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Algebraic D-Modules

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INTRODUCTION

$\mathcal{D}$-module is shorthand for a sheaf of modules over the sheaf of rings of germs of differential operators on a manifold, an algebraisation of the notion of system of linear partial differential equations. The theory of $\mathcal{D}$-modules was developed, in the seventies, first for holomorphic differential operators on complex manifolds. One of its main results is the Riemann-Hilbert correspondence, a far reaching generalization of the existence of meromorphic differential equations in one variable with regular singular points and prescribed monodromy. It attracted the attention of a number of outsiders, such as the undersigned, at first because of some remarkable applications outside the theory of PDE, in particular to infinite dimensional representations of semisimple real Lie groups or to finite dimensional representations of Weyl groups.

The main aim of these Notes is to present a reasonably self-contained account of a variant of that theory, due to A. Beilinson and J. Bernstein, in which the underlying manifold and the differential operators are algebraic. It suffices for the applications to representation theory, while being less demanding with regard to analysis than its holomorphic counterpart. It is the subject matter of Chapters VI, VII and VIII. The proofs of the main theorems make use of earlier results of P. Deligne on the one hand, of J. Bernstein on the other. Those are presented in Parts IV and V. Finally, III is a preliminary to IV and I and II supply or review some foundational material.

Chapters III to VIII are an outgrowth of a seminar held weekly in Bern in the spring 1984, for the Swiss academic mathematical community, while I and II emanate from a preparatory one-week meeting which took place in Les Plans-sur-Bex in March 1984, under the auspices of the "3ème cycle Romand".

We shall now outline more precisely the contents of the various chapters.

I, by P. Grivel, is devoted to derived categories. Those form the natural framework for the whole discussion of $\mathcal{D}$-modules and are an indispensable prerequisite. Chapter I provides a rather detailed introduction, aimed at the non-expert with some background in homological algebra.

Much of the formalism in the theory of $\mathcal{D}$-modules is patterned after that of coherent sheaves or modules in algebraic or analytic geometry. II, by B. Kaup, reviews some of the main definitions and facts pertaining to coherence, especially in the complex analytic case.

A first example of $\mathcal{D}$-module is given by the sheaves of germs of holomorphic sections of holomorphic vector bundles endowed with a flat holomorphic connection. III and IV are devoted to those. III, by A. Haefliger, reviews the theory of meromorphic ordinary differential equations in one variable with regular singular points. IV, by B. Malgrange, describes Deligne's generalization of it to connections on a smooth algebraic variety and solution of the Riemann-Hilbert problem in this case; it asserts in particular that any local system of finite dimensional vector spaces is the sheaf of germs of local horizontal sections of some flat connection which is regular in a suitable sense, generalizing the Fuchs condition [De]. This chapter provides a self-contained exposition of the part of [De] which is needed later.

The simplest case of the algebraic theory of $\mathcal{D}$-modules is that of $\mathcal{D}$-modules over complex affine n-space $\mathbf{A}^n$. The global algebraic differential operators on $\mathbf{A}^n$ form a "Weyl algebra" generated by a system of coordinates $\{x_i\}$ and the corresponding partial derivatives $\partial_i = \partial/\partial x_i$, subject to the relations $[\partial_i, x_j] = \delta_{ij}$. In the same way as the theory of quasi-coherent sheaves on an affine variety is equivalent to that of modules over the ring of regular functions, we need not yet sheafify and may consider instead modules over the Weyl algebra. This theory was developed by J. Bernstein, with important complements pertaining to homological dimension by J.E. Roos. Although motivated originally in part by an analytic problem, namely the meromorphic continuation of certain distributions depending holomorphically on a complex parameter in some half-plane, it has turned out to be completely algebraic. It is the topic of Chapter V, by F. Ehlers.

Those are then all the prerequisites necessary to deal with the theory of algebraic $\mathcal{D}$-modules in Chapters VI, VII and VIII, written by the undersigned. They are basically an elaboration of Notes by J. Bernstein [Be] of lectures he gave at the Federal School of Technology Zürich, Spring 1983, and then also, in a somewhat abbreviated form, at the Luminy Colloquium on differential systems and singularities (6/27–7/9 1983), with some complements or modifications he supplied later.

Chapter VI defines first the basic structural sheaf of the theory, namely the sheaf $\mathcal{D}_X$ of germs of algebraic linear differential operators, relates the $\mathcal{D}_X$-modules to flat connections or to systems of linear partial differential equations and introduces the categories $\mu(\mathcal{D}_X)$ (resp. $coh(\mathcal{D}_X)$, resp. $hol(\mathcal{D}_X)$) of $\mathcal{D}_X$-modules which are quasi-coherent over the structural sheaf $\mathcal{O}_X$ (resp. coherent over $\mathcal{D}_X$, resp. holonomic). The $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic by definition if its characteristic variety $SS(\mathcal{M})$ in the cotangent bundle $T^*(X)$ to X has the smallest possible dimension, namely the dimension of X, by a result of Bernstein (the more precise theorem stating that $SS(\mathcal{M})$ is involutive is not needed here and is not proved in these Notes). We let $D^b(\mathcal{D}_X)$, $D^b_{coh}(\mathcal{D}_X)$ and $D^b_h(\mathcal{D}_X)$ be the corresponding bounded derived categories. $\mathcal{D}_X$ is locally noetherian, coherent, of finite cohomological dimension (§1) and $\mu(\mathcal{D}_X)$ has enough injective or locally projective modules (§2), so that we can apply standard homological algebra, although the sheaf $\mathcal{D}_X$ does not have commutative stalks (except when X has dimension zero). However one has to distinguish between left and right $\mathcal{D}_X$-modules; §3 indicates how to go from one to the other. It also introduces a dualizing module and a duality functor $\mathfrak{D}_X$. The latter induces an involution on $hol(\mathcal{D}_X)$ and $D^b_h(\mathcal{D}_X)$ (3.6,3.7) as well as on $D^b_{coh}(\mathcal{D}_X)$ (9.5) and is the analogue of Verdier duality for constructible sheaves. §§4 and 8 are devoted to the notion of inverse and direct images of $\mathcal{D}$-modules, their derived functors and some of their basic properties, in general and also more specifically for locally closed embeddings or projections. §8 concludes with the $\mathcal{D}$-module version of the standard exact triangle associated in sheaf theory to a closed embedding (8.3) and a base change theorem (8.4).

Chapter VII establishes first some deeper properties of coherent or holonomic complexes of $\mathcal{D}_X$-modules (§§9, 10) and then turns to regular holonomic complexes (§§11, 12). A complex $F^{\cdot}$ of $\mathcal{D}_X$-modules is holonomic if $H^{\cdot}F^{\cdot}$ is a holonomic module and is regular holonomic (r.h.) if in addition the derived inverse images $H^i\phi^!F^{\cdot}$ with respect to any morphism $\phi : C \to X$ of a smooth curve into X are regular, where regularity on a curve amounts essentially to the Fuchs condition, garbed in a more sophisticated language. This latter definition is not the one of [Be] and was proposed by Bernstein orally in 1984. It is shown that the bounded derived categories $D^b_h(\mathcal{D}_X)$ of holonomic complexes and $D^b_{rh}(\mathcal{D}_X)$ or regular holonomic complexes are preserved under direct or inverse images and duality, (10.1, 12.2, 12.7) and that $F^{\cdot}$ is regular holonomic if and only if $H^{\cdot}F^{\cdot}$ and all its subquotients are regular holonomic (12.7). Furthermore, $D^b_{rh}(\mathcal{D}_X)$ is generated, as a triangulated category, by the direct images of irreducible regular connections on smooth locally closed affine varieties (12.2). Finally, it is proved in §12 that if X is an algebraic transformation space of the affine algebraic group G with finitely many orbits, then every G-equivariant coherent $\mathcal{D}_X$-module is regular holonomic (12.11), a fact which may be viewed as the starting point of the applications of $\mathcal{D}$-modules to representations of semisimple groups.

VIII is devoted to the Riemann-Hilbert correspondence. Roughly speaking, it asserts that $D^b_{rh}(\mathcal{D}_X)$ is equivalent to the category $D^b_{cs}(X)$ of constructible complexes of sheaves on X, by an equivalence which lets regular holonomic modules correspond to perverse sheaves (14.2). Here a complex of sheaves $S^{\cdot}$ on the underlying analytic variety X^a is constructible if X^a admits a stratification by locally closed smooth algebraic subvarieties on which $H^{\cdot}S^{\cdot}$ is locally constant in the analytic topology. The equivalence is established by means of the "de Rham functor" DR, which associates to $F^{\cdot}$ its de Rham complex on X^a, shifted by the dimension of X for technical reasons. §13 introduces DR, §14 states the main results, gives some consequences and some reference to mostly earlier results on the Riemann-Hilbert correspondence in the holomorphic case (14.9). In 14.5, the main theorem is subdivided into eight assertions, some valid more generally for holonomic complexes, which are proved in §§15 to 22.

The Bern Seminar and these Notes would not have come about without the help of J. Bernstein, J.E. Björk and B. Malgrange, to whom we are glad to express our gratefulness. First of all, all along and already during the planning of the Seminar, we benefited from the advice

of Malgrange, who moreover contributed the lectures on Deligne's work, and of Björk. In addition, I received personally enormous help from long explanations by Björk and Bernstein while I was learning this material or preparing to lecture on it. My thanks are also due to J. Bernstein, D. Milicic and M. Saito for many corrections to, or remarks on, a preliminary draft of Chapters VI, VII and VIII, which led to various improvements, besides eliminating a number of mistakes or misprints.

A. Borel

Princeton, September 1986

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I. CATEGORIES DERIVEES ET FONCTEURS DERIVES

Pierre-Paul GRIVEL

INTRODUCTION

A la fin des années cinquante des lacunes dans les fondements de l'algèbre homologique ne permettent pas de développer, en géométrie algébrique, une théorie cohomologique de la dualité des Modules cohérents sur les préschémas. Au début de la décennie suivante, J.L. Verdier, dans sa thèse (dont les détails n'ont jamais été publiés, voir [V]), comble ces lacunes en introduisant les catégories dérivées et les foncteurs dérivés, ces derniers permettant alors de définir un morphisme de trace. Grâce à l'efficacité de ces nouveaux outils on a pu obtenir depuis de nouveaux théorèmes de dualité (voir par exemple [V_1]).

Nous allons esquisser brièvement les idées qui sont à la base de la construction des catégories dérivées et des foncteurs dérivés.

Si $\mathbb{A}$ est une catégorie abélienne, soit $C(\mathbb{A})$ la catégorie abélienne des complexes d'objets de $\mathbb{A}$.

Dans beaucoup de situations rencontrées en algèbre homologique les complexes ne sont déterminés qu'à isomorphisme homologique près, un isomorphisme homologique étant un morphisme de complexes qui induit un isomorphisme en cohomologie. Par exemple si $X^\cdot$ et $Y^\cdot$ sont deux résolutions injectives d'un complexe $Z^\cdot$, on a un diagramme

$$X^\cdot \xleftarrow{u} Z^\cdot \xrightarrow{v} Y^\cdot$$

dans lequel les morphismes u et v induisent un isomorphisme en cohomologie.

Il est donc naturel de poser les deux problèmes suivants :

a) Existe-t-il une catégorie où, sans changer les objets $X^\bullet$ et $Y^\bullet$ (donc sans perdre d'information) ceux-ci deviennent isomorphes (c'est-à-dire les isomorphismes homologiques deviennent des isomorphismes)?

b) Si c'est le cas, soit $F : C(\mathbb{A}) \to C(\mathbb{B})$ un foncteur additif, où $\mathbb{A}$ et $\mathbb{B}$ sont des catégories ebéliennes. En général $F(u)$ et $F(v)$ ne sont plus des isomorphismes homologiques. Alors est-ce que F induit, d'une façon raisonnable, un foncteur entre les nouvelles catégories?

Remarquons d'emblée que dans le diagramme précédent les morphismes u et v ne sont déterminés qu'à homotopie près et que dans de nombreux cas les diagrammes que l'on rencontrera en développant la théorie ne seront commutatifs qu'à homotopie près. Il est donc naturel et nécessaire de se placer tout de suite dans la catégorie $K(\mathbb{A})$ dont les objets sont les complexes d'objets de $\mathbb{A}$ mais dont les morphismes sont les classes d'homotopie de morphismes de complexes. Malheureusement la catégorie $K(\mathbb{A})$ n'est plus abélienne en général. Cette perte de structure est cependant compensée par l'apparition d'une autre structure sur $K(\mathbb{A})$ qui va permettre de remplacer les suites exactes courtes par une notion nettement plus simple, comme on va le voir, celle des triangles distingués.

On appelle triangle distingué un diagramme

$$\begin{tikzcd} & Z^\bullet \arrow[dl] & \\ X^\bullet \arrow[rr, "u"'] & & Y^\bullet \arrow[ul] \end{tikzcd}$$

, où le morphisme $Z^\bullet \to X^\bullet$ est de degré +1, isomorphe au triangle standard

$$\begin{tikzcd} & C_u^\bullet \arrow[dl] & \\ X^\bullet \arrow[rr, "u"'] & & Y^\bullet \arrow[ul] \end{tikzcd}$$

construit à l'aide du cône $C_u^\bullet$ du morphisme u.

A tout triangle distingué est associé une suite exacte longue de cohomologie, dans laquelle le morphisme de connection entre le terme $H^i(Z^\bullet)$ et le terme $H^{i+1}(X^\bullet)$ est simplement induit par un morphisme de complexes.

De plus n'importe quel foncteur additif $F : \mathbb{A} \to \mathbb{B}$ entre catégories abéliennes induit évidemment un foncteur $F : K(\mathbb{A}) \to K(\mathbb{B})$ et celui-ci transforme les triangles distingués en triangles distingués.

Notons enfin qu'à une suite exacte courte $0 \to X^\bullet \xrightarrow{u} Y^\bullet \to Z^\bullet \to 0$ est associé le triangle

$$\begin{tikzcd} & C_u^\bullet \arrow[dl] & \\ X^\bullet \arrow[rr, "u"'] & & Y^\bullet \arrow[ul] \end{tikzcd}$$

dans lequel le cône $C_u^\bullet$ est homologiquement isomorphe à $Z^\bullet$ et même isomorphe à $Z^\bullet$ si la suite exacte est semi-scindée. Mais on a encore un meilleur résultat. En se plaçant dans la catégorie dérivée $D(\mathbb{A})$, que l'on va définir dans un moment, à la suite exacte courte $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ est naturellement associé le triangle

$$\begin{tikzcd} & Z^\bullet \arrow[dl] & \\ X^\bullet \arrow[rr] & & Y^\bullet \arrow[ul] \end{tikzcd}$$

.

Ces quelques remarques, essentiellement reprises de ([Bo],chap.5, §5), montrent les avantages des triangles et suggèrent que le cône $C_u^\bullet$ peut être considéré comme un substitut du conoyau (et aussi du noyau).

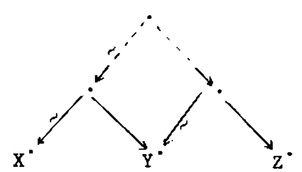
Revenons maintenant à la question a) posée précédemment. On doit résoudre un problème universel relatif aux isomorphismes homologiques. Puisque ceux-ci doivent être rendus inversibles il est clair qu'on ne peut y parvenir qu'en augmentant le nombre des flèches de la catégorie $K(\mathbb{A})$. La théorie de la localisation d'une catégorie, construite dans [GZ], apporte une bonne solution à ce problème.

On obtient en effet la catégorie dérivée $D(\mathbb{A})$ en localisant la catégorie $K(\mathbb{A})$ relativement au système multiplicatif des isomorphismes homologiques de $K(\mathbb{A})$ et on a un foncteur canonique $P_{\mathbb{A}} : K(\mathbb{A}) \to D(\mathbb{A})$ qui est universel pour la propriété de transformer les isomorphismes homologiques de $K(\mathbb{A})$ en des isomorphismes de $D(\mathbb{A})$.

Ainsi les objets de $D(\mathbb{A})$ sont les mêmes que ceux de $K(\mathbb{A})$, mais les morphismes sont les diagrammes

$$X^\bullet \overset{\sim}{\leftarrow} \cdot \to Y^\bullet$$

où la première flèche est un isomorphisme homologique; plus précisément ce sont des classes d'équivalence de tels diagrammes car si on remplace l'objet du milieu par un objet qui lui est homologiquement isomorphe on obtient encore un diagramme de même type. La composition des morphismes est définie à partir de diagrammes de la forme



L'aspect rébarbatif de ces catégories dérivées s'atténuera peut-être un peu si on remarque que lorsque la catégorie $\mathbb{A}$ admet suffisament d'objets injectifs, alors la catégorie $D^+(\mathbb{A})$, obtenue en localisant la catégorie $K^+(\mathbb{A})$ des complexes bornés inférieurement d'objets de $\mathbb{A}$, est équivalente à l'usuelle catégorie $K^+(I(\mathbb{A}))$ des complexes bornés inférieurement d'objets injectifs de $\mathbb{A}$.

Examinons maintenant la question b).

Soit $F : K(\mathbb{A}) \to K(\mathbb{B})$ un foncteur additif qui transforme les triangles distingués en triangles distingués.

Si F transforme les isomorphismes homologiques en isomorphismes homologiques (par exemple si F est induit d'un foncteur exact $F : \mathbb{A} \to \mathbb{B}$), il est clair, d'après la propriété universelle du foncteur de localisation, qu'il existe un unique foncteur $RF : D(\mathbb{A}) \to D(\mathbb{B})$ qui rend commutatif le diagramme

$$\begin{array}{ccc} K(\mathbb{A}) & \xrightarrow{F} & K(\mathbb{B}) \\ {\scriptstyle P_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle P_{\mathbb{B}}} \\ D(\mathbb{A}) & \xrightarrow[RF]{} & D(\mathbb{B}) \end{array}$$

Dans le cas général, sous des conditions suffisamment bonnes (par exemple si $\mathbb{A}$ admet suffisamment d'objets injectifs et si on considère les complexes bornés inférieurement), il existe un foncteur RF, unique à isomorphisme près, et un morphisme fonctoriel $\xi : P_{\mathbb{B}} \circ F \to RF \circ P_{\mathbb{A}}$ tel que le couple $(RF;\xi)$ soit universel pour cette propriété.

Le foncteur dérivé RF a la propriété suivante. Supposons que F est induit d'un foncteur $F : \mathbb{A} \to \mathbb{B}$ et que la catégorie $\mathbb{A}$ admet suffisamment

d'objets injectifs. Si X est un objet de $\mathbb{A}$, considéré comme objet de $D(\mathbb{A})$ par le foncteur d'inclusion évident $\mathbb{A} \to D(\mathbb{A})$, alors on a un isomorphisme canonique $H^i(RF(X)) = R^iF(X)$, où R^iF est le $i^{\text{ème}}$ foncteur dérivé de F usuel.

Supposons de nouveau que la catégorie $\mathbb{A}$ admet suffisamment d'objets injectifs et restreingnons-nous à la catégorie $D^+(\mathbb{A})$. Dans ce cas le foncteur RF est défini essentiellement de la façon suivante. Si $X^\cdot$ est un objet de $D^+(\mathbb{A})$ on choisit une résolution $X^\cdot \xrightarrow{\sim} I^\cdot$ de $X^\cdot$ par des objets injectifs de $\mathbb{A}$ et on pose $RF(X^\cdot) = F(I^\cdot)$.

Pour conclure on remarquera que la notion de catégories dérivées et de foncteurs dérivés fournit un cadre bien adapté pour faire de l'algèbre homologique. Aux raisonnements traditionnels qui nécessitent la lourde machinerie des résolutions sont substitués des opérations sur les foncteurs dérivés. Un exemple particulièrement intéressant est celui des suites spectrales, qui sont remplacées par une simple composition de foncteurs dérivés.

Dans l'exposé qui va suivre on n'a pas cherché à donner aux résultats la forme la plus générale possible. Son ambition est, au contraire, de permettre au lecteur non spécialisé de se familiariser assez rapidement avec le langage et les techniques des catégories dérivées.

Dans la bibliographie on trouvera, entre autres, une liste, qui ne se prétend nullement exhaustive, d'articles consacrés, en tout ou en partie, au sujet traité ici.

§ 1. LA LOCALISATION D'UNE CATEGORIE

1.1 Soit $\mathbb{C}$ une catégorie.

Un système multiplicatif dans $\mathbb{C}$ est une famille $\Sigma_{\mathbb{C}} \subset F\ell(\mathbb{C})$ de flèches de $\mathbb{C}$ qui vérifie les propriétés suivantes :

FR1 : a) Si $s,t \in \Sigma_{\mathbb{C}}$ et si st existe alors $st \in \Sigma_{\mathbb{C}}$.

b) Si $X \in \mathrm{Ob}\mathbb{C}$ alors $1_X \in \Sigma_{\mathbb{C}}$.

FR2 : Soit $X,Y,Z \in \mathrm{Ob}\mathbb{C}$. Si $u \in \mathrm{Hom}_{\mathbb{C}}(X;Y)$ (respectivement $u \in \mathrm{Hom}_{\mathbb{C}}(Y;X)$) et $s \in \mathrm{Hom}_{\mathbb{C}}(Z;Y) \cap \Sigma_{\mathbb{C}}$ (respectivement $s \in \mathrm{Hom}_{\mathbb{C}}(Y;Z) \cap \Sigma_{\mathbb{C}}$) il existe $W \in \mathrm{Ob}\mathbb{C}$ et des morphismes $v \in \mathrm{Hom}_{\mathbb{C}}(W;Z)$ (respectivement $v \in \mathrm{Hom}_{\mathbb{C}}(Z;W)$) et $t \in \mathrm{Hom}_{\mathbb{C}}(W;X) \cap \Sigma_{\mathbb{C}}$ (respectivement $t \in \mathrm{Hom}_{\mathbb{C}}(X;W) \cap \Sigma_{\mathbb{C}}$) tels que le diagramme

$$\begin{tikzcd} W \arrow[r, dashed, "v"] \arrow[d, dashed, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "u"'] & Y \end{tikzcd} \quad \text{(respectivement} \quad \begin{tikzcd} W & Z \arrow[l, dashed, "v"'] \\ X \arrow[u, dashed, "t"] & Y \arrow[l, "u"] \arrow[u, "s"'] \end{tikzcd} \text{)}$$

soit commutatif.

FR3 : Soit $X,Y \in \mathrm{Ob}\mathbb{C}$ et $u,v \in \mathrm{Hom}_{\mathbb{C}}(X;Y)$.

Les conditions suivantes sont équivalentes :

i) Il existe $Y' \in \mathrm{Ob}\mathbb{C}$ et $s \in \mathrm{Hom}_{\mathbb{C}}(Y;Y') \cap \Sigma_{\mathbb{C}}$ tels que $su = sv$.

ii) Il existe $X' \in \mathrm{Ob}\mathbb{C}$ et $t \in \mathrm{Hom}_{\mathbb{C}}(X';X) \cap \Sigma_{\mathbb{C}}$ tels que $ut = vt$.

$$X' \xrightarrow{t} X \overset{u}{\underset{v}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

S'il n'y a pas de confusion possible on écrira Σ au lieu de $\Sigma_{\mathbf{C}}$. Dans les diagrammes on écrira souvent $X \xrightarrow[\sim]{s} Y$ pour indiquer que $s \in \Sigma$.

1.2 THEOREME : *Si Σ est un système multiplicatif dans $\mathbf{C}$ il existe une catégorie $\mathbf{C}_\Sigma$ et un foncteur*

$$P_\Sigma : \mathbf{C} \to \mathbf{C}_\Sigma$$

qui vérifient les propriétés suivantes :

i) Si $s \in \Sigma$ alors $P_\Sigma(s)$ est un isomorphisme dans $\mathbf{C}_\Sigma$.

ii) Si $F : \mathbf{C} \to \mathbf{D}$ est un foncteur qui transforme les éléments de Σ en des isomorphismes de $\mathbf{D}$, alors il existe un unique foncteur $G : \mathbf{C}_\Sigma \to \mathbf{D}$ tel que le diagramme

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[dr, "P_\Sigma"'] & \mathbf{D} \\ & \mathbf{C}_\Sigma \arrow[u, dashed, "G"'] \end{tikzcd}$$

soit commutatif.

(La catégorie $\mathbf{C}_\Sigma$ s'appelle la localisation de $\mathbf{C}$ relativement au système multiplicatif Σ).

1.3 La démonstration du théorème 1.2 va occuper les numéros 1.4 à 1.13 suivants.

1.4 Soit $X,Y \in \mathrm{Ob}\mathbf{C}$. Notons $M^{\mathbf{C}}_{X,Y}$ l'ensemble des diagrammes de la forme

$$X \xleftarrow[\sim]{s} Z \xrightarrow{a} Y$$

où $Z \in \mathrm{Ob}\mathbf{C}$, $s \in \mathrm{Hom}_{\mathbf{C}}(Z;X) \cap \Sigma$ et $a \in \mathrm{Hom}_{\mathbf{C}}(Z;Y)$. On désignera un tel diagramme par un triple $(Z;s;a)$. S'il n'y a pas de confusion possible on écrira $M_{X,Y}$ au lieu de $M^{\mathbf{C}}_{X,Y}$.

Si $(Z;s;a)$ et $(V;t;b)$ sont des éléments de $M_{X,Y}$, un morphisme $p : (Z;s;a) \to (V;t;b)$ est un morphisme $p \in \mathrm{Hom}_{\mathbf{C}}(Z;V)$ tel que le diagramme

$$\begin{tikzcd} & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] \arrow[dd, dashed, "p"] & \\ X & & Y \\ & V \arrow[ul, "t", "\sim"'] \arrow[ur, "b"'] & \end{tikzcd}$$

soit commutatif.

Dans l'ensemble $M_{X,Y}$ on définit une relation, notée $\equiv$, de la façon suivante :

Si $(Z;s;a), (Z',s',a') \in M_{X,Y}$ alors $(Z;s;a) \equiv (Z';s';a')$ si et seulement si il existe $(T;r;b) \in M_{X,Y}$ et des morphismes $p : (T;r;b) \to (Z;s;a)$ et $p' : (T;r;b) \to (Z';s';a')$; autrement dit si on a un diagramme commutatif

(1.4.1)
$$\begin{tikzcd} & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\ X & T \arrow[l, dashed, "r"', "\sim"] \arrow[r, dashed, "b"] \arrow[u, dashed, "p"'] \arrow[d, dashed, "p'"] & Y \\ & Z' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & \end{tikzcd}$$

1.5. LEMME : *La relation $\equiv$ est une relation d'équivalence dans l'ensemble $M_{X,Y}$.*

Démonstration : La réfléxivité et la symétrie sont évidentes. Démontrons la transitivité. Supposons donc que $(Z;s;a) \equiv (Z';s';a')$ et que $(Z';s';a') \equiv (Z'';s'';a'')$. On a deux diagrammes commutatifs

$$\begin{tikzcd} & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\ X & T \arrow[l, "r"', "\sim"] \arrow[r, "b"] \arrow[u, "p"'] \arrow[d, "p'"] & Y \\ & Z' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} & Z' \arrow[dl, "s'"', "\sim"] \arrow[dr, "a'"] & \\ X & S \arrow[l, "t"', "\sim"] \arrow[r, "c"] \arrow[u, "q"'] \arrow[d, "q'"] & Y \\ & Z'' \arrow[ul, "s''", "\sim"'] \arrow[ur, "a''"'] & \end{tikzcd}$$

et il faut construire un diagramme commutatif

(1.5.1)
$$\begin{tikzcd} & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\ X & R \arrow[l, dashed, "x"', "\sim"] \arrow[u, dashed, "y"'] \arrow[r, dashed, "d"] \arrow[d, dashed, "y'"] & Y \\ & Z'' \arrow[ul, "s''", "\sim"'] \arrow[ur, "a''"'] & \end{tikzcd}$$

En utilisant FR2 on peut compléter le diagramme

$$T \xrightarrow[\sim]{r} X \xleftarrow[\sim]{t} S$$

en un diagramme commutatif

$$\begin{tikzcd} W \arrow[r, dashed, "r'"] \arrow[d, dashed, "t''"', "\wr"] & S \arrow[d, "t", "\wr"'] \\ T \arrow[r, "\sim", "r"'] & X \end{tikzcd}$$

On a $s'qr' = s'p't'$, donc d'après FR3 il existe $u \in \mathrm{Hom}_{\mathbb{C}}(R;W) \cap \Sigma$ tel que $qr'u = p't'u$

$$R \xrightarrow[\sim]{u} W \underset{p't'}{\overset{qr'}{\rightrightarrows}} Z' \xrightarrow[\sim]{s'} X$$

Alors les morphismes $x = rt'u \in \Sigma$, $d = bt'u$, $y = pt'u$ et $y' = q'r'u$ rendent commutatif le diagramme (1.5.1)

1.6. *Remarque :* Si il existe un morphisme $p : (Z;s;a) \to (Z';s';a')$ il est clair que $(Z;s;a) \equiv (Z';s';a')$. Plus précisément la relation $\equiv$ est la relation d'équivalence engendrée par la relation "il existe un morphisme $(Z;s;a) \to (Z';s';a')$ ou un morphisme $(Z';s';a') \to (Z;s;a)$" .

1.7 Dans l'ensemble $M_{X,Y}/\equiv$ on peut définir une opération de composition de la façon suivante.

Soit $X,Y,Z \in \mathrm{Ob}\mathbb{C}$. Soit $\alpha \in M_{X,Y}/\equiv$ représenté par $(U;s:a) \in M_{X,Y}$ et $\beta \in M_{Y,Z}/\equiv$ représenté par $(V;t;b) \in M_{Y,Z}$. D'après FR2 on peut compléter le diagramme $U \xrightarrow{a} Y \xleftarrow[\sim]{t} V$ en un diagramme commutatif

$$\begin{tikzcd} & & W \arrow[dl, dashed, "r"', "\sim"] \arrow[dr, dashed, "c"] & & \\ & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & & V \arrow[dl, "t"', "\sim"] \arrow[dr, "b"] & \\ X & & Y & & Z \end{tikzcd}$$

Ainsi $(W;sr;bc) \in M_{X,Z}$.

1.8. LEMME : *si* $(U';s';a') \equiv (U;s;a)$ *alors* $(W';s'r';bc') \equiv (W;sr;bc)$.

Démonstration : On a le diagramme commutatif

$$\begin{tikzcd} & & & W \arrow[dll, dashed, "r"', "\sim"] \arrow[d, dashed, "c"] & \\ & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & & V \arrow[dl, "t"', "\sim"] \arrow[dr, "b"] & \\ X & T \arrow[l, "u", "\sim"'] \arrow[u, "x"'] \arrow[r, "z"] \arrow[d] & Y & & Z \\ & U' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & & & \\ & & & W' \arrow[ull, "r'", "\sim"'] \arrow[uuu, dashed, "c'"'] & \end{tikzcd}$$

et il faut construire un diagramme commutatif

(1.8.1)
$$\begin{tikzcd} & W \arrow[dl, "sr"', "\sim"] \arrow[dr, "bc"] & \\ X & P \arrow[l, dashed, "f"', "\sim"] \arrow[u, dashed, "v"'] \arrow[r, dashed, "g"] \arrow[d, dashed, "v'"] & Z \\ & W' \arrow[ul, "s'r'", "\sim"'] \arrow[ur, "bc'"'] & \end{tikzcd}$$

En utilisant FR2 on peut compléter les diagrammes $T \xrightarrow{x} U \xleftarrow[\sim]{r} W$ et $T \xrightarrow{x'} U' \xleftarrow[\sim]{r'} W'$ en des diagrammes commutatifs

$$\begin{tikzcd} R \arrow[r, dashed, "q"] \arrow[d, dashed, "p''"', "\sim"] & W \arrow[d, "r", "\sim"'] \\ T \arrow[r, "x"'] & U \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} R' \arrow[r, dashed, "q'"] \arrow[d, dashed, "p'"', "\sim"] & W' \arrow[d, "r'", "\sim"'] \\ T \arrow[r, "x'"'] & U' \end{tikzcd}$$

puis le diagramme $R \xrightarrow[\sim]{p} T \xleftarrow[\sim]{p'} R'$ en un diagramme commutatif

$$\begin{tikzcd} Q \arrow[r, dashed, "k"] \arrow[d, "h"', "\sim"] & R' \arrow[d, "p'", "\sim"'] \\ R \arrow[r, "\sim", "p"'] & T \end{tikzcd}$$

On a $tc'q'k = tcqh$, donc d'après FR3 il existe $w \in \mathrm{Hom}_{\mathbf{C}}(P;Q) \cap \Sigma$ tel que $c'q'kw = cqhw$

$$P \overset{w}{\underset{\sim}{\dashrightarrow}} Q \underset{cqh}{\overset{c'q'k}{\rightrightarrows}} V \xrightarrow[\sim]{t} Y$$

Alors les morphismes $f = uphw \in \Sigma$, $g = bc'q'kw$, $v = qhw$ et $v' = q'kw$ rendent commutatif le diagramme (1.8.1).

1.9 Reprenons les notations de 1.7. On définit $\beta\alpha \in M_{X,Z}/\equiv$ comme étant la classe d'équivalence de $(W;sr;bc)$ dans $M_{X,Z}/\equiv$.

On vérifie immédiatement que cette opération est associative et que, pour tout $X \in \mathrm{Ob}\mathbf{C}$, la classe d'équivalence du triple $(X;1_X;1_X) \in M_{X,X}$ est l'élément unité pour cette opération.

1.10 On définit la catégorie localisée $\mathbf{C}_\Sigma$ de la façon suivante :

i) $\mathrm{Ob}\mathbf{C}_\Sigma = \mathrm{Ob}\mathbf{C}$

ii) Si $X,Y \in \mathrm{Ob}\mathbf{C}$ alors $\mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y) = M_{X,Y}/\equiv$. Ainsi $\mathrm{Hom}_{\mathbf{C}}(X;Y)$ est l'ensemble des classes d'équivalence des diagrammes de la forme

(1.10.1) $$X \xleftarrow{\sim} \cdot \longrightarrow Y$$

où la première flèche appartient à Σ.

iii) Si $\alpha \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$ et $\beta \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(Y;Z)$ alors $\beta\alpha \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Z)$ est la classe d'équivalence des diagrammes de la forme

$$\begin{tikzcd} & & \cdot \arrow[dl, dashed, "\sim"'] \arrow[dr] & & \\ & \cdot \arrow[dl, "\sim"'] \arrow[dr] & & \cdot \arrow[dl, "\sim"'] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

1.11 On définit un foncteur

$$P_\Sigma : \mathbf{C} \to \mathbf{C}_\Sigma$$

(noté simplement P s'il n'y a pas de risque de confusion) de la façon suivante : P_Σ est l'identité sur les objets et si $u \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$ alors $P_\Sigma(u)$ est la classe d'équivalence de l'élément $(X;1_X;u) \in M_{X,Y}$ dans $\mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$.

1.12 Soit $X,Y \in \mathrm{Ob}\mathbf{C}$ et $s \in \mathrm{Hom}_{\mathbf{C}}(X\ Y) \cap \Sigma$; alors $P(s)$ est représenté par l'élément $(X;1_X;s) \in M_{X,Y}$. Soit $\alpha \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(Y;X)$ le morphisme représenté par l'élément $(X;s;1_X) \in M_{Y,X}$. On a $\alpha \circ P(s) = 1_X$ et $P(s) \circ \alpha = 1_Y$, donc $P(s)$ est un isomorphisme dans $\mathbf{C}_\Sigma$.

1.13 Soit $F : \mathbf{C} \to \mathbf{D}$ un foncteur tel que si $s \in \Sigma$ alors $F(s)$ est un isomorphisme dans $\mathbf{D}$. On définit un foncteur $G : \mathbf{C}_\Sigma \to \mathbf{D}$ de la façon suivante. Si $X \in \mathrm{Ob}\mathbf{C}_\Sigma = \mathrm{Ob}\mathbf{C}$ on pose $G(X) = F(X)$. Soit $\alpha \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$ représenté par $(Z;s;a) \in M_{X,Y}$. Comme $s \in \Sigma$, le morphisme $F(s)$ est inversible. On pose alors $G(\alpha) = F(a) \circ F(s)^{-1}$. Pour montrer que ce morphisme est bien défini il faut vérifier que si $(Z';s';a') \equiv (Z;s;a)$ alors $F(a') \circ F(s')^{-1} = F(a) \circ F(s)^{-1}$.
Par hypothèse on a un diagramme commutatif

$$\begin{tikzcd} & Z \arrow[dl, dashed, "s"', "\sim"] \arrow[dr, "a"] & \\ X & T \arrow[l, "r"', "\sim"] \arrow[r, "b"] \arrow[u, dashed, "p"'] \arrow[d, dashed, "p'"] & Y \\ & Z' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & \end{tikzcd}$$

On a donc

$$F(a)F(s)^{-1}F(r) = F(a)F(p) = F(a')F(p') = F(a')F(s')^{-1}F(r).$$

Comme $F(r)$ est un isomorphisme on a la relation cherchée. Finalement $G\circ P = F$ car si $u \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$ on a $G(P(u)) = F(u)F(1_X)^{-1} = F(u)$.

Supposons que $G' : \mathbf{C}_\Sigma \to \mathbf{D}$ est un foncteur tel que $G'\circ P = F$. Si $\alpha \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$ est représenté par $(Z;s;a) \in M_{X,Y}$ on a $\alpha = P(a)\circ P(s)^{-1}$, cette écriture étant indépendante du choix du représentant de α; il suffit en effet de faire $F = P$ et $G = 1_{\mathbf{C}_\Sigma}$ dans la construction précédente. On a alors $G'(\alpha) = G'(P(a))\circ G'(P(s))^{-1} = F(a)\circ F(s)^{-1} = G(\alpha)$. Donc $G' = G$.

Ceci termine la démonstration du théorème 1.2.

1.14 *Remarque :* On peut donner une autre définition des morphismes de $\mathbf{C}_\Sigma$.

Soit $X, Y \in \mathrm{Ob}\mathbf{C}$. Notons $N^{\mathbf{C}}_{X,Y}$ (ou simplement $N_{X,Y}$) l'ensemble des diagrammes de la forme

$$X \longrightarrow \cdot \xleftarrow{\sim} Y \qquad (1.14.1)$$

où la deuxième flèche appartient à Σ.

En modifiant de façon évidente les considérations précédentes on peut définir une relation d'équivalence $\equiv'$ dans $N_{X,Y}$ et munir $N_{X,Y}/\equiv'$ d'une opération de composition. Si $(Z;s;a) \in M_{X,Y}$, il lui est associé, d'après FR2, un élément $(Z';a';s') \in N_{X,Y}$. Si $(Z'';a'';s'')$ est un autre élément de $N_{X,Y}$ qui est associé à $(Z;s;a)$ il est facile de démontrer que $(Z';a';s') \equiv' (Z'';a'';s'')$. Inversément si $(Z;a;s) \in N_{X,Y}$ il lui est associé un élément $(Z';s';a') \in M_{X,Y}$ et deux tels éléments associés à $(Z;a;s)$ sont équivalents pour la relation $\equiv$. Il en résulte qu'il existe une bijection canonique entre les ensembles $M_{X,Y}/\equiv$ et $N_{X,Y}/\equiv'$ et que cette bijection préserve les opérations de composition définies sur chacun de ces ensembles. On peut donc aussi poser $\mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y) = N_{X,Y}/\equiv'$. Dorénavant on pourra donc représenter un morphisme de X à Y dans $\mathbf{C}_\Sigma$ indifféremment par un diagramme du type (1.10.1) ou par un diagramme du type (1.14.1).

1.15 *Remarque :* On peut donner encore une autre interprétation de l'ensemble $\mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$. Soit $X \in \mathrm{Ob}\mathbf{C}$ et notons $\mathbf{I}_X$ la petite catégorie dont les objets sont les couples $(Z;s)$ formés d'un objet Z de $\mathbf{C}$ et d'un morphisme $s \in \mathrm{Hom}_{\mathbf{C}}(Z;X) \cap \Sigma$ et dont les morphismes de $(Z;s)$ à $(Z';s')$ sont les morphismes $f \in \mathrm{Hom}_{\mathbf{C}}(Z;Z')$ tels que le diagramme

$$\begin{tikzcd} Z \arrow[d, "f"'] \arrow[dr, "s", "\sim"'] & \\ Z' \arrow[r, "s'"', "\sim"] & X \end{tikzcd}$$

soit commutatif.

Maintenant soit $Y \in \mathrm{Ob}\mathbf{C}$ et considérons le foncteur contravariant $F_Y : \mathbf{I}^\circ_X \to \mathbf{Ens}$ défini de la façon suivante. Si $(Z;s) \in \mathrm{Ob}\mathbf{I}_X$ alors $F_Y(Z;s) = \{(Z;s;a) \in M_{X,Y} | a \in \mathrm{Hom}_{\mathbf{C}}(Z;Y)\}$; ainsi $F_Y(Z;s)$ est un sous-ensemble de $M_{X,Y}$ qui s'identifie à l'ensemble $\mathrm{Hom}_{\mathbf{C}}(Z;Y)$. Si $f : (Z;s) \to (Z';s')$ est un morphisme de $\mathbf{I}_X$ alors $F_Y(f) = \mathrm{Hom}_{\mathbf{C}}(f;1_Y) : \mathrm{Hom}_{\mathbf{C}}(Z';Y) \to \mathrm{Hom}_{\mathbf{C}}(Z;Y)$; on a donc des diagrammes commutatifs

$$\begin{tikzcd} & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a'f"] \arrow[dd] & \\ X & & Y \\ & Z' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & \end{tikzcd}$$

et par suite $(Z';s';a') \equiv (Z;s;a'f)$.

Pour tout $(Z;s) \in \mathrm{Ob}\mathbf{I}_X$ considérons l'application

$$\lambda_{(Z;s)} : F_Y(Z;s) \to \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$$

qui à un élément $(Z;s;a) \in F_Y(Z;s)$ associe sa classe d'équivalence dans $M_{X,Y}/\equiv$.

Si $f : (Z;s) \to (Z';s')$ est un morphisme de $\mathbf{I}_X$ il est immédiat, d'après ce qui précède, que le diagramme

$$\begin{tikzcd} F_Y(Z';s') \arrow[d, "F_Y(f)"'] \arrow[dr, "\lambda_{(Z';s')}"] & \\ F_Y(Z;s) \arrow[r, "\lambda_{(Z;s)}"'] & \mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y) \end{tikzcd}$$

est commutatif. Il en résulte que $\mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y)$ est un F_Y-cône inductif ([JP], chap VI, §1). Ce F_Y-cône inductif est universel. En effet soit $\{\mu_{(Z;s)} : F_Y(Z;s) \to M\}_{(Z;s)\in \mathrm{Ob}\,I_X}$ un F_Y-cône inductif; donc si $f : (Z;s) \to (Z';s')$ on a $\mu_{(Z';s')}(Z';s';a') = \mu_{(Z;s)}(Z;s;a'f)$.

Il faut construire une application $m : \mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y) \to M$ telle que, pour tout $(Z;s)\in \mathrm{Ob}\,I_X$, le diagramme

(1.15.1)
$$\begin{tikzcd} & \mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y) \arrow[d, dashed, "m"] \\ F_Y(Z;s) \arrow[ur, "\lambda_{(Z;s)}"] \arrow[r, "\mu_{(Z;s)}"'] & M \end{tikzcd}$$

soit commutatif.

Soit $\alpha\in \mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y)$; choisissons un représentant $(Z;s;a)$ de α et posons $m(\alpha) = \mu_{(Z;s)}(Z;s;a)$; cette définition est indépendante du choix du représentant. Ainsi l'application m est bien définie et elle est clairement l'unique application qui rend commutatif les diagrammes (1.15.1). On a donc ([JP], idem)

$$\mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y) = \varinjlim_{I_X} \mathrm{Hom}_{\mathfrak{C}}(Z;Y)$$

1.16 THEOREME : *Si* $\mathfrak{C}$ *est une catégorie additive alors* $\mathfrak{C}_\Sigma$ *est une catégorie additive et le foncteur canonique* P_Σ *est additif.*

1.17 Pour simplifier on démontrera ce théorème dans le cas particulier où le système multiplicatif Σ satisfait la condition supplémentaire suivante :

(1.17.1) Si $s\in\Sigma$ et si $su\in\Sigma$ alors $u\in\Sigma$.

Cette condition entraîne que, dans l'axiome FR2, si $u,s\in\Sigma$ alors $t,v\in\Sigma$. De plus, dans le diagramme (1.4.1), qui définit la relation d'équivalence $\equiv$, les morphismes p et p' sont dans Σ.

On notera que la condition (1.17.1) sera toujours satisfaite dans les catégories que nous aurons à localiser dans la suite de ce travail.

La démonstration du théorème repose maintenant essentiellement sur le lemme suivant.

1.18 LEMME : *Supposons que la catégorie* $\mathfrak{C}$ *est additive et que le système multiplicatif* Σ *satisfait la condition* (1.17.1). *Alors pour tout* $X,Y\in \mathrm{Ob}\,\mathfrak{C}_\Sigma$, *l'ensemble* $\mathrm{Hom}_{\mathfrak{C}_\Sigma}(X;Y)$ *est muni d'une structure de groupe abélien.*

Démonstration : Soit $(U;s;a)$, $(V;t;b)\in M_{X,Y}$. A l'aide de FR2 on peut compléter le diagramme $U \xrightarrow[\sim]{s} X \xleftarrow[\sim]{t} V$ en un diagramme commutatif et obtenir ainsi le diagramme

$$\begin{tikzcd} & & Y \\ V \arrow[urr, "b"] \arrow[d, "t"', "\wr"] & S \arrow[l, dashed, "v"', "\sim"] \arrow[d, dashed, "u", "\wr"'] \arrow[dl, dashed, "x"] \arrow[ur, dashed, "y"] & \\ X & U \arrow[l, "s", "\sim"'] \arrow[uur, "a"'] & \end{tikzcd}$$

où $x = su = tv\in\Sigma$ et $y = au+bv\in \mathrm{Hom}_{\mathfrak{C}}(S;Y)$. On obtient ainsi un élément $(S;x;y)\in M_{X,Y}$. Supposons que $(U';s';a') \equiv (U;s;a)$; on a donc un diagramme commutatif

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\ X & T \arrow[l, "r"', "\sim"] \arrow[u, dashed, "p"'] \arrow[r, dashed, "m"] \arrow[d, dashed, "p'"] & Y \\ & U' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] & \end{tikzcd}$$

Avec (U';s';a') et (V;t;b) on peut construire comme précédemment un élément $(S';x';y') \in M_{X,Y}$;on a donc un diagramme

$$\begin{tikzcd} & & Y \\ V \arrow[d, "t"', "\sim"] \arrow[urr, "b"] & S' \arrow[l, dashed, "v'"', "\sim"] \arrow[d, dashed, "u'", "\sim"'] \arrow[dl, dashed, "x'"] \arrow[ur, dashed, "y'"'] & \\ X & U' \arrow[l, "s'", "\sim"'] \arrow[uur, "a'"'] & \end{tikzcd}$$

où $x' = s'u' = t'v'$ et $y' = a'u' + bv'$. On affirme que $(S';x';y') \equiv (S;x;y)$; il faut donc démontrer que l'on a un diagramme commutatif

(1.18.1)
$$\begin{tikzcd} & S \arrow[dl, "x"', "\sim"] \arrow[dr, "y"] & \\ X & W \arrow[l, dashed, "z"', "\sim"] \arrow[u, dashed, "q"'] \arrow[r, dashed, "n"] \arrow[d, dashed, "q'"] & Y \\ & S' \arrow[ul, "x'", "\sim"'] \arrow[ur, "y'"'] & \end{tikzcd}$$

On commence par construire, avec (T;r;m) et (V;t;b) un élément $(\overline{S};\overline{x};\overline{y}) \in M_{X,Y}$;on a donc un diagramme

$$\begin{tikzcd} & & Y \\ V \arrow[d, "t"', "\sim"] \arrow[urr, "b"] & \overline{S} \arrow[l, "\overline{v}"', "\sim"] \arrow[d, "\overline{u}", "\sim"'] \arrow[dl, dashed, "\overline{x}"] \arrow[ur, dashed, "\overline{y}"'] & \\ X & T \arrow[l, "r", "\sim"'] \arrow[uur, "m"'] & \end{tikzcd}$$

où $\overline{x} = r\overline{u} = t\overline{v}$ et $\overline{y} = m\overline{u}+b\overline{v}$.

Maintenant à l'aide de FR2 on obtient les diagrammes commutatifs

$$\begin{tikzcd} N_1 \arrow[r, dashed, "i_1", "\sim"'] \arrow[d, dashed, "j_1"', "\wr"] & \overline{S} \arrow[d, "\overline{v}", "\wr"'] & N_1' \arrow[l, dashed, "i_1'"', "\sim"] \arrow[d, "j_1'", "\wr"'] \\ S \arrow[r, "\sim", "v"'] & V & S' \arrow[l, "\sim"', "v'"] \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[d, "u"', "\wr"] & N_2 \arrow[l, dashed, "j_2"', "\sim"] \arrow[d, dashed, "i_2", "\wr"'] \\ U & \overline{S} \arrow[l, "\sim", "p\overline{u}"'] \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} S' \arrow[d, "u'"', "\wr"] & N_2' \arrow[l, dashed, "j_2'"', "\sim"] \arrow[d, dashed, "i_2'", "\wr"'] \\ U' & \overline{S} \arrow[l, "\sim", "p'\overline{u}"'] \end{tikzcd}$$

On en déduit les diagrammes commutatifs

$$\begin{tikzcd} N_1 \arrow[d, "j_1"', "\wr"] & N_3 \arrow[l, dashed, "h_1"', "\sim"] \arrow[d, dashed, "h_2", "\wr"'] \\ S & N_2 \arrow[l, "\sim", "j_2"'] \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} N_1' \arrow[d, "j_1'"', "\wr"] & N_3' \arrow[l, dashed, "h_1'"', "\sim"] \arrow[d, dashed, "h_2'", "\wr"'] \\ S' & N_2' \arrow[l, "\sim", "j_2'"'] \end{tikzcd}$$

et finalement le diagramme commutatif

$$\begin{tikzcd} N_3' \arrow[d, "i_2'h_2'"', "\wr"] & M \arrow[l, "k'"', "\sim"] \arrow[d, dashed, "k", "\wr"'] \\ S & N_3 \arrow[l, "\sim", "i_2h_2"'] \end{tikzcd}$$

On a $\overline{x}i_1h_1k = \overline{x}i_1'h_1'k'$, donc d'après FR3 il existe $w \in \mathrm{Hom}_{\mathcal{C}}(W;M) \cap \Sigma$ tel que $i_1h_1kw = i_1'h_1'k'w$

$$W \overset{w}{\underset{\sim}{\dashrightarrow}} M \overset{i_1h_1k}{\underset{i_1'h_1'k'}{\rightrightarrows}} \overline{S} \overset{\overline{x}}{\underset{\sim}{\longrightarrow}} X$$

Alors les morphismes $z = xj_2h_2kw \in \Sigma$, $q = j_2h_2kw$, $q' = j_2'h_2'k'w$ et $n = yj_2h_2kw$ rendent commutatif le diagramme (1.18.1).

Cela étant, soit $\alpha,\beta \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ représentés respectivement par $(U;s;a)$ et $(V;t;b)$. On définit alors $\alpha+\beta \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ comme étant la classe d'équivalence de l'élément $(S;x;y)$ construit précédemment à partir de $(U;s;a)$ et $(V;t;b)$. On munit ainsi $\mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ d'une opération associative et commutative. L'élément nul $0 \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ est la classe de l'élément $(X;1_X;0) \in M_{X,Y}$. Enfin si $\alpha \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ est représenté par $(U;s;a)$, alors la classe d'équivalence de l'élément $(U;s;-a)$ définit l'opposé de α.

1.19 *Démonstration du théorème 1.16.* Il résulte du lemme 1.18 que $\mathrm{Hom}_{\mathbb{C}_\Sigma}(X;Y)$ est muni d'une structure de groupe abélien compatible avec la composition des morphismes. De plus il est immédiat que si $u,v \in \mathrm{Hom}_{\mathbb{C}}(X;Y)$ alors $P_\Sigma(u+v) = P_\Sigma(u)+P_\Sigma(v)$.

On vérifie aisément que l'objet nul dans $\mathbb{C}$ est l'objet nul dans $\mathbb{C}_\Sigma$.

Il reste donc à vérifier que la catégorie $\mathbb{C}_\Sigma$ admet des sommes finies. Soit $X,Y \in \mathrm{Ob}\,\mathbb{C}_\Sigma$. Regardés comme objets de $\mathbb{C}$, X et Y admettent une somme Z, c'est-à-dire on a dans $\mathbb{C}$ un diagramme

$$\begin{tikzcd} X \arrow[r, shift right, "q_X"'] & Z \arrow[l, shift right, "p_X"'] \arrow[r, shift left, "p_Y"] & Y \arrow[l, shift left, "q_Y"] \end{tikzcd}$$

avec $p_Xq_X = 1_X$, $p_Yq_Y = 1_Y$ et $q_Xp_X+q_Yp_Y = 1_Z$. Appliquant le foncteur P_Σ à ce diagramme on voit, compte tenu des propriétés de P_Σ, que Z est aussi la somme de X et Y dans $\mathbb{C}_\Sigma$.

1.20 PROPOSITION : *Soit* $u \in \mathrm{Hom}_{\mathbb{C}}(X;Y)$ *où* $\mathbb{C}$ *est une catégorie additive.*

Les affirmations suivantes sont équivalentes :

a) $P(u) = 0$

b) Il existe $s \in \mathrm{Hom}_{\mathbb{C}}(Z;X)\cap\Sigma$ *tel que* $us = 0$.

c) Il existe $t \in \mathrm{Hom}_{\mathbb{C}}(Y;Z')\cap\Sigma$ *tel que* $tu = 0$.

Démonstration : L'équivalence entre b) et c) résulte de FR3. Soit $s \in \mathrm{Hom}_{\mathbb{C}}(Z;X)\cap\Sigma$ tel que $us = 0$; on a $P(u)P(s) = 0$ donc $P(u) = 0$ puisque $P(s)$ est un isomorphisme.

Le morphisme $P(u)$ est représenté par $(X;1_X;u) \in M_{X,Y}$. Si $P(u) = 0$ on a donc $(X;1_X;u) \equiv (X;1_X;0)$ d'où un diagramme commutatif

$$\begin{tikzcd} & X \arrow[dl, "1_X"', "\sim"] \arrow[dr, "u"] & \\ X & Z \arrow[l, "s", "\sim"'] \arrow[u, dashed, "s"'] \arrow[d, dashed, "s"] \arrow[r, "0"] & Y \\ & X \arrow[ul, "1_X", "\sim"'] \arrow[ur, "0"'] & \end{tikzcd}$$

Ainsi a) implique b).

1.21 COROLLAIRE : *Soit*

$$\begin{tikzcd} X \arrow[r, "P(u)"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "P(u')"'] & Y' \end{tikzcd}$$

un diagramme commutatif dans $\mathbb{C}_\Sigma$.

Il existe un représentant $(R;s;a)$ *de* α *, un représentant* $(S;t;b)$ *de* β *et un morphisme* $m \in \mathrm{Hom}_{\mathbb{C}}(R;S)$ *tel que le diagramme dans* $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "u"] & Y \\ R \arrow[u, "s", "\wr"'] \arrow[r, "m"] \arrow[d, "a"'] & S \arrow[u, "t"', "\wr"] \arrow[d, "b"] \\ X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

soit commutatif.

Démonstration : Soit $(S;t;b)$ un représentant de β et $(R';s';a')$ un représentant de α. On a donc $\alpha = P_\Sigma(a')\circ P_\Sigma(s')^{-1}$ et

$\beta = P_\Sigma(b)\circ P_\Sigma(t)^{-1}$ si bien que l'hypothèse donne

$P_\Sigma(b)P_\Sigma(t)^{-1}P_\Sigma(u) = P_\Sigma(u')P_\Sigma(a')P_\Sigma(s')^{-1}$.

D'après FR2 on a un diagramme commutatif

$$\begin{tikzcd} R'' \arrow[r, dashed, "n"] \arrow[d, "r"', "\wr"] & S \arrow[d, "t", "\wr"'] \\ R' \arrow[r, "us'"'] & Y \end{tikzcd}$$

On en déduit le diagramme

$$\begin{tikzcd} X \arrow[r, "u"] & Y \\ R'' \arrow[u, "s'r"', "\wr"] \arrow[r, "n"] \arrow[d, "a'r"'] & S \arrow[u, "t", "\wr"'] \arrow[d, "b"] \\ X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

dans lequel le carré supérieur est commutatif. Compte tenu de l'hypothèse on a donc $P_\Sigma(u'a'r-bn) = P_\Sigma(u')P_\Sigma(a')P_\Sigma(r)-P_\Sigma(b)P_\Sigma(n)$
$= P_\Sigma(b)P_\Sigma(t)^{-1}P_\Sigma(u)P_\Sigma(s')P_\Sigma(r)-P_\Sigma(b)P_\Sigma(n)$
$= P_\Sigma(b)P_\Sigma(t)^{-1}P_\Sigma(t)P_\Sigma(s)-P_\Sigma(b)P_\Sigma(n) = 0$.

Donc il existe $r' \in \mathrm{Hom}_{\mathbf{C}}(R;R'')\cap\Sigma$ tel que $u'a'rr' = bnr'$. On pose $s = s'rr' \in \Sigma$, $a = a'rr'$ et $m = nr'$.

