HANDBOOK OF CONVEX GEOMETRY

Volume B

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Preface

One aim of this Handbook is to survey convex geometry, its many ramifications and its relations with other areas of mathematics. We believe that it will be a useful tool for the expert. A second aim is to give a high level introduction to most branches of convexity and its applications, showing the major ideas, methods and results. We hope that because of this feature the Handbook will act as an appetizer for future researchers in convex geometry. For them the many explicitly or implicitly stated problems should turn out to be a valuable source of inspiration. Third, the Handbook should be useful for mathematicians working in other areas as well as for econometrists, computer scientists, crystallographers, phycisists and engineers who are looking for geometric tools for their own work. In particular, mathematicians specializing in optimization, functional analysis, number theory, probability theory, the calculus of variations and all branches of geometry should profit from the Handbook.

The famous treatise "Theorie der konvexen Körper" by Bonnesen and Fenchel presented in 164 pages an almost complete picture of convexity as it appeared around 1930. While a similarly comprehensive report today seems to be out of reach, the Handbook deals with most of the more important topics of convexity and its applications. By comparing the Handbook with the survey of Bonnesen and Fenchel and with more recent collections of surveys of particular aspects of geometric convexity such as the AMS volume edited by Klee (1963), the Copenhagen Colloquium volume edited by Fenchel (1967), the two green Birkhäuser volumes edited by Tölke and Wills (1979) and Gruber and Wills (1983), respectively, and the New York Academy volume edited by Goodman, Lutwak, Malkevitch and Pollack (1985), the reader may see where progress was made in recent years.

During the planning stage of the Handbook, which started in 1986, we got generous help from many prominent convex geometers, in particular Peter McMullen, Rolf Schneider and Geoffrey Shephard. The discussion of the list of contents and of prospective authors turned out to be difficult. Both of us are obliged to the authors who agreed to contribute to the Handbook. In the cooperation with them we got much encouragement and the professional contacts furthered our good personal relations with many of them. The manuscripts which we finally received turned out to be much more diverse than we had anticipated. They clearly exhibit the most different characters and scientific styles of the authors and this should make the volume even more attractive.

There are several researchers in geometric convexity whom we invited to contribute to the Handbook but who for personal, professional or other reasons – regretfully – were not able to participate. The reader will also note that one area or another is missing in the list of contents. Examples are elementary geometry of normed planes, axiomatic convexity, and Choquet theory, but this should not diminish the usefulness of the Handbook.

Some fields such as computational and algorithmic aspects or lattice point results are dealt with in several chapters. The subjects covered are organized in five parts which in some sense reflect the fact that convexity is situated between analysis, geometry and discrete mathematics. This organization clearly has some disadvantages but we think that it will be helpful for the reader who wants to orient himself.

In the editing of the Handbook we received much help, in particular from Dr. M. Henk, Dr. J. Müller, Professor S. Hildebrandt, Professor L. Payne and Professor F. Schnitzer. We are most grateful to Ms. S. Clarius and Ms. E. Rosta who typed about 1000 letters to the authors, to colleagues whom we asked for advice and to the Publishers. We most gratefully acknowledge the friendly cooperations and expert support of Dr. A. Sevenster and Mr. W. Maas from Elsevier Science Publishers.

Finally we should like to express our sincere hope that the readers will appreciate the great effort of so many prominent authors and will find the Handbook useful for their scientific work.

Peter M. Gruber
Jörg M. Wills

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CHAPTER 3.2

Lattice Points

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HANDBOOK OF CONVEX GEOMETRY
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1. Preliminaries

1.1. Introduction

This survey deals with various lattice point problems, both, from a theoretical and from an algorithmic point of view. Lattice point problems trace back to Lagrange, Gauß, Dirichlet and Hermite. Most important contributions came from Minkowski, when he created in 1891 the field "Geometry of Numbers" by applying geometric tools to number-theoretic problems. The basic idea was to interpret integer solutions of equations or inequalities as points of the integer lattice. This approach was very fruitful and many eminent mathematicians – Weyl, Siegel, Mordell, Blichfeldt, van der Corput, Mahler, Davenport, Hlawka, W.M. Schmidt among them – contributed to this field; for historic outlines see Hlawka (1980) and Schwermer (1991).

Lattice points are also used in other areas including numerical analysis, computer science and, in particular, integer programming.

The central notions in this paper are, first, the *lattice point enumerator* $G_{\mathbb{L}}$, a functional that is defined for a given point lattice $\mathbb{L}$ in $\mathbb{R}^d$ and that associates with a given subset S of $\mathbb{R}^d$ the number $\operatorname{card}(S \cap \mathbb{L})$ of $\mathbb{L}$-points in S and, second, *convex lattice polytopes*, polytopes that are the convex hull of their $\mathbb{L}$-points. In particular, we are interested in properties of $\mathbb{L}$-polyhedra $I_{\mathbb{L}}(K)$ that are associated with given convex bodies K via $I_{\mathbb{L}}(K) = \operatorname{conv}(\mathbb{L} \cap K)$.

The integer programming problem is the task to maximize a linear functional over $I_{\mathbb{Z}^d}(F)$ where F is the region of points that satisfy a given system of rational linear equalities and linear inequalities. A basic question of interest is to give criteria for $I_{\mathbb{Z}^d}(F)$ to be nonempty. A related but more general problem is to give upper and lower bounds for $G_{\mathbb{L}}(I_{\mathbb{L}}(F))$, bounds on the number of such solutions for a given lattice $\mathbb{L}$.

In the geometry of numbers there has been some interest in bounds of the discrete functional $G_{\mathbb{L}}$ in terms of continuous functionals like the volume, the surface area and other quermassintegrals. For algorithmic purposes other functionals are of some relevance.

All these connections with other areas have been fruitful and some of the more recent contacts will be addressed in this article, in particular some of the computational aspects of the field. We will, however, not focus on aspects that are being covered by various books and survey articles like Cassels (1971), Gruber and Lekkerkerker (1987), Erdős, Gruber and Hammer (1989), Schrijver (1986, 1993a,b), Gruber (1979), Hlawka (1980), Keller (1954), Lang (1990) and Lagarias (1993) and, particularly, limit the overlap with other chapters of this Handbook as much as possible. These chapters are: 3.1 *Geometry of Numbers* by Gruber, 3.3 *Packing and Covering with Convex Sets* by G. Fejes Tóth and Kuperberg, 3.4 *Finite Packing and Covering* by Gritzmann and Wills, 2.8 *Convexity and Discrete Optimization* by Burkard and 2.7 *Mathematical Programming and Convex Geometry* by Gritzmann and Klee. In particular, we will not cover the following topics:

weighted lattice point enumerator;
lattice points on the boundary of convex bodies;
lattice points in "large" convex bodies; asymptotic results.

For surveys on these subjects see Betke and Wills (1979), Fricker (1982) and Walfisz (1957). Further there are far too many ramifications of 2-dimensional results to be mentioned here in detail; we restrict our attention to those which we deemed most closely related to the purpose of this article.

1.2. General definitions and notation

Let $\mathbb{R}^d$ denote the d-dimensional vector space over the reals; $\mathbb{E}^d$ is the d-dimensional Euclidean space. The Euclidean scalar product and unit ball are denoted by $\langle \cdot, \cdot \rangle$ and $\mathbb{B}^d$, respectively.

A *convex body* in $\mathbb{R}^d$ is a nonempty compact convex subset of $\mathbb{R}^d$. Let $\mathcal{K}^d$ denote the set of convex bodies of $\mathbb{R}^d$, let $\mathcal{K}_0^d$ be the set of convex bodies with nonempty interior and let $\hat{\mathcal{K}}_0^d$ denote the subset of convex bodies in $\mathcal{K}_0^d$ that are symmetric about the origin.

A *convex polytope* in $\mathbb{R}^d$ is a convex body that can be presented as the convex hull of finitely many points (or equivalently, as the intersection of finitely many closed halfspaces). The set of all convex polytopes in $\mathbb{R}^d$ will be denoted by $\mathcal{P}^d$; $\mathcal{P}_0^d$ and $\hat{\mathcal{P}}_0^d$ are abbreviations for $\mathcal{P}^d \cap \mathcal{K}_0^d$ and $\mathcal{P}^d \cap \hat{\mathcal{K}}_0^d$, respectively. Given $P \in \mathcal{P}^d$, let $\mathcal{F}_i(P)$ denote the set of all i-dimensional faces of P. For comprehensive treatments of combinatorial properties of convex polytopes see Grünbaum (1967) and chapter 2.3 by Bayer and Lee in this Handbook.

For computational purposes it is sometimes relevant to distinguish between the two different ways how polytopes may be given.

A *$\mathcal{V}$-presentation* of a polytope $P \subset \mathbb{R}^d$ consists of positive integers d and m, and m points $v_1, \ldots, v_m$ in $\mathbb{Q}^d$ such that $P = \operatorname{conv}\{v_1, \ldots, v_m\}$. An *$\mathcal{H}$-presentation* of P consists of positive integers d and m, a rational $m \times d$ matrix A, and a vector $b \in \mathbb{Q}^m$ such that $P = \{x \in \mathbb{R}^d\colon Ax \leqslant b\}$ Sometimes we use the shorter form $\mathcal{V}$- or $\mathcal{H}$-polytope when dealing with a polytope that is $\mathcal{V}$- or $\mathcal{H}$-presented, respectively, and we will also speak of an $\mathcal{H}$-polyhedron F to indicate that $F = \{x \in \mathbb{R}^d\colon Ax \leqslant b\}$ with rational A and b.

For $K \in \mathcal{K}^d$ let $V(K)$ denote K's volume; in particular, we set $\kappa_d = V(\mathbb{B}^d)$, the volume of the Euclidean unit ball. An important set of functionals on $\mathcal{K}^d$ are Minkowski's quermassintegrals $W_0, \ldots, W_d$ or their renormalization, the intrinsic volumes $V_0, \ldots, V_d$, given for $j = 0, 1, \ldots, d$ by $V_j = \kappa_{d-j}^{-1} \binom{d}{j} W_{d-j}$; (see Hadwiger 1957, McMullen 1975, 1977, McMullen and Schneider 1983, and chapter 1.8 by Schneider in this Handbook). The intrinsic volume V_i is continuous, additive, monotone and positive, invariant under rigid motions, homogeneous of degree i and independent of the dimension of the space in which K is embedded. In particular, $V_d = V$ is the volume, $V_{d-1} = \frac{1}{2}F$ is half the surface area and $V_0 = 1$.

A lattice $\mathbb{L}$ in $\mathbb{R}^d$ is a discrete subset $\mathbb{L}$ of $\mathbb{R}^d$ of the following form: there is a

basis $\{v_1, \ldots, v_d\}$ such that

$$\mathbb{L} = \left\{ \sum_{j=1}^{d} \nu_i v_i \colon \nu_1, \ldots, \nu_d \in \mathbb{Z} \right\}.$$

The set of all lattices in $\mathbb{R}^d$ will be denoted by $\mathcal{L}^d$. For a lattice $\mathbb{L}$ with basis $\{v_1, \ldots, v_d\}$, $\det(\mathbb{L})$ denotes the determinant of $\mathbb{L}$, the volume of the parallelotope $\sum_{i=1}^d [0,1] v_i$, and $C(\mathbb{L})$ denotes the *Dirichlet*-cell of 0, the set of all points of $\mathbb{R}^d$ which are not farther away from 0 than from any other lattice point. Observe that $C(\mathbb{L})$ is the closure of a fundamental region of $\mathbb{L}$; see chapters 3.1 by Gruber and 3.3 by G. Fejes Tóth and Kuperberg.

For a given lattice $\mathbb{L}$ let $\mathcal{P}^d(\mathbb{L})$ and $\mathcal{P}_0^d(\mathbb{L})$ denote the subsets of $\mathcal{P}^d$ and $\mathcal{P}_0^d$, respectively, of convex polytopes P with $\mathcal{F}_0(P) \subset \mathbb{L}$. The polytopes $P \in \mathcal{P}^d(\mathbb{L})$ are called convex $\mathbb{L}$-polytopes (or $\mathbb{L}$-integer polytopes). A convex $\mathbb{L}$-polyhedron P is the intersection of finitely many closed halfspaces such that $P = \operatorname{conv}(P \cap \mathbb{L})$. $\mathbb{L}$-rational polytopes and polyhedra are defined analogously. For $\mathbb{L} \in \mathcal{L}^d$ and an arbitrary convex set C the $\mathbb{L}$-hull (or the $\mathbb{L}$-integer hull) of C is given by $I_{\mathbb{L}}(C) = \operatorname{conv}(\mathbb{L} \cap C)$.

For a subset $S \subset \mathbb{R}^d$ the lattice point enumerator $G_{\mathbb{L}}(S)$ is defined by $G_{\mathbb{L}}(S) = \operatorname{card}(S \cap \mathbb{L})$. Typically, S will be a convex body but we will also consider cases where S is the boundary or the interior of a convex body. For $\mathbb{L} = \mathbb{Z}^d$ we write G instead of $G_{\mathbb{Z}^d}$.

As usual, the polar lattice $\mathbb{L}^*$ of a lattice $\mathbb{L}$ is defined by

$$\mathbb{L}^* = \{x \in (\mathbb{E}^d)^*\colon y \in \mathbb{L} \Longrightarrow \langle x, y \rangle \in \mathbb{Z}\}.$$

$\mathbb{L}^*$ is a lattice in the conjugate space $(\mathbb{E}^d)^*$. Since $\mathbb{E}^d$ is self-conjugate we will regard $\mathbb{L}^*$ as a lattice in $\mathbb{E}^d$. Then, clearly, $(\mathbb{L}^*)^* = \mathbb{L}$ and, in particular, $(\mathbb{Z}^d)^* = \mathbb{Z}^d$. For various interactions between $\mathbb{L}$ and $\mathbb{L}^*$ see McMullen (1984), Kannan and Lovász (1988) and Schnell (1992).

In sections 5 and 6 we will use the standard notation of complexity theory. We use the binary Turing machine model, hence, the size of the input of a problem – usually denoted by L – is the number of binary digits needed to encode the input data. The complexity classes that are relevant here are the classes P, NP-hard, NP-complete, #P-hard and #P-complete. For underlying concepts of theoretical computer science, definitions and for numerous results see Aho, Hopcroft and Ullman (1974) and Garey and Johnson (1979).

2. Centrally symmetric convex bodies

2.1. Minkowski's fundamental theorem

The starting point of the geometry of numbers and the predominant result of this section is Minkowski's fundamental theorem, Minkowski (1896).

Let $\mathbb{L} \in \mathcal{L}^d$, $K \in \hat{\mathcal{K}}_0^d$ *and suppose* $G_{\mathbb{L}}(\operatorname{int} K) = 1$. *Then*

$$V(K) \leqslant 2^d \det \mathbb{L}. \tag{2.1}$$

This estimate is tight; in particular, it holds with equality for lattice parallelotopes. In addition, the slack in (2.1) can be expressed explicitly:

$$\frac{1}{V(K)} \sum_{u \in \mathbb{L}^* \setminus \{0\}} \left| \int_{(1/2)K} e^{-\pi i u x} \, dx \right|^2 = 2^d \det \mathbb{L} - V(K).$$

This identity holds under the same assumptions as (2.1); it was derived by Siegel (1935) as a special multiple Fourier series instance of Parseval's identity from functional analysis.

Minkowski's theorem has numerous applications in the geometry of numbers (see chapter 3.1). Its impact shows in the number of ramifications, refinements and generalizations that it led to. In particular, Blichfeldt (1914) and van der Corput (1935, 1936) proved that for $\mathbb{L} \in \mathcal{L}^d$ and $K \in \hat{\mathcal{K}}^d$:

$$2 \left\lfloor \frac{V(K)}{2^d(\det \mathbb{L})} \right\rfloor + 1 \leqslant G_{\mathbb{L}}(K). \tag{2.2}$$

Equality occurs for suitable "quasi-1-dimensional prisms" of the form $\alpha C(\mathbb{L}) + \beta v_1$ with $0 < \alpha, \beta$ and $\alpha < 2$.

The following inequalities for the lattice point enumerator are closely related to (2.1); they are also due to Minkowski (1896).

Let $\mathbb{L} \in \mathcal{L}^d$, $K \in \hat{\mathcal{K}}_0^d$ *and suppose* $G_{\mathbb{L}}(\operatorname{int} K) = 1$. *Then*

$$G_{\mathbb{L}}(K) \leqslant 3^d. \tag{2.3}$$

If, in addition, K is strictly convex then $G_{\mathbb{L}}(K) \leqslant 2^{d+1} - 1$.

Further analogues of (2.1) can be found in Cassels (1971), Gruber and Lekkerkerker (1987), and Erdős, Gruber and Hammer (1989).

2.2. *Successive minima*

The most important improvement of (2.1) is Minkowski's theorem on successive minima. Given a lattice $\mathbb{L}$ and a convex body $K \in \hat{\mathcal{K}}_0^d$, then the successive minima $\lambda_1, \ldots, \lambda_d$ are defined by

$$\lambda_i(K, \mathbb{L}) = \inf\{\lambda > 0\colon \dim \operatorname{aff}(\lambda K \cap \mathbb{L}) = i\}, \quad i = 1, \ldots, d.$$

For brevity we write $\lambda_i(\mathbb{L})$ and λ_i for $\lambda_i(K, \mathbb{L})$, $\lambda_i(K, \mathbb{Z}^d)$, respectively. Then, (2.1) reads

$$\lambda_1(\mathbb{L})^d V(K) \leqslant 2^d \det \mathbb{L},$$

while Minkowski's (1896) generalization is the following result:

Given $\mathbb{L} \in \mathcal{L}^d$ *and* $K \in \hat{\mathcal{K}}_0^d$, *then*

$$\frac{2^d}{d!} \det \mathbb{L} \leqslant \lambda_1(\mathbb{L}) \times \cdots \times \lambda_d(\mathbb{L}) V(K) \leqslant 2^d \det \mathbb{L}. \tag{2.4}$$

The left inequality is simple, but the right one is a deep improvement of (2.1). For a generalization of (2.4) see Woods (1966) and for various proofs see Bambah, Woods and Zassenhaus (1965), Cassels (1971, p. 208) and Gruber and Lekkerkerker (1987, p. 59).

There are some recent analogues of (2.1), (2.2), (2.3) and (2.4), Henk (1990), Betke, Henk and Wills (1993):

$$\begin{aligned} &\lambda_{i+1} \times \cdots \times \lambda_d V(K) \leqslant 2^{d-i} V_i(K), \quad i = 1, \ldots, d, \\ &\frac{2^i}{i!} \leqslant \lambda_1 \times \cdots \times \lambda_i V_i(K), \quad i = 1, \ldots, d, \\ &G_{\mathbb{L}}(K) \leqslant \left\lfloor \frac{2}{\lambda_1(\mathbb{L})} + 1 \right\rfloor^d, \\ &\frac{1}{d!} \prod_{i=1}^{d} \left(\frac{2}{\lambda_i(\mathbb{L})} - 1 \right) \leqslant G_{\mathbb{L}}(K). \end{aligned} \tag{2.5}$$

The estimate (2.5) extends Minkowski's inequality (2.3). Betke, Henk and Wills (1993) conjecture that the stronger bound

$$G_{\mathbb{L}}(K) \leqslant \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(\mathbb{L})} + 1 \right\rfloor$$

holds and prove it for $d = 2$; along with analogous results for strictly convex bodies.

For details on reduction theory of quadratic forms and other classical results we refer to Cassels (1971), Gruber and Lekkerkerker (1987), Erdős, Gruber and Hammer (1989) and chapter 3.1 of this Handbook.

3. General convex bodies

3.1. *Lattice points and intrinsic volumes*

The classical theory of geometry of numbers deals mainly with centrally symmetric convex bodies; hence there are only a few classical results on general convex bodies. One such result is the simple mean value theorem:

$$V(K) = \int_{C(\mathbb{L})} G_{\mathbb{L}}(x + K) \, dx,$$

which holds for arbitrary $\mathbb{L} \in \mathscr{L}^d$ and $K \in \mathscr{K}^d$.

Mahler (1939) generalized Minkowski's fundamental theorem by relaxing the symmetry assumption on K. As a measure of asymmetry Mahler defined

$$\sigma(K) = \min\{\sigma \geqslant 1\colon -K \subset \sigma K\},$$

for all $K \in \mathscr{K}^d$ with $0 \in \operatorname{int} K$ and derived an inequality that was later improved by Sawyer (1954) to the following result:

Let $K \in \mathscr{K}^d$ and suppose $(\operatorname{int} K) \cap \mathbb{L} = \{0\}$. Then

$$V(K) \leqslant (1+\sigma(K))^d \left(1 - \left(1 - \frac{1}{\sigma(K)}\right)^d\right) \det \mathbb{L}. \tag{3.1}$$

Observe that when $K = -K$, (3.1) is just Minkowski's theorem (2.1). If $z(K)$ denotes the centroid of K then $z(K) = 0$ implies $\sigma(K) \leqslant d$ (cf. Bonnesen and Fenchel 1934, p. 53). Together with (3.1) this gives

$$V(K) \leqslant (d+1)^d \left(1 - \left(1 - \frac{1}{d}\right)^d\right) \det \mathbb{L}$$

for all $K \in \mathscr{K}^d$ with $z(K) = 0$ and $(\operatorname{int} K) \cap \mathbb{L} = \{0\}$. Ehrhart (1955a) conjectured, however, that the much stronger inequality

$$V(K) \leqslant \frac{(d+1)^d}{d!} \det \mathbb{L}$$

holds for all $K \in \mathscr{K}^d$ with $z(K) = 0$. This estimate would be tight but has only been verified for $d = 2$, Ehrhart (1955a).

In the remainder of this section we will focus on bounds for the lattice-point enumerator $G_{\mathbb{L}}$ in terms of quermassintegrals and closely related functionals. Observe that for finding upper bounds for $G_{\mathbb{L}}(K)$ in terms of monotone functionals it is enough to just consider convex lattice polytopes since $I_{\mathbb{L}}(K)$ has the same number of $\mathbb{L}$-lattice points as K.

The oldest general upper bound for the lattice point enumerator is due to Blichfeldt (1921).

Let $\mathbb{L} \in \mathscr{L}^d$, let $K \in \mathscr{K}^d$ and suppose $I_{\mathbb{L}}(K) \in \mathscr{P}_0^d$. Then

$$G_{\mathbb{L}}(K) \leqslant \frac{d!}{\det \mathbb{L}} V(K) + d. \tag{3.2}$$

Equality holds in (3.2) for instance for lattice simplices of volume $(1/d!) \det \mathbb{L}$. If $R_{\mathbb{L}}$ denotes the circumradius of $C(\mathbb{L})$ then

$$V(K) - R_{\mathbb{L}} F(K) \leqslant G_{\mathbb{L}}(K) \cdot \det \mathbb{L} \leqslant V\big(K + C(\mathbb{L})\big); \tag{3.3}$$

Gritzmann (1984, pp. 52, 88), Wills (1990a, p. 37).

Schnell (1992) and Wills (1991) gave the bounds

$$\frac{V_d(K)}{D_d(\mathbb{L})} - \tfrac{1}{2} d^{3/2} \frac{V_{d-1}(K)}{D_{d-1}(\mathbb{L})} \leqslant G_{\mathbb{L}}(K) \leqslant \sum_{i=0}^{d} i! \frac{V_i(K)}{D_i(\mathbb{L})}, \tag{3.4}$$

which involve the lattice functionals $D_i(\mathbb{L})$, $i = 0, \ldots, d$ defined by

$$D_0(\mathbb{L}) = 1$$

and for $i = 1, \ldots, d$

$$D_i(\mathbb{L}) = \min\{|\det(\mathbb{L}_i)|\colon \mathbb{L}_i \text{ is an } i\text{-dimensional sublattice of } \mathbb{L}\}.$$

Note that, in particular, $D_d(\mathbb{L}) = \det(\mathbb{L})$ and that $D_1(\mathbb{L})$ is the length of a shortest nonzero lattice vector. Neither of the two inequalities of (3.4) is tight. For $d = 2$, the factors $\frac{1}{2} d^{3/2}$ and $i!$ can be replaced by 1, and this is best possible; Oler (1961) [upper bound; see (3.9)] and Schnell and Wills (1991) (lower bound). Further, for $\mathbb{L} = \mathbb{Z}^d$ there is the asymptotically tight inequality

$$V(K) - \tfrac{1}{2} F(K) \leqslant G(K) \tag{3.5}$$

of Bokowski, Hadwiger and Wills (1972) for arbitrary $K \in \mathscr{K}^d$. The proof of (3.5) is based on previous work of Hadwiger (1972). Partial results can also be found in Nosarzewska (1948), Bender (1962), Wills (1968, 1970), Hadwiger (1970), Schmidt (1972) and Bokowski and Wills (1974). Some consequences of (3.5) were given by Hammer (1971) and Bokowski and Odlyzko (1973).

The right-hand inequality in (3.3) can be developed into mixed volumes (or via $C(\mathbb{L}) \subset R_{\mathbb{L}} \mathbb{B}^d$ and Steiner's formula into quermassintegrals). For $\mathbb{L} = \mathbb{Z}^d$, we obtain Davenport's (1951) inequality.

Let for $K \in \mathscr{K}^d$, K_j^i be the orthogonal projection of K into the i-dimensional coordinate subspace $\mathbb{E}_j^i \subset \mathbb{E}^d$, $i = 1, \ldots, d$; $j = 1, \ldots, \binom{d}{i}$. Then

$$G(K) \leqslant \sum_{i=0}^{d} \sum_{j=1}^{\binom{d}{i}} V_i(K_j^i). \tag{3.6}$$

(See Betke 1979.) Equality holds for lattice boxes. A simple consequence of (3.6) is the estimate

$$G(K) \leqslant V(K) + \sum_{i=0}^{d-1} V_i(Q), \quad K \in \mathscr{K}^d,$$

where Q denotes the smallest lattice box containing K. Ehrhart (1977) conjectured that the bound can be improved to

$$G(K) \leqslant V(K) + \tfrac{1}{2}F(K) + \sum_{i=0}^{d-2} V_i(Q), \quad K \in \mathcal{K}^d;$$

which holds for $d \leqslant 3$ (Ehrhart 1977), but is open for $d \geqslant 4$.

Another simple consequence of (3.6) is

$$G(K) \leqslant \sum_{i=0}^{d} \binom{d}{i} V_i(K), \quad K \in \mathcal{K}^d, \tag{3.7}$$

which holds with equality if and only if K is a lattice point. Obviously, the coefficients of V_0 and V_d are best possible. There were, however, many attempts to improve the other coefficients. Wills (1973) conjectured that

$$G(K) \leqslant \sum_{i=0}^{d} V_i(K). \tag{3.8}$$

This estimate holds with equality for lattice boxes. Further, it has been verified for dimensions $d = 2, 3$ (Nosarzewska 1948, Overhagen 1975), for rotation bodies when $d \leqslant 20$ (Hadwiger and Wills 1974) and for arbitrary lattice zonotopes (Betke and Gritzmann 1986). Moreover, Hadwiger (1975) gave integral representations of the functional $W = \sum_{i=0}^{d} V_i$ which inspired some work from the viewpoint of valuations (McMullen 1975; see also chapter 3.6 by McMullen in this Handbook). However, the conjecture turned out to be false. Hadwiger (1979) showed that for $d \geqslant 441$ there are lattice simplices for which (3.8) does not hold. Later Betke and Henk (1993) showed that it is false for suitable lattice cross polytopes already when $d \geqslant 207$. In fact, even the much weaker estimate

$$G(K) \leqslant V(K + \kappa_d^{-1/d}\mathbb{B}^d),$$

conjectured and proved for $d \leqslant 5$ by Bokowski (1975), is false for $d \geqslant 3.7 \cdot 10^{159}$, Höhne (1980). Another attempt to fill the gap between (3.7) and (3.8) is the following conjecture of Wills (1990b) which is closely related to a covering theorem by Santaló (1976, p. 274):

$$G(K) \leqslant \sum_{i=0}^{d} \frac{\kappa_i \kappa_{d-i}}{\kappa_d} V_i(K), \quad K \in \mathcal{K}^d.$$

3.2. Classes of lattices

Lattice point problems are closely related to lattice packing and lattice covering. In order to facilitate the transition between these areas the following classes of lattices are introduced for $C \in \mathcal{K}_0^d$.

$\mathcal{L}_p(C)$ is the subset of $\mathcal{L}^d$ of all lattices $\mathbb{L}$ which pack C, i.e., the translates of C in $\{g + C: g \in \mathbb{L}\}$ do not have interior points in common;

$\mathcal{L}_c(C)$ is the subset of $\mathcal{L}^d$ of all lattices $\mathbb{L}$ for which $\mathbb{L} + C$ covers $\mathbb{R}^d$, i.e., $\bigcup_{g \in \mathbb{L}} (g + C) = \mathbb{R}^d$.

Since the main interest focuses on bounds for the lattice point enumerator in terms of quermassintegrals only the case $C = \mathbb{B}^d$ has been studied thoroughly.

For $\mathbb{L} \in \mathcal{L}_p(\mathbb{B}^d)$ and $K \in \mathcal{K}^d$ there is the simple but useful upper bound

$$G_{\mathbb{L}}(K) \leqslant \kappa_d^{-1} V(K + \mathbb{B}^d) = \sum_{i=0}^{d} \frac{\kappa_{d-i}}{\kappa_d} V_i(K).$$

With $\delta_{\mathcal{L}}(\mathbb{B}^d)$ and $\vartheta_{\mathcal{L}}(\mathbb{B}^d)$ denoting the lattice packing and lattice covering density of $\mathbb{B}^d$, respectively, it is easy to see that the inequality

$$G_{\mathbb{L}}(K) \leqslant \sum_{i=0}^{d} \frac{\delta_{\mathcal{L}}(\mathbb{B}^d)}{\kappa_i} V_i(K)$$

would be the best possible bound of this kind. It has been shown by Gritzmann and Wills (1986) that this inequality holds for $\mathbb{L}$-zonotopes but is false in general for suitably high dimension. However, in dimension 2 it holds for general $K \in \mathcal{K}^2$ and $\mathbb{L} \in \mathcal{L}_p(\mathbb{B}^d)$ even in the Minkowski plane (Oler 1961). In the Euclidean case Oler's result reads as follows:

$$G_{\mathbb{L}}(K) \leqslant \frac{1}{2\sqrt{3}} V(K) + \frac{1}{2} V_1(K) + 1. \tag{3.9}$$

Observe that (3.9) follows from Pick's (1899) identity (4.1). In fact, (4.1) implies that for $K \in \mathcal{K}^2$ and $\mathbb{L} \in \mathcal{L}^2$

$$G_{\mathbb{L}}(K) \leqslant \frac{V(K)}{\det \mathbb{L}} + \frac{V_1(K)}{D_1(\mathbb{L})} + 1.$$

Now, (3.9) follows since for each $\mathbb{L} \in \mathcal{L}_p(\mathbb{B}^2)$ we have $\det \mathbb{L} \geqslant \kappa_2/\delta_{\mathcal{L}}(\mathbb{B}^2) = 2\sqrt{3}$ and $D_1(\mathbb{L}) \geqslant 2$. The inequality (3.9) is equivalent to the "lattice version" of a packing theorem of Groemer (1960); another proof is due to Folkman and Graham (1969); cf. also Zassenhaus (1961). Oler's result might possibly be the 2-dimensional case of the general inequality for $\mathbb{L} \in \mathcal{L}_p(\mathbb{B}^d)$ and $K \in \mathcal{K}^d$:

$$G_{\mathbb{L}}(P) \leqslant \sum_{i=0}^{d} \frac{\sigma_i}{\kappa_i} V_i(P). \tag{3.10}$$

Here σ_i denotes Rogers's (1964) packing constants, i.e. the ratio of the sum of the volumes of the intersection of $d+1$ unit balls centered at the vertices of a regular simplex of side 2 to the volume of the simplex. Gritzmann and Wills (1986) showed that (3.10) is valid for $\mathbb{L}$-zonotopes and conjecture that it holds in general.

Gritzmann (1984, 1986) showed that

$$G_{\mathbb{L}}(K) \leqslant \left((2+\sqrt{2})\sqrt{\frac{\pi}{2d}} + \frac{2}{\sqrt{d(d-1)}}\right)\sum_{i=0}^{d}\frac{\kappa_{d-i}}{\kappa_d}V_i(K);$$

again, for arbitrary $\mathbb{L} \in \mathscr{L}_p(\mathbb{B}^d)$ and $K \in \mathscr{K}^d$. (For a "finite packing interpretation" of this result see chapter 3.4.)

As in section 3.1, the lower bound problem is simpler and essentially solved by Wills (1989):

Given $\mathbb{L} \in \mathscr{L}_c(\mathbb{B}^d)$ *and* $K \in \mathscr{K}^d$, *then* $\vartheta^d_{\mathscr{L}}\kappa_d^{-1}\{V(K) - F(K)\} \leqslant G_{\mathbb{L}}(K)$.

3.3. Nonlinear inequalities

In the previous two sections the bounds for G were linear combinations of the V_i or related functionals. We now consider nonlinear inequalities for G in terms of functionals which are homogeneous of degree 1.

Let $D(K)$, $R(K)$, $r(K)$ and $w(K)$ denote the diameter, circumradius, inradius and width of $K \in \mathscr{K}^d$, respectively. We begin with the following trivial inequalities.

$$G(K) \leqslant (D(K)+1)^d,$$
$$G(K) < \kappa_d(R(K)+\tfrac{1}{2}\sqrt{d})^d,$$
$$\kappa_d(r(K)-\tfrac{1}{2}\sqrt{d})^d < G(K), \quad \text{if } r(K) \geqslant \tfrac{1}{2}\sqrt{d}.$$

There are various inequalities for lattice-point-free convex bodies which involve the width. Results of this kind are of some relevance for reducing a d-dimensional integer programming problem to lower-dimensional ones; see Lenstra (1983). Let

$$\omega_d = \max\{w(K): K \in \mathscr{K}^d \wedge G(K) = 0\}.$$

McMullen and Wills (1981) showed

$$(\sqrt{2}+1)(\sqrt{d-1} - \alpha) < \omega_d < d+1,$$

where $\alpha \approx 1.018$ is a constant. For $K \in \mathscr{K}^d$ let $d_i(K)$ denote the length of the projection of K onto the x_i-axis ("outer quermass") and let $s_i(K)$ denote the length of a maximal segment of K parallel to the x_i-axis ("inner quermass"). Then Scott (1979) (cf. also Wills 1990a) showed that for $K \in \mathscr{K}^d$ with $G(K) = 0$

$$\frac{1}{d_i(K)} + \sum_{j\neq i}\frac{1}{s_j(K)} \geqslant 1, \quad i = 1,\ldots,d,$$

with equality for suitable cross polytopes. McMullen and Wills (1981) proved under the same assumptions that

$$\frac{\sqrt{2}\omega_{d-1}}{w(K)} + \frac{1}{d_i(K)} \geqslant 1, \quad i = 1,\ldots,d,$$

$$\frac{\sqrt{2}\omega_{d-1}}{w(K)} + \frac{\sqrt{d}}{D(K)} \geqslant 1.$$

Possibly the last inequality can be improved to

$$\frac{\omega_{d-1}}{w(K)} + \frac{1}{D(K)} \geqslant 1,$$

a result confirmed by Scott (1973) for $d = 2$. Some sharp nonlinear inequalities for special convex bodies can be found in Erdős, Gruber and Hammer (1989) and Xu and Yau (1992).

A simple but useful inequality is

$$G(K) - V(K) \leqslant \prod_{i=1}^{d}(d_i(K)+1) - \prod_{i=1}^{d}d_i(K); \tag{3.11}$$

it holds with equality for lattice boxes. For applications of (3.11) and some closely related inequalities see Chalk (1980) and Niederreiter and Wills (1975).

The following results use the covering minimum μ_i, introduced by Kannan and Lovász (1988), which is defined for $\mathbb{L} \in \mathscr{L}^d$, $K \in \mathscr{K}^d$ and $i = 1,\ldots,d$ by

$$\mu_i(K,\mathbb{L}) = \inf\{t: tK + \mathbb{L} \text{ meets every } (d-i)\text{-flat of } \mathbb{R}^d\}.$$

The μ_i correspond to Minkowski's successive minima λ_i and, in fact, μ_d is the classical inhomogeneous minimum μ in geometry of numbers. $\mu_1(K,\mathbb{L})^{-1}$ is called the $\mathbb{L}$-width of K – or, when the lattice is specified by the context, simply *lattice width* of K. Kannan and Lovász (1988) prove that there is a positive constant γ such that for $\mathbb{L} \in \mathscr{L}^d$ and $K \in \mathscr{K}_0^d$

$$\left\lfloor\frac{1}{\mu_1(K,\mathbb{L})\gamma d^2}\right\rfloor^d - 1 \leqslant G_{\mathbb{L}}(K), \tag{3.12}$$

and if $K \in \hat{\mathscr{K}}_0^d$

$$\left\lfloor\left(\frac{1}{\mu_1(K,\mathbb{L})\gamma d} - d\right)^d\right\rfloor \leqslant G_{\mathbb{L}}(K). \tag{3.13}$$

The constant γ comes from the constant in the following theorem.

There is a positive constant β such that for $K \in \hat{\mathcal{K}}_0^d$

$$\beta^d \kappa_d^2 \leqslant V(K)V(K^*) \leqslant \kappa_d^2. \tag{3.14}$$

The upper bound in (3.14) is due to Santaló (1949), the lower bound was given by Bourgain and Milman (1985). Using the known formula for κ_d one sees that $V(K)V(K^*)$ is bounded below by $(\alpha d)^{-d}$, where α is a positive absolute constant. As a consequence of (3.14) and Minkowski's theorem (2.4) on successive minima Bourgain and Milman (1987) obtain

$$\prod_{i=1}^{d} \lambda_i(K,\mathbb{L})\lambda_i(K^*,\mathbb{L}^*) \leqslant \gamma^d d^d.$$

Their paper contains also various other results on the λ_i and μ_i.

Betke, Henk and Wills (1993) prove some inequalities for the μ_i and the V_i; an example is the result that there is a constant β with $0 < \beta \leqslant 1/d!$ such that for every $K \in \mathcal{K}_0^d$

$$\mu_1(K,\mathbb{Z}^d)\cdots\mu_d(K,\mathbb{Z}^d)V(K) \geqslant \beta.$$

4. Lattice polytopes

4.1. General lattice polytopes

The methods in the theory of lattice polytopes are mainly from combinatorics and linear algebra and several results do not require convexity. For the purpose of this subsection only, a polytope in $\mathbb{R}^d$ (or a polygon in $\mathbb{R}^2$) is the underlying point set of a simplicial cell complex (in the sense of Grünbaum 1967). Equivalently, a polytope can be defined as a finite union of convex polytopes of dimension at most d. A polytope is called proper, if it is the closure of its interior or, equivalently, if it can be represented as the finite union of convex d-polytopes. Given a lattice $\mathbb{L} \in \mathcal{L}^d$, a polytope P is called $\mathbb{L}$-polytope if there is a simplicial cell complex with underlying point set P whose vertices are all in $\mathbb{L}$. As usual the Euler-characteristic $\chi(P)$ of a polytope is the sum $\sum_{i=0}^{d}(-1)^i|\mathcal{F}_i(\mathcal{C})|$ where $\mathcal{C}$ is any simplicial cell complex with underlying point set P. The first result on lattice polygons is Pick's identity, Pick (1899).

If $P \subset \mathbb{E}^2$ is a lattice polygon whose boundary is a closed Jordan curve, then

$$G_{\mathbb{L}}(P) = \frac{V(P)}{\det \mathbb{L}} + \tfrac{1}{2}G_{\mathbb{L}}(\operatorname{bd} P) + 1. \tag{4.1}$$

(Compare also Varberg 1985.) Observe that Nosarzewska's inequality (3.8, for $d = 2$) is a simple consequence of (4.1). There are various proofs of (4.1) and many generalizations. For instance, Reeve (1957) showed

$$G_{\mathbb{L}}(P) = \frac{V(P)}{\det \mathbb{L}} + \tfrac{1}{2}G_{\mathbb{L}}(\operatorname{bd} P) + \chi(P) - \tfrac{1}{2}\chi(\operatorname{bd} P), \tag{4.2}$$

and Hadwiger and Wills (1975) proved

$$G_{\mathbb{L}}(P) = \frac{V(P)}{\det \mathbb{L}} + \tfrac{1}{2}E(P) + \chi(P),$$

where $E(P)$ denotes the number of segments between two consecutive lattice points on $\operatorname{bd} P$, and 1-dimensional parts of P are counted twice.

Ding and Reay (1987a) give a generalization of (4.1) to hexagonal tilings and indicate possible applications to computer graphics; for further results along these lines see Ding and Reay (1987b) and Ding, Kolodziejczyk and Reay (1988). Grünbaum and Shephard (1992) extend (4.1) to more general oriented polygons, a result that implies many of the previously known (plane) variants of (4.1).

Hadwiger and Wills (1975) showed the following.

Let $\mathbb{L} \in \mathcal{L}^2$, let P be an $\mathbb{L}$-polygon and $x \in \mathbb{R}^2 \backslash \mathbb{L}$. Then

$$G_{\mathbb{L}}(P) - G_{\mathbb{L}}(x + P) \geqslant \chi(P); \tag{4.3}$$

and for each $\mathbb{L} \in \mathcal{L}^2$ and $\chi \in \mathbb{Z}$ there are an $\mathbb{L}$-polygon P with $\chi(P) = \chi$ and an $x \in \mathbb{R}^2 \setminus \mathbb{L}$ such that (4.3) holds with equality.

No analogue of (4.3) is known for $d \geqslant 3$; but there are remarkable generalizations of (4.1) and (4.2) for arbitrary $\mathbb{L} \in \mathcal{L}^d$ and proper $\mathbb{L}$-polytopes P:

$$\begin{aligned}\frac{(d-1)d!}{2\det \mathbb{L}} V(P) &= (-1)^{d-1}\left(\chi(P) - \tfrac{1}{2}\chi(\operatorname{bd} P)\right) \\ &\quad + \sum_{j=0}^{d-2}\binom{d-1}{j}(-1)^j\left(G_{\mathbb{L}}((d-1-j)P) - \tfrac{1}{2}G_{\mathbb{L}}(\operatorname{bd}((d-1-j)P))\right),\end{aligned}$$

$$\frac{d!}{\det \mathbb{L}} V(P) = (-1)^d\chi(P) + \sum_{j=0}^{d-1}\binom{d}{j}(-1)^j G_{\mathbb{L}}((d-1-j)P).$$

These identities are due to Macdonald (1963); the case $d = 3$ was first proved by Reeve (1957, 1959).

4.2. Convex lattice polytopes: Equalities

Ehrhart (1967, 1968) discovered the "polynomiality" of the lattice point enumerator $G_{\mathbb{L}} : \mathcal{P}^d(\mathbb{L}) \to \mathbb{N}_0$:

For each $\mathbb{L} \in \mathcal{L}$ *there are functionals* $G_{\mathbb{L},i} : \mathcal{P}^d(\mathbb{L}) \to \mathbb{N}_0$ *such that for every* $P \in \mathcal{P}^d(\mathbb{L})$ *and* $n \in \mathbb{N}$

$$G_{\mathbb{L}}(nP) = \sum_{i=0}^{d} n^i G_{\mathbb{L},i}(P) \tag{4.4}$$

and

$$G_{\mathbb{L}}(\operatorname{relint}(nP)) = (-1)^{\dim P} \sum_{i=0}^{d} (-n)^i G_{\mathbb{L},i}(P). \tag{4.5}$$

Equation (4.5) is called the *reciprocity law*, and the polynomial in (4.4) is often referred to as *Ehrhart-polynomial*. It is quite important for various questions. In particular, the question when the dual of an integer polytope P (with $0 \in \operatorname{int}(P)$) is again an integer polytope can be answered in terms of conditions on the Ehrhart-polynomial, Hibi (1992). See the book by Stanley (1986) for more facts on the Ehrhart polynomial. For $\mathbb{L} = \mathbb{Z}^d$ we will use the abbreviation G_i rather than $G_{\mathbb{L},i}$.

In case of lattice zonotopes one can give an explicit formula for the Ehrhart polynomial; Stanley (1980, 1986). Stanley (1991a) used this fact to find a generating function for the number of degree sequences of simple n-vertex graphs; in fact, there is a one-to-one correspondence between these degree sequences and the integer points of a suitable zonotope.

The $G_{\mathbb{L},i}$ are valuations (cf. McMullen 1977 and McMullen's chapter 3.6 in this Handbook). Further they are invariant under unimodular transformations, i.e., transformations U of $\mathbb{R}^d$ with $U(x) = Ax + b$, where A is an integer $d \times d$-matrix, $\det A = \pm 1$ and $b \in \mathbb{L}$. These properties provide the framework of an analogy between the $G_{\mathbb{L},i}$ and the intrinsic volumes V_i which shows in the following two important theorems.

(Hadwiger 1951) *Every continuous and additive functional on* $\mathcal{K}^d$ *which is invariant under rigid motions is a linear combination of the* $d+1$ *intrinsic volumes* $V_0, \ldots, V_d$.

(Betke and Kneser 1985) *Every additive and unimodular invariant functional on* $\mathcal{P}^d(\mathbb{L})$ *is a linear combination of the* $d+1$ *functionals* $G_{\mathbb{L},0}, \ldots, G_{\mathbb{L},d}$.

In addition, for $i = 0, d-1, d$, V_i and G_i are very similar

$$G_d = V_d = V, \qquad G_0 = V_0 = 1$$

and

$$G_{d-1}(P) = \tfrac{1}{2} \sum_{F \in \mathcal{F}_{d-1}} \mu_F V_{d-1}(F) \leqslant \tfrac{1}{2} F(P),$$

where $1/\mu_F = \det(\mathbb{Z}^d \cap \operatorname{aff} F)$ is the determinant of the sublattice of $\mathbb{Z}^d$ induced by $\operatorname{aff} F$. As opposed to the corresponding intrinsic volumes, $G_1, \ldots, G_{d-2}$, however, do not admit a simple geometric interpretation. In particular they are neither monotone nor nonnegative. Let us point out that Stanley (1991b) introduced a different basis $h_0^*, \ldots, h_d^*$ via

$$(1-x)^{d+1} \sum_{n \geqslant 0} G_{\mathbb{L}}(nP) x^n = h_0^* + h_1^* x + \cdots + h_d^* x^d$$

for the same space whose elements are monotone and, hence, nonnegative; see also Hibi (1991).

As Wills (1982) pointed out there is a simple analogue of Minkowski's lattice point theorem (2.1) for G_{d-1}. In fact,

$$G_{d-1}(P) \leqslant d2^{d-1},$$

for all $P \in \hat{\mathcal{P}}_0^d$ with $G(\operatorname{int} P) = 1$.

It is open whether

$$G_i(P) \leqslant \binom{d}{i} 2^i$$

holds for $i = 1, \ldots, d-2$ under the same assumptions. Betke and McMullen (1985) showed that for each $i = 1, \ldots, d$ there are constants α_i, β_i such that for all $P \in \mathcal{P}^d$

$$G_i(P) \leqslant \alpha_i G_d(P) + \beta_i.$$

The following identity resembles the polynomial expansion of the Minkowski sums of convex bodies into mixed volumes and generalizes Ehrhart's polynomial expansion (4.4).

(Bernstein 1976, McMullen 1977) *Let* $\mathbb{L} \in \mathcal{L}^d$, $k \in \mathbb{N}$, $P_1, \ldots, P_k \in \mathcal{P}^d(\mathbb{L})$ *and* $n_1, \ldots, n_k \in \mathbb{N}$. *Then there are coefficient functionals* $G_{\mathbb{L}}(P_1, n_1, \ldots, P_k, n_k)$ *such that*

$$G_{\mathbb{L}}\left(\sum_{i=1}^{k} n_i P_i\right) = \sum_{\substack{j_1, \ldots, j_k = 0 \\ j_1 + \cdots + j_k \leqslant d}}^{d} n_1^{j_1} \cdots n_k^{j_k} G_{\mathbb{L}}(P_1, j_1, \ldots, P_k, j_k).$$

The number (of equivalence classes under the group of unimodular transformations) of convex lattice polygons and polytopes has been studied by Arnold (1980), Bárány and Pach (1992), and Bárány and Vershik (1992).

4.3. Convex lattice polytopes: Inequalities

Unlike the bounds in section 3 which hold for general $K \in \mathcal{K}^d$, the following inequalities are tailored to the case of convex lattice polytopes.

Using methods of Ehrhart (1955a) and Stanley (1976) on formal power series and polyhedral cell complexes Betke and McMullen (1985) extended Blichfeldt's inequality (3.2) as follows:

$$G_{\mathbb{L}}(nP) \leqslant \left(\frac{V(nP)}{\det \mathbb{L}} + \frac{n^{d-1}}{(d-1)!}\right) \prod_{i=1}^{d-1}\left(1+\frac{i}{n}\right),$$

$$\left(\frac{V(nP)}{\det \mathbb{L}} - \frac{n^{d}}{d!}\right) \prod_{|i|<d/2}\left(1+\frac{i}{n}\right) + \binom{n+d}{d} \leqslant G_{\mathbb{L}}(nP).$$

These bounds hold for all $\mathbb{L} \in \mathscr{L}^d$ and $P \in \mathscr{P}_0^d(\mathbb{L})$.

Ehrhart (1955b) and Scott (1976) proved that for $\mathbb{L}$-polygons P with $G_{\mathbb{L}}(\operatorname{int} P) = k \geqslant 1$,

$$V(P) \leqslant \begin{cases} 4.5 \cdot \det \mathbb{L}, & \text{if } k = 1, \\ 2(k+1) \cdot \det \mathbb{L}, & \text{if } k \geqslant 2. \end{cases}$$

The bounds are sharp. There is no direct analogue for $k = 0$. Perles, Wills and Zaks (1982) showed that there is a constant $\alpha \approx 0{,}5856$ such that for each $d \geqslant 3$ and $k \geqslant 1$ there is a $P \in \mathscr{P}^d(\mathbb{L})$ with $G_{\mathbb{L}}(\operatorname{int} P) = k$ and

$$V(P) \geqslant \frac{k+1}{d!} 2^{2^{d-\alpha}} \det \mathbb{L}. \tag{4.6}$$

The much harder problem of the existence of an upper bound for V was solved by Hensley (1983). Subsequently, Lagarias and Ziegler (1991) improved his bound and showed that

$$V(P) \leqslant k(7(k+1))^{2^{d+1}} \det \mathbb{L}, \tag{4.7}$$

whenever $P \in \mathscr{P}^d(\mathbb{L})$ and $G_{\mathbb{L}}(\operatorname{int} P) = k \geqslant 1$. Further, Lagarias and Ziegler (1991) conjecture that the examples for (4.6) are optimal and show that (4.6) and (4.7) can be generalized to rational convex polytopes. Via Blichfeldt's inequality (3.2) one obtains similar results for $G_{\mathbb{L}}(P)$.

Rabinowitz (1989) determined all convex lattice polygons (up to unimodular transformations) with at most one interior lattice point. Lattice simplices in $\mathbb{E}^3$ containing no lattice points except their vertices were studied among others by Reeve (1957), White (1964), Scarf (1985) and Reznick (1986). A relation of the Frobenius problem to "maximal lattice free bodies" was given by Scarf and Shallcross (1990) (cf. also Kannan 1989).

5. Lattice polyhedra in combinatorial optimization

As we will see in the last two sections combinatorial optimization problems are naturally related to some special lattice point problems which are usually formulated for $\mathbb{Z}^d$. Of course, most problems can also be phrased in terms of other lattices. However, from a theoretical point of view most problems studied in sections 5 and 6 are, indeed, affinely invariant. The same is true from a computational point of view if we assume that the occurring lattices $\mathbb{L}_d$ are related to $\mathbb{Z}^d$ by affine transformations A_d of size that is bounded by a polynomial in d. Hence it seems unnecessarily clumsy to formulate the following results for lattices other than the integer lattice. Therefore, with the exception of section 6.1 the results in sections 5 and 6 will be phrased in terms of $\mathbb{Z}^d$ only.

5.1. *The combinatorics of associated lattice polyhedra*

In this section we deal with the integer hull $I_{\mathbb{Z}^d}(F)$ of polyhedra F. It is easy to see that $I_{\mathbb{Z}^d}(F)$ is not a polyhedron in general. However, Meyer (1974) showed that for a rational polyhedron F, $I_{\mathbb{Z}^d}(F)$ is again a polyhedron. Hence, we will in the following only deal with rational polyhedra. In fact, we have the following stronger statement (cf. Schrijver 1986, p. 237).

Let A be an integer $m \times d$ matrix, $b \in \mathbb{Z}^m$ and let $F = \{x \in \mathbb{R}^d\colon Ax \leqslant b\}$. Further, let Δ be the maximum of the absolute values of the subdeterminants of the matrix $[A\ b]$. Then there are integer vectors $x_1, \ldots, x_n, y_1, \ldots, y_s$ with all components at most $(d+1)\Delta$ such that $I_{\mathbb{Z}^d}(F) = \operatorname{conv}\{x_1, \ldots, x_n\} + \operatorname{pos}\{y_1, \ldots, y_s\}$.

This result implies that if a rational system $Ax \leqslant b$ has an integer solution then it has one of size that is bounded by a polynomial in the size of the input A and b. Further, if c is rational and $\max\{\langle c, x\rangle\colon x \in I_{\mathbb{Z}^d}(F)\}$ is finite then the maximum is attained by a vector of polynomial size. This means that one can restrict all considerations in integer programming to polytopes P and associated lattice polytopes $I_{\mathbb{Z}^d}(P)$ and that the integer programming problem is in the class NP. For various other results along this line see Schrijver (1986).

Doignon (1973) and, independently, Bell (1977) and Scarf (1977) showed the Helly-type theorem that if each set of at most 2^d of the constraints in $Ax \leqslant b$ has an integer solution then there is an integer solution for the complete set $Ax \leqslant b$ of constraints.

The importance of integer hulls of $\mathscr{H}$-polytopes in combinatorial optimization stems from the fact that linear functionals can be maximized (or minimized) over rational polyhedra $F = \{x\colon Ax \leqslant b\}$ in polynomial time; indeed, this is the linear programming problem, Khachiyan (1979), Karmarkar (1984). Hence, if we could find in polynomial time a presentation of $I_{\mathbb{Z}^d}(F)$ in terms of linear inequalities we could solve the integer programming problem in polynomial time. This is particularly easy if b is integer and if A is totally unimodular, i.e., if each subdeterminant of A is in $\{-1, 0, 1\}$ hence, then, $F = \{x\colon Ax \leqslant b\}$ is already a lattice polyhedron, all vertices are integer. Therefore, in this case the integer programming problem is solved by any linear programming algorithm. This result is essentially a characterization of total unimodularity: Hoffman and Kruskal (1956) show the following theorem, for which a short proof was later provided by Veinott and Dantzig (1968).

An integer $m \times d$ matrix A is totally unimodular if and only if for each vector $b \in \mathbb{Z}^m$ the polyhedron $\{x: Ax \leqslant b \wedge x \geqslant 0\}$ is integer.

Note that this property holds, in particular, for network matrices; see Schrijver (1986, p. 272). As a consequence of Seymour's (1980) decomposition theorem total unimodularity can be tested in polynomial time; for more details see Schrijver (1986, p. 290).

Unfortunately, the situation is much worse, in general. Edmonds (1965) showed that there is no polynomial p such that for each $\mathscr{H}$-polyhedron $F = \{x \in \mathbb{R}^d: Ax \leqslant b\}$, the associated polyhedron $I_{\mathbb{Z}^d}(F)$ has at most $p(\text{size}(A,b))$ facets.

Let for $j \in \mathbb{N}$, T_j denote the triangle in $\mathbb{R}^2$ given by

$$T_j = \{x = (\xi_1, \xi_2) \in \mathbb{R}^2: \phi_{2j}\xi_1 + \phi_{2j+1}\xi_2 \leqslant \phi_{2j+1}^2 - 1 \wedge \xi_1, \xi_2 \geqslant 0\},$$

where ϕ_k denotes the kth Fibonacci-number. Rubin (1970) showed that $I_{\mathbb{Z}^d}(T_j)$ has $j+3$ vertices and therefore also $j+3$ facets. Hence, there is no bound on the number of facets of the polytope $I_{\mathbb{Z}^d}(P)$ in terms of the number of facets of a polytope P.

Hayes and Larman (1983) showed that the number of vertices of the knapsack polytope $P = \{x \in \mathbb{R}^d: \langle a,x\rangle \leqslant \beta \wedge x \geqslant 0\}$, with $a \in \mathbb{N}^d$, $\beta \in \mathbb{N}$, is at most $(\log_2(2 + 2\beta/\alpha))^d$, where α is the smallest component of a. Extension of their arguments yields an $O(m^d L^d)$ upper bound for arbitrary rational polyhedra of size L with m facets in fixed $\mathbb{R}^d$. (For some related results see Shevchenko 1981.) Strengthening this result Cook et al. (1992) showed that a rational polyhedron F in $\mathbb{R}^d$ presented as the set of solutions to a system of m linear inequalities of total size L can have at most $2m^d(6d^2L)^{d-1}$ vertices.

Rubin's (1970) result shows that the order in L of the above result is best possible for polygons. Recently, Bárány, Howe and Lovász (1992) proved that for any fixed $d \geqslant 2$ and for any $L \in \mathbb{N}$ there exists a rational polyhedron $F \in \mathbb{R}^d$ of size at most L and with at most $2d^2$ facets such that the number of vertices of $I_{\mathbb{Z}^d}(F)$ is at least γL^{d-1}, where γ is a constant depending only on n.

Similarly sharp results for the number of facets of associated lattice polyhedra are not known.

5.2. *Polyhedral combinatorics*

Associated lattice polyhedra and lattice polytopes play an important role for the algorithmic solution of combinatorial optimization problems. Here we will only give some paradigms to outline the concept. For more details we refer to the relevant literature on combinatorial optimization, particularly to Lawler et al. (1985) for results on the traveling salesman polytope, to Schrijver (1993a) and to Schrijver's (1993b) forthcoming book on polyhedral combinatorics. The general approach of polyhedral combinatorics is to apply linear programming techniques to combinatorial optimization problems by studying the structure of corresponding lattice polytopes. These polytopes are usually given as the convex hull of a finite set of points. Hence, in order to apply linear programming techniques we have to find a (possibly small) system of linear inequalities that presents the polytope P. This might seem a rather strange detour since, obviously, a linear functional can be maximized over a finite point set by evaluating it for each such point and taking the maximum. However, typically the number of points is exponential. This is also true for the number of facets but on the other hand an optimal vertex v of P is already characterized by $\dim P$ of its facets that are incident with v. Hence the underlying philosophy is that it might be possible to find an optimum solution without having to consider too many facets. Focusing on its theoretical aspects, polyhedral combinatorics is mainly concerned with the combinatorial and geometric study of special lattice polytopes. In many cases these polytopes evolve as follows.

Let $E = \{e_1, \ldots, e_n\}$, and let $\mathscr{S}$ be a subset of 2^E, the set of all subsets of E. With every $S \in 2^E$ we associate the incidence vector $x^S = (\xi_1^S, \ldots, \xi_n^S)$, where

$$\xi_i^S = \begin{cases} 0 & \text{if } e_i \notin S, \\ 1 & \text{if } e_i \in S, \end{cases} \qquad i = 1, \ldots, n.$$

Then we set

$$P_{\mathscr{S}} = \operatorname{conv}\{x^S \in \mathbb{R}^n: S \in \mathscr{S}\}.$$

To give a concrete example suppose $V = \{1, \ldots, d\}$ and let E denote the set of all edges of the complete graph $K_{|V|}$ on the vertex set V. Hence $n = \binom{d}{2}$. Further, let $\mathscr{S}^d$ denote the subset of 2^E of all Hamiltonian cycles (tours) in $K_{|V|}$. Then the polytope $P_{\mathscr{S}^d}$ is called the symmetric traveling salesman polytope. If we do the same in the complete directed graph on V and consider the set $\mathscr{D}^d$ of all directed Hamiltonian cycles then $P_{\mathscr{D}^d}$ is the asymmetric traveling salesman polytope. In principle, solving the symmetric or asymmetric traveling salesman problem is just a linear programming problem over $P_{\mathscr{S}^d} \subset \mathbb{R}^{\binom{d}{2}}$ or $P_{\mathscr{D}^d} \subset \mathbb{R}^{d(d-1)}$. Clearly, these polytopes are the convex hull of a suitable subset of the vertices of the cube $[0,1]^{\binom{d}{2}}$, $[0,1]^{d(d-1)}$, respectively.

The dimensions of the traveling salesman polytopes are for $d \geqslant 3$:

$$\dim P_{\mathscr{S}^d} = \tfrac{1}{2}d(d-3), \qquad \dim P_{\mathscr{D}^d} = (d-1)^2 - d,$$

cf. Grötschel and Padberg (1985).

Padberg and Rao (1974) further showed that for $d \geqslant 6$, the (graph-theoretic) diameter of the 1-skeleton of $P_{\mathscr{D}^d}$ is 2; for $3 \leqslant d \leqslant 5$ it is 1. (This does not imply that two pivot operations of the simplex algorithm would suffice; however, $2d-1$ pivot steps do suffice, Padberg and Rao 1974.)

There are many facets known for the traveling salesman polytopes – too many to be described here, see Grötschel and Padberg (1985). Some classes can be used in cutting plane approaches. However, there are also facets which are defined by properties that – unless P = NP – cannot be checked in polynomial-time. For a survey on algorithmic implications of polyhedral theory see Grötschel and Padberg (1985).

6. Computational complexity of lattice point problems

6.1. *Algorithmic problems in geometry of numbers*

Essentially any result in the geometry of numbers can be studied from an algorithmic point of view. Here, we only give some examples and refer to Schrijver (1986), Kannan (1987b), Grötschel, Lovász and Schrijver (1988) and chapter 3.1 by Gruber for further studies.

Recall, first, the definition of a reduced basis of a given lattice $\mathbb{L}$. Let $(v_1,\ldots,v_d)$ be an ordered basis of $\mathbb{L}$, let $(v_1^*,\ldots,v_d^*)$ be its Gram–Schmidt orthogonalization and let

$$v_i = \sum_{j=1}^{i} \mu_{ij} v_j^*, \quad i = 1,\ldots,d.$$

$(v_1,\ldots,v_d)$ is called *reduced* if the following two conditions hold:

$$|\mu_{ij}| \leqslant \tfrac{1}{2} \quad \text{for } 1 \leqslant j < i \leqslant d,$$

$$\|v_{i+1}^* + \mu_{i+1,i} v_i^*\|^2 \geqslant \tfrac{3}{4}\|v_i^*\|^2 \quad \text{for } i = 1,\ldots,d-1.$$

One of the fundamental results in the algorithmic geometry of numbers is Lovász' basis reduction algorithm which first appeared in Lenstra, Lenstra and Lovász (1982), (cf. also Grötschel, Lovász and Schrijver 1988, Gruber and Lekkerkerker 1987, and Kannan 1987b).

There is a polynomial-time algorithm that, for any given linearly independent vectors $v_1,\ldots,v_d \in \mathbb{Q}^d$, finds a reduced basis of the lattice $\mathbb{L}$ spanned by $v_1,\ldots,v_d$.

Needless to say how important basis reduction is in geometry of numbers, hence this result has numerous implications. For example, it leads to a polynomial-time algorithm for factorization of polynomials and to an approximate polynomial-time algorithm for simultaneous Diophantine approximation, Lenstra, Lenstra and Lovász (1982), but it can also be used in cryptography, Shamir (1984). A generalization of the basis reduction algorithm to Minkowski geometry was given by Lovász and Scarf (1990).

The basis reduction algorithm was, in particular, applied by Grötschel, Lovász and Schrijver (1988, p. 149) to give the following algorithmic version of Minkowski's fundamental theorem (2.1).

Let $v_1,\ldots,v_d \in \mathbb{Q}^d$ and let $\mathbb{L} \in \mathcal{L}^d$ be the lattice spanned by $v_1,\ldots,v_d$. Let $K \in \mathcal{K}_0^d$, let $r,R \in \mathbb{Q}$ such that $r\mathbb{B}^d \subset K \subset R\mathbb{B}^d$ and suppose

$$V(K) \geqslant \frac{2^{d(d-1)/4}\pi^{d/2}}{\Gamma(d/2+1)}(d+1)^d \det \mathbb{L}.$$

Further, let $\varepsilon \in \mathbb{Q}, \varepsilon > 0$. Then there is a rational arithmetic algorithm for finding a nonzero lattice point in K that uses as a subroutine a procedure that, given a point $x \in \mathbb{Q}^d$ either asserts that $x \in K + \varepsilon\mathbb{B}^d$ or asserts that

$$x \notin \mathrm{cl}\Big(\mathbb{R}^d \setminus \big((\mathbb{R}^d \setminus K) + \varepsilon\mathbb{B}^d\big)\Big).$$

Assuming that a call of the subroutine has unit complexity, the algorithm runs in time that is bounded by a polynomial in $d, \log r, \log R$ and $\log \varepsilon$.

Observe that the requirement for the volume of K is much stronger than in (2.1), and it is not known whether there is a similar algorithmic version of Minkowski's theorem with its original bound 2^d.

For more results on the algorithmic theory of convex bodies see Grötschel, Lovász and Schrijver (1988) and chapter 2.7 by Gritzmann and Klee. For some interesting recent algorithmic results concerning successive minima see Kannan, Lovász and Scarf (1990).

6.2. *NP-hard problems*

Karp (1972) showed that the feasibility problem of integer programming is NP-complete:

> Given a rational $m \times d$ matrix A and a vector $b \in \mathbb{Q}^m$, is there a point $x \in \mathbb{Z}^d$ such that $Ax \leqslant b$?

The problem remains NP-hard if all entries of A and b are in $\{0,1\}$ and x is required to be a 0–1-vector. Hence, integer programming is NP-hard in the strong sense, even over rational polytopes. Even the following variant of the knapsack problem is NP-hard.

> Given $a \in \mathbb{Q}^d$ with $a \geqslant 0$ and $\beta \in \mathbb{Q}$; does there exist a vector $x \in \{0,1\}$ such that $\langle a,x\rangle = \beta$?

A transformation from this problem can be used to show that the problem of deciding whether a given $\mathcal{V}$-polytope contains an integer point is also NP-complete (Freund and Orlin 1985)

However, the situation for $\mathcal{H}$-polytopes is even worse: Papadimitriou and Yannakakis (1982) (cf. also Schrijver 1986, p. 253) showed that the following problem is NP-complete.

Given an $\mathcal{H}$-polytope P and a vector $y \in \mathbb{Q}^d$; is $y \in I_{\mathbb{Z}^d}(P)$?

The following problems are also NP-complete; Schrijver (1986, p. 254), Papadimitriou (1978):

Let P be an $\mathcal{H}$-polytope. Given $y \in P \cap \mathbb{Z}^d$, is y not a vertex of $I_{\mathbb{Z}^d}(P)$? Given $x_1, x_2 \in P \cap \mathbb{Z}^d$, are x_1 and x_2 non-adjacent vertices of $I_{\mathbb{Z}^d}(P)$?

The last problem remains NP-complete even when $P \subset (0,1)^d$. In this case x_1, x_2 are vertices of P. Similar difficulties arise when a hyperplane is to be tested for being a facet of $I_{\mathbb{Z}^d}(F)$. Even the problem of deciding whether an $\mathcal{H}$-polytope has only integer vertices is NP-hard: Papadimitriou and Yannakakis (1990) show that the following problem is NP-complete.

Given an $\mathcal{H}$-polytope P, is $P \neq I_{\mathbb{Z}^d}(P)$?

Clearly, computing $G_{\mathbb{Z}^d}(P)$ for an $\mathcal{H}$-polytope P is NP-hard, even #P-complete (see Valiant 1979). This does, however, not directly imply that the problem of counting the number of lattice points of a lattice polytope is also #P-hard since the restriction to polytopes P with integer vertices changes the problem. However, as it is known (Valiant 1979) the problem of determining the number of perfect matchings in a bipartite graph is #P-complete and on the other hand the node–edge incidence matrix of a bipartite graph is totally unimodular. This implies that the corresponding polytope has 0–1-vertices. Hence the problem of counting the number of lattice points of a lattice $\mathcal{H}$-polytope is, indeed, #P-complete. For integer $\mathcal{V}$-polytopes the #P-hardness can be inferred from Ehrhart's result (4.4) and the fact that, by Dyer and Frieze (1988), computing the volume of a rational polytope is #P-hard, Dyer, Gritzmann and Hufnagel (1993). The same is true for the problem of counting the number of lattice points of a lattice zonotope, Dyer, Gritzmann and Hufnagel (1993).

The problem of counting lattice points of polytopes in fixed dimensions was studied by various authors; it can be solved in polynomial time for $d \leqslant 4$ (Zamanskii and Cherkasskii 1983, 1985, Dyer 1991).

For various additional results on the computational complexity of integer programming etc. see Schrijver (1986) and chapter 2.8 by Burkard.

6.3. *Polynomial time solvability*

As we have seen, counting the number of lattice points in an $\mathcal{H}$-polytope is a hard problem. Cook et al. (1992), however, describe an algorithm that determines the number of integer points in a polyhedron $\{x\colon Ax \leqslant b\}$ to within a multiplicative factor of $1+\varepsilon$ in time polynomial in m, L and $1/\varepsilon$ when the dimension d is fixed. This result has to be seen in connection with results of Zamanskii and Cherkasskii (1985) but also in conjunction with Lenstra's (1983) polynomial-time algorithm for integer programming in fixed dimension. Lenstra (1983) proved the following remarkable theorem.

For fixed $d \in \mathbb{N}$ there is a polynomial-time algorithm for the following problem: given $m \in \mathbb{N}$, a rational $m \times d$ matrix A and $b \in \mathbb{Q}^m$, find an integer solution of the system $Ax \leqslant b$ or decide that there is no such solution.

This result implies, in particular, that the integer programming problem can be solved in polynomial time when the dimension is fixed. Based on work of Lenstra, Lenstra and Lovász (1982) and making crucial use of Minkowski's theorem (2.1) Kannan (1987a) improved Lenstra's complexity bound by giving an integer programming algorithm whose complexity depends on the dimension d as $d^{O(d)}$. The problem of counting the number of lattice points of lattice polytopes in fixed $\mathbb{R}^d$ has some relevance for problems in computer algebra, cf. Gritzmann and Sturmfels (1993).

