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CONTENTS

PREFACE

The present volume represents a unique blending of two fields only recently recognized as related. On one hand lies the field of Combinatorics with roots (at least immediately traceable via generating functions to Umbral Calculus, the specialty at hand) in the 19th century writings of Boole on operator calculus. Both the foundations and much of the history of the Umbral Calculus are explored in great clarity in [R-K-0 and R-R] which have extensive bibliographies. On the other hand is the field of Hopf Algebras, which is usually traced to the paper of Milnor and Moore [M-M] but whose first general exposition is little more than a decade old [S].

For some years, Gian-Carla Rota wrote that this theory should be directly applicable to Combinatorics, especially the Umbral Calculus, but the first distinct attention given by specialists is probably the lecture Moss Sweedler save at the 13-th Dennison Algebra Conference, Dennison College, 1978. This did not really appear in print, nor did Sweedler's Hopf Algebra colleagues sieze the subject and carry it fur ther forward.

Both Rota and Sweedler, therefore, were pleased when the University of Oklahoma was able to support their joint appearance at a conference funded by the J. C. Karcher Foundation in May, 1978. The conference centered on lectures they gave, with S. A. Joni assisting Rota. Sweedler lectured first on elementary coalgebra theory aimed at combinatorists, Rota on elementary combinatorics aimed at the algebraists. Both lectures converged toward those who were or would work at the intersection. Sweedler and Warren Nichols prepared notes of Sweedler's talks and Joni and Rota of Rota's and a mimeographed version was circulated by the Oklahoma Mathematics Department. The present volume represents an attempt to make these more accessible.

The Sweedler notes here are essentially unchanged from those distributed by Oklahoma. They aim, in a direct and elementary way, to give the reader sufficient knowledge of coalgebra theory to understand the coalgebra formulation of special sequences of polynomials.

The Rota notes are reproduced with permission from [J-R], and represent a reworking of the original, with corrections and a few additions. They contain detailed applications not only to Umbral Calculus, but to partition studies, incidence algebras, lattice theory, and other traditional spheres of combinatoric interest. The notes form a broad survey for anyone who would like detailed and concrete examples of the areas already known to be amenable to a coalgebraic approach.

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Coalgebras and Bialgebras in Combinatorics

By S.A. Joni and G.-C. Rota†*

The following material is discussed in this paper: Incidence Coalgebras for PO sets; Reduced Boolean Coalgebras; Divided Powers Coalgebra; Dirichlet Coalgebra; Eulerian Coalgebra; Faà di Bruno Bialgebra; Incidence Coalgebras for Categories; The Umbral Calculus; Infinitesimal Coalgebras; Creation and Annihilation Operators; Point Lattice Coalgebras; Restricted Placements; Cleavages; and Hereditary Bialgebras.

Dedicated to William T. Tutte on his 60th birthday.

Forse altri canterà con miglior plettro

—*Ariosto*

I. Introduction

A great many problems in combinatorics are concerned with assembling, or disassembling, large objects out of pieces of prescribed shape, as in the familiar board puzzles. Even in the seemingly simple case of finite sets, very little is known on, say, the structure of families of sets subject to restrictions. The oldest result in this direction is Sperner's theorem, which gives the structure of all maximum size families of subsets of a finite set, subject to the restriction that no set in the family may be contained in another. On the blueprint of Sperner's theorem, a host of similar results have been developed, largely in the last fifteen years, but the proofs rely more on ingenuity than on general techniques.

In more complicated cases, our understanding is even more limited; rarely, except perhaps in number theory, has a branch of mathematics been so rich in relevant problems and so poor in general ideas as to how such problems may be attacked.

This paper grew out of an attempt to make some of the combinatorial problems of assemblage available to a public of algebraists. It originated from the realization that the notions of coalgebra, bialgebra, and Hopf algebra, recently introduced into mathematics, may give in a variety of cases a valuable formal framework for the study of combinatorial problems. Armed with this realization, we have assembled in this paper a variety of coalgebras and bialgebras which arise in combinatorics, in the hope of interesting both the combinatorist in search of a theoretical horizon, and the algebraist in search of examples which may point to new and general theorems.

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The modesty of our undertaking cannot be overemphasized. We have simply given a list of coalgebras and bialgebras as possible objects of investigation, and proved only a few elementary results whenever the proofs were indispensable to the understanding of the examples.

Several of the coalgebras described below are presented here for the first time, notably puzzles, closure coalgebras, infinitesimal coalgebras, hereditary bialgebras, rook coalgebras, and cleavages. Others are drawn from previous work on the subject by P. Doubilet, M. Henle, R. W. Lawvere, S. Roman, R. Stanley, and ourselves.

It must be stressed that the coalgebras of combinatorics come endowed with a distinguished basis, and many an interesting combinatorial problem can be formulated algebraically as that of transforming this basis into another basis with more desirable properties. Thus, a mere structure theory of coalgebras—or Hopf algebras—will hardly be sufficient for combinatorial purposes.

Most of the content of this paper was developed from the Hopf Algebras and Combinatorics lectures presented by G.-C. Rota during the Umbral Calculus Conference at the University of Oklahoma on May 15–19, 1978. The authors take this opportunity to thank Professor M. Marx and Professor Robert Morris of the Mathematics Department of the University of Oklahoma for giving them an opportunity to present these ideas to a responsive audience of coalgebraists, as well as for their gracious hospitality.

II. Notation and terminology

Very little knowledge is required to read this work. Most of the concepts basic enough to be left undefined in the succeeding sections will be introduced here.

A *partial ordering relation* (denoted by $\leq$) on a set P is one which is reflexive, transitive, and antisymmetric (that is, $a \leq b$ and $b \leq a$ imply $a=b$). A set P together with a partial ordering relation is a *partially ordered set*, or PO set for short. For $x \leq y$ in P, the *segment* (or *interval*) $[x,y]$ is the collection of all elements z in P such that $x \leq z \leq y$. A PO set is said to be *locally finite* if every segment is finite. All the PO sets we shall consider will be locally finite.

A PO set P is said to have a 0 or a 1 if it has a unique minimal or maximal element. An element y is said to cover x if the segment $[x,y]$ has two elements. An *atom* of P is an element which covers a minimal element.

An *ordered ideal* in a PO set P is a subset J which has the property that if $y \in J$ and $x \leq y$, then $x \in J$.

The *product* $P \times Q$ of two PO sets P and Q is the set of all ordered pairs (p,q) where $p \in P$ and $q \in Q$, with $(p,q) \geq (r,s)$ if and only if $p \geq r$ and $q \geq s$. The product of any number of PO sets is defined similarly.

A *lattice* is a PO set where the max and min of two elements (we call them join and meet, and write them $\vee$ and $\wedge$) are defined. A *sublattice* L' of a lattice

L' is a subset of L which is a lattice under the induced partial ordering such that the join and meet of any two elements in L' are the same as those in L. A *distributive lattice* is one in which for all p,q,r in L, $p\wedge(q\vee r)=(p\wedge r)\vee(p\wedge q)$ and $p\vee(q\wedge r)=(p\vee r)\wedge(p\vee q)$.

A *partition* π of a set S is a collection of pairwise disjoint nonempty subsets of S, called the *blocks* of π, whose union is S. The *lattice of partitions* $\Pi(S)$ is the set of all partitions of S ordered by *refinement*: a partition π is less than or equal to a partition σ (or π is a refinement of σ) if each block of π is contained in a block of σ. The 0 of $\Pi(S)$ is the partition having all blocks of size one, and the 1 is the partition with one block. For further study of lattices, the reader is referred to Birkhoff.

We come now to the definition of the *incidence algebra* $\mathcal{I}(P)$ of a locally finite PO set P over a field K. We shall assume throughout that K has characteristic zero. The members of $\mathcal{I}(P)$ are functions of two variables $f: P\times P\to K$ such that $f(x,y)=0$ unless $x\leqslant y$. The sum of two functions, as well as multiplication by scalars, is defined as usual. The product (or convolution) $f*g=h$ is defined by

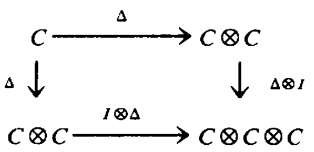
$$h(x,y) = \sum_{z\in P} f(x,z)g(z,y).$$

Since P is locally finite, the variable z in the above sum ranges over the finite segment $[x,y]$. It is immediate that this product is associative, and the unit element δ is

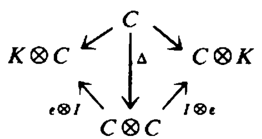
$$\delta(x,y) = \begin{cases} 1 & \text{if } x=y, \\ 0 & \text{otherwise}\end{cases}.$$

No further knowledge of the incidence algebra is required in the present paper; the reader is referred to [4] and [12] for studies of this algebra.

A *coalgebra* is a triple (C,Δ,ε) with C a K-vector space, $\Delta: C\to C\otimes C$ a map called *diagonalization* or *comultiplication*, and $\varepsilon: C\to K$ a map called the *counit* or *augmentation*, where Δ and ε satisfy the following commutative diagrams:



(coassociativity), (2.1)



(counitary property). (2.2)

Thus, coassociativity says $(I\otimes\Delta)\circ\Delta=(\Delta\otimes I)\circ\Delta$, or in words, after diagonalizing once, we can next diagonalize in either factor and obtain the same result. When we write "a coalgebra C," we mean "a coalgebra (C,Δ,ε)."

A *subcoalgebra* of a coalgebra C is a subspace W such that $\Delta(W)\subseteq W\otimes W$. A *coideal* of C is a subspace J such that $\Delta(J)\subseteq J\otimes C+C\otimes J$ and $\varepsilon(J)=0$. If $\sim$ is an equivalence relation on a basis of C such that the subspace J spanned by

$\{f-g: f\sim g\}$ is a coideal, then the quotient space $C/\sim$ can be endowed with the coalgebra structure of the quotient coalgebra of C modulo J.

A space B which is simultaneously an algebra and a coalgebra is said to be a *bialgebra* if the diagonalization Δ and counit ε are algebra maps.

Let C be a coalgebra, A an algebra, and set for $c\in C$

$$\Delta c = \sum_i c_{1_i}\otimes c_{2_i}.$$

We give $\mathrm{Hom}(C,A)$ an algebra structure by defining the product or convolution $f*g=h$ as follows:

$$h(c) = f*g(c) = \sum_i f(c_{1_i})g(c_{2_i}).$$

The unit of this algebra is $u\varepsilon$, where u is the unit of A.

Let H be a bialgebra, and let us define I in $\mathrm{Hom}(H,H)$ to be map $I(h)=h$ for all h in H. If it exists, the unique element S in H which is inverse under $*$ to I (i.e., $S*I=I*S=u\varepsilon$) is the *antipode* of H. A bialgebra with an antipode is a *Hopf algebra*. For a further study of bi-, co-, and Hopf algebras, the reader is referred to [27].

III. Section coefficients

We begin with the abstract concept of section coefficients. This concept arises as a natural generalization of the binomial coefficients. We shall see many examples in the later sections, particularly in Sec. IV–IX. Using section coefficients, one can give an alternative definition of coalgebras (with a specified basis) that does not involve commutative diagrams. Let $\mathcal{I}$ denote a set. *Section coefficients* $(i|j,k)$ of $\mathcal{I}$ arise by specifying and counting the number of ways an element i in $\mathcal{I}$ can be "cut up" into the ordered pair of pieces j,k(with j,k in $\mathcal{I}$). The *multisection coefficients* $(i|j,p,q)$ count the number of ways we can "cut" i into the ordered triple of pieces j,p,q. To get $(i|j,p,q)$ we could cut i into pieces j,k and then cut k into pieces p,q in all possible ways, and we want to get the same number if we cut i into pieces s,q and then cut s into pieces j,p in all possible ways. More precisely, section coefficients are a mapping

$$(i,j,k)\mapsto(i|j,k)\in\mathbb{Z}$$

satisfying

$$\text{Given } i, \text{ the number of ordered pairs } j,k \text{ such that } (i|j,k)\neq 0 \text{ is finite} \tag{3.1}$$

and

$$\sum_k (i|j,k)(k|p,q) = \sum_s (i|s,q)(s|j,p). \tag{3.2}$$

The common value of the two sides of (3.2) is denoted $(i|j,p,q)$. Iterating (3.2) allows us to define more general multisection coefficients $(i|j,k,\ldots,p,q)$.

Often, there exists a function $\varepsilon: \mathcal{G} \to K$ such that

$$\sum_j (i|j,k)\varepsilon(j) = \delta_{i,k},$$

$$\sum_k (i|j,k)\varepsilon(k) = \delta_{i,j}. \tag{3.3}$$

If $\mathcal{G}$ is a commutative semigroup (written additively), the section coefficients are called *bisection coefficients* if they satisfy

$$(i+j|p,q) = \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (i|p_1,q_1)(j|p_2,q_2). \tag{3.4}$$

In words, cutting up $i+j$ is the same as cutting i and j individually and piecing back together.

Example 3.1 (Binomial coefficients): The binomial coefficients are defined by

$$(n|j,k) = \begin{cases} \dfrac{n!}{j!k!} & \text{if} \quad n=j+k, \\ 0 & \text{otherwise.} \end{cases}$$

They count the number of ways a set with n element can be "cut up" into two disjoint sets of size j and $k(=n-j)$. Usually, we write $\binom{n}{j}$ for these coefficients. The condition (3.2) is easily seen to be satisfied, since

$$(n|j,p,q) = \begin{cases} \dfrac{n!}{j!p!q!} & \text{if} \quad j+p+q=n, \\ 0 & \text{otherwise,} \end{cases}$$

and for $j+p+q=n$,

$$(n|j,p,q) = \frac{n!}{j!(p+q)!}\,\frac{(p+q)!}{p!q!} = \frac{n!}{(j+p)!q!}\,\frac{(j+p)!}{j!p!}.$$

The well-known Vandermonde convolution identity

$$\binom{i+j}{p} = \sum_{p_1+p_2=p} \binom{i}{p_1}\binom{j}{p_2}$$

shows that the binomial coefficients are bisection coefficients.

Each collection of section coefficients satisfying (3.1), (3.2), and (3.3) gives rise to a coalgebra C in the following way: we associate to each i in $\mathcal{G}$ the variable x_i and let C be the free vector space spanned by the x_i's. The counit ε is

the function defined in (3.3), and the diagonalization Δ is defined by

$$\Delta x_i = \sum_{j,k} (i|j,k) x_j \otimes x_k. \tag{3.5}$$

In our examples it is often the case that there exists a unique "0" in $\mathcal{G}$ such that $(i|0,j)=(i|j,0)=\delta_{ij}$, and the counit ε is given by

$$\varepsilon(x_j) = \begin{cases} 1 & \text{if} \quad j=0 \\ 0 & \text{otherwise.} \end{cases}$$

The condition (3.2) gives that C is coassociative. C is cocommutative if and only if for all i,j,k, $(i|j,k)=(i|k,j)$. In addition, if the section coefficients are bisection coefficients, and we set $x_i x_j = x_{i+j}$, then C is a bialgebra. This is so because

$$\begin{aligned} \Delta(x_i)\Delta(x_j) &= \Big(\sum_{p_1,q_1} (i|p_1,q_1) x_{p_1} \otimes x_{q_1}\Big)\Big(\sum_{p_2,q_2} (j|p_2,q_2) x_{p_2} \otimes x_{q_2}\Big) \\ &= \sum_{p,q} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (i|p_1,q_1)(j|p_2,q_2) x_{p_1+p_2} \otimes x_{q_1+q_2} \\ &= \sum_{p,q} (i+j|p,q) x_p \otimes x_q \\ &= \Delta x_{i+j}. \end{aligned}$$

Good references for this section include [4], [11], [16], and [17].

IV. Incidence coalgebras for partially ordered sets

Many of the coalgebras arising from the study of combinatorial problems are incidence or reduced incidence coalgebras of locally finite PO sets. The duals of these coalgebras, namely the incidence and reduced incidence algebras for PO sets, have been objects of intensive study during the last fifteen years. In this section, we give the abstract setting, definitions, and some basic results. In Sec. V – IX we work out some of the fundamental examples.

Given a locally finite PO set $(P, <)$, the *incidence coalgebra* $\mathcal{C}(P)$ (over K, a field of characteristic zero) is the free vector space spanned by the indeterminates $[x,y]$, for all intervals (or segments) $[x,y]$ in P. The diagonalization Δ and counit ε are given by

$$\Delta[x,y] = \sum_{x<z<y} [x,z] \otimes [z,y] \tag{4.1}$$

and

$$\varepsilon([x,y]) = \begin{cases} 1 & \text{if} \quad x=y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Here, the section coefficients are

$$([x_1,x_2]|[y_1,y_2],[z_1,z_2]) = \begin{cases} 1 & \text{if} \quad x_1=y_1,\ x_2=z_2 \text{ and } y_2=z_1, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $\mathcal{C}(P)$ is coassociative. Moreover, it is cocommutative if and only if the order relation is trivial, i.e., no two elements of P are comparable.

Note that $\mathcal{C}^*(P)=\mathrm{Hom}(\mathcal{C}(P),K)$ is isomorphic to $\mathcal{I}(P)$, the incidence algebra of P, since if $f,g\in\mathcal{C}^*(P)$, then

$$f\circ g[x,y] = \sum_{x\le z\le y} f[x,z]\,g[z,y]$$

which is precisely the definition of $f*g$ in $\mathcal{I}(P)$.

It is frequently the case in enumeration problems that the full incidence coalgebra is not required; rather, we want to work with a smaller quotient coalgebra of $\mathcal{C}(P)$. These quotient coalgebras, called *reduced incidence coalgebras*, are obtained by taking suitable equivalence relations on P.

Definition 4.1. An equivalence relation $\sim$ on the segments of P is said to be *order compatible* if the subspace spanned by the collection $\{[x,y]-[u,v] \,|\, [x,y]\sim[u,v]\}$ is a coideal.

Whenever $\sim$ is order compatible, the quotient space $\mathcal{C}(P)/\sim$ is isomorphic to a quotient coalgebra of $\mathcal{C}(P)$ (see [27, p.22]). In general, there is no simple criteria expressible in terms of the partial ordering to decide when an equivalence relation on P is order-compatible. A useful sufficient condition due to D.A. Smith [4, p. 276] is the following.

Proposition 4.1. *An equivalence relation $\sim$ on the segments of P is order compatible if whenever $[x,y]\sim[u,v]$ there exists a bijection ϕ, depending in general on $[x,y]$, of $[x,y]$ onto $[u,v]$ such that $[x_1,y_1]\sim[\phi(x_1),\phi(y_1)]$ for all $x\le x_1\le y_1\le y$.*

Note that the linear dual $(\mathcal{C}(P)/\sim)^*$ is isomorphic to the reduced incidence algebra $\mathcal{I}(P)/\sim$.

If $\sim$ is an order compatible equivalence relation on P, we call the nonempty equivalence classes of $\mathcal{C}(P)/\sim$ *types*, and we think of $\mathcal{C}(P)/\sim$ as the vector space spanned by the variables x_α associated to each type α. Each such reduced incidence coalgebra gives rise to a collection of section coefficients $(\alpha|\beta,\gamma)$, where $(\alpha|\beta,\gamma)$ counts the number of distinct z in any interval $[x,y]$ of type α such that $[x,z]$ is of type β and $[z,y]$ is of type γ, and the diagonalization in $\mathcal{C}(P)/\sim$ is given by

$$\Delta x_\alpha = \sum (\alpha|\beta,\gamma)\,x_\beta\otimes x_\gamma,$$

where the sum ranges over all ordered pairs of types β,γ.

The *standard reduced incidence coalgebra* is obtained from the equivalence relation

$[x,y]\sim[u,v]$ if and only if $[x,y]$ is isomorphic to $[u,v]$.

One way of obtaining bialgebras of combinatorial interest is to form reduced incidence coalgebras. We shall return several times to the question of when a reduced incidence coalgebra is a bialgebra.

The following definition is motivated by the fact that the lattice of closed ideals of an incidence algebra is distributive [4].

Definition 4.2. A *combinatorial coalgebra* is a coalgebra whose lattice of subcoalgebras is distributive.

The characterization of all combinatorial coalgebras is an opening problem. At present, we can prove

Theorem 4.1. *Every (full) incidence coalgebra is a combinatorial coalgebra.*

Proof: Let W be a subcoalgebra of $\mathcal{C}(P)$. If $[x,y]$ is in W, then for all $x\le w\le z\le y$, $[w,z]$ is in W. This is seen as follows: If $x\le z\le y$, then the term $[x,z]\otimes[z,y]$ occurs in $\Delta[x,y]$. The occurrence of the segment $[x,z]$ (and $[z,y]$) is unique, and all segments are linearly independent. Thus, we must have $[x,z]$ and $[z,y]$ in W. Since $[x,z]\in W$, the same argument applies and gives that for all $x\le w\le z\le y$, we must have $[w,z]$ in W. Thus the collection of segments of W forms an order ideal in the PO set of all segments of P, $\mathrm{Seg}(P)$, ordered by inclusion. Conversely, if J is an order ideal in $\mathrm{Seg}(P)$,, then $\Delta J\subseteq J\otimes J$, so that the linear span of J forms a subcoalgebra of $\mathcal{C}(P)$. Therefore, the lattice of subcoalgebras of $\mathcal{C}(P)$ is isomorphic to the lattice of order ideals of $\mathrm{Seg}(P)$. A well-known theorem of Birkhoff states that the lattice of order ideals of any PO set is distributive, and our proof is complete.

V. Reduced Boolean coalgebras

The Boolean PO set (lattice) $\mathcal{B}$ consists of all finite sets of positive integers ordered by inclusion. The minimum element of this lattice is the empty set. The Boolean incidence coalgebra $\mathcal{C}(\mathcal{B})$ is spanned by all segments $[A,B]$ with

$$\Delta[A,B] = \sum_{A\subseteq C\subseteq B} [A,C]\otimes[C,B].$$

1. Boolean coalgebras

The *Boolean coalgebra* $\mathcal{B}$ is the coalgebra spanned by all sets of (positive) integers, with (for $A\in\mathcal{B}$)

$$\Delta A = \sum_{\substack{A_1\cap A_2=\varnothing \\ A_1\cup A_2=A}} A_1\otimes A_2 \tag{5.1}$$

and

$$\varepsilon(A) = \begin{cases} 1 & \text{if} \quad A=\varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in (5.1), A_1, A_2 is an ordered pair. This coalgebra is isomorphic to the reduced Boolean incidence coalgebra obtained by setting $[A,B]\sim[C,D]$ if and only if $B-A=D-C$. Thus, each set A represents the equivalence class of all segments $[B,C]$ such that $C-B=A$.

2. Binomial coalgebras

For each integer $s>0$, we define the binomial coalgebra B_s to be vector space $K[x_1,x_2,\ldots,x_s]$ with

$$\Delta x_1^{n_1}\cdots x_s^{n_s} = \sum_{(m_1,\ldots,m_s)\leq(n_1,\ldots,n_s)} \binom{n_1}{m_1}\cdots\binom{n_s}{m_s} x_1^{m_1}\cdots x_s^{m_s}\otimes x_1^{n_1-m_1}\cdots x_s^{n_s-m_s} \tag{5.2}$$

and

$$\varepsilon(x_1^{n_1}\cdots x_s^{n_s}) = \begin{cases} 1 & \text{if } n_1=\cdots=n_s=0, \\ 0 & \text{otherwise.}\end{cases}$$

Each binomial coalgebra is seen to be the Boolean incidence coalgebra modulo the coideal generated by a compatible $\sim$ as follows: For $s=1$, the (univariate) binomial coalgebra $B_1=K[x]$ is obtained by setting $[A,B]\sim[C,D]$ if and only if $|B-A|=|D-C|$. This is the standard reduced incidence coalgebra. Here the section coefficients are the binomial coefficients $\binom{n}{k}$.

For $s=2$, we set $[A,B]\sim[C,D]$ if and only if the numbers of even and odd integers in $B-A$ and $D-C$ are equal. For general s, we set $[A,B]\sim[C,D]$ if and only if for all $k=1,2,\ldots,s$,

$$|\{i\in B-A \mid i\equiv k \bmod s\}| = |\{j\in D-C \mid j\equiv k \bmod s\}|.$$

It is easy to verify that the binomial coalgebras are cocommutative bialgebras, and in fact, Hopf algebras with the antipode S given by $S(x_i)=-x_i$. In addition, the dual B_s^* is isomorphic to the algebra of formal exponential power series in s variables. A final heuristic remark: "$B_\infty \cong \mathfrak{B}$."

3. Polynomial sequences of Boolean and binomial type

A polynomial sequence $p_n(x)$ is said to be of *binomial type* if

$$\deg p_n(x) = n \qquad \text{for all } n, \tag{5.3}$$

and

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y). \tag{5.4}$$

Let us rephrase (5.4) in the landuage of bialgebras. The polynomial ring $K[x,y]$ is seen to be isomorphic to $K[x]\otimes K[x]$ under the mapping $x\mapsto x\otimes 1$ and $y\mapsto 1\otimes x$. By linearity, for any polynomials $q(x)$ and $r(y)$, $q(x)\mapsto q(x)\otimes 1$ and $r(y)\mapsto 1\otimes r(x)$. Thus, (5.4) can be restated

$$p_n(x\otimes 1+1\otimes x) = \sum_{k=0}^{n}\binom{n}{k} p_k(x)\otimes p_{n-k}(x). \tag{5.5}$$

A map p mapping the binomial coalgebra $K[x]$ to itself is a *coalgebra map* if $\Delta\circ p=(p\otimes p)\circ\Delta$. Thus, a polynomial sequence is of binomial type if and only if it is the image of $\{x^n\}$ under an invertible coalgebra map p. This is seen as follows. Let $p_n(x)$ denote the image of x^n under p. Since $K[x]$ is a bialgebra, we have

$$(\Delta\circ p)x^n = \Delta p_n(x) = p_n(\Delta x) = p_n(x\otimes 1+1\otimes x), \tag{5.6}$$

and clearly

$$((p\otimes p)\circ\Delta)x^n = \sum_{k=0}^{n}\binom{n}{k} p_k(x)\otimes p_{n-k}(x). \tag{5.7}$$

Therefore, if p is an invertible coalgebra map, $\deg p_n(x)=n$ and (5.5) holds, and conversely.

Multivariate polynomial sequences of binomial type, $\{p_{n_1\ldots n_s}(x_1,\ldots,x_s)\}$, are similarly seen to correspond to invertible coalgebra maps of B_s to itself.

Examples of sequences of polynomials of binomial type include x^n, $(x)_n = x(x-1)\cdots(x-n+1)$, $x(x-na)^{n-1}$, and the Laguerre, Gould, and exponential polynomial sequences. The reader is referred to [3] and [5] for further examples, and to [18] for their multivariate analogs.