1.22. COROLLAIRE : *Si* $u \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$ *est un monomorphisme (respectivement épimorphisme) alors* $P_\Sigma(u) \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(X;Y)$ *est un monomorphisme (respectivement épimorphisme).*

Démonstration : Soit $\alpha,\beta \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(Z;X)$ tels que $P_\Sigma(u)\alpha = P_\Sigma(u)\beta$. Compte tenu de FR2 on peut trouver des représentants de α et β de la forme $(W;s;a)$ et $(W;s;b)$ respectivement. On a donc
$P_\Sigma(u)P_\Sigma(a)P_\Sigma(s)^{-1} = P_\Sigma(u)P_\Sigma(b)P_\Sigma(s)^{-1}$ d'où $P_\Sigma(ua-ub) = 0$. Donc il existe $t \in \mathrm{Hom}_{\mathbf{C}_\Sigma}(V;W)\cap\Sigma$ tel que $uat = ubt$. Comme u est un monomorphisme on a donc $at = bt$ d'où $P(a) = P(b)$; donc $\alpha = \beta$. Ainsi $P_\Sigma(u)$ est un monomorphisme. La démonstration est analogue dans le cas d'un

épimorphisme.

1.23 Soit $\mathbf{B}$ une sous-catégorie pleine d'une catégorie $\mathbf{C}$. Si $\Sigma_{\mathbf{C}}$ est un système multiplicatif dans $\mathbf{C}$ posons $\Sigma_{\mathbf{B}} = F\ell(\mathbf{B})\cap\Sigma_{\mathbf{C}}$.

Si $\Sigma_{\mathbf{B}}$ est un système multiplicatif dans $\mathbf{B}$ on peut alors définir la catégorie localisée $\mathbf{B}_{\Sigma_{\mathbf{B}}}$ et le foncteur canonique $P_{\Sigma_{\mathbf{B}}} : \mathbf{B} \to \mathbf{B}_{\Sigma_{\mathbf{B}}}$.

Le foncteur d'inclusion $\mathbf{B} \to \mathbf{C}$ induit un unique foncteur F_Σ qui rend commutatif le diagramme

$$\begin{tikzcd} \mathbf{B} \arrow[r] \arrow[d, "P_{\Sigma_{\mathbf{B}}}"'] & \mathbf{C} \arrow[d, "P_{\Sigma_{\mathbf{C}}}"] \\ \mathbf{B}_{\Sigma_{\mathbf{B}}} \arrow[r, "F_\Sigma"'] & \mathbf{C}_{\Sigma_{\mathbf{C}}} \end{tikzcd}$$

En effet cela résulte de la propriété universelle du foncteur $P_{\Sigma_{\mathbf{B}}}$ puisque le foncteur composé $\mathbf{B} \to \mathbf{C} \xrightarrow{P_{\Sigma_{\mathbf{C}}}} \mathbf{C}_{\Sigma_{\mathbf{C}}}$ transforme évidemment les éléments de $\Sigma_{\mathbf{B}}$ en des isomorphismes de $\mathbf{C}_{\Sigma_{\mathbf{C}}}$.

1.24. PROPOSITION : *Supposons que l'une des conditions suivantes est satisfaite :*

i) Pour tout $s \in \mathrm{Hom}_{\mathbf{C}}(X;Y)\cap\Sigma_{\mathbf{C}}$ *où* $Y \in \mathrm{Ob}\mathbf{B}$, *il existe* $f \in \mathrm{Hom}_{\mathbf{C}}(Z;X)$, *où* $Z \in \mathrm{Ob}\mathbf{B}$, *tel que* $sf \in \Sigma_{\mathbf{B}}$.

ii) Pour tout $s \in \mathrm{Hom}_{\mathbf{C}}(X;Y)\cap\Sigma$ *où* $X \in \mathrm{Ob}\mathbf{B}$, *il existe* $f \in \mathrm{Hom}_{\mathbf{C}}(Y;Z)$, *où* $Z \in \mathrm{Ob}\mathbf{B}$, *tel que* $fs \in \Sigma_{\mathbf{B}}$.

Alors le foncteur $\Gamma_\Sigma : \mathbf{B}_{\Sigma_{\mathbf{B}}} \to \mathbf{C}_{\Sigma_{\mathbf{C}}}$ *est pleinement fidèle (si bien que* $\mathbf{B}_{\Sigma_{\mathbf{B}}}$ *s'identifie à une sous-catégorie pleine de* $\mathbf{C}_{\Sigma_{\mathbf{C}}}$ *).*

Démonstration : Soit $X,Y \in \mathrm{Ob}\mathbf{B}$; il faut montrer que l'application

$$\mathrm{Hom}_{\mathbf{B}_{\Sigma_{\mathbf{B}}}}(X;Y) \to \mathrm{Hom}_{\mathbf{C}_{\Sigma_{\mathbf{C}}}}(F_\Sigma(X);F_\Sigma(Y))$$

est bijective.

Soit $\alpha,\beta \in \mathrm{Hom}_{\mathbf{B}_{\Sigma_{\mathbf{B}}}}(X;Y)$ représentés respectivement par $(U;s;a)$, $(V;t;b) \in M^{\mathbf{B}}_{X;Y}$. Supposons que $F_\Sigma(\alpha) = F_\Sigma(\beta)$. Or $F_\Sigma(\alpha)$ et $F_\Sigma(\beta)$ sont représentés respectivement par $(U;s;a) \in M^{\mathbf{C}}_{X,Y}$ et $(V;t;b) \in M^{\mathbf{C}}_{X,Y}$ si bien que l'on a un diagramme commutatif dans $\mathbf{C}$

$$\begin{tikzcd}
 & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\
X & T \arrow[l, dashed, "r"', "\sim"] \arrow[u, dashed, "p"'] \arrow[r, dashed, "c"] \arrow[d, dashed, "p'"] & Y \\
 & V \arrow[ul, "t", "\sim"'] \arrow[ur, "b"'] &
\end{tikzcd}$$

D'après i) il existe $f : S \to T$ avec $S \in \mathrm{Ob}\mathbf{B}$ tel que $rf \in \Sigma_{\mathbf{B}}$.

On a alors un diagramme commutatif dans $\mathbf{B}$

$$\begin{tikzcd}
 & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\
X & S \arrow[l, dashed, "rf"', "\sim"] \arrow[u, dashed, "pf"'] \arrow[r, dashed, "cf"] \arrow[d, dashed, "p'f"] & Y \\
 & V \arrow[ul, "t", "\sim"'] \arrow[ur, "b"'] &
\end{tikzcd}$$

donc $\alpha = \beta$ dans $\mathbf{B}_{\Sigma_{\mathbf{B}}}$.

Maintenant soit $\beta \in \mathrm{Hom}_{\mathbf{C}_{\Sigma_{\mathbf{C}}}}(F_\Sigma(X);F_\Sigma(Y))$ représenté par $(Z;s;b) \in M^{\mathbf{C}}_{X,Y}$.

D'après i) il existe $f : W \to Z$ avec $W \in \mathrm{Ob}\mathbf{B}$ tel que $sf \in \Sigma_{\mathbf{B}}$. Soit $\alpha \in \mathrm{Hom}_{\mathbf{B}_{\Sigma_{\mathbf{B}}}}(X;Y)$ le morphisme représenté par $(W;sf;bf) \in M^{\mathbf{B}}_{X,Y}$. On a dans $\mathbf{C}$ le diagramme commutatif

$$\begin{tikzcd}
 & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "b"] & \\
X & W \arrow[l, dashed, "sf"', "\sim"] \arrow[u, dashed, "f"'] \arrow[r, dashed, "bf"] \arrow[d, dashed, "1_W"] & Y \\
 & W \arrow[ul, "sf", "\sim"'] \arrow[ur, "bf"'] &
\end{tikzcd}$$

donc $F_\Sigma(\alpha) = \beta$ dans $\mathbf{C}_{\Sigma_{\mathbf{C}}}$.

Dans le cas où la condition ii) est satisfaite on utilise la définition équivalente duale des morphismes dans la catégorie localisée.

§ 2. LES CATEGORIES TRIANGULEES

2.1 Soit $\mathfrak{C}$ une catégorie additive.

Un foncteur additif $T : \mathfrak{C} \to \mathfrak{C}$ qui est un automorphisme de catégorie sera appelé un foncteur de translation.

Soit $\mathfrak{C}$ et $\mathfrak{C}'$ deux catégories additives munies respectivement des foncteurs de translation T et T'. On dira qu'un foncteur additif $F : \mathfrak{C} \to \mathfrak{C}'$ est compatible avec les foncteurs de translation (ou est un foncteur gradué) si $F \circ T = T' \circ F$.

Si $F,G : \mathfrak{C} \to \mathfrak{C}'$ sont deux foncteurs gradués, alors un morphisme de foncteurs gradués est un morphisme de foncteurs $\varphi : F \to G$ tel que pour tout $X \in \mathrm{Ob}\mathfrak{C}$ on ait un diagramme commutatif

$$\begin{tikzcd} F(T(X)) \arrow[r, "\varphi(T(X))"] \arrow[d] & G(T(X)) \arrow[d] \\ T'(F(X)) \arrow[r, "T'(\varphi(X))"'] & T'(G(X)) \end{tikzcd}$$

dans lequel les flèches verticales sont des isomorphismes canoniques.

2.2. Pour tout $n \in \mathbb{Z}$ on notera T^n le $n^{\text{ième}}$ itéré de T. Considérons un foncteur $F : \mathfrak{C} \to \mathfrak{D}$; on posera $F^n = F \circ T^n$. En particulier pour tout $X,Y \in \mathrm{Ob}\mathfrak{C}$ on posera

$$\mathrm{Hom}^n(X;Y) = \mathrm{Hom}_{\mathfrak{C}}(X;T^n(Y))$$

Soit $X,Y,Z \in \mathrm{Ob}\mathfrak{C}$, $u \in \mathrm{Hom}^p(X;Y)$ et $v \in \mathrm{Hom}^q(Y;Z)$. On définira $v \circ u \in \mathrm{Hom}^{p+q}(X;Z)$ en posant $v \circ u = T^p(v) \circ u$.

2.3. Soit $\mathfrak{C}$ une catégorie additive munie d'un foncteur de translation T.

Un triangle dans $\mathfrak{C}$ est un sextuple $(X;Y;Z;u;v;w)$ où $X,Y,Z \in \mathrm{Ob}\mathfrak{C}$, $u \in \mathrm{Hom}^{\circ}(X;Y)$, $v \in \mathrm{Hom}^{\circ}(Y;Z)$ et $w \in \mathrm{Hom}^1(Z;X)$.

Un triangle est représenté par une suite dans $\mathfrak{C}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

ou par un diagramme

$$\begin{tikzcd} & Z \arrow[dl, "w"'] & \\ X \arrow[rr, "u"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

dans lequel le morphisme w est de degré 1.

2.4 Un morphisme du triangle $(X;Y;Z;u;v;w)$ dans le triangle $(X';Y';Z';u';v';w')$ est un triple $(f;g;h)$ de morphismes $f \in \mathrm{Hom}_{\mathfrak{C}}(X;X')$, $g \in \mathrm{Hom}_{\mathfrak{C}}(Y;Y')$ et $h \in \mathrm{Hom}_{\mathfrak{C}}(Z;Z')$ tels que le diagramme

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X') \end{tikzcd}$$

soit commutatif.

2.5 Une structure triangulée sur une catégorie additive $\mathfrak{C}$ consiste en la donnée d'un foncteur de translation T et d'une famille $\mathcal{T}$ de triangles, appelés les triangles distingués, qui vérifie les axiomes suivants :

TR1 : a) Tout triangle isomorphe à un triangle de $\mathcal{T}$ appartient à $\mathcal{T}$.

b) Tout morphisme $u \in \mathrm{Hom}_{\mathfrak{C}}(X;Y)$ est la base d'un triangle $(X;Y;Z;u;v;w) \in \mathcal{T}$.

c) Si $X \in \mathrm{Ob}\mathfrak{C}$ alors $(X;X;0;1_X;o;o) \in \mathcal{T}$.

TR2 : $(X;Y;Z;u;v;w) \in \mathcal{T}$ si et seulement si $(Y;Z;T(X);v;w;-T(u)) \in \mathcal{T}$.

TR3 : Si $(X;Y;Z;u;v;w) \in \mathcal{T}$ et $(X';Y';Z';u';v';w') \in \mathcal{T}$ et si $f \in \mathrm{Hom}_{\mathfrak{C}}(X;X')$ et $g \in \mathrm{Hom}_{\mathfrak{C}}(Y;Y')$ vérifient $gu = u'f$, alors il existe $h \in \mathrm{Hom}_{\mathfrak{C}}(Z;Z')$ tel que $(f;g;h) : (X;Y;Z;u;v;w) \to (X';Y';Z';u';v';w')$ soit un morphisme de triangles.

(2.5.1)
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dashed, "h"'] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X')
\end{tikzcd}$$

TR4 : Etant donnés $(X;Y;Z';u;v;w) \in \mathcal{T}$, $(Y;Z;X';u';v';w') \in \mathcal{T}$ et $(X;Z;Y';u'';v'';w'') \in \mathcal{T}$ tels que $u'' = u'u$, il existe $f \in \mathrm{Hom}_{\mathfrak{C}}(Z';Y')$ et $g \in \mathrm{Hom}_{\mathfrak{C}}(Y';Z')$ tels que $(Z';Y';X';f;g;T(v)w') \in \mathcal{T}$ et tels que $(1_X;u';f)$ et $(u;1_Z;g)$ soient des morphismes de triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "1_X"'] & Y \arrow[r, "v"] \arrow[d, "u'"'] & Z' \arrow[r, "w"] \arrow[d, dashed, "f"'] & T(X) \arrow[d, "1_{T(X)}"] \\
X \arrow[r, "u''"] \arrow[d, "u"'] & Z \arrow[r, "v''"] \arrow[d, "1_Z"'] & Y' \arrow[r, "w''"] \arrow[d, dashed, "g"'] & T(X) \arrow[d, "T(u)"] \\
Y \arrow[r, "u'"] & Z \arrow[r, "v'"] & X' \arrow[r, "w'"] \arrow[d, "T(v)w'"'] & T(Y) \arrow[dl, "T(v)"] \\
 & & T(Z') &
\end{tikzcd}$$

2.6. *Remarques :*

i) En vertu de TR1 b) et de TR2 tout morphisme de $\mathfrak{C}$ est le premier ou le deuxième côté d'un triangle distingué.

ii) En vertu de TR2 et TR3 si $(X;Y;Z;u;v;w) \in \mathcal{T}$ et $(X';Y';Z';u';v';w') \in \mathcal{T}$ et si $g \in \mathrm{Hom}_{\mathfrak{C}}(Y;Y')$ et $h \in \mathrm{Hom}_{\mathfrak{C}}(Z;Z')$ vérifient $hv = v'g$, alors il existe $f \in \mathrm{Hom}_{\mathfrak{C}}(X;X')$ tel que $(f;g;h)$ soit un morphisme de triangles.

2.7 *Remarque :* Une discussion détaillée des axiomes précédents, en particulier de l'axiome de l'octaèdre TR4, se trouve dans ([ES]; §1.1). On notera aussi que les triangles distingués sont parfois appelés des triangles exacts.

2.8 PROPOSITION : *La composition de deux morphismes consécutifs dans un triangle distingué*

$$\begin{tikzcd}
 & Z \arrow[dl, "w"'] & \\
X \arrow[rr, "u"'] & & Y \arrow[ul, "v"']
\end{tikzcd}$$

est le morphisme nul.

Démonstration : D'après TR2, il suffit de montrer que $vu = 0$. En vertu de TR2, TR1 c) et TR3 il existe un morphisme $h \in \mathrm{Hom}_{\mathfrak{C}}(T(X);0)$ tel que le diagramme

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "v"'] & Z \arrow[r, "w"] \arrow[d, "1_Z"'] & T(X) \arrow[r, "-T(u)"] \arrow[d, dashed, "h"] & T(Y) \arrow[d, "T(v)"] \\
Z \arrow[r, "1_Z"'] & Z \arrow[r, "o"'] & 0 \arrow[r, "o"'] & T(Z)
\end{tikzcd}$$

soit commutatif. On a donc $T(vu) = 0$ d'où la conclusion puisque T est un automorphisme de catégories.

2.9 Soit $\mathfrak{C}$ une catégorie triangulée et $\mathfrak{A}$ une catégorie abélienne. On dit qu'un foncteur additif $F : \mathfrak{C} \to \mathfrak{A}$ est un foncteur cohomologique si F associe à tout triangle distingué

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

dans $\mathbf{C}$, une suite exacte longue

$$\cdots \to F^n(X) \xrightarrow{u^*} F^n(Y) \xrightarrow{v^*} F^n(Z) \xrightarrow{w^*} F^{n+1}(X) \to \cdots$$

dans $\mathbf{A}$.

2.10 PROPOSITION : *Pout tout* $M \in \mathrm{Ob}\mathbf{C}$ *les foncteurs*

$$\mathrm{Hom}_{\mathbf{C}}(M;-) : \mathbf{C}^{\circ} \to \mathbf{Ab}$$

et

$$\mathrm{Hom}_{\mathbf{C}}(-;M) : \mathbf{C} \to \mathbf{Ab}$$

sont des foncteurs cohomologiques.

Démonstration : Soit $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ un triangle distingué dans $\mathbf{C}$. D'après TR2 il suffit de montrer que la suite

$$\mathrm{Hom}_{\mathbf{C}}(M;X) \xrightarrow{u_*} \mathrm{Hom}_{\mathbf{C}}(M;Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathbf{C}}(M;Z)$$

est exacte dans $\mathbf{Ab}$.

La proposition 2.8 montre déjà que $v_*u_* = 0$. Soit $\varphi \in \mathrm{Ker}v_*$; en vertu de TR1 c) et de la remarque 2.6. ii) il existe un morphisme $\psi \in \mathrm{Hom}_{\mathbf{C}}(M;X)$ tel que le diagramme

$$\begin{tikzcd}
M \arrow[r, "1_M"] \arrow[d, dashed, "\psi"'] & M \arrow[r, "0"] \arrow[d, "\varphi"'] & 0 \arrow[r, "0"] \arrow[d] & T(M) \arrow[d, dashed, "T(\psi)"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] \arrow[u, phantom, ""] & Z \arrow[r, "w"'] & T(X)
\end{tikzcd}$$

soit commutatif; on a donc $u_*(\psi) = \varphi$; ainsi $\mathrm{Ker}v_* \subset \mathrm{Im}u_*$. La démonstration est analogue pour le deuxième foncteur.

2.11 COROLLAIRE : *Si, dans la situation de l'axiome* TR3, *les morphismes* f *et* g *sont des isomorphismes, alors* h *est aussi un isomorphisme.*

Démonstration : Appliquons les foncteurs cohomologiques $\mathrm{Hom}_{\mathbf{C}}(Z';-)$ et $\mathrm{Hom}_{\mathbf{C}}(-;Z)$ au diagramme (2.5.1). On obtient ainsi les diagrammes commutatifs suivants dans lesquels les lignes sont exactes

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';X) \arrow[r] \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathbf{C}}(Z';Y) \arrow[r] \arrow[d, "g_*"'] & \mathrm{Hom}_{\mathbf{C}}(Z';Z) \arrow[r] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathbf{C}}(Z';T(X)) \arrow[r] \arrow[d, "T(f)_*"'] & \mathrm{Hom}_{\mathbf{C}}(Z';T(Y)) \arrow[r] \arrow[d, "T(g)_*"'] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';X') \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';Y') \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';Z') \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';T(X')) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z';T(Y')) \arrow[r] & \cdots
\end{tikzcd}$$

et

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_{\mathbf{C}}(T(Y);Z) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(T(X);Z) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Z;Z) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(Y;Z) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(X;Z) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(T(Y');Z) \arrow[r] \arrow[u, "T(g)^*"] & \mathrm{Hom}_{\mathbf{C}}(T(X');Z) \arrow[r] \arrow[u, "T(f)^*"] & \mathrm{Hom}_{\mathbf{C}}(Z';Z) \arrow[r] \arrow[u, "h^*"] & \mathrm{Hom}_{\mathbf{C}}(Y';Z) \arrow[r] \arrow[u, "g^*"] & \mathrm{Hom}_{\mathbf{C}}(X';Z) \arrow[r] \arrow[u, "f^*"] & \cdots
\end{tikzcd}$$

D'après le lemme des cinq h_* et h^* sont des isomorphismes. Il existe donc des morphismes $\varphi, \psi \in \mathrm{Hom}_{\mathbf{C}}(Z';Z)$ tels que $h_*(\varphi) = 1_Z$, et $h^*(\psi) = 1_Z$. Or si $h\varphi = 1_Z$, et $\psi h = 1_Z$ on en déduit que $\varphi = \psi$ donc h est un isomorphisme.

2.12 Soit $\mathbf{C}$ et $\mathbf{C}'$ deux catégories additives triangulées. On dit qu'un foncteur additif $F : \mathbf{C} \to \mathbf{C}'$ est un ∂-foncteur si F est un foncteur gradué et si F transforme les triangles distingués dans $\mathbf{C}$ en des triangles distingués dans $\mathbf{C}'$.

2.13 Soit $\mathbf{C}$ une catégorie additive, triangulée par la donnée d'un foncteur de translation T et d'une famille $\mathcal{T}$ de triangles distingués.

La catégorie opposée $\mathbf{C}^{\circ}$ est alors munie canoniquement d'une structure triangulée, définie de la façon suivante. Le foncteur de translation $\tilde{T}$ sur $\mathbf{C}^{\circ}$ est défini en posant $\tilde{T} = T^{-1}$. Il en résulte que pour tout $X,Y \in \mathrm{Ob}\mathbf{C} = \mathrm{Ob}\mathbf{C}^{\circ}$ et pour tout $n \in \mathbb{Z}$, on a un isomorphisme

$$\mathrm{Hom}_{\mathbf{C}^{\circ}}(Y;\tilde{T}^n(X)) = \mathrm{Hom}_{\mathbf{C}}(X;T^n(Y)).$$

La famille $\tilde{T}$ des triangles $(X;Y;Z;u;v;w)$ dans $\mathfrak{C}^{\circ}$ tels que $(Z;Y;X;w;v;u) \in T$ constitue alors la famille des triangles distingués dans $\mathfrak{C}^{\circ}$ comme on le vérifie aussitôt.

2.14 Soit $\mathfrak{C}$ une catégorie additive, triangulée par la donnée d'un foncteur de translation T et d'une famille T de triangles distingués.

Soit Σ un système multiplicatif dans $\mathfrak{C}$. Le système multiplicatif Σ est compatible avec la structure triangulée de $\mathfrak{C}$ si en plus de FR1, FR2 et FR3 la famille Σ vérifie les deux propriétés suivantes :

FR4 : $s \in \Sigma$ si et seulement si $T(s) \in \Sigma$.

FR5 : Si, dans la situation de l'axiome TR3, les morphismes $f,g \in \Sigma$, alors $h \in \Sigma$.

2.15 THEOREME : *Soit $\mathfrak{C}$ une catégorie additive triangulée et Σ un système multiplicatif dans $\mathfrak{C}$ compatible avec la structure triangulée de $\mathfrak{C}$. Alors la catégorie localisée $\mathfrak{C}_{\Sigma}$ est munie d'une structure triangulée telle que le foncteur canonique $P_{\Sigma} : \mathfrak{C} \to \mathfrak{C}_{\Sigma}$ soit un ∂-foncteur.*

Démonstration : D'après FR4 et la propriété universelle de P_{Σ} il existe un unique foncteur T_{Σ} tel que le diagramme

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "T"] \arrow[d, "P_{\Sigma}"'] & \mathfrak{C} \arrow[d, "P_{\Sigma}"] \\ \mathfrak{C}_{\Sigma} \arrow[r, dashed, "T_{\Sigma}"'] & \mathfrak{C}_{\Sigma} \end{tikzcd}$$

soit commutatif et T_{Σ} est évidemment un automorphisme de catégorie.

Alors T_{Σ} sera le foncteur de translation dans $\mathfrak{C}_{\Sigma}$. Maintenant soit T_{Σ} la famille des triangles $(X;Y;Z;\alpha;\beta;\gamma)$ dans $\mathfrak{C}_{\Sigma}$ qui sont isomorphes à l'image par le foncteur P_{Σ} d'un triangle distingué dans $\mathfrak{C}$. Le théorème 2.15 est alors une conséquence immédiate du résultat suivant .

2.16 LEMME : *La famille T_{Σ} vérifie les axiomes* TR1,TR2,TR3 *et* TR4.

Démonstration :

TR1 : a) et c) sont évidents.

b) Soit $\alpha \in \mathrm{Hom}_{\mathfrak{C}_{\Sigma}}(X;Y)$ représenté par $(Z;s;a) \in M_{X,Y}$. Il existe dans $\mathfrak{C}$ un triangle distingué

$$Z \xrightarrow{a} Y \xrightarrow{b} W \xrightarrow{c} T(Z)$$

On a alors dans $\mathfrak{C}_{\Sigma}$ un diagramme commutatif

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \arrow[r, "P_{\Sigma}(b)"] & W \arrow[r, "T_{\Sigma}P_{\Sigma}(s)P_{\Sigma}(c)"] & T_{\Sigma}(X) \\ Z \arrow[u, "P_{\Sigma}(s)"] \arrow[r, "P_{\Sigma}(a)"'] & Y \arrow[u, "1_Y"'] \arrow[r, "P_{\Sigma}(b)"'] & W \arrow[u, "1_W"'] \arrow[r, "P_{\Sigma}(c)"'] & T_{\Sigma}(Z) \arrow[u, "T_{\Sigma}(P_{\Sigma}(s))"'] \end{tikzcd}$$

dans lequel les flèches verticales sont des isomorphismes; donc $(X;Y;W;\alpha;P_{\Sigma}(b);T_{\Sigma}P_{\Sigma}(s)P_{\Sigma}(c)) \in T_{\Sigma}$.

TR2 est évident.

TR3 : Soit $(X;Y;Z;\alpha;\beta;\gamma) \in T_{\Sigma}$ et $(X';Y';Z';\alpha';\beta';\gamma') \in T_{\Sigma}$; il existe donc des triangles distingués $(A;B;C;u;v;w)$ et $(A';B';C';u';v';w')$ dans $\mathfrak{C}$ tels que, dans $\mathfrak{C}_{\Sigma}$, les diagrammes

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\xi"'] & Y \arrow[r, "\beta"] \arrow[d, "\eta"'] & Z \arrow[r, "\gamma"] \arrow[d, "\zeta"'] & T_{\Sigma}(X) \arrow[d, "T_{\Sigma}(\xi)"] \\ A \arrow[r, "P_{\Sigma}(u)"'] & B \arrow[r, "P_{\Sigma}(v)"'] & C \arrow[r, "P_{\Sigma}(w)"'] & T_{\Sigma}(A) \end{tikzcd}$$

et

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "\xi'"'] & Y' \arrow[r, "\beta'"] \arrow[d, "\eta'"'] & Z' \arrow[r, "\gamma'"] \arrow[d, "\zeta'"'] & T_\Sigma(X') \arrow[d, "T_\Sigma(\xi')"] \\ A' \arrow[r, "P_\Sigma(u')"'] & B' \arrow[r, "P_\Sigma(v')"'] & C' \arrow[r, "P_\Sigma(w')"'] & T_\Sigma(A') \end{tikzcd}$$

soient commutatifs et les flèches verticales soient des isomorphismes. Supposons que l'on a, dans $\mathbb{C}_\Sigma$, un diagramme

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & Y \arrow[r, "\beta"] \arrow[d, "\psi"'] & Z \arrow[r, "\gamma"] & T_\Sigma(X) \arrow[d, "T_\Sigma(\varphi)"] \\ X' \arrow[r, "\alpha'"'] & Y' \arrow[r, "\beta'"'] & Z' \arrow[r, "\gamma'"'] & T\ (X') \end{tikzcd}$$

dans lequel le premier carré est commutatif. On en déduit, dans $\mathbb{C}_\Sigma$, un diagramme

$$\begin{tikzcd} A \arrow[r, "P_\Sigma(u)"] \arrow[d, "\xi'\varphi\xi^{-1}"'] & B \arrow[r, "P_\Sigma(v)"] \arrow[d, "\eta'\psi\eta^{-1}"'] & C \arrow[r, "P_\Sigma(w)"] & T_\Sigma(A) \arrow[d, "T_\Sigma(\xi'\varphi\xi^{-1})"] \\ A' \arrow[r, "P_\Sigma(u')"'] & B' \arrow[r, "P_\Sigma(v')"'] & C' \arrow[r, "P_\Sigma(w')"'] & T_\Sigma(A') \end{tikzcd}$$

dans lequel le premier carré est commutatif.

D'après le corollaire 1.21 on peut trouver un représentant (R;s;a) de $\xi'\varphi\xi^{-1}$, un représentant (S;t;b) de $\eta'\psi\eta^{-1}$ et un morphisme $m \in \mathrm{Hom}_{\mathbb{C}}(R;S)$ tels que, dans le diagramme en traits pleins

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A) \\ R \arrow[u, "s"] \arrow[u, phantom, "\wr"'] \arrow[r, "m"] \arrow[d, "a"'] & S \arrow[u, "t"] \arrow[u, phantom, "\wr"'] \arrow[r, dashed] \arrow[d, "b"'] & M \arrow[u, dashed, "r"] \arrow[u, phantom, "\wr"'] \arrow[r, dashed] \arrow[d, dashed, "c"'] & T(R) \arrow[u, "T(s)"'] \arrow[d, "T(a)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, dashed, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

les carrés soient commutatifs.

D'après TR1 b) et TR3 on peut compléter ce diagramme en traits pleins en un diagramme de triangles et de morphismes de triangles et d'après FR5 on a $r \in \Sigma$. Soit $\rho \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(C;C')$ la classe de $(M;r;c) \in M_{C,C'}$.

Alors le morphisme $\Theta = \zeta'^{-1}\rho\zeta \in \mathrm{Hom}_{\mathbb{C}_\Sigma}(Z;Z')$ rend commutatif le diagramme

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & Y \arrow[r, "\beta"] \arrow[d, "\psi"'] & Z \arrow[r, "\gamma"] \arrow[d, dashed, "\theta"'] & T_\Sigma(X) \arrow[d, "T_\Sigma(\varphi)"] \\ X' \arrow[r, "\alpha'"'] & Y' \arrow[r, "\beta'"'] & Z' \arrow[r, "\gamma'"'] & T_\Sigma(X') \end{tikzcd}$$

TR4 se vérifie suivant le même principe.

2.17 Soit $\mathbb{B}$ une sous-catégorie pleine d'une catégorie triangulée $\mathbb{C}$. Alors $\mathbb{B}$ est une sous-catégorie triangulée de $\mathbb{C}$ si les deux conditions suivantes sont satisfaites :

a) $\mathbb{B}$ est stable par le foncteur de translation de $\mathbb{C}$

b) pour tout triangle de $\mathbb{C}$ dont les deux premiers sommets sont des objets de $\mathbb{B}$ alors le troisième sommet est aussi un objet de $\mathbb{B}$.

2.18 PROPOSITION : *Soit $\mathbb{C}$ et $\mathbb{C}'$ deux catégories triangulées et $F : \mathbb{C} \to \mathbb{C}'$ un ∂-foncteur plein. Si $X \to Y \to Z \to T(X)$ est un triangle distingué dans $\mathbb{C}'$ dont les deux premiers sommets (respectivement les deux derniers sommets) sont isomorphes à des objets de l'image de F, alors le troisième sommet (respectivement le premier sommet) est isomorphe à un objet de l'image de F.*

Démonstration : C'est une conséquence immédiate des propriétés TR1 b) et TR3, du corollaire 2.11, et des remarques 2.6.

§ 3. LA CATEGORIE C($\mathbb{A}$)

3.1 Soit $\mathbb{A}$ une catégorie.

Un $\mathbb{A}$-objet gradué est une famille $X^{\cdot} = \{X^i\}_{i\in\mathbb{Z}}$ d'objets X^i de $\mathbb{A}$; l'objet X^i s'appelle la composante de degré i de $X^{\cdot}$.

Soit $X^{\cdot}$ et $Y^{\cdot}$ deux $\mathbb{A}$-objets gradués et soit un entier $p\in\mathbb{Z}$. On note $\mathrm{Hom}^p(X^{\cdot};Y^{\cdot})$ l'ensemble des morphismes gradués de degré p, c'est-à-dire l'ensemble des familles $u = \{u^i\}_{i\in\mathbb{Z}}$ de morphismes $u^i \in \mathrm{Hom}_{\mathbb{A}}(X^i;Y^{i+p})$.

Soit $X^{\cdot},Y^{\cdot},Z^{\cdot}$ des $\mathbb{A}$-objets gradués, $u\in\mathrm{Hom}^p(X^{\cdot};Y^{\cdot})$ et $v\in\mathrm{Hom}^q(Y^{\cdot};Z^{\cdot})$. On définit $vu\in\mathrm{Hom}^{p+q}(X^{\cdot};Z^{\cdot})$ en posant $(vu)^i = v^{i+p}u^i$ pour tout $i\in\mathbb{Z}$.

3.2 Soit $\mathbb{A}$ une catégorie additive.

Un complexe d'objets de $\mathbb{A}$ est un couple $(X^{\cdot};d_X)$ formé d'un $\mathbb{A}$-objet gradué $X^{\cdot}$ et d'un morphisme gradué $d_X\in\mathrm{Hom}^1(X^{\cdot};X^{\cdot})$ tel que $d_Xd_X = 0$. Ce morphisme d_X s'appelle la différentielle du complexe. Lorsqu'aucune confusion n'en résulte on écrit d au lieu de d_X et on peut même omettre d'indiquer la différentielle dans la donnée du complexe . De plus un complexe est souvent représenté par une suite

$$\dots \to X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} X^{p+1} \to \dots .$$

Si $(X^{\cdot};d_X)$ et $(Y^{\cdot};d_Y)$ sont deux complexes d'objets de $\mathbb{A}$, un morphisme de complexes $u : (X^{\cdot};d_X) \to (Y^{\cdot};d_Y)$ est un morphisme gradué $u\in\mathrm{Hom}^{\circ}(X^{\cdot};Y^{\cdot})$ tel que $d_Yu = ud_X$.

Les complexes d'objets de $\mathbb{A}$ et les morphismes de complexes forment une catégorie que l'on désigne par C($\mathbb{A}$).

3.3 PROPOSITION : *Si la catégorie $\mathbb{A}$ est additive (respectivement abélienne) alors la catégorie C($\mathbb{A}$) est additive (respectivement abélienne).*

Démonstration : La démonstration, complètement élémentaire, est laissée au lecteur.

3.4 On note $C_0 : \mathbb{A} \to C(\mathbb{A})$ le foncteur additif pleinement fidèle qui à un objet X de $\mathbb{A}$ associe le complexe $(C_0(X);d_{C_0(X)})$ défini en posant

$$C_0(X)^i = \begin{cases} X & \text{si } i=0 \\ 0 & \text{si } i\neq 0 \end{cases} \quad \text{et} \quad d_{C_0(X)} = 0$$

3.5 Soit $\mathbb{A}$ une catégorie abélienne.

Pour chaque entier $k\in\mathbb{Z}$ on définit un foncteur additif $H^k : C(\mathbb{A}) \to \mathbb{A}$ en associant à chaque objet $(X^{\cdot};d_X)$ de C($\mathbb{A}$) l'objet

$$H^k(X^{\cdot}) = \frac{\mathrm{Ker}(d_X^k : X^k \to X^{k+1})}{\mathrm{Im}(d_X^{k-1} : X^{k-1} \to X^k)} \text{ de } \mathbb{A} .$$

On obtient aisni un δ-foncteur exact $H^* = \{H^k\}_{k\in\mathbb{Z}}$ appelé foncteur de cohomologie.

3.6 On dit qu'un objet $(X^{\cdot};d_X)$ de C($\mathbb{A}$) est acyclique si $H^k(X^{\cdot}) = 0$ pour tout $k\in\mathbb{Z}$.

3.7 On dit qu'un morphisme $u\in\mathrm{Hom}_{C(\mathbb{A})}(X^{\cdot};Y^{\cdot})$ est un isomorphisme homologique (en abrégé h.i.) si $H^k(u)\in\mathrm{Hom}_{\mathbb{A}}(H^k(X^{\cdot});H^k(Y^{\cdot}))$ est un isomorphisme pour tout $k\in\mathbb{Z}$.

On notera que les isomorphismes homologiques sont appelés des quasi-isomorphismes par certains auteurs.

(Dans les diagrammes on écrira $X^{\cdot} \xrightarrow[u]{\sim} Y^{\cdot}$ pour indiquer que u est un isomorphisme homologique).

3.8 Soit $\mathbb{A}$ une catégorie additive et soit $u,v \in Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$.

On dit que u est homotope à v (et on écrit $u\sim v$) s'il existe un morphisme gradué $k \in Hom^{-1}(X^\cdot;Y^\cdot)$ tel que $u-v = d_Y k + k d_X$. Ce morphisme k s'appelle un opérateur d'homotopie.

3.9 On vérifie aisément que la relation d'homotopie est une relation d'équivalence dans l'ensemble $Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$, compatible avec la structure de groupe abélien de $Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$. De plus l'ensemble $\Sigma_{C(\mathbb{A})}(X^\cdot;Y^\cdot) = \{u \in Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot) | u\sim 0\}$ est un sous-groupe du groupe $Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$.
Enfin la relation d'homotopie est compatible avec la composition des morphismes.

3.10 On dit qu'un morphisme $u \in Hom_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$ est une équivalence d'homotopie s'il existe un morphisme $v \in Hom_{C(\mathbb{A})}(Y^\cdot;X^\cdot)$ tel que $vu\sim 1_{X^\cdot}$ et $uv\sim 1_{Y^\cdot}$.

3.11 Supposons que la catégorie $\mathbb{A}$ est abélienne. Il est facile de voir que si $u\sim v$ alors $H^k(u) = H^k(v)$ pour tout $k \in \mathbb{Z}$. Il en résulte que si u est un isomorphisme homologique, alors tous les éléments qui se trouvent dans la classe d'homotopie de u sont aussi des isomorphismes homologiques.

3.12 Soit $\mathbb{A}$ une catégorie additive.

On désigne par $C^+(\mathbb{A})$ (respectivement $C^-(\mathbb{A})$, respectivement $C^b(\mathbb{A})$) la sous-catégorie pleine de $C(\mathbb{A})$ formée des objets $X^\cdot$ de $C(\mathbb{A})$ pour lesquels il existe un entier $k \in \mathbb{Z}$ tel que $X^i = 0$ pour tout $i < k$ (respectivement $i>k$, respectivement $|i|>k$). Les objets de $C^+(\mathbb{A})$ (respectivement $C^-(\mathbb{A})$, respectivement $C^b(\mathbb{A})$) s'appellent les complexes bornés inférieurement (respectivement bornés supérieurement, respectivement bornés). On notera que $C^b(\mathbb{A}) = C^+(\mathbb{A}) \cap C^-(\mathbb{A})$. Pour désigner l'une ou l'autre des catégories $C(\mathbb{A})$, $C^+(\mathbb{A})$, $C^-(\mathbb{A})$ ou $C^b(\mathbb{A})$ on écrira souvent $C^*(\mathbb{A})$ (l'indice * pouvant donc prendre les valeurs $\emptyset$, +, - ou b).

3.13 Les résultats de ce paragraphe se généralisent aisément aux multicomplexes.

On va développer brièvement le cas des bicomplexes. Soit $\mathbb{A}$ une catégorie additive.
Un $\mathbb{A}$-objet bigradué est une famille $X^{\cdot\cdot} = \{X^{i,j}\}_{i,j\in\mathbb{Z}}$ où $X^{i,j} \in Ob\mathbb{A}$.

On note $Hom^{p,q}(X^{\cdot\cdot},Y^{\cdot\cdot})$ le groupe abélien des morphismes bigradués de bidegré (p;q), c'est-à-dire l'ensemble des familles $u = \{u^{i,j}\}_{i,j\in\mathbb{Z}}$ de morphismes $u^{i,j} \in Hom_{\mathbb{A}}(X^{i,j};Y^{i+p,j+q})$.

Un bicomplexe d'objets de $\mathbb{A}$ est un triple $(X^{\cdot\cdot};d_1;d_2)$ formé d'un $\mathbb{A}$-objet bigradué $X^{\cdot\cdot}$ et de deux morphismes $d_1 \in Hom^{1,0}(X^{\cdot\cdot};X^{\cdot\cdot})$ et $d_2 \in Hom^{0,1}(X^{\cdot\cdot};X^{\cdot\cdot})$ tels que $d_1d_1 = 0$, $d_2d_2 = 0$ et $d_1d_2+d_2d_1 = 0$.

Un morphisme de bicomplexes $u : (X^{\cdot\cdot};d_1;d_2) \to (Y^{\cdot\cdot};d_1;d_2)$ est un morphisme bigradué $u \in Hom^{0,0}(X^{\cdot\cdot};X^{\cdot\cdot})$ tels que $d_1u = ud_1$ et $d_2u = ud_2$.

Les bicomplexes d'objets de $\mathbb{A}$ et les morphismes de bicomplexes forment une catégorie que l'on désigne par $C^{\cdot\cdot}(\mathbb{A})$.

3.14 On définit un foncteur additif $s : C^{\cdot\cdot}(\mathbb{A}) \to C(\mathbb{A})$ en associant à un objet $(X^{\cdot\cdot};d_1;d_2)$ de $C^{\cdot\cdot}(\mathbb{A})$ le complexe $(s(X^{\cdot\cdot});d)$ obtenu en posant $s(X^\cdot)^p = \bigoplus_{i+j=p} X^{i,j}$ et $d = d_1+d_2$.

§ 4. LE CONE D'UN MORPHISME

4.1 Soit $\mathbb{A}$ une catégorie additive. On définit un foncteur de translation $T : C^*(\mathbb{A}) \to C^*(\mathbb{A})$ ($* = \emptyset,+,-,b$) en associant à tout objet $(X^\cdot;d_X)$ de $C^*(\mathbb{A})$, le complexe $(T(X^\cdot);d_{T(X^\cdot)})$ obtenu en posant $T(X^\cdot)^i = X^{i+1}$ pour tout $i \in \mathbb{Z}$ et $d_{T(X^\cdot)} = -d_{X^\cdot}$.

4.2 Soit $X^\cdot, Y^\cdot \in \mathrm{Ob}C^*(\mathbb{A})$. Les $\mathbb{A}$-objets gradués $Y^\cdot$ et $T(X^\cdot)$ ont une somme directe $Y^\cdot \oplus T(X^\cdot)$ qui est caractérisée par le diagramme

$$Y^\cdot \underset{q_{Y^\cdot}}{\overset{p_{Y^\cdot}}{\rightleftarrows}} Y^\cdot \oplus T(X^\cdot) \underset{q_{X^\cdot}}{\overset{p_{X^\cdot}}{\rightleftarrows}} T(X^\cdot)$$

où les flèches sont des morphismes gradués de degré 0 qui vérifient les relations

$$p_{Y^\cdot} \cdot q_{Y^\cdot} = 1_{Y^\cdot} \ , \ p_{X^\cdot} \cdot q_{X^\cdot} = 1_{T(X^\cdot)}$$

$$p_{Y^\cdot} \cdot q_{X^\cdot} = 0, \ p_{X^\cdot} \cdot q_{Y^\cdot} = 0$$

$$q_{Y^\cdot} \cdot p_{Y^\cdot} + q_{X^\cdot} \cdot p_{X^\cdot} = 1_{Y^\cdot \oplus T(X^\cdot)}$$

On notera que l'on peut identifier $p_{X^\cdot}$ à un morphisme gradué $p_{X^\cdot} : Y^\cdot \oplus T(X^\cdot) \to X^\cdot$ de degré +1 et $q_{X^\cdot}$ à un morphisme gradué $q_{X^\cdot} : X^\cdot \to Y^\cdot \oplus T(X^\cdot)$ de degré -1.

4.3 Soit $u \in \mathrm{Hom}_{C^*(\mathbb{A})}(X^\cdot;Y^\cdot)$. On appelle cône de u l'objet $(C_u^\cdot;d_u)$ de $C^*(\mathbb{A})$ obtenu en posant

$$C_u^\cdot = Y^\cdot \oplus T(X^\cdot)$$

$$d_u(y;x) = (dy+u(x);-dx)$$

4.4 Le cône de u définit un triangle dans $C^*(\mathbb{A})$

(4.4.1) $$X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{q} C_u^\cdot \xrightarrow{p} T(X^\cdot)$$

où $q = q_{Y^\cdot}$ et $p = p_{X^\cdot}$.

4.5 LEMME : *La composition de deux morphismes consécutifs dans le triangle* $(X^\cdot;Y^\cdot;C_u^\cdot;u;q;p)$ *est homotope à* 0.

Démonstration : Par définition on a $pq = 0$. Pour voir que $qu \sim 0$ (respectivement $up \sim 0$) on prend comme opérateur d'homotopie le morphisme $q_{X^\cdot} : X^\cdot \to Y^\cdot \oplus T(X^\cdot)$ (respectivement $p_{Y^\cdot} : Y^\cdot \oplus T(X^\cdot) \to Y^\cdot$) considéré comme morphisme de degré -1.

4.6 LEMME : *Pour tout* $X^\cdot \in \mathrm{Ob}C^*(\mathbb{A})$ *le cône* $C^\cdot_{1_{X^\cdot}}$ *est homotope au complexe nul* $0^\cdot$.

Démonstration : Les morphismes $1_{C^\cdot_{1_{X^\cdot}}}$, $0 \in \mathrm{Hom}_{C^*(\mathbb{A})}(C^\cdot_{1_{X^\cdot}};C^\cdot_{1_{X^\cdot}})$ sont homotopes, l'opérateur d'homotopie $k \in \mathrm{Hom}^{-1}(X^\cdot \oplus T(X^\cdot);X^\cdot \oplus T(X^\cdot))$ étant donné par $k(x';x'') = (0;x')$.

4.7 PROPOSITION : *Soit* $u,v \in \mathrm{Hom}_{C^*(\mathbb{A})}(X^\cdot;Y^\cdot)$. *Si* $u \sim v$ *alors il existe un isomorphisme de complexes* $\pi : C_u^\cdot \to C_v^\cdot$ *tel que* $\pi q = q$ *et* $p\pi = p$.

Démonstration : Soit k l'opérateur d'homotopie entre u et v. Le morphisme π est défini en posant $\pi(y;x) = (y+k(x);x)$, son inverse étant le morphisme $\pi'(y;x) = (y-k(x);x)$.

4.8 Supposons dorénavant que $\mathbb{A}$ est une catégorie abélienne.

PROPOSITION : *Au triangle (4.4.1), défini par le cône* $C_u^\cdot$ *d'un morphisme* $u \in \mathrm{Hom}_{C^*(\mathbb{A})}(X^\cdot;Y^\cdot)$, *est associé une suite exacte longue de cohomologie*

$$\dots \to H^k(X^\cdot) \xrightarrow{u^*} H^k(Y^\cdot) \xrightarrow{q^*} H^k(C_u^\cdot) \xrightarrow{p^*} H^{k+1}(X^\cdot) \to \dots$$

dans $\mathbb{A}$.

Démonstration : A la suite exacte courte dans $C^*(\mathbb{A})$

$$0 \to Y^\cdot \xrightarrow{q} C_u^\cdot \xrightarrow{p} T(X^\cdot) \to 0$$

est associée la suite exacte longue de cohomologie dans $\mathbb{A}$

$$\dots \to H^k(Y^\cdot) \xrightarrow{q^*} H^k(C_u^\cdot) \xrightarrow{p^*} H^k(T(X^\cdot)) \xrightarrow{\delta} H^{k+1}(Y^\cdot) \to \dots$$

Soit $\xi \in H^k(T(X^\cdot)) = H^{k+1}(X^\cdot)$ représenté par un cocycle $x \in X^{k+1}$; on a $p(0;x) = x$ et $q(u(x)) = d_u(0;x)$; donc $\delta(\xi) = [u(x)]$. Ainsi $\delta = u^*$.

4.9 COROLLAIRE : *Si* $u \in \mathrm{Hom}_{C^*(\mathbb{A})}(X^\cdot;Y^\cdot)$ *est un isomorphisme homologique alors le cône* $C_u^\cdot$ *est un complexe acyclique.*

4.10 PROPOSITION : *Soit*

(4.10.1) $$0 \to X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} Z^\cdot \to 0$$

une suite d'objets et de morphisme de $C^*(\mathbb{A})$ *et soit* $C_u^\cdot$ *le cône du morphisme* u.

i) *si* $vu = 0$ *il existe* $m \in \mathrm{Hom}_{C^*(\mathbb{A})}(C_u^\cdot;Z^\cdot)$ *tel que* $mq = v$.

ii) *Si la suite (4.10.1) est exacte alors* m *est un isomorphisme homologique.*

iii) *Si de plus la suite (4.10.1) est semi-scindée, alors* m *est une équivalence d'homotopie.*

Démonstration :

i) Le morphisme m est défini en posant $m(y;x) = v(y)$.

ii) Considérons le diagramme commutatif

$$\begin{tikzcd}
0 \arrow[r] & X^\cdot \arrow[r, "u"] & Y^\cdot \arrow[r, "v"] & Z^\cdot \arrow[r] & 0 \\
& X^\cdot \arrow[u, "1_{X^\cdot}"] \arrow[r, "-u"'] & Y^\cdot \arrow[u, "-1_{Y^\cdot}"] \arrow[r, "-q"'] & C_u^\cdot \arrow[u, "m"'] \arrow[r, "-p"'] & T(X^\cdot)
\end{tikzcd}$$

Les suites exactes longues de cohomologie fournissent le diagramme commutatif

$$\begin{tikzcd}
\dots \arrow[r] & H^k(X^\cdot) \arrow[r, "u^*"] & H^k(Y^\cdot) \arrow[r, "v^*"] & H^k(Z^\cdot) \arrow[r, "\delta"] & H^{k+1}(X^\cdot) \arrow[r, "u^*"] & H^{k+1}(Y^\cdot) \arrow[r] & \dots \\
\dots \arrow[r] & H^k(X^\cdot) \arrow[u, "1_{H(X)}"] \arrow[r, "-u^*"'] & H^k(Y^\cdot) \arrow[u, "-1_{H(Y)}"] \arrow[r, "-q^*"'] & H^k(C_u^\cdot) \arrow[u, "m^*"] \arrow[r, "-p^*"'] & H^{k+1}(X^\cdot) \arrow[u, "1_{H(X)}"] \arrow[r, "-u^*"'] & H^{k+1}(Y^\cdot) \arrow[u, "1_{H(Y)}"] \arrow[r] & \dots
\end{tikzcd}$$

Il résulte alors du lemme des cinq que m^* est un isomorphisme.

iii) Soit $s : Z^\cdot \to Y^\cdot$ une application linéaire de degré 0 telle que $us = 1_Z$.

Soit $z \in Z^k$; comme $v(ds(z)-s(dz)) = 0$ il exsite un unique $x \in X^{k+1}$ tel que $u(x) = ds(z)-s(dz)$. On définit alors un morphisme $n \in \mathrm{Hom}_{C^*(\mathbb{A})}(Z^\cdot;C_u^\cdot)$ en posant $n(z) = (s(z);-x)$. Il est clair que $mn = 1_Z$.

Si $(y;x) \in C_u^k$ on a $nm(y;x) = (sv(y);-x')$ où $x' \in X^{k+1}$ est l'unique élément tel que $u(x') = dsv(y)-s(dv(y))$. Pour voir que $nm \sim 1_{C_u^\cdot}$ définissons un opérateur d'homotopie $k \in \mathrm{Hom}^{-1}(C_u^\cdot;C_u^\cdot)$ en posant $k(y;x) = (0;\bar{x})$ où $\bar{x} \in X^k$ est l'unique élément tel que $u(\bar{x}) = y-sv(y)$. On a $dk(y;x) = (u(\bar{x});-d\bar{x}) = (y-sv(y);-d\bar{x})$, et $kd(y;x) = k(dy+u(x);-dx) = (0;x'+x+d\bar{x})$ car $u(x'+x+d\bar{x}) = dy+u(x)-s(dv(y)) = dy+u(x)-sv(dy+u(x))$. Ainsi $(dk+kd)(y;x) = (y-sv(y);x+x') = (1_{C_u^\cdot}-nm)(y;x)$.

§ 5. LA CATEGORIE K($\mathbb{A}$)

5.1 Soit $\mathbb{A}$ une catégorie additive.

En vertu des résultats du No 3.9, on peut définir une catégorie K($\mathbb{A}$) dont les objets sont les complexes d'objets de $\mathbb{A}$ et dont les morphismes sont les classes d'homotopie des morphismes de complexes. Plus précisément ObK($\mathbb{A}$) = ObC($\mathbb{A}$) et pour tout $X^\cdot, Y^\cdot \in$ ObK($\mathbb{A}$) on pose

$$\mathrm{Hom}_{K(\mathbb{A})}(X^\cdot\ Y^\cdot) = \frac{\mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)}{\Sigma_{C(\mathbb{A})}(X^\cdot;Y^\cdot)} .$$

Un morphisme dans K($\mathbb{A}$) est un isomorphisme si et seulement si il admet un représentant qui est une équivalence d'homotopie dans C($\mathbb{A}$); mais alors tous ses représentants sont des équivalences d'homotopie.

5.2 PROPOSITION : *La catégorie* K($\mathbb{A}$) *est additive.*

Démonstration : On utilise les résultats du No 3.9.

5.3 On note $h : C(\mathbb{A}) \to K(\mathbb{A})$ le foncteur additif plein qui est l'identité sur les objets et qui à une flèche de C($\mathbb{A}$) associe sa classe d'homotopie dans K($\mathbb{A}$).

En composant ce foncteur avec le foncteur C_0 du No 3.4 on obtient un foncteur additif pleinement fidèle que l'on note encore $C_0 : \mathbb{A} \to K(\mathbb{A})$.

5.4 Supposons que $\mathbb{A}$ est une catégorie abélienne. D'après le No 3.11, le foncteur H^k du No 3.5 se factorise à travers h pour donner un foncteur encore noté H^k

$$\begin{tikzcd} C(\mathbb{A}) \arrow[rr, "H^k"] \arrow[dr, "h"'] & & \mathbb{A} \\ & K(\mathbb{A}) \arrow[ur, dashed, "H^k"'] & \end{tikzcd}$$

5.5 Un morphisme dans K($\mathbb{A}$) est un isomorphisme homologique si et seulement si il admet un représentant qui est un isomorphisme homologique dans C($\mathbb{A}$); mais alors tous ses représentants sont des isomorphismes homologiques.

5.6 Soit $u,u' \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$ deux représentants d'un morphisme $\varphi \in \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot)$. D'après la proposition 4.7 le cône $C^\cdot_u$ est isomorphe au cône $C^\cdot_{u'}$. Ainsi dans la catégorie K($\mathbb{A}$) le cône $C^\cdot_\varphi$ d'un morphisme φ n'est défini qu'à isomorphisme près.

5.7 *Remarque :* En général la catégorie K($\mathbb{A}$) n'est pas abélienne. En effet soit $\mathbb{A}$ une catégorie abélienne admettant des extensions non triviales, par exemple $\mathbb{A} = \mathbb{A}\mathrm{b}$ la catégorie des groupes abéliens.

Soit $u \in \mathrm{Hom}_{\mathbb{A}}(X;Y)$. A l'aide du foncteur C_0 on peut considérer $u \in \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot)$ où les complexes $X^\cdot$ et $Y^\cdot$ sont concentrés en degré 0. Supposons que ce morphisme u admet dans K($\mathbb{A}$) un conoyau $(Z^\cdot;\pi)$. Puisque le conoyau est unique à isomorphisme près il est clair qu'on peut supposer que le complexe $Z^\cdot$ est concentré en degré 0. On peut donc considérer le couple $(Z;\pi)$ dans $\mathbb{A}$ et par unicité du conoyau ce couple $(Z;\pi)$ est nécessairement le conoyau de u dans $\mathbb{A}$.

Le cône $C^\cdot_u$ de u dans K($\mathbb{A}$) est bien défini et concentré en degré -1 et 0.

Considérons le diagramme

$$\begin{tikzcd} X^\cdot \arrow[r, "u"] & Y^\cdot \arrow[r, "\pi"] \arrow[dr, "q"'] & Z^\cdot \arrow[d, dashed, shift left, "n"] \\ & & C^\cdot_u \arrow[u, dashed, shift left, "m"] \end{tikzcd}$$

On a $\pi u = 0$, donc d'après la proposition 4.10 i), il existe un morphisme $m : C^\cdot_u \to Z^\cdot$ tel que $mq = \pi$. De plus d'après le lemme 4.5 on a $qu = 0$, donc il existe un morphisme $n : Z^\cdot \to C_u^\cdot$ tel que $n\pi = q$.

Comme π est un épimorphisme il résulte de ces deux relations que $mn = 1_Z$. Donc $Z^\cdot$ est facteur direct dans $C_u^\cdot$ mais comme $Z^\cdot$ est concentré en degré 0 on en déduit que Z est facteur direct dans Y. Mais ceci contredit l'hypothèse que $\mathbb{A}$ admet des extensions non triviales.

5.8 La remarque précédente montre qu'il y a perte de structure lorsque l'on passe de la catégorie $C(\mathbb{A})$ à la catégorie $K(\mathbb{A})$. On peut cependant mettre sur $K(\mathbb{A})$ une nouvelle structure qui contiendra suffisamment d'informations pour l'usage que nous en ferons par la suite. Cette structure est celle de catégorie triangulée. Dans un certain sens la notion de triangles remplace la notion de suites exactes qui fait défaut dans $K(\mathbb{A})$.