6.4. *The complexity of computing upper and lower bound functionals*

In sections 2, 3 and 4.3 we have stated various inequalities involving $G_{\mathbb{L}}$ and some other functionals. In view of the hardness of counting lattice points, it is natural to ask whether these functionals can be computed easily. It turns out that functionals like the volume, the surface area, the diameter and the width can be computed for rational ($\mathcal{V}$- or $\mathcal{H}$-presented) polytopes in polynomial-time, if the dimension is fixed. For d being part of the input, computing the volume or surface area is #P-hard, Dyer and Frieze (1988) (see also Khachiyan 1992). The complexity of inner and outer radii like diameter, width, inradius and circumradius in finite-dimensional normed spaces has been studied by Gritzmann and Klee (1993) – and we refer to their paper for the precise statement (and some applications) of the following results. In Euclidean space the situation is roughly as follows: the diameter (and hence the circumradius) problem is NP-hard even for $\mathcal{H}$-presented parallelotopes (centered at the origin); see also Bodlaender et al. (1990). The width problem is NP-hard already for ($\mathcal{H}$- or $\mathcal{V}$-presented) simplices, the inradius problem is NP-hard already for $\mathcal{V}$-presented cross-polytopes. The following radii can be computed or at least approximated in polynomial time: the inradius for $\mathcal{H}$-polytopes, the width for symmetric $\mathcal{H}$-polytopes, the diameter and the circumradius for $\mathcal{V}$-polytopes. In ℓ_1 and in ℓ_∞ spaces some of the radius computations become easier.

The complexity of computing the lattice width of a polytope as used in (3.12) and (3.13) has not been determined, yet.

Let us close with the discouraging result of Cook et al. (1992) that for any polynomial $p : \mathbb{Z} \to \mathbb{Z}$ the following problem is NP-hard:

Given an $\mathcal{H}$-polytope P in $\mathbb{R}^d$; find positive integers α, β such that $\alpha \leqslant G_{\mathbb{Z}^d}(P) + 1 \leqslant \beta$ and $\beta \leqslant 2^{p(d)}\alpha$.

This result shows a dilemma for the problem of computing lower and upper bound functionals for the lattice point enumerator in variable dimensions: either the gap between the lower and the upper bound grows super-exponentially in the dimension or at least one of the two functionals is itself hard to compute.

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CHAPTER 3.6

Valuations and Dissections

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Contents

HANDBOOK OF CONVEX GEOMETRY
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Introduction

The concept of a valuation lies at the very heart of geometry, as does the closely related concept of a dissection. Indeed, the word "geometry" means "measuring earth", and involves the notion of dividing up plots of ground and calculating their areas. The Greek definition of area was built on the idea of dissecting planar polygonal regions into triangles, whose areas are given by the familiar formula, although a rigorous proof that polygonal regions of the same area admit equidissections into congruent triangles had to wait until comparatively recently (see section 5.3).

The comparable 3-dimensional problem of comparing volumes proved much more difficult to the Greeks. The technique finally adopted, that of the method of exhaustion due to Eudoxus Archimedes (though first applied less formally in the plane by Antiphon), involves a limiting process. Whether an elementary dissection argument would also work here was asked by Gauss, if not earlier, but it was only with Dehn's negative solution of Hilbert's third problem (see section 4.5) that the problem was, at least partially, settled. What is now understood by Hilbert's third problem is that of finding necessary and sufficient conditions on polytopal regions for such equidissections (under various restrictions on the motions allowed) to be possible.

In this article, we shall discuss the current state of knowledge of valuations and dissections. An earlier survey article (McMullen and Schneider 1983) covered the same ground as this, and we shall draw on it extensively. There is, though, one striking difference. Recently, the concept of the polytope algebra, introduced by McMullen (1989), has done for general translation invariant valuations what the earlier algebra of polytopes of Jessen and Thorup (1978) did for translation invariant simple valuations. (The later work did, however, rely heavily on the earlier.) We shall therefore base our treatment of the abstract foundation of the theory of valuations on the polytope algebra (see section 3).

1. The basic theory

We begin with a discussion of the basic theory of valuations and dissections, including the background notions of Euclidean space which we employ. It is worth remarking, though, that much of this basic theory works in the more general context of a finite-dimensional linear space over an arbitrary ordered field, and we only choose the Euclidean context for simplicity of treatment (see, in particular, McMullen 1989).

1.1. *Euclidean notions*

We shall work, for the most part, in d-dimensional Euclidean space $\mathbb{E}^d$. A general vector in $\mathbb{E}^d$ is written $x = (\xi_1, \ldots, \xi_d)$, where $\xi_j \in \mathbb{R}$, the real numbers, for $j = 1, \ldots, d$. We endow $\mathbb{E}^d$ with the *inner product* $\langle x, y \rangle := \sum_{j=1}^{d} \xi_j \eta_j$, where

$y = (\eta_1, \ldots, \eta_d)$, and corresponding *norm* $\|x\| := \sqrt{\langle x, x\rangle}$. Occasionally, though, we shall refer to a vector space V over an arbitrary (not necessarily Archimedean) ordered field $\mathbb{F}$. This too can be given an inner product, but there will not usually be a norm, because $\mathbb{F}$ will not be square-root-closed. However, the Gram–Schmidt orthogonalization process will enable us to define orthogonal projection onto subspaces of V.

We shall be as much concerned with affine as linear properties of $\mathbb{E}^d$; useful references here are Grünbaum (1967) or McMullen and Shephard (1971). In particular, the *vector* or *Minkowski sum* of $S, T \subseteq \mathbb{E}^d$ is defined by

$$S + T := \{x + y \mid x \in S, y \in T\},$$

the *translate* of S by $t \in \mathbb{E}^d$ is $S + t := S + \{t\}$, and the *scalar multiple* of S by $\lambda \in \mathbb{R}$ is

$$\lambda S := \{\lambda x \mid x \in S\}.$$

The *Hausdorff distance* $\rho(S,T)$ between two non-empty compact subsets S, T of $\mathbb{E}^d$ is defined by

$$\rho(S,T) := \min\{\rho \geqslant 0 \mid S \subseteq T + \rho B, T \subseteq S + \rho B\},$$

where $B = B^d := \{x \in \mathbb{E} \mid \|x\| \leqslant 1\}$ is the unit ball in $\mathbb{E}^d$. Continuity for functions on classes of compact subsets of $\mathbb{E}^d$ will usually be with respect to the Hausdorff metric.

1.2. Valuations

Let $\mathcal{S}$ be a family of sets in $\mathbb{E}^d$. We call a function φ on $\mathcal{S}$, taking values in some Abelian group, a *valuation*, or say that φ is *additive*, if $\varphi(S \cup T) + \varphi(S \cap T) = \varphi(S) + \varphi(T)$ whenever $S, T, S\cup T, S \cap T \in \mathcal{S}$. If $\emptyset \in \mathcal{S}$, we shall always suppose that $\varphi(\emptyset) = 0$. Our families $\mathcal{S}$ will usually be *intersectional*, which means that $S \cap T \in \mathcal{S}$ whenever $S, T \in \mathcal{S}$. For an intersectional family $\mathcal{S}$, we write $U\mathcal{S}$ for the family of finite unions of members of $\mathcal{S}$, and $\overline{U\mathcal{S}} := \{S \setminus T \mid S, T \in U\mathcal{S}\}$. Particular examples of intersectional families in $\mathbb{E}^d$ are the family $\mathcal{K}^d$ of compact convex sets or *convex bodies*, the family $\mathcal{P}^d$ of (convex) polytopes, and the family $\mathcal{C}^d$ of convex cones with apex the origin (zero vector) o. The family $U\mathcal{K}^d$ is called by Hadwiger (1957) the *convex ring*, although we shall avoid the term, and the members of $U\mathcal{P}^d$ are called *polyhedra*. In addition, a subscript $*$ on $\mathcal{K}^d$ or $\mathcal{P}^d$ will denote the subset of non-empty members of the appropriate family.

There are many examples of valuations, which we shall discuss in more detail in section 2. For now, let us note that the restriction of a measure to any family in $\mathbb{E}^d$ will yield a valuation on that family; in particular, volume is a valuation in every dimension. Surface area is also a valuation, as is the Euler characteristic on the three families just introduced. All these valuations φ are *translation invariant*, in that $\varphi(S + t) = \varphi(S)$ for each appropriate S and each $t \in \mathbb{E}^d$ (the definition has no force for the family $\mathcal{C}^d$). More generally, we shall say that a valuation φ on a class $\mathcal{S}$ which is permuted by a group G of affinities of $\mathbb{E}^d$ is *G-invariant* if $\varphi(\Phi S) = \varphi(S)$ for each $S \in \mathcal{S}$ and $\Phi \in G$. Then these valuations just mentioned are actually D-invariant, where D is the group of all isometries of $\mathbb{E}^d$.

An important subclass of valuations consists of those that are simple. In general, if L is a linear subspace of $\mathbb{E}^d$, we say that a valuation φ on a family $\mathcal{S}$ is *L-simple* if $\varphi(S) = 0$ whenever $S \in \mathcal{S}(L) := \{S \in \mathcal{S} \mid S \subseteq L\}$ satisfies $\dim S < \dim L$ (our class here will consist of convex sets, for which the dimension can be defined). The term *simple* alone will mean $\mathbb{E}^d$-simple.

A function φ on a suitable family $\mathcal{S}$ is called *Minkowski additive* if $\varphi(S + T) = \varphi(S) + \varphi(T)$ for all $S, T \in \mathcal{S}$. Then there is a fundamental relation, due to Sallee (1966).

Lemma 1.1. *If $S, T, S \cup T \in \mathcal{K}^d_*$, then*

$$(S \cup T) + (S \cap T) = S + T.$$

Hence every Minkowski additive function on $\mathcal{K}^d_*$ is a valuation.

Since $\mathcal{K}^d_*$ is a semigroup under Minkowski addition which has a cancellation law, we can imbed $\mathcal{K}^d_*$ in an Abelian group; we can thus interpret this observation as saying that the identity map from $\mathcal{K}^d_*$ into itself is a valuation.

A closely related example of a valuation is the *support functional* $h(K, \cdot)$, defined by

$$h(K,u) = \max\{\langle x, u\rangle \mid x \in K\}$$

for $K \in \mathcal{K}^d_*$ and $u \in \mathbb{E}^d$, since Minkowski addition of convex bodies corresponds to addition of their support functionals. (By the way, we might remark here that the Hausdorff distance ρ is given by

$$\rho(K,L) = \max\{|h(K,u) - h(L,u)| \mid u \in \Omega\},$$

where $\Omega = \Omega^{d-1} = \{x \in \mathbb{E}^d \mid \|x\| = 1\}$ is the *unit sphere*.) If $C \in \mathcal{K}^d_*$ is fixed, then the map φ_C from $\mathcal{K}^d_*$ into itself defined by $\varphi_C(K) := K + C$ is a valuation, since

$$(S \cup T) + C = (S + C) \cup (T + C)$$

for all $S, T \in \mathcal{K}^d_*$, while

$$(S \cap T) + C = (S + C) \cap (T + C)$$

provided that $S \cup T \in \mathcal{K}^d_*$ also (see Hadwiger 1957, p. 144). This remark will play an important role later. This observation is also an immediate consequence of the following.

Lemma 1.2. *If $f : \mathcal{K}^d \to \mathcal{K}^d$ is a map such that $f(S \cup T) = f(S) \cup f(T)$ and $f(S \cap T) = f(S) \cap f(T)$ if $S, T, S \cup T \in \mathcal{K}^d$, then $\varphi \circ f$ is a valuation on $\mathcal{K}^d$ whenever φ is.*

For polytopes, a variant notion of valuation is useful. We call a map φ on $\mathcal{P}^d$ a *weak valuation* if $\varphi(P) + \varphi(P \cap H) = \varphi(P \cap H^+) + \varphi(P \cap H^-)$ whenever $P \in \mathcal{P}^d$ and

H is a hyperplane bounding the two closed half-spaces H^+ and H^-. Sallee (1968) (see also Groemer 1978) has shown:

Lemma 1.3. *Every weak valuation on $\mathcal{P}^d$ gives rise to a valuation.*

However, examples show that weak valuations on $\mathcal{K}^d$ need not arise from valuations in this way (see McMullen and Schneider 1983, p. 173).

1.3. Extensions

While it was not important for Jessen and Thorup (1978), it turns out to be vital for the approach of McMullen (1989) to be able to extend valuations on $\mathcal{S}$ to valuations on $\overline{U\mathcal{S}}$, for $\mathcal{S} = \mathcal{P}^d$ or $\mathcal{K}^d$. The quickest way to see that such extensions are possible follows Groemer (1977a) (we slightly modify his argument). If $S \in \mathcal{S}$, its *characteristic function* $S^\dagger$ is given by

$$S^\dagger(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

The Abelian group of integer valued functions generated by the functions $S^\dagger$ with $S \in \mathcal{S}$ is denoted $X\mathcal{S}$. Then we have:

Lemma 1.4. *A valuation on $\mathcal{S}$ admits a unique extension to $\overline{U\mathcal{S}}$.*

With $\overline{U\mathcal{S}}$ replaced by $U\mathcal{S}$, the result is originally due to Volland (1957) (see also Perles and Sallee 1970). Since the basic idea of the proof is important, as well as simple, we shall outline it. First, note that, whenever $S, T \subseteq \mathbb{E}^d$, intersection is given in $X\mathcal{S}$ by

$$(S \cap T)^\dagger = S^\dagger T^\dagger$$

(the product is ordinary multiplication of functions), while complementation is given by

$$(\mathbb{E}^d \setminus S)^\dagger = 1 - S^\dagger.$$

(If necessary, we adjoin $\mathbb{E}^d$ to our intersectional family $\mathcal{S}$.) Thus union is given by

$$1 - (S_1 \cup \cdots \cup S_k)^\dagger = (1 - S_1^\dagger) \cdots (1 - S_k^\dagger)$$

(note that 1 occurs on both sides of the equation, so adjoining $\mathbb{E}^d$ to $\mathcal{S}$ is only a convenience), and this results in the *inclusion–exclusion principle*:

Lemma 1.5. *The extension of a valuation φ on $\mathcal{S}$ to $U\mathcal{S}$ is given by*

$$\varphi(S_1 \cup \cdots \cup S_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{i(i)<\cdots<i(j)} \varphi(S_{i(1)} \cap \cdots \cap S_{i(j)}),$$

with $S_1, \ldots, S_k \in \mathcal{S}$, and further to $\overline{U\mathcal{S}}$ by

$$\varphi(A \setminus B) = \varphi(A) - \varphi(A \cap B),$$

with $A, B \in U\mathcal{S}$.

1.4. Dissections

Let $\mathcal{S}$ be a subfamily of $U\mathcal{Q}^d$, where $\mathcal{Q}^d$ consists of the *polyhedral sets*, or intersections of finitely many closed half-spaces, in $\mathbb{E}^d$. (We shall therefore consider the unions of polyhedral sets from the outset; the particular subfamilies we shall usually be concerned with are those obtained from the polytopes or polyhedral cones.) A *dissection* of $P \in \mathcal{S}$ is an expression of the form

$$P = P_1 \uplus \cdots \uplus P_k$$

with $P_1, \ldots, P_k \in \mathcal{S}$, which means that $P = P_1 \cup \cdots \cup P_k$, with $\operatorname{int}(P_i \cap P_j) = \emptyset$ whenever $i \neq j$; that is, a dissection is a union of sets with pairwise disjoint interiors.

Let G be a group of affinities of $\mathbb{E}^d$, which permutes the members of $\mathcal{S}$. We say that two members P, Q of $\mathcal{S}$ are *G-equidissectable*, written $P \approx_G Q$, if there are dissections $P = P_1 \uplus \cdots \uplus P_k$ and $Q = Q_1 \uplus \cdots \uplus Q_k$, and elements $\Phi_1, \ldots, \Phi_k \in G$, such that $Q_i = \Phi_i P_i$ for $i = 1, \ldots, k$. We say that $P, Q \in \mathcal{S}$ are *G-equicomplementable*, written $P \sim_G Q$, if there are $P', P'', Q', Q'' \in \mathcal{S}$ such that $P'' = P \uplus P'$, $Q'' = Q \uplus Q'$, and $P' \approx_G Q'$, $P'' \approx_G Q''$. We shall discuss equidissectability and equicomplementability, and the relationship between them, in section 4.4 and section 4.5 below. However, the reader is surely familiar with various games and puzzles, such as tangrams and pentominoes, which involve dissections. It is clear from the definitions that, if $\mathcal{S}$ is such a class as above, and φ is a G-invariant simple valuation on $\mathcal{S}$, then $\varphi(P) = \varphi(Q)$ whenever $P \approx_G Q$ (or $P \sim_G Q$); the core of Hilbert's third problem is to find sufficient conditions for G-equidissectability (or equicomplementability) in terms of suitable families of G-invariant simple valuations.

2. The classical examples

We shall now more formally introduce the classical examples of valuations. In the first two subsections, we consider volume, moment, and related valuations, while in the third, we treat the lattice-point enumerator, which differs in a number of important respects from the other examples.

2.1. Volume and derived valuations

Any measure on a ring of subsets of $\mathbb{E}^d$ which contains $\mathcal{K}^d$ which is finite on $\mathcal{K}^d$ will yield a valuation on $\mathcal{K}^d$. In particular, Lebesgue measure gives the ordinary volume V; this is a simple valuation. However, there is a quite different approach to volume, beginning with an elementary notion of volume of polytopes (which can be characterized axiomatically; see section 5.3 below), and extending this to general convex bodies by continuity arguments. For further details, see Hadwiger (1957), Boltyanskiĭ (1978) or Böhm and Hertel (1980).

We introduce a little notation. We shall write $\kappa_d := V(B^d)$ for the volume of the unit ball in $\mathbb{E}^d$, and σ for Lebesgue measure on Ω^{d-1}, so that $\sigma(\Omega^{d-1}) = d\kappa_d =: \omega_{d-1}$ is the area of the unit sphere.

As we remarked in section 1.2, $\psi(K) := \varphi(K + C)$ is a valuation if φ is, and if $C \in \mathcal{K}^d_*$ is fixed. It is easy to show that, if $K_1,\ldots,K_k \in \mathcal{K}^d_*$ and $\lambda_1,\ldots,\lambda_k \geqslant 0$, then the volume of $\lambda_1K_1 + \cdots + \lambda_kK_k$ is a homogeneous polynomial in $\lambda_1,\ldots,\lambda_k$ of degree d, say

$$V(\lambda_1K_1 + \cdots + \lambda_kK_k) = \sum \lambda_{i(1)}\cdots\lambda_{i(d)}V(K_{i(1)},\ldots,K_{i(k)}),$$

with $V(K_{i(1)},\ldots,K_{i(k)})$ symmetric in the indices, and depending only on $K_{i(1)},\ldots, K_{i(k)}$. These coefficients are known as *mixed volumes*. We often write this expression in the form

$$V(\lambda_1K_1 + \cdots + \lambda_kK_k) = \sum \binom{d}{r_1\cdots r_k}\lambda_1^{r_1}\cdots\lambda_k^{r_k}V(K_1,r_1;\ldots;K_k,r_k),$$

where

$$\binom{d}{r_1\cdots r_k} := \begin{cases} \dfrac{d!}{r_1!\cdots r_k!} & \text{if } \sum_{j=1}^k r_j = d \text{ and } r_j \geqslant 0\ (j=1,\ldots,k), \\ 0 & \text{otherwise,} \end{cases}$$

is the *multinomial coefficient*. Here, and elsewhere in such expressions, we use the abbreviation

$$\varphi(K_1,r_1;\ldots;K_k,r_k) = \varphi(\underbrace{K_1,\ldots,K_1}_{r_1 \text{ times}},\ldots,\underbrace{K_k,\ldots,K_k}_{r_k \text{ times}}),$$

with $r_1 + \cdots + r_k = m$, for any function φ of m variables; we also write

$$\varphi(K_1,\ldots,K_p,\mathcal{C}) = \varphi(K_1,\ldots,K_p,L_{p+1},\ldots,L_m),$$

when $\mathcal{C} = (L_{p+1},\ldots,L_m)$ is a fixed $(m-p)$-tuple. With this convention, we then obtain

$$V(\lambda_1K_1 + \cdots + \lambda_kK_k,p;\mathcal{C}) = \sum \binom{p}{r_1\cdots r_k}\lambda_1^{r_1}\cdots\lambda_k^{r_k}V(K_1,r_1;\ldots;K_k,r_k;\mathcal{C}),$$

with $m = d$ in the above.

Since the mapping

$$K \mapsto V(\lambda K + \lambda_{p+1}K_{p+1} + \cdots + \lambda_dK_d)$$

is a valuation on $\mathcal{K}^d$ for any $p \in \{1,\ldots,d\}$ and any $(d-p)$-tuple $\mathcal{C} = (K_{p+1},\ldots,K_d)$, it follows by considering the coefficient of $(p!/d!)\lambda^p\lambda_{p+1}\cdots\lambda_d$ that the mapping φ, given by

$$\varphi(K) := V(K,p;\mathcal{C})$$

is also a valuation. In particular, the jth *quermassintegral* W_j, defined by

$$W_j(K) := V(K,d-j;B,j),$$

is a valuation, and corresponds to the *Steiner parallel formula*

$$V(K + \lambda B) = \sum_{j=0}^{d}\binom{d}{j}\lambda^jW_j(K).$$

We also define the normalized quermassintegral, or *intrinsic r-volume* V_r by

$$\kappa_{d-r}V_r := \binom{d}{r}W_{d-r};$$

since $W_d(K) = \kappa_d$ for all $K \in \mathcal{K}^d_*$, it follows that $V_0(K) = 1\,(= \chi(K)$, the *Euler characteristic*, see below) for all such K. More generally, the normalization, due to McMullen (1975a), is such that $V_r(K)$ is the ordinary r-dimensional volume of K if $\dim K = r$; note that V_r is *homogeneous of degree r*, in that $V_r(\lambda K) = \lambda^rV_r(K)$ for all K and all $\lambda \geqslant 0$. Further, $S(K) := dW_1(K) = 2V_{d-1}(K)$ is the *surface area* of K.

While the valuation property of the quermassintegrals was pointed out by Blaschke (1937), and played an important role in the work of Hadwiger (see section 5), that of the general mixed volumes is, strangely, not mentioned in any standard textbooks to date.

Without going into details, let us also remark that the intrinsic volumes occur in various integral-geometric formulae, which average the intrinsic volumes of sections (projections) of a convex body by affine (on linear) subspaces of $\mathbb{E}^d$. [See Hadwiger (1957) or Santaló (1976, sections 13, 14) for the exact results and their proofs, and Hadwiger (1956, 1957) for generalizations.]

The extension property of valuations, discussed in section 1.3, enables the intrinsic volumes to be extended to $\overline{U\mathcal{K}^d}$. In particular, the extension of V_0 is the *Euler characteristic* χ. The Euler characteristic is, of course, of considerable importance in other branches of mathematics, and so a substantial literature has been devoted to it. Within convexity, we mention the inductive construction (in $U\mathcal{K}^d$, and on dimension) due to Hadwiger (1955a); this idea was further explored in Hadwiger (1957, 1959, 1968b, 1969c) and Hadwiger and Mani (1972) (see also Hadwiger 1974a, concerning planar polygons). For applications to combinatorial geometry, see (in addition) Hadwiger (1947), Klee (1963), and Rota (1964, 1971); the last paper provided a very general theoretical treatment. The extension of χ to relatively open polytopes was considered by Lenz (1970) and Groemer (1972); see also Hadwiger (1969c, 1973) for special cases, and Groemer (1973, 1974, 1975) for further generalizations.

The extension of the higher intrinsic volumes has received somewhat less attention. However, the Gauss–Bonnet theorem describes the Euler characteristic of surfaces in local terms. The Lipschitz–Killing curvatures, though initially defined in analytic (differential geometric) terms, turn out in the piecewise linear case just to be the extended intrinsic volumes, though this has not hitherto been recognized (see, for example, Cheeger, Müller and Schrader 1984 and Budach 1989; the desired connexion is most easily obtained from the latter).

Closely related to surface area are the *mixed area functions*. Using their properties of Minkowski additivity and uniform continuity in each argument, the Riesz representation theorem shows that, for given $K_1,\ldots,K_{d-1} \in \mathcal{K}_*^d$, there is a unique (positive) measure $S(K_1,\ldots,K_{d-1};\cdot)$ on the Borel subsets of the unit sphere Ω, such that

$$V(L,K_1,\ldots,K_{d-1}) = \frac{1}{d}\int_\Omega h(L,u)\,\mathrm{d}S(K_1,\ldots,K_{d-1};u),$$

where $h(L,\cdot)$ is the support functional of $L \in \mathcal{K}_*^d$. This measure is due, independently, to Aleksandrov (1937) and Fenchel and Jessen (1938); see also Busemann (1958) (it is often called a *Fenchel–Jessen area measure*). In particular, one writes

$$S_p(K;\cdot) := S(K,p;B,d-p-1;\cdot)$$

for the *pth order area function* of K; we note that $S_p(K;\Omega) = dW_{d-p}(K)$.

If ω is a Borel set in Ω, then $S(K;\omega)$ is the ($(d-1)$-dimensional Hausdorff) measure of that part of the boundary of K at which there is an outer normal vector (to a support hyperplane) lying in ω. The general mixed area function is then given by

$$\begin{aligned}&S_{d-1}(\lambda_1K_1+\cdots+\lambda_kK_k;\cdot)\\&\quad=\sum\lambda_{i(1)}\cdots\lambda_{i(d-1)}S(K_{i(1)},\ldots,K_{i(d-1)};\cdot).\end{aligned}$$

It then follows that the mapping

$$K \mapsto S(K,p,\mathscr{C};\cdot)$$

is a valuation for each $p \in \{1,\ldots,d-1\}$ and each $(d-p-1)$-tuple $\mathscr{C} = (L_{p+1},\ldots,L_{d-1})$ in $\mathcal{K}_*^d$. In particular, each S_p is a valuation on $\mathcal{K}_*^d$ (with values in the vector space of signed Borel measures on Ω); this valuation property was first pointed out and used by Schneider (1975a).

The Fenchel–Jessen area measures can be obtained from a local version of the Steiner parallel formula. Variants on these measures are the *Federer measures*, which now depend additionally on a Borel set in $\mathbb{E}^d$ itself. As initially defined by Federer (1959), they applied to more general sets than convex surfaces. We shall not give any details here, but instead refer the reader to Federer's paper, and to Schneider (1978, 1979); the latter survey article contains other references. Subsequently, these ideas were further generalized by Wieacker (1982).

Schneider (1980) has shown how to extend the area functions and curvature measures to $U\mathcal{K}^d$; among other things, he obtains a variant of the Gauss–Bonnet theorem. Related notions (to this last) occur in Banchoff (1967, 1970) (see also Schneider 1977b).

2.2. Moments and derived valuations

The *moment vector* $z(K)$ of $K \in \mathcal{K}_*^d$ is defined by

$$z(K) := \int_K x\,\mathrm{d}V(x),$$

so that, if $\dim K = d$, then $g(K) := V(K)^{-1}z(K)$ is the centre of gravity of K. There is a polynomial expansion, exactly analoguous to that for volume, of the form

$$z(\lambda_1K_1+\cdots+\lambda_kK_k) = \sum\lambda_{i(0)}\cdots\lambda_{i(d)}\,z(K_{i(0)},\ldots,K_{i(d)}),$$

whose coefficients are called *mixed moment vectors*. (In case $d=3$, this expansion was already noticed by Minkowski (1911, section 23); the general theory was developed by Schneider 1972a,b.) As for mixed volumes, for each $p \in \{1,\ldots,d+1\}$ and $(d+1-p)$-tuple $\mathscr{C}$ of (fixed) convex bodies, the mapping

$$K \mapsto z(K,p;\mathscr{C})$$

is a valuation, which is homogeneous of degree p. Other properties of mixed volumes also carry over, but observe that, while they are invariant under translation of any of their arguments, the mixed moment vectors satisfy

$$z(K_0+t,K_1,\ldots,K_d) = z(K_0,K_1,\ldots,K_d) + \frac{1}{d+1}V(K_1,\ldots,K_d)t,$$

which is a special case of translation covariance (see section 5.5 below).

2.3. The lattice point enumerator

We call P a *lattice polytope* if its vertices lie in the lattice $\mathbb{Z}^d$. Of great importance in a number of areas outside valuation theory (in, for example, the theory of numbers – see Gruber 1979) is the *lattice point enumerator* G, defined by $G(P) := \mathrm{card}(\mathbb{Z}^d \cap P)$. In fact, $G(S)$ can clearly be defined for any subset S of $\mathbb{E}^d$, and is a valuation which is invariant under *lattice translations*, that is, those in $\mathbb{Z}^d$. Various connexions with other functionals on convex bodies are discussed by Betke and Wills (1979), but we shall confine our attention here to those aspects strictly related to valuation theory.

We denote by $\mathscr{P}_L^d$ the family of lattice polytopes in $\mathbb{E}^d$. Ehrhart (1967a) showed that there is a polynomial expansion

$$G(nP) = \sum_{i=0}^{d} n^iG_i(P),$$

with n a non-negative integer, when $P \in \mathscr{P}_L^d$; the coefficients $G_i(P)$ do not depend on n. Besides making applications of this to various counting problems, Ehrhart (1967b) also discovered the *reciprocity law*

$$G(\mathrm{relint}(nP)) = (-1)^{\dim P}\sum_{i=0}^{d}(-n)^iG_i(P),$$

where again $P \in \mathcal{P}_L^d$ and (this time) n is a positive integer.

The polynomial expansions for mixed volumes have analogues for the lattice point enumerator. If $P_1, \ldots, P_k \in \mathcal{P}_L^d$ and $n_1, \ldots, n_k$ are non-negative integers, then

$$G(n_1P_1 + \cdots + n_kP_k) = \sum n_1^{r_1} \cdots n_k^{r_k} G(P_1, r_1; \ldots; P_k, r_k),$$

where the sum extends over all non-negative integers $r_1, \ldots, r_k$ satisfying $r_1 + \cdots + r_k \leqslant d$. This was independently found by Bernshtein (1976), at around the same time that more general results were discovered by McMullen (1975, 1977).

Various *weighted* lattice point numbers have been considered by Macdonald (1963, 1971); if

$$\alpha(x,P) := \lim_{\rho \searrow 0} \frac{V(P \cap (\rho B + x))}{V(\rho B)}$$

is that proportion of a sufficiently small ball centred at $x \in \mathbb{Z}^d$ which belongs to $P \in \mathcal{P}_L^d$, then

$$A(P) := \sum_x \alpha(x, P)$$

gives a simple valuation which is invariant under lattice translations. He proved an analogous polynomial expansion, and investigated the coefficients. Hadwiger (1957, p. 69), replaced the ball by the lattice-oriented cube, and obtained a criterion for equidissectability of lattice polytopes under lattice translations.

3. The polytope algebra

In section 4.1 below, we shall discuss the algebraic structure which underlies the theory of translation invariant simple valuations; this was described, independently, by Jessen and Thorup (1978) and Sah (1979). This structure is that of a vector space over $\mathbb{R}$ (or, more generally, over whichever ordered, but not necessarily Archimedean, field is the base field). It is therefore not surprising that the analogous structure underlying general translation invariant valuations behaves somewhat similarly. What is, perhaps, surprising is that the polytope algebra, introduced by McMullen (1989), has a far richer structure, and, indeed, fails to be a real graded (commutative) algebra in only one trivial respect.

In the following sections, we shall describe the polytope algebra; as usual, however, proofs of the results will be omitted (details can be found in McMullen 1989).

3.1. The algebra structure

The *polytope algebra* Π is, initially, the Abelian group with a generator $[P]$, called the *class* of P, for each $P \in \mathcal{P} := \mathcal{P}^d$ (with $[\emptyset] = 0$); these generators satisfy the

relations (V) $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever $P, Q, P \cup Q \in \mathcal{P}$ (the condition ensures that $P \cap Q \neq \emptyset$ if $P, Q \neq \emptyset$), and (T) $[P + t] = [P]$ whenever $P \in \mathcal{P}$ and $t \in \mathbb{E}^d$. Of course, (V) (which governs addition) and (T) just reflect the valuation property and translation invariance. It is clear that we have:

Lemma 3.1. *A translation invariant valuation on $\mathcal{P}^d$ (into some Abelian group) induces a homomorphism on Π, and conversely.*

We invariably employ the same symbol for the valuation and homomorphism.

We at once introduce the *multiplication*, which is induced by Minkowski addition, by (M) $[P] \cdot [Q] = [P + Q]$ for $P, Q \in \mathcal{P}^d$, and extend it by linearity to Π. In view of

$$P + (Q_1 \cup Q_2) = (P + Q_1) \cup (P + Q_2),$$

which always holds, and

$$P + (Q_1 \cap Q_2) = (P + Q_1) \cap (P + Q_2),$$

whenever $P \cup Q \in \mathcal{P}$ (compare Hadwiger 1957, section 1.2.2), the extension by linearity is compatible with addition on Π, so that Π becomes a commutative ring, with unity $1 := [o] \doteq [t]$ for any $t \in \mathbb{E}^d$ (we write $[t]$ instead of $[\{t\}]$ for brevity).

An important role is played by *dilatation* (D) $\Delta(\lambda)[P] := [\lambda P]$ for $P \in \mathcal{P}^d$ and $\lambda \in \mathbb{R}$. It is clear that each $\Delta(\lambda)$ is a ring endomorphism of Π. We can now state the main structure theorem for Π.

Theorem 3.2. *The polytope algebra Π is almost a real graded (commutative) algebra, in the following sense:*

(a) *as an Abelian group, Π admits a direct sum decomposition*

$$\Pi = \bigoplus_{r=0}^{d} \Xi_r;$$

(b) *under multiplication,*

$$\Xi_r \cdot \Xi_s = \Xi_{r+s},$$

for $r, s = 0, \ldots, d$ (with $\Xi_r = \{0\}$ for $r > d$);

(c) *$\Xi_0 \cong \mathbb{Z}$, and for $r = 1, \ldots, d$, Ξ_r is a real vector space (with $\Xi_d \cong \mathbb{R}$);*

(d) *if $x, y \in Z_1 := \bigoplus_{r=1}^{d} \Xi_r$ and $\lambda \in \mathbb{R}$, then $(\lambda x)y = x(\lambda y) = \lambda(xy)$;*

(e) *the dilatations are algebra endomorphisms of Π, and, for $r = 0, \ldots, d$, if $x \in \Xi_r$ and $\lambda \geqslant 0$, then*

$$\Delta(\lambda)x = \lambda^r x,$$

with $\lambda^0 = 1$.

While we cannot prove Theorem 3.2 here, we can outline some of the ingredients of the proof. There are three stages. First, we establish the algebra structure over the rational numbers. Next, we introduce the real vector space structure on Ξ_1, and prove a special case of the algebra property (d). Last, after proving the separation Theorem 3.11, we extend the real vector space structure and the algebra property to the rest of Π.

The subgroup (actually subring) Ξ_0 of Π generated by the class 1 of a point clearly plays an anomalous role. The isomorphism $\Xi_0 \cong \mathbb{Z}$ is obvious. Writing Z_1 for the subgroup of Π generated by all elements of the form $[P]-1$ with $P \in \mathcal{P}^d_*$, we have:

Lemma 3.3. *As an Abelian group, $\Pi = \Xi_0 \oplus Z_1$. The projection from Π onto Ξ_0 is the dilatation $\Delta(0)$, and Z_1 is an ideal in Π, with $z \in Z_1$ if and only if $\Delta(0)z = 0$.*

If $a_0, a_1, \dots, a_k \in \mathbb{E}^d$ are such that $\{a_1, \dots, a_k\}$ is linearly independent, we write

$$T(a_1, \dots, a_k) = \operatorname{conv}\{a_0, a_0 + a_1, \dots, a_0 + \dots + a_k\},$$

which is a k-simplex, and define

$$s(a_1, \dots, a_k) := [T(a_1, \dots, a_k)] - [T(a_1, \dots, a_{k-1})],$$

with $s(\emptyset) = 1$. This is the class of a partly open simplex (lacking one facet), and these classes generate Π (an arbitrary polytope can be dissected into simplices – an easy proof is given by Tverberg 1974). The key tools are the analogues of the simplex dissection theorems of Hadwiger (1957).

Lemma 3.4. *For $\lambda, \mu \geqslant 0$,*

$$\Delta(\lambda + \mu)s(a_1, \dots, a_k) = \sum_{j=0}^{k} (\Delta(\lambda)s(a_1, \dots, a_j))(\Delta(\mu)s(a_{j+1}, \dots, a_k)).$$

Lemma 3.5. *For $k \geqslant 1$ and integer $n \geqslant 0$,*

$$\Delta(n)s(a_1, \dots, a_k) = \sum_{r=1}^{k} \binom{n}{r} z_r,$$

where

$$z_r = \sum_{0=j(0)<\dots<j(r)=k} \prod_{i=1}^{r} s(a_{j(i-1)+1}, \dots, a_{j(i)})$$

is independent of n.

Lemma 3.5 shows that, if $x \in \Pi$, there exist $y_0 \in \Xi_0$ and $y_1, \dots, y_d \in Z_1$, such that

$$\Delta(n)x = \sum_{r=0}^{d} \binom{n}{r} y_r$$

for all integer $n \geqslant 0$; these y_r are unique, since, in fact,

$$y_r = \sum_{n=0}^{r} (-1)^{r-n} \binom{r}{n} \Delta(n)x.$$

If we compare

$$\Delta(n)[P] = [nP] = [P]^n = (([P]-1)+1)^n = \sum_{r=0}^{n} \binom{n}{r} ([P]-1)^r$$

with this, we see that $([P]-1)^r = 0$ whenever $P \in \mathcal{P}^d_*$ and $r > d$. We now let Z_r be the subgroup of Π generated by all elements of the form $([P]-1)^j$, with $P \in \mathcal{P}^d_*$ and $j \geqslant r$. Writing the relation above as

$$\Delta(n)([P]-1) = \sum_{k=1}^{d} \binom{n}{k} ([P]-1)^k,$$

taking the jth power (with $j \geqslant r$), and using the fact that $\Delta(n)$ is a ring endomorphism, shows that, if $x \in Z_r$, then $\Delta(n)x - n^r x \in Z_{r+1}$. It is now a short step to show that Z_1 is uniquely divisible, that is, for each $x \in Z_1$, there is a unique $y \in Z_1$, such that $x = ny$ (we recall here that $Z_{d+1} = \{0\}$).

At the next stage, we introduce the concepts of *logarithm* and *exponential*. These are defined by

$$\log(1+z) = \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} z^{k-1},$$

$$\exp z = \sum_{k \geqslant 0} \frac{1}{k!} z^k$$

(with $z^0 = 1$), for every $z \in Z_1$. The nilpotence of Z_1 (which follows from its definition, and the nilpotence of elements $[P]-1$) shows that these are well-defined inverse functions on Z_1, with the usual properties of ordinary log and exp. If $P \in \mathcal{P}^d_*$, we write $\log P := \log[P]$; setting $z = [P]-1$, we recognize $\log P$ as the coefficient of n in the expansion of $[nP] = [P]^n$ given above. Indeed, since $\log[nP] = \log([P]^n) = n\log[P]$, and $\Delta(\lambda)\log P = \log(\lambda P)$ for rational λ, we deduce:

Lemma 3.6. *For $P \in \mathcal{P}^d_*$ and rational $\lambda \geqslant 0$, $\Delta(\lambda)\log P = \lambda \log P$.*

We now invert this relation. If $P \in \mathcal{P}^d_*$, $p = \log P$ and $\lambda \geqslant 0$ is rational, then, since $\Delta(\lambda)$ is a ring endomorphism, we have

$$[\lambda P] = \Delta(\lambda)[P] = \Delta(\lambda)\exp p = \exp(\Delta(\lambda)p) = \exp(\lambda p) = \sum_{r=0}^{d} \lambda^r \cdot \frac{1}{r!} p^r.$$

The sum terminates at $r = d$, because $p^{d+1} = 0$.

For $r = 1, \ldots, d$, we now define the *rth weight space* Ξ_r to be the subgroup of Π generated by all the elements p^r, with $p = \log P$ for some $P \in \mathcal{P}^d_*$. We then obtain (after a little more work) Theorem 3.2, except that the dilatations and scalar multiplications are, as yet, only by rationals. (Along the way, it is useful to characterize Ξ_r as the set of all $x \in \Pi$, such that $\Delta(\lambda)x = \lambda^r x$ for one single positive rational $\lambda \neq 1$.)

While it is not yet necessary, we now deal with Ξ_d. Comparison of the definition with Lemma 3.5 shows that Ξ_d is generated by all elements of the form $s(a_1)\cdots s(a_d)$, with $\{a_1, \ldots, a_d\}$ linearly independent. If $i \neq j$, it is easy to show that $s(a_i + \lambda a_j)s(a_j) = s(a_i)s(a_j)$ for any scalar λ, and, since $s(-a_j) = s(a_j)$, standard vector space theory shows that, if we choose a fixed basis $\{e_1, \ldots, e_d\}$ of $\mathbb{E}^d$, then $s(a_1)\cdots s(a_d) = s(\mu e_1)\cdots s(e_d)$, where $\mu = |\det(a_1, \ldots, a_d)|$, the determinant being with respect to the basis $\{e_1, \ldots, e_d\}$. Clearly also, $s((\mu + \nu)e_1) = s(\mu e_1) + s(\nu e_1)$ for $\mu, \nu \geqslant 0$. It now follows that the mapping

$$s(a_1)\cdots s(a_d) \mapsto |\det(a_1, \ldots, a_d)|$$

induces an isomorphism between the Abelian groups Ξ_d and $\mathbb{R}$. This isomorphism (together with the homomorphism it induces on Π, and the corresponding translation invariant valuation on $\mathcal{P}^d$) is called *volume*, and is denoted vol. More generally, each linear subspace L of $\mathbb{E}^d$ also admits a volume vol_L, which is unique up to positive scalar multiplication (see, for example, Hadwiger 1957, section 2.1.3).

The first weight space Ξ_1 admits an alternative characterization. Since Minkowski addition on $\mathcal{P}^d_*$ has a cancellation law, we can define an Abelian group $\mathcal{P}_T$, whose elements are the equivalence classes (P, Q) with $P, Q \in \mathcal{P}^d_*$ under the relation

$$(P, Q) \sim (P', Q') \Leftrightarrow P + Q' = P' + Q + t \quad \text{for some } t \in \mathbb{E}^d,$$

with addition $(P, Q) + (P', Q') = (P + P', Q + Q')$. Then, in fact:

Theorem 3.7. *The mapping* $\log : \mathcal{P}^d_* \to \Xi_1$ *induces an isomorphism between* $\mathcal{P}_T$ *and* Ξ_1.

Scalar multiplication on Ξ_1 can now be defined in one of two equivalent ways:

$$\lambda(P, Q) = \begin{cases} (\lambda P, \lambda Q) & \text{if } \lambda \geqslant 0, \\ (-\lambda Q, -\lambda P) & \text{if } \lambda < 0, \end{cases}$$

on $\mathcal{P}_T \cong \Xi_1$, or

$$\lambda x = \begin{cases} \Delta(\lambda)x & \text{if } \lambda \geqslant 0, \\ -\Delta(-\lambda)x & \text{if } \lambda < 0, \end{cases}$$

on Ξ_1 itself.

Checking that $(\lambda + \mu)x = \lambda x + \mu x$ is tedious rather than difficult; all the other vector space properties are obvious.

The next step involves proving a special case of Theorem 3.2(d). If we write $\Pi(L)$ for the subalgebra of Π generated by the polytopes in the linear subspace L of $\mathbb{E}^d$, and $\Xi_r(L)$ for its rth weight space, then we have:

Lemma 3.8. *If L and H are a complementary line and hyperplane in $\mathbb{E}^d$, E is a line segment in L and $e = \log E$, and $\lambda \in \mathbb{R}$, then $(\lambda e)x = e(\lambda x)$ for all $x \in \Xi_1(H)$.*

The method employed in McMullen (1989) to prove Lemma 3.8 closely followed that in Jessen and Thorup (1978) of the analogous result for the polytope group; for reasons of space, we shall not reproduce the details, even though the result is crucial for the discussion. We then appeal to the separation Theorem 3.11 (see section 3.3), to prove property (d) in full, after establishing it for $x, y \in \Xi_1$ when $d = 2$. The details are largely technical in nature, and do not merit much discussion. In the course of the proof, the remaining properties of Theorem 3.2 are also verified; once (d) is known, the rest follows relatively easily.

3.2. Negative dilatations and Euler-type relations

In section 3.1, the only dilatations considered were those by non-negative scalars λ. We now describe what happens when λ is allowed to be negative.

The *Euler map* * is defined on the generators $[P]$ of Π (with $P \in \mathcal{P}^d$) by (E) $[P]^* := \sum_F (-1)^{\dim F}[F]$, where the sum (here and elsewhere) extends over all faces F of P (including P itself).

Theorem 3.9. *The Euler map is an involutory algebra automorphism of Π. Moreover, for $r = 0, \ldots, d$, if $x \in \Xi_r$ and $\lambda < 0$, then*

$$\Delta(\lambda)x = \lambda^r x^*.$$

The proof of Theorem 3.9 uses the isomorphism Theorem 3.12, which, in turn, depends on the separation Theorem 3.11. In fact, the key result here is an abstract version of a theorem of Sommerville (1927) on polyhedral cones (we shall mention this in section 4.4 below); the idea of the proof is the same as that of the more concrete version proved for translation invariant valuations in McMullen (1977) (the result occurs without proof in McMullen 1975).

There is an amusing algebraic consequence of Theorem 3.9. It is clear that $x \in \Pi$ is invertible if and only if it is of the form $x = \pm(1+z)$ for some $z \in Z_1$. In particular, if $P \in \mathcal{P}^d_*$, then $[P]$ is invertible. Now, if $p = \log P$, then $[P]^{-1} = \exp(-p)$, and, in view of Theorem 3.9, $-p = (\Delta(-1)p)^*$. The algebra properties of the Euler map (which really follow from those of $\Delta(-1)$) now show that $[P]^{-1} = [-P]^*$; written in the form $[P] \cdot [-P]^* = 1 = [o]$, this has a nice interpretation as an equidecomposability result. (In fact, it can be shown not to depend on translation invariance – see section 3.5 below.)

Further properties of the Euler map, and its relations with the Euler characteristic, are given in McMullen (1989).

3.3. The separation theorem

If $o \neq u \in \mathbb{E}^d$, then the *face* of a polytope P *in direction* u is defined to be

$$P_u = \{v \in P \mid \langle v,u\rangle = h(P,u)\}.$$

Then we have:

Lemma 3.10. *The mapping $P \mapsto P_u$ induces an endomorphism $x \mapsto x_u$ of Π.*

Now let $U = (u_1,\dots,u_{d-r})$ be a $(d-r)$*-frame*, that is, an ordered orthogonal set of non-zero vectors in $\mathbb{E}^d$. Defining recursively $P_U = (P_{u_1,\dots,u_{d-r-1}})_{u_{d-r}}$, we see that we can define an induced endomorphism $x \mapsto x_U$. If

$$L = U^{\perp} := \{v \in \mathbb{E}^d \mid \langle v,u\rangle = 0 \text{ for all } u \in U\},$$

and we write $\mathrm{vol}_U := \mathrm{vol}_L$, then clearly the mapping f_U defined by

$$f_U(P) := \mathrm{vol}_U(P_U)$$

induces a (group) homomorphism $f_U : \Pi \to \mathbb{R}$, which we call a *frame functional of type r* (note that $\dim L = r$, so that f_U will be homogeneous of degree r). The frame functional of type d (with $U = \emptyset$) is just vol itself, and that of type 0 (which is, essentially, unique) is χ (or $\Delta(0)$). Then we have:

Theorem 3.11. *The frame functionals separate Π; that is, if $x \in \Pi$ is such that $f_U(x) = 0$ for all frames U, then $x = 0$.*

The key idea behind the proof, which may be found in McMullen (1989), is to use Lemma 3.8 to reduce the result to an inductive proof on the dimension. Again, this adapts the idea of Jessen and Thorup (1978).

The frame functionals are not independent, but although the family of relations or *syzygies* between them is conjectured in McMullen (1989), it is not yet completely established.

3.4. The cone group

Although we shall return to the question of spherical dissections in section 4.4 below, we need to begin the discussion of the cone group here. If L is a linear subspace of $\mathbb{E}^d$, the *cone group* $\widehat{\Sigma}(L)$ is defined to be the Abelian group generated by the *cone classes* $\langle K\rangle$ with $K \in \mathscr{C}(L) := \{K \in \mathscr{C}^d \mid K \subseteq L\}$, with the relations (V) (the valuation property) and (S) $\langle K\rangle = 0$ if $\dim K < \dim L$. Thus $\widehat{\Sigma}(L)$ is the abstract group underlying simple valuations on $\mathscr{C}(L)$. We further define the *full cone group* to be

$$\widehat{\Sigma} := \bigoplus_L \widehat{\Sigma}(L),$$

the sum, as usual, extending over all subspaces L of $\mathbb{E}^d$.

Let us write $P\|L$ to mean that the affine hull of the polyhedral set P is a translate of the linear subspace L, and say that P and L are *parallel*. If F is a face of a (non-empty) P, we define the *normal cone* $N(F,P)$ to P at F to be the set of outer normal vectors to hyperplanes which support P in F, and $n(F,P)$ to be its *intrinsic class*, that is, its class in $\widehat{\Sigma}(L)$, where $L\|N(F,P)$. We note that $\dim N(F,P) = d - \dim F$. We shall write $\mathrm{vol}\,P$ from now on to mean the volume of a polytope P measured relative to the subspace L parallel to P, with $\mathrm{vol}\,P = 1$ if P is a point. An important result, which is a consequence of the separation Theorem 3.11 is:

Theorem 3.12. *The mapping*

$$\sigma(P) := \sum_F \mathrm{vol}\,F \otimes n(F,P)$$

induces a monomorphism from Π into $\mathbb{R} \otimes \widehat{\Sigma}$.

If we define

$$\sigma_r(P) := \sum_{\dim F = r} \mathrm{vol}\,F \otimes n(F,P),$$

then, similarly, σ_r induces a monomorphism from Ξ_r into $\mathbb{R} \otimes \widehat{\Sigma}_{d-r}$, where

$$\widehat{\Sigma}_{d-r} := \bigoplus_{\dim L = d-r} \widehat{\Sigma}(L).$$

3.5. Translation covariance

Following McMullen (1983), we call a valuation φ on $\mathscr{P}^d$ taking values in the vector space $\mathscr{X}$ *translation covariant* if there exists a map $\Phi : \mathscr{P}^d \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{E}^d, \mathscr{X})$, such that $\varphi(P+t) = \varphi(P) + \Phi(P)t$ for all $P \in \mathscr{P}^d$ and $t \in \mathbb{E}^d$. With φ weakly continuous (see section 5.5 below), we can replace $\mathrm{Hom}_{\mathbb{Q}}$ by Hom.

The abstract theory underlying translation covariant valuations is, as yet, still in its infancy, and we outline the (unpublished) material only in the hope of stimulating further research. It is convenient, *for this subsection alone*, to change the notation somewhat. We now write Π_0 instead of Π, and define Π to be the Abelian group generated by the polytope classes under the relations (V) alone. As before, Π is actually a ring, with the multiplication given by (M); it still has a subring Ξ_0 generated by $1 := [o]$ which is isomorphic to $\mathbb{Z}$.

We let T be the ideal of Π generated by the elements of the form $[t] - 1$, with $t \in \mathbb{E}^d$. Then $\Pi_0 = \Pi/T$. The powers of T are also ideals of Π; we write $\Pi_k := \Pi/T^{k+1}$. Thus Π_1 is the abstract group underlying translation covariant valuations on $\mathscr{P}^d$. The Π_k for larger k correspond, in a similar way, to the theory of valuations which behave under translation like tensors.

It may be shown that each Π_k is almost (in the same sense as in Theorem 3.2) a graded commutative algebra over $\mathbb{Q}$, with graded terms $\Xi_{k,r}$ of degrees $r \leqslant d+k$. However, except in the relatively trivial case $d=1$ and $k=1$, we have not, so far, been able to extend the algebra properties to allow scalars in $\mathbb{R}$. The first weight space $\Xi_{k,1}$ is always a real vector space, and is, for $k \geqslant 1$, isomorphic to the space $\mathcal{P}_1$ defined like $\mathcal{P}_T$ in section 3.1, but with the equivalence relation not factoring out translations. The analogue of Lemma 3.8, though, has so far proved elusive.

It may be helpful to observe the following stability result. If we write Z for the ideal of Π generated by all the elements $[P]-1$, with $P \in \mathcal{P}^d_*$, then $T^j \subseteq Z^{d+j}$ for all $j \geqslant 0$. Define $\Xi_r := Z^r/Z^{r+1}$. Let Z_k be the corresponding ideal of Π_k, so that

$$Z_k = Z/(Z \cap T^{k+1}) \cong (Z + T^{k+1})/T^{k+1},$$

and hence, if $k \geqslant r$, then

$$\Xi_{k,r} = Z_k^r/Z_k^{r+1} \cong (Z^r + T^{k+1})/(Z^{r+1} + T^{k+1}) \cong Z^r/Z^{r+1} = \Xi_r.$$

What can be proved is that the Euler-type relation

$$\Delta(-1)x = (-1)^r x^*,$$

with $x \in \Xi_{k,r}$, remains valid for all k, and since

$$\bigcap_{k \geqslant 0} T^k = \emptyset,$$

it therefore follows that, if $P \in \mathcal{P}^d_*$, then $[P]^{-1} = [-P]^*$ in Π itself. Writing this out, and observing that we can identify $[P]$ with $P^\dagger$, we have the decomposability result:

Theorem 3.13. *If $P \in \mathcal{P}^d_*$, then*

$$\sum_F (-1)^{\dim F}(P-F)^\dagger = \{o\}^\dagger.$$

We end the section by remarking that Lawrence (unpublished) and Fischer and Shapiro (1992) have also investigated Π, which they call the *Minkowski ring*. In the former paper is shown the following. Let Y be an indeterminate, for $u \in \mathbb{E}^d$, define

$$\varepsilon_u(P) := Y^{h(P,u)},$$

with $P \in \mathcal{P}_*$, and extend to Π by linearity (it is clear from Lemma 1.1 that ε_u is a valuation). Then:

Theorem 3.14. *The valuations ε_u are multiplicative homomorphisms which separate Π.*

It is not clear how these valuations ε_u relate to the frame functionals (which are certainly not multiplicative).

The latter paper considers the subrings of Π generated by finitely many polytope classes, and their prime ideal structures.

3.6. Mixed polytopes

In section 2 we discussed mixed volumes and moment vectors. Not unnaturally, there are analogues in the polytope algebra Π itself (and, in fact, in the more general Π_k defined in section 3.5). The definition of a mixed polytope is straightforward: if $P_1, \dots, P_r \in \mathcal{P}^d_*$ and $p_i = \log P_i$ for $i = 1, \dots, r$, then the *mixed polytope* $m(p_1, \dots, p_r)$ is defined by

$$m(p_1, \dots, p_r) := \frac{1}{r!} \cdot p_1 \cdots p_r.$$

Thus the rth weight space component p_r of a polytope class $[P]$ is just $p_r = m(p, \dots, p)$.

A theory of mixed polytopes was attempted by Meier (1977), but it appears that, at one point, his argument is flawed. Another theory, but only within the context of the polytope group (see section 4.1), was propounded in McMullen and Schneider (1983, section 6).

If φ is now a translation invariant valuation on $\mathcal{P}^d$ which is homogeneous of degree r, then the corresponding mixed valuation is

$$\varphi(P_1, \dots, P_r) = \varphi(m(p_1, \dots, p_r)).$$

Since

$$(\lambda_1 p_1) \cdots (\lambda_r p_r) = (\lambda_1 \cdots \lambda_r) p_1 \cdots p_r,$$

we have the curious consequence that, for fixed polytopes $P_1, \dots, P_r$, the value of the mixed valuation $\varphi(\lambda_1 p_1, \dots, \lambda_r p_r)$ depends only on the product $\lambda_1 \cdots \lambda_r$, without any assumption on the continuity of φ.

We make one final remark in this subsection. It was observed by Groemer (1977a) that, if $K, L \in \mathcal{K}^d$ are such that $K \cup L$ is convex, then the mixed volume satisfies

$$V(K \cup L, K \cap L, \mathscr{C}) = V(K, L, \mathscr{C})$$

for any $(d-2)$-tuple $\mathscr{C}$ of convex bodies. An easier proof than the original was given in McMullen and Schneider (1983, section 3). However, the essence of the proof is algebraic, and in Π is almost trivial. Let $P, Q \in \mathcal{P}^d_*$ be such that $X := P \cup Q$ is convex, let $Y := P \cap Q$, and write $p := \log P$, and so on. Then the valuation property $[X] + [Y] = [P] + [Q]$ yields in Ξ_r:

$$x^r + y^r = p^r + q^r.$$

Hence

$$\begin{aligned} xy &= \tfrac{1}{2}((x+y)^2 - (x^2+y^2)) \\ &= \tfrac{1}{2}((p+q)^2 - (p^2+q^2)) \\ &= pq, \end{aligned}$$

which is the abstract version of the required property.

3.7. Relatively open polytopes

Schneider (1985) developed a theory of decomposition by translation, based on relatively open polytopes. We briefly describe this here, although we shall not set up the underlying abstract structure.

We know from section 3.1 that the class [relint P] of the relative interior of a polytope P exists in Π, and since

$$[P]=\sum_F [\operatorname{relint} F]$$

is obvious, Möbius inversion (see Rota 1964) yields

$$[\operatorname{relint} P]=\sum_F (-1)^{\dim P-\dim F}[F],$$

or:

Theorem 3.15. *The class of the relative interior of a polytope P is given by*

$$[\operatorname{relint} P]=(-1)^{\dim P}[P]^*.$$

In view of the fact that the mapping $[P]\mapsto[P]^*$ is an automorphism of Π, we see that $[\operatorname{relint} P]\mapsto(-1)^{\dim P}[P]$ also gives an automorphism – in retrospect this is transparent. Schneider (1985) gives a separation criterion based on relatively open polytopes (closely related to the isomorphisms of section 4.3); the simpler one given by McMullen (1989) is essentially the previous remark.

3.8. Invariance under other groups

When we impose invariance under bigger groups than translations on the polytope algebra, we not unnaturally lose some of its properties. In this subsection, we briefly discuss what is so far known to happen.

Let G be a group of affinities acting on $\mathbb{E}^d$, which contains the group $T\cong\mathbb{E}^d$ (as an additive group) of translations in $\mathbb{E}^d$. If we replace the translation invariance condition (T) in the definition of the polytope algebra Π by the stronger condition (G) $[\Phi P]=[P]$ for all $P\in\mathcal{P}^d$ and $\Phi\in G$, we obtain a new group Π_G. The polytope algebra Π itself is thus Π_T.

It is clear that Π_G is no longer a ring, unless $G=T$, because Minkowski addition is not compatible with affinities which are not translations. However, since Π_G is, as an Abelian group, a quotient of Π, a great deal of the structure of Π does survive.

Theorem 3.16. *For any group G of affinities of $\mathbb{E}^d$ which contains the translations, the polytope group Π_G has the following structure:*

(a) *Π_G has a direct sum decomposition*

$$\Pi_G=\bigoplus_{r=0}^{d}\Xi_r,$$

where $\Xi_0\cong\mathbb{Z}$ and, for $r=1,\dots,d$, Ξ_r is a real vector space;

(b) *dilatation acts on Ξ_r by*

$$\Delta(\lambda)(x)=\begin{cases}\lambda^r x & \text{if } \lambda\geqslant 0,\\ \lambda^r x^* & \text{if } \lambda<0\end{cases}$$

(with $\lambda^0=1$) if $x\in\Xi_r$, where * is the Euler map.

We obtain the direct sum decomposition, because it is obvious that dilatations commute with the endomorphisms of Π induced by affinities.

For most groups G, we can say little more than this. However, there are some special cases.

Theorem 3.17. *If G contains a dilatation by some $\lambda\neq\pm1$, then $\Pi_G\cong\mathbb{Z}$.*

Let A denote the group of all affinities of $\mathbb{E}^d$, and let EA denote the subgroup of *equiaffinities*, that is, the mappings of the form $v\mapsto\Phi v+t$, where Φ is a linear mapping with $\det\Phi=\pm1$. Then we have:

Theorem 3.18. (a) *$\Pi_A\cong\mathbb{Z}$;*

(b) *For $d\geqslant1$, $\Pi_{\mathrm{EA}}\cong\mathbb{Z}\oplus\mathbb{R}$.*

Part (a) is a consequence of the previous theorem, while in (b) the volume term additionally survives.

Finally, let TH be the group consisting of the translations and reflexions in points (mappings of the form $v\mapsto 2c-v$, where $c\in\mathbb{E}^d$). Since $\Delta(-1)$ acts as the identity on Π_{TH}, we have:

Theorem 3.19. *If $G\supseteq$ TH, then in Π_G, the rth weight space Ξ_r is generated by classes of polytopes of lower dimension than d if $r\not\equiv d$ modulo 2.*

4. Simple valuations and dissections

We now consider the groups which correspond to the simple valuations on $\mathcal{P}^d$. As in section 4.1, the translation invariant case is the fundamental one, about which most is known. However, since the rigid motion invariant case has provided much of the motivation for research in this area, we shall also devote a fair amount of space to it.

4.1. The algebra of polytopes

The *polytope group* $\widehat{\Pi}^d$ has a generator $\langle P\rangle$ for each $P\in\mathcal{P}^d$; these generators satisfy the relations (V) and (T) of the polytope algebra, together with (S) $\langle P\rangle=0$ if $\dim P<d$, which corresponds to simple valuations. From Theorems 3.2 and 3.9, we deduce at once the main structure theorem of Jessen and Thorup (1978) and Sah (1979).

Theorem 4.1. (a) $\widehat{\Pi}^0 \cong \mathbb{Z}$.

(b) *If $d > 0$, then $\widehat{\Pi}^d$ has a direct sum decomposition*

$$\widehat{\Pi}^d = \bigoplus_{r=1}^{d} \widehat{\Xi}_r$$

into real vector spaces. Moreover, dilatation acts on $\widehat{\Xi}_r$ by

$$\Delta(\lambda)x = \begin{cases} \lambda^r x & \text{if } \lambda \geqslant 0, \\ (-1)^d \lambda^r x & \text{if } \lambda < 0. \end{cases}$$

Of course, dilatations are compatible with the relations (S). For the negative dilatations, we observe that the Euler map in $\widehat{\Pi}^d$ acts as

$$\langle P \rangle^* = \sum_F (-1)^{\dim F} \langle F \rangle = (-1)^d \langle P \rangle.$$

[In Jessen and Thorup (1978), this behaviour under negative dilatations was assumed; in Sah (1979), by contrast, it was proved.]

For the separation result, we need another concept. If $U = (u_1, \ldots, u_{d-r})$ is a frame, and $E = (\varepsilon_1, \ldots, \varepsilon_{d-r})$ is a vector with entries $\varepsilon_i = \pm 1$, we write $EU := (\varepsilon_1 u_1, \ldots, \varepsilon_{d-r} u_{d-r})$, and

$$\operatorname{sgn} E := \prod_{j=1}^{d-r} \varepsilon_j.$$

A *Hadwiger functional of type r* is then a map of the form

$$h_U := \sum_E \operatorname{sgn} E \, f_{EU},$$

where U is a $(d-r)$-frame. As with the frame functionals, $h_\emptyset = \text{vol}$ is just ordinary volume. Then we have:

Theorem 4.2. *The Hadwiger functionals separate $\widehat{\Pi}^d$.*

This result, proved by Jessen and Thorup (1978) and Sah (1979), cannot be deduced from Theorem 3.11, but must be shown independently.

Just as there are syzygies between the frame functionals, so there are between the Hadwiger functionals. In Sah (1979), these syzygies were described, but the proof that they were the only ones was lacking. The proof was provided by Dupont (1982). In fact, there is a stronger result than Theorem 4.2.

Theorem 4.3. *Let $\mathscr{X}$ be a real vector space. Then all linear mappings $\varphi : \widehat{\Pi}^d \to \mathscr{X}$ are of the form*

$$\varphi = \sum_U f_U c_U,$$

where $U \mapsto c_U$ is an arbitrary function from frames into $\mathscr{X}$.

4.2. Cones and angles

We have already introduced the cone group $\widehat{\Sigma}$ in section 3.3. In order to proceed, we need to investigate it a little further.

First, we produce the analogue of the isomorphism Theorem 3.12. The subgroup of $\widehat{\Sigma}$ generated by the classes of cones which contain a line (and so have a face of apices of positive dimension) is denoted $\widehat{\Gamma}$. If $c \in \widehat{\Sigma}$, we write $\bar{c}$ for its image under the quotient map from $\widehat{\Sigma}$ onto $\widehat{\Sigma}/\widehat{\Gamma}$, and $\bar{n}(F,P) := \overline{n(F,P)}$ (with analogous notation employed subsequently). Then Theorem 4.2 implies:

Theorem 4.4. *The map $\bar{\sigma} : \mathscr{P}^d \to \mathbb{R} \otimes (\widehat{\Sigma}/\widehat{\Gamma})$, defined by*

$$\bar{\sigma}(P) = \sum_F \text{vol}(F) \otimes \bar{n}(F,P)$$

induces a monomorphism from $\widehat{\Pi}^d$ into $\mathbb{R} \otimes (\widehat{\Sigma}/\widehat{\Gamma})$.

We shall see the suggestive role this result plays in section 4.5. For the moment, we just note that the classes of the normal cones to the faces of a lower-dimensional polytope lie in $\widehat{\Gamma}$.

An important notion is that of angle. In $\mathbb{E}^d$, we have a natural notion of rotation invariant angle, but over more general fields $\mathbb{F}$, where we do not usually have a full rotation group, an alternative approach is necessary. If L is a linear subspace, then an *angle* on $\mathscr{C}(L)$ is an L-simple valuation ω_L (into the base field), such that $\omega_L(L) = 1$. We can choose an angle on each subspace L simultaneously, as follows (the notion was suggested by U. Betke): pick any d-polytope Q with $o \in \text{int}(Q)$, and define ω_L by

$$\omega_L(C) = \frac{\text{vol}(C \cap Q)}{\text{vol}(L \cap Q)}$$

for each $C \in \mathscr{C}(L)$, where $\text{vol} = \text{vol}_L$. (By the way, the same idea enables us to choose a particular scaling of all the volumes vol_L simultaneously – we just set $\text{vol}_L(L \cap Q) = 1$ for all L.)

If we have an angle ω_L for each subspace L, then we can define a homomorphism ω on $\widehat{\Sigma}$ by $\omega = \omega_L(c)$ if $c \in SP(L)$. We shall also refer to such a homomorphism ω as an angle.

The *angle cone* of a polyhedral set P at its face F is the cone $A(F,P) := \text{pos}(P - F)$ generated by P at (any relatively interior point of) F. We write $a(F,P)$ for the intrinsic class of $A(F,P)$ in $\widehat{\Sigma}$, and, if ω is an angle, we write $\alpha(F,P) := \omega(a(F,P))$, which we call an *inner angle*. We similarly write $\nu(F,P) := \omega(n(F,P))$, with $n(F,P)$ the class of the normal cone, which we call an *outer angle*. We observe that $\alpha(F,F) = 1 = \nu(F,F)$ for all (non-empty) faces F. Further, we call an inner angle α and outer angle ν *inverse* if

$$\sum_J (-1)^{\dim J - \dim F} \alpha(F,J) \nu(J,G) = \delta(F,G),$$

where

$$\delta(G,H) := \begin{cases} 1 & \text{if } F = G, \\ 0 & \text{if } F \neq G, \end{cases}$$

is the *delta function*.

The relationship between inner and outer angles is such that:

Lemma 4.5. *If α and ν are inverse inner and outer angles, then*

$$\sum_J (-1)^{\dim G - \dim J} \nu(F,J)\alpha(J,G) = \delta(F,G).$$

In fact, the best way to look at this is within the context of the *incidence algebra* of Rota (1964). This consists of the functions κ on ordered pairs of faces (taking values in the base field), such that $\kappa(F,G) = 0$ unless F is a face of G. Addition and multiplication of such functions are defined by

$$(\kappa + \lambda)(F,G) = \kappa(F,G) + \lambda(F,G),$$
$$(\kappa\lambda)(F,G) = \sum_J \kappa(F,J)\lambda(J,G).$$

These functions can be thought of as triangular matrices indexed by faces of polyhedral sets. The crucial result about angles is the following.

Lemma 4.6. *If ν is an outer angle, then there exists an inverse inner angle α, and conversely.*

When the angles are the ordinary normalized angles in $\mathbb{E}^d$, this result was first proved by McMullen (1975); the general result occurs in McMullen (1989).

Inner and outer angles can be used to find another relationship between the polytope algebra and the polytope groups (see section 4.3 below). The abstract results on which it depends are the following. In each case, $\mathcal{S}$ stands for $\mathcal{P}^d$ or $\mathcal{C}^d$, and in the latter case translation invariance is to be ignored.

Lemma 4.7. *Let $\mathcal{G}$ be an Abelian group, and for each subspace L, let $\psi_L : \mathcal{S}(L) \to \mathcal{G}$ be an L-simple translation invariant valuation. If $\psi : \mathcal{S} \to \mathcal{G}$ is defined by $\psi(P) := \psi_L(P)$ if $P \| L$, then the mapping $\varphi : \mathcal{S} \to \mathcal{G} \otimes \widehat{\Sigma}$ given by*

$$\varphi(P) := \sum_F \psi(F) \otimes n(F,P)$$

is a translation invariant valuation.

Lemma 4.8. *Let $\mathcal{G}$ be an Abelian group, and let $\varphi : \mathcal{S} \to \mathcal{G}$ be a translation invariant valuation. Then for each subspace L, the mapping $\psi_L : \mathcal{S}(L) \to \mathcal{G} \otimes \widehat{\Sigma}$ defined by*

$$\psi_L(P) := \begin{cases} \sum_F \varphi(F) \otimes (-1)^{\dim P - \dim F} a(F,P) & \text{if } P \| L, \\ 0 & \text{otherwise,} \end{cases}$$

is an L-simple translation invariant valuation.

The final lemma in this subsection is clear.

Lemma 4.9. *If $\mathcal{X}$ is a real vector space, and ω is an angle, then the mapping $\pi : \mathcal{X} \otimes \widehat{\Sigma} \to \mathcal{X}$ defined by $\pi(x \otimes c) := \omega(c)x$, with $x \in \mathcal{X}$ and $c \in \widehat{\Sigma}$, is a homomorphism.*

4.3. The polytope groups

Since a linear subspace L of $\mathbb{E}^d$ is itself a real vector space of the appropriate dimension, it also has associated with it a polytope group $\widehat{\Pi}(L)$; thus $\widehat{\Pi}^d = \widehat{\Pi}(\mathbb{E}^d)$. Before we discuss the connexion between the polytope groups and the polytope algebra, we shall mention a kind of multiplication, which is an analogue of the genuine multiplication in Π.

Theorem 4.10. *Let L and M be complementary linear subspaces of $\mathbb{E}^d$ of positive dimension. Then there is a natural embedding of $\widehat{\Pi}(L) \otimes \widehat{\Pi}(M)$ into $\widehat{\Pi}^d$. This embedding is compatible with the scalar multiplication in that, if $x \times y$ is the image of $x \otimes y$, with $x \in \widehat{\Pi}(L)$ and $y \in \widehat{\Pi}(M)$, then for each scalar λ,*

$$(\lambda x) \times y = x \times (\lambda y) = \lambda(x \times y).$$

The embedding is that which is obviously induced by the geometric direct sum. In fact, this theorem lies at the heart of the original proof of the structure Theorem 4.1 in Jessen and Thorup (1978) and Sah (1979).

