 2, 3, \dots].$$

$$8. \sum_{\substack{k=1 \\ k \neq 2n}}^{\infty} \frac{(-1)^k}{k^2 - 4n^2} = \frac{3}{16n^2}.$$

$$9. \sum_{\substack{k=1 \\ k \neq n/2}}^{\infty} \frac{1}{k^2 - \frac{n^2}{4}} = \frac{1}{2n^2} [5 + (-1)^n] \quad [n=1, 2, 3, \dots].$$

$$10. \sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 \pm a^2} = \frac{1}{2} \left\{ \beta(ia) + \beta(-ia) \right\} \left\{ \beta(a) + \beta(-a) \right\}.$$

$$11. \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1} = -0,269611. \quad 12. \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1/4} = -0,516128.$$

$$13. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(k^2 + a^2)} = \frac{1}{2a^2} \left\{ \psi(1+ia) + \psi(1-ia) - 2\psi(1) \right\} \left\{ \beta(1+ia) + \beta(1-ia) - 2\beta(1) \right\}.$$

$$14. \sum_{k=1}^{\infty} \frac{1}{k(k^2 + 1)} = 0,671866. \quad 15. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 + 1)} = -0,423567.$$

$$16. \sum_{k=1}^{\infty} \frac{1}{k(k^2 + 1/4)} = 0,993317. \quad 17. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 + 1/4)} = -0,708078.$$

$$18. \sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/4)} = 4(2 \ln 2 - 1). \quad 19. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 - 1/4)} = -1,227411.$$

$$20. \sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/9)} = \frac{27}{2} (\ln 3 - 1).$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/36)} = -108 + 54 \ln 3 + 72 \ln 2.$$

$$22. \sum_{k=1}^{\infty} \frac{1}{k^2(k^2 \pm a^2)} = \frac{1}{2a^4} \pm \frac{\pi^2}{6a^2} - \frac{\pi}{2a^3} \left\{ \operatorname{cth} \pi a \right\} \left\{ \operatorname{ctg} \pi a \right\}.$$

$$23. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2 \pm a^2)} = \frac{1}{2a^4} \mp \frac{\pi^2}{12a^2} - \frac{\pi}{2a^3} \left\{ \operatorname{sh} \pi a \right\}^{-1} \left\{ \sin \pi a \right\}.$$

$$24. \sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^3 + a^3} = \frac{1}{3} \beta(a) + \frac{1}{3} \beta(ae^{2\pi i/3}) + \frac{1}{3} \beta(ae^{4\pi i/3}).$$

$$25. \sum_{k=0}^{\infty} \frac{1}{(k^2 \pm a^2)^2} = \frac{1}{2a^4} + \frac{\pi}{4a^3} \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} + \frac{\pi^2}{4a^2} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2.$$

$$26. \sum_{k=0}^{\infty} \frac{1}{(k^2 \pm a^2)(k^2 \pm b^2)} = \frac{\pi}{2ab} \cdot \frac{1}{a^2 - b^2} \left(a \left\{ \frac{\operatorname{cth} \pi b}{\operatorname{ctg} \pi b} \right\} - b \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \right) - \frac{1}{a^2 b^2}.$$

$$27. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2 \pm a^2)^2} = \frac{1}{2a^4} + \frac{\pi}{4a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\} \left(1 + \pi a \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \right).$$

$$28. \sum_{k=1}^{\infty} \frac{k}{(k^2 \pm a^2)^2} = \frac{1}{4\alpha a} [\psi'(-\alpha a) - \psi'(\alpha a)] \quad [\alpha = \{i\}].$$

$$29. \sum_{k=1}^{\infty} \frac{k}{(k^2 + 1)^2} = 0,397117. \quad 30. \sum_{k=1}^{\infty} \frac{k}{(k^2 + 1/4)^2} = 0,825042.$$

$$31. \sum_{k=1}^{\infty} \frac{k}{(k^2 - 1/4)^2} = 2.$$

$$32. \sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/4)^2} = 16 \left(\frac{3}{2} - 2 \ln 2 \right).$$

$$33. \sum_{k=1}^{\infty} \frac{1}{k^2(k^2 \pm a^2)^2} = \frac{\pi^2}{6a^4} \pm \frac{1}{a^6} \mp \frac{3\pi}{4a^5} \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \mp \frac{\pi^2}{4a^4} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2.$$

$$34. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2 \pm a^2)^2} = \pm \frac{1}{a^6} - \frac{\pi^2}{12a^4} \mp \frac{\pi}{4a^5} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\} \left(3 + \pi a \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \right).$$

$$35. \sum_{k=1}^{\infty} \frac{k^2}{(k^2 \pm a^2)^2} = \frac{\pi}{4a} \left(\pm \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \mp \pi a \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \right).$$

$$36. \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{(k^2 \pm a^2)^2} = \frac{\pi}{4a} \left(\pm 1 \mp \pi a \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \right) \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}.$$

$$37. \sum_{k=1}^{\infty} \frac{1}{(k^2 \pm a^2)^3} = \mp \frac{1}{2a^6} \pm \frac{3\pi^2}{16a^4} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \pm$$

$$\pm \frac{3\pi}{16a^5} \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\} \pm \frac{\pi^3}{a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right\}.$$

$$38. \sum_{k=1}^{\infty} \frac{k^2}{(k^2 \pm a^2)^3} = \frac{\pi}{16a^3} \left\{ \operatorname{cth} \pi a \right\} \left\{ \operatorname{ctg} \pi a \right\} + \frac{\pi}{16a^2} \left\{ \operatorname{cosech} \pi a \right\}^2 \left(1 - 2\pi a \left\{ \operatorname{ctg} \pi a \right\} \right).$$

$$39. \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{(k^2 \pm a^2)^3} = \frac{\pi}{16a^3} \left\{ \operatorname{cosech} \pi a \right\} \left\{ \operatorname{cosec} \pi a \right\} \left(\mp \pi^2 a^2 - 2\pi^2 a^2 \left\{ \operatorname{cosech} \pi a \right\}^2 + 1 + \pi a \left\{ \operatorname{cth} \pi a \right\} \right).$$

5.1.26. Ряды вида $\sum \frac{(\pm 1)^k (2k+1)^{\pm m}}{[(2k+1)^2 \pm a^2]^n}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pm a^2} = \frac{\pi}{4a} \left\{ \operatorname{th} (\pi a/2) \right\} \left\{ \operatorname{tg} (\pi a/2) \right\}.$$

$$2. \sum_{\substack{k=0 \\ k \neq (n-1)/2}}^{\infty} \frac{1}{(2k+1)^2 - n^2} = \frac{1}{8n^2} [1 - (-1)^n].$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 - a^2} = \frac{1}{4a} \left[\frac{\pi}{\cos (\pi a/2)} - 2\beta \left(\frac{a+i}{2} \right) \right].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1} = 0,425841. \quad 5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1/4} = 0,71864956.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 - 1/4} = 1,246450.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{(2k+1)^2 \pm a^2} = \frac{\pi}{4} \left\{ \operatorname{sech} (\pi a/2) \right\} \left\{ \sec (\pi a/2) \right\}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 + 1]} = 0,547701.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 + 1/4]} = 0,85069754.$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 - 1/4]} = 1,386294.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 \pm a^2]} = \pm \frac{\pi}{4a^2} \left[1 - \left\{ \operatorname{sech} (\pi a/2) \right\} \left\{ \sec (\pi a/2) \right\} \right].$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 - a^2]} = \frac{1}{4a^2} \left[2\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1-a}{2}\right) - \psi\left(\frac{1+a}{2}\right) \right].$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 + 1]} = 0,47238807.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 - 1/4]} = 2 \ln 2.$$

$$15. \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 - 1]} = \frac{3}{4} - \frac{\pi}{4}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 [(2k+1)^2 \pm a^2]} = \pm \frac{\pi^2}{8a^2} \mp \frac{\pi}{4a^3} \left\{ \operatorname{th}(\pi a/2) \right\} \left\{ \operatorname{tg}(\pi a/2) \right\}.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + a^2]} = \frac{i}{4a^3} \left[\beta\left(\frac{1-ia}{2}\right) - \beta\left(\frac{1+ia}{2}\right) \right] - \frac{1}{4a^2} \beta'\left(\frac{1}{2}\right).$$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + 1]} = 0,490124.$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + 1/4]} = 0,789264.$$

$$20. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 - 1/4]} = 1,32193954.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 \pm a^2]^2} = \pm \frac{\pi}{8a^3} \left\{ \operatorname{th}(\pi a/2) \right\} \left\{ \operatorname{tg}(\pi a/2) \right\} \mp \frac{\pi^2}{16a^2} \left\{ \operatorname{sech}(\pi a/2) \right\}^2 \left\{ \sec(\pi a/2) \right\}.$$

$$22. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)^2 - 1/4]^2} = 1,766036.$$

$$23. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 \pm a^2]^2} = \frac{1}{16\alpha a} \left[\psi'\left(\frac{1-\alpha a}{2}\right) - \psi'\left(\frac{1+\alpha a}{2}\right) \right] \quad [\alpha = \{i\}]$$

$$24. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 + 1]^2} = 0,293986.$$

$$25. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 + 1/4]^2} = 0,689623.$$

$$26. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 - 1/4]^2} = 1,831931.$$

$$27. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{[(2k+1)^2 \pm a^2]^2} = \frac{\pi^2}{16a} \begin{Bmatrix} \operatorname{sech}(\pi a/2) & \operatorname{th}(\pi a/2) \\ \sec(\pi a/2) & \operatorname{tg}(\pi a/2) \end{Bmatrix}.$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 [(2k+1)^2 \pm a^2]^2} =$$

$$= \frac{\pi^2}{8a^4} - \frac{3\pi}{8a^5} \begin{Bmatrix} \operatorname{th}(\pi a/2) \\ \operatorname{tg}(\pi a/2) \end{Bmatrix} + \frac{\pi^2}{16a^4} \left\{ \operatorname{sech}(\pi a/2) \right\}^2.$$

$$29. \sum_{k=0}^{\infty} \frac{(2k+1)^2}{[(2k+1)^2 \pm a^2]^2} = \frac{\pi}{8a} \begin{Bmatrix} \operatorname{th}(\pi a/2) \\ \operatorname{tg}(\pi a/2) \end{Bmatrix} + \frac{\pi^2}{16} \left\{ \operatorname{sech}(\pi a/2) \right\}^2.$$

$$30. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 \pm a^2]^2} =$$

$$= \frac{\pi}{4a^4} \begin{Bmatrix} 1 - \operatorname{sech}(\pi a/2) \\ 1 - \sec(\pi a/2) \end{Bmatrix} \mp \frac{\pi^2}{16a^3} \begin{Bmatrix} \operatorname{sech}(\pi a/2) & \operatorname{th}(\pi a/2) \\ \sec(\pi a/2) & \operatorname{tg}(\pi a/2) \end{Bmatrix}.$$

$$31. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^3}{[(2k+1)^2 \pm a^2]^2} = \frac{\pi}{16} \begin{Bmatrix} \operatorname{sech}(\pi a/2) \\ \sec(\pi a/2) \end{Bmatrix} \left(4 \mp \pi a \begin{Bmatrix} \operatorname{th}(\pi a/2) \\ \operatorname{tg}(\pi a/2) \end{Bmatrix} \right).$$

$$32. \sum_{k=0}^{\infty} \frac{(2k+1)^2}{[(2k+1)^2 \pm a^2]^2} = \frac{\pi}{16a} \left(2 \begin{Bmatrix} \operatorname{th}(\pi a/2) \\ \operatorname{tg}(\pi a/2) \end{Bmatrix} + \pi a \begin{Bmatrix} \operatorname{sech}(\pi a/2) \\ \sec(\pi a/2) \end{Bmatrix}^2 \right).$$

$$33. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^2}{[(2k+1)^2 + a^2]^2} =$$

$$= \frac{\pi}{8ia} \operatorname{sech} \frac{\pi a}{2} - \frac{1}{4ia} \beta \left(\frac{1+ia}{2} \right) - \frac{1}{16} \left[\beta' \left(\frac{1-ia}{2} \right) + \beta' \left(\frac{1+ia}{2} \right) \right].$$

$$34. \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^2}{[(2k+1)^2 + 1]^2} = 0,184658.$$

$$35. \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^2}{[(2k+1)^2 + 1/4]^2} = 0,561256.$$

$$36. \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^2}{[(2k+1)^2 - 1/4]^2} = 1,687959.$$

$$37. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 \pm a^2]^3} =$$

$$= \frac{\pi}{64a^5} \left[-3\pi a \left\{ \frac{\operatorname{sech}(\pi a/2)}{\sec(\pi a/2)} \right\}^2 + 6 \left\{ \frac{\operatorname{th}(\pi a/2)}{\operatorname{tg}(\pi a/2)} \right\} + \pi^2 a^2 \left\{ \frac{\operatorname{sech}(\pi a/2)}{\sec(\pi a/2)} \right\}^2 \left\{ \frac{\operatorname{th}(\pi a/2)}{\operatorname{tg}(\pi a/2)} \right\} \right].$$

$$38. \sum_{k=0}^{\infty} \frac{(2k+1)^2}{[(2k+1)^2 \pm a^2]^3} =$$

$$= \pm \frac{\pi}{32a^3} \left\{ \frac{\operatorname{th}(\pi a/2)}{\operatorname{tg}(\pi a/2)} \right\} \mp \frac{\pi^2}{64a^2} \left[1 \mp \pi a \left\{ \frac{\operatorname{th}(\pi a/2)}{\operatorname{tg}(\pi a/2)} \right\} \right] \left\{ \frac{\operatorname{sech}(\pi a/2)}{\sec(\pi a/2)} \right\}^2.$$

5.1.27. Ряды вида $\sum \frac{(\pm 1)^k}{(k^4 \pm a^4)^m}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{k^4 + a^4} = \frac{1}{2a^4} + \frac{\pi}{2a^3 \sqrt{2}} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k^4 + a^4} = \frac{1}{2a^4} - \frac{\pi}{2\sqrt{2}a^3} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a} +$$

$$+ \frac{\pi}{2\sqrt{2}a^3} \frac{\operatorname{sh}(\pi a/\sqrt{2}) + \sin(\pi a/\sqrt{2})}{\operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{k^4 - a^4} = \frac{-1}{2a^4} - \frac{\pi}{4a^3} (\operatorname{ctg} \pi a + \operatorname{cth} \pi a).$$

$$4. \sum_{k=2}^{\infty} \frac{1}{k^4 - 1} = \frac{7}{8} - \frac{\pi}{4} \operatorname{cth} \pi.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{k^4 - a^4} = \frac{-1}{2a^4} - \frac{\pi}{4a^3} (\operatorname{cosec} \pi a + \operatorname{cosech} \pi a).$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(k^4 + a^4)^2} =$$

$$= \frac{1}{2a^8} + \frac{3\pi}{8\sqrt{2}a^7} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a} + \frac{\pi^2}{4a^6} \frac{\operatorname{sh} \sqrt{2} \pi a \sin \sqrt{2} \pi a}{(\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a)^2}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^4 + a^4)^2} =$$

$$= \frac{1}{2a^8} - \frac{3\pi}{8\sqrt{2}a^7} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a} - \frac{\pi^2}{4a^6} \frac{\operatorname{sh} \sqrt{2} \pi a \sin \sqrt{2} \pi a}{(\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a)^2} +$$

$$+ \frac{3\pi}{8\sqrt{2}a^7} \frac{\operatorname{sh}(\pi a/\sqrt{2}) + \sin(\pi a/\sqrt{2})}{\operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})} + \frac{\pi^2}{8a^6} \frac{\operatorname{sh}(\pi a/\sqrt{2}) \sin(\pi a/\sqrt{2})}{[\operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})]^2}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k^4 - a^4)^2} = \frac{1}{2a^8} + \frac{3\pi}{16a^7} (\operatorname{ctg} \pi a + \operatorname{cth} \pi a) + \frac{\pi^2}{16a^6} (\operatorname{cosec}^3 \pi a + \operatorname{cosech}^3 \pi a).$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^4 - a^4)^2} = \frac{1}{2a^8} + \frac{3\pi}{16a^7} (\operatorname{cosec} \pi a + \operatorname{cosech} \pi a) + \frac{\pi^2}{16a^6} (\operatorname{cosec} \pi a \operatorname{ctg} \pi a + \operatorname{cosech} \pi a \operatorname{cth} \pi a).$$

$$10. \sum_{k=0}^{\infty} \frac{2k + 2a + 1}{[k^2 + (2a + 1)k + a^2 + a + b^2]^2 + b^2} = \frac{1}{a^2 + b^2}.$$

$$11. \sum_{k=0}^{\infty} \frac{k^2 + (2a + 1)k + a^2 + a - b^2}{[k^2 + (2a + 1)k + a^2 + a + b^2]^2 + b^2} = \frac{a}{a^2 + b^2}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k [k^2 + (2a + 1)k + a^2 + a + b^2]}{[k^2 + (2a + 1)k + a^2 + a - b^2]^2 + (2k + 2a + 1)b^2} = \frac{1}{2(a^2 + b^2)}.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k [k^2 + (2a + 1)k + a^2 + a + b^2] (2k + 2a + 1)}{[k^2 + (2a + 1)k + a^2 + a - b^2]^2 + (2k + 2a + 1)b^2} = \frac{a}{a^2 + b^2}.$$

$$14. \sum_{k=1}^{\infty} \frac{k}{4k^4 + 1} = \frac{1}{4}. \quad 15. \sum_{k=1}^{\infty} \frac{2k^2 - 1}{4k^4 + 1} = \frac{1}{2}.$$

5.1.28. Ряды вида $\sum \frac{(\pm 1)^k}{[(2k + 1)^4 \pm a^4]^m}.$

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^4 + a^4} = \frac{\pi}{2\sqrt{2}a^3} \frac{\operatorname{sh} \sqrt{2}\pi a + \sin \sqrt{2}\pi a}{\operatorname{ch} \sqrt{2}\pi a - \cos \sqrt{2}\pi a} - \frac{\pi}{4\sqrt{2}a^3} \frac{\operatorname{sh} (\pi a/\sqrt{2}) + \sin (\pi a/\sqrt{2})}{\operatorname{ch} (\pi a/\sqrt{2}) - \cos (\pi a/\sqrt{2})}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^4 - a^4} = \frac{\pi}{8a^3} \left(\operatorname{tg} \frac{\pi a}{2} - \operatorname{th} \frac{\pi a}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^4 + 1} = 0,489093. \quad 4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^4 + 1/4} = 0,930130.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^4 - 1/4} = 1,055602$$

$$6. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^4 + a^4]^2} =$$

$$= \frac{3\pi}{8\sqrt{2}a^7} \frac{\operatorname{sh} \sqrt{2}\pi a + \sin \sqrt{2}\pi a}{\operatorname{ch} \sqrt{2}\pi a - \cos \sqrt{2}\pi a} - \frac{3\pi}{16\sqrt{2}a^7} \frac{\operatorname{sh}(\pi a/\sqrt{2}) + \sin(\pi a/\sqrt{2})}{\operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})} +$$

$$+ \frac{\pi^2}{4a^6} \frac{\operatorname{sh} \sqrt{2}\pi a \sin \sqrt{2}\pi a}{(\operatorname{ch} \sqrt{2}\pi a - \cos \sqrt{2}\pi a)^2} - \frac{\pi^2}{16a^6} \frac{\operatorname{sh}(\pi a/\sqrt{2}) \sin(\pi a/\sqrt{2})}{[\operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})]^2}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^4 - a^4]^2} =$$

$$= \frac{-3\pi}{32a^7} \left(\operatorname{tg} \frac{\pi a}{2} - \operatorname{th} \frac{\pi a}{2} \right) + \frac{\pi^2}{64a^6} \left(\sec^2 \frac{\pi a}{2} - \operatorname{sech}^2 \frac{\pi a}{2} \right).$$

5.1.29. Ряды вида $\sum b_n \frac{a^{mn}}{1 \pm a^{ln}}$.

Обозначение: $q = \exp[-\pi K'(k)/K(k)]$.

$$1. \sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1 + q^n} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [E(k) - k^2 K(k)].$$

$$2. \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} = \frac{1}{2\pi} K(k) - \frac{1}{4}.$$

$$3. \sum_{k=1}^{\infty} \frac{n q^n}{1 - q^{2n}} = \frac{1}{2\pi^2} K(k) [K(k) - E(k)] \quad [|q| < 1].$$

$$4. \sum_{n=0}^{\infty} \frac{(-1)^k q^{2n+1}}{1 - q^{2n+1}} = \frac{1}{2\pi} K(k) - \frac{1}{4} \quad [|q| < 1].$$

$$5. \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1 - q^n)^2} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [E(k) - k^2 K(k)].$$

$$6. \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [2E(k) - K(k)].$$

$$7. \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} = \frac{1}{24} + \frac{1}{6\pi^2} K(k) [(2 - k^2) K(k) - 3E(k)] \quad [|q| < 1].$$

$$8. \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2} = \frac{1}{2\pi^2} K(k) [E(k) - (1 - k^2) K(k)].$$

$$9. \sum_{n=1}^{\infty} \frac{q^{2n+1}}{(1 - q^{2n+1})^2} = \frac{1}{2\pi^2} K(k) [K(k) - E(k)] \quad [|q| < 1].$$

$$10. \sum_{k=1}^{\infty} \frac{q^{2k}}{1 \pm 2q^{2k} \cos 2x + q^{4k}} =$$

$$= \mp \frac{1}{4} \operatorname{cosec} 2x \left[\left\{ \frac{\theta_2(x, q)}{\theta_1(x, q)} \right\}^{-1} \frac{\partial}{\partial x} \left\{ \frac{\theta_2(x, q)}{\theta_1(x, q)} \right\} \pm \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \right] \quad [|q| < 1].$$

$$11. \sum_{k=1}^{\infty} \frac{q^{2k+1}}{1 \pm 2q^{2k+1} \cos 2x + q^{4k+2}} = \mp \frac{1}{4} \operatorname{cosec} 2x \left\{ \frac{\theta_3(x, q)}{\theta_4(x, q)} \right\}^{-1} \frac{\partial}{\partial x} \left\{ \frac{\theta_3(x, q)}{\theta_4(x, q)} \right\}$$

$$[|q| < 1].$$

$$12. \sum_{k=0}^{\infty} \frac{2^k}{1+q^{2^k}} \left\{ \frac{q^{2^k}}{1} \right\} = \frac{1}{q-1} \left\{ \frac{-q}{1} \right\} \quad \left[\left\{ \begin{array}{l} |q| < 1 \\ |q| > 1 \end{array} \right\} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{1-q^{2^{k+1}}} q^{2^k} = \frac{1}{1-q} \left\{ \frac{q}{1} \right\} \quad \left[\left\{ \begin{array}{l} |q| < 1 \\ |q| > 1 \end{array} \right\} \right].$$

5.1.30. Ряды вида $\sum \frac{a_k}{k^k}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^k} = 1,291285997. \quad 2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^k} = -0,78343051.$$

$$3. \sum_{k=1}^{\infty} \frac{k!}{k^k} = 1,87985386. \quad 4. \sum_{k=1}^{\infty} (-1)^k \frac{k!}{k^k} = -0,65583160.$$

5.1.31. Ряды вида $\sum \frac{a_k}{k!}$.

(См. также ряды вида $\sum \frac{a_k}{k!} x^k$ (в п. 5.2) при $x = \pm 1$.)

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+a)} = \gamma(a, 1).$$

$$2. \sum_{k=0}^{\infty} \frac{k^n}{k!} = e S_n,$$

$$S_{0,1} = 1, \quad S_2 = 2, \quad S_3 = 5, \quad S_4 = 15, \quad S_5 = 52, \quad S_6 = 203,$$

$$S_7 = 877, \quad S_8 = 4140, \quad S_{n+1} = \frac{1}{n+1} \sum_{k=0}^n \frac{1}{k!} S_{n-k}.$$

$$3. \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} = \frac{1}{e} A_n,$$

$$A_0 = 1, \quad A_1 = -1, \quad A_2 = 0, \quad A_3 = 1, \quad A_4 = 1, \quad A_5 = -2,$$

5.1.32. Ряды вида $\sum a_k \left(\sum b_m \right)^{N(k)}$.

$$1. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{1}{2} \ln^2 2 - \frac{\pi^2}{12}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left(\sum_{m=0}^k \frac{1}{2m+1} \right) = \frac{\pi^2}{16}.$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{1}{2} \ln^2 2.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k+1} \left(\sum_{m=1}^{2k} \frac{1}{m} \right) = \frac{\pi}{8} \ln 2.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) = \zeta(3).$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left(\sum_{m=1}^{2k-1} \frac{1}{m} \right) = \frac{1}{2} + 2 \ln 2.$$

$$7. \sum_{k=1}^{\infty} \frac{1}{(k+1)^3} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{\pi^4}{360}.$$

5.2. СТЕПЕННЫЕ РЯДЫ

5.2.1. Введение.

В этом разделе приведены суммы рядов вида $\sum a_k x^k$. Ряды, в которых a_k содержит тригонометрические функции, логарифмы и т. д., помещены в соответствующих разделах этой главы.

Многие ряды могут быть записаны в виде обобщенных гипергеометрических рядов

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x)$$

и суммируются тем самым к соответствующей обобщенной гипергеометрической функции. Отметим, что в теории степенных рядов естественно возникает понятие гамма-функции, частным случаем которой является факториальная функция ($\Gamma(k+1) = k!$), в связи с этим было сочтено целесообразным поместить в этой главе некоторые степенные ряды, коэффициенты которых содержат гамма-

функцию. При этом могут быть полезными следующие формулы (см. также приложение 1.6):

$$(a)_k = \frac{\Gamma(k+a)}{\Gamma(a)} = (-1)^k \frac{\Gamma(1-a)}{\Gamma(1-a-k)} = a(a+1) \dots (a+k-1),$$

$$k! = (1)_k, \quad (2k)! = (1)_k \left(\frac{1}{2}\right)_k 2^{2k}, \quad (2k+1)! = (1)_k \left(\frac{3}{2}\right)_k 2^{2k},$$

$$(3k)! = \frac{1}{2\pi} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (1)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k 3^{3k+1/2},$$

$$(2k)!! = (1)_k 2^k, \quad (2k+1)!! = \left(\frac{3}{2}\right)_k 2^k,$$

$$k+a = \frac{\Gamma(a+1)}{\Gamma(a)} \frac{(a+1)_k}{(a)_k}, \quad (a+1)(a+2) \dots (a+k) = (a+1)_k.$$

Если известна сумма ряда

$$\sum_{k=0}^{\infty} a_k x^k = S(x),$$

то с помощью почленного интегрирования и дифференцирования (в области сходимости) можно найти также следующие суммы:

$$\sum_{k=0}^{\infty} \frac{a_k}{nk+m} x^{nk+m} = \int_0^x x^{m-1} S(x^n) dx,$$

$$\sum_{k=0}^{\infty} (nk+m) a_k x^{nk+m} = x \frac{d}{dx} [x^{m+1} S(x^n)]$$

и

$$\sum_{k=0}^{\infty} k^m a_k x^k = \left(x \frac{d}{dx}\right)^m S(x).$$

Другие способы суммирования степенных рядов изложены в [20, 23, 42].

5.2.2. Ряды вида $\sum P(k) x^k$

Условие: $|x| < 1$.

$$1. \quad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

$$2. \quad \sum_{k=0}^{\infty} (\pm 1)^k k^n x^k = \left(x \frac{d}{dx}\right)^n \frac{1}{1 \mp x}.$$

$$3. \quad = \sum_{k=1}^n \frac{(\mp 1)^k x^k}{(1 \mp x)^{k+1}} \left[\sum_{m=1}^k (-1)^m \binom{k}{m} m^n \right].$$

$$4. \quad \sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}. \quad 5. \quad \sum_{k=1}^{\infty} k^2 x^k = \frac{x(x+1)}{(1-x)^3}.$$

$$6. \sum_{k=1}^{\infty} k^3 x^k = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

$$7. \sum_{k=1}^{\infty} k^4 x^k = \frac{x(x^3+6x^2-x-2)}{(1-x)^5}.$$

$$8. \sum_{k=1}^{\infty} k(k+1)x^k = \frac{2x}{(1-x)^3}. \quad 9. \sum_{k=0}^{\infty} (k+1)^2 x^k = \frac{1+x}{(1-x)^3}.$$

$$10. \sum_{k=0}^{\infty} (k+1)(k+2)\dots(k+n-1)x^k = \frac{(n-1)!}{(1-x)^n}.$$

5.2.3. Ряды вида $\sum \frac{x^k}{(k+a)^s}$.

$$1. \sum_{k=0}^{\infty} \frac{x^k}{(k+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-xe^{-t}} dt = \Phi(x, s, a)$$

[Re $a > 0$; $-1 \leq x < 1$; Re $s > 0$; если $x = 1$, то Re $s > 1$].

$$2. \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x\Phi(x, s, 1).$$

$$3. \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \text{Li}_n(x) \quad [n > 2, |x| \leq 1; n = 1, -1 \leq x < 1].$$

$$4. \sum_{k=0}^{\infty} \frac{x^{k+a}}{k+a} = \int_0^x \frac{x^{a-1}}{1-x} dx \quad [\text{Re } a > 0; 1 \leq x < 1].$$

$$5. \sum_{k=-\infty}^{\infty} \frac{1}{k+a} e^{ik\pi b} = \pi \operatorname{cosec} a\pi \exp[i(2m-b+1)\pi]$$

[m — целое число, для которого $2m < a < 2m+1$].

$$6. \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k+a} e^{ik\pi b} = \pi \operatorname{cosec} a\pi \exp[i(2m-b)\pi]$$

[m — целое число, для которого $2m-1 < a < 2m$].

5.2.4. Ряды вида $\sum \frac{(\pm 1)^k}{nk+m} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{1}{nk+m} x^k = \frac{1}{n} x^{-m/n} \left[2 \sum_{k=1}^{[(n-1)/2]} \sin \frac{2km}{n} \pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2k\pi}{n}}{1-x^{1/n} \cos \frac{2k\pi}{n}} - \right. \\ \left. - \sum_{k=1}^{[(n-1)/2]} \cos \frac{2km\pi}{n} \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2k\pi}{n} + 1 \right) - \right. \\ \left. - \ln(1-x^{1/n}) + (-1)^{m+1} \frac{1+(-1)^n}{2} \ln(1+x^{1/n}) \right]$$

[$|x| < 1$; $m \leq n$; $m, n = 1, 2, 3, \dots$].

$$\begin{aligned}
2. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{nk+m} x^k &= \frac{1}{x^{m/n}} \left[\sum_{k=1}^{[(m-1)/n]} (-1)^{k-1} \frac{x^{(m-kn)/n}}{m-kn} + \right. \\
&\quad + \frac{2}{n} \sum_{k=0}^{[(n-2)/2]} \sin \frac{2k+1}{n} m\pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2k+1}{n} \pi}{1 - x^{1/n} \cos \frac{2k+1}{n} \pi} - \\
&\quad - \frac{1}{n} \sum_{k=0}^{[(n-2)/2]} \cos \frac{2k+1}{n} m\pi \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2k+1}{n} \pi + 1 \right) + \\
&\quad \left. + (-1)^{m-1} \frac{1 - (-1)^n}{2n} \ln (1 + x^{1/n}) \right] \quad [m, n = 1, 2, 3, \dots; -1 < x \leq 1].
\end{aligned}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{k+m} x^{k+m} = -\ln(1-x) - \sum_{k=1}^{m-1} \frac{1}{k} x^k \quad [-1 \leq x < 1; m = 1, 2, 3, \dots].$$

$$4. \quad \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x) \quad [-1 \leq x < 1].$$

$$\begin{aligned}
5. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+m} x^{2k+m} &= \sum_{k=1}^{[(m-1)/2]} (-1)^{k-1} \frac{x^{m-2k}}{m-2k} + \\
&\quad + \sin \frac{m\pi}{2} \operatorname{arctg} x - \frac{1}{2} \cos \frac{m\pi}{2} \ln(1-x^2) \quad [|x| < 1].
\end{aligned}$$

$$\begin{aligned}
6. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+m} x^{3k+m} &= \sum_{k=1}^{[(m-1)/3]} (-1)^{k-1} \frac{x^{m-3k}}{m-3k} + \frac{2}{3} \sin \frac{m\pi}{3} \operatorname{arctg} \frac{x\sqrt{3}}{2-x} - \\
&\quad - \frac{1}{3} \cos \frac{m\pi}{3} \ln(x^2 - x + 1) + \frac{(-1)^{m-1}}{3} \ln(1+x) \quad [-1 < x \leq 1].
\end{aligned}$$

$$\begin{aligned}
7. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+m} x^{4k+m} &= \sum_{k=1}^{[(m-1)/4]} (-1)^{k-1} \frac{x^{m-4k}}{m-4k} + \\
&\quad + \frac{1}{2} \sin \frac{m\pi}{4} \operatorname{arctg} \frac{x}{\sqrt{2-x}} + \frac{1}{2} \sin \frac{3m\pi}{4} \operatorname{arctg} \frac{x}{\sqrt{2+x}} - \\
&\quad - \frac{1}{4} \cos \frac{m\pi}{4} \ln(x^2 - x\sqrt{2} + 1) + \frac{1}{4} \cos \frac{3m\pi}{4} \ln(x^2 + x\sqrt{2} + 1) \quad [|x| \leq 1].
\end{aligned}$$

$$8. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = \begin{cases} 2^{-1} \ln [(1+x)/(1-x)] \\ \operatorname{arctg} x \end{cases} \quad \left[\begin{cases} |x| < 1 \\ |x| \leq 1 \end{cases} \right].$$

$$\begin{aligned}
9. \quad \sum_{k=0}^{\infty} \frac{1}{3k+1} x^{3k+1} &= \\
&= \frac{1}{6} \ln(x^2 + x + 1) - \frac{1}{3} \ln(1-x) + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{x+2} \quad [-1 \leq x < 1].
\end{aligned}$$

$$10. \sum_{k=0}^{\infty} \frac{1}{3k+2} x^{3k+2} = \frac{1}{6} \ln(1-x^3) - \frac{1}{2} \ln(1-x) - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{x+2} \quad [-1 \leq x < 1].$$

$$11. \sum_{k=0}^{\infty} \frac{1}{4k+1} x^{4k+1} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \operatorname{arctg} x \quad [|x| < 1].$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+1} x^{4k+1} = \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} \quad [|x| \leq 1].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{4k+3} x^{4k+3} = \frac{1}{4} \ln \frac{1+x}{1-x} - \frac{1}{2} \operatorname{arctg} x \quad [|x| < 1].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+3} x^{4k+3} = \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} \quad [-1 < x \leq 1].$$

$$15. \sum_{k=0}^{\infty} \frac{1}{6k+1} x^{6k+1} = \frac{1}{6} \ln \frac{1+x}{1-x} + \frac{1}{12} \ln \frac{1+x+x^2}{1-x+x^2} + \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{1-x^2} \quad [|x| < 1].$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+1}}{6k+1} = \frac{1}{3} \operatorname{arctg} x + \frac{1}{6} \operatorname{arctg} \frac{x}{1-x^2} + \frac{1}{12\sqrt{3}} \ln \frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2} \quad [|x| \leq 1].$$

5.2.5. Ряды вида $\sum_k \frac{x^k}{(nk+m)(lk+j)}$.