A polynomial sequence indexed by the finite subsets of a set $\{p_A(x)\}$ is said to be of *Boolean type* if

$$p_A(x+y) = \sum_{A_1+A_2=A} p_{A_1}(x)p_{A_2}(y), \tag{5.8}$$

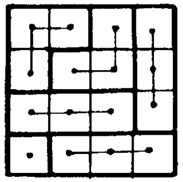
or equivalently, if $p_A(x)$ is the image of A under a coalgebra map from $\mathfrak{B}$ to $K[x]$. [Usually, we require that $\deg p_A(x)=|A|$.] Chromatic polynomials of graphs provide combinatorially interesting examples of polynomials of Boolean type. Given a graph G, the chromatic polynomial of G, $\mathcal{X}_G(x)$, counts the number of proper colorings (i.e. assignments of colors to the vertices of G so that no edge connects two vertices of the same color) of G with x colors. Given a subset H of the vertex set of G, we think of H as the full subgraph of G obtained by restricting the vertex set of G to H. Similarly, we denote by $G\setminus H$ the graph obtained by restricting the vertex set of G to $G-H$. Tutte, in [28], states

$$\mathcal{X}_G(x+y) = \sum_{H} \mathcal{X}_H(x)\mathcal{X}_{G\setminus H}(y). \tag{5.9}$$

This is not difficult to verify, since every proper coloring of G in $x+y$ different colors decomposes uniquely into the proper colorings of the subgraph H colored with the x colors and $G\setminus H$ colored with the y colors, and conversely. Polynomials of Boolean type were first studied by J. P. S. Kung and T. Zaslavsky.

4. Puzzles

Everyone is familiar with solitaire games where several flat pieces of wood or cardboard are to be assembled into a required shape, for example, a square, as in the following figure:



Little is known at present of the underlying mathematical theory that might lead, for example, to an algorithm for verifying that an assigned shape can be assembled out of a given set of pieces. We shall develop here the very first step in such a program, namely, the precise definition of a puzzle as a very special type of coalgebra. The definition of comultiplication is in fact a natural rendering of the combinatorial operation of cutting up an object into a set of pieces.

Before introducing the general definition, we shall describe the coalgebra associated with the puzzles in the above picture. We shall develop the construction in two steps. In the first step we define the placement coalgebra; in the second step we decribe a quotient coalgebra of the placement coalgebra, modulo a certain coideal. The quotient coalgebra will be called the puzzle or the piece coalgebra, and we shall see that the difficulty of the puzzle is carried in the structure of this coideal.

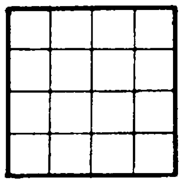
The *pieces* of the puzzle are

a  2 pieces

b  3 pieces (5.10)

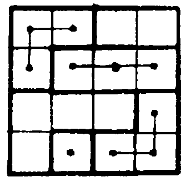
c  1 piece.

The *board* is the four-by-four square



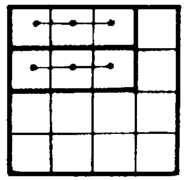
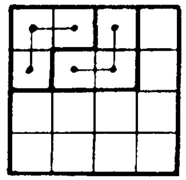
on which pieces are to be placed. The squares are labeled by Cartesian coordinates.

A *placement* of some of the pieces on the board is a subset of the board obtained by placing some of the pieces on the board without overlapping. For example the placement

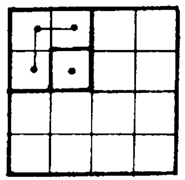
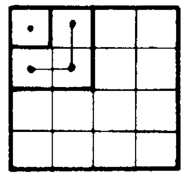
 (5.11)

is obtained by placing two pieces of shape a, a piece of shape b, and a piece of shape c, as indicated. In a placement, no more than the alloted number of pieces is allowed.

Two placements covering the same squares by distinct sets of pieces, or by pieces placed in different positions are considered to be different, for example,

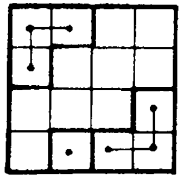
 and 

are distinct, as are

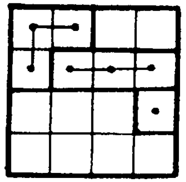
 and  .

The pieces in a placement need not be adjacent. To every placement p, specified by the occupied squares and the position of the pieces, we associate a variable $x(p)$, and we denote by V the free module over the integers spanned by the variables $x(p)$ and the variable 1, which denotes the trivial placement of no pieces.

We now define a comultiplication on the module V, as follows. If p and q are placements, it is clear what is meant by saying that q is a subplacement of p. The pieces used in q are a submultiset of the pieces in p, and they are placed in the same positions. For example,



is a subplacement of the placement given in (5.11), whereas



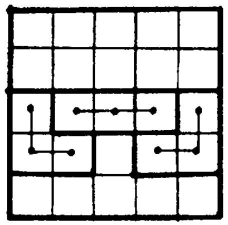
is *not* a subplacement. Thus, there is a partial ordering of placements, and we denote this PO set by P. P has a unique minimal element, the empty placement, but in general, it has no maximal element. Furthermore, for any placement p, the segment $\{q|q < p\}$ is a *Boolean algebra*; therefore, the PO set P is a *simplicial complex*. We are now ready to define the *placement coalgebra*. For any placement p, list all *ordered* pairs (q,r) such that

q and r are subplacements of p, (5.12a)
q and r do not overlap, (5.12b)
the union of q and r is the placement p. (5.12c)

Now set

$$\Delta x(p) = \sum x(q) \otimes x(r) \tag{5.13}$$

where the sum ranges over all such pairs. For example, if p is the placement



, (5.14)

and $x_1, x_2, \ldots, x_6$ are the placements shown in Fig. 1, then

$$\Delta x(p) = 1 \otimes x(p) + x_1 \otimes x_6 + x_2 \otimes x_4 + x_3 \otimes x_5 + x_4 \otimes x_2 + x_5 \otimes x_3 + x_6 \otimes x_1 + x(p) \otimes 1.$$

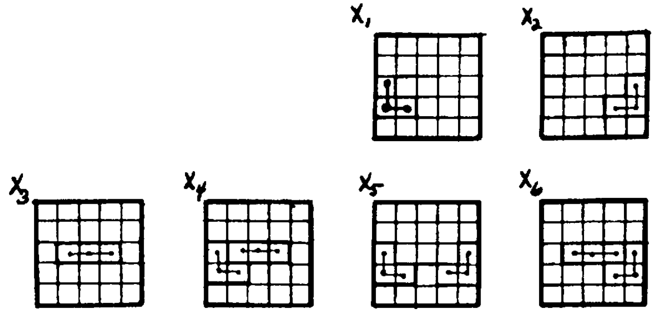


Figure 1.

It is intuitively clear that the comultiplication just defined is coassociative, in fact, it follows from the coassociativity of the Boolean coalgebra. The counit ε is defined by

$$\varepsilon(1) = 1 \text{ and } \varepsilon(x(p)) = 0 \quad \text{for all} \quad x(p) \neq 1. \tag{5.15}$$

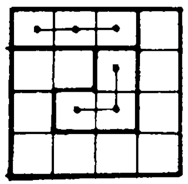
We now come to the definition of a puzzle, at least in the special case we are considering. To this end, we begin by defining an equivalence relation on placements. We shall say that $p \sim q$ when:

p and q are obtained by placing,
possibly in different positions, the same pieces
with the same multiplicity, (5.16a)

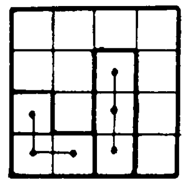
and

the placement q can be obtained from the placement p
by rigidly sliding and rotating (and possibly turning over,
depending on the rules of the game) placement p. (5.16b)

For example, any two placements of single pieces of the same shape are equivalent. As another example,



and



are equivalent.

It is immediate that the relation $\sim$ is an equivalence. An equivalence class will be called a *shape*. The equivalence classes corresponding to placements of a single piece will be called, appropriately enough, pieces.

The most important remark is that the submodule C of V generated by all elements

$$x(p)-x(q),$$

where $p\sim q$, is a coideal. Again, this is intuitively clear, but we shall verify it in detail. We have

$$\Delta x(p)=\sum_i x(p_{1i})\otimes x(p_{2i})$$

and

$$\Delta x(q)=\sum_i x(q_{1i})\otimes x(q_{2i}),$$

and it follows from the definition of equivalence that the families $\{(p_{1i},p_{2i})\}$ and $\{(q_{1i},q_{2i})\}$ of ordered pairs can be put into one-to-one correspondence in such a way that the entries are respectively equivalent. We can therefore write

$$x(p_{1i})\otimes x(p_{2i})-x(q_{1i})\otimes x(q_{2i})$$
$$=[x(p_{1i})-x(q_{1i})]\otimes x(p_{2i})+x(q_{1i})\otimes[x(p_{2i})-x(q_{2i})].$$

Thus, if $p\sim q$, then

$$\Delta(x(p)-x(q))=\sum_i [x(p_{1i})-x(q_{1i})]\otimes x(p_{2i})$$
$$+x(q_{1i})\otimes[x(p_{2i})-x(q_{2i})].$$

In other words, this shows that $\Delta C\subseteq C\otimes V+V\otimes C$, and thus proves that C is a coideal (see [27, p. 18]). We can therefore take the quotient coalgebra V/C. This coalgebra generated by shapes is called a *puzzle*. If p is the placement given in (5.14), then in the puzzle (or quotient coalgebra) we have $x_1\sim x_2$ and $x_4\sim x_6$. Thus (if we represent each equivalence class by its placement of smallest index) in the puzzle

$$\Delta x(p)=1\otimes x(p)+2(x_1\otimes x_4)+x_3\otimes x_5+x_5\otimes x_3+2(x_4\otimes x_1)+x(p)\otimes 1.$$

From the preceding example it is now easy to extract the general definition of a puzzle. One begins with a finite simplicial complex P, and one associates to P a placement coalgebra in the same way as we have done above: to every p in P, one associates the set of ordered pairs (q,r) such that $q\vee r=p$ and $q\wedge r=0$.

From this, one obtains the definition of the placement coalgebra in exactly the same way. A puzzle is now generally defined as the quotient of the placement coalgebra by a coideal defined by an equivalence relation among the elements of P.

The basic problem about puzzles is to determine how many distinct shapes cover the entire board. At present, too little is known about the structures of puzzles to even hazard a conjecture on how one might approach the problem.

VI. Divided powers coalgebra

Let N denote the lattice of nonnegative integers under natural ordering. The incidence coalgebra $\mathcal{C}(N)$ is spanned by all segments $[i,j]$ with

$$\Delta[i,j]=\sum_{i<k<j}[i,k]\otimes[k,j].$$

The *divided powers coalgebra* $\mathcal{D}$ is the vector space $K[x]$ with

$$\Delta x^n=\sum_{k=0}^{n} x^k\otimes x^{n-k}$$

and

$$\varepsilon(x^n)=\delta_{0,n}.$$

It is the standard reduced incidence coalgebra of $\mathcal{C}(N)$, and its dual $\mathcal{D}^*$ is isomorphic to the algebra of formal power series $k[[x]]$ (with the usual multiplication). Multivariate divided powers coalgebras are similarly defined to be the standard reduced incidence coalgebra of

$$\mathcal{C}(N^s)=\mathcal{C}(\underbrace{N\times\cdots\times N}_{s\text{ times}}).$$

VII. Dirichlet coalgebra

Let Z^+ denote the lattice of positive integers ordered by divisibility, i.e., $m<n$ if and only if m divides n. The 0 of this lattice is 1. The equivalence relation on the segments of $\mathcal{C}(Z^+)$ which gives the Dirichlet coalgebra is $[i,j]\sim[k,l]$ if and only if $j/i=l/k$. Alternatively, the Dirichlet coalgebra D is the vector space spanned by the variables $\{n^x: n=0,1,2,\ldots\}$, with

$$\Delta(n^x)=\sum_{pq=n} p^x\otimes q^x$$

and

$$\varepsilon(n^x)=\delta_{0,n}.$$

D has a natural algebra structure given by $n^x m^x = (nm)^x$. While D is not a bialgebra, the comultiplication is an algebra map when n and m are coprime, that is,

$$\Delta(n^x m^x) = \Delta(n^x)\Delta(m^x)$$

whenever the gcd of n and m equals 1.

The linear dual D^* is isomorphic to the algebra of formal Dirichlet series, the isomorphism being given by

$$f \mapsto \varphi(s) = \sum_n \frac{f(n^x)}{n^s}.$$

Multivariate Dirichlet coalgebras are obtained from the same equivalence relation on the incidence coalgebra $\mathcal{C}(Z^+ \times \cdots \times Z^+)$.

The standard reduced incidence coalgebra is a subcoalgebra of the Dirichlet coalgebra. Let $[i,j]$ and $[k,l]$ be two segments, and let

$$j/i = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \quad \text{and} \quad l/k = q_1^{\beta_1} \cdots q_r^{\beta_r}$$

be their respective prime factorizations. The segments $[i,j]$ and $[k,l]$ are isomorphic if and only if $s=r$, and as multisets, the collections $\{\alpha_i\}$ and $\{\beta_i\}$ are the same. In other words, given n, let $\text{shape}(n)=(\lambda_1,\lambda_2,\ldots)$ where λ_k is the number of distinct primes in the factorization of n which occur precisely k times. Then $[i,j] \sim [k,l]$ if and only if $\text{shape}(j/i)=\text{shape}(l/k)$.

VIII. Eulerian coalgebra

Let V denote the lattice of all finite-dimensional subspaces of a vector space of countable dimension over GF (q), ordered by inclusion. The minimal element of V is the trivial subspace. The standard reduced incidence coalgebra of $\mathcal{C}(V)$ is obtained by setting

$$[X,Y] \sim [S,T] \quad \text{if and only if} \quad \dim Y - \dim X = \dim T - \dim S.$$

The section coefficients count the number of subspaces of dimension k contained in a subspace of dimension n, which is given by the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}.$$

If we set $[n]_q! = (1-q)(1-q^2)\cdots(1-q^n)$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The *Eulerian coalgebra E* is the vector space $K[x]$ with

$$\Delta x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \otimes x^{n-k}$$

and

$$\varepsilon(x^n) = \delta_{0,n}.$$

It is cocommutative, and E^* is isomorphic to the algebra of formal Eulerian power series, the isomorphism being given by

$$f \mapsto \varphi(u) = \sum \frac{f(x^n)}{[n]_q!} u^n.$$

IX. The Faà di Bruno bialgebra

The Faà di Bruno coalgebra $\mathcal{F}$ is the standard reduced incidence coalgebra for the full lattice of partitions, Π. As such, it bears the same relationship to the lattice of partitions as does the binomial coalgebra to the lattice of subsets and the Eulerian coalgebra to the lattice of subspaces of a vector space over a finite field. In this section we shall show that this coalgebra is a bialgebra. (The proof is due to Doubilet [11].) Moreover, this bialgebra serves as a blueprint for the formulation and understanding of the general class of hereditary bialgebras presented in Sec. XVII.

The full lattice of partitions, Π, is the lattice of all set partitions of Z^+ (positive integers) having exactly one infinite block and finitely many finite blocks, ordered by refinement (see Sec. II).

Every segment $[\sigma,\tau]$ of Π is isomorphic to $\Pi_1^{\lambda_1} \times \Pi_2^{\lambda_2} \times \cdots \times \Pi_k^{\lambda_k} \cdots$, where Π_n is the lattice of partitions of an n-set, and λ_k equals the number of blocks of τ which consist of k blocks of σ. (This isomorphism can be seen by thinking of the ith block of σ as the "element" B_i, and $[\sigma,\tau]$ as a partition on the collection of B_i's, with σ as the finest partition.) To each segment $[\sigma,\tau]$ of Π, we associate the sequence $\lambda=(1,1,\ldots,1,2,\ldots,2,\ldots)$ of λ_1 ones, λ_2 twos, ..., sometimes written $\lambda=(1^{\lambda_1}2^{\lambda_2}\cdots)$ or equivalently, $x_1^{\lambda_1}x_2^{\lambda_2}\ldots=\mathbf{x}^{\lambda}$; λ, or $\mathbf{x}^{\lambda}$, is the *type* of $[\sigma,\tau]$, and clearly $[\sigma_1,\tau_1]$ is isomorphic to $[\sigma_2,\tau_2]$ if and only if they have the same type. The type $\lambda=(1^0 2^0 \cdots n^1 \ldots)=x_n$ is often written as n. We shall use the symbols $\alpha,\beta,\lambda,\nu,\mu$ to denote types.

The section coefficients $\begin{bmatrix} n \\ \alpha,\beta \end{bmatrix}$ count the number of partitions π contained in $[0,(1,2,\ldots,n)] \cong \Pi_n$ such that $[0,\pi]$ is of type α and $[\pi,(1,2,\ldots,n)]$ is of type β. Note that if $\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, we must have $\alpha_1+2\alpha_2+\ldots+n\alpha_n=n$ and $\beta=x_{\alpha_1+\alpha_2+\cdots+\alpha_n}$. These section coefficients, known as the *Faà di Bruno coefficients*, are given explicitly by

$$\begin{bmatrix} n \\ \alpha,\beta \end{bmatrix} = \frac{n!}{\alpha_1!\alpha_2!\ldots\alpha_n!(1!)^{\alpha_1}(2!)^{\alpha_2}\ldots(n!)^{\alpha_n}}. \tag{9.1}$$

The explicit coalgebra structure of $\mathcal{F}$ is as follows. As a vector space, $\mathcal{F}$ is isomorphic to $K[x_1, x_2, \ldots] = K[\mathbf{x}]$. The diagonalization Δ and counit ε are given by

$$\Delta \mathbf{x}^{\lambda} = \sum_{\alpha,\beta} \begin{bmatrix} \lambda \\ \alpha,\beta \end{bmatrix} \mathbf{x}^{\alpha} \otimes \mathbf{x}^{\beta} \tag{9.2}$$

and

$$\varepsilon(\mathbf{x}^{\lambda}) = \begin{cases} 1 & \text{if } \lambda = (0,0,0,\ldots) \text{ or } (1,0,0,\ldots) \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

If $\sigma \leq \pi$ and B is a block of π, then by $\sigma \cap B$ we mean the partition of B consisting of the blocks of σ contained in B. Let $[\sigma, \tau]$ be of type $x_m x_n$, i.e., τ has two blocks B and B', where B contains m blocks of σ, and B' contains n. Suppose π is such that $\sigma \leq \pi \leq \tau$, $[\sigma \cap B, \pi \cap B]$ is of type $x_1^{\alpha_1} x_2^{\alpha_2}, \ldots$, and $[\sigma \cap B', \pi \cap B']$ is of type $x_1^{\alpha'_1} x_2^{\alpha'_2} \cdots$. Then clearly $[\sigma, \pi]$ is of type $x_1^{\alpha_1 + \alpha'_1} x_2^{\alpha_2 + \alpha'_2} \cdots = \mathbf{x}^{\alpha} \mathbf{x}^{\alpha'}$. Similarly, if $[\pi \cap B, \tau \cap B]$ is of type $\mathbf{x}^{\beta}$ and $[\pi \cap B', \tau \cap B']$ is of type $\mathbf{x}^{\beta'}$, then $[\pi, \tau]$ is of type $\mathbf{x}^{\beta} \mathbf{x}^{\beta'}$. Thus, for all m, n, ν, μ,

$$\begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix} = \sum_{\substack{\alpha,\alpha',\beta,\beta' \\ \alpha+\alpha'=\nu,\, \beta+\beta'=\mu}} \begin{bmatrix} x_m \\ \alpha,\beta \end{bmatrix} \begin{bmatrix} x_n \\ \alpha',\beta' \end{bmatrix}, \tag{9.4}$$

where addition of sequences is defined by $(1^{\alpha_1} 2^{\alpha_2} \cdots) + (1^{\beta_1} 2^{\beta_2} \ldots) = (1^{\alpha_1 + \beta_1} 2^{\alpha_2 + \beta_2} \ldots)$, i.e. $\mathbf{x}^{\alpha} \mathbf{x}^{\beta} = \mathbf{x}^{\alpha + \beta}$. It follows that $\begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix}$ is the coefficient of $\mathbf{x}^{\nu} \otimes \mathbf{x}^{\mu}$ in $\Delta(x_n)\Delta(x_m)$. This is equivalent to

$$\sum_{\nu,\mu} \begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix} \mathbf{x}^{\nu} \otimes \mathbf{x}^{\mu} = \left(\sum_{\alpha,\beta} \begin{bmatrix} x_m \\ \alpha,\beta \end{bmatrix} \mathbf{x}^{\alpha} \otimes x^{\beta} \right) \left(\sum_{\alpha,\beta} \begin{bmatrix} x_n \\ \alpha,\beta \end{bmatrix} \mathbf{x}^{\alpha} \otimes \mathbf{x}^{\beta} \right). \tag{9.5}$$

More generally, $\begin{bmatrix} x_1^{\lambda_1} x_2^{\lambda_2} \cdots \\ \nu,\mu \end{bmatrix}$ is the coefficient of $\mathbf{x}^{\nu} \otimes \mathbf{x}^{\mu}$ in $\Delta(x_1)^{\lambda_1} \Delta(x_2)^{\lambda_2} \cdots$. But this is just

$$\Delta(x_1^{\lambda_1} x_2^{\lambda_2} \cdots) = \Delta(x_1)^{\lambda_1} \Delta(x_2)^{\lambda_2} \cdots. \tag{9.6}$$

In addition, it is clear from (9.3) that

$$\varepsilon(x_1^{\lambda_1} x_2^{\lambda_2} \ldots) = \varepsilon(x_1)^{\lambda_1} \varepsilon(x_2)^{\lambda_2} \cdots.$$

Hence, we have shown

THEOREM 9.1. *$\mathcal{F}$ is a bialgebra under ordinary multiplication and the coalgebra structure obtained from the standard reduced incidence coalgebra of $\mathcal{C}(\Pi)$.*

Note that F is non-cocommutative. By Theorem 9.1, the space of all K-liner maps from $\mathcal{F}$ to itself, $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, is an algebra with multiplication (or convolution) $*$ defined by

$$f * g(\mathbf{x}^{\lambda}) = \sum_{\alpha,\beta} \begin{bmatrix} \lambda \\ \alpha,\beta \end{bmatrix} f(\mathbf{x}^{\alpha}) g(\mathbf{x}^{\beta}). \tag{9.7}$$

A function f in $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ is said to be *multiplicative* if and only if for all λ, $f(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \ldots) = f(x_1)^{\lambda_1} \ldots f(x_n)^{\lambda_n} \ldots$. Any such function is determined by the values it takes on the segments Π_n. Let $\mathfrak{M}(\mathcal{F})$ denote the class of multiplicative functions. The following elementary result is fundamental [4].

PROPOSITION 9.1. *The convolution of two multiplicative functions is multiplicative.*

Thus, $\mathfrak{M}(\mathcal{F})$ is a subsemigroup of the multiplicative semigroup $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$. If $f \in \mathfrak{M}(\mathcal{F})$, let $f(n)$ denote $f(\Pi_n)$, that is, $f([\sigma, \tau])$ for all $[\sigma, \tau]$ of type n. For $f, g \in \mathfrak{M}(\mathcal{F})$, we get from (9.1) and (9.7) that

$$f * g(n) = \sum_{\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n} \frac{n! f(1)^{\alpha_1} \cdots f(n)^{\alpha_n} g(\alpha_1 + \ldots + \alpha_n)}{\alpha_1! \ldots \alpha_n! (1!)^{\alpha_1} \ldots (n!)^{\alpha_n}}. \tag{9.8}$$

THEOREM 9.2 (Doubilet, Rota, Stanley). *The semigroup $\mathfrak{M}(\mathcal{F})$ is anti-isomorphic to the algebra of all formal power series with zero constant term over $K[\mathbf{x}]$ in the variable u under the operation of functional composition. The anti-isomorphism is given by $f \mapsto f(u)$, where*

$$f(u) = \sum_{n=1}^{\infty} \frac{f(n)}{n!} u^n. \tag{9.9}$$

Thus $f * g(u) = g(f(u))$.

Proof: Clearly the map defined by (9.9) is a bijection, so we need only check that multiplication is preserved. Now

$$g(f(u)) = \sum_{k=1}^{\infty} \frac{g(k)}{k!} \left(\sum_{\nu=1}^{\infty} \frac{f(\nu)}{\nu!} u^{\nu} \right)^k. \tag{9.10}$$

The coefficient of u^n in the expansion of

$$\left(\sum_{\nu=1}^{\infty} \frac{f(\nu)}{\nu!} u^{\nu} \right)^k$$

is

$$\sum_{\substack{\nu_1 + \ldots + \nu_k = n \\ \nu_i \geq 1}} \frac{f(\nu_1) \cdots f(\nu_k)}{\nu_1! \ldots \nu_k!} = \sum \frac{k!}{\alpha_1! \ldots \alpha_n!} \frac{f(1)^{\alpha_1} \ldots f(n)^{\alpha_n}}{(1!)^{\alpha_1} \ldots (n!)^{\alpha_n}},$$

where the summation is taken over $\alpha_1+2\alpha_2+\ldots+n\alpha_n=n$ and $\alpha_1+\ldots+\alpha_n=k$, since there are $k!/\alpha_1!\ldots\alpha_n!$ ways of ordering the partition $\alpha_1+2\alpha_2+\ldots+n\alpha_n=n$. When we multiply (9.11) by $g(k)/k!$ and sum over all k, we obtain (9.8), and the proof follows.

$\mathcal{F}$ is not a Hopf algebra, since $\Delta(x_1)=x_1\otimes x_1$. It can be realized as a Hopf algebra in $K[\mathbf{x}]$ localized at x_1, with

$$\Delta\left(\frac{1}{x_1}\right)=\frac{1}{x_1}\otimes\frac{1}{x_1}.$$

A proof of the existence of the antipode is given by demonstrating that a certain recursion can be carried out. An explicit formula can be obtained using the Lagrange inversion theorem [19].

In $K[\mathbf{x}]$ localized at x_1, $\mathfrak{M}(\mathcal{F})$ is anti-isomorphic to the group of all invertible (under functional composition) formal power series. The inverse of any function can be obtained by composition of this function with the antipode S. For a more detailed discussion of the Hopf-algebra aspects of $\mathcal{F}$, we refer the reader to [11], [19].

X. Incidence coalgebras for categories

Certain enumeration problems (see [4, p. 283]) lead to counting over structures more complicated than a single PO set. The concept of a *Mobius category* [21], gives one such structure. To extend the notion of incidence and reduced incidence coalgebras for PO sets to these situations, we are led to define incidence coalgebras for categories.

A *locally finite* category is a category in which for each morphism f, the collection of pairs of morphisms $\{(f_1,f_2):f_1\circ f_2=f\}$ is finite.