We now write

$$\widehat{\Pi} := \bigoplus_L \widehat{\Pi}(L),$$

the sum, as usual, extending over all linear subspaces of $\mathbb{E}^d$. Our other isomorphism theorem for the polytope algebra is:

Theorem 4.11. $\Pi \cong \widehat{\Pi}$.

The isomorphism is easily described. Let α and ν be any pair of inverse inner and outer angles. First, we define the mapping $\varphi : \mathcal{P}^d \to \widehat{\Pi}$ by

$$\varphi(P) := \sum_F \nu(F,P)\langle F \rangle,$$

where $\langle F \rangle$ is the intrinsic class of F, that is, its class in $\widehat{\Pi}(L)$, where $L \| F$. There is no trouble with the vertices F^0 of P, even though $\widehat{\Pi}^0 \cong \mathbb{Z}$, since $\sum_{F^0} \nu(F^0,P) = 1$. Then φ induces a homomorphism from Π to $\widehat{\Pi}$ by Lemmas 4.7 and 4.9.

Next, for each subspace L of $\mathbb{E}^d$, we define a mapping $\psi_L : \mathcal{P}(L) \to \Pi$ by

$$\psi_L(P) = \begin{cases} \sum_F (-1)^{\dim P - \dim F} \alpha(F,P)[F] & \text{if } P \| L, \\ 0 & \text{otherwise.} \end{cases}$$

Again, we have no problems with the 0-components of the classes $[F]$, since

$$\sum_F (-1)^{\dim P - \dim F} \alpha(F,P) = \begin{cases} 1 & \text{if } \dim P = 0, \\ 0 & \text{if } \dim P > 0. \end{cases}$$

Then these ψ_L induce a homomorphism $\psi : \widehat{\Pi} \to \Pi$ by Lemmas 4.8 and 4.9.

Finally, the definition of inverse angles shows that φ and ψ are inverse homomorphisms, as required.

This proof closely parallels that in McMullen (1977) of the relationship between general and simple translation invariant valuations. However, this proof covers more than just the real-valued case considered there.

4.4. Spherical dissections

In preparation for the discussion of Hilbert's third problem in section 4.5, we must first discuss the analogous problem for spherical polytopes or cones (we can identify a spherical polytope on Ω^{d-1} with the cone in $\mathbb{E}^d$ which it spans, and so we shall usually consider the latter). In this section, we shall largely follow Sah (1979).

The group $\widehat{\Sigma}$ is, as a group, not of great interest. It is only when we impose additional relations on it that it begins to acquire some structure. As far as we are concerned, the most important case is the following. The group $\widehat{\Sigma}_O^d$ is $\widehat{\Sigma}(\mathbb{E}^d)$, with the additional relations (O) $\langle \Psi K \rangle = \langle K \rangle$ whenever $K \in \mathscr{C}^d$ and $\Psi \in O$, the orthogonal group. We define $\widehat{\Sigma}_O := \sum_{d \geqslant 0} \widehat{\Sigma}_O^d$, with $\widehat{\Sigma}_O^0 = \mathbb{Z}$. We then have a natural product $*$ on $\widehat{\Sigma}_O$, which is induced by orthogonal Cartesian product, and which is compatible with orthogonal transformations. Formally, if K_1 and K_2 are two cones, we define $\langle K_1 \rangle * \langle K_2 \rangle := \langle K_1 \times \Psi K_2 \rangle$, where Ψ is a suitable rotation taking K_2 into a subspace orthogonal to K_1.

Before we proceed further, let us make a remark. If SO is the subgroup consisting of the rotations in O, then we have (compare Lemma 4.31 below):

Lemma 4.12. *$\widehat{\Sigma}_O^d = \widehat{\Sigma}_{SO}^d$ for every dimension $d \geqslant 2$.*

Clearly, $\widehat{\Sigma}_O^1 \cong \mathbb{Z}$, and is generated by the class p (which stands for "point") of a half-line. The same notion which proves Lemma 4.12 also lies at the heart of:

Lemma 4.13. *The group $\widehat{\Sigma}_O$ is 2-divisible.*

We also have:

Lemma 4.14. *For $d = 2$ or 3, $\widehat{\Sigma}_O^d \cong \mathbb{R}$.*

The case $d = 2$ is obvious; while the second is fairly familiar, we shall justify it in Theorem 4.17 below.

In order to investigate $\widehat{\Sigma}_O$, we need to introduce some further concepts. The *recession cone* rec K of a polyhedral set K is defined to be

$$\operatorname{rec} K := \{x \in \mathbb{E}^d \mid x + y \in K \text{ for all } y \in K\}.$$

If K is a polytope, then $\operatorname{rec} K = \{o\}$, while if K is a cone with apex a, then $\operatorname{rec} K = K - a$. A common generalization of results of Brianchon (1837) and Gram (1874) (see also Shephard 1967), and Sommerville (1927) is the following, due to McMullen (1983):

Theorem 4.15. *Let K be a polyhedral set in $\mathbb{E}^d$. Then*

$$\sum_F (-1)^{\dim F} a(F,K) = (-1)^d \langle \operatorname{rec}(-K) \rangle.$$

As in section 3.4, $a(F,K)$ denotes the class of the angle cone $A(F,K)$. We have kept $\operatorname{rec}(-K)$ instead of $\operatorname{rec} K$, to emphasize the geometric nature of the dissection result.

In case K is a pointed polyhedral cone in $\mathbb{E}^d$, we can apply Lemma 4.13 to Theorem 4.15, to deduce:

Theorem 4.16. *If d is odd, and K is a pointed polyhedral cone in $\mathbb{E}^d$, then*

$$\langle K \rangle = \frac{1}{2} \sum_{F \neq o} (-1)^{\dim F - 1} a(F,K).$$

An *r-fold join* is a just a product $k_1 * \cdots * k_r$, where $k_i = \langle K_i \rangle$ is the class of a cone of dimension at least 1 for $i = 1, \ldots, r$. We write $\widehat{\Sigma}_r^d$ for the subgroup of $\widehat{\Sigma}^d$ generated by the r-fold joins, and $\widehat{\Sigma}_r := \bigcup_{d \geqslant 0} \widehat{\Sigma}_r^d$. Then

$$\widehat{\Sigma} = \widehat{\Sigma}_1 \supseteq \widehat{\Sigma}_2 \supseteq \cdots,$$

and

$$\widehat{\Sigma}_{r_1}^{d_1} * \widehat{\Sigma}_{r_2}^{d_2} \subseteq \widehat{\Sigma}_{r_1+r_2}^{d_1+d_2},$$

for all r_1, r_2, d_1, d_2.

We may observe that each term in the sum of Theorem 4.16 is a join. Indeed, if we define the *intrinsic inner cone* of K at its face F to be $B(F,K) := A(F,K) \cap F^{\perp}$, where $F^{\perp}$ is the orthogonal complementary subspace to F in $\mathbb{E}^d$, then for $d \geqslant 1$, we can express $A(F,K)$ as a non-trivial product

$$A(F,K) = B(F,K) \times \operatorname{lin} F.$$

A closely related angle cone is

$$\widetilde{A}(F,K) := B(F,K) \times \mathbb{E}^{\dim F - 1}$$

whenever $\dim F \geqslant 1$, which corresponds to the angle cone at $F \cap \Omega$ of the spherical polytope $K \cap \Omega$. The mapping $e : \widehat{\Sigma}^d \to \widehat{\Sigma}^{d-1}$ given by

$$e(K) := \sum_{F \neq o} (-1)^{\dim F - 1} \widetilde{a}(F, K),$$

where $\widetilde{a}(F,K) := \langle \widetilde{A}(F,K) \rangle$, is called by Sah (1979) the *Gauss–Bonnet map* (the reasons for this name are not altogether clear). When d is even, Sah (1981) shows that $e(\langle K \rangle) = 0$, and Theorem 4.16 can be written in the form

$$\langle K \rangle = p * e(\langle K \rangle).$$

Extending the notation of section 3.4, let us write $\widehat{\Gamma}^d := p * \widehat{\Sigma}^{d-1}$. The crucial result of Sah (1981) is:

Theorem 4.17. *If $j \geqslant 0$, then $\widehat{\Sigma}^{2j+1} = \widehat{\Gamma}^{2j+1} = p * \widehat{\Sigma}^{2j}$, and the map $e; \widehat{\Sigma}^{2j+1} \to \widehat{\Sigma}^{2j}$ is an isomorphism inverse to $x \mapsto p * x$.*

It follows that, if $\widehat{\Gamma} := \bigoplus_{d \geqslant 1} \widehat{\Gamma}^d$, then $\widehat{\Sigma}/\widehat{\Gamma}$ is evenly graded by degree.

We now introduce the graded volume map. Our normalization of (spherical) volume gives the total volume of Ω^{d-1} as 1, in contrast to Sah (1979), who follows Schläfli in assigning it volume 2^d, and is just the rotation invariant angle. A natural way of defining this for a polyhedral cone K with apex o is by

$$\operatorname{vol} K := \int_K \exp(-\pi \|x\|^2)\, dx,$$

where dx is ordinary Lebesgue measure in the subspace $\operatorname{lin} K$. Thus, the volume of a linear subspace is always 1. The *graded volume* of K is then defined by

$$\text{gr.vol}\langle K \rangle := \operatorname{vol} K \cdot T^{\dim K},$$

where T is an indeterminate. Further, Theorem 4.16 has the implication

$$\operatorname{vol} K = \tfrac{1}{2} \sum_{F \neq 0} \beta(F, K),$$

where $\beta(F,K) := \operatorname{vol} B(F,K) = \operatorname{vol} A(F,K)$, because of the normalization, and the elementary observation

$$\operatorname{vol}(K_1 \times K_2) = \operatorname{vol} K_1 \cdot \operatorname{vol} K_2$$

for orthogonal cones K_1 and K_2. Lemma 4.14 for $d = 3$ is now an immediate consequence of this and Theorem 4.17.

We now discuss various dissection results and their consequences. Let K be a polyhedral cone with apex o. First, when we note that each point $z \in \mathbb{E}^d$ admits a unique expression of the form $z = x + y$, where $x \in \operatorname{relint} F$ for some face F of K (possibly K itself) and $y \in N(F,K)$, we have:

Theorem 4.18. *Let K be a polyhedral cone in $\mathbb{E}^d$ with apex o. Then the cones F and $N(F,K)$, with F a face of K, are orthogonal, and $\mathbb{E}^d$ is dissected into the cones $F \times N(F,K)$.*

In $\widehat{\Sigma}$, this result leads to

$$\langle \mathbb{E}^d \rangle = \sum_F \langle F \rangle * n(F,K),$$

where $n(F,K) := \langle N(F,K) \rangle$.

The analogue for normal cones of $B(F,K)$ is the *intrinsic outer cone* $C(F,K) := N(F,K) \cap \operatorname{lin} K$, the normal cone to K at its face F in the subspace $\operatorname{lin} K$ which it spans; we write $c(F,K)$ for its class, and $\gamma(F,K)$ for its volume. Similarly, we write $Z(F,K)$ for the orthogonal complement of $\operatorname{lin} F$ in $\operatorname{lin} K$, and $z(F,K)$ for its class. Further, we define

$$m(F,K) := (-1)^{\dim Z(F,K)} z(F,K),$$

and the identity function i by

$$i(F,K) := \begin{cases} 1\ (\in \mathbb{Z}) & \text{if } F = K, \\ 0 & \text{if } F \neq K. \end{cases}$$

We can extend the notation of the incidence algebra of Rota (1964) to the multiplication $*$, and write

$$f * g(F,G) := \sum_J f(F,J) g(J,G),$$

where F, G are faces of a polyhedral cone, and the summation is over all faces J, with the understanding that $f(F,J) = 0$ unless F is a face of J, and so on. The Euler relation for polyhedral cones implies:

Theorem 4.19. $m * z = i = z * m$.

If we define $\widetilde{b}$ by

$$\widetilde{b}(F,K) := (-1)^{\dim K - \dim F} b(F,K),$$

and $\widetilde{c}$ similarly, we can now rephrase Theorems 4.15 (for cones) and 4.18 as:

Theorem 4.20. (a) $m * b = \widetilde{b}$;
(b) $b * c = z$.

Combining these two results, we have consequences of McMullen (1983) (compare also McMullen 1975b):

Theorem 4.21. (a) $\widetilde{b} * c = i = b * \widetilde{c}$;
(b) $c * \widetilde{b} = i\ (= \widetilde{c} * b)$.

The basic result here is part (a), from which (b) follows by use of the incidence algebra, and polarity, which we shall talk about below (the second part of (b) is actually the same as the first). An important consequence of this (obtained by replacing the cone classes by their volumes or angles) is that β and γ are inverse inner and outer angles.

Now we introduce polarity. We define the *polar* K° of a cone K by

$$K^\circ := \{x \in \mathbb{E}^d \mid \langle x, y\rangle \leqslant 0 \text{ for all } y \in K\}.$$

Note that $K^{\circ\circ} = K$. As an example, $N(F,K) = A(F,K)^\circ$ for every face F of a polyhedral cone K. Our first remark is that polarity is compatible with the valuation property [this originates in Sah (1979), from which what follows is taken, but see also Lawrence (1988)].

Lemma 4.22. *Let K_1, K_2 be polyhedral cones with apex o. Then*

$$\begin{aligned}(K_1 \cap K_2)^\circ &= K_1^\circ + K_2^\circ\\ &= K_1^\circ \cup K_2^\circ,\end{aligned}$$

if $K_1^\circ \cup K_2^\circ$ is convex.

Further, if $\dim K < d$, then K° has $(\operatorname{lin} K)^\perp$ as its (non-trivial) face of apices, and so $\langle K^\circ\rangle \in \widehat{\Gamma}^d$; the converse is also obviously true. There then follows:

Theorem 4.23. *Polarity induces an involutory automorphism § of $\widehat{\Sigma}^d/\widehat{\Gamma}^d$, defined by*

$$\langle K\rangle^{\S} := \langle K^\circ\rangle.$$

This automorphism is called the *antipodal map*, and it extends to $\widehat{\Sigma}/\widehat{\Gamma}$ in the natural way, if § is now defined intrinsically; thus $b(F,K)^{\S} = c(F,K)$ in $\widehat{\Sigma}/\widehat{\Gamma}$. The algebra (ring) structure on $\widehat{\Sigma}$, with multiplication $*$, now induces an algebra structure on $\widehat{\Sigma}/\widehat{\Gamma}$. In fact, we also have a co-algebra structure.

Before we describe this, however, let us introduce something more general. If $x \in \widehat{\Sigma}$, we write $\overline{x}$ for its image in $\widehat{\Sigma}/\widehat{\Gamma}$; we also define $\overline{b}$ by $\overline{b}(F,K) := \overline{b(F,K)}$, and so on. The *total spherical Dehn invariant* of a pointed polyhedral cone K (or of the corresponding spherical polytope $K \cap \Omega$) is

$$\Psi_S := \sum_F^{*} \langle F\rangle \otimes \overline{b}(F,K) \in \widehat{\Sigma} \otimes (\widehat{\Sigma}/\widehat{\Gamma}),$$

where the sum extends over all faces $F \neq \{o\}$ with $\dim K - \dim F$ even. In fact, the terms with $\dim K - \dim F$ odd drop out anyway, and, when $\dim K$ is even, that for $F = \{o\}$ is not needed, because the information carried in the term $\langle K\rangle \otimes \overline{\langle o\rangle}$ contains that in $\langle o\rangle \otimes \overline{\langle K\rangle}$.

The map Ψ_S then induces a map $\overline{\Psi}_S : \widehat{\Sigma}/\widehat{\Gamma} \to (\widehat{\Sigma}/\widehat{\Gamma}) \otimes (\widehat{\Sigma}/\widehat{\Gamma})$, defined on the generators $\overline{\langle K\rangle}$ of $\widehat{\Sigma}/\widehat{\Gamma}$ by

$$\overline{\Psi}_S(\overline{\langle K\rangle}) := \sum_F^{*} \overline{\langle F\rangle} \otimes \overline{b}(F,K).$$

This is the *comultiplication* on $\widehat{\Sigma}/\widehat{\Gamma}$. The *co-unit* or *augmentation* (which is dual to the unit) is the natural mapping whose kernel is the set of elements of $\widehat{\Sigma}/\widehat{\Gamma}$ of positive degree. With these algebra and co-algebra structures and the antipodal map, $\widehat{\Sigma}/\widehat{\Gamma}$ then becomes a Hopf algebra (see Sah 1979).

In discussing equidissectability, it is natural to look for a suitable family of separating homomorphisms, as in section 3.3 for the polytope algebra. The map Ψ_S separates $\widehat{\Sigma}^d$, but in a rather trivial way, since it is obviously injective. More relevant is the *total classical Dehn invariant*

$$\Phi_S := (\text{gr.vol} \otimes \text{id}) \circ \Psi_S;$$

thus

$$\Psi_S(K) = \sum_F^{**} (\operatorname{vol} F \cdot T^{\dim F}) \otimes \overline{b}(F,K) \in \mathbb{R}[T] \otimes (\widehat{\Sigma}/\widehat{\Gamma}),$$

with the same summation convention as above.

In the positive direction, we have:

Theorem 4.24. *Φ_S separates $\widehat{\Sigma}^d$ for $d = 2$ or 3.*

This is really just a restatement of Lemma 4.14. For $d \geqslant 4$, however, the situation is quite different, and the general equidissectability problem remains unsolved. For example, if $d = 4$, and the cone K has rational dihedral angles, then $\Phi_S(K) = \operatorname{vol} K \cdot T^4$, so that, if Φ_S does separate $\widehat{\Sigma}^4$, then K should be equidissectable with a product cone. This is far from obviously true; indeed, such cones K are a possible source of torsion in $\widehat{\Sigma}^4$ (see Sah 1979).

A partial result in this direction by Dupont and Sah (1982) is the following.

Theorem 4.25. *A polyhedral cone which is the fundamental cone for a finite orthogonal group in $\mathbb{E}^d$ is equidissectable with a $(d-1)$-fold product cone.*

What is actually shown is that the fundamental polyhedral cones for two such orthogonal groups of the same order are equidissectable, and the core of the argument lies in proving it for p-groups (Sylow subgroups).

4.5. Hilbert's third problem

We now come to Hilbert's third problem and its variants. In section 1.4, we introduced the concepts of G-equidissectability and G-equicomplementability of polytopes (or polyhedra) under a group of affinities G of $\mathbb{E}^d$. As examples of typical such groups G, we have the groups T, TH, A and EA introduced in section 3.8, as well as the group D of all isometries of $\mathbb{E}^d$, and its subgroup SD of direct isometries or rigid motions. In what follows, we let G be a group of affinities. We begin with two important results of Hadwiger (1957).

Lemma 4.26. *Let $P, Q \in \mathcal{P}^d$. Then $P \approx_G Q$ if and only if $P \sim_G Q$.*

This result holds, in fact, whenever the base field $\mathbb{F}$ is Archimedean. We say that a simple valuation φ on $\mathcal{P}^d$ is *G-invariant* if $\varphi(\Phi P) = \varphi(P)$ whenever $\Phi \in G$. Then we have:

Theorem 4.27. *Let $P, Q \in \mathcal{P}^d$. Then $P \approx_G Q$ if and only if $\varphi(P) = \varphi(Q)$ for all G-invariant simple valuations φ on $\mathcal{P}^d$.*

Over a non-Archimedean field, $\approx_G$ has to be replaced by $\sim_G$.

Hadwiger's proof of Theorem 4.27 is highly non-constructive, because it uses the axiom of choice to pick a basis of the polytope group $\widehat{\Pi}_G$, which is obtained from $\widehat{\Pi}^d$ in the same way that Π_G is obtained from Π, namely by imposing on $\widehat{\Pi}^d$ the extra relations (G) $\langle \Phi P \rangle = \langle P \rangle$ for all $P \in \mathcal{P}^d$ and $\Phi \in G$.

In fact, the variants of *Hilbert's third problem* reduce to finding, for a given group G, a "nice" family of G-invariant simple valuations which separates $\widehat{\Pi}_G$. We have seen that the Hadwiger functionals provide such a family when $G = T$ (Theorem 4.2). Another group is easily dealt with.

Theorem 4.28. *Two d-polytopes P and Q are* TH*-equidissectable if and only if $h_U(P) = h_U(Q)$ for every Hadwiger functional whose type is congruent to d modulo* 2.

The reason is that, if h_U is of type r, then $h_U(-P) = h_U(P)$ for all P if and only if $r \equiv d$ modulo 2.

Another easy result, which is a consequence of Theorem 3.18(b), is the following.

Theorem 4.29. *Two d-polytopes P and Q are* EA*-equidissectable if and only if $V(P) = V(Q)$.*

As a last preliminary result, we have a result proved by Gerwien (1833a), which was also observed by F. Bólyai.

Theorem 4.30. *Two planar polygons are D-equidissectable if and only if they have the same area.*

This follows from the fact that a triangle is D- (or even SD- or TH-) equidissectable with a parallelogram; any two parallelograms of the same area are T-equidissectable. Hadwiger and Glur (1951) later showed that, if a group G of affinities is such that two planar polygons of the same area are always G-equidissectable, then $G \supseteq$ TH.

We now come to the classical version of Hilbert's third problem. In contrast to the planar situation, it was early recognized (by Gauss, among others) that the 3-dimensional case was likely to be more difficult. Hilbert (1900) formally posed the question of finding two 3-polytopes of the same volume (even pyramids with the same height on the same base) which were not D-equidissectable. Modifying an earlier attempt by Bricard (1896), Dehn (1900, 1902) found an example before the problem was published.

Before describing the example, we make some remarks about the polytope groups $\widehat{\Pi}^d_D$; we now distinguish the dimension. First, if two polytopes are symmetric in a hyperplane, then it may be shown that they are SD-equidissectable. Thus:

Lemma 4.31. $\widehat{\Pi}^d_D = \widehat{\Pi}^d_{\mathrm{SD}}$.

In view of the presence of scaling by -1, we also have (compare Theorem 4.28 above):

Lemma 4.32. $\widehat{\Pi}^d_D = \bigoplus_{r \equiv d(2), r>0} \widehat{\Xi}^d_r$.

That is, there are no graded terms $\widehat{\Xi}^d_r$ unless $r \equiv d$ modulo 2.

There is a natural product structure on $\widehat{\Pi}_D := \sum_{d \geqslant 0} \widehat{\Pi}^d_D$, induced by orthogonal Cartesian product, which, as in Theorem 4.10, we denote by $\times$. It is often helpful to work with $\widehat{\Pi}_D$, rather than with the individual terms $\widehat{\Pi}^d_D$. Observe that each term $\widehat{\Xi}_r$ with $r > 1$ is a sum of non-trivial products.

There are conjectures about separating functionals for $\widehat{\Pi}_D$, which are exactly analogous to those for separation of $\widehat{\Sigma}^d$ which were described in section 4.4. First, we have the *total Euclidean Dehn invariant* of a polytope P, defined by

$$\Psi_E(P) := \sum_F^* \langle F \rangle \otimes \bar{b}(F, P) \in \widehat{\Pi}_D \otimes (\widehat{\Sigma}/\widehat{\Gamma});$$

as in section 4.4, such sums are over all faces F of P with $\dim F > 0$ and $\dim P - \dim F$ even. Similarly, the *total classical Euclidean Dehn invariant* is

$$\Phi_E := (\mathrm{gr.vol} \otimes \mathrm{id}) \circ \Psi_E,$$

so that

$$\Phi_E(P) = \sum_F^* \mathrm{vol}\, F \cdot T^{\dim F} \otimes \bar{b}(F, P).$$

Theorem 4.30 shows that Φ_E separates $\widehat{\Pi}^d_D$ for $d = 2$ (and the case $d = 1$ is trivial). It was the considerable achievement of Sydler (1965) to extend this to the case $d = 3$. Jessen (1968) simplified Sydler's proof by using the language of the algebra of polytopes (see section 4.1), and then (Jessen 1972) extended it further to the case $d = 4$. Thus we have:

Theorem 4.33. *If $d \leqslant 4$, then Φ_E separates $\widehat{\Pi}^d_D$.*

We shall not give any of the details of the proof here; in the crucial case $d = 3$, these involve clever dissection results for special tetrahedra. However, a disadvantage of the proof is that it involves an appeal to the axiom of choice at several stages. More recently, though, Dupont and Sah (1992) have produced a quite different approach, using the Eilenberg–MacLane homology of the classical groups, and Hochschild homology of the quaternions; this avoids both the special constructions of Sydler (which Jessen's proof retains) and the need for any appeal to the axiom of choice.

We should note that the case $d = 4$ follows directly from the case $d = 3$ and $\widehat{\Pi}_D^4 = \widehat{\Xi}_2^4 \oplus \widehat{\Xi}_4^4$, which says that every $x \in \widehat{\Pi}_D^4$ is equivalent to a prism $e \times y$, where e is the class of a line segment and $y \in \widehat{\Pi}_D^3$; Jessen (1972) gave a direct proof of this result.

We end this section with a few remarks. First, the antipodal map ${}^{\S}S$ on the Hopf algebra $\widehat{\Sigma}/\widehat{\Gamma}$ (see section 4.4) enables us to replace $\overline{b}(F,P)$ by $\overline{c}(F,P)$ in the definitions of the two Euclidean Dehn invariants Ψ_E and Φ_E. In view of the isomorphism Theorems 3.12 and 4.4, this change is very natural, although Dupont and Sah (1992) offer contrary evidence in favour of retaining the present definitions.

Next, the Dehn invariants are compatible with the product structure. Indeed, since the angle cone of the orthogonal product $P \times Q$ at its face $F \times G$ is $B(F,P) \times B(G,Q)$, we have

$$\Psi_D(P \times Q) = \Psi_D(P)\Psi_D(Q).$$

A more general question along these lines, which is prompted as well by the proof of Theorem 4.33, is the following: *is every polytope equivalent to a direct sum of products of odd-dimensional components*? For example,

$$\widehat{\Pi}_D^2 = \widehat{\Xi}_2^2 \cong \widehat{\Xi}_1^1 \otimes \widehat{\Xi}_1^1,$$

$$\widehat{\Pi}_D^4 = \widehat{\Xi}_2^4 \oplus \widehat{\Xi}_4^4 \cong (\widehat{\Xi}_1^1 \otimes \widehat{\Xi}_1^3) \oplus (\otimes^4 \widehat{\Xi}_1^1),$$

since the other term $\widehat{\Xi}_1^2 \otimes \widehat{\Xi}_1^2$ in $\widehat{\Xi}_2^4$ vanishes. The first open question here concerns the case $d = 6$.

The space of indecomposable elements of $\widehat{\Pi}_D^d$ is certainly the sum of the $\widehat{\Xi}_1^{2s+1}$ for $s \geqslant 0$. We might also ask: *is $\widehat{\Pi}_D$ isomorphic to a symmetric algebra based on the space of indecomposable elements*? In particular: *is $\widehat{\Pi}_D$ an integral domain, or a Hopf algebra*?

A final question is: *what are the images of the Dehn Invariants*? This question is particularly interesting in case $d = 3$.

5. Characterization theorems

Certain valuations with invariance or covariance properties can be characterized in various ways. In this section, we shall survey the known results of this kind, as well as discussing several related open problems.

5.1. Continuity and monotonicity

We first consider some general relationships between valuations, involving continuity and similar notions. The natural metric on subfamilies of convex bodies is the Hausdorff metric, introduced in section 1.1, although other closely connected metrics have been used from time to time. In what follows, all functions will take values in some real vector space; a *functional* is then a real-valued function.

We shall call a functional φ on a subfamily $\mathcal{S}$ of $\mathcal{K}^d$ *monotone* if $\varphi(K) \leqslant \varphi(K')$ whenever $K, K' \in \mathcal{S}$ satisfy $K \subseteq K'$. The convention $\varphi(\emptyset) = 0$ for valuations φ ensures that a monotone valuation on $\mathcal{S}$ is non-negative.

A useful result of McMullen (1977) is the following (the case $d = 2$ was proved by Hadwiger 1951b).

Theorem 5.1. *A monotone translation invariant valuation is continuous.*

The theorem is initially proved for polytopes, and uses the polynomial expansion for translation invariant valuations of rational multiples of polytopes. The extension to general convex bodies is routine.

A different concept of continuity is often more appropriate for polytopes; it is due to Hadwiger (1952d). Let $U = (u_1, \ldots, u_n)$ be a (for the moment) fixed set of unit outer normal vectors, and write $\mathcal{P}^d(U)$ for the family of polytopes of the form

$$P(y) = \{x \in \mathbb{E}^d \mid \langle x, u_i \rangle \leqslant \eta_i \ (i = 1, \ldots, n)\},$$

where $y = (\eta_1, \ldots, \eta_n)$. We call a function φ on $\mathcal{P}_*^d$ *weakly continuous* if, for each such U, the function φ_U defined by $\varphi_U(y) = \varphi(P(y))$ is continuous. We clearly have:

Lemma 5.2. *A continuous function on $\mathcal{P}_*^d$ is weakly continuous.*

As we saw in sections 3.1 and 3.5, translation invariant (or even covariant) valuations admit polynomial expansions with rational coefficients. To extend these expansions to real coefficients, weak continuity, rather than continuity, will suffice. In fact, we have:

Theorem 5.3. *The following conditions on a translation invariant or covariant valuation φ on $\mathcal{P}_*^d$ are equivalent:*

(a) *φ is weakly continuous;*

(b) *$\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ is a polynomial in $\lambda_1, \ldots, \lambda_k$ for all polytopes $P_1, \ldots, P_k$ and all real numbers $\lambda_1, \ldots, \lambda_k \geqslant 0$;*

(c) *for each U, the one-sided partial derivatives of φ_U exist.*

In addition, if φ is translation invariant, a further equivalent condition is

(d) *φ is continuous under dilatations; that is, the mapping Θ_P on $\mathbb{R}$ defined by $\Theta_P(\lambda) = \varphi(\lambda P)$ is continuous.*

The equivalence of conditions (a), (b) and (c) is due to McMullen (1977). The equivalence of (a) and (d), which was left open by Hadwiger (1952e), follows from the fact that $Z_1 \subset \Pi$ (the polytope algebra) is a real vector space. This latter equivalence for translation covariant valuations (which was inadvertently claimed in Theorem 11.2 of McMullen and Schneider 1983) would follow if it could be shown that Π_1 were a real algebra (see section 3.5); a stronger condition which is equivalent is that φ is continuous under translations as well as dilatations (compare McMullen 1983).

A final remark is often useful.

Theorem 5.4. *The mixed valuations derived from a (weakly) continuous translation invariant or covariant valuation are (weakly) continuous in each of their arguments.*

Whether this, or an analogous, result holds for monotone valuations is unknown.

5.2. Minkowski additive functions

A starting point for the characterization of more general valuations is often that of certain special types. We have already discussed the Euler characteristic in section 2.1, and we shall consider volume and moment in section 5.3 immediately following. A further important case is that of a *Minkowski additive* function φ, which means that $\varphi(K+K') = \varphi(K)+\varphi(K')$ for appropriate K and K'; because of the strong assumption, more specific results are available for such functions.

Lemma 1.1 says that a Minkowski additive function φ is a valuation. Further, for fixed K, we also have $\varphi(\lambda K) = \lambda\varphi(K)$ for rational $\lambda \geqslant 0$, and, if φ is continuous, this holds for real λ.

Since the *width* $w_u(K) = h(K,u)+h(K,-u)$ of K in the direction of the unit vector u obviously gives a translation invariant valuation, so does the *mean width* $\bar{b}$, given by

$$\bar{b}(K) = \frac{2}{\omega_{d-1}} \int_{\Omega} h(K,u)\,\mathrm{d}\sigma(u)$$

for $K \in \mathcal{K}_*^d$, which is a constant multiple of the intrinsic length V_1 (or quermass-integral W_{d-1}). In fact, V_1 admits the following characterization.

Theorem 5.5. *If $\varphi : \mathcal{K}_*^d \to \mathbb{R}$ is Minkowski additive, continuous and invariant under rigid motions, then $\varphi = \alpha V_1$, for some real constant α.*

The proof of Hadwiger (1957, p. 213), uses a rotation averaging process, and actually shows more: it is only necessary to assume that φ is continuous at the unit ball B.

However, it would be nice to have Theorem 5.5 for suitable subclasses of $\mathcal{K}_*^d$, in particular for $\mathcal{P}_*^d$. If φ is locally uniformly continuous on $\mathcal{P}_*^d$, then it can be extended uniquely to $\mathcal{K}_*^d$, and any additive and invariance properties carry over; by *local*, we mean that the uniform continuity (or other condition) holds for the elements in any fixed ball. It is natural to ask whether a continuous or uniformly bounded Minkowski additive functional on $\mathcal{P}_*^d$ must also be a constant multiple of V_1.

The vector-valued counterpart of mean width is the *Steiner point* s, which is defined on $\mathcal{K}_*^d$ by

$$s(K) = \frac{1}{\kappa_d} \int_{\Omega} h(K,u)\,\mathrm{d}\sigma(u);$$

this is clearly Minkowski additive. We call a mapping f on $\mathcal{K}_*^d$ or $\mathcal{P}_*^d$ taking values in $\mathbb{E}^d$ *rigid motion (translation) equivariant* if $f(\Phi K) = \Phi f(K)$ for every rigid motion Φ of $\mathbb{E}^d$ ($f(K+t) = f(K)+t$ for every translation $t \in \mathbb{E}^d$, respectively). Then we have the analogue of Theorem 5.5:

Theorem 5.6. *If $f : \mathcal{K}_*^d \to \mathbb{E}^d$ is Minkowski additive, continuous and rigid motion equivariant, then $f = s$, the Steiner point.*

Again, it need only be assumed that f is continuous at the unit ball. The first proof of Theorem 5.6 was by Schneider (1971); the case $d = 2$ was earlier shown by Shephard (1968b) using Fourier series, and Schneider's proof generalizes this (though not straightforwardly) by using spherical harmonics. A more elementary proof, which makes weaker assumptions, was given by Positel'skiĭ (1973). However, the application of spherical harmonics seems to be the proper tool in this context, since the method also treats other similar problems.

Earlier attempts to characterize the Steiner point, a problem which was first posed by Grünbaum (1963, p. 239), are worth mentioning. Grünbaum asked whether s can be characterized by Minkowski additivity and dilatation equivariance, but this was shown not to be the case by Sallee (1971) and (with an easier counter-example) by Schneider (1974a). Shephard (1968b) added the continuity assumption, and Meyer (1970) proved a weaker version of Theorem 5.6 by assuming uniform continuity; two other attempts to prove the general result (Schmitt 1968, Hadwiger 1969a) contained errors.

In case $d = 2$, elementary proofs were given by Hadwiger (1971) and Berg (1971), the latter obtaining additional results for polytopes. To describe them, let ν be any outer angle (see section 3.4), and define s_ν by

$$s_\nu(P) = \sum_{v \in \operatorname{vert} P} \nu(v,P)\,v,$$

where $\operatorname{vert} P$ denotes the set of vertices of the polytope P. If ν is the usual rotation invariant angle, then s_ν is the Steiner point. In general, s_ν is translation and dilatation covariant, and it may readily be shown that it is also Minkowski additive. Berg (1971) calls a Minkowski additive function $f : \mathcal{P}_*^d \to \mathbb{E}^d$ which is rigid motion and dilatation covariant an *abstract Steiner point*. If the angle ν is rotation invariant, then s_ν is an abstract Steiner point. Whether the converse holds when $d > 3$ is an open question; Berg (1971) established this for $d = 2$ and 3, showing additionally that (in these cases) a locally bounded abstract Steiner point is the usual Steiner point.

Another interesting family of Minkowski additive functions consists of those which take values in $\mathcal{K}_*^d$ itself. They should, perhaps, be called *endomorphisms*, but Schneider (1974a), answering certain questions raised by Grünbaum (1963), reserves this term for those which are also continuous and rigid motion equivariant, noting that such functions are compatible with the most natural geometric structures on $\mathcal{K}_*^d$. The cases $d = 2$ and $d \geqslant 3$ show different features, since the rotation group is only commutative if $d = 2$.

For $d = 2$, Schneider (1974a) characterizes endomorphisms as limits of functions of the form

$$\Phi(K) := \sum_{j=1}^{k} \lambda_j \Psi_j[K - s(K)] + s(K)$$

for $K \in \mathcal{K}_*^d$, where $\lambda_j \geqslant 0$ and $\Psi_j \in SO_2$, the rotation group, for $j = 1, \ldots, k$. More precisely, if we write $u(\alpha) = (\cos\alpha, \sin\alpha)$ for $\alpha \in [0, 2\pi)$ (with coordinate vectors relative to some orthonormal basis of $\mathbb{E}^2$), we have

Theorem 5.7. *Let Φ be an endomorphism of $\mathcal{K}_*^2$. Then there exists a (positive) measure ν on the Borel subsets of $[0, 2\pi)$, such that*

$$h(\Phi(K), u(\alpha)) = \int_0^{2\pi} h(K - s(K), u(\alpha + \beta))\ \mathrm{d}\nu(\beta) + \langle s(K), u(\alpha)\rangle$$

for $\alpha \in [0, 2\pi)$ and all $K \in \mathcal{K}_*^2$.

The proof, by Schneider (1974b), uses a characterization by Hadwiger (1951b) of continuous translation invariant Minkowski additive functionals on $\mathcal{K}_*^2$; clearly, any Borel measure ν on $[0, 2\pi)$ defines an endomorphism Φ in this way. Schneider gives additional results about the uniqueness of ν for a given Φ, and about the nature of Φ when its image contains a polygon. In particular, if Φ maps $\mathcal{K}_*^2$ *onto* itself, it takes the form $\Phi(K) = \lambda\Psi(K - s(K)) + s(K)$ for some $\lambda > 0$ and $\Psi \in SO_2$. The space of all endomorphisms is also shown to have the structure of a convex cone, and certain properties proved by Inzinger (1949) for special endomorphisms are extended (after some normalization) to all of them.

For $d \geqslant 3$, as yet only partial results have been obtained, by Schneider (1974a). If $q : [0, \infty) \to [0, \infty)$ is a function for which the following integrals exist and are finite, then for $K \in \mathcal{K}_*^d$ there is a unique $\Phi_q(K) \in \mathcal{K}_*^d$ for which

$$h(\Phi_q(K), u) = \int_{\mathbb{E}^d} h(K - s(K), u - \|u\|z)q(\|z\|)\,\mathrm{d}z + \langle s(K), u\rangle$$

for $u \in \mathbb{E}^d$ (support functions are here defined for all vectors in $\mathbb{E}^d$); this Φ_q is then an endomorphism of $\mathcal{K}_*^d$. Such endomorphisms have proved useful in the study of certain approximation problems; see Berg (1969) and Weil (1975b). Schneider (1974a) obtained various results which characterized certain kinds of endomorphism Φ of $\mathcal{K}_*^d$ (for brevity, we tacitly assume this notation below, and any K will lie in $\mathcal{K}_*^d$).

Theorem 5.8. (a) *Every such Φ is uniquely determined by the image of some suitable convex body, such as a triangle with an irrational angle.*

(b) *If Φ takes a body which is not a point onto a point, then $\Phi(K) = s(K)$ for all K. If Φ takes some body onto a segment, then*

$$\Phi(K) = \lambda[K - s(K)] + \mu[-K + s(K)] + s(K)$$

for some $\lambda, \mu \geqslant 0$ with $\lambda + \mu > 0$.

(c) *If Φ is surjective, then*

$$\Phi(K) = \lambda[K - s(K)] + s(K)$$

for some $\lambda \neq 0$.

(d) *If Φ satisfies $V_r(\Phi(K)) = V_r(K)$ for some $r = 2, \ldots, d$, then*

$$\Phi(K) = \varepsilon[K - s(K)] + s(K)$$

for some $\varepsilon = \pm 1$.

Writing $\mathcal{K}_d^d$ for the family of full-dimensional convex bodies, we further have:

Theorem 5.9. *Let $\Phi : \mathcal{K}_d^d \to \mathcal{K}_*^d$ be a continuous Minkowski additive mapping, such that $\Phi(\Psi(K)) = \Psi(\Phi(K))$ for every affinity Ψ of $\mathbb{E}^d$. Then*

$$\Phi(K) = K + \lambda[K - K]$$

for some $\lambda \geqslant 0$.

If invariance or equivariance with respect to some group of affinities of $\mathbb{E}^d$ is not assumed, then other stronger conditions must be imposed. For example, Schneider (1974c) has shown:

Theorem 5.10. *Let $\Phi : \mathcal{K}_*^d \to \mathcal{K}_*^d$ be a Minkowski additive function such that $V(\Phi(K)) = V(K)$ for each K. Then there is an equiaffinity Ψ of $\mathbb{E}^d$ such that $\Phi(K)$ is a translate of $\Psi(K)$ for each K.*

Finally, we remark that Valette (1974) has studied the continuous maps $\Phi : \mathcal{K}_*^d \to \mathcal{K}_*^d$ (with $d \geqslant 2$) which commute with affine maps, and satisfy the weaker condition $\Phi(K_1 + K_2) \supseteq \Phi(K_1) + \Phi(K_2)$ for $K_1, K_2 \in \mathcal{K}_*^d$.

5.3. *Volume and moment*

Until further notice, we shall take all valuations to be real-valued. We first deal with the characterizations of volume. From a geometric viewpoint, we should wish for something simpler than the fact that the essential uniqueness of Haar measure on $\mathbb{E}^d$ shows that Lebesgue measure (that is, volume) is the unique translation invariant (positive) measure φ on the Borel sets of $\mathbb{E}^d$ such that $\varphi(C) = 1$ for some fixed unit cube C. In particular, we should prefer to consider simple valuations on $\mathcal{P}^d$ or $\mathcal{K}^d$, rather than σ-additive functions on Borel sets. We shall, in fact, take $\mathcal{P}^d$ as our domain of definition, since the assumptions we have to impose will extend to uniqueness on $\mathcal{K}^d$ as well. Further, the extension of a simple valuation to $U\mathcal{P}^d$ will share any non-negativity or monotonicity property, or invariance under any group of affinities, of the original. Bearing these remarks in mind, we have:

Theorem 5.11. *If φ is a translation invariant, non-negative simple valuation on $\mathcal{P}^d$, then $\varphi = \alpha V$ for some $\alpha \geqslant 0$.*

Various proofs of this theorem, for example, that in Maak (1960) (see also Hadwiger 1955b), use techniques such as exhaustion or polyhedral approximation, and so are not strictly speaking elementary. Some continuity argument is necessary, since finite dissections alone will not suffice to compare volumes whose ratio is irrational. However, the limit process can be reduced to the essential uniqueness of monotone real-valued functions λ which satisfy Cauchy's equation $\lambda(\alpha+\beta) = \lambda(\alpha)+\lambda(\beta)$ (with $\alpha, \beta \in \mathbb{R}$), while the remainder of the proof does only use finite dissections. Such a proof was given by Hadwiger (1950d, 1957, section 2.1.3); and (1949a) for $d = 3$.

As shown by Schneider (1978), the analogous result holds in spherical spaces as well (for the notion of polytopes in these and hyperbolic spaces, see Böhm and Hertel 1980), when rotation invariance replaces translation invariance. An extension to compact homogeneous spaces was given by Schneider (1981), and a general approach which treats hyperbolic spaces also was outlined in McMullen and Schneider (1983, p. 226).

Returning to $\mathbb{E}^d$, we have the following alternative characterizations; the first is due to Hadwiger (1957, p. 79), the second to Hadwiger (1970), and the third to Hadwiger (1952e, 1957, p. 221).

Theorem 5.12. *A translation invariant valuation on $\mathcal{P}^d$ which is homogeneous of degree d is a constant multiple of volume.*

Theorem 5.13. *A non-negative simple valuation on $\mathcal{P}^d$ which is invariant under volume preserving linear mappings of $\mathbb{E}^d$ is a constant multiple of volume.*

Theorem 5.14. *A continuous rigid motion invariant simple valuation on $\mathcal{K}^d$ is a constant multiple of volume.*

For the last, one would like to replace $\mathcal{K}^d$ by $\mathcal{P}^d$, but it is so far unknown whether this is possible (the proof of Theorem 5.14 uses Theorem 5.5). Whether there is a characterization of the usual rotation invariant angle on convex cones (polyhedral or more general) analogous to Theorem 5.14 is also an open problem. We may observe that Theorem 4.17 gives a reduction from odd-dimensional polyhedral cones to products of lower-dimensional cones, but it is not clear that this remark is particularly helpful.

If $K \in \mathcal{K}^d_d$, then its *centroid* $c(K)$ is defined by $V(K)c(K) := z(K)$, the moment vector of K. We have the following counterparts to Theorems 5.11 and 5.14, which are due to Schneider (1973) and (1972b), respectively.

Theorem 5.15. *If $f : U\mathcal{P}^d_d \to \mathbb{E}^d$ is a translation equivariant function, such that Vf is a simple valuation and $f(P) \in \operatorname{conv} P$ for each P, then $f = c$.*

Theorem 5.16. *If $f : \mathcal{K}^d_d \to \mathbb{E}^d$ is a continuous rigid motion equivariant function such that Vf is a valuation, then $f = c$.*

5.4. Intrinsic volumes and moments

One of the central results in the theory of valuations is Hadwiger's famous characterization of linear combinations of quermassintegrals. Rephrasing this in the language of intrinsic volumes, it states:

Theorem 5.17. *If $\varphi : \mathcal{K}^d \to \mathbb{R}$ is a continuous rigid motion invariant valuation, then there are constants $\alpha_0, \ldots, \alpha_d$ such that*

$$\varphi(K) = \sum_{r=0}^{d} \alpha_r V_r(K)$$

for all $K \in \mathcal{K}^d$.

There is a variant on Theorem 5.17, with monotonicity replacing continuity.

Theorem 5.18. *If $\varphi : \mathcal{K}^d \to \mathbb{R}$ is a monotone rigid motion invariant valuation, then there are non-negative constants $\alpha_0, \ldots, \alpha_d$ such that*

$$\varphi(K) = \sum_{r=0}^{d} \alpha_r V_r(K)$$

for all $K \in \mathcal{K}^d$.

Blaschke (1937) was the first to produce a result of this kind (with $d = 3$), but he needed to make a somewhat artificial assumption about the "volume part" of a valuation. Hadwiger proved Theorem 5.17 for $d = 3$ in (1951a) (see also 1955b), and for general d in (1952e). Theorem 5.18 was proved in Hadwiger (1953a); it can also be deduced form Theorem 5.17 by means of Theorem 5.1. Both results appear in Hadwiger (1957, section 6.1.10) (see also Leichtweiß 1980).

Since Hadwiger's proof of Theorem 5.17 uses Theorem 5.14 (which in turn relies on Theorem 5.5), it is unclear whether $\mathcal{K}^d$ can be replaced by $\mathcal{P}^d$ in these theorems, possibly with local boundedness or non-negativity instead of continuity or monotonicity. It should be noted that these alternative conditions are inappropriate for general convex bodies; if $\varphi(K)$ is the sum of the $(d-1)$-dimensional volumes of the $(d-1)$-faces of the convex body K (or twice the $(d-1)$-dimensional volume of an at most $(d-1)$-dimensional body), then φ is a rigid motion invariant valuation which is locally bounded and non-negative, but which is clearly not a linear combination of intrinsic volumes. Theorem 5.17 has important applications to integral geometry. The basic idea, which is to show that certain integrals give continuous or monotone rigid motion invariant valuations, goes back to Blaschke (1937), but was systematically exploited by Hadwiger (1950e, 1955b, 1956, 1957) to derive both old and new integral geometric formulae. A different kind of application, to random sets, was made by Matheron (1975), and variants of the theorem are due to Groemer (1972) and Baddeley (1980).

It might be expected that analogues of Theorems 5.17 and 5.18 hold in spherical and hyperbolic space. For a polyhedral cone C with face of apices A, there are two analogues of the intrinsic volumes. Define $\beta(A,F)$ to be the ordinary normalized angle (in $\operatorname{lin} F$) of the face F of C, and let $\gamma(F,C)$ be the similarly normalized angle of the normal cone to C at F (these are the *intrinsic inner* and *outer angles*, see McMullen 1975). For $r=0,\ldots,d$, define

$$\varphi_r(C) := \sum_{F^r} \beta(A,F^r)\gamma(F^r,C),$$

where the sum extends over all r-faces F^r of C, and

$$\psi_r := \sum_{m\geqslant 0} \varphi_{d+1-r+2m}.$$

Clearly, φ_d is increasing, it can be shown that φ_{d-1} is also (see Shephard 1968d), and by duality, φ_0 and φ_1 are decreasing; however, for $2\leqslant r\leqslant d-2$, examples show that φ_r is neither increasing nor decreasing (see McMullen and Schneider 1983, section 3). Since $2\psi_r(C)$ is the normalized measure of the r-dimensional linear subspaces of $\mathbb{E}^d$ which do not meet C in the origin o alone, it is also increasing. Moreover, all these functions are continuous, and extend to general closed convex cones. It may be conjectured that continuous (monotone) rotation invariant valuations on $\mathscr{C}^d$ are linear combinations of the φ_r (ψ_r with non-negative coefficients, respectively); the first question has an affirmative answer if the corresponding characterization of spherical volume, mentioned in section 5.3, is valid.

5.5. *Translation invariance and covariance*

In a sense, the description of the polytope algebra in section 3, and particularly the isomorphism Theorem 3.12, tell us what a translation invariant valuation on $\mathscr{P}^d$ looks like; it is just the composition of σ with some group homomorphism. However, without some additional assumptions on the valuation, the resulting characterization will be too vague to be useful.

Natural conditions to impose include some form of continuity. We shall discuss the known results in this area. We begin with weak continuity. Throughout, $\mathscr{X}$ will denote a real vector space.

Theorem 5.19. *A function $\varphi:\mathscr{P}^d\to\mathscr{X}$ is a weakly continuous translation invariant valuation if and only if*

$$\varphi(P) = \sum_F \operatorname{vol} F\,\lambda(F,P),$$

where $\lambda:\hat{\Sigma}\to\mathscr{X}$ is a simple valuation, and vol *is ordinary volume.*

This follows directly from Theorem 3.12, but was proved by McMullen (1983) using the following result of Hadwiger (1952e) (the extension from the original paper stated here is straightforward).

Theorem 5.20. *A function $\varphi:\mathscr{P}^d\to\mathscr{X}$ is a weakly continuous translation invariant simple valuation if and only if*

$$\varphi(P) = \sum_U \operatorname{vol} P_U\,\eta(U),$$

where η is an odd function on frames, and vol *is ordinary volume.*

In view of the isomorphism Theorem 4.4 for the polytope group $\hat{\Pi}$, we could rephrase Theorem 5.20 in terms of mappings on $\mathbb{R}\otimes(\hat{\Sigma}/\hat{\Gamma})$. It should also be remarked that there are exactly analogous results for translation covariant valuations, which involve the moment vectors of faces as well as their volumes; see McMullen (1983) for details.

We now turn to continuity, and for simplicity confine our attention to real-valued valuations. The problem of characterizing continuous translation invariant valuations remains open; the supposed characterization of Betke and Goodey (1984) was unfortunately flawed. However, there are some partial results. For dimension $d=2$ there is a complete solution by Hadwiger (1949, 1951b).

Theorem 5.21. (a) *If φ is a continuous translation invariant valuation on $\mathscr{K}^2$, then*

$$\varphi(K) = \alpha + \int_{\Omega^1} g(u)\,\mathrm{d}S_1(K;u) + \beta V_2(K)$$

for some constants α,β and some continuous function g.

(b) *If φ is a locally bounded translation invariant valuation on $\mathscr{P}^2$, then the same expression for φ holds, with g a bounded function.*

Actually, Hadwiger did not express his results in terms of the area function $S_1(K;\cdot)$. The function g is uniquely determined, up to a function of the form $\langle v,\cdot\rangle$, with v a constant vector. If φ is just Minkowski additive in part (b), then the same result holds with $\alpha=\beta=0$.

For $d\geqslant 4$, no such explicit representations are known; the case $d=3$ is covered by the results below. If we use the fact that a continuous translation invariant valuation φ can be written as a sum $\varphi=\sum_{r=0}^d \varphi_r$, with φ_r (continuous and) homogeneous of degree r, then we can obviously investigate the individual φ_r. By Hadwiger (1957, p. 79), φ_d is a constant multiple of volume, and φ_0 is constant. The only complete solutions for any of the remaining cases are those of McMullen (1980) for $r=d-1$, and Goodey and Weil (1984) for $r=1$.

Theorem 5.22. *Let φ be a continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree $d-1$. Then there is a continuous function g on the unit sphere Ω, such that*

$$\varphi(K) = \int_\Omega g(u)\,\mathrm{d}S_{d-1}(K;u)$$

for each $K\in\mathscr{K}^d$.

As above, g is unique up to a function of the form $\langle v, \cdot \rangle$. The valuation can also be expressed as a limit

$$\varphi(K) = \lim_{i\to\infty}[V(K,d-1;L_i) - V(K,d-1;M_i)],$$

for suitable sequences $(L_i), (M_i)$ of convex bodies.

When $r = 1$, we have the following.

Theorem 5.23. *Let φ be a continuous translation invariant valuation on $\mathcal{K}^d$ which is homogeneous of degree 1. Then there are sequences $(L_i), (M_i)$ of convex bodies such that*

$$\varphi(K) = \lim_{i\to\infty}[V(K;L_i,d-1) - V(K;M_i,d-1)],$$

uniformly for all $K \subseteq mB$ and all $m > 0$.

For $r \in \{2,\ldots,d-2\}$, rather less is known. Clearly, suitable limits of mixed volumes will provide continuous translation invariant valuations, but it is open whether all such valuations can be obtained in this way. Goodey and Weil (1984) tried to relate such (mixed) valuations to distributions on tensor products of support functionals, of the form

$$\varphi(K_1,\ldots,K_r) = T(h(K_1,\cdot)\otimes\cdots\otimes h(K_r,\cdot)),$$

but an important part of their argument seems to be invalid.

With a stronger continuity assumption, McMullen (1980) (compare also Schneider 1974b) showed that a uniformly continuous translation invariant valuation on $\mathcal{K}^d$ which is homogeneous of degree 1 is of the form

$$\varphi(K) = V(K;L,d-1) - V(K;M,d-1)$$

for some convex bodies L, M; this can be deduced from the Riesz representation theorem.

A little more can be said about the case of monotone valuations.

Theorem 5.24. *Let $r = 1$ or $d-1$. If φ is a monotone translation invariant valuation on $\mathcal{K}^d$ which is homogeneous of degree r, then there exist convex bodies $L_{r+1},\ldots,L_d$, such that*

$$\varphi(K) = V(K,r;L_{r+1},\ldots,L_d).$$

The case $r = d$ is similar, with a non-negative multiple inserted (Theorem 5.11). For the theorem, the case $r = 1$ is due to Firey (1976), while the case $r = d-1$ was proved by McMullen (1990). It would be tempting to conjecture that the same result holds for all r, but the evidence to support this is meagre.

Finally, let us mention translation covariant valuations. McMullen (1983) proved the following analogue of Theorem 5.19. As before, $\mathcal{X}$ is a real vector space.

Theorem 5.25. *A function $\varphi : \mathcal{P}^d \to \mathcal{X}$ is a weakly continuous translation covariant valuation if and only if*

$$\varphi(P) = \sum_F (\operatorname{vol} F\, \lambda(F,P) + m(F)\, \Lambda(F,P)),$$

where vol is volume, m is moment, and $\lambda : \hat{\Sigma} \to \mathcal{X}$ and $\Lambda : \hat{\Sigma} \to \operatorname{Hom}(\mathbb{E}^d,\mathcal{X})$ are simple valuations.

There is a similar result for simple weakly continuous translation covariant valuations; compare Theorem 5.20.

5.6. Lattice invariant valuations

We call a function on subsets of $\mathbb{E}^d$ which is invariant under the translations of the integer lattice $\mathbb{Z}^d$ *lattice invariant*. A *unimodular* mapping of $\mathbb{E}^d$ is an affinity which leaves $\mathbb{Z}^d$ invariant; it is therefore the composition of a linear mapping whose matrix (with respect to the standard basis) has integer entries and determinant ± 1 with a lattice translation. The lattice point enumerator G, the derived functionals G_r and the weighted lattice point numbers A are examples of lattice invariant valuations, and the first two are also invariant under unimodular mappings. In this section, we consider various characterization theorems along the lines of, for example, Theorem 5.17, on the classes $\mathcal{P}^d_L$ of lattice polytopes and $\mathcal{P}^d_{\mathbb{Q}}$ of polytopes whose vertices have rational coordinate vectors. The first result is due to Betke (1979, unpublished a).

Theorem 5.26. *Let $\varphi : \mathcal{P}^d_L \to \mathbb{R}$ be a valuation which is invariant under unimodular mappings. Then there are constants $\alpha_0,\ldots,\alpha_d$ such that*

$$\varphi = \sum_{r=0}^{d} \alpha_r G_r.$$

Originally, Betke assumed that φ satisfied the stronger inclusion–exclusion principle, but Stein (1982) showed that this followed from the valuation property and lattice invariance; later, Betke (unpublished b) was able to remove this latter assumption.

A consequence of Theorem 5.26 and the method of its proof is the following description of the underlying abstract structure of valuations on $\mathcal{P}^d_L$ which are invariant under unimodular mappings.

Theorem 5.27. *The Abelian group Π_L generated by the equivalence classes of lattice polytopes under unimodular mappings, with addition defined by the valuation property* (V), *is isomorphic to $\mathbb{Z}^{d+1}$.*

In fact, the $d+1$ generators of the group are just the classes of the lattice polytopes $\operatorname{conv}\{o,e_1,\ldots,e_r\}$, for $r = 0,\ldots,d$. This result was proved by Betke and

Kneser (1985). Müller (1988) has extended these results to equidissectability with respect to more general crystallographic groups.

We finally discuss lattice invariant and covariant valuations on $\mathcal{P}^d_{\mathbb{Q}}$. McMullen (1983) proved the following analogue of Theorem 5.19.

Theorem 5.28. *A function* $\varphi : \mathcal{P}^d_{\mathbb{Q}} \to \mathbb{R}$ *is a lattice invariant valuation if and only if*

$$\varphi(P) = \sum_F \operatorname{vol} F \, \gamma(F, P),$$

where γ *is a real valued function on translates of normal cones, which depends only on the translation class of* $\operatorname{aff} F$ *modulo* $\mathbb{Z}^d$.

In McMullen (1978), the corresponding result for simple valuations was proved.

There are results on covariant valuations analogous to Theorem 5.25; these are mentioned in McMullen (1983).

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We have attempted to give a complete bibliography on valuations and dissections. As a consequence, some of the references listed here may not be mentioned in the text.

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CHAPTER 3.7

Geometric Crystallography

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Contents

HANDBOOK OF CONVEX GEOMETRY
Edited by P.M. Gruber and J.M. Wills

CHAPTER 4.1

Convexity and Differential Geometry

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Contents

HANDBOOK OF CONVEX GEOMETRY
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Introduction

In differential geometry intrinsic geometric properties of a differentiable m-dimensional manifold M as well as extrinsic properties with respect to an immersion of M into the d-dimensional Euclidean space $\mathbb{R}^d$ $(m < d)$ are studied. By such an immersion we mean a differentiable map $x : M \to \mathbb{R}^d$, given by $x = x(u^1, \dots, u^m)$ in the local coordinates $u^1, \dots, u^m$ of M, where the induced linear map $x_* : T_pM \to \mathbb{R}^d$ is of maximal rank or, equivalently, the partial derivatives $x_i := \partial x/\partial u^i$, $i = 1, \dots, m$, are linearly independent at each point $p \in M$. The map x defines a differentiable submanifold F of $\mathbb{R}^d$ which is called hypersurface in the special case $m = d - 1$. A differentiable hypersurface F may be oriented if M is covered by coordinate systems with coordinate changes of positive functional determinant everywhere. In this case the normalized vector product of $x_1, \dots, x_{d-1}$ represents the so-called "unit normal vector" n of F which is defined independently from the chosen coordinate system of M.

The importance of convexity in differential geometry consists in the fact that certain differentiability assumptions easily may be removed if a differentiable hypersurface F of $\mathbb{R}^d$ is "convex", i.e., the point set $x(M)$ lies in the boundary of some suitable closed convex set K of $\mathbb{R}^d$. By this way one obtains convex geometric generalizations of notions and theorems in differential geometry. Conversely, if the boundary of a closed convex set K of $\mathbb{R}^d$ is the point set of some differentiable hypersurface F, then it is not hard to compute convex geometric entities and to prove convex geometric theorems in a differential geometric manner.

In order to get an impression of the influence of the convexity of a differentiable hypersurface F with $x : M \to \mathbb{R}^d$ of differentiability class C_2 we choose the first $d - 1$ coordinates $x^1, \dots, x^{d-1}$ of a suitable Cartesian coordinate system $\{x^1, \dots, x^d\}$ in $\mathbb{R}^d$ as local parameters of F such that

$$x^d = f(x^1, \dots, x^{d-1}), \tag{1}$$

with f of C_2 is a local representation of F. Without loss of generality we may assume that H^-: $x^d \geq 0$ is a supporting halfspace of $x(M)$ with the additional property that H: $x^d = 0$ touches $x(M)$ at the origin. hence

$$f(0, \dots, 0) = f_i(0, \dots, 0) = 0 \quad \text{and} \quad f(x^1, \dots, x^{d-1}) \geq 0 \tag{2}$$

$(f_i := \partial f/\partial x^i$, $i = 1, \dots, d - 1)$. Moreover, f must be a convex function, and it is well known that this is equivalent to the fact that the quadratic form

$$f_{ij}\xi^i\xi^j \quad \left(f_{ij} := \frac{\partial^2 f}{\partial x^i\, \partial x^j}\,, \quad i, j = 1, \dots, d - 1\right) \tag{3}$$

is positive semidefinite everywhere.

This may be expressed in a more geometrical manner: the curvature k of a plane normal section of F is given as the quotient of the "second fundamental form" of F

$$h_{ij}\xi^i\xi^j \quad \left(h_{ij} := \left\langle \frac{\partial^2 x}{\partial x^i\, \partial x^j}, n \right\rangle = \frac{f_{ij}}{(1+\Sigma_{i=1}^{d-1}(f_i)^2)^{1/2}}, \right.$$
$$\left. i, j = 1, \ldots, d-1 \right) \tag{4}$$

and the (positive definite) "first fundamental form" of F

$$g_{ij}\xi^i\xi^j \quad (g_{ij} := \langle x_i, x_j\rangle, \quad i, j = 1, \ldots, d-1). \tag{5}$$

If the position of the cutting plane (given by $\xi^1, \ldots, \xi^{d-1}$) varies, k attains $d-1$ stationary values $k_1, \ldots, k_{d-1}$, the so-called "principal curvatures" of F. According to Lagrange's multiplicator method they are the roots of the characteristic equation

$$\det(h_{ij}g^{jk} - z\delta_i^k) = 0. \tag{6}$$

Their normalized elementary symmetric functions,

$$\begin{aligned} H_1 &:= \frac{1}{d-1}(k_1 + \cdots + k_{d-1}), \\ &\ \vdots \\ H_{d-1} &:= k_1 \cdots k_{d-1}, \end{aligned} \tag{7}$$

are defined as the "mean curvatures" of F (especially H_{d-1} is called the "Gaussian curvature" of F).

Now the positive semidefiniteness of $f_{ij}\xi^i\xi^j$ is equivalent to the positive semidefiniteness of the second fundamental form of F, eq. (4), or to the nonnegativity of all principal curvatures of F. We comprehend: *locally, a hypersurface F in $\mathbb{R}^d$ is convex iff (after changing the orientation of F eventually) all principal curvatures $k_1, \ldots, k_{d-1}$ of F are nonnegative. In this case also the Gaussian curvature H_{d-1} of F is nonnegative.*

1. Differential geometric characterization of convexity

A basic result on the global characterization of the convexity of a compact hypersurface in $\mathbb{R}^d$ is:

Theorem 1.1 (Hadamard 1897, pp. 352–353). *Let $x: M \to \mathbb{R}^d$ $(d \geq 3)$ be a compact oriented hypersurface F of class C_2 with positive Gaussian curvature:*

$$H_{d-1} > 0. \tag{8}$$

Then x is an embedding and $x(M)$ is equal to the boundary of a suitable compact convex body K in $\mathbb{R}^d$.

Proof. We consider the spherical map ν of F, defined by $x(p) \in x(M) \mapsto -n(p) \in S^{d-1}$. Taking into account, besides eqs. (8) and (7), a special point $p_0 \in M$ where a hyperplane of $\mathbb{R}^d$ touches the compact set $x(M)$ in $\mathbb{R}^d$, we can conclude that the principal curvatures $k_1, \ldots, k_{d-1}$ of F at $x(p_0)$ are all of the same sign, say $+1$, after changing the orientation of F eventually. This remains true for all points $p \in M$ [as no principal curvature of F changes its sign because of (8)] and justifies us to say that F is "one-sided curved". Moreover, the induced linear map $\nu_*: x_*(T_pM) \to T_{-n(p)}S^{d-1}$, defined by the Weingarten equations:

$$-n_i(p) = h_{ij}(p)g^{jk}(p)x_k(p) \quad \left(n_i := \frac{\partial n}{\partial u^i}, \quad i = 1, \ldots, d-1\right) \tag{9}$$

[compare (4) and (5)], has the eigenvalues $k_1, \ldots, k_{d-1}$ and a positive determinant

$$\det \nu_* = \det(h_{ij}g^{jk}) = k_1 \cdots k_{d-1} > 0 \tag{10}$$

everywhere, see (6). As a consequence the image of the spherical map ν itself is a covering of S^{d-1} without boundary or branch points. Therefore, S^{d-1} being simply connected for $d \geq 3$, this covering must be onefold. So each tangential hyperplane $H(p)$ of F with the "outer" unit normal $-n(p)$ is a supporting hyperplane of F. Namely otherwise, by a compactness argument of $x(M)$, there would exist a supporting hyperplane of F with the same outer normal vector $-n(p)$ and different from $H(p)$, which is impossible. This yields the fact that $x(M)$ coincides with the boundary of the compact convex intersection K of all closed supporting halfspaces $H^-(p)$ of F and that x is injective. □

Remark 1.2. Theorem 1.1 becomes wrong if we have $d = 2$ (regard the slightly disturbed k times traversed unit circle in the plane!).

The assumptions in Theorem 1.1 may be essentially weakened in the sense that the positivity of the principal curvatures k_i of F is replaced by their nonnegativity (i.e., the local convexity of F), with the exception of one point, and the compactness of F by its completeness. This is a trivial consequence of:

Theorem 1.3 (van Heijenoort 1952, p. 241). *If $x: M \to \mathbb{R}^d$ $(d \geq 3)$ is a complete, locally homeomorphic mapping F of the $(d-1)$-dimensional topological manifold M which is*

(i) *locally convex, and*

(ii) *"absolutely convex" at a point p of M (i.e., there is locally a supporting hyperplane of $x(M)$ which intersects $x(M)$ only at p),*

then x is homeomorphic and $x(M)$ is the boundary of a suitable convex body in $\mathbb{R}^d$.

We omit the proof of this theorem – where differential geometry is not involved – but make the following remark.

Remark 1.4. Theorem 1.3 becomes wrong if assumption (ii) is dropped. Counterexample: Cartesian product of $\mathbb{R}^{d-2}$ and the curve in $\mathbb{R}^2$ of Remark 1.2.

Remark 1.5. Before van Heijenoort, Stoker (1936) had just proven that Theorem 1.1 remains valid if $d=3$ and F is complete instead of being compact (with K not necessarily compact).

The last step of extensions of Theorem 1.1 was done in:

Theorem 1.6 (Sacksteder 1960). *Let $x: M \to \mathbb{R}^d$ $(d \geq 3)$ be a complete hypersurface F of class C_{d+1} with*

(i) *a (positive or negative) semidefinite second fundamental form (4) everywhere,*

or equivalently,

(ii) *nonnegative sectional curvature*

$$K(\xi, \eta) := \frac{R_{ijkl}\xi^i\eta^j\xi^k\eta^l}{(g_{ik}g_{jl} - g_{il}g_{jk})\xi^i\eta^j\xi^k\eta^l} \quad (\xi, \eta = \text{linearly independent}) \tag{11}$$

of the induced Riemannian metric (5) on M everywhere $[R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} =$ components of the Riemannian curvature tensor of the metric (5)].

Then either

(a) *$x(M)$ is the boundary of a convex body K in $\mathbb{R}^d$ (if $K(\xi,\eta) \not\equiv 0$) or*

(b) *$x(M)$ is a "hypercylinder" consisting of ∞^1 parallel $(d-2)$-flats in $\mathbb{R}^d$ (if $K(\xi,\eta) \equiv 0$).*

Proof (*Outline*). M is divided into the set M_0 of flat points $(h_{ij}\xi^i\xi^j \equiv 0)$ and its complement M_1 $(h_{ij}\xi^i\xi^j \not\equiv 0)$. Then for each (connected) component C of M_0 the unit normal vector $n|_C$ is constant. Namely, a theorem of Sard (1942, p. 888) says that $\nu(C) = -n(C)$ is a one-dimensional zero set of S^{d-1} and therefore totally disconnected as image of flat points of M where the rank of the induced linear map ν_*, given by (9), is 0. But simultaneously, $\nu(C)$ must be connected as continuous map of the connected set C. Thus $n(C) = n_0$ and $\langle n_0, x\rangle|_C = \text{const}$, whence

$$x(C) \subset H \tag{12}$$

for a suitable hyperplane H in $\mathbb{R}^d$. But eq. (12) and simple convexity arguments, applied to the convex hypersurfaces $x(D_\alpha)$ (D_α = component of M_1) imply that $x(C)$ *is a convex set in* $\mathbb{R}^d$. Conversely, a k-flat $x(L)$ on $x(M)$ has a constant unit normal vector. These two facts, together with the completeness of F and rather

complicated topological considerations permit to reduce the proof of Theorem 1.6 to Theorem 1.3 by factoring out eventually a suitable k-flat of $x(M)$ in case (a). □

Remark 1.7. The assumption of completeness of F is essential in Theorem 1.6 as seen in the following example: $d=3$ and F: $x^3 = (x^1)^3(1+(x^2)^2)$ with $(x^2)^2 < \frac{1}{2}$. A simple calculation shows that the second fundamental form of F is positive definite for $x^1 > 0$, zero for $x^1 = 0$ and negative definite for $x^1 < 0$, i.e., F cannot be locally convex at the origin. Indeed F is not complete because of the restriction for x^2.