Условие: $|x| \leq 1$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(nk+m)(lk+j)} x^k = \frac{1}{nj-lm} \left[x^{-j/l} \sum_{k=0}^{l-1} \exp\left(-\frac{2ikj\pi}{l}\right) \ln\left(1-x^{1/l} \exp \frac{2ik\pi}{l}\right) - x^{-m/n} \sum_{k=0}^{n-1} \exp\left(-\frac{2ikm\pi}{n}\right) \ln\left(1-x^{1/n} \exp \frac{2ik\pi}{n}\right) + l \sum_{k=1}^s \frac{1}{j-kl} x^{-k} \right]$$

[$m \leq n$; $nj-lm \neq 0$; s — целое число, для которого $sl < j \leq (s+1)l$].

$$2. \sum_{k=1}^{\infty} \frac{x^{k+n}}{k(k+n)} = \frac{1}{n} (1-x^n) \ln(1-x) + \frac{1}{n} \sum_{k=1}^n \frac{x^k}{k}.$$

$$3. \sum_{k=1}^{\infty} \frac{x^{k+n}}{(k+m)(k+n)} = \\ = \frac{1}{n-m} \left[\ln(1-x) - x^{n-m} \ln(1-x) + \sum_{k=1}^n \frac{1}{k} x^k - x^{n-m} \sum_{k=1}^m \frac{x^k}{k} \right].$$

$$4. \sum_{k=1}^{\infty} \frac{x^k}{k^2} = - \int_0^x \frac{\ln(1-t)}{t} dt = \text{Li}_2(x).$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^{k+1} = x + (1-x) \ln(1-x).$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} x^{k+2} = \frac{x}{2} + \frac{x^2}{4} + \frac{1}{2} (1-x^2) \ln(1-x).$$

$$7. \sum_{k=1}^{\infty} \frac{1}{k(k+3)} x^{k+3} = \frac{1}{18} [6x + 3x^2 + 2x^3 + 6(1-x^3) \ln(1-x)].$$

$$8. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(2k+1)} x^{2k+1} = 2x - 2 \left\{ \begin{matrix} \text{Arth } x \\ \text{arctg } x \end{matrix} \right\} - x \ln(1 \mp x).$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k(2k+3)} x^{2k+3} = \frac{2x}{3} + \frac{2x^3}{9} - \frac{x^3}{3} \ln(1-x^2) - \frac{2}{3} \text{arctg } x.$$

$$10. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)(2k+1)} x^{2k+2} = \pm 2x \left\{ \begin{matrix} \text{Arth } x \\ \text{arctg } x \end{matrix} \right\} + \ln(1 \mp x^2).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)} x^{3k+3} = \\ = \frac{\sqrt{3}}{2} x^2 \text{arctg } \frac{x\sqrt{3}}{x+2} + \frac{1}{4} (x^2+2) \ln(1-x^3) - \frac{3}{4} x^2 \ln(1-x).$$

$$12. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+2)(2k+1)} x^{k+2} = \frac{1}{3} \left[\pm x + \ln(1 \mp x) + 2x^{3/2} \left\{ \begin{matrix} \text{Arth } \sqrt{x} \\ \text{arctg } \sqrt{x} \end{matrix} \right\} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} x^{2k+1} = \frac{1}{2} \int_0^x \frac{1}{t} \ln \frac{1+t}{1-t} dx = \frac{1}{2} [\text{Li}_2(x) - \text{Li}_2(-x)].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)^2} = \int_0^x \frac{1}{t} \operatorname{arctg} t \, dt.$$

$$15. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)(2k+3)} x^{2k+3} = \frac{1}{2} (1 \mp x^2) \left\{ \operatorname{Arth} x \right\}_{\operatorname{arctg} x}^{\pm} \mp \frac{x}{2}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)} x^{3k+2} = \\ = \frac{1}{6} (x-1) [\ln(1-x^3) - 3 \ln(1-x)] + \frac{1}{\sqrt{3}} (x+1) \operatorname{arctg} \frac{x\sqrt{3}}{x+2}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+4)} x^{3k+4} = \\ = \frac{x}{2} + \frac{1}{12} (1-x^2) [3 \ln(1-x) - \ln(1-x^3)] - \frac{1}{2\sqrt{3}} (1+x^2) \operatorname{arctg} \frac{x\sqrt{3}}{x+2}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} x^{4k+3} = \frac{1}{8} \left[(1-x^2) \ln \frac{1-x}{1+x} + 2(x^2+1) \operatorname{arctg} x \right].$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)} x^{4k+3} = \\ = \frac{1}{2\sqrt{2}} (x^2-1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + \frac{1}{4\sqrt{2}} (1+x^2) \ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)} x^{4k+5} = \frac{x}{4} + \frac{1}{16} (x^4-1) \left(\ln \frac{1+x}{1-x} + 2 \operatorname{arctg} x \right).$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)} x^{4k+5} = \\ = -\frac{x}{4} + \frac{1}{16\sqrt{2}} (1+x^4) \left(\ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2 \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} \right).$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)} x^{4k+5} = \frac{x}{2} - \frac{1}{8} (x^2-1) \ln \frac{1+x}{1-x} - \frac{1}{4} (x^2+1) \operatorname{arctg} x.$$

$$23. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+5)} x^{4k+5} = \\ = \frac{x}{2} + \frac{1}{4\sqrt{2}} (x^2-1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \frac{1}{8\sqrt{2}} (x^2+1) \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)} x^{4k+7} = \frac{x^3}{12} + \frac{1}{16} (x^4 - 1) \left(\ln \frac{1+x}{1-x} - 2 \operatorname{arctg} x \right).$$

$$25. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+7)} x^{4k+7} = \\ = \frac{1}{16\sqrt{2}} (x^4 - 1) \left(2 \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} \right).$$

$$26. \sum_{k=0}^{\infty} \frac{1}{k^2+1} x^k = 1 - x \int_0^1 \frac{\sin \ln t}{1-xt} dt.$$

5.2.6. Ряды вида $\sum \frac{(\pm 1)^k x^k}{(kn_1+m_1)(kn_2+m_2)\dots(kn_j+m_j)}$, $j > 2$.
Условие: $|x| \leq 1$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} x^{k+1} = (x-1) \ln(1-x) + x [\operatorname{Li}_2(x) - 1].$$

$$2. \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} x^{k+1} = 2x + (1-x) \ln(1-x) - \operatorname{Li}_2(x).$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} x^{k+2} = \frac{3}{4} x^2 - \frac{x}{2} - \frac{1}{2} (1-x)^2 \ln(1-x).$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} x^{2k+1} = 3x - \left(x + \frac{1}{x} \right) \ln(1-x^2) + 2 \ln \frac{1-x}{1+x} \quad [|x| < 1].$$

$$5. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)(2k+1)(2k+3)} x^{2k+3} = \\ = \pm x \ln(1 \mp x^2) \mp x \pm (1 \pm x^2) \begin{Bmatrix} \operatorname{Arth} x \\ \operatorname{arctg} x \end{Bmatrix} \quad [|x| < 1].$$

$$6. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)(4k+3)(4k+5)} x^{4k+5} = \\ = \frac{1}{2} (1+x^2) \left\{ \ln \frac{(1+x)/(1-x)}{2^{-1/2} \ln [(x^2+x\sqrt{2}+1)/(x^2-x\sqrt{2}+1)]} \right\} \pm \\ \pm \ln(1 \mp x^4) \pm (1-x^2) \left\{ \operatorname{arctg} x \right. \\ \left. \pm 2^{-1/2} \operatorname{arctg} [x\sqrt{2}/(1-x^2)] \right\} \mp 2.$$

$$7. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)(2k+3)(2k+5)} x^{2k+5} = \frac{1}{24} \left[3(1 \mp x^2)^2 \begin{Bmatrix} \operatorname{Arth} x \\ \operatorname{arctg} x \end{Bmatrix} - 3x \pm 5x^3 \right].$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+5)} x^{4k+5} = \frac{1}{32} \left[(1-x^2)^2 \ln \frac{1+x}{1-x} + 2(1+x^2)^2 \operatorname{arctg} x - 4x \right].$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+5)} x^{4k+5} = \frac{1}{64} \left[\sqrt{2} (x^4 + 2x^2 - 1) \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} + 2\sqrt{2} (x^4 - 2x^2 - 1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + 8x \right].$$

$$10. \sum_{k=1}^{\infty} \frac{1}{k^2 (k+1)^2} x^{k+1} = 2(x-1) \ln(1-x) - 3x + (x+1) \operatorname{Li}_2(x).$$

$$11. \sum_{k=1}^{\infty} \frac{x^{k+2}}{k^2 (k+2)^2} = \frac{1}{4} (x^2+1) \operatorname{Li}_2(x) + \frac{1}{4} (x^2-1) \ln(1-x) - \frac{x}{2} - \frac{3x^2}{16}.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} x^{k+3} = \frac{1}{36} [11x^3 - 15x^2 + 6x + 6(1-x)^3 \ln(1-x)].$$

$$13. \sum_{k=1}^{\infty} \frac{x^{k+2}}{k^2 (k+1)^2 (k+2)^2} = \frac{1}{4} \left[(1+4x+4x^2) \operatorname{Li}_2(x) - 4x - \frac{23}{4} x^2 + 3(1-x^2) \ln(1-x) \right].$$

5.2.7. Ряды вида $\sum \frac{(\pm 1)^k x^k}{(nk+m)!}$, $\sum \frac{(\pm 1)^k x^k}{\Gamma(ka+b)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(nk+m)!} x^k = \frac{1}{nx^m} \left[\sum_{k=1}^n \frac{e^{\theta_k x^{1/n}}}{\theta_k^m} - n \sum_{k=1}^{[m/n]} \frac{x^{(m-nk)/n}}{(m-nk)!} \right]$$

$[\theta_k - \text{один из корней степени } n \text{ из } 1: \theta_k = \sqrt[n]{1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} = e^{2ik\pi/n}].$

$$2. \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x. \quad 3. \sum_{k=0}^{\infty} \frac{\ln^k a}{k!} x^k = a^x.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{k!} \begin{Bmatrix} \sin [(k\pi)/4] \\ \cos [(k\pi)/4] \end{Bmatrix} x^k = e^{x/\sqrt{2}} \begin{Bmatrix} \sin (x/\sqrt{2}) \\ \cos (x/\sqrt{2}) \end{Bmatrix}.$$

$$5. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k)!} x^{2k} = \begin{Bmatrix} \operatorname{ch} x \\ \cos x \end{Bmatrix}. \quad 6. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)!} x^{2k+1} = \begin{Bmatrix} \operatorname{sh} x \\ \sin x \end{Bmatrix}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^{[(k+\delta)/2]}}{(2k+1)!} x^{2k+1} = \sqrt{2} \begin{Bmatrix} \operatorname{sh} (x/\sqrt{2}) \cos (x/\sqrt{2}) \\ \operatorname{ch} (x/\sqrt{2}) \sin (x/\sqrt{2}) \end{Bmatrix} \quad [\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

8.
$$\sum_{k=0}^{\infty} \frac{1}{(3k)!} x^{3k} = \frac{1}{3} \left[e^x + 2e^{-x/2} \cos \left(\frac{\sqrt{3}}{2} x \right) \right].$$
9.
$$\sum_{k=0}^{\infty} \frac{1}{(3k+\varepsilon)!} x^{3k+\varepsilon} = \frac{1}{3} e^x - \frac{2}{3} e^{-x/2} \cos \left(\frac{\sqrt{3}}{2} x - \frac{(-1)^\varepsilon \pi}{3} \right) \quad [\varepsilon = 1 \text{ или } 2].$$
10.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k)!} x^{4k} = \begin{cases} 2^{-1} (\operatorname{ch} x + \cos x) \\ \cos (x/\sqrt{2}) \operatorname{ch} (x/\sqrt{2}) \end{cases}.$$
11.
$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1} = \frac{1}{2} (\operatorname{sh} x + \sin x).$$
12.
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)!} x^{4k+1} = \frac{1}{\sqrt{2}} \left(\operatorname{sh} \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}} + \operatorname{ch} \frac{x}{\sqrt{2}} \sin \frac{x}{\sqrt{2}} \right).$$
13.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k+2)!} x^{4k+2} = \begin{cases} 2^{-1} (\operatorname{ch} x - \cos x) \\ \sin (x/\sqrt{2}) \operatorname{sh} (x/\sqrt{2}) \end{cases}.$$
14.
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)!} x^{4k+3} = \frac{1}{\sqrt{2}} \left(\sin \frac{x}{\sqrt{2}} \operatorname{ch} \frac{x}{\sqrt{2}} - \cos \frac{x}{\sqrt{2}} \operatorname{sh} \frac{x}{\sqrt{2}} \right).$$
15.
$$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} = \frac{1}{2} (\operatorname{sh} x - \sin x).$$
16.
$$\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu + k\rho^{-1})} = E_\rho(x; \mu) \quad [\rho > 0].$$
17.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^{[k/2]}}{\Gamma(k/2 + 1)} x^k = e^{\pm x^2} \begin{cases} \operatorname{erfc}(-x) \\ 1 + \operatorname{erfi}(x) \end{cases}.$$
18.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{\Gamma(k + 3/2)} x^{2k+1} = e^{\pm x^2} \begin{cases} \operatorname{erf}(x) \\ \operatorname{erfi}(x) \end{cases}.$$
19.
$$\sum_{k=0}^{\infty} \frac{x^{4k+2\pm 1}}{\Gamma(2k+2\pm 1/2)} = \frac{1}{2} [e^{x^2} \operatorname{erf}(x) \mp e^{-x^2} \operatorname{erfi}(x)].$$
20.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{\Gamma(k + \nu + 1)} x^k = \frac{1}{\Gamma(\nu + 1)} x^{-\nu} e^{\pm x} \int_0^{x^\nu} e^{\mp t^{1/\nu}} dt.$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+v+1)} x^{2k+v} = U_v(2x, 0).$$

5.2.8. Ряды вида $\sum \frac{a_k}{(nk+m)!(k+a)^l} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+a)} x^k = x^{-a} \gamma(a, x).$$

$$2. \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(-1)^k}{k!(k-n)} x^k = \frac{1}{n!} (-x)^n [\psi(n+1) - \ln x] - x^n \Gamma(-n, x)$$

[|arg x| < π].

$$3. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k! k} x^k = -C - \ln x + \text{Ei}(\pm x) = \int_0^x \frac{1}{t} (e^{\pm t} - 1) dt.$$

$$4. \sum_{k=0}^{\infty} \frac{x^{k+n}}{k!(k+n)} = (-1)^n (n-1)! \left[1 - e^x \sum_{m=0}^{n-1} \frac{(-1)^m x^m}{m!} \right].$$

$$5. \sum_{k=0}^{\infty} \frac{1}{k!(k+1)} x^{k+1} = e^x - 1. \quad 6. \sum_{k=0}^{\infty} \frac{1}{k!(k+2)} x^{k+2} = (x-1)e^x + 1.$$

$$7. \sum_{k=1}^{\infty} \frac{k}{(k+n)!} x^{k+n} = (x-n)e^x + (n-x) \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{x^n}{(n-1)!}.$$

$$8. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!(2k+1)} x^{2k+1} = \frac{\sqrt{\pi}}{2} \left\{ \text{erfi}(x) \right\} / \left\{ \text{erf}(x) \right\}.$$

$$9. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)!(2k+1)} x^{2k+1} = \sqrt{\pi} x \left\{ \text{erfi}(x) \right\} / \left\{ \text{erf}(x) \right\} \mp e^{\pm x^2}$$

$$10. \sum_{k=0}^{\infty} \frac{x^k}{k! \sqrt{k+1}} = \frac{2}{\sqrt{\pi}} \int_0^x \exp(xe^{-t^2} - t^2) dt.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(k+a)} x^{2k} = 2x^{-2a} \left[\Gamma(2a) \cos \pi a - \int_x^{\infty} t^{2a-1} \cos t dt \right].$$

$$12. \sum_{k=1}^{\infty} \frac{(\pm 1)^k x^{2k}}{(2k)! k} = 2 \left\{ \text{chi}(x) \right\} / \left\{ \text{ci}(x) \right\} - 2 \ln x - 2C.$$

$$13. \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(2k)!(4k+1)} = \frac{\sqrt{\pi}}{4} [\operatorname{erf}(x) + \operatorname{erfi}(x)].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(k+a)} x^{2k} = 2x^{-2a} \left[-\Gamma(2a-1) \cos \pi a - \int_x^{\infty} t^{2a-2} \sin t \, dt \right].$$

$$15. \sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k+1}}{(2k+1)!(2k+1)} = \begin{Bmatrix} \operatorname{Shi}(x) \\ \operatorname{Si}(x) \end{Bmatrix} = \int_0^x \frac{1}{t} \begin{Bmatrix} \operatorname{sh} t \\ \sin t \end{Bmatrix} dt.$$

$$16. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)!(2k+1)^2} x^{2k+1} = - \int_0^1 \ln t \begin{Bmatrix} \operatorname{sh} xt \\ \sin xt \end{Bmatrix} dt.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)} = \sqrt{\frac{\pi}{2}} C(x^2).$$

$$18. \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(2k+1)!(4k+3)} = \frac{\sqrt{\pi}}{4} [\operatorname{erfi}(x) - \operatorname{erf}(x)].$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)} = \sqrt{\frac{\pi}{2}} S(x^2).$$

$$20. \sum_{k=0}^{\infty} \frac{x^{2k+2}}{(k+1)\Gamma(k+3/2)} = 2 \int_0^x e^{t^2} \operatorname{erf}(t) \, dt.$$

$$21. \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(2k+1)\Gamma(2k+3/2)} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \operatorname{erfi}(x).$$

$$22. \sum_{k=0}^{\infty} \frac{k}{(2k+2)!} x^{2k+1} = 2\sqrt{\pi} \int_0^{x/2} e^{t^2} \operatorname{erf}(t) \, dt.$$

$$23. \sum_{k=0}^{\infty} \frac{(1+1^2)(1+3^2)\dots[1+(2k+1)^2]}{(2k+2)!} x^{2k+2} = \operatorname{ch}(\arcsin x) \sec(\arcsin x) - 1.$$

$$24. \sum_{k=1}^{\infty} \frac{(1^2-v^2)(3^2-v^2)\dots[(2k-1)^2-v^2]}{(2k+1)!} x^{2k} = \frac{1}{vx} \sin(v \arcsin x) - 1.$$

5.2.9. Ряды вида $\sum \frac{k^n}{(k+m)!} x^k$.

Обозначение: $x = ae^{-a}$.

$$1. \sum_{k=0}^{\infty} \frac{(k+b)^n}{k!} x^k = \frac{d^n}{dt^n} \exp(bt + xe^t) \Big|_{t=0}.$$

$$2. \sum_{k=1}^{\infty} \frac{k^n}{k!} x^k = \left(x \frac{d}{dx} \right)^n e^x = S_n.$$

$$3. S_n = \sum_{k=1}^n \frac{(-1)^k}{k!} \left[\sum_{m=1}^k (-1)^m \binom{k}{m} m^n \right] x^k e^x,$$

$$S_1 = x e^x, \quad S_2 = x(x+1) e^x, \quad S_3 = x(x^2 + 3x + 1) e^x.$$

$$4. \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} x^k = S_n,$$

$$S_{-1} = a, \quad S_0 = \frac{a}{1-a}, \quad S_1 = \frac{a}{(1-a)^3}, \quad S_2 = \frac{a(2a+1)}{(1-a)^5}.$$

$$5. \sum_{k=1}^{\infty} \frac{k^k}{(k+1)!} x^k = \frac{e^a - 1}{a} - 1.$$

$$6. \sum_{k=1}^{\infty} \frac{k^{k+1}}{(k+1)!} x^k = - \frac{(1-a)e^a - 1}{a(1-a)}$$

$$[0 < |a| < 1].$$

$$7. \sum_{k=1}^{\infty} \frac{k^k}{(k+2)!} x^k = \frac{1 - 2a + 4ae^a - e^{2a}}{4a^2} - \frac{1}{2}.$$

$$8. \sum_{k=1}^{\infty} \frac{k^{k+1}}{(k+2)!} x^k = \frac{1 + 2ae^a - e^{2a}}{2a^2}.$$

5.2.10. Ряды вида $\sum \frac{a_k}{\Gamma(ka+b)\Gamma(kc+d)} x^k.$

$$1. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!(k+n)!} x^{2k} = x^{-n} \begin{Bmatrix} I_n(2x) \\ J_n(2x) \end{Bmatrix}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! \Gamma[1+(n-k)/2]} = \frac{1}{\sqrt{\pi}} 2^{(n+1)/2} e^{-x^2/8} D_{-n-1} \left(\frac{x\sqrt{2}}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{[(2k)!]^2} x^{4k} = \frac{1}{2} \begin{Bmatrix} J_0(2x) + I_0(2x) \\ 2 \operatorname{ber}(2x) \end{Bmatrix}.$$

$$4. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k)!(2k+1)!} x^{4k+1} = \frac{1}{2} \begin{Bmatrix} I_1(2x) + J_1(2x) \\ 2 \operatorname{bei}'(2x) \end{Bmatrix}.$$

$$5. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{[(2k+1)!]^2} x^{4k+2} = \frac{1}{2} \begin{Bmatrix} I_0(2x) - J_0(2x) \\ 2 \operatorname{bei}(2x) \end{Bmatrix}.$$

$$6. \sum_{k=0}^{\infty} \frac{(\pm 1)^{k+1}}{(2k+1)! (2k+2)!} x^{4k+3} = \frac{1}{2} \left\{ \frac{J'_0(2x) + I'_0(2x)}{2 \operatorname{ber}'(2x)} \right\}.$$

$$7. \sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{k! \Gamma(k+v+1)} = x^{-v} \left\{ \begin{matrix} I_v(2x) \\ J_v(2x) \end{matrix} \right\}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(v-k+1)} x^k = \frac{1}{\Gamma(v+1)} (1+x)^v.$$

$$9. \sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{\Gamma(k+3/2) \Gamma(k+v+3/2)} = x^{-v-1} \left\{ \begin{matrix} L_v(2x) \\ H_v(2x) \end{matrix} \right\}.$$

$$10. \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k/2+n/2+1) \Gamma(k/2-n/2+1)} = e^{-in\pi/2} [J_n(2ix) + iE_n(2ix)].$$

$$11. \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k/2+1) \Gamma(k/2+n+1)} = x^{-n} [J_n(2ix) - iH_n(2ix)].$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(k+v+1) \Gamma(k-v+1)} = J_{2v}(2x) \cos v\pi + E_{2v}(2x) \sin v\pi.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(k!)^2 (k+2)} = J'_2(2x).$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\mu) (k+v)} x^k = 2^{\mu-2v} x^{-v} \int_0^{2\sqrt{x}} t^{2v-\mu} J_{\mu-1}(t) dt \quad [\operatorname{Re} v > 0].$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+v+1)} \left\{ \begin{matrix} \sin\left(\frac{3v}{4} + \frac{3k}{2}\right) \pi \\ \cos\left(\frac{3v}{4} + \frac{3k}{2}\right) \pi \end{matrix} \right\} x^{3k} = x^{-v} \left\{ \begin{matrix} \operatorname{bei}_v(2x) \\ \operatorname{ber}_v(2x) \end{matrix} \right\}.$$

$$16. \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(-1)^k x^{2k}}{k! \Gamma(k+v+1) (k-n)} =$$

$$= \frac{2(-1)^n x^{2n}}{n! \Gamma(n+v+1)} \left[\frac{1}{2} \psi(n+1) + \frac{1}{2} \psi(n+v+1) - \ln x \right] - 2^{-\mu} x^{2n} \int_{2x}^{\infty} t^{\mu} J_v(t) dt$$

$$[\operatorname{Re} \mu < 1/2, \mu + v = -2n - 1].$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2) \Gamma(k+\mu) (k+v)} x^k = 2^{\mu-2v+1/2} x^{-v} \int_0^{2\sqrt{x}} t^{2v-\mu-1/2} H_{\mu-3/2}(t) dt$$

$$[\operatorname{Re} v > 0].$$

$$\begin{aligned}
 18. \quad & \sum_{\substack{k=0 \\ k \neq n-1}}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(k+3/2) \Gamma(k+\nu+3/2) (k-n+1)} = \\
 & = \frac{2(-1)^n x^{2n-2}}{\Gamma(n+1/2) \Gamma(n+\nu+1/2)} \left[\ln x - \frac{1}{2} \psi\left(n+\frac{1}{2}\right) - \frac{1}{2} \psi\left(n+\nu+\frac{1}{2}\right) \right] - \\
 & \quad - 2^{-\mu} x^{2n-2} \int_{2x}^{\infty} t^{\mu} H_{\nu}(t) dt \quad [\operatorname{Re} \mu < 1/2, \mu + \nu = -2n].
 \end{aligned}$$

5.2.11. Ряды вида $\sum \frac{a_k \Gamma(ka+b)}{\Gamma(kc+d)} x^k$.

$$1. \quad \sum_{k=0}^{\infty} \frac{(m+k)!}{(n+k)!} x^k = \frac{m!}{(n-1)! (1-x)^{m-n+1}} \sum_{k=0}^{m-n} (-1)^k \binom{m-n}{k} \frac{x^k}{n+k} \quad [n \leq m].$$

$$\begin{aligned}
 2. \quad & = \frac{m!}{(n-1)!} \binom{n-1}{m} \left[\frac{(-1)^{n-m}}{x^n} (1-x)^{n-m-1} \ln(1-x) + \right. \\
 & \quad \left. + \frac{1}{x^m} \sum_{k=1}^{n-m} (-1)^k \binom{n-m-1}{k-1} \sum_{l=1}^{m+k-1} \frac{x^{l-k}}{l} \right] \quad [n > m].
 \end{aligned}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} x^k = \frac{m!}{(1-x)^{m+1}}.$$

$$\begin{aligned}
 4. \quad & \sum_{k=0}^{\infty} \frac{(nk)!}{(nk+m+1)!} x^k = \\
 & = \frac{1}{(m-1)!} \sum_{k=0}^{m-1} (-1)^{k-1} \binom{m-1}{k} \frac{1}{x^{k+1}} \left[\sum_{l=1}^{[k/n]} \frac{x^{(k-ln+1)/n}}{k-ln+1} + \right. \\
 & \quad + \frac{2}{n} \sum_{l=1}^{[(n-1)/2]} \sin \frac{2l}{n} (k+1) \pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2l}{n} \pi}{1 - x^{1/n} \cos \frac{2l}{n} \pi} + \\
 & \quad + \frac{1}{n} \sum_{k=1}^{[(n-1)/2]} \cos \frac{2l}{n} (k+1) \pi \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2l}{n} \pi + 1 \right) + \\
 & \quad \left. + \frac{1}{n} \ln(1-x) + (-1)^{k-1} \frac{1+(-1)^n}{2n} \ln(1+x) \right].
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \sum_{k=0}^{\infty} \frac{k!}{(m+k)!} x^k = \\
 & = \frac{1}{(m-1)!} \left[\sum_{k=0}^{m-2} (-1)^k \frac{(1-x)^k}{(m-k-1) x^{k+1}} + (-1)^m \frac{(1-x)^{m-1}}{x^m} \ln(1-x) \right].
 \end{aligned}$$

6.
$$\sum_{k=0}^{\infty} (\pm 1)^k \frac{k!}{(2k)!} x^{2k} = 1 \pm \frac{\sqrt{\pi}}{2} x e^{\pm x^2/4} \begin{Bmatrix} \operatorname{erf}(x/2) \\ \operatorname{erfi}(x/2) \end{Bmatrix}.$$
7.
$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k k!}{(2k+1)!} x^{2k+1} = \sqrt{\pi} e^{\pm x^2/4} \begin{Bmatrix} \operatorname{erf}(x/2) \\ \operatorname{erfi}(x/2) \end{Bmatrix}.$$
8.
$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(4k)!} x^{4k} = 1 - \frac{\sqrt{\pi}}{2} x U_{3/2} \left(\frac{x^2}{2}, 0 \right).$$
9.
$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(4k+1)!} x^{4k+1} = \sqrt{\pi} U_{1/2} \left(\frac{x^2}{2}, 0 \right).$$
10.
$$\sum_{k=0}^{\infty} \frac{\Gamma(kv + \mu)}{\Gamma(kv + \mu + n + 1)} x^k = \frac{1}{n!} \int_0^1 \frac{t^{\mu-1}}{1 - xt^v} (1-t)^n dt \quad [|x| \leq 1; a, b > 0].$$
11.
$$\sum_{k=1}^{\infty} \frac{\Gamma(kv + 1)}{\Gamma(kv + k + 1)} x^k = x \int_0^1 t^v \exp [xt^v (1-t)] dt \quad [v > 0].$$
12.
$$\sum_{k=0}^{\infty} \frac{\Gamma(kv + 1)}{k!} x^k = \int_0^{\infty} \exp (xt^v - t) dt \quad [0 \leq v < 1].$$
13.
$$\sum_{k=0}^{\infty} \frac{\Gamma(kv)}{k!} (-x)^k = \frac{1}{v} \left[\int_0^{\infty} e^{-t} \operatorname{Ei} (-xt^v) dt - (1-v) \mathbf{C} - \ln x \right] \quad [\operatorname{Re} x > 0; 0 < \operatorname{Re} v < 1].$$
14.
$$\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{2} + v\right)}{k!} x^k = 2^{1-v} \Gamma(2v) e^{x^2/8} D_{-2v} \left(-\frac{x}{\sqrt{2}} \right).$$
15.
$$\sum_{k=0}^{\infty} (\pm 1)^k \frac{\Gamma(k + v)}{k! (k + a)^s} x^k = \frac{\Gamma(v)}{\Gamma(s)} \int_0^{\infty} e^{-at} t^{s-1} (1 \mp x e^{-t})^{-v} dt \quad [\operatorname{Re} v, \operatorname{Re} s > 0].$$
16.
$$\sum_{k=0}^{\infty} \frac{\Gamma(k + v)}{k!} x^k = \Gamma(v) (1-x)^{-v} \quad [|x| \leq 1; x \neq 1 \text{ при } v > 0].$$
17.
$$\sum_{k=0}^{\infty} \frac{(k + v) \Gamma(k + 2v)}{k!} x^k = \frac{v \Gamma(2v) (1+x)}{(1-x)^{2v+1}} \quad [|x| \leq 1; x \neq 1 \text{ при } v > 1/2].$$

5.2.12. Ряды вида $\sum \frac{a_k}{\Gamma(ka+b)\Gamma(kc+d)\Gamma(ke+f)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{4k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu+1)} = x^{-2\nu} \left\{ \begin{matrix} \text{ber}_{\nu}^2(2x) + \text{bei}_{\nu}^2(2x) \\ J_{\nu}(2x) I_{\nu}(2x) \end{matrix} \right\}.$$

$$2. \sum_{k=0}^{\infty} \frac{x^{4k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu+2)} = \\ = x^{-2\nu-1} [\text{ber}_{\nu}(2x) \text{bei}'_{\nu}(2x) - \text{ber}'_{\nu}(2x) \text{bei}_{\nu}(2x)].$$

$$3. \sum_{k=0}^{\infty} \frac{x^{4k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu)} = \\ = 2x^{1-2\nu} [\text{ber}_{\nu}(2x) \text{ber}'_{\nu}(2x) + \text{bei}_{\nu}(2x) \text{bei}'_{\nu}(2x)].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k \cos[(k+\nu)\pi/2]}{k! \Gamma(k/2+\nu/2+1) \Gamma(k/2-\nu/2+1)} x^{2k} = J_{\nu}(2x) I_{-\nu}(2x).$$

5.2.13. Ряды вида $\sum \frac{a_k \Gamma(ka+b)}{\Gamma(kc+d)\Gamma(ke+f)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} x^k = (1-4x)^{-1/2} \quad [-1/4 \leq x < 1/4].$$

$$2. \sum_{k=0}^{\infty} \frac{(2k+1)!}{(k!)^2} x^k = (1-4x)^{-3/2} \quad [-1/4 \leq x < 1/4].$$

$$3. \sum_{k=0}^{\infty} (-1)^{[k/2]} \frac{(2k)!}{(k!)^2} (\pm x)^k = (1+16x^2)^{-1/2} [(1+16x^2)^{1/2} \pm 4x]^{1/2}.$$

$$4. \sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2 k} x^k = \frac{1}{2} - x - \frac{1}{2} (1-4x)^{1/2} \quad [-1/4 \leq x < 1/4].$$

$$5. \sum_{k=1}^{\infty} (\pm 1)^k \frac{(2k)!}{(k!)^2 k^2} x^{2k} = -\frac{1}{2} \ln^2 |x| - 4 \int_0^{2x} \left\{ \begin{matrix} \text{Arsech } t \\ \text{Arcosech } t \end{matrix} \right\} \frac{dt}{t} \quad [|x| < 1/2].$$

$$6. \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 (2k-1)} x^k = -(1-4x)^{1/2} \quad [-1/4 \leq x < 1/4].$$

$$7. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{(k!)^2 (2k+1)^2} x^{2k+1} = \frac{x}{2} \left\{ \begin{matrix} \arcsin(2x) \\ \text{Arsh}(2x) \end{matrix} \right\} + \frac{1}{4} (1-4x^2)^{1/2} \quad [|x| < 1/2].$$

8. $\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 (2k-1)^2} x^{2k-1} = -\frac{1}{x} (1-4x^2)^{1/2} - 2 \arcsin(2x)$ [|x| < 1/2].
9. $\sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+1)!} x^{k+1} = \frac{1}{2} [1 - (1-4x)^{1/2}]$ [|x| < 1/4].
10. $\sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{(k!)^2 (2k+1)} x^{2k+1} = \frac{1}{2} \left\{ \arcsin(2x) \right\}$ [|x| ≤ 1/2].
11. $\sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+2)!} x^{k+2} = \frac{1}{12} [(1-4x)^{3/2} + 6x - 1]$ [|x| ≤ 1/4].
12. $\sum_{k=0}^{\infty} \frac{(2k+1)!}{k! (k+2)!} x^{k+2} = \frac{1}{4} [1 - 2x - (1-4x)^{1/2}]$ [|x| ≤ 1/4].
13. $\sum_{k=0}^{\infty} \frac{(2k+1)!}{[(k+1)!]^2} x^{k+1} = -\ln [1 + (1-4x)^{1/2}] + \ln 2$ [|x| ≤ 1/4].
14. $\sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+3)!} x^{k+3} = \frac{1}{120} [30x^2 - 10x + 1 - (1-4x)^{5/2}]$ [|x| ≤ 1/4].
15. $\sum_{k=0}^{\infty} \frac{(2k+1)!}{k! (k+3)!} x^{k+3} = \frac{1}{24} [6x - 1 - 6x^2 + (1-4x)^{3/2}]$ [|x| ≤ 1/4].
16. $\sum_{k=0}^{\infty} \frac{(2k+2)!}{k! (k+2)!} x^k = 4 (1-4x)^{-1/2} [1 + (1-4x)^{1/2}]^{-2}$ [-1/4 ≤ x < 1/4].
17. $\sum_{k=0}^{\infty} \frac{(2k+2)!}{k! (k+3)!} x^k = \frac{8}{3} [1 + (1-4x)^{1/2}]^{-3}$ [|x| < 1/4].
18. $\sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+1)! (2k+3)} x^{2k+3} = \frac{1}{8} [4x - 2x (1-4x^2)^{1/2} - \arcsin(2x)]$ [|x| ≤ 1/2].
19. $\sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k+1)!}{(k!)^2 (2k+3)} x^{2k+3} =$
 $= \pm \frac{1}{8} [2x (1-4x^2)^{-1/2} \mp \left\{ \arcsin(2x) \right\}]$ [|x| ≤ 1/2].
20. $\sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+3)!} x^k = 8 (1-4x)^{-1/2} [1 + (1-4x)^{1/2}]^{-3}$ [-1/4 ≤ x < 1/4].

$$21. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+2)!} x^k = (1-4x)^{-3/2} [(1-4x)^{1/2} + 1]^{-2} [8(1-4x)^{1/2} + 4]$$

$$[-1/4 \leq x < 1/4].$$

$$22. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+1)!} x^k = 6(1-4x)^{-5/2}$$

$$[-1/4 \leq x < 1/4].$$

$$23. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+4)!} x^{k+4} = \frac{1}{16} [1-2x-(1-4x)^{1/2}]^2$$

$$[|x| \leq 1/4].$$

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k (3k+1)!}{k! (2k+1)! (2k+1)} x^{2k+1} = \operatorname{arctg} y$$

$$[x=y(1+y); |x| < 2/3^{3/2}].$$

$$25. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(4k)!}{[(2k)!]^2} x^{2k} = \left[\frac{1+(1 \mp 16x^2)^{1/2}}{2(1 \mp 16x^2)} \right]^{1/2}$$

$$[|x| < 1/4].$$

$$26. \sum_{k=0}^{\infty} \frac{(4k+1)!}{[(2k)!]^2} x^{2k} = \frac{1}{2} [(1-4x)^{-3/2} + (1+4x)^{-3/2}]$$

$$[-1/4 \leq x < 1/4].$$

$$27. \sum_{k=0}^{\infty} (\pm 1)^k \frac{\Gamma(k+\nu+1/2)}{k! \Gamma(k+2\nu+1)} x^k = \sqrt{\pi} x^{-\nu} e^{\pm x/2} I_{\nu} \left(\frac{x}{2} \right).$$

$$28. \sum_{k=0}^{\infty} \frac{k! x^k}{\Gamma(k-\nu+3/2) \Gamma(k+\nu+3/2)} = \frac{1}{2} e^{x/2} \int_{-1}^1 e^{xt/2} P_{\nu-1/2}(t) dt.$$

$$29. \sum_{k=0}^{\infty} \frac{\Gamma(k\nu+\mu)}{k! \Gamma(k\nu-k+\mu)} x^k = \frac{y^{\mu}}{(1-\nu)y+\nu}$$

$$[x=(y-1)/y^{\nu}; |x| < |(v-1)^{\nu-1}/v^{\nu}|].$$

$$30. \sum_{k=0}^{\infty} \frac{\Gamma(k\nu+\mu)}{k! \Gamma(k\nu-k+\mu) (k\nu+\mu)} x^k = \frac{1}{\mu-1} y^{\mu-1}$$

$$[x=(y-1)/y^{\nu}; |x| < |(v-1)^{\nu-1}/v^{\nu}|].$$

$$31. \sum_{k=0}^{\infty} \frac{\Gamma(k+\mu)}{\Gamma(k+\mu-\nu+1/2) \Gamma(k+\mu+\nu+1/2)} x^k =$$

$$= \frac{\Gamma(\mu)}{\Gamma(\mu-\nu-1/2) \Gamma(\mu+\nu-1/2)} \left(\frac{x}{2} \right)^{-1-\mu} e^{x/2} h_{\mu-3/2, \nu} \left(\frac{x}{2} \right).$$

5.2.14. Ряды вида $\sum a_k \frac{\Gamma(ka+b) \Gamma(kc+d)}{\Gamma(ke+f)} x^k$.

$$1. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{(2k)!} x^{2k} = \frac{4}{4 \mp x^2} \pm 4x (4 \mp x^2)^{-3/2} \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases}$$

$$[|x| < 2]$$

$$2. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+1)!} x^{2k+1} = 4(4 \mp x^2)^{-1/2} \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases} \quad [|x| < 2].$$

$$3. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+2)!} x^{2k+2} = 2 \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases}^2 \quad [|x| \leq 2].$$

$$4. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+3)!} x^{2k+3} = \pm 4(4 \mp x^2)^{1/2} \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases} + 2 \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases}^2 \mp 4 \quad [|x| \leq 2].$$

$$5. \sum_{k=0}^{\infty} (\pm 1)^k \frac{k!(k+1)!}{(2k+3)!} x^{2k+3} = \pm 2x \mp 2(4 \mp x^2)^{1/2} \begin{cases} \arcsin(x/2) \\ \operatorname{Arsh}(x/2) \end{cases} \quad [|x| \leq 2].$$

5.2.15. Ряды вида $\sum a_k \frac{\Gamma(ka+b)\Gamma(kc+d)}{\Gamma(ke+f)\Gamma(kg+h)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k (k!)^2}{(2k)!(2k+\varepsilon)!} x^{2k} = 1 - \frac{\pi}{2} \left(\frac{x}{2}\right)^{1-\varepsilon} H_{\varepsilon}\left(\frac{x}{2}\right) \quad [\varepsilon = 0 \text{ или } 1].$$

$$2. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 x^{2k+1} = \pi H_0\left(\frac{x}{2}\right).$$

$$3. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 \frac{x^{2k+1}}{2k+1} = \frac{\pi^2}{4} - \pi \int_{x/2}^{\infty} H_0(t) \frac{dt}{t}.$$

$$4. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 \frac{x^{2k+2}}{2k+3} = 2\pi H_1\left(\frac{x}{2}\right).$$

$$5. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+1)!(2k+2)!} x^{2k+2} = 2\pi \int_0^{x/2} \begin{Bmatrix} L_0(t) \\ H_0(t) \end{Bmatrix} dt.$$

$$6. \sum_{k=0}^{\infty} (-1)^k \frac{k!(k+1)!}{[(2k+3)!]^2} x^{2k+2} = 2 \ln x - 4 + 2C + \pi \int_{x/2}^{\infty} H_0(t) \frac{dt}{t^2}.$$

5.2.16. Ряды вида $\sum a_k \frac{\Gamma(kb_1+c_1)\dots\Gamma(kb_l+c_l)}{\Gamma(kd_1+e_1)\dots\Gamma(kd_l+e_l)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^3} x^k = e^{2x} I_0(2x).$$

$$2. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{(k!)^4} x^{2k} = J_0^2(2x).$$

$$3. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{[k! (2k+1)!]^2} x^{2k+1} = \frac{1}{2} \int_0^{2x} \left\{ \begin{matrix} \arcsin t \\ \operatorname{Arsh} t \end{matrix} \right\} \frac{dt}{t} \quad [|x| < 1/2].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k (2k)! x^{2k}}{(k!)^2 \Gamma(k+\nu+1) \Gamma(k-\nu+1)} = J_{\nu}(2x) J_{-\nu}(2x).$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+\mu) x^{2k}}{k! \Gamma(k+\mu) \Gamma(k+\nu) \Gamma(k+\mu-\nu+1)} = x^{1-\mu} J_{\mu-\nu}(2x) J_{\nu-1}(2x).$$

$$6. \sum_{k=0}^{\infty} \frac{[(2k)!]^2}{(k!)^4} x^{2k} = \frac{2}{\pi} K(4x) \quad [|x| < 1/4].$$

$$7. \sum_{k=0}^{\infty} \frac{1}{2k-1} \frac{[(2k)!]^2}{(k!)^4} x^{2k} = -\frac{2}{\pi} E(4x) \quad [|x| < 1/4].$$

$$8. \sum_{k=0}^{\infty} \left[\frac{(2k)!}{k! (k+1)!} \right]^2 x^{2k+2} = \frac{1}{2\pi} (1+4x) E\left(\frac{4x}{1+4x}\right) - \frac{1}{4}.$$

$$9. \sum_{k=0}^{\infty} \frac{(2k)! (2k+1)!}{[k! (k+1)!]^2} x^{2k+2} = \frac{1}{4} - \frac{1}{2\pi} E(4x) \quad [0 \leq x \leq 1/4].$$

$$10. \sum_{k=0}^{\infty} \left[\frac{(2k+1)!}{k! (k+2)!} \right]^2 x^{2k+4} = -\frac{1}{16\pi} \left[\pi + 4\pi x^2 - 2(1+4x) E\left(\frac{4x}{1+4x}\right) \right] \quad [0 < x < 1/4].$$