Given a locally finite category M, the incidence coalgebra $\mathcal{C}(M)$ is the free vector space over K spanned by the indeterminates f, where f is a morphism of M with coalgebra structure given by

$$\Delta f=\sum_{f_1\circ f_2=f} f_1\otimes f_2. \tag{10.1}$$

and

$$\varepsilon(f)=\begin{cases}1 & \text{if } \ f=id_p \text{ for some object } p \text{ in } M,\\ 0 & \text{otherwise.}\end{cases} \tag{10.2}$$

Let $\sim$ denote an equivalence relation on the morphisms of M. The *subspace generated by* $\sim$ is the subspace of $\mathcal{C}(M)$ spanned by the collection $\{f-g: f\sim g\}$. We say $\sim$ is *compatible* if the subspace generated by $\sim$ is a coideal. A *reduced incidence coalgebra* $\mathcal{C}(M)/\sim$ is the quotient coalgebra of $\mathcal{C}(M)$ modulo the coideal generated by a compatible $\sim$ relation.

Given two morphisms f,g in M, we say that g *divides* f if there exists morphisms h,k in M such that $f=h\circ g\circ k$. Let $[f]$ denote the subcategory generated by $\{g|g$ divides $f\}$. The *standard reduced* incidence coalgebra is obtained via the following equivalence on the morphisms: We set $f\sim g$ if and only if $[f]$ is isomorphic (as a subcategory) to $[g]$. Clearly the subspace generated by this equivalence relation is a coideal.

The range of meaningful reduced incidence coalgebras for categories is much larger than those of PO sets. For example, the *inner reduced* incidence coalgebra arises by setting $f\sim g$ if and only if there exists an invertible morphism h in M such that $[f]\simeq h\circ[g]\circ h^{-1}$, and the *strongly reduced* incidence coalgebra arises by setting $f\sim g$ if and only if there exists a category isomorphism φ: $[f]\to[g]$ such that $\varphi(f)=g$.

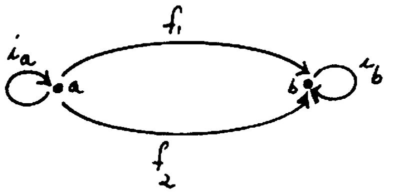
Example 10.1: Every locally finite PO set P can be viewed as a locally finite category as follows: the objects of M are the elements (or vertices) of P, and there is a unique morphism $f_{x,y}\, x\to y$ if and only if $x\leq y$. Clearly, if $x<z<y$, then $f_{x,z}\circ f_{z,y}=f_{x,y}$ and $[f_{x,y}]$ corresponds to the interval $[x,y]$. There are no invertible morphisms (other than the trivial ones, i.e. $f_{x,x}$), so that in this case, the inner reduced incidence coalgebra is the full incidence coalgebra. Moreover, in this case the standard reduced and strongly reduced incidence coalgebra are isomorphic.

Example 10.2: Every finite group G can be viewed as a locally finite category. The category has only one object, and each morphism f_g corresponds to an element of G. Composition of morphisms is given by $f_{g_1}\circ f_{g_2}=f_{g_1g_2}$, and if e is the identity element of G, f_e is the identity morphism. In contrast to the case of PO sets, the standard reduced incidence coalgebra for this category is isomorphic to the trivial category (consisting of one object and one morphism), whereas the inner reduced incidence coalgebra is isomorphic to the category of conjugacy classes of G. Indeed, in the inner reduced coalgebra, we have $f\sim g$ if and only if there exists an h such that $f=hgh^{-1}$, that is, f is conjugate to g. Let J denote the subspace generated by $\sim$. If $f\sim g$, then to each pair (f_1,f_2) such that $f_1f_2=f$ there correspond a unique pair (g_1,g_2) such that $g_1g_2=g$ and $f_i\sim g_i$. The correspondence is given explicitly by $g_i=h^{-1}f_ih$. Therefore

$$\begin{aligned}\Delta(f-g)&=\sum_i f_{1i}\otimes f_{2i}-\sum_i h^{-1}f_{1i}h\otimes h^{-1}f_{2i}h\\ &=\sum_i\left[\left(f_{1i}-h^{-1}f_{1i}h\right)\otimes f_{2i}+h^{-1}f_{1i}h\otimes\left(f_{2i}-h^{-1}f_{2i}h\right)\right]\\ &\subseteq J\otimes\mathcal{C}(M)+\mathcal{C}(M)\otimes J.\end{aligned}$$

Hence J is a coideal and $\mathcal{C}(M)/\sim$ is isomorphic to the category of conjugacy classes of G, as asserted. In the strongly reduced incidence category we have $f\sim g$ if and only if there exists a group automorphism φ such that $\varphi(f)=g$.

As we have seen in Sec. IV (Theorem 4.1), the lattice of subcoalgebras of the incidence coalgebra for PO sets is distributive. This is also trivially true for the lattice of subcoalgebras of the incidence coalgebra for a group G, because there are no proper subcoalgebras. It is, however, in general false. For example, let M be the category



where $i_a \circ f_j = f_j$ and $f_j \circ i_b = f_j$, $j = 1, 2$. The lattice of subcoalgebras of this category is not a distributive lattice. This is easily seen as follows: Let $L(A)$ denote the linear span of A, and set $M_1 = L(i_a, i_b)$; $M_2 = L(f_1, i_a, i_b)$; $M_3 = L(f_2, i_a, i_b)$; $M_4 = L(f_1 + f_2, i_a, i_b)$. Then each M_i is a subcoalgebra of $\mathcal{C}(M)$, and

$$M_2 \wedge (M_3 \vee M_4) = M_2,$$

whereas

$$(M_2 \wedge M_3) \vee (M_2 \wedge M_4) = M_1$$

In fact, the segment $[M_1, \mathcal{C}(M)]$ is isomorphic to the lattice of subspaces of a two-dimensional vector space over K, and it is well known that this lattice is not distributive.

XI. The umbral calculus

The binomial bialgebra has been studied in great detail, in particular with regard to applications to combinatorics, in a series of papers beginning with Mullin and Rota [2], followed by Kahaner, Odlyzko, and Rota [3] and finally Roman and Rota [5]. Elegant expositions of the results of Mullin and Rota were given by Aigner [6], Garsia [14], Liu [22], and several others. We shall summarize the main lines of this theory, keeping in mind that these results should act as blueprints for yet to be carried out generalizations to the more complex bialgebras and coalgebras arising in combinatorics, some of which are described in the rest of the present paper.

The comultiplication

$$\Delta x^n = \sum_k \binom{n}{k} x^k \otimes x^{n-k}. \tag{11.1}$$

on the algebra of polynomials $p(x)$ of one variable, defines a bialgebra structure. The dual algebra on linear functions L—where we denote by $\langle L | p(x) \rangle$ the action of the linear functional L on the polynomial $p(x)$—is seen to be

$$\langle L_1 L_2 | x^n \rangle = \sum_k \binom{n}{k} \langle L_1 | x^k \rangle \langle L_2 | x^{n-k} \rangle. \tag{11.2}$$

The dual algebra, with the augmentation ε acting as the identity, has been called the *umbral algebra* by Roman and Rota. The umbral algebra is isomorphic to the algebra of formal power series under the map

$$L \to \sum_n \langle L | x^n \rangle \frac{t^n}{n!}, \tag{11.3}$$

even in a topological sense. The formal power series thus associated to a linear functional is said to be its *indicator*.

The algebra of shift-invariant operators on polynomials is the algebra of all linear operators T mapping polynomials into polynomials, such that $TE^a = E^a T$, where E^a is the shift operator mapping $p(x) \to p(x+a)$, for all a. It turns out that the umbral algebra is also isomorphic to the algebra of shift invariant operators under the map sending the linear functional L to the operator Q given by

$$Qx^n = \sum_k \binom{n}{k} \langle L | x^k \rangle x^{n-k}.$$

A coalgebra isomorphism U, that is, a one-to-one onto linear operator on polynomials such that

$$\Delta U x^n = \sum_k \binom{n}{k} U x^k \otimes U x^{n-k} \tag{11.4}$$

has been called an *umbral operator* by Mullin and Rota. The adjoint of an umbral operator is an isomorphism of the umbral algebra, and conversely, with due respect to topology. The sequence $p_n(x) = Ux^n$, where U is an umbral operator, is said to be of *binomial type*, and is characterized by the identity

$$p_n(x+a) = \sum_k \binom{n}{k} p_k(a) p_{n-k}(x). \tag{11.5}$$

Sequences of binomial type are of frequent occurrence in combinatorics, and have motivated much of the work on the umbral calculus. For example the sequences $(x)_n = x(x-1)\cdots(x-n+1)$, $x(x-na)^{n-1}$, and the Laguerre polynomials are of binomial type.

A *delta functional* L is a linear functional such that $\langle L | 1 \rangle = 0$ and $\langle L | x \rangle \neq 0$. To every delta functional one can associate two polynomial sequences of binomial type: the *associated sequence* $p_n(x)$ uniquely defined by the biorthogonality requirements

$$\langle L^k | p_n(x) \rangle = n! \, \delta_{k,n}, \tag{11.6}$$

and the *conjugate sequence* $q_n(x)$, defined by

$$q_n(x) = \sum_k \langle L^k | x^n \rangle \frac{x^k}{k!}. \tag{11.7}$$

Conversely, every sequence of binomial type, $p_n(x)$, is the associated sequence and the conjugate sequence of unique delta functionals, say L and $\tilde{L}$, which are said to be *reciprocal.*

A shift-invariant operator Q associated to a delta functional L is said to be a *delta operator.* If $p_n(x)$ is the associated sequence of the linear functional L, then the identity $Qp_n(x)=np_{n-1}(x)$ shows that the sequence $p_n(x)$ is related to the delta operator Q in a manner analogous to D and x^n. This leads to the generalization to delta operators of several classical formulas of the calculus; as the simplest example, Taylor's formula generalizes to

$$p(x+a) = \sum_n \frac{p_n(a)}{n!} Q^n p(x). \tag{11.8}$$

For example, for the sequence $p_n(x)=(x)_n$, the delta operator Q is the difference operator Δ defined by $\Delta p(x)=p(x+1)-p(x)$. Every delta operator Q equals the product DP, where $Dp(x)=p'(x)$ is the ordinary derivative, and the inverse operator P^{-1} exists. The operator P is called the *transfer operator* of the sequence $p_n(x)$. We come now to the first basic fact of the umbral calculus, which is the transfer formula:

$$p_n(x) = xP^{-n}x^{n-1}, \tag{11.9}$$

where P is the transfer operator of the sequence of binomial type $p_n(x)$. This formula is closely related to the Lagrange inversion formula for formal power series [15].

To introduce the next basic fact, we consider the operator $\mathbf{x}$ mapping $p(x)$ to $xp(x)$. The operator $Q'=Q\mathbf{x}-\mathbf{x}Q$ is called the *Pincherlé derivative* of the operator Q, and is also shift-invariant if Q is. Now, if Q is the delta operator of the sequence $p_n(x)$, then the *recurrence formula*

$$p_n(x) = x(Q')^{-1}p_{n-1}(x) \tag{11.10}$$

gives another way of explicitly computing a sequence of binomial type.

We now come to the fundamental fact of the umbral calculus. If $p_n(x)$ is a sequence of binomial type, then its generating function is of the form

$$\sum_n \frac{p_n(x)}{n!} t^n = \exp\left[x\left(a_1 t + \frac{a_2}{2!}t^2 + \quad\right)\right] = e^{xf(t)} \tag{11.11}$$

for some formal power series $f(t)$ such that $a_0=0$ and $a_1\neq 0$, (a delta series, for short) and conversely. If $p_n(x)$ is the associated sequence for the delta functional L with indicator $g(t)$, then the series $f(t)$ and $g(t)$ are inverse in the sense of functional composition, that is, $f(g(t))=g(f(t))=t$. Furthermore, if $p_n(x)$ is the conjugate sequence of the delta functional $\tilde{L}$, then $f(t)$ is the indicator of $\tilde{L}$.

Functional composition is also related to umbral operators. It turns out that every umbral operator U is uniquely related to a delta series $u(t)$, and if L has

indicator $f(t)$, then the linear functional $U^*(L)$ has the indicator $f(u(t))$; the converse is also true.

The coalgebraic statement of this fact leads to the interpretation and rigorization of the classical technique of treating indices as exponents, from which the umbral calculus derives its name. If $p_n(x)=\sum_k a_{n,k}x^k$ and $q_k(x)$ are sequences of polynomials of binomial type, then the polynomial sequence

$$r_n(x) = \sum_k a_{n,k} q_k(x) = p_n(\mathbf{q}(x)) \tag{11.12}$$

is called the *umbral composition* of the sequences $p_n(x)$ and $q_n(x)$. It turns out that the sequence $r_n(x)$ is also of binomial type; furthermore, if the indicators of the delta functionals L and M with respect to which $p_n(x)$ and $q_n(x)$ are the associated sequences are, respectively, $f(t)$ and $g(t)$, then the corresponding indicator of the sequence $r_n(x)$ is the functional composition $f(g(t))$.

Among many other facts of the umbral calculus which cannot be mentioned here—but some of which will be found in the memoir of Roman and Rota—we mention the extension of the preceding results to other module actions of the umbral algebra; in fact, it would be of the utmost interest to classify all such module actions. For example, a natural action is defined on the ring of inverse formal power series

$$f(t) = \sum_{n>1} \frac{a_n}{t^n}$$

by sending t^{-n} to $(t+a)^{-n}$, thus defining the operator E^a, and then taking a suitable closure. In this way, one can define "inverse" analogs of all sequences of binomial type; for example,

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)},$$

leading to a generalization of the classical theory of factorial series.

XII. Infinitesimal coalgebras; the Newtonian coalgebra

Recall that a bialgebra A is a vector space which is simultaneously an algebra and a coalgebra such that the comultiplication Δ is an "endomorphism" of A (as an algebra). The analogy between endomorphisms and derivations leads us to define an *infinitesimal coalgebra A* to be a vector space which is simultaneously an algebra and a coalgebra (possibly without a counit) such that the comultiplication Δ is a derivation of A in the sense that for p,q in A,

$$\Delta(pq) = (\Delta p)(q\otimes 1) + (1\otimes p)(\Delta q). \tag{12.1}$$

In this section we shall present only one infinitesimal coalgebra, the Newtonian coalgebra. The study of this coalgebra should provide a prototype for the general study of infinitesimal coalgebras.

Let us recall the definition of the Newton divided differences. The 0th divided difference is

$$[f:x_0] = f(x_0).$$

The first divided difference is

$$[f:x_0,x_1] = \frac{f(x_0)-f(x_1)}{x_0-x_1},$$

the second

$$[f:x_0,x_1,x_2] = \frac{[f:x_0,x_1]-[f:x_1,x_2]}{x_0-x_2},$$

and the kth divided difference $[f:x_0,\ldots,x_k]$ is obtained by iteration.

A polynomial sequence $\{p_n(x)\}$ [with $p_0(x)\equiv 1$] is said to be of *Newtonian type* if

$$\frac{p_n(x)-p_n(y)}{x-y} = \sum_{k=0}^{n-1} p_k(x)p_{n-k-1}(y). \tag{12.2}$$

Two examples of such sequences are $\{x^n\}$, and $\{(x+a)^n\}$ for any a. There are two coalgebras within which we can study these polynomial sequences. As vector spaces and as algebras, both are isomorphic to $K[x]$. The first coalgebra we shall consider is the *Newtonian coalgebra*, denoted N. The comultiplication in N is

$$\Delta p(x) = \frac{p(x)-p(y)}{x-y} \tag{12.3}$$

and is easily checked to be coassociative. There is no counit in N. Moreover, it is immediate to verify that

$$\Delta(pq) = \Delta p\,(q\otimes 1) + (1\otimes p)\Delta q,$$

so that N is an infinitesimal coalgebra. The kth divided difference $[p:x_0,\ldots,x_k] = \Delta^k p(x)$. This coalgebra setting gives an elegant proof of Newton's formula, namely

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)\cdots(x-x_k)[f:x_0,\ldots,x_k].$$

In the dual algebra, the following striking relationship between divided differences and ordinary differentiation is seen. Let us set, for all p,

$$\langle \varepsilon_p | f(x)\rangle = f(p).$$

Then

$$\langle \varepsilon_p\varepsilon_q | f(x)\rangle = \frac{f(p)-f(q)}{p-q},$$

and

$$\langle \varepsilon_p^2 | f(x)\rangle = f'(p).$$

An extensive study of the theory within this setting has been pursued by S. Roman [25].

A different approach was taken by Garsia and Joni. Using the umbral machinery with the "differentiation" operator A defined by

$$Ax^n = x^{n-1}$$

and the "multiplication" operator B defined by

$$Bx^n = \frac{x^{n+1}}{n+1},$$

they define a polynomial sequence $\{q_n(x)\}$ $[q_0(x)\equiv 1]$ to be of *Newjonian type* if

$$\frac{xq_n(x)-yq_n(y)}{x-y} = \sum_{k=0}^{n} q_k(x)q_{n-k}(y). \tag{12.4}$$

Note that $q_n(0)=0$ for all $n>1$. Examples of such sequences are $\{x^n\}$, $\{x(x+a)^{n-1}\}$.

Here, the underlying coalgebra structure is given by

$$\Delta p(x) = \frac{xp(x)-yp(y)}{x-y}$$

and

$$\varepsilon(x^n) = \begin{cases} 1, & n=0, \\ 0 & \text{otherwise} \end{cases}$$

(that is, ε is evaluation at zero).

In this setting, all of the results within the umbral calculus, appropriately modified (see [13]) for the "differentiation" A and "multiplication" B, apply. For example it is not difficult to show that $\{q_n(x)\}$ is of Newjonian type if and only if

$$\sum_{n=0}^{\infty} q_n(x)u^n = \frac{1}{1-xf(u)},$$

where $f(u)$ is an invertible (under functional composition) formal power series.

It turns out that polynomial sequences of Newtonian and Newjonian type are essentially the same class of polynomial sequences. Indeed, we have

THEOREM 12.1. *A polynomial sequence* $\{p_n(x)\}$ *is of Newtonian type if and only if the polynomial sequence* $\{q_n(x)\}$ *defined by*

$$q_n(x) = \begin{cases} 1, & n=0, \\ xp_{n-1}(x), & n > 1 \end{cases}$$

is of Newjonian type.

The Newjonian coalgebra setting provided the machinery for the explicit computation of the "Newtonian analogs" of many of the classical polynomial sequences (e.g. Laguerre, Abel, exponential, Gould, etc.).

XIII. Creation and annihilation operators

The creation and annihilation operators we present here generalize those of quantum field theory.

Let $\{(i|j,k)\}$ be a collection of section coefficients satisfying the extra condition that for each ordered pair (j,k), the set $\{i:(i|j,k)\neq 0\}$ is finite, and let C be the coalgebra defined by these section coefficients and a given counit ε (see Sec. III). Creation and annihilation operators are linear maps from C to itself defined as follows:
for each $j \in \mathcal{G}$, *the creation operator* K_j is

$$K_j x_k = \sum_i (i|j,k)x_i \tag{13.1}$$

and the *annihilation operator* A_j is

$$A_j x_k = \sum_i (k|j,i)x_i. \tag{13.2}$$

If the section coefficients are all equal to zero or one, and if, in addition, for each j,k there is at most one i such that $(i|j,k)=1$, then the creation operator K_j acting on x_k gives the "piece" i obtained by piecing k and j together if it exists,

and zero otherwise. Similarly, the annihilation operator A_j acting on x_k gives the piece i which when added to the piece j is the piece k, if such a piece exists, and zero otherwise.

For example, let us look at the situation when C is an incidence coalgebra for a PO set. An easy computation gives

$$A_{[x,y]}[u,v] = \begin{cases} [y,v] & \text{if} \quad x=u \text{ and } x \leq y \leq v, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly

$$K_{[x,y]}[u,v] = \begin{cases} [x,v] & \text{if} \quad y=u, \\ 0 & \text{otherwise.} \end{cases}$$

In a puzzle, $K_j x_k$ gives a list (with multiplicities) of all possible x_i obtainable by piecing together x_j and x_k. Similarly, $A_j x_k$ gives a list (with multiplicities) of all possible pieces x_i such that x_i and x_j can be pieced together to form x_k.

Straightforward computations give

$$K_p A_j x_k = \sum_{i,q} (k|j,i)(q|p,i)x_q,$$

and

$$A_j K_p x_k = \sum_{i,q} (i|p,k)(i|j,q)x_q.$$

If the section coefficients are bisection coefficients, then C is a bialgebra, and in addition,

$$A_k x_{i+j} = \sum_{p_1+p_2=k} (A_{p_1}x_i)(A_{p_2}x_j).$$

PROPOSITION 13.1. *If* $\mathcal{G}$ *is a commutative semigroup (written additively), then*

$$K_j K_l = K_l K_j = K_{j+l} \quad \text{and} \quad A_j A_l = A_l A_j = A_{j+l}$$

if and only if for all j,k,l,q

$$\sum_p (k|j,p)(p|l,q) = \sum_p (k|l,p)(p|j,q)$$
$$= (k|j+l,q). \tag{13.3}$$

Proof:

$$K_jK_lx_q = K_j\Big(\sum_p (p|l,q)x_p\Big)$$

$$= \sum_{p,k} (p|l,q)(k|j,p)x_k$$

and

$$K_{j+l}x_q = \sum_k (k|j+l,q)x_k.$$

Therefore, $K_jK_l = K_{j+l}$ if and only if

$$(k|j+l,q) = \sum_p (p|l,q)(k|j,p).$$

The same argument using K_lK_j completes the proof for creation operators, and the analogous argument holds for annihilation operators.

The coalgebra C, considered as a vector space, has a natural inner product:

THEOREM 13.1. *The bilinear form*

$$\langle x_i|x_j\rangle_C = \sum_q (i|j,q)\varepsilon(x_q)$$

on C *is symmetric and nondegenerate.*

Proof: Since ε is the counit of C, we have

$$x_i = \sum_{j,q} (i|j,q)\varepsilon(x_q)x_j. \qquad (13.4)$$

Equating coefficients of the x's on both sides gives

$$\langle x_i|x_j\rangle_C = \sum_q (i|j,q)\varepsilon(x_q) = \delta_{i,j}.$$

THEOREM 13.2. *Relative to the symmetric form* $\langle\ ,\ \rangle_C$, A_j *and* K_j *are adjoint operators.*

Proof: We show that

$$\langle A_ix_j|x_k\rangle_C = \langle x_j|K_ix_k\rangle_C$$

for all i,j,k. Expanding the left side gives

$$\langle A_ix_j|x_k\rangle_C = \sum_p (j|i,p)\langle x_p,x_k\rangle_C = (j|i,k)$$

since $\langle x_p,x_k\rangle_C = \delta_{p,k}$. Similarly, the right-hand side gives

$$\langle x_j|K_ix_k\rangle_C = \sum_p (p|i,k)\langle x_j,x_p\rangle = (j|i,k),$$

as desired.

In the following examples we see how creation and annihilation operators cut up and piece together sets and partitions.

Example 13.1: For the binomial coalgebra, the creation and annihilation operators are easily seen to be

$$K_jx^k = \binom{j+k}{j}x^{k+j}$$

and

$$A_jx^k = \begin{cases} \binom{k}{j}x^{k-j} & \text{if } j \leq k,\\ 0 & \text{otherwise.}\end{cases}$$

Example 13.2: Let X and Y be subsets. The creation and annihilation operators for the Boolean coalgebra are

$$K_XY = \begin{cases} X\cup Y & \text{if } X\cap Y=\varnothing,\\ 0 & \text{otherwise}\end{cases}$$

and

$$A_XY = \begin{cases} Y-X & \text{if } X\subseteq Y,\\ 0 & \text{otherwise.}\end{cases}$$

Example 13.3: The creation and annihilation operators for the Faá di Bruno coalgebra are a bit more complicated than those of the previous two examples. Let α, β, and λ denote types of partitions. The creation operators K_α are, by definition,

$$K_\alpha \mathbf{x}^\beta = \sum_\lambda (\lambda|\alpha,\beta)\mathbf{x}^\lambda.$$

Since $(\lambda|\alpha,\beta)\neq 0$ only if $\beta_1+2\beta_2+\ldots+n\beta_n=\alpha_1+\alpha_2+\ldots+\alpha_n$, $\lambda_1+2\lambda_2+\ldots+n\lambda_n=\alpha_1+2\alpha_2+\ldots+n\alpha_n$, and $\lambda_1+\lambda_2+\ldots+\lambda_n=\beta_1+\beta_2+\ldots+\beta_n$, the types $\mathbf{x}^\lambda$ occurring in $K_\alpha\mathbf{x}^\beta$ [with multiplicites $(\lambda|\alpha,\beta)$] are seen to be the types obtainable by merging, in all possible ways, the blocks of a partition of type α so that the resulting partition has the same number of blocks as β. The multiplicities count the number of ways in which a given type can occur.

The annihilation operators A_α are

$$A_\alpha \mathbf{x}^\beta = \sum_\lambda (\beta|\alpha,\lambda)\mathbf{x}^\lambda,$$

so here we must have $\alpha_1+2\alpha_2+\ldots+n\alpha_n=\beta_1+2\beta_2+\ldots+n\beta_n$, $\lambda_1+2\lambda_2+\ldots+n\lambda_n=\alpha_1+\ldots+\alpha_n$, and $\lambda_1+\lambda_2+\ldots+\lambda_n=\beta_1+\beta_2+\ldots+\beta_n$ for the type $\mathbf{x}^\lambda$ to occur in $A_\alpha\mathbf{x}^\beta$. Thus we obtain a list of all the types of partitions (with appropriate multiplicities) of a set of size $\alpha_1+\ldots+\alpha_n$ with the same number of blocks as β.

XIV. Point-lattice coalgebras

Let $\mathfrak{L}$ be a finite point lattice, that is, a lattice in which every element is the supremum of a set of atoms. It is well known and easily proved that $\mathfrak{L}$ is isomorphic to the lattice of closed sets relative to the closure operation defined on subsets of the set $\mathcal{Q}$ of atoms by

$$\overline{A}=\{p\in\mathcal{Q}\,|\,p\leq \sup A\}\qquad \text{for}\quad A\subseteq\mathcal{Q}.$$

The closure operation enjoys the properties

$$A\subseteq\overline{A}, \tag{14.1a}$$

$$\overline{\overline{A}}=\overline{A}, \tag{14.1b}$$

$$\text{if}\quad A\subseteq B,\quad\text{then}\quad \overline{A}\subseteq\overline{B} \tag{14.1c}$$

(but *not*, in general $\overline{A\cup B}=\overline{A}\cup\overline{B}$). The complements of closed sets, called *open sets*, can be characterized even more simply by

(1) the union of any family of open sets is an open set,
(2) every open set is the union of the minimal nonempty open sets it contains.