Remark 1.8. Moreover, Theorem 1.6 fails to be true if the assumption of the nonnegativity of the sectional curvature (11) is replaced by the nonnegativity of the Gaussian curvature H_{d-1}, see (7)! Namely, Chern and Lashof (1958, pp. 10–12) gave the following counterexamples: (1) $d \geq 4$ even, $M = S^1 \times S^{d-2}$, $x(M)$: $(\sqrt{(x^1)^2 + \cdots + (x^{d-1})^2} - 2)^2 + (x^d)^2 = 1$; (2) $d \geq 5$ odd, $M = S^2 \times S^{d-3}$, $x(M)$: $(\sqrt{(x^1)^2 + \cdots + (x^{d-2})^2} - 2)^2 + (x^{d-1})^2 + (x^d)^2 = 1$.

Remark 1.9. Before Sacksteder, Chern and Lashof (1958, p.6) (see also Voss 1960, p. 125) gave proofs of Theorem 1.6 in the special case $d=3$ and $F=$ compact.

At the end of this section we would like to mention that there is also a possibility to characterize the convexity of an m-dimensional submanifold F with $x: M \to \mathbb{R}^d$ $(0 < m < d)$, regarded as hypersurface in its affine hull aff$(x(M))$, in a differential geometric manner. This was first done by Fenchel (1929) for $m=1$ and $d=3$, and, for general m and d, in the following theorem.

Theorem 1.10 (Chern and Lashof 1957, p. 307). *We suppose that $x: M \to \mathbb{R}^d$ is a compact oriented m-dimensional submanifold F of class C_∞ ith the generalized spherical map*

$$\nu: (x(p), n(p)) \mapsto -n(p) := -\sum_{\alpha=m+1}^{d} \lambda_\alpha \overset{\alpha}{n}(p)$$

$$\left(\overset{m+1}{n}, \ldots, \overset{d}{n} = \text{orthonormal normal vector fields of } F,\ \sum_{\alpha=m+1}^{d} (\lambda_\alpha)^2 = 1\right)$$

of the unit normal bundle B_ν of F into the unit sphere S^{d-1} about the origin. Then

(a) *The induced linear map $\nu_*: x_*(T_pM) \oplus T_{-n(p)}S^{d-m-1} \to T_{-n(p)}S^{d-1}$, given by*

$$-\overset{\alpha}{n}_i(p) = \overset{\alpha}{h}_{ij}(p)g^{jk}(p)x_k(p) + \sum_{\beta=m+1}^{d} \overset{\alpha\beta}{t}_i(p)\overset{\beta}{n}(p),$$

$$i = 1, \ldots, m,\ \alpha = m+1, \ldots, d,$$

$$\overset{\alpha}{h}_{ij} := \left\langle \frac{\partial^2 x}{\partial u^i \, \partial u^j}, \overset{\alpha}{n} \right\rangle, \quad \overset{\alpha\beta}{t}_i := -\left\langle \overset{\alpha}{n}_i, \overset{\beta}{n} \right\rangle = -\overset{\beta\alpha}{t}_i , \tag{13}$$

[*compare eq.* (9)] *has the determinant*

$$\det \nu_* := \det\Big(\sum_{\alpha=m+1}^{d} \lambda_\alpha \overset{\alpha}{h}_{ij} g^{jk} \Big) \tag{14}$$

[*compare eq.* (10)], *which is the ratio of the volume element* $d\omega^{d-1}$ *of* S^{d-1} *and the volume element* $d\omega^{d-m-1} \wedge dF$ *of* B_ν *relative to the map* ν_* ($d\omega^{d-m-1}$ = *volume element of the sphere* S^{d-m-1} *of the unit normal vectors of F at a fixed point*, dF = *volume element of F*),

(b) *If* $G(p, n(p)) := \det \nu_*(p)$ *is the so-called "Lipschitz–Killing curvature" of F at* $(p, n(p))$, *and*

$$K^*(p) := \int_{S^{d-m-1}(p)} |G(p, n(P))| \, d\omega^{d-m-1} , \tag{15}$$

then the "total absolute curvature"

$$\int_M K^* \, dF \tag{16}$$

of F attains its minimal value $2d\kappa_d$ (κ_d = *volume of the d-dimensional unit ball*) *iff* $x(M)$ *is the boundary of a compact convex body K in a subspace* $\mathbb{R}^{m+1}$ *of* $\mathbb{R}^d$.

The proof of Theorem 1.10(a) is obvious, and the proof of 1.10(b) needs a detailed study of the covering of S^{d-1} by the endpoints of the unit normal vectors of F. It is highly connected with integral geometry and with Morse theory for the nondegenerate critical points of differentiable functions. Finally it is noteworthy that:

Corollary 1.11 (of Theorem 1.10). *A compact and (suitably) oriented hypersurface F of class* C_∞ *in* $\mathbb{R}^d$ *bounds a compact convex body in* $\mathbb{R}^d$ *iff*

$$H_{d-1} \geq 0 \tag{17}$$

and

$$\deg \nu = +1 . \tag{18}$$

This follows easily from the fact that (17) and (18) imply $K^* = 2H_{d-1}$ [see eqs. (15), (14), (10) and (17)] and therefore

$$\int_M K^* \, dF = 2 \int_M H_{d-1} \, dF = 2(\deg \nu) d\kappa_d = 2d\kappa_d .$$

Corollary 1.11 fails to be true if the assumption (18) is omitted (see the counterexample in Remark 1.8).

2. Elementary symmetric functions of principal curvatures respectively principal radii of curvature at Euler points

We return now to the mean curvatures (7) of a convex hypersurface $F : M \to \mathbb{R}^d$ of class C_2. They may be geometrically characterized in the following manner: Steiner's formula of the volume for the shell U_λ, bounded by F: $x = x(u^1, \ldots, u^{d-1})$ and its outer parallel hypersurface F_λ: $x = x(u^1, \ldots, u^{d-1}) - \lambda n(u^1, \ldots, u^{d-1})$ ($\lambda = \text{const} > 0$) indicates that we have by means of eqs. (9), (6) and (7)

$$\begin{aligned}
(d-1)! V(U_\lambda) &= \int_0^\lambda \Big[\int_M (dx - \mu \, dn, \underbrace{\ldots}_{d-1}, dx - \mu \, dn, n) \Big] d\mu \\
&= \int_0^\lambda \Big[\int_M \sum_{\nu=0}^{d-1} \mu^{d-1-\nu} \binom{d-1}{\nu} (-dn, \underbrace{\ldots}_{d-1-\nu}, -dn, dx, \underbrace{\ldots}_{\nu}, dx, n) \Big] d\mu \\
&= \sum_{\nu=0}^{d-1} \binom{d-1}{\nu} \frac{1}{d-\nu} \lambda^{d-\nu} \int_M H_{d-1-\nu} (dx, \ldots, dx, n) \\
&= (d-1)! \frac{1}{d} \sum_{\nu=0}^{d-1} \binom{d}{\nu} \lambda^{d-\nu} \int_M H_{d-1-\nu} \, dF .
\end{aligned} \tag{19}$$

Here $dx := \Sigma_{i=1}^{d-1} (\partial x / \partial u^i) \, du^i$, $dn := \Sigma_{i=1}^{d-1} (\partial n / \partial u^i) \, du^i$, and the determinants with differential form valued column vectors have to be computed as in Leichtweiss (1973). Equation (19) establishes that *the mean curvature* $H_{d-1-\nu}$ *is the density of the* ν*th "curvature measure" in the sense of Federer of a convex body bounded by* F ($\nu = 0, \ldots, d-1$, $H_0 := 1$; see chapter 1.8). In the case of an oriented and compact M with F of class C_3 these mean curvatures are related by Minkowski's integral formulas

$$\begin{aligned}
\int_M H_{\nu-1} \, dF &= \frac{1}{(d-1)!} \int_M (-dn, \underbrace{\ldots}_{\nu-1}, -dn, dx, \underbrace{\ldots}_{d-\nu}, dx, n) \\
&= \frac{1}{(d-1)!} \int_M -(-dn, \underbrace{\ldots}_{\nu}, -dn, dx, \underbrace{\ldots}_{d-1-\nu}, dx, x) \\
&= \int_M H_\nu h \, dF , \quad \nu = 1, \ldots, d-1 ,
\end{aligned} \tag{20}$$

where

$$h := -\langle x, n \rangle \tag{21}$$

is the support function of F, a direct consequence of Stokes' theorem. The eqs. (20) are an important tool for proving results in the global differential geometry of hypersurfaces.

If our convex hypersurface F has positive Gaussian curvature H_{d-1} everywhere,

$$H_{d-1} > 0\,, \tag{22}$$

then all the principal curvatures $k_1, \ldots, k_{d-1}$ of F must be positive [see eq. (7)], and we may define their reciprocal values

$$\begin{aligned} R_1 &:= \frac{1}{k_1} > 0\,,\\ &\vdots\\ R_{d-1} &:= \frac{1}{k_{d-1}} > 0\,, \end{aligned} \tag{23}$$

as the "principal radii of curvature" of F. In this case we can locally use the spherical image $\nu(x(M)) \subset S^{d-1}$ of F as parameter manifold for F and represent the points of F by their unit normal vector n in the following manner:

$$x(-n) = \left(\frac{\partial H}{\partial y^1}(-n), \ldots, \frac{\partial H}{\partial y^d}(-n)\right). \tag{24}$$

Herein, H is the support function of a convex body, bounded by F:

$$\begin{aligned} &H(y^1, \ldots, y^d) := \|y\| h\left(\frac{y^1}{\|y\|}, \ldots, \frac{y^d}{\|y\|}\right),\\ &\quad y := (y^1, \ldots, y^d) \in \mathbb{R}^d \backslash \{0\} \end{aligned} \tag{25}$$

[see eq. (21)], convex, positively homogeneous of degree 1 and of class C_2 (see chapter 1.2). The homogeneity of H yields

$$\sum_{b=1}^{d} \frac{\partial H}{\partial y^b} y^b = H\,, \tag{26}$$

whence after partial differentiation

$$\sum_{b=1}^{d} \frac{\partial^2 H}{\partial y^a \partial y^b}(-n) n^b = 0\,, \quad a = 1, \ldots, d\,. \tag{27}$$

Therefore, the Hessian of H at $-n$ has the eigenvector n to the eigenvalue 0. Because of the relations

$$\begin{aligned} &x_i(-n) = \left(-\sum_{b=1}^{d} \frac{\partial^2 H}{\partial y^1 \partial y^b}(-n)(n^b)_i, \ldots, -\sum_{b=1}^{d} \frac{\partial^2 H}{\partial y^d \partial y^b}(-n)(n^b)_i\right),\\ &\quad i = 1, \ldots, d-1 \end{aligned} \tag{28}$$

– resulting from (24) by differentiation with respect to u^i ($i = 1, \ldots, d-1$) – the other $d-1$ eigenvalues of $(\mathrm{Hess}\,H)(-n)$ coincide with the eigenvalues $R_1(-n), \ldots, R_{d-1}(-n)$ of the linear map $(\nu^{-1})_* : T_{-n}S^{d-1} \to x_*(T_{p(-n)}M)$, induced by the (locally existing) inverse ν^{-1} of the spherical map ν [see eq. (23)].

From these facts we deduce that we have for the normalized elementary symmetric functions

$$\begin{aligned} P_1 &:= \frac{1}{d-1}(R_1 + \cdots + R_{d-1})\,,\\ &\vdots\\ P_{d-1} &:= R_1 \cdots R_{d-1}\,, \end{aligned} \tag{29}$$

defined as the "mean radii of curvature" of F, the equation:

$$\binom{d-1}{\nu} P_\nu(-n) = D_\nu(H)(-n)\,, \quad \nu = 1, \ldots, d-1\,,\ -n \in \nu(x(M))\,, \tag{30}$$

where $D_\nu(H)$ denotes the sum of all principal minors with ν rows of the matrix Hess H. Now eqs. (7), (29) and (23) prove

$$P_\nu = \frac{H_{d-1-\nu}}{H_{d-1}}\,, \quad \nu = 0, \ldots, d-1,\ P_0 := 1 \tag{31}$$

[see eq. (22)], whence (19) respectively (20) may be transformed into the "dual" formulas

$$V(U_\lambda) = \frac{1}{d} \sum_{\nu=0}^{d-1} \binom{d}{\nu} \lambda^{d-\nu} \int_{\nu(x(M))} P_\nu \, d\omega \tag{32}$$

($d\omega$ = volume element of S^{d-1}) and, for $d > 2$ (see Theorem 1.1) and F of class C_3,

$$\int_{S^{d-1}} P_{d-\nu} \, d\omega = \int_{S^{d-1}} P_{d-\nu-1} h \, d\omega\,, \quad \nu = 1, \ldots, d-1\,. \tag{33}$$

Equation (32) shows that the *mean radius of curvature P_ν is the density of the νth "surface area measure" in the sense of Aleksandrov of a convex body bounded by F* ($\nu = 0, \ldots, d-1$; see chapter 1.8).

A very important theorem about the influence of the infinitesimal behaviour of a convex hypersurface to its global behaviour is the following:

Theorem 2.1. (Blaschke 1956 (1916), p. 118). *A sphere S_R of radius $R > 0$ in $\mathbb{R}^d$ is rolling freely in the interior K of an "ovaloid" F, i.e., a hypersurface $x : M \to \mathbb{R}^d$ of class C_2 with positive Gaussian curvature bounding a compact convex body K, if the condition*

$$R \leq \min_{-n \in S^{d-1}} \{R_1(-n), \ldots, R_{d-1}(-n)\} \tag{34}$$

holds.

Proof. In the first step we prove Theorem 2.1 in the case $d=2$. Here, eq. (30) becomes

$$R_1(-n)=P_1(-n)=\left(\frac{\partial^2 H}{(\partial y^1)^2}+\frac{\partial^2 H}{(\partial y^2)^2}\right)(-n)=g(\varphi)+\frac{\mathrm{d}^2 g}{\mathrm{d}\varphi^2}(\varphi)\,, \tag{35}$$

after the introduction of the angle φ by

$$-n:=(\cos\varphi,\sin\varphi) \tag{36}$$

and of the auxiliary function g by

$$g(\varphi):=h(\cos\varphi,\sin\varphi)\,,\quad 0\leqslant\varphi\leqslant 2\pi\,, \tag{37}$$

see eq. (25). If we denote $R_1(-n)=R_1(\cos\varphi,\sin\varphi)$ by $r(\varphi)$, we find by the method of variation of constants the following solution of the inhomogeneous linear ordinary differential equation of second order (35):

$$g(\varphi)=\int_{\varphi_0}^{\varphi} r(\psi)\sin(\varphi-\psi)\,\mathrm{d}\psi+C_1\cos\varphi+C_2\sin\varphi\,, \tag{38}$$

with given initial conditions at φ_0 implying

$$C_1=\cos\varphi_0 g(\varphi_0)-\sin\varphi_0 g'(\varphi_0)\,,\qquad C_2=\sin\varphi_0 g(\varphi_0)+\cos\varphi_0 g'(\varphi_0)\,. \tag{39}$$

Now we compare g with the support function g_R of the circle $S_R(\varphi_0)$ of radius R touching the curve F from the inner side at an arbitrary point with the outer unit normal vector $(\cos\varphi_0,\sin\varphi_0)$. Clearly, g_R has the representation

$$g_R(\varphi)=\int_{\varphi_0}^{\varphi} R\sin(\varphi-\psi)\,\mathrm{d}\psi+C_1\cos\varphi+C_2\sin\varphi \tag{40}$$

[analogous to eq. (38)], with the same integration constants C_1 and C_2 because of (39). Therefore, subtraction of (40) from (38) gives

$$g(\varphi)-g_R(\varphi)=\int_{\varphi_0}^{\varphi}(r(\psi)-R)\sin(\varphi-\psi)\,\mathrm{d}\psi\,,\quad \varphi_0-\pi\leqslant\varphi\leqslant\varphi_0+\pi\,, \tag{41}$$

and the condition (34) or $R\leqslant r(\psi)$ for all ψ yields

$$g(\varphi)\geqslant g_R(\varphi)\,,\quad \varphi_0-\pi\leqslant\varphi\leqslant\varphi_0+\pi\,. \tag{42}$$

Equation (37) together with eq. (42) proves that the circle $S_R(\varphi_0)$ totally lies in the interior of F so that S_R "is rolling freely" there as the contact point of S_R and F may be chosen arbitrarily.

The second step in the proof of Theorem 2.1 consists in the reduction of dimension by an orthogonal projection π of $\mathbb{R}^d$ onto a suitable plane $\mathbb{R}^2$ in $\mathbb{R}^d$ containing the origin. After (21) and (25) the support function $H^{(\pi)}$ of the curve $F^{(\pi)}:=\pi\circ F$ in the plane $\mathbb{R}^2$ equals the restriction of the support function H of F to $\mathbb{R}^2$. Therefore, the radius of curvature $R_1^{(\pi)}(-n)$ of $F^{(\pi)}$ at a point with the unit normal vector $-n\in\mathbb{R}^2$ has the value

$$\begin{aligned}R_1^{(\pi)}(-n)&=\min_{\substack{z\in\mathbb{R}^2\\ \|z\|=1,\ \langle z,n\rangle=0}}\frac{\partial^2 H}{\partial y^a\,\partial y^b}(-n)z^a z^b\\ &\geqslant\min_{\substack{z\in\mathbb{R}^d\\ \|z\|=1,\ \langle z,n\rangle=0}}\frac{\partial^2 H}{\partial y^a\,\partial y^b}(-n)z^a z^b\\ &=\min\{R_1(-n),\dots,R_{d-1}(-n)\}\,,\end{aligned} \tag{43}$$

as $R_1(-n),\dots,R_{d-1}(-n)$ are the eigenvalues of $(\operatorname{Hess} H)(-n)$ different from the eigenvalue 0 for its eigenvector n. Now the assumption (34) implies $R\leqslant R_1^{(\pi)}(-n)$ so that $\pi(S_R)$ is rolling freely in the interior $\pi(K)$ of $F^{(\pi)}$. But consequently S_R is rolling freely in the interior of F because otherwise there would exist a sphere S_R touching F from the inner side at $x(p)$ and with a point $x(q)$ of F in the proper interior of S_R; and the projection π along the $(d-2)$-dimensional direction simultaneously orthogonal to the normals of F at $x(p)$ and $x(q)$ would produce a contradiction to $\pi(S_R)\subset\pi(K)$. □

Remark 2.2. Using the definition $R_i=1/k_i$ of the principal radii of curvature where the k_i are the extremal values of curvature of the normal sections of F by planes $(i=1,\dots,d-1)$ it can easily be seen that the condition (34) is also necessary for the assertion of Theorem 2.1.

Remark 2.3. By the same method of proof it follows that an ovaloid $\tilde{F}$ is rolling freely in the interior of another ovaloid F, iff

$$\max_{-n\in S^{d-1}}\{\tilde{R}_1(-n),\dots,\tilde{R}_{d-1}(-n)\}\leqslant\min_{-n\in S^{d-1}}\{R_1(-n),\dots,R_{d-1}(-n)\}$$

($\tilde{R}_1,\dots,\tilde{R}_{d-1}$ = principal radii of curvature of $\tilde{F}$) holds. This fact also possesses a local version. For further information see Schneider (1988, Theorem 1), and Brooks and Strantzen (1989).

At the end of this section we shall explain how the notion of principal curvatures or principal radii of curvature of a convex hypersurface F of class C_2 in $\mathbb{R}^d$, given by (1), can be extended to the case of an arbitrary convex hypersurface. Then f is no more continuously twice differentiable but only convex. However, a famous theorem of Aleksandrov (1939) says that *f must be nevertheless twice differentiable in a certain abstract sense* (*due to Frechet*) *almost everywhere* (see chapter 4.2). This implies the validity of Taylor's formula at the points $(u_{(0)}, f(u_{(0)}))$ of twice differentiability of f in the form

$$|f(u) - f(u_{(0)}) - \mathrm{d}f_{(u_{(0)})}(u - u_{(0)}) - \tfrac{1}{2}\,\mathrm{d}^2 f_{(u_{(0)})}(u - u_{(0)}, u - u_{(0)})|$$
$$\leq R(\|u - u_{(0)}\|)\|u - u_{(0)}\|^2\,, \tag{44}$$
$$(u := x^1, \ldots, x^{d-1}\,,\quad u_{(0)} := x^1_{(0)}, \ldots, x^{d-1}_{(0)}\,,$$
$$R : \mathbb{R}^+ \to \mathbb{R}^+ \text{a monotone increasing function with } \lim_{t\to+0} R(t) = 0)\,,$$

where $\mathrm{d}f_{(u_{(0)})}$ is a linear and $\mathrm{d}^2 f_{(u_{(0)})}$ is a positive semidefinite quadratic function on $\mathbb{R}^{d-1}$ (see Bangert 1979, Lemma 4.8).

In order to understand the geometrical meaning of (44) it is convenient to choose a coordinate system in $\mathbb{R}^d$ with $u_{(0)} = 0$, $f(u_{(0)}) = 0$, $\mathrm{d}f_{(u_{(0)})} = 0$ and $f(u) \geq 0$, and to consider the intersection D_h of a convex body K, bounded by F, with the hyperplane $x^d = h = \text{const} > 0$, expanded by the factor $1/\sqrt{2h}$ and projected orthogonally onto the supporting hyperplane $x^d = 0$. The boundary bd D_h of the convex body D_h is given in polar coordinates by

$$r = \frac{1}{\sqrt{2h}}\,\rho_h(v)\,, \tag{45}$$

where $\rho_h(v)$ fulfils

$$f(\rho_h(v)v) = h\,,\quad v \in \mathbb{R}^{d-1}\,,\ \|v\| = 1\,. \tag{46}$$

By this we conclude from (44) that $\lim_{h\to+0} D_h$ exists (in Hausdorff sense) with the representation of its boundary in the form

$$r = \lim_{h\to+0} \frac{1}{\sqrt{2h}}\,\rho_h(v) = \lim_{h\to+0} \frac{\rho_h(v)}{\sqrt{2f(\rho_h(v)v)}}$$
$$= \lim_{t\to \lim_{h\to+0}\rho_h(v)+0} \frac{1}{\sqrt{\dfrac{2f(tv)}{t^2}}} = \frac{1}{\sqrt{\mathrm{d}^2 f_{(0)}(v, v)}} \tag{47}$$

or, equivalently,

$$\mathrm{d}^2 f_{(0)}(u, u) = 1\,. \tag{48}$$

Therefore, the boundary is a quadric in $\mathbb{R}^{d-1}$ with the origin as midpoint, namely an ellipsoid if $\mathrm{d}^2 f_{(0)}$ is positive definite, or an elliptic cylinder if $\mathrm{d}^2 f_{(0)}$ is positive semidefinite but not positive definite or zero, or the empty set if $\mathrm{d}^2 f_{(0)}$ is zero. This leads to:

Definition 2.4 (Aleksandrov 1939). A point $x(p)$ of a general convex hypersurface F in $\mathbb{R}^d$ is called *normal* iff $\lim_{h\to+0} D_h$ exists and its boundary $I(p)$ is a quadric in the tangent hyperplane of F at $x(p)$ with $x(p)$ as its midpoint, the so-called "indicatrix of Dupin".

We have just seen that $x(p)$ is a normal point of a convex hypersurface F if the function f, representing F, is twice differentiable at this point. However, also the converse is true as shown by Aleksandrov (1939). It is convenient to define the *inverse of the squared length of the semiaxes of* $I(p)$ as the (generalized) "principal curvatures" of F at a normal point $x(p)$ [compare (47) in the C_2-case!]. Their product may be regarded as a (generalized) Gaussian curvature $H_{d-1}(x(p))$ of F which turns out to be the appropriately defined derivative at $x(p)$ of the (generalized) Gauss map ν as a set function (see Aleksandrov 1939). Finally, if in addition the (convex) support function H of a convex body, bounded by F, is twice differentiable at $-n(p)$, then the product of the "principal radii of curvature" (inverse to the principal curvatures) $1/H_{d-1}(x(p))$ is finite and equals to the entity $D_{d-1}(H)(-n(p))$ of (30) (see also Aleksandrov 1939). For further information we refer to a paper of Busemann and Feller (1936) and a paper of Schneider (1979).

3. Mixed discriminants and mixed volumes

In this section we want to express the volume of a compact convex body K respectively the mixed volume of such bodies $K_1, \ldots, K_m$ in $\mathbb{R}^d$, provided that they are "regular", i.e., they have C_2-boundaries F, respectively $F_1, \ldots, F_m$, with positive Gaussian curvatures, by suitable integrals over F, respectively $F_1, \ldots, F_m$, and also over the unit sphere S^{d-1}. This will produce certain "counterparts" of important theorems on mixed volumes (see chapter 1.2). We begin with the representation

$$V(K) = \frac{1}{d!}\int_K (\underbrace{\mathrm{d}x, \ldots, \mathrm{d}x}_{d}) = \frac{1}{d!}\int_F (\underbrace{\mathrm{d}x, \ldots, \mathrm{d}x}_{d-1}, -x) = \frac{1}{d}\int_F h\,\mathrm{d}F$$
$$= \frac{1}{d}\int_{S^{d-1}} HP_{d-1}\,\mathrm{d}\omega = \frac{1}{d}\int_{S^{d-1}} HD_{d-1}(H)\,\mathrm{d}\omega \tag{49}$$

of the volume of K, a consequence of Stokes' theorem after a suitable orientation of F [see also eqs. (21), (31) and (30)]. If we insert in (49) for x a linear combination $x = \Sigma_{l=1}^m \lambda_l x^{(l)}$ $(\lambda_1 \geq 0, \ldots, \lambda_m \geq 0)$ for corresponding points $x^{(1)}, \ldots, x^{(m)}$ (with the same unit normal vector n), the comparison of the coefficients of $\lambda_{l_1} \cdots \lambda_{l_d}$ yields

$$V(K_{l_1}, \ldots, K_{l_d}) = \frac{1}{d!}\int_F (\mathrm{d}x^{(l_1)}, \ldots, \mathrm{d}x^{(l_{d-1})}, -x^{(l_d)})\,,$$
$$1 \leq l_1 \leq \cdots \leq l_d \leq m\,. \tag{50}$$

In order to get this formula we used the symmetry of the integral in (50) with respect to the indices $l_1, \ldots, l_d$ arising from Stokes' theorem and the compactness of the oriented hypersurface F. Furthermore, the insertion of (28) into (50) gives

$$V(K_{l_1}, \ldots, K_{l_d}) = \frac{1}{d!} \int_F (\mathrm{d}x^{(l_1)}, \ldots, \mathrm{d}x^{(l_{d-1})}, n) h^{(l_d)}$$
$$= \frac{1}{d} \int_{S^{d-1}} D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-1})}) H^{(l_d)} \, \mathrm{d}\omega \,, \tag{51}$$

where $(d-1)! D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-1})})$ denotes the sum of all "mixed" principal minors with $d-1$ rows of the Hessians of the support functions $H^{(l_1)}, \ldots, H^{(l_{d-1})}$ of $K_{l_1}, \ldots, K_{l_{d-1}}$. We can easily realize (51) if we use the fact that this term equals the coefficient of $\lambda_0 \cdot \lambda_{l_1} \cdots \lambda_{l_{d-1}}$ in the determinant

$$\det\left[\lambda_0 \delta_{ab} + \lambda_{l_1} \frac{\partial^2 H^{(l_1)}}{\partial y^a \, \partial y^b} + \cdots + \lambda_{l_{d-1}} \frac{\partial^2 H^{(l_{d-1})}}{\partial y^a \, \partial y^b} \right],$$

and that this determinant is invariant against orthogonal transformations of $\mathbb{R}^d$, so that it suffices to prove (51) under the additional assumption $n = (0, \ldots, 0, 1)$ and therefore $\mathrm{d}n^d = 0$ as well $\partial^2 H / \partial y^a \, \partial y^d = 0$ [$a = 1, \ldots, d$; see eq. (27)] at the corresponding point. For all these reasons it is convenient to define:

Definition 3.1. Let $Q_1, \ldots, Q_t$ be arbitrary symmetric $(d \times d)$-matrices. Then the coefficients $D(Q_{s_1}, \ldots, Q_{s_d})$ of the expansion

$$\det(w_1 Q_1 + \cdots + w_t Q_t) = \sum_{s_1=1}^{t} \cdots \sum_{s_d=1}^{t} w_{s_1} \cdots w_{s_d} D(Q_{s_1}, \ldots, Q_{s_d}) \tag{52}$$

which are required to be symmetric in $s_1, \ldots, s_d$ are called *mixed discriminants* of $Q_1, \ldots, Q_t$.

These mixed discriminants $D(Q_{s_1}, \ldots, Q_{s_d})$ are depending only on the matrices $Q_{s_1}, \ldots, Q_{s_d}$, and we see easily that

$$D(Q, \ldots, Q) = \det Q \tag{53}$$

in the case $Q_1 = \cdots = Q_t =: Q$. Moreover, the mixed discriminants are linear in each argument:

$$D(\alpha_1 Q^{(1)}_{s_1} + \alpha_2 Q^{(2)}_{s_1}, Q_{s_2}, \ldots, Q_{s_d})$$
$$= \alpha_1 D(Q^{(1)}_{s_1}, Q_{s_2}, \ldots, Q_{s_d}) + \alpha_2 D(Q^{(2)}_{s_1}, Q_{s_2}, \ldots, Q_{s_d}) \,. \tag{54}$$

As we have just shown, there exists the relation

$$D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-1})}) = d \cdot D(\text{Hess } H^{(l_1)}, \ldots, \text{Hess } H^{(l_{d-1})}, E) \tag{55}$$

[$E = (d \times d)$ – unit matrix] for the integrand in (51). In the same way the equation

$$D_\nu(H^{(l_1)}, \ldots, H^{(l_\nu)}) = \binom{d}{\nu} D(\text{Hess } H^{(l_1)}, \ldots, \text{Hess } H^{(l_\nu)}, \underbrace{E, \ldots, E}_{d-\nu}) \tag{56}$$

[$1 \leq \nu \leq d-1$; see eq. (30)!] becomes obvious.

A first important property for mixed discriminants – a counterpart to the nonnegativity of the mixed volume – is expressed in:

Proposition 3.2. *If all the (symmetric) matrices $Q_1, \ldots, Q_t$ in Definition 3.1 are positive semidefinite (respectively positive definite), then all mixed discriminants $D(Q_{s_1}, \ldots, Q_{s_d})$ are nonnegative (positive) ($1 \leq s_1 \leq t, \ldots, 1 \leq s_d \leq t$).*

Proof. After suitable approximation of positive semidefinite matrices $Q_1, \ldots, Q_t$ by positive definite ones we can see that it suffices to prove this proposition for positive definite matrices. This will be done by induction with respect to d. The case $d = 1$ is trivial. We suppose Proposition 3.2 to be valid for $d - 1$. Then at first the application of Sylvesters' law of inertia for the positive definite matrix Q_{s_d} together with the simultaneous multiplication of all matrices $Q_1, \ldots, Q_t$ by T^* on the left and T on the right implies

$$D(Q_{s_1}, \ldots, Q_{s_d}) = (\det T)^{-2} D(Q'_{s_1}, \ldots, Q'_{s_{d-1}}, E) \,, \tag{57}$$

where $\det T \neq 0$ and $Q'_{s_1} := T^* Q_{s_1} T, \ldots, Q'_{s_{d-1}} := T^* Q_{s_{d-1}} T$, $E = T^* Q_{s_d} T$ [see eq. (52)]. However, $d! \cdot D(Q'_{s_1}, \ldots, Q'_{s_{d-1}}, E)$ is the sum of all "mixed" principal minors with $d-1$ rows of the positive definite matrices $Q'_{s_1}, \ldots, Q'_{s_{d-1}}$ and therefore positive by the inductive assumption so that also $D(Q_{s_1}, \ldots, Q_{s_d})$ must be positive because of (57). □

Remark 3.3. Proposition 3.2 and eqs. (51), (55) together with the relation

$$d \cdot D(\text{Hess } H^{(l_1)}, \ldots, \text{Hess } H^{(l_{d-1})}, E)$$
$$= D'(\text{Hess}' \, H^{(l_1)}, \ldots, \text{Hess}' \, H^{(l_{d-1})}) > 0 \,, \tag{58}$$

where D' denotes the mixed discriminant of the (positive definite) restrictions of Hess $H^{(l_1)}(-n), \ldots,$ Hess $H^{(l_{d-1})}(-n)$ to the $(d-1)$-dimensional sum of their eigenspaces corresponding to their positive eigenvalues (the radii of curvature of $F_{l_1}, \ldots, F_{l_{d-1}}$), imply the well-known property for regular compact convex bodies

$$V(K_{l_1}, \ldots, K_{l_d}) > 0. \tag{59}$$

Hereby the positivity of $H^{(l_d)}$ was used coming from the choice of the origin in the interior of the convex body K_{l_d}. The same argument yields the monotonicity property

$$V(K_{l_1}, \ldots, K_{l_{d-1}}, K_{l_d}) \leq V(K_{l_1}, \ldots, K_{l_{d-1}}, \tilde{K}_{l_d}) \quad \text{if } K_{l_d} \subset \tilde{K}_{l_d}, \tag{60}$$

with equality iff $K_{l_d} = \tilde{K}_{l_d}$.

We now proceed to the following counterpart of the Aleksandrov–Fenchel–Jessen inequalities for mixed volumes (see chapter 1.2).

Theorem 3.4 (Aleksandrov 1938). *If the matrices $Q_{s_1}, \ldots, Q_{s_{d-1}}$ are positive definite and Q_{s_d} is any (symmetric) matrix, then*

$$[D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_{s_{d-1}}, Q_{s_d})]^2 \geq D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_{s_{d-1}}, Q_{s_{d-1}}) \cdot D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_{s_d}, Q_{s_d}), \tag{61}$$

where the equality holds only if Q_{s_d} is proportional to $Q_{s_{d-1}}$.

Proof (*See also* Busemann 1958, pp. 53–56). After introduction of the quadratic form $g_{s_1 \cdots s_{d-2}}$ in the essential elements ξ_{ij} $(1 \leq i \leq j \leq d)$ of an arbitrary symmetric matrix X by:

$$\begin{aligned} g_{s_1 \cdots s_{d-2}}(X, X) &:= D(Q_{s_1}, \ldots, Q_{s_{d-2}}, X, X) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} (D(Q_{s_1}, \ldots, Q_{s_{d-2}}, X))_{ij} \xi_{ij} \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{l=1}^{d} (D(Q_{s_1}, \ldots, Q_{s_{d-2}}))_{ij,kl} \xi_{ij} \xi_{kl}, \end{aligned} \tag{62}$$

the inequality (61) takes the form

$$[g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q_{s_d})]^2 \geq g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q_{s_{d-1}}) \cdot g_{s_1 \cdots s_{d-2}}(Q_{s_d}, Q_{s_d}), \tag{63}$$

with equality iff

$$Q_{s_d} = \gamma \cdot Q_{s_{d-1}}, \quad \gamma \in \mathbb{R}. \tag{64}$$

This is equivalent to the implication

$$g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q) = 0 \Rightarrow g_{s_1 \cdots s_{d-2}}(Q, Q) \leq 0 \tag{65}$$

for a symmetric matrix Q, with equality iff

$$Q = 0. \tag{66}$$

Indeed, eqs. (63) and (62) together with Proposition 3.2 imply (65) and (66). Conversely, after applying (65) and (66) for

$$Q := Q_{s_d} - \frac{g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q_{s_d})}{g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q_{s_{d-1}})} \cdot Q_{s_{d-1}}$$

we get (63) and (64).

In the following, (65) and (66) will be proven by induction with respect to d. (65) and (66) may be easily shown in the case $d = 2$ if the positive definite (respectively symmetric) matrices Q_{s_1} (Q) have been simultaneously transformed into diagonal forms

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad a_{11} > 0, \ a_{22} > 0 \qquad \left(\text{respectively } \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}\right)$$

by a unimodular transformation, not changing the mixed discriminants. Indeed, here the implication

$$a_{11} b_{22} + a_{22} b_{11} = 0 \Rightarrow b_{11} b_{22} \leq 0 \tag{67}$$

is trivial, with equality iff

$$b_{11} = b_{22} = 0. \tag{68}$$

Now we assume (65) and (66) to be true for $d-1$ and show their validity for d by consideration of the nonnegative eigenvalues of $g_{s_1 \cdots s_{d-2}}$ with respect to the quadratic form

$$\sum_{i=1}^{d} \sum_{j=1}^{d} (\xi_{ij})^2. \tag{69}$$

At first this will be done for the special quadratic form

$$g^{(0)}(X, X) := D(E, \ldots, E, X, X) = \binom{d}{2}^{-1} \sum_{1 \leq i < j \leq d} (\xi_{ii} \xi_{jj} - (\xi_{ij})^2). \tag{70}$$

It results from an elementary calculation involving Lagrange's multiplicator method that $g^{(0)}$ has $1/d$ as single positive, and $-1/d(d-1)$ as $(\frac{1}{2}d(d+1) - 1)$-fold negative eigenvalue, so that 0 cannot be an eigenvalue of $g^{(0)}$. *More general, 0 is not an eigenvalue for* $g_{s_1 \cdots s_{d-2}}$. Otherwise we would have

$$g_{s_1 \cdots s_{d-2}}(Q_0, X) = D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0, X) = 0 \tag{71}$$

for a symmetric matrix $Q_0 \neq 0$ and an arbitrary symmetric matrix X [see eq. (62)], and this relation remains unchanged after a simultaneous unimodular matrix transformation of $Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0, X$ such that the positive definite $Q_{s_{d-2}}$ and the symmetric Q_0 are transformed into diagonal matrices with elements

$$a_{ij}^{(s_{d-2})} = a_{ii}^{(d-2)}\delta_{ij} \quad (a_{ii}^{(d-2)} > 0) \quad \text{and} \quad b_{ij}^{(0)} = b_{ii}\delta_{ij}\,, \qquad 1 \leq i \leq j \leq d\,. \tag{72}$$

So we may suppose the validity of (72) without loss of generality and deduce from (62) and (71) the equations $(D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0))_{ii} = 0$, $i = 1, \ldots, d$, where $d(D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0))_{ii}$ equals the mixed discriminant of $Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0$ with all elements with index i having been deleted. For this reason we have by the inductive assumption

$$(D(Q_{s_1}, \ldots, Q_{s_{d-3}}, Q_0, Q_0))_{ii} \leq 0\,, \quad 1 \leq i \leq d\,, \tag{73}$$

and by (71)

$$\sum_{i=1}^{d} (D(Q_{s_1}, \ldots, Q_{s_{d-3}}, Q_0, Q_0))_{ii} a_{ii}^{(d-2)} = D(Q_{s_1}, \ldots, Q_{s_{d-2}}, Q_0, Q_0) = 0\,.$$

But this and eq. (72) imply equality in all the inequalities (73), and therefore $Q_0 = 0$ by the inductive assumption which contradicts $Q_0 \neq 0$. Now a continuity argument, applied to the eigenvalues of the forms

$$g^{(\tau)}(X, X) := D((1-\tau)E + \tau Q_{s_1}, \ldots, (1-\tau)E + \tau Q_{s_{d-2}}, X, X)\,,$$
$$0 \leq \tau \leq 1\,,$$

connecting $g^{(0)}$ with $g_{s_1 \cdots s_{d-2}}$, shows that $g_{s_1 \cdots s_{d-2}}$ *has exactly one positive and no zero eigenvalue* as well as $g^{(0)}$ because no eigenvalue of $g^{(\tau)}$ passes through 0 when τ runs from 0 to 1.

At the end of the proof we interpret the essential elements ξ_{ij} $(1 \leq i \leq j \leq d)$ of a (nonvanishing) symmetric matrix X as projective coordinates of the points of a $(\frac{1}{2}d(d+1) - 1)$ – dimensional projective space P. Then, by the previously mentioned property of $g_{s_1 \cdots s_{d-2}}$,

$$g_{s_1 \cdots s_{d-2}}(X, X) = 0 \tag{74}$$

is the equation of a hyperellipsoid in P. Because of (62) and Proposition 3.2 we have $g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q_{s_{d-1}}) > 0$, which means that the point $Q_{s_{d-1}}$ lies in the component of the hyperellipsoid (74) which does not contain a full line of P. But then each point $Q \neq 0$ with $g_{s_1 \cdots s_{d-2}}(Q_{s_{d-1}}, Q) = 0$, i.e., each point Q conjugate to $Q_{s_{d-1}}$ with respect to (74) lies in the other component, whence $g_{s_1 \cdots s_{d-2}}(Q, Q) < 0$. Therefore, the implication (65) is true with equality iff (66) holds which completes the proof of Theorem 3.4. □

4. Differential geometric proof of the Aleksandrov–Fenchel–Jessen inequalities

In this section we shall apply Theorem 3.4 for a proof, formally analogous to that of Theorem 3.4 for mixed discriminants, of the following *Aleksandrov–Fenchel–Jessen inequalities for mixed volumes*:

$$\begin{aligned}&[V(K_{l_1}, \ldots, K_{l_{d-2}}, K_{l_{d-1}}, K_{l_d})]^2 \\ &\quad \geq V(K_{l_1}, \ldots, K_{l_{d-2}}, K_{l_{d-1}}, K_{l_{d-1}}) \cdot V(K_{l_1}, \ldots, K_{l_{d-2}}, K_{l_d}, K_{l_d})\,, \\ &\quad 1 \leq l_1, \ldots, l_d \leq m\,,\end{aligned} \tag{75}$$

where $K_1, \ldots, K_m$ are assumed to be regular compact convex bodies in $\mathbb{R}^d$ with the origin in the interior and equality occurs iff $K_{l_{d-1}}$ and K_{l_d} are homothetic:

$$K_{l_d} = \gamma \cdot K_{l_{d-1}} + a\,, \quad \gamma > 0\,, \ a \in \mathbb{R}^d\,, \tag{76}$$

see Aleksandrov (1938, §6) and Busemann (1958, pp. 56–59).

For this purpose Aleksandrov extends [similar to his first proof of eq. (75)] the notion of the mixed volume (51) as a functional over the d-fold Cartesian product of the space $C_2^{\text{conv}}(S^{d-1})$ of all those positive C_2-functions on the sphere S^{d-1} with convex positively homogeneous extensions H of degree 1 to $\mathbb{R}^d \backslash \{0\}$ whose Hessians have $d-1$ positive eigenvalues (orthogonal to the position vector of the argument) in the following manner: let Z be the positively homogeneous extension of degree 1 of an *arbitrary* C_2*-function on* S^{d-1} to $\mathbb{R}^d \backslash \{0\}$, and let $C_2(S^{d-1})$ be the space of all such functions. Then, by adding to Z a suitable positive multiple αH_0 of the special function $H_0(y) := (\Sigma_{a=1}^d (y^a)^2)^{1/2}$ with a Hessian of $d-1$ eigenvalues 1 on S^{d-1}, we get another function $H_1 \in C_2^{\text{conv}}(S^{d-1})$ (because of the compactness of S^{d-1}!) so that the representation

$$Z = H_1 - \alpha H_0\,, \quad H_1 \in C_2^{\text{conv}}(S^{d-1})\,, \ H_0 \in C_2^{\text{conv}}(S^{d-1}) \tag{77}$$

holds for Z. This fact permits us to define a bilinear function $q_{l_1 \cdots l_{d-2}}$ on the space $C_2(S^{d-1}) \times C_2(S^{d-1})$ by a well-defined bilinear extension of the mixed volume (51) considered as a bilinear form on $C_2^{\text{conv}}(S^{d-1}) \times C_2^{\text{conv}}(S^{d-1})$ with the fixed support functions $H^{(l_1)}, \ldots, H^{(l_{d-2})} \in C_2^{\text{conv}}(S^{d-1})$ of the fixed regular compact convex bodies $K_{l_1}, \ldots, K_{l_{d-2}}$. This extension has the property

$$q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, H^{(l_d)}) = V(K_{l_1}, \ldots, K_{l_d})\,, \tag{78}$$

and we shall prove, instead of (75) with the equality condition (76), the following stronger version:

$$\begin{aligned}&[q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, Z)]^2 \geq q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, H^{(l_{d-1})}) \cdot q_{l_1 \cdots l_{d-2}}(Z, Z)\,, \\ &Z \in C_2(S^{d-1})\,,\end{aligned} \tag{79}$$

with equality iff

$$Z = \gamma \cdot H^{(l_{d-1})} + \langle a, \cdot \rangle\,, \quad \gamma \in \mathbb{R}\,,\ a \in \mathbb{R}^d\,, \tag{80}$$

compare (78).

Now a first step of the proof is to show that (79) *and* (80) *are equivalent to the fact that*

$$q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, Z) = 0\,, \quad Z \in C_2(S^{d-1})\,, \tag{81}$$

implies

$$q_{l_1 \cdots l_{d-2}}(Z, Z) \leq 0\,, \tag{82}$$

with equality iff

$$Z = \langle a, \cdot \rangle\,. \tag{83}$$

Clearly, from (79) and (81) we may deduce (82) considering (78) and (59), and the insertion of (80) into (81) yields $\gamma = 0$, i.e., (83) because of the translation invariance of the mixed volume. Conversely,

$$\hat{Z} := Z - \frac{q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, Z)}{q_{l_1 \cdots l_{d-2}}(H^{(l_{d-1})}, H^{(l_{d-1})})} \cdot H^{(l_{d-1})} \tag{84}$$

fulfils (81) implying (82) for $\hat{Z}$, which is equivalent to (79). There equality occurs iff this is true for (82) with $\hat{Z}$, i.e., $\hat{Z} = \langle a, \cdot \rangle$ or (80) by definition (84).

Then a next step is the application of Hilbert's "parametrix method" (see Hilbert 1912, chapter 18 in the case $d = 3$), the essential idea of Aleksandrov's proof. This will be done by defining a "weighted" inner product on the space $C_2(S^{d-1})$ by:

$$\langle Z, W \rangle := \int_{S^{d-1}} Z \cdot W\, w \,\mathrm{d}\omega\,, \tag{85}$$

with the weight

$$w := \frac{D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-1})})}{H^{(l_{d-1})}} > 0\,; \tag{86}$$

and a linear differential operator L on $C_2(S^{d-1})$ by:

$$L(Z) := D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-2})}, Z)\,. \tag{87}$$

This operator is symmetric because of the relation

$$\begin{aligned}\int_{S^{d-1}} L(Z) \cdot W \,\mathrm{d}\omega &= \int_{S^{d-1}} D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-2})}, Z) \cdot W \,\mathrm{d}\omega \\ &= \int_{S^{d-1}} D_{d-1}(H^{(l_1)}, \ldots, H^{(l_{d-2})}, W) \cdot Z \,\mathrm{d}\omega = \int_{S^{d-1}} Z \cdot L(W) \,\mathrm{d}\omega\,,\end{aligned} \tag{88}$$

arising from (51) and the symmetry of the mixed volume with respect to $K_{l_1}, \ldots, K_{l_d}$ after extension of $C_2^{\mathrm{conv}}(S^{d-1})$ to $C_2(S^{d-1})$. Moreover, L is of elliptic type what means that the (symmetric) matrix C' of the cofactors of the elements $\partial^2 Z / \partial u'^i\, \partial u'^j$ $(i, j = 1, \ldots, d-1)$ in the expansion

$$\begin{aligned}L(Z) &= D'(\mathrm{Hess}'\, H^{(l_1)}, \ldots, \mathrm{Hess}'\, H^{(l_{d-2})}, \mathrm{Hess}'\, Z) \\ &= \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} (D'(\mathrm{Hess}'\, H^{(l_1)}, \ldots, \mathrm{Hess}'\, H^{(l_{d-2})}))_{ij} \cdot \frac{\partial^2 Z}{\partial u'^i\, \partial u'^j}\end{aligned} \tag{89}$$

[see eqs. (87), (55) and (58)], is positive definite. This can be seen as follows. By a suitable orthogonal transformation of the Cartesian coordinates $u'^1, \ldots, u'^{d-1}$ of the tangent hyperplane $T_{-n}S^{d-1}$ of S^{d-1} which acts on the Hessians of $H^{(l_1)}, \ldots, H^{(l_{d-2})}$ simultaneously by:

$$\begin{aligned}&\mathrm{Hess}''\, H^{(l_1)} = T'^*\, \mathrm{Hess}'\, H^{(l_1)}\, T'\,, \\ &\quad\vdots \\ &\mathrm{Hess}''\, H^{(l_{d-2})} = T'^*\, \mathrm{Hess}'\, H^{(l_{d-2})}\, T'\,,\end{aligned}$$

and for this reason on the matrix C' of cofactors (in the same manner as on the matrix of the cofactors of a determinant) by:

$$C'' = (\det T')^2 (T')^{-1} C' (T'^*)^{-1} = T'^* C' T'$$

we can make C'' a diagonal matrix. Therefore, C'' and thus C' are positive definite if the diagonal elements of C'' are positive:

$$(D''(\mathrm{Hess}''\, H^{(l_1)}, \ldots, \mathrm{Hess}''\, H^{(l_{d-2})}))_{ii} > 0\,, \quad i = 1, \ldots, d-1\,. \tag{90}$$

But these cofactors are, up to the factor $1/(d-1)$, the mixed discriminants of the matrices $\mathrm{Hess}''\, H^{(l_1)}, \ldots, \mathrm{Hess}''\, H^{(l_{d-2})}$ with suppressed elements with the index i, being all positive definite, so that (90) holds because of Proposition 3.2.

Now the principal result of Hilbert's parametrix method is contained in:

Proposition 4.1 [Hilbert (1912, p. 241) for $d = 3$ and the C_∞-case].

(a) *If L is any linear symmetric differential operator of second order and elliptic type with continuous coefficients on the space $C_2(S^{d-1})$, and if w is any continuous positive weight function on S^{d-1}, then the differential equation*

$$L(z) + \lambda w z = 0 \tag{91}$$

for $z := Z|_{S^{d-1}}$, *has a countable number of eigenvalues* $\lambda_1 \le \lambda_2 \le \cdots$ *with a [relatively to* $C_2(S^{d-1})$*] closed system of eigenfunctions* $z_1 = Z_1|_{S^{d-1}}$, $z_2 = Z_2|_{S^{d-1}}$, $\ldots$, *mutually orthogonal with respect to the inner product* (85):

$$\langle Z_\mu, Z_\nu \rangle = 0 \quad \text{if } \mu \neq \nu , \tag{92}$$

and normalized by

$$\langle Z_\nu, Z_\nu \rangle = 1 . \tag{93}$$

(b) *Hereby,* λ_ν *is characterized by the property*:

$$\lambda_\nu = \min_{\langle Z, Z\rangle = 1,\ \langle Z_1, Z\rangle = \cdots = \langle Z_{\nu-1}, Z\rangle = 0} \left(- \int_{S^{d-1}} Z \cdot L(Z)\, \mathrm{d}\omega \right), \tag{94}$$

attained for $Z = Z_\nu$.

(c) *If L and w depend analytically on a parameter t, then* λ_ν *depends continuously on t.*

Applying this result to our differential operator (87), together with the weight function (86) and the inner product (85), the implication (81) $\Rightarrow$ (82), with equality iff (83) holds, or equivalently,

$$\langle H^{(l_{d-1})}, Z \rangle = 0 \;\Rightarrow\; -\int_{S^{d-1}} Z \cdot L(Z)\, \mathrm{d}\omega \ge 0 , \tag{95}$$

with equality iff

$$Z(y) = \sum_{a=1}^{d} a_a y_a , \quad a_a = \text{const} , \tag{96}$$

will be proved if we can show:

$$\lambda_1 = -1 \quad \text{with eigenfunction } Z_1 = H^{(l_{d-1})} , \tag{97}$$

$$\lambda_2 = \cdots = \lambda_{d+1} = 0 \quad \text{with eigenfunctions } Z_2 = y_1, \ldots, Z_{d+1} = y_d \tag{98}$$

and

$$\lambda_{d+2} > 0 . \tag{99}$$

That $H^{(l_{d-1})}$ is eigenfunction of L for the eigenvalue -1 is a direct consequence of (87) and (86). Moreover, the linearly independent linear functions $y_1, \ldots, y_d$ with vanishing Hessians are eigenfunctions of L for the eigenvalue 0 because of (87). But this is not enough to prove (97), (98) and (99) because there might exist further nonpositive eigenvalues for L. Therefore, Aleksandrov considers at first the special case $K_{l_1} = \cdots = K_{l_{d-1}} = K_0 := B^d$, with $H^{(l_1)}(y) = \cdots = H^{(l_{d-1})}(y) = H_0(y) = (\Sigma_{a=1}^d (y^a)^2)^{1/2}$, and thus

$$\text{Hess}'\, H^{(l_1)}(-n) = \cdots = \text{Hess}'\, H^{(l_{d-1})}(-n) = E' . \tag{100}$$

The insertion of (100) into (89) leads to

$$L_0(Z) = \frac{1}{d-1} \sum_{i=1}^{d-1} \frac{\partial^2 Z}{(\partial u'^i)^2} := \frac{1}{d-1}\, \Delta' Z \tag{101}$$

as well as

$$w = 1 . \tag{102}$$

Now it is well known that the so-called "spherical harmonics", defined by:

$$s_n(y) := \frac{p_n(y)}{(\Sigma_{a=1}^d (y^a)^2)^{n/2}} , \quad y \neq 0 , \tag{103}$$

with suitably normalized harmonic homogeneous polynomials $p_n(y)$ of degree n in the coordinates $y^1, \ldots, y^d$,

$$\Delta p_n = 0 , \quad n = 0, 1, 2, \ldots , \tag{104}$$

form a closed system for the continuous functions on S^{d-1} (see Müller 1966). These spherical harmonics are mutually orthogonal with respect to the inner product (85) with $w = 1$. Their positively homogeneous extensions

$$Z_n^{(0)}(y) := \frac{p_n(y)}{(\Sigma_{a=1}^d (y^a)^2)^{(n-1)/2}} , \quad y \neq 0 , \tag{105}$$

of degree 1 to $\mathbb{R}^d \setminus \{0\}$ provide *all* (normalized) eigenfunctions of our differential operator L_0 because (101), (105), (104) and Euler's homogeneity relation yield

$$L_0(Z_n^{(0)}) := \frac{1}{d-1}\, \Delta' Z_n^{(0)} = \frac{1}{d-1}\, \Delta Z_n^{(0)} = \frac{(n-1)(-n-d+1)}{d-1} \cdot Z_n^{(0)} \tag{106}$$

on S^{d-1}. Therefore, we see that in fact $(n-1)(n+d-1)/(d-1)$, $n = 0, 1, 2, \ldots$, are all the possible eigenvalues for L_0. *Especially* $\lambda_1^{(0)} = -1$ *is the smallest eigenvalue of* L_0 *with multiplicity* 1 *and* $\lambda_2^{(0)} = \cdots = \lambda_{d+1}^{(0)} = 0$ *is the next one with multiplicity d; there exist no other nonpositive eigenvalues for* L_0. Finally, Aleksandrov applies the "continuity method" in order to complete his proof by showing (97), (98) and (99). As we have just seen this is true in the special case $L = L_0$. It remains to prove this fact for a general differential operator L. For this reason we consider the array of differential operators L_τ respectively of weights

w_τ, defined by (87) respectively (86) with respect to the regular compact convex bodies $K_\tau^{(l_1)} := (1-\tau)K_0 + \tau K_{l_1}, \dots, K_\tau^{(l_{d-1})} = (1-\tau)K_0 + \tau K_{l_{d-1}}$ $(0 \le \tau \le 1)$. Proposition 4.1(c) says that the eigenvalues $\lambda_\nu^{(\tau)}$ of L_τ together with w_τ ($\nu = 1, 2, \dots$) depend continuously on τ. Therefore, it suffices to prove that 0 is eigenvalue for all L_τ, w_τ *with multiplicity d* because then no higher eigenvalue $\lambda_\nu^{(0)}$ of L_0 with $\nu \ge d+2$ can move into the interval $(-\infty, 0]$ when τ runs from 0 to 1.

This will be done as follows. Let Z be an arbitrary solution of

$$L_\tau(Z) := D'(\text{Hess}'\, H_\tau^{(l_1)}, \dots, \text{Hess}'\, H_\tau^{(l_{d-2})}, \text{Hess}'\, Z) = 0\,, \tag{107}$$

see (55) and (58). Then (61) implies

$$D'(\text{Hess}'\, H_\tau^{(l_1)}, \dots, \text{Hess}'\, H_\tau^{(l_{d-3})}, \text{Hess}'\, Z, \text{Hess}'\, Z) \le 0 \quad \text{on } S^{d-1} \tag{108}$$

Now

$$\begin{aligned}&\int_{S^{d-1}} D'(\text{Hess}'\, H_\tau^{(l_1)}, \dots, \text{Hess}'\, H_\tau^{(l_{d-3})}, \text{Hess}'\, Z, \text{Hess}'\, Z) \cdot H_\tau^{(l_{d-2})}\, d\omega \\ &\quad = \int_{S^{d-1}} D'(\text{Hess}'\, H_\tau^{(l_1)}, \dots, \text{Hess}'\, H_\tau^{(l_{d-2})}, \text{Hess}'\, Z) \cdot Z\, d\omega = 0\,,\end{aligned} \tag{109}$$

because of (88) and (107), so that we may deduce the equality in (108) from (109) and $H_\tau^{(l_{d-2})} > 0$. But Theorem 3.4 says that this occurs only if the relation

$$\text{Hess}'\, Z = q \cdot \text{Hess}'\, H_\tau^{(l_{d-2})} \tag{110}$$

holds on S^{d-1} with $q \equiv 0$ as it can be seen by insertion of (110) into (107). Thus $\text{Hess}'\, Z = \text{Hess}\, Z \equiv 0$, whence $Z(y) = \langle a, y\rangle = \Sigma_{b=1}^d a_b y^b$, with $a_1, \dots, a_d =$ const, so that $Z_2^{(\tau)} = y^1, \dots, Z_{d+1}^{(\tau)} = y^d$ are all linearly independent eigenfunctions of L_τ for the eigenvalue 0. This completes Aleksandrov's proof. □

Remark 4.2. By an approximation argument it may be seen that the Aleksandrov–Fenchel–Jessen inequalities (75) hold for arbitrary compact convex bodies. But this argument gives no information for the equality case which has not yet been completely solved. For this topic we refer to chapter 1.2.

5. Uniqueness theorems for convex hypersurfaces

In literature numerous theorems about the uniqueness of a regular compact convex body in $\mathbb{R}^d$ with prescribed infinitesimal behaviour of its boundary (up to certain congruence) exist. Most typical result in this direction is the following theorem.

Theorem 5.1. [Aleksandrov (1937, Section 7) and Fenchel and Jessen (1938), both without differentiability assumptions]. *If the boundaries of two regular compact convex bodies* K_0 *and* K_1 *in* $\mathbb{R}^d$ *have the same* ν*th mean radius of curvature at corresponding points with equal unit normal,*

$$P_\nu^{(0)}(-n) = P_\nu^{(1)}(-n)\,, \quad -n \in S^{d-1} \tag{111}$$

for a fixed ν $(1 \le \nu \le d-1)$, *then* K_0 *and* K_1 *only differ by a translation.*

Proof for $\nu > 1$ (*Outline*) [Chern (1959) *in case of differentiability* C_∞]. In the same manner as the Aleksandrov–Fenchel–Jessen inequalities (75) imply the concavity of the Brunn–Minkowski-function

$$\Psi(t) := \Big(V(\underbrace{K_t, \dots, K_t}_{\nu+1}, \underbrace{B^d, \dots, B^d}_{d-1-\nu})\Big)^{1/\nu+1}\,,$$
$$K_t := (1-t)K_0 + tK_1\,, \quad 0 \le t \le 1$$

(see Leichtweiss 1980, Satz 24.1), the corresponding inequalities (61) for the (multilinear) mixed discriminants imply the concavity of the function

$$\begin{aligned}\Phi(t) &:= \left[\binom{d-1}{\nu} D'(\underbrace{\text{Hess}'\, H^{(t)}, \dots, \text{Hess}'\, H^{(t)}}_{\nu}, \underbrace{E', \dots, E'}_{d-1-\nu})(-n)\right]^{1/\nu} \\ &= \left[\binom{d}{\nu} D(\underbrace{\text{Hess}\, H^{(t)}, \dots, \text{Hess}\, H^{(t)}}_{\nu}, \underbrace{E, \dots, E}_{d-\nu})(-n)\right]^{1/\nu} \\ &= [D_\nu(H^{(t)})(-n)]^{1/\nu}\end{aligned} \tag{112}$$

[compare eq. (56)], where $H^{(t)} := (1-t)H^{(0)} + tH^{(1)}$ is the support function of K_t. Therefore, after applying (30) and (111), the inequalities

$$D_\nu(\underbrace{H^{(0)}, \dots, H^{(0)}}_{\nu-1}, H^{(1)}) \ge [D_\nu(H^{(0)})]^{(\nu-1)/\nu} \cdot [D_\nu(H^{(1)})]^{1/\nu} = D_\nu(H^{(1)}) \tag{113}$$

and (changing the part of K_0 and K_1)

$$D_\nu(H^{(0)}, \underbrace{H^{(1)}, \dots, H^{(1)}}_{\nu-1}) \ge [D_\nu(H^{(0)})]^{1/\nu} \cdot [D_\nu(H^{(1)})]^{(\nu-1)/\nu} = D_\nu(H^{(0)}) \tag{114}$$

hold (see Leichtweiss 1980, Hilfssatz 22.2). Hereby equality only occurs if $\Phi(t)$ is linear in t, i.e., if we have equality in the underlying inequalities (61), whence (because of $\nu > 1$), after using (111), $\text{Hess}'\, H^{(0)} = \text{Hess}'\, H^{(1)}$. For this reason, in order to prove Theorem 5.1, it suffices to show equality in (113) and (114). This may be done by subtracting the generalized Minkowski's integral formulas

$$\int_{S^{d-1}} h^{(0)}\binom{d-1}{\nu}^{-1} D_\nu^{(\nu-1,1)}\,d\omega = \int_{S^{d-1}} \binom{d-1}{\nu+1}^{-1} D_{\nu+1}^{(\nu,1)}\,d\omega = \int_{S^{d-1}} h^{(1)}\binom{d-1}{\nu}^{-1} D_\nu^{(\nu,0)}\,d\omega \qquad (115)$$

and vice versa

$$\int_{S^{d-1}} h^{(1)}\binom{d-1}{\nu}^{-1} D_\nu^{(1,\nu-1)}\,d\omega = \int_{S^{d-1}} \binom{d-1}{\nu+1}^{-1} D_{\nu+1}^{(1,\nu)}\,d\omega = \int_{S^{d-1}} h^{(0)}\binom{d-1}{\nu}^{-1} D_\nu^{(0,\nu)}\,d\omega\,, \qquad (116)$$

with

$$D_\nu^{(\lambda,\nu-\lambda)} = D_\nu(\underbrace{H^{(0)},\ldots,H^{(0)}}_{\lambda}, \underbrace{H^{(1)},\ldots,H^{(1)}}_{\nu-\lambda}),\quad 0\le\lambda\le\nu \qquad (117)$$

[compare eqs. (30) and (33)!], from each other. Namely, this yields

$$\int_{S^{d-1}} [h^{(0)}(D_\nu^{(\nu-1,1)} - D_\nu^{(0,\nu)}) + h^{(1)}(D_\nu^{(1,\nu-1)} - D_\nu^{(\nu,0)})]\,d\omega = 0\,, \qquad (118)$$

which indeed implies equality in (113) and (114) if we assume (without loss of generality) $h^{(0)} > 0$ and $h^{(1)} > 0$ on S^{d-1}. □

In the case $\nu = 1$, Chern gave a proof of Theorem 5.1 [which does not make use of the profound inequalities (113) and (114)] by another integral formula. This theorem was at first proved for $d = 3$ and $\nu = 1$ by Christoffel (1865), and for $d = 3$ and $\nu = 2$ by Minkowski (1903). Some important generalizations of it, replacing P_ν by a suitable function $\Phi(R_1,\ldots,R_{d-1},n,x)$ with $\partial\Phi/\partial R_i > 0$ for $i = 1,\ldots,d-1$, and using a maximum principle for the solutions of elliptic differential equations, are due to Aleksandrov (1962). For other generalizations with regard to parts of convex hypersurfaces, see Oliker (1979). We want to mention further that Theorem 5.1 is obviously equivalent to the following fact: *Two regular convex hypersurfaces F_0 and F_1 in $\mathbb{R}^d$ are congruent if they have equal "third fundamental form"*

$$e_{ij}\xi^i\xi^j\,,\quad e_{ij} = \left\langle \frac{\partial n}{\partial u^i}, \frac{\partial n}{\partial u^j}\right\rangle,\ i, j = 1,\ldots,d-1\,, \qquad (119)$$

and equal νth mean radius of curvature at corresponding points, because the first condition induces the congruence of the spherical maps of F_0 and F_1.

Corollary 5.2 (Süss 1929, Satz 2). *A regular convex hypersurface F in $\mathbb{R}^d$ with the property*

$$P_\nu = \text{const}\,,\quad \nu \text{ fixed}\,,\ 1\le\nu\le d-1\,, \qquad (120)$$

is a sphere.

"Dual" to Corollary 5.2 is the following theorem.

Theorem 5.3 [Liebmann (1990, pp. 107 and 109) in the case $d = 3$]. *A regular convex hypersurface F in $\mathbb{R}^d$ with the property*

$$H_\nu = \text{const}\,,\quad \nu \text{ fixed}\,,\ 1\le\nu\le d-1\,, \qquad (121)$$

is a sphere.

Proof (*in the C_3-case for $\nu < d-1$*). By Newton's formulas (see Hardy, Littlewood and Pòlya 1934, p. 104), there is

$$\frac{H_1}{H_0} \ge \frac{H_2}{H_1} \ge \cdots \ge \frac{H_{\nu+1}}{H_\nu}\,,\quad H_0 := 1\,, \qquad (122)$$

with equality only in the case $k_1 = \cdots = k_{d-1}$ or F = sphere. But (121) and the Minkowski-formulas (20) imply

$$\int_M (H_1H_\nu - H_{\nu+1})h\,dF = \int_M (H_\nu - H_{\nu+1}h)\,dF - H_\nu\int_M (H_0 - H_1h)\,dF = 0\,, \qquad (123)$$

whence indeed follows equality in (122) or $H_1H_\nu - H_{\nu+1} \ge 0$ if we assume, without loss of generality, $h > 0$. □

For a proof of Theorem 5.3 in the case $\nu = d-1$ by a slightly different integral formula, see Walter (1989, pp. 186–187). Furthermore, we want to draw attention to a paper of Walter (1985) with far reaching extensions of Theorem 5.3 involving special "isoparametric hypersurfaces" (whose principal curvatures are all constant).

Theorem 5.3 is a consequence of a reflection theorem and – more general – of a uniqueness theorem for hypersurfaces of another type as theorem 5.1:

Theorem 5.4 (Voss 1956, Satz VI). *If the boundaries F_0 and F_1 of two regular compact convex bodies K_0 and K_1 in $\mathbb{R}^d$ of differentiability class C_3 have the same "lower" and "upper" orthogonal projection along the direction $e \in \mathbb{R}^d$ on a hyperplane in $\mathbb{R}^d$ as well as equal νth mean curvature at corresponding points with respect to this projection,*

$$H_\nu^{(0)}(x^{(0)}) = H_\nu^{(1)}(x^{(1)})\,,\quad \nu \text{ fixed}\,,\ 1\le\nu\le d-1\,, \qquad (124)$$

where

$$x^{(1)} = x^{(0)} + \chi e\,, \tag{125}$$

then K_0 and K_1 only differ by a translation along (the constant vector) e.

The idea of the proof of this theorem consists in a modification of Steiner's continuous symmetrization: F_0 and F_1 are joined by a linear array F_t, given by

$$x^{(t)} := x^{(0)} + t\chi e\,, \quad 0 \leq t \leq 1\,, \tag{126}$$

compare eq. (125)! Then Theorem 5.4 turns out to be a direct consequence of the integral formula

$$\begin{aligned} 0 &= \int_M (H_\nu^{(1)} - H_\nu^{(0)})\langle \chi e, n^{(0)}\rangle \, \mathrm{d}F_0 \\ &= \int_0^1 \left[\int_M \frac{\partial H_\nu^{(t)}}{\partial t} \langle \chi e, n^{(t)}\rangle \, \mathrm{d}F_t\right] \mathrm{d}t \\ &= \frac{\nu}{(d-1)!} \int_0^1 \left[\int_M \left(-\mathrm{d}\frac{\partial n^{(t)}}{\partial t}, \underbrace{-\mathrm{d}n^{(t)}, \ldots, -\mathrm{d}n^{(t)}}_{\nu-1}, \underbrace{\mathrm{d}x^{(t)}, \ldots, \mathrm{d}x^{(t)}}_{d-1-\nu}, \chi e\right)\right] \mathrm{d}t \\ &= \frac{\nu}{(d-1)!} \int_0^1 \left[\int_M \left(\underbrace{-\mathrm{d}n^{(t)}, \ldots, -\mathrm{d}n^{(t)}}_{\nu-1}, \underbrace{\mathrm{d}x^{(t)}, \ldots, \mathrm{d}x^{(t)}}_{d-1-\nu}, \mathrm{d}\chi e, -\frac{\partial n^{(t)}}{\partial t}\right)\right] \mathrm{d}t \\ &= -\frac{\nu}{d-1} \int_M C_{(\nu)}^{ij}(\langle e, n^{(0)}\rangle)^2 \frac{\partial x}{\partial u^i}\frac{\partial \chi}{\partial u^j} \, \mathrm{d}F_0 \end{aligned} \tag{127}$$

[compare eq. (20)!], because hereby the matrix $(C_{(\nu)}^{ij})$ is positive definite and $\langle e, n^{(0)}\rangle$ vanishes only on the "shadow boundary" of $F_{(0)}$ with respect to projection along e.

Remark 5.5 (Aeppli 1959, *Satz* 10). Theorem 5.4 remains valid if we replace there the "parallel map" (125) by a "radial map"

$$x^{(1)} = \chi x^{(0)}\,, \quad \chi > 0\,, \tag{128}$$

the equality (124) by the equality of the (dilatation invariant) "reduced νth mean curvature" at corresponding points

$$\|x^{(0)}\|^\nu H_\nu^{(0)}(x^{(0)}) = \|x^{(1)}\|^\nu H_\nu^{(1)}(x^{(1)})\,, \quad \nu \text{ fixed}\,,\ 1 \leq \nu \leq d-1\,, \tag{129}$$

and the translation along e by a dilatation with the origin of $\mathbb{R}^d$ as the center – under the additional assumption that all "joining hypersurfaces"

$$x^{(t)} := (1 + t(\chi - 1))x^{(0)}\,, \quad 0 \leq t \leq 1 \tag{130}$$

[compare eq. (126)], be regular convex hypersurfaces.

It should be mentioned that two different common generalizations of Theorem 5.4 and Remark 5.5, involving 1-parameter transformation groups of a Riemannian target space $\mathbb{R}^d$, have been made by Hopf and Katsurada (1968a,b), based on Stokes' theorem and on a maximum principle. For further information on the topic of this section, especially for $d = 3$, see Huck et al. (1973).