5.9 Soit $\mathbb{A}$ une catégorie additive.

Il est évident que le foncteur de translation T dans $C(\mathbb{A})$ défini au No 4.1, induit un foncteur dans $K(\mathbb{A})$, encore noté T, qui sera le foncteur de translation dans $K(\mathbb{A})$, et qui vérifie la condition $hT = Th$, où $h : C(\mathbb{A}) \to K(\mathbb{A})$ est le foncteur défini au No 5.3.

5.10 Soit $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$. En appliquant le foncteur h au triangle $(X^\cdot;Y^\cdot;C_u^\cdot;u;q;p)$ dans $C(\mathbb{A})$ du No 4.4 on obtient un triangle dans $K(\mathbb{A})$ qu'on note encore, par abus de notations, $(X^\cdot;Y^\cdot;C_u^\cdot;u;q;p)$ et qu'on appelle le triangle standard construit sur le morphisme u. On remarquera que les triangles standards construits sur deux morphisme homotopes sont isomorphes dans $K(\mathbb{A})$.

5.11 Désignons par $\mathcal{T}$ la famille des triangles dans $K(\mathbb{A})$ qui sont isomorphes à un triangle standard. Autrement dit $(X^\cdot;Y^\cdot;Z^\cdot;\alpha;\beta;\gamma) \in \mathcal{T}$ si et seulement si, il existe $u \in \mathrm{Hom}_{C(\mathbb{A})}(X'^\cdot;Y'^\cdot)$ et un diagramme commutatif dans $K(\mathbb{A})$

$$\begin{tikzcd}
X^\cdot \arrow[r, "\alpha"] \arrow[d, "f"'] & Y^\cdot \arrow[r, "\beta"] \arrow[d, "g"'] & Z^\cdot \arrow[r, "\gamma"] \arrow[d, "h"'] & T(X^\cdot) \arrow[d, "T(f)"] \\
X'^\cdot \arrow[r, "u"'] & Y'^\cdot \arrow[r, "q"'] & C_u^\cdot \arrow[r, "p"'] & T(X'^\cdot)
\end{tikzcd}$$

dans lequel les flèches verticales sont des isomorphismes dans $K(\mathbb{A})$.

5.12 PROPOSITION : *La famille $\mathcal{T}$ vérifie les axiomes* TR1, TR2, TR3 *et* TR4.

Démonstration :

TR1 : a) est évident pour transitivité des isomorphismes.

b) il suffit de prendre le triangle standard $(X^\cdot;Y^\cdot;C_u^\cdot;u;q;p)$ où $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$ est un représentant de α.

c) il suffit de remarquer que, d'après les lemmes 4.5 et 4.6, le diagramme

$$\begin{tikzcd}
X^\cdot \arrow[r, "1_X"] \arrow[d, "1_{X^\cdot}"'] & X^\cdot \arrow[r, "0"] \arrow[d, "1_{X^\cdot}"'] & 0^\cdot \arrow[r, "0"] \arrow[d, "0"'] & T(X^\cdot) \arrow[d, "1_{T(X^\cdot)}"] \\
X^\cdot \arrow[r, "1_X"'] & X^\cdot \arrow[r, "q"'] & C_{1_{X^\cdot}}^\cdot \arrow[r, "p"'] & T(X^\cdot)
\end{tikzcd}$$

est commutatif dans $K(\mathbb{A})$ et la troisième flèche verticale est un isomorphisme dans $K(\mathbb{A})$.

TR2 : Pour démontrer l'implication dans le sens direct il faut montrer que si $(X^\cdot;Y^\cdot;C_u^\cdot;u;q;p)$ est le triangle standard construit sur le morphisme $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$, alors le triangle $(Y^\cdot;C_u^\cdot;T(X^\cdot);q;p;-T(u))$ est distingué. Soit $C_q^\cdot$ le cône du morphisme $q : Y^\cdot \to C_u^\cdot$. Considérons dans $C(\mathbb{A})$ le diagramme suivant

$$\begin{tikzcd}
Y^\bullet \arrow[r, "q"] \arrow[d, "1_{Y^\bullet}"'] & C^\bullet_u \arrow[r, "p"] \arrow[d, "1_{C^\bullet_u}"'] & T(X^\bullet) \arrow[r, "-T(u)"] \arrow[d, dashed, shift right, "r"'] & T(Y^\bullet) \arrow[d, "1_{T(Y^\bullet)}"] \\
Y^\bullet \arrow[r, "q"'] & C^\bullet_u \arrow[r, "\overline{q}"'] & C^\bullet_q \arrow[r, "\overline{p}"'] \arrow[u, dashed, shift right, "r'"'] & T(Y^\bullet)
\end{tikzcd}$$

où $\overline{q} = q_{C^\bullet_u} : C^\bullet_u \to C^\bullet_u \oplus T(Y^\bullet)$ et $\overline{p} = p_{T(Y^\bullet)} : T(Y^\bullet) \to C^\bullet_u \oplus T(Y^\bullet)$.

Les morphismes de complexes $r : T(X^\bullet) \to C^\bullet_q$ et $r' : C^\bullet_q \to T(X^\bullet)$ sont définis en posant $r(x) = (0;x;-u(x))$ et $r'(y;x;y') = x$.

On a évidemment $r'r = 1_{T(X^\bullet)}$.

De plus $rr' \sim 1_{C^\bullet_q}$, l'opérateur d'homotopie $k \in \mathrm{Hom}^{-1}(C^\bullet_q;C^\bullet_q)$ étant défini en posant $k(y;x;y') = (0;0;y)$. Ainsi r est une équivalence d'homotopie. Enfin le premier et le troisième carré du diagramme précédent sont commutatifs tandis que le deuxième carré est commutatif à homotopie près, l'opérateur d'homotopie entre rp et $\overline{q}$ étant le morphisme $k \in \mathrm{Hom}^{-1}(C^\bullet_u;C^\bullet_q)$ donné par $k(y;x) = (0;0;y)$. Pour démontrer l'implication dans le sens inverse on applique deux fois l'implication directe au triangle distingué $(Y^\bullet;Z^\bullet;T(X^\bullet);\beta;\gamma;-T(\alpha))$ et on remarque que $(T(X^\bullet);T(Y^\bullet);T(Z^\bullet);-T(\alpha);-T(\beta);-T(\gamma)) \in \mathcal{T}$ si et seulement si $(X^\bullet;Y^\bullet;Z^\bullet;\alpha;\beta;\gamma) \in \mathcal{T}$. En effet il est évident que si un triangle est distingué, il reste dinstingué lorsqu'on le suspend ou on le désuspend. Il ne reste donc plus qu'à voir que si $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\bullet;Y^\bullet)$ alors le diagramme dans $C(\mathbb{A})$

$$\begin{tikzcd}
X^\bullet \arrow[r, "u"] \arrow[d, "-1_{X^\bullet}"'] & Y^\bullet \arrow[r, "-q"] \arrow[d, "1_{Y^\bullet}"] & C^\bullet_u \arrow[r, "-p"] \arrow[d, dashed, "m"] & T(X^\bullet) \arrow[d, "-1_{T(X^\bullet)}"] \\
X^\bullet \arrow[r, "-u"'] & Y^\bullet \arrow[r, "q"'] & C^\bullet_{-u} \arrow[r, "p"'] & T(X^\bullet)
\end{tikzcd}$$

où $m : C^\bullet_u \to C^\bullet_{-u}$ est donné par $m(y,x) = (-y;x)$, est commutatif et les flèches verticales sont des isomorphismes.

TR3 : Soit $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\bullet;Y^\bullet)$ et $u' \in \mathrm{Hom}_{C(\mathbb{A})}(X'^\bullet;Y'^\bullet)$ et supposons qu'il existe des morphismes $f : X^\bullet \to X'^\bullet$ et $g : Y^\bullet \to Y'^\bullet$ dans $C(\mathbb{A})$ tels que, dans le diagramme

$$\begin{tikzcd}
X^\bullet \arrow[r, "u"] \arrow[d, "p"'] & Y^\bullet \arrow[r, "q"] \arrow[d, "g"] & C^\bullet_u \arrow[r, "p"] \arrow[d, dashed, "h"] & T(X^\bullet) \arrow[d, "T(f)"] \\
X'^\bullet \arrow[r, "u'"'] & Y'^\bullet \arrow[r, "q'"'] & C^\bullet_{u'} \arrow[r, "p'"'] & T(X'^\bullet)
\end{tikzcd}$$

le premier carré soit commutatif à homotopie près; notons $k : X^\bullet \to Y'^\bullet$ l'opérateur d'homotopie entre gu et $u'f$. Le morphisme de complexes $h : C^\bullet_u \to C^\bullet_{u'}$, défini en posant $h(y;x) = (g(y)+k(x);f(x))$, rend commutatif les deux derniers carrés du diagramme.

TR4 : Soit $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\bullet;Y^\bullet)$, $v \in \mathrm{Hom}_{C(\mathbb{A})}(Y^\bullet;Z^\bullet)$ et $w \in \mathrm{Hom}_{C(\mathbb{A})}(X^\bullet;Z^\bullet)$ tels que $w \sim vu$.

Considérons le diagramme

$$\begin{tikzcd}
X^\bullet \arrow[r, "u"] \arrow[d, "1_X"'] \arrow[dr, phantom, "1"] & Y^\bullet \arrow[r, "q"] \arrow[d, "v"] \arrow[dr, phantom, "2"] & C^\bullet_u \arrow[r, "p"] \arrow[d, dashed, "f"] \arrow[dr, phantom, "3"] & T(X^\bullet) \arrow[d, "1_{T(X^\bullet)}"] \\
X^\bullet \arrow[r, "w"] \arrow[d, "u"'] \arrow[dr, phantom, "4"] & Z^\bullet \arrow[r, "q''"] \arrow[d, "1_{Z^\bullet}"] \arrow[dr, phantom, "5"] & C^\bullet_w \arrow[r, "p''"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "6"] & T(X^\bullet) \arrow[d, "T(u)"] \\
Y^\bullet \arrow[r, "v"'] & Z^\bullet \arrow[r, "q'"] & C^\bullet_v \arrow[r, "p'"] \arrow[d, "T(q)p'"'] & T(Y^\bullet) \arrow[dl, "T(q)"] \\
 & & T(C^\bullet_u) &
\end{tikzcd}$$

dans lequel les carrés 1 et 4 sont commutatifs à homotopie près et soit $k : X^\bullet \to Z^\bullet$ l'opérateur d'homotopie de vu à w. Les morphismes de complexes $f : C^\bullet_u \to C^\bullet_w$ et $g : C^\bullet_w \to C^\bullet_v$, définis en posant $f(y;x) = (v(y)+k(x);x)$ et $g(z;x) = (z-k(x);u(x))$, rendent commutatifs les carrés 2,3,5 et 6 du diagramme précédent.

Considérons maintenant le diagramme

$$\begin{tikzcd} C^{\cdot}_u \arrow[r, "f"] \arrow[d, "1_{C^{\cdot}_u}"'] & C^{\cdot}_w \arrow[r, "g"] \arrow[d, "1_{C^{\cdot}_w}"'] & C^{\cdot}_v \arrow[r, "T(g)p'"] \arrow[d, dashed, shift right, "\omega"'] & T(C^{\cdot}_u) \arrow[d, "1_{T(C^{\cdot}_u)}"] \\ C^{\cdot}_u \arrow[r, "f"'] & C^{\cdot}_w \arrow[r, "\bar{q}"'] & C^{\cdot}_f \arrow[r, "\bar{p}"'] \arrow[u, dashed, shift right, "\theta"'] & T(C^{\cdot}_u) \end{tikzcd}$$

Le morphisme de complexes $\omega : C^{\cdot}_v \to C^{\cdot}_f$, défini en posant $\omega(z;y) = (z;0;y;0)$, rend commutatif le troisième carré du diagramme précédent. De plus le deuxième carré de ce diagramme commute à homotopie près, l'opérateur d'homotopie $\bar{k} \in \mathrm{Hom}^{-1}(C^{\cdot}_w;C^{\cdot}_f)$ de $\bar{q}$ à ωg étant donné par $\bar{k}(z;x) = (0;0;0;x)$. Enfin il reste à voir que ω est une équivalence d'homotopie. Le morphisme de complexes $\theta : C^{\cdot}_f \to C^{\cdot}_v$, donné par $\theta(z;x;y;x') = (z-k(x);u(x)+y)$, vérifie $\theta\omega = 1_{C^{\cdot}_v}$; de plus $\omega\theta \sim 1_{C^{\cdot}_f}$, l'opérateur d'homotopie $\tilde{k} \in \mathrm{Hom}^{-1}(C^{\cdot}_f;C^{\cdot}_f)$ étant donné par $\tilde{k}(z;x;y;x') = (0;0;0;x)$.

5.13 COROLLAIRE : *La catégorie* K(A) *est triangulée.*

5.14 PROPOSITION : *Supposons que la catégorie* A *est abélienne Le foncteur* H* : K(A) → A *est un foncteur cohomologique.*

Démonstration : C'est une conséquence de la proposition 4.8.

5.15 COROLLAIRE : *Si, dans la situation de l'axiome* TR3, *les morphismes* f *et* g *sont des isomorphismes homologiques, alors* h *est un isomorphisme homologique.*

5.16 Soit A une catégorie additive. On peut définir d'une manière évidente (voir 3.12) les catégories $K^+(A)$, $K^-(A)$ et $K^b(A)$ qui sont des sous-catégories pleines de K(A).

5.17 PROPOSITION : *Les catégories* K*(A) (* = +,-,b) *sont des sous-catégories triangulées de* K(A).

Démonstration : Il suffit de remarquer qu'un triangle standard dont les deux premiers sommets sont dans K*(A) a son troisième sommet dans K*(A).

§ 6. LA CATEGORIE D(A)

6.1 Soit A une catégorie abélienne. On désigne par His_{A} (ou simplement par His s'il n'y pas d'ambiguité) la famille de tous les isomorphismes homologiques de K(A). (voir 5.5).

6.2 PROPOSITION : *La famille* His *est un système multiplicatif dans* K(A), *compatible avec la structure triangulée de* K(A).

Démonstration :

FR1 est évident.

FR2 : D'après la remarque 2.6 i) on peut construire le triangle $(Z^{\cdot};Y^{\cdot};N^{\cdot};s;f;g)$ puis le triangle $(W^{\cdot};X^{\cdot};N^{\cdot};t;fu)$

$$\begin{tikzcd}
W^{\cdot} \arrow[rr, dashed, "v"] \arrow[d, dashed, "t"'] & & Z^{\cdot} \arrow[d, "s", "\wr"'] & \\
& N^{\cdot} \arrow[ul, dashed, "h"'] \arrow[rr, dashed, "1_{N^{\cdot}}"] & & N^{\cdot} \arrow[ul, dashed, "g"'] \\
X^{\cdot} \arrow[rr, "u"] \arrow[ur, dashed, "fu"] & & Y^{\cdot} \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

Comme le couple de morphismes $(u;1_{N^{\cdot}})$ vérifie la relation $1_{N^{\cdot}}.fu = fu$ il résulte de la remarque 2.6 ii) qu'il existe un morphisme $v : W^{\cdot} \to Z^{\cdot}$ tel que $(v;u;1_{N^{\cdot}})$ soit un morphisme de triangles; on a donc $ut = sv$. Pour montrer que t est un h.i. on considère les suites exactes longues de cohomologie associées à chacun des triangles précédents (prop. 5.14). De la première suite on déduit que $H^*(N^{\cdot}) = 0$ car s^* est un isomorphisme; mais alors cette information introduite dans la deuxièm suite montre que t^* est un isomorphisme.
La deuxième partie de la démonstration se déduit de la première

en se plaçant dans la catégorie opposée qui est aussi triangulée.

FR3 : Si on pose $w = u-v$ tout revient à montrer que les conditions suivantes sont équivalentes :

i') Il existe $s \in \mathrm{Hom}_{K(A)}(Y^{\cdot};Y^{\cdot\cdot}) \cap \mathrm{His}$ tel que $sw = 0$.

ii') Il existe $t \in \mathrm{Hom}_{K(A)}(X^{\cdot\cdot};X^{\cdot}) \cap \mathrm{His}$ tel que $wt = 0$.

Démontrons que i') implique ii'), l'implication réciproque s'en déduira en se plaçant dans la catégorie opposée.

$$\begin{tikzcd}
X^{\cdot\cdot} \arrow[d, dashed, "t"'] & & Z^{\cdot} \arrow[ll, dashed, "m"'] \arrow[dl, dashed, "x"'] \\
X^{\cdot} \arrow[r, "w"'] \arrow[urr, dashed, "z"] & Y^{\cdot} \arrow[r, "\sim", "s"'] & Y^{\cdot\cdot} \arrow[u, dashed, "y"']
\end{tikzcd}$$

D'après la remarque 2.6. i) on peut construire le triangle $(Z^{\cdot};Y^{\cdot};Y^{\cdot\cdot};x;s;y)$ et d'après la proposition 2.10 on a une suite exacte

$$\ldots \to \mathrm{Hom}_{K(A)}(X^{\cdot};Z^{\cdot}) \xrightarrow{x_*} \mathrm{Hom}_{K(A)}(X^{\cdot};Y^{\cdot}) \xrightarrow{s_*} \mathrm{Hom}_{K(A)}(X^{\cdot};Y^{\cdot\cdot}) \to \ldots$$

La condition i') signifie que $w \in \mathrm{Ker}\, s_*$, donc il existe $z \in \mathrm{Hom}_{K(A)}(X^{\cdot};Z^{\cdot})$ tel que $x_*(z) = w$.
De nouveau la remarque 2.6 i) montre que l'on peut construire le triangle $(X^{\cdot\cdot};X^{\cdot};Z^{\cdot};t;z;m)$ et la proposition 2.10 donne une suite exacte

$$\ldots \to \mathrm{Hom}_{K(A)}(Z^{\cdot};Y^{\cdot}) \xrightarrow{z^*} \mathrm{Hom}_{K(A)}(X^{\cdot};Y^{\cdot}) \xrightarrow{t^*} \mathrm{Hom}_{K(A)}(X^{\cdot\cdot};Y^{\cdot}) \to \ldots$$

Or $w \in \mathrm{Im}(z^*)$ donc $t^*(w) = 0$; ainsi on a $wt = 0$. Pour montrer que t est un h.i. on considère les suites exactes longues de cohomologie associées à chacun des triangles précédents (prop.5.14). De la première suite on déduit que $H^*(Z^{\cdot}) = 0$ car s^* est un isomorphisme mais alors la deuxième suite montre que t^* est un isomorphisme.

FR4 est évident.

FR5 résulte du corollaire 5.15.

6.3 *Remarques :* Le système multiplicatif His satisfait la condition supplémentaire (1.17.1), c'est-à-dire si $s \in$ His et si $su \in$ His alors $u \in$ His.
De plus les isomorphismes dans $K(\mathbb{A})$ sont évidemment des éléments de His.

6.4 On appelle catégorie dérivée de la catégorie abélienne $\mathbb{A}$, et on note $D(\mathbb{A})$, la localisation de $K(\mathbb{A})$ relativement au système multiplicatif His.
Ainsi $\mathrm{Ob}D(\mathbb{A}) = \mathrm{Ob}K(\mathbb{A})$ et si $X^\cdot, Y^\cdot \in \mathrm{Ob}D(\mathbb{A})$ alors $\mathrm{Hom}_{D(\mathbb{A})}(X^\cdot;Y^\cdot)$ est l'ensemble des classes d'équivalence des diagrammes dans $K(\mathbb{A})$ de la forme $X^\cdot \underset{\sim}{\leftarrow} \cdot \rightarrow Y^\cdot$ où la première flèche est un h.i. La composition des morphismes est définie par la classe d'équivalence des diagrammes dans $K(\mathbb{A})$ de la forme

$$\begin{tikzcd} & & \cdot \arrow[dl, dashed, "\sim"'] \arrow[dr, dashed] & & \\ & \cdot \arrow[dl, "\sim"'] \arrow[dr] & & \cdot \arrow[dl, "\sim"'] \arrow[dr] & \\ X^\cdot & & Y^\cdot & & Z^\cdot \end{tikzcd}$$

On note $P_{\mathbb{A}}$ (ou simplement P s'il n'y a pas d'ambiguité) le foncteur canonique de localisation $K(\mathbb{A}) \to D(\mathbb{A})$.

6.5 THEOREME : *La catégorie* $D(\mathbb{A})$ *est additive et munie d'une structure triangulée.*
Le foncteur canonique $P : K(\mathbb{A}) \to D(\mathbb{A})$ *est additif et est un* ∂*-foncteur.*
Il transforme les isomorphismes homologiques de $K(\mathbb{A})$ *en des isomorphismes de* $D(\mathbb{A})$ *et il est universel pour cette propriété.*

Démonstration : Cela résulte des théorèmes 1.2, 1.16 et 2.15.

6.6 Pour chaque $k \in \mathbb{Z}$ le foncteur $H^k : K(\mathbb{A}) \to \mathbb{A}$ transforme évidemment les isomorphismes homologiques de $K(\mathbb{A})$ en des isomorphismes de $\mathbb{A}$. D'après le théorème 6.5. il existe donc un unique foncteur, que l'on notera encore H^k, qui rend commutatif le diagramme suivant

$$\begin{tikzcd} K(\mathbb{A}) \arrow[r, "H^k"] \arrow[d, "P"'] & \mathbb{A} \\ D(\mathbb{A}) \arrow[ur, dashed, "H^k"'] & \end{tikzcd}$$

6.7 PROPOSITION : *Un morphisme* $\alpha \in \mathrm{Hom}_{D(\mathbb{A})}(X^\cdot;Y^\cdot)$ *est un isomorphisme dans* $D(\mathbb{A})$ *si, et seulement si, il admet un représentant* $(Z^\cdot;s;a) \in M_{X^\cdot,Y^\cdot}$ *où s et a sont tous les deux des isomorphismes homologiques (et par suite tous les représentants de* α *ont cette propriété).*

Démonstration : Soit $\alpha \in \mathrm{Hom}_{D(\mathbb{A})}(X^\cdot;Y^\cdot)$ représenté par un élément $(Z^\cdot;s;a) \in M_{X^\cdot,Y^\cdot}$ où $s, a \in$ His. Notons β la classe d'équivalence de l'élément $(Z^\cdot;a;s) \in M_{Y^\cdot,X^\cdot}$. On a $\beta\alpha = 1_{X^\cdot}$ et $\alpha\beta = 1_{Y^\cdot}$; donc α est un isomorphisme dans $D(\mathbb{A})$.

Inversément supposons que α est un isomorphisme dans $D(\mathbb{A})$ et soit $(Z^\cdot;s;a)$ un représentant de α. Comme on l'a déjà remarqué on peut écrire $\alpha = P(a) \circ P(s)^{-1}$. D'après 6.6 on a $H^k(\alpha) = H^k(a) \circ H^k(s)^{-1}$; or $H^k(\alpha)$ et $H^k(s)$ sont des isomorphismes dans $\mathbb{A}$, donc $H^k(a)$ est un isomorphisme (pour tout $k \in \mathbb{Z}$) et ainsi $a \in$ His.

6.8 PROPOSITION : *Soit* $u \in \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot)$*. Si* $P(u) = 0$ *alors* $u^* = 0$ *(où* u^* *est le morphisme induit par u sur la cohomologie).*

Démonstration : Cela résulte de la proposition 1.20.

6.9 Soit $D : \mathbb{A} \to D(\mathbb{A})$ le composé du foncteur $C_0 : \mathbb{A} \to K(\mathbb{A})$ (voir 5.3) avec le foncteur canonique $P : K(\mathbb{A}) \to D(\mathbb{A})$.

PROPOSITION : *Le foncteur D est pleinement fidèle.*

Démonstration : Soit $X,Y\in \mathrm{Ob}\mathbb{A}$: il faut montrerque le morphisme de groupes abéliens

$$(6.9.1)\qquad \mathrm{Hom}_{\mathbb{A}}(X;Y) \to \mathrm{Hom}_{D(\mathbb{A})}(D(X);D(Y))$$

est bijectif.

Soit $u\in \mathrm{Hom}_{\mathbb{A}}(X;Y)$ tel que $D(u) = 0$; on a donc $(u) = 0$, et d'après la proposition 6.8 on a $u^* = 0$. Mais $u^* = u$ car u est concentré en degré 0. Le morphisme (6.9.1) est donc injetif.

Soit $\alpha\in \mathrm{Hom}_{D(\mathbb{A})}(D(X);D(Y))$ représenté par $(Z^{\cdot};;a)\in M_{D(X);D(Y)}$ Comme $D(X) = X$ et $D(Y) = Y$ sont concentrés en degé 0 on a un isomorphisme $s^* : H^0(Z^{\cdot}) \to X$ et un morphisme $a^* : H^0(Z^{\cdot}) \to Y$. Posons $u = a^*s^{*-1}\in \mathrm{Hom}_{\mathbb{A}}(X;Y)$.

Notons $Z^{\cdot}_{\leq 0}$ le complexe $Z^{\cdot}$ tronqué en degrés poitifs c'est-à-dire le complexe obtenu en posant

$$(Z^{\cdot}_{\leq 0})^p = \begin{cases} \mathrm{Ker}(d : Z^0 \to Z^1) & \text{si } p=0 \\ 0 & \text{si } p\geq 1 \\ Z^p & \text{si } p\leq -1 \end{cases}$$

L'inclusion $i : Z^{\cdot}_{\leq 0} \to Z^{\cdot}$ est un h.i. et le diagramme

$$\begin{tikzcd} (Z^{\cdot}_{\leq 0})^0 \arrow[r, "i"] \arrow[d, "j"'] & Z^0 \arrow[d, "s"] \\ H^0(Z^{\cdot}) \arrow[r, "s^*"'] & X \end{tikzcd}$$

est commutatif.

On en déduit que le diagramme

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & Z^{\cdot}_{\leq 0} \arrow[u, dashed] \arrow[l, dashed, "si"'] \arrow[r, dashed, "ai"] \arrow[d, dashed, "si"] & Y \\ & X \arrow[ul, "1_X"] \arrow[ur, "u"'] & \end{tikzcd}$$

est commutatif.

On a donc $P(u) = \alpha$ et aisni le morphisme (6.9.1) est surjectif.

6.10 Soit $\mathbb{B}$ la sous-catégorie pleine de $D(\mathbb{A})$ formée par les objets $X^{\cdot}$ de $D(\mathbb{A})$ qui sont acycliques en tout degré non nul. Il est clair que si $X\in \mathrm{Ob}\mathbb{A}$ alors $D(X)\in \mathrm{Ob}\mathbb{B}$; on peut donc considérer le foncteur $D : \mathbb{A} \to \mathbb{B}$ induit par le foncteur $D : \mathbb{A} \to D(\mathbb{A})$.

COROLLAIRE : *Le foncteur* $D : \mathbb{A} \to \mathbb{B}$ *est une équivalence de catégorie.*

Démonstration : On sait déjà que D est pleinement fidèle. Il reste à montrer que pour tout $Y^{\cdot}\in \mathrm{Ob}\mathbb{B}$ il existe $X\in \mathrm{Ob}\mathbb{A}$ et un isomorphisme $\alpha\in \mathrm{Hom}_{D(\mathbb{A})}(D(X);Y^{\cdot})$.

Posons $X = H^0(Y^{\cdot})$ et $Z^{\cdot} = Y^{\cdot}_{\leq 0}$;l'inclusion $i : Z^{\cdot} \to Y^{\cdot}$ et la projection $j : Z^{\cdot} \to D(X)$ sont des h.i. puisque $H^p(Y^{\cdot}) = 0$ pour $p\neq 0$. Le morphisme $\alpha : D(X) \to Y^{\cdot}$ représenté par $(Z^{\cdot};j;i)$ est donc un isomorphisme en vertu de la proposition 6.7.

6.11 *Remarque :* Dans la catégorie $D(\mathbb{A})$, munie du foncteur de translation T, on peut évidemment définir le cône $C^{\cdot}_{\omega}$ d'un morphisme $\omega\in \mathrm{Hom}_{D(\mathbb{A})}(X^{\cdot};Y^{\cdot})$.

Si $u\in \mathrm{Hom}_{K(\mathbb{A})}(X^{\cdot};Y^{\cdot})$ il est clair que l'on a, dans $D(\mathbb{A})$, un isomorphisme $P(C^{\cdot}_u) = C^{\cdot}_{P(u)}$.

Si $q : Y^{\cdot} \to Y^{\cdot}\oplus T(X^{\cdot})$ et $p : Y^{\cdot}\oplus T(X^{\cdot}) \to T(X^{\cdot})$ sont les morphismes canoniques dans $K(\mathbb{A})$, posons $\varepsilon = P(q)$ et $\pi = P(p)$ et considérons dans $D(\mathbb{A})$ le triangle standard $X^{\cdot} \xrightarrow{\omega} Y^{\cdot} \xrightarrow{\varepsilon} C^{\cdot}_{\omega} \xrightarrow{\pi} T(X^{\cdot})$ construit sur ω. C'est un triangle distingué. En effet si $(Z^{\cdot};s;a)\in M_{X^{\cdot},Y^{\cdot}}$ est un représentant de ω on a un diagramme commutatif

$$\begin{tikzcd} X^{\cdot} \arrow[r, "\omega"] & Y^{\cdot} \arrow[r, "\varepsilon"] & C^{\cdot}_{\omega} \arrow[r, "\pi"] & T(X^{\cdot}) \\ Z^{\cdot} \arrow[u, "P(s)"] \arrow[r, "P(a)"'] & Y^{\cdot} \arrow[u, "1_{Y^{\cdot}}"] \arrow[r, "P(q)"'] & C^{\cdot}_a \arrow[u, "1_Y\oplus P(T(s))"'] \arrow[r, "P(p)"'] & T(Z^{\cdot}) \arrow[u, "T(P(s))"'] \end{tikzcd}$$

dans lequel les flèches verticales sont des isomorphismes. Ainsi les triangles standards dans D(A) sont isomorphes aux images par P des triangles standards dans K(A). Il en résulte qu'un triangle est distingué dans D(A) si et seulement si il est isomorphe à un triangle standard dans D(A).

6.12 *Remarques :* En général la catégorie D(A) n'est pas abélienne.

En effet soit un monomorphisme $u : X^{\cdot} \to Y^{\cdot}$ dans K(A); alors $P(u) : X^{\cdot} \to Y^{\cdot}$ est encore un monomorphisme dans D(A) en vertu du corollaire 1.22.

Supposons que la catégorie D(A) est abélienne; on a alors une suite exacte

$$0 \to X^{\cdot} \xrightarrow{P(u)} Y^{\cdot} \to \mathrm{Coker}P(u) \to 0$$

D'un autre côté on a, d'après la remarque 6.11, un triangle distingué

$$X^{\cdot} \xrightarrow{P(u)} Y^{\cdot} \xrightarrow{\varepsilon} C^{\cdot}_{P(u)} \xrightarrow{\pi} T(X^{\cdot}) \qquad (6.12.1)$$

Mais d'après la proposition 4.10 ii) on a dans D(A) un isomorphisme $m : C^{\cdot}_{P(u)} \to \mathrm{Coker}P(u)$.

En appliquant la proposition 2.10 au triangle distingué (6.12.1) on obtient une suite exacte longue de groupes abéliens

$$\dots \to \mathrm{Hom}^p_{D(A)}(M^{\cdot};X^{\cdot}) \xrightarrow{P(u)_*} \mathrm{Hom}^p_{D(A)}(M^{\cdot};Y^{\cdot}) \xrightarrow{\varepsilon_*} \mathrm{Hom}^p_{D(A)}(M^{\cdot};C^{\cdot}_{P(u)})$$
$$\xrightarrow{\pi_*} \mathrm{Hom}^{p+1}_{D(A)}(M^{\cdot};Y^{\cdot}) \to \dots$$

Or $P(u)_*$ est injectif, donc $\mathrm{Hom}_{D(A)}(M^{\cdot};C_{P(u)}) = 0$ pour tout $M' \in \mathrm{Ob}D(A)$.

Il en résulte que $\mathrm{Coker}P(u) = 0$;donc P(u) est un isomorphisme.

Or ceci est visiblement absurde; par exemple si $X^{\cdot} = 0^{\cdot}$ et $Y^{\cdot}$ n'est pas acyclique, et si $u : 0^{\cdot} \to Y^{\cdot}$ est le monomorphisme évident, alors u n'est évidemment pas un h.i. et P(u) n'est pas un isomorphisme dans D(A).

6.13 Posons $\mathrm{His}^* = \mathrm{His} \cap F\ell(K^*(A))$ $(* = +,-,b)$.

PROPOSITION : *La famille* His* *est un système multiplicatif dans* K*(A) .

Démonstration : Il suffit de recopier la démonstration de la proposition 6.2 en prenant tous les triangles avec leurs sommets dans K*(A), ce qui est possible en vertu de la proposition 5.17.

6.14 D'après le No. 1.23 on peut donc définir les catégories dérivées D*(A) et les foncteurs canoniques de localisation $P^*_A : K^*(A) \to D^*(A)$ $(* = +,-,b)$.

Soit $F^* : D^*(A) \to D(A)$ les foncteurs induits par les foncteurs d'inclusion $K^*(A) \to K(A)$.

6.15 PROPOSITION : *Les foncteurs* $F^* : D^*(A) \to D(A)$ $(* = +,-,b)$ *sont pleinement fidèles.*

Démonstration : Soit $s \in \mathrm{Hom}_{K(A)}(X^{\cdot};Y^{\cdot}) \cap \mathrm{His}$ où $X^{\cdot} \in \mathrm{Ob}K^+(A)$; on peut supposer $X^i = 0$ pour $i < 0$. Comme s induit un isomorphisme sur la cohomologie on a donc $H^i(Y^{\cdot}) = 0$ pour $i < 0$.
Posons

$$Z^i = \begin{cases} 0 & \text{si } i<0 \\ \mathrm{Coker}(d : Y^{-1} \to Y^0) & \text{si } i=0 \\ Y^i & \text{si } i>0 \end{cases}$$

Alors $Z^{\cdot} \in \mathrm{Ob}K^+(A)$ et le morphisme évident $Y^{\cdot} \to Z^{\cdot}$ est un h.i.

La condition ii) de la proposition 1.24 est donc satisfaite.
Maintenant soit $s \in \mathrm{Hom}_{K(A)}(X^{\cdot};Y^{\cdot}) \cap \mathrm{His}$ où $Y^{\cdot} \in \mathrm{Ob}K^-(A)$; on peut supposer $Y^i = 0$ pour $i > 0$. Comme s induit un isomorphisme sur la cohomologie on a donc $H^i(X^{\cdot}) = 0$ pour $i > 0$.
Posons

$$Z^i = \begin{cases} 0 & \text{si } i>0 \\ \mathrm{Ker}(d : X^0 \to X^1) & \text{si } i=0 \\ X^i & \text{si } i<0 \end{cases}$$

Alors $Z^\cdot \in \mathrm{Ob}K^-(\mathbb{A})$ et le morphisme évident $Z^\cdot \to X^\cdot$ est un h.i.

La condition i) de la proposition 1.24 est donc satisfaite. Le cas de $K^b(\mathbb{A})$ se déduit des précédents.

§ 7. LES RESOLUTIONS

7.1 Les notions usuelles de résolutions injectives ou projectives et les conséquences qui en résultent en algèbre homologique peuvent se généraliser de la façon suivante.

Soit $\mathbb{A}$ une catégorie abélienne.

On considère une sous-classe $\Theta(\mathbb{A}) \subset \mathrm{Ob}\mathbb{A}$, qui contient les objets nuls et qui est stable par sommes directes (si $X,Y \in \Theta(\mathbb{A})$ alors $X \oplus Y \in \Theta(\mathbb{A})$) et stable par isomorphisme (si $X \in \Theta(\mathbb{A})$ est isomorphe à $Y \in \mathrm{Ob}\mathbb{A}$, alors $Y \in \Theta(\mathbb{A})$). On écrira simplement Θ au lieu de $\Theta(\mathbb{A})$ s'il n'y a pas de risque de confusion.

7.2 On note $C^*(\Theta(\mathbb{A}))$ (respectivement $K^*(\Theta(\mathbb{A}))$) ($* = \emptyset,+,-,b$) la sous-catégorie pleine de $C^*(\mathbb{A})$ (respectivement $K^*(\mathbb{A})$) engendrée par les complexes dont les objets en tout degré sont dans $\Theta(\mathbb{A})$.

7.3 Soit $A^\cdot \in \mathrm{Ob}C^*(\mathbb{A})$.
On appelle Θ-résolution à droite (respectivement à gauche) de $A^\cdot$, un isomorphisme homologique $A^\cdot \to X^\cdot$ (respectivement $X^\cdot \to A^\cdot$) où $X^\cdot \in \mathrm{Ob}C^*(\Theta(\mathbb{A}))$.

En particulier soit $A \in \mathrm{Ob}\mathbb{A}$ et identifions A avec le complexe $C_0(A)$ concentré en degré 0 (3.4). Alors une Θ-résolution à droite (respectivement à gauche) de A est un isomorphisme homologique $A \to X^\cdot$ (respectivement $X^\cdot \to A$) où $X^\cdot \in C^+(\Theta(\mathbb{A}))$ et $X^i = 0$ pour $i < 0$ (respectivement $X^\cdot \in C^-(\Theta(\mathbb{A}))$ et $X^i = 0$ pour $i > 0$).

7.4 On dit que la catégorie $\mathbb{A}$ admet suffisamment de Θ-objets à droite (respectivement à gauche) si pour tout $A \in \mathrm{Ob}\mathbb{A}$ il existe $X \in \Theta(\mathbb{A})$ et un monomorphisme $A \to X$ (respectivement un épimorphisme $X \to A$).

7.5 THEOREME : *Si la catégorie $\mathbb{A}$ admet suffisamment de θ-objets à droite (respectivement à gauche) alors chaque objet de $C^+(\mathbb{A})$ (respectivement de $C^-(\mathbb{A})$) admet une θ-résolution à droite (respectivement à gauche).*

En particulier chaque objet de $\mathbb{A}$ admet une θ-résolution à droite (respectivement à gauche).

Démonstration : On fait la démonstration dans le cas où $\mathbb{A}$ admet suffisamment de θ-objets à droite.

Soit $X^\cdot \in C^+(\mathbb{A})$; sans restreindre la généralité on peut supposer $X^i = 0$ si $i < 0$. On va construire un complexe $I^\cdot \in C^+(\theta(\mathbb{A}))$ et un monomorphisme $i : X^\cdot \to I^\cdot$ qui soit un h.i. Supposons le complexe construit jusqu'au degré p, avec $p \geq 0$, et que i^q est un monomorphisme pour $q \leq p$ et un h.i. pour $q \leq p-1$.

Considérons le diagramme suivant

$$\begin{tikzcd}
\cdots \arrow[r] & X^{p-1} \arrow[r, "d"] \arrow[d, tail, "i^{p-1}"'] & X^p \arrow[r, "d"] \arrow[d, tail, "i^p"'] & X^{p+1} \arrow[r] \arrow[d, dashed, tail, "i^{p+1}"] \arrow[ddr, dashed] & \cdots \\
\cdots \arrow[r] & I^{p-1} \arrow[r, "d'"'] & I^p \arrow[r, "d'"] \arrow[drr, dashed, bend right] & \frac{I^p}{Im(d')} \wedge X^{p+1} \arrow[dr, dashed] & \\
 & & & & I^{p+1}
\end{tikzcd}$$

où $\frac{I^p}{Im(d')} \wedge X^{p+1}$ désigne la somme amalgamée de X^{p+1} et $\frac{I^p}{Im(d')}$ sur X^p, i^{p+1} est le morphisme canonique et d' est le composé de la projection $I^p \to \frac{I^p}{Im(d')}$ avec le morphisme canonique.

Il est immédiat que $d'i^p = i^{p+1}d$ et que $d'd' = 0$. Montrons que i^{p+1} est injectif. Soit $x \in X^{p+1}$ tel que $i^{p+1}(x) = [(0;x)] = 0$. Il existe donc $z \in X^p$ tel que $(0;x) = ([i^p(z)];dz)$ et par suite on a $dz = x$ et il existe $y \in I^{p-1}$ tel que $i^p(z) = d'y$, mais cette dernière égalité signifie qu'il existe $u \in X^{p-1}$ tel que $[i^{p-1}(u)] = [y]$ et $du = z$. Il en résulte que $x = dz = ddu = 0$.

Montrons maintenant que $(i^p)^*$ est un isomorphisme. Soit $\alpha \in H^p(X^\cdot)$ représenté par un cocycle $x \in X^p$; supposons que $(i^p)^*(\alpha) = 0$. Il existe donc $y \in I^{p-1}$ tel que $d'y = i^p(x)$ mais ceci signifie de nouveau qu'il existe $z \in X^{p-1}$ tel que $dz = x$ et $[i^{p-1}(z)] = [y]$. Donc $\alpha = 0$.

Maintenant soit $\beta \in H^p(I^\cdot)$ représenté par un cocycle $y \in I^p$; il existe donc $x \in X^p$ tel que $dx = 0$ et $[i^p(x)] = [y]$ c'est-à-dire $y = i^p(x) + d'z$ avec $z \in I^{p-1}$; si α est la classe de x dans $H^p(X^\cdot)$ on a donc $(i^p)^*(\alpha) = \beta$.

Finalement il existe par hypothèse $I^{p+1} \in \theta(\mathbb{A})$ et un monomorphisme $\frac{I^p}{Im(d')} \wedge X^{p+1} \to I^{p+1}$. En notant encore i^{p+1} et d' le composé de i^{p+1} et d' avec ce monomorphisme, on a donc prolongé la construction du complexe $I^\cdot$ et du morphisme i jusqu'au degré p+1.

7.6 *Remarques :* On peut montrer que si $I^\cdot$ et $J^\cdot$ sont deux θ-résolutions à droite (respectivement à gauche) de $X^\cdot \in C^+(\mathbb{A})$ (respectivement $X^\cdot \in C^-(\mathbb{A})$) alors $I^\cdot$ et $J^\cdot$ sont isomorphes dans la catégorie dérivée $D^+(\mathbb{A})$ (respectivement $D^-(\mathbb{A})$).

7.7 THEOREME : *Supposons que la catégorie $\mathbb{A}$ admet suffisamment de θ-objets à droite (respectivement à gauche) et que la condition suivante est satisfaite :*

Si $0 \to U \to V \to W \to 0$ (respectivement $0 \to W \to V \to U \to 0$) est une suite exacte courte dans $\mathbb{A}$ où $U \in \theta(\mathbb{A})$ et $V \in \theta(\mathbb{A})$, alors $W \in \theta(\mathbb{A})$.

Dans ces conditions si $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ est une suite exacte courte dans $\mathbb{A}$ alors il existe des θ-résolutions à droite (respectivement à gauche), $I^\cdot, J^\cdot$ et $K^\cdot$ de X,Y et Z telles que dans le diagramme commutatif

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, tail] & Y \arrow[r, "v"] \arrow[d, tail] & Z \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & I^\cdot \arrow[r, "\varphi"] & J^\cdot \arrow[r, "\psi"] & K^\cdot \arrow[r] & 0
\end{tikzcd}$$

(respectivement

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & 0 \\ 0 \arrow[r] & I^{\cdot} \arrow[r, "\varphi"] \arrow[u] & J^{\cdot} \arrow[r, "\psi"] \arrow[u] & K^{\cdot} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$
)

la deuxième ligne soit une suite exacte dans $C^+(\theta(\mathbb{A}))$ *(respectivement* $C^-(\theta(\mathbb{A}))$).

Démonstration : On fait la démonstration dans le cas où $\mathbb{A}$ admet suffisamment de θ-objets à droite.

D'après le théorème 7.5 il existe une θ-résolution à droite $I^{\cdot}$ de X.

Supposons que l'on a effectué la construction des complexes $J^{\cdot}$ et $K^{\cdot}$ et des morphismes φ et ψ jusqu'au degré p, avec $p \geq 2$, et considérons le diagramme

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & I^{p-2} \arrow[r, "\varphi^{p-2}"] \arrow[d, "d'"] & J^{p-2} \arrow[rr, "\psi^{p-2}"] \arrow[d, "d''"] & & K^{p-2} \arrow[r] \arrow[d, "d'''"] & 0 \\
0 \arrow[r] & I^{p-1} \arrow[r, "\varphi^{p-1}"] \arrow[d, "d'"] & J^{p-1} \arrow[rr, "\psi^{p-1}"] \arrow[d, "d''"] & & K^{p-1} \arrow[r] \arrow[d, "d'''"] & 0 \\
0 \arrow[r] & I^{p} \arrow[r, "\varphi^{p}"] \arrow[d, "d'"] & J^{p} \arrow[rr, "\psi^{p}"] \arrow[d, "d''"] \arrow[ddr, dashed, "d''"] & & K^{p} \arrow[r] \arrow[dd, dashed, "d'''"] & 0 \\
0 \arrow[r] & I^{p+1} \arrow[r, "\varphi^{p+1}"] \arrow[d] \arrow[dr, dashed, "\varphi^{p+1}"'] & I^{p+1} \wedge \frac{J^p}{\mathrm{Im}(d'')} \arrow[dr, dashed] & & & \\
 & \vdots & & J^{p+1} \arrow[r, "\psi^{p+1}"'] & K^{p+1} \arrow[r] & 0
\end{tikzcd}$$

où φ^{p+1} et d'' sont les morphismes canoniques.

Il est immédiat que $\varphi^{p+1}d' = d''\varphi^p$. Le morphisme φ^{p+1} est injectif car soit $x \in I^{p+1}$ tel que $\varphi^{p+1}(x) = 0$; il existe donc $y \in I^p$ tel que

$d'y = x$ et $[\varphi^p(y)] = 0$, c'est-à-dire il existe $z \in J^{p-1}$ tel que $\varphi^p(y) = d''z$. On en déduit $d'''\psi^{p-1}(z) = \psi^p(d''z) = \psi^p(\varphi^p(y)) = 0$ donc il existe $u \in K^{p-2}$ tel que $d'''u = \psi^{p-1}(z)$ puisque $H^{p-1}(K^{\cdot}) = 0$. Il existe $v \in J^{p-2}$ tel que $\psi^{p-2}(v) = u$. On a $\psi^{p-1}(z) = d'''\psi^{p-2}(v) = \psi^{p-1}(d''v)$; donc $z - d''(v) \in \mathrm{Ker}\psi^{p-1} = \mathrm{Im}\varphi^{p-1}$ d'où il existe $w \in I^{p-1}$ tel que $\varphi^{p-1}(w) = z - d''(v)$. On a $\varphi^p(d'w) = d''\varphi^{p-1}(w) = d''z = \varphi^p(y)$ d'où $d'w = y$ puisque φ^p est injectif. Il en résulte que $x = d'y = d'd'w = 0$.

Il est immédiat que $d''d'' = 0$. Montrons que $H^p(J^{\cdot}) = 0$. Soit $y \in J^p$ tel que $d''y = 0$; il existe donc $x \in I^p$ tel que $d'x = 0$ et $[\varphi^p(x)] = [y]$ c'est-à-dire il existe $z \in J^{p-1}$ tel que $\varphi^p(x) = y + d''z$. De plus il existe $u \in I^{p-1}$ tel que $d'u = x$. On a alors $d''(\varphi^{p-1}(u) - z) = \varphi^p(d'u) - d''z = \varphi^p(x) - d''z = y$.

Par hypothèse il existe $J^{p+1} \in \theta(\mathbb{A})$ et un monomorphisme $I^{p+1} \wedge \frac{J^p}{\mathrm{Im}(d'')} \to J^{p+1}$; notons encore d'' et φ^{p+1} les composés de d'' et φ^{p+1} avec ce monomorphisme.

Posons $K^{p+1} = \mathrm{Coker}\,\varphi^{p+1}$ et ψ^{p+1} la projection canonique. On a donc une suite exacte

$$0 \to I^{p+1} \xrightarrow{\varphi^{p+1}} J^{p+1} \xrightarrow{\psi^{p+1}} K^{p+1} \to 0$$

où $I^{p+1} \in \theta(\mathbb{A})$ et $J^{p+1} \in \theta(\mathbb{A})$, donc d'après l'hypothèse on a aussi $K^{p+1} \in \theta(\mathbb{A})$.

Le morphisme $d''' : K^p \to K^{p+1}$ est défini en posant, pour $z \in K^p$, $d'''(z) = \psi^{p+1}(d''y)$ où $y \in J^p$ est un élément tel que $\psi^p(y) = z$. On a évidemment $d'''\psi^p = \psi^{p+1}d''$. Il est immédiat que $d'''d''' = 0$ car si $z \in K^{p-1}$ on a $d'''z = \psi^p(d''y)$ où $y \in J^{p-1}$ avec $\psi^{p-1}(y) = z$; on a donc $d'''(d'''z) = \psi^{p+1}(d''d''y) = 0$.

Il reste à montrer que $H^p(K^{\cdot}) = 0$. Soit $z \in K^p$ tel que $d'''z = 0$; donc $d'''z = \psi^{p+1}(d''y) = 0$ où $y \in J^p$ tel que $\psi^p(y) = z$; ainsi il existe $x \in I^{p+1}$ tel que $\varphi^{p+1}(x) = d''y$, mais ceci signifie qu'il existe $\bar{x} \in I^p$ tel que $d'\bar{x} = x$ et $[\varphi^p(\bar{x})] = [y]$. Il existe donc $\bar{y} \in J^{p-1}$ tel que $d''\bar{y} = y - \varphi^p(\bar{x})$. Posons $\bar{z} = \psi^{p-1}(\bar{y})$; on a $d'''\bar{z} = d'''\psi^{p-1}(\bar{y}) = \psi^p(d''\bar{y}) = \psi^p(y - \varphi^p(\bar{x})) = z$. La démonstration commence sans difficulté en degré 0,1 et 2 en adaptant de façon évidente le raisonnement précédent.

7.8 THEOREME : *Supposons que la catégorie* $\mathbb{A}$ *admet suffisamment de θ-objets à droite (respectivement à gauche) et que de plus les deux conditions suivantes sont satisfaites :*

a) Si $0 \to X \to Y \to Z \to 0$ *(respectivement* $0 \to Z \to Y \to X \to 0$*) est une suite exacte courte dans* $\mathbb{A}$ *avec* $X \in \theta(\mathbb{A})$*, alors* $Y \in \theta(\mathbb{A})$ *si et seulement si* $Z \in \theta(\mathbb{A})$.

b) Il existe un entier $n \geq 0$ *tel que si*

$$X^0 \to X^1 \to X^2 \to \dots \to X^{n-1} \to X^n \to 0$$

(respectivement

$$0 \to X^n \to X^{n-1} \to \dots \to X^2 \to X^1 \to X^0)$$

est une suite exacte dans $\mathbb{A}$ *avec* $X^i \in \theta(\mathbb{A})$ *pour* $0 \leq i \leq n-1$*, alors* $X^n \in \theta(\mathbb{A})$.

Alors chaque objet de $C(\mathbb{A})$ *admet une θ-résolution à droite (respectivement à gauche).*

Démonstration : La démonstration, que l'on fera dans le cas où $\mathbb{A}$ admet suffisamment de θ-objets à droite, comporte trois étapes.

i) Etant donné un complexe $X^\cdot \in C(\mathbb{A})$ et un entier $i_0 \in \mathbb{Z}$, il existe un complexe $X_0^\cdot \in C(\mathbb{A})$ qui est homologiquement isomorphe à $X^\cdot$ et tel que $X_0^j \in \theta(\mathbb{A})$ pour $j \geq i_0$.

En effet considérons le complexe $X^\cdot_{<i_0} \in C^+(\mathbb{A})$ défini en posant

$$(X^\cdot_{<i_0})^j = \begin{cases} 0 & \text{si } j < i_0 \\ X^j & \text{si } j \geq i_0 \end{cases}$$

D'après le théorème 7.5 il existe $I^\cdot \in C^+(\theta(\mathbb{A}))$ et un isomorphis homologique $u : X^\cdot_{<i_0} \to I^\cdot$ tel que $u^j : X^j_{<i_0} \to I^j$ soit un monomorphisme.

Considérons alors le complexe $X_0^\cdot \in C(\mathbb{A})$ défini en posant

$$X_0^j = \begin{cases} X^j & \text{si } j \leq i_0-1 \\ I^j & \text{si } j \geq i_0 \end{cases}$$, la différentielle d_{X_0} étant donnée par

$$d^j_{X_0} = \begin{cases} d^j_{X^\cdot} & \text{si } j \leq i_0-2 \\ u^{i_0} d^{i_0}_{I^\cdot} & \text{si } j = i_0-1 \\ d^j_{I^\cdot} & \text{si } j \geq i_0 \end{cases}$$

Le morphisme $v : X^\cdot \to X_0^\cdot$ défini en posant $v^j = \begin{cases} 1_{X^\cdot} & \text{si } j \leq i_0-1 \\ u^j & \text{si } j \geq i_0 \end{cases}$

est évidemment un h.i. et chaque v^j est un monomorphisme.

ii) Soit $X_1^\cdot \in C(\mathbb{A})$ un complexe tel qu'il existe un entier i_1 pour lequel on a $X_1^j \in \theta(\mathbb{A})$ lorsque $j \geq i_1$. Alors pour tout entier $i_2 < i_1$ il existe un complexe $X_2^\cdot \in C(\mathbb{A})$, homologiquement isomorphe à $X_1^\cdot$, tel que $X_2^j \in \theta(\mathbb{A})$ si $j \geq i_2$ et $X_2^j = X_1^j$ si $j \geq i_1+n+1$.

En effet en vertu de i) il existe un complexe $X'^\cdot \in C(\mathbb{A})$ tel que $X'^j \in \theta(\mathbb{A})$ si $j \geq i_2$, et un isomorphisme homologique $u : X_1^\cdot \to X'^\cdot$ tel que chaque u^j soit un monomorphisme. Posons $Y^\cdot = \operatorname{Coker} u$ et considérons la suite exacte courte

$$0 \to X_1^\cdot \xrightarrow{u} X'^\cdot \xrightarrow{v} Y^\cdot \to 0.$$

Comme u est un h.i., il résulte de la suite exacte longue de cohomologie que $Y^\cdot$ est un complexe acyclique. De plus dans les suites exactes courtes

$$0 \to X_1^j \to X'^j \to Y^j \to 0$$

on a $X_1^j \in \theta(\mathbb{A})$ et $X'^j \in \theta(\mathbb{A})$ dès que $j \geq i_1$, donc d'après l'hypothèse a) on a $Y^j \in \theta(\mathbb{A})$ dès que $j \geq i_1$.

Considérons alors, pour tout entier $k \geq i_1$, les suites exactes

$$Y^k \to Y^{k+1} \to \dots \to Y^{k+n} \to B^{k+n}(Y^\cdot) \to 0$$

où $B^j(Y^\cdot) = \operatorname{Im}(d^j_{Y^\cdot})$; il résulte de l'hypothèse b) que $B^j(Y^\cdot) \in \theta(\mathbb{A})$ pour tout entier $j \geq i_1+n$.

Définissons maintenant un complexe $X_2^{\cdot}$ en posant

$$X_2^j = \begin{cases} X'^j & \text{si } j \le i_1+n-1 \\ B^{i_1+n}(X'^{\cdot}) \wedge X_1^{i_1+n} & \text{si } j = i_1+n \\ X_1^j & \text{si } j \ge i_1+n+1 \end{cases}$$

où $B^j(X'^{\cdot}) \wedge X_1^j$ est la somme amalgamée définie par le diagramme suivant

$$\begin{tikzcd} X_1^{j-1} \arrow[r, "d_{X^{\cdot}}^{j-1}"] \arrow[d, "d_{X'^{\cdot}}^{j-1} u^{j-1}"'] & X_1^j \arrow[d, dashed, "\psi"] \\ B^j(X'^{\cdot}) \arrow[r, dashed, "\varphi"'] & B^j(X'^{\cdot}) \wedge X_1^j \end{tikzcd}$$

La différentielle de $d_{X_2^{\cdot}}$ est donnée par

$$d_{X_2^{\cdot}}^j = \begin{cases} d_{X'^{\cdot}}^j & \text{si } j \le i_1+n-2 \\ \varphi d_{X'^{\cdot}}^{i_1+n-1} & \text{si } j = i_1+n-1 \\ \bar{d} & \text{si } j = i_1+n \\ d_{X_1^{\cdot}}^j & \text{si } j \ge i_1+n+1 \end{cases}$$

où $\bar{d}$ est induit par le morphisme naturel $B^{i_1+n}(X'^{\cdot}) \oplus X_1^{i_1+n} \to X_1^{i_1+n+1}$ provenant de $d_{X_1^{\cdot}}$.