Thus, every point lattice can be represented as the family of all open sets in a closure relation where the join in the lattice is set-theoretic union. In the following we shall assume that $\mathfrak{L}$ is so represented by a fixed set $\mathcal{Q}$. We shall further assume that $\mathfrak{L}$ has a unique minimal element, which is represented by the empty set. This representation of $\mathfrak{L}$ allows us to define a very interesting coalgebra structure on $\mathcal{Q}$. As a vector space, this coalgebra $\mathcal{C}(\mathfrak{L})$ is isomorphic to the free vector space over K with basis consisting of all open sets of $\mathcal{Q}$. For each open set $A\subseteq\mathcal{Q}$, the diagonalization is

$$\Delta A=\sum_{\substack{A_1,A_2\ \text{open}\\ A_1\cap A_2=\varnothing\\ A_1\cup A_2=A}} A_1\otimes A_2, \tag{14.3}$$

and the counit is

$$\varepsilon(A)=\begin{cases}1 & \text{if}\quad A=\varnothing\\ 0 & \text{otherwise.}\end{cases} \tag{14.4}$$

Since the union of open sets is again an open set, it follows immediately that the above diagonalization (14.3) is coassociative. Since point lattices occur in many combinatorial investigations, the study of this class of coalgebras should prove very interesting. We give three examples of point-lattice coalgebras.

Example 14.1 *(The Boolean coalgebra):* Finite-dimensional Boolean coalgebras arise from the point lattice of subsets of $\{1,2,\ldots,n\}$. This lattice can be represented as follows: the minimal nonempty open sets are the sets consisting of one element, i.e., the sets $\{j\}$, for $1\leq j\leq n$. Thus, every subset is an open set. Hence, for each A,

$$\Delta A=\sum_{\substack{A_1\cap A_2=\varnothing\\ A_1\cup A_2=A}} A_1\otimes A_2,$$

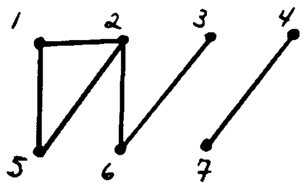
so that these coalgebras are isomorphic to subcoalgebras of the Boolean coalgebra defined in Sect. V.

Example 14.2 *(The $n\times n$ board):* Let $\mathcal{Q}$ denote the collection of the n^2 squares $\{a_{ij}\}_{i,j=1}^n$ on an $n\times n$ square board. Our point lattice $\mathfrak{L}$ is represented by the following family of open subsets of $\mathcal{Q}$: the minimal nonempty open sets are the *lines* of the board, where a line, by definition, is either a row or column. The open sets consist of all possible unions of lines, so each open set A is uniquely determined by the two subsets of $\{1,2,\ldots,n\}$

$$R(A)=\{i\,|\,\text{row } i \text{ is in } A\}\quad\text{and}\quad C(A)=\{j\,|\,\text{column } j \text{ is in } A\}.$$

Two open sets A_1 and A_2 can have $A_1\cap A_2=\varnothing$ if and only if either $|R(A_1)|=|R(A_2)|=0$ or $|C(A_1)|=|C(A_2)|=0$. Thus, our comultiplication Δ breaks up open sets which are unions of rows or unions of columns, and leaves intact any open set which is a combination of both rows and columns.

Example 14.3 *(Graphs):* Let $\mathcal{G}=(V,E)$ be an undirected graph with vertex set V, $|V|<\infty$, and edge set E. Here, our point lattice is the family of open subsets of V defined as follows: the minimal nonempty open sets are (unordered) pairs of vertices p and q such that there is an edge in E connecting p and q. We shall sometimes write (p,q) to denote such an edge. An open set A is a subset of V such that for each $p\in A$, there exists a $q\in A$ such that (p,q) is an edge in E. (Note that q need not be unique.) Our comultiplication gives all ways of dividing an open vertex set A into two disjoint sets A_1 and A_2 such that each vertex in A_i, $i=1,2$, remains connected to some other vertex in A_i. For example, let $\mathcal{G}$ be the graph given by the following figure:



$V=\{1,2,\ldots,7\}.$

Then $\{1,5,2\}\otimes\{3,6,4,7\}$ occurs in ΔV, whereas $\{1,2,3,5\}\otimes\{4,6,7\}$ does not.

An element p in any lattice $\mathfrak{L}$ is said to be a *join-irreducible* element if it cannot be expressed as the join of two incomparable elements of $\mathfrak{L}$. Every element in $\mathfrak{L}$ is the supremum of a set of join irreducibles, and $\mathfrak{L}$ is isomorphic to the lattice of closed sets relative to the closure operation defined on subsets of the set of join irreducibles J by

$$\bar{A}=\{p\in J\,|\,p<\sup A\}\qquad\text{for}\quad A\subseteq J.$$

Thus, the construction given for the point-lattice coalgebras extends in the obvious way to a construction for general lattices.

XV. Restricted placements

A fundamental concept in the study of permutations with restricted positions is that of a *non-taking subset* of a board. A non-taking subset of a board is a collection of squares $\{a_{ij}\}$ such that no two squares have the same row or column index. They are best visualized as follows: if we place a rook on each square in a given set A, then A is non-taking if and only if no rook can "take" any other rook, that is, no two rooks are in the same row or column.

In this section we shall give a very general setting for the construction of non-taking sets; non-taking sets of boards arise as one special case. Another special case gives totally unconnected collections of vertices in graphs, which are closely related to the problem of colorings of graphs. Within this context we are lead to a very natural interpretation of Möbius inversion for a large class of lattices, and a coalgebra closely associated with enumerations of non-taking sets.

In order that this paper may be reasonably self-contained, we give a brief sketch of Möbius inversion for an arbitrary locally finite PO set P. The reader is referred to [1] for a more complete discussion. In enumeration, we often wish to calculate $f(y)$, a function on P, and it turns out to be much easier to calculate

$$g(x)=\sum_{y>x}f(y).$$

As an example, if P is the lattice of subsets of $\{1,2,\ldots,n\}$, and $f(A)$ is the

number of permutations of $\{1,2,\ldots,n\}$ whose set of fixed points is precisely A, then it is easy to see that

$$g(A)=\sum_{B\supseteq A}f(B)=\text{the number of permutations whose set of fixed points contains } A$$

$$=(n-|A|)!.$$

We obtain the values of f (in terms of the values of g) via Möbius inversion. The *zeta function* is the function in the incidence algebra $\mathcal{I}(P)$ defined by

$$\zeta(x,y)=\begin{cases}1 & \text{if}\quad x<y,\\ 0 & \text{otherwise.}\end{cases}$$

The inverse of ζ (under $*$), μ, is the *Möbius function*. That is, μ satisfies, for all $x<y$,

$$\sum_{x<z<y}\mu(x,z)\zeta(z,y)=\sum_{x<z<y}\zeta(x,z)\mu(z,y)$$

$$=\delta_{x,y}=\begin{cases}1 & \text{if}\quad x=y,\\ 0 & \text{otherwise.}\end{cases}\tag{15.1}$$

THEOREM 15.1 (Möbius inversion). *Let f and g be functions on a given PO set P such that*

$$g(x)=\sum_{y>x}f(y).\tag{15.2}$$

Then

$$f(x)=\sum_{y}\mu(x,y)g(y).\tag{15.3}$$

Proof: Equation (15.2) states that

$$g(y)=\sum_{z>y}f(z)=\sum_{z}\zeta(y,z)f(z).$$

Thus, multiplying both sides by $\mu(x,y)$ and summing over y gives

$$\sum_{y}\mu(x,y)g(y)=\sum_{z}\sum_{y}\mu(x,y)\zeta(y,z)f(z)$$

$$=\sum_{z}\delta_{x,z}f(z)=f(x).$$

As in Sec. XIV, we shall assume that we are given a point lattice $\mathcal{L}$, represented as a family of open subsets of a set $\mathcal{A}$. We shall call nonempty minimal open sets *forbidden sets*. Given $\mathcal{L}$, we construct a new lattice $\mathrm{St}(\mathcal{L})$, the *lattice of stars of* $\mathcal{L}$, as follows: for each $p \in \mathcal{A}$, the *star of* p, $\mathrm{st}(p)$, is the union of all forbidden (minimal nonempty open) sets containing p. If A is any subset of $\mathcal{A}$, we set

$$\mathrm{st}(A) = \bigcup_{p \in A} \mathrm{st}(p).$$

We say that an element S in $\mathcal{L}$ is a *star* if and only if $S = \mathrm{st}(A)$ for some $A \subseteq \mathcal{A}$. (Note that A is, in general, not unique.) If $A \subseteq B$, then $\mathrm{st}(A) \subseteq \mathrm{st}(B)$. We say that A *generates* S if $\mathrm{st}(A) = S$ and for all $A' \subsetneq A$, $\mathrm{st}(A') \subsetneq \mathrm{st}(A)$. The lattice $\mathrm{St}(\mathcal{L})$ consists of all stars of $\mathcal{L}$, ordered by inclusion, where the join is set-theoretic union. $\mathrm{St}(\mathcal{L})$ is, in general, not a sublattice of $\mathcal{L}$. Indeed, the meets in the two lattices are not necessarily the same, since if S and T are stars, then in $\mathrm{St}(\mathcal{L})$, their meet will be the maximal star contained in $S \cap T$, whereas in $\mathcal{L}$, their meet is the maximal open set contained in $S \cap T$. Moreover, $\mathrm{St}(\mathcal{L})$ need not be a point lattice. We shall give an example of this later in this section.

A subset A of $\mathcal{A}$ will be said to be *non-taking* when for all $p \neq q$ in A, $p \notin \mathrm{st}(q)$ and $q \notin \mathrm{st}(p)$. We define two functions f and g on $\mathrm{St}(\mathcal{L})$ as follows: given a star S, let $g(S)$ be the number of non-taking sets whose star contains S, and let $f(S)$ be the number of non-taking sets whose star equals S. Clearly,

$$g(S) = \sum_{T \supseteq S} f(T), \tag{15.4}$$

where T ranges over all stars. Hence, by Möbius inversion,

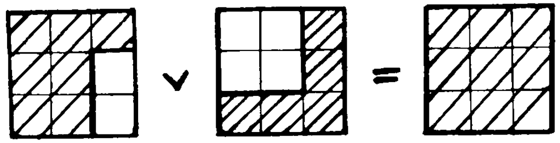
$$f(S) = \sum_{T \supseteq S} \mu(S,T) g(T), \tag{15.5}$$

and we have exhibited a combinatorial interpretation of Möbius inversion over any lattice of stars.

For our first example, let us return to the problem of rooks on an $n \times n$ board. As in Example 14.2, our point lattice $\mathcal{L}$ is represented by the family of open subsets of $\mathcal{A}$ (where $\mathcal{A}$ is the collection of squares $\{a_{ij}\}$ of the board) whose forbidden sets are lines. The minimal nonempty stars of $\mathcal{L}$ are the unions of the two lines through each square a_{ij}. Thus, the number of atoms of $\mathrm{St}(\mathcal{L})$ is n^2, and since every star is a union of these minimal stars, $\mathrm{St}(\mathcal{L})$ is a point lattice. A non-taking set $A \subseteq \mathcal{A}$ is, by definition, a set such that for each $a_{ij} \neq a_{pq}$ in A, $a_{pq} \notin \mathrm{st}(a_{ij})$ and $a_{ij} \notin \mathrm{st}(a_{pq})$. Clearly there is a bijection between these sets and all possible placements on the $n \times n$ board of non-taking rooks. Recall that for an open set A, $R(A) = \{i \mid \text{row } i \text{ is in } A\}$ and $C(A) = \{j \mid \text{column } j \text{ is in } A\}$. Let $r(A) = |R(A)|$ and $c(A) = |C(A)|$. If A generates the star S, then $|A| = \max(r(S), c(S))$. The number of sets generating S equals the number of maps from a set with $\max(r(S), c(S))$ elements *onto* a set with $\min(r(S), c(S))$ elements. Moreover, A is non-taking if and only if $r(\mathrm{st}(A)) = c(\mathrm{st}(A))$. Thus, if S is a star such that $r(S) = c(S) = m$, then S is generated by $m!$ non-taking sets of size m. Therefore, we have shown

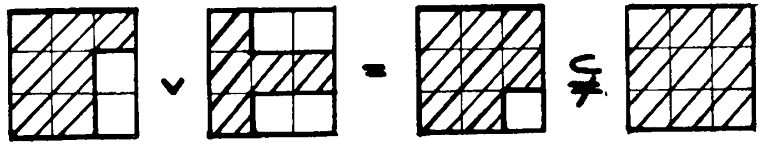
$$f(S) = \begin{cases} m! & \text{if } r(S) = c(S) = m, \\ 0 & \text{otherwise.} \end{cases} \tag{15.6}$$

For $n > 3$, the lattice $\mathrm{St}(\mathcal{L})$ does not satisfy the chain condition. This is easily seen, since



(15.7)

while



(15.8)

In general, T is a successor of S in $\mathrm{St}(\mathcal{L})$ if and only if

$$r(S) < r(T) < r(S+1),$$

and

$$c(S) < c(T) < c(S+1). \tag{15.9}$$

Thus the number of successors of S is given by

$$[n - r(S)] + [n - c(S)] + [n - r(S)][n - c(S)]$$
$$= [n + 1 - r(S)][n + 1 - c(S)] - 1.$$

Moreover, if for $S \subseteq W$, we set $\mathrm{Succ}(S, W) = \{T \subseteq W \mid T \text{ is a successor of } S\}$, then $|\mathrm{Succ}(S, W)|$ is

$$\nu(S, W) = [r(W) + 1 - r(S)][c(W) + 1 - c(S)] - 1. \tag{15.10}$$

Let us set, for $S \subseteq W$ and $2 < k < \nu(S, W)$,

$$c(S, W; k) = |\{Y \subseteq \mathrm{Succ}(S, W) : |Y| = k \text{ and } \sup Y = W\}|. \tag{15.11}$$

The cross-cut theorem [1] for Möbius functions of lattices gives

$$\mu(S, W) = \sum_{k > 2} (-1)^k c(S, W; k). \tag{15.12}$$

While the constants in (15.11) are not tremendously difficult to calculate in any given interval, no closed formula is known at this time.

Finally, let us calculate the values of $g(S)$. If T is a star generated by a non-taking set and $T \supseteq S$, then $r(T)=c(T)=m \geqslant \max(r(S),c(S))$, and there are precisely

$$\binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}$$

such T. Hence, there are

$$\binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}m!$$

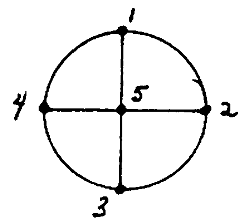
non-taking sets of size m whose star contains S. Therefore, we have

$$g(S) = \sum_{m=\max(r(S),c(S))}^{n} \binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}m!. \qquad (15.13)$$

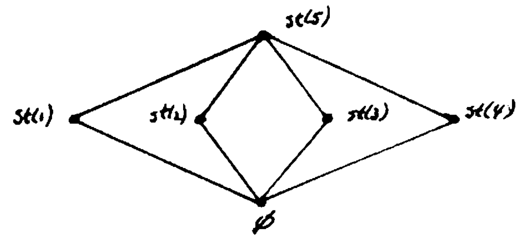
Let us now turn to the case of graphs. As in Example 14.3, given a finite graph $\mathcal{G}=(V,E)$, our lattice $\mathcal{L}$ is represented by the family of open subsets of V whose forbidden sets consist of two-point subsets $\{p,q\}$ such that (p,q) is an (undirected) edge in E. The minimal nonempty stars will be a collection of $\mathrm{st}(p)$'s, but not every $\mathrm{st}(p)$ is necessarily minimal. A two-subset $\{p,q\}$ is nontaking if and only if $p \notin \mathrm{st}(q)$ and $q \notin \mathrm{st}(p)$, that is (p,q) is not an edge of $\mathcal{G}$. Thus, nontaking subsets correspond to collections of vertices where no two are connected by an edge of $\mathcal{G}$.

A proper coloring of a graph is a placement of colors, one on each vertex of $\mathcal{G}$, such that no edge connects two vertices of the same color. Clearly, the maximum number of vertices we can color with one color is equal to $\max_A \{|A| : A \text{ in non-taking}\}$. The minimum number of colors needed to properly color a graph is equal to the smallest k such that there exists a collection of pairwise disjoint non-taking subsets $A_1, A_2, \ldots, A_k$ whose union is V.

Since the class of all finite graphs is extremely general, one would not expect to be able to obtain general formulas for the functions f and g. However, in many specific cases, they are very simple. As an example, let $\mathcal{G}$ be the graph



There are six stars in V, and $\mathrm{St}(\mathcal{L})$ is



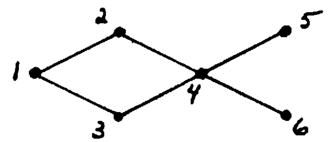
The non-taking subsets are $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,3\}$, and $\{2,4\}$. Thus,

$$f(\mathrm{st}(j)) = \begin{cases} 1, & 0 \leqslant j \leqslant 4, \\ 3, & j=5, \end{cases} \qquad (15.14)$$

and

$$g(\mathrm{st}(j)) = \begin{cases} 8, & j=0, \\ 4, & 1 \leqslant j \leqslant 4, \\ 3, & j=5. \end{cases} \qquad (15.15)$$

Now consider the graph



Here the lattice $\mathrm{St}(\mathcal{L})$ is *not* a point lattice. Indeed the minimal nonempty stars are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{4,5\}$, and $\{4,6\}$. Thus, $\mathrm{st}(4)=\{2,3,4,5,6\}$ is not a minimal star, and is not the join of the minimal nonempty stars contained in it.

Let us recall that our point lattice $\mathcal{L}$ is represented by a family of open subsets of the set $\mathcal{Q}$. For some sets $A \subseteq \mathcal{Q}$, the enumeration of the non-taking subsets contained in A can be reduced to counting the number of non-taking subsets contained in certain subsets of A. The precise determination of when this occurs leads us to the following definition. We say that two subsets A and B *split* (or form a splitting of) a subset W if $A \cup B = W$, $A \cap \mathrm{st}(B)=\varnothing$, and $\mathrm{st}(A) \cap B = \varnothing$. Let A and B split W, and suppose S is a non-taking subset of W. Clearly, $S \cap A$ and $S \cap B$ are non-taking subsets of A and B, respectively. Conversely, if S is a non-taking subset of A, and T is a non-taking subset of B, then $S \cup T$ is a non-taking subset of W. To see this let $s \in S$, $t \in T$, and $s \in \mathrm{st}(t)$. Then $s \in A \cap \mathrm{st}(B)$, but $A \cap \mathrm{st}(B)$ is empty. Let $r(W;k)$ denote the number of non-taking subsets of W of size k. Then for any splitting A,B of W, we have shown

$$r(W,n) = \sum_{k=0}^{n} \binom{n}{k} r(A,k) r(B,n-k). \tag{15.16}$$

This identity is a generalization of those given by Henle for morphs relative to certain dissects [17]. If $\mathcal{A}$ itself admits a nontrivial splitting A,B, then $\mathcal{L}$ is the direct product of the lattices of open subsets of A and B.

The *Henle coalgebra* $\mathcal{H}(\mathcal{L})$, associated to a point lattice $\mathcal{L}$, studies the splits of $\mathcal{A}$. More precisely, $\mathcal{H}(\mathcal{L})$ is the vector space over K with basis consisting of all subsets of $\mathcal{A}$. The diagonalization Δ and counit ε are given by

$$\Delta A = \sum_{\substack{(A_1,A_2) \text{ ordered} \\ \text{splits of } A}} A_1 \otimes A_2 \tag{15.17}$$

and

$$\varepsilon(A) = \begin{cases} 1 & \text{if } A = \varnothing, \\ 0 & \text{otherwise} \end{cases}. \tag{15.18}$$

That the comultiplication Δ is coassociative is easily verified. The full $n \times n$ rook board admits no nontrivial splittings. However, if W is the subset shown in Fig. 2, then

$$\Delta W = \varnothing \otimes W + W_1 \otimes W_2 + W_2 \otimes W_1 + W \otimes \varnothing.$$

A graph $\mathcal{G}$ with admit a splitting if and only if it has more than one connected component.

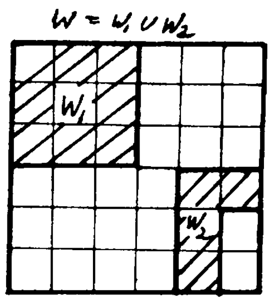


Figure 2.

XVI. Cleavages

We shall next discuss a class of coalgebras that generalizes the classical notion of the shuffle algebra to partially ordered sets. A family Σ of PO sets is called an SBC family (suitable for building cleavages) when it satisfies the following condition:

if P is a partially ordered set in Σ,
and if Q is a partially ordered subset of P in Σ,
then the partially ordered subset P-Q also belongs to Σ. (16.1)

Under this condition we define a *cleavage* of a PO set P in Σ as an ordered pair (Q,R) of subsets of P—with the inherited partial order—such that $Q \cap R = \varnothing$, $Q \cup R = P$, and Q and R belong to Σ.

The *cleavage coalgebra* of the family Σ, $C(\Sigma)$, is now defined as follows. Associate a variable x_P to each P in Σ, including 1 for the empty PO set, and let $C(\Sigma)$ be the vector space having the X_P's as a basis. Set

$$\Delta x_P = \sum_{(Q,R)} x_Q \otimes x_R, \tag{16.2}$$

where the sum ranges over all cleavages (Q,R) of P. The counit $\varepsilon(x_P)$ is zero unless $x_P = 1$, and $\varepsilon(1) = 1$; the verification of coassociativity is immediate.

We shall call a family $\tilde{\Sigma}$ of *types* (i.e. isomorphism classes) of partially ordered sets a *reduced SBC family* when it satisfies the following condition:

if α is a type in $\tilde{\Sigma}$,
if P is a partially ordered set of type α,
if Q is a partially ordered subset of P,
and if the type β of Q belongs to $\tilde{\Sigma}$, then the type γ
of the partially ordered subset $P - Q$ belongs to $\tilde{\Sigma}$. (16.3)

Clearly, if Σ is an SBC family, and if $\tilde{\Sigma}$ is the family of types (or isomorphism classes) of Σ, then $\tilde{\Sigma}$ is a reduced SBC family. The *reduced cleavage coalgebra* of the family $\tilde{\Sigma}$ is the vector space $C(\tilde{\Sigma})$ freely spanned by the variables x_α associated to each type α, including 1 for the empty PO set, with

$$\Delta x_\alpha = \sum (\alpha | \beta, \gamma) x_\beta \otimes x_\gamma, \tag{16.4}$$

where the sum ranges over all ordered pairs (β,γ) of types in $\tilde{\Sigma}$ such that a partially ordered set P of type α contains a cleavage of type (β,γ).

The section coefficients $(\alpha|\beta,\gamma)$ are integers counting the number of cleavages of type (β,γ) in a partially ordered set of type α. The counit is the obvious one, and again the verification of coassociativity is immediate.

If Σ is an SBC family and $\tilde{\Sigma}$ is the family of types of Σ, then the reduced cleavage coalgebra $C(\tilde{\Sigma})$ is isomorphic to the quotient of the cleavage coalgebra

$C(\Sigma)$ modulo the coideal generated by $x_P - x_Q$ for all isomorphic PO sets P, Q in Σ.

Examples of SBC and reduced SBC families are not abundant, and we shall only give three.

Example 16.1. Let Σ be the family of all finite linearly ordered subsets; $\tilde{\Sigma}$ is the family of all types of finite linearly ordered subsets. The cleavage coalgebra $C(\Sigma)$ is isomorphic to the Boolean coalgebra. The reduced cleavage coalgebra $C(\tilde{\Sigma})$ turns out to be isomorphic to the shuffle coalgebra. It is well known that this coalgebra is a bialgebra, where the noncommutative multiplication is simply juxtaposition.

Example 16.2. Let Σ be the family of all finite forests, and $\tilde{\Sigma}$ the reduced family consisting of all types of finite forests, considered as PO sets. Clearly Σ is an SBC family. $\tilde{\Sigma}$ defines an interesting reduced cleavage coalgebra, the tree coalgebra, which does not seem to have been studied. We do not know whether the tree coalgebra can be significantly turned into a bialgebra.

Example 16.3: Let Σ be the family of all finite PO sets; $\tilde{\Sigma}$, the reduced family of all types of PO sets. The associated reduced cleavage coalgebra, we conjecture, should have some notable universal mapping characterization, generalizing the universal properties of the shuffle coalgebra.

Several SBC subfamilies (reduced SBC subfamilies) of PO sets defined by restricting the length or width of the PO sets (types) allowed give subcoalgebras of the cleavage (reduced cleavage) coalgebra. For example, one can take all PO sets (types of PO sets) with the property that in each P, no chain exceed in length an integer n prescribed in advance.

The cleavage and reduced cleavage coalgebras can be viewed as generalizations of the incidence and reduced incidence coalgebras. Very probably, other coalgebras "in between" these two extremal cases can be defined.

XVII. Hereditary bialgebras

We come now to the description of a class of bialgebras—indeed, of Hopf algebras—which are probably the richest in structure and combinatorial applications. They are obtained from hereditary classes of matroids, a notion which we proceed to discuss briefly.

Recall that a *matroid* $M(S)$ on a (finite) set S is a closure relation defined on the subsets of S which enjoys the MacLane-Steinitz exchange property: if A is any subset of S, $\overline{A}$ its closure, and p, q elements of S such that $q \in \overline{A \cup p}$ but $q \notin \overline{A}$, then $p \in \overline{A \cup q}$. We shall need only a few elementary concepts from the theory of matroids; further details can be found in the books by Crapo and Rota [9] and by Welsh. The *direct sum* of two matroids $M(S_1)$ and $M(S_2)$ on disjoint sets S_1 and S_2 is defined as $M(S_1 + S_2)$ by setting $\overline{A_1 \cup A_2} = \overline{A}_1 \cup \overline{A}_2$, where $A_i \subseteq S_i$. A matroid is said to be *connected* when it is not isomorphic to a nontrivial direct sum of two matroids. Every matroid $M(S)$ is uniquely the direct sum of connected matroids $M(S_i)$ obtained from the blocks S_i of a suitable partition of the set S. A segment of a matroid is defined as follows. Let

A and B be closed sets of $M(S)$, and let $A \subseteq B$. The *segment* $M(A,B;S)$ is the matroid defined as the set $B - A$ with, for $C \subseteq B - A$, the closure $\tilde{C}$ of C to be $\tilde{C} = \overline{C \cup A} - A$.