5.2.17. Ряды вида $\sum_k a_k \left(\sum_l \frac{(\pm 1)^l}{nl+m} \right) x^k$.

$$1. \sum_{k=1}^{\infty} \left(\sum_{m=1}^k \frac{1}{m} \right) x^k = \frac{1}{x-1} \ln(1-x) \quad [|x| < 1].$$

$$2. \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-1} \frac{1}{m} \right) x^k = \frac{1}{2} \ln^2(1-x) \quad [-1 \leq x < 1]$$

$$3. \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-1} \frac{(-1)^{m-1}}{m} \right) x^k = \operatorname{Li}_2\left(\frac{1-x}{2}\right) - \frac{\pi^2}{12} + \ln 2 \left[\frac{1}{2} \ln 2 - \ln(1-x) \right] \quad [|x| < 1].$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{2k-1} (-1)^m \frac{1}{m} \right) x^{2k} = \ln(1+x) \ln(1-x) \quad [|x| < 1].$$

$$5. \sum_{k=1}^{\infty} \frac{(\pm 1)^{k+1}}{k} \left(\sum_{m=0}^{k-1} \frac{1}{2m+1} \right) x^{2k} = \begin{cases} \operatorname{Arth} x \\ \operatorname{arctg} x \end{cases}^2 \quad [|x| < 1].$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{2k+1} = \ln(1+x^2) \operatorname{arctg} x - x \operatorname{arctg}^2 x + \int_0^x \operatorname{arctg}^2 t \, dt \quad [|x| < 1].$$

$$7. \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left(\sum_{m=1}^{2k} \frac{1}{m} \right) x^{2k+1} = -\frac{1}{2} \operatorname{arctg} x \ln(1+x^2) \quad [|x| < 1].$$

$$8. \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\sum_{m=0}^{2k} \frac{(-1)^m}{2m+1} \right) x^{2k+2} = \frac{1}{2} \operatorname{arctg} x \ln \frac{1+x}{1-x} \quad [|x| < 1].$$

$$9. \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{k+1} = \\ = \frac{1}{2} \ln x \ln^2(1-x) + \ln(1-x) \operatorname{Li}_2(1-x) - \operatorname{Li}_3(1-x) + \operatorname{Li}_3(1).$$

$$10. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k!} \left(\sum_{m=1}^k \frac{1}{m} \right) x^k = e^{\pm x} [\mathbf{C} + \ln x - \operatorname{Ei}(\mp x)].$$

$$11. \sum_{k=1}^{\infty} \frac{x^{2k}}{k!} \sum_{m=0}^{k-1} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} e^{x^2} \left[3 + \frac{2}{\sqrt{\pi}} \operatorname{erf}(x) \right]^2.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^k m^3 \right) x^k = \frac{1}{4} x e^x (x^3 + 8x^2 + 14x + 4).$$

$$13. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{2k} = \left(\mathbf{C} + \frac{1}{2} \ln x^2 \right) \left\{ \begin{matrix} I_0(2x) + K_0(2x) \\ J_0(2x) - (\pi/2) Y_0(2x) \end{matrix} \right\}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k)!]^2} \left(\sum_{m=1}^{2k} \frac{1}{m} \right) x^{2k} = \ker(2x) + \left(\mathbf{C} + \frac{1}{2} \ln x \right) \operatorname{ber}(2x) - \frac{\pi}{4} \operatorname{bei}(2x).$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)!]^2} \left(\sum_{m=1}^{2k+1} \frac{1}{m} \right) x^{2k+2} = \\ = \operatorname{kei}(2x) + \left(\mathbf{C} + \frac{1}{2} \ln x \right) \operatorname{bei}(2x) + \frac{\pi}{4} \operatorname{ber}(2x).$$

$$16. \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+2)!} \left(\sum_{m=1}^k \frac{1}{m^2} \right) x^{2k+2} = \frac{2}{3} \operatorname{arcsin}^4 \frac{x}{2}.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+1)!} \left(\sum_{m=1}^{k+1} \frac{1}{m} \right) x^{2k} = \frac{1}{2x} [(\ln x + 2C) J_1(2x) - \pi Y_1(2x)] - \frac{1}{2x^2} J_0(2x).$$

5.2.18. Разные ряды вида $\sum a_k x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k x^{\mu+2k+1}}{[(\mu+1)^2 - \nu^2][(\mu+3)^2 - \nu^2] \dots [(\mu+2k+1)^2 - \nu^2]} = s_{\mu, \nu}(x).$$

$$2. \sum_{k=0}^{\infty} \frac{(2\mu+1)(2\mu+3) \dots (2\mu+2k+1)}{[(\mu+1)^2 - \nu^2][(\mu+2)^2 - \nu^2] \dots [(\mu+k+1)^2 - \nu^2]} x^{\mu+k+1} = (2\mu+1) e^x h_{\mu, \nu}(x).$$

$$3. \sum_{k=1}^{\infty} a^{\sqrt{k}} x^k = \frac{x \ln a}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{(x-e^t) t^{3/2}} e^{-(\ln^2 a)/(4t)} dt \quad [|x| < 1; 0 < a < 1].$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k^k} x^k = x \int_0^1 t^{-x^t} dt.$$

$$5. \sum_{k=0}^{\infty} \left[\sum_{m=0}^k \frac{(-1)^m}{2m+1} \binom{p}{k-m} \right] x^{2k+1} = (1+x^2)^p \operatorname{arctg} x.$$

$$6. \sum_{k=1}^{\infty} (-1)^{k-1} \left\{ \sum_{m=0}^{[(k-1)/l]} \frac{(-1)^{lm}}{k-lm} \binom{p}{m} \right\} x^k = (1+x^l)^p \ln(1+x).$$

$$7. \sum_{k=1}^{\infty} \left\{ \sum_{m=0}^{[k/2]} \frac{(-1)^m}{k-m} \binom{k-m}{m} \right\} x^k = -\ln(1-x+x^2).$$

$$8. \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k)!} \left[\sum_{m=1}^n (-1)^m \binom{2n}{n-m} m^{2k} \right] x^{2k} = 2^{2n-1} \sin^{2n} \frac{x}{2}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[\sum_{m=0}^n \binom{n}{m} (n-2m)^{2k} \right] x^{2k} = 2^n \cos^n x.$$

$$10. \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\sum_{m=0}^n (-1)^m \binom{2n+1}{n-m} (2m+1)^{2k+1} \right] x^{2k+1} = 2^{2n} \sin^{2n+1} x.$$

$$11. \sum_{k=n}^{\infty} \frac{1}{k!} \binom{k}{l} \left[\sum_{m=1}^n (-1)^m \binom{n}{m} m^k \right] x^k = \frac{x^l}{l!} \sum_{k=1}^n (-1)^k \binom{n}{k} k^l e^{kx}.$$

$$12. \sum_{k=1}^{\infty} \left\{ \sum_{m=0}^{[k/n]} (-1)^m \binom{k-nm+p-1}{p-1} \binom{p}{m} \right\} x^k = (1+x+x^2+\dots+x^{n-1})^p - 1.$$

$$13. \sum_{k=1}^{\infty} \frac{x^k}{a^k - 1} = \frac{x}{a^2 - 1} {}_2\Phi_1(a, a; a^2; x).$$

$$14. \sum_{k=0}^{\infty} \frac{x^k}{a^k + 1} = \frac{1}{2} {}_2\Phi_1(a, -1; -a; x).$$

$$15. \sum_{k=0}^{\infty} \frac{x^k}{a^{2k+1} - 1} = \frac{1}{a-1} {}_2\Phi_1(a^2, a; a^3; x).$$

$$16. \sum_{k=0}^{\infty} \frac{x^k}{a^k + b} = \frac{1}{b} \int_0^{\infty} \frac{\sin(t \ln b) - x \sin(t \ln ab)}{1 - 2x \cos(t \ln a) + x^2} \operatorname{cosech} \pi t \, dt + \frac{1}{2b(1-x)} \quad [a, b > 0].$$

5.3. РЯДЫ С ПОКАЗАТЕЛЬНОЙ И ГИПЕРБОЛИЧЕСКИМИ ФУНКЦИЯМИ

5.3.1. Ряды вида $\sum a_k \frac{1}{e^{nk\pi} \pm 1}$.

$$1. \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{\pi k} - (-1)^k} = \frac{1}{4n} (2^{2n} - 1) B_{2n} - \frac{1}{4\pi} \delta_{n,1}$$

$$2. \sum_{k=1}^{\infty} \frac{k^{4n+1}}{(-1)^{k+1} e^{\pi k} - 1} = \frac{B_{4n+2}}{8n+4}.$$

$$3. \sum_{k=1}^{\infty} \frac{k^{4n+1}}{e^{2\pi k} - 1} = \frac{1}{2} \frac{B_{4n+2}}{4n+2} \quad [n > 0].$$

$$4. \sum_{k=1}^{\infty} \frac{k}{e^{2\pi k} - 1} = \frac{1}{24} - \frac{1}{8\pi}.$$

$$5. \sum_{k=1}^{\infty} \frac{k^5}{(e^{2\pi k} - 1)(4k^4 + n^4)} = \frac{1}{96} + \frac{\pi n^2}{64} \cdot \frac{(-1)^n + \operatorname{ch} n\pi}{(-1)^n - \operatorname{ch} n\pi} + \frac{n^3}{32} \sum_{k=1}^n \frac{1}{2k^2 - 2kn + n^2}.$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k^{4n+3} (e^{2\pi k} - 1)} = -\frac{1}{2} \zeta(4n+3) - \frac{(2\pi)^{4n+3}}{(4n+4)! 4} \sum_{k=0}^{2n+2} \binom{4n+4}{2k} B_{2k} B_{4n-2k+4}.$$

$$7. \sum_{k=0}^{\infty} \frac{(2k+1)^{4n+1}}{e^{(2k+1)\pi} + 1} = (2^{4n+1} - 1) \frac{B_{4n+2}}{8n+4}.$$

$$8. \sum_{k=0}^{\infty} \frac{2k+1}{[e^{(2k+1)\pi} + 1][(2k+1)^4 + 4n^4]} =$$

$$= \frac{\pi}{32n^2} \frac{(-1)^n - \operatorname{ch} n\pi}{(-1)^n + \operatorname{ch} n\pi} + \frac{1}{8n} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^2 - 2n(2k+1) + 2n^2}.$$

5.3.2. Ряды вида $\sum a_k q^{(ka+b)^2}$.

Условие: $|q| < 1$.

$$1. \sum_{k=-\infty}^{\infty} q^{(k+a)^2} = \theta_3(\pi a, e^{\pi^2/\ln q}) \ln^{-1/2} \frac{1}{q}.$$

$$2. \sum_{k=0}^{\infty} (\pm 1)^k q^{k^2} = \frac{1}{2} + \frac{1}{2} \left\{ \begin{matrix} \theta_3(0, q) \\ \theta_4(0, q) \end{matrix} \right\}.$$

$$3. \sum_{k=0}^{\infty} q^{k(k+1)} = \frac{1}{2q^{1/4}} \theta_2(0, q).$$

$$4. \sum_{k=1}^{\infty} (\pm 1)^k k q^{k^2} = \mp \frac{1}{\sqrt{\pi}} (-\ln q)^{-3/2} \int_0^{\infty} t \exp\left(\frac{t^2}{\ln q}\right) \left\{ \begin{matrix} \operatorname{ctg} t \\ \operatorname{tg} t \end{matrix} \right\} dt \quad [0 < q < 1].$$

$$5. \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} = \frac{1}{2} q^{-1/4} \frac{\partial}{\partial a} \theta_1(a, q) \Big|_{a=0}.$$

$$6. \sum_{k=0}^{\infty} q^{(2k+1)^2} = \frac{1}{2} \theta_2(0, q^4).$$

$$7. \sum_{k=0}^{\infty} (-1)^k q^{(2k+1)^2} = \frac{2q}{\sqrt{\pi}} \int_0^{\infty} \frac{q^t \cos 4t \sqrt{-\ln q} + 1}{q^3 + 2q^t \cos 4t \sqrt{-\ln q} + 1} e^{-t^2} dt \quad [0 \leq q < 1].$$

$$8. \sum_{k=0}^{\infty} (2k+1) q^{(2k+1)^2} = -\frac{1}{2\sqrt{\pi}} (-\ln q)^{-3/2} \int_0^{\infty} \exp\left(\frac{t}{\ln q}\right) \operatorname{cosec} 2\sqrt{t} dt.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k^n} e^{-k^2} = S_n,$$

$$S_0 = 0,38631860, \quad S_2 = 0,37247207, \quad S_4 = 0,26902569.$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} e^{-k^2} = T_n,$$

$$T_0 = 0,34968710, \quad T_2 = 0,36331424, \quad T_4 = 0,36673624.$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} e^{-(2k+1)^2} = U_n,$$

$$U_0 = 0,36800285, \quad U_1 = 0,36792058, \quad U_2 = 0,36789315, \quad U_4 = 0,36788096.$$

$$12. \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)^2} = 0,36775603.$$

$$13. \sum_{k=0}^{\infty} (4\pi k^2 - 1) e^{-\pi k^2} = \frac{3}{2}.$$

$$14. \sum_{k=0}^{\infty} e^{-\pi(2k+1)^2} = \frac{(2^{1/4} - 1)}{2^{11/4} \pi^{3/4}} \Gamma\left(\frac{1}{4}\right) = 0,04521392.$$

$$15. \sum_{k=-\infty}^{\infty} (\pm 1)^k a^{2k} q^{k^2} = \left\{ \begin{matrix} \theta_3(-i \ln a, q) \\ \theta_4(-i \ln a, q) \end{matrix} \right\}.$$

$$16. \sum_{k=-\infty}^{\infty} (\pm 1)^k a^{2k} q^{k(k+1)} = \frac{1}{a} \left\{ \begin{matrix} \theta_2(-i \ln a, q) \\ i\theta_1(-i \ln a, q) \end{matrix} \right\}.$$

5.3.3. Ряды, содержащие $\operatorname{sh} a_k x$ и $\operatorname{ch} a_k x$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k} r^k \left\{ \begin{matrix} \operatorname{sh} kx \\ \operatorname{ch} kx \end{matrix} \right\} = \operatorname{Arth} \left(\frac{r \operatorname{sh} x}{1 - r \operatorname{ch} x} \right) \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} - \frac{1}{2} \ln(1 - 2r \operatorname{ch} x + r^2) \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}$$

$$[|a|e^{|\operatorname{Re} x|} < 1].$$

$$2. \sum_{k=0}^{\infty} \frac{1}{k!} r^k \left\{ \begin{matrix} \operatorname{sh} kx \\ \operatorname{ch} kx \end{matrix} \right\} = e^{r \operatorname{ch} x} \left\{ \begin{matrix} \operatorname{sh}(r \operatorname{sh} x) \\ \operatorname{ch}(r \operatorname{sh} x) \end{matrix} \right\}.$$

$$3. \sum_{k=0}^{\infty} e^{-a^2 k^2} \operatorname{ch} kx = \frac{1}{2} + \frac{\sqrt{\pi}}{2a} e^{x^2/(4a^2)} \theta_3\left(\frac{\pi x}{2a^2}, e^{-\pi^2/a^2}\right).$$

$$4. \sum_{n=1}^{\infty} \frac{\operatorname{ch} nx}{\operatorname{ch} 2nx + a} = \frac{1}{x \sqrt{2(a+1)}} K(k) \operatorname{dn} u - \frac{1}{2(a+1)}$$

$$\left[x = \frac{\pi K'(k)}{K(k)}, a = \operatorname{ch} \frac{\pi u}{K'(k)}; |\operatorname{Im} \operatorname{Arch} a| < \operatorname{Re} x \right].$$

$$5. \sum_{k=0}^{\infty} \frac{r^k}{\operatorname{ch}(kx + a) + b} =$$

$$= \frac{2}{\sqrt{1-b^2}} \int_0^{\infty} \frac{\cos at - r \cos(a-b)t}{1 - 2r \cos xt + r^2} \operatorname{sh}(t \operatorname{arccos} b) \operatorname{cosech} \pi t dt$$

$$[|b| < 1].$$

5.3.4. Ряды вида $\sum a_n \operatorname{cosech} b_n x$.

Обозначение. $x = \frac{\pi K'(k)}{K(k)}$, $k^2 < 1$.

$$1. \sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{cosech} \left(n - \frac{1}{2} \right) x = \frac{k}{\pi} K(k).$$

$$2. \sum_{n=1}^{\infty} n \operatorname{cosech} nx = \frac{1}{\pi^2} K(k) [K(k) - E(k)].$$

$$3. \sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{cosech} nx = \frac{1}{\pi^2} K(k) E(k) - \frac{1-k^2}{\pi^2} K^2(k).$$

$$4. \sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{cosech} (2n-1)x = -\frac{1}{8} \ln(1-k^2).$$

$$5. \sum_{n=1}^{\infty} \operatorname{cosech}^2 nx = \frac{1}{6} + \frac{2(2-k^2)}{3\pi^2} K^2(k) - \frac{2}{\pi^2} K(k) E(k).$$

$$6. \sum_{n=1}^{\infty} \operatorname{cosech}^2 \left(n - \frac{1}{2} \right) x = \frac{2}{\pi^2} K(k) [K(k) - E(k)].$$

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\operatorname{cosech} na - \operatorname{cosech} \frac{n\pi}{a} \right) = \frac{1}{2} \left(\frac{\pi}{a} - a \right) \quad [a > 0].$$

5.3.5. Ряды вида $\sum a_k \operatorname{cosech} k\pi$.

$$1. \sum_{k=1}^{\infty} k \operatorname{cosech} k\pi = \frac{1}{32\pi^3} \Gamma^2 \left(\frac{1}{4} \right) - \frac{1}{4\pi}.$$

$$2. \sum_{k=1}^{\infty} (-1)^k k \operatorname{cosech} k\pi = -\frac{1}{4\pi}.$$

$$3. \sum_{k=1}^{\infty} (-1)^k k^{2n+1} \operatorname{cosech} k\pi = 0 \quad [n=1, 2, 3, \dots],$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4n+3}} \operatorname{cosech} k\pi =$$

$$= \pi^{4n+3} \sum_{k=0}^{2n+2} \frac{(-1)^k}{(2k)! (4n-2k+4)!} (1-2^{2k-1}) (1-2^{4n-2k+3}) B_{2k} B_{4n-2k+4}$$

$$[n=1, 2, 3, \dots].$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \operatorname{cosech} k\pi = \frac{\pi^3}{360}.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7} \operatorname{cosech} k\pi = \frac{13\pi^2}{453\,600}.$$

$$7. \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^4 + a^4} \operatorname{cosech} k\pi = \frac{\pi}{2a^2} [\operatorname{ch}(\sqrt{2}\pi a) - \cos(\sqrt{2}\pi a)]^{-1} - \frac{1}{4\pi a^4}.$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^k k^5}{k^4 - a^4} \operatorname{cosech} k\pi = -\frac{\pi a^2}{4} \operatorname{cosec} \pi a \operatorname{cosech} \pi a.$$

$$9. \sum_{k=1}^{\infty} \operatorname{cosech}^2 k\pi = \frac{1}{6} - \frac{1}{2\pi}.$$

$$10. \sum_{k=0}^{\infty} \operatorname{cosech}^2 \frac{2k+1}{2} \pi = -\frac{1}{2\pi} + \frac{1}{16\pi^3} \Gamma^4\left(\frac{1}{4}\right).$$

$$11. \sum_{k=1}^{\infty} \frac{1}{k^2 + (k+1)^2} \operatorname{cosech} k\pi \operatorname{cosech} (k+1)\pi = \\ = \frac{1}{2\pi} \operatorname{cosech} \pi \left(2 + 2\pi \operatorname{cth} \pi - \pi^2 \operatorname{th}^2 \frac{\pi}{2} \right).$$

$$12. \sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{cosech} k\pi \operatorname{cth} k\pi = \frac{1}{12}.$$

$$13. \sum_{k=1}^{\infty} \operatorname{cosech}^4 k\pi = -\frac{11}{90} + \frac{1}{3\pi} + \frac{1}{1920\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$

$$14. \sum_{k=0}^{\infty} \operatorname{cosech}^4 \frac{2k+1}{2} \pi = \frac{1}{3\pi} - \frac{1}{24\pi^3} \Gamma^4\left(\frac{1}{4}\right) + \frac{1}{192\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$

5.3.6. Ряды вида $\sum a_n \operatorname{sech} b_n x$.

Обозначение: $x = \frac{\pi K'(k)}{K(x)}$, $k^2 < 1$.

$$1. \sum_{n=-\infty}^{\infty} \operatorname{sech} (nx + a) = \frac{2}{x} K(k) \operatorname{dn} \left[\frac{2a}{\pi} K'(k) \right]$$

$$[|\operatorname{Im}(2a/x)| < \operatorname{Re}(\pi/x)].$$

$$2. \sum_{n=-\infty}^{\infty} \operatorname{sech} nx = \frac{2}{\pi} K(k).$$

$$3. \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} nx = \frac{2(1-k^2)^{1/2}}{\pi} K(k).$$

$$4. \sum_{n=-\infty}^{\infty} \operatorname{sech} \left(n - \frac{1}{2} \right) x = \frac{2k}{\pi} K(k).$$

$$5. \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 nx = \frac{4}{\pi^2} K(k) E(k).$$

$$6. \sum_{n=1}^{\infty} \operatorname{sech}^2 \left(n - \frac{1}{2} \right) x = \frac{2}{\pi^2} K(k) E(k) - \frac{2(1-k^2)}{\pi^2} K^2(k).$$

$$7. \sum_{n=0}^{\infty} 2^{\pm 2n} \operatorname{sech}^2 2^{\pm n} x = \pm \operatorname{cosech}^2 x + \frac{1}{x^2} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

5.3.7. Ряды вида $\sum a_k \operatorname{sech} b_k \pi$.

$$1. \sum_{k=0}^{\infty} (-1)^k (2k+1)^{4n+3} \operatorname{sech} \frac{2k+1}{2} \pi = 0.$$

$$2. \sum_{k=0}^{\infty} (-1)^k (2k+1)^{6n-1} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = 0 \quad [n=1, 2, 3, \dots]$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{4n+1}} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{1}{4(4n)!} \left(\frac{\pi}{2} \right)^{4n+1} \sum_{k=0}^{2n} \binom{4n}{2k} E_{2k} E_{4n-2k}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^5} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi^5}{768}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^9} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{23\pi^9}{1720320}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{6n+1}} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} =$$

$$= (-1)^n \frac{\pi^{6n+1}}{2} \sum_{k=0}^{3n} \frac{2^{2k+1} - 1}{(2k+1)! (6n-2k)! (k+1)} B_{2k+2} B_{6n-2k} \cos \frac{2k+1}{3} \pi$$

[n=1, 2, 3, ...]

- $$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = \frac{\pi}{24}.$$
- $$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^7} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = \frac{\pi^7}{23\,040}.$$
- $$10. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^3}{(2k+1)^4 - a^4} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8} \sec \frac{\pi a}{2} \operatorname{sech} \frac{\pi a}{2}.$$
- $$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^4 - a^4]} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8a^4} \left(\sec \frac{\pi a}{2} \operatorname{sech} \frac{\pi a}{2} - 1 \right).$$
- $$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^4 + a^4]} \operatorname{sech} \frac{2k+1}{2} \pi =$$
- $$= \frac{\pi}{8a^4} \left[1 - \frac{2}{\cos(\pi a/\sqrt{2}) + \operatorname{ch}(\pi a/\sqrt{2})} \right].$$
- $$13. \sum_{k=1}^{\infty} \operatorname{sech}^2 k\pi = \frac{1}{2\pi} - \frac{1}{2} + \frac{1}{16\pi^3} \Gamma^2\left(\frac{1}{4}\right).$$
- $$14. \sum_{k=0}^{\infty} \operatorname{sech}^2 \frac{2k+1}{2} \pi = \frac{1}{2\pi}.$$
- $$15. \sum_{k=1}^{\infty} \operatorname{sech}^4 k\pi = -\frac{1}{2} + \frac{1}{3\pi} + \frac{1}{24\pi^3} \Gamma^4\left(\frac{1}{4}\right) + \frac{1}{192\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$
- $$16. \sum_{k=0}^{\infty} \operatorname{sech}^4 \frac{2k+1}{2} \pi = \frac{1}{3\pi} - \frac{1}{384\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$

5.3.8. Ряды вида $\sum a_k \operatorname{th} b_k$.

- $$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{4n+3}} \operatorname{th} \frac{2k+1}{2} \pi =$$
- $$= \frac{1}{2} \pi^{4n+3} \sum_{k=0}^{2n} \frac{(-1)^k (2^{2k} - 1) (2^{4n-2k+2} - 1)}{(2k+2)! (4n-2k+2)!} B_{2k+2} B_{4n-2k+2}.$$
- $$2. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \operatorname{th} \frac{2k+1}{2} \pi = \frac{\pi^3}{32}.$$
- $$3. \sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^4 - a^4} \operatorname{th} \frac{2k+1}{2} \pi = \frac{\pi}{8a^2} \operatorname{tg} \frac{\pi a}{2} \operatorname{th} \frac{\pi a}{2}.$$

$$4. \sum_{k=1}^{\infty} \frac{1}{2^k} \operatorname{th} \frac{x}{2^k} = \operatorname{cth} x - \frac{1}{x}.$$

$$5. \sum_{k=0}^{\infty} 2^k (1 - \operatorname{th} 2^k x) = \operatorname{cth} x - 1$$

[$x > 0$].

$$6. \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \operatorname{th}^2 \frac{x}{2^k} = \operatorname{cth}^2 x - \frac{1}{x^2} - \frac{2}{3}.$$

5.3.9. Ряды вида $\sum a_k \operatorname{cth} k\pi$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \operatorname{cth} k\pi = \frac{1}{2} (2\pi)^{2n+1} \sum_{k=1}^n \frac{(-1)^{k+1}}{(2k)! (2n-2k+2)!} B_{2k} B_{2n-2k+2} + \\ + \frac{(-1)^n 2^{2n+1} \pi^{2n+2}}{(2n+2)!} B_{2n+2}.$$

$$2. \sum_{k=1}^{\infty} \frac{1}{k^3} \operatorname{cth} k\pi = \frac{7}{180} \pi^3. \quad 3. \sum_{k=1}^{\infty} \frac{1}{k^7} \operatorname{cth} k\pi = \frac{19}{56700} \pi^7.$$

$$4. \sum_{k=1}^{\infty} \frac{k}{k^4 + a^4} \operatorname{cth} k\pi = \frac{\pi}{4a^2} \frac{\operatorname{ch}(\pi a \sqrt{2}) + \cos(\pi a \sqrt{2})}{\operatorname{ch}(\pi a \sqrt{2}) - \cos(\pi a \sqrt{2})} - \frac{1}{4\pi a^4}.$$

$$5. \sum_{k=1}^{\infty} \frac{k}{k^4 - a^4} \operatorname{cth} k\pi = \frac{1}{4\pi a^4} - \frac{\pi}{4a^2} \operatorname{ctg} \pi a \operatorname{cth} \pi a.$$

5.4. ТРИГОНОМЕТРИЧЕСКИЕ РЯДЫ

5.4.1. Введение.

В этом разделе помещены ряды, содержащие различные тригонометрические функции, в том числе ряды Фурье. Отметим, что в некоторых случаях тригонометрические ряды сводятся к рядам другого вида, например, к степенным рядам (5.2), к рядам, содержащим показательную и гиперболические функции (5.3).

При суммировании тригонометрических рядов могут оказаться полезными следующие формулы.

$$\sum_{k=1}^{\infty} F(k) \sin kx = \frac{1}{2} \sin x \int_0^{\infty} \frac{f(t) dt}{\operatorname{ch} t - \cos x} \quad [0 \leq x \leq 2\pi],$$

$$\sum_{k=1}^{\infty} F(k) \cos kx = \frac{1}{2} \int_0^{\infty} \frac{\cos x - e^{-t}}{\operatorname{ch} t - \cos x} f(t) dt \quad [0 \leq x \leq 2\pi],$$

где

$$F(x) = \int_0^{\infty} f(t) e^{-xt} dt.$$

Ряды, содержащие логарифмы, приведены в 5.5.1.

5.4.2. Ряды вида $\sum \frac{(\pm 1)^k}{k^s} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

1. $\sum_{k=1}^{\infty} \frac{1}{k^s} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \left(e^t \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right)}{1 - 2e^t \cos x + e^{2t}} dt$
 $[x \neq 0, \operatorname{Re} s > 0 \text{ или } |x| \geq 0, \operatorname{Re} s > 1].$
2. $= \frac{(2\pi)^s}{4\Gamma(s)} \begin{Bmatrix} \operatorname{cosec}(\pi s/2) \\ \sec(\pi s/2) \end{Bmatrix} \left[\zeta\left(1-s, \frac{x}{2\pi}\right) \mp \zeta\left(1-s, 1-\frac{x}{2\pi}\right) \right]$
 $[\operatorname{Re} s > 1; 0 < x \leq 2\pi].$
3. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \left(e^t \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right)}{1 + 2e^t \cos x + e^{2t}} dt$
 $[\operatorname{Re} s > 0].$
4. $= \frac{(2\pi)^s}{4\Gamma(s)} \begin{Bmatrix} \operatorname{cosec}(\pi s/2) \\ \sec(\pi s/2) \end{Bmatrix} \left[\zeta\left(1-s, \frac{x+\pi}{2}\right) \mp \zeta\left(1-s, \frac{\pi-x}{2}\right) \right]$
 $[\operatorname{Re} s > 1; -\pi < x < \pi].$
5. $\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin kx = \frac{(-1)^{n-1}}{2(2n+1)!} (2\pi)^{2n+1} B_{2n+1} \left(\frac{x}{2\pi} \right)$
 $[0 \leq x \leq 2\pi \text{ при } n=1, 2, 3, \dots; 0 < x < 2\pi \text{ при } n=0].$
6. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n+1}} \sin kx = \frac{(-1)^{n-1}}{2(2n+1)!} (2\pi)^{2n+1} B_{2n+1} \left(\frac{x+\pi}{2\pi} \right)$
 $[-\pi < x \leq \pi; n=0, 1, 2, \dots].$
7. $\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \cos kx = \frac{(-1)^{n-1}}{2(2n)!} (2\pi)^{2n} B_{2n} \left(\frac{x}{2\pi} \right)$
 $[0 \leq x \leq 2\pi; n=1, 2, 3, \dots].$
8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} \cos kx = \frac{(-1)^{n-1}}{2(2n)!} (2\pi)^{2n} B_{2n} \left(\frac{x+\pi}{2\pi} \right)$
 $[-\pi \leq x \leq \pi].$
9. $\sum_{k=1}^{\infty} \frac{1}{k} \begin{Bmatrix} \sin(kx+a) \\ \cos(kx+a) \end{Bmatrix} = \pm \frac{\pi-x}{2} \begin{Bmatrix} \cos a \\ \sin a \end{Bmatrix} - \ln \left(2 \sin \frac{x}{2} \right) \begin{Bmatrix} \sin a \\ \cos a \end{Bmatrix}$
 $[0 < x < 2\pi].$
10. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \begin{Bmatrix} \sin(kx+a) \\ \cos(kx+a) \end{Bmatrix} = \pm \frac{x}{2} \begin{Bmatrix} \cos a \\ \sin a \end{Bmatrix} + \ln \left(2 \cos \frac{x}{2} \right) \begin{Bmatrix} \sin a \\ \cos a \end{Bmatrix}$
 $[-\pi < x < \pi].$
11. $\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k^2} \sin kx = - \int_0^x \ln \begin{Bmatrix} 2 \sin(t/2) \\ 2 \cos(t/2) \end{Bmatrix} dt$
 $\left[\begin{Bmatrix} 0 \leq x < \pi \\ -\pi < x < \pi \end{Bmatrix} \right].$
12. $\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k^2} \cos kx = \frac{1}{12} \begin{Bmatrix} 3x^2 - 6\pi x + 2\pi^2 \\ 3x^2 - \pi^2 \end{Bmatrix}$
 $\left[\begin{Bmatrix} 0 \leq x \leq 2\pi \\ -\pi \leq x \leq \pi \end{Bmatrix} \right].$

$$13. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k^3} \sin kx = \frac{x}{12} \begin{cases} x^2 - 3\pi x + 2\pi^2 \\ x^2 - \pi^2 \end{cases} \quad \left[\begin{cases} 0 \leq x \leq 2\pi \\ -\pi \leq x \leq \pi \end{cases} \right].$$

5.4.3. Ряды вида $\sum \frac{(\pm 1)^k}{(k+a)^s} \begin{Bmatrix} \sin(kx+b) \\ \cos(kx+b) \end{Bmatrix}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm e^{-t} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}}{1 - 2e^{-t} \cos x + e^{-2t}} dt$$

[Re a > 0, Re s > 0 при x ≠ 0, Re s > 1 при x = 0].

$$2. \sum_{k=0}^{\infty} \frac{1}{k+a} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} =$$

$$= \beta(a) \begin{Bmatrix} \sin(\pi-x)a \\ \cos(\pi-x)a \end{Bmatrix} + \frac{1}{2} \int_x^{\pi} \begin{Bmatrix} \sin[-xa + (a-1/2)t] \\ \cos[-xa + (a-1/2)t] \end{Bmatrix} \operatorname{cosec} \frac{t}{2} dt.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} =$$

$$= \mp \beta(a) \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} \pm \frac{1}{2} \int_0^x \begin{Bmatrix} \cos[-xa + (a-1/2)t] \\ \sin[-xa + (a-1/2)t] \end{Bmatrix} \sec \frac{t}{2} dt.$$

$$4. \sum_{k=-\infty}^{\infty} \frac{1}{k+a} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \pi \operatorname{cosec} \pi a \begin{Bmatrix} \sin[(2n+1)\pi-x]a \\ \cos[(2n+1)\pi-x]a \end{Bmatrix}$$

[2πn < x < 2π(n+1)].

$$5. \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k+a} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \pi \operatorname{cosec} \pi a \begin{Bmatrix} \sin(2n\pi-x)a \\ \cos(2n\pi-x)a \end{Bmatrix}$$

[(2n-1)π < x < (2n+1)π].

$$6. \sum_{k=-\infty}^{\infty} \frac{1}{kx+a} \begin{Bmatrix} \sin(kx+a) \\ \cos(kx+a) \end{Bmatrix} = \frac{\pi}{x \sin(\pi a/x)} \begin{Bmatrix} \sin[(2n+1)\pi a/x] \\ \cos[(2n+1)\pi a/x] \end{Bmatrix}$$

[2nπ < x < (2n+2)π].

$$7. \sum_{k=0}^{\infty} \frac{1}{km+n} \begin{Bmatrix} \sin(kx+a) \\ \cos(kx+a) \end{Bmatrix} =$$

$$= \frac{\mp 1}{2m^2} \sum_{k=0}^{m-1} [(2k-m)\pi + x] \begin{Bmatrix} \cos[a - (x+2k\pi)n/m] \\ \sin[a - (x+2k\pi)n/m] \end{Bmatrix} -$$

$$- \frac{1}{m} \sum_{k=0}^{m-1} \begin{Bmatrix} \sin[a - (x+2k\pi)n/m] \\ \cos[a - (x+2k\pi)n/m] \end{Bmatrix} \ln \left(2 \left| \sin \frac{x+2k\pi}{2m} \right| \right) +$$

$$+ \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{1}{km-n} \begin{Bmatrix} \sin(a-kx) \\ \cos(a-kx) \end{Bmatrix}$$

[m > 0; 0 < x < 2π; δ(n-m) = {1, n > m; 0, n ≤ m}].

$$\begin{aligned}
 8. \quad \sum_{k=0}^{\infty} \frac{1}{km-n} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\} = \\
 = \frac{\mp 1}{2m^2} \sum_{k=0}^{m-1} [(2k-m)\pi + x] \left\{ \begin{matrix} \cos[a + (x+2k\pi)n/m] \\ \sin[a + (x+2k\pi)n/m] \end{matrix} \right\} - \\
 - \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \begin{matrix} \sin[a + (x+2k\pi)n/m] \\ \cos[a + (x+2k\pi)n/m] \end{matrix} \right\} \ln \left[2 \left| \sin \frac{x+2k\pi}{2m} \right| \right] + \\
 + \sum_{k=0}^{[n/m]} \frac{1}{km-n} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\} \quad [m > 0; 0 < x < 2\pi; n/m \neq 0, 1, 2, \dots].
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{km+n} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\} = \\
 = \frac{\mp 1}{2m^2} \sum_{k=0}^{m-1} [(2k-m+1)\pi + x] \left\{ \begin{matrix} \cos[a - (x+2k\pi+\pi)n/m] \\ \sin[a - (x+2k\pi+\pi)n/m] \end{matrix} \right\} - \\
 - \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \begin{matrix} \sin[a - (x+2k\pi+\pi)n/m] \\ \cos[a - (x+2k\pi+\pi)n/m] \end{matrix} \right\} \ln \left(2 \sin \left| \frac{x+(2k+1)\pi}{2m} \right| \right) + \\
 + \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{(-1)^k}{km-n} \left\{ \begin{matrix} \sin(a-kx) \\ \cos(a-kx) \end{matrix} \right\} \\
 [m > 0; -\pi < x < \pi; \delta(n-m) = \begin{cases} 1, & n > m \\ 0, & n \leq m \end{cases}].
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{km-n} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\} = \\
 = \frac{\mp 1}{2m^2} \sum_{k=0}^{m-1} [(2k-m+1)\pi + x] \left\{ \begin{matrix} \cos[a + (x+2k\pi+\pi)n/m] \\ \sin[a + (x+2k\pi+\pi)n/m] \end{matrix} \right\} - \\
 - \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \begin{matrix} \sin[a + (x+2k\pi+\pi)n/m] \\ \cos[a + (x+2k\pi+\pi)n/m] \end{matrix} \right\} \ln \left(2 \sin \left| \frac{x+(2k+1)\pi}{2m} \right| \right) + \\
 + \sum_{k=0}^{[n/m]} \frac{(-1)^k}{km-n} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\} \quad [m > 0; -\pi < x < \pi; n/m \neq 0, 1, 2, \dots].
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \sum_{k=1}^{\infty} \frac{1}{k+n} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \frac{1}{2} (\pi - x) \left\{ \begin{matrix} \cos nx \\ \sin nx \end{matrix} \right\} - \\
 - \sum_{k=1}^n \frac{1}{k} \sin kx \left\{ \begin{matrix} \cos nx \\ \sin nx \end{matrix} \right\} \pm \left[\ln \left(2 \sin \frac{x}{2} \right) + \sum_{k=1}^n \frac{1}{k} \cos kx \right] \left\{ \begin{matrix} \sin nx \\ \cos nx \end{matrix} \right\} \\
 [n = 1, 2, 3, \dots; 0 < x < 2\pi].
 \end{aligned}$$

$$12. \sum_{k=1}^{\infty} \frac{(-1)^k}{k+n} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = (-1)^{n+1} \left[\sum_{k=1}^n \frac{(-1)^k}{k} \sin kx + \frac{x}{2} \right] \begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} \pm$$

$$\pm (-1)^n \left[\ln \left(2 \cos \frac{x}{2} \right) + \sum_{k=1}^n \frac{(-1)^k}{k} \cos kx \right] \begin{Bmatrix} \sin nx \\ \cos nx \end{Bmatrix}$$

[n = 1, 2, 3, ...; -\pi < x < \pi].

$$13. \sum_{k=0}^{\infty} \frac{1}{2k+1} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{\pi}{4} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} \mp \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix} \ln \operatorname{ctg}^2 \frac{x}{4} \quad [0 < x < 2\pi].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = -\frac{1}{4} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} \ln \operatorname{ctg}^2 \left(\frac{x+\pi}{4} \right) \mp \frac{\pi}{4} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix}$$

[-\pi < x < \pi].