6. Convexity and relative geometry

In the so-called "relative differential geometry" of convex hypersurfaces, a regular convex hypersurface $F : M \to \mathbb{R}^d$ of differentiability class C_3, given by $x = x(u^1, \ldots, u^{d-1})$, is referred to a regular convex "gauge hypersurface" $N : M \to \mathbb{R}^d$ of differentiability class C_2, containing the origin in the "interior" and given by $-y = -y(u^1, \ldots, u^{d-1})$, by means of the so-called "Peterson map" between (equally oriented) points of F and N with parallel tangent hyperplanes:

$$-y_i := -\frac{\partial y}{\partial u^i} = B_i^k x_k\,, \quad i = 1, \ldots, d-1 \tag{131}$$

[compare eq. (9)], with

$$(x_1, \ldots, x_{d-1}, y) > 0\,. \tag{132}$$

(Thus Euclidean differential geometry is the special case $y = n$ of relative differential geometry!) Besides the "normalization vector" y of F we consider also its "conormal vector" X, defined by:

$$\langle X, x_i\rangle = \langle X, y_i\rangle = 0\,, \quad i = 1, \ldots, d-1\,, \tag{133}$$

and

$$\langle X, y\rangle = 1\,. \tag{134}$$

Because of the obvious symmetry relations

$$B_{ij} = B_i^k G_{kj} = \left\langle \frac{\partial X}{\partial u^j}, y_i\right\rangle = -\left\langle X, \frac{\partial^2 y}{\partial u^i\,\partial u^j}\right\rangle = \left\langle \frac{\partial X}{\partial u^i}, y_j\right\rangle = B_j^k G_{ki} = B_{ji} \tag{135}$$

for the coefficients of the (quadratic) "third fundamental form"

$$B_{ij}\xi^i\xi^j \tag{136}$$

of (F, N), compare eq. (119), with respect to the coefficients

$$G_{ij} := \langle X, \partial^2 x/\partial u^i\, \partial u^j \rangle = G_{ji}$$

of its *positive definite* (quadratic) "second fundamental form"

$$G_{ij}\xi^i\xi^j \tag{137}$$

[compare eq. (4)], the roots of the characteristic equation $\det(B_i^k - z\delta_i^k) = 0$ are all real. They are called the "*relative principal curvatures*" ${}_rk_1, \ldots, {}_rk_{d-1}$ of (F, N), and their inverses, ${}_rR_1, \ldots, {}_rR_{d-1}$, the "*relative principal radii of curvature*" of (F, N) with the normalized elementary symmetric functions ${}_rH_\nu$ respectively ${}_rP_\nu$ $(\nu = 1, \ldots, d-1)$. As we have assumed also the gauge hypersurface N to be regular convex, all the entities ${}_rk_\nu$, ${}_rR_\nu$, ${}_rH_\nu$, ${}_rP_\nu$ $(\nu = 1, \ldots, d-1)$ are positive in view of the positive definiteness of the matrix $(B_{ij}) = (\langle X, -\partial^2 y/\partial u^i\, \partial u^j \rangle)$, see eq. (135). They are related by the "relative geometric Minkowski's integral formulas"

$$\int_M {}_rH_{\nu-1}\, \mathrm{d}_rF = \int_M {}_rH_\nu\, {}_rh\, \mathrm{d}_rF\,, \quad \nu = 1, \ldots, d-1\,, \tag{138}$$

$$\int_M {}_rP_{d-\nu}\, \mathrm{d}_rN = \int_M {}_rP_{d-\nu-1}\, {}_rh\, \mathrm{d}_rN\,, \quad \nu = 1, \ldots, d-1\,, \tag{139}$$

involving the "relative geometric support function"

$${}_rh := -\langle X, x \rangle \tag{140}$$

[compare eq. (21)] and the "relative surface area"

$$\int_M \mathrm{d}_rF = \frac{1}{(d-1)!} \int_M (\underbrace{\mathrm{d}x, \ldots, \mathrm{d}x}_{d-1}, y) \tag{141}$$

of F, respectively the "relative surface area"

$$\int_M \mathrm{d}_rN = \frac{1}{(d-1)!} \int_M (\underbrace{-\mathrm{d}y, \ldots, -\mathrm{d}y}_{d-1}, y) \tag{142}$$

of N [compare eqs. (20) and (33)!]. These Minkowski's formulas may be proved exactly as in the Euclidean case after replacing the Euclidean unit normal vector n by the normalization vector y. As an important consequence we note:

Theorem 6.1 [Süss (1927, p. 69) for $d = 3$]. *A pair (F, N) of regular compact convex hypersurfaces of differentiability class (C_3, C_2) with constant relative νth mean curvature ${}_rH_\nu$ for a fixed ν with $1 \leq \nu \leq d-1$, is a "relative sphere", i.e., a pair of homothetic hypersurfaces F and N* (*compare Theorem* 5.3!).

The proof of this theorem is totally analogous to the proof of Theorem 5.3; it uses the Minkowski's formulas (138). Theorem 6.1 remains valid with ${}_rH_\nu$ having been replaced by ${}_rP_\nu$ $(1 \leq \nu \leq d-1)$ [compare Corollary 5.2 and use Minkowski's formulas (139)!]. We remark that this fact is a trivial consequence of the following relative geometric generalization of Theorem 5.1.

Theorem 6.2. *The hypersurface F of the pair (F, N) of regular compact convex hypersurfaces of class (C_3, C_2) is uniquely determined up to a translation by N and the relative νth mean radius of curvature ${}_rP_\nu$ for an arbitrarily fixed ν with $1 \leq \nu \leq d-1$.*

This theorem may be proved totally analogous to Theorem 5.1 for $\nu > 1$; for $\nu = 1$ there exists (in the C_∞-case) a proof, applying a suitable integral formula, due to Oliker and Simon (1984, Theorem 3.1). Finally, we cite in this context:

Theorem 6.3 [Schneider 1967, Satz 4.4 (4.1)]. *A pair (F, N) of regular compact convex hypersurfaces of class (C_3, C_3) is uniquely determined up to a nondegenerate affinity by its third (second) fundamental form* (136) (*eq.* (137)), *the relative $(d-1)$th (first)mean curvature ${}_rH_{d-1}$ $({}_rH_1)$ and the "Tschebycheff-vector"*

$$V_i := G^{jk}A_{ijk}\,, \quad i = 1, \ldots, d-1\,, \tag{143}$$

where (G^{jk}) is inverse to (G_{ij}) and $A_{ijk} := \langle X, x_{i;j;k} \rangle$ are the (symmetric) coefficients of the "cubic fundamental form"

$$A_{ijk}\xi^i\xi^j\xi^k \tag{144}$$

of (F, N) $(x_{i;j;k}$ denote covariant derivatives with respect to the Riemannian metric $\mathrm{d}s^2 = G_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j$!).

7. Convexity and affine differential geometry

In the so-called "equiaffine differential geometry" the special conormal vector

$$X := \frac{[x_1, \ldots, x_{d-1}]}{[\det((x_1, \ldots, x_{d-1}, x_{ij}))]^{1/(d+1)}} \tag{145}$$

is assigned to a regular convex hypersurface F in $\mathbb{R}^d$ of differentiability class C_3, invariant against orientation preserving parameter transformations of F. Then the envelope of the hyperplanes $\langle X, z \rangle = 1$, reflected at the origin, may be used as a gauge hypersurface $-C$ (reflected "curvature image") for F (we assume for the moment that $-C$ must not be regular convex). Its position vector $-y$ [compare eqs. (134) and (133)] is called the negative "affine normal vector" of F, as $-C$ transforms itself in a homogeneous equivariant manner when F is transformed by an equiaffine (i.e., volume preserving affine) map of $\mathbb{R}^d$. From eqs. (145) and (134) we conclude for the "affine surface area" of F [compare eq. (141)]

$$\int_M \mathrm{d}_a F = \frac{1}{(d-1)!} \int_M (\mathrm{d}x, \underbrace{\ldots}_{d-1}, \mathrm{d}x, y)$$
$$= \int_M (\det(G_{ij}))^{1/2} \, \mathrm{d}u^1 \cdots \mathrm{d}u^{d-1} , \tag{146}$$

whence after logarithmic partial differentiation of the integrands in (146),

$$V_i = 0 , \quad i = 1, \ldots , d-1 \tag{147}$$

("apolarity conditions"). So we have natural specializations of Theorem 6.3 in the equiaffine differential geometry (e.g., see Schneider 1967, Satz 4.3). Now the notions of relative mean curvatures (respectively relative mean radii of curvature) immediately translate to equiaffine differential geometry and we have the following theorem.

Theorem 7.1. *A regular compact convex hypersurface F of class C_5 with constant affine νth mean curvature ${}_aH_\nu$ for a fixed ν with $1 \leq \nu \leq d-1$, is an ellipsoid (compare Theorem 5.3!).*

This follows from the fact that such a hypersurface must be a "proper affine sphere", i.e., a hypersurface with homothetic F and $-C$ (compare Theorem 6.1), and therefore an ellipsoid by a famous theorem of Blaschke (1923, §74 and §77) for $d = 3$ and Deicke (1953) for general d. For a short proof, see Schneider (1967, pp. 395–396), for further information, Simon (1985).

There are special results for regular compact convex hypersurfaces *F with regular convex reflected curvature image* $-C$ which we will denote as being "*of elliptic type*". For an example, we cite the following theorem.

Theorem 7.2 (Leichtweiss 1990, Satz 1). *A regular compact convex hypersurface F_0 of class C_4 and elliptic type, containing another one F_1 (not necessarily of elliptic type) in its interior, has a bigger affine surface area* (146) *than F_1 unless F_0 and F_1 coincide.*

Finally, we mention affine geometrical interpretations of the notions "affine normal vector" and "affine surface area", given by Blaschke (1923, §43 and §47) for $d = 3$, and by Leichtweiss (1989, 1986) for general d, that are connected with each other in a certain sense.

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CHAPTER 4.2

Convex Functions

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Contents

HANDBOOK OF CONVEX GEOMETRY
Edited by P.M. Gruber and J.M. Wills

A real valued function f defined on an interval I of the real line is said to be convex if for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y).$$

We say f is strictly convex if the inequality is strict for all x and y, $x \neq y$. In terms of a graph, the definition requires that if P, Q, and R are any three points on the graph of f with Q between P and R, then Q is on or below the chord PR (fig. 1). The definition can be taken as a statement about the slopes of the segments pictured in fig. 1,

$$\text{slope } PQ \leq \text{slope } PR \leq \text{slope } QR.$$

The papers of Jensen (1905, 1906) are generally cited as the first systematic study of the class of convex functions, but earlier work that noted properties of such functions is summarized in Roberts and Varberg (1973, p. 8).

It is easily seen that f is convex if and only if the set of points above its graph, its *epigraph*, (fig. 1) is a convex set in the plane. The study of convex functions is therefore subsumed by the study of convex sets, so that most studies of convexity (including this handbook) focus on convex sets. Convex functions do arise naturally, however, in optimization, analytic inequalities, functional analysis, and applied mathematics, so that there has arisen a vast literature that treats convexity in language familiar to analysts, that of functions. Our purpose is to survey that literature, according to the following outline:

1. Basic notions: Mid-convexity and continuity; Lower semi-continuity and closure of convex functions; Conjugate convex functions.

2. Differentiability of convex functions: Functions defined on $\mathbb{R}$; A function defined on $\mathbb{R}^2$; Functions defined on a linear space $\mathscr{L}$; Differentiable convex functions.

3. Inequalities: Classical inequalities obtained from convex functions; Matrix inequalities.

We conclude this introductory section with a briefly annotated bibliography of surveys of our topic, the complete citations of which can be found in the references listed at the end of the article.

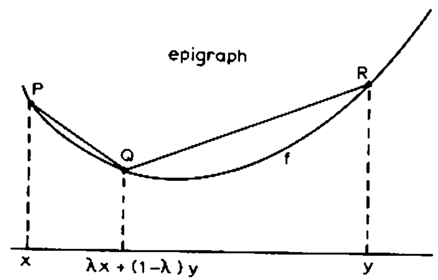

Figure 1.

(Beckenbach 1948). This article from the Bulletin of the AMS gives the flavor of some later, more extensive, frequently cited, but unpublished notes by the author.

(Ekeland and Temam 1976). Translated from the French, this was one of the first general works to introduce convex analysis into the calculus of variations.

(Fenchel 1953). Another set of frequently quoted but unpublished notes used as the basis of lectures at Princeton, these notes are given special mention for their influence on Rockafellar's *Convex Analysis*.

(Giles 1982). This text, acknowledging its debt to the unpublished 1978 notes of Phelps, focused on making accessible to graduate students the research on differentiability of convex functions.

(Moreau 1966). These are again lecture notes that were influential in the development of what is now called convex analysis.

(Phelps 1989). Perhaps prompted by the Giles book, Phelps finally brought up to date and then published the 1978 notes he had used at the University of London.

(Roberts and Varberg 1973). This text, written for an undergraduate audience, gives accessible proofs to most of the fundamental properties of convex functions.

(Rockafellar 1970b). Restricted to convex functions on $\mathbb{R}^n$, this carefully written book stands as the most complete reference on the topic. It makes extensive use of the notion of conjugate functions.

(Van Tiel 1984). Here is another book aimed at the undergraduate which provides a handy place to look for proofs of the basic properties of convex functions.

1. Basic notions

1.1. Midconvexity and continuity

Jensen, in his classic papers, said a function f was convex if it satisfied an inequality we shall take as the definition of midconvexity; f is *midconvex* on an interval I if for every $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \leq \tfrac{1}{2}[f(x) + f(y)] .$$

Examples of discontinuous functions satisfying this inequality were known to Jensen, so he naturally addressed the question of what minimal additional conditions would guarantee the continuity of a midconvex function. He first established what is now known as Jensen's inequality.

Theorem 1. *f is midconvex on I if and only if*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$$

for any n points $x_i \in I$ and any n nonnegative rational α_i such that $\Sigma_{i=1}^{n} \alpha_i = 1$.

He then showed that a midconvex function defined and bounded on an open interval would be continuous there.

This set in motion a series of papers that still continue, in which people strive for minimal additional conditions to imply continuity. It is known, for example, that a midconvex function defined on $[a, b]$ will be continuous on (a, b) if it is bounded above on a set M of Lebesgue measure $m(M) > 0$, or if it is bounded above on a second category Baire set. Roberts and Varberg (1973, chapter 7) summarize and give references to papers giving conditions that, along with midconvexity, imply continuity.

Results of this kind also occur for functions defined on spaces other than the real line. If U is a convex set in a linear space $\mathscr{L}$, and if $f : U \to \mathbb{R}$ is a real valued function defined on U, then the definitions of both convexity and midconvexity still make sense. Of course, to talk about the continuity of f, we need some sort of topology on $\mathscr{L}$, and most convex analysis is carried out with the understanding that $\mathscr{L}$ is a normed linear space. That seems the right context in which to end our discussion of the continuity of midconvex functions. A midconvex function defined on an open set U in a normed linear space is continuous on U if it is bounded above in a neighborhood of a single point of U.

1.2. Lower semi-continuity and closure of convex functions

If a convex function is defined on a nonempty open set in a locally convex linear topological space $\mathscr{L}$, then it is quite easy to describe the continuity properties of f.

Theorem 2. *Let $f : U \to \mathbb{R}$ be convex on a nonempty open set $U \subset \mathscr{L}$. If f is bounded above in a neighborhood of just one point p of U, then f is continuous on U* (Roberts and Varberg 1973, p. 67).

It is possible, of course, that a convex function may fail to be bounded above in a neighborhood of even a single point; well-known examples of discontinuous linear functionals defined on infinite dimensional linear spaces all fail to be bounded above in a neighborhood of any point. One might summarize with the observation that a convex function defined on an open set U is either continuous on U, or wildly discontinuous there. In particular, if $U \subset \mathscr{L} = \mathbb{R}^n$, then f is bounded in a neighborhood of every point $p \in U$, and is continuous on U (Roberts and Varberg 1973, p. 93).

Two difficulties that enter into this otherwise tidy situation can be illustrated by letting U be the half plane subset of $\mathbb{R}^3$

$$U = \{(x, y, 0) : x > 0\}$$

and defining the convex function $f : U \to \mathbb{R}$ by:

$$f(x, y, z) = y^2/x .$$

There is a natural sense in which U has interior points, but since no point of U

has, in the topology of $\mathbb{R}^3$, a neighborhood in U, it is not meaningful to talk about the continuity of f until we embed U in its affine hull (the entire xy plane in our example). Then we can say that f is continuous on U in the relative topology. Though easily done, this procedure introduces the complications of phrasing everything in terms of the relevant topology, and it is commonly avoided by the expedient described below of taking any convex function to be defined on all $\mathscr{L}$.

There are, in fact, far more compelling reasons than the one just mentioned for extending to all of $\mathscr{L}$ the definition of a convex function originally defined only on a subset of $\mathscr{L}$. It is very useful in any setting, such as optimization, where we are working with a large set of functions, to have them all defined on a common domain; in convex programming problems, it enables us to build constraints into the objective function to be minimized by defining the function to be infinity outside of its feasible set; and it is absolutely essential to have our functions globally defined if we are to take full advantage of the duality we shall meet when we introduce conjugate convex functions in the next section.

To properly set the stage for this program, we need to consider functions $f: U \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = \{\mathbb{R} \cup (+\infty) \cup (-\infty)\}$, possibly taking the values $\pm\infty$. This requires careful, but common sense arithmetic rules involving $+\infty$ and $-\infty$, and we must modify our definition to say that f is convex if and only if for any x and y in U for which there are real numbers α and β with $f(x) < \alpha$, $f(y) < \beta$,

$$f[\lambda x + (1-\lambda)y] \leq \lambda\alpha + (1-\lambda)\beta$$

when $0 < \lambda < 1$. In this setting, we say that the *effective domain* of f is $\operatorname{dom}(f) = \{x \in \mathscr{L}: f(x) < +\infty\}$, and a convex function is called *proper* if $\operatorname{dom}(f) \neq \emptyset$. We shall avoid some technical difficulties with the understanding that all the convex functions we discuss are proper. Proper convex functions never take the value of $-\infty$.

Now given a function $f: U \to \mathbb{R}$ convex on $U \subset \mathscr{L}$, it would be possible to extend f to all of $\mathscr{L}$ and preserve convexity by simply defining $f(x) = \infty$ for any $x \not\in U$. Quite obviously, however, we would sacrifice whatever continuity properties f might have had. It is natural to wonder, in the case of a convex function continuous on U (perhaps with an appropriately chosen relative topology), whether we might do better, and it is here that our example above illustrates another difficulty. Since an approach to $(0,0)$ along the parabolic path $x = y^2/m$ results in a limit of m for any $m > 0$, no definition of $f(0,0)$ will make f continuous at $(0,0)$. We face the fact that even in the relative topology that enables us to talk about the continuity of f on U, we may still be unable to extend f to the closure of U so as to retain the continuity of f.

It is easy to show (Fenchel 1949) that if a convex function f is defined at a limit point p of its domain U, then

$$\liminf_{u \to p} f(u) \leq f(p)\,.$$

This suggests that lower semi-continuity is the right goal to have in mind when

extending the definition of a convex function to include the limit points of its domain.

This works particularly well when $U \subset \mathbb{R}^n$. Then given a convex function $f: U \to \bar{\mathbb{R}}$ that never takes the value $-\infty$, we define clf to be identical with f on the interior of U, and we define it at the limit points p of U by:

$$\operatorname{clf}(p) = \liminf_{u \to p} f(u)$$

when this limit is finite; otherwise we set $f(p) = \infty$. To complete the definition, let clf be the constant $-\infty$ for the case in which f assumes the value $-\infty$ at some point. This function is lower semi-continuous.

Theorem 3. *Let $f: U \to \bar{\mathbb{R}}$ be a convex function that is not identically $+\infty$ on $U \subset \mathbb{R}^n$. Then* clf *is a lower semi-continuous convex function that agrees with f except possibly at relative boundary points of the* $\operatorname{dom}(f)$ (Rockafellar 1970b, p. 56).

Apart from necessary fussiness over details, we may conclude that any convex function can, with some possible redefinitions on the boundary for which we have a constructive method, be taken to be lower semi-continuous; and since all this will be true if we extend clf to all of $\mathbb{R}^n$ by setting $\operatorname{clf}(x) = +\infty$ for all x not in the closure of U, we may take any convex function to be globally defined.

For functions $f: \mathscr{L} \to \bar{\mathbb{R}}$, we define the epigraph of f to be the set $\operatorname{epi}(f) = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}\colon \alpha \geq f(x)\}$. It is still true that f is convex if and only if its epigraph, epi f is a convex subset $\mathscr{L} \times \bar{\mathbb{R}}$. It can also be shown that f is lower semi-continuous if and only if epi f is a closed subset of $\mathscr{L} \times \bar{\mathbb{R}}$. For this reason, clf is called the *closure* of f.

At this point we see an argument for subordinating the study of convex functions to an epigraphical viewpoint that immerses the study of functions in a study of their epigraphs. In their argument for this approach, Rockafellar and Wets (1984) point out that the graph of $f: \mathbb{R}^n \to \bar{\mathbb{R}}$ is not well defined as a subset of $\mathbb{R}^{n+1}$ because $f(x)$ may be $+\infty$; the graph is really a subset of $\mathbb{R}^n \times \bar{\mathbb{R}}$. The epigraph, however, does lie entirely in $\mathbb{R}^{n+1}$.

A basic reference for the epigraphical perspective is the monograph of Attouch (1984), which considers various convergence notions for convex functions in terms of their epigraphs. One important concept here for the case when the underlying space is reflexive is Mosco convergence (Mosco 1969). A promising approach for general normed linear spaces is the Attouch–Wets convergence, where convergence of epigraphs means uniform convergence on bounded subsets of $\mathscr{L} \times \mathbb{R}$ of the distance functions for the epigraphs. This reduces to ordinary Hausdorff metric convergence when restricted to closed and bounded convex sets and is well suited for estimation and approximation. Also, this notion of convergence is stable with respect to duality without reflexivity (Beer 1990).

Corresponding to a nonempty convex set U in a normed linear space $\mathscr{L}$, there are four globally defined convex functions that play a prominent role in the literature:

The *indicator function*

$$I(x|U)=\begin{cases}0 & \text{if } x\in U,\\ +\infty & \text{if } x\notin U.\end{cases}$$

The *gauge* function

$$G(x|U)=\inf\{\lambda\geq 0 \mid x\in\lambda U\}.$$

The *support* function

$$S(\ell, U)=\sup\{\langle \ell, x\rangle \mid x\in U\} \quad \text{for } U\subset\mathbb{R}^n,\ \ell\in\mathscr{L}^*.$$

The *distance* function

$$D(x|U)=\inf\{\|x-y\| \mid y\in U\}.$$

1.3. Conjugate convex functions

The two-variable version of the geometric mean–arithmetic mean inequality as stated in Theorem 18 below can be written, for $x>0$, $y>0$, in the form

$$xy\leq f(x)+g(y), \tag{1}$$

where f and g are convex functions defined, with $p>0$, $q>0$ and $1/p+1/q=1$, by:

$$f(x)=\frac{1}{p}x^p, \qquad g(y)=\frac{1}{q}y^q.$$

Our interest is in inequalities of the form (1).

Another inequality of this form from classical analysis involves integrals. Consider a function $h:[0,\infty)\to[0,\infty)$ that is strictly increasing and continuous with $h(0)=0$ and $\lim_{t\to\infty}h(t)=\infty$; in such a case, h^{-1} exists and has the same property as h. This allows us to define two convex functions

$$f(x)=\int_0^x h(t)\,dt, \qquad g(y)=\int_0^y h^{-1}(t)\,dt$$

for which Young's inequality then says

$$xy\leq f(x)+g(y).$$

Fenchel began a seminal paper (1949) by calling attention to Young's inequality and to one more example. If we set the support function S and the gauge function G defined at the end of the previous section for $U\subset\mathbb{R}^n$ equal to f and g, respectively, they satisfy the inequality (1). Generalizing from these observations, he then proved a theorem that has turned out to be a fundamental tool in the study of convex functions.

Theorem 4. *Let f be a convex function defined on a convex point set $\mathscr{F}\subset\mathbb{R}^n$ so as to be lower semi-continuous and such that $\lim_{x\to p}f(x)=\infty$ for each boundary point p of $\mathscr{F}$ that does not belong to $\mathscr{F}$. Then there exists a unique convex function g defined on a convex set $\mathscr{G}$ with exactly the same properties and such that*

$$x_1y_1+\cdots+x_ny_n\leq f(x_1,\ldots,x_n)+g(y_1,\ldots,y_n).$$

Moreover, the relationship is dual in the sense that if we begin with g defined on the convex set $\mathscr{G}$, we obtain f and $\mathscr{F}$.

The functions f and g are called *conjugate* convex functions.

Beginning with a convex function f having the required continuity properties, the unique conjugate function guaranteed by Theorem 4 is denoted by f^*. The usual procedure is simply to define, for $y\in\mathscr{L}$,

$$f^*(y)=\sup_{x\in\mathscr{F}}\{\langle x,y\rangle-f(x)\}$$

and then show that it has the desired properties. The notation $\langle x, y\rangle$ is used for $y(x)$ to emphasize the duality.

If one is not so careful about the lower semi-continuity of f, it is still possible to define the conjugate f^*, but it can then be proved that f^* is a closed convex function; that is, its epigraph will be a closed set in $\mathscr{L}\times\mathbb{R}$. There is no hope, then, of achieving the complete duality of having $f^{**}=f$ unless f is a closed function to begin with. It is for this reason that we find the conventions mentioned at the end of the last section to be convenient.

Though pairs of functions satisfying (1) entered into an earlier paper by Birnbaum and Orlicz (1931), Fenchel (1949) gave the first general treatment of conjugate convex functions, and his work has had far reaching ramifications to which we can only allude here.

We shall see when we ask about the existence of the derivatives of convex functions that one of the most satisfying answers comes in the form of the relationship between the subdifferential of f^* and the inverse of the subdifferential of f; $\partial(f^*)=(\partial f)^{-1}$.

The convex programming problem, which is to minimize a convex function f over a constraint set K, can be replaced, according to the Rockafellar–Fenchel Theorem (Holmes 1972, p. 68) with the dual problem of maximizing the sum of the conjugate of f and the indicator function of the set K. Rockafellar (1970b, 1974) has developed these ideals very fully.

After exploring the role of the duality of conjugate convex functions in the calculus of variations and in minimax theory, Ekeland and Temam (1976) take up applications to numerical analysis, control theory, mechanics, and economics.

This list does not exhaust the applications that are made of conjugate convex functions. In addition to the authors already mentioned, Ioffe and Tikhomirov (1968) provide a good survey of the applications of conjugacy.

2. Differentiability

We begin our consideration of the differentiability of convex functions with a careful look at functions of a single variable, first because easily drawn graphs often expose the heart of proofs that can be carried to more abstract settings, and secondly because the properties we shall discover help us anticipate what is true in general. In section 2.2, we use a particular function of two variables to introduce related concepts central to our survey in section 2.3 of differentiability of convex functions defined on Banach spaces. In section 2.4 we turn from questions of existence to look at what can be proved about convex functions known to be differentiable throughout an open set.

2.1. Functions defined on $\mathbb{R}$

The characterization of convexity in terms of slopes of secant chords says for the four points shown in fig. 2 that

$$\text{slope } PQ \leq \text{slope } PR \leq \text{slope } QR \leq \text{slope } QS \leq \text{slope } RS .$$

In particular,

$$\frac{f(x)-f(y)}{x-y} \leq \frac{f(z)-f(y)}{z-y} . \tag{2}$$

Since slope $PR \leq$ slope QR, it is clear that slope QR increases as $x \uparrow y$. Similarly, slope RS decreases as $z \downarrow y$. Thus the quotient on the left side of (2) increases as

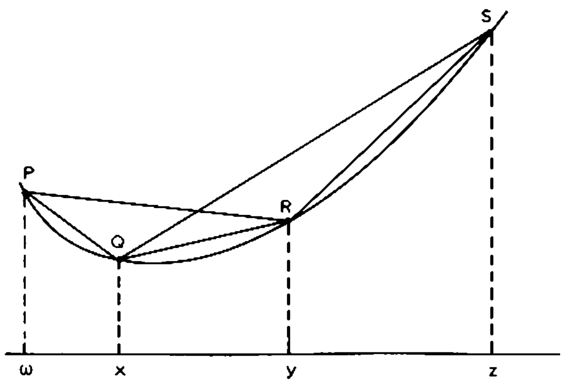

Figure 2.

$x \uparrow y$, the one on the right decreases as $z \downarrow y$, and we have established our first theorem.

Theorem 5. *At an arbitrary point y interior to its interval of definition, a convex function f has both a left derivative* $f'_-(y)$ *and a right derivative* $f'_+(y)$*; moreover,* $f'_-(y) \leq f'_+(y)$.

When $f'_-(y) = f'_+(y) = m$, then f is differentiable at y, and a line with slope m is tangent to the graph at $R(y, f(y))$. Otherwise we may choose any m satisfying $f'_-(y) < m < f'_+(y)$ and draw a line with slope m through R that lies entirely under the graph of f. Such a line is said to be a *line of support* for f at y.

Referring once again to our statement comparing slopes of secant lines, we see that

$$f'_+(w) \leq \frac{f(x)-f(w)}{x-w} \leq \frac{f(y)-f(x)}{y-x} \leq f'_-(y)$$

with all inequalities strict if f is strictly convex. Thus, drawing on Theorem 5, we can write

$$f'_-(w) \leq f'_+(w) \leq f'_-(y) \leq f'_+(y) .$$

Theorem 6. *If* $f: I \to \mathbb{R}$ *is convex (strictly convex), then* $f'_-(x)$ *and* $f'_+(x)$ *exist and are increasing (strictly increasing) on their respective domains.*

Further analysis of fig. 2 establishes quickly that

$$\lim_{x \uparrow y} f'_+(x) = f'_-(y) \quad \text{and} \quad \lim_{z \downarrow y} f'_+(z) = f'_+(y) .$$

From these two facts, we conclude that $f'_-(y) = f'_+(y)$ if and only if f'_+ is continuous at y. Stated another way, the derivative fails to exist at precisely those points where the increasing function f'_+ is discontinuous. Since an increasing function can be discontinuous on at most a countable set, we have proved the following theorem.

Theorem 7. *If* $f: I \to \mathbb{R}$ *is convex on an interval I, the set E where f' fails to exist is countable. Moreover, f' is continuous on* $I - E$.

Finally, let us note that if the convex function f is differentiable at x_0, then for every $x > x_0$ in the domain of f,

$$f[(1-\alpha)x_0 + \alpha x] \leq (1-\alpha)f(x_0) + \alpha f(x) .$$

If we set $h = \alpha(x - x_0)$, we see that

$$\frac{f(x_0+h)-f(x_0)}{h} \leq \frac{f(x)-f(x_0)}{x-x_0} .$$

Consideration of the case $x < x_0$ leads to the same inequality, and taking limits as $h \to 0$ gives us our last theorem of this section.

Theorem 8. *If f is convex on an interval I and differentiable at x_0, then for any $x \in I$,*

$$f(x) - f(x_0) \geqslant f'(x_0)(x - x_0) .$$

2.2. Related concepts

Many of the very deep results about the differentiation of convex functions are couched in terms that can be nicely illustrated with a function of two variables. Consider the graph (fig. 3) of the function

$$f(x, y) = \begin{cases} x^2 + y^2 , & x^2 + y^2 \geqslant 2 , \\ 2 , & x^2 + y^2 \leqslant 2 . \end{cases}$$

If $P(x_0, y_0, f(x_0, y_0))$ is a point on the graph, then

$$z = A(x, y) = f(x_0, y_0) + a(x - x_0) + b(y - y_0)$$

is a plane that meets the graph of f at P. It is instructive to look at two such planes:

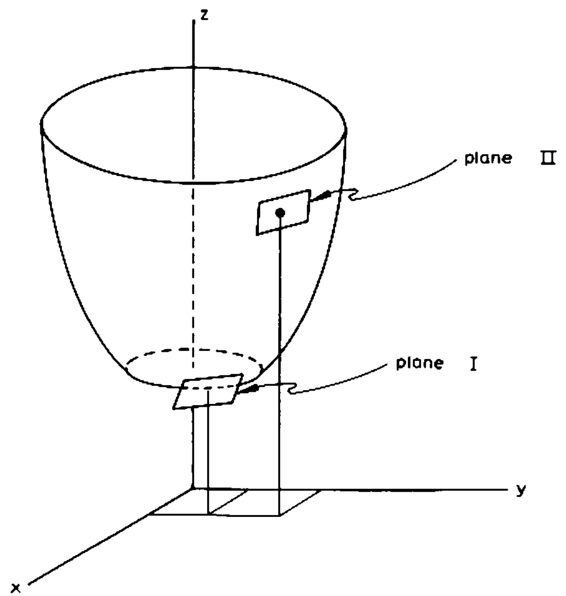

Figure 3.

Plane I, through $(1, 1, 2)$, $\quad z = 2 + \frac{3}{2}(x - 1) + \frac{3}{2}(y - 1)$,

Plane II, through $(1, 2, 5)$, $\quad z = 5 + 2(x - 1) + 4(y - 2)$.

Both meet the graph of f in exactly one point, and otherwise lie below it. A plane that meets the graph of $z = f(x, y)$ in at least one point and never rises above it is called a *support plane*. Our example illustrates the following facts about support planes.

(1) A plane of support may meet the graph of $z = f(x, y)$ in many points; $z = 2$ is a support plane that meets the graph in fig. 3 in infinitely many points.

(2) There may be many planes of support passing through the same point on the graph of $z = f(x, y)$; besides Plane I and the plane $z = 2$, there are obviously many other planes of support to the graph of fig. 3 at $(1, 1, 2)$.

(3) The plane of support at a point (x_0, y_0, z_0) might be unique, as in Plane II, the plane tangent to the graph at $(1, 2, 5)$ in our example.

(4) A function $f(x, y)$ is convex if and only if it has at least one plane of support of each point (x_0, y_0, z_0) on its graph.

The most obvious generalization of the derivative of a function of one variable to a function of two variables (not necessarily convex) is the so-called directional derivative. Starting at $\boldsymbol{p}_0(x_0, y_0)$ and moving in the direction of the unit vector $\boldsymbol{v} = [r \; s]$, we define:

$$f'(\boldsymbol{p}_0)(\boldsymbol{v}) = \lim_{t \to 0} \frac{f(\boldsymbol{p}_0 + t\boldsymbol{v}) - f(\boldsymbol{p}_0)}{t} .$$

In our example, with $\boldsymbol{p}_0(1, 2)$ and $\boldsymbol{v} = [\frac{4}{5} \; \frac{3}{5}]$, it may be shown that $f'(\boldsymbol{p}_0)(\boldsymbol{v}) = 4$. This is the slope of the line formed by the intersection of the tangent Plane II and the plane through $(1, 2)$ that contains $\boldsymbol{v}$ and is perpendicular to the xy plane.

The directional derivatives for the choice $\boldsymbol{v} = [1 \; 0]$ and $\boldsymbol{v} = [0 \; 1]$, when they exist, are called the partial derivatives $\partial f / \partial x(x_0, y_0)$ and $\partial f / \partial y(x_0, y_0)$. When the directional derivative exists at $\boldsymbol{p}_0$ for every choice of $\boldsymbol{v}$, we say f is *Gateaux* differentiable at $\boldsymbol{p}_0$.

In general, Gateaux differentiability does not guarantee all that we would like to be true of a differentiable function. The Gateaux derivative of f may exist at a point even if f is discontinuous there; and $f'(\boldsymbol{p}_0)(\boldsymbol{v})$ need not be linear in $\boldsymbol{v}$. All this changes if it is known that f is convex; convexity imposes orderlines. If a convex function is Gateaux differentiable at a point, it is continuous there, and $f'(\boldsymbol{p}_0)(\boldsymbol{v})$ will be linear in $\boldsymbol{v}$.

If the convex function f has continuous partial derivatives in a neighborhood of (x_0, y_0), then the support plane, $A(x, y)$ as defined above, is unique. It is the plane tangent to the graph at (x_0, y_0, z_0), and the coefficients a and b that determine the plane are the components of the gradient vector ∇f of elementary calculus which, with an eye to the future we shall call

$$\mathrm{d}f = \left[\frac{\partial f}{\partial x}(x_0, y_0) \quad \frac{\partial f}{\partial y}(x_0, y_0) \right] .$$

We turn now to a second generalization of the derivative which, though it brings us to the same place for convex functions defined on $U \subset \mathbb{R}^n$, will prove to be a stronger concept for functions defined on a normed linear space.

Consider again the function $A(x, y)$ defined above. It is said to be *affine*. It is the sum of a constant $f(x_0, y_0)$ and a linear function, a fact that can be emphasized by writing it in the form

$$A(x, y) = f(x_0, y_0) + [a\ b]\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

We now adopt the viewpoint that the derivative of f at (x_0, y_0) is the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$, represented by the matrix $[a\ b]$. The special thing about this linear transformation is, of course that it is a linear transformation that closely approximates $A(x, y) - f(x_0, y_0)$. That is the idea that gives rise to our definition.

Using $\Delta x = x - x_0$ and $\Delta y = y - y_0$, we say that f is Frechet differentiable at (x_0, y_0) if there exists a linear transformation L such that

$$f(x, y) = f(x_0, y_0) + L(\Delta x, \Delta y) + |(\Delta x, \Delta y)|\varepsilon(x_0, y_0, \Delta x, \Delta y),$$

where $\varepsilon(x_0, y_0, \Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to 0$. The linear transformation L, if it exists, is easily shown to be unique; it is called the *Frechet derivative* $f'(x_0, y_0)$ of f at (x_0, y_0).

As we have already said, for a convex function f defined on $U \subset \mathbb{R}^n$, the Frechet derivative is identical to the Gateaux derivative:

$$\mathrm{d}f = \left[\frac{\partial f}{\partial x}(x_0, y_0) \quad \frac{\partial f}{\partial y}(x_0, y_0)\right] = f'(x_0, y_0).$$

There is yet a third way to look at this expression. We may regard d as a special case of a set valued operator ∂ that maps a point (x_0, y_0) into a set of linear transformations, any one of which will define a plane of support. Thus, when f is differentiable at (x_0, y_0), $\partial f(x_0, y_0)$ is the single transformation $\mathrm{d}f(x_0, y_0)$, but when f is not differentiable, $\partial f(x_0, y_0)$ is many valued. In our example,

$$\partial(1, 1) = [t\ t], \quad \text{where } t \in [0, 2].$$

The set valued operator ∂ is called the *subdifferential* of f at (x_0, y_0), and a particular member of the set, such as $[\frac{3}{4}\ \frac{3}{4}]$ is called a subgradient. Our example makes it clear that a function will have a subdifferential at points where it may not have a derivative in the sense of either Gateaux or Frechet. It is this fact that makes the subdifferential such a useful tool in convex analysis.

We have seen that a convex function of a single real variable has, by virtue of its convexity, numerous differentiability properties. It is already clear from the example in this section, which is nondifferentiable on $x^2 + y^2 = 2$, that we cannot hope to establish differentiability on all but a countable set. Under what conditions will a convex function on $\mathscr{L}$ have a Gateaux differential? a Frechet derivative? When will these derivatives be equal? How can we characterize sets of nondifferentiability? These questions have led to a rich, deep, and largely satisfying literature to which we now turn.

2.3. *Functions defined on a linear space $\mathscr{L}$*

A function $f : U \to \mathbb{R}$ defined on an open set U in a linear topological space $\mathscr{L}$ is said to be Gateaux differentiable at $x_0 \in U$ provided that

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists for every $v \in \mathscr{L}$. The limit, when it exists, is called the Gateaux differential $\mathrm{d}f(x_0)$.

Though not true of the Gateaux differential in general, $\mathrm{d}f(x_0)$ turns out to be a linear functional on $\mathscr{L}$ if f is a convex function; and while $\mathrm{d}f(x_0)$ may be a discontinuous linear functional (as, for example, when f itself is a discontinuous linear functional) it can be shown to be continuous if the convex function f is continuous at x_0.

Under what conditions will f have a Gateaux differential at x_0? For convex functions defined on $U \subset \mathbb{R}^n$, the answer is easy: it is sufficient that the n partial derivatives exist and are finite. To answer this in greater generality, we need to extend to $\mathscr{L}$ some of the concepts first met in $\mathbb{R}^2$. We say $f : U \to \mathbb{R}$ has *support* at $x_0 \in U$ if there exists an affine function $A : \mathscr{L} \to \mathbb{R}$ such that $A(x_0) = f(x_0)$ and $A(v) \leq f(v)$ for every $v \in \mathscr{L}$. With what we know about convex functions of a single real variable and a standard application of the Hahn–Banach Theorem, it can be shown that f is convex on U if and only if f has support at each $x \in U$. If the convex function f is continuous, then the support functions will be continuous; the converse is more tricky to prove, but it will be true if $\mathscr{L}$ is a normed linear space.

Theorem 9. *The convex function $f : U \to \mathbb{R}$ has a Gateaux differential at $x_0 \in U$ if and only if f has unique support at x_0.*

The existence of a convex function on ℓ^∞ that is everywhere continuous and nowhere Gateaux differentiable (Phelps 1989, p. 13) shows that we will not extend to an arbitrary space $\mathscr{L}$ the fact that a convex function of a single real variable is differentiable except on at most a countable set. The usual response is to put conditions on $\mathscr{L}$ as well as on f, and the most familiar theorem of this sort is due to Mazur (1933).

Theorem 10. *If $\mathscr{L}$ is a separable Banach space and f is a continuous convex function defined on an open set $U \subset \mathscr{L}$, then the set of points x where the Gateaux differential $\mathrm{d}f(x)$ exists is a dense G_δ set in U.*

Another not quite equivalent definition of a derivative gets a lot of attention in the literature. We say that $f: U \to \mathbb{R}$ is Frechet differentiable at $x_0 \in U$ if there exists a linear functional $L: \mathcal{L} \to \mathbb{R}$ such that for sufficiently small $v \in \mathcal{L}$,

$$f(x_0 + v) = f(x_0) + L(v) + \|v\| \varepsilon(x_0, v),$$

where $\varepsilon(x_0, v) \to 0$ as $v \to 0$. When such a functional L exists, it is, as was noted above, unique; it is called the *Frechet derivative* and is designated by $f'(x_0)$.

If the Frechet derivative $f'(x_0)$ exists, then so does the Gateaux differential $\mathrm{d}f(x_0)$ and $f'(x_0) = \mathrm{d}f(x_0)$, but not conversely. There are examples of convex functions that are Gateaux differentiable at every nonzero point but nowhere Frechet differentiable. Consequently, Frechet and Gateaux derivatives are sometimes referred to as strong and weak derivatives respectively.

A word about continuity is in order. A linear transformation $f: \mathcal{L} \to \mathbb{R}$ is both Gateaux and Frechet differentiable at any x_0 in $\mathcal{L}$, and $f = \mathrm{d}f(x_0) = f'(x_0)$. Since there are, on infinite dimensional spaces $\mathcal{L}$, well-known examples of discontinuous linear (and so convex) functions, it follows that neither the Gateaux nor the Frechet derivative of a convex function needs to be continuous.

If, however, the convex function is continuous at x_0, then the existence of $\mathrm{d}f(x_0)$ is enough to also guarantee that $\mathrm{d}f(x_0)$ is a continuous linear functional; similarly with $f'(x_0)$. Moreover, for the Frechet derivative, it is a two way street. The derivative $f'(x_0)$ of a convex function is a continuous linear transformation if and only if the function f is continuous at x_0.

It is enough for our purposes to say that caution must be exercised, even when $\mathcal{L} = \mathbb{R}^n$, where the existence of the Gateaux differential $\mathrm{d}f(x_0)$ does not by itself guarantee the continuity of f. We shall simplify our own exposition by assuming henceforth that our function $f: U \to \mathbb{R}$ is convex and continuous on $U \subset \mathcal{L}$. This will guarantee that when either $\mathrm{d}f(x_0)$ or $f'(x_0)$ exists, it will be a member of the dual space $\mathcal{L}^*$ of continuous linear functionals on $\mathcal{L}$.

If $U \subset \mathbb{R}^n$, it can be proved that a continuous convex function $f: U \to \mathbb{R}$ has a Frechet derivative almost everywhere (in the sense of Lebesgue) (Roberts and Varberg 1973, p. 116). In the context of general linear spaces, attempts to characterize sets on which $f: U \to \mathbb{R}$ is differentiable, Gateaux or Frechet, have led to deep and revealing connections between convex functions, the geometry of Banach spaces, measure theory, and the study of monotone operators. Progress on the general question is generally dated from the work of Asplund (1968) who took a two pronged approach. He examined spaces more general than separable Banach spaces in which the results of Mazur's theorem still held, and he also examined a sub-class of Banach spaces in which the conclusion of Mazur's Theorem held for Frechet differentiability. He called these latter strong differentiability spaces, but common terminology now associates his name with both classes of spaces.

A Banach space $\mathcal{L}$ is an *Asplund space* if every continuous convex function defined on a nonempty open convex set $U \subset \mathcal{L}$ is Frechet differentiable at each point of a dense G_δ subset of D; and it is a *weak Asplund space* if Frechet is replaced by Gateaux in the definition.

In this terminology, Mazur's Theorem says that if a Banach space $\mathcal{L}$ is separable, then $\mathcal{L}$ is a weak Asplund space. Asplund proved that if the dual $\mathcal{L}^*$ of a Banach space $\mathcal{L}$ is separable, then $\mathcal{L}$ is an Asplund space. The converse was proved later (Namioka and Phelps 1975), and provides a very satisfying characterization.

Theorem 11. *A separable Banach space is an Asplund space if and only if its dual space $\mathcal{L}^*$ is separable.*

In fact, more can be said. A Banach space is an Asplund space if and only if every separable closed subspace has a separable dual.

The rich interconnections mentioned above have made it possible to characterize Asplund spaces in other ways. It turns out that a Banach space $\mathcal{L}$ is an Asplund space if and only if $\mathcal{L}^*$ has the Raydon–Nikodym Property; that is, roughly speaking, the classical Raydon–Nikodym Theorem holds for suitably restricted vector-valued measures with values in $\mathcal{L}^*$. This idea is developed very fully by Diestral and Uhl (1977) and by Bourgin (1983).

Another characterization figures into what we shall say below, so we shall state it as a theorem here (Giles 1982, p. 202; Phelps 1989, p. 80).

Theorem 12. *A Banach space $\mathcal{L}$ is an Asplund space if and only if every nonempty weak* compact convex subset of $\mathcal{L}^*$ is the weak* closed convex hull of its weak* strongly exposed points.*

Characterization of weak Asplund spaces turns out to be more difficult. Asplund showed that if the norm of a given Banach space can be replaced by an equivalent norm that has a strictly convex dual norm, then $\mathcal{L}$ is a weak Asplund space; and Preiss, Phelps and Namioka (1990) showed very recently that the existence of an equivalent Gateaux differentiable norm on a Banach space $\mathcal{L}$ implies that $\mathcal{L}$ is a weak Asplund space. Complete characterization, however, remains a research problem.

A case can be made (Phelps 1989, chapter 6) for trying instead to characterize *Gateaux differentiability spaces*. These are defined to be Banach spaces in which any convex function defined on a nonempty open set U is Gateaux differentiable on a set G that is dense in U. This differs from a weak Asplund space only in that the set G does not need to be a G_δ set, but the difference is enough to allow an analogue to Theorem 12. The Banach space will be a Gateaux differentiability space under exactly the same conditions specified in Theorem 12 except that one speaks about exposed points rather than strongly exposed points.

We turn finally to the topic of the subdifferential of a convex function $f: U \to \mathbb{R}$ defined on a set U of a linear space $\mathcal{L}$. Since the concept finds its application not only in the treatment of nondifferentiable functions, but in the theory of optimization where discontinuous convex functions arise in natural ways, we abandon in this context the understanding we have had that the functions under consideration are continuous.

Roughly speaking, the subdifferential is an operator that maps each $x \in U$ into a subset $\partial f(x)$ of $\mathscr{L}^*$ that contains all the possible candidates for $f'(x)$. More formally,

$$\partial f(x_0) = \{L \in \mathscr{L}^*: A(x) = f(x_0) + L(x - x_0) \leq f(x) \text{ for all } x \in U\}\ .$$

There can be points for which $\partial f(x)$ is empty, even when f is convex (Rockafellar 1970a, p. 215), but the important fact is that $\partial f(x)$ is most frequently nonempty, in which case we say that f is *subdifferentiable* at x. When it is recalled that any convex function is, or can by re-definitions at certain boundary points be made lower semi-continuous, one sees the importance of the Bronsted–Rockafellar Theorem which guarantees that if $\mathscr{L}$ is a Banach space, then ∂f exists on a dense subset of the effective domain of f. Of course much more can be said when f is continuous at $x_0 \in U$. Then $\partial f(x_0)$ is nonempty, convex, and weak* compact in $\mathscr{L}^*$ (Giles 1982, p. 132).

Let x and y be two points in the interior of an open set $U \subset \mathscr{L}$ on which f is convex, and choose two subgradients $L_x \in \partial f(x)$ and $L_y \in \partial f(y)$. Then

$$L_x(y - x) \leq f(y) - f(x)\ ,$$
$$L_y(x - y) \leq f(x) - f(y)$$

and addition gives

$$(L_y - L_x)(y - x) \geq 0\ .$$

A set valued function $T: \mathscr{L} \to \mathscr{L}^*$ is called a *monotone* operator if for all x and y whenever $L_x \in T(x)$ and $L_y \in T(y)$, $(L_y - L_x)(y - x) \geq 0$, so the calculation above shows that a subdifferential is a monotone operator. Subdifferentials may therefore be studied in the context of what is known about monotone operators, a program that has been carried out by Phelps (1989) for functions defined on a normed linear space $\mathscr{L}$, and in great detail for the case in which $\mathscr{L} = \mathbb{R}^n$ by Rockafellar (1970b). Rockafellar (1970a) has in fact been able to characterize those monotone operators which are subdifferentials of convex functions, and recent work by Simons (1991) offers a much shorter proof.

The convex function f is Gateaux differentiable at x_0 if and only if $\partial f(x_0)$ consists of exactly one $L \in \mathscr{L}^*$; and it is Frechet differentiable at x_0 if and only if in addition to the uniqueness of $L \in \partial f(x_0)$, the operator ∂f is norm to norm upper-semi-continuous at x_0 (Giles 1982, p. 122). The great utility of the subdifferential comes, of course, in those situations where it enables one to say something about a nondifferentiable function. This occurs in optimization, for instance (Phelps 1989, p. 44).

Theorem 13. *A proper lower semi-continuous convex function f has a global minimum at x_0 if and only if $\mathcal{O} \in \partial f(x_0)$.*

2.4. Differentiable convex functions

When a function $f: U \to \mathbb{R}$ is convex on $U \subset \mathscr{L}$ and Frechet differentiable at $x_0 \in U$, then $\partial f(x_0)$ consists of the single linear functional $f'(x_0)$, and for any other $x \in U$,

$$f(x) - f(x_0) \geq f'(x_0)(x - x_0)\ .$$

This inequality characterizes convex functions that are differentiable throughout U (Roberts and Varberg 1973, p. 98).

Theorem 14. *If if is differentiable throughout $U \subset \mathscr{L}$, then f is convex if and only if*

$$f(x) - f(x_0) \geq f'(x_0)(x - x_0)$$

for all x and $x_0 \in U$. Moreover, f is strictly convex if and only if the inequality is strict whenever $x \neq x_0$.

If f is continuous and differentiable throughout an open set $U \subset \mathscr{L}$, the computations performed above for differentials show that f' is a monotone operator. This also characterizes convex functions that are differentiable throughout U.

Theorem 15. *Let $f: U \to \mathbb{R}$ be continuous and differentiable on the open convex set $U \subset \mathscr{L}$. Then f is convex if and only if*

$$[f'(x) - f'(y)](x - y) \geq 0$$

and f is strictly convex if and only if the inequality is strict whenever $x \neq y$.

The convexity of functions $f: U \to \mathbb{R}$ that are twice differentiable on $U \subset \mathscr{L}$ can also be characterized in terms of their second derivatives. The Frechet derivative is a mapping $f': U \to \mathscr{L}^*$ from one normed space to another, and as such may itself be differentiable, in which case $f''(x)$ is a bilinear transformation on $\mathscr{L}$, commonly written $f''(x)(h, k)$. Under reasonably general conditions (Dieudonne 1960, p. 175), $f''(x)$ is symmetric, enabling us to ask whether $f''(x)$ is nonnegative definite.

Theorem 16. *Let f be twice differentiable throughout $U \subset \mathscr{L}$. Then f is convex if and only if $f''(x)$ is nonnegative definite for every $x \in U$; and if $f''(x)$ is positive definite on U, then f is strictly convex.*

This is of course what one would expect when thinking intuitively about the graph of a convex function of a single variable. The tangent line will rotate counterclockwise as it moves along the curve.

3. Inequalities

3.1. Classical inequalities

A rich study at which we can only hint in the available space uses convex functions to derive well-known and not so well-known inequalities with great elegance. One chain of inequalities with their proofs will give the idea.

Theorem 17 (Jensen's Inequality). *Let f be a convex function defined on the (possibly infinite) open interval (a, b), and let $x_i \in (a, b)$. If $\alpha_i > 0$ and $\Sigma_1^n \alpha_i = 1$, then*

$$f\Big(\sum_1^n \alpha_i x_i\Big) \le \sum_1^n \alpha_i f(x_i) .$$

Proof. Let $x_0 = \Sigma_1^n \alpha_i x_i$; the convex function f has support at x_0, so there is a support function $L(x) = f(x_0) + m(x - x_0)$ that satisfies $L(x_i) \le f(x_i)$ for each i. Multiply each side by α_i and sum, remembering that $\Sigma_1^n a_i = 1$, to obtain the inequality. □

We will now use Jensen's inequality with the choice of $f(t) = \exp(t)$.

Theorem 18 (The Geometric Mean–Arithmetic Mean Inequality). *If $x_i \ge 0$, $\alpha_i > 0$, and $\Sigma_1^n \alpha_i = 1$, then*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \le \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n .$$

Proof. Since the inequality is obvious if any $x_i = 0$, we need only consider the case where all $x_i > 0$, enabling us to define $t_i = \ln x_i$. Then

$$x_i^{\alpha_i} = \exp(\alpha_i \ln x_i) = \exp(\alpha_i t_i) ,$$

$$\begin{aligned} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} &= \exp(\alpha_1 t_1 + \cdots + \alpha_n t_n) \\ &\le \alpha_1 \exp(t_1) + \cdots + \alpha_n \exp(t_n) \\ &= \alpha_1 x_1 + \cdots + \alpha_n x_n . \end{aligned}$$ □

The choice of $\alpha_i = 1/n$ for each i gives us the GM–AM inequality in its classic form and with this inequality available, it is but a short jump to the inequalities of Hölder, Cauchy–Bunyakovski–Schwarz, and Minkowski. We shall move on to a less well-known inequality in the chain we are pursuing.

Theorem 19. *If $x_i \ge 0$, $y_i \ge 0$, and n is a positive integer, then*

$$\Big[\prod_1^n (x_i + y_i)\Big]^{1/n} \ge \Big(\prod_1^n x_i\Big)^{1/n} + \Big(\prod_1^n y_i\Big)^{1/n} .$$

Proof. Again we need only concern ourselves with the case where $x_i + y_i > 0$ for all i. Some algebraic manipulation and the classic form of the GM–AM inequality gives us

$$\begin{aligned} \frac{\Big(\prod_1^n x_i\Big)^{1/n} + \Big(\prod_1^n y_i\Big)^{1/n}}{\Big[\prod_1^n (x_i + y_i)\Big]^{1/n}} &= \Big(\prod_1^n \frac{x_i}{x_i + y_i}\Big)^{1/n} + \Big(\prod_1^n \frac{y_i}{x_i + y_i}\Big)^{1/n} \\ &\le \frac{1}{n} \sum_1^n \frac{x_i}{x_i + y_i} + \frac{1}{n} \sum_1^n \frac{y_i}{x_i + y_i} = 1 . \end{aligned}$$ □

3.2. Matrix inequalities

We shall conclude this section by deriving some matrix inequalities. Let Ω_n be the class of doubly stochastic matrices, that is matrices with nonnegative entries in which the sum of elements in any row or any column is 1. Ω_n is easily shown to be a convex set in the space of all $n \times n$ matrices, and it can be shown that the extreme points of Ω_n are the permutation matrices, that is matrices obtained by some permutation of the rows of the identity matrix. If, for a fixed vector $\boldsymbol{v} \in \mathbb{R}^n$, we let

$$K_v = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = S\boldsymbol{v}, S \in \Omega_n\}$$

and observe that K_v is a convex set in $\mathbb{R}^n$, we can obtain an optimization theorem for matrices.

Theorem 20. *Given a vector $\boldsymbol{v} \in \mathbb{R}^n$ and a convex function $f : U \to \mathbb{R}$ where U contains K_v, define $g : \Omega_n \to \mathbb{R}$ by:*

$$g(S) = f(S\boldsymbol{v}) = f(\langle s_1, \boldsymbol{v}\rangle, \ldots, \langle s_n, \boldsymbol{v}\rangle) ,$$

where s_i is the ith row vector of S. Then g must be convex, and it assumes its maximum value at a permutation matrix P.

Proof. Verification of the convexity of g follows in a straightforward way from the definition. Now let $P_1 \cdots P_m$ be the $m = n!$ permutation matrices in Ω_n. Since they are the extreme points of Ω_n, any $S \in \Omega_n$ may be written in the form $S = \Sigma_1^m \alpha_i P_i$ for some choice of $\alpha_i \ge 0$ where $\Sigma_1^m \alpha_i = 1$. The convexity of g then enables us to write

$$g(S) \le \sum_1^m \alpha_i g(P_i) \le \sum_1^m \alpha_i g(P_v) = g(P_v) ,$$

where P_v is one of the P_i, chosen so that $g(P_v) = \max\{g(P_i)\}$. □

We are now in a position to prove a general inequality for convex functions of symmetric matrices, from which numerous matrix inequalities may be easily derived.

Theorem 21. *Let A be a real symmetric $n \times n$ matrix with eigenvalues $r_1, \ldots, r_n$. Set $v = (r_1, \ldots, r_n)$ and let $f : U \to \mathbb{R}$ be convex on a set U that contains K_v. Then for any orthonormal set $\{p_1, \ldots, p_n\}$,*

$$f(\langle Ap_1, p_1\rangle, \ldots, \langle Ap_n, p_n\rangle) \leq f(P_v v),$$

where P_v is the permutation matrix whose existence is asserted by Theorem 20.

Proof. Let $\{u_1, \ldots, u_n\}$ be the normalized eigenvectors of A corresponding respectively to $r_1, \ldots, r_n$. Define S to be the matrix with entries $s_{ij} = \langle u_i, p_j\rangle^2$. S is clearly symmetric and it is an exercise to show it doubly stochastic. Next we note that for any j,

$$\begin{aligned} Ap_j &= A[\langle u_1, p_j\rangle u_1 + \cdots + \langle u_n, p_j\rangle u_n] \\ &= \langle u_1, p_j\rangle r_1 u_1 + \cdots + \langle u_n, p_j\rangle r_n u_n \end{aligned}$$

so

$$\begin{aligned} \langle Ap_j, p_j\rangle &= \langle u_1, p_j\rangle^2 r_1 + \cdots + \langle u_n, p_j\rangle^2 r_n \\ &= s_{1j} r_1 + \cdots + s_{nj} r_n = \langle s_j, v\rangle, \end{aligned}$$

where s_j is the jth row vector of S.

$$f(\langle Ap_1, p_1\rangle, \ldots, \langle Ap_n, p_n\rangle) = f(\langle s_1, v\rangle, \ldots, \langle s_n, v\rangle) = f(Sv).$$

Since f is convex on a set that contains K_v, we know from Theorem 20 that $f(Sv) \leq f(P_v v)$.

To prove our last result, it will be convenient to know something about the function f defined for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, the positive orthant where $x_i \geq 0$ for all i, by:

$$f(x) = f(x_1, \ldots, x_n) = (x_1 \cdot x_2 \cdots x_n)^{1/n}.$$

According to Theorem 19, for $\alpha + \beta = 1$ and $x, y \in \mathbb{R}^n_+$,

$$\begin{aligned} f(\alpha x + \beta y) = \left[\prod_1^n (\alpha x_i + \beta y_i)\right]^{1/n} &\geq \left(\prod_1^n \alpha x_i\right)^{1/n} + \left(\prod_1^n \beta y_i\right)^{1/n} \\ &\geq \alpha f(x) + \beta f(y) \end{aligned}$$

so $-f$ is convex on $\mathbb{R}^n_+$. □

Theorem 22 (Hadamard's Determinant Theorem). *If A is a real nonnegative definite matrix with entries a_{ij}, then $\det A \leq a_{11} \cdot a_{22} \cdots a_{nn}$.*

Proof. Let the eigenvalues of A be $r_1, \ldots, r_n$ and form $v = (r_1, \ldots, r_n)$. The fact that the determinant of a matrix is equal to the product of its eigenvalues can be expressed in terms of the function f above by writing $\det A = [f(v)]^n$; indeed, $\det A = f(Pv)$ for any permutation matrix P. Our appeal is to Theorem 18. The domain of f surely contains K_v; we take the orthonormal set $\{p_1, \ldots, p_n\}$ to be the standard basis, meaning that p_j has a 1 in the jth position as its only nonzero entry, and that $\langle Ap_j, p_j\rangle = a_{jj}$. We conclude for the convex $-f$ that

$$-f(\langle Ap_1, p_1\rangle, \ldots, \langle Ap_n, p_n\rangle) \leq -f(P_v v)$$

which is equivalent to $a_{11} \cdot a_{22} \cdots a_{nn} \geq \det A$. □

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CHAPTER 4.3

Convexity and Calculus of Variations

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HANDBOOK OF CONVEX GEOMETRY
Edited by P.M. Gruber and J.M. Wills

CHAPTER 5.1

Integral Geometry

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Contents

HANDBOOK OF CONVEX GEOMETRY
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Integral geometry is concerned with the study, computation, and application of invariant measures on sets of geometric objects. It has its roots in some questions on geometric probabilities. The early development, where the names of Crofton, Sylvester, Poincaré, Lebesgue and others play a role, is subsumed in the book of Deltheil (1926); see also Stoka (1968). Integral geometry, as considered here, was essentially promoted by Wilhelm Blaschke and his school in the mid-thirties; lectures of Herglotz (1933) mark the beginning of this period. Standard sources are the books by Blaschke (1955) (first published in 1935 and 1937; see also vol. 2 of his collected works, Blaschke 1985), Santaló (1953), Hadwiger (1957 chapter 6), and in particular the comprehensive work of Santaló (1976).

From its very beginning, and more so in the work of Hadwiger, integral geometry was closely connected to the geometry of convex bodies. In the following, we restrict ourselves essentially to those parts of integral geometry which are related to convexity. In contrast to the existing monographs, we prefer a measure-theoretic approach; in particular, Federer's (1959) curvature measures and the area measures related to the theory of mixed volumes play an essential role. The article will, therefore, be different in spirit from the books listed above.

For different views on integral geometry and some more recent developments, we refer to the books of Matheron (1975) and Ambartzumian (1982, 1990). Some connections to convexity appear there, too, but these cannot be taken into consideration in the present article. To a certain amount, this article is continued by chapter 5.2 on "Stochastic Geometry".

1. Preliminaries: Spaces, groups, and measures

In this section, we give a brief account of some notation, concepts and results concerning the main spaces occurring in the integral geometry of Euclidean spaces.

We work in d-dimensional real Euclidean vector space $\mathbb{E}^d$ with the usual scalar product $\langle\cdot,\cdot\rangle$, the Euclidean norm $\|\cdot\|$, the induced topology and the corresponding σ-algebra $\mathcal{B}(\mathbb{E}^d)$. (Generally, $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of a topological space X.) For $m \in \mathbb{N} \cup \{0\}$, the m-dimensional Hausdorff (outer) measure λ_m on $\mathbb{E}^d$ is defined by

$$\lambda_m(A) := 2^{-m}\kappa_m \lim_{\delta\to 0} \inf\{\sum_{j=1}^{\infty}(\operatorname{diam} M_j)^m\colon\ A \subset \bigcup_{j=1}^{\infty} M_j, \operatorname{diam} M_j \leqslant \delta\}$$

whenever $A \subset \mathbb{E}^d$, where κ_m is the volume of the unit ball B^m in $\mathbb{E}^m$ and diam denotes the diameter. The restriction of λ_m to $\mathcal{B}(\mathbb{E}^d)$ is a measure which coincides for $m = d$ with d-dimensional Lebesgue measure and for $m = 0$ with the counting measure. Moreover, the restriction of λ_m to the σ-algebra of Borel sets of an m-dimensional C^1-submanifold of $\mathbb{E}^d$ coincides with the classical measures of arc length, surface area, etc., used in differential geometry. In particular, the restriction of λ_{d-1} to $\mathcal{B}(S^{d-1})$ is the spherical Lebesgue measure on S^{d-1}, the unit sphere of

$\mathbb{E}^d$. The measure λ_m is obviously invariant under Euclidean isometries. A detailed investigation of Hausdorff measures can be found in Federer (1969).

The group G_d of rigid motions, that is, of all orientation preserving isometries of $\mathbb{E}^d$, will be endowed with the compact-open topology which is generated by the family

$$\{\{g \in G_d\colon gK \subset U\}\colon K \subset \mathbb{E}^d \text{ compact}, U \subset \mathbb{E}^d \text{ open}\}.$$

With this topology, G_d is a topological group, locally compact, σ-compact and Hausdorff, and the action of G_d on $\mathbb{E}^d$ is continuous (see, e.g., Kobayashi-Nomizu 1963); moreover, there is no other topology sharing all these properties. In the following, all subgroups of G_d will be endowed with the trace of this topology. The most important of them are the group of translations, which may be identified with $\mathbb{E}^d$, and the compact group $\mathrm{SO}(d)$ of rotations, which may be identified with the group of orthogonal $(d\times d)$-matrices with determinant 1 considered as a topological subspace of $\mathbb{R}^{d^2}$. In particular, G_d may be identified with the semidirect product $\mathbb{E}^d \times \mathrm{SO}(d)$ with the product topology, the identifying mapping being

$$\gamma : \mathbb{E}^d \times \mathrm{SO}(d) \to G_d, \qquad \gamma(t,\rho) := \beta_{t,\rho}$$

where $\beta_{t,\rho}(x) := \rho x + t$ for $x \in \mathbb{E}^d$. Recall that the law of composition of the semidirect product $\mathbb{E}^d \times \mathrm{SO}(d)$ is given by

$$(t_1,\rho_1)(t_2,\rho_2) := (t_1 + \rho_1 t_2, \rho_1\rho_2).$$

All these groups have a left Haar measure (nonzero regular Borel measure invariant under left translations), unique up to normalization, which is also invariant under right translations. The Haar measure on $\mathbb{E}^d$ normalized so that the unit cube has measure 1 coincides with Lebesgue measure. We shall denote by ν the Haar measure on $\mathrm{SO}(d)$ satisfying $\nu(\mathrm{SO}(d)) = 1$. Finally, μ will be the Haar measure on G_d with $\mu(\{g \in G_d\colon g0 \in C\}) = 1$ for a unit cube C. Thus μ is the image measure of $\lambda_d \otimes \nu$ under the map γ.

The set $\mathcal{F}^d$ of all closed subsets of $\mathbb{E}^d$ is endowed with the topology generated by the family

$$\{\mathcal{F}_U\colon U \subset \mathbb{E}^d \text{ open}\} \cup \{\mathcal{F}^K\colon K \subset \mathbb{E}^d \text{ compact}\}$$

where

$$\mathcal{F}_A := \{F \in \mathcal{F}^d\colon F \cap A \neq \emptyset\}, \qquad \mathcal{F}^A := \{F \in \mathcal{F}^d\colon F \cap A = \emptyset\}$$

for $A \subset \mathbb{E}^d$. With this topology, sometimes called the hit-or-miss topology (or topology of closed convergence), $\mathcal{F}^d$ is compact, Hausdorff and separable, and the natural action of G_d on $\mathcal{F}^d$ given by

$$(g,F) \mapsto gF := \{gx\colon x \in F\}, \quad F \in \mathcal{F}^d,$$

is continuous. A sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}^d\backslash\{\emptyset\}$ converges to $F \in \mathcal{F}^d\backslash\{\emptyset\}$ if and only if each $x \in F$ is the limit of some sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in F_n$ for all

$n \in \mathbb{N}$ and, for each sequence
with $y_k \in F_{n_k}$ for all $k \in \mathbb{N}$ satis
see that the map $(F,F') \mapsto F \cup$
the σ-algebra $\mathcal{B}(\mathcal{F}^d)$ is generate
by the family $\{\mathcal{F}^K\colon K \subset \mathbb{E}^d$ co
$\{(F_1,F_2) \in \mathcal{F}^d \times \mathcal{F}^d\colon F_1 \cap F_2 \in \mathcal{F}$
is measurable. Similarly, since fo
is open in $\mathcal{F}^d$, the map $F \mapsto$
be found in Matheron (1975) a
compact convex subsets of $\mathbb{E}^d$, th
the topology induced by the Ha

The space $\mathcal{E}_q^d$ of all q-flats (q-
as a topological subspace of $\mathcal{F}^d$ a
Moreover, with the natural acti
Haar measure which is unique
Haar measure on $\mathcal{E}_q^d$ satisfying
computations, the following repr
fixed and let $E_q^\perp$ be the orthogon

$$\gamma_q : E_q^\perp \times \mathrm{SO}(d) \to \mathcal{E}_q^d,$$

Then γ_q is surjective and continu
measure $(\lambda_{d-q}|_{\mathcal{B}(E_q^\perp)}) \otimes \nu$ under
we have

$$\int_{\mathcal{E}_q^d} f \,\mathrm{d}\mu_q = \int_{\mathrm{SO}(d)} \int_{E_q^\perp} f$$

2. Intersection formulae

The most familiar type of integr
of a fixed and a "moving" geome
formula for convex bodies provi
positions of a moving convex bo
Crofton's intersection formula do
of all k-dimensional flats meeting
appearing in the results, the que
replace the characteristic function
respect to invariant measures. Th
since they are valid in local versio
we shall now present these inters

For a convex body $K \in \mathcal{K}$
Federer curvature measures of
$\Phi_d(K,\cdot) := \lambda_d(K \cap \cdot)$. Thus, $\Phi_i($
the cited article for its geometric

Theorem 2.1. *For convex bodies $K, K' \in \mathcal{K}^d$, Borel sets $\beta, \beta' \in \mathcal{B}(\mathbb{E}^d)$, and for $j \in \{0, \dots, d\}$,*

$$\int_{G_d} \Phi_j(K \cap gK', \beta \cap g\beta')\, \mathrm{d}\mu(g) = \sum_{k=j}^{d} \alpha_{djk} \Phi_{d+j-k}(K, \beta) \Phi_k(K', \beta') \tag{2.1}$$

with

$$\alpha_{djk} = \frac{\binom{k}{j}\kappa_k \kappa_{d+j-k}}{\binom{d}{k-j}\kappa_j \kappa_d} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{d+j-k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}.$$

Theorem 2.1 contains, in particular, the *complete system of kinematic formulae*, namely

$$\int_{G_d} V_j(K \cap gK')\, \mathrm{d}\mu(g) = \sum_{k=j}^{d} \alpha_{djk} V_{d+j-k}(K) V_k(K'). \tag{2.2}$$

Here

$$V_j(K) = \Phi_j(K, \mathbb{E}^d) = \frac{\binom{d}{j}}{\kappa_{d-j}} W_{d-j}(K) \tag{2.3}$$

is the jth intrinsic volume of K, and $W_m(K)$ is the mth quermassintegral of K. Of course, formula (2.2) could equivalently be written in terms of $W_0, \dots, W_d$. The intrinsic volume V_0 is equal to the Euler characteristic χ, hence a special case of (2.2) is the *principal kinematic formula*

$$\int_{G_d} \chi(K \cap gK')\, \mathrm{d}\mu(g) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} V_{d-k}(K) V_k(K'). \tag{2.4}$$

Observe that the left side is equal to $\mu(\{g \in G_d \colon K \cap gK' \neq \emptyset\})$.