In the following we denote by Greek letters isomorphism classes, or *types*, of matroids. The lattice of closed sets of a matroid is called a *geometric lattice*. Two non-isomorphic matroids may have isomorphic geometric lattices. In fact, among all non-isomorphic types of matroids having isomorphic geometric lattices, there is one which is canonically associated with the geometric lattice L as follows: the set S is the set of atoms of the lattice L (that is, elements covering the minimum element), and for $A \subseteq S$, one sets $\overline{A} = \{p \in S : p \leqslant \sup A\}$. This matroid is called the *combinatorial geometry* associated to the geometric lattice L.

The geometric lattice of the segment matroid $M(A,B;S)$ is isomorphic to the segment $[A,B]$ in the geometric lattice L of the matroid $M(S)$.

We come now to our main notion. A *hereditary class* H of matroids is a family of types of combinatorial geometries with the following properties:

(1) If α and β belong to H, then the direct sum $\alpha + \beta$ is a combinatorial geometry, and it belongs to H. The geometric lattice of $\alpha + \beta$ is the product, in the sense of partially ordered sets, of the geometric lattices of α and β.

(2) If $M(A,B;S)$ is a segment of a matroid $M(S)$ and the type of $M(S)$ is in H, then the combinatorial geometry of the type of the matroid $M(A,B;S)$ also belongs to H.

(3) If α belongs to H and α is isomorphic to the nontrivial direct sum $\alpha = \alpha_1 + \alpha_2$ of combinatorial geometries, then $\alpha_i \in H$.

Let H be a hereditary class, with types α, β, γ in H. The section coefficient $(\alpha | \beta, \gamma)$ of H is defined to be the number of closed sets A in a matroid $M(S)$ of type α such that the segment $M(\varnothing, A; S)$ is of type β and the segment $M(A, S; S)$ is of type γ. It is easy to see that this number depends only on the types α, β, γ.

We have the important

PROPOSITION 17.1. *The section coefficients of a hereditary class of matroids are section coefficients.*

Proof: We have to prove the identity

$$\sum_{\gamma} (\alpha | \beta, \gamma)(\gamma | \pi, \sigma) = (\alpha | \beta, \pi, \sigma)$$
$$= \sum_{\delta} (\alpha | \delta, \sigma)(\delta | \beta, \pi).$$

Let $(\alpha | \beta, \pi, \sigma)$ be the number of pairs of closed sets $A \subseteq B$ of a matroid $M(S)$ of type α such that $M(\varnothing, A; S)$ is of type β, $M(A,B;S)$ is of type π, and $M(B,S;S)$ is of type σ. The first sum is obtained by fixing A and letting B vary, whereas the second sum is obtained by fixing B and letting A vary.

More important is the

THEOREM 17.1. *The section coefficients associated with a hereditary class of matroids H are bisection coefficients.*

Proof: We define a semigroup structure on the hereditary class H by taking direct sums as addition. For clarity, we shall prove the bilinear identity for the special case $(\alpha|\beta,\gamma)$ where $\alpha=\alpha_1+\alpha_2$ and the α_i are connected (nontrivial) types; the general case is similar. Thus, we need to show that

$$(\alpha_1+\alpha_2|\beta,\gamma) = \sum_{\substack{\beta_1+\beta_2=\beta \\ \gamma_1+\gamma_2=\gamma}} (\alpha_1|\beta_1,\gamma_1)(\alpha_2|\beta_2,\gamma_2).$$

To this end, let $A\subseteq S$ be a closed set of $M(S)$, and let $M(S)$ be the direct sum of $M(S_1)$ and $M(S_2)$. Then the matroid $M(\varnothing,A;S)$ is isomorphic to the direct sum of the matroid $M(\varnothing,A_1;S_1)$ and $M(\varnothing,A_2;S_2)$ where $A_i=A\cap S_i$. Similarly, the matroid $M(A,S;S)$ is isomorphic to the direct sum of $M(A_1,S_1;S_1)$ and $M(A_2,S_2;S_2)$. Counting, we obtain the desired identity.

Thus, associating the variable x_α to each type of the hereditary class H, we obtain a bialgebra where the underlying algebra is the polynomial algebra in the variables x_α for which α is a *connected* type, and the comultiplication is defined by

$$\Delta x_\alpha = \Sigma(\alpha|\beta,\gamma)x_\beta\otimes x_\gamma.$$

The augmentation is defined in the obvious way.

The bialgebras obtained by this construction will be called *hereditary bialgebras*. We list some of the examples previously discussed.

(1) The Boolean algebra of subsets of finite sets turns out to be a hereditary bialgebra, which is in fact the binomial bialgebra.

(2) The Faà di Bruno bialgebra is the hereditary bialgebra obtained by taking the bond closure (see [1]) on graphs which are direct sums of complete graphs.

(3) The Eulerian coalgebra is also associated—although rather trivially—with a hereditary bialgebra. One takes all direct sums of matroids whose geometric lattices are the lattices of all subspaces of a vector space over a finite field. If α is connected, then $\Delta(x_\alpha)$ agrees with the definition already given.

Other notable hereditary classes of matroids are (4) all finite sets of points in projective space over a fixed field; (5) all series-parallel networks, (6) all graphs, (7) all planar graphs; (8) all unimodular matroids.

Each hereditary bialgebra leads to a generalization of the umbral calculus, for which the umbral calculus in one variable, outlined in Sec. XI, is the blueprint. We believe the development of such "hereditary" calculi to be one of the most promising prospects of present-day combinatorics.

In the preceding theorem (Theorem 17.1), an essential role is played by the very special factorization properties of matroids. Thus, the notion of hereditary bialgebras can be extended to any family of PO sets where one can prove the factorization properties required to make the above proof work. One such class is the class of semimodular lattices. The discovery of the most general such class, if any, may well lead to a class of bialgebras sharing a simple axiomatic definition.

In closing, we remark that the detailed study of hereditary bialgebras should have as some of its goals the extension to hereditary bialgebras of the exponential formula of the binomial bialgebra, as well as generalizations of the Lagrange inversion formula.

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Hopf Algebras and Combinatorics

by Warren Nichols and Moss Sweedler

The following material is discussed in this paper: Coalgebras from Formulas; Grouplikes; Primitives; Coalgebra Definitions and Examples; Coalgebra Maps; Convolution Algebras; Coproduct of Subspaces; Comodules; Sheffer Sequences; Coalgebra Maps to $\mathcal{D}^{\infty}$; Coradical Filtration; Conjugate Sequences; Associated Sequences; Algebra-Coalgebra Interactions; Bialgebras; Measuring; Comeasuring; Hopf Algebras; Antipodes; Pincherle Derivative; Recurrence Formula; Transfer Formula; Dual Coalgebras.

I. Introduction

These are a revision by the second author of notes taken by the first author at lectures by the second author.

I shall try to give a seductive overview of coalgebra and Hopf algebra theory and some of its relations to combinatorics, especially to the umbral calculus. Skimming through many definitions, terms, examples, results but few proofs. I hope that by the end of these lectures you will have a sense of familiarity with coalgebras and Hopf algebras and you will be motivated to really learn about them.

Thanks to Warren Nichols for taking these notes and for cleaning up where I was too terse, incomplete, slip-shod or wrong. Warren has made helpful additions and deletions, including an interesting example of a non-subcoalgebra.

References in the form [p.**] are to Hopf Algebras by Moss E. Sweedler, W.A. Benjamin, New York, 1969.

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II. Coalgebras from formulas

We first encounter coalgebras as their diagonalization Δ arises from generic addition formulas. Some examples are: (i) the trigometric identities $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$; (ii) the exponential law . $e^{x+y} = e^x e^y$; (iii) sequences $\{p_n(x)\}$ of polynomials in one variable, with degree $p_n = n$, satisfying the relations $p_n(a+b) = \Sigma_{i=0}^n \binom{n}{i} p_i(a)p_{n-i}(b)$, these are known as sequences of binomial type; and (iv) the divided power sequences of polynomials $\{d_n(x)\}$, satisfying $d_n(a+b) = \Sigma_{i=0}^n d_i(a)d_{n-i}(b)$, which can be obtained in characteristic zero by replacing $p_n(x)$ with $p_n(x)/n!$.

Let k be any field. There is an algebra map $\Delta: k[x] \to k[x] \otimes k[x]$ defined by sending x to $x\otimes 1 + 1\otimes x$; we then have $\Delta(1) = 1\otimes 1$, and $\Delta(x^n) = (\Delta x)^n = (x\otimes 1 + 1\otimes x)^n$. For any polynomial $f(x)$, we have $\Delta(f(x)) = f(\Delta(x))$. Note that $f(x\otimes 1) = f(x)\otimes 1$, and $f(1\otimes x) = 1\otimes f(x)$.

Give $k[x]$ the ideal topology determined by $\langle x \rangle$, (i.e. an element of $k[x]$ is "small" if it is divisible by a large power of x). Give $k[x] \otimes k[x]$ the ideal topology determined by $\langle x\otimes 1, 1\otimes x \rangle$. Then Δ is continuous, and passing to the completion yields a map

$$\hat{\Delta}: k[[x]] \to \widehat{k[x] \otimes k[x]} = k[[x \otimes 1,\ 1 \otimes x]]$$

which extends Δ. Write Δ for $\hat{\Delta}$. Note that $k[[x]] \otimes k[[x]] \subset k[[x\otimes 1, 1\otimes x]]$.

If k has characteristic zero, using the power series expansion of $\sin(x)$ and $\cos(x)$ we have

$$\begin{aligned}\Delta(\sin(x)) &= \sin(\Delta(x)) = \sin(x\otimes 1 + 1\otimes x)\\ &= \sin(x\otimes 1)\cos(1\otimes x) + \cos(x\otimes 1)\sin(1\otimes x)\\ &= (\sin(x)\otimes 1)(1\otimes \cos(x)) + (\cos(x)\otimes 1)(1\otimes \sin(x))\\ &= \sin(x)\otimes\cos(x) + \cos(x)\otimes\sin(x)\ .\end{aligned}$$

Similarly, $\Delta(\cos(x)) = \cos(x)\otimes\cos(x) - \sin(x)\otimes\sin(x)$. For the power series expansion of e^x: $\Delta(e^x) = e^{\Delta(x)} = e^{x\otimes 1 + 1\otimes x} = e^{x\otimes 1}e^{1\otimes x} = (e^x\otimes 1)(1\otimes e^x) = e^x \otimes e^x$. (More generally, $\Delta(e^{\lambda x}) = e^{\lambda x}\otimes e^{\lambda x}$ for any $\lambda \in k$.)

With sequences of binomial type, we need not pass to the completion: $\Delta(p_n(x)) = p_n(x\otimes 1 + 1\otimes x) = \Sigma_{i=0}^n \binom{n}{i} p_i(x\otimes 1)p_{n-i}(1\otimes x) = \Sigma_{i=0}^n \binom{n}{i} p_i(x)\otimes p_{n-i}(x)$. The calculation for sequences of divided powers is similar. The upshot is that each addition formula naturally gives a formula for the action of Δ.

Next we view coalgebra diagonalization Δ arising from operator formulas. The first of these is the formula for an algebra map σ from algebras A to B: $\sigma(a\alpha) = \sigma(a)\sigma(\alpha)$ for all $a,\alpha \in A$. Another formula arises from the derivations of an algebra A-- the maps $d: A \to A$ which satisfy $d(a\alpha) = d(a)\alpha + ad(\alpha)$ for all $a,\alpha \in A$. More generally, consider the higher derivations $\{d^{(n)}\}$ which satisfy the Leibnitz formula $d^{(n)}(a\alpha) = \Sigma_{i=0}^n d^{(i)}(a)d^{(n-i)}(\alpha)$. If $d: A \to A$ is a derivation and the characteristic is zero, setting $d^{(n)} = d^n/n!$ gives $\{d^{(n)}\}$, a higher derivation.

Let $\overline{H}$ be a vector space, and A,B algebras. Suppose we are given a map "-": $\overline{H} \to \mathrm{Hom}(A,B)$, sending $\overline{h} \in \overline{H}$ to $h: A \to B$. Define two <u>evaluation</u> maps: $\overline{H}\otimes A \to B$ by $\overline{h}\otimes a \to h(a)$, and $\overline{H}\otimes\overline{H}\otimes A\otimes A \to B$ by $\overline{h}_1 \otimes \overline{h}_2\otimes a\otimes\alpha$ to $h_1(a)h_2(\alpha)$.

A map $\Delta: \overline{H} \to \overline{H} \otimes \overline{H}$ which makes the diagram commute:

$$\begin{array}{ccc} \overline{H}\otimes A\otimes A & \xrightarrow{\Delta\otimes I\otimes I} & \overline{H}\otimes\overline{H}\otimes A\otimes A \\ \downarrow{\scriptstyle I\otimes \text{mult}} & & \downarrow{\scriptstyle \text{evaluation}} \\ \overline{H}\otimes A & \xrightarrow{\text{evaluation}} & B \end{array}$$

gives an operator formula as follows: if $\Delta(\overline{h}) = \Sigma \overline{h}_{i,1}\otimes \overline{h}_{i,2}$, then

OPERATOR FORMULA $\quad h(a\alpha) = \Sigma h_{i,1}(a)h_{i,2}(\alpha)$ for all $a,\alpha \in A$.

In this situation, we say that H <u>measures</u> A to B.

An example, suppose that $\sigma: A \to B$ is an algebra map. Let $\overline{H}$ be a one dimensional space with basis $\overline{\sigma}$. Define $\Delta: \overline{H} \to \overline{H} \otimes \overline{H}$ by $\Delta(\overline{\sigma}) = \overline{\sigma}\otimes\overline{\sigma}$. Then the formula $\sigma(a\alpha) = \sigma(a)\sigma(\alpha)$ shows that $\overline{H}$ measures A to B if

$\overline{H} \to \mathrm{Hom}(A,B)$, $\overline{\sigma} \to \sigma$.

When $A = B$, and $d: A \to A$ is a derivation, let $\overline{H}$ have basis $\overline{I}, \overline{d}$ and define $\Delta(\overline{I}) = \overline{I} \otimes \overline{I}$, and $\Delta(\overline{d}) = \overline{I} \otimes \overline{d} + \overline{d} \otimes \overline{I}$. Then $\overline{H}$ measures A to itself if $\overline{H} \to \mathrm{Hom}(A,A)$, $\overline{I} \to I$, $\overline{d} \to d$. Here I is the identity map in $\mathrm{Hom}(A,A)$.

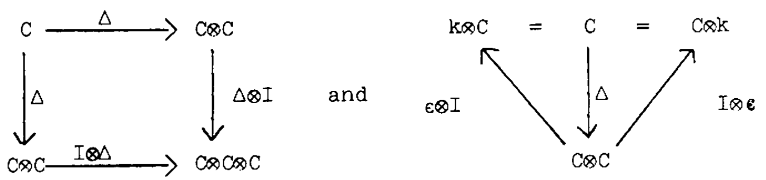
Given a higher derivation $\{d^{(n)}\}$ of A, let $\{\overline{d^{(n)}}\}$ be a basis for $\overline{H}$, and set $\Delta(\overline{d^{(n)}}) = \Sigma_{i=0}^{n} \overline{d^{(i)}} \otimes \overline{d^{(n-i)}}$. Then $\overline{H}$ measures A to itself if $\overline{H} \to \mathrm{Hom}(A,A)$, $\overline{d^{(n)}} \to d^{(n)}$. Higher derivations are especially useful in characteristic $p > 0$, where derivations vanish on p^{th} powers.

As you can see from the examples measuring unifies homomorphisms, derivations and higher derivations. A technique afforded by measuring is to turn operator statements into coalgebra statements and then to deduce facts about the operators from coalgebra results.

Of particular importance in the study of coalgebras are elements g satisfying the conditions $\Delta(g) = g \otimes g$ and (ϵ to be explained shortly) $\epsilon(g) = 1$. The elements are called grouplike. Their existence may depend in part upon the field k; for instance, if $i = \sqrt{-1} \in k$, then $\cos(x) \pm i \sin(x) = e^{\pm ix}$ are grouplike. Also, primitive elements are important. An element d is g-primitive if $\Delta(d) = g \otimes d + d \otimes g$ where g is grouplike.

III. Coalgebra definitions and examples

A coalgebra is a vector space C over a field K, together with linear maps $\Delta : C \to C \otimes C$ (the coproduct) and $\epsilon : C \to k$ (the counit), making the diagrams commute:



The first diagram says that Δ is coassociative; the second, that ϵ is a left and right counit for Δ.

An important tool for coalgebra manipulations is the Σ notation. When working with elements $x_1, x_2, \ldots, x_n$ of an associative algebra, it is convenient to write $x_1 x_2 \ldots x_n$ in place of $x_1(x_2(\ldots x_{n-1}(x_n)\ldots))$ and to think of $x_1 x_2 \ldots x_n$ as denoting the result of multiplication performed in any association whatsoever. Σ notation is the appropriate dual procedure for coalgebras.

Write $\overset{n}{\otimes} C$ for the n-fold tensor product of C with itself. Define $\Delta_n : C \to \overset{n+1}{\otimes} C$ by: $\Delta_1 = \Delta$, and $\Delta_n = (\Delta \otimes I \otimes \ldots \otimes I) \circ \Delta_{n-1}$. Thus, $\Delta_n(c)$ is obtained from $\Delta_{n-1}(c)$ by applying Δ to the first factor. Write $\Delta_n(c) = \Sigma_{(c)} c_1 \otimes \ldots \otimes c_{n+1}$. We emphasize that the Σ is a dummy summation sign serving merely to remind us that $\Delta_n(c)$ is a sum of elements of the form $x_1 \otimes x_2 \otimes \ldots \otimes x_{n+1}$ where each $x_i \in C$. The symbols $c_1, \ldots, c_{n+1}$ do not denote particular elements, but are merely place-holders.

Σ notation is often used in connection with multilinear functions. Suppose that V is a vector space, and $f : \overset{n}{\times} C = C \times \ldots \times C \to V$ a multilinear map. Let $F : \overset{n}{\otimes} C \to V$ be the induced map. Write $\Sigma_{(c)} f(c_1, \ldots, c_n)$ for $F(\Delta_{n-1}(c))$. Again, the Σ is dummy and the c_i's are place-holders. If we express $\Delta_{n-1}(c) = \Sigma_j c_{1,j} \otimes c_{2,j} \otimes \ldots \otimes c_{n,j}$, where $c_{i,j} \in C$ for all i,j, then $F(\Delta_{n-1}(c)) = \Sigma_j f(c_{1,j}, \ldots, c_{n,j})$.

For example, let C be a coalgebra, A an algebra, and $g,h: C \to A$ linear maps. The notation $\Sigma_{(c)} g(c_1)h(c_2)$ is interpreted as follows: $C \times C \to A$ sending (d,e) to $g(d)h(e)$ is a bilinear map. This induces a linear map $F : C \otimes C \to A$, given by $F = \mathrm{mult} \circ (g \otimes h)$. Then $\Sigma_{(c)} g(c_1)h(c_2) = F(\Delta(c))$. More explicitly, if we write $\Delta(c) = \Sigma_{i=1}^{n} c_{1,i} \otimes c_{2,i}$ where $c_{1,i}, c_{2,i} \in C$ for all i, then

$\Sigma_{(c)} g(c_1)h(c_2) = \Sigma_{i=1}^{n} g(c_{1,i})h(c_{2,i})$.

Σ notation can be used to convey concisely the information in commutative diagrams. For example, $\Delta: C \to C \otimes C$ is coassociative iff $\Sigma_{(c)} \Delta(c_1)\otimes c_2 = \Sigma_{(c)} c_1\otimes\Delta(c_2)$ for all $c \in C$. Also, $\varepsilon : C \to k$ is a counit for Δ iff $\Sigma_{(c)} \varepsilon(c_1)c_2 = c = \Sigma_{(c)} c_1\varepsilon(c_2)$.

We will say that C is cocommutative if $\Delta = T\circ\Delta$, where $T : C\otimes C \to C\otimes C$ is the "twist" map sending $d\otimes e$ to $e\otimes d$. In Σ notation this condition is $\Sigma_{(c)} c_1\otimes c_2 = \Sigma_{(c)} c_2\otimes c_1$. Many of the coalgebras important in combinatories are cocommutative.

Here are a number of examples of coalgebras.

1. The trigonometric coalgebra $\mathcal{T}$. This coalgebra has basis s,c with $\Delta(s) = s\otimes c + c\otimes s$, $\Delta(c) = c\otimes c - s\otimes s$, $\varepsilon(s) = 0$, $\varepsilon(c) = 1$. (Think of s,c as the functions $\sin(x)$, $\cos(x)$, then ε corresponds to evaluation at zero.)

2. The divided power coalgebras $\mathcal{D}^N$, $\mathcal{D}^\infty$. Here $\mathcal{D}^\infty$ has basis $\{d_i\}_{i \geq 0}$, with $\Delta(d_n) = \Sigma_{i=0}^n d_i\otimes d_{n-i}$, $\varepsilon(d_n) = \delta_{0,n}$. To get $\mathcal{D}^N$ take only $d_0, d_1, \ldots, d_N$. Notice that d_0 is grouplike and d_1 is d_0-primitive.

3. The coalgebras $\mathcal{B}^N$, $\mathcal{B}^\infty$ of binomial type. These are similar to the divided power coalgebras. $\mathcal{B}^\infty$ has basis $\{b_i\}_{i \geq 0}$, with $\Delta(b_n) = \Sigma_{i=0}^n \binom{n}{i} b_i\otimes b_{n-i}$, $\varepsilon(b_n) = \delta_{0,n}$. Again, b_0 is grouplike and b_1 is b_0-primitive. In characteristic zero, $\mathcal{B}^\infty$ is isomorphic to $\mathcal{D}^\infty$, with d_n corresponding to $b_n/n!$. To get $\mathcal{B}^N$ take only $b_0, \ldots b_N$.

4. For any set S form a coalgebra with the elements s of S as a basis by setting $\Delta(s) = s\otimes s$, $\varepsilon(s) = 1$.

5. For each positive integer n, the comatric coalgebra $M^c(n,k)$ has basis $\{x_{ij}\}_{i,j=1}^n$, with $\Delta(x_{ij}) = \Sigma_{k=1}^n x_{ik}\otimes x_{kj}$, and $\varepsilon(x_{ij}) = \delta_{ij}$. In contrast to the examples above, $M^c(n,k)$ is not cocommutative if $n \geq 2$.

6. We can form a coalgebra C with basis g,h,ℓ by declaring g,h to be grouplike, and setting $\Delta(\ell) = g\otimes\ell + \ell\otimes h$, $\varepsilon(\ell) = 0$. This coalgebra lives inside $M^c(2,k)$, with $g = x_{11}$, $h = x_{22}$, $\ell = x_{12}$. It is not cocommutative.

7. The incidence coalgebra. Let $\mathcal{S}$ be a partially ordered set. Let C have basis $\{(x,y) \in \mathcal{S} \times \mathcal{S} : x \leq y$, x near $y\}$,

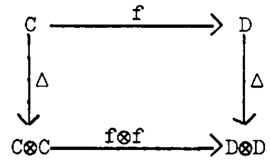
"x near y" means that $\{z \in \mathcal{S} : x \leq z \leq y\}$ is finite.

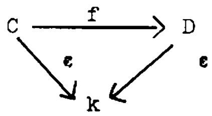
Define $\Delta((x,y)) = \sum_{x \leq z \leq y} (x,z)\otimes(z,y)$, and $\varepsilon((x,y)) = \delta_{x,y}$. Note that example 6 is an incidence coalgebra, with $\mathcal{S} = \{x,y\}$, $x < y$, $g = (x,x)$, $h = (y,y)$, $\ell = (x,y)$.

8. Let V be any vector space. Let $C = k\oplus V$. Declare $g = (1,0)$ to be grouplike. Identifying $v \in V$ with $(0,v) \in C$, set $\Delta(v) = g\otimes v + v\otimes g$, and $\varepsilon(v) = 0$. Thus the elements of V are g-primitive.

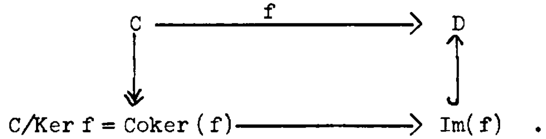
IV. Coalgebra Maps

Let C,D be coalgebras. A coalgebra map $f : C \to D$ is a linear transformation which makes the diagrams below commute:





This can be written $\Delta(f(c)) = \Sigma_{(c)} f(c_1)\otimes f(c_2)$, and $\varepsilon(f(c)) = \varepsilon(c)$. Subcoalgebras correspond to injective coalgebra maps; a subspace V of C defines a subcoalgebra of C if $\Delta(V) \subseteq V\otimes V$, for in this case the Δ and ε for C can be used to define a coalgebra structure on V. A coalgebra quotient of C is given by a surjective coalgebra map; the kernel of such a map is called a coideal. A subspace K of C is a coideal if and only if $\Delta(K) \subseteq K\otimes C + C\otimes K$ and $\varepsilon(K) = 0$; in this case, C/K has a unique coalgebra structure so that the canonical projection $C \to C/K$ is a coalgebra map. In general, any coalgebra map $f : C \to D$ has a unique factorization:



Now a digression for budding coalgebraists to point out difficulties when k is not a field but merely a commutative ring. Others please skip this digression by proceeding to the beginning of section V, page 57.

Since the tensor product of injections may fail to be injective, we cannot identify the tensor product of submodules of C with a submodule of $C \otimes C$.

First an example where V is a submodule of a coalgebra C, and there may be more than one coalgebra structure on V with the natural inclusion $V \hookrightarrow C$ a coalgebra map. Take $k = \mathbb{Z}$, $C = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, with $g = (1,0)$ grouplike, and $x = (0,1)$ g-primitive. Let V be the submodule spanned by g and $2x$. Then $V \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$-module. V can be made into a coalgebra by letting g be grouplike, and $2x$ g-primitive. Alternatively define a coproduct Δ' on V, with g still grouplike, but now with $\Delta'(2x) = g \otimes (2x) + (2x) \otimes (2x) + (2x) \otimes g$ in $V \otimes V$, and $\epsilon(2x) = 0$. This second coalgebra structure has an additional grouplike element $g + (2x)$, and thus is not isomorphic to the first structure. However, for each coalgebra structure on V the inclusion $V \hookrightarrow C$ is a coalgebra map, since in $C \otimes C$ the term $2x \otimes 2x = 4x \otimes x = 0 \otimes x = 0$.

Second, an example where V is a submodule of C with $\Delta(V) \subseteq \mathrm{Im}(V \otimes V \to C \otimes C)$, yet V has no coalgebra structure with $V \hookrightarrow C$ a coalgebra map. This example was supplied by Warren Nichols. The example is without a counit. (A counit could easily be adjoined.) Let $k = \mathbb{Z}$, $C = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $x = (1,0)$, $z = (0,1)$. Define $\Delta : C \to C \otimes C$ by: $\Delta(x) = 0$, $\Delta(z) = 4x \otimes x$. (Since $4x \otimes x$ has order 2 in $C \otimes C$, Δ is well-defined.) Let $y = 2x$, $V = \mathbb{Z}y + \mathbb{Z}z \subset C$. Since $\Delta(z) = y \otimes y$, we have $\Delta(V) \subseteq \mathrm{Im}(V \otimes V \to C \otimes C)$. However, the map $\Delta : V \to C \otimes C$ has no lifting to $V \otimes V$, since every preimage of $\Delta(z)$ in $V \otimes V$ has order 4.