5.4.4. Ряды вида $\sum \frac{(\pm 1)^k}{(kn_1+m_1)(kn_2+m_2)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \pm (\pi - x) \sin \frac{x}{2} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix} \pm$$

$$\pm 2 \sin \frac{x}{2} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} \ln \left(2 \sin \frac{x}{2} \right) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [0 \leq x \leq 2\pi].$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = -x \cos \frac{x}{2} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} \pm$$

$$\pm 2 \cos \frac{x}{2} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix} \ln \left(2 \cos \frac{x}{2} \right) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [-\pi \leq x \leq \pi].$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \mp \frac{1}{2} [1 + (x - \pi) \sin x] \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} -$$

$$- \sin x \ln \left(2 \sin \frac{x}{2} \right) \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \frac{1}{4} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [0 \leq x \leq 2\pi].$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \begin{Bmatrix} \sin (k+1)x \\ \cos (k+1)x \end{Bmatrix} =$$

$$= \mp 2 \sin \frac{x}{2} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} \ln \left(2 \sin \frac{x}{2} \right) + \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - (\pi - x) \sin \frac{x}{2} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix}$$

[0 \leq x \leq 2\pi].

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \begin{Bmatrix} \sin (k+1)x \\ \cos (k+1)x \end{Bmatrix} = \mp x \cos \frac{x}{2} \begin{Bmatrix} \cos (x/2) \\ \sin (x/2) \end{Bmatrix} +$$

$$+ \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - 2 \cos \frac{x}{2} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix} \ln \left(2 \cos \frac{x}{2} \right) \quad [-\pi \leq x \leq \pi].$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \pm \frac{1}{2} (1 - x \sin x) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - \\ - \sin x \ln \left(2 \cos \frac{x}{2} \right) \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \frac{1}{4} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [-\pi \leq x \leq \pi].$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{4} \sin \frac{x}{2} \ln \operatorname{ctg}^2 \frac{x}{4} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \pm \\ \pm \frac{\pi}{4} \sin \frac{x}{2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mp \frac{1}{2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \quad [0 < x < 2\pi].$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{4} \cos \frac{x}{2} \ln \operatorname{tg}^2 \frac{x+\pi}{4} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \mp \\ \mp \frac{\pi}{4} \cos \frac{x}{2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \pm \frac{1}{2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \quad [-\pi < x < \pi].$$

5.4.5. Ряды вида $\sum \frac{(\pm 1)^k}{(k^2 \pm a^2)^m} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^2 \pm a^2} \cos kx = \pm \frac{\pi}{2a} \begin{Bmatrix} \operatorname{ch}[(\pi-x)a] \operatorname{cosech} \pi a \\ \cos[(\pi-x)a] \operatorname{cosec} \pi a \end{Bmatrix} \mp \frac{1}{2a^2} \quad [0 \leq x \leq 2\pi].$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 \pm a^2} \cos kx = \mp \frac{\pi}{2a} \begin{Bmatrix} \operatorname{ch} ax \operatorname{cosech} \pi a \\ \cos ax \operatorname{cosec} \pi a \end{Bmatrix} \pm \frac{1}{2a^2} \quad [-\pi \leq x \leq \pi].$$

$$3. \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{k^2 - n^2} \cos kx = \frac{1}{2n^2} + \frac{\cos nx}{4n^2} - \frac{(\pi-x) \sin nx}{2n} \quad [0 < x < 2\pi].$$

$$4. \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 - 1} \cos kx = \frac{1}{2} - \frac{1}{4} \cos x - \frac{x \sin x}{2} \quad [-\pi \leq x \leq \pi].$$

$$5. \sum_{k=1}^{\infty} \frac{k}{k^2 \pm a^2} \sin kx = \frac{\pi}{2} \begin{Bmatrix} \operatorname{sh}(\pi-x)a \operatorname{cosech} \pi a \\ \sin(\pi-x)a \operatorname{cosec} \pi a \end{Bmatrix} \quad [0 < x < 2\pi].$$

$$6. \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{k}{k^2 - n^2} \sin kx = \frac{1}{2} (\pi - x) \cos nx - \frac{1}{4n} \sin nx \\ [0 < x < 2\pi; n = 1, 2, 3, \dots].$$

$$7. \sum_{k=2}^{\infty} \frac{k}{k^2 - 1} \cos kx = -\frac{1}{2} - \frac{\cos x}{4} - \cos x \ln \left(2 \sin \frac{x}{2} \right) \quad [0 < x < 2\pi].$$

$$8. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 \pm a^2} \sin kx = \frac{\pi}{2} \begin{cases} \operatorname{sh} ax & \operatorname{cosech} \pi a \\ \sin ax & \operatorname{cosec} \pi a \end{cases} \quad [-\pi < x < \pi].$$

$$9. \sum_{k=2}^{\infty} \frac{(-1)^k k}{k^2 - 1} \sin kx = \frac{\sin x}{4} + \frac{x \cos x}{2} \quad [-\pi < x < \pi].$$

$$10. \sum_{k=1}^{\infty} \frac{k}{4k^2 - 1} \cos kx = -\frac{1}{4} - \frac{1}{4} \cos \frac{x}{2} \ln \operatorname{tg} \frac{x}{4} \quad [0 < x < 2\pi].$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k^2 \pm a^2)^2} \cos kx = \\ = \frac{1}{2a^4} + \frac{\pi}{4a^3} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases} \left[\begin{cases} \operatorname{ch} (\pi - x) a \\ \cos (\pi - x) a \end{cases} \pm ax \begin{cases} \operatorname{sh} (\pi - x) a \\ \sin (\pi - x) a \end{cases} \right] + \\ + \frac{\pi^2}{4a^2} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}^2 \begin{cases} \operatorname{ch} ax \\ \cos ax \end{cases}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2 \pm a^2)^2} \cos kx = \\ = \frac{1}{2a^4} + \frac{\pi}{4a^3} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases} \left[\begin{cases} \operatorname{ch} ax \\ \cos ax \end{cases} \mp a(x + \pi) \begin{cases} \operatorname{sh} ax \\ \sin ax \end{cases} \right] + \\ + \frac{\pi^2}{4a^2} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}^2 \begin{cases} \operatorname{ch} (x + \pi) a \\ \cos (x + \pi) a \end{cases}.$$

$$13. \sum_{k=1}^{\infty} \frac{k}{(k^2 \pm a^2)^2} \sin kx = \\ = \mp \frac{\pi^2}{4a} \begin{cases} \operatorname{sh} ax \\ \sin ax \end{cases} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}^2 \mp \frac{\pi x}{4a} \begin{cases} \operatorname{ch} (\pi - x) a \\ \cos (\pi - x) a \end{cases} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k k}{(k^2 \pm a^2)^2} \sin kx = \\ = \mp \frac{\pi^2}{4a} \begin{cases} \operatorname{sh} (x + \pi) a \\ \sin (x + \pi) a \end{cases} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}^2 \mp \frac{\pi}{4a} (x + \pi) \begin{cases} \operatorname{ch} ax \\ \cos ax \end{cases} \begin{cases} \operatorname{cosech} \pi a \\ \operatorname{cosec} \pi a \end{cases}.$$

$$15. \sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{(k^2 + a^2)^{1/2} \pm k}{k^2 + a^2} \right]^{1/2} \sin \frac{k\pi}{3} = \left(\frac{6}{\pi} \right)^{1/2} \int_0^{\infty} \frac{1}{1 + 2 \operatorname{ch} t^2} \begin{cases} \cos at^2 \\ \sin at^2 \end{cases} dt.$$

5.4.6. Ряды вида $\sum a_k \begin{cases} \sin (2k+1) x \\ \cos (2k+1) x \end{cases}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \begin{cases} \sin (2k+1) x \\ \cos (2k+1) x \end{cases} = \frac{1}{\Gamma(s)} \begin{cases} \sin x \\ \cos x \end{cases} \int_0^{\infty} \frac{t^{s-1} e^{-at} (1 \pm e^{-t})}{1 - 2e^{-t} \cos 2x + e^{-2t}} dt$$

[Re a , Re $s > 0$].

$$2. \sum_{k=1}^{\infty} \frac{1}{k^s} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \left(e^t \begin{Bmatrix} \sin 3x \\ \cos 3x \end{Bmatrix} - \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right)}{e^{2t} - 2e^t \cos 2x + 1} dt$$

[Re $s > 0$].

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \sin (2k+1) x = \frac{(-1)^n \pi^{2n+1}}{4 (2n)!} E_{2n} \left(\frac{x}{\pi} \right)$$

[$0 < x < \pi$; $n = 0, 1, 2, \dots$].

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n}} \sin (2k+1) x = \frac{(-1)^{n-1} \pi^{2n}}{4 (2n-1)!} E_{2n-1} \left(\frac{x}{\pi} + \frac{1}{2} \right)$$

[$-\pi/2 \leq x \leq \pi/2$; $n = 1, 2, 3, \dots$].

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} \cos (2k+1) x = \frac{(-1)^n \pi^{2n}}{4 (2n-1)!} E_{2n-1} \left(\frac{x}{\pi} \right)$$

[$0 \leq x \leq \pi$; $n = 1, 2, 3, \dots$].

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \cos (2k+1) x = \frac{(-1)^n \pi^{2n+1}}{4 (2n)!} E_{2n} \left(\frac{x}{\pi} + \frac{1}{2} \right)$$

[$-\pi/2 < x < \pi/2$; $n = 0, 1, 2, \dots$].

$$7. \sum_{k=0}^{\infty} \frac{1}{2k+1} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \frac{\pi}{4} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \frac{1}{4} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \ln \operatorname{ctg}^2 \frac{x}{4}, \quad [0 < x < \pi].$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \frac{\pi}{4} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \frac{1}{4} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \ln \operatorname{ctg}^2 \left(\frac{x+\pi}{4} \right)$$

[$-\pi/2 < x < \pi/2$].

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin (2k+1) x = -\frac{1}{2} \int_0^x \ln \operatorname{tg} \frac{t}{2} dt \quad [0 \leq x < \pi].$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos (2k+1) x = \frac{\pi}{8} (\pi - 2|x|) \quad [-\pi \leq x < \pi].$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin (2k+1) x = \begin{cases} \pi x/4 \\ \pi (\pi - x)/4 \end{cases} \quad \begin{matrix} [-\pi/2 \leq x \leq \pi/2] \\ [\pi/2 \leq x \leq 3\pi/2] \end{matrix}$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos (2k+1) x = -\frac{1}{2} \int_0^{\pi/2-x} \ln \operatorname{tg} \frac{t}{2} dt \quad [-\pi/2 \leq x \leq \pi/2].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \sin (2k+1) x = \frac{x\pi}{8} (\pi - |x|) \quad [-\pi \leq x \leq \pi].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \cos (2k+1) x = \frac{\pi}{32} (\pi^2 - 4x^2) \quad [-\pi/2 \leq x \leq \pi/2].$$

$$15. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sin (2k+1) x = \sin x - 2 \sin x \ln (2 \sin x) \quad [0 < x < \pi].$$

$$16. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \sin (2k+1) x = \sin x - 2x \cos x \quad [-\pi/2 < x < \pi/2].$$

$$17. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \cos (2k+1) x = \cos x - \pi \sin x + 2x \sin x \quad [0 < x < \pi].$$

$$18. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \cos (2k+1) x = \cos x - 2 \cos x \ln (2 \cos x) \quad [-\pi/2 < x < \pi/2].$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(2k+3)} \sin (2k+1) x = \frac{\pi}{8} \sin 2x \quad [-\pi/2 < x < \pi/2].$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+3)} \cos (2k+1) x = -\frac{\pi}{8} \sin 2x \quad [0 \leq x \leq \pi].$$

$$21. \sum_{k=0}^{\infty} \frac{(2k+1)}{(2k-1)(2k+3)} \sin (2k+1) x = \frac{\pi}{4} \cos 2x \quad [0 < x < \pi].$$

$$22. \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2k+1}{(2k-1)(2k+3)} \cos (2k+1) x = \frac{\pi}{4} \cos 2x \quad [-\pi/2 \leq x \leq \pi/2].$$

$$23. \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(4k^2-1)(2k+3)} \cos (2k+1) x = \frac{\pi}{4} \cos^2 x \quad [-\pi/2 \leq x \leq \pi/2].$$

$$24. \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^2 \pm a^2} \sin (2k+1) x = \frac{\pi}{4a} \begin{cases} \operatorname{sh} ax & \operatorname{sech} (a\pi/2) \\ \sin ax & \sec (a\pi/2) \end{cases} \quad [-\pi/2 < x < \pi/2].$$

$$25. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pm a^2} \cos (2k+1) x = \frac{\pi}{4a} \begin{cases} \operatorname{sh} (\pi - 2x) a & \operatorname{sech} (a\pi/2) \\ \sin (\pi - 2x) a & \sec (a\pi/2) \end{cases} \quad [0 \leq x \leq \pi].$$

$$26. \sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^2 \pm a^2} \sin (2k+1) x = \frac{\pi}{4} \begin{cases} \operatorname{ch} (\pi - 2x) a & \operatorname{sech} (a\pi/2) \\ \cos (\pi - 2x) a & \sec (a\pi/2) \end{cases} \quad [0 < x < \pi].$$

$$27. \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{(2k+1)^2 \pm a^2} \cos (2k+1) x = \frac{\pi}{4} \begin{cases} \operatorname{ch} ax \operatorname{sech} (a\pi/2) \\ \cos ax \sec (a\pi/2) \end{cases} \\ [-\pi/2 < x < \pi/2].$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(ka+b)(ka+b+1)\dots(ka+b+n)} \begin{Bmatrix} \sin [(2k+1)x] \\ \cos [(2k+1)x] \end{Bmatrix} = \\ = \frac{1}{n!} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \int_0^1 \frac{t^{b-1} (1-t)^n (1 \pm t^a)}{1 - 2t^a \cos 2x + t^{2a}} dt \quad [a, b > 0; 0 < x < \pi].$$

$$29. \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \begin{Bmatrix} \sin [(2k+1)x] \\ \cos [(2k+1)x] \end{Bmatrix} = \frac{1}{\sqrt{2} \sin x} \begin{Bmatrix} \sin [(2x+\pi)/4] \\ \cos [(2x+\pi)/4] \end{Bmatrix} - \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \\ [0 < x < \pi].$$

$$30. \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \begin{Bmatrix} \sin [(2k+1)x] \\ \cos [(2k+1)x] \end{Bmatrix} = \frac{1}{\sqrt{2} \cos x} \begin{Bmatrix} \sin (x/2) \\ \cos (x/2) \end{Bmatrix} - \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \\ [-\pi/2 < x < \pi/2].$$

5.4.7. Ряды вида $\sum \frac{1}{k!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = e^{\cos x} \begin{Bmatrix} \sin (\sin x) \\ \cos (\sin x) \end{Bmatrix}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \mp e^{-\cos x} \begin{Bmatrix} \sin (\sin x) \\ \cos (\sin x) \end{Bmatrix}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \begin{Bmatrix} \sin [\sin (x/2)] \operatorname{sh} [\cos (x/2)] \\ \cos [\sin (x/2)] \operatorname{ch} [\cos (x/2)] \end{Bmatrix}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \mp \begin{Bmatrix} \sin [\cos (x/2)] \operatorname{sh} [\sin (x/2)] \\ \cos [\cos (x/2)] \operatorname{ch} [\sin (x/2)] \end{Bmatrix}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \cos \frac{x}{2} \begin{Bmatrix} \sin [\sin (x/2)] \operatorname{ch} [\cos (x/2)] \\ \cos [\sin (x/2)] \operatorname{sh} [\cos (x/2)] \end{Bmatrix} \mp \\ \mp \sin \frac{x}{2} \begin{Bmatrix} \cos [\sin (x/2)] \operatorname{sh} [\cos (x/2)] \\ \sin [\sin (x/2)] \operatorname{ch} [\cos (x/2)] \end{Bmatrix}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \cos \frac{x}{2} \begin{Bmatrix} \cos [\cos (x/2)] \operatorname{sh} [\sin (x/2)] \\ \sin [\cos (x/2)] \operatorname{ch} [\sin (x/2)] \end{Bmatrix} \mp \\ \mp \sin \frac{x}{2} \begin{Bmatrix} \sin [\cos (x/2)] \operatorname{ch} [\sin (x/2)] \\ \cos [\cos (x/2)] \operatorname{sh} [\sin (x/2)] \end{Bmatrix}.$$

5.4.8. Ряды вида $\sum a_k \left\{ \begin{smallmatrix} \sin (kx+a) \\ \cos (kx+a) \end{smallmatrix} \right\}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} \left\{ \begin{smallmatrix} \sin kx \\ \cos kx \end{smallmatrix} \right\} =$$

$$= \frac{1}{2} (1 - \cos x) \left[\mp \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\} \ln (2 - 2 \cos x) \mp (\pi - x) \left\{ \begin{smallmatrix} \cos x \\ \sin x \end{smallmatrix} \right\} \right] + \frac{3}{4} \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\} \pm \frac{1}{2} \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\}$$

[0 < x < 2π].

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)} \left\{ \begin{smallmatrix} \sin kx \\ \cos kx \end{smallmatrix} \right\} =$$

$$= \frac{1}{2} (1 + \cos x) \left[\mp x \left\{ \begin{smallmatrix} \cos x \\ \sin x \end{smallmatrix} \right\} \pm \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\} \ln (2 + 2 \cos x) \right] + \frac{3}{4} \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\} \mp \frac{1}{2} \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\}$$

[-π < x < π].

$$3. \sum_{k=0}^{\infty} \frac{1}{(kr+b)(kr+b+1) \dots (kr+b+n)} \left\{ \begin{smallmatrix} \sin (kx+a) \\ \cos (kx+a) \end{smallmatrix} \right\} =$$

$$= \frac{1}{n!} \int_0^1 \frac{t^{b-1} (1-t)^n}{1 - 2t^r \cos x + t^{2r}} \left[\left\{ \begin{smallmatrix} \sin a \\ \cos a \end{smallmatrix} \right\} \pm t^r \left\{ \begin{smallmatrix} \sin (x-a) \\ \cos (x-a) \end{smallmatrix} \right\} \right] dt \quad [r, b > 0].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kr+b)(kr+b+1) \dots (kr+b+n)} \left\{ \begin{smallmatrix} \sin (kx+a) \\ \cos (kx+a) \end{smallmatrix} \right\} =$$

$$= \frac{1}{n!} \int_0^1 \frac{t^{b-1} (1-t)^n}{1 + 2t^r \cos x + t^{2r}} \left[\left\{ \begin{smallmatrix} \sin a \\ \cos a \end{smallmatrix} \right\} \mp \left\{ \begin{smallmatrix} \sin (x-a) \\ \cos (x-a) \end{smallmatrix} \right\} \right] dt \quad [r, b > 0].$$

$$5. \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \cos kx = \left(2 \sin \frac{x}{2} \right)^{-1/2} \cos \frac{\pi - x}{4} - 1 \quad [0 < x < 2\pi].$$

$$6. \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \cos kx = \left(2 \cos \frac{x}{2} \right)^{-1/2} \cos \frac{x}{4} - 1 \quad [-\pi < x < \pi].$$

$$7. \sum_{k=0}^{\infty} \frac{[(2k)!]^2}{(n!)^4 2^{4k}} \left\{ \begin{smallmatrix} \sin (2k+1/2) x \\ \cos (2k+1/2) x \end{smallmatrix} \right\} = \frac{1}{\pi} \left\{ \begin{smallmatrix} K [\sin (x/2)] \\ K [\cos (x/2)] \end{smallmatrix} \right\}$$

[0 < x < π].

$$8. \sum_{k=0}^{\infty} \frac{(2k)! (2k+1)!}{(k!)^3 (k+1)! 2^{4k}} \left\{ \begin{smallmatrix} \sin (2k+3/2) x \\ \cos (2k+3/2) x \end{smallmatrix} \right\} =$$

$$= \pm \frac{4}{\pi} \left\{ \begin{smallmatrix} E [\sin (x/2)] \\ K [\cos (x/2)] \end{smallmatrix} \right\} \mp \frac{2}{\pi} \left\{ \begin{smallmatrix} K [\sin (x/2)] \\ E [\cos (x/2)] \end{smallmatrix} \right\}$$

[0 < x < π].

$$9. \sum_{k=0}^{\infty} \frac{(\mu)_k (\mu + \nu - 1)_k}{k! (\nu)_k} \left\{ \begin{matrix} \sin (\mu + \nu + 2k - 1) x \\ \cos (\mu + \nu + 2k - 1) x \end{matrix} \right\} =$$

$$= \frac{\Gamma(\nu) \pi^{\pm 1/2}}{2\Gamma(\mu + \nu - 1)} (2 \sin x)^{1/2 - \mu} \left\{ \begin{matrix} P_{\nu - 3/2}^{\mu - 1/2}(\cos x) \\ 2Q_{\nu - 3/2}^{\mu - 1/2}(\cos x) \end{matrix} \right\} \quad [0 < x < \pi].$$

$$10. \sum_{k=0}^{\infty} \frac{(2k + 2m)! (k + m + n)! (k + n + 1)!}{k! (k + m)! (2k + 2n + 2)!} \cos (2k + m + n + 1) x =$$

$$= 2^{-2-m-2n} (\sin x)^{-m} Q_n^m(\cos x) \quad [0 < x < \pi; m, n = 0, 1, 2, \dots].$$

$$11. \sum_{k=1}^{\infty} \frac{\sin kx}{k^4 + 4a^4} = \frac{\pi}{16a^4} \left[1 - \frac{\operatorname{ch} ax \cos a(\pi - x) + \cos ax \operatorname{ch} a(\pi - x)}{\operatorname{ch} \pi a + \cos \pi a} \right]$$

$$[0 \leq x \leq \pi].$$

$$12. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{\Gamma(\nu + 1 + k) \Gamma(\nu + 1 - k)} \cos kx =$$

$$= \frac{2^{2\nu-1}}{\Gamma(2\nu + 1)} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\}^{2\nu} - \frac{1}{2\Gamma^2(\nu + 1)} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}; \left\{ \begin{matrix} -\pi < x < \pi \\ 0 < x < 2\pi \end{matrix} \right\} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{\Gamma(\nu + k + 3/2) \Gamma(\nu - k + 1/2)} \left\{ \begin{matrix} \sin (2k + 1) x \\ \cos (2k + 1) x \end{matrix} \right\} =$$

$$= \frac{2^{2\nu-1}}{\Gamma(2\nu + 1)} \left\{ \frac{\sin x}{\cos x} \right\}^{2\nu} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}; \left\{ \begin{matrix} 0 < x < \pi \\ -\pi/2 < x < \pi/2 \end{matrix} \right\} \right].$$

Б.4.9. Ряды вида $\sum \frac{r^k}{(k+a)^s} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\}$.

Условие: $|r| < 1$.

$$1. \sum_{k=0}^{\infty} \frac{r^k}{(k+a)^s} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \frac{1}{2} \begin{Bmatrix} i \\ 1 \end{Bmatrix} [\Phi(re^{-ix}, s, a) \mp \Phi(re^{ix}, s, a)].$$

$$2. \sum_{k=1}^{\infty} \frac{r^k}{k^s} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \frac{r}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \left(e^t \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} - r \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \right)}{1 - 2re^t \cos x + r^2 e^{2t}} dt$$

$[x \neq 0, \operatorname{Re} s > 0]$ или $[|x| \geq 0, \operatorname{Re} s > 1]$.

$$3. \sum_{k=0}^{\infty} r^k \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = (1 - 2r \cos x + r^2)^{-1} \left[\pm r \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} + \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \right].$$

$$4. \sum_{k=0}^{\infty} r^{2k+1} \left\{ \begin{matrix} \sin (2k+1)x \\ \cos (2k+1)x \end{matrix} \right\} = r(1 \pm r^2) [(1 + r^2)^2 - 4r^2 \cos^2 x]^{-1} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}.$$

$$5. \sum_{k=0}^{\infty} (-1)^k r^{2k+1} \left\{ \begin{matrix} \sin (2k+1)x \\ \cos (2k+1)x \end{matrix} \right\} = r(1 \mp r^2) [(1 + r^2)^2 - 4r^2 \sin^2 x]^{-1} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}.$$

$$6. \sum_{k=1}^{\infty} r^k \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{r}{(1 - 2r \cos x + r^2)^{1/2}} \left[(1 - r^2) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - 2r (1 - r \cos x) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right].$$

$$7. \sum_{k=0}^{\infty} \frac{r^k}{km+n} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} =$$

$$= \pm \frac{1}{mr^{n/m}} \sum_{k=0}^{m-1} \begin{Bmatrix} \cos [n(x+2k\pi)/m] \\ \sin [n(x+2k\pi)/m] \end{Bmatrix} \operatorname{arctg} \frac{r^{1/m} \sin [(x+2k\pi)/m]}{1 - r^{1/m} \cos [(x+2k\pi)/m]} \mp$$

$$\pm \frac{1}{2mr^{n/m}} \sum_{k=0}^{m-1} \begin{Bmatrix} \sin [n(x+2k\pi)/m] \\ \cos [n(x+2k\pi)/m] \end{Bmatrix} \ln \left(1 - 2r^{1/m} \cos \frac{x+2k\pi}{m} + r^{2/m} \right) \mp$$

$$\mp \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{1}{r^k (km-n)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$$

* $\delta(n-m) = 1$ при $n > m$, $\delta(n-m) = 0$ при $n \leq m$, $m > 0$; $0 < x < 2\pi$.

$$8. \sum_{k=0}^{\infty} \frac{r^k}{km-n} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} =$$

$$= \frac{r^{n/m}}{m} \sum_{k=0}^{m-1} \begin{Bmatrix} \cos [n(x+2k\pi)/m] \\ \sin [n(x+2k\pi)/m] \end{Bmatrix} \operatorname{arctg} \frac{r^{1/m} \sin [(x+2k\pi)/m]}{1 - r^{1/m} \cos [(x+2k\pi)/m]} \mp$$

$$- \frac{r^{n/m}}{2m} \sum_{k=0}^{m-1} \begin{Bmatrix} \sin [n(x+2k\pi)/m] \\ \cos [n(x+2k\pi)/m] \end{Bmatrix} \ln \left(1 - 2r^{1/m} \cos \frac{x+2k\pi}{m} + r^{2/m} \right) \mp$$

$$+ \sum_{k=0}^{[n/m]} \frac{r^k}{km-n} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} \quad [n/m \neq 0, 1, 2, \dots; m > 0; 0 < x < 2\pi].$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{km \pm n} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \sum_{k=0}^{\infty} \frac{r^k}{km \pm n} \begin{Bmatrix} \sin k(x+\pi) \\ \cos k(x+\pi) \end{Bmatrix} \quad [m > 0; -\pi < x < \pi].$$

$$10. \sum_{k=1}^{\infty} \frac{(\pm 1)^k r^k}{k+n} \sin kx = \frac{(\pm 1)^k}{r^n} \left[\sum_{k=1}^{n-1} \frac{(\pm 1)^{n-k-1} r^{n-k}}{n-k} \sin kx + \right.$$

$$\left. + \cos nx \operatorname{arctg} \frac{r \sin x}{1 \mp r \cos x} \mp \frac{1}{2} \sin nx \ln (1 \mp 2r \cos x + r^2) \right].$$

$$11. \sum_{k=1}^{\infty} \frac{(\pm 1)^k r^k}{k+n} \cos kx = \frac{(\pm 1)^n}{r^n} \left[\mp \sum_{k=1}^n \frac{(\pm 1)^{n-k-1} r^{n-k-1}}{n-k+1} \cos (k-1)x + \right.$$

$$\left. + \sin nx \operatorname{arctg} \frac{r \sin x}{1 \mp r \cos x} \mp \frac{1}{2} \cos nx \ln (1 \mp 2r \cos x + r^2) \right].$$

$$12. \sum_{k=1}^{\infty} \frac{r^k}{k} \sin kx = \operatorname{arctg} \left(\frac{r \sin x}{1 - r \cos x} \right).$$

$$13. \sum_{k=1}^{\infty} \frac{r^k}{k} \cos kx = -\frac{1}{2} \ln (1 - 2r \cos x + r^2).$$

$$14. \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \frac{1}{2} \cos \frac{x}{2} \left\{ \begin{matrix} \operatorname{arctg} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \sin (x/2) \\ \cos (x/2) \end{matrix} \right\} \right] \\ \operatorname{Arth} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \sin (x/2) \\ \cos (x/2) \end{matrix} \right\} \right] \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2} \sin \frac{x}{2} \left\{ \begin{matrix} \operatorname{Arth} \left[\frac{2r}{1 \pm r^2} \left\{ \begin{matrix} \cos (x/2) \\ \sin (x/2) \end{matrix} \right\} \right] \\ \operatorname{arctg} \left[\frac{2r}{1 \pm r^2} \left\{ \begin{matrix} \cos (x/2) \\ \sin (x/2) \end{matrix} \right\} \right] \end{matrix} \right\}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \frac{1}{2} \cos \frac{x}{2} \left\{ \begin{matrix} \operatorname{Arth} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \sin (x/2) \\ \cos (x/2) \end{matrix} \right\} \right] \\ \operatorname{arctg} \left[\frac{2r}{1 \pm r^2} \left\{ \begin{matrix} \sin (x/2) \\ \cos (x/2) \end{matrix} \right\} \right] \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2} \sin \frac{x}{2} \left\{ \begin{matrix} \operatorname{arctg} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \cos (x/2) \\ \sin (x/2) \end{matrix} \right\} \right] \\ \operatorname{Arth} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \cos (x/2) \\ \sin (x/2) \end{matrix} \right\} \right] \end{matrix} \right\}.$$

$$16. \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \left\{ \begin{matrix} \sin (2k+1)x \\ \cos (2k+1)x \end{matrix} \right\} = \frac{1}{2} \operatorname{arctg} \left[\frac{2r}{1 \mp r^2} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right].$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1} \left\{ \begin{matrix} \sin (2k+1)x \\ \cos (2k+1)x \end{matrix} \right\} = \frac{1}{2} \operatorname{arctg} \left[\frac{2r}{1 \pm r^2} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} \right].$$

5.4.10. Р я д ы в и д а $\sum_{k=0}^{\infty} \frac{r^k}{k!} \left\{ \begin{matrix} \sin (kx+a) \\ \cos (kx+a) \end{matrix} \right\}.$

$$1. \sum_{k=0}^{\infty} \frac{r^k}{k!} \left\{ \begin{matrix} \sin (kx+a) \\ \cos (kx+a) \end{matrix} \right\} = e^{r \cos x} \left\{ \begin{matrix} \sin (r \sin x + a) \\ \cos (r \sin x + a) \end{matrix} \right\}.$$

$$2. \sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \left\{ \begin{matrix} \operatorname{sh} [r \cos (x/2)] \sin [r \sin (x/2)] \\ \operatorname{ch} [r \cos (x/2)] \cos [r \sin (x/2)] \end{matrix} \right\}.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{(2k)!} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \mp \left\{ \begin{matrix} \operatorname{sh} [r \sin (x/2)] \sin [r \cos (x/2)] \\ \operatorname{ch} [r \sin (x/2)] \cos [r \cos (x/2)] \end{matrix} \right\}.$$

$$4. \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \left\{ \begin{matrix} \operatorname{ch} [r \cos (x/2)] \sin [r \sin (x/2)] \\ \operatorname{sh} [r \cos (x/2)] \cos [r \sin (x/2)] \end{matrix} \right\} \cos \frac{x}{2} \mp$$

$$\mp \left\{ \begin{matrix} \operatorname{sh} [r \cos (x/2)] \cos [r \sin (x/2)] \\ \operatorname{ch} [r \cos (x/2)] \sin [r \sin (x/2)] \end{matrix} \right\} \sin \frac{x}{2}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{(2k+1)!} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\} = \left[\begin{matrix} \operatorname{sh} [r \sin (x/2)] \cos [r \cos (x/2)] \\ \operatorname{ch} [r \sin (x/2)] \sin [r \cos (x/2)] \end{matrix} \right] \cos \frac{x}{2} \mp$$

$$\mp \left\{ \begin{matrix} \operatorname{ch} [r \sin (x/2)] \sin [r \cos (x/2)] \\ \operatorname{sh} [r \sin (x/2)] \cos [r \cos (x/2)] \end{matrix} \right\} \sin \frac{x}{2}.$$

$$6. \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \begin{Bmatrix} \sin (r \sin x) \operatorname{ch} (r \cos x) \\ \cos (r \sin x) \operatorname{sh} (r \cos x) \end{Bmatrix}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{(2k+1)!} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \begin{Bmatrix} \cos (r \cos x) \operatorname{sh} (r \sin x) \\ \sin (r \cos x) \operatorname{ch} (r \sin x) \end{Bmatrix}.$$

$$8. \sum_{k=0}^{\infty} \frac{r^k}{(nk)!} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{n} \exp \left(r^{1/n} \cos \frac{x}{n} \right) \begin{Bmatrix} \sin [r^{1/n} \sin (x/n)] \\ \cos [r^{1/n} \sin (x/n)] \end{Bmatrix} \mp \\ \mp \frac{1}{2n} [1 + (-1)^n] \exp \left(-r^{1/n} \cos \frac{x}{n} \right) \begin{Bmatrix} \sin [r^{1/n} \sin (x/n)] \\ \cos [r^{1/n} \sin (x/n)] \end{Bmatrix} + \\ + \frac{1}{n} \sum_{k=1}^{[(n-1)/2]} \left[\exp \left(r^{1/n} \cos \frac{2k\pi - x}{n} \right) \begin{Bmatrix} \sin \left(r^{1/n} \sin \frac{2k\pi + x}{n} \right) \\ \cos \left(r^{1/n} \sin \frac{2k\pi + x}{n} \right) \end{Bmatrix} \mp \right. \\ \left. \mp \exp \left(r^{1/n} \cos \frac{2k\pi - x}{n} \right) \begin{Bmatrix} \sin \left(r^{1/n} \sin \frac{2k\pi - x}{n} \right) \\ \cos \left(r^{1/n} \sin \frac{2k\pi - x}{n} \right) \end{Bmatrix} \right].$$

$$9. \sum_{k=0}^{\infty} \frac{r^{2k}}{(k!)^2} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} i \\ 1 \end{Bmatrix} [I_0(2re^{-ix/2}) \mp I_0(2re^{ix/2})].$$

$$10. \sum_{k=0}^{\infty} \frac{r^{2k+1}}{k!(k+1)!} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} i \\ 1 \end{Bmatrix} [I_1(2re^{-ix}) \mp I_1(2re^{ix})].$$

5.4.11. Ряды вида $\sum a_k q^{a(k)} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

Условие: $|q| < 1$.

$$1. \sum_{k=0}^{\infty} (\mp 1)^k q^{k(k+1)} \begin{Bmatrix} \sin (2k+1) x \\ \cos (2k+1) x \end{Bmatrix} = \frac{1}{2} q^{-1/4} \begin{Bmatrix} \theta_1(x, q) \\ \theta_2(x, q) \end{Bmatrix}.$$

$$2. \sum_{k=0}^{\infty} (\pm 1)^k q^{k^2} \cos kx = \frac{1}{2} + \frac{1}{2} \begin{Bmatrix} \theta_3(x/2, q) \\ \theta_4(x/2, q) \end{Bmatrix}.$$

5.4.12. Ряды вида $\sum a_k e^{-ka} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$.

$$1. \sum_{k=1}^{\infty} (\pm 1)^k e^{-ka} \sin kx = -\frac{1}{2} \frac{\sin x}{\cos x \mp \operatorname{ch} a}.$$

$$2. \sum_{k=1}^{\infty} (\pm 1)^k e^{-ka} \cos kx = \frac{1}{2} \frac{\operatorname{sh} a}{\operatorname{ch} a \mp \cos x} - \frac{1}{2}.$$

$$3. \sum_{k=0}^{\infty} (\pm 1)^k e^{-(2k+1)a} \sin (2k+1)x = \frac{\sin x}{\operatorname{ch} 2a \pm \cos 2x} \begin{Bmatrix} \operatorname{ch} a \\ \operatorname{sh} a \end{Bmatrix}.$$

$$4. \sum_{k=0}^{\infty} (\pm 1)^k e^{-(2k+1)a} \cos (2k+1)x = \frac{\cos x}{\operatorname{ch} 2a \mp \cos 2x} \begin{Bmatrix} \operatorname{sh} a \\ \operatorname{ch} a \end{Bmatrix}.$$

$$5. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k} e^{-ka} \sin kx = -\operatorname{arctg} \frac{\sin x}{\cos x \mp e^a}.$$

$$6. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k} e^{-ka} \cos kx = \frac{a}{2} - \frac{1}{2} \ln [2 (\operatorname{ch} a \mp \cos x)].$$

$$7. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{2k+1} e^{-(2k+1)a} \begin{Bmatrix} \sin (2k+1)x \\ \cos (2k+1)x \end{Bmatrix} = \frac{1}{2} \operatorname{arctg} \left[\frac{1}{\operatorname{sh} a} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right].$$

$$8. \sum_{k=0}^{\infty} \frac{(\mp 1)^k}{2k+1} e^{-(2k+1)a} \begin{Bmatrix} \sin (2k+1)x \\ \cos (2k+1)x \end{Bmatrix} = \\ = \frac{1}{4} \ln \left(\operatorname{ch} a + \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right) - \frac{1}{4} \ln \left(\operatorname{ch} a - \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right).$$

5.4.13. Ряды вида $\sum a_k r^k \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}^m$.

$$1. \sum_{k=1}^{\infty} \frac{r^k}{k} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}^{2m} = \\ = \frac{+1}{2^{2m}} \sum_{k=1}^m (-1)^{k-1} \binom{2m}{m-k} \ln (1 - 2r \cos kx + r^2) - \frac{1}{2^{2m}} \binom{2m}{m} \ln (1-r) \\ [-1 \leq r < 1].$$

$$2. \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} r^k}{k} \sin^{2m+1} kx = \\ = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{m-k} \operatorname{arctg} \frac{r \sin (2k+1)x}{1 \mp r \cos (2k+1)x} \quad [-1 \leq r \leq 1].$$

$$3. \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} r^k}{k} \cos^{2m+1} kx = \\ = -\frac{1}{2^{2m+1}} \sum_{k=0}^m \binom{2m+1}{m-k} \ln [1 \mp 2r \cos (2k+1)x + r^2] \quad [-1 \leq r \leq 1].$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k-1} r^k}{k} \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\}^{2m} =$$

$$= \frac{1}{2^{2m}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \ln(1 + 2r \cos 2kx + r^2) + \frac{1}{2^{2m}} \binom{2m}{m} \ln(1+r)$$

[$-1 < r \leq 1$].

$$5. \sum_{k=0}^{\infty} \frac{r^k}{k!} \left\{ \begin{matrix} \sin(kx+a) \\ \cos(kx+a) \end{matrix} \right\}^m = (-1)^{[m/2]} 2^{1-m} m! \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k! (m-k)!} \times$$

$$\times \exp[r \cos(m-2k)x] \cos[a \sin[(m-2k)x] + (m-2k)a - \frac{\pi}{4} \left\{ \frac{1 - (-1)^m}{0} \right\}].$$

5.4.14. Ряды вида $\sum \frac{a_n q^n}{1 \pm q^{mn}} \left\{ \begin{matrix} \sin nx \\ \cos nx \end{matrix} \right\}$.