Considérons les suites

$$0 \to X_1^j \xrightarrow{\psi} B^j(X'^{\cdot}) \wedge X_1^j \xrightarrow{\theta} B^j(Y^{\cdot}) \to 0$$

où θ est induit par le morphisme naturel $B^j(X'^{\cdot}) \oplus X_1^j \to B^j(Y^{\cdot})$ provenant de $v|_{B^j(X'^{\cdot})}$. Ces suites sont exactes. En effet il est immédiat que ψ est injectif, que θ est surjectif et que $\theta\psi = 0$. Soit alors $a \in B^j(X'^{\cdot}) \wedge X_1^j$ représenté par $(x';x) \in B^j(X'^{\cdot}) \oplus X_1^j$. Si $\theta(a) = 0$ on a $v(x') = 0$ donc il existe $x'' \in X_1^j$ tel que $u(x'') = x'$. Comme u est un h.i. et que la classe de cohomologie

de x' est nulle, il existe $\bar{x} \in X_1^{j-1}$ tel que $d\bar{x} = x''$. Posons $y = x - x'' \in X_1^j$; on a $\psi(y) = [(0;x-x'')] = a$; en effet $(x';x'')$ est équivalent à $(0;0)$ dans $B^j(X'^{\cdot}) \wedge X_1^j$ car $d\bar{x} = x''$ et $du(\bar{x}) = u(d\bar{x}) = u(x'') = x'$. Comme $X_1^j \in \theta(A)$ et $B^j(Y^{\cdot}) \in \theta(A)$ dès que $j \ge i_1+n$, il résulte de l'hypothèse a) que $B^j(X'^{\cdot}) \wedge X_1^j \in \theta(A)$ dès que $j \ge i_1+n$. Maintenant on définit un morphisme de complexes $w : X_1^{\cdot} \to X_2^{\cdot}$ en posant

$$w^j = \begin{cases} u^j & \text{si } j \le i_1+n-1 \\ \psi & \text{si } j = i_1+n \\ 1_{X_1^j} & \text{si } j \ge i_1+n+1 \end{cases}$$

Pour montrer que w est un h.i. il suffit donc de vérifier que ψ induit un isomorphisme sur la cohomologie de degré i_1+n, ce qui est immédiat si on revient aux définitions.

iii) Soit un complexe $X^{\cdot} \in C(A)$. Considérons la suite décroissante d'entiers $i_k = -k$ pour $k \ge 0$. En appliquant l'étape i) avec l'entier $i_0 = 0$ puis en appliquant successivement l'étape ii) avec les entiers $i_k = -k$ pour $k \ge 1$ on construit un système de complexes et d'isomorphismes homologiques

$$X^{\cdot} \to X_0^{\cdot} \to X_1^{\cdot} \to X_2^{\cdot} \to \dots$$

qui vérifient la propriété suivante : $X_k^j \in \theta(A)$ pour $j \ge -k$ et pour chaque $k \ge 0$ on a $X_{k+p}^{n-k+1} = X_{k+p+1}^{n-k+1}$ pour $p \ge 0$. On pose $I^{\cdot} = \varinjlim_k X_k^{\cdot}$; il est clair que $I^{\cdot} \in C(\theta(A))$ et qu'on a un isomorphisme homologique $X^{\cdot} \to I^{\cdot}$.

§ 8. LA CATEGORIE $D^+(I(\mathbb{A}))$

8.1 Soit $\mathbb{A}$ une catégorie abélienne et soit $\theta(\mathbb{A})$ une sous-classe de $\mathrm{Ob}\mathbb{A}$ qui satisfait les hypothèses du No 7.1.

PROPOSITION : *Les catégories* $K^*(\theta(\mathbb{A}))$ $(* = \emptyset,+,-,b)$ *sont des sous-catégories triangulées de* $K(\mathbb{A})$.

Démonstration : Cela résulte immédiatement des hypothèses faites sur la classe $\theta(\mathbb{A})$.

8.2 Posons $\mathrm{His}^*_{\theta(\mathbb{A})} = \mathrm{His} \cap F\ell(K^*(\theta(\mathbb{A})))$.

PROPOSITION : *La famille* $\mathrm{His}^*_{\theta(\mathbb{A})}$ *est un système multiplicatif dans* $K^*(\theta(\mathbb{A}))$, *compatible avec la structure triangulée de* $K^*(\theta(\mathbb{A}))$.

Démonstration : Il suffit de recopier la démonstration de la proposition 6.2 en prenant tous les triangles avec leurs sommets dans $K^*(\theta(\mathbb{A}))$ ce qui est possible en vertu de la proposition 8.1.

8.3 D'après le No 1.23, on peut donc définir les catégories dérivées $D^*(\theta(\mathbb{A}))$ et les foncteurs canoniques de localisation $P^*_{\theta(\mathbb{A})} : K^*(\theta(\mathbb{A})) \to D^*(\theta(\mathbb{A}))$. On note $Q^*_{\theta(\mathbb{A})} : D^*(\theta(\mathbb{A})) \to D^*(\mathbb{A})$ le foncteur induit par le foncteur d'inclusion $K^*(\theta(\mathbb{A})) \to K^*(\mathbb{A})$ et $P^*_{\theta(\mathbb{A})} : K^*(\theta(\mathbb{A})) \to D^*(\mathbb{A})$ le foncteur composé $K^*(\theta(\mathbb{A})) \to K^*(\mathbb{A}) \xrightarrow{P^*_{\mathbb{A}}} D^*(\mathbb{A})$

8.4 On va étudier plus en détails les différents foncteurs précédents lorsque l'on prend pour $\theta(\mathbb{A})$ la classe $I(\mathbb{A})$ des objets injectifs de $\mathbb{A}$.

8.5 LEMME : *Si* $X^\cdot \in K(\mathbb{A})$ *est un complexe acyclique et si* $Y^\cdot \in K^+(I(\mathbb{A}))$, *alors* $\mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot) = 0$.

Démonstration : Soit $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$; il faut définir un opérateur d'homotopie $k \in \mathrm{Hom}^{-1}(X^\cdot;Y^\cdot)$ tel que $dk+kd = u$. Sans restreindre la généralité on peut supposer $Y^p = 0$ pour $p < 0$. Il est clair que $k^p = 0$ si $p \leq 0$. Supposons que l'on a défini k^p pour $p \leq q$ avec $q \geq 0$.

Soit $x \in \mathrm{Im}(d : X^q \to X^{q+1}) \subset X^{q+1}$ et soit $x' \in X^q$ tel que $dx' = x$. Si $x'' \in X^q$ tel que $dx'' = x$ il existe $x''' \in X^{q-1}$ tel que $dx''' = x'-x''$.

On a $u(x')-dk^q(x') = u(x'')-dk^q(x'')$; en effet $u(x'-x'')-dk^q(x'-x'') = du(x''')-d(u(x''')-dk^{q-1}(x''')) = 0$. On peut donc définir $k^{q+1} : \mathrm{Im}(d) \to Y^q$ en posant $k^{q+1}(x) = u(x')-dk^q(x')$ et comme Y^q est injectif on peut étendre cette application en une application $k^{q+1} : X^{q+1} \to Y^q$ qui satisfait évidemment la condition requise.

8.6 LEMME : *Si* $u \in \mathrm{Hom}_{C^+(I(\mathbb{A}))}(X^\cdot;Y^\cdot)$ *est un isomorphisme homologique alors* u *est une équivalence d'homotopie.*

Démonstration : Considérons le triangle standard

$$X^\cdot \xrightarrow[\sim]{u} Y^\cdot \xrightarrow{q} C^\cdot_u \xrightarrow{p} T(X^\cdot)$$

Comme u est un h.i., le cône $C^\cdot_u$ est acyclique, donc d'après le lemme 8.5 p est homotope à zéro; il existe donc une application $k \in \mathrm{Hom}^{-1}(C^\cdot_u;T(X^\cdot))$ telle que

$$dk+kd = p \qquad (8.6.1)$$

L'application $v = kq : Y^\cdot \to X^\cdot$ est de degré 0; en multipliant les deux membres de (8.6.1) par q on obtient $dv = vd$; donc $v \in \mathrm{Hom}_{C^+(I(\mathbb{A}))}(Y^\cdot;X^\cdot)$.

L'application $k' = kq_{X^\cdot} : X^\cdot \to X^\cdot$ est de degré -1; en multipliant les deux membres de (8.6.1) par $q_{X^\cdot}$ et en tenant compte de la définition de $d_{C^\cdot_u}$ on obtient $dk' + k'd = vu - 1_{X^\cdot}$. Donc $vu \sim 1_{X^\cdot}$.

Il est clair que v est un h.i. car $v^*u^* = 1_{H(X^\cdot)}$ et u^* est un isomorphisme.

On peut donc appliquer à v le même raisonnement que précédemment pour obtenir un morphisme $w \in \mathrm{Hom}_{C^+(I(A))}(X^\cdot;Y^\cdot)$ tel que $wv \sim 1_Y$.

On en déduit que $w \sim u$ et par suite que $1_{Y^\cdot} \sim uv$.

8.7 COROLLAIRE : *Dans la catégorie $K^+(I(A))$ il y a équivalence entre isomorphisme homologique et isomorphisme.*

8.8 PROPOSITION : *Le foncteur canonique*

$$P^+_{I(A)} : K^+(I(A)) \to D^+(I(A))$$

est un isomorphisme de catégories.

Démonstration : D'après le corollaire 8.7 le foncteur $1_{K^+(I(A))}$ transforme les isomorphismes homologiques en isomorphismes, donc d'après la propriété universelle du foncteur $P^+_{I(A)}$ il existe un unique foncteur $G : D^+(I(A)) \to K^+(I(A))$ tel que $G \circ P^+_{I(A)} = 1_{K^+(I(A))}$.

La propriété universelle du foncteur $P^+_{I(A)}$ montre aussi que $P^+_{I(A)} \circ G = 1_{D^+(I(A))}$.

8.9 PROPOSITION : *Le foncteur*

$$Q^+_{I(A)} : D^+(I(A)) \to D^+(A)$$

est pleinement fidèle.

Démonstration : Soit $s \in \mathrm{Hom}_{K^+(A)}(X^\cdot;Y^\cdot) \cap \mathrm{His}^+$ où $X^\cdot \in \mathrm{Ob}K^+(I(A))$.

La première partie de la démonstration du lemme 8.6, qui n'utilise que le fait que $X^\cdot$ est un complexe borné inférieurement d'objets injectifs, montre qu'il existe $f \in \mathrm{Hom}_{K^+(A)}(Y^\cdot;X^\cdot)$ tel que $fs = 1_{X^\cdot} \in \mathrm{His}^+_{I(A)}$.

La condition ii) de la proposition 1.24 est donc satisfaite.

8.10 THEOREME : *Le foncteur*

$$\overrightarrow{P}^+_{I(A)} : K^+(I(A)) \to D^+(A)$$

est pleinement fidèle.

De plus, si la catégorie A admet suffisamment d'objets injectifs, alors $\overrightarrow{P}^+_{I(A)}$ est une équivalence de catégorie.

Démonstration : Comme $\overrightarrow{P}^+_{I(A)}$ est composé des foncteurs $P^+_{I(A)}$ et $Q^+_{I(A)}$, la première partie résulte des proposition 8.8 et 8.9.

Maintenant soit $Y^\cdot \in \mathrm{Ob}D^+(A)$. Si la catégorie A admet suffisamment d'objets injectifs alors il existe $X^\cdot \in \mathrm{Ob}K^+(I(A))$ et un isomorphisme homologique $s : Y^\cdot \to X^\cdot$ dans $K^+(A)$. Mais alors $\overrightarrow{P}^+_{I(A)}(s) = P^+_A(s) : Y^\cdot \to \overrightarrow{P}^+_{I(A)}(X^\cdot)$ est un isomorphisme dans $D^+(A)$.

8.11 On obtient des résultats analogues aux précédents en considérant la sous-catégorie Proj(A) des objets projectifs de A et en remplaçant l'indice + par l'indice − . On laisse au lecteur le soin d'énoncer et de démontrer ces résultats.

§ 9. LES FONCTEURS DERIVES

9.1 Soit $\mathbb{A}$ une catégorie abélienne.

Si $K^*(\mathbb{A})$ est une sous-catégorie triangulée de $K(\mathbb{A})$, il est facile de vérifier que la famille $His^* = His \cap F\ell K^*(\mathbb{A})$ est un système multiplicatif dans $K^*(\mathbb{A})$. On peut donc localiser la catégorie $K^*(\mathbb{A})$ relativement à His^*. Supposons que le foncteur $K^*(\mathbb{A})_{His^*} \to D(\mathbb{A})$, induit par le foncteur d'inclusion $K^*(\mathbb{A}) \to K(\mathbb{A})$, soit pleinement fidèle. Dans ce cas la catégorie localisée triangulée $K^*(\mathbb{A})_{His^*}$ peut s'identifier à une sous-catégorie pleine de $D(\mathbb{A})$; on la note alors $D^*(\mathbb{A})$ et on note $P^*_{\mathbb{A}} : K^*(\mathbb{A}) \to D^*(\mathbb{A})$ le foncteur de localisation. Par exemple il résulte de la proposition 6.15 que les catégories $K^+(\mathbb{A})$, $K^-(\mathbb{A})$ et $K^b(\mathbb{A})$ vérifient l'hypothèse précédente.

9.2 Soit $\mathbb{A}$ et $\mathbb{B}$ deux catégories abéliennes.

Soit $K^*(\mathbb{A})$ une sous-catégorie triangulée de $K(\mathbb{A})$ qui satisfait les hypothèses du No 9.1 et soit $F : K^*(\mathbb{A}) \to K(\mathbb{B})$ un ∂-foncteur.

On appelle foncteur dérivé à droite de F un couple $(R^*F;\xi_F)$, où $R^*F : D^*(\mathbb{A}) \to D(\mathbb{B})$ est un ∂-foncteur et $\xi_F : P_{\mathbb{B}} \circ F \to R^*F \circ P^*_{\mathbb{A}}$ est un morphisme de foncteurs de $K^*(\mathbb{A})$ à $D(\mathbb{B})$, qui satisfait la propriété universelle suivante : pour tout ∂-foncteur $G : D^*(\mathbb{A}) \to D(\mathbb{B})$ et pour tout morphisme de foncteurs $\zeta : P_{\mathbb{B}} \circ F \to G \circ P^*_{\mathbb{A}}$ il existe un unique morphisme de foncteurs $\eta : R^*F \to G$ tel que $\zeta = (\eta \circ P^*_{\mathbb{A}}) \circ \xi_F$.

De même on appelle foncteur dérivé à gauche de F un couple $(L^*F;\xi_F)$, où $L^*F : D^*(\mathbb{A}) \to D(\mathbb{B})$ est un ∂-foncteur et $\xi_F : L^*F \circ P^*_{\mathbb{A}} \to P_{\mathbb{B}} \circ F$ est un morphisme de foncteurs de $K^*(\mathbb{A})$ à $D(\mathbb{B})$, qui satisfait la propriété universelle suivante : pour tout ∂-foncteur $G : D^*(\mathbb{A}) \to D(\mathbb{B})$ et pour tout morphisme de foncteurs $\zeta : G \circ P^*_{\mathbb{A}} \to P_{\mathbb{B}} \circ F$ il existe un unique morphisme de foncteurs $\eta : G \to L^*F$ tel que $\zeta = \xi_F \circ (\eta \circ P^*_{\mathbb{A}})$.

9.3 *Remarque :* La définition précédente signifie que R^*F (respectivement L^*F) est un objet qui représente le foncteur $G \to Hom(P_{\mathbb{B}} \circ F; G \circ P^*_{\mathbb{A}})$ (respectivement $G \to Hom(G \circ P^*_{\mathbb{A}}; P_{\mathbb{B}} \circ F)$), de la catégorie des ∂-foncteurs de $K^*(\mathbb{A})$ dans $D(\mathbb{B})$, dans la catégorie $\mathbf{Ens}$.

9.4 De la remarque précédente il résulte immédiatement que l'on a les deux propriétés suivantes :

a) Si R^*F (respectivement L^*F) existe, il est unique à isomorphisme près.

b) Si $\varphi : F \to G$ est un morphisme de foncteurs de $K^*(\mathbb{A})$ à $K(\mathbb{B})$ et si R^*F et R^*G (respectivement L^*F et L^*G) existent, alors il existe un unique morphisme de foncteurs $R^*\varphi : R^*F \to R^*G$ (respectivement $L^*\varphi : L^*F \to L^*G$) tel que $\xi_G \circ (P_{\mathbb{B}} \circ \varphi) = (R^*\varphi \circ P^*_{\mathbb{A}}) \circ \xi_F$ (respectivement $(P_{\mathbb{B}} \circ \varphi) \circ \xi_F = (L^*\varphi \circ P^*_{\mathbb{A}}) \circ \xi_G$).

9.5 THEOREME D'EXISTENCE : *Soit $\mathbb{A}$ et $\mathbb{B}$ deux catégories abéliennes et $K^*(\mathbb{A})$ une sous-catégorie triangulée de $K(\mathbb{A})$ qui satisfait les hypothèses du No 9.1 . Soit $F : K^*(\mathbb{A}) \to K(\mathbb{B})$ un ∂-foncteur.*

Supposons qu'il existe une sous-catégorie triangulée $\Gamma(\mathbb{A})$ de $K^*(\mathbb{A})$ qui vérifie les deux conditions suivantes :

a) Pour tout $X^{\cdot} \in ObK^*(\mathbb{A})$ il existe $M^{\cdot} \in Ob\Gamma(\mathbb{A})$ et un isomorphisme homologique $X^{\cdot} \to M^{\cdot}$ (respectivement $M^{\cdot} \to X^{\cdot}$).

b) Si $M^{\cdot} \in Ob\Gamma(\mathbb{A})$ est acyclique, alors $F(M^{\cdot}) \in ObK(\mathbb{B})$ est aussi acyclique.

Alors le foncteur dérivé à droite $(R^*F;\xi_F)$ (respectivement à gauche $(L^*F;\xi_F)$) existe et pour tout $M^{\cdot} \in Ob\Gamma(\mathbb{A})$ le morphisme $\xi_F(M^{\cdot}) : P_{\mathbb{B}}(F(M^{\cdot})) \to R^*F(P^*_{\mathbb{A}}(M^{\cdot}))$ (respectivement $\xi_F(M^{\cdot}) : L^*F(P^*_{\mathbb{A}}(M^{\cdot})) \to P_{\mathbb{B}}(F(M^{\cdot}))$ est un isomorphisme dans $D(\mathbb{B})$.

Démonstration : On va faire la démonstration pour R^*F; la démonstration est analogue pour L^*F en se plaçant dans les catégories opposées.

Le diagramme suivant contient les différents foncteurs qui interviennent dans la démonstration.

$$\begin{tikzcd}
K^*(A) \arrow[rr, "F"] \arrow[dddd, "P^*_A"'] & & K(A) \arrow[dddd, "P_B"] \\
& \Gamma(A) \arrow[ul] \arrow[ur] \arrow[d, "\overline{P}"'] & \\
& \overline{\Gamma(A)} \arrow[ddl, dashed, bend right, "Q"'] \arrow[ddr, dashed, "\overline{F}"] & \\
& & \\
D^*(A) \arrow[rr, "R^*F"'] \arrow[uur, dashed, "U"'] & & D(B)
\end{tikzcd}$$

i) Puisque $\Gamma(A)$ est une sous-catégorie triangulée, la famille $\overline{\text{His}} = \text{His} \cap F\ell\Gamma(A)$ est un système multiplicatif dans $\Gamma(A)$; on peut donc définir la catégorie localisée $\overline{\Gamma(A)} = \Gamma(A)_{\overline{\text{His}}}$ et le foncteur de localisation $\overline{P} : \Gamma(A) \to \overline{\Gamma(A)}$.

ii) Notons encore F la restriction du foncteur F à $\Gamma(A)$. Soit $s \in \text{Hom}_{\Gamma(A)}(X^\cdot;Y^\cdot) \cap \text{His}$. D'après TR1 b) il existe un triangle distingué $X^\cdot \xrightarrow{s} Y^\cdot \to Z^\cdot \to T(X^\cdot)$ et d'après la proposition 5.14 $Z^\cdot$ est acyclique puisque s est un h.i. Il résulte de l'hypothèse b) que $F(Z^\cdot)$ est acyclique et comme F est un ∂-foncteur on a $F(s) \in \text{Hom}_{K(B)}(F(X^\cdot);F(Y^\cdot)) \cap \text{His}$.

iii) D'après la propriété universelle du foncteur $\overline{P}$ il existe donc un unique foncteur $\overline{F} : \overline{\Gamma(A)} \to D(B)$ tel que $\overline{F} \circ \overline{P} = P_B \circ F$.

iv) Soit $s \in \text{Hom}_{K^*(A)}(X^\cdot;Y^\cdot) \cap \text{His}$ où $X^\cdot \in \text{Ob}\Gamma(A)$. D'après l'hypothèse a) il existe un isomorphisme homologique $f : Y^\cdot \to M^\cdot$ où $M^\cdot \in \text{Ob}\Gamma(A)$. Ainsi la condition ii) de la proposition 1.24 est satisfaite si bien que le foncteur $Q : \overline{\Gamma(A)} \to D^*(A)$, induit par le foncteur d'inclusion $\Gamma(A) \to K^*(A)$, est pleinement fidèle.

Maintenant soit $Y^\cdot \in \text{Ob}D^*(A)$. D'après l'hypothèse a) il existe un isomorphisme homologique $s : Y^\cdot \to X^\cdot$ avec $X^\cdot \in \text{Ob}\Gamma(A)$. Mais alors $Q(s) = P^*_A(s) : Y^\cdot \to Q(X^\cdot)$ est un isomorphisme dans $D^*(A)$. Il en résulte que le foncteur Q est une équivalence de catégorie. Notons $U : D^*(A) \to \overline{\Gamma(A)}$ un foncteur quasi-inverse à Q; il existe donc des isomorphismes de foncteurs

$$\alpha : 1_{\overline{\Gamma(A)}} \to U \circ Q$$

$$\beta : 1_{D^*(A)} \to Q \circ U$$

v) On pose $R^*F = \overline{F} \circ U : D^*(A) \to D(B)$.
Il faut maintenant définir un morphisme de foncteurs

$$\xi_F : P_B \circ F \to R^*F \circ P^*_A = \overline{F} \circ U \circ P^*_A$$

Si $X^\cdot \in \text{Ob}K^*(A)$ alors $U \circ P^*_A(X^\cdot) \in \text{Ob}\overline{\Gamma(A)}$ donc il existe $M^\cdot \in \text{Ob}\Gamma(A)$ tel que $\overline{P}(M^\cdot) = U \circ P^*_A(X^\cdot)$. L'isomorphisme dans $D^*(A)$

$$\beta(P^*_A(X^\cdot)) : P^*_A(X^\cdot) \to Q \circ U \circ P^*_A(X) = Q \circ \overline{P}(M^\cdot)$$

est représenté par un diagramme dans $K^*(A)$

$$X^\cdot \xrightarrow[\sim]{t} V^\cdot \xleftarrow[\sim]{s} M^\cdot$$

où on peut supposer $V^\cdot \in \text{Ob}\Gamma(A)$ d'après l'hypothèse a). On en déduit un diagramme dans $K(B)$

$$F(X^\cdot) \xrightarrow{F(t)} F(V^\cdot) \xleftarrow[\sim]{F(s)} F(M^\cdot)$$

où $F(s)$ est un h.i. d'après ii). On définit alors $\xi_F(X^\cdot) : P_B \circ F(X^\cdot) \to P_B \circ F(M^\cdot) = \overline{F} \circ U \circ P^*_A(X^\cdot)$ comme étant la classe d'équivalence de ce diagramme; autrement dit $\xi_F(X^\cdot) = (P_B(F(s)))^{-1} \circ P_B(F(t))$. Cette définition est indépendante du choix du représentant de $\beta(P^*_A(X^\cdot))$. On notera tout de suite que si $X^\cdot \in \text{Ob}\Gamma(A)$ alors $F(t)$ est aussi un h.i., donc $\xi_F(X^\cdot)$ est un isomorphisme dans ce cas.
Il faut vérifier que la définition précédente est fonctorielle, c'est-à-dire si $f \in \text{Hom}_{K^*(A)}(X^\cdot;Y^\cdot)$ alors on a un diagramme commutatif dans $D(B)$

$$\begin{tikzcd}
P_B \circ F(X^\cdot) \arrow[r, "\xi_F(X^\cdot)"] \arrow[d, "P_B(F(f))"'] & R^*F \circ P^*_A(X^\cdot) \arrow[d, "R^*F(P^*_A(f))"] \\
P_B \circ F(Y^\cdot) \arrow[r, "\xi_F(Y^\cdot)"'] & R^*F \circ P^*_A(Y^\cdot)
\end{tikzcd}$$

Soit $N^\cdot \in \mathrm{Ob}\Gamma(A)$ tel que $\overline{P}(N^\cdot) = U \circ P^*_A(Y^\cdot)$ et soit $Y^\cdot \xrightarrow[\sim]{t'} W^\cdot \xleftarrow[\sim]{s'} N^\cdot$ un représentant de $\beta(P^*_A(Y^\cdot))$. En appliquant deux fois l'axiome FR2 on peut construire le diagramme suivant dans lequel tous les objets, sauf $X^\cdot$ et $Y^\cdot$, sont dans $\Gamma(A)$

$$\begin{tikzcd}
X^\cdot \arrow[rr, "t", "\sim"'] \arrow[dd, "f"'] & & V^\cdot \arrow[dl, dashed, "f'"'] & & M^\cdot \arrow[ll, "s"', "\sim"] \arrow[dl, dashed, "a"] \\
& P^\cdot \arrow[rr, dashed, "u", "\sim"'] & & Q^\cdot & \\
Y^\cdot \arrow[rr, "\sim", "t'"'] \arrow[ur, dashed, "r", "\sim"'] & & W^\cdot \arrow[ur, dashed, "r'", "\sim"'] & & N^\cdot \arrow[ll, "\sim"', "s'"] \arrow[ul, "b"']
\end{tikzcd}$$

On en déduit le diagramme

$$\begin{tikzcd}
F(X^\cdot) \arrow[rr, "F(t)"] \arrow[dd, "F(f)"'] & & F(V^\cdot) \arrow[rr, "F(s)", "\sim"'] \arrow[dl, "F(f')"'] & & F(M^\cdot) \arrow[dl, "F(a)"] \\
& F(P^\cdot) \arrow[rr, "F(u)", "\sim"'] & & F(Q^\cdot) & \\
F(Y^\cdot) \arrow[rr, "F(t')"'] \arrow[ur, "F(r)"] & & F(W^\cdot) \arrow[ur, "\sim", "F(r')"'] & & F(N^\cdot) \arrow[ll, "\sim", "F(s')"'] \arrow[ul, "F(b)"', "\sim"]
\end{tikzcd}$$

Remarquons que $R^*F \circ P^*_A(X^\cdot) = P_B \circ F(M^\cdot)$ et $R^*F \circ P^*_A(Y^\cdot) = P_B \circ F(N^\cdot)$ si bien que le morphisme $R^*F(P^*_A(f)) : R^*F \circ P^*_A(X^\cdot) \to R^*F \circ P^*_A(Y^\cdot)$ est représenté par le diagramme $F(M^\cdot) \xrightarrow{F(a)} F(Q^\cdot) \xleftarrow[\sim]{F(b)} F(N^\cdot)$.

On a alors

$R^*F(P^*_A(f)) \circ \xi_F(X^\cdot) = P_BF(b)^{-1} \circ P_BF(a) \circ P_BF(s)^{-1} \circ P_BF(t)$

$= (P_BF(r') \circ P_BF(s'))^{-1} \circ P_BF(u) \circ P_BF(f') \circ P_BF(s) \circ P_BF(s)^{-1} \circ P_BF(t)$

$= P_BF(s')^{-1} \circ P_BF(r')^{-1} \circ P_BF(u) \circ P_BF(f') \circ P_BF(t)$

$= P_BF(s')^{-1} \circ P_BF(r')^{-1} \circ P_BF(u) \circ P_BF(r) \circ P_BF(f)$

$= P_BF(s')^{-1} \circ P_BF(r')^{-1} \circ P_BF(r') \circ P_BF(t') \circ P_BF(f)$

$= P_BF(s')^{-1} \circ P_BF(t') \circ P_BF(f) = \xi_F(Y^\cdot) \circ P_BF(f)$.

vi) Soit $(G;\zeta)$ où $G : D^*(A) \to D(B)$ est un ∂-foncteur et $\zeta : P_B \circ F \to G \circ P^*_A$ est un morphisme de foncteurs. S'il existe un morphisme de foncteurs $\eta : R^*F \to G$ tel que $\zeta = (\eta \circ P^*_A) \circ \xi_F$ alors pour tout $M^\cdot \in \mathrm{Ob}\Gamma(A)$ on a $\zeta(M^\cdot) = \eta(M^\cdot) \circ \xi_F(M^\cdot)$ donc $\eta(M^\cdot) = \zeta(M^\cdot) \circ \xi_F(M^\cdot)^{-1}$ puisque $\xi_F(M^\cdot)$ est un isomorphisme dans ce cas.

L'égalité précédente définit donc η sur la sous-catégorie $\overline{\Gamma(A)}$ de $D^*(A)$. Maintenant si $X^\cdot \in \mathrm{Ob}D^*(A) = \mathrm{Ob}K^*(A)$ alors il existe un isomorphisme homologique $s : X^\cdot \to M^\cdot$ où $M^\cdot \in \mathrm{Ob}\Gamma(A)$. Comme $P^*_A(s)$ est un isomorphisme on peut poser

$\eta(X^\cdot) = G(P^*_A(s))^{-1} \circ \eta(M^\cdot) \circ R^*F(P^*_A(s))$.

Comme ζ et ξ_F sont des morphismes de foncteurs il est immédiat que η est un morphisme de foncteurs sur la sous-catégorie $\overline{\Gamma(A)}$. Maintenant soit $f \in \mathrm{Hom}_{D^*(A)}(X^\cdot;Y^\cdot)$ et des isomorphismes homologiques $s : X^\cdot \to M^\cdot$ et $t : Y^\cdot \to N^\cdot$. En appliquant deux fois l'axiome FR2 on construit, comme dans v), un morphisme $g \in \mathrm{Hom}_{D^*(A)}(M^\cdot;N^\cdot)$ tel que, dans le diagramme

$$\begin{tikzcd}
R^*F(M^\cdot) \arrow[rrr, "\eta(M^\cdot)"] \arrow[ddd, "R^*F(g)"'] & & & G(M^\cdot) \arrow[ddd, "G(g)"] \\
& R^*F(X^\cdot) \arrow[r, "\eta(X^\cdot)"] \arrow[ul, "R^*F(P^*_A(s)"'] \arrow[d, "R^*F(f)"'] & G(X^\cdot) \arrow[ur, "G(P^*_A(s))"'] \arrow[d, "G(f)"] & \\
& R^*F(N^\cdot) \arrow[r, "\eta(Y^\cdot)"'] \arrow[dl, "R^*F(P^*_A(t))"] & G(Y^\cdot) \arrow[dr, "G(P^*_A(t))"'] & \\
R^*F(N^\cdot) \arrow[rrr, "\eta(N^\cdot)"'] & & & G(N^\cdot)
\end{tikzcd}$$

les carrés latéraux soient commutatifs. Comme le carré extérieur est commutatif il en résulte que le carré intérieur est commutatif.

Enfin il est évident par construction que η est l'unique morphisme de fonteurs tel que $\zeta = (\eta \circ P^*_A) \circ \xi_F$.

9.6 Soit $K^*(A)$ et $K^\#(A)$ deux sous-catégories triangulées de $\mathcal{K}(A)$ qui satisfont les hypothèses du No 9.1 et telles que $K^\#(A)$ soit une sous-catégorie pleine de $K^*(A)$.

Soit $F : K^*(A) \to K(A)$ un ∂-foncteur et notons $F_{|K^\#(A)}$ la restriction de F à $K^\#(A)$ c'est-à-dire la composition du foncteur d'inclusion $K^\#(A) \to K^*(A)$ avec le foncteur F.

Supposons que R^*F et $R^\#(F_{|K^\#(A)})$ (respectivement L^*F et $L^\#(F_{|K^\#(A)})$) existent et notons $(R^*F)_{|D^\#(A)}$ (respectivement $(L^*F)_{|D^\#(A)}$) la restriction de R^*F (respectivement L^*F) à $D^\#(A)$.

Alors il existe un morphisme de foncteurs

$$\rho : R^\#(F_{|K^\#(A)}) \to (R^*F)_{|D^\#(A)}$$

(respectivement

$$\rho : (L^*F)_{|D^\#(A)} \to L^\#(F_{|K^\#(A)}) \quad)$$

tel que $(\xi_F)_{|K^\#(A)} = (\rho \circ P_A^\#) \circ \xi_{(F_{|K^\#(A)})}$ (respectivement $(\xi_F)_{|K^\#(A)} = \xi_{(F_{|K^\#(A)})} \circ (\rho \circ P_A^\#)$).

En effet remarquons tout d'abord que le diagramme commutatif d'inclusions

$$\begin{tikzcd}
K^*(A) \arrow[dr, "I^*"] & \\
& K(A) \\
K^\#(A) \arrow[uu, "I"] \arrow[ur, "I^\#"'] &
\end{tikzcd}$$

induit un diagramme commutatif

$$\begin{tikzcd}
D^*(A) \arrow[dr, "Q^*"] & \\
& D(A) \\
D^\#(A) \arrow[uu, "Q"] \arrow[ur, "Q^\#"'] &
\end{tikzcd}$$

dans lequel les foncteurs Q^* et $Q^\#$ sont pleinement fidèles par hypothèse; on en déduit aisément que le foncteur Q est aussi pleinement fidèle.

Il suffit alors d'appliquer la propriété universelle du foncteur dérivé $(R^\#(F_{|K^\#(A)}); \xi_{(F_{|K^\#(A)})})$ au couple $((R^*F)_{|D^\#(A)}; (\xi_F)_{|K^\#(A)})$ pour obtenir le morphisme ρ . On procède d'une façon analogue pour les foncteurs dérivés à gauche.

9.7 PROPOSITION : *Soit* $K^*(A)$ *et* $K^\#(A)$ *deux sous-catégories triangulées de* $K(A)$ *qui satisfont les hypothèses du No 9.1 et telles que* $K^\#(A)$ *soit une sous-catégorie pleine de* $K^*(A)$.

Soit $F : K^*(A) \to K(B)$ *un* ∂*-foncteur.*

Supposons qu'il existe une sous-catégorie triangulée $\Gamma(A)$ *de* $K^*(A)$ *qui vérifie les trois conditions suivantes :*

a) Pour tout $X^\cdot \in \mathrm{Ob}K^*(A)$ *il existe* $M^\cdot \in \mathrm{Ob}\Gamma(A)$ *et un isomorphisme homologique* $X^\cdot \to M^\cdot$ *(respectivement* $M^\cdot \to X^\cdot$*).*

b) Si $M^\cdot \in \mathrm{Ob}\Gamma(A)$ *est acyclique, alors* $F(M^\cdot) \in \mathrm{Ob}K(B)$ *est aussi acyclique.*

c) Pour tout $X^\cdot \in \mathrm{Ob}K^\#(A)$ *il existe* $M^\cdot \in \mathrm{Ob}\Gamma(A) \cap \mathrm{Ob}K^\#(A)$ *et un isomorphisme homologique* $X^\cdot \to M^\cdot$ *(respectivement* $M^\cdot \to X^\cdot$*).*

Alors les foncteurs dérivés R^*F *et* $R^\#(F_{|K^\#(A)})$ *(respectivement* L^*F *et* $L^\#(F_{|K^\#(A)})$*) existent et le morphisme de foncteurs*

$\rho \;:\; R^\#(F_{|K^\#(A)}) \to (R^*F)_{|D^\#(A)}$ *(respectivement*

$\rho \;:\; (L^*F)_{|D^\#(A)} \to L^\#(F_{|K^\#(A)})$*) est un isomorphisme.*

Démonstration : L'existence de R^*F relève immédiatement du théorème 9.5. Maintenant d'après les hypothèses b) et c) la sous-catégorie triangulée $\Gamma(A) \cap K^\#(A)$ de $K^\#(A)$ vérifie évidemment la condition a) du théorème 9.5 et la condition b) de ce théorème pour le foncteur $F_{|K^\#(A)}$. Il en résulte que le foncteur $R^\#(F_{|K^\#(A)})$ existe.

Soit $X^\cdot \in \mathrm{Ob}D^\#(A) = \mathrm{Ob}K^\#(A)$; d'après l'hypothèse c) il existe $M^\cdot \in \mathrm{Ob}(\Gamma(A) \cap K^\#(A))$ et un isomorphisme $X^\cdot = P_A(M^\cdot)$ dans $D^\#(A)$, et d'après le No 9.6 on a $\xi_F(M^\cdot) = (\rho \circ P_A(M^\cdot)) \circ \xi_{(F_{|K^\#(A)})}(M^\cdot)$.

Mais alors d'après la dernière partie du théorème 9.5 le morphisme $\rho(X^{\cdot}) = \xi_F(M^{\cdot})\circ\xi_{(F_{|K*(\mathbb{A})})}(M^{\cdot})^{-1}$ est un isomorphisme.

Le cas des foncteurs dérivés à gauche se démontre d'une façon analogue.

9.8 PROPOSITION : *Soit* $\mathbb{A}$ *et* $\mathbb{B}$ *deux catégories abéliennes et* $F : K^+(\mathbb{A}) \to K(\mathbb{B})$ *(respectivement* $F : K^-(\mathbb{A}) \to K(\mathbb{B})$*) un* ∂*-foncteur.*

Supposons que la catégorie $\mathbb{A}$ *admet suffisamment d'objets injectifs (respectivement projectifs) et que le foncteur* F *transforme les objets acycliques de* $K^+(I(\mathbb{A}))$ *(respectivement de* $K^-(\mathrm{Proj}(\mathbb{A}))$*) en des objets acycliques de* $K(\mathbb{B})$.
Alors le foncteur dérivé R^+F *(respectivement* L^-F*) existe.*

Démonstration : La catégorie $K^+(\mathbb{A})$ est une sous-catégorie triangulée de $K(\mathbb{A})$ qui, d'après la proposition 6.15 satisfait les hypothèses du No 9.1. Soit $K^+(I(\mathbb{A}))$ la catégorie des complexes bornés inférieurement d'objets injectifs de $\mathbb{A}$;c'est une sous-catégorie triangulée de $K^+(\mathbb{A})$ qui satisfait évidemment la condition a) du théorème 9.5 ([S], chap. III,§2). Maintenant si $M^{\cdot} \in \mathrm{Ob}K^+(I(\mathbb{A}))$ est acyclique il résulte de l'hypothèse que $F(M^{\cdot})$ est acyclique. La condition b) du théorème 9.5 est donc aussi satisfaite. Ainsi le foncteur R^+F existe.

L'existence de L^-F se démontre d'une façon analogue.

9.9 *Remarque* : Il est peut-être utile de donner une construction explicite du foncteur R^+F lorsque F est induit d'un foncteur additif $F : \mathbb{A} \to \mathbb{B}$.

Soit $X^{\cdot} \in \mathrm{Ob}D^+(\mathbb{A}) = \mathrm{Ob}K^+(\mathbb{A})$. En prenant une résolution injective de chacun des termes du complexe $X^{\cdot}$ on obtient un complexe double $J^{\cdot\cdot}$ d'objets injectifs de $\mathbb{A}$ et on en déduit un isomorphisme homologique de $X^{\cdot}$ dans le complexe simple $sJ^{\cdot\cdot}$ associé à $J^{\cdot\cdot}$.
Alors $R^+F(X^{\cdot}) = P_{\mathbb{B}}F(sJ^{\cdot\cdot})$.
Si $\alpha \in \mathrm{Hom}_{D^+(\mathbb{A})}(X^{\cdot};Y^{\cdot})$ est représenté par le diagramme $X^{\cdot} \xleftarrow[\sim]{s} Z^{\cdot} \xrightarrow{a} Y^{\cdot}$, en prenant des résolutions injectives de $X^{\cdot}$,$Y^{\cdot}$ et $Z^{\cdot}$ on obtient un diagramme commutatif

$$\begin{tikzcd} X^{\cdot} \arrow[d, "\wr"'] & Z^{\cdot} \arrow[l, "s"', "\sim"] \arrow[r, "a"] \arrow[d, "\wr"] & Y^{\cdot} \arrow[d, "\wr"] \\ sJ^{\cdot\cdot} & sK^{\cdot\cdot} \arrow[l, "s'"] \arrow[r, "a'"'] & sI^{\cdot\cdot} \end{tikzcd}$$

dans lequel s' est un isomorphisme (corollaire 8.7). Alors $R^+F(\alpha) = P_{\mathbb{B}}(F(a')\circ F(s')^{-1}) \in \mathrm{Hom}_{D(\mathbb{B})}(R^+F(X^{\cdot});R^+F(Y^{\cdot}))$.

9.10 Dans les numéros suivants on va donner quelques autres cas d'existence des foncteurs dérivés.

PROPOSITION : *Soit* $\mathbb{A}$ *et* $\mathbb{B}$ *deux catégories abéliennes et* $F : \mathbb{A} \to \mathbb{B}$ *un foncteur additif.*
Supposons qu'il existe une sous-classe $\theta(\mathbb{A}) \subset \mathrm{Ob}\mathbb{A}$ *qui vérifie les hypothèses du No 7.1 et qui satisfait les conditions suivantes :*

i) La catégorie $\mathbb{A}$ *admet suffisamment de* θ*-objets à droite (respectivement à gauche).*

ii) Si $0 \to X \to Y \to Z \to 0$ *(respectivement* $0 \to Z \to Y \to X \to 0$*) est une suite exacte courte dans* $\mathbb{A}$ *avec* $X \in \theta(\mathbb{A})$ *, alors* $Y \in \theta(\mathbb{A})$ *si et seulement si* $Z \in \theta(\mathbb{A})$.

iii) Si $0 \to X \to Y \to Z \to 0$ *est une suite exacte courte dans* $\mathbb{A}$ *avec* $X,Y,Z \in \theta(\mathbb{A})$*, alors la suite* $0 \to F(X) \to F(Y) \to F(Z) \to 0$ *est exacte.*

Notons encore F l'extension de F au ∂*-foncteur* $K^+(\mathbb{A}) \to K(\mathbb{B})$ *(respectivement* $K^-(\mathbb{A}) \to K(\mathbb{B})$*). Alors le foncteur dérivé* R^+F *(respectivement* L^-F*) existe.*

Démonstration : On va faire la démonstration pour le foncteur dérivé à droite; elle est analogue dans le cas du foncteur dérivé à gauche en se plaçant dans la catégorie opposée.

Soit $K^+(\theta(\mathbb{A}))$ la sous-catégorie triangulée de $K^+(\mathbb{A})$ engendrée par $\theta(\mathbb{A})$ (7.2 et 8.1).

D'après la condition i) de l'hypothèse et le théorème 7.5, la condition a) du théorème 9.5 est satisfaite. Il reste donc à vérifier que la condition b) de ce théorème est aussi satisfaite. Supposons que $M^\cdot \in \mathrm{Ob}K^+(\theta(\mathbf{A}))$ est acyclique. Sans restreindre la généralité on peut supposer que $M^i = 0$ pour $i < 0$.

De la suite exacte longue

$$0 \to M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \to \dots$$

on déduit, pour chaque $p \geq 0$, les suites exactes courtes

$$0 \to \mathrm{Ker}\, d^p \to M^p \to \mathrm{Ker}\, d^{p+1} \to 0.$$

Alors, par récurrence sur p, il résulte de la condition ii) de l'hypothèse, que $\mathrm{Ker}\, d^p \in \theta(\mathbf{A})$ pour tout $p \geq 0$, et par suite, d'après la condition iii) de l'hypothèse, les suites

$$0 \to F(\mathrm{Ker}\, d^p) \to F(M^p) \to F(\mathrm{Ker}\, d^{p+1}) \to 0$$

sont exactes. En mettant ces suites exactes bout-à-bout on en déduit évidemment que la suite

$$0 \to F(M^0) \to F(M^1) \to F(M^2) \to F(M^3) \to \dots$$

est exacte donc que $F(M^\cdot)$ est acyclique.

9.11 Avant d'énoncer le résultat suivant il faut introduire quelques définitions.

Soit $\mathbf{A}$ et $\mathbf{B}$ deux catégories abéliennes, $K^*(\mathbf{A})$ une sous-catégorie triangulée de $K(\mathbf{A})$ qui satisfait les hypothèses du No 9.1 et $F : K^*(\mathbf{A}) \to K(\mathbf{B})$ un ∂-foncteur.

Supposons que R^*F (respectivement L^*F) existe.

Pour tout entier $i \in \mathbb{Z}$ on pose $R^iF = H^i(R^*F)$ (respectivement $L^iF = H^i(L^*F)$).

Par exemple si le foncteur F est induit d'un foncteur additif $F : \mathbf{A} \to \mathbf{B}$ et si les hypothèses des propositions 9.8 ou 9.10 sont satisfaites alors les foncteurs R^iF (respectivement L^iF) existent. Dans ce cas on pose les définitions suivantes :

Le foncteur F est de dimension cohomologique finie à droite (respectivement à gauche) s'il existe un entier $n \geq 0$ tel que $R^iF(D(Y)) = 0$ (respectivement $L^{-i}F(D(Y)) = 0$) pour tout entier $i > n$ et tout $Y \in \mathrm{Ob}\mathbf{A}$.

Un objet $X \in \mathrm{Ob}\mathbf{A}$ est F-acyclique à droite (respectivement à gauche) si $R^iF(D(X)) = 0$ (respectivement $L^{-i}F(D(X)) = 0$) pour tout entier $i > 0$.

9.12 *Remarque :* Soit $F : \mathbf{A} \to \mathbf{B}$ un foncteur additif et supposons que la catégorie $\mathbf{A}$ admet suffisamment d'objets injectifs (respectivement projectifs).

D'après ce qui précède il est évident que le foncteur R^iF (respectivement L^iF) défini ci-dessus coïncide avec le foncteur usuel R^iF (respectivement L^iF) défini par exemple dans ([CE], chap. V, §3 ou [S] chap. III, §2).

9.13 PROPOSITION : *Soit $\mathbf{A}$ et $\mathbf{B}$ deux catégories abéliennes et $F : \mathbf{A} \to \mathbf{B}$ un foncteur additif.*

Supposons que l'une ou l'autre des hypothèses suivantes est satisfaite.

I. La catégorie $\mathbf{A}$ admet suffisamment d'objets injectifs (respectivement projectifs), tout $X \in I(\mathbf{A})$ est F-acyclique à droite (respectivement tout $X \in \mathrm{Proj}(\mathbf{A})$ est F-acyclique à gauche) et F est exact à gauche (respectivement exact à droite).

II. Il existe une sous-calsse $\theta(\mathbf{A}) \subset \mathrm{Ob}\mathbf{A}$ qui vérifie les hypothèses du No 7.1 et qui satisfait les conditions i), ii) et iii) de la proposition 9.10.

Si le foncteur F est de dimension cohomologique finie, alors le foncteur dérivé RF (respectivement LF) existe et on a un isomorphisme $(RF)_{|D^+(\mathbf{A})} = R^+F$ (respectivement $(LF)_{|D^-(\mathbf{A})} = L^-F$).

Démonstration : On fait la démonstration dans le cas de l'hypothèse I et des foncteurs dérivés à droite. Elle est analogue dans les autres cas.

Soit $\Omega(\mathbb{A})$ la sous-classe de $\mathrm{Ob}\mathbb{A}$ formé par les objets de $\mathbb{A}$ qui sont F-acyclique à droite.

On va montrer pour commencer que $\Omega(\mathbb{A})$ satisfait les hypothèses du théorème 7.8.

Remarquons tout d'abord qu'il résulte de l'hypothèse que si $X \in I(\mathbb{A})$ alors $X \in \Omega(\mathbb{A})$ (Dans le cas II cela résulte facilement d'un argument analogue à celui de la proposition 9.10).

On en déduit immédiatement que la catégorie $\mathbb{A}$ admet suffisamment de Ω-objets à droite.

Soit $0 \to X \to Y \to Z \to 0$ une suite exacte courte dans $\mathbb{A}$ avec $X \in \Omega(\mathbb{A})$.

On peut trouver ([CE], chap. V, §1) des résolutions injectives $0 \to X \to I^{\cdot}$, $0 \to Y \to J^{\cdot}$ et $0 \to Z \to K^{\cdot}$ telles que la suite de complexes $0 \to I^{\cdot} \to J^{\cdot} \to K^{\cdot} \to 0$ soit exacte. (Dans le cas II on utilisera le théorème 7.7).

Comme F est un foncteur additif il transforme les sommes directes finies en sommes directes finies et par suite F est exact sur les objets injectifs.

Il en résulte que la suite de complexes

$$0 \to F(I^{\cdot}) \xrightarrow{\varphi} F(J^{\cdot}) \xrightarrow{\psi} F(K^{\cdot}) \to 0$$

est exacte ; de plus, d'après l'hypothèse faite sur X, on a $H^i(F(I^{\cdot})) = 0$ pour $i \geq 1$.

Supposons que le foncteur F est de dimension cohomologique inférieure à n; on a donc $R^iF(M) = 0$ pour $i > n$ et pour tout $M \in \mathrm{Ob}\mathbb{A}$.

Supposons tout d'abord que $Y \in \Omega(\mathbb{A})$. On a donc $H^i(F(J^{\cdot})) = 0$ pour $i \geq 1$ et d'autre part on a $H^i(F(K^{\cdot})) = 0$ pour $i > n$. Supposons qu'on a démontré que $H^i(F(K^{\cdot})) = 0$ pour $i \geq p+1$ avec $p \geq 1$; c'est un exercice facile de chasse dans le diagramme de voir que $H^p(F(K^{\cdot})) = 0$. Il en résulte que $H^i(F(K^{\cdot})) = 0$ pour $i \geq 1$ et par suite $Z \in \Omega(\mathbb{A})$.

Supposons maintenant que $Z \in \Omega(\mathbb{A})$. On a donc $H^i(F(K^{\cdot})) = 0$ pour $i \geq 1$ et d'autre part on a $H^i(F(J^{\cdot})) = 0$ pour $i > n$. Un argument analogue au précédent permet de voir que $H^i(F(J^{\cdot})) = 0$ pour $i \geq 1$ par suite $Y \in \Omega(\mathbb{A})$.

La condition a) du théorème 7.8 est donc satisfaite. Considérons maintenant une suite exacte dans $\mathbb{A}$

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \to \dots \to X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} 0$$

avec $X_p \in \Omega(\mathbb{A})$ pour $0 \leq p \leq n-1$. Il faut montrer que $X_n \in \Omega(\mathbb{A})$.

Posons $Z_j = \mathrm{Ker}\, \alpha_j$ $(j = 0,1,\dots,n)$; alors pour chaque entier $0 \leq p \leq n-1$ on a une suite exacte

$$0 \to Z_p \to X_p \to Z_{p+1} \to 0$$

On peut trouver des résolutions injectives $0 \to Z_p \to I^{\cdot}_p$ et $0 \to X_p \to J^{\cdot}_p$ telles que chacune des suites

$$0 \to I^{\cdot}_p \to J^{\cdot}_p \to I^{\cdot}_{p+1} \to 0$$

soit exacte. D'après les hypothèses faites sur F il en résulte que chacune des suites

$$0 \to F(I^{\cdot}_p) \to F(J^{\cdot}_p) \to F(I^{\cdot}_{p+1}) \to 0$$

est encore exacte.

Comme par hypothèse on a $H^i(F(J^{\cdot}_p)) = 0$ pour $i \geq 1$ et $0 \leq p \leq n-1$, il résulte de la suite exacte longue de cohomologie que l'on a $H^i(F(I^{\cdot}_p)) = H^{i+q}(F(I^{\cdot}_{p-q}))$ pour $i \geq 1$, $i \leq p \leq n$ et $0 \leq q \leq p$.

Par hypothèse on a $H^i(F(I^{\cdot}_0)) = 0$ pour $i > n$ et aussi $H^i(F(I^{\cdot}_n)) = 0$ pour $i > n$; il reste donc à montrer que $H^i(F(I^{\cdot}_n)) = 0$ pour $1 \leq i \leq n$. Or pour $1 \leq i \leq n$ on a $H^i(F(I^{\cdot}_n)) = H^{i+n}(F(I^{\cdot}_0)) = 0$ puisqu'alors $i+n \geq n+1$.

Ainsi la condition b) du théorème 7.8 est satisfaite.

Cela étant, soit $K(\Omega(\mathbb{A}))$ la sous-catégorie triangulée de $K(\mathbb{A})$ engendrée par la classe $\Omega(\mathbb{A})$.

Puisque $\Omega(\mathbb{A})$ vérifie les hypothèses du théorème 7.8 il en résulte que la condition a) du théorème 9.5 est satisfaite.

Maintenant supposons que $M^{\cdot} \in \mathrm{Ob}K(\Omega(\mathbb{A}))$ est acyclique, c'est-à-dire que l'on a une suite exacte longue dans $\mathbb{A}$

$$\dots \to M^{p-1} \xrightarrow{d_{p-1}} M^p \xrightarrow{d_p} M^{p+1} \xrightarrow{d_{p+1}} \dots$$

dont les termes sont F-acycliques.

Pour tout entier $p \in \mathbb{Z}$ considérons la suite exacte

$$\dots \to M^{p-1} \to M^p \to \operatorname{Ker} d_{p+1} \to 0$$

On a $M^i \in \Omega(\mathbb{A})$ pour $i \le p$; donc $\operatorname{Ker} d_{p+1} \in \Omega(\mathbb{A})$ en vertu de ce qui précède.

D'après ([G], Appendice, corollaire A3) la suite

$$\dots \to F(M^{p-1}) \to F(M^p) \to F(\operatorname{Ker} d_{p+1}) \to 0$$

est exacte.

Il en résulte que la suite

$$\dots \to F(M^{p-1}) \to F(M^p) \to F(M^{p+1}) \to \dots$$

est exacte, et la condition b) du théorème 9.5 est donc aussi satisfaite.

Ainsi le foncteur RF existe.

Finalement les catégories triangulées $K^+(\mathbb{A})$ et $K(\mathbb{A})$ satisfont les hypothèses du No 9.1, $K^+(\mathbb{A})$ est une sous-catégorie pleine de $K(\mathbb{A})$ et il est évident que la sous-catégorie triangulée $K(\Omega(\mathbb{A}))$ de $K(\mathbb{A})$ vérifie les conditions de la proposition 9.7. On a donc un isomorphisme $R^+F = (RF)_{|D^+(\mathbb{A})}$.

9.14 Il reste à étudier le comportement des foncteurs dérivés vis à vis de la composition des foncteurs.

Soit $\mathbb{A},\mathbb{B},\mathbb{C}$ trois catégories abéliennes.

Soit $K^*(\mathbb{A})$ (respectivement $K^{\#}(\mathbb{B})$) une sous-catégorie triangulée de $K(\mathbb{A})$ (respectivement $K(\mathbb{B})$) qui satisfait les hypothèses du No 9.1 et soit $F : K^*(\mathbb{A}) \to K(\mathbb{B})$ et $G : K^{\#}(\mathbb{B}) \to K(\mathbb{C})$ deux ∂-foncteurs.

Supposons que $F(K^*(\mathbb{A})) \subset K^{\#}(\mathbb{B})$; il en résulte que si le foncteur R^*F (respectivement L^*F) existe alors on a $R^*F(D^*(\mathbb{A})) \subset D^{\#}(\mathbb{B})$ (respectivement $L^*F(D^*(\mathbb{A})) \subset D^{\#}(\mathbb{B})$).

LEMME : *Supposons que les foncteurs dérivés* $(R^*F;\xi_F)$, $(R^{\#}G,\xi_G)$ *et* $(R^*(G\circ F);\xi_{G\circ F})$ *(respectivement* $(L^*F;\xi_F)$, $(L^{\#}G;\xi_G)$ *et* $(L^*(G\circ F);\xi_{G\circ F})$) *existent.*

Alors il existe un unique morphisme de foncteurs

$$\eta : R^*(G\circ F) \to R^{\#}G\circ R^*F$$

(respectivement $\qquad \eta : L^{\#}G\circ L^*F \to L^*(G\circ F))$

tel que $(\eta\circ P^*_{\mathbb{A}})\circ\xi_{G\circ F} = (R^{\#}G\circ\xi_F)\circ(\xi_G\circ F)$ *(respectivement* $\xi_{G\circ F}\circ(\eta\circ P^*_{\mathbb{A}}) = (\xi_G\circ F)\circ(L^{\#}G\circ\xi_F)$).

Démonstration : Il suffit d'appliquer la propriété universelle du foncteur dérivé $(R^*(G\circ F);\xi_{G\circ F})$ au couple $(R^{\#}G\circ R^*F;\zeta)$ où $\zeta = (R^{\#}G\circ\xi_F)\circ(\xi_G\circ F) : P_{\mathbb{C}}\circ(G\circ F) \to (R^{\#}G\circ R^*F)\circ P^*_{\mathbb{A}}$. La démonstration est analogue dans l'autre cas.

9.15 PROPOSITION : *Supposons qu'il existe des sous-catégories triangulées* $\Gamma(\mathbb{A})$ *de* $K^*(\mathbb{A})$ *et* $\Gamma(\mathbb{B})$ *de* $K^{\#}(\mathbb{B})$ *qui vérifient les conditions du théorème 9.5 pour les foncteurs F et G respectivement et telles que* $F(\Gamma(\mathbb{A})) \subset \Gamma(\mathbb{B})$. *Alors les foncteurs dérivés* $R^*F, R^{\#}G$ *et* $R^*(G\circ F)$ *(respectivement* $L^*F, L^{\#}G$ *et* $L^*(G\circ F)$) *existent et le morphisme canonique*

$$\eta : R^*(G\circ F) \to R^{\#}G\circ R^*F$$

(respectivement $\qquad \eta : L^{\#}G\circ L^*F \to L^*(G\circ F))$

est un isomorphisme de foncteurs.

Démonstration : On fait la démonstration dans le cas des foncteurs dérivés à droite; elle est analogue dans l'autre cas.