Turning to quotients, let K be a submodule of C. The kernel of the natural map $C \otimes C \to C/K \otimes C/K$ is $\mathrm{Im}(K \otimes C \to C \otimes C) + \mathrm{Im}(C \otimes K \to C \otimes C)$. Thus, a submodule K of C is the kernel of a <u>surjective</u> coalgebra map iff $\Delta(K) \subseteq \mathrm{Im}(K \otimes C \to C \otimes C) + \mathrm{Im}(C \otimes K \to C \otimes C)$, and $\epsilon(K) = 0$, so the characterization of coideals is still correct. However, the kernel of a <u>non-surjective</u> coalgebra map may not be a coideal. Again with $k = \mathbb{Z}$, let $D = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, with $d_0 = (1,0,0)$, $d_1 = (0,1,0)$, $d_2 = (0,0,1)$ being a sequence of divided powers. Take $E = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, with $e_0 = (1,0)$, $e_1 = (0,1)$ also a sequence of divided powers. Define $f : D \to E$ by: $f(d_0) = e_0$, $f(d_1) = 2e_1$, $f(d_2) = 0$. Then f is a coalgebra map; we check only $(f \otimes f)\Delta(d_2) = (f \otimes f)(d_0 \otimes d_2 + d_1 \otimes d_1 + d_2 \otimes d_0) = 0 + 2e_1 \otimes 2e_1 + 0 = 4e_1 \otimes e_1 = 0 = \Delta(f(d_2))$. $\mathrm{Ker}\, f = \mathbb{Z}d_2$ is not a coideal of C, since $\Delta(d_2) \not\subseteq d_2 \otimes C + C \otimes d_2$.

V. The convolution algebra

Returning to the setting k a field, let C be a coalgebra, A an algebra. ("An algebra is defined by taking the defining diagrams for a coalgebra and reversing arrows." -- old coalgebraists joke.) Algebras are assumed to be associative and have unit.

$\mathrm{Hom}(C,A)$ has an algebra structure, with the product (called "convolution") defined by $(f*g)(c) = \Sigma_{(c)} f(c_1)g(c_2)$ for $f,g \in \mathrm{Hom}(C,A)$ and $c \in C$. In terms of maps, $f*g = (\mathrm{mult}) \circ (f \otimes g) \circ \Delta$. The unit $\underline{\epsilon}$ of $\mathrm{Hom}(C,A)$ is given by $\epsilon(c) = \epsilon(c)1$. The algebra structure is functorial: if $\sigma : A \to B$ is an algebra map, and $\gamma : C \to D$ is a coalgebra map, then $\mathrm{Hom}(\gamma,\sigma)$ is an algebra map where $\mathrm{Hom}(\gamma,\sigma) : \mathrm{Hom}(D,A) \to \mathrm{Hom}(C,B)$, is defined by $\mathrm{Hom}(\gamma,\sigma)(f) = \sigma \circ f \circ \gamma$, for $f \in \mathrm{Hom}(D,A)$. When A is the ground field k, we get the <u>dual algebra</u> $C^* = \mathrm{Hom}(C,k)$. There is a natural algebra injection $C^* \otimes A \to \mathrm{Hom}(C,A)$, defined by $(f \otimes a)(c) = f(c)a$ for $f \in C^*$, $a \in A$, $c \in C$. This map is an algebra isomorphism when C or A is finite dimensional.

Let A be an algebra, $f : C \to D$ a coalgebra map. (The case $A = k$ is the most important.) We have an algebra map $\mathrm{Hom}(f,A) : \mathrm{Hom}(D,A) \to \mathrm{Hom}(C,A)$, sending $h \in \mathrm{Hom}(D,A)$ to $h \circ f \in \mathrm{Hom}(C,A)$. Identify the kernel of this map as $(\mathrm{Im}(f))^{\perp}$, the maps from D to A which vanish of $f(C)$. The image of $\mathrm{Hom}(f,A)$ is $(\mathrm{Ker}\, f)^{\perp}$, the maps from C to A vanishing on $\mathrm{Ker}\, f$. Thus if C is a subcoalgebra of D, $C^{\perp}$ is an ideal of $\mathrm{Hom}(D,A)$; and if K is a coideal of C, $K^{\perp}$ is a subalgebra of $\mathrm{Hom}(C,A)$.

We take $\perp$ --perp-- of both subspaces of C and subspaces of $\mathrm{Hom}(C,A)$. For a subspace X of C: $X^{\perp} = \{g \in \mathrm{Hom}(C,A) : X \subset \mathrm{Ker}\, g\} \subset \mathrm{Hom}(C,A)$. For a subspace Y of $\mathrm{Hom}(C,A)$: $Y^{\perp} = \cap_{y \in Y} \mathrm{Ker}\, y$. In general some subspaces of Hom(C,A) will not be the $\perp$ of any subspace of C. Taking the $\perp$'s of finite dimensional subspaces of C as a basis of open neighborhoods of zero in Hom(C,A) induces a topology of a topological group on Hom(C,A). In this topology $\perp$'s of subspaces of C are closed and if A = k, the ground field, then the closed subspaces of Hom(C,A) are precisely the $\perp$'s of subspaces of C.

For general algebras A if $J \subseteq \mathrm{Hom}(C,A)$ is an ideal, then $J^{\perp}$ is a subcoalgebra of C. If $U \subseteq \mathrm{Hom}(C,A)$ is a subalgebra, then $U^{\perp}$ is a coideal in C if A = k but $U^{\perp}$ need not be a coideal in C when $A \neq k$. In fact for $C = \mathcal{D}^2$ and $A = k[x]/\langle x^2 \rangle$, the subspace U of Hom(C,A) spanned by ε and the linear map $f: C \to A$ determined by $d_0 \to 0$, $d_1 \to \bar{x}$, $d_2 \to 0$ is a subalgebra where $U^{\perp} = kd_2 \subset C$ and $U^{\perp}$ is not a coideal in C.

When A = k $\perp$-ing gives a bijective correspondence between the set of subcoalgebras of C and the set of closed ideals in Hom(C,k). $\perp$-ing also gives a bijective correspondence between the set of coideals of C and the set of closed subalgebras of Hom(C,k).

Checking our examples:

First $\mathcal{D}^{\infty}$. Define elements $\{t^n\}$ of $\mathrm{Hom}(\mathcal{D}^{\infty},A)$ by: $t^n(d_m) = \delta_{n,m}1$. Then $\mathrm{Hom}(\mathcal{D}^{\infty},A) \cong A[[t]]$ as vector spaces. We have $(t^r * t^s)(d_m) = \Sigma_{i=0}^{m} t^r(d_i)t^s(d_{m-i}) = \delta_{r+s,m}1$. Thus $t^r * t^s = t^{r+s}$, and $\mathrm{Hom}(\mathcal{D}^{\infty},A) \cong A[[t]]$ as algebras. Note that $\mathrm{Hom}(\mathcal{D}^N,A) \cong A[t]/(t^{N+1})$.

Similarly, $\mathrm{Hom}(\mathcal{B}^{\infty},A) \cong A\{\{t\}\}$ as algebras. Here we write the formal symbol $\frac{t^i}{i!}$ for the element in $\mathrm{Hom}(\mathcal{B}^{\infty},A)$ with $\frac{t^i}{i!}(b_j) = \delta_{ij}$; we have $\frac{t^i}{i!} * \frac{t^j}{j!} = \binom{i+j}{i} \frac{t^{i+j}}{(i+j)!}$. When characteristic k = 0,

$$A\{\{t\}\} \cong A[[t]],\ \frac{t^i}{i!} \longleftrightarrow \frac{1}{i!}\, t^i .$$

Let $\mathcal{T}$ be the trigonometric coalgebra. Write $\mathrm{Hom}(\mathcal{T},A) = A\varepsilon \oplus As^*$, where ε is the unit and s^* is defined by: $s^*(c) = 0$, $s^*(s) = 1$. We have $(s^* * s^*)(s) = s^*(s)s^*(c) + s^*(c)s^*(s) = 0$, and $(s^* * s^*)(c) = s^*(c)s^*(c) - s^*(s)s^*(s) = -1$. Thus, $s^* * s^* = -\varepsilon$. This shows that $\mathrm{Hom}(\mathcal{T},A) = A \oplus A\sqrt{-1}$

If $i = \sqrt{-1} \in k$, $\mathcal{T} = k(c+is) \oplus k(c-is)$ is the sum of one-dimensional subcoalgebras. ("The dual of a coalgebra tends to contain what is needed to split it."--old coalgebraists proverb.)

If C is a coalgebra with a basis of grouplike elements indexed by the elements of a set S, then $\mathrm{Hom}(C,A) \cong \prod_{s \in S} A$ as algebras.

We now take $C = M^c(n,k)$, the comatric coalgebra. C has basis $\{x_{ij}\}_{1 \leq i,j \leq n}$. Hom(C,A) has dual A-basis $\{e_{ij}\}$. We have $\mathrm{Hom}(C,A) \cong M(n,A) = M(n,k) \otimes A$, where M(n,-) is the $n \times n$ matrix algebra over "-". Note that duality allows us to identify certain important subspaces of C. For example, the subspace $J = \Sigma_{i > j} kx_{ij}$ is the $\perp$ of the upper-triangular matrices, and thus a coideal. The upper-triangular matrices are isomorphic to Hom(C/J,A) as an algebra.

When C has basis g, h, ℓ with g, h grouplike and $\Delta(\ell) = g \otimes \ell + \ell \otimes h$, we can realize Hom(C,A) as upper-triangular matrices over A, with g, h, ℓ dual to the matrix elements e_{11}, e_{22}, e_{12}.

VI. The coproduct of subspaces

Let C be a coalgebra, $U, V \subset C$ subspaces. Define $U \Delta V = \mathrm{Ker}(C \xrightarrow{\Delta} C \otimes C \to C/U \otimes C/V) = \Delta^{-1}(C \otimes V + U \otimes C)$. ($U \Delta V$ is often written $U \wedge V$ in the literature, and called the wedge.) The coideal condition $\Delta(U) \subseteq U \otimes C + C \otimes U$ can be written $U \subset U \Delta U$. The significance of the operation is clear from its dual: in Hom(C,A), we have $(U \Delta V)^{\perp} = U^{\perp} * V^{\perp}$ (sums of products). The wedge will be useful later in describing the coradical filtration.

VII. Comodules

Let C be a coalgebra. A right C-comodule is a pair (M, ψ), where M is a vector space, and $\psi: M \to M \otimes C$ is a linear map such that the diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M \otimes C \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi\otimes I} \\ M\otimes C & \longrightarrow & M \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\psi} & M \otimes C \\ & \searrow\!\!\!\cong & \downarrow{\scriptstyle I\otimes\varepsilon} \\ & & M \otimes k \end{array}$$

Write $\psi(m) = \Sigma_{(m)} m_1 \otimes m_2$. (This is the Σ notation for comodules.) The lowest number in $\psi_n(m) = \Sigma_{(m)} m_1 \otimes \cdots \otimes m_{n+1}$ holds an M-place.) Note that (C,Δ) is a right C-comodule.

For $U \subset M$, $V \subset C$ write

$$U \psi V = \mathrm{Ker}(M \xrightarrow{\psi} M\otimes C \to M/U \otimes C/V).$$

Let M,N be right C-comodules. Then $f:M \to N$ is a comodule map if the diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle\psi_M} & & \downarrow{\scriptstyle\psi_N} \\ M\otimes C & \xrightarrow{f\otimes I} & N\otimes C \end{array} .$$

We look at some examples of comodules.

Let C be the comatric coalgebra $M^c(n,k)$. Let M have basis $\{v_i\}_{i=1}^n$. Set $\psi(v_i) = \Sigma_{j=1}^n v_j \otimes x_{ji}$. Note that, for each t we have a comodule injection $f_t:M \to C$ defined by $f_t(v_j) = x_{tj}$.

Let C be the incidence coalgebra on the partially ordered set $\mathcal{S}$. For each $t \in \mathcal{S}$, let M_t be a vector space with basis $\{(u):u \in \mathcal{S},\ u \geq t,\ u \text{ near } t\}$. Define $\psi_t:M_t \to M_t \otimes C$ by: $\psi_t((u)) = \sum_{t \leq v \leq u} (v)\otimes(v,u)$. Then (M_t, ψ_t) is a right C-comodule. We have a comodule injection $f_t:M \to C$, defined by $f_t((v)) = (t,v)$.

Here is an example related to factor sequences. The coalgebra is $\mathcal{B}^\infty$. The comodule Q^∞ has basis $\{q_{-n}\}_{n=0}^\infty$. Set

$$\psi(q_{-n}) = \Sigma_{i=0}^n \binom{-i}{n-i} q_{-i} \otimes b_{n-i}$$

where as usual $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ and $\binom{0}{k} = \delta_{0,k}$. The tie-in with factor sequences will be described at the end of this section and at the end of (XI,3).

VIII. Sheffer sequences and comodule maps

Consider $\mathcal{B}^\infty$ with the usual basis $\{b_i\}_0^\infty$ and let $\{e_i\}_0^\infty \subset \mathcal{B}^\infty$ be a sequence satisfying:

$$\Delta e_n = \Sigma_{i=0}^n \binom{n}{i} e_i \otimes b_{n-i}. \qquad *$$

This is the notion of $\{e_i\}$ being Sheffer with respect to $\{b_i\}$. Defining $f:\mathcal{B}^\infty \to \mathcal{B}^\infty$, $f(b_n) = e_n$, $(*)$ insures that f is a comodule map where $\mathcal{B}^\infty$ is considered as a right $\mathcal{B}^\infty$-comodule. Conversely if $g:\mathcal{B}^\infty \to \mathcal{B}^\infty$ is a right $\mathcal{B}^\infty$-comodule map and $e_i = g(b_i)$ then $(*)$ is satisfied. Hence Sheffer sequences may be identified with comodule maps from $\mathcal{B}^\infty$ to itself. In fact if $\{b_i'\}$ is any basis for $\mathcal{B}^\infty$ which is a sequence of binomial type and $g:\mathcal{B}^\infty \to \mathcal{B}^\infty$ a comodule map then $\{g(b_i')\}$ is the Sheffer sequence for $\{b_i'\}$ equivalent to g.

All this carries over to $\mathcal{D}^\infty$. Call $\{f_i\} \subset \mathcal{D}^\infty$ Sheffer for $\{d_i\}$ if

$$\Delta(f_n) = \Sigma_{i=0}^n f_i \otimes d_{n-i}. \qquad **$$

In characteristic zero $(*)$ and $(**)$ relate by the correspondence $e_i/i! \longleftrightarrow f_i$, $b_j/j! \longleftrightarrow d_j$. As with $\mathcal{B}^\infty$ the Sheffer sequences correspond to the comodule maps from $\mathcal{D}^\infty$ to itself, with $\mathcal{D}^\infty$ considered as a right $\mathcal{D}^\infty$-comodule.

Turning to duality considerations. Let M be a right C-comodule. Recall that $C^* = \mathrm{Hom}_k(C,k)$ is an algebra. For $f \in C^*$, $m \in M$, define $f\cdot m = \Sigma_{(m)} m_1 f(m_2) = (I\otimes f)\psi(m) \in M\otimes k = M$. This makes M into a left C^*-module.

Note that M is a locally finite C^*-module i.e. M is the union of its finite-dimensional submodules. Indeed, if $\psi(m) = \Sigma_{i=1}^n m_i \otimes c_i$, then $f\cdot m = \Sigma_{i=1}^n m_i f(c_i) \in C^*\cdot m \subset \Sigma_{i=1}^n k m_i$. It is true for $C = \mathcal{D}^\infty$ -- but not for coalgebras in general -- that every locally finite C^*-module arises in the above manner from a C-comodule.

Let M,N be right C-comodules. A map $f:M \to N$ is a comodule map iff it is a C^*-module map. A subspace K of M is a C^*-submodule iff it is a C-subcomodule i.e. $\psi(K) \subseteq K\otimes C$. (There are problems when not working over a field.) If K is a subcomodule, then M/K has a natural comodule structure, and the isomorphism theorems hold.

If U is a left C^*-module, then U has a largest C^*-submodule $U^{\square}$ which arises from a C-comodule structure.

Recall that C is a right C-comodule, and thus a left C^*-module. For $c^*\epsilon C^*$, $c\epsilon C$ write this action as $c^*\rightharpoonup c = \Sigma_{(c)}\, c_1 c^*(c_2) = \Sigma_{(c)}\, c_1\langle c^*|c_2\rangle$. (The "$\rightharpoonup$" notation avoids confusion with other actions which will be introduced; the star on an element of C^* is purely decorative, and should not be confused with the notation for a dual basis.) Note that $\langle c^*|c\rangle = \langle \epsilon|c^*\rightharpoonup c\rangle$. More generally, the following "contragradience" formula holds: for $c^*, d^* \in C^*$, $c \in C$ we have $\langle c^* * d^*|c\rangle = \langle c^*|d^*\rightharpoonup c\rangle$. Indeed, $\langle c^*|d^*\rightharpoonup c\rangle = \langle c^*|\Sigma_{(c)}\, c_1\langle d^*|c_2\rangle\rangle = \Sigma_{(c)}\, \langle c^*|c_1\rangle\langle d^*|c_2\rangle = \langle c^* * d^*|c\rangle$.

Left C-comodules are analogous to right C-comodules. C is a left C-comodule, and thus a right C^*-module; write the action as $c\leftharpoonup c^* = \Sigma_{(c)}\langle c^*|c_1\rangle c_2$. This gives the "contragradience" formula $\langle c^* d^*|c\rangle = \Sigma_{(c)}\langle d^*|c\leftharpoonup c^*\rangle$. The coassociativity of Δ gives that C is a C^*-bimodule: we have $c^*\rightharpoonup(c\leftharpoonup d^*) = \Sigma_{(c)}\langle d^*|c_1\rangle c_2\langle c^*|c_3\rangle = (c^*\rightharpoonup c)\leftharpoonup d^*$.

Suppose that V is a sub-bimodule of C. Since V is a left C^*-module, we have $\Delta(V) \subseteq V\otimes C$. Similarly $\Delta(V) \subseteq C\otimes V$, so $\Delta(V) \subseteq (V\otimes C) \cap (C\otimes V) = V\otimes V$. Thus, V is a subcoalgebra. Conversely, subcoalgebras are sub-bimodules.

Local finiteness shows that if W is a finite-dimensional subspace of C, then the submodules $C^*\rightharpoonup W$ and $W\leftharpoonup C^*$ are finite-dimensional. For $c \in C$, we have that $c \in C^*\rightharpoonup c \leftharpoonup C^*$ a finite-dimensional sub-bimodule. Thus <u>C is the union of its finite-dimensional subcoalgebras</u>; this is often called the <u>Fundamental Theorem of Coalgebras</u>. $C^*\rightharpoonup c\leftharpoonup C^*$ is the subcoalgebra generated by c; we see that it can be recovered from $\Delta_2(c)$.

If A is an algebra, N a <u>left</u> A-module, then N^* is a <u>right</u> A-module via $\langle n^*a|n\rangle = \langle n^*|an\rangle$ for $n^* \in N^*$, $a \in A$, $n \in N$. This is called the contragradient action of A on N^*. In particular, if M is a <u>right</u> C-comodule, then M^* is a right C^*-module. This action can be obtained more directly as follows: given our structure map $\psi: M \to M\otimes C$, form $\psi^*: (M\otimes C)^* \to M$. In general, for any vector spaces V,W we have an injection $\varphi: V^*\otimes W^* \to (V\otimes W)^*$, induced by $\varphi(v^*\otimes w^*)(v\otimes w) = \langle v^*|v\rangle\langle w^*|w\rangle$. ($\varphi$ is an isomorphism if either V or W is finite-dimensional.) Then $\psi^*\circ\varphi: M^*\otimes C^* \to (M\otimes C)^* \to M^*$ is the contragradient action. When $M = C$ (as a right C-comodule) the contragradient action of C^* on $M^* = C^*$ is simply right multiplication.

When M is a right C-comodule so that M^* is a right C^*-module then M^{**} is a left C^*-module. For $M \subset M^{**}$ the left action of C^* is the original $c^*\rightharpoonup m = \Sigma_{(m)}\, m_1\langle c^*|m_2\rangle$.

If $M \to N$ is a right C-comodule map, then $N^* \to M^*$ is a right C^*-module map. In particular, if V is a right subcomodule of C, then $V^{\perp} = \mathrm{Ker}(C^* \to V^*)$ is a right ideal of C^*. Conversely, the $\perp$ of a right ideal of C^* is a right subcomodule of C. The analogous results hold on the left.

The left and right C^*-module structure on C interplay to give umbral theorems on shift-invariant operators. Here is an example.

If M is an A-bimodule, there is a map R from A to the set $\mathrm{End}_A M$ of endmorphisms of M as a left A-module, given by $R(a)(m) = ma$ for $a \in A$, $m \in M$.

Let C be a coalgebra, and consider $\mathrm{End}_{C^*}C$. (Recall that $f:C \to C$ lies in $\mathrm{End}_{C^*}C$ iff f is a map of right C-comodules iff $\Delta(f(c)) = \Sigma_{(c)}\, f(c_1)\otimes c_2$ for all $c \in C$.) We assert: $R: C^* \to \mathrm{End}_{C^*}C$ is an isomorphism. The inverse map $E: \mathrm{End}_{C^*}C \to C^*$ is given by $E(f) = \epsilon\circ f$.

If $c^* \in C^*$, then $\langle ER(c^*)|c\rangle = \langle \epsilon|R(c^*)c\rangle = \langle\epsilon|c\leftharpoonup c^*\rangle = \langle c^*|c\rangle$. Thus $ER(c^*) = c^*$.

If $f \in \mathrm{End}_{C^*}C$, $c \in C$, then $RE(f)(c) = c\leftharpoonup E(f) = \Sigma_{(c)}\, \epsilon(f(c_1))c_2 = (\epsilon\otimes id)\Delta(f(c)) = f(c)$. Thus $RE(f) = f$

and $R: C^* \xrightarrow{\cong} \mathrm{End}_{C^*}C$. We can describe R by $R(c^*) = \leftharpoonup c^*$ for $c^* \in C^*$.

Apply the above to the specific case $C = \mathcal{D}^\infty$, $C^* = k[[t]]$. Since C is cocommutative, the left and right C^*-module structures on C coincide; i.e. $t\rightharpoonup\ = \ \leftharpoonup t$.

We have $t\rightharpoonup d_n = \Sigma_{i=0}^{n} d_i\langle t|d_{n-i}\rangle = d_{n-1}$. $(t\rightharpoonup d_0 = 0)$. If we formally write $d_n = \frac{x^n}{n!}$ then $\mathcal{D}^\infty = k\{x\}$ and $t\rightharpoonup$ is the operator $\frac{d}{dx}$. An operator $T: k[x] \to k[x]$ is called _shift-invariant_ if it commutes with $\frac{d}{dx}$.

If an operator $f : \mathcal{D}^\infty \to \mathcal{D}^\infty$ commutes with $t\rightharpoonup$, then it commutes with $t^n\rightharpoonup$ for all n. Since $t^n\rightharpoonup d_m = 0$ if $n > m$, it follows that f commutes with the action of every element of $k[[t]]$. Thus $\mathrm{End}_{k[[t]]}\mathcal{D}^\infty$ consists precisely of the shift-invariant operators. Our results thus tell that every shift-invariant operator $f: \mathcal{D}^\infty \to \mathcal{D}^\infty$ can be realized as $f = (\epsilon f)\rightharpoonup\ = \ \leftharpoonup(\epsilon f)$.

Let $f: \mathcal{D}^\infty \to \mathcal{D}^\infty$ be a comodule map i.e. shift-invariant operator. Since $f = \epsilon f\rightharpoonup$ we have that f must satisfy $f(d_n) = \Sigma_{i=0}^{n} d_i\epsilon(f(d_{n-i}))$. Conversely, this relation characterizes the comodule maps.

Restating our results, every $L \in k[[t]] = \mathcal{D}^{\infty *}$ defines a shift-invariant operator $L\rightharpoonup$ on $\mathcal{D}^\infty$. Each shift-invariant operator f arises in this manner from a unique L, given by

$$L = \epsilon\circ f = \Sigma_{i=0}^{\infty} \epsilon(f(d_i))t^i .$$

Recall that Sheffer sequences correspond to comodule maps. Thus, each sequence $\{e_n\}$ which is Sheffer relative to $\{d_n\}$ is defined by a unique $L \in k[[t]]$, given by the condition $L\rightharpoonup d_n = e_n$.

Let us now look at the factor sequence example where Q^∞ is a right comodule for $\mathcal{B}^\infty$, with $\{q_{-n}\}_{n=0}^{\infty}$, $\{b_i\}_0^\infty$ the respective bases and $\psi(q_{-n}) = \Sigma_{i=0}^{n} \binom{-i}{n-i}q_{-i} \otimes b_{n-i}$.

Dualize. Let $\{A_{-n}\}$ be the (topological) basis dual to $\{q_{-n}\}$, and $\{\frac{t^i}{i!}\}$ dual to $\{b_i\}$. Q^∞ is a right $\mathcal{B}^{\infty *}$-module, with the action given by

$$\langle A_{-m}\cdot\frac{t^\ell}{\ell!} \mid q_{-n}\rangle = (A_{-m}\otimes\frac{t^\ell}{\ell!})\psi q_{-n} = \langle A_{-m}|\frac{t^\ell}{\ell!}\rightharpoonup q_{-n}\rangle$$

$$= \binom{\ell-n}{\ell}\langle A_{-m}|q_{\ell-n}\rangle = \binom{\ell-n}{\ell}\delta_{-m,\ell-n} = \binom{-m}{\ell}\delta_{-m-\ell,-n}$$

$$= \binom{-m}{\ell}\langle A_{-m-\ell}|q_{-n}\rangle .$$

Thus

$$A_{-m}\cdot\frac{t^\ell}{\ell!} = \binom{-m}{\ell}A_{-m-\ell} .$$

A typical element of $Q^{\infty *}$ can be written $\Sigma_{i=0}^{\infty}\lambda_i A_{-i}$. If we think of A_{-n} as U^{-n} then t acts as $\frac{d}{dU}$. Note that $Q^{\infty *}$ is _not_ a locally finite $\mathcal{B}^{\infty *}$-module. In fact kA_{-0} is the only non-zero finite dimensional submodule. This shows that when M is a C-comodule, M is a locally finite C^*-module but M^* may not be.