Обозначения: $q = e^{-\pi K'(k)/K(k)}$, $u = \frac{2K(k)x}{\pi}$, $Q_0 = \prod_{n=0}^{\infty} (1 - q^{2n})$.

Условия: $|q| < 1$, $|k| < 1$.

$$1. \sum_{k=1}^{\infty} \frac{(\pm 1)^n q^n}{1 + q^n} \sin 2nx = \pm \frac{1}{4} \left\{ \begin{matrix} \operatorname{ctg} x \\ \operatorname{tg} x \end{matrix} \right\} \mp \frac{1}{2\pi} K(k) \left\{ \begin{matrix} \operatorname{cn} u \operatorname{dn} u \operatorname{ns} u \\ (1 - k^2) \operatorname{sn} u \operatorname{nc} u \operatorname{nd} u \end{matrix} \right\}$$

[$|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$2. \sum_{n=1}^{\infty} \frac{(\pm 1)^n q^n}{1 + (-q)^n} \sin 2nx = \mp \frac{1}{4} \left\{ \begin{matrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{matrix} \right\} \pm \frac{1}{2\pi} K(k) \left\{ \begin{matrix} \operatorname{sn} u \operatorname{dn} u \operatorname{nc} u \\ \operatorname{cn} u \operatorname{nd} u \operatorname{ns} u \end{matrix} \right\}$$

[$2|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$3. \sum_{k=1}^{\infty} \frac{(\pm 1)^n q^n}{1 - q^{2n}} \sin 2nx = \frac{1}{4} \frac{\{\theta_4(x, q)\}^{-1}}{\{\theta_3(x, q)\}} \frac{\partial}{\partial x} \frac{\{\theta_4(x, q)\}}{\{\theta_3(x, q)\}}$$

[$2|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$4. \sum_{k=1}^{\infty} \frac{(\pm 1)^n q^n}{1 + q^{2n}} \cos 2nx = \frac{1}{2\pi} K(k) \left\{ \frac{\operatorname{dn} u}{1 - k^2 \operatorname{nd} u} \right\} - \frac{1}{4}$$

[$2|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$5. \sum_{k=1}^{\infty} \frac{(\pm 1)^n q^{2n}}{1 + q^{2n}} \sin 2nx = \pm \frac{1}{4} \left\{ \begin{matrix} \operatorname{ctg} x \\ \operatorname{tg} x \end{matrix} \right\} \mp \frac{1}{2\pi} K(k) \left\{ \frac{\operatorname{cs} u}{1 - k^2 \operatorname{sc} u} \right\}$$

[$|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$6. \sum_{n=1}^{\infty} \frac{(\pm 1)^n q^{2n}}{1 - q^{2n}} \sin 2nx = \frac{1}{4} \frac{\{\theta_1(x, q)\}^{-1}}{\{\theta_2(x, q)\}} \frac{\partial}{\partial x} \frac{\{\theta_1(x, q)\}}{\{\theta_2(x, q)\}} \mp \frac{1}{4} \left\{ \begin{matrix} \operatorname{ctg} x \\ \operatorname{tg} x \end{matrix} \right\}$$

[$2|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

$$7. \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos 2nx = \frac{1}{2\pi^2} K^2(k) (1 - k^2 \operatorname{sn}^2 u) - \frac{1}{2\pi^2} K(k) E(k)$$

[$2|\operatorname{Im} x| < \operatorname{Re}(-\ln q)$].

8.
$$\sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1 - q^{2n}} \cos 2nx = \frac{1}{2\pi^2} \{K(k) [K(k) - E(k)] - [kK(k) \operatorname{cn} u \operatorname{nd} u]\}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
9.
$$\sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \cos 2nx = \frac{1}{8} \operatorname{cosec}^2 x + \frac{1}{2\pi^2} K(k) \{K(k) [1 - \operatorname{ns}^2 u] - E(k)\}$$

 $[| \operatorname{Im} x | < \operatorname{Re}(\frac{1}{2} \ln q)].$
10.
$$\sum_{n=1}^{\infty} \frac{q^n}{n(1 + q^{2n})} \sin 2nx = \frac{1}{2} \operatorname{am} u - \frac{x}{2}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
11.
$$\sum_{n=1}^{\infty} \frac{q^n}{n(1 + q^n)} \cos 2nx = \frac{1}{2} \ln \left(\frac{1}{2} k^{1/2} q^{-1/4} \operatorname{cosec} x \operatorname{sn} u \right)$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
12.
$$\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n [1 + (-1)^n q^n]} \cos 2nx =$$

$$= \frac{1}{2} \ln \left\{ \frac{\operatorname{sn} u}{\operatorname{cn} u} \right\} - \frac{1}{2} \ln \left\{ \frac{\sin x}{\sqrt{1 - k^2} \cos x} \right\} - \frac{1}{2} \ln (2q^{1/4} k^{-1/2})$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
13.
$$\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n(1 - q^{2n})} \cos 2nx = \frac{1}{2} \ln Q_0 - \frac{1}{2} \ln \left\{ \frac{\theta_4(x/\pi)}{\theta_3(x/\pi)} \right\}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
14.
$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{n(1 - q^{2n})} \cos 2nx = \frac{1}{2} \ln 2Q_0 + \frac{1}{8} \ln q + \frac{1}{2} \left\{ \ln \sin x - \ln \theta_1(x/\pi) \right\}$$

$$- \frac{1}{2} \left\{ \ln \cos x - \ln \theta_2(x/\pi) \right\}$$

 $[| \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
15.
$$\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + q^{2n+1}} \sin (2n+1)x = \frac{kK(k)}{2\pi \sqrt{q}} \left\{ \frac{\operatorname{sn} u}{\sqrt{1 - k^2} \operatorname{sd} u} \right\}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
16.
$$\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 + q^{2n+1}} \cos (2n+1)x = \frac{kK(k)}{2\pi \sqrt{q}} \left\{ \frac{\operatorname{cn} u}{\operatorname{cd} u} \right\}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(\frac{1}{2} \ln q)].$
17.
$$\sum_{n=1}^{\infty} \frac{q^{2n+1}}{1 + q^{2n+1}} \sin (2n+1)x = \pm \frac{1}{4} \operatorname{cosec} x + \frac{1}{2\pi} K(k) \left\{ \frac{\operatorname{ds} u}{\operatorname{ns} u} \right\}$$

 $[| \operatorname{Im} x < \operatorname{Re}(-\ln q)].$
18.
$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1 + q^{2n+1}} \cos (2n+1)x = \pm \frac{1}{4} \sec x + \frac{1}{2\pi} K(k) \left\{ \frac{\sqrt{1 - k^2} \operatorname{nc} u}{\operatorname{dc} u} \right\}$$

 $[2 | \operatorname{Im} x < \operatorname{Re}(-\ln q)].$

5.4.15. Ряды вида $\sum a_k \prod \sin kx, \prod \cos ky_l$.

$$1. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k} \sin kx \sin ky = \frac{1}{4} \ln \left\{ \frac{\sin [(x+y)/2] \operatorname{cosec} [(x-y)/2]}{\cos [(x+y)/2] \sec [(x-y)/2]} \right\}^2$$

$$\left[\begin{array}{l} \{x \pm y \neq 0, 2\pi, 4\pi, \dots\} \\ \{x \pm y \neq \pi, 3\pi, 5\pi, \dots\} \end{array} \right].$$

$$2. \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k} \cos kx \cos ky = -\frac{1}{4} \ln [4 (\cos x \mp \cos y)^2]$$

$$\left[\begin{array}{l} \{x \pm y \neq 0, 2\pi, 4\pi, \dots\} \\ \{x \pm y \neq \pi, 3\pi, 5\pi, \dots\} \end{array} \right].$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \cos ky = \begin{cases} -1/2, & 0 < x < y, \\ (\pi - 2y)/4, & x = y, \\ (\pi - x)/2, & y < x < \pi \end{cases} \quad [0 < y < \pi].$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx \sin ky = \begin{cases} x(\pi - y)/2, & -y \leq x \leq y, \\ y(\pi - x)/2, & y \leq x \leq 2\pi - y \end{cases} \quad [0 < y < \pi].$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \sin kx \sin ky = \frac{xy}{2} \quad [|x \pm y| \leq \pi].$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k^2} \cos kx \cos ky = \frac{1}{12} \begin{cases} 3x^2 + 3(y - \pi)^2 - \pi^2, & 0 \leq x \leq y, \\ 3y^2 + 3(x - \pi)^2 - \pi^2, & y \leq x \leq \pi \end{cases} \quad [0 < y < \pi].$$

$$7. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \cos ky = \frac{1}{12} \begin{cases} 3(x^2 + y^2) - \pi^2, & -(\pi - y) \leq x \leq \pi - y, \\ 3(x - \pi)^2 + 3(y - \pi)^2 - \pi^2, & \pi - y \leq x \leq \pi + y \end{cases}$$

$$[0 < y < \pi].$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sin kx \cos ky = \frac{x}{12} (\pi^2 - x^2 - 3y^2) \quad [|x \pm y| \leq \pi].$$

$$9. \sum_{k=1}^{\infty} \frac{a^k}{k} \sin kx \sin ky = \frac{1}{4} \ln \frac{4a \sin^2 [(x+y)/2] + (a-1)^2}{4a \sin^2 [(x-y)/2] + (a-1)^2} \quad [0 < a < 1].$$

$$10. \sum_{k=1}^{\infty} \frac{(\pm 1)^k q^k}{k(1-q^{2k})} \sin kx \sin ky = \frac{1}{4} \ln \frac{\theta_i [(x+y)/2, q]}{\theta_i [(x-y)/2, q]} \quad \left[i = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right].$$

$$11. \sum_{k=1}^{\infty} \frac{(\pm 1)^k q^{2k}}{k(1-q^{2k})} \sin kx \sin ky =$$

$$= \frac{1}{4} \ln \frac{\theta_i [(x+y)/2, q]}{\theta_i [(x-y)/2, q]} - \frac{1}{4} \ln \left\{ \frac{\sin [(x+y)/2] \operatorname{cosec} [(x-y)/2]}{\cos [(x+y)/2] \sec [(x-y)/2]} \right\}$$

$$\left[i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; |\operatorname{Im} x| + |\operatorname{Im} y| < -\frac{1}{2} \operatorname{Re} \ln q \right].$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \sin^2 ky = \begin{cases} \pi/4, & 0 < x < 2y, \\ 0, & 2y < x < \pi \end{cases} \quad [0 < x; y < \pi].$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin ky \sin ky \cos kz = \\ = \frac{1}{4} \begin{cases} 2(\pi - y)x, & 0 < x < y - z, \\ (x + y - z)\pi - 2xy, & y - z < x < y + z, \\ 2(\pi - x)y, & y + z < x < \pi \end{cases} \quad [0 < z < y < \pi/2].$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} \sin kx \sin ky \sin kz = \frac{xyz}{2}.$$

[Точка (x, y, z) лежит внутри многогранника $|x| + |y| + |z| < \pi$].

$$15. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2 kx \sin^2 ky = \frac{\pi x}{2} \quad [0 \leq x \leq y \leq \pi/2].$$

$$16. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin^4 kx \sin^2 ky = \frac{\pi x}{8} \quad [0 \leq y \pm 2x \leq \pi].$$

$$17. \sum_{k=1}^{\infty} \frac{1}{k^4} \sin^4 kx \sin^2 ky = \frac{\pi x^3}{6}.$$

$$18. \sum_{k=1}^{\infty} \frac{1}{k^n} \sin^n kx = -\frac{\pi^n}{n!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} B_n \left[(n-2k) \frac{x}{2\pi} \right] - \\ - \frac{1}{4} (-1)^{n/2} [1 + (-1)^n] \left[\left(\frac{n}{2} \right)! \right]^2 \pi^n B_n \\ [0 < x < 2\pi \text{ при } n=1; 0 \leq x \leq 2\pi/n \text{ при } n > 1].$$

$$19. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin^{n+2m} kx = \frac{x^n}{2} \delta_{m,0} \quad [|x| < \pi/(2m+n); n=1, 2, 3, \dots].$$

$$20. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \prod_{j=1}^{n+2m} \sin kx_j = \frac{1}{2} \delta_{m,0} \prod_{j=1}^n x_j \\ \left[n \geq 1; \sum_{j=1}^n |x_j| < \pi \text{ при } m=0; \sum_{j=1}^{n+2m} |x_j| \leq \pi \text{ при } m > 0 \right]$$

$$21. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \left(\prod_{j=1}^{n+2m} \sin kx_j \right) \left(\prod_{l=1}^p \cos ky_l \right) = \frac{1}{2} \delta_{m,0} \prod_{j=1}^n x_j \\ \left[n \geq 1; \sum_{j=1}^{n+2m} |x_j| + \sum_{l=1}^p |y_l| < \pi \right]$$

5.4.16. Разные ряды, содержащие тригонометрические функции.

$$1. \sum_{k=1}^{\infty} 3^k \sin^3 \frac{x}{3^k} = \frac{3}{4} (x - \sin x).$$

$$2. \sum_{k=1}^{\infty} 2^{2k} \sin^4 \frac{x}{2^k} = x^2 - \sin^2 x.$$

$$3. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \sin \frac{m\pi}{2k+1} \right) x^k = \frac{x(12-x)}{(4-x)^2} \quad [|x| < 4].$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^k}{3^k} \cos^3 3^k x = \frac{3}{4} \cos x.$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}} \cos^4 2^k x = \frac{1}{20} - \frac{1}{4} \cos^2 2x.$$

$$6. \sum_{k=1}^{\infty} \frac{\cos kx \sec^k x}{2^k k^2} = \frac{\pi^2}{12} - \frac{x^2}{2} - \frac{1}{2} \ln^2 (2 \cos x) \quad [-\pi/3 < x < \pi/3].$$

$$7. \sum_{k=1}^{\infty} \frac{2^k \cos^k x}{k^2} \cos kx = \left(x - \frac{\pi}{2} \right)^2 \quad [\pi/3 < x < 2\pi/3].$$

$$8. \sum_{k=1}^{\infty} \prod_{m=1}^{2k} \cos^l \frac{m\pi}{2k+1} = \frac{1}{2^l - 1} [1 + (-1)^l 2^{2l}].$$

$$9. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \cos \frac{m\pi}{2k+1} \right) x^k = -\frac{x}{4+x} \quad [|x| < 4].$$

$$10. \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \sec^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{x^2}.$$

$$11. \sum_{n=-\infty}^{\infty} (\pm 1)^n \operatorname{cosec} (nx + a) = \pm \frac{2k}{\pi} K(k) \begin{cases} \operatorname{sn} u \\ \operatorname{cn} u \end{cases}$$

$[x = -i\pi K'(k)/K(k), u = iK'(k) - 2aK(k)/\pi; |\operatorname{Im}(u/K(k))| < \operatorname{Re}(K'(k)/K(k))].$

$$12. \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \operatorname{cosec} (k\pi \sqrt{2}) =$$

$$= \frac{(-1)^n \pi^{2n+1}}{2(\sqrt{2}-1)[1+(\sqrt{2}-1)^{2n}]} \sum_{k=0}^{n+1} \frac{(2^{2k}-2)(2^{2n-2k+2}-2)}{(2k)!(2n-2k+2)!} (\sqrt{2}-1)^{2k} B_{2k} B_{2n-2k+2}$$

$[n=1, 2, 3, \dots]$

$$13. \sum_{k=1}^{\infty} \frac{1}{k^3} \operatorname{cosec} (k\pi \sqrt{2}) = -\frac{13\pi^3}{360 \sqrt{2}}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{2^k} \operatorname{tg} \frac{x}{2^k} = \frac{1}{x} - \operatorname{ctg} x.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \operatorname{tg} \frac{2k+1}{2} \pi \sqrt{3} = \frac{\pi^3}{96} (3 - 2\sqrt{3}).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \operatorname{tg}^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{x^2} - \frac{1}{3}.$$

$$17. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \operatorname{tg} \frac{m\pi}{2k+1} \right) x^k = -\frac{x(3+x)}{(1+x)^2} \quad [x < 1].$$

$$18. \sum_{l=-\infty}^{\infty} (-1)^l \operatorname{ctg} (lx+a) = \frac{2}{\pi} K(k) \operatorname{dn} u \quad [\text{см условие к 5.4.16 11}].$$

$$19. \sum_{k=1}^{\infty} \frac{1}{k^3} \operatorname{ctg} \left(\frac{1+\sqrt{5}}{2} k\pi \right) = -\frac{\pi^3}{45 \sqrt{5}}.$$

$$20. \sum_{k=1}^{\infty} \prod_{n=1}^{2k} \operatorname{ctg} \frac{n\pi}{2k+1} = \frac{\pi}{4} - 1.$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k} \cos kx \operatorname{tg} \frac{k\pi}{2n+1} = \frac{\pi}{2} \begin{cases} -1/(2n+1), & x \text{ кратно } 2\pi/(2n+1), \\ (-1)^m - 1/(2n+1), & x \text{ не кратно } 2\pi/(2n+1) \end{cases}$$

[m — целая часть $(2n+1)x/(2\pi)$; $n \geq 1$; $0 < x < 2\pi$].

5.5. РЯДЫ С ЛОГАРИФИЧЕСКОЙ И ОБРАТНЫМИ ТРИГОНОМЕТРИЧЕСКИМИ ФУНКЦИЯМИ

5.5.1. Ряды вида $\sum a_k \ln b_k$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^s} \ln ka = \zeta(s) \ln a - \zeta'(s).$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} \ln k = 2^{1-s} \zeta(s) \ln 2 + (1 - 2^{1-s}) \zeta'(s).$$

$$3. \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \ln k = C \ln 2 - \frac{\ln^2 2}{2}. \quad 4. \sum_{k=2}^{\infty} \frac{(-1)^k}{k+1} \ln k = 0,096615.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} \ln(2k+1) = -2^{-s} \zeta(s) \ln 2 - (1-2^{-s}) \zeta'(s).$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \ln(2k+1) a = \frac{\pi}{4} \ln a - \frac{\pi C}{4} - \frac{1}{4} \int_0^{\infty} \ln t \operatorname{sech} t dt.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \ln[(2k+1)^2 a^2 + 1] = \frac{\pi}{2} \int_1^{\infty} \frac{1}{t} \operatorname{sech} \frac{t\pi}{2a} dt.$$

$$8. \sum_{k=1}^{\infty} \frac{1}{k} x^k \ln(k+1) = - \int_0^1 \ln \left(\frac{1-xt}{1-x} \right) \frac{dt}{\ln t} \quad [|x| < 1].$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^{a-1} - t^{b-1}) \ln(1+xt) \frac{dt}{\ln t}.$$

$$10. \sum_{k=1}^n (-1)^k \binom{n}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^{a-1} - t^{b-1}) (1-xt)^n \frac{dt}{\ln t}.$$

$$11. \sum_{k=1}^{\infty} \binom{n+k-1}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^a - t^b) (1-xt)^{-n} \frac{dt}{t \ln t}.$$

$$12. \sum_{k=1}^{\infty} \frac{\Gamma(k+a)}{k!} \ln(k+1) = \Gamma(a) \int_0^1 (1-t)^{-a} \frac{dt}{\ln t} \quad [a \leq -1].$$

$$13. \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k! 2^k (2k+1)} \ln(k+1) a = \frac{\pi}{2} \ln \frac{a}{2} - \int_0^1 \frac{\arccos t}{\ln t} dt.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \ln(2k+1) = \frac{1}{8} (\pi - 2C - 2 \ln 2\pi) \left[\zeta\left(\frac{1}{2}, \frac{1}{4}\right) - \zeta\left(\frac{1}{2}, \frac{3}{4}\right) \right].$$

$$15. \sum_{k=1}^{\infty} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = Ca + \ln \Gamma(a+1).$$

$$16. \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = \ln(\pi a \operatorname{cosec} a\pi).$$

$$17. \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (-1)^{k+1} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = \ln \left(\frac{2}{\pi a} \operatorname{tg} \frac{a\pi}{2} \right).$$

$$18. \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln \frac{k+a+1}{k+a} \right) = C + \ln(a+1).$$

$$19. \sum_{k=0}^{\infty} \left(\frac{1}{k+a} - \ln \frac{k+a+1}{k+a} \right) = \ln a - \psi(a).$$

$$20. \sum_{k=1}^{\infty} \left(\frac{1}{k} - 2 \ln \frac{2k+2}{2k+1} \right) = C - \ln \frac{\pi}{4}.$$

$$21. \sum_{k=0}^{\infty} \left[1 - (k+1) \ln \frac{2k+3}{2k+1} \right] = \frac{1}{2} (\ln 2 - 1).$$

$$22. \sum_{k=0}^{\infty} [2 - (2k+2a+1)] \ln \frac{k+a+1}{k+a} = \ln 2\pi + (2a-1) \ln a - 2 \ln \Gamma(a) - 2a.$$

$$23. \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{2k-a+1} \ln(2k-a+1)\pi - \frac{1}{2k+a+1} \ln(2k+a+1)\pi \right] =$$

$$= \frac{\pi}{2} (\ln \pi - C) \left\{ \frac{\operatorname{tg}(\pi a/2)}{\sec(\pi a/2)} \right\} - \int_0^{\pi/4} (\ln \ln \operatorname{ctg} t) \left\{ \frac{\sec 2t}{1} \right\} [\operatorname{ctg}^a t - \operatorname{tg}^a t] dt.$$

$$24. \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \ln ka =$$

$$= \frac{1}{2} (x - \pi) \left(C - \ln \frac{a}{2\pi} \right) + \frac{\pi}{2} \ln \left| \frac{1}{\pi} \sin \frac{x}{2} \right| + \pi \ln \Gamma \left(\frac{x}{2\pi} \right) \quad [0 < x < 2\pi].$$

$$25. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx \ln ka = \frac{x}{2} \left(C - \ln \frac{a}{2\pi} \right) + \frac{\pi}{2} \ln \Gamma \left[\frac{(\pi+x)/(2\pi)}{(\pi-x)/(2\pi)} \right]$$

$$[-\pi < x < \pi].$$

$$26. \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x \ln(2k+1)a =$$

$$= \frac{\pi}{4} \left(\ln \frac{a}{4\pi} - C + \ln \operatorname{tg} \frac{x}{2} - 2 \ln \Gamma \left[\frac{(\pi+x)/(2\pi)}{x/(2\pi)} \right] \right) \quad [0 < x < \pi].$$

$$27. \sum_{k=0}^{\infty} \sin(2k+1)x \ln \frac{k}{k+1} = \frac{\pi}{2} \cos x + \left[\psi \left(\frac{x}{\pi} \right) + C + \ln 2\pi \right] \sin x$$

$$[0 < x < \pi].$$

5.5.2. Ряды вида $\sum a_k \operatorname{arctg} b_k$.

$$1. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \operatorname{arctg} ka = -\frac{a}{2} + \frac{\pi}{2} \int_1^{\infty} \frac{1}{t} \operatorname{cosech} \frac{t\pi}{a} dt.$$

$$2. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{a}{2k+1} = \frac{1}{2} \operatorname{arctg} \left(\operatorname{sh} \frac{a\pi}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \left[\operatorname{arctg} \left(\frac{a}{k+b} \right) - \frac{a}{k+b} \right] = a\psi(b) - \arg \Gamma(b+ia)$$

[$b+ia \neq 0, -1, -2, \dots$].

$$4. \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (\pm 1)^k \left[\operatorname{arctg} \left(\frac{a}{k+b} \right) - \frac{a}{k} \right] = \begin{cases} \operatorname{arctg} (\operatorname{th} a\pi \operatorname{ctg} b\pi) \\ \operatorname{arctg} (\operatorname{sh} a\pi \operatorname{cosec} b\pi) \end{cases} - \operatorname{arctg} \frac{a}{b}.$$

$$5. \sum_{k=1}^{\infty} (\pm 1)^k \operatorname{arctg} \frac{2ab}{k^2 + a^2 - b^2} = \operatorname{arctg} \frac{a}{b} - \begin{cases} \operatorname{arctg} (\operatorname{th} a\pi \operatorname{ctg} b\pi) \\ \operatorname{arctg} (\operatorname{sh} a\pi \operatorname{cosec} b\pi) \end{cases}.$$

$$6. \sum_{k=0}^{\infty} \operatorname{arctg} \frac{a}{k^2 + (2b+1)k + a^2 + b^2 + b} = \operatorname{arctg} \frac{a}{b}.$$

$$7. \sum_{k=1}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+1)a}{k^2 + 1 - a^2} = \operatorname{arctg} a - \pi.$$

$$8. \sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{arctg} \frac{4ka}{4k^2 - a^2 - 1} = \operatorname{arctg} a.$$

$$9. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+2b+1)a}{k^2 + (2b+1)k + b^2 + b - a^2} = \operatorname{arctg} \frac{a}{b}.$$

$$10. \sum_{k=1}^{\infty} \operatorname{arctg} \frac{ac}{(k+b)^2 - c(k+b) + a^2} = \arg [\Gamma(b+ia) \Gamma(b-c+1-ia)].$$

$$11. \sum_{k=1}^{\infty} (\pm 1)^k \operatorname{arctg} \frac{2}{k^2} = \pm \frac{\pi}{4} \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}.$$

$$12. \sum_{k=1}^{\infty} \operatorname{arctg} \frac{1}{k^2 + k + 1} = \frac{\pi}{4}.$$

$$13. \sum_{k=0}^{\infty} \operatorname{arctg} \frac{2ab}{(2k+1)^2 + a^2 - b^2} = \operatorname{arctg} (\operatorname{th} a\pi \operatorname{tg} b\pi).$$

$$14. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+1)a}{(2k+1)^2 - a^2} = \operatorname{arctg} \left(\operatorname{sh} \frac{a\pi}{4} \sec \frac{\sqrt{3}a\pi}{4} \right).$$

$$15. \sum_{k=0}^{\infty} (-1)^k (2k+1) \operatorname{arctg} \frac{8n^2}{(2k+1)^2} = 4n (-1)^{n-1} \operatorname{arctg} (e^{-n\pi}).$$

$$16. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} q^{n+1/2} = \frac{1}{4} \arcsin k \quad \left[|q| < 1; q = \exp \left\{ -\frac{\pi K'(k)}{K(k)} \right\} \right].$$

5.6. КРАТНЫЕ РЯДЫ

$$1. \sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2l)^k} = \ln 2. \quad 2. \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{1}{(l+a)^k} = \frac{1}{1+a}.$$

$$3. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-1)^{2k}} = \frac{1}{4} \ln 2.$$

$$4. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-1)^{2k+1}} = \frac{\pi}{8} - \frac{1}{2} \ln 2.$$

$$5. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-2)^{2k}} = \frac{\pi}{8}.$$

$$6. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{k+l}}{(k+l)^2} k! = \frac{1}{6} \left(\ln 2 - \frac{1}{4} \right).$$

$$7. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} x^{k+l} = \frac{1}{1-2x} \quad [-1 < x < 1/2].$$

$$8. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(k+l+m)!}{k!l!m!} x^{k+l+m} = \frac{1}{1-3x} \quad [-1 < x < 1/3].$$

$$9. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{k^2(k^2 - kl + l^2)} = \frac{\pi^4 \sqrt{3}}{30}.$$

$$10. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{k^6(k^2 + l^2)} = \frac{13\pi^8}{28 \cdot 350}.$$

$$11. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k+1)^2 [(2k+1)^2 + (2l+1)^2]} = \frac{\pi^4}{32}.$$

$$12. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k+1)^6 [(2k+1)^2 + (2l+1)^2]} = \frac{7\pi^9}{23\,040}.$$

$$13. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{l^2 - k^2}{(l^2 + k^2)^2} = \frac{\pi}{4}. \quad 14. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l-1)^2 - (2k-1)^2}{[(2l-1)^2 + (2k-1)^2]^2} = \frac{\pi}{16}.$$

$$15. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l-1)^2 - (2k)^2}{[(2l-1)^2 + (2k)^2]^2} = \frac{1}{16} \left[\pi + \pi^2 - \frac{1}{8\pi} \Gamma^4 \left(\frac{1}{4} \right) \right].$$

$$16. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l)^2 - (2k-1)^2}{[(2l)^2 + (2k-1)^2]^2} = \frac{1}{16} \left[\pi + \pi^2 + \frac{1}{8\pi} \Gamma^4 \left(\frac{1}{4} \right) \right].$$

$$17. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_k (\beta)_l (\beta')_l}{(\gamma)_{k+l} k! l!} x^k y^l = F_1(\alpha, \beta, \beta', \gamma; x, y) \quad [|x|, |y| < 1].$$

$$18. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l} (\beta)_k (\beta')_l}{(\gamma)_k (\gamma')_l k! l!} x^k y^l = F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) \quad [|x| + |y| < 1].$$

$$19. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_k (\alpha')_l (\beta)_k (\beta')_l}{(\gamma)_{k+l} k! l!} x^k y^l = F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) \quad [|x|, |y| < 1].$$

$$20. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l} (\beta)_{k+l}}{(\gamma)_k (\gamma')_l k! l!} x^k y^l = F_4(\alpha, \beta, \gamma, \gamma'; x, y) \quad [| \sqrt{x} | + | \sqrt{y} | < 1].$$

Другие ряды вида (17) — (20) см. в [1], т. 1, стр. 219.

$$21. \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(k)}{\Gamma(2l+k+2)} x^{2l+1} = \text{Si}(x).$$

$$22. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \binom{k+l-1}{k} \binom{k+l-1}{l} \frac{x^k y^l}{k+l-1} = \frac{1}{2} [1 - x - y - \sqrt{1 - 2(x+y) + (x-y)^2}].$$

$$23. \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{k_1^2 k_2^2 \dots k_n^2} = \frac{\pi^{2n}}{(2n+1)!}.$$

$$24. \sum_{k_1, k_2, \dots, k_n=0}^{\infty} \frac{(-1)^{k_1+k_2+\dots+k_n}}{k_1! k_2! \dots k_n! \Gamma(k_1+k_2+\dots+k_n+\nu+1)} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} = (x_1+x_2+\dots+x_n)^{-\nu-2} J_{\nu}(2\sqrt{x_1+x_2+\dots+x_n}).$$

Глава 6. ПРОИЗВЕДЕНИЯ

6.1. КОНЕЧНЫЕ ПРОИЗВЕДЕНИЯ

6.1.1. Степенная и алгебраические функции.

1. $\prod_{k=1}^n (k+a)^s = \{(a+1)_n\}^s.$ 2. $\prod_{k=1}^n k^s = (n!)^s.$
3. $\prod_{k=1}^n (2k+1)^s = 2^{-ns} \left[\frac{(2n+1)}{n!} \right]^s.$ 4. $\prod_{k=1}^n \left(1 + \frac{a}{k+b} \right) = \frac{(a+b+1)_n}{(b+1)_n}.$
5. $\prod_{k=1}^n \left(1 - \frac{a^2}{k^2} \right) = \frac{1}{(n!)^2} (1+a)_n (1-a)_n.$
6. $\prod_{k=1}^n \left[1 - \frac{a^2}{(2k+1)^2} \right] = \left[\left(\frac{3}{2} \right)_n \right]^2 \left(\frac{3+a}{2} \right)_n \left(\frac{3-a}{2} \right)_n.$
7. $\prod_{k=2}^n \frac{k^3-1}{k^3+1} = \frac{2}{3} \left(1 + \frac{1}{n^2+n} \right).$
8. $\prod_{k=1}^n (k+a)^k = \Gamma^n (n+a+1) \prod_{k=1}^n \frac{1}{\Gamma(k+a)}.$
9. $\prod_{k=0}^{n-1} (1+x^{2^k}) = \frac{1-x^{2^n}}{1-x}.$ 10. $\prod_{k=1}^n (a - e^{2\pi k i / n}) = a^n - 1.$

6.1.2. Тригонометрические функции.

1. $\prod_{k=0}^{n-1} \begin{pmatrix} \sin [x + (k\pi/n)] \\ \cos [x + (k\pi/n)] \end{pmatrix} = 2^{1-n} \begin{pmatrix} \sin n x \\ \sin [n (2x + \pi)/2] \end{pmatrix}.$
2. $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = 2^{1-n} n.$

$$3. \prod_{k=1}^{[(n-1)/2]} \left\{ \frac{\sin(k\pi/n)}{\cos(k\pi/n)} \right\} = 2^{1-n/2} \left\{ \frac{1+(-1)^n n^{1/2}}{2} + \frac{1-(-1)^n}{2} \right\}.$$

$$4. \prod_{k=0}^{n-1} \left\{ \frac{\sin[x+2\pi k/n]}{\cos[x+2\pi k/n]} \right\} = (-2)^{1-n} \left\{ \frac{\cos[n(2x-\pi)/2] - \cos(n\pi/2)}{-\cos(n\pi/2) + \cos nx} \right\}.$$

$$5. \prod_{k=0}^{[(n-1)/2]} \left\{ \frac{\sin[(2k+1)\pi/(2n)]}{\cos[(2k+1)\pi/(2n)]} \right\} = 2^{(1-n)/2} \left\{ \frac{1-(-1)^n n^{1/2}}{2} + \frac{1+(-1)^n}{2} \right\}.$$

$$6. \prod_{k=1}^{2n-1} \cos \frac{k\pi}{n} = 2^{1-2n} [(-1)^n - 1].$$

$$7. \prod_{k=0}^{n-1} \left[a^2 - 2ab \cos \left(x + \frac{2\pi k}{n} \right) + b^2 \right] = a^{2n} - 2a^n b^n \cos nx + b^{2n}.$$

$$8. \prod_{k=1}^{n-1} \left(a^2 - 2a \cos \frac{\pi k}{n} + 1 \right) = \frac{a^{2n} - 1}{a^2 - 1}.$$

$$9. \prod_{k=1}^n \left(a^2 - 2a \cos \frac{2\pi k}{2n+1} + 1 \right) = \frac{a^{2n+1} - 1}{a - 1}.$$

6.2. БЕСКОНЕЧНЫЕ ПРОИЗВЕДЕНИЯ

6.2.1. Степенная и алгебраические функции.

$$1. \prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right) = \infty. \quad 2. \prod_{k=2}^{\infty} \left(1 - \frac{1}{k} \right) = 0.$$

$$3. \prod_{k=1}^{\infty} \left[1 + (-1)^k \frac{a}{k+b} \right] = 2^{-b} \sqrt{\pi} \Gamma \left[\frac{b+1}{(a+b+2)/2}, \frac{(b-a+1)/2} \right].$$

$$4. \prod_{k=1}^{\infty} \left[1 + (-1)^k \frac{a}{2k+1} \right] = \frac{\sqrt{2}}{a+1} \sin \frac{(a+1)\pi}{2}.$$

$$5. \prod_{k=1}^{\infty} \left[1 - \frac{a^2}{(k+b)^2} \right] = \Gamma \left[\frac{b+1}{b+a+1}, \frac{b+1}{b-a+1} \right].$$

$$6. \prod_{k=1}^{\infty} \left(1 + \frac{a^2}{k^2} \right) = \frac{1}{\pi a} \left\{ \frac{\operatorname{sh} \pi a}{\sin \pi a} \right\}. \quad 7. \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2} \right) = \frac{1}{2}.$$

$$8. \prod_{k=0}^{\infty} \left[1 \pm \frac{a^2}{(2k+1)^2} \right] = \begin{cases} \operatorname{ch} (\pi a/2) \\ \cos (\pi a/2) \end{cases}. \quad 9. \prod_{k=1}^{\infty} \left[1 - \frac{1}{(2k+1)^2} \right] = \frac{\pi}{4}.$$

$$10. \prod_{k=-\infty}^{\infty} \left[1 \pm \frac{a^2}{(k+b)^2} \right] = 1 \pm \operatorname{cosec}^2 \pi b \left\{ \frac{\operatorname{sh} \pi a}{\sin \pi a} \right\}^2.$$

$$11. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k+b} \right) \left(1 - \frac{a}{k+c} \right) = \Gamma \left[\begin{matrix} b+1, c+1 \\ a+b+1, c-a+1 \end{matrix} \right].$$

$$12. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k+b} \right) \left(1 - \frac{a}{k-b} \right) = \frac{b}{a+b} \sin (a+b) \pi \operatorname{cosec} \pi a.$$

$$13. \prod_{k=1}^{\infty} \left(1 \pm \frac{1}{k^3} \right) = \frac{1}{3\pi} \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} \operatorname{ch} \frac{\pi \sqrt{3}}{2}.$$

$$14. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k} \right) \left(1 + \frac{b+1}{k+c} \right) \left(1 - \frac{a+b+1}{k+b} \right) = -\frac{\sin \pi a}{\pi} \Gamma \left[\begin{matrix} b, c \\ b+c+2 \end{matrix} \right].$$

$$15. \prod_{k=0}^{\infty} \left(1 - \frac{a}{2k+1} \right) \left(1 - \frac{a}{2k+2} \right) \left(1 + \frac{a}{k+1} \right) = 2^{-a} \frac{\sin \pi a}{\pi a}.$$

$$16. \prod_{k=1}^{\infty} \frac{k(k+\alpha+\beta)}{(k+\alpha)(k+\beta)} = \frac{\alpha\beta}{\alpha+\beta} B(\alpha, \beta) \Gamma(1), \quad [\alpha, \beta \neq 0, -1, -2, \dots].$$

$$17. \prod_{k=1}^{\infty} \left(1 \pm \frac{a^4}{k^4} \right) = \frac{1}{2\pi^2 a^2} \left\{ \frac{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a}{2 \sin \pi a \operatorname{sh} \pi a} \right\}.$$

$$18. \prod_{k=0}^{\infty} \left[1 \pm \frac{a^4}{(2k+1)^4} \right] = \frac{1}{2} \left\{ \frac{\cos (\pi a/\sqrt{2}) + \operatorname{ch} (\pi a/\sqrt{2})}{2 \cos (\pi a/2) \operatorname{ch} (\pi a/2)} \right\}.$$

$$19. \prod_{k=1}^{\infty} \left[1 + \frac{1}{(4k-1)^2-1} \right] \left[1 - \frac{1}{(4k+1)^2} \right] = \frac{16}{\pi^2} \Gamma^4 \left(\frac{5}{4} \right).$$

$$20. \prod_{k=1}^{\infty} \left(1 - \frac{x^n}{k^n} \right) = -\frac{1}{x^n} \prod_{k=0}^{n-1} \frac{1}{\Gamma(-a^k x)} \quad [a = e^{2\pi i/n}; n=2, 3, 4, \dots].$$

$$21. \prod_{k=1}^{\infty} \left[1 - \frac{(-1)^k}{(2k+1)^{2n+1}} \right] = \frac{1}{2(2n)!} \left(\frac{\pi}{2} \right)^{2n+1} |E_{2n}|.$$

$$\begin{aligned}
 22. \quad & \prod_{k=1}^{\infty} \left[1 + \left(\frac{a}{k+b} \right)^n \right] \left[1 + \left(\frac{a}{k-b} \right)^n \right] = \\
 & = \frac{b^n}{a^n + b^n} (2 \sin \pi b)^{-n} \exp \left(2\pi a \cos \frac{\pi}{n} \right) \prod_{k=1}^{[n/2]} [1 - 2\alpha_k \cos (b + \beta_k) + \alpha_k^2]^{1/2} \times \\
 & \quad \times [1 - 2\alpha_k \cos (b - \beta_k) + \alpha_k^2]^{1/2} \\
 & \quad [0 < b < 1; \alpha_k = \exp (-2\pi a \sin [(2k-1)\pi/n]), \beta_k = 2\pi a \cos [(2k-1)\pi/n]].
 \end{aligned}$$

$$23. \quad \prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^s \left(1 - \frac{s}{k+a} \right) = \frac{\Gamma(a+1)}{\Gamma(a-s+1)}.$$

$$24. \quad \prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^s \left(1 + \frac{s}{k} \right)^{-1} = \Gamma(s+1).$$