Il résulte immédiatement des hypothèses que les foncteurs dérivés R^*F et $R^{\#}G$ existent.

De plus, si $M' \in \operatorname{Ob}\Gamma(\mathbb{A})$ est acyclique alors $G\circ F(M')$ est aussi acyclique puisque d'après l'hypothèse $F(M') \in \operatorname{Ob}\Gamma(\mathbb{B})$. Donc le foncteur dérivé $R^*(G\circ F)$ existe aussi.

D'après le lemme du No 9.14 on a donc un morphisme de foncteurs

$$\eta : R^*(G\circ F) \to R^{\#}G\circ R^*F$$

Pour montrer que η est un isomorphisme reprenons les notations introduites dans la démonstration du théorème 9.5.

Remarquons tout d'abord que la restriction $F : \Gamma(\mathbb{A}) \to \Gamma(\mathbb{B})$ du foncteur F induit un foncteur $\overline{F} : \overline{\Gamma(\mathbb{A})} \to \overline{\Gamma(\mathbb{B})}$ tel que $\overline{F}\circ\overline{P}_{\Gamma(\mathbb{A})} = \overline{P}_{\Gamma(\mathbb{B})}\circ F$. Maintenant en tenant compte de la propriété universelle du foncteur $\overline{P}_{\Gamma(\mathbb{A})}$ on voit que $\overline{G\circ F} = \overline{G}\circ\overline{F}$.

Il suffit maintenant d'utiliser la définition des foncteurs dérivés donnée dans le théorème 9.5.

On a, en notant $U_{\Gamma(\mathbb{A})}$ (respectivement $U_{\Gamma(\mathbb{B})}$) un foncteur quasi-inverse à $Q_{\Gamma(\mathbb{A})} : \overline{\Gamma(\mathbb{A})} \to D^*(\mathbb{A})$ (respectivement $Q_{\Gamma(\mathbb{B})} : \overline{\Gamma(\mathbb{B})} \to D^*(\mathbb{B})$), $R^*G\circ R^*F = \overline{G}\circ U_{\Gamma(\mathbb{B})}\circ\overline{F}\circ U_{\Gamma(\mathbb{A})} = \overline{G}\circ\overline{F}\circ U_{\Gamma(\mathbb{A})} = \overline{G\circ F}\circ U_{\Gamma(\mathbb{A})} = R^*(G\circ F)$, car $U_{\Gamma(\mathbb{B})}\circ\overline{F}\circ U_{\Gamma(\mathbb{A})} = \overline{F}\circ U_{\Gamma(\mathbb{A})}$ puisque $\overline{F} : \overline{\Gamma(\mathbb{A})} \to \overline{\Gamma(\mathbb{B})}$.

§ 10. LES FONCTEURS DERIVES DE $Hom^{\cdot}$

10.1 Soit $\mathbb{A}$ une catégorie abélienne.

On obtient un bifoncteur

$$Hom^{\cdot}(-;-) : K(\mathbb{A})^{\circ}\times K(\mathbb{A}) \to K(\mathbb{A}b)$$

de la façon suivante.

Si $X^{\cdot}, Y^{\cdot} \in ObK(\mathbb{A})$ alors $Hom^{\cdot}(X^{\cdot};Y^{\cdot})$ est le complexe de groupes abéliens défini en posant, pour tout $n \in \mathbb{Z}$,

$$Hom^{n}(X^{\cdot};Y^{\cdot}) = \prod_{p\in\mathbb{Z}} Hom_{\mathbb{A}}(X^{p};T^{n}(Y^{\cdot})^{p}),$$

la différentielle

$$d : Hom^{n}(X^{\cdot};Y^{\cdot}) \to Hom^{n+1}(X^{\cdot};Y^{\cdot})$$

étant définie en posant, pour tout $\varphi \in Hom^{n}(X^{\cdot};Y^{\cdot})$,

$$d\varphi = d_{Y}\circ\varphi-(-1)^{n}\varphi\circ d_{X}.$$

Si $X^{\cdot}, X'^{\cdot}, Y^{\cdot}, Y'^{\cdot} \in ObK(\mathbb{A})$ et si $u \in Hom_{K(\mathbb{A})}(X'^{\cdot};X^{\cdot})$ et $v \in Hom_{K(\mathbb{A})}(Y^{\cdot};Y'^{\cdot})$, on définit un morphisme de complexes

$$Hom^{\cdot}(u;v) : Hom^{\cdot}(X^{\cdot};Y^{\cdot}) \to Hom^{\cdot}(X'^{\cdot};Y'^{\cdot})$$

en posant, pour tout $n \in \mathbb{Z}$ et tout $\varphi \in Hom^{n}(X^{\cdot};Y^{\cdot})$,

$$Hom^{n}(u;v)(\varphi) = T^{n}(v)\circ\varphi\circ u$$

(On vérifie aisément que le foncteur $C(\mathbb{A})^{\circ}\times C(\mathbb{A}) \to C(\mathbb{A}b)$ passe aux quotients).

10.2 LEMME : *Pour tout entier* $n \in \mathbb{Z}$ *et tout* $X^\cdot, Y^\cdot \in \mathrm{Ob}K(\mathbb{A})$ *on a*

$$H^n(\mathrm{Hom}^\cdot(X^\cdot;Y^\cdot)) = \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;T^n(Y^\cdot))$$

Démonstration : Soit $\varphi \in \mathrm{Hom}^n(X^\cdot;Y^\cdot)$. Si φ est un cocycle on a $d_Y \circ \varphi = (-1)^n \varphi \circ d_X$ ce qui signifie que $\varphi \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;T^n(Y^\cdot))$.

Maintenant s'il existe $\psi \in \mathrm{Hom}^{n-1}(X^\cdot;Y^\cdot)$ tel que $d\psi = \varphi$ on a $d_Y \circ \psi - (-1)^n \psi \circ d_X = \varphi$ ce qui signifie que φ est homotope à 0 dans $\mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$.

Il en résulte que le morphisme naturel, qui à la classe de cohomologie représentée par le cocycle φ associe la classe d'homotopie $[\varphi] \in \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot)$, est un isomorphisme de $H^n(\mathrm{Hom}^\cdot(X^\cdot;Y^\cdot))$ sur $\mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;Y^\cdot)$.

10.3 On va étudier maintenant le comportement du foncteur $\mathrm{Hom}^\cdot(-;-)$ vis-à-vis des triangles. Pour cela on commence par établir un résultat concernant le cône d'un morphisme.

LEMME : *Soit* $W^\cdot \in \mathrm{Ob}C(\mathbb{A})$ *et* $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$.

Posons $\overline{u} = \mathrm{Hom}^\cdot(1_{W^\cdot};u) : \mathrm{Hom}^\cdot(W^\cdot;X^\cdot) \to \mathrm{Hom}^\cdot(W^\cdot;Y^\cdot)$ *et* $\underline{u} = \mathrm{Hom}^\cdot(u;1_{W^\cdot}) : \mathrm{Hom}^\cdot(Y^\cdot;W^\cdot) \to \mathrm{Hom}^\cdot(X^\cdot;W^\cdot)$.

Alors on a $C^\cdot_{\overline{u}} = \mathrm{Hom}^\cdot(W^\cdot;C^\cdot_u)$ *et* $C^{\cdot 0}_{\underline{u}} = \mathrm{Hom}^\cdot(C^\cdot_u;W^\cdot)$ *où* $C^0_{\underline{u}}$ *est le cône de* $\underline{u}$ *dans la catégorie opposée.*

Démonstration : Par définition on a, pour chaque $n \in \mathbb{Z}$,

$$C^n_{\overline{u}} = \mathrm{Hom}^n(W^\cdot;Y^\cdot) \oplus \mathrm{Hom}^{n+1}(W^\cdot;X^\cdot)$$

et $\mathrm{Hom}^n(W^\cdot;C^\cdot_u) = \mathrm{Hom}^n(W^\cdot;Y^\cdot \oplus T(X^\cdot)) = \mathrm{Hom}^n(W^\cdot;Y^\cdot) \oplus \mathrm{Hom}^{n+1}(W^\cdot;X^\cdot)$.

On a donc des isomorphismes naturels

$$\Phi_n : C^n_{\overline{u}} \to \mathrm{Hom}^n(W^\cdot;C^\cdot_u)$$

et on vérifie immédiatement qu'ils commutent avec les différentielles.

La démonstration est analogue dans le deuxième cas.

10.4 PROPOSITION : *Le foncteur* $\mathrm{Hom}^\cdot(-;-)$ *est un bi-∂-foncteur.*

Démonstration : Soit $W^\cdot \in \mathrm{Ob}K(\mathbb{A})$; il faut montrer que les foncteurs $\mathrm{Hom}^\cdot(W^\cdot;-) : K(\mathbb{A}) \to K(\mathbb{A}b)$ et $\mathrm{Hom}^\cdot(-;W^\cdot) : K(\mathbb{A})^\circ \to K(\mathbb{A}b)$ sont des ∂-foncteurs. On va faire la vérification pour le premier d'entre eux ; elle est analogue pour le second en se plaçant dans la catégorie opposée.

Remarquons tout d'abord que pour tout $X^\cdot \in \mathrm{Ob}K(\mathbb{A})$ on a $\mathrm{Hom}^\cdot(W^\cdot;T(X^\cdot)) = T\mathrm{Hom}^\cdot(W^\cdot;X^\cdot)$.

Maintenant le fait qu'un triangle distingué de $K(\mathbb{A})$ est transformé par $\mathrm{Hom}^\cdot(W^\cdot;-)$ en un triangle distingué de $K(\mathbb{A}b)$ résulte immédiatement du lemme 10.3.

10.5 LEMME : *Soit* $X^\cdot \in \mathrm{Ob}K(\mathbb{A})$ *(respectivement* $X^\cdot \in \mathrm{Ob}K^-(\mathrm{Proj}(\mathbb{A}))$) *et* $Y^\cdot \in \mathrm{Ob}K^+(I(\mathbb{A}))$ *(respectivement* $Y^\cdot \in \mathrm{Ob}K(\mathbb{A})$ *).*

Supposons que l'une ou l'autre des hypothèses suivantes est satisfaite :

I. $X^\cdot$ *est acyclique*

II. $Y^\cdot$ *est acyclique.*

Alors $\mathrm{Hom}^\cdot(X^\cdot;Y^\cdot)$ *est acyclique.*

Démonstration : Compte tenu du fait qu'un objet $Y^\cdot$ est injectif ou acyclique si et seulement si l'objet $T^n(Y^\cdot)$ est injectif ou acyclique pour tout $n \in \mathbb{Z}$, il suffit, en vertu du lemme 10.2, de vérifier que tout morphisme $\varphi : X^\cdot \to Y^\cdot$ est homotope à 0 dans $C(\mathbb{A})$.

Sous l'hypothèse I, la conclusion est donc un conséquence immédiate du lemme 8.5.

Supposons maintenant que l'hypothèse II est satisfaite. Il faut construire un opérateur d'homotopie $k \in \mathrm{Hom}^{-1}(X^\cdot;Y^\cdot)$ tel que $dk+kd = \varphi$.

Sans restreindre la généralité on peut supposer $Y^i = 0$ pour $i < 0$; il est clair alors que $k^i = 0$ pour $i \leq 0$.

Supposons que l'on a déjà construit k^i pour $i \leq p-1$ avec $p \geq 1$. Il faut donc construire $k^p : X^p \to Y^{p-1}$ tel que $dk^{p-1}+k^p d = \varphi$. Remarquons que si $x \in X^{p-1}$ est un cocycle on a $dk^{p-1}(x) = \varphi(x)$.

Soit $x \in \mathrm{Im}(d : X^{p-1} \to X^p)$ et soit $x' \in X^{p-1}$ tel que $dx' = x$; si $x'' \in X^{p-1}$ avec $dx'' = x$ alors $x'-x''$ est un cocycle de $X^\cdot$ et comme $Y^\cdot$ est acyclique il existe $y \in Y^{p-2}$ tel que $dy = \varphi(x'-x'')$. De la remarque précédente il résulte de plus que $dk^{p-1}(x'-x'') = dy$.

On peut alors définir $k^p : \mathrm{Im}(d) \to Y^{p-1}$ en posant $k^p(x) = \varphi(x')-dk^{p-1}(x')$.

Comme Y^{p-1} est injectif on peut étendre cette application en une application $k^p : X^p \to Y^{p-1}$ qui satisfait évidemment la condition requise.

La démonstration est analogue dans le cas où on considère des objets projectifs.

10.6 Supposons maintenant que la catégorie $\mathbf{A}$ admet suffisammen d'objets injectifs.

Soit $W^\cdot \in \mathrm{Ob}K(\mathbf{A})$. D'après le lemme 10.5 le ∂-foncteur

$$\mathrm{Hom}^\cdot(W^\cdot;-) : K^+(\mathbf{A}) \to K(\mathbf{Ab})$$

satisfait les hypothèses de la proposition 9.8. Il admet donc un foncteur dérivé, que l'on note

$$R^+_{II}\mathrm{Hom}^\cdot(W^\cdot;-) : D^+(\mathbf{A}) \to D(\mathbf{Ab})$$

pour indiquer qu'on prend le dérivé par rapport au second argument. On en déduit un foncteur

$$R^+_{II}\mathrm{Hom}^\cdot(-;-) : K(\mathbf{A})^0 \times D^+(\mathbf{A}) \to D(\mathbf{Ab})$$

d'où, si $Y^\cdot \in \mathrm{Ob}D^+(\mathbf{A})$, un foncteur

$$R^+_{II}\mathrm{Hom}^\cdot(-;Y^\cdot) : K(\mathbf{A})^0 \to D(\mathbf{Ab})$$

qui transforme les isomorphismes homologiques de $K(\mathbf{A})$ en des isomorphismes de $D(\mathbf{Ab})$. En effet si $s : X^\cdot \to X'^\cdot$ est un h.i. dans $K(\mathbf{A})$ et si $0 \to Y^\cdot \to J^\cdot$ est une résolution injective de $Y^\cdot$, alors le morphisme induit $\mathrm{Hom}^\cdot(X'^\cdot;J^\cdot) \to \mathrm{Hom}^\cdot(X^\cdot;J^\cdot)$ est évidemment un h.i. dans $K(\mathbf{Ab})$ en vertu du lemme 10.5.

Alors du théorème 1.2 on déduit l'existence d'un unique foncteur que l'on note, pour des raisons évidentes,

$$R_I R^+_{II}\mathrm{Hom}^\cdot(-;Y^\cdot) : D(\mathbf{A})^0 \to D(\mathbf{Ab})$$

d'où finalement un foncteur

$$R_I R^+_{II}\mathrm{Hom}^\cdot(-;-) : D(\mathbf{A})^0 \times D^+(\mathbf{A}) \to D(\mathbf{Ab})$$

10.7 Si la catégorie $\mathbf{A}$ admet suffisamment d'objets projectifs un raisonnement analogue au précédent permet, compte tenu de la variance du foncteur $\mathrm{Hom}^\cdot(-;-)$ par rapport au premier argument, de construire un foncteur

$$R_{II}R^-_I\,\mathrm{Hom}^\cdot(-;-) : D^-(\mathbf{A})^0 \times D(\mathbf{A}) \to D(\mathbf{Ab})$$

10.8 PROPOSITION : *Supposons que la catégorie* $\mathbf{A}$ *admet suffisamment d'objets injectifs et suffisamment d'objets projectifs. Alors les foncteurs*

$$R_I R^+_{II}\,\mathrm{Hom}^\cdot(-;-) : D^-(\mathbf{A})^0 \times D^+(\mathbf{A}) \to D(\mathbf{Ab})$$

et

$$R_{II}R^-_I\,\mathrm{Hom}^\cdot(-;-) : D^-(\mathbf{A})^0 \times D^+(\mathbf{A}) \to D(\mathbf{Ab})$$

sont canoniquement isomorphes.

Démonstration : Notons ξ_{II} (respectivement ξ_I) les morphismes de foncteurs associés à $R^+_{II}\mathrm{Hom}^\cdot(-;-)$ (respectivement $R^-_I\mathrm{Hom}^\cdot(-;-)$) (voir 9.2).

Alors compte tenu des constructions 10.6 et 10.7, du théorème 1.2 et de 9.2 on a

$$\xi_{II} : P_{\mathbf{Ab}}\circ\mathrm{Hom}^\cdot(-;-) \to R_I R^+_{II}\mathrm{Hom}^\cdot(-;-)\circ(P^-_{\mathbf{A}}\times P^+_{\mathbf{A}})$$

$$\xi_{I} : P_{\mathbf{Ab}}\circ\mathrm{Hom}^\cdot(-;-) \to R_{II} R^-_{I}\mathrm{Hom}^\cdot(-;-)\circ(P^-_{\mathbf{A}}\times P^+_{\mathbf{A}})$$

D'après la propriété universelle de ξ_{II} relativement à ξ_I (respectivement de ξ_I relativement à ξ_{II}) il existe un unique morphisme de foncteurs $\eta_{II} : R_I R_{II}^+ \operatorname{Hom}^\cdot(-;-) \to R_{II} R_I^- \operatorname{Hom}^\cdot(-;-)$ (respectivement $\eta_I : R_{II} R_I^- \operatorname{Hom}^\cdot(-;-) \to R_I R_{II}^+ \operatorname{Hom}^\cdot(-;-)$) tel que $(\eta_{II} \circ (P_A^- \times P_A^+)) \circ \xi_{II} = \xi_I$ (respectivement $(\eta_I \circ (P_A^- \times P_A^+)) \circ \xi_I = \xi_{II}$) (voir 9.2). Mais alors, de nouveau à cause de la propriété universelle on a $\eta_I \circ \eta_{II} = 1_{R_I R_{II}^+ \operatorname{Hom}^\cdot(-;-)}$ et $\eta_{II} \circ \eta_I = 1_{R_{II} R_I^- \operatorname{Hom}^\cdot(-;-)}$ donc η_{II} est un isomorphisme de foncteurs.

10.9 PROPOSITION : *Supposons que la catégorie $\mathbb{A}$ admet suffisamment d'objets injectifs. Alors pour tout $X^\cdot \in \operatorname{Ob} D(\mathbb{A})$, $Y^\cdot \in \operatorname{Ob} D^+(\mathbb{A})$ et pour tout $n \in \mathbb{Z}$ on a*

$$H^n(R_I R_{II}^+ \operatorname{Hom}^\cdot(X^\cdot;Y^\cdot)) = \operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;T^n(Y^\cdot))$$

Démonstration : Soit un isomorphisme homologique $s : Y^\cdot \to I^\cdot$ où $I^\cdot \in K^+(I(\mathbb{A}))$. On a $\operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;T^n(Y^\cdot)) = \operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;T^n(I^\cdot)) = \operatorname{Hom}_{K(\mathbb{A})}(X^\cdot;T^n(I^\cdot))$.

En effet le premier isomorphisme est évident car s induit un isomorphisme dans $D(\mathbb{A})$. Pour démontrer le second isomorphisme il suffit évidemment de montrer que

$$\operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;I^\cdot) = \operatorname{Hom}_{K(\mathbb{A})}(X^\cdot;I^\cdot)$$

Soit $\alpha \in \operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;I^\cdot)$ représenté par un diagramme $X^\cdot \xrightarrow{a} Z^\cdot \xleftarrow[\sim]{t} I^\cdot$ d'objets et de morphismes de $K(\mathbb{A})$. Comme $I^\cdot$ est borné inférieurement l'argument utilisé dans la démonstration de la proposition 6.15 montre qu'il existe un isomorphisme homologique $Z^\cdot \to W^\cdot$ où $W^\cdot \in K^+(\mathbb{A})$, et comme $\mathbb{A}$ admet suffisamment d'objets injectifs on a aussi un isomorphisme homologique $W^\cdot \to J^\cdot$ où $J^\cdot \in K^+(I(\mathbb{A}))$.

Finalement on peut donc représenter α par un diagramme $X^\cdot \xrightarrow{b} J^\cdot \xleftarrow[\sim]{r} I^\cdot$, mais alors r est un isomorphisme d'après le corollaire 8.7.

La correspondance qui à α associe $r^{-1}b : X^\cdot \to I^\cdot$ définit alors l'isomorphisme cherché.

Finalement d'après le lemme 10.2 on a $\operatorname{Hom}_{K(\mathbb{A})}(X^\cdot;T^n(I^\cdot)) = H^n(\operatorname{Hom}^\cdot(X^\cdot;I^\cdot))$.

Mais d'après la construction 10.6 et d'après 6.6 on a $H^n(\operatorname{Hom}^\cdot(X^\cdot;I^\cdot)) = H^n(R_{II}^+ \operatorname{Hom}^\cdot(X^\cdot;Y^\cdot)) = H^n(R_I R_{II}^+ \operatorname{Hom}^\cdot(X^\cdot;Y^\cdot))$.

10.10 Pour tout $X^\cdot, Y^\cdot \in \operatorname{Ob} D(\mathbb{A})$ et pour tout $n \in \mathbb{Z}$ on pose

$$\operatorname{Ext}^n(X^\cdot;Y^\cdot) = \operatorname{Hom}_{D(\mathbb{A})}(X^\cdot;T^n(Y^\cdot))$$

et $\operatorname{Ext}^n(X^\cdot;Y^\cdot)$ s'appelle le $n^{\text{ième}}$ hyperext de $X^\cdot$ et $Y^\cdot$. Sous les hypothèses de la proposition 10.9 on vient donc de montrer que

$$\operatorname{Ext}^n(X^\cdot;Y^\cdot) = H^n(R_I R_{II}^+ \operatorname{Hom}^\cdot(X^\cdot;Y^\cdot))$$

On notera qu'il résulte immédiatement des diverses constructions que nous avons faites que si $\mathbb{A}$ admet suffisamment d'objets injectifs, alors pour tout $X,Y \in \operatorname{Ob}\mathbb{A}$, le groupe $\operatorname{Ext}^n(D(X);D(Y))$ coïncide avec le groupe usuel $\operatorname{Ext}^n(X;Y)$ défini par exemple dans ([CE], chap. VI, §1).

§ 11. LES FONCTEURS DERIVES DE $\otimes$

11.1 Soit $\mathbb{A}$ une catégorie abélienne dans laquelle la notion de produit tensoriel est définie (par exemple la catégorie des faisceaux de R-modules sur un espace topologique).

On dit qu'un objet $M \in \mathrm{Ob}\mathbb{A}$ est plat si pour toute suite exacte courte $0 \to X \to Y \to Z \to 0$ dans $\mathbb{A}$ la suite $0 \to X\otimes M \to Y\otimes M \to Z\otimes M \to 0$ est encore exacte. On note $P(\mathbb{A})$ la sous-classe de $\mathrm{Ob}\mathbb{A}$ formé par les objets plats de $\mathbb{A}$; cette classe $P(\mathbb{A})$ satisfait les hypothèses du No 7.1 ([B], chap. 10, §1, no3).

11.2 On définit un bifoncteur

$$\otimes : K(\mathbb{A})\times K(\mathbb{A}) \to K(\mathbb{A})$$

de la façon suivante.

Si $X^\cdot, Y^\cdot \in \mathrm{Ob}K(\mathbb{A})$ alors $X^\cdot\otimes Y^\cdot$ est le complexe d'objets de $\mathbb{A}$ défini en posant, pour tout $n \in \mathbb{Z}$,

$$(X^\cdot\otimes Y^\cdot)^n = \bigoplus_{p+q=n} X^p\otimes Y^q,$$

la différentielle

$$d : (X^\cdot\otimes Y^\cdot)^n \to (X^\cdot\otimes Y^\cdot)^{n+1}$$

étant donnée par

$$d = d_{X^\cdot}\otimes 1_{Y^\cdot} + 1_{X^\cdot}\otimes d_{Y^\cdot}.$$

Si $X^\cdot, X'^\cdot, Y^\cdot, Y'^\cdot \in \mathrm{Ob}K(\mathbb{A})$ et si $u \in \mathrm{Hom}_{K(\mathbb{A})}(X^\cdot;X'^\cdot)$ et $v \in \mathrm{Hom}_{K(\mathbb{A})}(Y^\cdot;Y'^\cdot)$ on définit d'une façon évidente un morphisme de complexes

$$u\otimes v : X^\cdot\otimes Y^\cdot \to X'^\cdot\otimes Y'^\cdot$$

(On vérifie aisément que le foncteur $C(\mathbb{A})\times C(\mathbb{A}) \to C(\mathbb{A})$ passe aux quotients).

11.3 LEMME : *Soit* $W^\cdot \in \mathrm{Ob}C(\mathbb{A})$ *et* $u \in \mathrm{Hom}_{C(\mathbb{A})}(X^\cdot;Y^\cdot)$; on a

$$C^\cdot_{u\otimes 1_{W^\cdot}} = C^\cdot_u\otimes W^\cdot$$

Démonstration : On remarque tout d'abord que $T(W^\cdot\otimes -) = W^\cdot\otimes T(-)$ et $T(-\otimes W^\cdot) = T(-)\otimes W^\cdot$.

On a donc
$$\begin{aligned} C^\cdot_{u\otimes 1_{W^\cdot}} &= (Y^\cdot\otimes W^\cdot)\oplus T(X^\cdot\otimes W^\cdot) \\ &= (Y^\cdot\oplus T(X^\cdot))\otimes W^\cdot \\ &= C^\cdot_u\otimes W^\cdot \end{aligned}$$

Il existe donc un isomorphisme naturel d'objets gradués

$$C^\cdot_{u\otimes 1_{W^\cdot}} \to C^\cdot_u\otimes W^\cdot$$

et on vérifie immédiatement que c'est un isomorphisme de complexes.

11.4 PROPOSITION : *Le foncteur $\otimes$ est un bi-∂-foncteur.*

Démonstration : Soit $W^\cdot \in \mathrm{Ob}K(\mathbb{A})$; il faut montrer que les foncteurs $-\otimes W^\cdot : K(\mathbb{A}) \to K(\mathbb{A})$ et $W^\cdot\otimes - : K(\mathbb{A}) \to K(\mathbb{A})$ sont des ∂-foncteurs. On a déjà remarqué que le foncteur de translation T commute avec ces foncteurs. Maintenant le fait qu'un triangle distingué de $K(\mathbb{A})$ est transformé par ces foncteurs en un triangle distingué résulte immédiatement du lemme 11.3.

11.5 LEMME : *Soit* $X^\cdot \in \mathrm{Ob}K(\mathbb{A})$ *(respectivement* $X^\cdot \in \mathrm{Ob}K^-(\mathbb{A})$*) et* $Y^\cdot \in \mathrm{Ob}K^b(P(\mathbb{A}))$ *(respectivement* $Y^\cdot \in \mathrm{Ob}K^-(P(\mathbb{A}))$*).*

Supposons que l'une ou l'autre des hypothèses suivantes est satisfaite :

I. $Y^\cdot$ *est acyclique*

II. $X^\cdot$ *est acyclique.*

Alors $X^\cdot\otimes Y^\cdot$ *est acyclique.*

Démonstration : On va utiliser un argument standard de suites spectrales. Considérons le complexe double $(K^{\cdot\cdot};d';d'')$ obtenu en posant $K^{pq} = X^p \otimes Y^q$, $d' = d_{X^\cdot} \otimes 1_{Y^\cdot}$, $d'' = 1_{X^\cdot} \otimes d_{Y^\cdot}$ et remarquons que $(X^\cdot \otimes Y^\cdot;d) = (sK^{\cdot\cdot};d'+d'')$. Sans restreindre la généralité on peut supposer $K^{pq} = 0$ pour $q < 0$ et $q > q_0 \geq 0$ (respectivement $K^{pq} = 0$ pour $p > 0$ ou $q > 0$). Il en résulte que les deux suites spectrales associées à ce complexe double sont convergentes ; on a donc ([Go],chap. 1, §4)

$${}_{I}E_2^{pq} = {}'H^p\, {}''H^q(K^{\cdot\cdot}) \Rightarrow H^n(X^\cdot \otimes Y^\cdot)$$

$${}_{II}E_2^{pq} = {}''H^p\, {}'H^q(K^{\cdot\cdot}) \Rightarrow H^n(X^\cdot \otimes Y^\cdot)$$

Supposons tout d'abord que $Y^\cdot$ est acyclique ; on a donc une suite exacte $\dots \to Y^{q-1} \xrightarrow{d^{q-1}} Y^q \xrightarrow{d^q} Y^{q+1} \to \dots$ qui est dans tous les cas bornée supérieurement. On en déduit les suites exactes courtes

$$0 \to \operatorname{Ker} d^q \to Y^q \to \operatorname{Ker} d^{q+1} \to 0$$

Puisque Y^q est un objet plat pour tout $q \in \mathbb{Z}$, on en déduit, par récurrence sur q, que $\operatorname{Ker} d^q$ est un objet plat pour tout q ([B], chap. 10, §4, No 6, corollaire 1). Il en résulte que la suite

$$0 \to X^p \otimes \operatorname{Ker} d^q \to X^p \otimes Y^q \to X^p \otimes \operatorname{Ker} d^{q+1} \to 0$$

est exacte pour tout p et tout q ([B], chap. 10, §4, No 6, théorème 2).

Ainsi la suite

$$\dots \to X^p \otimes Y^{q-1} \to X^p \otimes Y^q \to X^p \otimes Y^{q+1} \to \dots$$

est exacte, c'est-à-dire pour tout p le complexe $X^p \otimes Y^\cdot$ est acyclique. On a donc ${}_{I}E_2^{pq} = 0$ d'où $H^n(X^\cdot \otimes Y^\cdot) = 0$.

Supposons maintenant que $X^\cdot$ est acyclique. De la suite exacte

$$\dots \to X^{p-1} \xrightarrow{d^{p-1}} X^p \xrightarrow{d^p} X^{p+1} \to \dots$$

on déduit les suites exactes courtes

$$0 \to \operatorname{Ker} d^p \to X^p \to \operatorname{Ker} d^{p+1} \to 0$$

Puisque Y^q est un objet plat pour tout q, les suites

$$0 \to \operatorname{Ker} d^p \otimes Y^q \to X^p \otimes Y^q \to \operatorname{Ker} d^{p+1} \otimes Y^q \to 0$$

sont encore exactes pour tout p et tout q. Ainsi la suite

$$\dots \to X^{p-1} \otimes Y^q \to X^p \otimes Y^q \to X^{p+1} \otimes Y^q \to \dots$$

est exacte c'est-à-dire pour tout q le complexe $X^\cdot \otimes Y^q$ est acyclique. On a donc ${}_{II}E_2^{pq} = 0$ d'où $H^n(X^\cdot \otimes Y^\cdot) = 0$.

11.6 Supposons dorénavant que la catégorie $\mathbb{A}$ admet suffisamment d'objets plats (à gauche).

La catégorie $K^-(P(\mathbb{A}))$ est une sous-catégorie triangulée de $K^-(\mathbb{A})$; il suffit en effet de remarquer que si $u \in \operatorname{Hom}_{K(P(\mathbb{A}))}(X^\cdot;Y^\cdot)$ alors le cône $C_u^\cdot \in K(P(\mathbb{A}))$.

Puisque la catégorie $\mathbb{A}$ admet suffisamment d'objets plats il résulte du théorème 7.5 que la condition a) du théorème 9.5 est satisfaite.

De plus soit $W^\cdot \in \operatorname{Ob} K^-(\mathbb{A})$; d'après le lemme 11.5 le ∂-foncteur $W^\cdot \otimes - : K^-(\mathbb{A}) \to K(\mathbb{A})$ satisfait la condition b) du théorème 9.5. Il admet donc un foncteur dérivé à gauche, et on en déduit un foncteur

$$L_{II}^-(-\otimes-) : K^-(\mathbb{A}) \times D^-(\mathbb{A}) \to D(\mathbb{A})$$

Donc, si $Y^\cdot \in \operatorname{Ob} D^-(\mathbb{A})$, on a un foncteur

$$L_{II}^-(-\otimes Y^\cdot) : K^-(\mathbb{A}) \to D(\mathbb{A})$$

qui transforme les isomorphismes homologiques de $K^-(\mathbb{A})$ en des isomorphismes de $D(\mathbb{A})$ en vertu du lemme 11.5. D'après le théorème 1.2 on en déduit donc un foncteur

$$L_I L_{II}^-(-\otimes Y^\cdot) : D^-(\mathbb{A}) \to D(\mathbb{A})$$

d'où finalement un foncteur

$$L_I L_{II}^-(-\otimes-) : D^-(\mathbb{A}) \times D^-(\mathbb{A}) \to D(\mathbb{A}).$$

11.7 On peut évidemment construire d'une manière analogue un foncteur

$$L_{II}L_{I}^{-}(-\otimes-) : D^{-}(\mathbb{A})\times D^{-}(\mathbb{A}) \to D(\mathbb{A})$$

et un argument semblable à celui utilisé dans la démonstration de la proposition 10.8 prouve que les foncteurs $L_{I}L_{II}^{-}(-\otimes-)$ et $L_{II}L_{I}^{-}(-\otimes-)$ sont canoniquement isomorphes.

11.8 Pour tout $X^{\cdot},Y^{\cdot} \in \mathrm{Ob}D^{-}(\mathbb{A})$ et pour tout $n \in \mathbb{Z}$ on pose

$$\mathrm{Tor}_n(X^{\cdot};Y^{\cdot}) = H^{-n}(L_{I}L_{II}^{-}(X^{\cdot}\otimes Y^{\cdot}))$$

et $\mathrm{Tor}_n(X^{\cdot};Y^{\cdot})$ s'appelle le $n^{\text{ième}}$ hypertor de $X^{\cdot}$ et $Y^{\cdot}$. On notera que si $X,Y \in \mathrm{Ob}\mathbb{A}$, alors $\mathrm{Tor}_n(D(X);D(Y))$ coïncide avec le $\mathrm{Tor}_n(X;Y)$ usuel défini par exemple dans ([CE], chap. VI, §1).

§ 12. LES CLASSES GENERATRICES DE $D^b(\mathbb{A})$

12.1 Soit $\mathbb{A}$ une catégorie abélienne.

On peut généraliser les définitions des No 3.4, 5.3 et 6.9. Pour tout entier $n \in \mathbb{Z}$ considérons le foncteur additif pleinement fidèle $C_n = T^n \circ C_0 : \mathbb{A} \to C(\mathbb{A})$, où T est le foncteur de translation défini dans le No 4.1. Il induit un foncteur pleinement fidèle $C_n : \mathbb{A} \to K(\mathbb{A})$ et finalement on obtient un foncteur pleinement fidèle $D_n : \mathbb{A} \to D(\mathbb{A})$. Ce foncteur induit une équivalence entre la catégorie $\mathbb{A}$ et la sous-catégorie pleine de $D(\mathbb{A})$ formée par les objets $X^{\cdot}$ de $D(\mathbb{A})$ qui sont acycliques en tous degrés différents de n.

12.2 Maintenant soit G une sous-classe de $\mathrm{Ob}\mathbb{A}$ qui contient l'objet nul 0.

Posons $G_1 = \bigcup_{n \in \mathbb{Z}} D_n(G)$; G_1 est donc la sous-classe de $\mathrm{Ob}D(\mathbb{A})$, en fait de $\mathrm{Ob}D^b(\mathbb{A})$, dont les éléments sont les complexes dont toutes les composantes sont nulles sauf une qui est un élément de G, la différentielle étant évidemment nulle.

Dans la catégorie triangulée $D^b(\mathbb{A})$, tout élément $X^{\cdot} \in G_1$ est sommet d'un triangle distingué dont les deux autres sommets sont aussi des éléments de G_1; il suffit en effet de considérer le triangle

$$X^{\cdot} \to X^{\cdot} \to 0^{\cdot} \to T(X^{\cdot}).$$

12.3 On définit alors, inductivement, pour chaque entier $p \geq 2$, une sous-classe G_p de $\mathrm{Ob}D^b(\mathbb{A})$ de la façon suivante.

Un objet $X^{\cdot}$ de $D^b(\mathbb{A})$ appartient à G_p si, et seulement si, il existe un triangle distingué dans $D^b(\mathbb{A})$ dont $X^{\cdot}$ est l'un des sommets et dont les deux autres sommets sont des éléments de G_{p-1}. On vérifie aisément qu'on a une inclusion $G_{p-1} \to G_p$. On peut donc considérer la suite $G_1 \to G_2 \to G_3 \to \dots$ de sous-classes de $\mathrm{Ob}D^b(\mathcal{A})$.

12.4 On dit que la sous-classe G de $\mathrm{Ob}\mathcal{A}$ est une classe génératrice de $D^b(\mathcal{A})$ si tout objet $Y^\bullet$ de $D^b(\mathcal{A})$ est isomorphe, dans $D^b(\mathcal{A})$, à un élément $X^\bullet \in \varinjlim_p G_p$.

12.5 Avant de donner un exemple de classe génératrice on introduit la définition suivante.

Pour tout $X^\bullet \in \mathrm{Ob}D(\mathcal{A})$ posons

$$n(X^\bullet) = \{k \in \mathbb{Z} \mid H^k(X^\bullet) \neq 0\} .$$

Si $X^\bullet \in \mathrm{Ob}D^+(\mathcal{A})$ (respectivement $X^\bullet \in \mathrm{Ob}D^-(\mathcal{A})$), alors l'ensemble $n(X^\bullet)$ est borné inférieurement (respectivement borné supérieurement). Si l'ensemble $n(X^\bullet)$ est fini, en particulier si $X^\bullet \in \mathrm{Ob}D^b(\mathcal{A})$, le nombre $\ell(X^\bullet)$ des éléments de $n(X^\bullet)$ s'appelle la longueur cohomologique de $X^\bullet$.

Par exemple $X^\bullet \in \mathrm{Ob}D(\mathcal{A})$ est acyclique si et seulement si $\ell(X^\bullet) = 0$.

12.6 LEMME : *Soit* $X^\bullet \in \mathrm{Ob}D^*(\mathcal{A})$ $(* = +,-,b)$ *tel que* $\mathrm{Card}\, n(X^\bullet) \geq 2$. *Alors il existe un triangle distingué dans* $D^*(\mathcal{A})$

$$Z^\bullet \to X^\bullet \to Y^\bullet \to T(Z^\bullet)$$

tel que $n(Y^\bullet) \subsetneq n(X^\bullet)$ *et* $n(Z^\bullet) \subsetneq n(X^\bullet)$.

Démonstration : On fait la démonstration dans le cas où $X^\bullet \in \mathrm{Ob}D^+(\mathcal{A})$, laissant au lecteur le soin de l'adapter dans le cas où $X^\bullet \in \mathrm{Ob}D^-(\mathcal{A})$; enfin le cas où $X^\bullet \in \mathrm{Ob}D^b(\mathcal{A})$ se déduit évidemment des cas précédents.

Sans restreindre la généralité on peut supposer que $X^i = 0$ pour $i < 0$

Posons $n = \inf n(X^\bullet) \in \mathbb{N}$.

On définit un complexe $W^\bullet$ en posant

$$W^i = \begin{cases} 0 & \text{si } i \leq n-1 \\ \mathrm{Coker}(d : X^{n-1} \to X^n) & \text{si } i = n \\ X^i & \text{si } i \geq n+1 \end{cases}$$

Il est clair qu'on a $n(W^\bullet) = n(X^\bullet)$ et le morphisme évident $j : X^\bullet \to W^\bullet$ est un h.i. .

On définit maintenant un complexe $Y^\bullet$ en posant

$$Y^i = \begin{cases} 0 & \text{si } i \leq n \\ \mathrm{Coker}(d : X^n \to X^{n+1}) & \text{si } i = n+1 \\ X^i & \text{si } i \geq n+2 \end{cases}$$

D'après le choix de n on a nécessairement $n(Y^\bullet) \subsetneq n(X^\bullet)$.

On définit enfin un complexe $Z^\bullet$ en posant

$$Z^i = \begin{cases} 0 & \text{si } i \leq n-1 \text{ ou } i \geq n+2 \\ \mathrm{Coker}(d : X^{n-1} \to X^n) & \text{si } i = n \\ \mathrm{Im}(d : X^n \to X^{n+1}) & \text{si } i = n+1 \end{cases}$$

On a $H^i(Z^\bullet) = \begin{cases} H^n(X^\bullet) & \text{si } i=n \\ 0 & \text{si } i \neq n \end{cases}$; donc $n(Z^\bullet) \subset n(X^\bullet)$, $\ell(Z^\bullet) = 1$.

On obtient ainsi une suite exacte de complexes dans $C^+(\mathcal{A})$

$$0 \to Z^\bullet \xrightarrow{u} W^\bullet \xrightarrow{v} Y^\bullet \to 0$$

et d'après la proposition 4.10 i) et ii) il existe un isomorphisme homologique $m : C_u^\bullet \to Y^\bullet$.

On en déduit alors un diagramme de triangles

$$\begin{tikzcd}
Z^\bullet \arrow[r] \arrow[d, "1_{Z^\bullet}"'] & W^\bullet \arrow[r] \arrow[d, "1_{W^\bullet}"'] & C_u^\bullet \arrow[r] \arrow[d, "m"'] & T(Z^\bullet) \arrow[d, "1_{T(Z^\bullet)}"] \\
Z^\bullet \arrow[r] & W^\bullet \arrow[r] & Y^\bullet \arrow[r] & T(Z^\bullet) \\
Z^\bullet \arrow[r] \arrow[u, "1_{Z^\bullet}"] & X^\bullet \arrow[r] \arrow[u, "j"] & Y^\bullet \arrow[r] \arrow[u, "1_{Y^\bullet}"] & T(Z^\bullet) \arrow[u, "1_{T(Z^\bullet)}"']
\end{tikzcd}$$

dans lequel les flèches verticales sont au moins des h.i.

La dernière ligne de ce diagramme induit un triangle distingué dans $D^+(\mathcal{A})$, qui est le triangle cherché.

12.7 *Remarque :* Si $\ell(X^\bullet) = 1$ il existe un triangle distingué $Z^\bullet \to X^\bullet \to Y^\bullet \to T(Z^\bullet)$ avec $\ell(Y^\bullet) = 0$ et $\ell(Z^\bullet) = 1$; si $\ell(X^\bullet) = 0$ on a $\ell(Y^\bullet) = \ell(Z^\bullet) = 0$. Il suffit en effet dans les deux cas de considérer le triangle $X^\bullet \to X^\bullet \to 0^\bullet \to T(X^\bullet)$.

12.8 PROPOSITION : *La classe pleine* $G = \mathrm{Ob}\mathbb{A}$ *est une classe génératrice de* $D^b(\mathbb{A})$.

Démonstration : Soit $X^\bullet \in \mathrm{Ob}D^b(\mathbb{A})$.

Si $\ell(X^\bullet) = 0$ il est clair que $X^\bullet$ est homologiquement isomorphe à l'élément $0^\bullet \in G_1$.

Si $\ell(X^\bullet) = 1$ alors $X^\bullet$ est homologiquement isomorphe à un élément de G_1. En effet il existe un entier n tel que $H^n(X^\bullet) \neq 0$ et $H^i(X^\bullet) = 0$ pour $i \neq n$. Il suffit alors de considérer le diagramme commutatif

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[r, "d"] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}\, d \arrow[r, "\bar{d}"] & X^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathrm{Ker}\, \bar{d} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

dans lequel les flèches verticales sont des h.i. La dernière ligne de ce diagramme fournit un élément de G_1 isomorphe à $X^\bullet$ dans $D^b(\mathbb{A})$.

Si $\ell(X^\bullet) = 2$ il existe, d'après le lemme 12.6, un triangle distingué $Z^\bullet \to X^\bullet \to Y^\bullet \to T(Z^\bullet)$ avec $\ell(Y^\bullet) < 2$ et $\ell(Z^\bullet) < 2$. D'après ce qui précède $Y^\bullet$ et $Z^\bullet$ sont homologiquement isomorphes à des éléments de G_1. On en déduit que $X^\bullet \in G_2$.

Par récurrence on montre immédiatement que, pour tout entier $p \geq 3$, si $\ell(X^\bullet) = p$ alors $X^\bullet \in G_p$.

12.9 PROPOSITION : *Soit* $\mathbb{A}$ *et* $\mathbb{B}$ *deux catégories abéliennes et* $F : D^b(\mathbb{A}) \to D^b(\mathbb{B})$ *un ∂-foncteur pleinement fidèle. Supposons que* F *est induit d'un foncteur additif de* $\mathbb{A}$ *dans* $\mathbb{B}$ *dont l'image contient une classe génératrice* G *de* $D^b(\mathbb{B})$. *Alors* F *est une équivalence de catégorie.*

Démonstration : Par récurrence et en utilisant la proposition 2.18 on voit facilement que tout objet de $\varinjlim_p G_p$ est isomorphe à un objet de l'image de F. Il en résulte alors immédiatement que si $Y^\bullet \in \mathrm{Ob}\, D^b(\mathbb{B})$, on peut trouver $X^\bullet \in \mathrm{Ob}\, D^b(\mathbb{A})$ et un isomorphisme $\alpha \in \mathrm{Hom}_{D^b(\mathbb{B})}(F(X^\bullet); Y^\bullet)$.

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II. COHERENT $\mathcal{D}$-MODULES

by B. Kaup

In this text, we define and give some basic properties of coherent sheaves. Then, following the paper [6] of B. MALGRANGE, we prove that the sheaf $\mathcal{D}$ of linear partial differential operators with holomorphic coefficients is coherent (as a sheaf of rings) and give a characterization of coherent $\mathcal{D}$-modules, using good filtrations.

One important aspect of this text is to demonstrate the interplay (via the theorems 2.2, 2.3 and 2.4) and the analogy (see § 1) between coherent modules over coherent sheaves of rings and finitely generated modules over noetherian rings.

§ 1 COHERENCE

Coherent modules over a ring

In order to clarify the analogy between the notions "noetherian" and "coherent", we first collect some definitions and propositions concerning noetherian modules.

Let R be a (not necessarily commutative) ring with 1; let M,M',M",... be R-modules (R-module always means left R-module).

1.1 DEFINITION. M *is finitely generated if there exists an exact sequence* $R^p \longrightarrow M \longrightarrow 0$.

M is noetherian (as an R-module) if each sub-R-module N of M is finitely generated.
R is noetherian (as a ring) if R is noetherian as (left) R-module.

1.2 PROPOSITION. *Let M',M" be noetherian R-modules and* $M' \longrightarrow M \longrightarrow M''$ *an exact sequence. Then M is noetherian.*

1.3 PROPOSITION. *Let R be noetherian, M an R-module. Then M is noetherian iff M is finitely generated.*

1.4 REMARK. Let $\phi: M' \longrightarrow M''$ be a homomorphism of finitely generated R-modules. If R is noetherian, then, by 1.2 and 1.3, also Im ϕ, Ker ϕ, Coker ϕ are finitely generated. If R is not noetherian, one has to impose stronger conditions on M' and M" (see 1.9).

1.5 DEFINITION. *M is finitely presented if there exists an exact sequence* $R^q \longrightarrow R^p \longrightarrow M \longrightarrow 0$.
M is coherent (as an R-module) if the two following conditions are satisfied:

i) *M is finitely generated.*

ii) *Every finitely generated submodule* $N \subset M$ *is finitely presented.*

R is coherent (as a ring) if R is coherent as a left R-module.

1.6 LEMMA. *For an R-module M, the following conditions are equivalent.*

ii) *Every finitely generated submodule* $N \subset M$ *is finitely presented.*

iii) *For each homomorphism* $\phi: R^p \longrightarrow M$, *Ker* ϕ *is finitely generated.*

Proof (of ii) $\Rightarrow$ iii)). Let $\phi: R^p \longrightarrow M$ be a homomorphism. Then $\phi(R^p) \subset M$ is finitely generated; hence, there exists an exact sequence $R^r \xrightarrow{\sigma} R^s \xrightarrow{\psi} \phi(R^p) \longrightarrow 0$. Let $R^p \underset{\beta}{\overset{\alpha}{\rightleftarrows}} R^s$ be homomorphisms verifying $\psi\alpha = \phi$, $\phi\beta = \psi$. Then $\gamma: R^r \oplus R^p \longrightarrow \mathrm{Ker}\,\phi$, $(a,b) \longmapsto b + \beta(\sigma(a) - \alpha(b))$ is surjective.

1.7 PROPOSITION. *Let* $M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow M_3 \longrightarrow M_4$ *be an exact sequence, where* M_i *is coherent for all i. Then M is coherent.*

Proof. M is finitely generated:
There exists a diagram of the form ($\twoheadrightarrow$ indicates a surjection)

$$\begin{tikzcd}
R^{p_2} \arrow[d, "\chi"] & R^p \arrow[r, two heads, "\psi"] \arrow[d, dashed, "\omega"] & \mathrm{Ker}(\phi_3\circ\tau) \arrow[r, phantom, "\subset"] & R^{p_3} \arrow[d, "\tau"] \arrow[dr, "\phi_3\circ\tau"] & \\
M_2 \arrow[r, "\phi_2"'] & M \arrow[rr, "\phi"'] & & M_3 \arrow[r, "\phi_3"'] & M_4
\end{tikzcd}$$

(M_4 is coherent; hence, $\mathrm{Ker}(\phi_3\circ\tau)$ is finitely generated by 1.6). Since $\phi_3\tau\psi = 0$, there exists $\omega: R^p \longrightarrow M$ such that the diagram commutes. Then $R^{p_2} \oplus R^p \longrightarrow M$, $(r,s) \longmapsto \phi_2\chi(r) + \omega(s)$, is a surjection.
Now let $\beta: R^p \longrightarrow M$ be a homomorphism. We must show that Ker β is finitely generated. First, we may assume that $M_1 = \mathrm{Ker}\,\phi_2 \subset M_2$. There is a commutative diagram

$$\begin{tikzcd}
R^{p_1} \arrow[d, "\chi"] & R^q \arrow[r, two heads, "\psi"] \arrow[d, dashed, "\omega"'] & \mathrm{Ker}(\phi\beta) \arrow[r, phantom, "\subset"] & R^p \arrow[d, "\beta"'] \arrow[dr, "\phi\beta"] & \\
\mathrm{Ker}\,\phi_2 = M_1 \arrow[r, phantom, "\subset"] & M_2 \arrow[rr, "\phi_2"] & & M \arrow[r, "\phi"] & M_3 .
\end{tikzcd}$$

Define $\pi: R^{p_1} \oplus R^q \longrightarrow R^q$ and $\tau: R^{p_1} \oplus R^q \longrightarrow M_2$ by $\pi(r,s) := s$, $\tau(r,s) := \chi(r) + \omega(s)$. Ker τ is finitely generated, since M_2 is coherent. Hence, Ker β is finitely generated, since Ker $\beta = \psi\pi(\mathrm{Ker}\,\tau)$.

1.8 PROPOSITION. *If R is coherent, then M is coherent iff M is finitely presented.*

Proof. If R is coherent, then (by induction and 1.7) R^p is coherent for $p \geq 1$. If M is finitely presented, there exists an exact sequence $R^q \longrightarrow R^p \longrightarrow M \longrightarrow 0 \longrightarrow 0$, hence, M is coherent by 1.7.

1.9 PROPOSITION. *Let* $M,N,M' \subset M$, $M'' \subset M$ *be coherent R-modules. Then*

a) $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$, $\operatorname{Coker}\phi$ *are coherent for every homomorphism* $\phi: M \longrightarrow N$.

b) $M \oplus N$, $M' + M''$, $M' \cap M''$, M/M' *are coherent.*

c) $M \otimes_R N$ *and* $\operatorname{Hom}_R(M,N)$ *are coherent if* R *is commutative.*

Proof. As an example, we prove that $\operatorname{Hom}_R(M,N)$ is coherent: a finite presentation $R^q \longrightarrow R^p \longrightarrow M \longrightarrow 0$ of M induces the exact sequence

$$0 \longrightarrow 0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \underset{\substack{\| \\ N^p}}{\operatorname{Hom}_R(R^p,N)} \longrightarrow \underset{\substack{\| \\ N^q}}{\operatorname{Hom}_R(R^q,N)}$$

Hence, $\operatorname{Hom}_R(M,N)$ is coherent by 1.7, since N^p and N^q are coherent.

Coherent sheaves

Let $X \subset \mathbb{C}^n$ be an open subset, R a sheaf on X of (not necessarily commutative) rings, M a left R-module on X.

For U open in X, there is a canonical bijection between p-tuples $(h_1,\dots,h_p) \in \Gamma(U,M)^p$ and homomorphisms $\phi: R^p_{|U} \longrightarrow M_{|U}$:
$(h_1,\dots,h_p) \longmapsto (\phi: (f_1,\dots,f_p) \longmapsto (f_1h_1,\dots,f_ph_p))$.
The inverse mapping is given by $\phi \longmapsto (\phi(e_1),\dots,\phi(e_p))$, where $e_i = (0,\dots,1,\dots,0) \in \Gamma(U,R^p)$ is the i-th unit vector.
We say that $h_1,\dots,h_p \in \Gamma(U,M)$ generate $M_{|U}$, if the corresponding homomorphism $R^p_{|U} \longrightarrow M_{|U}$ is surjective (i.e. $h_{1,x},\dots,h_{p,x}$ generate M_x for all $x \in U$).

1.10 DEFINITION. M *is locally finitely generated (resp. locally finitely presented), if for each* $x \in X$, *there exists an exact sequence* $R^p_{|U} \longrightarrow M_{|U} \longrightarrow 0$ *(resp.* $R^q_{|U} \longrightarrow R^p_{|U} \longrightarrow M_{|U} \longrightarrow 0$*) over an open neighbourhood* U *of* x *in* X. M *is coherent (as an R-module) if the following two conditions are satisfied:*

i) M *is locally finitely generated.* ii) *For every* U *open in* X, *every locally finitely generated submodule* $N \subset M_{|U}$ *is locally finitely presented.*
R *is coherent (as a sheaf of rings) if* R *is coherent as an* R*-module.*

1.11 LEMMA. *Condition* ii) *in* 1.10 *is equivalent to the following condition (see* 1.6*):*
iii) *For every* U *open in* X *and every homomorphism* $\phi: R^p_{|U} \longrightarrow M_{|U}$, $\operatorname{Ker}\phi$ *is locally finitely generated.*

Proofs of 1.12, 1.13 and 1.14 are obtained by replacing "finitely generated" (resp. "finitely presented") by "locally finitely generated" resp. "locally finitely presented" in the proofs of 1.7, 1.8 and 1.9:

1.12 PROPOSITION. *Let* $M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow M_3 \longrightarrow M_4$ *be exact,* M_i *coherent for* $i = 1,\dots,4$. *Then* M *is coherent.*

1.13 PROPOSITION. *If* R *is coherent, then* M *is coherent iff* M *is locally finitely presented.*

1.14 PROPOSITION. *Let* $M,N,M' \subset M, M'' \subset M$ *be coherent* R*-modules. Then*

a) $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$, $\operatorname{Coker}\phi$ *are coherent for every homomorphism* $\phi: M \longrightarrow N$.

b) $M \oplus N$, $M' + M''$, $M' \cap M''$, M/M' *are coherent.*

c) $M \otimes_R N$ *and* $\operatorname{Hom}_R(M,N)$ *are coherent if* R *is commutative.*

1.15 EXAMPLE. For $X := \mathbb{C}$, $R := {}_{\mathbb{C}}\mathcal{O}$ = sheaf of holomorphic functions, define $I \subset R$ by $I_z := R_z$ for $0 \neq z \in \mathbb{C}$, $I_0 := (0) \subset R_0$, $M := R/I$. In the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow M \longrightarrow 0 ,$$

I is not locally finitely generated (but M is!). Hence, neither I nor M is coherent, (since R is coherent, see 2.3).

The utility of the notion of coherence is shown by the following proposition of the type "punctual $\longrightarrow$ local":

1.16 PROPOSITION. *Let M be a locally finitely generated R-module over X; $x_0 \in X$; U an open neighbourhood of x_0 in X; $f_1,\dots,f_p \in \Gamma(U,M)$ such that the germs $f_{1,x_0},\dots,f_{p,x_0}$ generate M_{x_0} as an R_{x_0}-module. Then there exists an open neighbourhood V of x_0 in U such that $f_{1|V},\dots,f_{p|V}$ generate $M_{|V}$. In particular, if $M_{x_0} = 0$, then $M_x = 0$ for x near x_0.*

1.17 PROPOSITION. *Let $\phi: M \longrightarrow N$ be a homomorphism between coherent modules. If, for some x_0, $\phi_{x_0}: M_{x_0} \longrightarrow N_{x_0}$ is injective (resp. surjective, resp. bijective), then $\phi_x: M_x \longrightarrow N_x$ is injective (resp. surjective, resp. bijective) for x near x_0.*

1.18 PROPOSITION. *Let M and N be R-modules on X. If M is coherent and R commutative, then the canonical homomorphism*

$$\Phi_x: [Hom_R(M,N)]_x \longrightarrow \mathrm{Hom}_{R_x}(M_x,N_x)$$

is bijective for all $x \in X$.

Proof. An exact sequence $R^q \longrightarrow R^p \longrightarrow M \longrightarrow 0$ near x provides in a canonical manner a commutative diagram with exact lines

$$\begin{tikzcd}
 & & (N^p)_x \arrow[d, phantom, "\shortparallel"] & (N^q)_x \arrow[d, phantom, "\shortparallel"] \\
0 \arrow[r] & Hom_R(M,N)_x \arrow[r] \arrow[d, "\mu"] & Hom_R(R^p,N)_x \arrow[r] \arrow[d, "\alpha"] & Hom_R(R^q,N)_x \arrow[d, "\beta"] \\
0 \arrow[r] & \mathrm{Hom}_{R_x}(M_x,N_x) \arrow[r] & \mathrm{Hom}_{R_x}(R^p_x,N_x) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}_{R_x}(R^q_x,N_x) \arrow[d, phantom, "\shortparallel"] \\
 & & (N^p)_x & (N^q)_x
\end{tikzcd}$$

α and β are isomorphisms; hence, μ is an isomorphism.