It is an essential feature of the kinematic formulae (2.1) that on the right side the bodies K and K' appear separated. This simplifying effect is a consequence of the integration over the group of rigid motions. If the integration extends only over the group of translations, the result can only be expressed in terms of measures that depend simultaneously on K and K':

Theorem 2.2. *For convex bodies $K, K' \in \mathcal{K}^d$, Borel sets $\beta, \beta' \in \mathcal{B}(\mathbb{E}^d)$, and for $j \in \{0, \dots, d\}$,*

$$\begin{aligned} &\int_{\mathbb{E}^d} \Phi_j(K \cap (K' + t), \beta \cap (\beta' + t))\, \mathrm{d}\lambda_d(t) \\ &= \Phi_j(K, \beta)\Phi_d(K', \beta') + \sum_{k=j+1}^{d-1} \Phi_k^{(j)}(K, K', \beta \times \beta') \\ &\quad + \Phi_d(K, \beta)\Phi_j(K', \beta') \end{aligned} \tag{2.5}$$

with unique maps $\Phi_k^{(j)} : \mathcal{K}^d \times \mathcal{K}^d \times \mathcal{B}(\mathbb{E}^d \times \mathbb{E}^d) \to \mathbb{R}$ having the following properties:

(a) *$\Phi_k^{(j)}(K, K', \cdot)$ is a finite measure,*

(b) *the map $(K, K') \mapsto \Phi_k^{(j)}(K, K', \cdot)$ from $\mathcal{K}^d \times \mathcal{K}^d$ into the space of finite measures on $\mathbb{E}^d \times \mathbb{E}^d$ with the weak topology is continuous,*

(c) *$\Phi_k^{(j)}(\cdot, K, \alpha)$ and $\Phi_k^{(j)}(K, \cdot, \alpha)$ are additive $(K \in \mathcal{K}^d, \alpha \in \mathcal{B}(\mathbb{E}^d \times \mathbb{E}^d))$,*

(d) *$\Phi_k^{(j)}(\cdot, K, \cdot \times \beta)$ is positively homogeneous of degree k, $\Phi_k^{(j)}(K, \cdot, \beta \times \cdot)$ is positively homogeneous of degree $d + j - k$ $(K \in \mathcal{K}^d, \beta \in \mathcal{B}(\mathbb{E}^d))$.*

In contrast to Theorem 2.1, the result of Theorem 2.2 on the translative case is mainly of a qualitative character, since no explicit representation of the measure $\Phi_k^{(j)}(K, K', \cdot)$ is available, except in special cases (see below).

A counterpart to formula (2.1), with the "moving convex body" gK' replaced by a "moving flat", is given by the following result.

Theorem 2.3. *For a convex body $K \in \mathcal{K}^d$, a number $k \in \{0, \dots, d\}$, a Borel set $\beta \in \mathcal{B}(\mathbb{E}^d)$, and for $j \in \{0, \dots, k\}$,*

$$\int_{\mathcal{E}_k^d} \Phi_j(K \cap E, \beta \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{djk} \Phi_{d+j-k}(K, \beta) \tag{2.6}$$

(with α_{djk} as in Theorem 2.1).

Again, some special cases are worth noting. With $\beta = \mathbb{E}^d$, we obtain *Crofton's intersection formula*

$$\int_{\mathcal{E}_k^d} V_j(K \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{djk} V_{d+j-k}(K). \tag{2.7}$$

The case $j = 0$ of (2.6),

$$\alpha_{d0k} \Phi_{d-k}(K, \beta) = \int_{\mathcal{E}_k^d} \Phi_0(K \cap E, \beta \cap E)\, \mathrm{d}\mu_k(E), \tag{2.8}$$

interprets the curvature measure Φ_{d-k}, up to a constant factor, as an integral-geometric mean value of the "Gaussian" curvature measure Φ_0. The specialization $\beta = \mathbb{E}^d$ yields

$$\begin{aligned} \alpha_{d0k} V_{d-k}(K) &= \int_{\mathcal{E}_k^d} \chi(K \cap E)\, \mathrm{d}\mu_k(E) \\ &= \mu_k(\{E \in \mathcal{E}_k^d \colon K \cap E \neq \emptyset\}). \end{aligned} \tag{2.9}$$

We give some hints to the literature and to proofs. The principal kinematic formula goes back, in different degrees of generality, to Blaschke, Santaló, Chern and Yien. One finds references in Blaschke (1955), Santaló (1953) and, in particular, Santaló (1976). In the latter book, formulae of type (2.2) are proved for domains with smooth boundaries, where the V_j are expressed as curvature integrals. For convex bodies (and, more generally, for sets of the convex ring, see section 5 below), Hadwiger (1950, 1951, 1956) has proved (2.2) and other integral-geometric formulae in an elegant way, making use of his axiomatic characterization of the linear combinations of the quermassintegrals. This method is also employed in chapter 6 of his book, Hadwiger (1957). From there (but with different notations) we quote a general version of the principal kinematic formula for convex bodies.

Theorem 2.4. *If $\varphi : \mathcal{K}^d \to \mathbb{R}$ is an additive and continuous function, then*

$$\int_{G_d} \varphi(K \cap gK')\, \mathrm{d}\mu(g) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} \varphi_{d-k}(K) V_k(K')$$

for $K, K' \in \mathcal{K}^d$, where

$$\varphi_{d-k}(K) := \frac{1}{\alpha_{d0k}} \int_{\mathcal{E}_k^d} \varphi(K \cap E)\, \mathrm{d}\mu_k(E).$$

From (2.4) and (2.9) it is clear that for $\varphi = \chi$ this reduces to the principal kinematic formula.

A short proof of formula (2.4) was also given by Mani-Levitska (1988).

Theorems 2.1 and 2.3 in their general forms for curvature measures are due to Federer (1959), who proved them for sets of positive reach. A shorter proof for this general version of (2.1) was given by Rother and Zähle (1990). For convex bodies, considerably simpler approaches are possible. Schneider (1978a) gave a proof by a method similar to that of Hadwiger, proving and applying an axiomatic characterization of the curvature measures. A slightly simpler proof was given in Schneider (1980b), using uniqueness results for Lebesgue measures and external angles in the case of convex polytopes, and approximation to obtain the general case. A method of Federer (1959) to deduce (2.6) from (2.1) was extended in Schneider (1980b) to obtain a common generalization of Theorems 2.1 and 2.3, namely a kinematic formula for a fixed convex body and a moving convex cylinder.

A still different approach to Theorem 2.1, this time via the translative case and thus leading also to Theorem 2.2, was followed in Schneider and Weil (1986). We briefly sketch the main ideas. After showing the measurability of the map $g \mapsto \Phi_j(K \cap gK', \beta \cap g\beta')$, one may write

$$\int_{G_d} \Phi_j(K \cap gK', \beta \cap g\beta')\, \mathrm{d}\mu(g) = \int_{SO(d)} I(\rho)\, \mathrm{d}\nu(\rho)$$

with

$$I(\rho) := \int_{\mathbb{E}^d} \Phi_j(K \cap (\rho K' + t), \beta \cap (\rho\beta' + t))\, \mathrm{d}\lambda_d(t)$$

for $\rho \in SO(d)$. Assuming first that K and K' are polytopes one obtains, by direct computation and for ν-almost all ρ, the formula

$$I(\rho) = \Phi_j(K,\beta)\Phi_d(K',\beta') + \Phi_d(K,\beta)\Phi_j(K',\beta') + \sum_{k=j+1}^{d-1} \sum_{F \in \mathcal{F}_{d+j-k}(K)} \sum_{F' \in \mathcal{F}_k(K')} \gamma(F, \rho F', K, \rho K')[F, \rho F'] \times \lambda_{d+j-k}(\beta \cap F)\lambda_k(\beta' \cap F').$$

Here $\mathcal{F}_m(K)$ denotes the set of m-faces of the polytope K. The number $\gamma(F, \rho F', K, \rho K')$ is the external angle of the polytope $K \cap (\rho K' + t)$ at its face $F \cap (\rho F' + t)$, where $t \in \mathbb{E}^d$ is chosen so that relint $F \cap$ relint$(\rho F' + t) \neq \emptyset$. Finally, $[F, \rho F']$ is a number depending only on the relative positions of the affine hulls of F and $\rho F'$. (The assumption of general relative position made in Schneider and Weil (1986) is superfluous.) Choosing for ρ the identity, we see that the equality for $I(\rho)$ proves the translative formula (2.5) for polytopes and at the same time gives an explicit representation, in this case, for the measures $\Phi_k^{(j)}(K, K', \cdot)$ appearing in Theorem 2.2. The general assertions of Theorem 2.2 are then obtained by approximation. The proof of Theorem 2.1 next requires the computation of the integral

$$\int_{SO(d)} \gamma(F, \rho F', K, \rho K')[F, \rho F']\, \mathrm{d}\nu(\rho).$$

This can conveniently be achieved in an indirect way, using the uniqueness of spherical Lebesgue measure to show that the integral must be proportional to the product of the external angles of K at F and of K' at F'. This yields formula (2.1) for polytopes, except that the numerical values of the coefficients have to be determined by an additional argument. The proof is then completed by approximation. Details are found in Schneider and Weil (1986). There one also finds a result similar to Theorem 2.2 which holds for convex bodies and translates of convex cylinders, and as a special case a translative Crofton formula for curvature measures.

We add some remarks on translative formulae in special cases. For $j = d$ and $j = d - 1$, formula (2.5) follows from general formulae of measure theory; see Groemer (1977, 1980a), Schneider (1981b). The global case of (2.5), that is, the case $\beta = \beta' = \mathbb{E}^d$, can be written in the form

$$\int_{\mathbb{E}^d} V_j(K \cap (K' + t))\, \mathrm{d}\lambda_d(t) = V_j(K)V_d(K') + \sum_{k=j+1}^{d-1} V^{(j)}_{k,d+j-k}(K, K') + V_d(K)V_j(K') \tag{2.10}$$

with $V^{(j)}_{k,d+j-k}(K, K') := \Phi_k^{(j)}(K, K', \mathbb{E}^d \times \mathbb{E}^d)$. In special cases, some more information on the functionals $V^{(j)}_{k,d+j-k}(K, K')$ is available. Investigations referring to the cases

$d=2$ and $d=3$ are found in Blaschke (1937), Berwald and Varga (1937), Miles (1974) (cf. also Firey 1977). For $j=0$ one obtains (see Groemer 1977 for some extensions)

$$\int_{\mathbb{E}^d} V_0(K\cap(K'+t))\,\mathrm{d}\lambda_d(t)$$
$$=\lambda_d(\{t\in\mathbb{E}^d:\ K\cap(K'+t)\neq\emptyset\})$$
$$=V_d(K+\check{K}')=\sum_{k=0}^{d}\binom{d}{k}V(\underbrace{K,\ldots,K}_{k},\underbrace{\check{K}',\ldots,\check{K}'}_{d-k}),\tag{2.11}$$

where $\check{K}':=\{-x:\ x\in K'\}$ and V denotes the mixed volume. Mixed volumes also appear in the following translative Crofton formula. Let $k\in\{1,\ldots,d-1\}$, $E_k\subset\mathbb{E}^d$ a k-dimensional linear subspace and $E_k^\perp$ its orthogonal complement; let B_k denote a k-dimensional unit ball in E_k. Then

$$\int_{E_k^\perp} V_j(K\cap(E_k+t))\,\mathrm{d}\lambda_{d-k}(t)$$
$$=\frac{1}{\kappa_{k-j}}\binom{d}{k-j}V(\underbrace{K,\ldots,K}_{d+j-k},\underbrace{B_k,\ldots,B_k}_{k-j})\tag{2.12}$$

for $j=0,\ldots,k$ (see Schneider 1981a).

The functionals $V^{(j)}_{k,d+j-k}(K,K')$ appearing in (2.10) can be expressed as integrals of mixed volumes, in the form

$$V^{(j)}_{k,d+j-k}(K,K')$$
$$=c_{djk}\int_{\mathscr{E}^d_{d-j}} V(\underbrace{K\cap E,\ldots,K\cap E}_{k-j},\underbrace{\check{K}',\ldots,\check{K}'}_{d+j-k})\,\mathrm{d}\mu_{d-j}(E)\tag{2.13}$$

with

$$c_{djk}=\binom{d}{j}\binom{d}{k-j}\frac{\kappa_d}{\kappa_j\kappa_{d-j}}.$$

The special case $d=3$, $k=2$, $j=1$ appears in Berwald and Varga (1937). The general case, and its extension to corresponding formulae for cylinders, is due to Goodey and Weil (1987). They also have different representations for $V^{(j)}_{k,d+j-k}$ in the case of centrally symmetric bodies, involving measures on Grassmannians or projection bodies.

The kinematic formula (2.2) can be iterated. Since on the right side of (2.2) there appear only intrinsic volumes, which can serve as integrands on the left side, one can use induction to obtain the formula

$$\int_{G_d}\cdots\int_{G_d} V_j(K_0\cap g_1K_1\cap\cdots\cap g_mK_m)\,\mathrm{d}\mu(g_1)\cdots\,\mathrm{d}\mu(g_m)$$
$$=\sum_{\substack{k_0,\ldots,k_m=j\\k_0+\cdots+k_m=dm+j}}^{d} c^{(j)}_{k_0,k_1,\ldots,k_m}V_{k_0}(K_0)V_{k_1}(K_1)\cdots V_{k_m}(K_m)\tag{2.14}$$

for $K_0,K_1,\ldots,K_m\in\mathcal{K}^d$, $m\in\mathbb{N}$, $j\in\{0,\ldots,d\}$, with explicitly known constants $c^{(j)}_{k_0,k_1,\ldots,k_m}$ (see, e.g., Streit 1970). Iterated versions of the translative formula (2.10) are easy to obtain for $d=2$ (Blaschke 1937, Miles 1974) and for $j=d-1$, d (Streit 1973, 1975), but not so in the general case. The latter has been investigated by Weil (1990). He showed that one has an expression

$$\int_{\mathbb{E}^d}\cdots\int_{\mathbb{E}^d} V_j(K_0\cap(K_1+t_1)\cap\cdots\cap(K_m+t_m))\,\mathrm{d}\lambda_d(t_1)\cdots\,\mathrm{d}\lambda_d(t_m)$$
$$=\sum_{\substack{k_0,\ldots,k_m=j\\k_0+\cdots+k_m=dm+j}}^{d} V^{(j)}_{k_0,k_1,\ldots,k_m}(K_0,K_1,\ldots,K_m),\tag{2.15}$$

by which a variety of mixed functionals $V^{(j)}_{k_0,k_1,\ldots,k_m}$ is introduced. Weil investigated the properties of these functionals and showed, in particular, that, for fixed j, they can be computed from the finitely many functionals $V^{(j)}_{k_1,\ldots,k_d}$, where $k_1,\ldots,k_d\in\{j,\ldots,d\}$ and $k_1+\cdots+k_d=d(d-1)+j$. However, explicit geometric descriptions are only known in special cases. The mixed functionals $V^{(j)}_{k_0,k_1,\ldots,k_m}$ satisfy in turn integral-geometric formulae; see Weil (1990) and also the short survey in Weil (1989b).

As a by-product of studies in translative integral geometry, Goodey and Weil (1987) and Weil (1990) obtained some Crofton-type formulae for mixed volumes, among them, for $j\in\{1,\ldots,d-1\}$,

$$\int_{\mathscr{E}^d_j} V(\underbrace{K\cap E,\ldots,K\cap E}_{j},\underbrace{L,\ldots,L}_{d-j})\,\mathrm{d}\mu_j(E)$$
$$=\binom{d}{j}^{-2}\frac{\kappa_j\kappa_{d-j}}{\kappa_d}V_d(K)V_{d-j}(L)\tag{2.16}$$

for $K,L\in\mathcal{K}^d$ and

$$\int_{\mathscr{E}^d_{d-j+1}}\int_{\mathscr{E}^d_{j+1}} V(\underbrace{K\cap E,\ldots,K\cap E}_{j},\underbrace{L\cap F,\ldots,L\cap F}_{d-j})$$
$$\times\,\mathrm{d}\mu_{j+1}(E)\,\mathrm{d}\mu_{d-j+1}(F)$$
$$=\frac{d(d-1)\alpha_{d0(j+1)}}{4\binom{d}{j}\kappa_{d-2}}V(\Pi K,\Pi L,B^d,\ldots,B^d)\tag{2.17}$$

for centrally symmetric convex bodies $K, L \in \mathcal{K}^d$, where ΠK denotes the projection body of K.

3. Minkowski addition and projections

The formulae of the preceding section refer to the intersection of a fixed convex body and a moving convex set. Similar formulae exist for other geometric operations, namely Minkowski addition and projection. Some global formulae of this type are immediate consequences of the principal kinematic formula, and we mention these first.

For convex bodies $K, K' \in \mathcal{K}^d$ and a rotation $\rho \in \mathrm{SO}(d)$, we integrate the trivial relation

$$V_d(K+\rho K') = \int_{\mathbb{E}^d} \chi(K \cap (\rho \check{K}' + t))\, \mathrm{d}\lambda_d(t)$$

over $\mathrm{SO}(d)$ and then use (2.4) to get

$$\int_{\mathrm{SO}(d)} V_d(K+\rho K')\, \mathrm{d}\nu(\rho) = \int_{G_d} \chi(K \cap g\check{K}')\, \mathrm{d}\mu(g)$$

$$= \sum_{k=0}^{d} \alpha_{d0k} V_k(K) V_{d-k}(K'). \tag{3.1}$$

Replacing K by $K + \varepsilon B^d$, expanding and comparing coefficients of equal powers of ε, we obtain

$$\int_{\mathrm{SO}(d)} V_j(K+\rho K')\, \mathrm{d}\nu(\rho) = \sum_{k=0}^{j} \beta_{djk} V_k(K) V_{j-k}(K') \tag{3.2}$$

with

$$\beta_{djk} = \frac{\binom{d-k}{j-k}\kappa_{d-k}\kappa_{d+k-j}}{\binom{d}{j-k}\kappa_{d-j}\kappa_d}. \tag{3.3}$$

More generally, in (3.1) we may write V_k, V_{d-k} as mixed volumes (with numerical factors), replace K and K' by Minkowski combinations of convex bodies, expand both sides and compare terms of equal degrees of homogeneity. Thus we obtain

$$\int_{\mathrm{SO}(d)} V(K_1, \dots, K_m, \rho K_{m+1}, \dots, \rho K_d)\, \mathrm{d}\nu(\rho)$$

$$= \frac{1}{\kappa_d} V(K_1, \dots, K_m, \underbrace{B^d, \dots, B^d}_{d-m}) V(\underbrace{B^d, \dots, B^d}_{m}, K_{m+1}, \dots, K_d). \tag{3.4}$$

To treat local versions of these and further formulae, we use the generalized curvature measures Θ_j (see chapter 1.8). Recall that they can be defined by

$$\lambda_d(M_\varepsilon(K,\eta)) = \frac{1}{d}\sum_{j=0}^{d-1} \varepsilon^{d-j}\binom{d}{j}\Theta_j(K,\eta) \tag{3.5}$$

for $K \in \mathcal{K}^d$, $\eta \in \mathcal{B}(\Sigma)$, and $\varepsilon > 0$. Here $\Sigma = \mathbb{E}^d \times S^{d-1}$, and the set $M_\varepsilon(K,\eta)$ is defined as follows. For $x \in \mathbb{E}^d$, we denote by $p(K,x)$ the unique point in $K \in \mathcal{K}^d$ nearest to x and by $r(K,x) := \|x - p(K,x)\|$ its distance from x. If $x \notin K$, the unit vector pointing from $p(K,x)$ to x is defined by $u(K,x) := (x - p(K,x))/r(K,x)$. Thus, for $x \in \mathbb{E}^d \setminus K$, the pair $(p(K,x), u(K,x))$ is a *support element* of K, by which we mean that $p(K,x)$ is a boundary point of K and $u(K,x)$ is an exterior unit normal vector to K at this point. The set of all support elements of K is denoted by $\mathrm{Nor}\,K$. Now the set $M_\varepsilon(K,\eta)$ appearing in (3.5) is defined by

$$M_\varepsilon(K,\eta) := \{x \in \mathbb{E}^d\colon\ 0 < r(K,x) \leqslant \varepsilon,\ (p(K,x), u(K,x)) \in \eta\}. \tag{3.6}$$

Special cases of the generalized curvature measures $\Theta_0(K,\cdot), \dots, \Theta_{d-1}(K,\cdot)$ are Federer's curvature measures and the area measures of lower order, which in the literature appear in two different normalizations:

$$\Theta_j(K, \beta \times S^{d-1}) = C_j(K,\beta) = d\binom{d}{j}^{-1}\kappa_{d-j}\Phi_j(K,\beta), \tag{3.7}$$

$$\Theta_j(K, \mathbb{E}^d \times \omega) = S_j(K,\omega) = d\binom{d}{j}^{-1}\kappa_{d-j}\Psi_j(K,\omega) \tag{3.8}$$

for $\beta \in \mathcal{B}(\mathbb{E}^d)$ and $\omega \in \mathcal{B}(S^{d-1})$. In the following, it seems preferable to formulate the local formulae in terms of Θ_j, C_j, S_j instead of the renormalized versions.

For sets $\eta, \eta' \subset \Sigma$ we define

$$\eta * \eta' := \{(x + x', u) \in \Sigma\colon\ (x,u) \in \eta,\ (x',u) \in \eta'\}.$$

This operation includes the behaviours of sets of boundary points and of normal vectors of convex bodies under addition: if $\eta \subset \mathrm{Nor}\,K$ and $\eta' \subset \mathrm{Nor}\,K'$, then $\eta * \eta' \subset \mathrm{Nor}(K+K')$. For $\beta, \beta' \subset \mathbb{E}^d$ and $\omega, \omega' \subset S^{d-1}$ we have

$$(\beta \times \omega) * (\beta' \times \omega') = (\beta + \beta') \times (\omega \cap \omega').$$

The following theorem contains a rather general local version of (3.2).

Theorem 3.1. *If $K, K' \in \mathcal{K}^d$ are convex bodies, $\eta \subset \mathrm{Nor}\,K$ and $\eta' \subset \mathrm{Nor}\,K'$ are Borel sets of support elements, and $j \in \{0, \dots, d-1\}$, then*

$$\int_{\mathrm{SO}(d)} \Theta_j(K+\rho K', \eta * \rho\eta')\, \mathrm{d}\nu(\rho) = \frac{1}{d\kappa_d}\sum_{k=0}^{j}\binom{j}{k}\Theta_k(K,\eta)\Theta_{j-k}(K',\eta'). \tag{3.9}$$

Special cases are

$$\int_{SO(d)} C_j(K+\rho K', \beta+\rho\beta')\,\mathrm{d}\nu(\rho) = \frac{1}{d\kappa_d}\sum_{k=0}^{j}\binom{j}{k} C_k(K,\beta)C_{j-k}(K',\beta') \tag{3.10}$$

for Borel sets $\beta \subset \operatorname{bd} K$, $\beta' \subset \operatorname{bd} K'$, and

$$\int_{SO(d)} S_j(K+\rho K', \omega\cap\rho\omega')\,\mathrm{d}\nu(\rho) = \frac{1}{d\kappa_d}\sum_{k=0}^{j}\binom{j}{k} S_k(K,\omega)S_{j-k}(K',\omega') \tag{3.11}$$

for $\omega, \omega' \in \mathcal{B}(S^{d-1})$.

In the same way as formula (3.4) was deduced from (3.1) we may derive, from the case $j = d-1$ of formula (3.11), a rotational mean value formula for the mixed area measure S, namely

$$\begin{aligned}\int_{SO(d)} & S(K_1,\dots,K_m,\rho K_{m+1},\dots,\rho K_{d-1},\omega\cap\rho\omega')\,\mathrm{d}\nu(\rho) \\ &= \frac{1}{d\kappa_d} S(K_1,\dots,K_m,\underbrace{B^d,\dots,B^d}_{d-1-m},\omega) \\ &\quad\times S(\underbrace{B^d,\dots,B^d}_{m},K_{m+1},\dots,K_{d-1},\omega')\end{aligned} \tag{3.12}$$

for $K_1,\dots,K_{d-1} \in \mathcal{K}^d$ and $\omega,\omega' \in \mathcal{B}(S^{d-1})$.

The case $j = d-1$ of (3.9) can also be used to obtain more general formulae involving arbitrary functions. For example, let $v(K,x)$ be the exterior unit normal vector of the convex body K at x if x is a regular boundary point of K (otherwise, $v(K,x)$ remains undefined); let $f: S^{d-1} \to \mathbb{R}$ be a nonnegative measurable function. Then

$$\begin{aligned}\int_{SO(d)} & \int \mathbf{1}_{\beta+\rho\beta'}(x) f(v(K+\rho K',x))\,\mathrm{d}C_{d-1}(K+\rho K',x)\,\mathrm{d}\nu(\rho) \\ &= \sum_{k=0}^{d-1}\binom{d-1}{k}\int \mathbf{1}_\beta(x) f(u)\,\mathrm{d}\Theta_k(K,(x,u))C_{d-1-k}(K',\beta')\end{aligned} \tag{3.13}$$

for $K, K' \in \mathcal{K}^d$ and Borel sets $\beta \subset \operatorname{bd} K$, $\beta' \subset \operatorname{bd} K'$; here $\mathbf{1}$ denotes the indicator function.

Some of the rotational mean value formulae can be specialized to yield projection formulae. In the following, E denotes a fixed k-dimensional linear subspace of $\mathbb{E}^d$, where $k \in \{1,\dots,d-1\}$. The image of a set $A \subset \mathbb{E}^d$ under orthogonal projection onto E is denoted by $A \mid E$. The mixed volume in a k-dimensional linear subspace will be denoted by $v^{(k)}$. If $K_1,\dots,K_k \in \mathcal{K}^d$ are convex bodies and if $U \subset E^\perp$ is a convex body with $\lambda_{d-k}(U) = 1$, then one shows in the theory of mixed volumes that

$$v^{(k)}(K_1 \mid E,\dots,K_k \mid E) = \binom{d}{k} V(K_1,\dots,K_k,\underbrace{U,\dots,U}_{d-k}).$$

Hence, from (3.4) we can infer that

$$\int_{SO(d)} v^{(k)}(K_1 \mid \rho E,\dots,K_k \mid \rho E)\,\mathrm{d}\nu(\rho) = \frac{\kappa_k}{\kappa_d} V(K_1,\dots,K_k,\underbrace{B^d,\dots,B^d}_{d-k}). \tag{3.14}$$

A special case can be written in the form

$$\int_{SO(d)} V_j(K \mid \rho E)\,\mathrm{d}\nu(\rho) = \beta_{d(d+j-k)j} V_j(K), \tag{3.15}$$

valid for $j \in \{0,\dots,k\}$, and further specialization gives

$$\int_{SO(d)} \lambda_k(K \mid \rho E)\,\mathrm{d}\nu(\rho) = \frac{\kappa_{d-k}\kappa_k}{\binom{d}{k}\kappa_d} V_k(K) = \frac{\kappa_k}{\kappa_d} W_{d-k}(K). \tag{3.16}$$

(This explains the name "quermassintegral", since the k-dimensional measure of the projection, $\lambda_k(K \mid \rho E)$, can in German be called a "Quermaß".)

The general formula (3.15) is often called *Kubota's integral recursion*. The case $k = d-1$ of (3.16) is *Cauchy's surface area formula*. The case $k = 1$ of (3.16) shows that

$$V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}}\bar{b}(K), \tag{3.17}$$

where $\bar{b}$ is the mean width.

From the local formula (3.9) one may also derive a local formula for projections. For $\eta \subset \Sigma$ we write

$$\eta \mid E := \{(x \mid E, u) \colon (x,u) \in \eta \text{ and } u \in E\}.$$

Theorem 3.2. *If $K \in \mathcal{K}^d$ is a convex body, $\eta \subset \operatorname{Nor} K$ is a Borel set of support elements, and $E \subset \mathbb{E}^d$ is a k-dimensional linear subspace ($k \in \{1,\dots,d-1\}$), then*

$$\int_{SO(d)} \Theta_j^{(k)}(K \mid \rho E, \eta \mid \rho E)\,\mathrm{d}\nu(\rho) = \frac{k\kappa_k}{d\kappa_d}\Theta_j(K,\eta) \tag{3.18}$$

for $j \in \{0,\dots,k-1\}$, where $\Theta_j^{(k)}$ is the generalized curvature measure taken with respect to the subspace ρE.

By specialization, we obtain the formulae

$$\int_{SO(d)} C_j^{(k)}(K \mid \rho E, \beta \mid \rho E)\, d\nu(\rho) = \frac{k\kappa_k}{d\kappa_d} C_j(K,\beta) \tag{3.19}$$

for Borel sets $\beta \subset \operatorname{bd} K$, and

$$\int_{SO(d)} S_j^{(k)}(K \mid \rho E, \omega \cap \rho E)\, d\nu(\rho) = \frac{k\kappa_k}{d\kappa_d} S_j(K,\omega) \tag{3.20}$$

for $\omega \in \mathcal{B}(S^{d-1})$; here $C_j^{(k)}$ and $S_j^{(k)}$ are computed in ρE.

In a similar way as (3.14) was deduced, one may obtain a corresponding mean value formula for mixed area measures. With E as above, we have

$$\int_{SO(d)} s^{(k)}(K_1 \mid \rho E, \dots, K_{k-1} \mid \rho E, \omega \cap \rho E)\, d\nu(\rho)$$

$$= \frac{k\kappa_k}{d\kappa_d} S(K_1, \dots, K_{k-1}, \underbrace{B^d, \dots, B^d}_{d-k}, \omega) \tag{3.21}$$

for $\omega \in \mathcal{B}(S^{d-1})$, where $s^{(k)}$ denotes the mixed area measure in ρE.

We give some hints to the literature. Rotation integrals for Minkowski addition of type (3.2) first appear in Hadwiger (1950), obtained in a different way; see also Hadwiger (1957, section 6.2.4). Theorems 3.1 and 3.2 are due to Schneider (1986); the special cases (3.11), (3.20) were proved before by Schneider (1975a) and the special cases (3.10), (3.19) by Weil (1979b), but in different and more indirect ways. These formulae are essential tools for the formulae of the next section. The equation (3.13) was applied by Papaderou-Vogiatzaki and Schneider (1988) to a question on geometric collision probabilities. The projection formulae (3.15) are classical, see, e.g., Hadwiger (1957).

Finally, we mention a formula that combines intersection and projection. Such a formula exists for a moving convex cylinder meeting a fixed convex body; one projects the intersection orthogonally into a generating subspace of the cylinder. Let a convex body $K \in \mathcal{K}^d$, a q-dimensional linear subspace E of $\mathbb{E}^d$, where $q \in \{0, \dots, d-1\}$, and a convex body $C \subset E^\perp$ be given. Then $Z = C + E$ is a convex cylinder with generating subspace E. For $j \in \{0, \dots, q\}$ we have

$$\int_{SO(d)} \int_{E^\perp} V_j((K \cap \rho(Z+t)) \mid \rho E)\, d\lambda_{d-q}(t)\, d\nu(\rho)$$

$$= \sum_{k=j}^{d+j-q} \gamma_{djkq} V_k(K) V_{d+j-q-k}(C) \tag{3.22}$$

with

$$\gamma_{djkq} = \frac{\binom{q}{j}\kappa_q \kappa_k \kappa_{d-k}}{\binom{d}{k}\kappa_{q-j}\kappa_d \kappa_j}.$$

A proof (with different notations) can be found in Schneider (1981a); the case where C is a ball in $E^\perp$ was treated earlier by Matheron (1976).

4. Distance integrals and contact measures

The principal kinematic formula (2.4), now written in terms of quermassintegrals, thus

$$\int_{G_d} \chi(K \cap gK')\, d\mu(g) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \binom{d}{k} W_k(K) W_{d-k}(K'), \tag{4.1}$$

refers to the set of rigid motions g for which $K \cap gK' \neq \emptyset$. One can also integrate over the complementary set of motions if one introduces suitable functions of the distance between K and gK'.

The distance, $r(K,L)$, of a compact set $K \subset \mathbb{E}^d$ and a closed set $L \subset \mathbb{E}^d$ is defined by

$$r(K,L) := \min\{\| x - y \| : x \in K, y \in L\}.$$

Now let $f : [0,\infty) \to [0,\infty)$ be a measurable function for which $f(0) = 0$ and

$$M_k(f) := k \int_0^\infty f(r) r^{k-1}\, dr < \infty \qquad \text{for } k = 1, \dots, d.$$

Then, for convex bodies $K, L \in \mathcal{K}^d$,

$$\int_{G_d} f(r(K,gL))\, d\mu(g)$$

$$= \frac{1}{\kappa_d} \sum_{k=1}^{d} \sum_{j=d+1-k}^{d} \binom{d}{k} \binom{k}{d-j} M_{k+j-d}(f) W_k(K) W_j(L). \tag{4.2}$$

This was first proved by Hadwiger (1975a). Similar results for moving flats are due to Bokowski, Hadwiger and Wills (1976). In his proof of (4.2), Hadwiger assumed monotonicity for f and then deduced the result from his axiomatic characterization of the quermassintegrals. In a more direct way, (4.2) can be obtained as follows. First assume that f is the indicator function of the interval $(a,b]$, where $0 \leqslant a < b$. Then

$$\int_{G_d} f(r(K,gL))\, d\mu(g)$$

$$= \mu(\{g \in G_d : (K + bB^d) \cap gL \neq \emptyset\})$$

$$- \mu(\{g \in G_d : (K + aB^d) \cap gL \neq \emptyset\})$$

$$= \frac{1}{\kappa_d} \sum_{k=0}^{d} \binom{d}{k} [W_k(K + bB^d) - W_k(K + aB^d)] W_{d-k}(L)$$

by (4.1). The application of the Steiner formula for the quermassintegral W_k of a parallel body now leads to (4.2) for functions f of the special type considered.

The extension to more general functions is then achieved by standard arguments of integration theory. By essentially this method, different generalizations of (4.2) were proved by Schneider (1977) and Groemer (1980b).

Local versions of (4.2) make sense in different ways. For example, one may integrate only over those rigid motions g for which a pair of points realizing the distance of K and gL, or the direction of the difference of these points, belongs to a specified set. First we state a simpler result of the latter type. For a convex body K and a closed convex set L, let $x \in K$ and $y \in L$ be points at distance $r(K,L)$. Then the unit vector pointing from K to L is defined by $u(K,L) := (y-x)/r(K,L)$. This vector is unique, although the pair (x,y) is not necessarily unique. If $\alpha, \beta \subset \mathcal{B}(S^{d-1})$ are Borel sets on the unit sphere, if

$$M(K,L;\alpha,\beta) := \{g \in G_d\colon\ K \cap gL = \emptyset \text{ and } u(K,gL) \in \alpha \cap g_0\check{\beta}\},$$

where $g_0 \in \mathrm{SO}(d)$ denotes the rotation part of $g \in G_d$, and if f is as in (4.2), then

$$\int_{M(K,L;\alpha,\beta)} f(r(K,gL))\,\mathrm{d}\mu(g)$$

$$= \frac{1}{d^2\kappa_d}\sum_{k,j=0}^{d-1}\binom{d}{k}\binom{d-k}{j}M_{d-k-j}(f)S_k(K,\alpha)S_j(L,\beta). \tag{4.3}$$

Formulae of this type were first obtained by Hadwiger (1975b). A short proof of (4.3), using (3.11), was given by Schneider (1977). Since (4.3) can be interpreted as involving the indicator functions of α and β, it is not surprising that further generalizations, involving more general functions, are possible.

We describe some rather general formulae, due mainly to Weil (1979a,b, 1981). These concern integrals of the types

$$\int_{K\cap gL=\emptyset} f(g)\,\mathrm{d}\mu(g) \quad\text{and}\quad \int_{K\cap E=\emptyset} h(E)\,\mathrm{d}\mu_q(E),$$

for functions f and h depending in different ways on the geometric situation. For given convex bodies $K, L \in \mathcal{K}^d$, Weil (1979b) established a decomposition of the form

$$\mu\,|_{\{g\colon\ K\cap gL=\emptyset\}} = \int_0^\infty \mu^{(r)}(K,L;\cdot)\,\mathrm{d}r, \tag{4.4}$$

where $\mu^{(r)}(K,L;\cdot)$ is a finite Borel measure concentrated on the set of rigid motions g for which $r(K,gL) = r$. He deduced that, for a μ-integrable function f on G_d,

$$\int_{K\cap gL=\emptyset} f(g)\,\mathrm{d}\mu(g)$$

$$= \int_0^\infty \int_{\mathrm{SO}(d)} \int_{\Gamma^d} f(\gamma(t,\rho))\,\mathrm{d}C_{d-1}(K + rB^d + \rho\check{L}, t)\,\mathrm{d}\nu(\rho)\,\mathrm{d}r. \tag{4.5}$$

From this, the following result can be derived.

Theorem 4.1. *Let $f : (0,\infty) \times S^{d-1} \times \mathrm{SO}(d) \to \mathbb{R}$ be a measurable function for which the integrals in (4.6) are finite; then*

$$\int_{K\cap gL=\emptyset} f(r(K,gL), u(K,gL), g_0)\,\mathrm{d}\mu(g)$$

$$= \sum_{j=0}^{d-1}\binom{d-1}{j}\int_0^\infty \int_{\mathrm{SO}(d)}\int_{S^{d-1}} f(r,u,\rho)\,\mathrm{d}S_j(K+\rho\check{L},u)\,\mathrm{d}\nu(\rho)r^{d-j-1}\,\mathrm{d}r. \tag{4.6}$$

For more restricted functions, the integration over the rotation group on the right side disappears:

Theorem 4.2. *Let $f : (0,\infty) \times S^{d-1} \times S^{d-1} \to \mathbb{R}$ be a measurable function for which the integrals in (4.7) are finite, then*

$$\int_{K\cap gL=\emptyset} f(r(K,gL), u(K,gL), g_0^{-1}u(gL,K))\,\mathrm{d}\mu(g)$$

$$= d\kappa_d \sum_{j=0}^{d-1}\sum_{k=0}^{j}\binom{d-1}{j}\binom{j}{k}$$

$$\times \int_0^\infty \int_{S^{d-1}}\int_{S^{d-1}} f(r,u,v)\,\mathrm{d}S_k(K,u)\,\mathrm{d}S_{j-k}(L,v)r^{d-j-1}\,\mathrm{d}r. \tag{4.7}$$

The following theorem contains a counterpart for variable q-flats instead of a moving convex body.

Theorem 4.3. *If $q \in \{0,\dots,d-1\}$ and if $h : (0,\infty) \times S^{d-1} \to \mathbb{R}$ is a measurable function for which the integrals in (4.8) are finite, then*

$$\int_{K\cap E=\emptyset} h(r(K,E), u(K,E))\,\mathrm{d}\mu_q(E)$$

$$= (d-q)\kappa_{d-q}\sum_{j=0}^{d-q-1}\binom{d-q-1}{j}\int_0^\infty \int_{S^{d-1}} h(r,u)\,\mathrm{d}S_j(K,u)r^{d-q-j-1}\,\mathrm{d}r. \tag{4.8}$$

One also has a translative version of Theorem 4.2:

Theorem 4.4. *If $h : (0,\infty) \times S^{d-1} \to \mathbb{R}$ is a measurable function for which the integrals in (4.9) are finite, then*

$$\int_{K\cap(L+t)=\emptyset} h(r(K,L+t),u(K,L+t))\,\mathrm{d}\lambda_d(t)$$

$$=\sum_{i,k=0}^{d-1}\binom{d-1}{i+k}\binom{i+k}{i}$$

$$\times\int_0^\infty\int_{S^{d-1}} h(r,u)\,\mathrm{d}S(\underbrace{K,\dots,K}_{i},\underbrace{L,\dots,L}_{k},\underbrace{B^d,\dots,B^d}_{d-1-i-k},u)r^{d-1-i-k}\,\mathrm{d}r.$$

(4.9)

So far, we have integrated functions that involve the positive distance of a convex body K from a moving convex set gL and the unit vector pointing from K to gL, but not the boundary points realizing the distance. If one wants to take the latter into account, the difficulty arises that a pair (x,y) of points realizing the distance of K and gL, where $K\cap gL=\emptyset$, is in general not unique. However, for μ-almost all $g\in G_d$, the distance of the disjoint convex bodies K and gL is realized by a unique pair. This was proved in Schneider (1978b), and the corresponding result for flats in Schneider (1978a). Once this is known, one can prove analogues of some of the results above. For a convex body K and a closed convex set L with $K\cap L=\emptyset$ define $x(K,L):=x$ if $x\in K$ is such that $\|x-y\|=r(K,L)$ for some $y\in L$ and the pair (x,y) is unique.

Theorem 4.5. *Let $f:(0,\infty)\times\mathrm{bd}\,K\times\mathrm{bd}\,L\to\mathbb{R}$ be a measurable function for which the integrals in* (4.10) *are finite, then*

$$\int_{K\cap gL=\emptyset} f(r(K,gL),x(K,gL),g^{-1}x(gL,K))\,\mathrm{d}\mu(g)$$

$$=d\kappa_d\sum_{j=0}^{d-1}\sum_{k=0}^{j}\binom{d-1}{j}\binom{j}{k}$$

$$\times\int_0^\infty\int_{\mathrm{bd}\,K}\int_{\mathrm{bd}\,L} f(r,x,y)\,\mathrm{d}C_k(K,x)\,\mathrm{d}C_{j-k}(L,y)r^{d-j-1}\,\mathrm{d}r. \qquad (4.10)$$

Theorem 4.6. *If $q\in\{0,\dots,d-1\}$ and if $h:(0,\infty)\times\mathrm{bd}\,K\to\mathbb{R}$ is a measurable function for which the integrals in* (4.11) *are finite, then*

$$\int_{K\cap E=\emptyset} h(r(K,E),x(K,E))d\mu_q(E)$$

$$=(d-q)\kappa_{d-q}\sum_{j=0}^{d-q-1}\binom{d-q-1}{j}\int_0^\infty\int_{\mathrm{bd}\,K} h(r,x)\,\mathrm{d}C_j(K,x)r^{d-q-j-1}\,\mathrm{d}r.$$

(4.11)

It should be clear by now that common generalizations of these results are

possible by extending them to support elements, generalized curvature measures, and convex cylinders.

The integrals involving the distance of a convex body and a moving convex set are closely related to contact measures. The contact measure of two convex bodies K and L is a measure which is concentrated on the set of rigid motions g for which K and gL are in contact, that is, touch each other, and which is derived from the Haar measure on G_d in the following natural way. For convex bodies $K,L\in\mathcal{K}^d$ we define

$$G_0(K,L):=\{g\in G_d\colon gL \text{ touches } K\}.$$

(gL touches K if $K\cap L\neq\emptyset$, but gL and K can be separated weakly by a hyperplane.) If $K\cap L=\emptyset$, there is a unique translation $\tau=\tau(K,L)$ by a vector of length $r(K,L)$ (namely, $-r(K,L)u(K,L)$) such that $K\cap\tau L\neq\emptyset$. If now $\alpha\in\mathcal{B}(G_d)$ is a Borel set in the motion group and if $\varepsilon>0$, we define

$$A_\varepsilon(K,L,\alpha):=\{g\in G_d\colon 0<r(K,gL)\leqslant\varepsilon,\ \tau(K,gL)\circ g\in\alpha\}.$$

This set is a Borel set, and for its Haar measure one obtains

$$\mu(A_\varepsilon(K,L,\alpha))=\frac{1}{d}\sum_{j=1}^{d-1}\varepsilon^{d-j}\binom{d}{j}\int_{\mathrm{SO}(d)} C_j(K+\rho\check{L},T(\alpha,\rho))\,\mathrm{d}\nu(\rho)$$

with

$$T(\alpha,\rho):=\{t\in\mathbb{E}^d\colon \gamma(t,\rho)\in\alpha\cap G_0(K,L)\}.$$

Hence, the limit

$$\varphi(K,L,\alpha):=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu(A_\varepsilon(K,L,\alpha))$$

exists and is given by

$$\varphi(K,L,\alpha)=\int_{\mathrm{SO}(d)} C_{d-1}(K+\rho\check{L},T(\alpha,\rho))\,\mathrm{d}\nu(\rho). \qquad (4.12)$$

Thus, $\varphi(K,L,\cdot)$ is a finite Borel measure on G_d, which is concentrated on the set $G_0(K,L)$ of rigid motions bringing L into contact with K. It is called the *contact measure* of K and L and was introduced in this way by Weil (1979a), who extended and unified formerly treated special cases. Using a different approach, Weil (1979b) also showed that this contact measure is the weak limit

$$\varphi(K,L,\cdot)=w-\lim_{r\to 0}\mu^{(r)}(K,L,\cdot),$$

where $\mu^{(r)}(K,L,\cdot)$ is defined by the disintegration (4.4).

For suitable sets of rigid motions defined by special touching conditions of geometric significance, the contact measure can be expressed in terms of curvature measures. For $K,K'\in\mathcal{K}^d$, $\omega,\omega'\in\mathcal{B}(S^{d-1})$, $\beta,\beta'\in\mathcal{B}(\mathbb{E}^d)$ and $\varepsilon>0$ we define the following sets. $M_0(K,K';\omega,\omega')$ is the set of rigid motions $g\in G_0(K,K')$ for which the unit normal vector u, pointing from K to gK', of a separating hyperplane of K and gK' satisfies $u \in \omega \cap g_0\check{\omega}'$. $L_0(K,K';\beta,\beta')$ is the set of motions $g \in G_0(K,K')$ for which $\beta \cap \operatorname{bd} K \cap g(\beta' \cap \operatorname{bd} K') \neq \emptyset$. Further,

$$M_\varepsilon(K,K';\omega,\omega') := \{g \in G_d\colon 0 < r(K,gK') \leqslant \varepsilon,\ u(K,gK') \in \omega,\ u(gK',K) \in g_0\omega'\},$$

$$L_\varepsilon(K,K';\beta,\beta') := \{g \in G_d\colon 0 < r(K,gK') \leqslant \varepsilon,\ x(K,gK') \in \beta,\ x(gK',K) \in g\beta'\}.$$

(Observe that the latter set is only defined up to a set of μ-measure zero.) The sets $M_0(K,K';\omega,\omega'), L_0(K,K';\beta,\beta')$ are $\varphi(K,K',\cdot)$-measurable, and

$$\varphi(K,K',M_0(K,K';\omega,\omega')) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu(M_\varepsilon(K,K';\omega,\omega')) = \frac{1}{d\kappa_d}\sum_{j=0}^{d-1}\binom{d-1}{j}S_j(K,\omega)S_{d-1-j}(K',\omega'). \tag{4.13}$$

Here the first equality follows from the definition of the contact measure and the second from a special case of (4.3) and thus of Theorem 4.2. Further,

$$\varphi(K,K',L_0(K,K';\beta,\beta')) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu(L_\varepsilon(K,K';\beta,\beta')) = \frac{1}{d\kappa_d}\sum_{j=0}^{d-1}\binom{d-1}{j}C_j(K,\beta)C_{d-1-j}(K',\beta'), \tag{4.14}$$

from a special case of Theorem 4.5. A common generalization of (4.13) and (4.14), involving generalized curvature measures, can be obtained if the touching conditions are formulated in terms of Borel sets of support elements.

In a similar way, contact measures for a convex body and a moving q-flat can be treated. Let $K \in \mathcal{K}^d$ and $q \in \{0,\dots,d-1\}$ be given. Proceeding in obvious analogy to the above, one constructs a natural measure $\varphi_q(K,\cdot)$ on $\mathfrak{B}(\mathcal{E}_q^d)$, concentrated on the set of q-flats touching K, and finds that it is given by

$$\varphi_q(K,\alpha) = \frac{1}{d\kappa_d}\int_{SO(d)} C_{d-q-1}^{(d-q)}(K \mid \rho L_q^\perp, T_q(\alpha,\rho))\, \mathrm{d}\nu(\rho), \tag{4.15}$$

where $\alpha \in \mathfrak{B}(\mathcal{E}_q^d)$, $L_q \subset \mathbb{E}^d$ is a fixed q-dimensional linear subspace, and

$$T_q(\alpha,\rho) := \{t \in \rho L_q^\perp\colon \rho L_q + t \in \alpha,\ \rho L_q + t \text{ touches } K\}.$$

We mention only one result analogous to (4.13), (4.14), this time formulated for generalized curvature measures. For $\eta \in \mathfrak{B}(\Sigma)$ and $\varepsilon > 0$, let $N_\varepsilon^q(K,\eta)$ be the set of all q-flats $E \in \mathcal{E}_q^d$ for which there exist points $x \in K$ and $y \in E$ for which $0 < \|x-y\| = r(K,E) \leqslant \varepsilon$ and $(x,u(K,E)) \in \eta$. Further, let $N_0^q(K,\eta)$ be the set of all q-flats E touching K at a point x and lying in a supporting hyperplane of K with outer unit normal vector u such that $(x,u) \in \eta$. Then $N_0^q(K,\eta)$ is $\varphi_q(K,\cdot)$-measurable (though not necessarily a Borel set, see Burton 1980), and

$$\varphi_q(K,N_0^q(K,\eta)) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu_q(N_\varepsilon^q(K,\eta)) = \frac{(d-k)\kappa_{d-k}}{d\kappa_d}\Theta_{d-q-1}(K,\eta). \tag{4.16}$$

The latter limit relation gives a direct geometric interpretation of the generalized curvature measures. For η of the special kind $\mathbb{E}^d \times \omega$ one obtains a representation of $S_{d-q-1}(K,\omega)$, and for $\eta = \beta \times S^{d-1}$ one of $C_{d-q-1}(K,\beta)$. These special cases are due to Firey (1972) and Schneider (1978a), respectively; the general case was mentioned in Schneider (1980a, Theorem 4.12).

In a common generalization of the cases of touching convex bodies and touching flats, one may define a contact measure for a convex body and a, possibly unbounded, closed convex set; see Weil (1989a), where similar results are obtained.

In the literature, results on contact measures have been studied in connection with so-called *collision* or *touching probabilities*. For example, let d-dimensional convex bodies $K, K' \in \mathcal{K}^d$ and subsets β, β' of their respective boundaries be given. Let K' undergo random motion in such a way that it touches K. What is the probability that the bodies touch at a point belonging to the prescribed boundary sets? (Of course, the same motions are applied to K' and β'.) A reasonable way to make this question precise is to choose the completion of the probability space

$$(G_0(K,K'), \mathfrak{B}(G_0(K,K')), \varphi(K,K',\cdot)/\varphi(K,K',G_0(K,K')))$$

as an underlying probabilistic model. If this is done and if β, β' are Borel sets, then (4.14) yields the value

$$\frac{\sum_{j=0}^{d-1}\binom{d-1}{j}C_j(K,\beta)C_{d-1-j}(K',\beta')}{\sum_{j=0}^{d-1}\binom{d-1}{j}C_j(K,K)C_{d-1-j}(K',K')}$$

for the probability of a collision at the preassigned boundary sets.

Contributions to this field of touching probabilities are due to Firey (1974, 1979), McMullen (1974), Molter (1986), Schneider (1975a,b, 1976, 1978b, 1980b), Schneider and Wieacker (1984), and Weil (1979a,b, 1981, 1982, 1989a).

5. Extension to the convex ring

Several of the integral-geometric formulae considered earlier in this article are not restricted to convex bodies. According to the class of sets envisaged, extensions require, say, methods of differential geometry or geometric measure theory. A class of sets which, at our present stage of considerations, is technically easy to treat, but which on the other hand is sufficiently general for applications (see, e.g., chapter 5.2), is provided by the convex ring $\mathcal{R}^d$. This is the set of all finite unions of convex bodies in $\mathbb{E}^d$ (for formal reasons, we assume that also $\emptyset \in \mathcal{R}^d$). A natural and useful extension of the curvature measures, and thus of the intrinsic volumes, to the convex ring is achieved if one exploits their additivity property.

Recall that a function $\varphi : \mathcal{R}^d \to A$ from $\mathcal{R}^d$ into some Abelian group A is called *additive* if $\varphi(\emptyset) = 0$ and

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{5.1}$$

for all $K, L \in \mathcal{R}^d$ (compare chapter 3.6). For such a function, the *inclusion–exclusion principle* says that

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}) \tag{5.2}$$

for $K_1, \ldots, K_m \in \mathcal{R}^d$. In particular, the values of the function φ on $\mathcal{R}^d$ are uniquely determined by its values on $\mathcal{K}^d$. For a more concise notation, let $S(m)$ be the set of nonempty subsets of $\{1, \ldots, m\}$ and write $|v| := \operatorname{card} v$ for $v \in S(m)$ as well as

$$K_v := K_{i_1} \cap \cdots \cap K_{i_r} \quad \text{for } v = \{i_1, \ldots, i_r\} \in S(m)$$

if $K_1, \cdots, K_m$ are given. Then (5.2) can be written in the form

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v). \tag{5.3}$$

Let $j \in \{0, 1, \ldots, d-1\}$. The generalized curvature measure Θ_j is additive on $\mathcal{K}^d$, that is, it satisfies

$$\Theta_j(K \cup L, \cdot) + \Theta_j(K \cap L, \cdot) = \Theta_j(K, \cdot) + \Theta_j(L, \cdot)$$

whenever $K, L \in \mathcal{K}^d$ are such that $K \cup L$ is convex (see chapter 1.8). One can show that the map Θ_j can be extended from $\mathcal{K}^d \times \mathcal{B}(\Sigma)$ to $\mathcal{R}^d \times \mathcal{B}(\Sigma)$ such that $\Theta_j(\cdot, \eta)$ is additive for each $\eta \in \mathcal{B}(\Sigma)$. If then K is a set of the convex ring, represented as $K = K_1 \cup \cdots \cup K_m$ with convex bodies $K_1, \ldots, K_m$, we have

$$\Theta_j(K, \cdot) = \sum_{v \in S(m)} (-1)^{|v|-1} \Theta_j(K_v, \cdot), \tag{5.4}$$

which shows that $\Theta_j(K, \cdot)$ is a finite signed measure on $\mathcal{B}(\Sigma)$. The possibility of additive extension could be deduced from a general theorem of Groemer (1978), using the weak continuity of the generalized curvature measures. The following approach proceeds in a more explicit way and yields additional information.

For $K \in \mathcal{R}^d$ and $q, x \in \mathbb{E}^d$ the *index* of K at q with respect to x is defined by

$$j(K,q,x) := \begin{cases} 1 - \lim\limits_{\delta \to +0} \lim\limits_{\varepsilon \to +0} \chi(K \cap B(x, \|x-q\| - \varepsilon) \cap B(q, \delta)) & \text{if } q \in K, \\ 0 & \text{if } q \notin K, \end{cases}$$

where χ denotes the Euler characteristic and $B(z, \rho)$ is the closed ball with centre z and radius ρ. Then $j(\cdot, q, x)$ is additive, and for convex K one simply has

$$j(K,q,x) = \begin{cases} 1 & \text{if } q = p(K,x), \\ 0 & \text{else} \end{cases}$$

(recall that $p(K,x)$ is the point of K nearest to x). Next, for $K \in \mathcal{R}^d$, a Borel set $\eta \in \mathcal{B}(\Sigma)$, a number $\varepsilon > 0$ and for $x \in \mathbb{E}^d$ one defines

$$c_\varepsilon(K, \eta, x) := \sum_{*} j(K \cap B(x, \varepsilon), q, x),$$

where the sum $\sum_*$ extends over the points $q \in \mathbb{E}^d$ for which $q \neq x$ and $(q, u) \in \eta$ for $u := (x-q)/\|x-q\|$ (only finitely many summands are not zero). If K is convex, then $c_\varepsilon(K, \eta, \cdot)$ is the indicator function of the set $M_\varepsilon(K, \eta)$ defined by (3.6). The function $c_\varepsilon(\cdot, \eta, x)$ is additive on $\mathcal{R}^d$, hence the function $\mu_\varepsilon(\cdot, \eta)$ defined by

$$\mu_\varepsilon(K, \eta) := \int_{\mathbb{E}^d} c_\varepsilon(K, \eta, x) \, d\lambda_d(x)$$

for $K \in \mathcal{R}^d$ is additive, too. From (3.5) it follows that

$$\mu_\varepsilon(K, \eta) = \frac{1}{d} \sum_{j=0}^{d-1} \varepsilon^{d-j} \binom{d}{j} \sum_{v \in S(m)} (-1)^{|v|-1} \Theta_j(K_v, \eta),$$

if $K = \bigcup_{i=1}^m K_i$ with $K_i \in \mathcal{K}^d$. Since the left side does not depend on the special representation of K, the same is true for

$$\Theta_j(K, \eta) := \sum_{v \in S(m)} (-1)^{|v|-1} \Theta_j(K_v, \eta).$$

This defines the generalized curvature measures $\Theta_0, \ldots, \Theta_{d-1}$ on the convex ring $\mathcal{R}^d$. The defining Steiner type formula

$$\int_{\mathbb{E}^d} c_\varepsilon(K, \eta, x) \, d\lambda_d(x) = \frac{1}{d} \sum_{j=0}^{d-1} \varepsilon^{d-j} \binom{d}{j} \Theta_j(K, \eta)$$

is the immediate generalization of (3.5), with the measure of the local parallel set $M_\varepsilon(K, \eta)$ replaced by the integral of the additive extension $c_\varepsilon(K, \eta, \cdot)$ of its indicator function.

The measures $C_j(K,\cdot)$, $\Phi_j(K,\cdot)$ on $\mathfrak{B}(\mathbb{E}^d)$ and $S_j(K,\cdot)$, $\Psi_j(K,\cdot)$ on $\mathfrak{B}(S^{d-1})$ are now obtained by specialization, verbally in the same way as in (3.7), (3.8). One also defines $\Phi_d(K,\beta) := \lambda_d(K\cap\beta)$ for $\beta\in\mathfrak{B}(\mathbb{E}^d)$.

In the way described here, the additive extensions of the generalized curvature measures to the convex ring were constructed in Schneider (1980a).

Once this additive extension has been achieved, the generalization of some of the integral-geometric formulae is immediate. We demonstrate this for Theorem 2.1. Let $K\in\mathfrak{R}^d$ be a set of the convex ring. It has a representation $K = K_1\cup\cdots\cup K_m$ with $K_1,\ldots,K_m\in\mathcal{K}^d$. First let $K'\in\mathcal{K}^d$ be a convex body. For $j\in\{0,\ldots,d\}$ and $g\in G_d$ we have

$$\Phi_j(K\cap gK',\cdot) = \Phi_j((K_1\cap gK')\cup\cdots\cup(K_m\cap gK'),\cdot)$$

$$= \sum_{v\in S(m)} (-1)^{|v|-1}\Phi_j(K_v\cap gK',\cdot)$$

by (5.4). Now (2.1) yields, for $\beta,\beta'\in\mathfrak{B}(\mathbb{E}^d)$,

$$\int_{G_d}\Phi_j(K\cap gK',\beta\cap g\beta')\,\mathrm{d}\mu(g)$$

$$= \sum_{v\in S(m)}(-1)^{|v|-1}\int_{G_d}\Phi_j(K_v\cap gK',\beta\cap g\beta')\,\mathrm{d}\mu(g)$$

$$= \sum_{v\in S(m)}(-1)^{|v|-1}\sum_{k=j}^{d}\alpha_{djk}\Phi_{d+j-k}(K_v,\beta)\Phi_k(K',\beta')$$

$$= \sum_{k=j}^{d}\alpha_{djk}\Phi_{d+j-k}(K,\beta)\Phi_k(K',\beta').$$

In a second step, K' can be replaced by a set of the convex ring, in precisely the same way. Thus formula (2.1) is valid if K and K' are elements of the convex ring.

In a strictly analogous way, Theorem 2.3 extends to the convex ring. Extensions are also possible for Theorem 2.2, and for Theorem 2.4 if φ is additive on $\mathfrak{R}^d$ and continuous on $\mathcal{K}^d$. The role of additivity for the extension of integral-geometric formulae to the convex ring was mainly emphasized by Hadwiger (1957).

The interpretation of the generalized curvature measures for convex bodies that is given by (4.16) can be extended to the measures Θ_{d-q-1} on the convex ring, if the measure of the set $N^q_\varepsilon(K,\eta)$ is replaced by the μ_q-integral of a suitable multiplicity function. This can be achieved using a more general version of the index introduced above; see Schneider (1988b). There one also finds an extension of the projection formula (3.18) to the convex ring, where a notion of tangential projections with multiplicities plays a role.

6. Translative integral geometry and auxiliary zonoids

Some translative integral-geometric formulae for convex bodies have already been mentioned in section 2. In an essentially different way, convex bodies play an unexpected and useful role in translative integral geometry for quite general rectifiable sets. This is due to the fact that each finite measure on the space of m-dimensional linear subspaces of $\mathbb{E}^d$ induces, in a natural way, a pseudo-norm on $\mathbb{E}^d$ which is the support function of a zonoid. Thus one can exploit classical results on convex bodies to treat several extremal problems of translative integral geometry and of stochastic geometry. For applications of the latter type we refer to chapter 5.2. Auxiliary zonoids in the sense to be discussed were first introduced by Matheron (1975) (although a special result of this type occurs already in Blaschke 1937). We present the method in a rather general version.

By $\mathcal{L}^d_m$ we denote the space of m-dimensional linear subspaces of $\mathbb{E}^d$, topologized as usual (as a subspace of $\mathcal{F}^d$, the space of closed subsets of $\mathbb{E}^d$, see section 1). For $m\in\{1,\ldots,d-1\}$, a subspace $L\in\mathcal{L}^d_m$ and $x\in\mathbb{E}^d$ we write $r_L(x) := r(L,\{x\})$ for the distance of x from L. Then r_L is the support function of the convex body $B^d\cap L^\perp$. Hence, if τ is some finite (Borel) measure on $\mathcal{L}^d_m$, then there is a unique convex body $\Pi^m(\tau)$ such that

$$h(\Pi^m(\tau),\cdot) = \tfrac{1}{2}\int_{\mathcal{L}^d_m} r_L(\cdot)\,\mathrm{d}\tau(L), \tag{6.1}$$

where h denotes the support function. Since each $B^d\cap L^\perp$ is a zonoid, $\Pi^m(\tau)$ is a zonoid (see chapter 4.9). For the mean width of $\Pi^m(\tau)$ one finds (using Fubini's theorem and (56) on p. 217 of Hadwiger 1957)

$$\bar b(\Pi^m(\tau)) = \frac{(d-m)\kappa_{d-m}}{d\kappa_d}\,\frac{\kappa_{d-1}}{\kappa_{d-m-1}}\,\tau(\mathcal{L}^d_m). \tag{6.2}$$

A second zonoid is obtained by putting $\Pi_m(\tau) := \Pi^{d-m}(\tau^\perp)$, where $\tau^\perp$ is the image measure of τ under the map $L\mapsto L^\perp$ from $\mathcal{L}^d_m$ on to $\mathcal{L}^d_{d-m}$. In the case $m=d-1$, $h(\Pi_m(\tau),\cdot)$ is essentially the spherical Radon transform of $h(\Pi^m(\tau),\cdot)$ (see, e.g., Wieacker 1984, Lemma 1).

Since stationary (i.e., translation invariant) measures on the space $\mathcal{E}^d_m$ of m-flats, or more generally on the space of m-dimensional surfaces, under weak assumptions induce finite measures on $\mathcal{L}^d_m$, the foregoing simple construction has far-reaching consequences.

A subset $M\subset\mathbb{E}^d$ is called *m-rectifiable* $(1\leqslant m\leqslant d-1)$ if it is the image of some bounded subset of $\mathbb{E}^m$ under a Lipschitz map, and *countably m-rectifiable* if it is the union of a countable family of m-rectifiable sets (see Federer 1969 for details). If M is countably m-rectifiable and λ_m-measurable, then there are countably many m-dimensional C^1-submanifolds $N_1,N_2,\ldots$ of $\mathbb{E}^d$ such that $\lambda_m(M\setminus\bigcup_{i\in\mathbb{N}}N_i) = 0$. Suppose that, moreover, $\lambda_m(M)<\infty$ and let T_xN_i denote the tangent space of N_i at $x\in N_i$. Then, defining

$$\tau_M(S) := \lambda_m\Big(\bigcup_{i\in\mathbb{N}}\{x\in M\cap N_i:\ T_xN_i\in S\}\Big) \tag{6.3}$$

for $S \in \mathfrak{B}(\mathcal{L}_m^d)$, we get a finite measure τ_M on $\mathcal{L}_m^d$, which can be shown to depend only on M. Hence, we can define

$$\Pi^m(M) := \Pi^m(\tau_M), \qquad \Pi_m(M) := \Pi_m(\tau_M).$$

Since $\tau_M(\mathcal{L}_m^d) = \lambda_m(M)$, (6.2) implies that the mean widths of the two zonoids $\Pi^m(M)$ and $\Pi_m(M)$ are, up to numerical factors, equal to $\lambda_m(M)$. As an example, if $K \in \mathcal{K}^d$ is a convex body with interior points, then bd K is countably $(d-1)$-rectifiable and $\Pi^{d-1}(\operatorname{bd} K)$ is the usual projection body of K. If K is a line segment, then $\Pi_1(K)$ is a translate of K. Clearly, if M and M' are countably m-rectifiable and λ_m-measurable subsets of $\mathbb{E}^d$ with finite λ_m-measure, then

$$\Pi^m(M \cup M') + \Pi^m(M \cap M') = \Pi^m(M) + \Pi^m(M'), \tag{6.4}$$

and the same relation holds for Π_m.

The preceding construction can be considerably generalized. Let

$$\mathcal{LC}_m := \{F \in \mathcal{F}^d:\ F \cap K \text{ is countably } m\text{-rectifiable } \forall K \in \mathcal{K}^d\}$$

be the space of locally countably m-rectifiable closed sets. If θ is a stationary σ-finite measure on $\mathcal{LC}_m$ and $\lambda_m(\cdot \cap [0,1]^d)$ is θ-integrable, then there is a unique convex body $\Pi^m(\theta)$ satisfying

$$\lambda_d(A) h(\Pi^m(\theta), x) = \int_{\mathcal{LC}_m} h(\Pi^m(F \cap A), x)\, \mathrm{d}\theta(F) \tag{6.5}$$

for all $x \in \mathbb{E}^d$, whenever $A \in \mathfrak{B}(\mathbb{E}^d)$ and $\lambda_d(A) < \infty$. The same holds for Π_m instead of Π^m.

The following lemma (Wieacker 1989) is often useful in exploiting the translation invariance in the proof of integral-geometric formulae for stationary σ-finite measures.

Lemma 6.1. *Let $\mathcal{T} \subset \mathcal{F}^d$ be a translation invariant measurable subset, θ a stationary σ-finite measure on $\mathcal{T}$, and ξ a measure on $\mathbb{E}^d$. If the map $K \mapsto \xi(K)$ is measurable on the space of nonempty compact subsets of $\mathbb{E}^d$, then*

$$\int_{\mathcal{T}} \xi(F)\, \mathrm{d}\theta(F) = \int_{\mathcal{T}} \int_{\mathbb{E}^d} \xi((F \cap]0,1]^d) + x)\, \mathrm{d}\lambda_d(x)\, \mathrm{d}\theta(F).$$

The general construction of auxiliary zonoids described here includes some special cases appearing in the literature.

Example 6.1. Let M be a countably m-rectifiable closed set with $\lambda_m(M) < \infty$, and let θ be the image measure of λ_d under the map $x \mapsto M + x$ from $\mathbb{E}^d$ into $\mathcal{LC}_m$. Then one easily shows that $\Pi^m(M) = \Pi^m(\theta)$.

Example 6.2. Let θ be a stationary measure on $\mathcal{LC}_m$ which is concentrated on $\mathcal{E}_m^d$ and locally finite in the sense that $\theta(\mathcal{F}_K \cap \mathcal{E}_m^d) < \infty$ for each compact subset K of $\mathbb{E}^d$. Then there is a unique finite measure $\theta^\dagger$ on $\mathcal{L}_m^d$ such that

$$\theta(A) = \int_{\mathcal{L}_m^d} \int_{L^\perp} \mathbf{1}_A(L + x)\, \mathrm{d}\lambda_{d-m}(x)\, \mathrm{d}\theta^\dagger(L) \tag{6.6}$$

for each Borel subset A of $\mathcal{E}_m^d$, where $\mathbf{1}_A$ denotes the indicator function of A. This is a consequence of Proposition 3.2.2 and its corollary in Matheron (1975) (an extension to convex cylinders is treated in Schneider 1987, Lemma 3.3). Hence, by (6.1), we may associate with each locally finite, stationary measure θ on $\mathcal{E}_m^d$ the zonoids $\Pi^m(\theta^\dagger)$ and $\Pi_m(\theta^\dagger)$, and it turns out that $\Pi^m(\theta) = \Pi^m(\theta^\dagger)$ and $\Pi_m(\theta) = \Pi_m(\theta^\dagger)$. From (6.1) and (6.3) it is now easy to see that, for a locally finite stationary measure θ on $\mathcal{E}_{d-1}^d$ and $x \in \mathbb{E}^d$, we have $2h(\Pi^{d-1}(\theta), x) = \theta(\mathcal{F}_{[0,x]} \cap \mathcal{E}_{d-1}^d)$. We shall see that this is a special case of a more general result.

Example 6.3. Let θ be a stationary measure on the space of non-empty compact sets with $\theta(\{K \in \mathcal{K}^d:\ K \cap [0,1]^d \neq \emptyset\}) < \infty$, and let Q be the set of all $K \in \mathcal{K}^d$ the circumsphere of which has centre 0. Then there is a unique finite measure $\theta^\natural$ on Q such that θ is the image measure of $\theta^\natural \otimes \lambda_d$ under the map $(K, x) \mapsto K + x$ from $Q \times \mathbb{E}^d$ into $\mathcal{K}^d$. Now, if θ is concentrated on $\mathcal{LC}_m \cap \mathcal{K}^d$ and if $\lambda_m(\cdot \cap [0,1]^d)$ is θ-integrable, then Example 1 shows that

$$h(\Pi^m(\theta), x) = \int_Q h(\Pi^m(K), x)\, \mathrm{d}\theta^\natural(K) \tag{6.7}$$

for all $x \in \mathbb{E}^d$. Thus, $\Pi^m(\theta)$ is in a certain sense the $\theta^\natural$-integral of the function which associates with each $K \in \mathcal{LC}_m \cap Q$ the convex body $\Pi^m(K)$. A formula analogous to (6.7) and a similar remark hold for $\Pi_m(\theta)$.