IX. Coalgebra maps to $\mathcal{D}^\infty$ and the coradical filtration

Let C be a coalgebra, $\alpha: C \to \mathcal{D}^\infty$ a coalgebra map. Write $\alpha(c) = \Sigma_{i=0}^{\infty}\alpha_i(c)d_i$, where $\alpha_i \in C^*$. Since $\epsilon(\alpha(c)) = \epsilon(c)$, we have $\alpha_0 = \epsilon$. Since α is a coalgebra map, we must have $(\alpha\otimes\alpha)\Delta(c) = \Delta(\alpha(c))$ for all $c \in C$. Thus,
$\Sigma_{i,j=0}^{\infty}\Sigma_{(c)}\alpha_i(c_1)\alpha_j(c_2)d_i\otimes d_j = \Sigma_{n=0}^{\infty}\Sigma_{i+j=n}\alpha_n(c)d_i\otimes d_j$.
Comparing coefficients of $d_i\otimes d_j$, shows that $\alpha_{i+j} = \alpha_i * \alpha_j$ for all i, j. Thus $\alpha_n = \alpha_1^n$.

Note that since for each $c \in C$, $\alpha(c)$ is an actual element of $\mathcal{D}^\infty$ -- not an infinite sum --we must have $\alpha_n(c) = 0$ for large n. Thus $(\alpha_1)^n(c) = 0$ for some n -- n depending on c. Call such α_1 _locally nilpotent_. The above calculation shows that any locally nilpotent $\alpha_1 \in C^*$ determines a coalgebra map $\alpha: C \to \mathcal{D}^\infty$ via the formula $\alpha(c) = \Sigma_{i=0}^{\infty}(\alpha_1)^i(c)d_i$, and conversely.

Here is another interpretation. We shall characterize the coalgebra maps $C \to \mathcal{D}^\infty$ as certain exponentials. For an algebra A let $A\{x\}$ be the divided polynomial ring over A which has free A-module basis consisting of the formal symbols $\{\frac{x^i}{i!}\}_{i=0}^{\infty}$ which commute with elements of A and multiply $(\frac{x^i}{i!})(\frac{x^j}{j!}) = \binom{i+j}{j}\frac{x^{i+j}}{(i+j)!}$. $A\{\{x\}\}$ denotes the divided power series ring over A which is similar to $A\{x\}$ except $\{\frac{x^i}{i!}\}$ is a topological basis, i.e. infinite sums $\Sigma_{i=0}^{\infty} a_i \frac{x^i}{i!} \in A\{\{x\}\}$ are permitted. $A\{x\} \subset A\{\{x\}\}$ as a subalgebra. For $a \in A$ define $\exp ax \equiv \Sigma a^i \frac{x^i}{i!} \in A\{\{x\}\}$. If $a, \alpha \in A$ commute then $(\exp ax)(\exp \alpha x) = \exp(a+\alpha)x$. If $a \in A$ is nilpotent then $\exp ax \in A\{x\}$.

Let us identify $\mathcal{D}^\infty$ with $k\{x\}$ by $d_i \longleftrightarrow \frac{x^i}{i!}$. There are "bialgebra" reasons for this which will appear in (XI.1). For a coalgebra C there is a natural algebra map

$$E: \mathrm{Hom}(C, k\{x\}) \longleftrightarrow C^*\{\{x\}\}$$
$$f \longrightarrow \Sigma(f_i)\frac{x^i}{i!}$$

where for $f \in \mathrm{Hom}(C, k\{x\})$, $\{f_i\} \subset C^*$ is defined by

$$f(c) = \Sigma f_i(c)\frac{x^i}{i!} \qquad c \in C .$$

$f \in \mathrm{Hom}(C, k\{x\})$ is a coalgebra map if and only if $f_i = f_1^i$ and f_1 is locally nilpotent. In this case $E(f) = \exp f_1 x$ and $\exp f_1 x$ "almost" lies in $C^*\{x\}$ in that the local nilpotence of f_1 implies that $\Sigma f_1^i(c)\frac{x^i}{i!}$ is only a finite sum for each $c \in C$. In particular if C is finite dimensional then $\exp f_1 x$ does lie in $C^*\{x\}$. So the coalgebra maps $C \to k\{x\} = \mathcal{D}^\infty$ are precisely the exponentials $\exp f_1 x$ with f_1 locally nilpotent.

When we write a coalgebra map $\alpha: C \to \mathcal{D}^\infty = k\{x\}$ as $\alpha = \exp hx = \Sigma_{i=0}^{\infty} h^i \frac{x^i}{i!}$ for h locally nilpotent then h can be recovered from α by $h(c) = \frac{d}{dx}\alpha(c)|_0$. Also the dual algebra map $\alpha^*: \mathcal{D}^{\infty *} = k[[t]] \to C^*$ sends t to

$h \in C^*$ and this determines α^* or recovers h from α^*.

Still another view of coalgebra maps $C \to \mathcal{D}^\infty$ is given in section XIII.

What are the locally nilpotent elements of C^*? The answer depends (in large part) upon the coradical of C.

A coalgebra is called "simple" if it has no proper subalgebras. By the fundamental theorem of coalgebras, every simple coalgebra is finite-dimensional. The coradical C_0 of a coalgebra C is the sum of the simple subcoalgebras of C. C_0 is the sum of the simple left subcomodules of C, and also the sum of the simple right subcomodules. $C_0^{\perp}$ is the Jacobson radical of C^*. The coradical of $\mathcal{D}^\infty$ is kd_0. The coradical of $\mathcal{B}^\infty$ is kb_0.

Define inductively $C_n = C_{n-1} \wedge C_0 = \Delta^{-1}(C_{n-1} \otimes C + C \otimes C_0)$. Then each C_n is a subcoalgebra. $C_{n+1} \supset C_n$, and $C = \bigcup_n C_n$. $\Delta(C_n) \subset \Sigma_{i=0}^{n} C_i \otimes C_{n-i}$. $\{C_n\}$ is called the coradical filtration of C.

If $g \in C$ is grouplike then kg is a simple subcoalgebra--being one dimensional--so that $g \in kg \subset C_0$. If $d \in C$ is g-primitive then $d \in C_1$, in fact $d \in kg \wedge kg$.

If M is a right C-module the coradical filtration on M is given by

$$M_0 = \{0\} \wedge C_0$$

$$M_i = M_{i-1} \wedge C_0 .$$

The filtration satisfies $M_0 \subset M_1 \subset M_2 \subset \ldots$ each M_i is a subcomodule, $M = \bigcup_i M_i$ and $\psi(M_n) \subset \Sigma_{i=0}^{n} M_i \otimes C_{n-i}$. Incidentally the coNakayama lemma for comodules goes: if M is a right C-comodule with subcomodule N and $N \wedge C_0 = N$ then $N = M$.

For the coalgebra $\mathcal{D}^\infty$ the coradical filtration is given by $\mathcal{D}^\infty_n = \mathcal{D}^n$, the subspace of $\mathcal{D}^\infty$ with basis $\{d_0, \ldots d_n\}$. In characteristic zero the picture is the same for $\mathcal{B}^\infty$; i.e. $\mathcal{B}^\infty_n = \mathcal{B}^n$. This follows from the characteristic zero coalgebra isomorphism of $\mathcal{D}^\infty$ and $\mathcal{B}^\infty$. In positive characteristic p the element b_{p^n} has diagonalization

$\Delta b_{p^n} = \Sigma_{i=0}^{p^n} \binom{p^n}{i} b_i \otimes b_{p^n - i} = b_0 \otimes b_{p^n} \otimes b_0$ since the "intermediate" binomial coefficients are zero mod p. Hence b_{p^n} is b_0-primitive and lies in $\mathcal{B}_1^\infty$. We go no further with the coradical filtration of $\mathcal{B}^\infty$ in positive characteristic.

Back to a general coalgebra C and locally nilpotent elements. The coradical filtration satisfies

$$\Delta_{n-1}(C_n) \subset \sum_{\substack{e_1+\dots+e_m=n \\ e_i \geq 0}} C_{e_1} \otimes \dots \otimes C_{e_m}$$

If $m > n$ then C_0 must appear in each term of the sum. Thus if $h \in \mathrm{Hom}(C,A)$ for some algebra A and $h(C_0) = \{0\}$ it follows $h^m(C_n) = \{0\}$ for $m > n$. Thus the condition $C_0 \subset \mathrm{Ker}\ h$ is sufficient to insure that h is locally nilpotent. However, the condition is not necessary: $M^c(2,k)$ is a simple coalgebra so $M^c(2,k) = M^c(2,k)_0$ gives the coradical filtration. The matrix element $e_{1,2}$ (dual to $x_{1,2}$) does not vanish on $M^c(2,k)_0$ yet $e_{1,2}{}^2 = 0$. When C is cocommutative an element of C^* is locally nilpotent if and only if it vanishes on C_0. The reason is that if $h \in C^*$ "lives" on C_0 it must "live" on some simple subcoalgebra $D \subset C_0$. Thus the inclusion $D \to C$ inducing an algebra map $C^* \to D^*$ maps h to a non-zero element of D^*. Since D^* is a finite dimensional simple commutative algebra it is a field and contains no non-zero nilpotents. If $h \in C^*$ were locally nilpotent it would map to a nilpotent in D^* since D is finite dimensional.

In general an element $h \in \mathrm{Hom}(C,A)$ is locally nilpotent if and only if for each finite dimensional coalgebra $D \subset C$ there is n -- depending on D -- with $D \subset \mathrm{Ker}\ f^n$.

When g is a grouplike in C we have $h^n(g) = h(g)^n$.

Thus the condition $h(g) = 0$ for g grouplike in C is clearly needed for h to be locally nilpotent. Remember grouplikes in C lie in C_0.

X. Conjugate sequences and associated sequences

Let us now consider the special case $C = \mathcal{D}^\infty$. We have a coalgebra map $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$, given by $\alpha(d_n) = e_n$. Then $\Delta(e_n) = \Delta(\alpha(d_n)) = (\alpha \otimes \alpha)\Delta(d_n) = \Sigma_{i=0}^n e_i \otimes e_{n-i}$. α may not be injective. In general if C is a coalgebra with $C_0 = kg$ where g is grouplike then a coalgebra map $\alpha: C \to D$ is injective iff it is injective on the g-primitives. Thus our $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$ is injective iff $e_1 \neq 0$.

By previous considerations α is given by $d \to \Sigma_{i=0}^\infty L^i(d) d_i$ where $L \in \mathcal{D}^{\infty *} = k[[t]]$ and $L(d_0) = 0$ i.e. $L \in \langle t \rangle$ the maximal ideal. We have $e_n = \alpha(d_n) = \Sigma_{i=0}^\infty L^i(d_n) d_i$. This is the formula for a conjugate sequence although it is usually assumed that L is a <u>delta</u>, which means that $L \in \langle t \rangle$, $L \notin \langle t^2 \rangle$. Thus a conjugate sequence of $\{d_i\}$ is the image of $\{d_i\}$ under a coalgebra map. I.e. $\{\alpha(d_i)\}$ is the conjugate sequence to $\{d_i\}$ arising from α.

Conversely if $\{c_i\} \subset \mathcal{D}^\infty$ is a not necessarily linearly independent sequence of divided powers, i.e. $\Delta c_n = \Sigma_{i=0}^n c_i \otimes c_{n-i}$, $\varepsilon(c_i) = \delta_{1,0}$, then $\beta: \mathcal{D}^\infty \to \mathcal{D}^\infty$, $d_i \to c_i$ is a coalgebra map and $\{c_i\}$ is the conjugate sequence to $\{d_i\}$ arising from β. We have established the bijective correspondences

Conjugate sequences. Sequences of divided powers in $\mathcal{D}^\infty$.	$\longleftrightarrow$	Coalgebra maps from $\mathcal{D}^\infty$ to itself.	$\longleftrightarrow$	The set $\langle t \rangle$.
$\{\alpha(d_i)\}$	$\longleftrightarrow$	α	$\longleftrightarrow$	$\alpha^*(t)$

Thus each $L \in \langle t \rangle$ defines a sequence of divided powers. This sequence of divided powers is linearly independent iff $e_1 \neq 0$ iff $L \notin \langle t^2 \rangle$.

For a coalgebra map $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$ set $L_\alpha = \alpha^*(t) \in \langle t \rangle$. Here L_α may not be a delta; i.e. perhaps $L_\alpha \in \langle t^2 \rangle$. For $F \in k[[t]]$ and $L \in \langle t \rangle$ the functional evaluation $F(L)$ converges and gives a well defined element of $k[[t]]$. So it makes sense to describe $\alpha^*: k[[t]] \to k[[t]]$ by $\alpha^*(F) = F(L_\alpha)$. Let $\beta: \mathcal{D}^\infty \to \mathcal{D}^\infty$ be another coalgebra map. Then $(\beta\circ\alpha)^*(t) = \alpha^*(\beta^*(t)) = \alpha^*(L_\beta) = L_\beta(L_\alpha)$. Thus $L_{\beta\circ\alpha} = L_\beta(L_\alpha)$.

In general if $\gamma: C \to D$ is a coalgebra map it may happen that $\gamma(C_n) \not\subset D_n$. For example let γ be the dual of the inclusion of the 3x3 upper-triangular matrices into $M(3,k)$. $M^c(3,k) = C$ is simple implies $C = C_0$. Being dual to an inclusion implies γ is onto. But $D = D_3 \neq D_0$. However $\gamma(C_n) \subset D_n$ when C is cocommutative. The first step $\gamma(C_0) \subset D_0$ follows from the fact that a subalgebra of a finite-dimensional simple commutative algebra is simple; the other inclusions follow from this. In particular any coalgebra map $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$ will preserve the coradical filtration. Thus if $\{e_n\}$ is a sequence of divided powers we must have degree $e_n \leq n$ from all n.

Let $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$ be a coalgebra map. Then α is of the form $\alpha(d) = \Sigma_{i=0}^\infty L^i(d) d_i$ where $L \in \mathcal{D}^{\infty *} = k[[t]]$ and $L(d_0) = 0$. If $L \neq 0$ we can write $L = a_n t^n + a_{n+1} t^{n+1} + \dots$ where $n \geq 1$ and $a_n \neq 0$. It is easy to see that $L^i(d_m) = 0$ if $ni > m$ and $L^i(d_{ni}) = (a_n)^i$. Thus $\alpha(d_{ni}) = (a_n)^i d_i$ + lower terms and α is surjective unless $L = 0$. In this case $\alpha(d) = \epsilon(d) d_0$ for all d. Recall that $\alpha^*: k[[t]] \to k[[t]]$ sends t to $L \in \langle t \rangle$. The dual fact is that α^* is injective unless $L = 0$.

Now turning to associated sequences.

Let $\alpha: \mathcal{D}^\infty \to \mathcal{D}^\infty$ be a coalgebra map. Write $\alpha(d) = \Sigma_{i=0}^\infty L^i(d) d_i$. Then α is injective iff α is bijective iff L is a delta. In this case we have an inverse coalgebra

map α^{-1}. Define $f_n = \alpha^{-1}(d_n)$. Then $\{f_n\}$ is a linearly independent sequence of divided powers to the delta L. We have $\langle L^i | f_j \rangle = \langle \alpha^*(t^i) | f_j \rangle = \langle t^i | \alpha(f_j) \rangle = \langle t^i | d_j \rangle = \delta_{ij}$. Thus $\{f_j\}$ is the basis of $\mathcal{D}^\infty$ which is dual to $\{L^j\}$.

The picture for associated sequences is given by the bijective correspondences

Associated sequences. Sequences of divided powers in $\mathcal{D}^\infty$ which are linearly independent.	$\longleftrightarrow$	Coalgebra isomorphisms from $\mathcal{D}^\infty$ to itself.	$\longleftrightarrow$	The set $\langle t \rangle - \langle t^2 \rangle$.
$\{\alpha^{-1}(d_i)\}$	$\longleftrightarrow$	α	$\longleftrightarrow$	$\alpha^*(t)$

XI. Algebra-coalgebra interactions

Let A be an algebra. Then $A \otimes A$ is an algebra with unit $1 \otimes 1$ and multiplication determined by $(a \otimes b)(c \otimes d) = ac \otimes bd$. If A is equipped with algebra maps $\Delta: A \to A \otimes A$ and $\epsilon: A \to k$ so that (A, Δ, ϵ) is a coalgebra, then A is called a bialgebra. Dually, a bialgebra is a coalgebra equipped with multiplication and unit maps which are coalgebra maps.

For example $\mathcal{D}^\infty$ is a bialgebra with $d_i d_j = \text{mult}(d_i \otimes d_j) = \binom{i+j}{i} d_{i+j}$. The unit is d_0. The identification $d_i = \frac{x^i}{i!}$ provides an algebra isomorphism $\mathcal{D}^\infty \cong k\{x\}$. The corresponding coalgebra structure on $k\{x\}$ is given by $\epsilon(\frac{x^i}{i!}) = \delta_{i,0}$, $\Delta(\frac{x^n}{n!}) = \Sigma_{i=0}^n \frac{x^i}{i!} \otimes \frac{x^{n-i}}{(n-i)!}$. Thus $k\{x\}$ is a bialgebra with this coalgebra structure. This identification of $\mathcal{D}^\infty$ with $k\{x\}$ appeared in section IX in the exponential interpretation of coalgebra maps $C \to \mathcal{D}^\infty$. $\mathcal{B}^\infty$ is a bialgebra with $b_i b_j = \text{mult}(b_i \otimes b_j) = b_{i+j}$. The unit is b_0. In characteristic zero the coalgebra isomorphism $\mathcal{D}^\infty \cong \mathcal{B}^\infty$, $d_i \longleftrightarrow b_i/i!$ is a bialgebra isomorphism.

The algebra $k[x]$ can be given a bialgebra structure by declaring $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$. Since Δ and ϵ

are to be algebra maps they extend uniquely to all of $k[x]$. In any characteristic $k[x] \cong \mathcal{B}^{\infty}$, $x^n \longleftrightarrow b_n$ and we take this for an identification.

2. Measuring

Let A, B be algebras, C a coalgebra. Let $\rho: C\otimes A \to B$ be a map. There is a vector space isomorphism $\mathrm{Hom}(C\otimes A, B) \cong \mathrm{Hom}(A, \mathrm{Hom}(C,B))$ in which ρ corresponds to the map ρ_2 defined by $\rho_2(a) = \rho(-\otimes a) \in \mathrm{Hom}(C,B)$ where $\rho_2(a): C \to B$, $c \mapsto \rho(c\otimes a)$. Say that (C,ρ) <u>measures</u> <u>A</u> <u>to</u> <u>B</u> if $\rho_2: A \to \mathrm{Hom}(C,A)$ is an algebra map.

We can give this condition in terms of the map ρ itself. Let us write $\rho(c\otimes a) = c(a) \in B$. The unit condition for an algebra map gives us $c(1_A) = \varepsilon(c)1_B$. The multiplicative condition shows that for all $a, \alpha \in A$ the operator formula $c(a\alpha) = \Sigma_{(c)}\, c_1(a)c_2(\alpha)$ holds.

Note that if c is grouplike, then c acts as a homomorphism from A to B. If c is g-primitive (with g grouplike) then c is a derivation from A to B with respect to the homomorphism given by g.

For an example let A be an algebra in characteristic zero and $\partial: A \to A$ a derivation. Define $\rho: \mathcal{D}^{\infty}\otimes A \to A$ by $\rho(d_i\otimes a) = \partial^i(a)/i!$. $\mathcal{D}^{\infty}$ measures A to A.

For another example let H be a bialgebra. H acts on itself by right and left translation. This gives us contragradient actions on H^*: for $g, h \in H$, $f \in H^*$, we have $\langle f^* \leftharpoonup g | h\rangle = \langle f^* | gh\rangle = \langle h \rightharpoonup f^* | g\rangle$.

This action $f^* \leftharpoonup g$ of H on H^* must not be confused with the action $f^* \rightharpoonup g = \Sigma\, g_1\langle f^* | g_2\rangle$ of H^* on H which is the contragradient action of H^* on $H \subset H^{**}$.

Let us verify that H measures H^* to itself. Let $g, h \in H$. Recall that the unit 1_{H^*} of H^* is ε. We have

$$\langle h \rightharpoonup \varepsilon | g\rangle = \langle \varepsilon | gh\rangle \quad \text{by definition of } \rightharpoonup$$
$$= \langle \varepsilon | g\rangle\langle \varepsilon | h\rangle \quad \text{since } \varepsilon \text{ is an algebra map}$$
$$= \langle \varepsilon(h)\,\varepsilon | g\rangle$$

Thus $h \rightharpoonup 1_{H^*} = \varepsilon(h)1_{H^*}$.

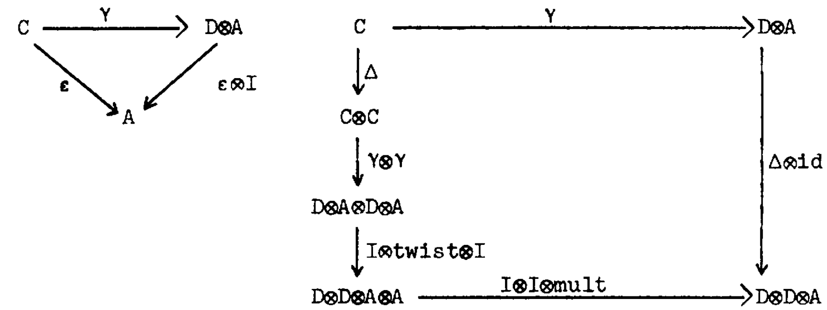
Now let $f^*, \ell^* \in H^*$

$$\langle h \rightharpoonup (f^*\ell^*) | g\rangle = \langle f^*\ell^* | gh\rangle \quad \text{by definition of } \rightharpoonup$$
$$= \langle f^*\otimes\ell^* | \Delta(gh)\rangle \quad \text{by definition of multiplication in } H^*$$
$$= \langle f^*\otimes\ell^* | \Sigma_{(g),(h)}\, g_1h_1\otimes g_2h_2\rangle \quad \text{since } \Delta \text{ is an algebra map}$$
$$= \Sigma_{(g),(h)} \langle f^* | g_1h_1\rangle\langle \ell^* | g_2h_2\rangle$$
$$= \Sigma_{(g),(h)} \langle h_1 \rightharpoonup f^* | g_1\rangle\langle h_2 \rightharpoonup \ell^* | g_2\rangle \quad \text{by definition of } \rightharpoonup$$
$$= \langle \Sigma_{(h)}\,(h_1 \rightharpoonup f^*)(h_2 \rightharpoonup g^*) | g\rangle \quad \text{by definition of multiplication in } H^*$$

Thus $h \rightharpoonup (f^*\ell^*) = \Sigma_{(h)}(h_1 \rightharpoonup f^*)(h_2 \rightharpoonup \ell^*)$ and we are done.

3. Comeasuring

This is useful for factor sequences. Let C, D be coalgebras, A an algebra and $\gamma: C \to D\otimes A$. Define $\gamma_1: D^* \to \mathrm{Hom}(C,A)$ by $\gamma_1(d^*)(c) = (d^*\otimes I)\gamma(c) \in k\otimes A = A$. Say that (A,γ) <u>comeasures</u> C to D if γ_1 is an algebra map. This is equivalent to requiring the following diagrams to commute:



This can be interpreted in the following way. For any vector space M let us consider $M\otimes A$ to be an A-bimodule via

$$a\cdot(m\otimes\alpha) = m\otimes(a\alpha) = (m\otimes a)\cdot\alpha \quad \text{for} \quad m\in M,\ a,\alpha\in A$$

If N is another vector space, identify $(M\otimes A)\otimes_A(N\otimes A)$ with $M\otimes N\otimes A$ via

$$(m\otimes a)\otimes_A(n\otimes\alpha) \longleftrightarrow m\otimes n\otimes a\alpha$$

For any coalgebra E define

$$\Delta_A = \Delta\otimes I : E\otimes A \longrightarrow E\otimes E\otimes A = (E\otimes A)\otimes_A (E\otimes A)$$

$$\epsilon_A = \epsilon\otimes I : E\otimes A \longrightarrow k\otimes A = A$$

When A is commutative this is the <u>base extension</u> of coalgebra structure from k to A.

For the comeasuring $\gamma : C \to D\otimes A$ define

$$\gamma' : C\otimes A \longrightarrow D\otimes A \quad \text{by} \quad \gamma'(c\otimes a) = \gamma(c)a$$

$$\gamma' : C\otimes A \longrightarrow D\otimes A \qquad \gamma''(c\otimes a) = a\gamma(c)$$

γ' is a right A-module map, γ'' is a left A-module map and so $\gamma'\otimes\gamma'' : (C\otimes A)\otimes_A (C\otimes A) \to (D\otimes A)\otimes_A (D\otimes A)$ is well defined. Note $\gamma' = \gamma''$ if A is commutative.

Commutativity of the first comeasuring diagram is equivalent to commutativity of the diagram

$$\begin{array}{ccc} C\otimes A & \xrightarrow{\gamma' \text{ or } \gamma''} & D\otimes A \\ & \searrow^{\epsilon_A} \quad \swarrow_{\epsilon_A} & \\ & A & \end{array}$$

By the "or" in the diagram we mean that commutativity of the above diagram with the top arrow γ' is equivalent to commutativity of the first comeasuring diagram and commutativity of the above diagram with the top arrow γ'' is equivalent to commutativity of the first comeasuring diagram. We shall use "or" in the same fashion in the next diagram. Commutativity of the second comeasuring diagram is equivalent to commutativity of:

$$\begin{array}{ccccc} C & \xrightarrow{(c \to c\otimes 1)} & C\otimes A & \xrightarrow{\gamma' \text{ or } \gamma''} & D\otimes A \\ \left(\begin{smallmatrix} c \\ \downarrow \\ c\otimes 1 \end{smallmatrix}\right)\big\downarrow & & & & \big\downarrow \Delta_A \\ C\otimes A & & & & \\ \big\downarrow & & & & \big\downarrow \\ (C\otimes A)\otimes_A (C\otimes A) & & \xrightarrow{\gamma'\otimes\gamma''} & & (D\otimes A)\otimes_A (D\otimes A) \end{array}$$

In case A is commutative the second comeasuring diagram being commutative is equivalent to commutativity of the diagram:

$$\begin{array}{ccc} C\otimes A & \xrightarrow{\gamma' = \gamma''} & D\otimes A \\ \big\downarrow \Delta_A & & \big\downarrow \Delta_A \\ (C\otimes)\otimes_A (C\otimes A) & \xrightarrow{\gamma'\otimes\gamma'} & (D\otimes A)\otimes_A (D\otimes A) \end{array}$$

Thus when A is commutative $\gamma : C \to D\otimes A$ being a comeasuring is equivalent to $\gamma' : C\otimes A \to D\otimes A$ being a coalgebra map of <u>coalgebras defined over A</u>, i.e. an <u>A-coalgebra</u> map.