1.19 DEFINITION. *Let M be an R-module on X. Then*

$$\mathrm{supp}\, M := \{x \in X;\ M_x \neq 0\}$$

is the support of M.

1.20 PROPOSITION. *If M is locally finitely generated, then* supp M *is closed in X.*

Let X be open in $\mathbb{C}^n$, $\mathcal{O}$ the sheaf of holomorphic functions on X. A subset $A \subset X$ is called *analytic in* X if A is closed in X and if for every $a \in A$ there exists an open neighbourhood U of x in X and $f_1,\dots,f_p \in \Gamma(U,\mathcal{O})$ such that

$$U \cap A = \{x \in U;\ f_1(x) = \dots = f_p(x) = 0\}.$$

1.21 PROPOSITION. *Let $I \subset \mathcal{O}$ be a finitely generated sheaf of ideals. Then $Z(I) := \{x \in X;\ I_x \neq \mathcal{O}_x\} = \{x \in X;\ f(x) = 0\ \forall f \in I_x\}$ is analytic in X.*

Proof. If $f_1,\dots,f_p$ generate $I_{|U}$, then $Z(I) \cap U = \{x \in U;\ f_1(x) = \dots = f_p(x) = 0\}$.

1.22 PROPOSITION. *Let M be a coherent $\mathcal{O}$-module. Then $\mathrm{Ann}_{\mathcal{O}} M \subset \mathcal{O}$ is coherent; hence, $\mathrm{supp}\, M = Z(\mathrm{Ann}_{\mathcal{O}} M)$ is analytic in X.*

Proof. $\mathrm{Ann}_{\mathcal{O}} M = \mathrm{Ker}(\mathcal{O} \longrightarrow Hom_{\mathcal{O}}(M,M))$ is coherent

$$f \longmapsto (a \longmapsto fa)$$

by 1.14, since $\mathcal{O}$ is coherent by 2.3.

§ 2 BASIC THEOREMS ON COHERENT $\mathcal{O}$-MODULES

2.1 DEFINITION. *For $a \in \mathbb{C}^n$ and $r \in (\mathbb{R}_{\geq 0})^n$, the set*

$$K := \{z \in \mathbb{C}^n;\ |z_j - a_j| \leq r_j \text{ for } j = 1,\dots,n\}$$

is called a closed polydisk.

Remark. If $r = (0,\dots,0)$, then K is a point. Let $\mathcal{M}$ be a sheaf on U (open in $\mathbb{C}^n$) and $K \subset U$. Then each section $s \in \mathcal{M}(K) = \Gamma(K,\mathcal{M})$ extends to an open neighbourhood V of K in U.

In this text, $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$, K a closed polydisk and X open in $\mathbb{C}^n$.

2.2 THEOREM (Frisch). *$\mathcal{O}(K)$ is noetherian for each closed polydisk*

2.3 THEOREM (Oka). *$\mathcal{O}$ is coherent.*

2.4 THEOREM (Cartan-Oka).

(A) *Let $\mathcal{M}$ be a coherent $\mathcal{O}$-module on U and $K \subset U$ a closed polydisk. Then there exists an exact sequence*

$$\mathcal{O}^p_{|K} \longrightarrow \mathcal{M}_{|K} \longrightarrow 0.$$

(B) *Let $K \subset \mathbb{C}^n$ be a closed polydisk. Then the functor*

$$\mathcal{M} \longmapsto \mathcal{M}(K)$$

($\mathcal{M}$ a coherent $\mathcal{O}$-module on a neighbourhood of K) is exact.

For proofs of these theorems, see [3], [4], [5].

In the rest of this paragraphe we give some applications. With the notation of (A), an immediate consequence of (A) is

2.5 PROPOSITION. *$\mathcal{M}(K)$ generates $\mathcal{M}_{|K}$, i.e. for every $x \in K$ and $h \in \mathcal{M}_x$, there exist $h_1,\dots,h_p \in \mathcal{M}(K)$ and $f_1,\dots,f_p \in \mathcal{O}_x$ such that $h = \Sigma f_j h_{j,x}$.*

2.6 PROPOSITION. *Let $K \subset X$ be a closed polydisk, and $\mathcal{M}$, a coherent $\mathcal{O}$-module over X. Then, for every $k \in \mathbb{N}$, $\mathcal{M}_{|K}$ admits a resolution of length k, i.e., there exists an exact sequence*

$$\mathcal{O}^{p_k}_{|K} \xrightarrow{k} \mathcal{O}^{p_{k-1}}_{|K} \longrightarrow \dots \longrightarrow \mathcal{O}^{p_o}_{|K} \xrightarrow{\phi_o} \mathcal{M}_{|K} \longrightarrow 0.$$

Proof by induction and (A).

2.7 PROPOSITION. *Let $\mathcal{M}$ be an $\mathcal{O}$-module on $X \subset \mathbb{C}^n$ and $K \subset X$ a closed polydisk.*

a) *If $\mathcal{M}$ is coherent, then*

$$(*)\begin{cases} \mathcal{M}(K) \text{ is a finitely generated } \mathcal{O}(K) \text{ module and the natural homomorphism} \\ \mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{M}(K) \longrightarrow \mathcal{M}_x \text{ is an isomorphism for all } x \in K. \end{cases}$$

b) *If $(*)$ is satisfied, then $\mathcal{M}_{|\mathring{K}}$ is coherent.*

Proof. Suppose that $\mathcal{M}$ is coherent. Then, by (2.6), there exists an exact sequence $\mathcal{O}^q_{|K} \longrightarrow \mathcal{O}^p_{|K} \longrightarrow \mathcal{M}_{|K} \longrightarrow 0$; by (B), the induced sequence

$$\mathcal{O}^q(K) \longrightarrow \mathcal{O}^p(K) \longrightarrow \mathcal{M}(K) \longrightarrow 0$$

is exact. Hence, $\mathcal{M}(K)$ is a finitely generated $\mathcal{O}(K)$-module.
Now consider the canonical commutative diagram

$$(**)\qquad \begin{array}{ccccccc} \mathcal{O}^q(K)\otimes_{\mathcal{O}(K)}\mathcal{O}_x & \longrightarrow & \mathcal{O}^p(K)\otimes_{\mathcal{O}(K)}\mathcal{O}_x & \longrightarrow & \mathcal{M}(K)\otimes_{\mathcal{O}(K)}\mathcal{O}_x & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \mathcal{O}^q_x & \longrightarrow & \mathcal{O}^p_x & \longrightarrow & \mathcal{M}_x & \longrightarrow & 0 \end{array}$$

The lines are exact, α and β are isomorphisms, hence γ is an isomorphism.
Now suppose $(*)$. Since $\mathcal{O}(K)$ is noetherian and $\mathcal{M}(K)$ finitely generated, there is an exact sequence

$$\mathcal{O}^q(K) \xrightarrow{\phi} \mathcal{O}^p(K) \xrightarrow{\psi} \mathcal{M}(K) \longrightarrow 0 .$$

That sequence generates (use $(**)$), for every $x \in K$, an exact sequence

$$\mathcal{O}^q_x \xrightarrow{\phi_x} \mathcal{O}^p_x \longrightarrow \mathcal{M}_x \longrightarrow 0$$

and thus an exact sequence

$$\mathcal{O}^q_{|K} \longrightarrow \mathcal{O}^p_{|K} \longrightarrow \mathcal{M}_{|K} \longrightarrow 0 .$$

Hence, $\mathcal{M}_{|\mathring{K}}$ is coherent.

2.8 PROPOSITION. *Let* $M_1 \subset M_2 \subset \ldots \subset M_k \subset \ldots \subset M$ *be coherent* $\mathcal{O}$*-modules on* $X \subset \mathbb{C}^n$. *Then, for every compact* $L \subset X$, *there exists* $k \in \mathbb{N}$ *such that* $M_{k|L} = M_{k+p|L}$ *for every* $p \in \mathbb{N}$.

Proof. We may assume that $L = K$ is a closed polydisk. Since $\mathcal{O}(K)$ is noetherian and $M_1(K) \subset M_2(K) \subset \ldots \subset M(K)$ are all finitely generated by 2.8, there is a $k \in \mathbb{N}$ such that $M_k(K) = M_{k+p}(K)$ for all $p \in \mathbb{N}$. The assertion follows by 2.7 .

2.9 DEFINITION. *Let* M *be an* $\mathcal{O}$*-module on* X. *An* $\mathcal{O}$*-exhaustion of* M *is a sequence* $M_0 \subset M_1 \subset \ldots \subset M_k \subset \ldots \subset M$ *of coherent sub-*$\mathcal{O}$*-modules such that* $M = \bigcup_k M_k$.

2.10 PROPOSITION. *Let* M, M' *be* $\mathcal{O}$*-modules on* X *that admit* $\mathcal{O}$*-exhaustions. If* $\phi: M \longrightarrow M'$ *is surjective, then* $M(K) \longrightarrow M'(K)$ *is surjective for every closed polydisk* $K \subset X$.

Proof. Let $(M_k)_{k\in\mathbb{N}}$ and $(M'_k)_{k\in\mathbb{N}}$ be $\mathcal{O}$-exhaustions. Since $M'(K) = \bigcup_k M'_k(K)$, it is sufficient to show that for every $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that $\phi(M_\ell(K)) \supset M'_k(K)$: as $M'_k = \bigcup_\ell (M'_k \cap \phi(M_\ell))$ is the increasing union of coherent $\mathcal{O}$-modules, there is (by 2.8) an $\ell \in \mathbb{N}$ such that $M'_{k|K} \subset (\phi\, M_\ell)_{|K}$. By (B) we have that $(\phi M_\ell)(K) = \phi(M_\ell(K))$; hence $M'_k(K) \subset (\phi\, M_\ell)(K) = \phi(M_\ell(K))$.

2.11 EXAMPLE. $M := \mathcal{O}[X_1,\ldots,X_m]$ is a non-coherent $\mathcal{O}$-module that admits an $\mathcal{O}$-exhaustion: put $M_k := \{P \in M;\ \text{degree}\ (P) \le k\}$.

2.12 EXAMPLE. For an open subset $U \subset \mathbb{C}^n$ and $m \in \mathbb{N}$ define

$$\mathcal{D}_m(U) := \{ \sum_{|\alpha| \le m} f_\alpha\, D^\alpha;\ f_\alpha \in \mathcal{O}(U)$$

(here $\alpha := (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$, $D^\alpha := D_1^{\alpha_1} \circ \ldots \circ D_n^{\alpha_n}$, $D_k := \frac{\partial}{\partial z_k}$).

The associated sheaf $\mathcal{D}_m$ is a coherent $\mathcal{O}$-module; hence, $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an $\mathcal{O}$-exhaustion of the sheaf $\mathcal{D} := \bigcup_{m\in\mathbb{N}} \mathcal{D}_m$ of linear differential operators.

Each $P \in \mathcal{D}(U)$ is a $\mathbb{C}$-linear homomorphism $P: \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$; P is uniquely determined by this homomorphism.
$\mathcal{D}$ is (in a natural way) a sheaf of (not commutative) rings: for $P, Q \in \mathcal{D}$, $P \cdot Q$ has the property that $(P \cdot Q)(f) = P(Q(f))$ for all $f \in \mathcal{O}$.
Since $\mathcal{D}_k \cdot \mathcal{D}_\ell \subset \mathcal{D}_{k+\ell}$ and $\mathcal{D}_0 = \mathcal{O}$, one has the following structures:

$\mathcal{D}$ is a left-$\mathcal{O}$-module: $f \cdot (\Sigma f_\alpha D^\alpha) = \Sigma (f f_\alpha) D^\alpha$;

$\mathcal{O}$ is a left-$\mathcal{D}$-module: $(\Sigma f_\alpha D^\alpha) \cdot f = \Sigma f_\alpha D^\alpha(f)$.

In § 3 we shall see that $\mathcal{O}[X_1,\ldots,X_m]$ and $\mathcal{D}$ are coherent as sheaves of rings. Note , that $\mathcal{O}[X_1,\ldots,X_m]$ and $\mathcal{D}$ can be defined in a natural way on complex analytic manifolds.

§ 3 $\mathcal{D}$ IS COHERENT

The main results of this paragraph are

3.1 THEOREM. $\mathcal{D}(K)$ *is noetherian for every closed polydisk* K.

3.2 THEOREM. $\mathcal{D}$ *is coherent as a sheaf of rings.*

Note, that all results and proofs in this paragraph remain valid if one replaces $\mathcal{D}$ by $\mathcal{O}[X_1,\ldots,X_m]$, $\mathcal{D}_m$ by $\mathcal{O}[X_1,\ldots,X_r]_m := \{P;\ \text{degree}\ (P) \le m\}$, etc. Hence, we obtain

3.3 THEOREM. $\mathcal{O}[X_1,\ldots,X_r]$ *is coherent as a sheaf of rings.*

($\mathcal{O}[X_1,\ldots,X_r]\,(K) = \mathcal{O}(K)\,[X_1,\ldots,X_r]$ is noetherian by 5.1 since $\mathcal{O}(K)$ is noetherian.)

For the proofs, we introduce the following notation:

3.4 DEFINITION. $\operatorname{gr} \mathcal{D} := \bigoplus_{k\geq 0} \mathcal{D}_k/\mathcal{D}_{k-1}$ $(\mathcal{D}_{-1} := (0))$.

$\operatorname{gr}\mathcal{D}$ is, in a natural manner, a graded $\mathcal{O}$-algebra:
for $\overline{P} \in \mathcal{D}_k/\mathcal{D}_{k-1}$, $\overline{Q} \in \mathcal{D}_\ell/\mathcal{D}_{\ell-1}$, define $\overline{P}\cdot\overline{Q} := \overline{P\cdot Q} \in \mathcal{D}_{k+\ell}/\mathcal{D}_{k+\ell-1}$

3.5 PROPOSITION. *The natural homomorphism*

$$\operatorname{gr}\mathcal{D} \longrightarrow \mathcal{O}[X_1,\dots,X_n]$$

$$\mathcal{D}_k/\mathcal{D}_{k-1} \ni \sum_{|\alpha|=k} f_\alpha D^\alpha \longmapsto \sum_{|\alpha|=k} f_\alpha X^\alpha$$

is an isomorphism of graded $\mathcal{O}$-algebras.

Proof. Use the following formulas:

$$D^\alpha\cdot f = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta(f)\, D^{\alpha-\beta}\ ,\ (P\cdot f)_k = f\cdot P_k \text{ for } f \in \mathcal{O},\ P \in \mathcal{D}_k$$

(for $Q = \sum_{|\alpha|\leq k} g_\alpha D^\alpha$ define $Q_k := \sum_{|\alpha|=k} g_\alpha D^\alpha$).

Proof of **3.1** By (B), $(\mathcal{D}_k/\mathcal{D}_{k-1})(K) = \mathcal{D}_k(K)/\mathcal{D}_{k-1}(K)$ for all k; hence, $\operatorname{gr}(\mathcal{D}(K)) = (\operatorname{gr}\mathcal{D})(K) = \mathcal{O}(K)[X_1,\dots,X_k]$. By (F), $\mathcal{O}(K)$ is noetherian. Hence, by 5.1, $\operatorname{gr}(\mathcal{D}(K))$ is noetherian; thus $\mathcal{D}(K)$ is noetherian by 5.3 .

Proof of **3.2** Let $U \subset \mathbb{C}^n$ be open, $\phi: \mathcal{D}^p_{|U} \longrightarrow \mathcal{D}_{|U}$ be a homomorphism, $\mathcal{K} := \ker\phi \subset \mathcal{D}^p_{|U}$. We must show that $\mathcal{K}$ is locally finitely generated. To that end let $K \subset U$ be a closed polydisk. $\mathcal{K}(K)$ is finitely generated as a $\mathcal{D}(K)$ module: the exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{D}^p_{|U}$ yields the exact sequence $0 \longrightarrow \mathcal{K}(K) \longrightarrow \mathcal{D}^p(K)$. Since $\mathcal{D}(K)$ is noetherian, $\mathcal{K}(K)$ is finitely generated. - We want to show that $\mathcal{K}_{|K}$ is finitely generated as an $\mathcal{D}_{|K}$-module. Since $\mathcal{K} = \bigcup_{\ell\in\mathbb{N}} \mathcal{K}_\ell$ $(\mathcal{K}_\ell := \mathcal{K}\cap\mathcal{D}^p_\ell)$, it is sufficient to prove that, for all ℓ, $\mathcal{K}_{\ell|K}$ is generated by $\mathcal{K}_\ell(K)$ as an $\mathcal{O}$-module. To do that, we may suppose that U is connected. There are $k \in \mathbb{N}$ and $P_1,\dots,P_k \in \mathcal{D}_k(U)$ such that

$\phi(Q_1,\dots,Q_p) = Q_1P_1+\dots+Q_pP_p$ for all $Q_1,\dots,Q_p \in \mathcal{D}$. Hence, $\phi(\mathcal{D}^p_\ell) \subset \mathcal{D}_{\ell+k}$ for all ℓ. Define $\phi_\ell := \phi: \mathcal{D}^p_\ell \longrightarrow \mathcal{D}_{\ell+k}$. Then $\mathcal{K}_\ell = \mathcal{K}\cap\mathcal{D}^p_\ell = \ker\phi_\ell$ is coherent as an $\mathcal{O}$-module; by (2.5), $\mathcal{K}_\ell(K)$ generates $\mathcal{K}_{\ell|K}$ as an $\mathcal{O}$-module.

§ 4 COHERENT $\mathcal{D}$-MODULES AND GOOD FILTRATIONS

By X we always denote an open subset of $\mathbb{C}^n$. Let M be a $\mathcal{D}$-module.

4.1 DEFINITION. *A filtration on M is a sequence*

$$M_0 \subset M_1 \subset \dots \subset M_k \subset \dots \subset M$$

of sub-$\mathcal{O}$-modules of M satisfying $M = \bigcup M_k$ and $\mathcal{D}_\ell \cdot M_k \subset M_{k+\ell}$ for all $\ell,k \in \mathbb{N}$. A filtered $\mathcal{D}$-module is a $\mathcal{D}$-module endowed with a filtration.

4.2 EXAMPLE. $\mathcal{D}$ is filtered by the canonical filtration $(\mathcal{D}_k)_{k\in\mathbb{N}}$.

4.3 DEFINITION. *Let $(M_k)_{k\in\mathbb{N}}$ be a filtration on M. Then*

$$\operatorname{gr} M := \bigoplus M_k/M_{k-1} \text{ (with } M_{-1} := 0)$$

is called the associated graded gr $\mathcal{D}$*-module; the* gr $\mathcal{D}$*-structure is given by*

$$\overline{P}\cdot\overline{F} := \overline{P\cdot F} \in M_{k+\ell}/M_{k+\ell-1}$$

for $\overline{P} \in \mathcal{D}_\ell/\mathcal{D}_{\ell-1}$, $\overline{F} \in M_k/M_{k-1}$.

Of course, gr M depends on the filtration.

The main results of this paragraph are the following two theorems:

4.4 THEOREM and DEFINITON. *A filtration $(M_k)_{k\in\mathbb{N}}$ on a $\mathcal{D}$-module M on X is called "good" if one of the following equivalent conditions is satisfied:*

i) gr M *is* (gr $\mathcal{D}$)*-coherent*

ii) *All M_k are $\mathcal{O}$-coherent and for every $x \in X$, there exist $k_0(x) \in \mathbb{N}$ and a* *neighbourhood* U *of* x *such that*

$\mathcal{D}_\ell \cdot M_{k_0|U} = M_{k_0+\ell|U}$ *for all* $\ell \in \mathbb{N}$.

4.5 THEOREM. *The following conditions are equivalent for a* $\mathcal{D}$*-module* M *on* X:

iii) M *is* $\mathcal{D}$*-coherent.*

iv) *Every* $x \in X$ *has an open neighbourhood* $U \subset X$ *such that* $M_{|U}$ *admits a good filtration.*

v) *There exists a cover of* X *by closed polydisks* K *that verify*

$$\begin{cases} M(K) \text{ is a finitely generated } \mathcal{D}(K)\text{-module and } \mathcal{O}_x \otimes_{\mathcal{O}(K)} M(K) \longrightarrow M_x \\ \text{is an isomorphism for all } x \in K. \end{cases}$$

Before we prove 4.4, let us take ii) as definition for a good filtration.

4.6 LEMMA. *Let* X *be connected and* $\mathcal{D}^q \xrightarrow{\psi} \mathcal{D}^p \xrightarrow{\phi} M \longrightarrow 0$ *exact on* X. *Then* $(M_k)_{k\in\mathbb{N}}$ *with* $M_k := \phi(\mathcal{D}^p_k)$, *is a good filtration on* M.

Proof. To prove that M_k is $\mathcal{O}$-coherent, it is sufficient to prove that $K := \psi(\mathcal{D}^q) \cap \mathcal{D}^p_k$ is $\mathcal{O}$-coherent, since $0 \longrightarrow \psi(\mathcal{D}^q) \cap \mathcal{D}^p_k \longrightarrow \mathcal{D}^p_k \longrightarrow M_k \longrightarrow 0$ is exact and $\mathcal{D}^p_k$ is $\mathcal{O}$-coherent. As in the proof of 3.2 find $s \in \mathbb{N}$ such that $\psi(\mathcal{D}^q_\ell) \subset \mathcal{D}^p_{\ell+s}$ for all ℓ. Then, for all ℓ, $\psi(\mathcal{D}^q_\ell)$ and $\psi(\mathcal{D}^q_\ell) \cap \mathcal{D}^p_k$ are $\mathcal{O}$-coherent; hence, $K = \bigcup_\ell \psi(\mathcal{D}^q_\ell) \cap \mathcal{D}^p_k$ is $\mathcal{O}$-coherent by 2.8. Since $\mathcal{D}_\ell\, \mathcal{D}^p_0 = \mathcal{D}^p_\ell$ for all ℓ (use $P \cdot 1 = P$ and $1 \in \mathcal{D}_0$), we have that $\mathcal{D}_\ell\, M_0 = M_\ell$ for all ℓ.

Together with 4.6, the following lemma is a kind of "Theorem B" for coherent $\mathcal{D}$-modules and sufficiently small K:

4.7 LEMMA. *Let* $K \subset \mathbb{C}^n$ *be a closed polydisk. Then the functor* $M \longmapsto M(K)$, *defined on all* $\mathcal{D}$*-modules* M, *which are defined in a neighbourhood of* K *and admit a good filtration, is exact.*

Proof. Good filtrations are $\mathcal{O}$-exhaustions, hence 4.7 follows from 2.10.

The following lemma is an immediate consequence of (B):

4.8 LEMMA. *Let* M *be a* $\mathcal{D}$*-module on* X, $(M_k)_{k\in\mathbb{N}}$, *a filtration on* M *by coherent* $\mathcal{O}$*-modules and* $K \subset X$, *a closed polydisk. Then the canonical homomorphism* $\mathrm{gr}\, M(K) \longrightarrow (\mathrm{gr}\, M)(K)$ *is an isomorphism. (*$M(K)$ *is a filtered* $\mathcal{D}(K)$*-module with respect to the canonical filtrations* $M(K)_k := M_k(K)$, $\mathcal{D}(K)_k := \mathcal{D}_k(K)$; $\mathrm{gr}\, M(K)$ *is a* $\mathrm{gr}\, \mathcal{D}(K)$*-module.)*

Now we prove 4.4 and 4.5.

(v) $\Longrightarrow$ (iii) We show that M is coherent on $\mathring{K}$ if (*) is satisfied. By 3.1, $\mathcal{D}(K)$ is noetherian, hence, there exists an exact sequence

$$(+) \quad \mathcal{D}^q(K) \longrightarrow \mathcal{D}^p(K) \longrightarrow M(K) \longrightarrow 0$$

that, for every $x \in K$, yields the exact sequence

$$\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{D}^q(K) \longrightarrow \mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{D}^p(K) \longrightarrow \mathcal{O}_x \otimes_{\mathcal{O}(K)} M(K) \longrightarrow 0.$$

By hypothesis, $\mathcal{O}_x \otimes_{\mathcal{O}(K)} M(K) \simeq M_x$. By 2.7, we have that $\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{D}_\ell(K) \simeq \mathcal{D}_{\ell,x}$ and thus $\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{D}^p(K) \cong \mathcal{D}^p_x$. Since all isomorphisms are natural, (+) induces an exact sequence

$$\mathcal{D}^q \longrightarrow \mathcal{D}^p \longrightarrow M \longrightarrow 0 \text{ on } K$$

and $M_{|\mathring{K}}$ is $\mathcal{D}$-coherent, since $\mathcal{D}$ is coherent.

(iii) $\Longrightarrow$ (iv) This is 4.6.

For the rest of the proof define $\tilde{M}_k := M_k / M_{k-1}$ for all k ($M_{-1}) := 0$).

(ii) $\Longrightarrow$ (i) By "(v) $\Longrightarrow$ (iii)" for $\mathrm{gr}\, \mathcal{D}$ instead of $\mathcal{D}$ it is sufficient to prove that, for a sufficiently small closed polydisk K, the following holds:

$$(*)\begin{cases}(\mathrm{gr}\,\mathcal{M})(K) \text{ is a finitely generated } (\mathrm{gr}\,\mathcal{D})(K)\text{-module and}\\ \mathcal{O}_x \otimes_{\mathcal{O}(K)} (\mathrm{gr}\,\mathcal{M})(K) \longrightarrow (\mathrm{gr}\,\mathcal{M})_x \text{ is an isomorphism for all } x \in K.\end{cases}$$

Since all $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are $\mathcal{O}$-coherent, $\mathcal{O}_x \otimes_{\mathcal{O}(K)} \tilde{\mathcal{M}}_k(K) \longrightarrow \tilde{\mathcal{M}}_{k,x}$ is an isomorphism for all $x \in K$ and all k; hence,
$\mathcal{O}_x \otimes_{\mathcal{O}(K)} \bigoplus_k \tilde{\mathcal{M}}_k(K) \longrightarrow \bigoplus_k \tilde{\mathcal{M}}_{k,x}$ is an isomorphism for all x.
Since $\bigoplus_k \tilde{\mathcal{M}}_{k,x} = (\mathrm{gr}\,\mathcal{M})_x$ and (by 4.8) $\bigoplus_k \tilde{\mathcal{M}}_k(K) = (\mathrm{gr}\,\mathcal{M})(K)$, the second assertion of (*) follows. To prove the first one, we note that $(\mathcal{D}_\ell \cdot \mathcal{M}_{k_0})_{|K} = \mathcal{M}_{k_0+\ell|K}$ implies $\mathcal{D}_\ell(K) \cdot \mathcal{M}_{k_0}(K) = \mathcal{M}_{k_0+\ell}(K)$: Consider $\mathcal{D}_\ell$ as a *right* $\mathcal{O}$-module; then the surjection $\mathcal{D}_\ell \otimes_{\mathcal{O}} \mathcal{M}_{k_0} \longrightarrow \mathcal{D}_\ell \cdot \mathcal{M}_{k_0} = \mathcal{M}_{k_0+\ell}$ induces the surjection $(\mathcal{D}_\ell \otimes_{\mathcal{O}} \mathcal{M}_{k_0})(K) \longrightarrow (\mathcal{D}_\ell \cdot \mathcal{M}_{k_0})(K) = \mathcal{M}_{k_0+\ell}(K)$. Since $\mathcal{D}_\ell$ is free as $\mathcal{O}$-module, $(\mathcal{D}_\ell \otimes_{\mathcal{O}} \mathcal{M}_{k_0})(K) = \mathcal{D}_\ell(K) \otimes_{\mathcal{O}(K)} \mathcal{M}_{k_0}(K)$ and thus $\mathcal{D}_\ell(K) \cdot \mathcal{M}_{k_0}(K) = \mathcal{M}_{k_0+\ell}(K)$. - Hence, $(\mathrm{gr}\,\mathcal{M})(K) = \mathrm{gr}\,\mathcal{M}(K)$ is a finite $\mathrm{gr}\,\mathcal{D}(K) = (\mathrm{gr}\,\mathcal{D})(K)$-module.

(iv) $\Rightarrow$ (v) We may suppose that $(\mathcal{M}_k)_{k\in\mathbb{N}}$ is a good filtration on $\mathcal{M}$. Let $K \subset X$ be a closed polydisk. Then the canonical isomorphisms $\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{M}_k(K) \longrightarrow \mathcal{M}_{k,x}$ for $k \in \mathbb{N}$, $x \in K$, induce the isomorphisms

$$\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{M}(K) = \varinjlim_k (\mathcal{O}_x \otimes_{\mathcal{O}(K)} \mathcal{M}_k(K)) \simeq \varinjlim_k \mathcal{M}_{K;x} = \mathcal{M}_x .$$

By "(ii) $\Rightarrow$ (i)", gr $\mathcal{M}$ is a coherent (gr $\mathcal{D}$)-module. Hence, (by "(iii) $\Rightarrow$ (v)" for gr $\mathcal{D}$ instead of $\mathcal{D}$), $\mathrm{gr}\,\mathcal{M}(K) = (\mathrm{gr}\,\mathcal{M})(K)$ is finitely generated as $\mathrm{gr}\,\mathcal{D}(K) = (\mathrm{gr}\,\mathcal{D})(K)$-module, if K is sufficiently small. As in the proof of (ii) $\Rightarrow$ (i), we see that $(\mathcal{D}_\ell\, \mathcal{M}_{k_0})_{|K} = \mathcal{M}_{k_0+\ell|K}$ implies $\mathcal{D}_\ell(K)\cdot\mathcal{M}_{k_0}(K) = \mathcal{M}_{k_0+\ell}(K)$. Hence, by 5.4, $\mathcal{M}(K)$ is finitely generated as $\mathcal{D}(K)$-module, if K is sufficiently small.

(i) $\Rightarrow$ (ii) In order to prove that all M_k are $\mathcal{O}$-coherent, it is sufficient to prove that $\tilde{M}_k = M_k/M_{k-1}$ are $\mathcal{O}$-coherent (since $M_{-1} = 0$ and $0 \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow \tilde{M}_k \longrightarrow 0$ is exact). To prove that, it is (by 2.7) sufficient to show that for all sufficiently small closed polydisks $K \subset X$:

$$(*)\begin{cases}\tilde{\mathcal{M}}_k(K) \text{ is a finitely generated } \mathcal{O}(K)\text{-module and}\\ \mathcal{O}_x \otimes_{\mathcal{O}(K)} \tilde{\mathcal{M}}_k(K) \longrightarrow \tilde{\mathcal{M}}_{k,x} \text{ is an isomorphism for all } x \in K.\end{cases}$$

Since gr $\mathcal{M}$ is a coherent gr $\mathcal{D}$-module, we conclude (by "(iii) $\Rightarrow$ (v)" for gr $\mathcal{D}$ instead of $\mathcal{D}$) that, if K is small:

a) $(\mathrm{gr}\,\mathcal{M})(K)$ is a finitely generated $(\mathrm{gr}\,\mathcal{D})(K)$-module and

b) $\mathcal{O}_x \otimes_{\mathcal{O}(K)} (\mathrm{gr}\,\mathcal{M})(K) \xrightarrow{\sim} (\mathrm{gr}\,\mathcal{M})_x$ for all $x \in K$.

Using $(\mathrm{gr}\,\mathcal{M})(K) = \bigoplus_k \tilde{\mathcal{M}}_k(K)$ and $(\mathrm{gr}\,\mathcal{M})_x = \bigoplus_k \tilde{\mathcal{M}}_{k,x}$, b) implies $\mathcal{O}_x \otimes_{\mathcal{O}(K)} \tilde{\mathcal{M}}_k(K) \cong \tilde{\mathcal{M}}_{k,x}$ for all k and x, and a) implies (by 5.2) that $\tilde{\mathcal{M}}_k(K)$ is a finitely generated $\mathcal{O}(K)$-module for all k. By 5.4, there exists k_0 such that $\mathcal{D}_\ell(K) \cdot \mathcal{M}_{k_0}(K) = \mathcal{M}_{k_0+\ell}(K)$ for all ℓ. Since $\mathcal{D}_\ell(K) \cdot \mathcal{M}_{k_0}(K) \subset (\mathcal{D}_\ell \cdot \mathcal{M}_{k_0})(K) \subset \mathcal{M}_{k_0+\ell}(K)$, that implies $(\mathcal{D}_\ell \cdot \mathcal{M}_{k_0})(K) = \mathcal{M}_{k_0+\ell}(K)$; thus $(\mathcal{D}_\ell \cdot \mathcal{M}_{k_0})_{|K} = \mathcal{M}_{k_0+\ell|K}$ for all ℓ by (2.7).

§ 5 SUPPLEMENT

In this supplement, we collect some algebraic results.

5.1 THEOREM. *Let* R *be a commutative noetherian ring. Then* $R[X_1,\dots,X_m]$ *is noetherian for every* $m \in \mathbb{N}$.

For the rest of this paragraph, we use the following notation: $R = \bigcup_{k\in\mathbb{N}} R_k$ is a filtered ring and $M = \bigcup_{k\in\mathbb{N}} M_k$ is a filtered R-module, i.e. $R_k \subset R_{k+1}$, $M_k \subset M_{k+1}$ and $R_k \cdot R_\ell \subset R_{k+\ell}$, $R_k \cdot M_\ell \subset M_{k+\ell}$ for all $k,\ell \in \mathbb{N}$. In particular, R_0 is a subring of R and M_k is an R_0-module for $k \in \mathbb{N}$. We assume that R_k is finitely generated as R_o-modul for $k \in \mathbb{N}$. We define $\mathrm{gr}\,R := \bigoplus_k R_k/R_{k-1}$ $(R_{-1} := 0)$; then $\mathrm{gr}\,M := \bigoplus_k M_k/M_{k-1}$ (with $M_{-1} := 0$) is a gr R-module. For $m \in M$, define $\tilde{m} \in \mathrm{gr}\,M$ by $\tilde{m} := 0$ if $m = 0$, $\tilde{m} := [m] \in M_k/M_{k-1}$ if $m \in M_k$, $m \notin M_{k-1}$.

5.2 PROPOSITION. *Suppose* $m_1,\dots,m_k \in M$. *Then* $m_1,\dots,m_k$ *generate* M *as an* R*-module if* $\tilde{m}_1,\dots,\tilde{m}_k$ *generate* gr M *as an* gr R*-module.*

5.3 PROPOSITION. *If* gr R *is noetherian, then* R *is noetherian.*

5.4 PROPOSITION. *Suppose that* R_o *and* R *are noetherian and that* gr M *is finitely generated as a* gr R*-module. Then, for all* k, M_k *is finitely generated as an* R_o*-module and there exists* $k_o \in \mathbb{N}$ *such that* $R_\ell \cdot M_{k_o} = M_{k_o+\ell}$ *for all* ℓ.

Proof of 5.2 It is sufficient to prove that for every $0 \neq m \in M_\ell$, there exist $r_i \in R$ such that $m - \sum_{i=1}^{k} r_i m_i \in M_{\ell-1}$. By hypothesis, there are $r_i \in R_{\ell-d_i}$ such that $\tilde{m} = \sum \tilde{r}_i \tilde{m}_i$. Then $m - \sum_{i=1}^{k} r_i m_i \in M_{\ell-1}$.

Proof of 5.3 Let $I \subset R$ be an ideal, $I_k := I \cap R_k$. Then the ideal $\mathrm{gr}\,I = \bigoplus I_k/I_{k-1}$ of gr R is finitely generated; hence, by 5.2, I is finitely generated.

Proof of 5.4 Define $\tilde{M}_k := M_k/M_{k-1}$ and $\tilde{R}_k := R_k/R_{k-1}$ for $k \in \mathbb{N}$. Since gr M is generated by its homogeneous elements, there exist $m_1,\dots,m_\ell \in M$, such that $\tilde{m}_1,\dots,\tilde{m}_\ell$ generate gr M. Suppose $\tilde{m}_\lambda \in \tilde{M}_{p_\lambda}$. Then, for all k, $\tilde{M}_k = \sum_\lambda \tilde{R}_{k-p_\lambda} \cdot \tilde{m}_\lambda$. Since all $\tilde{R}_j$ are finitely generated R_o-modules, also $\tilde{M}_k$ is finitely generated as an R_o-module. Put $k_o := \max(p_1,\dots,p_\ell)$. Then $R_k M_{k_o} = M_{k_o+k}$ for all k.

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III. LOCAL THEORY OF MEROMORPHIC CONNECTIONS IN DIMENSION ONE (FUCHS THEORY)

A. HAEFLIGER

This talk is an elementary account of the classical theory of Fuchs of systems of differential equations with regular singular points. I have followed closely the instructions and advice of B. Malgrange.

The following notation will be used throughout this talk.

$D_\varepsilon = \{z : z \in \mathbb{C}, |z| < \varepsilon)\}$ is the open disk of radius ε centered at 0 in $\mathbb{C}$.

$D^\bullet_\varepsilon = \{z : z \in \mathbb{C}, 0 < |z| < \varepsilon\}$ is the punctured disk.

$\mathcal{O}$ = ring of germs at 0 of holomorphic functions.

$K = \mathcal{O}[1/z]$ is the field of germs at 0 of meromorphic functions.

$\widetilde{\mathcal{O}}(D^\bullet_\varepsilon)$ is the ring of multivalued holomorphic functions on $D^\bullet_\varepsilon$, i.e. the ring of holomorphic functions on the universal covering $D^\bullet_\varepsilon = \{t \in C, |e^{2i\pi t}| < \varepsilon\}$ of $D^\bullet_\varepsilon$.

$\widetilde{\mathcal{O}} = \lim\limits_{\varepsilon \to 0} \widetilde{\mathcal{O}}(D^\bullet_\varepsilon)$. We have the inclusions $\mathcal{O} \subset K \subset \widetilde{\mathcal{O}}$

$\mathcal{D}$ (resp. $\mathcal{D}[1/z]$) will denote the ring of differential operators of the form $P = \sum a_i(z)(\frac{d}{dz})^{n-i}$, where $a_i(z) \in \mathcal{O}$ (resp. $a_i(z) \in K$).

§ 1. SYSTEMS OF HOMOGENOUS LINEAR DIFFERENTIAL EQUATIONS

1.1 *Statement of Fuchs' theorem*

Basic definition. A multivalued function $f \in \mathcal{O}$ has *moderate growth* if, for any sector $S = \{z : \theta_0 \le \arg z \le \theta_1, 0 < |z| < \varepsilon\}$, where ε is small enough, there is an integer $k > 0$ and a constant $C > 0$ such that

$$|f(z)| < C\, 1/|z|^k \quad \text{for all } z \text{ in } S$$

$\mathcal{O}^{mod}$ will denote the subring of $\mathcal{O}$ consisting of functions with moderate growth.

For instance, a uniform holomorphic function f on the punctured disk has moderate growth iff 0 is not an essential singularity of f (i.e. f is the restriction to D_ε^* of a meromorphic function on D_ε). The functions $z^\alpha (\alpha \in \mathbb{C})$ and $(\log z)^m$ have moderate growth.

Note that $\mathcal{D}[1/z]$ operates on $\mathcal{O}^{mod}$ (use the Cauchy inequalities to check it).

In a paper [1] published in 1866 and inspired by Riemann's memoir [6] on the hypergeometric function (1857), Fuchs proved the following remarkable theorem.

1.1.1. THEOREM OF FUCHS: *Let* $P = \sum_0^n a_k(z)(\frac{d}{dz})^{n-k}$ *be a differential operator with* $a_k(z) \in K$ *and* $a_0(z) \neq 0$.

The multivalued solutions $u \in \mathcal{O}$ *of the differential equation*

$$1) \qquad Pu = 0$$

have moderate growth iff the order of the pole of a_i/a_0 *at 0 is at most i, for* $i = 1,2,\dots,n$.

If these equivalent conditions are satisfied, then the differential equation $Pu = 0$ is said to be *regular* at 0 or one says that 0 is *a regular singular point.*

It is a good exercise to check this equivalence for $n = 1$.

A proof of this theorem will be given after the proof of the corresponding theorem for *system of n linear homogeneous first order differential equations in n unknowns and meromorphic coefficients* (cf. 1.3.1 and 1.3.2).

Such a system is of the form

$$\frac{du_i}{dz}(z) = \sum_{j=1}^n a_{ij}(z)u_j(z) \quad i = 1,2,\dots,n$$

where $a_{ij}(z) \in K$. It can also be written

$$\frac{dU}{dz}(z) = A(z)U(z)$$

where $A(z) = (a_{ij}(z)) \in \mathrm{End}(n,K)$ and $U(z)$ is a column vector with components $u_1(z),\dots,u_n(z)$. So using matrix notation such a system can be treated formally as a single first order equation.

The equation 1) can be converted into such a system of the form

$$1)' \qquad \begin{aligned} du_i/dz &= u_{i+1} \quad i = 1,2,\dots,n-1 \\ du_n/dz &= -a_n/a_0 u_1 - \dots - a_1/a_0 u_n \end{aligned}$$

by letting $u_i = (d/dz)^{i-1}u$, $i = 1,\dots,n$.

It will be seen later (2.3.2) that any system 2) is equivalent to a system of the form 1)'.

Another way of converting 1) into a system 2) is to define the operator $\partial = z\, d/dz$ and to rewrite 1) (after multiplication by $z^n a_0(z)^{-1}$) in the form

$$\partial^n u + \sum_{i=1}^n b_i(z)\partial^{n-i}u = 0.$$

If we set $v_i = \partial^{i-1}u$, then this equation is equivalent to the system

$$\partial v_i = v_{i+1} \quad i = 1,\dots,n-1$$
$$\partial v_n = -b_n(z)v_1 - \dots - b_1(z)v_n$$

or in matrix form

$$\frac{d}{dz} V(z) = B(z)V(z)$$

where B is the matrix

$$\begin{pmatrix} 0 & 1 & \dots\dots\dots & 0 \\ 0 & 0 & \dots\dots\dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots\dots 0 & 1 \\ -b_n & -b_{n-1} \dots & -b_2 & b_1 \end{pmatrix}$$

Note that Fuchs'condition of regularity, namely that a_i/a_0 has a pole of order at most i at 0, is equivalent to the condition that $b_1,\dots,b_n$ are holomorphic.

1.2. *General form of multivalued solutions and monodromy*

Choose $\varepsilon > 0$ so small that the coefficients $a_{ij}(z)$ of 2) are meromorphic functions of D_ε with no pole outside of zero.

Since the universal covering $\tilde{D}^{\cdot}_\varepsilon$ of $D^{\cdot}_\varepsilon$ is simply connected, the solutions of the lifted system (obtained by the change of variables $z = e^{2i\pi t}$) form a vector space over $\mathbb{C}$ of dimension n.

Let $\tilde{S}(t) = (\tilde{U}^1(t),\dots,\tilde{U}^n(t))$ be n linearly independent solutions. Each $\tilde{U}^k(t)$ is a column vector with components in $\mathcal{O}(\tilde{D}^{\cdot}_\varepsilon)$.

As the coefficients are of period one, if we replace t by t+1, we get another basis for the vector space of solutions, so there is matrix $C \in GL(n,\mathbb{C})$ such that

$$\tilde{S}(t+1) = \tilde{S}(t)C$$

The matrix C is the *monodromy matrix* of the system and the corresponding linear transformation of the space of solutions is called the *monodromy*.

Let Γ be a matrix such that $e^{2i\pi\Gamma} = C$ (such a matrix exists and is unique if one imposes the condition that the eigenvalues λ of Γ are such that $-1 \le \mathrm{Re}(\lambda) < 0$).

Then the matrix $\tilde{S}(t)e^{-2i\pi\Gamma t}$ is of period 1 in t, so replacing t by $1/2i\pi \log z$, we get a matrix $\Sigma(z)$ whose coefficients are holomorphic in $D^{\cdot}_\varepsilon$. The columns of the matrix $S(z) = \Sigma(z)e^{\Gamma\log z}$ form a basis for the vector space of multivalued solutions of 2) (a so called *fundamental system of solutions*). By "solution" of 2), we shall always mean "multivalued solutions".

So we have proved the following proposition.

1.2.1 PROPOSITION : *Any fundamental system of solutions of the equation*

$$\frac{dU}{dz} = A(z)U$$

is of the form

$$S(z) = \Sigma(z)e^{\Gamma\log z}$$

(which can also be written as $\tilde{S}(t) = \Sigma(e^{2i\pi t})e^{2i\pi\Gamma t}$) where $\Sigma(z)$ is a matrix with holomorphic coefficients outside of 0 and Γ is a matrix with constant coefficients such that $e^{\Gamma} = C$ is the monodromy matrix.

If we change the basis of the vector space of solutions, then the monodromy matrix C, as well as the matrix Γ, is conjugated by the matrix of the change of basis. Hence we can choose a basis such that Γ is in Jordan form, with k blocks of the form

$$\Gamma_{\alpha_i} = \begin{pmatrix} \alpha_i & 1 & 0 \\ 0 & \alpha_i & 1 \\ 0 & 0 & \alpha_i \end{pmatrix}$$

of size n_i, where $\alpha_i \in \mathbb{C}$. Then $e^{\Gamma\log z}$ is a matrix with blocks along the diagonal of the form

$$e^{\Gamma_{\alpha_i}\log z} = \begin{pmatrix} z^{\alpha_i} & z^{\alpha_i}\log z & z^{\alpha_i}\log^2 z & \cdots \\ 0 & z^{\alpha_i} & z^{\alpha_i}\log z & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Hence an element of this basis will be of the form

$$\sum_{r=1}^{s} z^{\alpha_i}(\log z)^{r-1} h_r(z)$$

where $1 \leq s \leq n_i$ and $h(z)$ is a column vector with components holomorphic outside of 0.

1.2.2 COROLLARY : *For any n-th order differential equation* $Pu = 0$ *there is a solution of the form* $z^{\alpha}h(z)$, *where* $h(z)$ *is holomorphic outside of* 0, *and* α *is such that* $e^{2i\pi\alpha}$ *is an eigenvalue of the monodromy.*

1.3 *Systems with regular singularities*

Definition. Two systems

$$\frac{d}{dz} U = AU \quad \text{and} \quad \frac{d}{dz} V = BV$$

are *equivalent* if there is a matrix $M(z) \in Gl(n,K)$ such that

$$B = \frac{d}{dz} M \cdot M^{-1} + MAM^{-1}.$$

This means that if U is a solution of the first equation, then $MU = V$ is a solution of the second one. It is clear that two equivalent systems have conjugated monodromy.

1.3.1 THEOREM : *For a system*

$$*) \qquad \frac{d}{dz} U(z) = A(z)U(z) \text{ where } A(z) \in End(n,K)$$

the following three conditions are equivalent :

i) *) *is equivalent to a system of the form*

$$\frac{d}{dz} V(z) = \frac{B(z)}{z} V(z)$$

where $B(z)$ *is a matrix with holomorphic coefficients.*

ii) *) *is equivalent to a system of the form*

$$\frac{d}{dz} V(z) = \frac{\Gamma}{z} V(z)$$

where Γ *is a matrix with constant coefficients*

iii) *Any (multivalued) solution* U *of* *) *has moderate growth (i.e. each component of* U *has moderate growth).*

A system satisfying one of those conditions is said to be *regular* (or 0 is a *regular singular point* for *)).

Proof :

iii) ⇒ ii). By 1.2.1, a fundamental system of solutions is of the form $S(z) = \Sigma(z)e^{\Gamma\log z}$.

By hypothesis $S(z)$ has moderate growth and, as $e^{-\Gamma\log z}$ has moderate growth, it follows that $\Sigma(z) = S(z)e^{-\Gamma\log z}$ has also moderate growth. But this implies that the coefficients of $\Sigma(z)$ are meromorphic at 0.

If we replace U by $V = \Sigma^{-1}(z)U$, then one gets the equivalent system

$$\frac{d}{dz} V = \frac{\Gamma}{z} V.$$

ii) ⇒ i) is obvious.

i) ⇒ iii). Let V be a solution of the system $\frac{d}{dz} V = \frac{B(z)}{z} V$; in the sector $S = \{z = re^{i\theta}\, \theta_0 \leq \theta \leq \theta_1,\ 0 < r \leq r_0\}$, we have

$$\frac{d}{dz} V(z) \leq \frac{C}{r} \|V(z)\|$$

where $\|V(z)\|^2 = \sum_i |V_i(z)|^2$ and $C = \sup_{z \in S} \|B(z)\|$.

As $\partial V/\partial r = \frac{dV}{dz} e^{i\theta}$, we can rewrite this as

$$\|\partial V/\partial r(re^{i\theta})\| \le \frac{C}{r}\|V(re^{i\theta})\|$$

Obviously $V(re^{i\theta}) - V(r_0e^{i\theta}) = -\int_r^{r_0} \partial V/\partial r\, dr$ implies that

$$\|V(re^{i\theta})\| \le \|V(r_0e^{i\theta})\| + \int_r^{r_0} \frac{C}{s}\|V(se^{i\theta})\|ds.$$

By Gronwall inequality (cf [3]), we have

$$\|V(re^{i\theta})\| \le \|V(r_0e^{i\theta})\| (\frac{r_0}{r})^C$$

so

$$\|V(re^{i\theta})\| \le \sup_{\theta_0\le\theta\le\theta_1} \|Vr_0e^{i\theta})\| (\frac{r_0}{r})^C$$

i.e. V has moderate growth.

1.3.2 COROLLARY : *Two regular systems are equivalent iff their monodromy are conjugate.*

Two regular systems are equivalent to systems of the form

$$\frac{dU}{dz} = \frac{\Gamma}{z} U \quad \text{and} \quad dU = \frac{\Gamma'}{z} U$$

where Γ,Γ' are matrices with constant coefficients and eigenvalues with real part in the interval $[-1,0[$ (cf. 1.2 and 1.3.1). The monodromy matrices are $e^{2i\pi\Gamma}$ and $e^{2i\pi\Gamma'}$; they are conjugate iff Γ is conjugate to Γ'.

1.3.3 *Proof of the theorem of Fuchs* (cf. Goursat [2]).

Suppose that the coefficients of $Pu = 0$ verify Fuchs conditions, namely that a_i/a_0 has a pole of order at most i at 0. Then the system associated to $Pu = 0$ as indicated at the end of 1.1 satisfies condition i) of 1.3.1. This implies that the solutions u have moderate growth.

Conversely assume that all solutions of $Pu = 0$ have moderate growth. We shall prove by induction on n that the coefficients $z^i a_i(z)$ are holomorphic. As observed at the end of 1.1 we can

rewrite the equation $Pu = 0$ under the form $Qu = \partial^n u + \sum_1^n b_i(z)\partial^{n-i}u=0$, where $\partial = z\frac{d}{dz}$, and we have to prove that the coefficients $b_i(z)$ are holomorphic.

By 1.2.2, we know there is a solution of the form $u(z) = z^\alpha h(z)$. As u has moderate growth, h(z) also, hence h(z) is a meromorphic function. After changing α by adding a suitable integer, we can assume that h(z) is holomorphic and that $h(0) \ne 0$.

If $v \in \tilde{\mathcal{O}}$, then $w(z) = u(z)v(z)$ is a solution of the equation $Qw = 0$ iff ∂v is a solution of the equation $R(\partial v) = 0$ of order n-1, where

$$R = \partial^{n-1}+c_1(z)\partial^{n-2}+\ldots+c_{n-1}(z)$$

and the coefficients are of the form $c_i(z) = b_i(z) + \sum_{k<i} c_{ik}(z)b_k(z)$ and $c_{ik}(z)$ is holomorphic at 0 (in this formula $b_0(z) = 1$ and $c_n(z) = 0$).

By hypothesis, all solutions $w = uv$ of $Q(uv) = 0$ have moderate growth, so also all solutions ∂v of $R(\partial v) = 0$ has moderate growth (note that any multivalued function can be written as ∂v and v has moderate growth iff ∂v has moderate growth). By induction hypothesis, this implies that the coefficients $c_i(z)$ are holomorphic, so also the coefficients $b_i(z)$.

1.4 *Computation of the monodromy of a regular system*

1.4.1 PROPOSITION : *Let* $\frac{dU}{dz} = \frac{B(z)}{z}U$ *be a regular system, where* B(z) *is a matrix with holomorphic entries. Assume that the difference of two eigenvalues of* B(0) *is never a non zero integer. Then the monodromy matrix is conjugate to* $\exp(2i\pi B(0))$; *equivalently the given system is equivalent to the system* $\frac{dV}{dz} = \frac{B(0)}{z} V$.

1.4.2 LEMMA : *Let* $\frac{dU}{dz} = A(z)U$ *be a regular system. Then any formal solution* $\sum_{k\ge k_0} z^k U_k$ *of the system* $\frac{dU}{dz} = A(z)U + C(z)$, *where* C(z) *is a given column vector whose components are meromorphic, is convergent.*

Proof : Using 1.3.1 one can assume that $A(z) = z^{-1}\Gamma$, where Γ is a constant matrix. Let $U(z) = \sum_{k\geq k_0} z^k U_k$ be a formal solution. If $C(z) = \sum z^k C_k$ then

$$kU_k = \Gamma U_k + C_{k-1}$$

For k large enough, the matrix $(k-\Gamma)$ is invertible and the norm $\|(k-\Gamma)^{-1}\|$ tends to zero when k tends to infinity. Hence, for large k,

$$\|U_k\| = \|(k-\Gamma)^{-1}B_{k-1}\| < \|C_{k-1}\|$$

so the series $\sum z^k U_k$ converges.

1.4.3 *Remark :* In [4], Malgrange proves that conversely, if any formal solution of $\frac{dU}{dz} = A(z)U + C(z)$ for any meromorphic C(z) is convergent, then the system $\frac{dU}{dz} = A(z)U$ is regular. This gives a fourth characterization of a regular system to add to the characterizations of 1.3.1.

Proof of 1.4.1 : We first observe that the given system is formally equivalent to the system $\frac{dV}{dz} = \frac{B(0)}{z} V$. In other words (cf. 1.3), there is a formal series $M(z) = \sum_{k\geq 0} z^k M_k$, with M_0 the identity matrix, such that

$$*) \qquad \frac{dM}{dz} = \frac{B(0)}{z} M - M \frac{B(z)}{z} .$$

If $B(z) = \sum z^k B_k$, then *) is equivalent to the system of equations

$$(kI-B_0)M_k + M_k B_0 = \sum_{v=0}^{k-1} M_i B_{k-i} .$$

By induction on k, we assume that the M_i's for $i<k$ are already determined. Then the above equation has a unique solution because $kI-B_0$ and B_0 have no common eigenvalue as a consequence of the following fact.

1.4.4 SUBLEMMA : *Let* P *and* Q *be* (n,n) *-matrices with complex coefficients. Then the matrix equation* PX-XQ = Y *has a unique solution* X *for any* Y *iff* P *and* Q *have no* eigenvalue in common.

To finish the proof of 1.4.1, we can consider *) as a system of n^2 linear meromorphic differential equation of first order whose unknowns are the entries of M. This system is regular, hence by 1.4.2, every formal solution is convergent.

Proof of the sublemma (cf.[7]). It is sufficient to check that any solution X of the matrix equation $PX-XQ = 0$ is zero. For any column vector $v \neq 0$ and $\lambda \in \mathbb{C}$, this implies $(P-\lambda I)^k Xv = X(Q-\lambda I)^k v$. So if $(Q-\lambda I)^k v = 0$, then λ is an eigenvalue of Q and $Xv = 0$, otherwise λ would also be an eigenvalue of P. One can find a basis $v^1,\dots,v^n$, scalars $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and integers $k_1,\dots,k_n$ such that $(Q-\lambda_i I)^{k_i} v^i=0$.

Hence $Xv^i = 0$ for all i, so $X = 0$.

1.4.5 *The general case.* When B(0) does not satisfy the conditions of proposition 1.4.1, one can make a finite number of equivalences (called shearing transformations) which modify the eigenvalues of B(0) by integers, and so one can reduce the general case to the case of 1.4.1 (cf. [4] p. 161).

After conjugating B(z) by a matrix with constant coefficients we can assume that B(0) is of the form

$$B(0) = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$$

where C and D are square matrices of order p and q respectively, C having only one eignevalue λ_1 distinct from the eignevalues $\lambda_2,\dots,\lambda_k$ of D.

If the equivalence is given by the matrix

$$M(z) = \begin{pmatrix} z^{-1}I_p & 0 \\ 0 & I_q \end{pmatrix}$$

then the matrix $\frac{\widetilde{B}(z)}{z}$ of the new system is of the form

$$\tilde{B}(z) = \begin{pmatrix} C-I_p & z^{-1}F \\ zG & D \end{pmatrix}$$

so $\tilde{B}(0) = \begin{pmatrix} C-I_p & F_1 \\ 0 & D \end{pmatrix}$ has eigenvalues $\lambda_1-1, \lambda_2,\dots,\lambda_k$.

This shows that when two eigenvalues of B(0) differs by a large integer, then one needs to know high order terms in the developpment of B(z) to determine the monodromy.

For instance, for the system

$$z\frac{du_1}{dz} = z^n u_2, \quad z\frac{du_2}{dz} = -nu_2$$

the eigenvalues of B(0) are 0 and $-n$. This system is equivalent to

$$z\frac{dv_1}{dz} = v_2, \quad z\frac{dv_2}{dz} = 0$$

(put $v_1 = u_1$ and $v_2 = z^n u_2$).