Now we turn to the relations between intersection problems of translative integral geometry and associated convex bodies, in particular to *Poincaré-type formulae* and extremal problems. We need some more notation. For $i = 1, \dots, n$ let L_i be an m_i-dimensional linear subspace of $\mathbb{E}^d$. By $D(L_1, \dots, L_n)$ we denote the $(m_1 + \dots + m_n)$-dimensional volume of $K_1 + \dots + K_n$, where $K_i \subset L_i$ is a compact set of λ_{m_i}-measure one; clearly $D(L_1, \dots, L_n)$ depends only on $L_1, \dots, L_n$. Further, if M_i is a countably m_i-rectifiable Borel subset of $\mathbb{E}^d$ with $\lambda_{m_i}(M_i) < \infty$, for $i = 0, \dots, n$, and if $m := m_0 + \dots + m_n \geqslant nd$, then we use the notation

$$I(M_0, m_0; \dots; M_n, m_n)$$
$$:= \int_{\mathbb{E}^d} \cdots \int_{\mathbb{E}^d} \lambda_{m-nd}(M_0 \cap (M_1 + y_1) \cap \dots \cap (M_n + y_n))\, \mathrm{d}\lambda_d(y_1) \cdots \mathrm{d}\lambda_d(y_n).$$

The following theorem shows that the value of this translative intersection integral depends only on the measures $\tau_{M_0}, \dots, \tau_{M_n}$ defined by (6.3).

Theorem 6.2. *For $i = 0, \dots, n < d$, let M_i be a countably m_i-rectifiable Borel subset of $\mathbb{E}^d$ with $\lambda_{m_i}(M_i) < \infty$, and suppose that $m := m_0 + \dots + m_n \geqslant nd$. Then*

$$I(M_0, m_0; \dots, M_n, m_n)$$
$$= \int_{\mathcal{L}_{m_0}^d} \cdots \int_{\mathcal{L}_{m_n}^d} D(L_0^\perp, \dots, L_n^\perp)\, \mathrm{d}\tau_{M_0}(L_0) \cdots \mathrm{d}\tau_{M_n}(L_n).$$

For the proof, one first assumes that $m = nd$ and applies Federer's area formula to the Lipschitz map $f : M_0 \times \cdots \times M_n \to (\mathbb{E}^d)^n$ defined by $f(x_0, \ldots, x_n) := (x_0 - x_1, \ldots, x_0 - x_n)$, observing that

$$\lambda_0(M_0 \cap (M_1 + y_1) \cap \cdots \cap (M_n + y_n)) = \lambda_0(f^{-1}(\{(y_1, \ldots, y_n)\}))$$

in this case. To prove the general case, let M_{n+1} be some m_{n+1}-dimensional cube in $\mathbb{E}^d$, where $m_{n+1} := (n+1)d - m$, and compute the integral

$$\int_{SO(d)} I(M_0, m_0; \ldots; M_n, m_n; \rho(M_{n+1}), m_{n+1}) \, d\nu(\rho)$$

with the formula obtained in the first case.

Very special cases of this statement go back to Berwald and Varga (1937) (a detailed proof of the general assertion can be found in Wieacker 1984).

For a countably $(d-1)$-rectifiable Borel subset M of $\mathbb{E}^d$ with $\lambda_{d-1}(M) < \infty$, Theorem 6.2 implies

$$2h(\Pi^{d-1}(M), x) = I([0, x], 1; M, d-1).$$

Hence, if θ is a translation invariant σ-finite measure on $\mathcal{LE}_{d-1}$ and $\lambda_{d-1}(\cdot \cap [0,1]^d)$ is θ-integrable, we infer from Lemma 6.1 that

$$2h(\Pi^{d-1}(\theta), x) = \int_{\mathcal{LE}_{d-1}} \lambda_0([0, x] \cap F) \, d\theta(F)$$

for all $x \in \mathbb{E}^d$.

The connection between translative integral geometry of general surfaces and the theory of convex bodies is now established by the observation that in some cases the integral in Theorem 6.2 can be expressed as a mixed volume of auxiliary convex bodies. We give two typical examples.

Theorem 6.3. *Let $\theta_1, \ldots, \theta_m$ be translation invariant σ-finite measures on $\mathcal{LE}_{d-1}$ satisfying $\theta_i(\mathcal{F}_K) < \infty$ for all $K \in \mathcal{K}^d$ and $i = 1, \ldots, m$. If $\lambda_{d-1}(\cdot \cap [0,1]^d)$ is θ_i-integrable for $i = 1, \ldots, m$, then*

$$\int_{\mathcal{LE}_{d-1}} \cdots \int_{\mathcal{LE}_{d-1}} \lambda_{d-m}(F_1 \cap \cdots \cap F_m \cap A) \, d\theta_1(F_1) \cdots d\theta_m(F_m)$$

$$= \frac{d!}{(d-m)!\kappa_{d-m}} V(\Pi^{d-1}(\theta_1), \ldots, \Pi^{d-1}(\theta_m), B^d, \ldots, B^d)\lambda_d(A)$$

for each bounded Borel subset A of $\mathbb{E}^d$.

For the proof, we take $m_0 = \cdots = m_n = d-1$ in Theorem 6.2; then the multiple integral there can be considered as an integral over $(S^{d-1})^{n+1}$, with τ_{M_i} corresponding to the generating measure of the zonoid $\Pi^{d-1}(M_i)$. Hence, the statement of Theorem 6.3 follows from Lemma 6.1, known results about zonoids, and the linearity of the mixed volume in each argument (more details in Wieacker 1986). A very special case of this theorem was proved by Goodey and Woodcock (1979).

As a consequence of Theorem 6.3, each inequality for mixed volumes leads to an inequality for the integral on the left-hand side in Theorem 6.3. For example $(\theta = \theta_1 = \cdots = \theta_m)$,

$$\int_{\mathcal{LE}_{d-1}} \cdots \int_{\mathcal{LE}_{d-1}} \lambda_{d-m}(F_1 \cap \cdots \cap F_m \cap A) \, d\theta(F_1) \cdots d\theta(F_m)$$

$$\leqslant \frac{d!\kappa_d}{(d-m)!\kappa_{d-m}} \left(\frac{\kappa_{d-1}}{d\kappa_d} \int_{\mathcal{LE}_{d-1}} \lambda_{d-1}(F \cap [0,1]^d) \, d\theta(F) \right)^m \lambda_d(A),$$

with equality if and only if $\Pi^{d-1}(\theta)$ is a ball (which, for instance, is the case if θ is rigid motion invariant).

If $\Pi^{d-1}(\theta)$ has interior points, then by Minkowski's theorem there is a uniquely determined centrally symmetric convex body $\Psi^{d-1}(\theta)$, centred at the origin, the area measure of which is the generating measure of $\Pi^{d-1}(\theta)$. This associated convex body appears in the following intersection formula.

Theorem 6.4. *Suppose that η and θ are translation invariant σ-finite measures on $\mathcal{LE}_m$ and $\mathcal{LE}_{d-1}$, respectively, and that $\lambda_m(\cdot \cap [0,1]^d)$ is η-integrable and $\lambda_{d-1}(\cdot \cap [0,1]^d)$ is θ-integrable. Then*

$$\int_{\mathcal{LE}_{d-1}} \int_{\mathcal{LE}_m} \lambda_{m-1}(F_1 \cap F_2 \cap A) \, d\eta(F_1) \, d\theta(F_2)$$

$$= 2dV(\Psi^{d-1}(\theta), \ldots, \Psi^{d-1}(\theta), \Pi_m(\eta))\lambda_d(A)$$

for each bounded Borel set $A \subset \mathbb{E}^d$.

The proof can be found in Wieacker (1989). Since the mixed volume in Theorem 6.4 is essentially the Minkowski area of $\Psi^{d-1}(\theta)$ with respect to $\Pi_m(\eta)$, the isoperimetric inequality of Minkowski geometry shows that, on the set of all convex bodies K of given positive volume, the integral

$$\int_{\mathcal{LE}_m} \lambda_{m-1}(F \cap \operatorname{bd} K) \, d\eta(F)$$

attains a minimum at K if and only if K is homothetic to $\Pi_m(\eta)$.

Further extremal problems in a similar spirit are treated in Schneider (1982, 1987), Wieacker (1984, 1986, 1989). Applications to stochastic processes of geometric objects are described in chapter 5.2.

7. Lines and flats through convex bodies

The present section is devoted to an entirely different facet of the close relations between integral geometry and convex bodies, this time of a more classical type. We consider various questions related to flats meeting a convex body. In this context, it is often convenient to formulate the results in terms of random flats, probabilities of geometric events, and expectations of geometric random variables. In essence, however, the results to be discussed here are either interpretations of integral-geometric identities, or inequalities obtained by methods from convex geometry. For a survey on related investigations of a more probabilistic flavour, see chapter 5.2.

Let $K \in \mathcal{K}^d$ be a convex body. A *random r-dimensional flat through* K (or *meeting* K) is a measurable map X from some probability space into the space $\mathcal{E}_r^d$ of r-flats such that $X \cap K \neq \emptyset$ with probability 1. The random flat X is called *uniform* if its distribution, which is a probability measure on $\mathcal{B}(\mathcal{E}_r^d)$, can be obtained from a translation invariant measure on $\mathcal{B}(\mathcal{E}_r^d)$ by restricting it to the flats meeting K and by normalizing the restriction to a probability measure. If the distribution of X can be derived, in this way, from a rigid motion invariant measure on $\mathcal{B}(\mathcal{E}_r^d)$, then X is called an *isotropic uniform random flat*, often abbreviated by IUR flat. In the case $r = 0$, both notions coincide, and we talk of a *uniform random point in* K.

In the following, the central setup will be that of a given finite number of independent random flats through K. These flats determine other geometric objects as well as geometric functions, and one may ask for various probabilities, expectations, or distributions connected with these. Integral-geometric identities may be useful to transform the problem, and often methods from convex geometry then lead to sharp inequalities.

We exclude from the following survey convex hulls of $d + 1$ or more random points; for these, chapter 5.2 gives a unified treatment.

Let $K \in \mathcal{K}^d$ be a convex body with interior points. First we consider an isotropic uniform random line X through K. Let σ_K denote the length of the random secant $X \cap K$. The random variable σ_K has found considerable interest in the literature. The moments of its distribution are essentially (up to a factor involving the surface area of K) the *chord power integrals* of K, defined by

$$I_k(K) := \frac{d\kappa_d}{2} \int_{\mathcal{E}_1^d} \lambda_1(L \cap K)^k \, \mathrm{d}\mu_1(L) \tag{7.1}$$

for $k \in \mathbb{N}_0$ (where $0^0 := 0$). (The factor before the integral occurs because in the older literature a different normalization of μ_1 is chosen.) In particular, (2.7) shows that

$$I_0(K) = \frac{\kappa_{d-1}}{2} S(K), \tag{7.2}$$

where S denotes the surface area, and

$$I_1(K) = \frac{d\kappa_d}{2} V_d(K). \tag{7.3}$$

From relation (7.5) below it follows that

$$I_{d+1}(K) = \frac{d(d+1)}{2} V_d(K)^2. \tag{7.4}$$

For the ball B_ρ^d of radius ρ one has

$$I_k(B_\rho^d) = \frac{2^{k-1}\pi^{d-1/2} k\Gamma(\frac{1}{2}k)}{\Gamma(\frac{1}{2}d)\Gamma(\frac{1}{2}(k+d+1))} \rho^{k+d-1};$$

see Santaló (1986). There one also finds the representation

$$I_k(K) = k(k-1) \int_0^\infty \ell^{k-2} M_K(\ell) \, \mathrm{d}\ell$$

where $M_K(\ell)$ denotes the (suitably normalized) kinematic measure of the set of all line segments of length ℓ contained in K.

A number of inequalities satisfied by the chord power integrals I_k are known; for these and for references, see Santaló (1976, pp. 48 and 238, 1986); see also Hadwiger (1957, section 6.4.6), and Voss (1984).

Blaschke (1955, p. 52), posed the following questions. If positive numbers $c_0, c_1, c_2, \ldots$ are given, what are the necessary and sufficient conditions in order that there exists a convex domain $K \in \mathcal{K}^2$ for which $I_k(K) = c_k$ for $k \in \mathbb{N}_0$? If K exists, to what extent is it determined by the numbers c_k? Mallows and Clark (1970) constructed two noncongruent convex polygons with the same chord length distribution. Gates (1982) showed how triangles and quadrangles may be reconstructed from their chord length distributions. A thorough study of the plane case was made by Waksman (1985). He was able to show that a convex polygon which is generic (in a precise sense, roughly saying that the polygon is sufficiently asymmetric) can, in fact, be reconstructed from its chord length distribution. Some more information on the distribution of the chord length for general convex bodies is contained in papers by Sulanke (1961), Gečiauskas (1987).

A classical integral-geometric transformation (see (7.9) below) relates the chord power integrals to distance power integrals of point pairs:

$$\int_K \int_K \|x_1 - x_2\|^k \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \frac{2}{(d+k)(d+k+1)} I_{d+k+1}(K) \tag{7.5}$$

for $k = -d+1, -d+2$; here $\mathrm{d}x_i$ stands for $\mathrm{d}\lambda_d(x_i)$. Formula (7.5) goes back to Crofton for $d = 2$ and $k = 0$; the general case was proved independently by Chakerian (1967) and Kingman (1969). In a generalization of (7.5) obtained by Piefke (1978b), the integrations on the left side may be extended over two different convex bodies, and the integrand can be of a more general form. For example,

$$\int_K \int_K f(\|x_1 - x_2\|) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = d\kappa_d \int_{\mathcal{E}_1^d} g(\lambda_1(L \cap K)) \, \mathrm{d}\mu_1(L),$$

if f is, say, continuous and g is determined from

$$g''(t) = f(t)t^{d-1} \quad \text{for } t \geqslant 0, \qquad g(0) = g'(0) = 0.$$

Piefke (1978a) also showed how the probability densities of the secant length σ_K and of the distance between two independent uniform random points in K can be computed from each other. This together with results of Coleman (1969) and Hammersley (1950) yields the explicit distributions of both random variables in the case of a 3-dimensional cube and a d-dimensional ball. Many special results and references on the distance between two random points in plane regions can be found in the paper by Sheng (1985); see also Gečiauskas (1976).

From the viewpoint of applications, an isotropic uniform random line through K need not be the most natural type of random line meeting the convex body K. We mention two other and equally natural ways of generating random lines through K. Let X_1, X_2 be independent uniform random points in K. With probability 1, they span a unique line. The random line through K thus generated is called λ-random. In this context, an isotropic uniform random line through K has been called μ-random. Another simple generation of random lines through K proceeds as follows. Choose a uniform random point in K and through that point a line with direction given by a unit vector which is chosen at random, independent from the point and with uniform distribution on the sphere S^{d-1}. The random line thus obtained is called ν-random. If the probability distribution of a μ-random, ν-random, λ-random line through K is denoted, respectively, by P_μ, P_ν, P_λ, then P_λ and P_ν are absolutely continuous with respect to P_μ, and for the corresponding Radon-Nikodym derivatives one has

$$\frac{\mathrm{d}P_\nu}{\mathrm{d}P_\mu}(L) = \frac{\kappa_{d-1}}{d\kappa_d}\frac{S(K)}{V_d(K)}\lambda_1(L\cap K), \tag{7.6}$$

$$\frac{\mathrm{d}P_\lambda}{\mathrm{d}P_\mu}(L) = \frac{\kappa_{d-1}}{d(d+1)}\frac{S(K)}{V_d(K)^2}\lambda_1(L\cap K)^{d+1}. \tag{7.7}$$

Essentially, these results can be found in Kingman (1965, 1969).

These and some other different types of random lines through convex bodies, as well as the induced random secants, were investigated in papers by Kingman (1965, 1969), Coleman (1969), Enns and Ehlers (1978, 1980, 1988), and Ehlers and Enns (1981); see also Warren and Naumovich (1977).

Enns and Ehlers (1978) had conjectured that, for all convex bodies K with given volume, the expectation of the length of a ν-random secant of K is maximal precisely when K is a ball. This was proved, independently, by Davy (1984), Schneider (1985), and Santaló (1986). All three authors establish the following more general result. For $k \in \mathbb{N}$, let $M_k(K)$ denote the expectation of $\lambda_1(L\cap K)^k$, where L is a ν-random line through K. If B denotes a ball with $V_d(K) = V_d(B)$, then

$$M_k(K)\begin{cases} \leqslant M_k(B) & \text{for } 1\leqslant k < d, \\ = M_k(B) & \text{for } k = d, \\ \geqslant M_k(B) & \text{for } k > d. \end{cases} \tag{7.8}$$

Equality holds for $k \neq d$ only if K is a ball. The proof makes use of (7.5) and of the following result, which can be obtained by Steiner symmetrization.

Theorem 7.1. *Let f be a decreasing measurable function on $(0,\infty)$ such that $x^{d-1} \times |f(x)|$ is integrable over all finite intervals. Then among all convex bodies K with fixed volume, the double integral*

$$\int_K \int_K f(\|x-y\|)\,\mathrm{d}x\,\mathrm{d}y$$

achieves its maximum value for the ball (and only for the ball, if f is strictly decreasing).

This was proved by Carleman (1919) (see also Blaschke 1918) for $d = 2$; extensions are due to Groemer (1982), Davy (1984) (where the above formulation appears), Pfiefer (1982, 1990).

Some of the results for random lines through a convex body K extend to random r-flats through K, for $r \in \{1,\dots,d-1\}$. Here the following integral-geometric transformation, going back to Blaschke and Petkantschin is useful (see Santaló 1976, p. 201, but also Kingman 1969 and Miles 1971, 1979). For any integrable function $f : (\mathbb{E}^d)^{r+1} \to \mathbb{R}$,

$$\int_{\mathbb{E}^d}\cdots\int_{\mathbb{E}^d} f(x_1,\dots,x_{r+1})\,\mathrm{d}x_1\cdots\mathrm{d}x_{r+1}$$
$$= c_{dr}(r!)^{d-r}\int_{\mathscr{E}_r^d}\int_E\cdots\int_E f(x_1,\dots,x_{r+1})\lambda_r(\mathrm{conv}\{x_1,\dots,x_{r+1}\})^{d-r}$$
$$\times\,\mathrm{d}x_1^E\cdots\mathrm{d}x_{r+1}^E\,\mathrm{d}\mu_r(E), \tag{7.9}$$

where

$$c_{dr} = \frac{\omega_d\cdots\omega_{d-r+1}}{\omega_1\cdots\omega_r}, \qquad \omega_m := m\kappa_m,$$

and $\mathrm{d}x_i^E$ indicates integration with respect to r-dimensional Lebesgue measure in the r-flat E.

If we choose a uniform random point in K and through that point independently a random r-flat with uniformly distributed direction (specified by an element of the Grassmannian $\mathscr{L}_r^d$), then we obtain a ν-random r-flat meeting K. With respect to the distribution of an isotropic uniform random r-flat through K, the distribution of a ν-random r-flat has a density proportional to $\lambda_r(E\cap K)$ for $E \in \mathscr{E}_r^d$. Such r-weighted r-flats through K, as they have been called, play an important role in stereology (Davy and Miles 1977, Miles and Davy 1976). Some of the inequalities of (7.8) can be extended to ν-random r-flats, see Schneider (1985).

Now we consider a finite number of independent uniform random flats through K. The case of points we mention only briefly. An extensive study of the case of up to $d+1$ points was made by Miles (1971). For more points, in particular the asymptotic behaviour of the convex hull of n points for $n \to \infty$, we refer to chapter 5.2 and to the survey by Schneider (1988a). In some of the treatments, integral-geometric transforms of type (7.9) play an essential role. We mention two further problems on random points in convex bodies where integral-geometric methods have been applied. Hall (1982) derived a formula for the probability that three uniform random points in the ball B^d form an acute triangle. If B^d is replaced by an arbitrary convex body K, Hall conjectured that the corresponding probability is maximized when K is a ball. The following problem was treated by Affentranger (1990). If $m+1$ $(2 \leqslant m \leqslant d-1)$ independent uniform random points in a convex body K are given, they determine, with probability one, a unique $(m-1)$-dimensional sphere C_{m-1} containing the points. Affentranger showed that the probability that $C_{m-1} \subset K$ is maximized if and only if K is a ball; for this case, the explicit value was computed.

For a thorough investigation of uniform random flats, we refer to Miles (1969). Here we consider only a few, mostly later, results. Let $E_1, \dots, E_s$ be independent isotropic uniform random flats through K, where $2 \leqslant s \leqslant d$,

$$1 \leqslant \dim L_i = r_i \leqslant d-1 \quad \text{for } 1, \dots, s,$$

$$m := r_1 + \dots + r_s - (s-1)d \geqslant 0.$$

With probability 1, the intersection $X := E_1 \cap \dots \cap E_s$ is a flat of dimension m. It is a matter of integral geometry to compute the distribution of this random m-flat. For example, the probability $p(K)$ that X meets K can be expressed in terms of quermassintegrals of K, and from the Aleksandrov–Fenchel inequalities it then follows that $p(K)$ is maximal precisely when K is a ball; see Schneider (1985), also Santaló (1976, III.14.2), and Miles (1969, p. 231). As another example, consider the case $d=2$, $s=2$, $r_1 = r_2 = 1$. The distribution of the intersection point X of two independent IUR lines through $K \in \mathcal{K}^2$ is given by

$$\operatorname{Prob}\{X \in \beta\} = \frac{2}{L(K)^2}\left[\pi\lambda_2(\beta \cap K) + \int_{\beta \setminus K} (\omega(x) - \sin\omega(x))\, \mathrm{d}\lambda_2(x)\right]$$

for $\beta \in \mathcal{B}(\mathbb{E}^2)$ (cf. Santaló 1976, I.4.3). Here $L(K)$ is the perimeter of K and $\omega(x)$ denotes the angle between the two supporting rays of K emanating from $x \in \mathbb{E}^2 \setminus K$. In particular, the distribution of X is uniform inside K. The probability that $X \in K$ is given by $2\pi V_2(K)/L(K)^2 \leqslant \frac{1}{2}$, by the isoperimetric inequality.

More generally, we may consider n independent IUR lines through K and ask for the distribution of the number of intersection points inside K. Only some partial results are known, see Sulanke (1965), Gates (1984).

For random flats through K which generally do not intersect, for dimensional reasons, one can study probabilities connected to the closeness of the flats. For example, n lines meeting K may be called *ρ-close*, for some given number $\rho > 0$, if there is a point in K at distance at most ρ from each of the lines. Hadwiger and Streit (1970) determined the probability that n independent IUR lines through $K \in \mathcal{K}^3$ are ρ-close, and they treated similar questions for points and planes. The proof makes use of iterated kinematic formulae for cylinders. Next, let L_1, L_2 be two independent IUR flats through $K \in \mathcal{K}^d$, with $\dim L_1 = r$, $\dim L_2 = s$, and $r+s \leqslant d-1$. With probability 1, there is a unique pair of points $x_1 \in L_1$, $x_2 \in L_2$ with smallest distance. Let $p(K)$ denote the probability that $x_1, x_2 \in K$. Then $p(K)$ is maximal if and only if K is a ball. This was proved for $d=3$, $r=s=1$ by Knothe (1937), for $r+s=d-1$ by Schneider (1985), and in general by Affentranger (1988).

New problems arise if we consider uniform random flats through K which are not necessarily isotropic. For example, let $H_1, \dots, H_n$ be independent, identically distributed uniform random hyperplanes through K. There is an even probability measure φ on the unit sphere determining the orientation of H_i (φ is the distribution of the unit normal vector of H_i if K is a ball). We assume that φ is not concentrated on a great subsphere. For $2 \leqslant n \leqslant d$, let $p_n(K, \varphi)$ denote the probability that $H_1 \cap \dots \cap H_n$ meets K. Let ω be the rotation invariant probability measure on S^{d-1}. Then

$$p_n(K, \omega) \leqslant p_n(B^d, \omega)$$

(Miles 1969), a special case of a result mentioned above. The inequality

$$p_n(B^d, \varphi) \leqslant p_n(B^d, \omega) \tag{7.10}$$

was conjectured by Miles (1969, p. 224), and proved by Schneider (1982). There it is also proved that, for given φ,

$$p_d(K, \varphi) \leqslant p_d(M_\varphi, \varphi)$$

for a convex body M_φ which is unique up to homothety. For $d=2$, one obtains $p_2(K,\varphi) \leqslant \frac{1}{2} = p_2(B^2, \omega)$, with equality for a unique homothety class of convex bodies determined by φ. However, for $d \geqslant 3$ it is shown that $p_d(K, \varphi) > p_d(B^d, \omega)$ is possible, and the maximum value of $p_d(K, \varphi)$ remains unknown. The probabilities $p_n(K, \varphi)$ also appear in the following formula. If $H_1, \dots, H_n$ are as above (but allowing arbitrary $n \geqslant 1$), these hyperplanes determine, in the obvious way, a decomposition of the interior of K into relatively open convex cells. The random variable ν_k is defined as the number of k-dimensional cells of this decomposition $(k = 0, \dots, d)$. Then its expectation is given by

$$E(\nu_k) = \sum_{j=d-k}^{d} \binom{j}{d-k}\binom{n}{j} p_j(K, \varphi),$$

as shown by Schneider (1982), extending special results of Santaló. In particular, if the hyperplanes are isotropic, then

$$E(\nu_k) = \sum_{j=d-k}^{d} \binom{j}{d-k}\binom{n}{j} \frac{j!}{2^j} \frac{V_j(K)}{V_1(K)^j},$$

which becomes maximal if and only if K is a ball.

Interesting new intersection phenomena arise for lower-dimensional flats. For example, the extremal property of the isotropic distribution expressed by (7.10) does not necessarily extend. Mecke (1988a,b) considers the case of two independent, identically distributed r-dimensional flats meeting the ball B^d in $\mathbb{E}^d$ for $d = 2r$ $(r \geqslant 2)$. With probability 1, the two flats have a unique intersection point X. Mecke was able to find all uniform distributions for which the probability $\operatorname{Prob}\{X \in B^d\}$ becomes maximal; they are not rotation-invariant.

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CHAPTER 5.2

Stochastic Geometry

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HANDBOOK OF CONVEX GEOMETRY
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Preliminaries

The roots of Stochastic Geometry can be traced back to the famous *needle problem* of Buffon in 1733. He asked for the probability that a needle of length L, randomly thrown onto a grid of parallel lines in the plane (with distance $D > L$), hits one of the lines. By using a suitable parametrisation of the needle and a subsequent elementary integration, Buffon showed this probability to be $2L/\pi D$ [he published this result only in 1777; see Miles and Serra (1978), for further historical remarks]. The potential danger in using parametrisations of geometrical objects, when dealing with problems of probabilistic type, was pointed out by an example of Bertrand in 1888 (it is usually referred to as Bertrand's paradox, although the apparent contradiction has a simple reason). Bertrand considered the probability for a random chord of the unit circle to be longer than $\sqrt{3}$. According to three different parametrisations of secants, he obtained three different answers using the corresponding uniform measures in the parameter space. Of course, the different solutions belong to different experiments to obtain a random secant of the unit circle. One of the solutions is a natural one from the mathematical point of view, since it is related to the (up to a normalising constant) unique motion invariant measure on the space $\mathscr{E}_1^2$ of lines in the plane $\mathbb{R}^2$. It is remarkable that similar problems occurred in $\mathbb{R}^3$ in stereological applications. Here, expectation formulae for random two-dimensional sections through solid particles in $\mathbb{R}^3$ are used in practice, but again they depend on the performance of the experiment. The methods frequently used to slice a particle randomly are different from the one obtained from the motion invariant measure on $\mathscr{E}_2^3$ (see section 8).

In view of Bertrand's paradox it seemed natural to connect probabilistic questions of geometric type to measures invariant under the group of rigid motions. This obviously spanned a bridge to Integral Geometry, and for a long time Geometrical Probability was just viewed as an application of integral geometric formulae (see chapter 5.1). A fairly complete overview of this period with numerous further references is given by Santaló (1976). In this survey, we do not try to copy most of the material which is already described in Santaló's book, but concentrate on the numerous new aspects and results in Geometrical Probability and Stochastic Geometry.

The direct application of integral geometric measures forces some limitations on the probabilistic problems to be considered. First, only a finite number of random objects are allowed. More seriously, the shapes of the objects have to be fixed, only their position and orientation may be random. Finally, since the invariant measure μ on the motion group G^d is infinite, reference sets have to be introduced in order to obtain compact subsets of G^d on which then μ can be normalised to give a probability measure. Consequently, the resulting distributions of the random geometric objects have limited invariance properties with respect to rigid motions. For example, the question, "Is the triangle spanned by three uniformly distributed random points in the plane more likely to be acute or obtuse?" only makes sense, if the points are chosen from a given bounded set $K \subset \mathbb{R}^2$. But of course, then the result will depend on the shape of K.

These problems were overcome, when *random sets* were introduced and combined with the already existing notion of *point processes* (this event also marked the transition from Geometrical Probability to Stochastic Geometry). Two models of random sets were presented independently by Kendall (1974) and Matheron (1972). Kendall's approach is slightly more general, but Matheron's model of *random closed sets* (RACS) is easier to follow, and found more applications in practical fields, like Image Analysis and Stereology (see Matheron 1975). Of course, for applications in stochastic processes, a more general notion of random set is necessary.

Since our goal is in applications of convexity, we limit ourselves to random sets in the class $\mathcal{K}^d$ of convex bodies, the convex ring $\mathcal{R}^d$, and the extended convex ring $\mathcal{S}^d$,

$$\mathcal{S}^d = \{K \subset \mathbb{R}^d\colon\ K \cap K' \in \mathcal{R}^d \text{ for all } K' \in \mathcal{K}^d\}\,,$$

as well as to point processes on these set classes. More general RACS and point processes are treated in section 6, since there convex bodies appear as secondary notions. Moreover, we will concentrate on results connected with convexity. Therefore, point processes of flats are not treated as a separate topic but included in sections 6 and 7, and many results on processes of flats which are of a nonconvex nature do not appear here. We refer to Stoyan, Kendall and Mecke (1987) for further information and references, also concerning statistical questions and applications. References are also found in Mecke et al. (1990), a book which is close to some parts of the following presentation.

We close this introductory section with some notation that will be used in the following. Besides the set classes $\mathcal{K}^d$, $\mathcal{R}^d$, and $\mathcal{S}^d$, which we have already introduced, we need the systems $\mathcal{F}^d$ and $\mathcal{C}^d$ of all closed and compact subsets of $\mathbb{R}^d$, respectively. Moreover, $\mathcal{P}^d$ denotes the polytopes in $\mathcal{K}^d$. We further use the groups G^d and SO^d with their Haar measures μ and ν, and the homogeneous spaces $\mathcal{E}_k^d$ of affine k-subspaces and $\mathcal{L}_k^d$ of linear k-subspaces with their invariant measures μ_k and ν_k. For details, and in particular for the topologies used on these spaces, we refer to chapter 5.1.

If X is a topological space, we denote by $\mathcal{B}(X)$ its Borel σ-algebra. All the above-mentioned spaces are supplied with their Borel σ-algebra, and measurability always refers to this Borel structure.

As in chapter 5.1, V_j will denote the jth intrinsic volume, and Φ_j the jth curvature measure, $j = 0, \ldots, d$. Here $\Phi_d(K, \cdot)$ is the Lebesgue measure λ_d restricted to K. We will also write V for the volume V_d, S for the surface area $2V_{d-1}$, W for the mean width (which is proportional to V_1), and χ for the Euler characteristic V_0. In the planar case, we use A for the area and L for the perimeter. For a polyhedral set Q, f_i denotes the number of i-faces. Also, in $\mathbb{R}^3$ we use the integral mean curvature M instead of V_1. κ_d is the volume of the unit ball $B^d \subset \mathbb{R}^d$.

For m-dimensional sets $A \subset \mathbb{R}^d$ with appropriate regularity properties, we use λ_m to denote the m-dimensional Hausdoff measure (on A). If A belongs to $\mathcal{R}^d$ or $\mathcal{S}^d$, this coincides with $\Phi_m(A, \cdot)$. Also, for $m = d-1$, and if A is the boundary of a set K in $\mathcal{R}^d$ or $\mathcal{S}^d$, we have $\lambda_{d-1} = 2\Phi_{d-1}(K, \cdot)$ on A.

Throughout the article, the letters $\mathbb{P}$ and $\mathbb{E}$ will denote probability measures and expectations.

1. Random points in a convex body

Random points, i.e., $\mathbb{R}^d$-valued random variables, are the simplest random objects in geometry. They can be used in different ways, to generate a great variety of random geometrical objects (random planes, random segments, random polytopes, random tessellations, etc.), and they have applications in several domains, for instance, in statistics, computer geometry and pattern analysis (see, e.g., Eddy and Gale 1981, Dwyer 1988, Ronse 1989, Grenander 1973, 1977). There is a very extensive literature on random points in a convex body. From the viewpoint of convex geometry it seems that the most important investigations in this field are those concerning the convex hull of random points in a convex body. Here we shall only consider the case where the random points are independently and uniformly distributed in the body, results concerning other generating procedures can be found in the excellent surveys of Schneider (1988) and Buchta (1985).

Let $K \subset \mathbb{R}^d$ be a convex body with interior points, and let $X_1, \ldots, X_n$ be n independently and uniformly distributed random points in K. Independently means that the joint distribution $\mathbb{P}$ of $X_1, \ldots, X_n$ is given by the product measure $\mathbb{P}_{X_1} \otimes \cdots \otimes \mathbb{P}_{X_n}$ of the distributions $\mathbb{P}_{X_i}$ of X_i. Uniformly means that the random points X_i all have the same distribution $\mathbb{P}_{X_i} = \Phi_d(K, \cdot)/V(K)$. Then the convex hull $Q_n := \mathrm{conv}\{X_1, \ldots, X_n\}$ is a random polytope, i.e., a random element of $\mathcal{P}^d$ (see section 4 for more details). For any measurable nonnegative function $g : \mathcal{P}^d \to [0, \infty]$, $g \circ Q_n$ is a random variable and we write $\mathbb{E}_n(g)$ for the expected value of $g \circ Q_n$. Some functions g are of particular interest, for instance, for $d \geq 3$, the volume V, the surface area S, the mean width W or the number f_i of i-dimensional faces. For $d = 2$, interesting functions are the area A, the perimeter L, and the vertex or edge number $f_0 = f_1$.

In most cases, the explicit computation of $\mathbb{E}_n(g)$ for one of the above-mentioned functionals g is complicated, even for simple convex bodies K. For example, for a tetrahedron K in $\mathbb{R}^3$, $\mathbb{E}_n(V)$ is still unknown, and the formulae derived by Buchta (1984b) for the explicit computation of $\mathbb{E}_n(A)$ in the case of a planar polygon K may give an idea of the difficulties which occur in this type of computation. Nevertheless, a number of explicit results are known in the case where K is the unit ball B^d of $\mathbb{R}^d$ (Hostinsky 1925, Kingman 1969, Buchta and Müller 1984, Affentranger 1988), and in the plane some results concerning the distribution and the moments of $A(Q_3)$ have been obtained in the cases where K is a triangle, a parallelogram or an ellipse (Reed 1974, Alagar 1977, Henze 1983). For an arbitrary convex body K in $\mathbb{R}^d$, some relations between the rth normalised moment $M_r(n, K) := \mathbb{E}_n(V^r)/V(K)^r$ of the volume of Q_n (an affine invariant parameter of K) and other expected values are easy to obtain; for example, the identity

$$\mathbb{P}(Q_n \text{ is a } d\text{-simplex}) = \binom{n}{d+1} M_{n-d-1}(d+1, K)$$

[in the planar case this relates an old problem of Sylvester to the computation of $M_1(3, K)$], or Efron's identity $\mathbb{E}_n(f_0) = n(1 - M_1(n-1, K))$. Both of them are direct consequences of Fubini's theorem. A further relation of this type is the identity $2\mathbb{E}_{d+2}(V) = (d+2)\mathbb{E}_{d+1}(V)$ proved by Buchta (1986). From the geometrical point of view, the most significant result concerning $M_r(n, K)$ up to now is probably the following theorem of Groemer (1974) (see also Schöpf 1977), which provides a characterisation of ellipsoids.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^d$ with interior points, and let $n > d$, $r \in \mathbb{N}$. Then $M_r(n, K)$ is minimal if and only if K is an ellipsoid.*

We give a brief outline of Groemer's proof. Using the affine invariance of $M_r(n, K)$, the existence of a "minimum body" can be proved with standard compactness arguments. For $Y = (y_1, \ldots, y_n) \in (\mathbb{R}^{d-1})^n$ and $Z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, define:

$$V(Y, Z) = V(\operatorname{conv}\{(y_1, z_1), \ldots, (y_n, z_n)\}),$$

where (y_i, z_i) is considered as a point of $\mathbb{R}^d$. From the convexity of the map $Z \mapsto V(Y, Z)$, we infer that

$$\int_{|z_1 - p_1| \le a_1} \cdots \int_{|z_n - p_n| \le a_n} V(Y, Z)^r \, dz_1 \cdots dz_n \geq \int_{|z_1| \le a_1} \cdots \int_{|z_n| \le a_n} V(Y, Z)^r \, dz_1 \cdots dz_n$$

whenever $Y \in ((\mathbb{R}^{d-1})^n$, $p_1, \ldots, p_n \in \mathbb{R}$ and $a_1, \ldots, a_n > 0$, with equality if and only if $(p_1, \ldots, p_n) = (0, \ldots, 0)$. Now, if K is not an ellipsoid, then there is a line L such that the set $Q(K, L)$ of the midpoints of the segments $K \cap (L + x)$, $x \in K - L$, is not contained in a hyperplane. Hence, for the convex body K' obtained from K by Steiner symmetrisation with respect to a hyperplane orthogonal to L, the above inequality implies $M_r(n, K') < M_r(n, K)$.

The convexity of the map $Z \mapsto V(Y, Z)$ has been used recently by Dalla and Larman (1991) to prove that, in the plane, $M_1(n, K)$ is maximal if K is a triangle, thus extending an old result of Blaschke (1917). However, the conjecture that in higher dimensions $M_1(n, K)$ is maximal if and only if K is a simplex, is still open.

More general results have been obtained for the asymptotic behaviour of $\mathbb{E}_n(g)$ as n tends to infinity. Problems of this type (in the plane) were first investigated in two classical papers of Rényi and Sulanke (1963, 1964). Up to now, most of the explicit computations of the asymptotic value of $\mathbb{E}_n(g)$ are based on (extensions or modifications of) their method, except in the case $g = W$ where an identity of Efron (1965) leads to a simpler proof. These results are collected in the following theorem, as long as they concern a large class of convex bodies and hold in arbitrary dimension. We use $\approx$ to denote asymptotic equality, as n tends to infinity.

Theorem 1.2. *Let K be a convex body in $\mathbb{R}^d$ with interior points. If* bd K *is of class C^3 and has positive Gauss–Kronecker curvature k everywhere, then*

$$\mathbb{E}_n(W) = W(K) - c_1(d) \int_{\partial K} k^{(d+2)/(d+1)}(x) \, d\lambda_{d-1}(x) \left(\frac{n}{V(K)}\right)^{-2/(d+1)} + O(n^{-3/(d+1)}),$$

$$\mathbb{E}_n(V) = V(K) - c_2(d) \int_{\partial K} k^{1/(d+1)}(x) \, d\lambda_{d-1}(x) \left(\frac{n}{V(K)}\right)^{-2/(d+1)} + O(n^{-3/(d+1)} \log^2 n),$$

$$\mathbb{E}_n(f_0) \approx c_2(d) \int_{\partial K} k^{1/(d+1)}(x) \, d\lambda_{d-1}(x) \left(\frac{n}{V(K)}\right)^{(d-1)/(d+1)},$$

$$\mathbb{E}_n(f_{d-1}) \approx c_3(d) \int_{\partial K} k^{1/(d+1)}(x) \, d\lambda_{d-1}(x) \left(\frac{n}{V(K)}\right)^{(d-1)/(d+1)},$$

with explicitly given constants $c_1(d)$, $c_2(d)$ and $c_3(d)$ depending only on d. If K is a polytope, then

$$W(K) - \mathbb{E}_n(W) \approx c_4(K) n^{-1/d},$$

with a constant $c_4(K)$ depending only on the shape of K in arbitrarily small neighbourhoods of the vertices of K, and if K is a simple polytope, then

$$\mathbb{E}_n(f_0) = \frac{d}{(d+1)^{d-1}} f_0(K) \log^{d-1} n + O(\log^{d-2} n),$$

$$\mathbb{E}_n(f_{d-1}) = \frac{d^d}{d!} f_{d-1}(K) M_1(\Delta_{d-1}) \log^{d-1} n + O(\log^{d-2} n),$$

$$\mathbb{E}_n(V) = V(K) - V(K) f_0(K) \frac{d}{(d+1)^{d-1}} \frac{\log^{d-1} n}{n} + O\left(\frac{\log^{d-2} n}{n}\right),$$

where $M_1(\Delta_{d-1})$ is the normalised expected volume of a random simplex in a $(d-1)$-dimensional simplex Δ_{d-1}. [The value of $M_1(\Delta_{d-1})$ is affine invariant and hence does not depend on the special choice of the simplex Δ_{d-1}.]

The main idea in the proof of these results may be formulated as follows. Let g be a bounded real function on the set of all oriented $(d-1)$-dimensional polytopes contained in K. For any polytope $Q \subset K$, define

$$g(Q) = \sum_{F \text{ facet of } Q} g(F)\,, \tag{1}$$

where the orientation of a facet F is given by the outer normal of Q at F (e.g., f_{d-1}, S and V are functions of this type). Suppose that $g \circ Q_n$ is integrable. Since Q_n is almost surely a simplicial polytope and the points are independently and uniformly distributed, $\mathbb{E}_n(g)$ can be expressed as in integral over K^d. The transformation of this integral into an integral over all hyperplanes meeting K via the Blaschke–Petkantschin identity [see Santaló (1976, p. 201) or chapter 5.1, section 7], leads to the relation

$$\mathbb{E}_n(g) = \frac{d\kappa_d}{2}\binom{n}{d} V(K)^{-d} \int_{\mathscr{E}^d_{d-1}} g_K(E)\Big(1 - \frac{V(K_E)}{V(K)}\Big)^{n-d} \,\mathrm{d}\mu_{d-1}(E) + \mathrm{o}(\varepsilon^n)\,, \tag{2}$$

with some $\varepsilon < 1$. Here K_E is the part of K cut out by E with the smaller volume $V(K_E) < \frac{1}{2}V(K)$ (both parts cut out by E have almost surely a different volume). Also, g_K is given by

$$g_K(E) := (d-1)! \int_{K\cap E} \cdots \int_{K \cap E} g(\operatorname{conv}\{x_1, \ldots, x_d\}) \times \lambda_{d-1}(\operatorname{conv}\{x_1, \ldots, x_d\})\,\mathrm{d}\lambda_{d-1}(x_1) \cdots \mathrm{d}\lambda_{d-1}(x_d)\,,$$

and the orientation of $\operatorname{conv}\{x_1, \ldots, x_d\}$ is given by the outer normal of the face $K \cap E$ of $K \backslash K_E$. From here on, one has to use the special properties of g and bd K to get a result expressed in terms of geometric parameters of K. For instance, if K is a smooth convex body in $\mathbb{R}^d$ (i.e., if bd K is of class C^3 and has positive curvature k everywhere), then a local Taylor approximation of bd K can be used to evaluate (2) in the case $g = f_{d-1}$ (Raynaud 1970, Wieacker 1978). Unfortunately, this local Taylor approximation is not good enough to obtain precise asymptotic results for $\mathbb{E}_n(V)$ or $\mathbb{E}_n(S)$, except in the case where K is a ball (Wieacker 1978; see also Affentranger 1992, and Meilijson 1990). In $\mathbb{R}^3$, the asymptotic behaviour of $\mathbb{E}_n(V)$ can easily be deduced from the asymptotic behaviour of $\mathbb{E}_n(f_2)$, since f_0 is a.s. related to f_2 via Euler's polyhedron theorem, and $\mathbb{E}_n(f_0)$ is related to $\mathbb{E}_{n-1}(V)$ via Efron's relation (Wieacker 1978). The asymptotic results concerning $\mathbb{E}_n(V)$ and $\mathbb{E}_n(f_0)$ in the case where K is smooth are due to Bárány (1992), who reduced the problem to the case of a ball (the reduction is not trivial). For a simple polytope K, (2) has been evaluated in the case where $g(F) = \eta_F^q \lambda_{d-1}(F)^q$ for some given $q \in \mathbb{N}$, where η_F is the distance between F and the supporting hyperplane of K parallel to F and oriented in the same direction. Some results about the position of the vertices of Q_n for large n

and an Efron-type argument show that one of these functionals has the same asymptotic behaviour as $\mathbb{E}_n(V)$, thus leading to the asymptotic value of $\mathbb{E}_n(V)$ and $\mathbb{E}_n(f_0)$ (Affentranger and Wieacker 1991; weaker results have been given by Dwyer 1988, and van Wel 1989). An extension of the above result on $\mathbb{E}_n(f_0)$ to arbitrary d-polytopes has been announced recently by Bárány and Buchta (1990). The computation of $\mathbb{E}_n(W)$ is due to Schneider and Wieacker (1980) for a smooth convex body (an extension to more general distributions was obtained by Ziezold 1984), and Schneider (1987a) for polytopes K.

Rényi and Sulanke (1963) obtained slightly more precise results for the planar case, since then $\mathbb{E}_n(f_{d-1})$ is also the expected number of vertices of Q_n. If K is a smooth convex body in the plane (i.e., bd K is of class C^3 and has positive curvature k everywhere), the asymptotic value of $\mathbb{E}_n(A)$ can be obtained from Theorem 1.2 via Efron's relation (Efron 1965). In this case, Rényi and Sulanke (1964) obtained the following relation by a direct computation:

$$\mathbb{E}_n(A) = A(K) - (\tfrac{2}{3})^{1/3}\Gamma(\tfrac{5}{3}) \int_{\partial K} k^{1/3}(x)\,\mathrm{d}\lambda_1(x) \Big(\frac{n}{A(K)}\Big)^{-2/3} + \mathrm{O}(n^{-1})\,. \tag{3}$$

For a polygon K with interior points in the plane, Rényi and Sulanke (1963) obtained the more precise result

$$\mathbb{E}_n(f_0) = \tfrac{2}{3} f_0(K)(\log n + C) + c_5(K) + \mathrm{o}(1)\,, \tag{4}$$

where C is Euler's constant, and $c_5(K)$ is a constant depending only on K and given explicitly (see also Ziezold 1970). The asymptotic value of $\mathbb{E}_n(A)$ was computed by Rényi and Sulanke in the case where K is a square, and by Buchta (1984a) for arbitrary polygons in the plane. The method of Rényi and Sulanke has also been used to compute the asymptotic value of $\mathbb{E}_n(L)$ (which is essentially the mean width) in the planar case (Rényi and Sulanke 1964, Buchta 1984a). Since all these results concern only smooth convex bodies or polytopes, the following estimates, which hold for arbitrary convex bodies and are in a certain sense best possible (see Theorem 1.4), are useful.

Theorem 1.3. *For each convex body K in $\mathbb{R}^d$ with interior points there are positive constants $c_6(K)$, $c_7(K)$, $c_8(d)$, $c_9(d)$ such that for large n:*

(a) $c_6(K) n^{-2/(d+1)} \le W(K) - \mathbb{E}_n(W) \le c_7(K) n^{-1/d}$,

(b) $c_6(K) \dfrac{\log^{d-1} n}{n} \le V(K) - \mathbb{E}_n(V) \le c_7(K) n^{-2/(d+1)}$,

(c) $c_8(d) \log^{d-1} n \le \mathbb{E}_n(f_i) \le c_9(d) n^{(d-1)/(d+1)}$.

In this theorem, (a) is due to Schneider (1987a), (b) to Bárány and Larman (1988), and (c) to Bárány (1989). The proof of Bárány and Larman is based on the following interesting idea. For a convex body K in $\mathbb{R}^d$ and $\varepsilon > 0$ they define

$$K_\varepsilon := \{x \in K\colon V(K \cap H) \leq \varepsilon \text{ for some half-space } H \text{ with } x \in \operatorname{bd} H\}\,, \tag{5}$$

and they show that

$$c_{10}V(K_{1/n}) < V(K) - \mathbb{E}_n(V) < c_{11}(d)V(K_{1/n}) \tag{6}$$

if $V(K) = 1$ and n is large enough. One of the main steps in the proof of this relation is the construction of an economic cap covering for K_ε. As a consequence, they obtain the first inequality in (b) from a lower bound for $V(K_{1/n})$, while the second one follows from Theorem 1.1 and the above-mentioned results for a ball. Similarly, Bárány deduced (c) from the relation

$$c_{12}(d)nV(K_{1/n}) \leq \mathbb{E}_n(f_i) \leq c_{13}(d)nV(K_{1/n}) \tag{7}$$

for $i = 0, \ldots, d-1$, $V(K) = 1$, and n sufficiently large.

Bárány (1989) obtained also an analogue of (7) for the intrinsic volume V_j, $j = 1, \ldots, d-1$. The following theorem is proved in Bárány and Larman (1988) and Bárány (1989). Here, $h(n) = \Theta(f(n))$ means that $h(n) = O(f(n))$ and $f(n) = O(h(n))$, as $n \to \infty$.

Theorem 1.4. *Let K be a convex body in $\mathbb{R}^d$ with interior points. If* bd K *is of class C^3 and has positive Gauss–Kronecker curvature k everywhere, then*

$$\mathbb{E}_n(f_i) = \Theta(n^{(d-1)/(d+1)}) \quad \textit{for } i = 0, \ldots, d-1\,,$$

$$V_j(K) - \mathbb{E}_n(V_j) = \Theta(n^{-2/(d+1)}) \quad \textit{for } j = 1, \ldots, d\,.$$

If K is a polytope, then

$$\mathbb{E}_n(f_i) = \Theta(\log^{d-1} n) \quad \textit{for } i = 0, \ldots, d-1\,,$$

$$V(K) - \mathbb{E}_n(V) = \Theta\Big(\frac{\log^{d-1} n}{n}\Big)\,,$$

$$V_j(K) - \mathbb{E}_n(V_j) = \Theta(n^{-1/(d-j+1)}) \quad \textit{for } j = 1, \ldots, d-1\,.$$

For a smooth convex body a more precise result concerning $\mathbb{E}_n(V_j)$ (with a sketch of the proof) and a conjecture concerning the asymptotic value of $\mathbb{E}_n(f_i)$ can be found in Bárány (1992). Another interesting theorem proved by Bárány (1989) concerns the expected Hausdorff distance $\delta^{\mathrm{H}}(K, Q_n)$ of K and Q_n: if bd K is of class C^2 with positive Gauss–Kronecker curvature everywhere, then

$$\mathbb{E}(\delta^{\mathrm{H}}(K, Q_n)) = \Theta\Big(\Big(\frac{\log n}{n}\Big)^{2/(d+1)}\Big)\,.$$

The preceding theorems show that the shape of K strongly influences the rate of convergence and that with respect to the asymptotic behaviour of f_i, W and V, the polytopes and the smooth convex bodies are in a certain sense extreme cases. However, the relation between the boundary structure of K and the rate of convergence of the expected values considered here seems to be fairly complicated, and for convex bodies which are neither smooth nor polytopes rather less is known. As a consequence of a general theorem of Gruber (1983) and the previous results, it turns out that for most convex bodies (in the sense of Baire categories, see chapter 4.10) the asymptotic behaviour of $W(K) - \mathbb{E}_n(W)$, $V(K) - \mathbb{E}_n(V)$ and $\mathbb{E}_n(f_i)$ is highly irregular. For instance, most convex bodies K have the property, that for any $\varepsilon > 0$, there are strictly increasing sequences $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$W(K) - \mathbb{E}_{p_i}(W) < p_i^{-(2/(d+1))+\varepsilon} \quad \text{and} \quad W(K) - \mathbb{E}_{q_i}(W) > q_i^{-(1/d)-\varepsilon}$$

for all $i \in \mathbb{N}$ (see, e.g., Schneider 1987a). Similar results for $V(K) - \mathbb{E}_n(V)$ and $\mathbb{E}_n(f_i)$ can be found in Bárány and Larman (1988), and in Bárány (1989). Since most convex bodies are strictly convex and have a boundary of class C^1, this shows that, in the range between C^1 and C^3, a small change of the smoothness properties may strongly influence the rate of convergence (see also Bárány and Larman 1988, Theorem 4). An interesting result concerning a special type of convex bodies that are neither smooth nor polytopes is due to Dwyer (1990). He proved that in the case where K is a product of lower dimensional balls, $K = B^{d_1} \times \cdots \times B^{d_k}$, with $d_1 = \cdots = d_m > d_{m+1} \geq \cdots \geq d_k$ and $d_1 + \cdots + d_k = d$, we have

$$\mathbb{E}_n(V) = \Theta(n^{(d_1-1)/(d_1+1)} \log^{m-1} n)\,. \tag{8}$$

Here the main idea, previously used by Bentley et al. (1978) and by Devroye (1980), is the following. Suppose that the random points $x_1, \ldots, x_n$ are independently and uniformly distributed in K. If d orthogonal hyperplanes are chosen through x_1, they divide K into 2^d convex sets. Let w be the probability content of the smallest of these 2^d sets, then the probability that x_1 is a vertex of $\operatorname{conv}\{x_1, \ldots, x_n\}$ is bounded above by $2^d(1-w)^{n-1}$. On the other hand, if $\tilde{w}$ is the probability content of a closed halfspace bounded by a hyperplane through x_1, then the probability that x_1 is a vertex of $\operatorname{conv}\{x_1, \ldots, x_n\}$ is bounded below by $(1-\tilde{w})^{n-1}$.

While the previous methods use essentially analytical and geometrical tools, a more stochastical approach due to Groeneboom (1988) provided very strong results in the case where K is a polygon or unit disk in the plane, among others a central limit theorem for f_0 and the asymptotic behaviour of the variance of f_0.

All results mentioned so far concern the case where the points are randomly distributed in the interior of the convex body K. The case where some of the points (or all of them) are randomly chosen on the boundary of K and the remaining ones in the interior, was investigated by several authors (Miles 1971a, Mathai 1982, Buchta, Müller and Tichy 1985, Affentranger 1988, Müller 1989, 1990). Some further results concern the asymptotic behaviour as the dimension tends to infinity (Miles 1971a, Ruben 1977, Mathai 1982, Buchta 1986, Bárány and Füredi 1988). Almost sure approximation of convex bodies by random polytopes (or more generally of smooth curves) in the plane was treated by Drobot (1982), Stute (1984), and Schneider (1988).

Further results and an extensive literature concerning other problems about random points may be found in the books of Santaló (1976) and Hall (1988). More recent contributions are due to Affentranger (1989, 1990, 1992), Dette and Henze (1989, 1990), Dwyer (1990), Meilijson (1990), Bárány and Vitale (1992), and Carnal and Hüsler (1991).

2. Random flats intersecting a convex body

A natural generalisation of the notion of a random point in a convex body is the notion of a random flat meeting a convex body. Such random flats have already been considered in chapter 5.1, section 7. Here we shall only mention a few asymptotic results which are closely related to the theory of Poisson processes of hyperplanes (see section 4, for the notion of Poisson processes). Some of them are in a certain sense dual to the results of the preceding section. Instead of considering the convex hull of random points, one may also consider the intersection of random closed halfspaces generated by random hyperplanes. Since a random polyhedral set generated in this way may be viewed as the solution set of a finite system of random linear inequalities, this type of random polyhedral sets is of interest in the average case analysis of linear programming algorithms (concerning this aspect of the problem, see, e.g., Prékopa 1972, Schmidt and Mattheiss 1977, Kelly and Tolle 1981, Borgwardt 1987, Buchta 1987a,b).

Let K and C be convex bodies in $\mathbb{R}^d$, C being contained in the interior of K, and let $\mathscr{H} := \{H \in \mathscr{E}^d_{d-1}\colon H \cap K \neq \emptyset, H \cap C = \emptyset\}$ be the set of all hyperplanes meeting K but not C. A *random hyperplane* X in $\mathscr{H}$ is a measurable map from some probability space into the space $\mathscr{E}^d_{d-1}$ such that $X \in \mathscr{H}$ almost surely. X is thus a special random closed set (see section 4). The random hyperplane X is called uniform (respectively uniform and isotropic) if its probability distribution is obtained from a translation invariant (motion invariant) measure on $\mathscr{B}(\mathscr{E}^d_{d-1})$ after restriction to $\mathscr{B}(\mathscr{H})$ and normalisation. For a random hyperplane X in $\mathscr{H}$ we shall denote by X^+ the random closed halfspace bounded by X and containing C. In the following, $X_1, \ldots, X_n$ are independent, identically distributed random hyperplanes in $\mathscr{H}$, and Q_n is the random polyhedral set $X_1^+ \cap \cdots \cap X_n^+$. As in the preceding section, $\mathbb{E}_n(f_0)$ denotes the expected number of vertices of Q_n.

The random polyhedral set Q_n was first investigated by Rényi and Sulanke (1968) in the case where $d = 2$ and the random hyperplanes are uniform and isotropic. They proved that $\mathbb{P}(Q_n \not\subseteq K) = O(\gamma^n)$, $0 < \gamma < 1$, as $n \to \infty$. Further, in the case where the boundary of C is smooth enough with positive and bounded curvature k, they proved that

$$\mathbb{E}_n(f_0) = (\tfrac{2}{3})^{1/3}\Gamma(\tfrac{5}{3})\Big(\frac{n}{L_2 - L_1}\Big)^{1/3} \int_0^{L_1} k^{2/3}(x)\, d\lambda_1(x) + O(1)$$

as $n \to \infty$. Here, L_1 and L_2 are the perimeters of C and K. In the case where C is a convex polygon with $f_0(C)$ vertices, they obtained

$$\mathbb{E}_n(f_0) = \tfrac{2}{3} f_0(C) \log n + O(1)$$

for $n \to \infty$. While Rényi and Sulanke gave a direct proof of these results, Ziezold (1970) showed that they can be deduced from the corresponding results (Theorem 1.2) in the preceding section by means of a duality relation. However, the reduction of such results for random intersections to analogous results for random convex hulls is difficult, in general. In particular, the image distribution is a rather complicated one, and in some cases a direct proof may be easier. Similar results for higher dimensions and more general situations have been obtained recently by Kaltenbach (1990). In the case where the boundary of C is of class C^3 with positive Gauss–Kronecker curvature k, one of his results implies

$$\mathbb{E}_n(f_0) = c(d)\Big(\frac{n}{W(K) - W(C)}\Big)^{(d-1)/(d+1)} \int_{\partial K} k^{d/(d+1)}(x)\, d\lambda_{d-1}(x) + O(n^{(d-2)/(d+1)})$$

for $n \to \infty$, with an explicitly given constant $c(d)$ depending only on the dimension d. Kaltenbach also investigated the behaviour of other functionals like the volume of the part of Q_n contained in a ball centred at the origin and containing K.

The case where C reduces to the origin and K is the unit ball is of particular interest. For uniform and isotropic random hyperplanes in $\mathbb{R}^d$, Schmidt (1968) proved that $\mathbb{E}_n(f_0)$ converges as $n \to \infty$. The limit was computed by Rényi and Sulanke (1968) in the case $d = 2$, and by Sulanke and Wintgen (1972) for arbitrary dimension. Schneider (1982) considered the case where the distribution of the random hyperplanes comes from a translation invariant, but not necessarily isotropic measure τ on $\mathscr{E}^d_{d-1}$ with $\tau(A) = \mathbb{P}(X \in A)$, for $A \in \mathscr{B}(\mathscr{H})$. Under the assumption that the random hyperplanes are not almost surely parallel to a line, he proved that

$$\lim_{n\to\infty} \mathbb{E}_n(f_0) = 2^{-d} d!\, V_d(\Pi^{d-1}(\tau)) V_d(\Pi^{d-1}(\tau)^*)\,,$$

where $\Pi^{d-1}(\tau)$ is a zonoid associated with τ (see chapter 5.1, section 6, for details) and $\Pi^{d-1}(\tau)^*$ is the polar body of $\Pi^{d-1}(\tau)$. Since the right-hand side is essentially the volume product of $\Pi^{d-1}(\tau)$, this implies that

$$2^d \leq \lim_{n\to\infty} \mathbb{E}_n(f_0) \leq 2^{-d} d!\, \kappa_d^2\,,$$

with equality on the left if and only if $\Pi^{d-1}(\tau)$ is a parallelotope and equality on the right if and only if $\Pi^{d-1}(\tau)$ is an ellipsoid (in particular, this is the case when the random hyperplanes are isotropic). Here, the lower bound and the corresponding equality case follow from results of Reisner (1985, 1986), while the upper bound is a consequence of the Blaschke–Santaló inequality (the equality case was treated by Saint Raymond 1981). In particular it follows that the limit is maximal in the isotropic case. These results coincide with the corresponding results for stationary Poisson hyperplane processes (see Theorem 7.2), and in fact the asymptotic behaviour of the above-described model is closely related to the behaviour of a stationary Poisson hyperplane process. A comparison of both models may be found in Kaltenbach (1990).

Related problems concerning almost sure approximation of planar convex bodies with smooth boundaries by circumscribed polytopes generated by independently and identically distributed random tangent lines are studied in Carlsson and Grenander (1967), and Schneider (1988).

3. Random convex bodies

A *random compact set* or a *random convex body* X is a random element of the measurable spaces $\mathscr{C}^d$ or $\mathscr{K}^d$, i.e., X is given by a probability measure on $\mathscr{C}^d$ or $\mathscr{K}^d$, which we call the *distribution* $\mathbb{P}_X$ of X.

Since the spaces $\mathscr{C}^d$ and $\mathscr{K}^d$ carry a linear structure, i.e., they are convex cones with respect to Minkowski addition and multiplication by nonnegative scalars, results for random elements X of $\mathscr{C}^d$ or $\mathscr{K}^d$, analogous to the classical results for real random variables, can be expected. Here, the main difference is that $\mathscr{K}^d$ can be embedded into a Banach space, e.g., into the space $C(S^{d-1})$ of continuous functions on S^{d-1} (see chapter 1.9), using the support function. Therefore, results for $C(S^{d-1})$-valued random elements can be transferred to random convex bodies. This is not possible directly for random compact sets, since the semigroup $\mathscr{C}^d$ is not embeddable into a group. Therefore, and in view of the goals of this Handbook, the considerations in this section will concentrate on random convex bodies, results for nonconvex sets will be mentioned at the end.

Some of the usual probabilistic notions like joint distribution and (stochastic) independence transfer to random compact sets immediately. For others, like the expectation, we need additional explanations. If the random compact set X is viewed as a measurable mapping from a basic probability space $(\Omega, \mathscr{A}, \mathbb{P})$ into $(\mathscr{C}^d, \mathscr{B}(\mathscr{C}^d))$, the *expectation* $\mathbb{E}X$ can be defined by:

$$\mathbb{E}X = \{\mathbb{E}\xi\colon\ \xi : \Omega \to \mathbb{R}^d \text{ integrable, } \xi(\omega) \in X(\omega) \text{ for almost all } \omega \in \Omega\}\,.$$

A measurable mapping $\xi : \Omega \to \mathbb{R}^d$ with $\xi(\omega) \in X(\omega)$, $\omega \in \Omega$, is called a selection of X, so $\mathbb{E}X$ is the set built by the mean vectors of all integrable selections of X. $\mathbb{E}X$ is compact if and only if $\mathbb{E}d(X, \{0\}) < \infty$. Moreover, if the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ is nonatomic, $\mathbb{E}X$ is convex (see Aumann 1965). This indicates a disadvantage in the definition of $\mathbb{E}X$, due to the use of selections the

expectation will depend on the structure of the underlying probability space and not on the distribution alone. To overcome this difficulty, we therefore assume in the following that $(\Omega, \mathscr{A}, \mathbb{P})$ is nonatomic (and fulfills $\mathbb{E}d(X, \{0\}) < 0$) (for more information about expectations, see Vitale 1988, 1990; different aspects of means for random sets are discussed in Stoyan 1989).

For a random compact set X, the convex hull conv X is a random convex body (see section 4) and then $\mathbb{E}(\text{conv}\, X)$ is a convex body, too. Moreover (in view of our assumptions), $\mathbb{E}X = \mathbb{E}(\text{conv}\, X)$. For a random convex body X, the support function h_X is a random element of $C(S^{d-1})$. Since the supremum norm $\|h_X\|_\infty$ obeys $\|h_X\|_\infty = d(X, \{0\})$, we have $\mathbb{E}\|h_X\|_\infty < \infty$. So the expectation $\mathbb{E}h_X$ exists in the (usual) weak sense, this means

$$\varphi(\mathbb{E}h_X) = \int_\Omega \varphi(h_{X(\omega)})\, d\mathbb{P}(\omega)$$

for all linear functionals $\varphi \in C'(S^{d-1})$ (see Araujo and Giné 1980). As one would expect, $\mathbb{E}h_X$ is the support function of $\mathbb{E}X$,

$$\mathbb{E}h_X = h_{\mathbb{E}X}\,.$$

Thus if $X_1, X_2, \ldots$ is a sequence of random convex bodies, then $h_{X_1}, h_{X_2}, \ldots$ is a sequence of random elements of the Banach space $C(S^{d-1})$ and distributional properties of the random bodies $X_1, X_2, \ldots$ like independence or identical distribution transfer immediately to the sequence of (random) support functions $h_{X_1}, h_{X_2}, \ldots$. Therefore, results for Banach-space-valued random variables can be used. This is the key to most of the results mentioned in the following. For example, the strong law of large numbers in $C(S^{d-1})$ (Araujo and Giné 1980) immediately gives the following *Strong Law of Large Numbers* for random convex bodies.

Theorem 3.1. *Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random convex bodies (such that $\mathbb{E}d(X_1, \{0\}) < \infty$), then almost surely*

$$\frac{1}{n}(X_1 + \cdots + X_n) \to EX_1$$

as $n \to \infty$.

This theorem was first obtained by Artstein and Vitale (1975) who thus initiated a variety of subsequent results of a similar nature. Variants of Theorem 3.1 and generalisations are due to Cressie (1978), Hess (1979), Giné, Hahn and Zinn (1983), Puri and Ralescu (1983) and Hiai (1984, 1985).

As a second basic result from probability theory which can be transferred to random convex bodies by the above method we present the *Central Limit Theorem*. To formulate it, we need the covariance Γ_X of a random element X in $C(S^{d-1})$. Γ_X is the mapping $\Gamma_X : C'(S^{d-1}) \times C'(S^{d-1}) \to \mathbb{R}$ defined by

$$\Gamma_X(\varphi, \psi) = \mathbb{E}[\varphi(X - \mathbb{E}X)\psi(X - \mathbb{E}X)], \quad \varphi, \psi \in C'(S^{d-1}).$$

For a random convex body X we set

$$\Gamma_X = \Gamma_{h_X}.$$

We also denote by $\underset{\mathscr{D}}{\rightarrow}$ the convergence in distribution.

Theorem 3.2. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random convex bodies (such that $\mathbb{E}d(X_1, \{0\}) < \infty$), then*

$$n^{1/2}d\left(\frac{1}{n}(X_1 + \cdots + X_n), \mathbb{E}X_1\right) \underset{\mathscr{D}}{\rightarrow} \|Z\|_\infty$$

as $n \to \infty$, where Z is a centred Gaussian $C(S^{d-1})$-variable with $\Gamma_Z = \Gamma_{X_1}$.

This result was proved independently in Weil (1982) and Giné, Hahn and Zinn (1983), versions for more special distributions are due to Cressie (1979) and Lyashenko (1982) (also, unpublished manuscripts on the Central Limit Theorem of Eddy and Vitale are listed in Giné, Hahn and Zinn 1983). Both results, Theorems 3.1 and 3.2, hold as well for random compact sets X_i. The reason is that the summation of sets is a convexifying operation. This has been made precise in a theorem of Shapley–Folkmann–Starr (see, e.g., Arrow and Hahn 1971), which is used in Artstein and Vitale (1975), and Weil (1982).

Further limit theorems for random compact sets or random convex bodies, which will not be mentioned in detail, are a law of the iterated logarithm (Giné, Hahn and Zinn 1983), ergodic theorems (Hess 1979, Schürger 1983), and the characterisations of infinitely divisible and stable random bodies (Mase 1979, Giné and Hahn 1985a,b,c). For the latter, it is important to mention that the Gaussian case does not play the same important role for random convex bodies as in classical probability theory. The fact that the space $\mathscr{K}^d$ of convex bodies is a convex cone, implies that any Gaussian measure on $\mathscr{K}^d$ is degenerated (Lyashenko 1983, Vitale 1983a).

Further results concern extensions to closed sets, sets in Banach spaces and convexification of compact sets (Artstein and Hart 1981, Artstein and Hansen 1985, Puri and Ralescu 1985, Puri, Ralescu and Ralescu 1986). General surveys are given by Giné, Hahn and Zinn (1983), Vitale (1983b) and Cressie (1984).

For random convex bodies X, the usual geometric functionals (intrinsic volumes, mixed volumes) become real random variables and some of their relations transfer into expectation formulae. We mention only two of them, the *Brunn–Minkowski Theorem* for random convex bodies,

$$V^{1/d}(\mathbb{E}X) \geq \mathbb{E}V^{1/d}(X),$$

proved by Vitale (1990), and the *generalised Steiner formula* (Matheron 1975, Stoyan, Kendall and Mecke 1987)

$$\mathbb{E}V(X + K) = \sum_{k=0}^{d} \alpha_{d0k}\mathbb{E}V_k(X)V_{d-k}(K). \tag{9}$$

Here, X is assumed to have a rotation invariant distribution and $K \in \mathscr{K}^d$ is arbitrary. Equation (9) is a simple consequence of

$$\mathbb{E}V(X + K) = \int_{SO_d} \mathbb{E}V(\vartheta X + K)\,d\nu(\vartheta) = \mathbb{E}\int_{G_d} \chi(gX \cap K)\,d\mu(g)$$

and the *Principal Kinematic Formula* (see chapter 5.1). The coefficients α_{d0k} are also determined by the latter.