6.2.2. Произведения, содержащие $(1 \pm x^n)^m$.
 Обозначение: $x = \exp \{-\pi K'(k)/K(k)\}$, $0 < k^2 < 1$.

$$1. \quad \prod_{n=1}^{\infty} (1 - x^{2n}) = \left[\frac{4}{x} \pi^{-6} k^3 (1 - k^2) K^6(k) \right]^{1/12}.$$

$$2. \quad \prod_{n=1}^{\infty} (1 + x^n)^6 = \frac{1}{2} x^{-1/2} k^{1/2} (1 - k^2)^{-1/2}.$$

$$3. \quad \prod_{n=1}^{\infty} (1 - x^n)^6 = 4\pi^{-3} x^{-1/4} k^{1/2} (1 - k^2) K^3(k).$$

$$4. \quad \prod_{n=1}^{\infty} (1 - x^{2n})^3 = \frac{1}{2} x^{-1/4} \frac{d}{da} \theta_1(a, x) \Big|_{a=0}.$$

$$5. \quad \prod_{n=1}^{\infty} (1 + x^{2n})^6 = \frac{1}{4} x^{-1/2} k (1 - k^2)^{-1/4}.$$

$$6. \quad \prod_{n=1}^{\infty} (1 - x^{2n})^6 = 2\pi^{-3} x^{-1/2} k (1 - k^2)^{1/2} K^3(k).$$

$$7. \quad \prod_{n=0}^{\infty} (1 + x^{2n+1})^6 = 2x^{1/4} k^{-1/2} (1 - k^2)^{-1/2}.$$

$$8. \quad \prod_{n=1}^{\infty} (1 - x^{2n+1})^6 = 2x^{1/4} k^{-1/2} (1 - k^2)^{1/2}.$$

$$9. \prod_{n=1}^{\infty} (1 + ax^{2n} + x^{4n}) = \frac{1}{\pi} x^{-1/6} (2k)^{1/3} (1 - k^2)^{1/6} \times \\ \times \operatorname{cosec} \left[\frac{1}{2} \arccos \left(-\frac{a}{2} \right) \right] K(k) \theta_1 \left[\frac{1}{2} \arccos \left(-\frac{a}{2} \right), x \right] \left[\frac{\partial}{\partial b} \theta_1(b, x) \Big|_{b=0} \right]^{-1}. \\ = (16xk^{-2})^{-1/6} (1 - k^2)^{-1/12} \sec \left(\frac{1}{2} \arccos \frac{a}{2} \right) \theta_2 \left(\frac{1}{2} \arccos \frac{a}{2}, x \right) \theta_2^{-1}(0, x).$$

$$10. \prod_{n=0}^{\infty} (1 + ax^{2n+1} + x^{4n+2}) = \\ = 2^{1/3} x^{1/12} k^{-1/6} (1 - k^2)^{-1/12} \theta_3 \left(\frac{1}{2} \arccos \frac{a}{2}, x \right) \theta_3^{-1}(0, x). \\ = 2^{1/3} x^{1/12} k^{-1/6} (1 - k^2)^{-1/6} \theta_4 \left[\frac{1}{2} \arccos \left(-\frac{a}{2} \right), x \right] \theta_4^{-1}(0, x).$$

$$11. \prod_{n=0}^{\infty} (1 + x^{n/2}) (1 - x^{n+1}) = \left(\frac{2}{\pi} \right)^{1/2} x^{-1/16} [kK(k)]^{1/2}.$$

$$12. \prod_{k=1}^{\infty} (1 - x)^{2k} (1 + x^{2k-1})^2 = \theta_4(0, x).$$

$$13. \prod_{n=0}^{\infty} \frac{1 - abx^n}{1 - bx^n} = {}_1\Phi_0(a, z). \quad 14. \prod_{n=1}^{\infty} \frac{1 - x^n}{1 + x^n} = \theta_4(0, x).$$

$$15. \prod_{n=1}^{\infty} \frac{1 - x^n}{1 - x^{n-1/2}} = [(2\pi)^{-1} x^{-1/8} k^{1/2} K(k)]^{1/2}.$$

$$16. \prod_{n=1}^{\infty} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right)^2 = \frac{2}{\pi} (1 - k^2)^{1/4} K(k).$$

$$17. \prod_{n=1}^{\infty} \left(\frac{1 + x^{2n}}{1 - x^{2n}} \right)^2 = \frac{1}{2} x^{-1/4} k^{1/2} (1 - k^2)^{-1/4}.$$

$$18. \prod_{n=1}^{\infty} \left(\frac{1 - x^{2n}}{1 - x^{2n-1}} \right)^2 = \frac{1}{\pi} x^{-1/4} k^{1/2} K(k).$$

$$19. \prod_{n=1}^{\infty} \left(\frac{1 + x^{2n}}{1 + x^{2n-1}} \right)^4 = \frac{k}{4} x^{-1/2}. \quad 20. \prod_{n=0}^{\infty} \left(\frac{1 - x^{2n+1}}{1 + x^{2n+1}} \right)^4 = (1 - k^2)^{1/2}.$$

6.2.3. Показательная функция.

$$1. \prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1 - x}$$

[|x| < 1].

$$2. \prod_{k=1}^{\infty} (1 + ae^{-2k}) = \frac{1}{2} (1 + \operatorname{cth} a)$$

[$a > 0$].

$$3. \prod_{k=1}^{\infty} \left(1 + \frac{2}{k}\right)^{(-1)^{k+1}k} = \frac{\pi^2}{12} \prod_{k=2}^{\infty} \left(1 + \frac{2}{k}\right)^{(-1)^k k} = \frac{\pi}{2e}.$$

$$4. \prod_{k=1}^{\infty} (1 - e^{-2\pi k/\sqrt{3}}) = 3^{1/4} e^{-\pi\sqrt{3}/18}.$$

$$5. \prod_{k=0}^{\infty} [1 + e^{-(2k+1)\pi}] = 2^{1/4} e^{-\pi/24}.$$

$$6. \prod_{k=1}^{\infty} \left(1 - \frac{a}{k+b}\right)^{a/k} = \frac{\Gamma(b+1)}{\Gamma(b-a+1)} e^{Ca}$$

[$\operatorname{Re} b, \operatorname{Re}(b-a) > 0$].

$$7. \prod_{k=0}^{\infty} \left(1 + \frac{a}{k+b}\right)^{-a/(k+b)} = \frac{\Gamma(b)}{\Gamma(a+b)} e^{a\psi(b)}.$$

$$8. \prod_{k=-\infty}^{\infty} \left(1 + \frac{a}{k+b}\right)^{-a/k} = \frac{b}{a+b} \sin[\pi(a+b)] \operatorname{cosec} \pi b.$$

$$9. \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(1 \pm \frac{a}{k}\right)^{a/k} = \frac{1}{\pi a} \sin \pi a.$$

$$10. \prod_{k=-\infty}^{\infty} \left(1 - \frac{a}{2k+1}\right)^{a/(2k+1)} = \cos \frac{\pi a}{2}.$$

6.2.4. Тригонометрические функции.

$$1. \prod_{k=1}^{\infty} \sec \frac{\pi}{k} = 8,70003663.$$

$$2. \prod_{k=1}^{\infty} \frac{1}{2\pi} (2k+1) \sin \frac{2\pi}{2k+1} = 0,18055054.$$

$$3. \prod_{k=1}^{\infty} \left(1 - \frac{\pi^2}{2k^2}\right) \sec \frac{\pi}{k} = 0,86885742.$$

$$4. \prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \frac{\sin x}{x}$$

[$|x| < 1$].

$$5. \prod_{k=1}^{\infty} \left(1 - \frac{4}{3} \sin^2 \frac{x}{3^k}\right) = \frac{\sin x}{x}.$$

Приложение I. НЕКОТОРЫЕ ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ И ИХ СВОЙСТВА

I. 1. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

$$\begin{aligned}\sin x &= -\sin(-x) = \pm \sqrt{1 - \cos^2 x} = \pm \sqrt{\frac{1}{2}(1 - \cos 2x)} = \\ &= 2 \cos^2 \left(\frac{x}{2} - \frac{\pi}{4} \right) - 1 = \frac{2 \operatorname{tg}(x/2)}{1 + \operatorname{tg}^2(x/2)} = \frac{e^{ix} - e^{-ix}}{2i}.\end{aligned}$$

$$\begin{aligned}\cos x &= \cos(-x) = \pm \sqrt{1 - \sin^2 x} = \pm \sqrt{\frac{1}{2}(1 + \cos 2x)} = \\ &= 2 \cos^2 \frac{x}{2} - 1 = \frac{1 - \operatorname{tg}^2(x/2)}{1 + \operatorname{tg}^2(x/2)} = \frac{e^{ix} + e^{-ix}}{2}.\end{aligned}$$

$$\begin{aligned}\operatorname{tg} x &= -\operatorname{tg}(-x) = \frac{\sin x}{\cos x} = \pm \sqrt{\sec^2 x - 1} = \frac{\sin 2x}{1 + \cos 2x} = \\ &= \frac{1 - \cos 2x}{\sin 2x} = \pm \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} = \frac{2 \operatorname{tg}(x/2)}{1 - \operatorname{tg}^2(x/2)} = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}.\end{aligned}$$

$$\begin{aligned}\operatorname{ctg} x &= -\operatorname{ctg}(-x) = \frac{\cos x}{\sin x} = \pm \sqrt{\cos^2 x - 1} = \frac{1 + \cos 2x}{\sin 2x} = \\ &= \frac{\sin 2x}{1 - \cos 2x} = \pm \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}} = \frac{1 - \operatorname{tg}^2(x/2)}{2 \operatorname{tg}(x/2)} = i \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}.\end{aligned}$$

$$\sec x = \sec(-x) = \frac{1}{\cos x} = \pm \sqrt{1 + \operatorname{tg}^2 x} = \frac{1 + \operatorname{tg}^2(x/2)}{1 - \operatorname{tg}^2(x/2)}.$$

$$\operatorname{cosec} x = -\operatorname{cosec}(-x) = \frac{1}{\sin x} = \pm \sqrt{1 + \operatorname{ctg}^2 x} = \frac{1 + \operatorname{tg}^2(x/2)}{2 \operatorname{tg}(x/2)}.$$

$$\sin^2 x + \cos^2 x = 1. \quad \sec^2 x - \operatorname{tg}^2 x = 1. \quad \cos^2 x - \operatorname{ctg}^2 x = 1.$$

2. Формулы дифференцирования.

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \operatorname{tg} x}{dx} = \sec^2 x, \quad \frac{d \sec x}{dx} = \frac{\sin x}{\cos^2 x},$$

$$\frac{d \cos x}{dx} = -\sin x, \quad \frac{d \operatorname{ctg} x}{dx} = -\operatorname{cosec}^2 x, \quad \frac{d \operatorname{cosec} x}{dx} = -\frac{\cos x}{\sin^2 x}.$$

3. Формулы Эйлера и Муавра.

$$e^{z+ix} = e^z (\cos x + i \sin x),$$

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \quad [n = 0, \pm 1, \pm 2, \dots].$$

4. Связь с гиперболическими функциями.

$$\sin(ix) = i \operatorname{sh} x, \quad \cos(ix) = \operatorname{ch} x,$$

$$\operatorname{tg}(ix) = i \operatorname{th} x, \quad \operatorname{ctg}(ix) = -i \operatorname{cth} x,$$

$$\operatorname{sec}(ix) = \operatorname{sech} x, \quad \operatorname{cosec}(ix) = -i \operatorname{cosech} x.$$

5. Формулы сложения.

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm iy) = \sin x \operatorname{ch} y \pm i \operatorname{sh} y \cos x,$$

$$\cos(x \pm iy) = \cos x \operatorname{ch} y \mp i \sin x \operatorname{sh} y,$$

$$\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}, \quad \operatorname{ctg}(x \pm y) = \frac{1 \mp \operatorname{tg} x \operatorname{tg} y}{\operatorname{tg} x \pm \operatorname{tg} y},$$

$$\operatorname{tg}(x \pm iy) = \frac{\operatorname{tg} x \pm i \operatorname{th} y}{1 \mp i \operatorname{tg} x \operatorname{th} y} = \frac{\sin 2x \pm i \operatorname{sh} 2y}{\cos 2x \mp \operatorname{ch} 2y},$$

$$\operatorname{ctg}(x \pm iy) = \frac{1 \pm i \operatorname{ctg} x \operatorname{ctg} y}{\pm \operatorname{ctg} x \mp i \operatorname{cth} y} = \frac{\sin 2x \mp i \operatorname{sh} 2y}{\operatorname{ch} 2y - \cos 2x}.$$

6. Формулы приведения.

$$\sin(x \pm n\pi) = (-1)^n \sin x, \quad \cos(x \pm n\pi) = (-1)^n \cos x,$$

$$\operatorname{tg}(x \pm n\pi) = \operatorname{tg} x, \quad \operatorname{ctg}(x \pm n\pi) = \operatorname{ctg} x,$$

$$\sin\left(x \pm \frac{2n+1}{2}\pi\right) = \pm (-1)^n \cos x, \quad \cos\left(x \pm \frac{2n+1}{2}\pi\right) = \mp (-1)^n \sin x,$$

$$\operatorname{tg}\left(x \pm \frac{2n+1}{2}\pi\right) = -\operatorname{ctg} x, \quad \operatorname{ctg}\left(x \pm \frac{2n+1}{2}\pi\right) = -\operatorname{tg} x,$$

$$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\sin x \pm \cos x),$$

$$\cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\cos x \mp \sin x),$$

$$\operatorname{tg}\left(x \pm \frac{\pi}{4}\right) = \frac{\sin x \pm \cos x}{\cos x \mp \sin x} = \frac{\operatorname{tg} x \pm 1}{1 \mp \operatorname{tg} x},$$

$$\operatorname{ctg}\left(x \pm \frac{\pi}{4}\right) = \frac{\cos x \mp \sin x}{\sin x \pm \cos x} = \frac{\operatorname{ctg} x \mp 1}{1 \pm \operatorname{ctg} x}.$$

7. Суммы и разности тригонометрических функций.

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2},$$

$$\cos x \pm \cos y = \pm 2 \begin{cases} \cos [(x+y)/2] \cos [(x-y)/2] \\ \sin [(x+y)/2] \sin [(x-y)/2] \end{cases},$$

$$\operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \operatorname{ctg} x \pm \operatorname{ctg} y = -\frac{\sin(x \mp y)}{\sin x \sin y},$$

$$\operatorname{tg} x - \operatorname{ctg} y = -\frac{\cos(x+y)}{\cos x \sin y},$$

$$a \cos x + b \sin x = r \sin(x + \varphi) = r \cos(x - \psi),$$

где

$$r = \sqrt{a^2 + b^2}, \quad \sin \varphi = a/r, \quad \cos \varphi = b/r, \quad \sin \psi = b/r, \quad \cos \psi = a/r,$$

$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y) \sin(x-y),$$

$$\sin^2 x - \cos^2 y = -\cos(x+y) \cos(x-y),$$

$$\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x.$$

8. Произведения тригонометрических функций.

$$\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)],$$

$$\cos x \cos y = \frac{1}{2} [\cos(x-y) + \cos(x+y)],$$

$$\sin x \cos y = \frac{1}{2} [\sin(x-y) + \sin(x+y)].$$

9. Степени тригонометрических функций.

$$\begin{cases} \sin x \\ \cos x \end{cases}^{2n} = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos 2(n-k)x + \frac{1}{2^{2n}} \binom{2n}{n},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^{n-k} \binom{2n+1}{k} \begin{cases} \sin(2n-2k+1)x \\ \cos(2n-2k+1)x \end{cases},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^2 = \mp \frac{1}{2} \cos 2x + \frac{1}{2},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^3 = \mp \frac{1}{4} \begin{cases} \sin 3x \\ \cos 3x \end{cases} + \frac{3}{4} \begin{cases} \sin x \\ \cos x \end{cases},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^4 = \frac{1}{8} \cos 4x \mp \frac{1}{2} \cos 2x + \frac{3}{8},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^5 = \frac{1}{16} \begin{cases} \sin 5x \\ \cos 5x \end{cases} \mp \frac{5}{16} \begin{cases} \sin 3x \\ \cos 3x \end{cases} + \frac{5}{8} \begin{cases} \sin x \\ \cos x \end{cases},$$

$$\begin{cases} \sin x \\ \cos x \end{cases}^6 = \mp \frac{1}{32} \cos 6x + \frac{3}{16} \cos 4x \mp \frac{15}{32} \cos 2x + \frac{5}{16}.$$

10. Тригонометрические функции кратных аргументов.

$$\sin nx = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \sin^{2k+1} x \cos^{n-2k-1} x,$$

$$= \sin x \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1}{k} 2^{n-2k-1} \cos^{n-2k-1} x,$$

$$\cos nx = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \sin^{2k} x \cos^{n-2k} x,$$

$$= 2^{n-1} \cos^n x + \frac{n}{2} \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \frac{(-1)^{k+1}}{k+1} \binom{n-k-2}{k} 2^{n-2k-2} \cos^{n-2k-2} x,$$

$$\begin{aligned}\sin 2nx &= 2n \cos x \left[\sin x + \sum_{k=1}^n (-1)^k \frac{(n^2-1^2)(n^2-2^2)\dots(n^2-k^2)}{(2k-1)!} 4^k \sin^{2k-1} x \right], \\ &= (-1)^{n-1} \cos x \left[2^{2n-1} \sin^{2n-1} x + \right. \\ &\quad \left. + \sum_{k=1}^n (-1)^k \frac{(2n-k-1)(2n-k-2)\dots(2n-2k)}{k!} 2^{2n-2k-1} \sin^{2n-2k-1} x \right],\end{aligned}$$

$$\begin{aligned}\cos 2nx &= 1 + \sum_{k=1}^n (-1)^k \frac{n^2(n^2-1^2)\dots[n^2-(k-1)^2]}{(2k)!} 4^k \sin^{2k} x, \\ &= (-1)^n \left[2^{2n-1} \sin^{2n} x - 2^{2n-3} n \sin^{2n-2} x + \right. \\ &\quad \left. + n \sum_{k=2}^n (-1)^k \frac{(2n-k-1)(2n-k-3)\dots(2n-2k+1)}{k!} 2^{2n-2k} \sin^{2n-2k} x \right],\end{aligned}$$

$$\begin{aligned}\sin (2n+1)x &= (2n+1)x \times \\ &\times \left[\sin x + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \sin^{2k+1} x \right], \\ &= (-1)^n \left[2^{2n} \sin^{2n+1} x - 2^{2n-2} (2n+1) \sin^{2n-1} x + \right. \\ &\quad \left. + \frac{2n+1}{2} \sum_{k=2}^n (-1)^k \frac{(2n-k)(2n-k-1)\dots(2n-2k+2)}{k!} 2^{2n-2k+1} \sin^{2n-2k+1} x \right],\end{aligned}$$

$$\begin{aligned}\cos (2n+1)x &= \cos x \times \\ &\times \left[1 + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!} \sin^{2k} x \right], \\ &= (-1)^n \cos x \left[2^{2n} \sin^{2n} x - \right. \\ &\quad \left. - \sum_{k=1}^n \frac{(2n-k)(2n-k-1)\dots(2n-2k+1)}{k!} 2^{2n-2k} \sin^{2n-2k} x \right],\end{aligned}$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2 \cos^2 x - 1,$$

$$\begin{cases} \sin 3x \\ \cos 3x \end{cases} = \pm 3 \begin{cases} \sin x \\ \cos x \end{cases} \mp 4 \begin{cases} \sin x \\ \cos x \end{cases}^3,$$

$$\sin 4x = \cos x (4 \sin x - 8 \sin^3 x),$$

$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1,$$

$$\begin{cases} \sin 5x \\ \cos 5x \end{cases} = 5 \begin{cases} \sin x \\ \cos x \end{cases} - 20 \begin{cases} \sin x \\ \cos x \end{cases}^3 + 16 \begin{cases} \sin x \\ \cos x \end{cases}^5,$$

$$\sin 6x = \cos (6 \sin x - 32 \sin^3 x + 32 \sin^5 x),$$

$$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1,$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}, \quad \operatorname{tg} 3x = \frac{3 \operatorname{tg} x - \operatorname{tg}^3 x}{1 - 3 \operatorname{tg}^2 x}, \quad \operatorname{tg} 4x = \frac{4 \operatorname{tg} x - 4 \operatorname{tg}^3 x}{1 - 6 \operatorname{tg}^2 x + \operatorname{tg}^4 x}.$$

II. Тригонометрические функции некоторых углов.

$$1. \sin n\pi = 0, \quad \cos n\pi = (-1)^n, \quad \operatorname{tg} n\pi = 0.$$

$$2. \begin{cases} \sin (n\pi/2) \\ \cos (n\pi/2) \end{cases} = (-1)^{[n/2]} \frac{1 \mp (-1)^n}{2}, \quad \operatorname{tg} \frac{n\pi}{2} = \frac{1 - (-1)^n}{1 + (-1)^n}.$$

$$3. \sin \frac{n\pi}{3} = (-1)^{[n/3]} \frac{\sqrt{3}}{4} [1 \mp (-1)^{[n/3]}] \quad \left[\begin{matrix} n - \text{четное} \\ n - \text{нечетное} \end{matrix} \right],$$

$$\cos \frac{n\pi}{3} = (-1)^{[(n+1)/3]} \frac{3 \mp (-1)^{[(n+1)/3]}}{4} \quad \left[\begin{matrix} n - \text{четное} \\ n - \text{нечетное} \end{matrix} \right],$$

$$\operatorname{tg} \frac{n\pi}{3} = (-1)^{[(n+2)/3]} \frac{\sqrt{3}}{2} [-1 \mp (-1)^{[(n+1)/3]}] \quad [n - \text{четное}],$$

$$= (-1)^{[n/3]} \frac{\sqrt{3}}{2} [1 \mp (-1)^{[(n+1)/3]}] \quad [n - \text{нечетное}],$$

$$4. \sin \frac{n\pi}{4} = (-1)^{[n/4]} \frac{1 - (-1)^{[n/2]}}{2} \quad [n - \text{четное}],$$

$$= (-1)^{[(n-1)/4]} \frac{1}{\sqrt{2}} \quad [n - \text{нечетное}],$$

$$\cos \frac{n\pi}{4} = (-1)^{[n/4]} \frac{1 + (-1)^{[n/2]}}{2} \quad [n - \text{четное}],$$

$$= (-1)^{[(n+1)/4]} \frac{1}{\sqrt{2}} \quad [n - \text{нечетное}],$$

$$\operatorname{tg} \frac{n\pi}{4} = (-1)^{[n/4]} \frac{1 - (-1)^{[n/2]}}{1 + (-1)^{[n/2]}} \quad [n - \text{четное}],$$

$$= (-1)^{[(2n+1)/4]} \quad [n - \text{нечетное}],$$

$$5. \sin \frac{n\pi}{5} = (-1)^{[n/5]} \frac{1 - (-1)^{[4n/5]}}{8} \sqrt{10 - (-1)^{[n/2]} 2\sqrt{5}} \quad [n - \text{четное}],$$

$$= (-1)^{[(n+2)/5]} \frac{1 + (-1)^{[(4n-1)/5]}}{8} \sqrt{10 - (-1)^{[n/2]} 2\sqrt{5}} \quad [n - \text{нечетное}],$$

$$\cos \frac{n\pi}{5} = (-1)^{[(n+2)/5]} \frac{\sqrt{5 + (-1)^{[n/2]}}}{4} \quad [n \neq 0, \pm 5, \pm 10, \dots].$$

$$6. \sin \frac{n\pi}{6} = (-1)^{[n/6]} \frac{\sqrt{3}}{4} [1 - (-1)^{[(n+1)/3]}] \quad [n - \text{четное}],$$

$$= (-1)^{[n/6]} \frac{3 - (-1)^{[(n+1)/3]}}{4} \quad [n - \text{нечетное}],$$

$$\cos \frac{n\pi}{6} = (-1)^{[(n+2)/6]} \frac{3 + (-1)^{[(n+1)/3]}}{4} \quad [n - \text{четное}],$$

$$= (-1)^{[(n+1)/6]} \frac{\sqrt{3}}{2} [1 - (-1)^{[(n+1)/3]}] \quad [n - \text{нечетное}],$$

$$\operatorname{tg} \frac{n\pi}{6} = (-1)^{[n/3]} \frac{\sqrt{3}}{2} [1 - (-1)^{[(n+1)/3]}] \quad [n - \text{четное}],$$

$$= (-1)^{[n/3]} \frac{2}{\sqrt{3} [1 + (-1)^{[(n+1)/3]}]} \quad [n - \text{нечетное}].$$

$$7. \sin \frac{n\pi}{8} = (-1)^{[(n+1)/8]} \frac{1}{2} \sqrt{2 - (-1)^{[n/4]} - (-1)^{[(3n+1)/4]}} \quad [n - \text{четное}],$$

$$= (-1)^{[n/8]} \frac{1}{2} \sqrt{2 - (-1)^{[(n+1)/4]} \sqrt{2}} \quad [n - \text{нечетное}],$$

$$\cos \frac{n\pi}{8} = (-1)^{[(n+5)/8]} \frac{1}{2} \sqrt{2 + (-1)^{[n/4]} + (-1)^{[(3n+1)/4]}} \quad [n - \text{четное}],$$

$$= (-1)^{[(n+3)/8]} \frac{1}{2} \sqrt{2 + (-1)^{[(n+1)/4]} \sqrt{2}} \quad [n - \text{четное}],$$

$$\operatorname{tg} \frac{n\pi}{8} = (-1)^{[(2n+1)/8]} \frac{2 - (-1)^{[n/4]} - (-1)^{[(3n+1)/4]}}{(-1)^{[n/4]} - (-1)^{[(3n+1)/4]}}, \quad [n - \text{четное}],$$

$$= (-1)^{[(2n+1)/8]} [\sqrt{2} - (-1)^{[(n+1)/4]}] \quad [n - \text{нечетное}].$$

$$8. \sin \frac{\pi}{10} = \frac{1}{4} (\sqrt{5} - 1), \quad \sin \frac{3\pi}{10} = \frac{1}{4} (\sqrt{5} + 1),$$

$$\cos \frac{\pi}{10} = \frac{1}{4} \sqrt{10 + 2\sqrt{5}}, \quad \cos \frac{3\pi}{10} = \frac{1}{4} \sqrt{10 - 2\sqrt{5}},$$

$$\operatorname{tg} \frac{\pi}{10} = \frac{1}{\sqrt{5}} \sqrt{5 - 2\sqrt{5}}, \quad \operatorname{tg} \frac{3\pi}{10} = \frac{1}{\sqrt{5}} \sqrt{5 + 2\sqrt{5}},$$

$$\operatorname{ctg} \frac{\pi}{10} = \sqrt{5 + 2\sqrt{5}}, \quad \operatorname{ctg} \frac{3\pi}{10} = \sqrt{5 - 2\sqrt{5}}.$$

$$9. \begin{cases} \sin (\pi/12) \\ \cos (\pi/12) \end{cases} = \frac{1}{4} (\sqrt{6} \mp \sqrt{2}), \quad \begin{cases} \sin (5\pi/12) \\ \cos (5\pi/12) \end{cases} = \frac{\sqrt{3} \pm 1}{2\sqrt{2}},$$

$$\begin{cases} \operatorname{tg} (\pi/12) \\ \operatorname{ctg} (\pi/12) \end{cases} = 2 \mp \sqrt{3}, \quad \begin{cases} \operatorname{tg} (5\pi/12) \\ \operatorname{ctg} (5\pi/12) \end{cases} = 2 \pm \sqrt{3},$$

$$\begin{cases} \sin (7\pi/12) \\ \cos (7\pi/12) \end{cases} = \frac{1}{2\sqrt{2}} (1 \pm \sqrt{3}), \quad \begin{cases} \sin (11\pi/12) \\ \cos (11\pi/12) \end{cases} = \frac{1}{2\sqrt{2}} (-1 \pm \sqrt{3}),$$

$$\begin{cases} \operatorname{tg} (7\pi/12) \\ \operatorname{ctg} (7\pi/12) \end{cases} = -2 \mp \sqrt{3}, \quad \begin{cases} \operatorname{tg} (11\pi/12) \\ \operatorname{ctg} (11\pi/12) \end{cases} = -2 \pm \sqrt{3}.$$

$$10. \sin \frac{\pi}{15} = \frac{1}{8} [-\sqrt{3}(\sqrt{5} - 1) + \sqrt{10 + 2\sqrt{5}}],$$

$$\cos \frac{\pi}{15} = \frac{1}{8} (\sqrt{5} - 1 + \sqrt{3}\sqrt{10 + 2\sqrt{5}}),$$

$$\operatorname{tg} \frac{\pi}{15} = \frac{1}{4} (3 - \sqrt{5}) (2\sqrt{3} - \sqrt{10 - 2\sqrt{5}}),$$

$$\operatorname{ctg} \frac{\pi}{15} = \frac{1}{4} (1 + \sqrt{5}) (2\sqrt{3} + \sqrt{10 - 2\sqrt{5}}),$$

$$\sin \frac{2\pi}{15} = \frac{1}{8} [\sqrt{3}(\sqrt{5} + 1) - \sqrt{10 - 2\sqrt{5}}],$$

$$\cos \frac{2\pi}{15} = \frac{1}{8} (\sqrt{5} + 1 + \sqrt{3}\sqrt{10 - 2\sqrt{5}}),$$

$$\operatorname{tg} \frac{2\pi}{15} = \frac{1}{4} (3 + \sqrt{5}) (-2\sqrt{3} + \sqrt{10 + 2\sqrt{5}}),$$

$$\operatorname{ctg} \frac{2\pi}{15} = \frac{1}{4} (\sqrt{5} - 1) (2\sqrt{3} + \sqrt{10 + 2\sqrt{5}}).$$

$$11. \quad \left\{ \begin{array}{l} \sin (\pi/16) \\ \cos (\pi/16) \end{array} \right\} = \frac{1}{2} \sqrt{2 \mp \sqrt{2 + \sqrt{2}}},$$

$$\left\{ \begin{array}{l} \operatorname{tg} (\pi/16) \\ \operatorname{ctg} (\pi/16) \end{array} \right\} = \sqrt{2} \sqrt{2 + \sqrt{2}} \mp (\sqrt{2} + 1),$$

$$\left\{ \begin{array}{l} \sin (3\pi/16) \\ \cos (3\pi/16) \end{array} \right\} = \frac{1}{2} \sqrt{2 \mp \sqrt{2 - \sqrt{2}}},$$

$$\left\{ \begin{array}{l} \operatorname{tg} (3\pi/16) \\ \operatorname{ctg} (3\pi/16) \end{array} \right\} = \sqrt{2} \sqrt{2 - \sqrt{2}} \mp (\sqrt{2} - 1).$$

$$12. \quad \left\{ \begin{array}{l} \sin (\pi/20) \\ \cos (\pi/20) \end{array} \right\} = \frac{\sqrt{2}}{8} (\sqrt{5} + 1 \mp \sqrt{10 - 2\sqrt{5}}),$$

$$\left\{ \begin{array}{l} \operatorname{tg} (\pi/20) \\ \operatorname{ctg} (\pi/20) \end{array} \right\} = \sqrt{5} + 1 \mp \sqrt{5 + 2\sqrt{5}},$$

$$\left\{ \begin{array}{l} \sin (3\pi/20) \\ \cos (3\pi/20) \end{array} \right\} = \frac{\sqrt{2}}{8} (\sqrt{10 + 2\sqrt{5}} + 1 \mp \sqrt{5}),$$

$$\left\{ \begin{array}{l} \operatorname{tg} (3\pi/20) \\ \operatorname{ctg} (3\pi/20) \end{array} \right\} = \sqrt{5} - 1 \mp \sqrt{5 - 2\sqrt{5}}.$$

$$13. \quad \left\{ \begin{array}{l} \sin (\pi/24) \\ \cos (\pi/24) \end{array} \right\} = \frac{1}{2} \sqrt{2 \mp \sqrt{2 + \sqrt{3}}},$$

$$\left\{ \begin{array}{l} \operatorname{tg} (\pi/24) \\ \operatorname{ctg} (\pi/24) \end{array} \right\} = (\sqrt{2} \mp 1) (\sqrt{3} \mp \sqrt{2}),$$

$$\left\{ \begin{array}{l} \sin (5\pi/24) \\ \cos (5\pi/24) \end{array} \right\} = \frac{1}{2} \sqrt{2 \mp \sqrt{2 - \sqrt{3}}},$$

$$\left\{ \begin{array}{l} \operatorname{tg} (5\pi/24) \\ \operatorname{ctg} (5\pi/24) \end{array} \right\} = (\sqrt{2} \pm 1) (\sqrt{3} \mp \sqrt{2}).$$

$$14. \quad \sin \frac{\pi}{30} = \frac{1}{8} [-(\sqrt{5} + 1) + \sqrt{3}\sqrt{10 - 2\sqrt{5}}],$$

$$\cos \frac{\pi}{30} = \frac{1}{8} [\sqrt{3}(\sqrt{5} + 1) + \sqrt{10 - 2\sqrt{5}}],$$

$$\operatorname{tg} \frac{\pi}{30} = \frac{1}{4} (\sqrt{5} - 1) (\sqrt{10 + 2\sqrt{5}} - 2\sqrt{3}),$$

$$\operatorname{ctg} \frac{\pi}{30} = \frac{1}{4} (3 + \sqrt{5}) (\sqrt{10 + 2\sqrt{5}} + 2\sqrt{3}),$$

$$\sin \frac{7\pi}{30} = \frac{1}{8} [-(\sqrt{5} - 1) + \sqrt{3}\sqrt{10 + 2\sqrt{5}}],$$

$$\cos \frac{7\pi}{30} = \frac{1}{8} [\sqrt{3} (\sqrt{5}-1) + \sqrt{10+2\sqrt{5}}],$$

$$\operatorname{tg} \frac{7\pi}{30} = \frac{1}{4} (\sqrt{5}+1) (2\sqrt{3}-\sqrt{10-2\sqrt{5}}),$$

$$\operatorname{ctg} \frac{7\pi}{30} = \frac{1}{4} (3-\sqrt{5}) (2\sqrt{3}+\sqrt{10-2\sqrt{5}}).$$

$$15. \begin{cases} \sin (3\pi/40) \\ \cos (3\pi/40) \end{cases} = \frac{1}{8} [\sqrt{2 \pm \sqrt{2}} \sqrt{10-2\sqrt{5}} \mp \sqrt{2 \mp \sqrt{2}} (1+\sqrt{5})],$$

$$\begin{cases} \operatorname{tg} (3\pi/40) \\ \operatorname{ctg} (3\pi/40) \end{cases} = \frac{1}{4} (\pm 1 \mp \sqrt{5} + 2\sqrt{2}) [\sqrt{10-2\sqrt{5}} \mp \sqrt{2} (\sqrt{5}-1)].$$

$$16. \begin{cases} \sin (\pi/60) \\ \cos (\pi/60) \end{cases} = \frac{\sqrt{2}}{8} [\sqrt{(2 \pm \sqrt{3})(3-\sqrt{5})} \mp \sqrt{(2 \mp \sqrt{3})(5+\sqrt{5})}],$$

$$\begin{cases} \operatorname{tg} (\pi/60) \\ \operatorname{ctg} (\pi/60) \end{cases} = \frac{1}{4} (2 \mp \sqrt{3}) (\pm 1 + 2\sqrt{3} \mp \sqrt{5}) (\sqrt{10-2\sqrt{5}} \mp 2).$$

1.2. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

$$\operatorname{sh} x = -\operatorname{sh} (-x) = \pm \sqrt{\operatorname{ch}^2 x - 1} = \pm \sqrt{\frac{1}{2} (\operatorname{ch} 2x - 1)} = \frac{e^x - e^{-x}}{2},$$

$$\operatorname{ch} x = \operatorname{ch} (-x) = \sqrt{\operatorname{sh}^2 x + 1} = \sqrt{\frac{1}{2} (\operatorname{ch} 2x + 1)} = 2 \operatorname{ch}^2 \frac{x}{2} - 1 = \frac{e^x + e^{-x}}{2},$$

$$\operatorname{th} x = -\operatorname{th} (-x) = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x + 1} = \frac{\operatorname{ch} 2x - 1}{\operatorname{sh} 2x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\operatorname{cth} x = -\operatorname{cth} (-x) = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x - 1} = \frac{\operatorname{ch} 2x + 1}{\operatorname{sh} 2x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\operatorname{ch} x}, \quad \operatorname{cosech} x = \frac{1}{\operatorname{sh} x},$$

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1, \quad \operatorname{th}^2 x + \operatorname{sech}^2 x = 1, \quad \operatorname{cth}^2 x - \operatorname{cosech}^2 x = 1,$$

$$(\operatorname{sh} x + \operatorname{ch} x)^n = \operatorname{sh} nx + \operatorname{ch} nx \quad [n=0, \pm 1, \pm 2, \dots].$$

2. Формулы дифференцирования.

$$\frac{d \operatorname{sh} x}{dx} = \operatorname{ch} x, \quad \frac{d \operatorname{cosech} x}{dx} = -\frac{\operatorname{ch} x}{\operatorname{sh}^2 x},$$

$$\frac{d \operatorname{ch} x}{dx} = \operatorname{sh} x, \quad \frac{d \operatorname{th} x}{dx} = \operatorname{sech}^2 x,$$

$$\frac{d \operatorname{sech} x}{dx} = -\frac{\operatorname{sh} x}{\operatorname{ch}^2 x}, \quad \frac{d \operatorname{cth} x}{dx} = -\operatorname{cosech}^2 x.$$

3. Связь с тригонометрическими функциями.

$$\operatorname{sh} (ix) = i \sin x, \quad \operatorname{ch} (ix) = \cos x, \quad \operatorname{th} (ix) = i \operatorname{tg} x, \quad \operatorname{cth} (ix) = -i \operatorname{ctg} x.$$

4. Формулы сложения.

$$\operatorname{sh} (x \pm y) = \operatorname{sh} x \operatorname{ch} y \pm \operatorname{sh} y \operatorname{ch} x,$$

$$\operatorname{ch} (x \pm y) = \operatorname{ch} x \operatorname{ch} y \pm \operatorname{sh} x \operatorname{sh} y,$$

$$\operatorname{sh}(x \pm iy) = \operatorname{sh} x \cos y \pm i \sin y \operatorname{ch} x,$$

$$\operatorname{ch}(x \pm iy) = \operatorname{ch} x \cos y \pm i \operatorname{sh} x \sin y,$$

$$\operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x \operatorname{th} y} = \frac{\operatorname{sh} 2x \pm \operatorname{sh} 2y}{\operatorname{ch} 2x \pm \operatorname{ch} 2y},$$

$$\operatorname{th}(x \pm iy) = \frac{\operatorname{th} x \pm i \operatorname{tg} y}{1 \pm i \operatorname{th} x \operatorname{tg} y} = \frac{\operatorname{sh} 2x \pm i \sin 2y}{\operatorname{ch} 2x \pm \cos 2y},$$

$$\operatorname{cth}(x \pm iy) = \frac{\operatorname{sh} 2x \mp i \sin 2y}{\operatorname{ch} 2x - \cos 2y}.$$

5. Суммы и разности гиперболических функций.

$$\operatorname{sh} x \pm \operatorname{sh} y = 2 \operatorname{sh} \frac{x \pm y}{2} \operatorname{ch} \frac{x \mp y}{2},$$

$$\operatorname{ch} x \pm \operatorname{ch} y = 2 \begin{cases} \operatorname{ch} [(x+y)/2] \operatorname{ch} [(x-y)/2] \\ \operatorname{sh} [(x+y)/2] \operatorname{sh} [(x-y)/2] \end{cases},$$

$$\operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x \operatorname{ch} y}, \quad \operatorname{cth} x \pm \operatorname{cth} y = \pm \frac{\operatorname{sh}(x \pm y)}{\operatorname{sh} x \operatorname{sh} y},$$

$$\operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{ch}^2 x - \operatorname{ch}^2 y = \operatorname{sh}(x+y) \operatorname{sh}(x-y),$$

$$\operatorname{sh}^2 x + \operatorname{ch}^2 y = \operatorname{ch}(x+y) \operatorname{ch}(x-y).$$

6. Произведения гиперболических функций.

$$\operatorname{sh} x \operatorname{sh} y = \frac{1}{2} [\operatorname{ch}(x+y) - \operatorname{ch}(x-y)],$$

$$\operatorname{ch} x \operatorname{ch} y = \frac{1}{2} [\operatorname{ch}(x+y) + \operatorname{ch}(x-y)],$$

$$\operatorname{sh} x \operatorname{ch} y = \frac{1}{2} [\operatorname{sh}(x+y) + \operatorname{sh}(x-y)].$$

7. Степени гиперболических функций.