In the factor sequence example A is commutative.

Recall Q^∞ has basis $\{q_{-n}\}_{n\geq 0}$ and is a comodule for $\mathcal{B}^\infty = k[x]$ where $\psi(q_{-n}) = \Sigma_{i=0}^{n} \binom{-i}{n-i} q_{-i}\otimes x^{n-i}$. Give Q^∞ a coalgebra structure by declaring $\{q_{-n}\}_{n\geq 0}$ to be a sequence of divided powers, i.e. $\epsilon(q_{-n}) = \delta_{0,n}$ $\quad \Delta(q_{-n}) = \Sigma_{i=0}^{n} q_{-i}\otimes q_{i-n}$. If $\{A_{-n}\}$ is a (topological) dual basis to $\{q_{-n}\}$, then $A_{-n}\cdot A_{-m} = A_{-(n+m)}$. Thus we may identify $Q^{\infty *}$ with $k[[U^{-1}]]$ with $\{U^{-n}\}_{n\geq 0}$ a topological dual basis to $\{q_{-n}\}_{n\geq 0}$. $\mathrm{Hom}(Q^\infty, k[x]) = k[x][[U^{-1}]]$ as an algebra. The map

$$\begin{array}{ccc} \psi_1 : Q^{\infty *} & \longrightarrow & \mathrm{Hom}(Q^\infty, k[x]) \\ \| & & \| \\ k[[U^{-1}]] & & k[x][[U^{-1}]] \end{array}$$

is given by

$$\psi_1(U^{-n}) = \Sigma_{j=0}^{\infty} \binom{-n}{j} U^{-n-j} x^j$$

which is the same as the usual factor sequence definition $(U+x)^{-n} = \Sigma_{j=0}^{\infty} \binom{-n}{j} U^{-n-j} x^j$. As observed earlier -- for $k[x]^* = k\{\{t\}\}$ where $t \in k[x]^*$ satisfies $\langle t | x^n \rangle = \delta_{n,1}$ -- t acts on $k[[U^{-1}]]$ as $\frac{d}{du}$.

It is sometimes useful to think of $k[[U^{-1}]]$ as being a "complete" $k[x]$-comodule, with structure map $\hat{\psi}_1 : k[[U^{-1}]] \to k[[U^{-1}]]\hat{\otimes} k[x] \cong k[x][[U^{-1}]]$ defined by $\hat{\psi}_1(U^{-n}) = \Sigma_{j=0}^{\infty} \binom{-n}{j} U^{-n-j} \hat{\otimes} x^j$.

XII. Hopf algebras and formulas

A Hopf algebra is a bialgebra H which is equipped with a map $S:H \to H$ such that $\Sigma_{(h)} h_1 S(h_2) = \epsilon(h) = \Sigma_{(h)} S(h_1)h_2$ for all $h \in H$. S is an algebra and a coalgebra anti-homomorphism [p. 74]. It plays the role for a Hopf algebra that the inverse map plays for a group; indeed, if H is a Hopf algebra and $g \in H$ is grouplike, then $S(g) = g^{-1}$. S is called the antipode of H.

Let $c \in H$, $g^*, h^* \in H^*$. If c is grouplike then since the contragradient action of H on H^* is a measuring, $c\rightharpoonup$ is an algebra map and

$$* \quad c^{-1}\rightharpoonup((c\rightharpoonup g^*)*h^*) = [c^{-1}\rightharpoonup(c\rightharpoonup g^*)]*[c^{-1}\rightharpoonup h^*] = g^**(c^{-1}\rightharpoonup h^*)$$

Using Σ notation here gives a generalization of this result when c is not necessarily grouplike. The technique more-or-less consists of writing in the appropriate "$\Sigma_{(c)}$" and subscripts and changing the "(c^{-1})" to "$S(c)$" in (*). This is a powerful method of generalizing to Hopf algebras, formulas and constructions for groups. It is one of the useful aspects of Σ notation.

The first and last terms of (*) become

$$\text{XII.I} \qquad \Sigma_{(c)} S(c_1)\rightharpoonup((c_2\rightharpoonup g^*)*h^*) = g^**(S(c)\rightharpoonup h^*)$$

Here is the verification following the line of reasoning at (*). Using that $\rightharpoonup$ is a measuring and that S is a coalgebra antihomorphism, the left side becomes

$$\begin{aligned}
&\Sigma_{(c)}[S(c_2)\rightharpoonup(c_3\rightharpoonup g^*)]*[S(c_1)\rightharpoonup h^*] \\
&= \Sigma_{(c)}[S(c_2)c_3\rightharpoonup g^*]*[S(c_1)\rightharpoonup h^*] \\
&= \Sigma_{(c)}[\epsilon(c_2)1\rightharpoonup g^*]*[S(c_1)\rightharpoonup h^*] \qquad \text{by the antipode condition} \\
&= \Sigma_{(c)}[1\rightharpoonup g^*]*[S(c_1)\epsilon(c_2)\rightharpoonup h^*] = g^**(S(c)\rightharpoonup h^*).
\end{aligned}$$

A second genreal formula is

$$\text{XII.II} \qquad (h^*\rightharpoonup a)b = \Sigma_{(b)} (S(b_2)\rightharpoonup h^*)\rightharpoonup(ab_1) \quad \text{for } a,b\in H,\ h^*\in H^*$$

It has been used to develop a theory of the integral for Hopf algebras. The proof is similar to that of (XII.I).

Let us now consider $\mathcal{B}^\infty = k[x]$ where k is a field of characteristic zero so $\mathcal{B}^\infty \cong \mathcal{D}^\infty$. The antipode is given by $S(x) = -x$ thus $S(x^n) = (-1)^n x^n$. Note that $\langle x\rightharpoonup t^n | x^m/m!\rangle = \langle t^n | (m+1)x^{m+1}/(m+1)!\rangle = (m+1)\delta_{n,m+1}$. Thus $x\rightharpoonup t^n = nt^{n-1}$ and for any $H \in k[[t]] = k[x]^*$ $x\rightharpoonup H = \frac{d}{dt} H$. Write $H^{(n)}$ for $x^n\rightharpoonup H$ the n^{th} derivative of H with respect to t.

With $x^n = c$, $L = g^*$, $M = h^*$, (XII.I) becomes

$$\Sigma_{i=0}^n (-1)^i\binom{n}{i} x^i\rightharpoonup((x^{n-i}\rightharpoonup L)M) = L((-1)^n x^n\rightharpoonup M) = (-1)^n(x^n\rightharpoonup M)L .$$

Thus $\Sigma_{i=0}^n (-1)^i\binom{n}{i}(L^{(n-i)}M)^{(i)} = (-1)^n M^{(n)}L$. This gives

$M^{(n)}L = \Sigma_{i=0}^n (-1)^{n-i}\binom{n}{i}(L^{(n-i)}M)^{(i)}$ and hence by switching index

$$M^{(n)}L = \Sigma_{j=0}^n (-1)^j\binom{n}{j}(L^{(j)}M)^{(n-j)} \qquad \text{XII.III}$$

For (XII.II), let $b = x^n$, $h^* = L$. Then

$$(L\rightharpoonup a)x^n = \Sigma_{i=0}^n (-1)^{n-i}\binom{n}{i}(x^{n-i}\rightharpoonup L)\rightharpoonup(ax^i)$$

Switching index and using $x\rightharpoonup = \frac{d}{dt}$

$$(L\rightharpoonup a)x^n = \Sigma_{j=0}^n(-1)^j\binom{n}{j}L^{(j)}\rightharpoonup(ax^{n-j}) \qquad \text{XII.IV}$$

Recall that $L\rightharpoonup$ is a shift-invariant operator -- it commutes with $D = \frac{d}{dx}$. Recall that every shift invariant operator U is uniquely of the form $L\rightharpoonup$ where $L = \epsilon\circ U$. Write x^ℓ for the operator on $k[x]$ which multiplies each element by x. It is easily checked that $Dx^\ell - x^\ell D = I$ and using this it is easily checked that $Ux^\ell - x^\ell U$ is shift-invariant for any shift invariant operator U. Define $\Box U$ as $Ux^\ell - x^\ell U$. This is the Pincherle Derivative of U. By what we have already observed $\Box U$ is shift-invariant if U is. For $U = L\rightharpoonup$

$$\epsilon(\Box U(x^n)) = \epsilon(U(x^{n+1})) = \langle L|x^{n+1}\rangle = \langle x\rightharpoonup L|x^n\rangle = \langle \tfrac{d}{dt} L|x^n\rangle$$

This shows that if $U = L\rightharpoonup$ then $\Box U = (\frac{d}{dt} L)\rightharpoonup$. Let us write $\boxed{n}U$ for the n^{th} Pincherle derivative of U. Thus if $U = L\rightharpoonup$

then $\boxed{n}U = L^{\boxed{n}}\rightharpoonup$. Formula XII.IV now becomes

$$\text{XII.V} \qquad U(a)x^n = \Sigma_{j=0}^{n} (-1)^j \binom{n}{j} \boxed{j}U(ax^{n-j})$$

for any $a \in k[x]$ and any shift invariant operator U.

Now some standard combinatorial proofs in the new language.

Let $\{e_i\}$ be the associated sequence of divided powers to a delta L. Let α be the coalgebra map corresponding to L; thus, $\alpha(d) = \Sigma_{i=0}^{\infty} L^i(d)x^i/i!$ and $\{e_i\}$ is the unique sequence of divided powers with $\alpha(e_i) = x^i/i!$. Recall that $\langle L^i | e_j\rangle = \delta_{i,j}$ i.e. $\{L^i\}$ is the topological dual basis to $\{e_i\}$. Since $\langle L^i | e_n\rangle = \langle L^{i-1} | L \rightharpoonup e_n\rangle$, this characterization of $\{e_n\}$ is equivalent to:

$$\epsilon(e_n) = \delta_{0,n} \quad \text{and} \quad L\rightharpoonup e_n = e_{n-1}$$

XII.VI Recurrence Formula: $e_n = \frac{1}{n} x(L'^{-1}\rightharpoonup e_{n-1})$

where $L' = \frac{d}{dt}L$. To verify this we apply L^i to $\frac{1}{n}x(L'^{-1}\rightharpoonup e_{n-1})$

$$\langle L^i | x(L'^{-1}\rightharpoonup e_{n-1}) \tfrac{1}{n}\rangle = \langle x\rightharpoonup L^i | L'^{-1}\rightharpoonup e_{n-1}\rangle \tfrac{1}{n}$$

$$= \langle \tfrac{i}{n} L^{i-1}L' | L'^{-1}\rightharpoonup e_{n-1}\rangle = \langle \tfrac{i}{n} L^{i-1}L' L'^{-1} | e_{n-1}\rangle$$

$$= \langle \tfrac{i}{n} L^{i-1} | e_{n-1}\rangle = \delta_{i,n} .$$

This completes the verification that $\{L^i\}$ is a topological dual basis to the sequence defined inductively by the recurrence formula XII.VI with initial term $e_0 = 1$. Hence the sequence defined by XII.VI is the original $\{e_i\}$ sequence.

Since L is a delta it lies in $\langle t\rangle - \langle t^2\rangle$ and $L = tP$ for <u>invertible</u> $P \in k[[t]]$.

XII.VII Transfer Formula: $e_n = \frac{1}{n}x(P^{-n}\rightharpoonup \frac{x^{n-1}}{(n-1)!})$

For verification using (XII.IV) with P^{-n} for L, x^{n-1} for a and x for x^n gives

$$x(P^{-n}\rightharpoonup x^{n-1}) = \Sigma_{j=0}^{1} (-1)^j \binom{1}{j} P^{-n\textcircled{j}}\rightharpoonup x^{n-j}$$

$$= P^{-n}\rightharpoonup x^n - P^{-n}{}'\rightharpoonup x^{n-1}$$

Divide by $n!$ and apply L^i which equals t^iP^i:

$$\langle t^iP^i | P^{-n}\rightharpoonup \tfrac{x^n}{n!}\rangle - \tfrac{1}{n}\langle t^iP^i | P^{-n}{}'\rightharpoonup \tfrac{x^{n-1}}{(n-1)!}\rangle$$

$$= \langle t^iP^iP^{-n} | \tfrac{x^n}{n!}\rangle - \langle t^iP^i \tfrac{P^{-n}{}'}{n} | \tfrac{x^{n-1}}{(n-1)!}\rangle$$

$$= \langle P^{i-n} | t^i\rightharpoonup \tfrac{x^n}{n!}\rangle + \langle P^{i-n-1}P' | t^i\rightharpoonup \tfrac{x^{n-1}}{(n-1)!}\rangle$$

When $i > n$ this yields zero and when $i=n$ this yields $\langle \epsilon | 1\rangle + 0$ which equals 1. Assume $i < n$ and use $t^i\rightharpoonup x^\ell/\ell! = x^{\ell-i}/(\ell-i)!$ for $i \leq \ell$. The last term set off above becomes

$$\langle P^{i-n} | \tfrac{x^{n-i}}{(n-i)!}\rangle + \langle P^{i-n-1}P' | \tfrac{x^{n-i-1}}{(n-i-1)!}\rangle$$

$$= \langle \tfrac{x}{n-i} \rightharpoonup P^{i-n} | \tfrac{x^{n-i-1}}{(n-i-1)!}\rangle + \langle P^{i-n-1}P' | \tfrac{x^{n-i-1}}{(n-i-1)!}\rangle$$

$$= \tfrac{i-n}{n-i} \langle P^{i-n-1}P' | \tfrac{x^{n-i-1}}{(n-i-1)!}\rangle + \langle P^{i-n-1}P' | \tfrac{x^{n-i-1}}{(n-i-1)!}\rangle$$

This is zero showing that the transfer formula XII.VII gives a dual basis to $\{L^i\}$.

XIII. The dual coalgebra

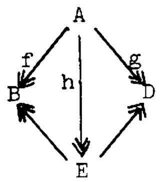
If C is a coalgebra then C^* is an algebra. Beginning with an algebra A to what extent is the dual a coalgebra? If A is finite dimensional then A^* is a coalgebra. If A is infinite dimensional there is a canonical largest subcoalgebra of A^* which is denoted A° and is called the coalgebra dual of A.

The concept of the coalgebra dual is necessary for an understanding of why $t\rightharpoonup$ acts as a derivation on $\mathfrak{D}^\infty = k\{x\}$ or $\mathfrak{B}^\infty = k[x]$, where $\mathfrak{D}^{\infty *} = k[[t]]$ and $\mathfrak{B}^{\infty *} = k\{\{t\}\}$. In the factor sequence example at the end of (XI.3) t acts on $k[[U^{-1}]]$ as the derivation $\frac{d}{dU}$. The coalgebra dual provides the reason.

If B is a finite dimensional algebra the full dual B^* has a natural coalgebra structure in this manner: dualize the multiplication map $B\otimes B \to B$ to get a map $B^* \to (B\otimes B)^*$. Since the natural map $B^*\otimes B^* \to (B\otimes B)^*$ is an isomorphism when B is finite dimensional we have the composite $B^* \to (B\otimes B)^* \xrightarrow{\cong} B^*\otimes B^*$ which is our coproduct Δ. The counit is the dual of the algebra unit map $k \to B$.

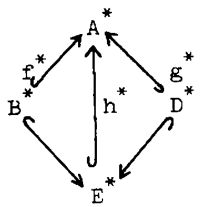
Now assume that A is possibly infinite dimensional. If B is a finite dimensional algebra and $f: A \twoheadrightarrow B$ is an algebra surjection then the dual map $B^* \hookrightarrow A^*$ is injective and allows us to define a coalgebra structure on the subspace $\operatorname{Im}(B^*) = (\operatorname{Ker} f)^{\perp}$ of A^*. By taking other choices for B and f we obtain other coalgebra structures on subspaces of A^*. We would like to show that this process is consistent i.e. that any two such coalgebras agree on their intersection.

Let $f: A \twoheadrightarrow B$ and $g: A \twoheadrightarrow D$ be surjections to finite dimensional algebras. $B \times D$ is a finite dimensional algebra and $a \mapsto (f(a), g(a))$ defines an algebra map from A to $B \times D$. Let E be the image of this map and $h: A \twoheadrightarrow E$ the algebra surjection defined by the above rule. We now have a commutative diagram



of surjective algebra maps where the maps out of $E \subset B \times D$ are induced by the projections: $B \longleftarrow B \times D \longrightarrow D$.

Dualizing we obtain the compatibility diagram



Both $B^* \hookrightarrow E^*$ and $D^* \hookrightarrow E^*$ are coalgebra maps; hence the coalgebra structures on subspaces of A^* are consistent.

We can define a coalgebra structure on $A^{\circ} = \cup_{\mathcal{I}} I^{\perp} \subset A^*$ where $\mathcal{I} = \{I \subset A : I$ is a two sided ideal and A/I is finite dimensional$\}$ by giving each $I^{\perp}$ the coalgebra structure dual to A/I. A° is called the coalgebra dual to A. The duality is not perfect. For example if A is a field which is infinite dimensional as a vector space over k then $A^{\circ} = \{0\}$ since

$\{0\}$ is the only finite dimensional quotient algebra of A.

If A and B are algebras and $f: A \to B$ is an algebra map then $f^*: B^* \to A^*$ satisfies $f^*(B^{\circ}) \subset A^{\circ}$. Let f° denote f^* restricted from B° to A°. The map f° is a coalgebra map.

Begin with a coalgebra C to obtain an algebra C^* and form $C^{*\circ}$. The natural map $C \to C^{**}$ has image in $C^{*\circ}$. The map $C \to C^{*\circ}$ is a coalgebra map.

The natural pairing $A^* \otimes A \xrightarrow{\langle | \rangle} k$ induces a map $A^{\circ} \otimes A \xrightarrow{\langle | \rangle} k$ and A° measures A to k. If A is a bialgebra then A^* is an algebra from the coalgebra structure on A and A° is not only a coalgebra but a subalgebra of A^*. With its algebra and coalgebra structures A° is a bialgebra. If A is a Hopf algebra with antipode S then the transpose $S^*: A^* \to A^*$ carries A° to itself. $S^*|A^{\circ}$ makes A° into a Hopf algebra.

When A is a bialgebra A is a left A^*-module via $\rightharpoonup$ from the coalgebra structure on A. Since A° is a subalgebra of A^* $\rightharpoonup$ gives A a left A°-module structure. A° measures A to A.

Suppose $C \xrightarrow{\gamma} D \otimes A$, where (A, γ) comeasures C to D, as in (XI. 3). Dualizing gives $(D \otimes A)^* \xrightarrow{\gamma^*} C^*$. Form the composite $\varkappa$:

$$\varkappa: D^* \otimes A^{\circ} \to D^* \otimes A^* \to (D \otimes A)^* \xrightarrow{\gamma^*} C^*$$

C^* and D^* are algebras since C and D are coalgebras. By $\varkappa: D^* \otimes A^{\circ} \to C^*$, A° measures D^* to C^*.

First example take $A = \mathfrak{D}^{\infty *} = k[[t]]$. Every non-zero ideal I of A is of the form $\langle t^n \rangle$ for some n. There is the natural vector space embedding $\mathfrak{D}^{\infty} \hookrightarrow \mathfrak{D}^{\infty **} = A^*$. Within A^* $\langle t^n \rangle^{\perp} = \operatorname{span}\{d_0, \ldots, d_{n-1}\}$. The coalgebra structure on $\langle t^n \rangle^{\perp}$ induced by the map $(A/I)^* \hookrightarrow A^*$ is the usual divided power structure. Thus $\mathfrak{D}^{\infty * \circ} = A^{\circ} = \mathfrak{D}^{\infty}$.

The fact that $\mathfrak{D}^{\infty * \circ} = \mathfrak{D}^{\infty}$ enables us to say still more about coalgebra maps $C \to \mathfrak{D}^{\infty}$. If α is such a coalgebra map then $\alpha^*: \mathfrak{D}^{\infty *} \to C^*$ is an algebra map. The composite

$$C \to C^{*\circ} \xrightarrow{\alpha^{*\circ}} \mathfrak{D}^{\infty * \circ} = \mathfrak{D}^{\infty}$$

is α. Beginning with an algebra map $f:\mathfrak{D}^{\infty *} = k[[t]] \to C^*$ yields a coalgebra map $\beta: C \to \mathfrak{D}^\infty$ as the composite

$$C \to C^{*o} \xrightarrow{f^o} \mathfrak{D}^{\infty * o} = \mathfrak{D}^\infty$$

and $\beta^* = f$. Hence there is a bijective correspondence

$$\left\{\begin{matrix}\text{coalgebra}\\ \text{maps from } C\\ \text{to } \mathfrak{D}^\infty\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{algebra maps}\\ \text{from } \mathfrak{D}^{\infty *}=k[[t]]\\ \text{to } C^*\end{matrix}\right\}$$

$$\alpha \longrightarrow \alpha^*$$

$$(C \to C^{*o} \xrightarrow{f^o} \mathfrak{D}^\infty) \longleftarrow f$$

This ties in with the "exponential" characterization of coalgebra maps C to $\mathfrak{D}^\infty$ in section IX. A coalgebra map $\alpha: C \to \mathfrak{D}^\infty$ was shown to be of the form $\exp hx$ with $h \in C^*$, h locally nilpotent. The corresponding algebra map $\mathfrak{D}^{\infty *}=k[[t]] \to C^*$ is determined by $t \to h$.

The second example of the dual coalgebra is the case $A = k[x] = \mathcal{B}^\infty$. Identify A^* with $k\{\{t\}\}$ as before. $k\varepsilon = k1 = \langle x\rangle^{\perp} \subset A^o$ and $\Delta(1) = 1\otimes 1$. $A = k[x] \twoheadrightarrow k[x]/\langle x^{n+1}\rangle = B$ dualizes to $B^* \hookrightarrow A^*$ and actually $B^* \hookrightarrow A^o$ since B is a finite dimensional algebra. If B has basis $1, \bar{x}, \ldots \bar{x}^n$ and B^* has dual basis $\varepsilon = T_0, T_1, \ldots, T_n$ then $\Delta T_m = \Sigma_{i=0}^{n} T_i \otimes T_{m-i}$. Under $B^* \to A^*$, T_m maps to $\frac{t^m}{m!}$ which shows $\frac{t^m}{m!} \in A^o$ and $\Delta \frac{t^m}{m!} = \Sigma_{i=0}^{m} \frac{t^i}{i!} \otimes \frac{t^{m-i}}{(m-i)!}$. The span of $\{\frac{t^i}{i!}\}$ in A^o is the divided polynomial algebra $k\{t\}$.

As already mentioned A^o measures A to A under the $\rightharpoonup$ action of A^o on A. Since $t = \frac{t^1}{1!} \in A^o$ and $\Delta t = 1\otimes t + t \otimes 1$ it follows that $t\rightharpoonup$ acts as a derivation. It is easily checked that $t \rightharpoonup x = 1$ from which it follows that $t\rightharpoonup$ must be the derivation $\frac{d}{dx}$.

Turn to the comeasuring example at the end of (XI, 3). $\psi: Q^\infty \to Q^\infty \otimes k[x]$ yields the *measuring* $Q^{\infty *}\otimes k[x]^o \to Q^{\infty *}$ i.e. $k[[U^{-1}]]\otimes k[x]^o \to k[[U^{-1}]]$. Since $t \in k[x]^o$ with $\Delta t = 1\otimes t + t\otimes 1$ it follows that t acts as a derivation. It is easily checked

that t carries U^{-1} to $-U^{-2}$ and t acts continuously from which it follows that t acts as $\frac{d}{dU}$.

The grouplike elements of A^o act as algebra maps from $A = k[x]$ to k. There is one such element for each $\lambda \in k$; it is the unique algebra map which sends x to λ, and is represented by the power series $e^{\lambda t} = \Sigma_{i=0}^{\infty} \lambda^i (\frac{t^i}{i!})$. Grouplike elements are always linearly independent. The k-span Γ of $\{e^{\lambda t}: \lambda \in k\}$ is the *group algebra* of k^+ the underlying additive group of k.

If k is algebraically closed A^o is generated by $k\{t\}$ and Γ; in fact $A^o \cong k\{t\}\otimes\Gamma$ as Hopf algebras. This follows from the Kostant structure theorem for cocommutative pointed Hopf algebras [pp. 176-7]. However A^o is larger in the non-algebraically closed case. For example in characteristic zero if $i = \sqrt{-1} \notin k$ then $\langle x^2+1\rangle^{\perp} = k \cos t + k \sin t$ is not in $k\{t\}\Gamma$. Of course if $i \in k$ we have $\sin t = \frac{i}{2}(e^{-it} - e^{it})$ and $\cos t = \frac{1}{2}(e^{it} + e^{-it}) \in \Gamma$.

In characteristic zero $\mathfrak{D}^\infty \cong \mathcal{B}^\infty = k[x]$ and the previous analysis applies. In *positive characteristic* p each d_i is p-power nilpotent for $i > 0$. Hence $\mathfrak{D}^\infty$ is local with unique maximal ideal spanned by $\{d_i\}_{i=1}^{\infty}$. This implies that the coradical of $\mathfrak{D}^{\infty o}$ is $k\varepsilon = k1 \subset \mathfrak{D}^{\infty *} = k[[t]]$. In a fashion similar to the demonstration for $\mathcal{B}^\infty$ it follows that t lies in $\mathfrak{D}^{\infty o}$ with $\Delta t = 1\otimes t + t\otimes 1$ and t act as $\frac{d}{dx}$ on $\mathfrak{D}^\infty = k\{x\}$. Since $\mathfrak{D}^{\infty o}$ is a subalgebra of $k[[t]] = \mathfrak{D}^{\infty *}$ it follows that $k[t] \subset \mathfrak{D}^{\infty o}$. However $k[t]$ is *not* all of $\mathfrak{D}^{\infty o}$ for example $v = t + t^p + t^{p^2} + t^{p^3} + \ldots$ lies in $\mathfrak{D}^{\infty o}$ and $\Delta v = 1\otimes v + v \otimes 1$.

XIV. Epilogue

In my opinion the umbral calculus Hopf algebra interplay has the -- as yet unrealized -- potential to prove its worth in a major way. The coalgebra point of view helps to explain what is happening in many umbral calculus situations. Yet the verifications of the recurrence formula XII.VI and transfer formula XII.VII were merely the old proofs in the new language. The formulas XII.I, XII.II and resulting XII.III, XII.IV and XII.V do take advantage of the Hopf algebra setting for their proofs but I don't know what they help count.

This is not meant to be discouragement but rather encouragement for work demonstrating the value of the combinatorics coalgebra interplay. An important application of the umbral calculus in which coalgebra or Hopf algebra techniques are essential would do the trick just fine.

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