4. Random sets

A *random closed set* (RACS) X is a random element of $(\mathscr{F}^d, \mathscr{B}(\mathscr{F}^d))$, i.e., a measurable mapping $X: (\Omega, \mathscr{A}, \mathbb{P}) \to (\mathscr{F}^d, \mathscr{B}(\mathscr{F}^d))$, where $(\Omega, \mathscr{A}, \mathbb{P})$ is an abstract probability space. The image measure of $\mathbb{P}$ under X is the distribution $\mathbb{P}_X$ of X, it is a probability measure on $(\mathscr{F}^d, \mathscr{B}(\mathscr{F}^d))$. Two random closed sets X, X' with the same distribution $\mathbb{P}_X = \mathbb{P}_{X'}$ are called *equivalent*. For a RACS X, there is an analogue of the classical distribution function for real random variables. This is the *capacity functional* T_X of X, defined on $\mathscr{C}^d$ by

$$C \mapsto T_X(C) := \mathbb{P}_X(\mathscr{F}_C) = \mathbb{P}(X \cap C \neq \emptyset).$$

Here, we have used the abbreviation $\mathscr{F}_C = \{F \in \mathscr{F}^d : F \cap C \neq \emptyset\}$. The following uniqueness result is part of a more general theorem of Choquet (see also Kendall 1974 and Matheron 1975).

Theorem 4.1. *Two random closed sets X and X' have the same distribution if and only if $T_X = T_{X'}$.*

Other familiar probabilistic notions (joint distribution, independence, etc.) can be transferred to random closed sets X in the obvious way. Moreover, the following geometric transformations map random closed sets X (respectively X, Y) into random closed sets since they are either continuous or have a certain semi-continuity property, and hence are measurable (for details, see Matheron 1975):

$$(X, Y) \mapsto X \cup Y, \qquad (X, Y) \mapsto X \cap Y,$$

$$(\alpha, X) \mapsto \alpha X, \quad \alpha \in \mathbb{R}, \qquad (g, X) \mapsto gX, \quad g \in G^d,$$

$$(X, Y) \mapsto X + Y \quad \text{for } Y \text{ compact},$$

$$X \mapsto \operatorname{conv} X \quad \text{for } X \text{ compact},$$

$$X \mapsto \operatorname{bd} X.$$

If the distribution $\mathbb{P}_X$ of a random closed set X is concentrated on one of the measurable subsets $\mathcal{S}^d$, $\mathcal{C}^d$, $\mathcal{R}^d$, $\mathcal{K}^d$, $\mathcal{E}_k^d$ and $\mathcal{L}_k^d$, we will speak of a *random $\mathcal{S}^d$-set*, *random compact set*, *random $\mathcal{R}^d$-set*, *random convex body*, *random k-flat* and *random (k-dimensional) subspace*, respectively. Some of these notions have already been mentioned and used in previous sections; they all appear now as special cases of a RACS.

From a theoretical as well as practical point of view, the main interest is in random closed sets X that have certain invariance properties against geometric transformations $\varphi : \mathcal{F}^d \to \mathcal{F}^d$. We call X *φ-invariant*, if $\varphi(X)$ and X have the same distribution, i.e., if $\mathbb{P}_X$ is invariant under φ. In particular, X is called *stationary* if X is φ-invariant for all translations φ, and X is called *isotropic* if X is φ-invariant for all rotations φ.

Throughout this section, we assume that X is stationary. If we also require $X \neq \emptyset$ (almost surely), then X is almost surely unbounded. In view of the geometric aspect of this Handbook, it is therefore natural to concentrate on (stationary) random $\mathcal{S}^d$-sets X. The main goal in the following is to define mean values $D_j(X)$ of the intrinsic volumes V_j for X (we call these mean values *quermass densities*) and to transfer classical integral geometric formulae (see chapter 5.1) to such random $\mathcal{S}^d$-sets.

For a stationary random $\mathcal{S}^d$-set X and $j \in \{0, \dots, d\}$, the curvature measure $\Phi_j(X, \cdot)$ is a random signed Radon measure. Also we may consider $V_j(X \cap K)$ for all $K \in \mathcal{K}^d$ and get a real random variable. In order that the expectations $\mathbb{E}\Phi_j(X, \cdot)$ and $\mathbb{E}V_j(X \cap K)$ exist, integrability conditions have to be fulfilled by X. Here we use a simple, but surely not optimal condition. For $K \in \mathcal{R}^d$ let $N(K)$ be the smallest number n such that $K = K_1 \cup \cdots \cup K_n$, with $K_i \in \mathcal{K}^d$. The mapping $N : \mathcal{R}_d \to \mathbb{N}_0$ is measurable. We now make the general assumption that

$$\mathbb{E}2^{N(X \cap K)} < \infty$$

for all $K \in \mathcal{K}^d$. This ensures that all expectations which appear in the following will exist.

For example, $\mathbb{E}\Phi_j(X, \cdot)$ is now again a signed Radon measure, and, because of the stationarity, translation invariant. Hence

$$\mathbb{E}\Phi_j(X, \cdot) = c\lambda_d ,$$

with a constant $c \in \mathbb{R}$, which can serve as quermass density.

Another approach could be to consider

$$\lim_{i \to \infty} \frac{\mathbb{E}V_j(X \cap K_i)}{V_d(K_i)} ,$$

where $K_i \nearrow \mathbb{R}^d$, if this limit exists and is independent of the sequence K_i.

As a third approach, one can use a set C_0 from a lattice tesselation of $\mathbb{R}^d$, e.g., $C_0 = [0, 1]^n$ (or any other box of unit volume), and subtract half the value of the boundary of C_0. More precisely, let

$$\partial^+ C_0 = \{x \in C_0 : \max_{i=1,\dots,d} x_i = 1\}$$

be the "upper right" boundary of C_0. Since $\partial^+ C_0 \in \mathcal{R}^d$,

$$\mathbb{E}[V_j(X \cap C_0) - V_j(X \cap \partial^+ C_0)]$$

is another candidate for the quermass density of X.

The following result shows that all three approaches are equivalent.

Theorem 4.2. *Let X be a stationary $\mathcal{S}^d$-set. Then for $j = 0, \dots, d$ there exists a number $D_j(X)$ such that*

$$\mathbb{E}\Phi_j(X, \cdot) = D_j(X) \cdot \lambda_d , \tag{10}$$

$$\lim_{r \to \infty} \frac{\mathbb{E}V_j(X \cap rK)}{V_d(rK)} = D_j(X) \quad \text{for all } K \in \mathcal{K}^d \text{ with } V_d(K) > 0 , \tag{11}$$

and

$$\mathbb{E}[V_j(X \cap C_0) - V_j(X \cap \partial^+ C_0)] = D_j(X) . \tag{12}$$

We call $D_j(X)$ the *jth quermass density* of X. The following formulae for quermass densities are the counterpart of the two basic formulae from integral geometry, the *Principal Kinematic Formula* and the *Crofton Formula* (see chapter 5.1, in particular for the explicit value of the coefficients).

Theorem 4.3. *Let X be a stationary and isotropic $\mathcal{S}^d$-set and let $K \in \mathcal{K}^d$. Then, for $j = 0, \dots, d$,*

$$\mathbb{E}V_j(X \cap K) = \sum_{k=j}^{d} \alpha_{djk} V_k(K) D_{d+j-k}(X) . \tag{13}$$

Theorem 4.4. *Let X be a stationary and isotropic $\mathcal{S}^d$-set and let $L \subset \mathbb{R}^d$ be a q-dimensional subspace. Then $X \cap L$ is also a stationary and isotropic $\mathcal{S}^d$-set (in L), and for $j = 0, \dots, q$*

$$D_j(X \cap L) = \alpha_{djq} D_{d+j-q}(X) . \tag{14}$$

The proofs are similar. We give a short outline in the case of Theorem 4.3. From the Principal Kinematic Formula and Fubini's theorem we have

$$(V_d(rB^d))^{-1} \int_{G_d} \mathbb{E}V_j(X \cap rB^d \cap gK)\, d\mu(g)$$

$$= (V_d(rB^d))^{-1} \sum_{k=j}^{d} \alpha_{djk} V_k(K) \mathbb{E}V_{d+j-k}(X \cap rB^d) .$$

The right side converges for $r \to \infty$, because of (11), towards the right side of (13). For the integration on the left side we may asymptotically (for $r \to \infty$) concentrate on those $g \in G^d$, for which $gK \subset rB^d$. For these g,

$$\mathbb{E}V_j(X \cap rB^d \cap gK) = \mathbb{E}V_j(X \cap gK) = \mathbb{E}V_j(X \cap K)\,,$$

hence the integrand is constant. The result therefore follows from

$$\lim_{r\to\infty} \frac{\mu(\{g \in G_d\colon gK \subset rB^d\})}{V_d(rB^d)} = 1\,.$$

These results were obtained in Weil and Wieacker (1984), and Weil (1984); for more general classes of sets, see Zähle (1986).

A theory of random closed sets can be developed in any suitable topological space, and in fact Theorem 4.1 is given in Matheron (1975) in this more general setting. We will use it in the particular case of the locally compact space $\overline{\mathscr{F}}^d = \mathscr{F}^d \setminus \{\emptyset\}$.

5. Point processes

For point processes on $\mathscr{F}^d$ two closely connected approaches exist. They can either be described geometrically as random collections of sets in $\mathscr{F}^d$ or analytically as locally finite random measures. We shortly survey both developments.

For the first approach, we call a set $\eta \subset \overline{\mathscr{F}}^d$ *locally finite*, if

$$\operatorname{card}\{F \in \eta\colon F \cap C \neq \emptyset\} < \infty$$

for all $C \in \mathscr{C}^d$. (Here card denotes the number of elements.) Let N be the class of all locally finite subsets $\eta \subset \overline{\mathscr{F}}^d$ and let $\mathscr{N}$ be the σ-algebra on N, generated by the "counting mappings" $\gamma_{\mathscr{F}}: \eta \mapsto \operatorname{card}(\eta \cap \mathscr{F})$, where $\mathscr{F}$ runs through the σ-algebra $\mathscr{B}(\overline{\mathscr{F}}^d)$. A *(simple) point process* X on $\mathscr{F}^d$ is then a measurable mapping $X: (\Omega, \mathscr{A}, \mathbb{P}) \to (N, \mathscr{N})$, where $(\Omega, \mathscr{A}, \mathbb{P})$ denotes an abstract probability space.

The distribution $\mathbb{P}_X$ is the image of $\mathbb{P}$ under X. Two point processes X, X' on $\mathscr{F}^d$ with the same distribution are again called *equivalent*.

It should be emphasised that a realisation $X(\omega)$ of a point process X on $\mathscr{F}^d$ is a collection of sets, hence it has a spatial component but no temporal interpretation. Therefore, X can also be called a *random field of closed sets*. Due to the definition, sets $F \in X(\omega)$ can occur only once (i.e., with multiplicity one), therefore, these point processes X are called *simple*. Simple point processes X on $\mathscr{F}^d$ are just locally finite random subsets of $\overline{\mathscr{F}}^d$, since $\mathscr{N}$ coincides with the Borel-σ-algebra $\mathscr{B}(\overline{\mathscr{F}}^d)$ restricted to N (see Ripley 1976). Hence all tools and results which can be formulated generally for random sets immediately transpose to point processes. In particular, this shows how the geometric transformations from the list in the previous section act on point processes and also, stationarity and isotropy for point processes is defined.

More specifically, for two point processes X, X' on $\mathscr{F}^d$ the *union* (or *superposition*) $X \cup X'$ is again a point process on $\mathscr{F}^d$, and for a point process X on $\mathscr{F}^d$ and $\mathscr{F} \in \mathscr{B}(\overline{\mathscr{F}}^d)$ the *intersection* $X \cap \mathscr{F}$ is again a point process on $\mathscr{F}^d$ (the *restriction* of X to $\mathscr{F}$). The *section process* $X \cap F$, for $F \in \overline{\mathscr{F}}^d$ is of a different nature, it consists of the sets $F' \cap F$, $F' \in X$.

If the point process X is concentrated on one of the sets $\mathscr{C}^d$, $\mathscr{R}^d$ or $\mathscr{K}^d$, we call it a *particle process*, point processes on $\mathscr{E}^d_k$ are called *processes of flats* (*k-flats*), and processes on $\mathbb{R}^d$ are called *ordinary point processes*. Also, the meaning of *line process*, *hyperplane process*, *process of curves* (*fibre process*) is now evident.

The following uniqueness result for point processes follows from the general version of Theorem 4.1, which goes back to Choquet, Kendall, and Matheron (see Matheron 1975).

Theorem 5.1. *Two point processes X, X' on $\mathscr{F}^d$ have the same distribution if and only if*

$$\mathbb{P}(X \cap \mathscr{F} = \emptyset) = \mathbb{P}(X' \cap \mathscr{F} = \emptyset)$$

for all compact $\mathscr{F} \subset \overline{\mathscr{F}}^d$.

The set-theoretic approach described so far, has the disadvantage that some natural operations lead to point processes which cannot be described as (simple) random collections any more. For example, for a particle process X, the centres (Steiner point, centre of gravity, etc.) of the particles $K \in X$ build an ordinary point process $\tilde{X}$ on $\mathbb{R}^d$ where multiple points are possible. Here, the random measure approach is more appropriate; it also embeds the theory of point processes in the theory of random measures.

We call a Borel measure φ on $\overline{\mathscr{F}}^d$ a *locally finite counting measure* if

$$\varphi(\{F \in \overline{\mathscr{F}}^d\colon F \cap C \neq \emptyset\}) \in \mathbb{N}_0$$

for all $C \in \mathscr{C}^d$. Let M be the collection of all locally finite counting measures on $\overline{\mathscr{F}}^d$, and let $\mathscr{M}$ be the σ-algebra on M, generated by the "evaluation mappings" $\gamma_{\mathscr{F}}: \varphi \mapsto \varphi(\mathscr{F})$, where $\mathscr{F}$ runs through the σ-algebra $\mathscr{B}(\overline{\mathscr{F}}^d)$. $\varphi \in M$ is called *simple* if $\varphi(\{F\}) \leq 1$ for all $F \in \overline{\mathscr{F}}^d$. There is an isomorphism between the simple measures $\varphi \in M$ and the sets $\eta \in N$ given by the representation

$$\varphi = \sum_{F \in \eta} \delta_F\,,$$

where δ_F denotes the Dirac measure in $F \in \overline{\mathscr{F}}^d$. This isomorphism also preserves the measurability structure, i.e., the σ-algebra $\mathscr{M}$, restricted to the simple measures, is isomorphic to the σ-algebra $\mathscr{N}$. As an extension of the definition given before, we therefore define a (*general*) *point process* X on $\mathscr{F}^d$ as a measurable mapping $X: (\Omega, \mathscr{A}, \mathbb{P}) \to (M, \mathscr{M})$, where $(\Omega, \mathscr{A}, \mathbb{P})$ denotes an abstract probability space.

It is clear that Theorem 5.1 is no longer true for general point processes (but general uniqueness results for random measures apply, see Kallenberg 1986). *Stationarity* and *isotropy* are defined for general point processes as in the simple case, based on the corresponding action of G^d on M.

In the following, we will concentrate on simple point processes without further mention; nonsimple processes can only occur as secondary processes (like the process of centres). We will however use both approaches to point processes simultaneously, i.e., we will not distinguish strictly between simple random measures in M and their corresponding random set in N. This allows us, e.g., to write $X(\mathcal{F})$ for the number of elements of X which lie in $\mathcal{F} \in \mathcal{B}(\overline{\mathcal{F}}^d)$, on one hand, and $F \in X$ for the elements F of X, on the other.

A basic notion for (simple or general) point processes X is the *intensity measure* Λ, a counterpart to the classical expectation for random variables. Λ is a measure on $\overline{\mathcal{F}}^d$ defined by

$$\Lambda(\mathcal{F}) = \mathbb{E}X(\mathcal{F}), \quad \mathcal{F} \in \mathcal{B}(\overline{\mathcal{F}}^d).$$

$\Lambda(\mathcal{F})$ is thus the mean number of sets of X lying in $\mathcal{F}$. We assume throughout that Λ is *locally finite*, i.e., obeys

$$\Lambda(\{F \in \overline{\mathcal{F}}^d : F \cap C \neq \emptyset\}) < \infty$$

for all $C \in \mathcal{C}^d$. For stationary (isotropic) X, Λ is translation (rotation) invariant.

A simple but rather useful result is the following. For indicator functions $f = 1_{\mathcal{F}}$ it is a direct consequence of the definition of the intensity measure, for general functions $f \geq 0$ it follows from the monotone convergence theorem of measure theory.

Theorem 5.2 (Campbell). *For a point process X on $\mathcal{F}^d$ and a measurable function $f : \mathcal{F}^d \to \mathbb{R}_+$,*

$$\omega \mapsto \sum_{F \in X(\omega)} f(F)$$

is measurable and

$$\mathbb{E} \sum_{F \in X} f(F) = \int_{\mathcal{F}^d} f \, \mathrm{d}\Lambda. \tag{15}$$

If X is a point process on $\mathcal{F}^d$, we may consider the *union set* Y_X defined by

$$Y_X = \bigcup_{F \in X} F.$$

Y_X is a RACS, and if X is stationary (isotropic), then Y_X is stationary (isotropic). The capacity functional of Y_X follows directly from the distribution of X, since

$$T_{Y_X}(C) = 1 - \mathbb{P}(Y_X \cap C = \emptyset) = 1 - \mathbb{P}(X \cap \mathcal{F}_C = \emptyset), \quad C \in \mathcal{C}^d.$$

The most important class of point processes is given by the Poisson processes. A point process X on $\mathcal{F}^d$ with intensity measure Λ is called a *Poisson process*, if $\operatorname{card}(X \cap \mathcal{F})$ is, for each $\mathcal{F} \in \mathcal{B}(\mathcal{F}^d)$ with $\Lambda(\mathcal{F}) < \infty$, a Poisson random variable with mean $\Lambda(\mathcal{F})$, i.e.,

$$\mathbb{P}(\operatorname{card}(X \cap \mathcal{F}) = k) = \mathrm{e}^{-\Lambda(\mathcal{F})} \frac{(\Lambda(\mathcal{F}))^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

For the general theory of Poisson processes and their most important properties, see Daley and Vere-Jones (1988), or Karr (1986). Here, we only mention some existence and uniqueness results.

Theorem 5.3. *Let Λ be a locally finite measure on $\mathcal{F}^d$. Then there is (up to equivalence) a unique Poisson process X on $\mathcal{F}^d$ with intensity measure Λ. X is stationary (isotropic) if and only if Λ is translation invariant (rotation invariant).*

It follows from Theorem 5.1 that the condition

$$\mathbb{P}(\operatorname{card}(X \cap \mathcal{F}) = 0) = \mathrm{e}^{-\Lambda(\mathcal{F})}, \quad \mathcal{F} \in \mathcal{B}(\mathcal{F}^d),$$

already characterises a Poisson process. This can even be generalised slightly.

Theorem 5.4. *A point process X on $\mathcal{F}^d$ is a Poisson process (with intensity measure Λ) if and only if*

$$\mathbb{P}(X \cap \mathcal{F}_C = \emptyset) = \mathrm{e}^{-\Lambda(\mathcal{F}_C)}$$

for all $C \in \mathcal{C}^d$.

In particular, this means that the Poisson property of a point process X on $\mathcal{F}^d$ is already determined by the union set Y_X! For further results on Poisson processes of sets, see Matheron (1975) and Stoyan, Kendall and Mecke (1987).

In view of the theme of this Handbook, we are mainly interested in particle processes on $\mathcal{K}^d$ or $\mathcal{R}^d$. For such processes, the union set Y_X is a random $\mathcal{S}^d$-set.

We first give a decomposition of the intensity measure of a stationary point process X on $\mathcal{C}^d$. Let $z : \mathcal{C}^d \to \mathbb{R}^d$ be a mapping which supplies each set $C \in \mathcal{C}^d$ with a center $z(C)$ in a motion covariant manner. We will use as $z(C)$ the midpoint of the circumsphere of C (on $\mathcal{R}^d$, the Steiner point is another reasonable choice). z is easily seen to be continuous on $\mathcal{C}^d$. Let

$$\mathcal{C}_0^d = \{C \in \mathcal{C}^d : z(C) = 0\}$$

be the set of centred particles (the sets $\mathcal{R}_0^d$ and $\mathcal{K}_0^d$ are defined analogously). Then

$$\varphi : \mathcal{C}^d \to \mathcal{C}_0^d \times \mathbb{R}^d, \qquad C \mapsto (C - z(C), z(C)),$$

is a homeomorphism. Let $\Lambda' = \Lambda \circ \varphi^{-1}$ be the image of Λ under φ, Λ' is thus a measure on $\mathscr{C}_0^d \times \mathbb{R}^d$. For stationary X, Λ' is translation invariant in the second coordinate, hence it is of the form

$$\Lambda' = \rho \otimes \lambda_d .$$

Since Λ was assumed to be locally finite, ρ is a finite measure on $\mathscr{C}_d^0$. In case $\Lambda \not\equiv 0$, ρ can be normalised to a probability measure.

Theorem 5.5. *Let X be a stationary point process on $\mathscr{C}^d$ with intensity measure $\Lambda \not\equiv 0$. Then there is a $\lambda \in (0, \infty)$ and a probability measure $\mathbb{P}_0$ on $\mathscr{C}_0^d$ with*

$$\Lambda = \lambda \cdot (\mathbb{P}_0 \otimes \lambda_d) \circ \varphi .$$

λ and $\mathbb{P}_0$ are uniquely determined by Λ. If X is isotropic, then $\mathbb{P}_0$ is rotation invariant.

We call λ the *intensity* of X and $\mathbb{P}_0$ the *shape distribution*. The interpretation of the latter is obvious, $\mathbb{P}_0$ is the distribution of a "typical" particle of X. For λ, the interpretation is given in the next theorem.

Theorem 5.6. *Let X be a stationary point process on $\mathscr{C}^d$. Then*

$$\lambda = \frac{1}{\kappa_d} \mathbb{E}\,\mathrm{card}\{C \in \mathscr{C}^d \colon C \in X, z(C) \in B^d\}$$

and

$$\lambda = \lim_{r \to \infty} \frac{1}{V_d(rK)} \mathbb{E}\,\mathrm{card}(X \cap \mathscr{F}_{rK})$$

for all $K \in \mathscr{K}^d$ with $V_d(K) > 0$.

The proof of the first equation follows from

$$\mathbb{E}\,\mathrm{card}(X \cap \{C \in \mathscr{C}^d \colon z(C) \in B^d\}) = \lambda \cdot \lambda_d(B^d)\mathbb{P}_0(\mathscr{C}_0^d) = \lambda \cdot \kappa_d .$$

To prove the second, we assume $K \in \mathscr{K}_0^d$. Then

$$\mathbb{E}\,\mathrm{card}(X \cap \mathscr{F}_{rK}) = \lambda \int_{\mathscr{C}_0^d} \lambda_d(A_r(C))\,\mathrm{d}\mathbb{P}_0(C),$$

where

$$A_r(C) = \{x \in \mathbb{R}^d \colon (C + x) \cap rK \neq \emptyset\} .$$

Since

$$\lim_{r \to \infty} \frac{\lambda_d(A_r(C))}{V_d(rK)} = 1$$

for all $C \in \mathscr{C}_0^d$, the result follows.

For Poisson processes further results are true.

Theorem 5.7. *Let $\lambda \in [0, \infty)$ and let $\mathbb{P}_0$ be a probability measure on $\mathscr{C}_0^d$. Then there is (up to equivalence) a unique stationary Poisson process X on $\mathscr{C}^d$ with intensity λ and shape distribution $\mathbb{P}_0$. X is isotropic, if and only if $\mathbb{P}_0$ is rotation invariant.*

An important property of (general) Poisson point processes is that the different points are independent. For a stationary Poisson process X on $\mathscr{C}^d$ this implies a simple but very useful procedure to simulate X, e.g., on a computer. First, an ordinary Poisson process $\tilde{X}$ of intensity λ in $\mathbb{R}^d$ is simulated (this involves the determination of a random number according to the appropriate Poisson distribution and afterwards a simulation of uniformly distributed points in a region). Then, to each point x of $\tilde{X}$ a random set X_x with distribution $\mathbb{P}_0$ is added independently. The resulting configuration is a realisation of X.

We now concentrate on processes X on $\mathscr{R}^d$ and aim to introduce quermass densities $D_j(X)$ of X. Again, we need an integrability condition. Let

$$\mathbb{E}2^{N(X_0)} < \infty ,$$

where X_0 is a random set with distribution $\mathbb{P}_{X_0} = \mathbb{P}_0$. For brevity, we sometimes denote the expectation $\mathbb{E}f(X_0)$ by $\bar{f}$.

In contrast to section 4, we now have the possibility of a direct definition of $D_j(X)$. We call $D_j(X) = \lambda \cdot \mathbb{E}V_j(X_0)$ the *jth quermass density* of X, $j = 0, \ldots, d$. The following theorem shows that the other approaches from section 4 lead to the same quantity.

Theorem 5.8. *Let X be a stationary point process on $\mathscr{R}^d$. Then, for $j = 0, \ldots, d$,*

$$\mathbb{E} \sum_{C \in X} \Phi_j(C, \cdot) = D_j(X) \cdot \lambda_d , \tag{16}$$

$$\lim_{r \to \infty} \frac{\mathbb{E} \sum_{C \in X} V_j(C \cap rK)}{V_d(rK)} = D_j(X) \quad \text{for all } K \in \mathscr{K}^d \text{ with } V_d(K) > 0 , \tag{17}$$

$$\mathbb{E} \sum_{C \in X} [V_j(C \cap C_0) - V_j(C \cap \partial^+ C_0)] = D_j(X) . \tag{18}$$

The proofs are easier here, since eq. (15) can be used. If X is ergodic, the second equation holds almost surely, i.e., without the expectation sign (Nguyen and Zessin 1979).

The transfer of the integral geometric formulae now proceeds without problems. We get the following versions of the Principal Kinematic Formula and the Crofton Formula.

Theorem 5.9. *Let X be a stationary and isotropic point process on $\mathcal{R}^d$, and let $K \in \mathcal{K}^d$. Then, for $j = 0, \ldots, d$,*

$$\mathbb{E} \sum_{C \in X} V_j(C \cap K) = \sum_{k=j}^{d} \alpha_{djk} V_k(K) D_{d+j-k}(X). \tag{19}$$

Theorem 5.10. *Let X be a stationary and isotropic point process on $\mathcal{R}^d$, and let $L \subset \mathbb{R}^d$ be a q-dimensional subspace. Then $X \cap L$ is also a stationary and isotropic point process on $\mathcal{R}^q$ (in L), and for $j = 0, \ldots, q$*

$$D_j(X \cap L) = \alpha_{djq} D_{d+j-q}(X). \tag{20}$$

Both formulae follow from

$$\mathbb{E} \sum_{C \in X} \Phi_j(C \cap K, A) = \sum_{k=j}^{d} \alpha_{djk} \Phi_k(K, A) D_{d+j-k}(X), \quad A \in \mathcal{B}_d,$$

which is a consequence of eq. (15) for $f(C) = \Phi_k(C, \cdot)$. Equation (19) is obtained with $A = \mathbb{R}^d$. For eq. (20), let $K = B^q$ be the unit ball in L and A the relative interior of B^q. Then

$$\mathbb{E} \sum_{C \in X} \Phi_j(C \cap K, A) = \mathbb{E} \sum_{C \in X \cap L} \Phi_j(C \cap K, A) = D_j(X \cap L) \cdot \kappa_q$$

and

$$\Phi_k(K, A) = 0, \quad k = 0, \ldots, q-1; \qquad \Phi_q(K, A) = \kappa_q.$$

The last results have been derived in a more general setting, for point processes of cylinders, in Weil (1987). There, formulae are also given for stationary, nonisotropic processes. Point processes on more general classes of sets are treated in Zähle (1982, 1986).

For a stationary Poisson process X on $\mathcal{C}^d$, the union set Y_X is called a *Boolean model*. Here, the capacity functional T_{Y_X} can be calculated,

$$T_{Y_X}(C) = \mathbb{P}(Y_X \in \mathcal{F}_C) = 1 - \mathbb{P}(X \cap \mathcal{F}_C = \emptyset) = 1 - e^{-\Lambda(\mathcal{F}_C)}$$

with

$$\Lambda(\mathcal{F}_C) = \lambda \int_{\mathcal{C}_0^d} \int_{\mathbb{R}^d} 1_{\mathcal{F}_C}(K + x)\, d\lambda_d(x)\, d\mathbb{P}_0(K)$$

$$= \lambda \int_{\mathcal{C}_0^d} \lambda_d(K + \check{C})\, d\mathbb{P}_0(K)$$

(where $\check{C}$ is the set C reflected in the origin). If X is moreover an isotropic process on $\mathcal{K}^d$ and if $K \in \mathcal{K}^d$, we can simplify the latter formula because of (9).

Theorem 5.11. *Let X be a stationary and isotropic Poisson process on $\mathcal{K}^d$ and $K \in \mathcal{K}^d$. Then*

$$-\ln(1 - T_{Y_X}(K)) = \sum_{k=0}^{d} \alpha_{d0k} V_k(K) D_{d-k}(X). \tag{21}$$

For isotropic Boolean models (in $\mathcal{S}^d$), a connection between the quermass densities of Y_X and those of X can also be given. The derivation of this result uses the additivity of V_j, the independence properties of Poisson processes X, and the corresponding product form of the moment measures of X.

Theorem 5.12. *Let X be a stationary Poisson process on $\mathcal{R}^d$, $j \in \{0, \ldots, d\}$ and $K \in \mathcal{K}^d$. Then*

$$\mathbb{E}V_j(Y_X \cap K) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \lambda^k \int_{\mathcal{R}_0^d} \cdots \int_{\mathcal{R}_0^d} F_j(K, K_1, \ldots, K_k)\, d\mathbb{P}_0(K_1) \cdots d\mathbb{P}_0(K_k), \tag{22}$$

with

$$F_j(K, K_1, \ldots, K_k) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} V_j(K \cap (K_1 + x_1) \cap \cdots \cap (K_k + x_k))\, d\lambda_d(x_1) \cdots d\lambda_d(x_k). \tag{23}$$

Moreover, for isotropic X, we have

$$F_j(K, K_1, \ldots, K_k) = \int_{G^d} \cdots \int_{G^d} V_j(K \cap g_1 K_1 \cap \cdots \cap g_k K_k)\, d\mu(g_1) \cdots d\mu(g_k). \tag{24}$$

By iteration of the Principal Kinematic Formula it is therefore possible, in the isotropic case, to express $\mathbb{E}V_j(Y_X \cap K)$ by (22) and (24) in terms of the quermassintegrals of K and the quermass densities of X. Theorem 4.2 then implies corresponding formulae for the quermass densities $D_j(Y_X)$. Here we give only the latter formulae.

Corollary 5.13. *The quermass densities of the Boolean model Y_X fulfill*

$$D_d(Y_X) = 1 - e^{-D_d(X)},$$

$$D_{d-1}(Y_X) = D_{d-1}(X)\, e^{-D_d(X)},$$

and

$$D_j(Y_X) = e^{-D_d(X)}\Big[D_j(X) + \sum_{k=2}^{d-j} \frac{(-1)^{k-1}}{k!} \sum_{\substack{m_1,\dots,m_k=j+1 \\ m_1+\cdots+m_k=(k-1)d+j}}^{d-1} c^{(j)}_{m_1,\dots,m_k} D_{m_1}(X)\cdots D_{m_k}(X)\Big],$$

with

$$c^{(j)}_{m_1,\dots,m_k} = \frac{d!\kappa_d}{j!\kappa_j}\prod_{i=1}^{k}\frac{m_i!\kappa_{m_i}}{d!\kappa_d}$$

for $j = 0,\dots,d-2$.

The most interesting cases for applications are of course $d = 2$ and $d = 3$ and there the results look less complicated. For $d = 2$ we use the notation A_A, L_A and χ_A to denote the area density, density of the boundary length and density of the characteristic of Y_X, as well as $\bar{A}$, $\bar{L}$ and $\bar{\chi}$ for the integrals of these functionals with respect to $\mathbb{P}_0$. In three dimensions a similar notation is used. The resulting formulae are then the following:

$$A_A = 1 - e^{-\lambda\bar{A}},$$

$$L_A = \lambda\bar{L}\, e^{-\lambda\bar{A}}, \tag{25}$$

$$\chi_A = e^{-\lambda\bar{A}}\Big(\lambda\bar{\chi} - \frac{1}{4\pi}\lambda^2\bar{L}^2\Big),$$

for $d = 2$ and

$$V_V = 1 - e^{-\lambda\bar{V}},$$

$$S_V = \lambda\bar{S}\, e^{-\lambda\bar{V}},$$

$$M_V = e^{-\lambda\bar{V}}\Big(\lambda\bar{M} - \frac{\pi^2}{32}\lambda^2\bar{S}^2\Big), \tag{26}$$

$$\chi_V = e^{-\lambda\bar{V}}\Big(\lambda\bar{\chi} - \frac{1}{4\pi}\lambda^2\bar{M}\bar{S} + \frac{\pi}{384}\lambda^3\bar{S}^3\Big),$$

for $d = 3$.

Our approach in this latter part of the section followed that of Weil and Wieacker (1984), but similar considerations in different generality are due to Matheron (1975), Miles (1976), Davy (1976, 1978), A. Kellerer (1983, 1985), H. Kellerer (1984), and Zähle (1986). In the nonisotropic case, (22) and (23) can also be used, but then iterations of the translative version of the Principal Kinematic Formula are necessary. The resulting formulae for the quermass densities of Y_X look similar to those in Corollary 5.13 but involve mixed densities (Weil 1990). In two and three dimensions and for Poisson processes X on $\mathcal{K}^d$ with some symmetry conditions these mixed functionals can be expressed as mixed volumes of convex mean bodies associated with X. This allows the application of classical inequalities to formulate and solve some extremal properties of Boolean models (Weil 1988). For example, for a stationary Poisson process X on $\mathcal{K}^2$, the smallest value of the density χ_V of the Euler characteristic of the union set Y_X is obtained if X is almost surely a process of homothetic equilateral triangles (Betke and Weil 1991). It is open, whether these are the only extremal processes. If the particles of the Poisson process X are all convex, a similar formula holds, without isotropy conditions, for the density χ_V^+ (respectively χ_A^+) of the "lower points of convexity" (specific convexity number) of Y_X (Stoyan, Kendall and Mecke 1987, p. 78).

6. Random surfaces

In the preceding sections, convex bodies were used to construct random sets and point processes. Random surfaces are implicit in this theory, e.g., they occur as boundaries bd X of random $\mathcal{S}^d$-sets X. Here, in this section, we will discuss a different connection between convex geometry and random surfaces, which is based on the integral geometric results discussed in chapter 5.1, section 6, and in which convex bodies occur as a secondary notion associated with a random surface X. Therefore, more general random closed sets X will be considered that have realisations in the space $\mathcal{LC}_m$ of locally countable m-rectifiable closed subsets of $\mathbb{R}^d$.

A random closed set $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{F}^d$ is called a *random m-surface* (*random hypersurface* if $m = d-1$ and *random curve* if $m = 1$) if $X \in \mathcal{LC}_m$ almost surely and $\mathbb{E}\lambda_m(X \cap K) < \infty$ for all $K \in \mathcal{K}^d$. If X is stationary, then its distribution $\mathbb{P}_X$ is a stationary (i.e., translation invariant) probability measure on $\mathcal{LC}_m$ and all results about stationary σ-finite measures on $\mathcal{LC}_m$ mentioned in chapter 5.1, section 6, apply also to X. In particular, we may associate with X the two auxiliary zonoids $\Pi^m(X) := \Pi^m(\mathbb{P}_X)$ and $\Pi_m(X) := \Pi_m(\mathbb{P}_X)$ defined there. The most important real parameter of a stationary random m-surface X is the density $D_m(X)$ defined by $\mathbb{E}\lambda_m(X \cap A) = D_m(X)\lambda_d(A)$, for $A \in \mathcal{B}(\mathbb{R}^d)$. This generalises the quermass density $D_m(X)$ used in previous sections for random $\mathcal{S}^d$-sets. It is natural to call $D_m(X)$ the (m-dimensional) surface area density of X; it is related to the above-mentioned zonoids by the identities

$$\tfrac{1}{2}D_m(X) = \frac{\kappa_{d-m-1}}{(d-m)\kappa_{d-m}} V_1(\Pi^m(X)) = \frac{\kappa_{m-1}}{m\kappa_m} V_1(\Pi_m(X)). \tag{27}$$

The support functions $h(\Pi^m(X),\cdot)$ and $h(\Pi_m(X),\cdot)$ of the zonoids $\Pi^m(X)$ and $\Pi_m(X)$ are closely related to the "rose of length of orthogonal intersections" and the "rose of number of intersections" considered by Pohlmann, Mecke and Stoyan (1981). They describe in a certain sense mean first order properties of the random surface. For $u \in S^{d-1}$ we have

$$\mathbb{E}\lambda_{m-1}(X \cap A \cap L(u)) = 2h(\Pi_m(X), u)\lambda_{d-1}(A \cap L(u)) \tag{28}$$

whenever $A \in \mathcal{B}(\mathbb{R}^d)$ and $L(u)$ is a hyperplane orthogonal to u. Moreover, a slightly modified version of Theorem 6.4 in chapter 5.1 and the isoperimetric inequality for the Minkowski area relative to a convex body show that, among all convex bodies K with fixed positive volume, $\mathbb{E}\lambda_{m-1}(X \cap \text{bd } K)$ attains a minimum if and only if K is homothetic to $\Pi_m(X)$. For $m < d-1$, there is a similar but more complicated interpretation of $h(\Pi^m(X), u)$ involving projected thick sections (Wieacker 1989). In the case $m = d-1$, where the role of $\Pi^m(X)$ is particularly important, there is a natural analogue of (28). We shall first consider this case.

Let X be a stationary random hypersurface. Then, denoting by $[a, b]$ the segment joining the points $a, b \in \mathbb{R}^d$, we have

$$h(\Pi^{d-1}(X), x) = \tfrac{1}{2}\mathbb{E}\,\text{card}(X \cap [0, x]) \tag{29}$$

for each $x \in \mathbb{R}^d$ (see Wieacker 1986, for details). In particular, if $\Pi^{d-1}(X)$ is not degenerated, then (29) implies that the map $x \mapsto \frac{1}{2}\mathbb{E}\,\text{card}(X \cap [0, x])$ is a norm in $\mathbb{R}^d$, the unit ball of which is the polar body $\Pi^{d-1}(X)^*$ of $\Pi^{d-1}(X)$. The behaviour of X in the Minkowski geometry corresponding to this norm is in some sense similar to the behaviour of a stationary and isotropic random hypersurface in Euclidean geometry. For a unit vector u, $h(\Pi^{d-1}(X), u)$ may also be viewed as the intersection density of X in the direction u. If the intersection density in a given direction is large and the random hypersurface is considered as opaque, then one may expect that the visible distance in the same direction is not too large. This observation leads to a second interpretation of $h(\Pi^{d-1}(X), u)$. More precisely, for $u \in S^{d-1}$ and $r \geq 0$ let $\varphi_u(r) := \mathbb{P}([0, ru] \cap X = \emptyset)$ be the probability that the visible distance from 0 in the direction u is at least r. A straightforward argument shows that the function φ_u is convex, and that the right derivative $\varphi_u'(0)$ of φ_u at the origin 0 exists. It turns out that

$$\varphi_u'(0) = -2h(\Pi^{d-1}(X), u)$$

for all $u \in S^{d-1}$. This shows that, for each $\varepsilon > 0$, $\Pi^{d-1}(X)$ is uniquely determined by the values of the capacity functional T_X of X on the set $\{[0, x]: \|x\| < \varepsilon\}$. For any opaque subset $A \subset \mathbb{R}^d$ let $S_A := \{y \in \mathbb{R}^d: [0, y] \cap A = \emptyset\}$ be the open star-shaped set of all points which are visible from the origin. Then, $\lambda_d(S_X)$ is a random variable and by Fubini's theorem we have

$$\mathbb{E}(\lambda_d(S_X)) = \int_{\mathbb{R}^d} \mathbb{P}([0, x] \cap X = \emptyset)\, d\lambda_d(x)$$
$$= \int_{S^{d-1}} \int_0^\infty r^{d-1}\varphi_u(r)\, dr\, d\lambda_{d-1}(u)\,.$$

A rough estimation of $\varphi_u(r)$ yields

$$\mathbb{E}(\lambda_d(S_X)) \geq \frac{1}{2^d(d+1)} V_d(\Pi^{d-1}(X)^*)\,.$$

Here the best numerical constant in the inequality is still unknown. Similar results for a stationary RACS (the boundary of which is a random hypersurface) and proofs may be found in Wieacker (1986). More precise results for special types of random hypersurfaces are given in Theorem 6.2 and in the next section. Visibility properties have been studied also in Serra (1982), and Yadin and Zacks (1985).

Random hypersurfaces may be used to generate lower-dimensional random sets. For instance, the intersection of a stationary random hypersurface with an m-dimensional C^1 submanifold (or more generally with some countably m-rectifiable Borel subset) M of $\mathbb{R}^d$ is an $(m-1)$-dimensional random closed subset of M. The expected λ_{m-1}-measure of this random set depends only on M and $\Pi^{d-1}(X)$.

Theorem 6.1. *Let X be a stationary random hypersurface. Then, for any m-dimensional submanifold M of class C^1, we have*

$$\mathbb{E}\lambda_{m-1}(X \cap A) = \frac{1}{\kappa_{m-1}} \int_A \int_{S(T_yM)} h(\Pi^{d-1}(X), x)\, d\lambda_{m-1}(x)\, d\lambda_m(y)$$

for each Borel subset A of M, where T_yM is the tangent space of M at y and $S(T_yM) := T_yM \cap S^{d-1}$.

Since the integral over $S(T_yM)$ is essentially the mean width of the orthogonal projection of $\Pi^{d-1}(X)$ on T_yM, Theorem 6.1 may be viewed as an analogue of (27). It should be also noticed that the integral over $S(T_yM)$ as a function of y is constant in M if $\Pi^{d-1}(X)$ is a ball (for instance if X is isotropic) or if M is an affine subspace of $\mathbb{R}^d$. In both cases, the measures $\mathbb{E}\lambda_{m-1}(X \cap \cdot)$ and λ_m, when restricted to $\mathcal{B}(M)$, are proportional. We may also consider the intersection of several random hypersurfaces. If a random m-surface Y is almost surely the intersection of $d-m$ independent stationary random hypersurfaces $X_1, \ldots, X_{d-m}$, then a slightly modified version of Theorem 6.3 in chapter 5.1 shows that $\Pi_m(Y)$ is essentially the mixed projection body of $\Pi^{d-1}(X_1), \ldots, \Pi^{d-1}(X_{d-m})$ and B^d $(m-1$ times).

The case where the random hypersurface X is generated by a Poisson process on $\mathcal{LC}_{d-1}$ (see section 5) is of particular interest. Suppose that X is the union set X_Y where Y is a stationary Poisson process on $\mathcal{LC}_{d-1}$ with intensity measure Λ. Then we have $\Pi^{d-1}(X_Y) = \Pi^{d-1}(\Lambda)$, where $\Pi^{d-1}(\Lambda)$ is defined as in chapter 5.1, section 6. For $\omega \in \Omega$ and $k \in \mathbb{N}$ let

$$X_Y^k(\omega) := \{x \in \mathbb{R}^d: Y(\omega)(\mathcal{F}_{(x)}) \geq d-k\}$$

be the set of points belonging to at least $d-k$ elements of Y. Then X_Y^k is a

k-dimensional random closed set and we have

$$D_k(X_Y^k) = V_{d-k}(\Pi^{d-1}(X_Y)), \tag{30}$$

where the density $D_k(X_Y^k)$ is again defined by $\mathbb{E}\lambda_k(X_Y^k \cap A) = D_k(X_Y^k)\lambda_d(A)$, $A \in \mathcal{B}(\mathbb{R}^d)$. Hence, eq. (27) and the Minkowski–Fenchel–Aleksandrov inequalities for the intrinsic volumes imply

$$D_k(X_Y^k) \leq \binom{d}{k} \frac{\kappa_d}{\kappa_k} \left(\frac{\kappa_{d-1}}{d\kappa_d} D_{d-1}(X_Y)\right)^{d-k}, \tag{31}$$

with equality if and only if $\Pi^{d-1}(X_Y)$ is a ball (this is the case, for instance, if the Poisson process Y is isotropic). In the case where X is generated by a Poisson process of hyperplanes, (30) is due to Matheron (1975) and (31) is due to Thomas (1984) (the general case is treated in Wieacker 1986). These densities, and in particular $D_0(X_Y^0)$, measure in a certain sense the denseness of the process. If the elements of the Poisson process are the boundaries of nondegenerate convex bodies, more precise results can be obtained. For a particle process X, we denote by bd X the corresponding process of boundary sets. If X is a Poisson process then also bd X, and bd X has the same invariance properties as X.

Theorem 6.2. *Let Z be a stationary Poisson process of nondegenerate convex bodies with locally finite intensity measure Λ and such that $0 < \mathbb{E}\lambda_{d-1}(\text{bd } Z \cap [0,1]^d) < \infty$. Then the conditional expectation $\mathbb{E}(\lambda_d(S_{X_Z}) \mid 0 \notin X_Z)$ fulfills*

$$\mathbb{E}(\lambda_d(S_{X_Z}) \mid 0 \notin X_Z) = d!V_d(\Pi^{d-1}(X_{\text{bd } Z})^*)$$
$$\geq d!\kappa_d \left(\frac{\kappa_{d-1}}{d\kappa_d} D_{d-1}(X_{\text{bd } Z})\right)^{-d},$$

with equality if and only if $\Pi^{d-1}(X_{\text{bd } Z})$ is a ball (in particular if Z is isotropic). Moreover, we have

$$4^d \leq D_0(X^0_{\text{bd } Z})\mathbb{E}(\lambda_d(S_{X_Z}) \mid 0 \notin X_Z) \leq d!\kappa_d^2,$$

with equality on the left if and only if $\Pi^{d-1}(X_{\text{bd } Z})$ is a parallelotope, and equality on the right if and only if $\Pi^{d-1}(X_{\text{bd } Z})$ is an ellipsoid.

The lower bound for the mean visible volume is a consequence of (27) and Jensen's inequality. In the second assertion, both inequalities and the corresponding equality cases follow as in section 2. Similar results have been obtained for the random mosaics generated by Poisson hyperplane processes. A common generalisation of these results for Poisson processes of cylinders and further inequalities involving the intensity measure are due to Schneider (1987b). Random surfaces dividing the space into convex polytopes (random mosaics), and in particular stationary Poisson hyperplane networks, are treated separately in section 7.

For stationary random m-surfaces with $1 < m < d-1$, it is generally difficult to get results of this type, even in the case where the random m-surface is generated by a Poisson process of m-flats. Here analytical methods seem to be more successful (see Matheron 1975, Mecke and Thomas 1986, Goodey and Howard 1990a,b). The case $m = 1$ is more accessible. In particular, if X is the union set of a stationary Poisson process of straight lines, then X is uniquely determined by $\Pi_1(X)$ (up to equivalence). Moreover, if we denote by Y the random hypersurface of all points which have distance r from at least one line of the process generating X, then Y is the union set of a stationary Poisson process of boundaries of cylinders, and we have

$$\Pi^{d-1}(Y) = 2\kappa_{d-2}r^{d-2}\Pi^1(X).$$

Hence, Schneider's results about Poisson processes of cylinders (Schneider 1987b) lead to several stochastic interpretations of the parameters of $\Pi^1(X)$. For random curves a few inequalities have been obtained (Wieacker 1989). In the planar case, additional results are due to the fact that $\Pi^1(X)$ is obtained by rotating $\Pi_1(X)$ by the angle $\frac{1}{2}\pi$. For instance, if C is a fixed curve given as the image $C = f([0,1])$ of some injective Lipschitzian function $f: [0,1] \to \mathbb{R}^2$ and if Y is a stationary Poisson process of translates of C with finite intensity λ, then (30) may be combined with an inequality for $V_2(\Pi_1(X))$ to get

$$D_0(X_Y^0) \geq 2V_2(\text{conv } C).$$

Here equality holds, for instance, if C is a half circle. Further results about random curves and random surfaces of intermediate dimension which are closely related to convex geometry may be found in Wieacker (1989).

Concerning the classical part of the theory which is not particularly related to convex geometry, we refer to Stoyan, Kendall and Mecke (1987, chapter 9) and the literature quoted there. Random processes of Hausdorff rectifiable closed sets were first investigated in Zähle (1982). While the theory described here is based on first order tangential properties, a second order theory based on Federer's sets with positive reach is also due to Zähle (1986). A very different aspect of the theory of random surfaces is treated in Wschebor (1985).

7. Random mosaics

A hypersurface $F \in \mathcal{LC}_{d-1}$ is called a *mosaic* if $\mathbb{R}^d \setminus F$ is a locally finite union of bounded, disjoint open convex sets the closures of which are called the cells of the mosaic. Here, "locally finite" means that each compact subset of $\mathbb{R}^d$ meets only a finite number of cells. From the definition it follows that the cells are convex polytopes, and the k-faces of these polytopes are called k-faces of the mosaic. A random hypersurface $X: (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{LC}_{d-1}$ is called a *random mosaic* if X is almost surely a mosaic (the terms "mosaic" and "random mosaic" are sometimes used in a more general sense). Since X is also uniquely determined by the random measure ξ_X satisfying

$$\xi_X(\omega, U) := \text{card}\{K \in U:\ K \text{ is a cell of } X(\omega)\}$$

for each $\omega \in \Omega$ and each $U \in \mathcal{B}(\mathcal{K}^d)$, a random mosaic may also be viewed as a particle process (concentrated on the space $\mathcal{P}^d$). We call ξ_X the point process associated with the random mosaic X. Other particle processes connected with X are the processes of the k-faces, $k \in \{0, \ldots, d-1\}$. However, for $k \leq d-2$, they do not determine X uniquely.

Classical examples of random mosaics are obtained from a stationary Poisson point process Y with finite intensity in $\mathbb{R}^d$. We may, for instance, associate with each point $y \in Y(\omega)$ its Voronoi cell, i.e., the set of all points $z \in \mathbb{R}^d$ which fulfill $\|z - y\| \leq \|z - \tilde{y}\|$ for all $\tilde{y} \in Y(\omega)$. The random mosaic obtained in this way is called the *Voronoi mosaic* associated with the Poisson process Y, it is stationary and isotropic. This type of random mosaic and related models in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been investigated, for instance, by Meijering (1953), Gilbert (1962), and Miles (1970). Examples of stationary random mosaics which are not necessarily isotropic are the nondegenerate Poisson hyperplane networks treated at the end of this section. The role of convex geometry in the theory of stationary random mosaics has two different aspects. On the one hand, each realisation of a random mosaic may be viewed as an aggregate of convex polytopes and hence as a natural object of study in convex geometry. On the other hand, a stationary random mosaic is a stationary random hypersurface, and hence the auxiliary zonoids introduced in the preceding section may be used to describe its behaviour in many situations. Here the emphasis will be on the relations between both aspects.

Let X be a stationary random mosaic and assume that the intensity measure of the associated point process ξ_X on $\mathcal{P}^d$ is locally finite. Thus, the expected number of cells meeting a compact set is finite. Then two random polytopes may be associated with X. On the one hand, the shape distribution $\mathbb{P}_0$ of ξ_X defined in section 5 is the distribution of a random set Q_X called the typical cell of the random mosaic. Typical k-faces, $k = 0, \ldots, d-1$, may be defined in a similar way. These notions have their origin in the theory of Palm measures, which proved very useful in the investigation of random mosaics (see, e.g., Mecke 1980, and Møller 1989). For $d = 2$ and $d = 3$ many relationships between mean values concerning the typical faces and other mean values concerning the stationary random mosaics are due to Mecke (1984a) (extensions to random mosaics with not necessarily convex cells may be found in Stoyan 1986 and Weiss and Zähle 1988). On the other hand, since X is stationary, the origin 0 belongs almost surely to exactly one cell of X denoted by R_X. R_X is a random polytope called the 0-cell of X. Between the distribution of the typical cell and the distribution of the 0-cell we have the relation

$$\int_\Omega f(R_X(\omega))\, d\mathbb{P}(\omega) = \left(\int_{\mathcal{P}^d} V_d(K)\, d\mathbb{P}_0(K)\right)^{-1} \int_{\mathcal{P}^d} f(K) V_d(K)\, d\mathbb{P}_0(K)$$

for each measurable and translation invariant function $f \geq 0$ on $\mathcal{P}^d$ (for the special case of a stationary Poisson hyperplane network see the remarks concerning the number law and the volume law in Matheron 1975, section 6.2). On the other hand, if Λ denotes the intensity measure of ξ_X and f is a nonnegative measurable function on $\mathcal{P}^d$, then we also have

$$\int_{\mathcal{P}^d} f(K)\, d\Lambda(K) = \int_\Omega V_d(R_X(\omega))^{-1} \int_{\mathbb{R}^d} f(R_X(\omega) + y)\, d\lambda_d(y)\, d\mathbb{P}(\omega)\,. \tag{32}$$

In fact, R_X is just the closure of the random open set S_X defined in the last section. Since the random closed set $Y := \mathbb{R}^d \setminus S_X$ is (up to equivalence) uniquely determined by its capacity functional T_Y (Theorem 4.1) and

$$T_Y(C) = 1 - \mathbb{P}(C \subset S_X) = T_X(\text{conv}(\{0\} \cup C))$$

for each $C \in \mathcal{C}^d$, it follows that R_X is uniquely determined by the values of T_X on the set of convex bodies containing the origin and vice versa. Consequently, the zonoid $\Pi^{d-1}(X)$ associated with the random hypersurface X (see section 6) is uniquely determined by R_X, and from (32) we infer that

$$2\Pi^{d-1}(X) = \mathbb{E}(V_d(R_X)^{-1}\Pi^{d-1}(\text{bd}\, R_X))\,, \tag{33}$$

where $\Pi^{d-1}(\text{bd}\, R_X)$ is the zonoid associated with bd R_X (see chapter 5.1) and on the right-hand side the expectation of the random set $V_d(R_X)^{-1}\Pi^{d-1}(\text{bd}\, R_X)$ is defined as in section 3. Since the mean width of $\Pi^{d-1}(X)$ is essentially the surface area density of X, this implies

$$D_{d-1}(X) = \mathbb{E}\left(\frac{V_{d-1}(R_X)}{V_d(R_X)}\right).$$

Further, since $\Pi^{d-1}(\text{bd}\, R_X)$ is the projection body of R_X in the usual sense, (33) gives some information concerning the relation between R_X and the inverse projection body $\Psi^{d-1}(X)$ of $\Pi^{d-1}(X)$. $\Psi^{d-1}(X)$ is the unique centrally symmetric convex body centred at the origin, the surface area measure of which is the generating measure of $\Pi^{d-1}(X)$ (see chapter 4.9). The relation between both convex bodies is rather complicated because of the factor $V_d(R_X)^{-1}$ in (33).

Theorem 7.1. *If X is a stationary random mosaic in $\mathbb{R}^d$, then*

$$2^d V_d(\Psi^{d-1}(X))^{d-1} \geq \mathbb{E}(V_d(R_X))^{-1}\,,$$

with equality if and only if for some convex polytope K, X is almost surely the boundary of a tiling of $\mathbb{R}^d$ by translates of K, and in this case K is homothetic to $\Psi^{d-1}(X)$.

This inequality, which has no analogue in the general case of a stationary random surface, is of interest because lower bounds involving $V_d(\Psi^{d-1}(X))$ appear in several extremal problems related to Theorem 6.4 in chapter 5.1. Both theorems together and classical results about mixed volumes lead to further inequalities for random mosaics. For a simple rectifiable curve K, for instance, we get a sharp inequality involving the mean number of intersection points of X and K, the volume of the convex hull conv K of K and the expected volume of the 0-cell, namely

$$\mathbb{E}(\mathrm{card}(X \cap K))^d \geq \frac{d! V_d(\mathrm{conv}\, K)}{d^d \mathbb{E}(V_d(R_X))} .$$

Further results of this type and a proof of the theorem may be found in Wieacker (1989).

In the important special case of a nondegenerate stationary Poisson hyperplane network more precise results may be obtained. A random hypersurface is called a *stationary Poisson hyperplane network* if it is the union set of a stationary Poisson process of hyperplanes in $\mathbb{R}^d$ with locally finite intensity measure. Nondegenerate means that the intensity measure of the Poisson process is not concentrated on the set of hyperplanes parallel to a line. This implies that, for almost all realisations of the process, all cells are bounded. If X is a nondegenerate and stationary Poisson hyperplane network in $\mathbb{R}^d$, then we may use Theorem 5.4 to compute the capacity functional T_X of X. The intensity measure Λ of the underlying hyperplane process can be decomposed into a *directional distribution* $\mathbb{P}_0$ and an intensity λ similar to Theorem 5.5 (compare also Example 2 in chapter 5.1, section 6). If we represent $\mathbb{P}_0$ as an even measure on the unit sphere, $\Lambda_0 = \lambda \mathbb{P}_0$ is precisely the generating measure of $\Pi^{d-1}(X)$ and hence the area measure of $\Psi^{d-1}(X)$. Therefore, we get

$$T_X(K) = 1 - \exp(-dV(\Psi^{d-1}(X), \ldots, \Psi^{d-1}(X), K - K)) ,$$

for all $K \in \mathcal{K}^d$. Here we have used the notation $K - K := \{x - y\colon x, y \in K\}$. This result and the isoperimetric inequality for the Minkowski area of $\Psi^{d-1}(X)$ (relative to a convex body K), where equality holds if and only if $\Psi^{d-1}(X)$ and K are homothetic, lead to a characterisation of $\Psi^{d-1}(X)$ (up to a homothety) as the solution of an extremal problem.

Theorem 7.2. *Let X be a nondegenerate and stationary Poisson hyperplane network in $\mathbb{R}^d$. Then, among all convex bodies K with given volume and containing the origin in their interior, $\mathbb{P}(K \subset R_X)$ attains a maximum if and only if K is homothetic to $\Psi^{d-1}(X)$.*

From the viewpoint of convex geometry, nondegenerate stationary Poisson hyperplane networks are of particular interest. On the one hand, a nondegenerate stationary Poisson hyperplane network X is uniquely determined (up to equivalence) by the intensity measure of the generating Poisson process of hyperplanes, and this intensity measure is uniquely determined by the generating measure of the zonoid $\Pi^{d-1}(X)$. On the other hand, if Z is a nondegenerate zonoid centred at the origin, then we may use the generating measure of Z to construct a nondegenerate stationary Poisson hyperplane network X satisfying $\Pi^{d-1}(X) = Z$. Hence, if we do not distinguish between random closed sets having the same distribution, then the map $X \mapsto \Pi^{d-1}(X)$ provides a bijection between the space of nondegenerate stationary Poisson hyperplane networks and the space of nondegenerate zonoids centred at the origin. It readily follows that a nondegenerate stationary Poisson hyperplane network X is isotropic if and only if $\Pi^{d-1}(X)$ is a ball. Moreover, many geometrical parameters of a nondegenerate zonoid may be expressed in terms of parameters of the corresponding Poisson hyperplane network and conversely. We give a few examples. For a nondegenerate and stationary Poisson hyperplane network X and for $k = 0, \ldots, d-1$ we shall denote by X^k the union set of all k-faces of the cells of X and by $\mathrm{skel}_k\, R_X$ the set of all k-extreme points of R_X. Then X^k is a random k-surface and from (30) we have $D_k(X^k) = V_{d-k}(\Pi^{d-1}(X))$ [moreover, if in inequality (31) Y is a stationary Poisson process of hyperplanes, then equality holds if and only if Y is isotropic]. The following theorem is an analogue of Theorem 6.2.

Theorem 7.3. *Let X be a nondegenerate and stationary Poisson hyperplane network in $\mathbb{R}^d$. Then*

$$\mathbb{E}(V_d(R_X)) = d!2^{-d} V_d(\Pi^{d-1}(X)^*) \geq d! \kappa_d \left(\frac{2\kappa_{d-1}}{d\kappa_d} D_{d-1}(X) \right)^{-d} ,$$

with equality if and only if X is isotropic. Moreover, for $k = 0, \ldots, d-1$, we have

$$\begin{aligned} \mathbb{E}(\lambda_k(\mathrm{skel}_k\, R_X)) &= D_k(X^k)\mathbb{E}(V_d(R_X)) \\ &= d!2^{-d} V_{d-k}(\Pi^{d-1}(X)) V_d(\Pi^{d-1}(X)^*) . \end{aligned}$$

In particular, $(2^d/d)\mathbb{E}(\lambda_0(\mathrm{skel}_0\, R_X))$ is the volume product of $\Pi^{d-1}(X)$, and hence

$$2^d \leq \mathbb{E}(\lambda_0(\mathrm{skel}_0\, R_X)) \leq d! 2^{-d} \kappa_d^2 ,$$

with equality on the left side if and only if $\Pi^{d-1}(X)$ is a parallelotope, and equality on the right side if and only if $\Pi^{d-1}(X)$ is an ellipsoid.

The last statement is closely related to Schneider's results about random polytopes generated by anisotropic hyperplanes (Schneider 1982; see also section 2). Many interesting relations of this type, for instance concerning the expected intrinsic volumes of Q_X, may be found in Matheron (1975, chapter 6). For a convex body K containing the origin, Kaltenbach (1990) studied the asymptotic behaviour of the conditional expectation $\mathbb{E}(V_d(R_X) \mid K \subset R_X)$ as the intensity measure of the Poisson process generating X tends to infinity in a suitable way.

This question is closely related to the problem considered in section 2, and also here the asymptotic behaviour strongly depends on the boundary structure of K.

Since we had to omit many contributions of continuing importance, some remarks concerning the literature about random mosaics are in order here. The survey of Miles (1972) and the collection of papers edited by Harding and Kendall (1974) give an idea of the progress in the sixties and in the early seventies. A detailed investigation of stationary and isotropic Poisson hyperplane networks can be found in the important papers of Miles (1961, 1971b). The treatment of the anisotropic case in the path-breaking work of Matheron (1975) was the starting point for many of the developments in this section and section 6. Random mosaics generated by hyperplanes were investigated by Mecke (1984b) (some of the relations proved there have an interesting deterministic analogue, as was shown by Schneider 1987c). A good account of the applications and the statistical analysis of random mosaics with many references is given in Stoyan, Kendall and Mecke (1987, chapter 10), see also Mecke et al. (1990). A unified exposition of the theory of random mosaics including results on Voronoi mosaics and Delauney mosaics (the dual of Voronoi mosaics) can be found in the paper of Møller (1989). For a different approach based on ergodic theory, see, for instance, Cowan (1980). Far-reaching generalisations for random cell complexes and generalised sets are due to Zähle (1988).

8. Stereology

Stereological problems are currently the main field of applications of stochastic geometry and stereological questions had a strong influence on the development of the theory described in some of the last sections. In the most general formulation, stereology deals with the determination (or estimation) of characteristic geometric properties of (usually three-dimensional) objects by investigations of sections, projections, intersections with test sets, or other transformed images. Such problems are inherent to most of the experimental sciences, whenever a direct investigation of a three-dimensional feature is not possible. Examples are biology, medicine, geology, metallurgy, forestry, to name only a few. Frequently, two-dimensional images are treated in a similar way with two- or one-dimensional test sets. Here similar problems occur in Image Analysis and Spatial Statistics, two fields which have also strong connections to stereology and stochastic geometry. Therefore, and for mathematical simplicity, we will formulate the following presentation in the general d-dimensional setting. We will however only sketch some of the basic stereological problems, for further details we refer to the literature (Weil 1983, Jensen et al. 1985, Stoyan, Kendall and Mecke 1987, Mecke et al. 1990, Stoyan 1990, and Baddeley 1991, are some of the more mathematically oriented references).

To mention a typical stereological example: the direct determination of the specific alveolar surface area S_V of the human (or animal) lung is practically impossible due to the complicated structure of the lung tissue. To estimate the quantity S_V, in pathology usually small (cubical) pieces of lung tissue are cut randomly in (parallel) thin slices which are then examined under the microscope. The resulting 2-dimensional image is then either treated directly or by imposing randomly placed grids of lines or segments. This allows the determination of the mean boundary length per unit area, L_A, of the planar microscopical image. Integral geometric formulae and their random versions now give the connection between the observed values of L_A and the quantity S_V which is to be estimated. The models of stochastic geometry also make precise the conditions under which certain estimators will work.

The stereological literature usually distinguishes between two dual mathematical idealisations of the practical problem, a so-called *designed-based* and a *model-based* approach. In the first, the object of investigation is assumed to be a fixed set (in $\mathbb{R}^3$) which is intersected by randomly chosen planes. In the second approach, the set itself is assumed to be random (with certain invariance properties). Then, sectioning planes with fixed orientations can be used. In the first case, integral geometric formulae can be applied directly after some probabilistic modifications. In the second case, the corresponding results for random sets or point processes have to be used. We shortly describe both situations.

Let $K_0 \subset \mathbb{R}^d$ be a convex body with inner points and let $K \subset K_0$, $K \in \mathcal{R}_d$. We assume that K_0 is a reference set of known shape and size (in applications K_0 usually is a cube or ball), whereas certain geometric quantities of K are to be estimated (the assumption $K \in \mathcal{R}_d$ is not a serious restriction for applications). For this purpose, a random q-dimensional section of K_0 is taken (i.e., a random q-flat X_q intersecting K_0) and the section $X_q \cap K$ is observed. If, for example, X_q is a *uniform isotropic random* flat, then the distribution of X_q is the invariant measure μ_q on the space $\mathscr{E}_q^d$ of q-flats, restricted to $\{E \in \mathscr{E}_q^d: E \cap K_0 \neq \emptyset\}$ and normalised. In that case, the *Crofton Formulae* (see chapter 5.1) lead directly to the following expectation formula for the intrinsic volumes,

$$\mathbb{E}V_j(K \cap X_q) = \frac{\alpha_{djq}}{\alpha_{d0q}} \frac{V_{d+j-q}(K)}{V_{d-q}(K_0)}, \tag{34}$$

$0 \leq j \leq q \leq d-1$. Thus,

$$\frac{\alpha_{d0q}}{\alpha_{djq}} V_{d-q}(K_0) V_j(K \cap X_q)$$

is an unbiased estimator of $V_{d+j-q}(K)$. A disadvantage of formula (34) is that it requires the determination of $V_{d-q}(K_0)$, on the other hand one quite often wants to estimate a quermassintegral of K per unit d-volume of K_0, i.e., an estimator for $V_{d+j-q}(K)/V_d(K_0)$. Here the above formula implies

$$\frac{\mathbb{E}V_j(K \cap X_q)}{\mathbb{E}V_q(K_0 \cap X_q)} = \alpha_{djq} \frac{V_{d+j-q}(K)}{V_d(K_0)}, \tag{35}$$

but of course $f(X_q) = V_j(K \cap X_q)/V_q(K_0 \cap X_q)$ is not an unbiased estimator of the right-hand side. If however, X_q is chosen to be a *volume-weighted random* flat (which is determined by a uniformly distributed random point in K_0 and an independently chosen uniform direction; see chapter 5.1 for more details), then

$$\mathbb{E}\,\frac{V_j(K\cap X_q)}{V_q(K_0\cap X_q)}=\alpha_{djq}\,\frac{V_{d+j-q}(K)}{V_d(K_0)}\,, \tag{36}$$

i.e., the estimator $f(X_q)$ is unbiased.

A number of other formulae from integral geometry have similar stereological interpretations and, after a suitable normalisation of the corresponding invariant measure, give unbiased estimators for certain stereological quantities. As a further example, we mention only the formula for projected thick sections (see chapter 5.1), which is the appropriate model for microscopical images, and which allows the estimation of particle number, a quantity which is not directly accessible with ordinary planar sections.

The model-based approach is simpler since the results from sections 4 and 5 can be used directly, the difficulties with formulae (35) and (36) do not occur. The basic assumption is that the underlying structure is a bounded part of a realisation of a stationary and isotropic random set or particle point process X. Here, the quotients of intrinsic volumes are replaced by the quermass densities. Thus, for a fixed q-dimensional subspace $L\subset\mathbb{R}^d$, the Crofton formula

$$D_j(X\cap L)=\alpha_{djq}D_{d+j-q}(X)\,, \tag{37}$$

[formula (14) in section 4 and (20) in section 5] replaces (35) and (36). In this case, both sides of (37) are expectations. Formulae (10), (11) and (12) [respectively (16), (17) and (18)] show different possibilities for unbiased or asymptotically unbiased estimation of the quermass densities. For example, in the case of a random set X and a convex body K of volume 1, $\Phi_j(X,\operatorname{int}K)$ and $V_j(X\cap C_0)-V_j(X\cap\partial^+C_0)$ are unbiased estimators of $D_j(X)$, whereas the estimator $r^{-d}V_j(X\cap rK)$ is asymptotically unbiased. If $V_j(X\cap K)$ is used instead of $\Phi_j(X,\operatorname{int}K)$, the effects of bd K (the so-called *edge effects*) lead to an error, the mean of which is given by (13). Moreover, if all the intrinsic volumes $V_j(X\cap K)$, $j=0,\ldots,d$, are evaluated, the linear system

$$\mathbb{E}V_j(X\cap K)=\sum_{k=j}^{d}\alpha_{djk}V_k(K)D_{d+j-k}(X)\,,\quad j=0,\ldots,d\,,$$

can be solved for the unknowns $D_i(X)$, $i=0,\ldots,d$, and another set of unbiased estimators for the quermass densities results. Of course, for the stereological situation described earlier, these estimators are applied correspondingly to the section $X\cap L$. Also, there is no principal difference between random sets and particle processes, so overlapping particle systems can be treated in the same way with the results from section 5. We mention that the section formula (37) contains the so-called *Fundamental formulae of stereology*:

$$V_V=A_A=L_L=\chi_X\,,\qquad S_V=\frac{4}{\pi}\,L_A=2\chi_L\,,\qquad M_V=2\pi\chi_A\,, \tag{38}$$

where we have used the notation from the end of section 5.

An important problem for applications is the determination of mean particle quantities for overlapping particle systems X when only the union set $Y=\bigcup_{K\in X}K$ is observable. Here, the formulae (25) and (26) are applicable, provided the underlying assumption of a Boolean model Y (i.e., a stationary and isotropic Poisson process X) is realistic. This is the case, whenever the particles are independently and uniformly distributed in a certain region. If all the particles are simply connected (hence $\bar{\chi}=1$) and if the quermass densities of Y on the left-hand side of (25) and (26) are estimated by the above-mentioned methods, (25) allows successively the estimation of $\lambda\bar{A}$, $\lambda\bar{L}$, and λ, and similarly (26) can be used.

Another important aspect of this method is that Boolean models Y are basic models of random sets Y which can be modified to match real set-valued data. The unknown parameters are then the intensity γ and the shape distribution $\mathbb{P}_0$. The method described above allows the estimation of γ and of some mean values of $\mathbb{P}_0$. It is an interesting problem to get more information on $\mathbb{P}_0$ (in particular, in the nonisotropic case). Here, integral geometric formulae, other than the basic ones, have to be developed.

We finally emphasise that the results described so far give mean values hence first-order information on random sets and point processes. This is due to the nature of the underlying integral geometric results. There are also some less geometric methods to obtain higher-order informations or distributions, but generally the determination of variances, e.g., is a major open problem.

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