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^{2n} = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \operatorname{ch} 2(n-k)x + \frac{(-1)^n}{2^{2n}} \binom{2n}{n},$$

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \left\{ \begin{matrix} \operatorname{sh}(2n-2k+1)x \\ \operatorname{ch}(2n-2k+1)x \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^2 = \frac{1}{2} (\operatorname{ch} 2x \mp 1), \quad \left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^3 = \frac{1}{4} (\operatorname{sh} 3x \mp 3 \operatorname{sh} x),$$

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^4 = \frac{1}{8} (\operatorname{ch} 4x \mp 4 \operatorname{ch} 2x \mp 3),$$

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^5 = \frac{1}{16} (\operatorname{sh} 5x \mp 5 \operatorname{sh} 3x \mp 10 \operatorname{sh} x),$$

$$\left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\}^6 = \frac{1}{32} (\operatorname{ch} 6x \mp 6 \operatorname{ch} 4x \mp 15 \operatorname{ch} 2x \mp 10).$$

8. Гиперболические функции кратных аргументов.

$$\begin{aligned}
 \operatorname{sh} nx &= \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \operatorname{sh}^{2k+1} x \operatorname{ch}^{n-2k-1} x, \\
 &= \operatorname{sh} x \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} \binom{n-k-1}{k} \operatorname{ch}^{n-2k-1} x, \\
 \operatorname{ch} nx &= \sum_{k=0}^{[n/2]} \binom{n}{2k} \sin^{2k} x \cos^{2n-2k} x, \\
 &= 2^{n-1} \cos^n x + \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}}{k+1} \binom{n-k-2}{k-2} 2^{n-2k-2} \operatorname{ch}^{n-2k-2} x, \\
 \operatorname{sh} 2x &= 2 \operatorname{sh} x \operatorname{ch} x, \quad \operatorname{ch} 2x = 2 \operatorname{ch}^2 x - 1, \\
 \begin{cases} \operatorname{sh} 3x \\ \operatorname{ch} 3x \end{cases} &= 4 \begin{cases} \operatorname{sh} x \\ \operatorname{ch} x \end{cases}^3 \pm 3 \begin{cases} \operatorname{sh} x \\ \operatorname{ch} x \end{cases}, \\
 \operatorname{sh} 4x &= \operatorname{ch} x (4 \operatorname{sh} x + 8 \operatorname{sh}^3 x), \quad \operatorname{ch} 4x = 8 \operatorname{ch}^4 x - 8 \operatorname{ch}^2 x + 1, \\
 \begin{cases} \operatorname{sh} 5x \\ \operatorname{ch} 5x \end{cases} &= 16 \begin{cases} \operatorname{sh} x \\ \operatorname{ch} x \end{cases}^5 \pm 20 \begin{cases} \operatorname{sh} x \\ \operatorname{ch} x \end{cases}^3 \pm 5 \begin{cases} \operatorname{sh} x \\ \operatorname{ch} x \end{cases}, \\
 \operatorname{sh} 6x &= \operatorname{ch} x (32 \operatorname{sh}^5 x + 32 \operatorname{sh}^3 x + 6 \operatorname{sh} x), \\
 \operatorname{ch} 6x &= 32 \operatorname{ch}^6 x - 48 \operatorname{ch}^4 x + 18 \operatorname{ch}^2 x - 1.
 \end{aligned}$$

1.3. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

Главные значения обратных тригонометрических функций определяются неравенствами

$$-\pi/2 \leq \arcsin x \leq \pi/2; \quad 0 \leq \arccos x \leq \pi \quad [-1 \leq x \leq 1]$$

$$-\pi/2 < \operatorname{arctg} x < \pi/2; \quad 0 < \operatorname{arcctg} x < \pi \quad [-\infty < x < \infty].$$

$$\arcsin(-x) = -\arcsin x, \quad \arccos(-x) = \pi - \arccos x,$$

$$\operatorname{arctg}(-x) = -\operatorname{arctg} x, \quad \operatorname{arcctg}(-x) = \pi - \operatorname{arcctg} x,$$

$$\sin(\arcsin x) = x, \quad \cos(\arccos x) = x,$$

$$\operatorname{tg}(\operatorname{arctg} x) = x, \quad \operatorname{ctg}(\operatorname{arcctg} x) = x.$$

$$\arcsin(\sin x) = x - 2n\pi \quad [2n\pi - \pi/2 \leq x \leq 2n\pi + \pi/2],$$

$$= -x + (2n+1)\pi \quad [(2n+1)\pi - \pi/2 \leq x \leq 2(n+1)\pi + \pi/2].$$

$$\arccos(\cos x) = x - 2n\pi \quad [2n\pi \leq x \leq (2n+1)\pi],$$

$$= -x + 2(n+1)\pi \quad [(2n+1)\pi \leq x \leq 2(n+1)\pi].$$

$$\operatorname{arctg}(\operatorname{tg} x) = x - n\pi \quad [n\pi - \pi/2 < x < n\pi + \pi/2],$$

$$\operatorname{arcctg}(\operatorname{ctg} x) = x - n\pi \quad [n\pi < x < (n+1)\pi].$$

$$\lim_{x \rightarrow \pm \infty} \operatorname{arctg} x = \pm \pi/2, \quad \lim_{x \rightarrow \pm \infty} \operatorname{arcctg} x = \pi/2, \mp \pi/2.$$

2. Связь с логарифмической и обратными гиперболическими функциями.

$$\arcsin z = -i \ln (iz + \sqrt{1-z^2}) = -i \operatorname{Arsh} (iz),$$

$$\arccos z = -i \ln (z + \sqrt{z^2-1}) = -i \operatorname{Arch} z,$$

$$\operatorname{arctg} z = -\frac{i}{2} \ln \frac{1+iz}{1-iz} = -i \operatorname{Arth} (iz),$$

$$\operatorname{arcctg} z = -\frac{i}{2} \ln \frac{iz-1}{iz+1} = i \operatorname{Arcth} (iz).$$

3. Формулы дифференцирования.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx} \operatorname{arctg} x = \frac{1}{1+x^2}, \quad \frac{d}{dx} \operatorname{arcctg} x = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}, \quad \left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} x < \left\{ \frac{\pi/2}{\pi} \right\} \right],$$

$$\frac{d}{dx} \operatorname{arccosec} x = \frac{-1}{x\sqrt{x^2-1}}, \quad \left[\left\{ \frac{-\pi/2}{0} \right\} < \operatorname{arccosec} x < \left\{ \frac{0}{\pi/2} \right\} \right].$$

4. Связь между обратными тригонометрическими функциями.

$$\arcsin x = \frac{\pi}{2} - \arccos x,$$

$$= \arccos \sqrt{1-x^2}$$

$$= -\arccos \sqrt{1-x^2}$$

$$= \operatorname{arctg} \frac{x}{\sqrt{1-x^2}}$$

$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x}$$

$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} - \pi$$

$$= \frac{1}{2} \arcsin (2x \sqrt{1-x^2})$$

$$= \frac{\pi}{2} - \frac{1}{2} \arcsin (2x \sqrt{1-x^2})$$

$$= -\frac{\pi}{2} - \frac{1}{2} \arcsin (2x \sqrt{1-x^2})$$

$$[0 \leq x \leq 1],$$

$$[-1 \leq x \leq 0],$$

$$[x^2 < 1],$$

$$[0 < x \leq 1],$$

$$[-1 \leq x < 0],$$

$$[x^2 \leq 1/2],$$

$$[1/\sqrt{2} < x \leq 1],$$

$$[-1 \leq x < -1/\sqrt{2}],$$

$$\arccos x = \frac{\pi}{2} - \arcsin x,$$

$$= \arcsin \sqrt{1-x^2}$$

$$= \pi - \arcsin \sqrt{1-x^2}$$

$$= \operatorname{arctg} \frac{\sqrt{1-x^2}}{x}$$

$$[0 \leq x \leq 1],$$

$$[-1 \leq x \leq 0],$$

$$[0 < x \leq 1],$$

$$\arccos x = \pi - \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [-1 \leq x < 0],$$

$$= \operatorname{arccotg} \frac{x}{\sqrt{1-x^2}} \quad [-1 \leq x \leq 1],$$

$$= \frac{1}{2} \arccos (2x^2 - 1) \quad [0 \leq x \leq 1],$$

$$= \pi - \frac{1}{2} \arccos (2x^2 - 1) \quad [-1 \leq x < 0].$$

$$\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arccotg} x,$$

$$= \arcsin \frac{x}{\sqrt{1+x^2}},$$

$$= \arccos \frac{1}{\sqrt{1+x^2}} \quad [x \geq 0],$$

$$= -\arccos \frac{1}{\sqrt{1+x^2}} \quad [x \leq 0],$$

$$= \frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} \quad [x > 0],$$

$$= -\frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} \quad [x < 0],$$

$$= \operatorname{arccotg} \frac{1}{x} \quad [x > 0],$$

$$= \operatorname{arccotg} \frac{1}{x} - \pi \quad [x < 0],$$

$$= \frac{\pi}{4} - \operatorname{arctg} \frac{1-x}{1+x} \quad [x > -1],$$

$$= -\frac{3}{4}\pi - \operatorname{arctg} \frac{1-x}{1+x} \quad [x < -1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} \quad [|x| < 1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} + \frac{\pi}{2} \quad [x > 1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} - \frac{\pi}{2} \quad [x < -1],$$

$$= -\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2x}{1+x^2} \quad [x < -1],$$

$$= \frac{1}{2} \arcsin \frac{2x}{1+x^2} \quad [-1 \leq x \leq 1],$$

$$= \frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2x}{1+x^2} \quad [x > 1],$$

$$= \frac{1}{2} \arccos \frac{1-x^2}{1+x^2} \quad [x \geq 0],$$

$$= -\frac{1}{2} \arccos \frac{1-x^2}{1+x^2} \quad [x \leq 0].$$

$$\begin{aligned}
 \operatorname{arccotg} x &= \frac{\pi}{2} - \operatorname{arctg} x, \\
 &= \arcsin \frac{1}{\sqrt{1+x^2}} & [x > 0], \\
 &= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} & [x < 0], \\
 &= \arccos \frac{1}{\sqrt{1+x^2}}, \\
 &= \operatorname{arctg} \frac{1}{x} & [x > 0], \\
 &= \pi + \operatorname{arctg} \frac{1}{x} & [x < 0].
 \end{aligned}$$

Б. Суммы и разности обратных тригонометрических функций.

$$\begin{aligned}
 \arcsin x + \arcsin y &= \arcsin (x \sqrt{1-y^2} + y \sqrt{1-x^2}) & [xy \leq 0 \text{ или } x^2 + y^2 \leq 1], \\
 &= \pm \pi - \arcsin (x \sqrt{1-y^2} + y \sqrt{1-x^2}) & \left[\begin{aligned} & \left\{ \begin{aligned} x, y > 0 \\ x, y < 0 \end{aligned} \right\}, & x^2 + y^2 > 1 \end{aligned} \right], \\
 &= \pm \arccos [\sqrt{(1-x^2)(1-y^2)} - xy] & \left[\begin{aligned} & \left\{ \begin{aligned} x, y \geq 0 \\ x, y \leq 0 \end{aligned} \right\} \end{aligned} \right], \\
 &= \operatorname{arctg} \frac{x \sqrt{1-y^2} + y \sqrt{1-x^2}}{\sqrt{(1-x^2)(1-y^2)} - xy} & [xy \leq 0 \text{ или } x^2 + y^2 < 1], \\
 &= \operatorname{arctg} \frac{x \sqrt{1-y^2} + y \sqrt{1-x^2}}{\sqrt{(1-x^2)(1-y^2)} - xy} \pm \pi & \left[\begin{aligned} & \left\{ \begin{aligned} x, y > 0 \\ x, y < 0 \end{aligned} \right\}, & x^2 + y^2 > 1 \end{aligned} \right],
 \end{aligned}$$

$$\arccos x + \arccos y = 2\pi \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \pm \arccos [xy - \sqrt{(1-x^2)(1-y^2)}] \quad \left[\begin{aligned} & \left\{ \begin{aligned} x+y \geq 0 \\ x+y < 0 \end{aligned} \right\} \end{aligned} \right],$$

$$\arccos x - \arccos y = \mp \arccos [xy + \sqrt{(1-x^2)(1-y^2)}] \quad \left[\begin{aligned} & \left\{ \begin{aligned} x \geq y \\ x < y \end{aligned} \right\} \end{aligned} \right],$$

$$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \frac{x+y}{1-xy} \quad (xy < 1),$$

$$= \pm \pi + \operatorname{arctg} \frac{x+y}{1-xy} \quad \left[\begin{aligned} & \left\{ \begin{aligned} x > 0 \\ x < 0 \end{aligned} \right\}, & xy > 1 \end{aligned} \right],$$

$$\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \frac{x-y}{1+xy} \quad (xy > -1),$$

$$= \pm \pi + \operatorname{arctg} \frac{x-y}{1+xy} \quad \left[\begin{aligned} & \left\{ \begin{aligned} x > 0 \\ x < 0 \end{aligned} \right\} & xy < -1 \end{aligned} \right].$$

1.4. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1. Связь с логарифмической и обратными тригонометрическими функциями

$$\operatorname{Arsh} z = \ln (z + \sqrt{z^2 + 1}) = -i \arcsin (iz),$$

$$\operatorname{Arch} z = \ln (z + \sqrt{z^2 - 1}) = i \arccos z,$$

$$\operatorname{Arth} z = \frac{1}{2} \ln \frac{1+z}{1-z} = -i \operatorname{arctg} (iz),$$

$$\operatorname{Arcth} z = \frac{1}{2} \ln \frac{z+1}{z-1} = -i \operatorname{arctg} (-iz).$$

2. Функции отрицательного аргумента.

$$\operatorname{Arsh} (-z) = -\operatorname{Arsh} z, \quad \operatorname{Arch} (-z) = \operatorname{Arch} z,$$

$$\operatorname{Arth} (-z) = -\operatorname{Arth} z, \quad \operatorname{Arcth} (-z) = -\operatorname{Arcth} z.$$

3. Формулы дифференцирования.

$$\frac{d}{dx} \operatorname{Arsh} \frac{x}{a} = \frac{1}{\sqrt{x^2 + a^2}},$$

$$\frac{d}{dx} \operatorname{Arch} \frac{x}{a} = \frac{\pm 1}{\sqrt{x^2 - a^2}}$$

$$[\operatorname{Arch} (x/a) \geq 0, \quad x/a > 1],$$

$$\frac{d}{dx} \operatorname{Arth} \frac{x}{a} = \frac{a}{a^2 - x^2}$$

$$[x^2 < a^2],$$

$$\frac{d}{dx} \operatorname{Arcth} \frac{x}{a} = \frac{a}{a^2 - x^2}$$

$$[x^2 > a^2].$$

4. Связь между обратными гиперболическими функциями.

$$\begin{aligned} \operatorname{Arsh} x &= \operatorname{Arch} \sqrt{x^2 + 1}, & \operatorname{Arch} x &= \operatorname{Arsh} \sqrt{x^2 - 1}, \\ &= \operatorname{Arth} \frac{x}{\sqrt{x^2 + 1}}, & &= \operatorname{Arth} \frac{\sqrt{x^2 - 1}}{x}, \end{aligned}$$

$$\operatorname{Arth} x = \operatorname{Arsh} \frac{x}{\sqrt{1 - x^2}},$$

$$= \operatorname{Arch} \frac{1}{\sqrt{1 - x^2}},$$

$$= \operatorname{Arcth} \frac{1}{x}.$$

5. Суммы и разности обратных гиперболических функций.

$$\operatorname{Arsh} x \pm \operatorname{Arsh} y = \operatorname{Arsh} (x \sqrt{1 + y^2} \pm y \sqrt{1 + x^2}),$$

$$\operatorname{Arch} x \pm \operatorname{Arch} y = \operatorname{Arch} [xy \pm \sqrt{(x^2 - 1)(y^2 - 1)}],$$

$$\operatorname{Arth} x \pm \operatorname{Arth} y = \operatorname{Arth} \frac{x \pm y}{1 \pm xy},$$

$$\begin{aligned} \operatorname{Arsh} x \pm \operatorname{Arch} y &= \operatorname{Arsh} [xy \pm \sqrt{(x^2 + 1)(y^2 - 1)}], \\ &= \operatorname{Arch} [y \sqrt{x^2 + 1} \pm x \sqrt{y^2 - 1}], \end{aligned}$$

$$\operatorname{Arth} x \pm \operatorname{Arcth} y = \operatorname{Arth} \frac{xy \pm 1}{y \pm x},$$

$$= \operatorname{Arcth} \frac{y \pm x}{xy \pm 1}.$$

1.5. БИНОМИАЛЬНЫЕ КОЭФФИЦИЕНТЫ

$$\binom{a}{k} = (-1)^k \frac{(-a)_k}{k!} = \frac{a(a-1)\dots(a-k+1)}{k!} \quad [k=1, 2, 3, \dots].$$

$$\binom{a}{0} = 1, \quad \binom{a}{-k} = 0 \quad [k=1, 2, 3, \dots; a \neq -1, -2, -3, \dots].$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad [k=1, 2, \dots, n].$$

$$\binom{n}{k} = 0 \quad [k=-1, -2, -3, \dots \text{ или } k > n].$$

$$\binom{n}{k} = \binom{n}{n-k} \quad [n \geq k \geq 0].$$

$$\binom{a}{k} = (-1)^k \binom{-a+k-1}{k}, \quad \binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

$$\binom{a}{b} = \binom{a}{a-b} \quad [b \neq -k, k+a, \text{ где } k=-1, -2, -3, \dots].$$

$$\binom{a}{b+1} = \frac{a}{b+1} \binom{a-1}{b} = \frac{a-b}{b+1} \binom{a}{b}, \quad \binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1},$$

$$\binom{n+a}{n} \binom{b}{n+a} = \binom{b}{n} \binom{b-n}{a} = \binom{b}{a} \binom{b-a}{n},$$

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^{2n}} \binom{2n}{n} = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$

$$\binom{1/2}{n} = \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} = (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!! n},$$

$$\binom{n+1/2}{2n+1} = (-1)^n 2^{-4n-1} \binom{2n}{n}, \quad \binom{2n+1/2}{n} = 2^{-2n} \binom{4n+1}{2n},$$

$$\binom{n}{1/2} = \frac{2^{2n+1}}{\pi} \binom{2n}{n}^{-1}, \quad \binom{n}{n/2} = \frac{2^{2n}}{\pi} \binom{n}{(n-1)/2}.$$

1.6. СИМВОЛ ПОХГАММЕРА $(a)_k$

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} = (-1)^k \frac{\Gamma(1-a)}{\Gamma(1-a-k)},$$

$$(a)_0 = 1, \quad (a)_{-k} = \frac{\Gamma(a-k)}{\Gamma(a)} = \frac{(-1)^k}{(1-a)_k} \quad [a \neq 1, 2, \dots, k; k=1, 2, 3, \dots].$$

$$(n)_k = \frac{(n+k-1)!}{(n-1)!}, \quad (1)_k = k!,$$

$$(1/2)_k = 2^{-2k} \frac{(2k)!}{k!}, \quad (3/2)_k = 2^{-2k} \frac{(2k+1)!}{k!},$$

$$(a)_{n+k} = (a)_n (a+n)_k, \quad (a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k},$$

$$(a)_{nk} = \left(\frac{a}{n}\right)_k \left(\frac{a+1}{n}\right)_k \dots \left(\frac{a+n-1}{n}\right)_k n^{nk}, \quad (a)_{2k} = \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k 2^{2k},$$

$$(a+mk)_{nk} = \frac{(a)_{mk+nk}}{(a)_{mk}}, \quad (a+k)_k = \frac{(a)_{2k}}{(a)_k}, \quad (a+m)_k = \frac{(a)_k (a+k)_m}{(a)_m},$$

$$(a - mk)_{nk} = (-1)^{mk} (a)_{nk-mk} (1-a)_{mk} \quad [n \geq m],$$

$$= (-1)^{nk} \frac{(1-a)_{mk}}{(1-a)_{mk-nk}} \quad [n < m].$$

$$(a - k)_k = (-1)^k (1-a)_k, \quad (a - m)_k = \frac{(1-a)_m (a)_k}{(1-a-k)_m},$$

$$(a + mk)_{n-lk} = \frac{(a)_n (a+n)_{mk-lk}}{(a)_{mk}}, \quad (a + k)_{n-k} = \frac{(a)_n}{(a)_k},$$

$$(a - mk)_{n-lk} = \frac{(-1)^{lk} (a)_n (1-a)_{mk}}{(1-a-n)_{lk+mk}}, \quad (a - k)_{n-k} = \frac{(-1)^k (a)_n (1-a)_k}{(1-a-n)_{2n}}.$$

Приложение II. НЕКОТОРЫЕ СПЕЦИАЛЬНЫЕ ФУНКЦИИ И ИХ СВОЙСТВА

II.1. ГАММА-ФУНКЦИЯ $\Gamma(z)$

О п р е д е л е н и е: $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad [\operatorname{Re} z > 0].$

При $\operatorname{Re} z \leq 0$ функция $\Gamma(z)$ определяется как аналитическое продолжение интеграла.

$\Gamma(z)$ — аналитическая функция во всей z -плоскости, за исключением точек $z=0, -1, -2, \dots$, в которых она имеет простые полюсы.

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!, \quad \Gamma(1) = \Gamma(2) = 1.$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}, \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$$

$$\Gamma\left(n + \frac{1}{3}\right) = \frac{1 \cdot 4 \cdot 7 \dots (3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right), \quad \Gamma\left(\frac{1}{3}\right) = 2,678938 \dots$$

$$\Gamma\left(n + \frac{2}{3}\right) = \frac{2 \cdot 5 \cdot 8 \dots (3n-1)}{3^n} \Gamma\left(\frac{2}{3}\right), \quad \Gamma\left(\frac{2}{3}\right) = 1,354118, \dots$$

$$\Gamma\left(n + \frac{1}{4}\right) = \frac{1 \cdot 5 \cdot 9 \dots (4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right), \quad \Gamma\left(\frac{1}{4}\right) = 3,625600 \dots$$

$$\Gamma\left(n + \frac{3}{4}\right) = \frac{3 \cdot 7 \cdot 11 \dots (4n-1)}{4^n} \Gamma\left(\frac{3}{4}\right), \quad \Gamma\left(\frac{3}{4}\right) = 1,225417 \dots$$

$$\Gamma(z) \Gamma(-z) = -\frac{\pi}{z \sin \pi z}, \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z}.$$

$$\frac{\Gamma(z+1)}{\Gamma(w+1) \Gamma(z-w+1)} = \binom{z}{w}, \quad \frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n.$$

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right),$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

$$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi} \Gamma(z) \Gamma\left(z + \frac{1}{3}\right) \Gamma\left(z + \frac{2}{3}\right).$$

$$|\Gamma(ix)|^2 = \frac{\pi}{x \operatorname{sh} \pi x}, \quad \left| \Gamma\left(\frac{1}{2} + ix\right) \right|^2 = \frac{\pi}{\operatorname{ch} \pi x},$$

$$|\Gamma(1+ix)|^2 = \frac{\pi x}{\operatorname{sh} \pi x}, \quad \Gamma\left(\frac{1}{4} + ix\right) \Gamma\left(\frac{3}{4} - ix\right) = \frac{\pi \sqrt{2}}{\operatorname{ch} \pi x + i \operatorname{sh} \pi x}.$$

II.2. ФУНКЦИЯ $\psi(z)$

Определение: $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

$\psi(z)$ — аналитическая функция во всей z -плоскости, за исключением точек $z=0, -1, -2, \dots$, в которых она имеет простые полюсы.

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{k+z}, \quad \psi(z-n) = \psi(z) - \sum_{k=1}^n \frac{1}{z-k},$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

$$\psi(z) - \psi(-z) = -\pi \operatorname{ctg} \pi z - \frac{1}{z}, \quad \psi(z) - \psi(1-z) = -\pi \operatorname{ctg} \pi z.$$

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \operatorname{ctg} \pi z, \quad \psi\left(\frac{1}{2} + z\right) - \psi\left(\frac{1}{2} - z\right) = \pi \operatorname{tg} \pi z.$$

$$\psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n, \quad \psi(2z) = \frac{1}{2} \psi(z) + \frac{1}{2} \psi\left(z + \frac{1}{2}\right) + \ln 2.$$

$$\operatorname{Re} \psi(ix) = \operatorname{Re} \psi(1-ix), \quad \operatorname{Im} \psi(ix) = \frac{1}{2x} + \frac{\pi}{2} \operatorname{cth} \pi x.$$

$$\operatorname{Im} \psi(1+ix) = -\frac{1}{2x} + \frac{\pi}{2} \operatorname{cth} \pi x, \quad \operatorname{Im} \psi\left(\frac{1}{2} + ix\right) = \frac{\pi}{2} \operatorname{th} \pi x.$$

$$\psi(n+1) = -\mathbf{C} + \sum_{k=1}^n \frac{1}{k}, \quad \psi(1) = -\mathbf{C}.$$

$$\psi\left(\frac{1}{2} + n\right) = -\mathbf{C} - 2 \ln 2 + 2 \sum_{k=0}^{n-1} \frac{1}{2k+1}, \quad \psi\left(\frac{1}{2}\right) = -\mathbf{C} - 2 \ln 2.$$

$$\psi\left(\frac{1}{3}\right) = -\mathbf{C} - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3, \quad \psi\left(\frac{2}{3}\right) = -\mathbf{C} + \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3.$$

$$\psi\left(\frac{1}{4}\right) = -\mathbf{C} - \frac{\pi}{2} - 3 \ln 2, \quad \psi\left(\frac{3}{4}\right) = -\mathbf{C} + \frac{\pi}{2} - 3 \ln 2.$$

$$\psi\left(\frac{p}{q}\right) = -C - \ln(2q) - \frac{\pi}{2} \operatorname{ctg} \frac{p\pi}{q} + 2 \sum_{k=1}^{[(q-1)/2]} \cos \frac{2pk\pi}{q} \ln \sin \frac{k\pi}{q}$$

[$p = 1, 2, \dots, q-1; q = 2, 3, 4, \dots$].

$$\psi(a) = -C + (a-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+a)}$$

[см. 5.1.7, 5.1.8].

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}}.$$

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad [n=1, 2, 3, \dots]$$

$$\psi'(1) = \frac{\pi^2}{6}, \quad \psi^{(m)}(n+1) = (-1)^m m! \left[-\zeta(m+1) + \sum_{k=1}^n \frac{1}{k^{m+1}} \right],$$

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}, \quad \psi^{(n)}(z+m) = \psi^{(n)}(z) + (-1)^n n! \sum_{k=0}^{m-1} \frac{1}{(k+z)^{n+1}}$$

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1} - 1) \zeta(n+1) \quad [n=1, 2, 3, \dots]$$

$$\psi'\left(\frac{1}{2} \pm n\right) = \frac{\pi^2}{2} \mp 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}, \quad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}.$$

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1},$$

$$\psi^{(n)}(1-z) + (-1)^{n+1} \psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \operatorname{ctg} \pi z.$$

II.3. ФУНКЦИЯ $\beta(z)$

Определение:

$$\beta(z) = \frac{1}{2} \left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right) \right],$$

$$\beta(z) = \int_0^1 \frac{t^{z-1}}{1+t} dt \quad [\operatorname{Re} z > 0],$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+z}$$

[см. 5.1.5].

$$\beta(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+z)(2k+z+1)},$$

[см. 5.1.9].

$$\beta(z+1) = \frac{1}{z} - \beta(z), \quad \beta(1-z) = \frac{2\pi^2}{\sin \pi z} - \beta(z),$$

$$\beta(nz) = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \beta\left(z + \frac{k}{n}\right),$$

$$\beta(n+1) = (-1)^n \ln 2 + (-1)^n \sum_{k=1}^n \frac{(-1)^k}{k}, \quad \beta(1) = \ln 2.$$

$$\beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin(p\pi/q)} - \sum_{k=0}^{\lfloor (q-1)/2 \rfloor} \cos \frac{(2k+1)p\pi}{q} \ln \left(2 - 2 \cos \frac{(2k+1)\pi}{q}\right)$$

[$p=1, 2, 3, \dots, q-1; q=2, 3, 4, \dots$].

II.4. ДЗЕТА-ФУНКЦИЯ РИМАНА $\zeta(z)$

Определение:

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \quad [\operatorname{Re} z > 1].$$

При $\operatorname{Re} z \leq 1, z \neq 1$, $\zeta(z)$ определяется как аналитическое продолжение этого ряда.

$$\zeta(2n) = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}| \quad [n=1, 2, 3, \dots],$$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \quad [n=1, 2, 3, \dots],$$

$$\zeta(-2n) = 0 \quad [n=1, 2, 3, \dots],$$

$$\zeta'(0) = -\frac{1}{2} \ln 2\pi,$$

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad [n=2, 3, 4, \dots; \text{см. 5.1.2}].$$

$$\lim_{z \rightarrow 1} \left[\zeta(z) - \frac{1}{z-1} \right] = \mathbf{C}.$$

II.5. МНОГОЧЛЕНЫ БЕРНУЛЛИ $B_n(x)$

Определение:

$$\frac{t e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad [|t| < 2\pi].$$

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad [\text{см. II.6}].$$

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \quad B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

$$B_n(0) = B_n, \quad B_n(1) = (-1)^n B_n, \quad B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n}) B_n.$$

II.6. ЧИСЛА БЕРНУЛЛИ B_n

Определение:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

$$B_n = \sum_{k=0}^n \binom{n}{k} B_k, \quad B_0 = 1, \quad B_n = B_n(0).$$

$$B_{2n} = \frac{(-1)^{n-1} (2n)!}{2^{2n-1} \pi^{2n}} \zeta(2n), \quad B_{2n+1} = 0 \quad [n=1, 2, 3, \dots]$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

II.7. МНОГОЧЛЕНЫ ЭЙЛЕРА $E_n(x)$

Определение:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} \quad |t| < \pi.$$

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x,$$

$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \quad E_4(x) = x^4 - 2x^3 + x,$$

$$E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}, \quad E_6(x) = x^6 - 3x^5 + 5x^3 - 3x.$$

II.8. ЧИСЛА ЭЙЛЕРА E_n

Определение:

$$\frac{1}{\operatorname{ch} t} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}.$$

$$E_n = 2^n E_n\left(\frac{1}{2}\right), \quad E_{2n+1} = 0,$$

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

$$E_{10} = -50521, \quad E_{12} = 2702765.$$

П р и л о ж е н и е III. ТАБЛИЦА ФУНКЦИЙ V_i и $V_i^\pm$

Обозначение:

 $r = p/q$; p и q — взаимно простые натуральные числа; $m = 2p + q + 1$; $n = p + q + 1$.