By 1.4.1, the monodromy of this system is not the identity and to determine it, we had to know the developpment of B(z) up to order n.

§ 2. REFORMULATION IN TERMS OF MEROMORPHIC CONNECTIONS (OR $\mathcal{D}$-MODULES)

2.1 *Meromorphic connections and systems*

A germ at $0 \in \mathbb{C}$ of a meromorphic connection (we shall say simply "meromorphic connection") is a finite dimensional vector space M over $K = \mathcal{O}[1/z]$ with an operator $\nabla : M \to M$ which is $\mathbb{C}$-linear and such that

$$\nabla(hu) = \frac{dh}{dz}u + h\nabla u$$

for any $h \in K$ and $u \in M$.

A meromorphic connection M is naturally a $\mathcal{D}[1/z]$-module (so in particular a $\mathcal{D}$-module), the action of a differential operator $P = \sum_i a_i(z)\left(\frac{d}{dz}\right)^{n-i}$ with meromorphic coefficients on $u \in M$ being defined by $Pu = \sum_i a_i(z)\nabla^{n-i}u$. Conversely any $\mathcal{D}$-module which is a finite dimensional vector space over K is a meromorphic connection.

If $e_1,\dots,e_n$ is a basis of M over K, and if

$$\nabla e_i = -\sum_j a_{ji}(z)e_j \qquad A = (a_{ij}(z)) \in \mathrm{End}(n,K),$$

for $u = \sum u_i(z)e_i$, then

$$\nabla u = \sum_i \left(\frac{du_i}{dz} - \sum_j a_{ij}(z)u_j(z)\right)e_i.$$

Hence the equation $\nabla u = 0$ is equivalent to the system of n homogenous differential equation of order 1 in n unknowns

$$\frac{dU}{dz} = AU.$$

Obviously this gives a 1-1 correspondance between equivalence classes of systems of n equations and isomorphisms classes of meromorphic connections of dimension n over K.

2.2. *Constructions of meromorphic connections*

Given a meromorphic connection (M,∇), its dual (M^*,∇^*) is obtained by taking the dual $M^* = \mathrm{Hom}_K(M,K)$ and defining ∇^* by $\langle\nabla^*\varphi,u\rangle = -\langle\varphi,\nabla u\rangle$.

If (M_1,∇_1) and (M_2,∇_2) are two meromorphic connections, we can define in an obvious way their direct sum $(M_1\oplus M_2,\nabla_1\oplus\nabla_2)$, as well as their tensor product $(M_1\otimes M_2),\nabla_1\otimes 1 + 1\otimes\nabla_2)$, and $(\mathrm{Hom}_K(M_1,M_2),\nabla)$, where $\langle\nabla\varphi,u\rangle = \nabla_2\langle\varphi,u\rangle - \langle\varphi,\nabla_1 u\rangle$.

A morphism of (M_1,∇_1) in (M_2,∇_2) is a K-linear map $\varphi : M_1 \to M_2$ such $\nabla_2\varphi = \varphi\nabla_1$.

The above constructions are often used in the classical theory of systems. For instance the system associated to the dual M^* of M is the adjoint of the system associated to M (with respect to dual basis).

If M_α is the meromorphic connection of dimension one with a basis e such that $z\nabla e = -\alpha e$, then the system associated to $M\otimes M_\alpha$ is the one obtained from the system $\frac{dU}{dz} = AU$ associated to M by posing $V = z^\alpha U$ and writing the system of equations satisfied by V. In the proof of 1.4.1, one considers the system associated to $\mathrm{Hom}_K(M,M)$.

2.3 *Construction of a cyclic element*

Let (M,∇) be a meromorphic connection of dimension n over K. A cyclic element v of M is an element such that $v, \nabla v,\dots,\nabla^{n-1}v$ generate M as a vector space over K.

2.3.1 PROPOSITION : *For any connection M, there is a cyclic element.*

The following construction has been communicated orally to B. Malgrange by N. Katz.

Choose a basis $e_1,\dots,e_n$ *of* M and $\varepsilon>0$ small enough so that $\nabla e_i = -\sum_j a_{ij}(z)e_j$, where $a_{ij}(z)$ is meromorphic in D_ε with possibly a pole only at 0.

Let $a\in D-\{0\}$. Define

$$\bar{e}_i = \sum_{k=1}^{n-i} \frac{(a-z)^k}{k!}\nabla^k e_i$$

and note that $\nabla^k \bar{e}_i$ is a linear combination of the basis elements $e_1,\dots,e_n$ with coefficients meromorphic in D_ε and vanishing at a for $k\le n-i$.

Let $v = \bar{e}_1 + (z-a)\bar{e}_2+\dots+\frac{(z-a)^{n-1}}{(n-1)!}\bar{e}_n$.

Then $v, \nabla v,\dots,\nabla^{n-1}v$, evaluated at a, are linearly independant. More precisely $v\wedge\nabla v\wedge\dots\wedge\nabla^{n-1}v = h(z)e_1\wedge\dots\wedge e_n$, where h(z) is a meromorphic function on D_ε with $h(a) = 1$. Hence the germ of h(z) at 0 is non zero so that $v, \nabla v,\dots,\nabla^{n-1}v$ are linearly independent over K, hence v is cyclic.

2.3.2 COROLLARY : *Any meromorphic connection* M *of dimension* n *over* K *is associated to a differential equation* Pu = 0 *of order* n *and meromorphic coefficients. Equivalently, any system of n meromorphic first order linear differential equations in n unknowns is equivalent to a system associated to an* n^{th} *order linear differential equation.*

Proof. Let ν be a cyclic vector of the dual M^* of M. Then

$$\nabla^{*n}\nu = -\sum_1^n b_i\nabla^{*n-i}\nu$$

so the matrix of ∇^* with respect to the basis $\nu,\nabla\nu,\dots\nabla^{n-1}\nu$ is of the form

$$\begin{pmatrix} 0 & 0 & \dots\dots\dots & -b_n \\ 1 & 0 & \dots\dots\dots & -b_{n-1} \\ 0 & 1 & \dots\dots\dots & -b_{n-2} \\ \dots & \dots\dots & & \\ 0 & 0 & \dots\dots\ 1 & -b_1 \end{pmatrix}$$

The matrix of ∇ for the dual basis is

$$\begin{pmatrix} 0 & -1 & \cdots\cdots\cdots & \\ & 0 & -1 \cdots\cdots\cdots & \\ & & & \\ b_n & b_{n-1} & \cdots\cdots\cdots & b_1 \end{pmatrix}$$

The associated system (cf.2.1) is

$$\frac{du_1}{dz} = u_2, \ldots \frac{du_{n-1}}{dz} = u_n$$

$$\frac{du_n}{dz} = -b_n u_1 - \ldots - b_1 u_n$$

This is the system corresponding to the differential equation

$$Pu = \left(\frac{d}{dz}\right)^n u + b_1\left(\frac{d}{dz}\right)^{n-1} u + \ldots + b_n u = 0.$$

2.4 *Regularity :* The condition i) of regularity in 13.1 can be rephrased as follows.

2.4.1 DEFINITION : *A meromorphic connection* M *is regular if there is a basis* $e_1,\ldots,e_n$ *of* M *over* K *such that*

$$z\nabla e_i = -\sum_j b_{ij}(z)e_j \quad \textit{where} \quad b_{ij}(z) \in \mathcal{O}.$$

By theorem 1.3.1, this is equivalent to the existence of a basis for which the $b_{ij}(z)$ are constant. We can then find another basis for which the matrix $B = (b_{ij})$ with constant coefficients is under Jordan form with eigenvalues having real part in [-1,0[. So theorem 1.3.1 has the following consequence.

2.4.2 COROLLARY : *Any meromorphic connection* M *is isomorphic to a direct sum of meromorphic connections of the form* $M_{\alpha,\ell}$ *, where* $M_{\alpha,\ell}$ *is the meromorphic connection with a basis* $e_1,\ldots,e_\ell$ *such that*

$$z\nabla e_i = \alpha e_i + e_{i+1} \quad \text{for} \quad i<\ell \quad \text{and} \quad z\nabla e_\ell = \alpha e_\ell$$

$M_{\alpha,\ell}$ *can be realized as the vector subspace of* $\tilde{\mathcal{O}}$ *over* K *with basis*

$$e_{\ell j} = z^\alpha \frac{(\log z)^{j-1}}{(j-1)!}$$

2.4.3 COROLLARY : *Any regular connection* M *is finitely generated as a* $\mathcal{D}$*-module. In particular* M *is a holonomic* $\mathcal{D}$*-module (cf. Pham* [5]*).*

Indeed the basis above generates $M_{\alpha,\ell}$ as a $\mathcal{D}$-module.

In fact, Kashiwava has proved that this corollary is also true without the hypothesis of regularity.

It is clear that M is regular iff its dual M* is regular.

2.4.4 PROPOSITION : *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of meromorphic connections. Then* M *is regular iff* M' *and* M'' *are regular.*

Proof : Let $e_1,\ldots,e_n$ be a basis of M such that $e_1,\ldots,e_k$ is a basis of M'. The matrix A of the system

1) $$\frac{dU}{dz} = AU$$

corresponding to M with this basis is of the form

$$A = \begin{pmatrix} A' & B \\ 0 & A'' \end{pmatrix}$$

where A' and A" are the matrices of the system corresponding to M' and M" respectively. So 1) is equivalent to the two systems

$$1)' \qquad \frac{dU'}{dz} = A'U' + BU''$$

$$1)'' \qquad \frac{dU''}{dz} = A''U''$$

where U' (resp. U") is the column vector whose components are the first k components (resp. the last (n-k) components) of U.

If M is regular, then any solution of 1) has moderate growth by 1.3.1. So in particular any solution of 1)' with U" = 0, namely any solution of $\frac{dU'}{dz} = A'U'$ has moderate growth, as well as any solution of 1)". Hence M' and M" are regular by 1.3.1.

Conversely, if M' and M" are regular, we can choose the basis e_i such that the coefficients of A' and A" have poles of order at most one at 0. By 1.3.1, any solution U" of 1)" will have moderate growth, so also the coefficients of BU". We have to show that the solutions U' of 1)' have moderate growth. Hence any solution of 1) will have moderate growth, so M is regular by 1.3.1.

Let S(z) be a matrix whose columns form a fundamental system of multivalued solutions of the adjoint system $\frac{dS}{dz} = -A'S$ with $S(z_0)$ the identity matrix for some z_0 in D^*_ε. Then the multivalued solution U'(z) of the system $\frac{dU'}{dz} = A'U' + BU''$ taking the value $U'(z_0)$ in z_0 is given by

$$U'(z) = S^{-1}(z)\, U'(z_0) + \int_{z_0}^{z} S(w)B(w)U''(w)\,dw$$

The second member has moderate growth because S(z) has moderate growth as solution of a regular system.

For a more algebraic proof using 2.4.2, see Pham [5].

2.4.5 *Inverse image of a meromorphic connection.*

Let $h : (\mathbb{C},0) \to (\mathbb{C},0)$ be a germ at 0 of a non zero holomorphic map (with h(0) = 0) sending z on w = h(z). It induces a ring homomorphism $h^* : \mathcal{O}_w \to \mathcal{O}_z$ defined by $h^*f = f\circ h$, where $\mathcal{O}_z$ (resp. $\mathcal{O}_w$) is the ring of germs of holomorphic functions at the origin of $\mathbb{C}$ considered as the source (resp. the target) of h.

If M is a meromorphic connection at $0 \in \mathbb{C}$ (thought as the target of h), then h*M is the meromorphic connection on $\mathcal{O}_z \otimes_{\mathcal{O}_w} M$ with ∇ defined by

$$\nabla(f\otimes u) = \frac{df}{dz}\otimes u + f\,\frac{dh}{dz}\otimes\nabla u.$$

Note that the natural inclusion $\mathcal{O}_z \otimes_{\mathcal{O}_w} M \to \mathcal{O}_z[1/z] \otimes_{\mathcal{O}_w} M$ is an isomorphism so that h*M is a vector space over $K = \mathcal{O}_z[1/z]$.

If $e_1,\ldots,e_n$ is a basis of M, then $1\otimes e_1,\ldots,1\otimes e_n$ is a basis of h*M over K, and if $\nabla e_i = -\sum_j a_{ji}(w)e_j$, then

$$\nabla(1\otimes e_i) = -\frac{dh}{dz}\otimes \sum_j a_{ji}(w)e_j$$
$$= -\sum_j \frac{dh}{dz}\, a_{ji}(h(z))(1\otimes e_j).$$

The system associated to M is $\frac{dV}{dw} = A(z)V$ and the system associated to h*M is the system $\frac{dU}{dz} = \frac{dh}{dz}A(h(z))U$ obtained by the change of variables w = h(z).

PROPOSITION : *The meromorphic connection* M *is regular iff* h*M *is regular.*

Indeed the coefficients of the matrix A(w) have poles of order at most one iff the same is true for the coefficients of the matrix $\frac{dh}{dz}A(h(z))$.

2.4.6 *Direct image of a meromorphic connection.*

Let (N,∇) be a meromorphic connection at $0 \in \mathbb{C}$ (thought as the source of $h : (\mathbb{C},0) \to (\mathbb{C},0)$). The direct image h_*N is the meromorphic connection at $0 \in \mathbb{C}$ (thought of as the target of h) whose underlying vector space is N considered as a vector space over $\mathcal{O}_w[1/w]$ via h* (restriction of scalars) and with the operator ∇_* defined by

$$\nabla_* n = \frac{1}{h'}\nabla n,$$

where h' is the germ at 0 of the derivative $\frac{dh}{dz}$ of h.

For $g \in \mathcal{O}_w[1/w]$, we have

$$\nabla_*(g\cdot n) = \frac{1}{h'}\nabla(g\circ h\circ n) = \frac{1}{h'}\frac{dg}{dw}h'n + \frac{1}{h'}g\circ h\,\nabla n$$
$$= \frac{dg}{dw}n + g\nabla n.$$

PROPOSITION : *The meromorphic connection* N *is regular iff* h_*N *is regular.*

Proof. We first make a change of coordinates such that $w = h(z) = z^k$. Assume that N is regular. Then there is a basis $e_1,\dots,e_n$ of N such that

$$z\nabla e_i = -\sum b_{ji}e_j$$

with $b_{ji} \in \mathcal{O}_z$.

The nk vectors $e_i^r = z^r e_i$, where $r = 0,\dots,k-1$ and $i = 1,\dots,n$, form a basis of N over $\mathcal{O}_w\,[1/w]$ and it is easy to check that $w\nabla_W e_i^r$ is a linear combination of vectors of this basis with coefficients holomorphic in w.

Conversely, assume that h_*N is regular. Then h^*h_*N is regular by 2.4.5. The map $N \to h^*h_*N$ mapping n on $\sum_{i=0}^{k-1} z^{-i}\otimes z^i n$ is an injective morphism of meromorphic connections. By 2.4.4, N is also regular.

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IV. REGULAR CONNECTIONS, AFTER DELIGNE

by Bernard Malgrange

§1. INTRODUCTION

Let X be a complex analytic manifold ; in this paper, it will always be assumed connected and countable union of compacts. Denote by $\mathcal{O}_X$ (resp. Ω_X^p) the sheaf of holomorphic functions (resp. holomorphic forms of degree p) on X. Let F be a holomorphic vector bundle, e.g. a locally free sheaf of finite rank, on X.

We recall that a *connection* on F is a $\mathbb{C}_X$ linear map $\nabla : F \to \Omega_X^1 \otimes_{\mathcal{O}_X} F$ satisfying for $f \in F$, $\varphi \in \mathcal{O}_X$: $\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f$. A connection extends, for all $p \geq 0$ to a unique map $\nabla : \Omega_X^p \otimes F \to \Omega_X^{p+1} \otimes F$ (we omit the subscript $\mathcal{O}_X$) satisfying $\nabla(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla f$. In this paper, *all connections will be supposed integrable*, e.g. satisfying $\nabla^2 = 0$; it is equivalent to say that the curvature $\nabla^2 : F \to \Omega_X^2 \otimes F$ vanishes.

A morphism of connections $(F,\nabla) \to (G,\nabla)$ is of course a morphism of $\mathcal{O}_X$-modules commuting with ∇.

For a connection (F,∇), denote by F^∇ the sheaf of *horizontal sections* of F, e.g. of the $f \in F$ satisfying $\nabla f = 0$. The usual theorem on existence and uniqueness of local solutions of differential equations implies that F^∇ is locally isomorphic to $\mathbb{C}_X^m$ with $m = \mathrm{rank}(F)$.

THEOREM (1.1). *The functor $(F,\nabla) \mapsto F^\nabla$ is an equivalence between (integrable) connections on X and local systems on X (e.g. sheaves locally isomorphic to $\mathbb{C}_X^m$ for some m).*

Actually, a quasi-inverse is obtained in the following way : if V is a local system on X, take $F = \mathcal{O}_X \otimes_{\mathbb{C}_X} V$, and $\nabla(\varphi \otimes v) = d\varphi \otimes v$,

$\varphi \in \mathcal{O}_X$, $v \in V$.

Remark (1.2). It would not have changed anything if we had replaced "vector bundle" by "coherent $\mathcal{O}_X$-module" ; actually it is classical that a coherent $\mathcal{O}_X$-module F provided with a connection, integrable or not, is locally free (sketch of the proof : choose a minimal system of generators of F at a point $a \in X$; to prove that they have no relation, consider such a relation in which the order at a of some coefficient is minimal, and differentiate it).

Now, let a be a point of X , and $\pi(X,a)$ the fundamental group of X with base point a . There is a standard equivalence of categories "local systems on X" $\leftrightarrow$ "finite dimensional representations over $\mathbb{C}$ of $\pi(X,a)$" .

On the other hand, let X be a smooth connected projective algebraic variety over $\mathbb{C}$, let X^{an} be the underlying analytic manifold, and $i : X^{an} \to X$ the canonical map ; according to GAGA , the functor $F \mapsto F^{an} = \mathcal{O}_{X^{an}} \underset{i^{-1}(\mathcal{O}_X)}{\otimes} i^{-1}F$ is an equivalence between vector bundles on X and X^{an} .

We will show that, in this correspondence , the connections (resp. integrable connections) correspond to each other.

To prove this result, we need another definition of the connections : let p_1,p_2 be the projections $X^2 \to X$, and put $P^1 = \mathcal{O}_{X^2}/J^2$, where J is the ideal defining the diagonal. For F a vector bundle on X , put $P^1(F) = p_{1_*}(P^1 \underset{\mathcal{O}_X^2}{\otimes} p_2^* F)$. If G is another vector bundle, there is a well-known 1-1 correspondence "differential operators of degree ≤ 1 from F to G" $\leftrightarrow$ "$\mathcal{O}_X$-linear maps $P^1(F) \to G$" . In particular, a connection ∇ is a differential operator from F to $\Omega^1_X \otimes F$, as it follows from its expression in a local basis of F ; therefore, it corresponds to an $\mathcal{O}_X$-linear map $P^1(F) \xrightarrow{\tilde{\nabla}} \Omega^1 \otimes F$. Conversely, given such a map $\tilde{\nabla}$, one checks that it defines a connection if and only if it splits the obvious exact sequence $0 \to \Omega^1_X \otimes F \to P^1(F) \to F \to 0$, or in other words, if $\tilde{\nabla}|\Omega^1_X \otimes F = \text{identity}$.

Now, the same result holds in the analytic category, and one has $P^1(F^{an}) = P^1(F)^{an}$; therefore, by GAGA, one has

$$\mathcal{H}om_X(P^1(F),\Omega^1_X \otimes F) = \mathcal{H}om_{X^{an}}(P^1(F^{an}), \Omega^1_{X^{an}} \otimes F^{an}) \quad ;$$

In particular, this gives a 1-1 correspondence "connections on F" $\leftrightarrow$ "connections on F^{an}" , and one checks that the curvatures correspond to each other. This proves the result.

Finally, we obtain an equivalence "integrable connections on X" $\leftrightarrow$ "local systems on X^{an}" , e.g. an algebraic description of the category of finite-dimensional representations over $\mathbb{C}$ of $\pi(X^{an},a)$.

One of the aims of these notes is to obtain a similar result for X smooth algebraic, but not necessarily projective, or not even proper. This is attained by adding a certain condition of *regularity at infinity* ; this condition will first be explained in the analytic context ; the translation via GAGA in algebraic terms will be given in section 7.

2. Meromorphic extensions.

Let X be a complex analytic manifold and Y be a divisor, e.g. a closed analytic subset of X such that, $\forall y \in Y, \operatorname{codim}_y Y = 1$; denote by $\mathcal{O}_X[Y]$ the sheaf of meromorphic functions on X , with poles on Y ; locally, if φ is an equation of Y , one has $\mathcal{O}_X[Y] = \mathcal{O}_X[\varphi^{-1}]$; from this, and the coherence of $\mathcal{O}_X$, one deduces immediately that $\mathcal{O}_X[Y]$ is coherent.

DEFINITION (2.1). *If* F *is a vector bundle over* $X-Y$, *we call meromorphic extension of* F *an* $\mathcal{O}_X[Y]$*-coherent module* $\tilde{F}$ *provided with an isomorphism* $\tilde{F}|X-Y = F$.

We will use the following terminology, which, maybe , is not the best possible : we call such an $\tilde{F}$ a "*vector bundle over* $X-Y$, *meromorphic on* Y". On the other hand, if ∇ is a connection on $F = \tilde{F}|X-Y$, we say that ∇ is meromorphic on Y (with respect to the extension $\tilde{F}$) if ∇ extends to a connection on $\tilde{F}$; if this extension exists, it is unique because of the property (2.3) below; it will also be denoted by ∇ . Finally, we call simply *connection* on $X-Y$; *meromorphic* on Y a pair $(\tilde{F},\nabla)$, with ∇ meromorphic with respect to $\tilde{F}$.

Let F be a vector bundle on $X-Y$, and $\tilde{F}$ a meromorphic extension of F ; locally on X , there exists obviously a $\mathcal{O}_X$-coherent module G such that $\tilde{F} = \mathcal{O}_X[Y] \underset{\mathcal{O}_X}{\otimes} G$

DEFINITION (2.2). *We say that* $\widetilde{F}$ *is "effectively meromorphic", or simply "effective" if there exists an* $\mathcal{O}_X$*-coherent module* G *such that* $F = \mathcal{O}_X[Y] \otimes_{\mathcal{O}_X} G$.

It is likely that there exist vector bundles on $X-Y$, meromorphic over Y , which are not effective ; however, I do not know any example ; note also that, in algebraic geometry, the similar phenomenon cannot occur.

The following property will be constantly used.

(2.3). A coherent $\mathcal{O}_X[Y]$-module $\widetilde{F}$ with support on Y vanishes ; for, let locally φ be an equation of Y , and G a coherent $\mathcal{O}_X$-module such that $\widetilde{F} = \mathcal{O}_X[Y] \otimes_{\mathcal{O}_X} G$; by the Nullstellensatz, one has $\varphi^p G = 0$ for some p ; therefore $\widetilde{F} = 0$.

Another useful property is the following one :

(2.4). If $\widetilde{F}$ is a vector bundle on $X-Y$, meromorphic on Y , then $\widetilde{F}$ as $\mathcal{O}_X[Y]$ module is reflexive, in fact, let $\widetilde{F} \to F^{\vee\vee}$ be the canonical morphism of F into its bidual ; the kernel and the cokernel have support on Y , and therefore vanish.

Actually, it is easy to prove a more precise result : for every $y \in Y$, $\widetilde{F}_y$ is $\mathcal{O}_{X,y}[Y]$-projective, and even stably free ; we will not use these results.

Remark (2.5). We can more generally consider extensions of bundles with connection along any closed analytic subset Y , and not only divisors ; this situation reduces to the preceding one for the following reason : Suppose first that everywhere, one has $\operatorname{codim} Y \geq 2$. Then the fundamental groups of X and $X-Y$ coincide ; therefore by theorem (1.1) any connection on $X-Y$ extends in a unique way to X ; the general case reduces immediately to that one first:extend the connection to the points of Y where $\operatorname{codim} Y \geq 2$; then we are reduced to the case of a divisor.

3. Regularity.

As in section 2, let Y be a divisor of X and $(\widetilde{F},\nabla)$ a connection on $X-Y$, meromorphic on Y . Let Z be a (connected) analytic manifold, and u a morphism $Z \to X$; we say that Z , or u , cuts properly Y if $u^{-1}(Y)$ is a divisor of Z ; it is equivalent to say that $u(Z) \not\subset Y$. In that case, u^*F is a vector bundle on $Z-u^{-1}(Y)$ and $u^*\widetilde{F} = \mathcal{O}_Z \otimes_{u^{-1}(\mathcal{O}_X)} u^{-1}\widetilde{F}$ is a meromorphic extension of it.

To define $u^*\nabla$, let us choose local coordinates $(z_1,..,z_p)$ near $a \in Z$, and $(x_1,...,x_n)$ near $b = u(a)$; then u is defined by the equations $x_i = u_i(z_1,..,z_p)$; if $f \in \widetilde{F}_b$, and $\nabla f = \Sigma\, dx_i \otimes f_i$, $f_i \in \widetilde{F}_b$, we put ("chain rule") $(u^*\nabla)\, u^*f = \Sigma\, du_i \otimes u^*f_i$.

One verifies that the second number depends only on u^*f and defines a connection on $(u^*\widetilde{F})_a$ (e.g.,by reducing the question to the cases of a projection and a closed immersion) ; this is also independent of the choice of coordinates, and gives an integrable connection by standard calculations. Finally, if V (resp. W) is the local system corresponding to ∇ on $X-Y$ (resp. $u^*\nabla$ on $Z-u^{-1}(Y)$) , one has $W = u^{-1}V$.

The preceding construction is actually a special case of the "inverse image of $\mathcal{D}$-modules" (see the chapter of these notes on this subject).

Choose in particular for Z a standard disc $D = \{|t| < r\} \subset \mathbb{C}$; let u be a mapping $D \to X$ which cuts Y properly ; shrinking D if necessary, we can assume that $u^{-1}(Y) = \{0\}$ and that $u^*\widetilde{F}$ is free on $\mathcal{O}_D[t^{-1}]$ with basis, say $e_1,...,e_m$. The connection $u^*\nabla$ is given by its equations $(u^*\nabla)e_j = dt \otimes \sum_i m_{ij}e_i$, m_{ij} meromorphic at 0 with pole at 0 ; $M = (m_{ij})$ is called the *matrix* of the connection $u^*\nabla$, and $M\,dt$ the *form* of the connection.

If $f = \varphi_1 e_1 + .. + \varphi_m e_m$ is a section of $u^*\widetilde{F}$, φ_i meromorphic with poles at 0 , one has

$$\nabla f = dt \otimes \Sigma\, \psi_i\, e_i$$

with $\psi_i = \dfrac{d\varphi_i}{dt} + \Sigma\, m_{ij}\varphi_j$; therefore, to give a connection with pole at 0 in a small disk is equivalent to give a linear differential system with pole at 0 .

We will say that $u^*\nabla$ is *regular* if this system has a regular singularity at 0 , e.g. if we can choose $e_1,..,e_m$ (possibly in a smaller disk ; actually, this is not necessary) such that M has a simple pole.

DEFINITION (3.1). *We say that* (F,∇) *is regular, or has a regular singularity on* Y *, if, for any disk* D *and map* $u : D \to X$ *with*

$u^{-1}(Y) = \{0\}$, u^* *is regular.*

In dimension 1, this definition is equivalent to the usual one : if $D' \xrightarrow{v} D$ is a map of disks, with $v^{-1}(0) = 0$ and if ∇ is a regular connection on D in the usual sense, then $v^*\nabla$ is also regular ; for the form of $v^*\nabla$ is the inverse image of that of ∇ ; if the latter one has a simple pole, the first one will have the same property.

Let $(\tilde{F},\nabla)$, $(\tilde{G},\nabla)$ etc... be connections on $X-Y$, meromorphic on Y . A morphism $(\tilde{F},\nabla) \to (\tilde{G},\nabla)$ is a morphism of $\mathcal{O}_X[Y]$-modules commuting with ∇ .

On $\tilde{F} \underset{\mathcal{O}_X[Y]}{\otimes} \tilde{G}$ and $\mathcal{H}om_{\mathcal{O}_X[Y]}(\tilde{F},\tilde{G})$, we define a connection by

$$\nabla(f\otimes g) = \nabla f\otimes g + f\otimes \nabla g$$

and

$$\nabla(v(f)) = (\nabla v)(f) + v(\nabla f) \quad ;$$

therefore, the morphisms $(\tilde{F},\nabla) \to (\tilde{G},\nabla)$ are the horizontal sections $\Gamma(X,\mathcal{H}om(\tilde{F},\tilde{G})^\nabla)$, a fact which will be very useful.

PROPOSITION (3.2). i) *Suppose* $0 \to (\tilde{F},\nabla) \to (\tilde{G},\nabla) \to (\tilde{H},\nabla)\to 0$ *is exact. If* $(\tilde{F},\nabla)$ *and* $(\tilde{H},\nabla)$ *are regular,* $(\tilde{G},\nabla)$ *is regular and conversely.*

ii) *If* $(\tilde{F},\nabla)$ *and* $(\tilde{G},\nabla)$ *are regular,* $(\tilde{F}\otimes\tilde{G},\nabla)$ *and* $(\mathcal{H}om(\tilde{F},\tilde{G}),\nabla)$ *are regular.*

iii) *Let* Z *be an analytic manifold, and* u *a morphism* $Z \to X$ *cutting properly* Y . *If* $(\tilde{F},\nabla)$ *is regular,* $(u^*\tilde{F},u^*\nabla)$ *is regular.*

i) follows from the special case of dimension one . To prove ii) note first the following result; if (Z,u) cuts Y properly, one has

$$u^*(\tilde{F}\otimes\tilde{G}) = u^*\tilde{F}\otimes u^*\tilde{G} \qquad \text{(obvious)}$$

and

$$u^*\mathcal{H}om(\tilde{F},\tilde{G}) = \mathcal{H}om(u^*\tilde{F},u^*\tilde{G})$$

in fact, there is a natural morphism of the former to the latter; as they coincide outside of Y , they coincide everywhere by (2.3).

Using these formulae, we are reduced to prove ii) in the case of dimension 1 ; in that case, choose basis of $\tilde{F}$ and $\tilde{G}$ near 0 in

which the matrices of the connections have simple poles ; the matrices of $(\tilde{F}\otimes\tilde{G},\nabla)$ and $(\mathcal{H}om(\tilde{F},\tilde{G}),\nabla)$ in the corresponding basis have obviously simple poles.

iii) is trivial.

Remark (3.3). Let V and W the local systems corresponding to $(\tilde{F},\nabla)$ and $(\tilde{G},\nabla)$ on $X-Y$; then the local systems corresponding to $\tilde{F}\otimes\tilde{G}$ (resp. $\mathcal{H}om(\tilde{F},\tilde{G})$) is $V\underset{\mathbb{C}_X}{\otimes} W$ (resp. $\mathcal{H}om_{\mathbb{C}_X}(V,W)$) ; in particular, the dual V^* corresponds to the dual $F^* = \mathcal{H}om(F,\mathcal{O}_X[Y])$, where $\mathcal{O}_X[Y]$ is provided with the connection d ; as it is obviously regular, F^* is also regular.

4. Divisors with normal crossings.

In this section, Y denotes a divisor of X with normal crossings.

Let G be a vector bundle on X (and not only on $X-Y$); consider a connection on $G|X-Y$, meromorphic on Y with respect to $\mathcal{O}_X[Y] \underset{\mathcal{O}_X}{\otimes} G$; locally near $a \in Y$, choose coordinates $x_1,\dots,x_n$, such that Y has for equation $x_1 \dots x_p = 0$ $(1 \leq p \leq n)$, and a basis $g_1,\dots,g_m$ of G . The *form* $\omega = \Sigma N^k dx_k$, $N^k = (n^k_{ij})$ of this connection with respect to the x's and e's is defined by the formula

$$\nabla g_j = \sum_{i,k} n^k_{ij} \; dx_k \otimes g_i$$

DEFINITION (4.1). *We say that* ∇ *has a logarithmic pole near* a *with respect to* G *if* N^k *is holomorphic for* $k \geq p+1$ *, and* $N^k = \frac{M_k}{x_k}$ *with* M_k *holomorphic for* $k \leq p$.

This definition does not depend on the coordinates ; it does not depend also on $(g_1,\dots,g_n)$; for, if we have another basis $(g'_1,\dots,g'_n)$ of G , one has $g'_j = \Sigma S_{ij}\, g_i$, with $S = (s_{ij})$ holomorphic invertible, and the new form of the connection ω' is given by $\omega' = S^{-1}\omega S + S^{-1}dS$; therefore, ω' verifies also the conditions (4.1). But, of course, this definition depends on G , and not only on $\mathcal{O}_X[Y] \otimes_{\mathcal{O}_X} G$.

Let $Y_1,\dots,Y_p$ be given by $x_1 = 0,\dots,x_p = 0$ respectively ; we define the *residue* of ∇ on Y_k in the basis $\{g_i\}$ as the matrix $M_k|Y_k$; this is independant of the choice of coordinates; for, if $x'_1,\dots,x'_n$ is another system of coordinates such that Y is defined by

$x'_1 \dots x'_k = 0$, we can reorder them so that $Y_k = \{x'_k = 0\}$; then one has $x'_k = x_k \varphi_k$, with φ_k invertible and
$\frac{dx'_k}{x'_k} = \frac{dx_k}{x_k}$ + (a holomorphic form) ; then the result follows.

If we change the basis, we see that the residue on Y_k is invariantly defined as a section of $\mathrm{End}\, G|Y_k$. The integrability condition is $d\omega + \omega\wedge\omega = 0$, or

$$\frac{\partial N_i}{\partial x_j} - \frac{\partial N_j}{\partial x_j} = [N_i, N_j] \quad ;$$

on $Y_i \cap Y_j$, we develop this relation in Laurent series in x_i and x_j ; if we consider the coefficient of $\frac{1}{x_i x_j}$, we get the following result

(4.2) On $Y_i \cap Y_j$, one has $[\mathrm{Res}_{Y_i} \nabla , \mathrm{Res}_{Y_j} \nabla] = 0$

Now, to fix the notations, suppose $i = 1$; on $Y_1 - \bigcup_{j\neq 1} Y_j$ we have

$$\omega = \frac{M_1}{x_1} dx_1 + \sum_2^n N_i\, dx_i \ ;$$

put $\overline{M}_1 = M_1|Y_1$ (= $\mathrm{Res}_{Y_1} \nabla$) and $\overline{N}_j = N_j|Y_1$, $j \geq 2$. The integrability conditions give

$$(4.3.1) \quad \frac{\partial \overline{N}_j}{\partial x_k} - \frac{\partial \overline{N}_k}{\partial x_j} = [\overline{N}_j, \overline{N}_k] \quad , \ j,k \geq 2$$

$$(4.3.ii) \quad \frac{\partial \overline{M}_1}{\partial x_j} = [\overline{M}_1, \overline{N}_j] \quad , \ j \geq 2 \ .$$

The first equation says that the connection defined by the form $\sum_2^n \overline{N}_j\, dx_j$ on $G|Y_1$ is integrable, the second one that $\overline{M}_1$ is a horizontal section of the extension of this connection to *End* $G|Y$; therefore, if we suppose Y_1 connected, which we may assume by replacing the given neighborhood of a by a suitable one, we see that, at the different points of $Y_1 - \bigcup_{j\neq 1} Y_j$, the values of $\overline{M}_1 = \mathrm{Res}_{Y_1}$ are conjugate to each other. In particular, they have the same eigenvalues ; by continuity, this last property remains true on the whole of Y_1 .

Strictly speaking, the previous result is not necessary for the next statement ; but it makes it perhaps a little bit clearer.

Fix a section τ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ (f.i, the section whose image is $\{z|\ 0 \leq \mathrm{Re}\, z < 1\}$) . The main result of this section is the following

THEOREM (4.4). *Let* Y *be a divisor with normal crossings of* X , F *a vector bundle on* X-Y *and* ∇ *a connection on* F . *There exists a locally free extension* G *of* F *on* X , *unique up to unique isomorphism such that*

i) ∇ *has a logarithmic pole with respect to* G .

ii) *The eigenvalues of the residues of* ∇ *with respect to* G *belong to the image of* τ .

Because of the assertion on uniqueness, it is sufficient to prove the result locally ; therefore we can suppose that X is a polydisk $\{|x_i| < r_i\}$, and Y is defined by $x_1 \dots x_p = 0$, with $1 \leq p \leq n$ (the case $p=0$ is trivial).

i) Existence.

Fix $a \in X$-Y ; according to section 1 , (F,∇) is determined by its monodromy, i.e. by the representation of $\pi(X\text{-}Y,a)$ which it defines. Now, $\pi(X\text{-}Y,a)$ is isomorphic to $\mathbb{Z}^p$, and is generated by the loops from a around $Y_1,\dots,Y_p$; therefore to give a representation of $\pi(X\text{-}Y,a)$ is equivalent to give the corresponding matrices $C_1,\dots,C_p \in \mathrm{Gl}(m,\mathbb{C})$ (m = rank F) , with $[C_i,C_j] = 0$. Then, it is sufficient to find a (G,∇) satisfying i) and ii) and having $(C_1,\dots,C_p)$ as monodromy.

LEMMA (4.5). *There exist unique matrices* $\Gamma_1,\dots,\Gamma_p \in \mathrm{End}(\mathbb{C}^m)$ *such that*

i) $\exp(-2\pi i\, \Gamma_j) = C_j$

ii) *The eigenvalues of* Γ_j *belong to the image of* τ .

Moreover, the Γ_j*'s commute.*

The proof is left to the reader.

Now, to obtain the result, we take $G = \mathcal{O}_X^m$, with the connection form given in the canonical basis by $\omega = \sum_i^p \Gamma_j \frac{dx_j}{x_j}$.

ii) Uniqueness.

It is sufficient to prove the following : let G be as before ; let (G',∇') satisfy i) and ii), and suppose that, on X-Y , we have an isomorphism $(G,\nabla) \sim (G',\nabla')$; then u extends to an isomorphism on X , obviously unique.

Shrinking X if necessary, we can suppose that G' is free on X ; choose a basis of G' , and let ω' be the corresponding connection form. The isomorphism is given by a holomorphic invertible matrix S on X-Y which satisfies $\omega' = S^{-1}\omega S + S^{-1}ds$; we have to prove that S and S^{-1} are holomorphic on X .

Let us give for instance the proof for S ; to do that, let us rewrite the preceding equation in the form : $dS = S\omega' - \omega S$.
By Hartog's theorem, it is sufficient to prove that S is holomorphic on the regular part of Y , e.g. the union of the $Y_i - \bigcup_{j\neq i} Y_j$; take for instance a point $b \in Y_1 - \bigcup_{j\neq 1} Y_j$; near b , we have

$$\omega' = \frac{M'_1}{x_1}\, dx_1 + \sum_2^n N'_i\, dx_i ,$$

with M'_1 , N'_i holomorphic, and we have

$$x_1 \frac{\partial S}{\partial x_1} = SM'_1 - \Gamma_1 S$$

Therefore, taking norms, one has, near b on $X-Y = X-Y_1$:

$$|x_1|.\|\frac{\partial S}{\partial x_1}\| \le C\, \|S\|$$

now as in the case of dimension 1, an elementary estimate proves that S is meromorphic near b .

To prove that S is actually holomorphic, we develop it in Laurent series in x_1 : $S = \Sigma S_k x_1^k$, with S_k holomorphic on Y_1 near b ; let $p = \inf\{k \mid S_k \neq 0\}$, $\overline{M}'_1 = M'_1(0,x_2,..,x_n)$; we have

$$pS_p = S_p \overline{M}'_1 - \Gamma_1 S_p$$

or

$$(\Gamma_1 + p.I)S_p = S_p\, \overline{M}'_1 ;$$

now, Γ_1 and $\overline{M}_1$ have their eigenvalues contained in the image of τ ; therefore, the following statement implies that $p=0$, which yields the result.

LEMMA (4.6). *Let* $A,B,S \in \mathrm{End}(\mathbb{C}^m)$ *satisfying* $AS = SB$, $S \neq 0$; *then* A *and* B *have a common eigenvalue.*

See chapter on local meromorphic connections in dimension one, sublemma 1.4.4.

5. The Riemann-Hilbert correspondence.

Here X is a complex analytic manifold, and Y a divisor of X . The following theorems will be proved together.

THEOREM (5.1). *The functor* $(\widetilde{F},\nabla) \mapsto (\widetilde{F},\nabla)|X-Y$ *is an equivalence "connection on* X-Y *meromorphic and regular on* Y" $\leftrightarrow$ *"connections on* X-Y *".*

Because of the theorem (1.1), this result can also be stated as the following equivalence (the "Riemann-Hilbert correspondence") : "connections on X-Y , meromorphic and regular on Y" $\leftrightarrow$ "local systems on X-Y" .

THEOREM (5.2). *Let* $(\widetilde{F},\nabla)$ *be a connection on* Y *, meromorphic on* Y *. Let* U *be an open set of* Y_{reg} *, intersecting all the components of* Y*,* D *be a disk, and* v *an isomorphism of* $U \times D$ *on a neighborhood of* U *, inducing the identity on* U $(= U \times \{0\})$ *. Suppose that, for any* $y \in U$ *, the restriction of* $v^*\nabla$ *to* $y \times D$ *is regular. Then* ∇ *is regular.*

THEOREM (5.3). i) *If* $(\widetilde{F},\nabla)$ *is a connection on* X-Y *, meromorphic and regular on* Y *, then* F *is effectively meromorphic.*

ii) *Suppose we have a diagram*

$$\begin{array}{ccc} Y_1 & \hookrightarrow & X_1 \\ \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & X \end{array}$$

with π *proper,* $\pi^{-1}(Y) = Y_1$ *,* $\pi : X_1 - Y_1 \to X - Y$ *biholomorphic.*

Then, if $(\widetilde{F},\nabla)$ *is a connection on* X_1 *, regular and meromorphic on* Y_1 *,* $(\pi_*\widetilde{F},\pi_*\nabla)$ *has the same properties.*

We will first prove the following lemma, with contains the "essential surjectivity" part of (5.1).

LEMMA (5.4). *Let* (F,∇) *be a connection on* $X-Y$ *; there exists a meromorphic extension* $\widetilde{F}$ *of* F *such that*

i) $\widetilde{F}$ *is effective.*

ii) ∇ *is meromorphic and regular with respect to* $\widetilde{F}$ *.*

The proof uses the desingularization theorem of Hironaka, in its global analytic version : according to that theorem, one can find an analytic manifold X_1 and a proper mapping π such that $\pi^{-1}(Y) = Y_1$ is a divisor with normal crossings and $\pi : X_1-Y_1 \to X-Y$ is biholomorphic. Lift (F,∇) to X_1-Y_1 ; by (4.4) , there exists a locally free $\mathcal{O}_{X_1}$-module G such that ∇ has logarithmic poles with respect to G . According to Grauert's direct image theorem, π_*G is a $\mathcal{O}_X$-coherent extension of F ; if we take $\widetilde{F} = \mathcal{O}_X[Y]_{\mathcal{O}_X}\otimes \pi_*G$ then $\widetilde{F}$ is an effective meromorphic extension of F ; ∇ is obviously meromorphic with respect to $\widetilde{F}$; to end the proof, we have only to verify that ∇ is regular with respect to $\widetilde{F}$.

Let D be a disk, and u a morphism $D \to X$, with $u^{-1}(Y) = \{0\}$. We prove first the following assertion :

(5.5) *There exist a morphism* $v : D \to X_1$ *such that* $\pi\circ v = u$.

In fact, such a property of lifting of disks is true for any proper map $X_1 \to X$ ("valuative criterion of properness") ; in the special case we have here, it is especially simple ; consider the analytic space $D_1 = (D \underset{X}{\times} X_1)_{red}$; then the projection $\pi : D_1 \to D$ is proper, and $D_1 - \pi^{1}(0) \to D - \{0\}$ is biholomorphic ; call D_1' the component in D_1 of $D_1 - \pi^{-1}(0)$ and $\overline{D}_1'$ its normalization ; $\overline{D}_1'$ is a smooth curve, the projection $\overline{D}_1' \xrightarrow{\overline{\pi}} D$ is proper and $D_1' - \overline{\pi}(0) \to D - \{0\}$ is biholomorphic ; therefore $\overline{\pi}$ is biholomorphic, and (5.5) follows.

Put $\widetilde{G} = \mathcal{O}_{X_1}[Y_1]\otimes_{\mathcal{O}_{X_1}} G$; $v^*(\widetilde{G},\nabla)$ is regular, since its matrix has a simple pole when expressed in a basis whose elements belong to

v^*G ; on the other hand the natural morphism $\pi^*\pi_*G \to G$ is an isomorphism outside Y_1 ; therefore it induces an isomorphism $\pi^*\widetilde{F} \to \widetilde{G}$, cf. (2.3) ; then, we have $u^*(\widetilde{F},\nabla) = v^*(\widetilde{G},\nabla)$. This proves the regularity of $u^*(\widetilde{F},\nabla)$, and the lemma.

Remark (5.6). The preceding arguments also show that (5.3.i) $\Rightarrow$ (5.3.ii) (The main point is to prove the coherence of $\pi_*\widetilde{F}$, which follows from the fact that $\widetilde{F}$ is effective and Grauert's theorem).

The rest of the proof will be a consequence of the following lemma

LEMMA (5.7). *In the hypotheses of (5.2), all the horizontal sections* $f \in \Gamma(X-Y,\widetilde{F}^{\nabla})$ *are meromorphic, e.g. are sections of* $\widetilde{F}$ *on* X .

Take such an f ; from the theory of regular singularities in one variable, we know that, $\forall y \in U$, $v^*f|y\otimes D$ is meromorphic ; therefore, we are reduced to prove the following assertion :

(5.8) *Take* $f \in \Gamma(X-Y,\widetilde{F})$ *, and suppose that, for every* $y \in U$, $v^*f|y\times D$ *is meromorphic ; then* f *is meromorphic.*

We will reduce the proof to the scalar case, e.g. the case $\widetilde{F} = \mathcal{O}_X[Y]$, by using the following trick: by (2.4), $\widetilde{F}$ is reflexive ; therefore, to prove that f is meromorphic at $b \in Y$, it is sufficient to prove the following result : for any $g \in F_b^{\vee}$, $\langle f,g\rangle$ is meromorphic at b . The proof is done in three steps.

i) f is meromorphic on U.

Take $b \in U$, $g \in F_b^{\vee}$, and put $h = \langle f,g \rangle$; we can suppose that h is defined in a neighborhood V of b isomorphic to a poly-disk $\{|x_i| < r_i\}$, in which Y is defined by $x_1 = 0$; we have to prove that h is meromorphic; to see that, develop h in Laurent series $h = \sum_{-\infty}^{+\infty} h_p(x_2,\dots,x_n)x_1^p$.

By the hypothesis, for $(x_2,.,x_n)$ fixed, h is meromorphic in x_1 , e.g. $h_p = 0$ for $p << 0$; the set W_q of points of $V \cap Y$ where $h_p = 0$ for $p \leq q$ is closed, and the union of these closed sets as $q \to -\infty$ is $V \cap Y$; by Baire's theorem, one of these sets, say W_{q_o} , has an interior point ; by analytic continuation, one has $W_{q_o} = V \cap Y$;

This proves the result.

ii) f is meromorphic on Y_{reg}.

Call $U' \subset Y_{reg}$ the set of points b where f is meromorphic ; U' is obviously open, and contains a point of each component of Y by i) and the hypothesis. Therefore, it is sufficient to prove that it is closed. Choose $b \in \overline{U}'$, $g \in F_b^\vee$; by hypothesis, $< f,g >$ is meromorphic on an open set of Y adherent to b ; by analytic continuation, as in the end of the preceding proof, it is meromorphic at b .

iii) f is meromorphic on Y.

Take $b \in Y_{sing}$, and $g \in F_b^\vee$; $< f,g >$ is meromorphic near b except on a set of codimension 2, by ii). By Hartogs, it is meromorphic at b . This ends the proof of (5.8) and (5.7).

We can now prove the theorems.

PROOF OF (5.1). The essential surjectivity has already been proved; it remains to prove that the functor is fully faithful ; this is equivalent to say that any morphism $(\widetilde{F},\nabla) \to (\widetilde{G},\nabla)$ on $X-Y$ extends to a morphism on X (the uniqueness is obvious). This is also equivalent to say that any section $f \in \Gamma(X-Y, \mathcal{H}om(\widetilde{F},\widetilde{G})\nabla)$ extends to an element of $\Gamma(X, \mathcal{H}om(\widetilde{F},\widetilde{G})^\nabla)$, e.g. is meromorphic with respect to $\mathcal{H}om\,(\widetilde{F},\widetilde{G})$; this follows from (3.2.ii) and (5.7).

PROOF OF (5.2). AND (5.3). Take $(\widetilde{F},\nabla)$ satisfying the hypotheses of (5.2), and take $(\widetilde{G},\nabla)$ the regular meromorphic extension of $(\widetilde{F},\nabla)|X-Y$ given by (5.4) ; one has an isomorphism $(\widetilde{F},\nabla) \to (\widetilde{G},\nabla)$ on $X-Y$, e.g. a section $f \in \Gamma(X-Y, \mathcal{H}om\,(\widetilde{F},\widetilde{G})^\nabla)$; applying (5.7) to $\mathcal{H}om(\widetilde{F},\widetilde{G})$, we see that f is meromorphic ; therefore f extends to a morphism $(\widetilde{F},\nabla) \to (\widetilde{G},\nabla)$ on X ; the same argument applied to f^{-1} shows that f is an isomorphism $(\widetilde{F},\nabla) \to (\widetilde{G},\nabla)$; therefore $(\widetilde{F},\nabla)$ is regular; this proves(5.2).

Moreover F is effective, since G is effective; this proves (5.3.i) . As (5.3.i) $\Rightarrow$ (5.3.ii) has already been proved in (5.6), this ends the proofs.

Remark (5.9). As in the case of dimension one, there is another definition of regularity in terms of growth of horizontal sections near Y ; we shall only sketch this point.

Given a vector bundle F on $X-Y$, provided with a meromorphic extension $\widetilde{F}$, we refer to Deligne, loc.cit, def. II. 2.10 for the notion of "multivalued section of F with moderate growth along Y". Then, given a connection ∇ on F , meromorphic with respect to $\widetilde{F}$, the following properties are equivalent :

i) ∇ is regular.

ii) Any horizontal section of F has a moderate growth along Y .

The proof that ii) $\Rightarrow$ i) is a consequence of the special case of dimension one and of (5.2). To prove conversely that i) $\Rightarrow$ ii) , one considers first the case where Y has normal crossings ; in that case, by (5.1) and (4.4) there exists a vector bundle G over X such that $G \otimes_{\mathcal{O}_X} \mathcal{O}_X[Y] = \widetilde{F}$ and that ∇ has a logarithmic pole along Y with respect to G . Then the proof of ii) follows from an elementary estimate, as in dimension one (see Deligne, th. II.1.19). Finally, one reduces the general case to this one by desingularization.

Remark (5.10). Suppose Y is a divisor with normal crossings, and denote by $\theta_{X,Y}$ the set of vector fields on X leaving Y invariant. Let (E,∇) be a connection on $X-Y$, meromorphic on Y ; then the following properties are equivalent

i) E is regular.

ii) E is a union of coherent $\mathcal{O}_X$-modules, stable under $\theta_{X,Y}$.

PROOF. i) $\Rightarrow$ ii). Choose a section τ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$; by (5.1) and (4.4), there exists a unique locally free $\mathcal{O}_X$-subsheaf G_τ of E , with $E = G_\tau \otimes_{\mathcal{O}_X} \mathcal{O}_X[Y]$, such that ∇ has a logarithmic pole with respect to G_τ and the eigenvalues of the residues of ∇ belong to the image of τ . It is easy to verify that E is the union of the G_τ's ; since G_τ is stable by $\theta_{X,Y}$, this proves the result.
ii) $\Rightarrow$ i). By theorem (5.2), it suffices to prove the result in the one dimensional case ; in that case, this follows from the definition of

the regularity.

6. Comparison theorem.

As before, Y is a divisor of X ; we note $X-Y \xrightarrow{j} X \xleftarrow{i} Y$ the canonical injections. Consider a connection (F,∇) on $X-Y$, meromorphic and regular on Y ; its de Rham complex is the complex $(\Omega^{\cdot}_X \underset{\mathcal{O}_X}{\otimes} F,\nabla)$; we will denote it by DRF .

In $D_b(\mathbb{C}_X)$, one has a natural morphism (the "attachment map")

(6.1). $DRF \to Rj_* \, j^* \, DRF$.

THEOREM (6.2). (Grothendieck-Deligne). *The morphism* (6.1) *is an isomorphism.*

i) Before giving the proof, let us comment on the result. The complex $j^*DRF = DRj^*F$ is the de Rham complex of $F|X-Y$; by Poincaré lemma, its cohomology sheaf is $F^{\nabla}|X-Y$, which will be denoted by V, in degree 0, and 0 in degree $\neq 0$. Therefore, in $D_b(X-Y,\mathbb{C}_{X-Y})$, one has $j^*DRF = V$, and therefore $\mathbb{H}^*(X,Rj_*j^*DRF) = H^*(X-Y,V)$. Therefore the theorem implies that one has a natural isomorphism.

(6.3). $\mathbb{H}^*(X,DRF) \xrightarrow{\sim} H^*(X-Y,V)$.

Take in particular $F = \mathcal{O}_X[Y]$, $\nabla = d$, which is obviously regular; then we obtain the result of Grothendieck :

$$\mathbb{H}^*(X,DR\ \mathcal{O}_X[Y]) = H^*(X-Y,\mathbb{C}) .$$

Note also that the result has been already proved in degree 0 by (5.7), since $\mathbb{H}^{\circ}(X,DRF)=H^{\circ}(X,F^{\nabla})$.

ii) We give another way of calculating Rj_*j^*DRF which will be useful later ; the complex j^*DRF is the complex on $X-Y$ $(\Omega^{\cdot}_{X-Y} \underset{\mathcal{O}_{X-Y}}{\otimes} j^*F,\nabla)$; now, I claim that the $\Omega^p_{X-Y} \otimes j^*F$ are acyclic for j_* ; in fact, take a point $a \in Y$, and a Stein neighborhood U of a ; $U-Y$ is also Stein, and therefore, one has, for $q \neq 0$,

$$H^q(U-Y,\Omega^p_{X-Y} \otimes j^*F) = 0 ;$$

using this result for a fundamental system of Stein neighborhoods of a,

we find $R^q j_*(\Omega^p_{X-Y} \otimes j^*F)_a = 0$ for $q \neq 0$, which is the required result.

By a standard argument, one deduces from this the following result

Rj_*j^*DRF is isomorphic in $D_b(X,\mathbb{C}_X)$ to the complex

$$(j_*(\Omega^{\cdot}_{X-Y} \underset{\mathcal{O}_{X-Y}}{\otimes} j^*F),\nabla) = (\Omega^{\cdot}_X \underset{\mathcal{O}_X}{\otimes} j_*j^*F,\nabla) ,$$

which will be denoted by $DR(j_*j^*F)$; it is easily verified that the morphism $DRF \to DR(j_*j^*F)$ corresponding to (6.1) is the morphism deduced from the attachment map $F \to j_*j^*F$ by applying "DR" .

More explicitly, take $a \in Y$; F_a is the space of sections of F near a, e.g. of the sections of F on $X-Y$ near a which are meromorphic ; and $(j_*j^*F)_a$ is the space of sections of F on $X-Y$ near a with possibly essential singularity on Y. Therefore, the result can be thought of as a theorem of comparison between "moderate cohomology" and "cohomology without growth conditions".

iii) We will show that the proof can be reduced to the case where Y has normal crossings (here, we will not yet use ii) . According to Hironaka's desingularization theorem, one can find an analytic manifold X_1 and a proper mapping $\pi : X_1 \to X$ such that $\pi^{-1}(Y) = Y_1$ is a divisor with normal crossings, and $\pi : X_1-Y_1 \to X-Y$ is biholomorphic. Put $(F_1,\nabla) = \pi^*(F,\nabla)$; (F_1,∇) is regular by (3.2). We assume that the theorem is true for (F_1,∇) and will prove it for (F,∇) .

Denote by $X_1-Y_1 \xrightarrow{j_1} X_1 \xleftarrow{i_1} Y$ the canonical morphisms. We have an isomorphism in $D_b(X_1,\mathbb{C}_{X_1})$:

$$DRF_1 \xrightarrow{\sim} Rj_{1*} j_1^* \, DRF_1$$

applying $R\pi_*$, we have an isomorphism

$$R\pi_* \, DRF_1 \longrightarrow R\pi_* Rj_{1*} \, j_1^* \, DRF_1$$

if we identify $X-Y$ with X_1-Y_1, we have

$$R\pi_* \, Rj_{1*} = Rj_* \text{ , and } j_1^* \, DRF_1 = j^* \, DRF \ ;$$

therefore, the second number is equal to Rj_*j^*DRF .

To end the proof, we have now to identify $R\pi_*DRF_1$ with DRF .