$$V_1^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_1^+ y^s F_{n-1} \left(\begin{matrix} 1, \Delta(p, a_1), \Delta(q, b_1); z \\ \Delta(p, c_1), \Delta(q, 1+j) \end{matrix} \right),$$

$$V_2^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_2^+ y^t F_{n-1} \left(\begin{matrix} 1, \Delta(p, a_2), \Delta(q, b_2); z \\ \Delta(p, 1+h), \Delta(q, c_2) \end{matrix} \right),$$

$$V_1^- = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_1^- y^u F_{n-1} \left(\begin{matrix} 1, \Delta(p, a_3), \Delta(q, b_3); z^{-1} \\ \Delta(p, 1+h), \Delta(q, c_3) \end{matrix} \right),$$

$$V_2^- = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_2^- y^v F_{n-1} \left(\begin{matrix} 1, \Delta(p, a_4), \Delta(q, b_4); z^{-1} \\ \Delta(p, c_4), \Delta(q, 1+j) \end{matrix} \right),$$

$$V_3^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_3^+ y^s F_q \left(\begin{matrix} 1, \Delta(q, a_1); z \\ \Delta(q, 1+j) \end{matrix} \right),$$

$$V_4^+ = \sum_{h=0}^{p-1} E_4^+ y^t F_q \left(\begin{matrix} 1, \Delta(q, a_2); z \\ \Delta(q, b_2) \end{matrix} \right),$$

$$V_3^- = \sum_{h=0}^{p-1} E_3^- y^u F_q \left(\begin{matrix} 1, \Delta(q, a_3); z^{-1} \\ \Delta(q, b_3) \end{matrix} \right),$$

$$V_4^- = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_4^- y^v F_q \left(\begin{matrix} 1, \Delta(q, a_4); z^{-1} \\ \Delta(q, 1+j) \end{matrix} \right),$$

$$V_5^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_5^+ y^s F_{m-1} \left(\begin{matrix} 1, \Delta(p, a_1), \Delta(p, b_1), \Delta(q, c_1); z \\ \Delta(p, d_1), \Delta(p, e_1), \Delta(q, 1+j) \end{matrix} \right),$$

$$V_6^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_6^+ y^t F_{m-1} \left(\begin{matrix} 1, \Delta(p, a_2), \Delta(p, b_2), \Delta(q, c_2); z \\ \Delta(p, 1+h), \Delta(p, d_2), \Delta(q, e_2) \end{matrix} \right),$$

$$V_7^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_7^+ y^t F_{m-1} \left(\begin{matrix} 1, \Delta(p, a_3), \Delta(p, b_3), \Delta(q, c_3); z \\ \Delta(p, 1+h), \Delta(p, d_3), \Delta(q, e_3) \end{matrix} \right),$$

$$V_5^- = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_5^- y^u F_{m-1} \left(\begin{matrix} 1, \Delta(p, a_4), \Delta(p, b_4), \Delta(q, c_4); z^{-1} \\ \Delta(p, 1+h), \Delta(p, d_4), \Delta(q, e_4) \end{matrix} \right),$$

$$V_6^- = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_6^- y^{a_1} {}_mF_{m-1} \left(\begin{matrix} 1, \Delta(p, a_5), \Delta(p, b_5), \Delta(q, c_5) \\ \Delta(p, 1+h), \Delta(p, d_5), \Delta(q, e_5) \end{matrix}; z^{-1} \right),$$

$$V_7^- = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_7^- y^b {}_mF_{m-1} \left(\begin{matrix} 1, \Delta(p, a_6), \Delta(p, b_6), \Delta(q, c_6) \\ \Delta(p, d_6), \Delta(p, e_6), \Delta(q, 1+j) \end{matrix}; z^{-1} \right),$$

$$V_{7+k}^+ = E_{7+k}^+ y^{s_k} {}_4F_3 \left(\begin{matrix} a_k, b_k, c_k, d_k \\ e_k, f_k, g_k \end{matrix}; z \right) \quad [k=1, 2, 3, 4],$$

$$V_{3+k}^- = E_{3+k}^- y^{s_k} {}_4F_3 \left(\begin{matrix} a_k, b_k, c_k, d_k \\ e_k, f_k, g_k \end{matrix}; z^{-1} \right) \quad [k=5, 6, 7, 8],$$

$$V_{12}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{12}^+ \frac{\sin [\gamma (lr + \alpha - 1)]}{\sin [(lr + \alpha) \pi]} y^{lr + \alpha - 2},$$

$$V_{12}^+ = \sum_{l=0}^{\infty} \Gamma \left(\frac{\alpha - 1 - l}{r} \right) E_{12}^+ \sin (l\gamma) (-1)^l y^{l-1}.$$

$$V_{12}^- = \sum_{l=0}^{\infty} \Gamma \left(\frac{\alpha + l}{r} \right) E_{12}^- \sin [(l+1)\gamma] (-1)^l y^{-l-2}.$$

$$V_{13}^- = -\frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \frac{(1-\rho)_l (-1)^l}{l!} E_{13}^- \frac{\sin [\gamma (\alpha - 1 + r(\rho - l - 1))]}{\sin [(\alpha + r(\rho - l - 1)) \pi]} y^{\alpha + r(\rho - l - 1) - 2},$$

$$V_{14}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{14}^+ \frac{\sin [(lr + \alpha - 1)\gamma]}{(1-\sigma)_l \sin [(lr + \alpha) \pi] \sin \gamma} z^{lr + \alpha - 2},$$

$$V_{13}^+ = \sum_{l=0}^{\infty} (-1)^l E_{13}^+ \frac{\sin (l\gamma)}{\sin \gamma} \Gamma \left[\frac{(\alpha - l - 1)/r}{1 - \sigma - (\alpha - l - 1)/r} \right] z^{l-1},$$

$$V_{14}^- = \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \sin \gamma} E_{14}^- \frac{\sin [(\alpha - lr/2 - hr/2 - 1)\gamma]}{\sin [(\alpha - (lr + hr)/2) \pi]} \Gamma \left[\frac{(l+h)/2}{1 - \sigma - (l+h)/2} \right] z^{\alpha - r(l+h)/2 - 2},$$

$$V_{15}^- = \sum_{l=0}^{\infty} (-1)^l E_{15}^- \frac{\sin [(l+1)\gamma]}{\sin \gamma} \Gamma \left[\frac{(\alpha + l)/2, h - 2(\alpha + l)/r}{1 - \sigma - (\alpha + l)/r} \right] z^{-l-2},$$

$$V_{16}^+ = \pi \sum_{l=0}^{\infty} (-1)^l \frac{\sin (\lambda + l\lambda) \sin [\gamma (1 - \alpha - lr)]}{\sin \lambda \sin \gamma \sin [(\alpha + lr) \pi]} y^{lr + \alpha - 2},$$

$$V_{12}^+ = \frac{\pi}{r} \sum_{l=0}^{\infty} (-1)^l \frac{\sin (l\gamma) \sin [\lambda r^{-1} (\alpha - l - 1 - r)]}{\sin \lambda \sin [(\alpha - l - 1) \pi / r] \sin \gamma} y^{l-1}.$$

$$V_{16}^{-} = \pi \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\lambda l) \sin[\gamma(\alpha - r - lr - 1)]}{\sin \lambda \sin \gamma \sin[(\alpha - r - lr)\pi]} y^{\alpha - r - lr - 2},$$

$$V_{17}^{-} = \frac{\pi}{r} \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\gamma + l\gamma) \sin[\lambda r^{-1}(r - \alpha - l)]}{\sin \lambda \sin \gamma \sin[(\alpha + l)\pi/r]} y^{-l-2},$$

$$V_{18}^{+} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{13}^{+} \frac{(\alpha + lr - 1) \cos[\gamma(\alpha + lr - 2)]}{\sin^2 \gamma \sin[(\alpha + lr)\pi]} y^{\alpha + lr - 4},$$

$$V_{19}^{+} = \sum_{l=0}^{\infty} (-1)^l E_{19}^{+} \Gamma\left(\frac{\alpha - l - 1}{r}\right) \frac{l \cos(\gamma l - \gamma)}{\sin^2 \gamma} y^{l-3},$$

$$V_{19}^{-} = \sum_{l=0}^{\infty} (-1)^l E_{18}^{-} \Gamma\left(\frac{\alpha + l}{r}\right) \frac{(l+1) \cos(\gamma l + 2\gamma)}{\sin^2 \gamma} y^{-l-4},$$

$$V_{19}^{-} = \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l (1-\rho)_l [\alpha - r(1-\rho+l) - 1] \cos[\gamma(\alpha - r(1-\rho+l) - 2)]}{l! \sin[(\alpha - r(1-\rho+l))\pi]} y^{r(1-\rho+l) - \alpha - 4},$$

$$V_{20}^{+} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{20}^{+} \frac{(lr + \alpha - 1) \cos[\gamma(lr + \alpha - 2)]}{\sin^2 \gamma \sin[(lr + \alpha)\pi] (1-\sigma)_\rho} z^{\alpha + lr - 4},$$

$$V_{21}^{+} = \sum_{l=0}^{\infty} (-1)^l E_{21}^{+} \frac{l \cos(l\gamma - \gamma)}{\sin^2 \gamma} \Gamma\left[\frac{(\alpha - l - 1)/r}{1 - \sigma - (\alpha - l - 1)/r}\right] z^{l-3},$$

$$V_{20}^{-} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{20}^{-} \times \\ \times \frac{(\alpha - r(h+l)/2 - 1) \cos[\gamma(\alpha - r(h+l)/2 - 2)]}{\sin^2 \gamma \sin[(\alpha - r(h+l)/2)\pi]} \Gamma\left[\frac{(l+h)/2}{1 - \sigma - (l+h)/2}\right] z^{r(h+l)/2 + 4 - \alpha},$$

$$V_{21}^{-} = \sum_{l=0}^{\infty} (-1)^l E_{21}^{-} \frac{(l+1) \cos[(l+2)\gamma]}{\sin^2 \gamma} \Gamma\left[\frac{(\alpha + l)/r}{1 - \sigma - (\alpha + l)/r}\right] z^{-l-4},$$

$$V_{22}^{+} = \frac{\pi}{2} \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\lambda l + \lambda) \cos[\gamma(\alpha + lr - 2)] (1 - \alpha - lr)}{\sin \lambda \sin^2 \gamma \sin[(\alpha + lr)\pi]} y^{\alpha + lr - 4},$$

$$V_{23}^{+} = \frac{\pi}{2r} \sum_{l=0}^{\infty} (-1)^l \frac{l \cos(\gamma' - \gamma) \sin[\lambda r^{-1}(\alpha - l - 1 - r)]}{\sin \lambda \sin^2 \gamma \sin[(\alpha - l - 1)\pi/r]} y^{l-3},$$

$$V_{22}^{-} = \frac{\pi}{2} \sum_{l=0}^{\infty} (-1)^l \frac{(\alpha - r - rl - 1) \sin(\lambda l) \cos[\gamma(\alpha - r - rl - 2)]}{\sin \lambda \sin^2 \gamma \sin[(\alpha - lr - r)\pi]} y^{\alpha - r - rl - 4},$$

$$V_{23} = \frac{\pi}{2r} \sum_{l=0}^{\infty} (-1)^l \frac{(l+1) \cos(\gamma l + 2\gamma) \sin[\lambda r^{-1}(r - \alpha - l)]}{\sin \lambda \sin^2 \gamma \sin[(\alpha + l)\pi/r]} y^{-l-2}.$$

$$V_{24}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{24}^+ \frac{\sin[\gamma(lr + \lambda r + \alpha - 1)]}{\sin[(lr + \lambda r + \alpha)\pi]} y^{\alpha + lr + \lambda r - 2}.$$

$$V_{25}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{25}^+ \frac{(lr + \lambda r + \alpha - 1) \cos[\gamma(lr + \lambda r + \alpha - 2)]}{\sin[(lr + \lambda r + \alpha)\pi]} y^{\alpha + lr + \lambda r - 2}.$$

$$V_{26} = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{26} y^{u_s h} {}_{q+1}F_{q+p} \left(\begin{matrix} 1, \Delta(q, a); \\ \Delta(p, 1+h), \Delta(q, b) \end{matrix} \middle| z \right).$$

$$V_{27} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{27} y^{u_s v_1} {}_{q+1}F_{q+p} \left(\begin{matrix} 1, \Delta(q, a_1); \\ \Delta(p, b_1), \Delta(q, 1+j) \end{matrix} \middle| z \right).$$

$$V_{27+k} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{27+k} y^{u_s v_k} {}_{2q+1}F_{2q+p} \left(\begin{matrix} 1, \Delta(q, a_k), \Delta(q, b_k); \\ \Delta(p, c_k), \Delta(q, d_k), \Delta(q, 1+j) \end{matrix} \middle| z \right) \\ [k=1, 2].$$

$$V_{28} = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{28} y^{u_s h} {}_{2q+1}F_{2q+p} \left(\begin{matrix} 1, \Delta(q, a_3), \Delta(q, b_3); \\ \Delta(p, 1+h), \Delta(q, c_3), \Delta(q, d_3) \end{matrix} \middle| z \right).$$

$$V_{29} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{29} y^{u_s} \tilde{b}^{2\gamma+2h} \times \\ \times {}_{q+1}F_{q+2p} \left(\begin{matrix} 1, \Delta(q, a_1); \\ \Delta(q, b_1), \Delta(p, c_1), \Delta(p, 1+h) \end{matrix} \middle| z \right).$$

$$V_{30} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{30} y^{u_s} \tilde{b}^{v_2} \times \\ \times {}_{q+1}F_{q+2p} \left(\begin{matrix} 1, \Delta(q, a_2); \\ \Delta(q, 1+j), \Delta(p, b_2), \Delta(p, c_2) \end{matrix} \middle| z \right).$$

$$V_{31} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{31} y^{u_s} \tilde{b}^{2\gamma+2h} {}_1F_{2p} \left(\begin{matrix} 1; \\ \Delta(p, 1+h), \Delta(p, a_1) \end{matrix} \middle| z \right).$$

$$V_{32} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} E_{32} y^{u_s} \tilde{b}^{v_1} {}_1F_{2p} \left(\begin{matrix} 1; \\ \Delta(p, a_2), \Delta(p, b_2) \end{matrix} \middle| z \right).$$

$$V_{33} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{33} y^{u_s} \tilde{b}^{2\gamma+2h} \times \\ \times {}_{2q+1}F_{2q+2p} \left(\begin{matrix} 1, \Delta(q, a_1), \Delta(q, b_1); \\ \Delta(q, c_1), \Delta(q, d_1), \Delta(p, e_1), \Delta(p, 1+h) \end{matrix} \middle| z \right).$$

$$V_{36} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{36} y^u \tilde{b}^{v_2} \times \\ \times {}_{2q+1}F_{2q+2p} \left(1, \Delta(q, a_2), \Delta(q, b_2); \Delta(q, 1+j), \Delta(q, c_2), \Delta(p, d_2), \Delta(p, e_2) \right)^z,$$

$$V_{37} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{37} y^u \tilde{b}^{v_3} \times \\ \times {}_{2q+1}F_{2q+2p} \left(1, \Delta(q, a_3), \Delta(q, b_3); \Delta(q, 1+j), \Delta(q, c_3), \Delta(p, d_3), \Delta(p, e_3) \right)^z,$$

$$V_{38} = y^u \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{38} \frac{\tilde{b}^{v_1} \sin a_l}{\Gamma(c_l) \sin b_l},$$

$$V_{39} = y^u \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{39} \Gamma \left[\begin{matrix} e_l, f_l \\ g_l \end{matrix} \right] \tilde{b}^{v_2} \sin d_l,$$

$$V_{40} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{40} s^u \varphi^{v_1} {}_{q+1}F_{2p} \left(1, \Delta(q, a_1); \Delta(p, b_1), \Delta(p, 1+h) \right)^z,$$

$$V_{41} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{41} s^u \varphi^{v_2} {}_{2p+1}F_q \left(1, \Delta(p, a_2), \Delta(p, b_2); \Delta(q, 1+j) \right)^z,$$

$$V_{42} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{42} s^u \varphi^{v_1} {}_{2p+q}F_{2p+q} \left(1; \Delta(p, 1+h), \Delta(p, a_1), \Delta(q, b_1) \right)^z,$$

$$V_{43} = \sum_{k=0}^{[(n-1)/2]} \varepsilon \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{43} s^u \varphi^{v_2} {}_{2p+q}F_{2p+q} \left(1; \Delta(p, a_2), \Delta(p, b_2); \Delta(q, 1+j) \right)^z.$$

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УКАЗАТЕЛЬ ОБОЗНАЧЕНИЙ ФУНКЦИЙ И ПОСТОЯННЫХ

$A(x)$ — алгебраическая функция или степенная функция с произвольным показателем

$\operatorname{am} u \equiv \operatorname{am}(u, k)$ — эллиптическая функция Якоби, $u = \int_0^{\operatorname{am} u} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$

$\arccos z$, $\operatorname{arccosec} z$, $\operatorname{arcsec} z$, $\arcsin z$, $\operatorname{arccotg} z$, $\operatorname{arctg} z$ — обратные тригонометрические функции

$\operatorname{Arch} z$, $\operatorname{Arcosech} z$, $\operatorname{Arsech} z$, $\operatorname{Arsh} z$, $\operatorname{Arcth} z$, $\operatorname{Arth} z$ — обратные гиперболические функции

$\arg z$ — аргумент комплексного числа z ($z = |z|e^{i \arg z}$)

B_n — числа Бернулли

$B_n(z)$ — многочлены Бернулли

$\operatorname{bei}_\nu(z)$, $\operatorname{ber}_\nu(z)$, $\operatorname{bei}(z) = \operatorname{bei}_0(z)$, $\operatorname{ber}(z) = \operatorname{ber}_0(z)$ ($\operatorname{ber}_\nu(z) + i \operatorname{bei}_\nu(z) = J_\nu(e^{3\pi i/4} z)$) — функции Кельвина

$C = 0,5772156649 \dots$ — постоянная Эйлера

$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt$ — косинус-интеграл Френеля

$C(x, \alpha) = \int_x^\infty t^{\alpha-1} \cos t dt$ [$\operatorname{Re} \alpha < 1$] — обобщенный косинус-интеграл Френеля

$C_n^\lambda(z) = \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, n+2\lambda; \lambda + \frac{1}{2}; \frac{1-z}{2}\right)$ — многочлены Гегенбауэра

$\operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u}$, $\operatorname{cn} u = \cos(\operatorname{am} u)$ — эллиптические функции Якоби

$\operatorname{ch} z$ — гиперболическая функция

$\operatorname{chi}(x) = C + \ln x + \int_0^x \frac{\operatorname{ch} t - 1}{t} dt$ — интегральный гиперболический косинус

$\operatorname{ci}(x) = - \int_x^\infty \frac{\cos t}{t} dt$ — интегральный косинус

$\operatorname{Cl}_2(z) = - \int_0^z \ln\left(2 \sin \frac{x}{2}\right) dx$ — интеграл Клаузена

$\csc z$, $\operatorname{cosec} z = \frac{1}{\sin z}$, $\operatorname{ctg} z$ — тригонометрические функции

$\operatorname{cosech} z = \frac{1}{\operatorname{sh} z}$, $\operatorname{cth} z$ — гиперболические функции

$$D(k) = \int_0^{\pi/2} \frac{\sin^2 \varphi d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = D\left(\frac{\pi}{2}, k\right) - \text{полный эллиптический интеграл}$$

$$D(\varphi, k) = \int_0^{\varphi} \frac{\sin^2 \varphi d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} - \text{эллиптический интеграл}$$

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \text{функция параболического цилиндра}$$

$$\operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}, \quad \operatorname{dn} u = \sqrt{1-k^2 \sin^2(\operatorname{am} u)}, \quad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u} - \text{эллиптические функции Якоби}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} d\varphi - \text{полный эллиптический интеграл 2-го рода}$$

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2 \sin^2 \varphi} d\varphi - \text{эллиптический интеграл 2-го рода}$$

E_n — числа Эйлера

$E_n(z)$ — многочлены Эйлера

$$E_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu\varphi - z \sin \varphi) d\varphi - \text{функция Вебера}$$

$$E_\rho(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})} \quad [\rho > 0] - \text{функция Миттаг-Леффлера}$$

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt - \text{интегральная показательная функция}$$

$$e = 2,718281828459 \dots$$

$e^z = \exp z$ — показательная функция

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt - \text{интеграл вероятности}$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt - \text{дополнительный интеграл вероятности}$$

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt - \text{интеграл вероятности мнимого аргумента}$$

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} - \text{эллиптический интеграл 1-го рода}$$

$$F(a, b; c; z) \equiv {}_1F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} - \text{гипергеометрическая функция}$$

Гаусса

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) \equiv {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!} -$$

обобщенная гипергеометрическая функция

$${}_1F_1(a; b, z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!} \text{ — вырожденная гипергеометрическая функция}$$

$$F_l(\dots; z, \zeta) \quad [l=1, 2, 3, 4] \text{ — функции Аппеля}$$

$$F_1(a, b, b'; c; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_{k+l} (b)_k (b')_l}{(c)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$F_2(a, b, b'; c, c'; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_{k+l} (b)_k (b')_l}{(c)_k (c')_l} \frac{z^k \zeta^l}{k! l!} \quad [|z| + |\zeta| < 1]$$

$$F_3(a, a', b, b'; c; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_k (a')_l (b)_k (b')_l}{(c)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$F_4(a, b; c, c'; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_{k+l} (b)_{k+l}}{(c)_k (c')_l} \frac{z^k \zeta^l}{k! l!} \quad [\sqrt{|z|} + \sqrt{|\zeta|} < 1]$$

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0,9159655942 \dots \text{ — постоянная Каталана}$$

$${}_v(z) = \frac{2}{V\pi} \left(\frac{z}{2}\right)^{v+1} \frac{1}{\Gamma(v+3/2)} {}_1F_2(1; 3/2, v+3/2; -z^2/4) \text{ — функция Струве}$$

$$H_v^{(1)}(z) = J_v(z) + iY_v(z), \quad H_v^{(2)}(z) = J_v(z) - iY_v(z) \text{ — функции Бесселя 3-го рода (функции Ганкеля 1-го и 2-го рода)}$$

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \text{ — многочлены Эрмита}$$

$$h_{\mu, \nu}(z) = \frac{z^{\mu+1} e^{-z}}{(\mu+1)^2 - \nu^2} {}_2F_2(1, \mu+3/2; \mu-\nu+2, \mu+\nu+2; 2z)$$

$$H_{\mu, \nu}(z) = h_{\mu, \nu}(z) - \frac{V\pi \Gamma(\mu-\nu+1) \Gamma(\mu+\nu+1)}{z^{\mu+1} \Gamma(\mu+3/2)} \left[I_{\nu}(z) + \frac{\sin(\nu-\mu)\pi}{\pi \cos \mu\pi} K_{\nu}(z) \right]$$

$$\operatorname{hei}_{\nu}(z), \operatorname{her}_{\nu}(z), \quad \operatorname{hei}(z) = \operatorname{hei}_0(z), \quad \operatorname{her}(z) = \operatorname{her}_0(z) \quad (\operatorname{her}_{\nu}^{\sim}(z) + i \operatorname{hei}_{\nu}(z) = H_{\nu}^{(1)}(e^{3\pi i/4} z)) \text{ — функции Кельвина}$$

$$I_{\nu}(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^{\nu} {}_0F_1\left(\nu+1, \frac{z^2}{4}\right) = e^{-\nu\pi i/2} J_{\nu}(ze^{\pi i/2}) \text{ — модифицированная функция Бесселя 1-го рода (функция Бесселя мнимого аргумента)}$$

$$\operatorname{Im} z \text{ — мнимая часть комплексного числа } z = x + iy \quad (\operatorname{Im} z = y)$$

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)} {}_0F_1\left(\nu+1, -\frac{z^2}{4}\right) \text{ — функция Бесселя 1-го рода}$$

$$J_{\nu}(z) = \frac{1}{\pi} \int_0^{\pi} \cos(\nu\varphi - z \sin \varphi) d\varphi \text{ — функция Ангера}$$

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = F\left(\frac{\pi}{2}, k\right) \text{ — полный эллиптический интеграл 1-го рода}$$

$$K_\nu(z) = \frac{\pi [I_{-\nu}(z) - I_\nu(z)]}{2 \sin \nu \pi} \quad [\nu \neq n], \quad K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z) \quad [n = 0, \pm 1, \pm 2, \dots]$$

функция Макдональда (модифицированная функция Бесселя 3-го рода)

k, k' $[0 \leq k \leq 1; k' = \sqrt{1-k^2}]$ — модуль и дополнительный модуль в эллиптических интегралах

$$k_\nu(z) = \frac{1}{\Gamma\left(\frac{\nu}{2} + 1\right)} W_{\nu/2, 1/2}(2z) \text{ — функция Бейтмена}$$

$\text{kei}_\nu(z), \text{ker}_\nu(z), \text{kei}(z) = \text{kei}_0(z), \text{ker}(z) = \text{ker}_0(z)$ $\{\text{ker}_\nu(z) + i \text{kei}_\nu(z) = e^{-\nu \pi i/2} K_\nu(e^{\pi i/4} z)\}$ — функции Кельвина

$$L(x) = - \int_0^x \ln \cos t \, dt \text{ — функция Лобачевского}$$

$$L_\nu(z) = e^{-(\nu+1)\pi i/2} H_\nu(e^{\pi i/2} z) \text{ — модифицированная функция Струве}$$

$$L_n(z) = L_n^0(z) \text{ — многочлены Лагерра}$$

$$L_n^\nu(z) = \frac{z^{-\nu} e^z}{n!} \frac{d^n}{dz^n} (z^{n+\nu} e^{-z}) \text{ — обобщенные многочлены Лагерра}$$

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad [|z| < 1]$$

$$= \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} dt}{e^t - z} \quad [\text{Re } s > 0; |\arg(1-z)| < \pi] \text{ — полилогарифм порядка } s$$

$$\text{li}(z) = \text{Ei}(\ln z), \quad \text{li}(x) = \int_0^x \frac{dt}{\ln t} \text{ — интегральный логарифм, } \ln z = \ln|z| + i \arg z \text{ — натураль-$$

ный логарифм $[z = |z| e^{i(\arg z + 2\pi k)}, k = 0, \pm 1, \pm 2, \dots]$

$M_{\kappa, \mu}(z) = z^\mu + 1/2 e^{-z/2} {}_1F_1(1/2 - \kappa + \mu; 2\mu + 1; z)$ — вырожденная гипергеометрическая функция Уиттекера

$$\text{sc } u, \text{ nd } u, \text{ ns } u, \text{ nc } u = \frac{1}{\text{cn } u}, \text{ nd } u = \frac{1}{\text{dn } u}, \text{ ns } u = \frac{1}{\text{sn } u} \text{ — эллиптические функции}$$

Якоби

$$O_n(z) = \frac{n}{4} \sum_{k=0}^{[n/2]} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n-1} \text{ — многочлены Неймана}$$

$$P_n(z) = \frac{2^{-n}}{n!} \frac{d^n}{dz^n} (z^2 - 1)^n \text{ — многочлены Лежандра}$$

$$P_\nu(z) = P_\nu^0(z) = {}_2F_1\left(-\nu, 1+\nu; 1; \frac{1-z}{2}\right) \quad [|\arg(1+z)| < \pi] \text{ — функция Лежандра 1-го рода}$$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right) \quad [|\arg(z \pm 1)| < \pi],$$

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, 1+\nu; 1-\mu; \frac{1-x}{2}\right) \quad [-1 < x < 1] \text{ — присоединенная функция Лежандра 1-го рода}$$

$$P_n^{(\alpha, \beta)}(z) = \frac{(-1)^n}{2^n n!} (1-z)^{-\alpha} (1+z)^{-\beta} \frac{d^n}{dz^n} [(1-z)^{\alpha+n} (1+z)^{\beta+n}] \text{ — многочлены Якоби}$$

$$Q_\nu(z) = Q_\nu^0(z) \text{ — функция Лежандра 2-го рода}$$

$$Q_\nu^\mu(z) = \frac{e^{i\mu\pi} \sqrt{\pi}}{2^{\nu+1}} \Gamma\left[\frac{\mu+\nu+1}{\nu+3/2}\right] z^{-\mu-\nu-1} (z^2-1)^{\mu/2} {}_2F_1\left((\mu+\nu+1)/2, (\mu+\nu)/2+1; \nu+3/2; z^{-2}\right) \quad [|\arg(z \pm 1)|, |\arg z| < \pi],$$

$$Q_v^\mu(x) = \frac{1}{2} e^{-i\mu\pi} \left[e^{-i\mu\pi/2} Q_v^\mu(x+i0) + e^{i\mu\pi/2} Q_v^\mu(x-i0) \right] =$$

$$= \frac{\pi}{2 \sin \mu\pi} \left[P_v^\mu(x) \cos \mu\pi - \Gamma \left[\frac{v+\mu+1}{v-\mu+1} \right] P_v^{-\mu}(x) \right] \quad [-1 < x < 1] - \text{присоединенная функция Лежандра 2-го рода}$$

$R(x)$ — рациональная функция

$\operatorname{Re} z$ — действительная часть комплексного числа $z = x + iy$ ($\operatorname{Re} z = x$)

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt - \text{синус-интеграл Френеля}$$

$$S(x, \alpha) = \int_x^\infty t^{\alpha-1} \sin t dt \quad [\operatorname{Re} \alpha < 1] - \text{обобщенный синус-интеграл Френеля}$$

$$S_{\mu, \nu}(z) = s_{\mu, \nu}(z) + 2^{\mu-1} \Gamma \left[\frac{\nu, (\mu+\nu+1)/2}{(1+\nu-\mu)/2} \right] (z/2)^{-\nu} {}_2F_1(1-\nu; -z^2/4) + 2^{\mu-1} \times$$

$$\times \Gamma \left[\frac{-\nu, (1+\mu-\nu)/2}{(1-\mu-\nu)/2} \right] (z/2)^\nu {}_2F_1(1+\nu; -z^2/4) - \text{функция Ломмеля}$$

$$s_{\mu, \nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2(1; (\mu-\nu+3)/2, (\mu+\nu+3)/2; -z^2/4) - \text{функция Ломмеля}$$

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt - \text{интегральный синус}$$

$$\operatorname{si}(x) = \operatorname{Si}(x) - \frac{\pi}{2} = - \int_x^\infty \frac{\sin t}{t} dt - \text{интегральный синус}$$

$$\sec z = \frac{1}{\cos z} - \text{тригонометрическая функция}$$

$$\operatorname{sech} z = \frac{1}{\operatorname{ch} z} - \text{гиперболическая функция}$$

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$$

$\operatorname{sh} z$ — гиперболическая функция

$$\operatorname{shi} x = \int_0^x \frac{\operatorname{sh} t}{t} dt = -i \operatorname{Si}(ix) - \text{интегральный гиперболический синус}$$

$\sin z$ — тригонометрическая функция

$\operatorname{sn} u = \sin(\operatorname{am} u)$ — эллиптическая функция Якоби

$$T_n(z) = \cos(n \arccos z) = \frac{1}{2} [(z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n] - \text{многочлены Чебышева 1-го рода}$$

$$\operatorname{tg} z = \frac{\sin z}{\cos z} - \text{тригонометрическая функция}$$

$$\operatorname{th} z = \frac{\operatorname{sh} z}{\operatorname{ch} z} - \text{гиперболическая функция}$$

$$U_n(z) = \frac{\sin[(n+1) \arccos z]}{\sqrt{1-z^2}} - \text{многочлены Чебышева 2-го рода}$$

$$U_\nu(z, \zeta) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{\zeta} \right)^{\nu+2k} J_{\nu+2k}(\zeta) - \text{функция Ломмеля двух переменных}$$

$$W_{\kappa, \mu}(z) = z^{\mu+1/2} e^{-z/2} \Psi(1/2 - \kappa + \mu, 2\mu + 1; z) - \text{вырожденная гипергеометрическая функция Уиттекера}$$

$$Y_\nu(z) = \frac{\cos \nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi} \quad [\nu \neq n], \quad Y_n(z) = \lim_{\nu \rightarrow n} Y_\nu(z) \quad [n = 0, \pm 1, \pm 2, \dots] \rightarrow$$

функция Неймана (функция Бесселя 2-го рода)

$y_n(z)$ — многочлены Бесселя

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \text{ — бета-функция}$$

$$\beta(z) = \frac{1}{2} \left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right) \right]$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad [\operatorname{Re} z > 0] \text{ — гамма-функция}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \text{ — дополнительная неполная гамма-функция}$$

$$\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt \text{ — неполная гамма-функция}$$

$$\Gamma \left[\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \right] = \frac{\prod_{k=1}^m \Gamma(a_k)}{\prod_{l=1}^n \Gamma(b_l)}$$

$$\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \dots, \frac{a+k-1}{k}$$

$$\Delta(2, a) = \frac{a}{2}, \frac{a+1}{2}$$

$$\delta_{m,n} = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases} \text{ — символ Кронекера}$$

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad [\operatorname{Re} s > 1] \text{ — дзета-функция Римана}$$

$$\zeta(s, v) = \sum_{k=0}^{\infty} \frac{1}{(v+k)^s} \quad [\operatorname{Re} s > 1; v \neq 0, -1, -2, \dots] \text{ — обобщенная дзета-функция}$$

Римана

$$\theta_0(v, x) = 1 + 2 \sum_{k=0}^{\infty} (-1)^k e^{-k^2 \pi^2 x} \cos 2kv\pi,$$

$$\theta_1(v, x) = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(k+1/2)^2 \pi^2 x} \sin (2k+1)v\pi,$$

$$\theta_2(v, x) = 2 \sum_{k=0}^{\infty} e^{-(k+1/2)^2 \pi^2 x} \cos (2k+1)v\pi,$$

$$\theta_3(v, x) = 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi^2 x} \cos 2kv\pi,$$

$$\theta_4(v, x) \equiv \theta_0(v, x) \text{ — эллиптические тэта-функции}$$

$$\lambda(x, \alpha) = \int_0^\alpha x^{-t} \Gamma(t+1) dt$$

$$\Psi(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt$$

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\varphi}{(1 - n \sin^2 \varphi) \sqrt{1 - k^2 \sin^2 \varphi}} \text{ — эллиптический интеграл 3-го рода}$$

$$\Phi(z, s, v) = \sum_{k=0}^{\infty} \frac{z^k}{(v+k)^s} \quad [|z| < 1; v \neq 0, -1, -2, \dots]$$

$${}_r\Phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_{q,k} (a_2)_{q,k} \dots (a_r)_{q,k}}{(q)_{q,k} (b_1)_{q,k} \dots (b_s)_{q,k}} z^k \text{ — базис-}$$

ный гипергеометрический ряд

$$(a)_{q,k} = (1-a)(1-aq) \dots (1-aq^{k-1}) \quad [k=1, 2, 3, \dots],$$

$$(c)_{q,0} = 1.$$

$$\Psi(a, b; z) = \Gamma \left[\frac{1-c}{1+a-c} \right] {}_1F_1(a; c; z) + \Gamma \left[\frac{c-1}{a} \right] z^{1-c} {}_1F_1(1+a-c; 2-c; z) \text{ — вырож-}$$

денная гипергеометрическая функция

$$\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)} \text{ — пси-функция}$$

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi(z)$$

УКАЗАТЕЛЬ ОБОЗНАЧЕНИЙ СИМВОЛОВ

$(a)_k = a(a+1) \dots (a+k-1)$ [$k=1, 2, 3, \dots$], $(a)_0 = 1$ — символ Похгаммера

$n! = 1 \cdot 2 \cdot 3 \dots (n-1)n$, $0! = 1! = 1$

$(2n)!! = 2 \cdot 4 \cdot 6 \dots (2n) = 2^n n!$

$(2n+1)!! = 1 \cdot 3 \cdot 5 \dots (2n+1) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(n + \frac{3}{2}\right) = \left(\frac{3}{2}\right)_n 2^n$

$n!! = \begin{cases} (2k)!! & n=2k, \\ (2k+1)!! & n=2k+1, \end{cases} \quad 0!! = (-1)!! = 1$

$\binom{n}{k} = \frac{n(n-1) \dots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \frac{(-1)^k (-n)_k}{k!}$ — биномиальные коэффи-

циенты

$\binom{n}{0} = 1$

$\operatorname{Re} a, \operatorname{Re} b > c$ означает $\operatorname{Re} a > c$ и $\operatorname{Re} b > c$

$[x] = n$ [$n \leq x < n+1$, $n=0, \pm 1, \pm 2, \dots$] — целая часть числа x

$\bar{z} = x - iy$ [$z = x + iy$]

$|z| = \sqrt{x^2 + y^2}$

$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$

$\prod_{k=m}^n a_k = a_m a_{m+1} \dots a_n$ [$n \geq m$]
 $= 1$ [$n < m$]

$\prod_{k=1}^{\infty} a_k(x) = \lim_{n \rightarrow \infty} \prod_{k=1}^n a_k(x)$

$\sum_{k=m}^n a_k = a_m + a_{m+1} + \dots + a_n$ [$n \geq m$]
 $= 0$ [$n < m$]

$\sum_{k=1}^{\infty} a_k(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k(x)$

УКАЗАТЕЛЬ РАЗЛОЖЕНИЙ НЕКОТОРЫХ ФУНКЦИЙ В РЯДЫ

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- $(1+x)^n$ 4.2.3.1, 5.1.29.12—13, 5.2.2.1—10, 5.2.10.8, 5.2.11.17
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 $e^r \operatorname{ch} x \operatorname{ch}(r \operatorname{sh} x)$ 5.3.3.2
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 $(1-x^n) \ln(1-x)$ 5.2.5.2—7
 $(1-x)^{-1} \ln(1-x)$ 5.2.17.3
 $\ln(1-x+x^2)$ 5.2.18.7
 $\ln \frac{1+x}{1-x}$ 5.2.4.8, 5.2.4.13
 $\ln(1-2r \cos x + r^2)$ 5.4.9.13

$$(1+x^2)^p \ln(1+x) \quad 5.2.18.6$$

$$\ln^2(1-x) \quad 5.2.17.2$$

$$\ln(1+x) \ln(1-x) \quad 5.2.17.4$$

$$\ln \sin x \quad 5.4.2.9$$

$$\ln \cos x \quad 5.4.2.10$$

$$\ln \operatorname{ctg} x \quad 5.4.6.7-8$$

$$\sin x \ln \sin x \quad 5.4.6.15$$

$$\cos x \ln \cos x \quad 5.4.6.18$$

$$\cos x \ln \sin \frac{x}{2} \quad 5.4.5.7$$

$$\cos x \ln \operatorname{tg} \frac{x}{2} \quad 5.4.5.10$$

$$\sin x \ln \operatorname{ctg}^2 x \quad 5.4.3.13-14$$

$$\cos x \ln \operatorname{ctg}^2 x \quad 5.4.3.13-14, 5.4.5.10$$

$$\ln^2 \cos x \quad 5.4.16.6$$

$$\arcsin x \quad 5.2.13.10$$

$$(1+x^2)^{1/2} \arcsin x \quad 5.2.14.2, 5.2.14.5$$

$$\arcsin^2 x \quad 5.2.14.3$$

$$\arcsin^4 x \quad 5.2.17.16$$

$$\operatorname{arctg} x^n \quad 4.4.9.6-7$$

$$\operatorname{arctg}^2 x \quad 5.2.17.5$$

$$(1+x^2)^p \operatorname{arctg} x \quad 5.2.18.5$$

$$\ln(1+x^2) \operatorname{arctg} x \quad 5.2.17.7$$

$$\ln \left(\frac{1+x}{1-x} \right) \operatorname{arctg} x \quad 5.2.17.8$$

$$\operatorname{arctg} \frac{r \sin x}{1-r \cos x} \quad 5.4.9.12$$

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$$(1+x^2)^{1/2} \operatorname{Arsh} x \quad 5.2.14.2, 5.2.14.5$$

$$\operatorname{Arsh}^2 x \quad 5.2.11.3$$

$$\operatorname{Arth} x \quad 5.2.5.8-15, 5.2.6.7$$

$$\operatorname{Arth}^2 x \quad 5.2.17.5$$

СПЕЦИАЛЬНЫЕ ФУНКЦИИ

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$$\operatorname{bei}(x) \quad 5.2.10.5, 5.2.17.14-15$$

$$\operatorname{bei}'(x) \quad 5.2.10.4$$

$$x^{-\nu} \operatorname{lei}_{\nu}(x) \quad 5.2.10.15$$

$$\operatorname{ber}(x) \quad 5.2.10.3, 5.2.17.14-15$$

$$\operatorname{ber}'(x) \quad 5.2.10.6$$

$$x^{-\nu} \operatorname{ter}_{\nu}(x) \quad 5.2.10.15$$

$$x^{-2\nu} [\operatorname{ter}_{\nu}^2(x) + \operatorname{bei}_{\nu}^2(x)] \quad 5.2.12.1$$

$$x^{-2\nu-1} [\operatorname{ber}_{\nu}(x) \operatorname{ter}_{\nu}'(x) - \operatorname{ber}_{\nu}'(x) \operatorname{tei}_{\nu}(x)] \quad 5.2.12.2$$

$$x^{1-2\nu} [\operatorname{ber}_{\nu}(x) \operatorname{ber}_{\nu}'(x) + \operatorname{bei}_{\nu}(x) \operatorname{bei}_{\nu}'(x)] \quad 5.2.12.3$$

$$C(x) \quad 5.2.8.17$$

$$C_n^{\nu}(\cos x) \quad 4.4.1.26$$

$$x^n C_n^{\nu} \left(\frac{1}{x} \right) \quad 4.1.7.20$$

$$C_{2n}^{\nu-n}(x), C_{2n+1}^{\nu-n-1}(x) \quad 4.1.7.24, 4.2.10.17$$

$$C_n^{-\nu-1/2}(1-2x) \quad 4.2.10.15$$

$$x^n C_n^{1-\nu-n} \left(\frac{1}{x} \right) \quad 4.2.10.19$$

$$x^n C_n^{\nu-n+1/2} \left(1 - \frac{2}{x} \right) \quad 4.2.10.13$$

$$x^n C_{2n}^{1+\nu-2n} \left(\sqrt{1 - \frac{1}{x}} \right) \quad 4.2.10.14$$

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$$H_0(x) \quad 5.2.15.3, 5.2.15.6-8$$

$$x H_0(x) \quad 5.2.15.1$$

$$H_1(x) \quad 5.2.15.2, 5.2.15.5$$

$$x^{-\nu-1} H_{\nu}(x) \quad 5.2.10.10$$

$$x^n \left[E_{n+1} \left(\frac{1}{x} \right) + H_{n+1} \left(\frac{1}{x} \right) \right] \quad 4.1.7.17$$

$$x^n [E_{-n-1}(x) + H_{-n-1}(x)] \quad 4.1.7.18, 4.2.3.33$$

$$H_n(x) \quad 4.1.7.13$$

$$x^n H_n \left(\frac{1}{x} \right) \quad 4.1.7.12, 4.2.7.45$$

$$e^x h_{\mu, \nu}(x) \quad 5.2.13.2$$

$$J_n(x) \quad 5.1.25.1, 5.2.10.1$$

$$J_{\nu}'(x) \quad 5.2.10.13$$

$$x^{-\nu} J_{\nu}(x) \quad 5.2.10.7$$

$$J_{\nu}^2(x) \quad 5.2.16.2$$

$$J_{\nu}(x) J_{-\nu}(x) \quad 5.2.16.4$$

$$x^{1-\mu} J_{\mu-\nu}(x) J_{\nu-1}(x) \quad 5.2.16.5$$

$$x^{2n+1} [J_{n+1/2}^2(x) + J_{-n-1/2}^2(x)] \quad 4.1.7.21$$

$$J_n(ix) + i E_n(ix) \quad 5.2.10.10$$

$$J_{\nu}(x) \cos \pi \nu + E_{\nu}(x) \sin \pi \nu \quad 5.2.10.12$$

$$x^{-n} [J_n(ix) - i H_n(ix)] \quad 5.2.13.11$$

$$I_n(x) \quad 5.2.10.1$$

$$x^{-\nu} I_{\nu}(x) \quad 5.2.10.7$$

$$e^{\pm x} I_{\nu}(x) \quad 5.2.14.27, 5.2.16.1$$

$$J_0(x) \pm I_0(x) \quad 5.2.10.3, 5.2.10.5$$

$$J_1(x) + I_1(x) \quad 5.2.10.4$$

$$J_0'(x) + I_0'(x) \quad 5.2.10.6$$

$$I_{\nu}(x) J_{-\nu}(x) \quad 5.2.12.4$$

$$x^{-2\nu} J_{\nu}(x) I_{\nu}(x) \quad 5.2.12.1$$

$$x^{2n+1} [I_{-n-1/2}^2(x) - I_{n+1/2}^2(x)] \quad 4.1.7.21$$

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УКАЗАТЕЛЬ ИНТЕГРАЛЬНЫХ ПРЕДСТАВЛЕНИЙ НЕКОТОРЫХ СПЕЦИАЛЬНЫХ ФУНКЦИЙ ЧЕРЕЗ ЭЛЕМЕНТАРНЫЕ

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