Note first that the natural morphism $\pi^*(\Omega^p_X \otimes F) \to \Omega^p_{X_1} \otimes F_1$ is an isomorphism since both are $\mathcal{O}_{X_1}[Y_1]$-coherent and coincide on $X_1 - Y_1$; now, $R^q\pi_* \ \pi^*(\Omega^p_X \otimes F)$ is coherent by Grauert's theorem since $\Omega^p_X \otimes F$ is locally effective on X (in fact, it is globally effective by 5.3, but this is not necessary here) ; for $q \neq 0$, it vanishes outside of Y , and therefore vanishes. As in ii), we conclude from this that $R\pi_*DRF_1$ is isomorphic to the complex $(\pi_*\pi^*(\Omega^p_X \otimes F),\nabla)$; finally the natural morphism $\Omega^p_X \otimes F \to \pi_*\pi^*(\Omega^p_X \otimes F)$ is an isomorphism, since it is an isomorphism outside of Y . This gives the result (modulo a verification of commutativity of diagram which I will omit).

iv) Now, we prove the result when Y has normal crossings. We can suppose that X is the germ at 0 of $\mathbb{C}^n$, and Y is defined by $x_1 \dots x_p = 0$, with $1 \leq p \leq n$ (the case $p=0$ is trivial). Now, the (F,∇) are classified by their monodromy around Y , and we can use the normal forms obtained in the proof of (4.4). By recurrence , we can suppose (F,∇) simple, e.g. of rank 1 ; in that case, we take $F = \mathcal{O}_X[Y]$ with the connection form $\omega = \sum_1^p \lambda_i \frac{dx_i}{x_i}$, $\lambda_i \in \mathbb{C}$, $0 \leq \mathrm{Re}\lambda_i < 1$; if we write $\delta_i = \frac{\partial}{\partial x_i} + \frac{\lambda_i}{x_i}$, the de Rham complex DRF at 0 is the Koszul complex $K(\delta_1,..,\delta_p;\mathcal{O}_X[Y]_o)$; write $A = \mathcal{O}_X[Y]_o$, $B = (j_*j^*\mathcal{O}_X)_o$; according to ii) , we have to prove that the natural morphism $K(\delta_1,\dots,\delta_p;A) \to K(\delta_1,\dots,\delta_p;B)$ is an isomorphism on the cohomology. The first complex is also the double complex $K(\delta_2,\dots,\delta_n;\ A \xrightarrow{\delta_1} A)$, and we have the same property with A replaced by B .

To calculate the cohomology of $A \xrightarrow{\delta_1} A$, we develop f in Laurent series $f = \sum f_k(x_2,\dots,x_n)x_1^k$, with $f_k \in A'$ (A' the space similar to A with only the variables $(x_2,\dots,x_n)$) , and some suitable conditions of convergence.

One has $\delta_1 f = \sum(k+\lambda_1)f_k\, x_1^{k-1}$; therefore, we have two cases :

1st case. $\lambda_1 \neq 0$. Here, δ_1 is an isomorphism, and the cohomology vanishes. The same result holds for B ,and the result is proved.

2nd case. $\lambda_1 = 0$. The kernel and the cokernel are isomorphic to A' , and $A \longrightarrow A$ is quasi-isomorphic to $A' \longrightarrow A'$; the same result is true for B ; by recurrence , this proves the theorem.

Remark (6.4). As theorem (6.1) is local on X , we have in fact used "only" the local version of Hironaka's theorem. On the other hand, (5.1) reduces easily to the similar local statement on X . Finally, the only point where the global desingularization theorem is really needed is in the proof of (5.3).

7. Algebraic case.

(7.1) REGULARITY. Let X be a smooth algebraic variety of finite type over $\mathbb{C}$, and (F,∇) a vector bundle on X with an integrable connection. We denote by X^{an} , F^{an} , etc. the corresponding analytic objects. X(resp. X^{an}) will be provided with the Zariski (resp. the usual transcendental) topology. We will define the notion of *regularity* , or regularity at infinity, of (F,∇).

i) Suppose that X is Zariski open in some $\overline{X}$ smooth projective, with $Y = \overline{X}-X$ a divisor of X ; note j the inclusion $X \to \overline{X}$; then $(j_*F)^{an}$ is a vector bundle on $(\overline{X}-Y)^{an}$ meromorphic on Y^{an} ; we say that (F,∇) is regular if $((j_*F)^{an},\nabla)$ is regular.

This definition is independent of $\overline{X}$; for, suppose we have another projective smooth completion $X \xrightarrow{j_1} \overline{X}_1$; there exist another one $X \xrightarrow{j_2} \overline{X}_2$ and morphisms $\overline{X} \xleftarrow{k} \overline{X}_2 \xrightarrow{k_1} \overline{X}_1$ such that the following diagram is commutative

$$\begin{tikzcd}
 & \overline{X} & \\
X \arrow[ur, "j"] \arrow[rr, "j_2"] \arrow[dr, "j_1"'] & & \overline{X}_2 \arrow[ul, "k"'] \arrow[dl, "k_1"] \\
 & \overline{X}_1 &
\end{tikzcd}$$

Actually, we can take for $\overline{X}_2$ a suitable desingularization of the closure of X in $\overline{X} \times \overline{X}_1$. Then the result follows from (3.2.iii) and (5.3.ii).

ii) According the Hironaka's theorem, the preceding construction can be done if X is affine. Now, in the general case, we will say that (F,∇) is regular if, for any affine Zariski-open subspace U of X, $(F,\nabla)|U$ is regular. It is easily verified that this is compatible with the definition given in 1) ; also, (F,∇) is regular iff, for some affine open covering $\{U_i\}$, $(F,\nabla)|U_i$ is regular.

iii) There is also a definition of regularity which is similar to (3.1).

PROPOSITION (7.1.1). *Given an integrable connection* (F,∇) *over* X, *the following properties are equivalent*

a) (F,∇) *is regular.*

b) *For any morphism* $u : C \to X$ *of a smooth curve into* X, $u^*(F,\nabla)$ *is regular.*

b) $\Rightarrow$ a) We can suppose that X is affine ; we take a smooth projective compactification $\overline{X}$ such that $Y = \overline{X}-X$ is a divisor. Then the result follows from (5.2), since one can arrange that the maps $v|y \times D$ are restrictions of morphisms of curves.

a) $\Rightarrow$ b) is trivial.

(7.2) MAIN THEOREMS.

THEOREM (7.2.1). ("Riemann-Hilbert correspondence" in the algebraic case) *The functor* $(F,\nabla) \mapsto (F^{an},\nabla)$ *is an equivalence "regular connections on* X" $\leftrightarrow$ "*integrable connections on* X^{an}".

Via (1.1), this gives an algebraic description of the local systems on X^{an} for the usual topology ; therefore this answers the question posed at the end of section 1.

It is sufficient to prove the result locally for the Zariski topology. So, we can suppose that X is affine. By Hironaka's theorem, there exists a smooth projective completion $X \xrightarrow{j} \overline{X}$ such that $Y = \overline{X}-X$ is a divisor.

On $\overline{X}$, all $\mathcal{O}_{\overline{X}}[Y]$-coherent modules are effective, e.g. generated by a $\mathcal{O}_{\overline{X}}$-coherent submodule (this is a general property of quasi-coherent modules in algebraic geometry), and the category of $\mathcal{O}_{\overline{X}}[Y]$ coherent modules is obviously equivalent to the category of $\mathcal{O}_X$-coherent modules. Via GAGA, this implies the equivalence "vector bundles over X" $\leftrightarrow$ "vector bundles over X^{an}, effectively meromorphic on Y^{an}". In this correspondence, the connections correspond to each other by the argument given in section 1. Therefore, the result follows from (5.1) and (5.3.i).

The de Rham complex DRF of (F,∇) is defined as in the analytic case. In $D_b(\mathbb{C})$, there is a natural morphism $\mathbb{H}^*(X,DRF) \to \mathbb{H}^*(X^{an},DRF^{an})$ which can be e.g. defined as follows : take $\{U_i\}$ an affine Zariski-open covering of X ; for $I = (i_o,.,i_p)$ put $U_I = U_{i_o} \cap \ldots \cap U_{i_p}$; as $\Omega^q_X \otimes_{\mathcal{O}_X} F$ is acyclic for $\Gamma(U_I,.)$, $\mathbb{H}^*(X,DRF)$ is isomorphic in $D_b(\mathbb{C})$ to the double complex $(\Gamma(U_I,\Omega^{\cdot}_X \otimes_{\mathcal{O}_X} F), (\nabla,\delta))$, where δ is the Čech differential ; similarly, as U_I^{an} is Stein, $\mathbb{H}^*(X^{an},DRF^{an})$ is isomorphic to the analytic version of this double complex. Then the morphism is given by the natural map $\Gamma(U_I,\Omega^q_X \otimes F) \to \Gamma(U_I^{an},\Omega^q_{X^{an}} \otimes F^{an})$. One verifies that this morphism does not depend on the chosen covering.

THEOREM (7.2.2). *Suppose* (F,∇) *is regular. The morphism* $\mathbb{H}^*(X,DRF) \to \mathbb{H}^*(X^{an},DRF^{an})$ *is an isomorphism.*

It is sufficient to prove the theorem for the U_I's, because this means that the preceeding morphism of double complexes is a homology isomorphism for the first differential ∇. Therefore, we can suppose that X is affine. We take a smooth projective completion $X \xrightarrow{j} \overline{X}$ as before. Then we have

$$\begin{aligned}\mathbb{H}^*(X,DRF) &= \mathbb{H}^*(\overline{X},Rj_*DRF)\\ &= \mathbb{H}^*(\overline{X},DRj_*F) \quad \text{(as in 6, ii)}\\ &= \mathbb{H}^*(\overline{X}^{an},DR(j_*F)^{an}) \quad \text{(GAGA)}\\ &= \mathbb{H}^*(X^{an},DRF^{an}) \quad (6.3)\end{aligned}$$

This proves the result.

EXAMPLE. (Grothendieck). Take $(F,\nabla) = (\mathcal{O}_X,d)$; then, one has

$$\mathbb{H}^*(X,(\Omega_X,d)) = H^*(X^{an},\mathbb{C}) .$$

However, it is well known that (Ω_X, d) does not verify in general the Poincaré lemma. In fact, in general, with the Zariski topology on X, one has $H^*(X,\mathbb{C}) \neq H^*(X^{an},\mathbb{C})$.

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V. THE WEYL ALGEBRA

F. Ehlers

§ 1 BERNSTEIN'S INEQUALITY AND SOME CONSEQUENCES

Let K be some fixed field of characteristic zero (usually it will be $\mathbb{C}$ or $\mathbb{R}$). The Weyl algebra $A_n(K) = A_n$ is the algebra of differential operators $\Sigma_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$ with polynomial coefficients acting on $K[x_1,\ldots,x_n]$ by formal differentiation. ($x = (x_1,\ldots,x_n)$; $\partial = (\partial_1,\ldots,\partial_n)$; $\partial_j = \partial/\partial x_j$; $x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$; $\partial^\beta = \partial_1^{\beta_1}\cdot\ldots\cdot\partial_n^{\beta_n}$; $a_{\alpha\beta} \in K$; $a_{\alpha\beta} \neq 0$ for at most finitely many α,β.) One can define $A_n(K)$ as the K algebra generated by $x_1,\ldots,x_n,\partial_1,\ldots,\partial_n$ subject to the relations $[x_i,x_k] = [\partial_i,\partial_k] = 0$, $[\partial_i,x_k] = \delta_{ik}$, where $[\ ,\ ]$ denotes the commutator and δ_{ik} the Kronecker symbol. There is an automorphism $F : A_n \to A_n$ defined by $x_i \mapsto \partial_i$, $\partial_i \mapsto -x_i$ playing the role of a formal Fourier transformation.

1.1 PROPOSITION. *$A_n(K)$ is a simple algebra, i.e. any two sided ideal in A_n is $\{0\}$ or A_n.*

Proof. Let $I \subset A_n$ be a two sided ideal and $0 \neq P = \Sigma a_{\alpha\beta} x^\alpha \partial^\beta \in I$. If x_j occurs in P with the highest order $m > 0$, then $0 \neq [\partial_j,P] \in I$ and x_j occurs in $[\partial_j,P]$ with the highest order $m - 1$. (Use $[\partial_j,x_j^k] = kx_j^{k-1}$.) By applying $[\partial_j,\ ]$ m times to P one obtains a nonzero element in I without the variable x_j. Using all the $[\partial_j,\ ]$, $[x_j,\ ]$ gives $1 \in I$. Thus $I = A_n$.

1.2 PROPOSITION. *Let A be a simple ring of infinite length as a left A-module. Then every left A-module of finite length is cyclic i.e. it can be generated by one element.*

(*Proof.* [Bj], pp. 30,31.)

1.3 COROLLARY. *A left $A_n(K)$-module of finite length is cyclic.*

1.4 *Filtration of* $A_n(K)$. A filtration of an algebra R is an increasing sequence of linear subspaces

$$\{0\} = S_{-1} \subset S_0 \subset S_1 \subset S_2 \subset \ldots \subset R \quad \text{with}$$

$$\bigcup_j S_j = R \quad \text{and } S_i \cdot S_j \subset S_{i+j} \quad \text{for all } i,j .$$

S_o is a subalgebra of R . We shall use two filtrations of A_n :

1) $S_j = T_j = \{\Sigma a_{\alpha\beta} x^\alpha \partial^\beta \mid |\alpha| + |\beta| \leqq j\}$, $|\gamma| := \gamma_1 + \ldots + \gamma_n$,

2) $S_j = \Sigma_j = \{\Sigma a_{\alpha\beta} x^\alpha \partial^\beta \mid |\beta| \leqq j\}$.

The first filtration is called the Bernstein filtration. Its advantage is that the T_j are finite dimensional. The second filtration can be generalized to algebras of differential operators on varieties; the Σ_j are finite K[x]-modules. ($K[x] = \Sigma_0$.) Observe that $S_i \cdot S_j = S_{i+j}$ for $i,j \geq 0$ in both cases.

Let $F.$ be one of these filtrations. There is an associated graded algebra defined by

$$gr^F A_n = \bigoplus_{j=0}^{\infty} F_j/F_{j-1} , \; [a]\cdot[b] = [ab] \text{ for homogeneous } [a], [b] .$$

1.5 PROPOSITION. $gr^F A_n$ *is canonically isomorphic to the polynomial algebra* $K[\bar{x}_1,\ldots,\bar{x}_n,\bar{\partial}_1,\ldots,\bar{\partial}_n]$ *in 2n variables,* $\bar{x}_j, \bar{\partial}_j \in F_1/F_0$.

The proof uses $\deg[x_j,\partial_j] < \deg x_j + \deg \partial_j$ for T, and Σ.

From now on, all modules without explicit declaration as right modules are left modules.

1.6 *Filtrations of A_n-modules.* Let M be an A_n-module. A filtration of M is an increasing sequence of K-subspaces

$$\{0\} = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \ldots \subset M , \text{ such that}$$

1) $\bigcup_j \Gamma_j = M$,

2) $F_i \cdot \Gamma_j \subset \Gamma_{i+j}$ for all i,j ,

3) the F_0-modules Γ_j (by 2) are finite.

There is an associated graded module

$$gr^\Gamma M = \bigoplus_{j=0}^{\infty} \Gamma_j/\Gamma_{j-1} \quad \text{over } gr^F A_n$$

with the module structure defined by $[a]\cdot[u] = [au]$ for homogeneous $[a] \subset gr^F A_n$, $[u] \in gr^\Gamma M$.

1.7 DEFINITIONS . 1) Γ is called a good filtration of M if $gr^\Gamma M$ is finitely generated over gr A_n .
2) Let $v \in \Gamma_j - \Gamma_{j-1}$. Then $\sigma(v) = [v] \in \Gamma_j/\Gamma_{j-1}$ is called the symbol of v .

1.8 PROPOSITION. 1) *If* $gr^\Gamma M$ *is finitely generated over* gr A_n *, then M is finitely generated over* A_n .
2) Γ *is good, if and only if there exists some* j_0 *such that* $F_i \cdot \Gamma_j = \Gamma_{i+j}$ *for all* $i \geqq 0$, $j \geqq j_0$.
3) *If* Γ *and* Ω *are filtrations of M and* Γ *is good, then there exists a* j_1 *, such that* $\Gamma_j \subset \Omega_{j_1+j}$ *for all* j . *If* Γ *and* Ω *are both good, then there exists a* j_2 *, such that* $\Omega_{j-j_2} \subset \Gamma_j \subset \Omega_{j+j_2}$ *for all* j .

Proof. Put $\tilde{\Gamma}_k := \bigoplus_{j\leqq k} \Gamma_j/\Gamma_{j-1}$, $\tilde{F}_k := \bigoplus_{j\leqq k} F_j/F_{j-1}$.

2) ' $\Rightarrow$ ' : Γ being good means that $gr^{\Gamma}M$ is generated by Γ_{j_0} for some j_0 . Looking at homogeneous components one sees that $\tilde{F}_i \cdot \tilde{\Gamma}_i = \tilde{\Gamma}_{i+j}$ for $i \geqq 0$, $j \geqq j_0$. Thus $F_i \cdot \Gamma_j + \Gamma_{i+j-1} = \Gamma_{i+j}$ for these i,j . Now use induction on i :

$$F_{i-1} \cdot \Gamma_j = \Gamma_{i+j-1} \Rightarrow F_i \cdot \Gamma_j = F_i \cdot \Gamma_j + \Gamma_{i+j-1} = \Gamma_{i+j} .$$

' $\Leftarrow$ ' : Obviously the condition implies that $gr^{\Gamma}M$ is generated by Γ_{j_0} .

1) Follows from 2 .

3) If $v_1,\dots,v_r$ generate M , then $\Gamma_j := \Sigma_{l=1}^{r} F_j v_l$ defines a filtration of M with the property of 2 .

4) Let j_0 be such that $F_i \cdot \Gamma_j = \Gamma_{i+j}$ for $i \geqq 0$, $j \geqq j_0$. Since Γ_{j_0} if finite over the (commutative) noetherian ring F_0 and Ω is a filtration of M , $\Gamma_{j_0} \subset \Omega_{j_1}$ for some j_1 . Then

$$\Gamma_j \subset \Gamma_{j_0+j} = F_j \cdot \Gamma_{j_0} \subset F_j \cdot \Omega_{j_1} \subset \Omega_{j_1+j} \quad \text{for } j \geqq 0 .$$

1.9 PROPOSITION . *If M is finitely generated over A_n , then M is noetherian.*

Proof. Let $N \subset M$ be an A_n-submodule and $\Gamma.$ a good filtration of M . (use 1.8.3) Let $\Gamma'_. = \Gamma. \cap N$ be the induced filtration of N . Then $gr^{\Gamma'}N$ is a gr A_n-submodule of the finitely generated module $gr^{\Gamma}M$. Since gr A_n is noetherian as a polynomial ring, $gr^{\Gamma'}N$ is finitely generated, therefore N is finitely generated over A_n by 1.8.1 .

For the remainder of **1.** , $F.$ is the Bernstein filtration $T.$.

1.10 *Dimensions and multiplicities* . Let M be a finitely generated $A_n(K)$-module and Γ a good filtration of M . There is a polynomial in t :

$$\chi(M,\Gamma,t) = \frac{m}{d!} t^d + \text{monomials in t of smaller degree ,}$$

$$m, d \in \mathbb{N} , m > 0 ,$$

such that

$$\dim \Gamma_j = \sum_{i=0}^{j} \Gamma_i/\Gamma_{i-1} = \chi(M,\Gamma,j) \text{ for } j >> 0 .$$

This is the Hilbert polynomial for $gr^{\Gamma}M$. Since gr A_n is a polynomial ring of 2n variables, $d \leq 2n$.

If Γ,Ω are good filtrations of M , then by 1.8.4

$$\chi(M,\Omega,j-j_2) \leq \chi(M,\Gamma,j) \leq \chi(M,\Omega,j+j_2) .$$

As the leading term of χ is not affected by a shift of j , the numbers d = d(M), m = m(M) depend on M only. They are called the dimension and the multiplicity of M, resp.

Examples. 1) $M = A_n$: $\dim T_j = \binom{2n+j}{2n}$, d = 2n , m = 1 .

2) $M = K[x_1,\dots,x_n] = A_n/\Sigma_{j=1}^{n} A_n\partial_j$, $\Gamma.$ = filtration by degrees: $\dim \Gamma_j = \binom{n+j}{n}$, d = n , m = 1 .

3) $M = A_n/\Sigma_{j=1}^{n} A_n x_j \cong K[\partial_1,\dots,\partial_n]$: d = n , m = 1 as in 2. In fact this is the formal Fourier transform of K[x] .

1.11 PROPOSITION . *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A_n-modules. Let M be finitely generated and Γ a good filtration of M . Let $\Gamma'_. := \Gamma. \cap M'$ and $\Gamma''_. = \mathrm{Im}\,\Gamma.$ be the induced filtrations on M' and M", respectively. Then*

$$0 \to gr^{\Gamma'}M' \to gr^{\Gamma}M \to gr^{\Gamma''}M'' \to 0$$

is an exact sequence of gr A_n-modules., Γ' and Γ'' are good filtrations, and we have

(1) $\chi(M,\Gamma,t) = \chi(M'.\Gamma',t) + \chi(M'',\Gamma'',t)$;

(2) $d(M) = \max(d(M'), d(M''))$;

(3) *If* $d(M') = d(M'')$, *then* $m(M) = m(M') + m(M'')$.

The proof follows from the fact, that $gr^{\Gamma}M$ is Noetherian, and therefore Γ' and Γ'' are good filtrations.

1.12 THEOREM (Bernstein). *If* $M \neq \{0\}$ *is a finitely generated* $A_n(K)$*-module, then* $d(M) \geqq n$.

Proof (A. Joseph).Let $\Gamma.$ be a filtration of M with $\Gamma_0 \neq \{0\}$.

LEMMA. *The K-linear map*

$$T_i \to \mathrm{Hom}_K(\Gamma_i,\Gamma_{2i}) \ , \ a \mapsto (u \mapsto a\cdot u)$$

is injective .

Proof of the lemma. We proceed by induction on i . The case i = 0 is just the assumption on Γ . Let $0 \neq a \in T_i$. We have to show that $a\cdot\Gamma_i \neq \{0\}$.
Assume, on the contrary, $a\cdot\Gamma_i = \{0\}$, and that the lemma is proved for i-1 . Then a cannot just be constant, so there is some x_j or ∂_j occuring in the expression for a . In the first case, $0 \neq [a,\partial_j] \in T_{i-1}$ and $[a,\partial_j]\cdot\Gamma_{i-1} = \{0\}$, contradicting the i-1 case. In the second case, use x_j instead of ∂_j . This proves the lemma.

Now let Γ be a good filtration of M and χ be the corresponding Hilbert polynomial. From the lemma we obtain the estimate

$$\dim T_i = \frac{1}{(2n)!}i^{2n} + \text{terms of lower order}$$

$$\leqq \chi(i)\cdot\chi(2i) = \dim \mathrm{Hom}_K(\Gamma_i,\Gamma_{2i}) \quad \text{for } i >> 0 .$$

Thus, $\deg(\chi) \geqq n$.

1.13 DEFINITION. *A finitely generated* $A_n(K)$*-module M is holonomic if* $M \neq 0$ *and* $d(M) = n$ *or if* $M = (0)$.

1.14 COROLLARY. *Submodules and quotients of holonomic modules are holonomic.*

1.15 PROPOSITION. *Let M be an* A_n*-module and* $\Gamma.$ *a filtration of M with* $\dim_K \Gamma_j \leqq \frac{c}{n!}j^n + c_1(j+1)^{n-1}$. *Then M is holonomic and* $m(M) \leqq c$. *In particular, M is finitely generated.*

Proof. Let $N \subset M$ be any finitely generated submodule and $\Omega.$ a good filtration of N . By 1.8.4 there is a j_1 such that $\Omega_j \subset \Gamma_{j+j_1} \cap N \subset \Gamma_{j+j_1}$. From this and Bernstein's theorem we get $\alpha(N) = n$ and then $m(N) \leqq c$. If $N_1 \subset N_2 \subset M$, $N_1 \neq N_2$, are both finitely generated, then $m(N_2) - m(N_1) = m(N_2/N_1)$ by 1.11 and 1.12 again. The length of any strictly increasing chain of finitely generated submodules is therefore bounded by c , so M itself is finitely generated.

1.16 *Applications.*

1.16.1 If M is holonomic, then its length is bounded by m(M) .
(Thus, holonomic modules are cyclic (1.2))
Proof: 1.11 , 1.12 .

1.16.2 THEOREM. *Let* $p \in K[x_1,\dots,x_n]$ *and let* $M = K[x,p(x)^{-1}]$ *have the* A_n*-structure defined by formal differentiation of rational functions. Then M is holonomic.*

Proof. Put m = deg p and $\Gamma_j = \{q(x)p(x)^{-j} \mid \deg q \leqq (m+1)j\} \subset M.\Gamma.$ is a filtration of M . (Proof of $\bigcup\Gamma_j = M$: $q\cdot p^{-j} = q\cdot p^{1}\cdot p^{-(j+1)} \in \Gamma_{j+1}$ <=> $\deg q + ml \leqq (m+1)(j+1)$ <=> $\deg q \leqq j+1+mj$, so $q\cdot p^{-j} \in \Gamma_{j+1}$ for l = deg q .) Now use 1.15 .

1.16.3 THEOREM. (Generalization of 1.16.2). *Let* $p \subset K[x_1,\dots,x_n]$ *and* M *be a holonomic* A_n*-module. Then* $M[p^{-1}] = K[x,p^{-1}] \otimes_{K[x]} M$ *is holonomic. (The* A_n*-structure is defined by the product rule.)*

Proof. 1.16.3 is proved as 1.16.2 .

1.16.4 *Existence of Bernstein's polynomial. Let* $p \in K[x_1,\dots,x_n]$, $\deg p = m$, and let λ be transcendental over $K[x_1,\dots,x_n]$. Consider the Weyl algebra $A_n(K(\lambda))$ over the function field $K(\lambda)$ and the $A_n(K(\lambda))$-module $N = K(\lambda)[x_1,\dots,x_n,p(x)^{-1}]\cdot p^\lambda$. Here p^λ can thought of as a formal symbol, the A_n-action on N is defined by $\partial_j p^\lambda = \lambda p^{-1}\cdot p^\lambda$ and the product rule (and formal differentiation of rational functions). Let $M = A_n(K(\lambda))\cdot p^\lambda$ be the submodule of N generated by p^λ . As in 1.16.2, $\Gamma_j := \{q\cdot p^{-j}\cdot p^\lambda \mid \deg q \leq (m+1)j\}$ is a filtration of N with a holonomic estimate (1.15). Therefore, N and M are holonomic. Since M has a finite length, the descending sequence $M_0 \supset M_1 \supset \dots \supset M_j = A_n(K(\lambda))\cdot p^j\cdot p^\lambda$, terminates: $M_k = M_{k+1}$ for some k. This means $p^k\cdot p^\lambda = D_1(\lambda)\cdot p^{k+1}\cdot p^\lambda$, $D_1(\lambda) \in A_n(K(\lambda))$. Since λ is transcendental, we can replace λ by $\lambda-k$ and get $p^\lambda = D_1(\lambda-k)\cdot p^{\lambda+1}$. Let $B(\lambda)$ a common multiple of the denominators of the coefficients of $D_1(\lambda-k)$ and put $D(\lambda) = B(\lambda)D_1(\lambda-k)$. Then

$$D(\lambda)\cdot p^{\lambda+1} = B(\lambda)\cdot p^\lambda \quad , D(\lambda) \in A_n(K)[\lambda] \ , B(\lambda) \in K[\lambda] \ .$$

$B(\lambda)$ and $D(\lambda)$ are not uniquely defined by this functional equation. The $B(\lambda)$ form an ideal whose generator $b(\lambda)$ with leading coefficient 1 is called the Bernstein polynomial of $p(x)$.

Remark. The functional equation implies $N = M$.

1.16.5 THEOREM. *Let* $p \in K[x_1,\dots,x_n]$ *. Then*

$$M = A_n(K)[\lambda]\cdot p^\lambda / A_n(K)[\lambda]\cdot p\cdot p^\lambda$$

is a holonomic $A_n(K)$*-module. The miltiplication with* λ *has a minimal polynomial:* $b(\lambda)$.

Proof. The first statement follows from the second one and that is just

a reformulation of the results of 1.16.4 .

1.16.6 Let

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n,\mathbb{C}) \mid p_{\alpha\beta}(f) < \infty \text{ for all } \alpha,\beta\}$$

$$p_{\alpha\beta} = (1+|x|^2)^\alpha\cdot|\partial^\beta f| \ , \alpha \in \mathbb{N} \ , \beta \in \mathbb{N}^n \ ,$$

be the space of tempered functions with its topology defined by the seminorms $p_{\alpha\beta}$. The topological dual space $S'(\mathbb{R}^n)$ is the space of tempered distributions with its weak topology. Let $p(x) \in \mathbb{R}[x_1,\dots,x_n]$ have nonnegative values on $\mathbb{R}^n$ and let $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$. Define

$$T_\lambda(\phi) = \int_{\mathbb{R}^n} p(x)^\lambda \phi(x)dx \ , \ \phi \in S(\mathbb{R}^n) \ .$$

Then $T_\lambda \in S'$ and $\lambda \mapsto T_\lambda$ is holomorphic on $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$. From $D(\lambda)\cdot p^{\lambda+1} = b(\lambda)\cdot p^\lambda$ we obtain by partial integration

$$T_\lambda(\phi) = b(\lambda)^{-1}\int (D(\lambda)\cdot p^{\lambda+1})\phi dx$$

$$= b(\lambda)^{-1}\int p^{\lambda+1}\cdot D(\lambda)^*\phi dx$$

$$= b(\lambda)^{-1}b(\lambda+1)^{-1}\int p^{\lambda+2}\cdot D(\lambda+1)^*\cdot D(\lambda)^*\phi dx$$

$$= \dots$$

This shows that $\lambda \mapsto T_\lambda$ has a meromorphic extension to all of $\mathbb{C}$ with poles in $\{\lambda \mid b(\lambda+j) = 0 \text{ for some } j \in \mathbb{N}\}$.

1.16.7 *Distributional inversion of polynomials.* In 1.16.6, T_λ has a Laurent expansion at $\lambda = -1$:

$$T_\lambda = \sum_{j \geqq -N} S_j(\lambda+1)^j \ , \ S_j \in S' \ .$$

Now T_λ is regular at $\lambda = 0$ and $T_0 = 1$: $\int p^\lambda\phi dx \to \int \phi dx$ for $\lambda \in \mathbb{R}_{>0}$, $\lambda \to 0$. Therefore

$$T_{\lambda+1} = p\cdot T_\lambda = \sum_{j\geq -N} p\cdot S_j(\lambda+1)^j$$

is regular at $\lambda = -1$, thus $p\cdot S_j = 0$ for $j < 0$ and $p\cdot S_0 = 1$.

If $p(x) \in \mathbb{C}[x_1,\ldots,x_n]$ is arbitrary, then by defining S_0 with $|p(x)|^2$ instead of $p(x)$ we get $1 = |p|^2\cdot S_0 = p\cdot(\bar{p}\cdot S_0)$. So $S = \bar{p}\cdot S_0 \in S'$ satisfies

$$p\cdot S = 1 , \quad S \in S' .$$

1.16.8 *Fundamental solutions of PDEs with constant coefficients.* Write a differential operator D with constant coefficients in the form $D = p(\frac{1}{i}\partial)$, $p(y) \in \mathbb{C}[y_1,\ldots,y_n]$, and let δ be the Dirac distribution, $\delta(\phi) = \phi(0)$. We can use the $S \in S'(\mathbb{R}^n)$, $p\cdot S = 1$, from 1.16.7 to construct a fundamental solution for $Du = f$, i.e. an $E \in S'(\mathbb{R}^n)$ with $DE = \delta$:

Define the Fourier transformations on S and S' by

$$\phi(x) = \int e^{-i<x,y>}\tilde{\phi}(y)dy , \quad \phi \in S ,$$

$$\tilde{T}(\phi) = T(\tilde{\phi}) \qquad T \in S' ,$$

and let $E = S$. Using $p\cdot S = 1$ we obtain

$$(p(\tfrac{1}{i}\partial)\tilde{S})(\phi) = \tilde{S}(p(i\partial)\phi) = S(p(i\partial)\phi) = S(p(y)\tilde{\phi})$$

$$= \int\tilde{d}y = \phi(0) = \delta(\phi) .$$

§ 2 SOME HOMOLOGICAL ALGEBRA

2.1 Let M be a finitely generated (left) A_n-module and $\Gamma.$ a good filtration of M with respect to the Bernstein filtration $T.$ of A_n . The dimension d(M) of 1.10 can be defined in a more geometric way: If $I \subset \mathrm{gr}\,A_n$ is the annihilator ideal of $\mathrm{gr}^\Gamma M$ and $J(M) = \sqrt{I}$, then

$$d(M) = \mathrm{Krulldim}\ \mathrm{gr}\ A_n/I = \mathrm{Krdim}\ \mathrm{gr}\ A_n/J(M) .$$

This definition can also be used in the case of the usual filtration $\Sigma.$ of A_n . If $I' \subset \mathrm{gr}^\Sigma A_n$ is the annihilator ideal of $\mathrm{gr}^\Gamma M$, $\Gamma.$ being some Σ-good filtration of M , then $J'(M) = \sqrt{I'}$ (and J(M)) depends only on M (and the choice of the A_n-filtration). (See 2.2 below.)

2.1.1 DEFINITION. 1) The characteristic variety ch(M) of M is the variety defined by J'(M).
2) The (Σ-) dimension of M is defined by

$$\delta(M) = \mathrm{Krdim}\ \mathrm{gr}^\Sigma A_n/J'(M)$$

(If K is algebraically closed, $\delta(M)$ is the dimension of ch(M) .)

One of the aims of this paragraph is to prove the equality $\delta(M) = d(M)$. Then Bernstein's theorem 1.12 says dim ch(M) $\geq$ n in the case of $\bar{K} = K$. This is also an immediate consequence of the much more difficult theorem of the involutivity of ch(M) with respect to some natural symplectic structure on Spec $\mathrm{gr}^\Sigma A_n$. The method of proving the equality consists in giving a homological definition for both d(M) and $\delta(M)$ which makes no use of any filtrations on A_n .

2.2. For the remainder of this paragraph we shall work with a general unitary ring R instead of A_n equipped with a filtration ($\{0\} = \Sigma_{-1} \subset \Sigma_0 \subset \Sigma_1 \subset \ldots$, $\bigcup_i \Sigma_i = R$, $\Sigma_i\cdot\Sigma_j \subset \Sigma_{i+j}$ for $i,j \geq 0$; Σ_0 is a subring of R.) We assume for the associated ring $S = \mathrm{gr}^\Sigma R$ the following properties:

(i) S is commutative and noetherian.

(ii) S is regular of pure dimension n. (I.e the global homological dimension $\mathrm{gl\ hd}(S_{\mathfrak{m}}) = n$ for all localisation $S_{\mathfrak{m}}$ at maximal ideals $\mathfrak{m} \in S$.)

Because of (i), this is equivalent to

$$\dim_{S/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = n = \mathrm{Krdim}\, S_{\mathfrak{m}} \quad \text{for all } \mathfrak{m} .$$

This includes the case of A_n, with $\mathrm{gl\ hd}(\mathrm{gr}\, A_n) = 2n$ of course, but also algebras of differential operators on affine varieties. (See the subsequent chapters.) Another case is the ring of differential operators with germs of holomorphic functions as coefficients.

For left or right R-modules, the notions of filtrations and good filtrations are defined as in 1.6 and 1.7. The propositions 1.8 and 1.9 carry over to this more general situation.

Let M be a finitely generated (left) R-module and $\Gamma.$ a good filtration of M. Let $I = \mathrm{Ann}_S\, \mathrm{gr}^{\Gamma} M \subset S$ and $J(M) = \sqrt{I}$.

LEMMA. $J(M)$ *does not depend on the choice of* Γ.

Proof. I is obviously homogeneous and so is $J(M)$. Given a homogeneous element $\sigma(a) \in S$, $\sigma(a)$ being the symbol of some $a \in R$, $\sigma(a) \in J(M)$ means $a^k\Gamma_j \subset \Gamma_{j-1}$ for some k and all j. Since then $a^{pk}\Gamma_j \subset \Gamma_{j-p}$ for all p, $\sigma(a) \in J(M)$ is equivalent to $\forall r \exists k : a^k\Gamma_j \subset \Gamma_{j-r}$. Because of the compatibility of good filtrations (1.8.4) this is independent of Γ.

2.2.1 DEFINITION. 1) $d(M) := \mathrm{Krdim}\,(S/J(M))$,

2) $j(M) := \min\{j \mid \mathrm{Ext}^j_R(M,R) \neq 0\}$.

2.2.2 THEOREM. $d(M) + j(M) = n$.

The equality $d(M) = \delta(M)$ in 2.1 is an immediate consequence of this theorem, which is proved in the next section.

2.2.3 PROPOSITION. *Let* N *be a finitely generated* S*-module. Let* $d(N) := \mathrm{Krdim}\,(S/\mathrm{Ann}_S N)$ *and* $j(N) := \min\{j \mid \mathrm{Ext}^j_S(N,S) \neq 0\}$. *Then*

1) $\mathrm{Ext}^j(N,S) = 0$ *for* $j < n - d(N)$,

2) $d(\mathrm{Ext}^j(N,S)) \leq n - j$ *for all* j,

3) $d(\mathrm{Ext}^{j(N)}(N,S)) = n - j(N)$.

Proof. 1) and 2) are proposition 5.23, ch.2, in [Bj]. 3) is a consequence of 1) and 2) : Assume '<' in 3). Let $E = \oplus_{j=0}^{n} \mathrm{Ext}^j(N,S)$. The assumption, 1) and 2) imply $d(E) < d(N)$, so there exists an $a \in J(E) - J(N)$. Since N is finitely generated and S noetherian, $\mathrm{Ext}^{\cdot}_S(N,\cdot)$ commutes with localisations. Therefore $N_a \neq 0$ and $\mathrm{Ext}^{\cdot}_{S_a}(N_a,S_a) = 0$. Now let K be any finitely generated S_a-module. From the short exact sequence $0 \to L \to S_a^k \to k \to 0$ and its $\mathrm{Ext}^{\cdot}_{S_a}(N_a,\cdot)$-cohomology sequence we see $\mathrm{Ext}^j_{S_a}(N_a,K) \cong \mathrm{Ext}^{j+1}_{S_a}(N_a,L)$. The assumptions on S imply $\mathrm{Ext}^j_{S_a}(N_a,K) = 0$ for $j > n$. By recursion on j we get $\mathrm{Ext}^j_{S_a}(N_a,K) = 0$ for all j and all finitely generated K. (Observe, that L is finitely generated.) A special case of this is $\mathrm{Hom}_{S_a}(N_a,N_a) = 0$, thus $N_a = 0$. This proves 3).

This proposition gives $d(N) + j(N) = n$ for N a finitely generated S-module. If M is a finitely generated R-module equipped with some good filtration, then $d(M) = d(\mathrm{gr}\, M)$ just by definitions. So theorem 2.2.2 reduces to $j(M) = j(\mathrm{gr}\, M)$.

2.3. *The spectral sequence of a good filtration.*

2.3.1. Let M be a finitely generated left R-module. Given any good filtration $\Gamma.$ of M, there exists $u_1,\dots,u_r \in M$, such that

$$\Gamma_j = \Sigma_{j-m_1}u_1 + \dots + \Sigma_{j-m_r}u_r \quad \text{for all } j ;$$

$m_k = \deg \sigma(u_k)$. (Namely, choose homogeneous generators v_k of $\mathrm{gr}\, M$ and u_k such that $v_k = \sigma(u_k)$. Then '$\supset$' is obvious and the surjectivity of

$$\Sigma_{j-m_1}u_1 + \dots + \Sigma_{j-m_r}u_r \to \Gamma_j/\Gamma_{j-1}$$

for all j and l is proved by induction on l as in the proof of 1.8.2.)
Now let $\varepsilon_1,\dots,\varepsilon_r$ be formal generators of degrees m_k for the free R-module $F_0 = \oplus_{k=1}^{r} R\varepsilon_k$. There is a good filtration of F_0 :

$$\Gamma_j(F_0) = \Sigma_{j-m_1}\varepsilon_1 + \dots + \Sigma_{j-m_r}\varepsilon_r .$$

Let K_0 be the kernel of the canonical projection $F_0 \to M$, $\varepsilon_k \mapsto u_k$. Since gr R is noetherian, the induced filtration $\Gamma.(K) = K_0 \cap \Gamma(F_0)$ is good. (gr K_0 = ker(gr $F_0 \to$ gr M) is finitely generated.) Applying this procedure to K_0 instead of M, and so on, one obtains a filtered free resolution of M :

$$\dots \to F_2 \to F_1 \to F_0 \to M \to 0$$

with the properties

1) gr F_j is free over S of the same rank as F_j ,

2) $\dots \to$ gr $F_1 \to$ gr $F_0 \to$ gr $M \to 0$ is exact .

The dual module $N^\vee = \mathrm{Hom}_R(N,R)$ and, more generally, all the $\mathrm{Ext}^j_R(N,R)$ are right R-modules via the right R-module structure of R. If N has the good filtration $\Gamma.(N)$, then $\Gamma_j(N^\vee) := \{\phi \in N^\vee \mid \phi(\Gamma_k(N)) \subset \Sigma_{k+j}$ for all k$\}$ is a good filtration of $N^\vee$. (Proof: Let j_0 be such that $\Sigma_1\Gamma_{j_0}(N) = \Gamma_{j_0+k}(N)$ for all $l \geqq 0$. Then $\Gamma_j(N^\vee) = \{\phi \in N^\vee \mid \phi(\Gamma_{j_0}(N)) \subset \Sigma_{j_0+j}\}$. Therefore $\bigcup_j \Gamma_j(N^\vee) = N^\vee$, because $\Gamma_{j_0}(N)$ is finite over Σ_0, and Σ_0 is noetherian because gr R is. From the injectivity of the canonical gr R-homomorphism gr $N^\vee \to \mathrm{Hom}_{\mathrm{gr}\,R}(\mathrm{gr}\,N, \mathrm{gr}\,R)$ and the fact that gr R is noetherian we see that gr $N^\vee$ is finitely generated.)

We are now going to study the spectral sequence $E_r^\cdot$ of the filtered complex $F^\vee_\cdot$, the cohomology of which is $\mathrm{Ext}^\cdot_R(M,R)$. Since the $F_j^\vee$ are free and finitely generated, the natural maps

$$E_0^j = \mathrm{gr}\,(F_j^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{gr}\,R}(\mathrm{gr}\,F_j, \mathrm{gr}\,R)$$

are isomorphisms. So we get

$$E_1^j \cong \mathrm{Ext}^j_{\mathrm{gr}\,R}(\mathrm{gr}\,M, \mathrm{gr}\,R) ,$$

$$E_\infty^j \cong \mathrm{gr}\,\mathrm{Ext}^j_R(M,R) .$$

For $j < j(\mathrm{gr}\,M)$ we have, by definition of $j(\mathrm{gr}\,M)$, $E_1^j = 0$, and so $E_\infty^j = 0$, because E_∞^j is a subquotient of E_1^j. Thus we get $j(M) \geqq j(\mathrm{gr}\,M)$, and one half of the wanted equality is proved.

In order to prove the other inequality we have to show that $E_\infty^{j(\mathrm{gr}\,M)} \neq 0$.

LEMMA. $E_r^\cdot$ *is convergent, i.e.* $E^j_{r(j)} = E^j_\infty$ *for some* r(j) .

Proof. a) (A general convergence criterion:) Let $d^j : F_j^\vee \to F_{j+1}^\vee$ be the differentials and let $Z^j = \ker d^j$, $B^j = \mathrm{Im}\, d^{j-1}$. Then $E_r^j = E_\infty^j$, if

$$d^j(F_j^\vee) \cap \Gamma_k F_{j+1}^\vee = d^j(\Gamma_{k+r-1}F_j^\vee) \cap \Gamma_k F_{j+1}^\vee$$

and the analogous equality for j-1 instead of j hold.
b) The good filtrations on the $F_j^\vee$ induce two good filtrations on the B^j : The first one (Γ) comes from the inclusion $B^j \subset F_j^\vee$ and the second one (Γ') from the projection $d^{j-1} : F_{j-1}^\vee \to B^j$. From the compatibility of good filtrations (1.8.4) we get an r with $\Gamma_k(B^{j+1}) \subset \Gamma'_{k+r-1}(B^{j+1})$ and $\Gamma_k(B^j) \subset \Gamma'_{k+r-1}(B^j)$ for all k . This is just the condition in a).

Now we use the regularity of S = gr R and proposition 2.2.3. It says:

(i) $E_1^j = 0$ for $j < j(\mathrm{gr}\,M)$,
(ii) $d(E_1^j) \leqq m - j$,

(iii) $d(E^{j(\mathrm{gr}\,M)}) = n - j(\mathrm{gr}\,M)$. Call $j(\mathrm{gr}\,M) = j_0$. Since $E_1^{j_0-1} = 0$, $E_2^{j_0}$ is a submodule of $E_1^{j_0}$, and since $d(E_1^{j_0}) = n - j_0 > d(E_1^{j_0+1})$ we get $d(E_2^{j_0}) = n - j_0$. We also have $E_2^j = 0$ for $j < j_0$ and $d(E_2^j) \leqq n - j$ for all j , because E_2^j is a subquotient of E_1^j . So the properties (i), (ii), (iii) hold for E_2 and in the same way for all E_r^j . Since $E_r^\bullet$ converges we have proved the following theorem.

2.3.2 THEOREM. *Let* M *be a finitely generated left* R-*module. Then*

1) $$\mathrm{Ext}_R^j(M,R) = 0 \text{ for } j < j(\mathrm{gr}\ M) ,$$

2) $$d(\mathrm{Ext}_R^j(M,R)) \leqq n - j \text{ for all } j ,$$

3) $$d(\mathrm{Ext}_R^{j(\mathrm{gr}\ M)}(M,R)) = n - j(\mathrm{gr}\ M) .$$

An immediate consequence of this is $j(M) = j(\mathrm{gr}\ M)$. Of course, the results of 2.3 hold for right R-modules as well.

2.3.3. THEOREM. *If* M *is holonomic, then* $M^{**} \cong M$.

Proof. From 2.4.3 and the fact that R is noetherian we get a resolution

$$0 \to P_n \to \dots \to P_0 \to M \to 0$$

by finitely generated projective R-modules. Then 2.4.1 yields an exact sequence

$$0 \to P_0^\vee \to \dots \to P_n^\vee \to M^* \to 0 .$$

2.3.4. *Remark.* From 2.3.3. and Bourbaki Algèbre 10, §8, No3, Cor. of Prop. 4, it follows that the homological dimension of A_n is at most n . From the next theorem, for example, it follows that it is exactly n.

2.4. *Duality for holonomic modules.* Let gr R = S have gl hd(S) = 2n instead of n and assume that Bernstein inequality $d(M) \geqq n$ for all finitely generated R-modules $M \neq 0$. The finitely generated (left or right) R-modules of dimension n are called holonomic.

2.4.1 THEOREM - DEFINITION. *A finitely generated left (right)* R- *module* $M \neq 0$ *is holonomic if and only if the right (left)* R-*modules* $\mathrm{Ext}_R^j(M,R) = 0$ *for* $j \neq n$. *In this case,* $M^* = \mathrm{Ext}_R^n(M,R)$ *is called the dual of* M . *It is holonomic.*

Proof. The first statement follows from 2.3.2 (ii), Bernstein's inequality, and 2.2.2 . (Remember that n was replaced by 2n.) The second statement follows from 2.3.2 .

2.4.2 THEOREM. $M \mapsto M^*$ *is an exact functor from the category of holonomic left (right)* R-*modules to the category of right (left)* R-*modules.*

Proof. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of holonomic modules then

$$0 \to \mathrm{Ext}_R^n(M_3,R) \to \mathrm{Ext}_R^n(M_2,R) \to \mathrm{Ext}_R^n(M_1,R) \to 0$$

is the only nontrivial part of the long exact $\mathrm{Ext}_R^\bullet(\cdot,R)$-cohomology sequence.

2.4.3 THEOREM. *If* M,N *are left (right)* R-*modules and* M *is finitely generated, then* $\mathrm{Ext}_R^j(M,N) = 0$ *for* $j > n$.

Proof. This is true for N = R by 2.3.2 (ii) and Bernstein's inequality. If N is finitely generated, there is an exact sequence

$$0 \to K \to F \to N \to 0$$

with F free of finite rank and K finitely generated. From the $\mathrm{Ext}_R^\bullet(M,\cdot)$-cohomology sequence we get isomorphisms

$$\mathrm{Ext}_R^j(M,N) \cong \mathrm{Ext}_R^{j+1}(M,K) \quad \text{for } j > n .$$

As in the first part of 2.3.1 , gr $\mathrm{Ext}^j_R(M,L)$ $(=E^j_\infty)$ is a subquotient of $\mathrm{Ext}^j_{\mathrm{gr}\,R}(\mathrm{gr}\,M,\mathrm{gr}\,L)$ $(=E^j_1)$ for any finitely generated R-module L and suitable good filtrations of M and L . The latter Ext-groups are all trivial for $j > 2n$ by the assumptions on R. By recursion on j we get the wanted result for a finitely generated N . For the general result one uses that direct limits commute with Ext in the second argument.

If P is a finite projective left (right) module then $P^\vee = \mathrm{Hom}_R(P,R)$ is a finite projective right (left) module. Repeating the first step with M* instead of M leads to

$$0 \to P_n^{\vee\vee} \to \dots \to P_0^{\vee\vee} \to M^{**} \to 0 \ .$$

Projective modules are reflexive, thus

$$0 \to P_n \to \dots \to P_0 \to M^{**} \to 0$$

is exact. This sequence of the P_j is the same as the first one.

2.4.4. *Remark.* From the theorem above it follows, that $M \mapsto M^*$ is an equivalence of categories for holonomic modules.

§ 3 INVERSE AND DIRECT IMAGES OF A_n-MODULES UNDER POLYNOMIAL MAPPINGS.

3.1 *Preservation theorems.* As in the preceding sections modules are usually left modules.

Let M be a holonomic $A_n(K)$-module and H the hyperplane $\{x \in K^n \mid x_n = 0\}$. Let A_{n-1} be the subalgebra of A_n generated by x_j , ∂_j , $j < n$.

3.1.1 DEFINITION. $\Gamma_{[H]}(M) = \{u \in M \mid x_n^j u = 0 \text{ for some } j\}$ is the A_n-submodule of M of elements supported by H . (By 1.14 it is also holonomic.)

3.1.2 THEOREM. *Let* M_0 *be the* A_{n-1}*-submodule* $\mathrm{Ker}_M x_n = \{u \in M \mid x_n u = 0\}$. *Then* $1 \otimes u \mapsto u$ *defines an isomorphism*

$$A_n/A_n x_n \otimes_{A_{n-1}} M_0 \xrightarrow{\sim} \Gamma_{[H]}(M) \ .$$

($A_n/A_n x_n$ *is considered as a left* A_n*-module and right* A_{n-1}*-module.*)

Proof. Surjectivity: We show $\{u \in M \mid x_n^k u = 0\} \subset A_n M_0$ by induction on k . If $x_n^k u = 0$, then

$$0 = \partial_n(x_n^k u) = x_n^{k-1}(ku + x_n \partial_n u) .$$

From the induction hypotheses we have $x_n u \in A_n M_0$ and $ku + x_n \partial_n u \in A_n M_0$, therefore

$$(k-1)u = ku + x_n \partial_n u - \partial_n x_n u \in A_n M_0 \ .$$

The case $k = 1$ is trivial .
Injectivity: The right A_{n-1}-module A_n has the following direct sum decomposition

$$A_n = A_n x_n \oplus A_{n-1} \oplus \partial_n A_{n-1} \oplus \partial_n^2 A_{n-1} \oplus \dots \ ,$$

so we have

$$A_n/A_n x_n = A_{n-1} \oplus \partial_n A_{n-1} \oplus \partial_n^2 A_{n-1} \oplus \dots \quad .$$

Thus the assertion means that the following sum of A_{n-1}-modules is a direct one and $\partial_n \cdot : \partial_n^k M_0 \to \partial_n^{k+1} M_0$ is injective.

3.1.3 $\quad \Gamma_{[H]}(M) = M_0 \oplus \partial_n M_0 \oplus \partial_n^2 M_0 \oplus \dots \quad .$

Proof. Let $u_0 + \partial_n u_1 + \dots + \partial_n^k u_k = 0$, $u_j \in M_0$. Multiplication with x_n gives

$$u_1 + 2\partial_n u_2 + \dots + k\partial_n^{k-1} u_k = 0 \quad .$$

(Use $x_n \partial_n^j = \partial_n^j x_n - j\partial_n^{j-1}$.) Successive multiplication by x_n shows that all $u_j = 0$.

3.1.4 *Remarks.* 1) This proof shows $x_n \cdot \Gamma_{[H]}(M) = \Gamma_{[H]}(M)$.

2) If M' is any finitely generated A_{n-1}-module, then

$$M'' = A_n/A_n x_n \otimes_{A_{n-1}} M'$$

is a finitely generated A_n-module with $d(M'') = d(M') + 1$ and $m(M'') = m(M')$. (In order to see this, write M'' in the form $M'' \cong A_1/A_1 x_n \otimes_K M'$. Here A_1 is the subalgebra of A_n generated by x_n , ∂_n , and the A_n-action on the right hand side is given by the A_1-action on the first and the A_{n-1}-action on the second factor, only. If Γ' is any good filtration of M' and $A_1/A_1 x_n \cong K[\partial_n]$ is filtered or graded by degrees, then $\operatorname{gr} M'' \cong K[\partial_n] \otimes \operatorname{gr} M'$.)

This last remark also proves

3.1.5 THEOREM. $\operatorname{Ker}_M x_n$ *is holonomic.*

3.1.6 THEOREM. *The map* $M' \mapsto A_n/A_n x_n \otimes_{A_{n-1}} M'$ *defines an equivalence of the category of holonomic* A_{n-1}*-modules and the category of* A_n*-modules supported by* H . (I.e. $M'' = \Gamma_{[H]}(M'')$.)

3.1.7 THEOREM. *The* A_{n-1}*-module* $M/x_n M$ *is holonomic.*

Proof. 1) Let $\bar{M} = M/\Gamma_{[H]}(M)$. The canonical A_{n-1}-morphism $M/x_n M \to \bar{M}/x_n \bar{M}$ is an isomorphism : $[u] \mapsto 0$ means $u \in x_n M + \Gamma_{[H]}(M)$; by 3.1.4.1 $u \in x_n M$ follows .

2) We show that $\bar{M}/x_n \bar{M}$ is holonomic. Since the multiplication by x_n is injective on $\bar{M}$, the canonical map $\bar{M} \to \bar{M}[x_n^{-1}]$ is an injective A_n-morphism. Let $N = \bar{m}[x_n^{-1}]/\bar{M}$. Then $u \mapsto x_n^{-1} u$ defines an A_{n-1}-isomorphism $\bar{M}/x_n \bar{M} \xrightarrow{\sim} \operatorname{Ker}_N x_n$. Now the assertion follows from part 1 , 1.14 and 3.1.5 .

3.1.8 THEOREM. $\operatorname{Ker}_M \partial_n$ *and* $M/\partial_n M$ *are holonomic.*

Proof. This follows from 3.1.5 and 3.1.7 by Fourier-transformation: For any A_n-module N define $F(N)$ to be the A_n-module which is identical to N as a K-vector space but with a different A_n-structure. Namely, let $F : A_n \to A_n$ be the automorphism defined by $x_j \mapsto \partial_j$, $\partial_j \mapsto -x_j$, and define $D \cdot u_{F(N)} := F^{-1}(D) \cdot u_N$. The index of u indicates an element of which module u is considered as. The Fourier-transformation leaves Bernstein's filtration invariant, therefore it preserves holonomicity of A_n-modules.

3.2 *Inverse images.* Let $P = (p_1, \dots, p_n) : K^m \to K^n$ be a polynomial mapping. Then $K[x_1, \dots, x_m]$ is a $K[y_1, \dots y_n]$-module via P . Let M be an A_n-module.

3.2.1 DEFINITION. The inverse image of M under P is the A_m-module

$$P^*M = K[x] \otimes_{K[y]} M$$

with the structure defined by

$$x_j\cdot(q(x)\otimes u) = x_j q(x)\otimes u$$

$$\partial_{x_j}\cdot(q(x)\otimes u) = \frac{\partial q}{\partial x_j}\otimes u + \sum_{i=1}^{n} q\,\frac{\partial p_i}{\partial x_j}\otimes \partial_{y_i} u\ .$$

3.2.2 PROPOSITION. *If* $Q : K^l \to K^m$ *is a second polynomial mapping, then* $(P\circ Q)^*M = Q^*P^*M$.

Proof. This is just a combination of the corresponding statement for modules over polynomial rings and the chain rule.

In order to study P^* , we write P as a composition of an embedding and a projection :

$$P = \pi\circ i$$

$$i : K^m \to K^{m+n}\ ,\ i(x) = (x,P(x))$$

$$\pi : K^{m+n} \to K^n\ ,\ \pi(x,y) = y$$

<u>π^*</u> : $\pi^*M = K[x,y]\otimes_{K[y]}M \cong K[x]\otimes_K M$. On the right hand side x_j , ∂_{x_j} act on $K[x]$ only and y_j , ∂_{y_j} act on M only. If M is finitely generated with the good filtration Γ. and Δ. is the filtration of $K[x]$ by degrees, then $\Omega_l = \Sigma_{j+k=l}(\Delta_j\otimes\Gamma_k)$ is a good filtration of π^*M . From $gr^\Omega\,\pi^*M = gr^\Delta K[x]\otimes_K gr^\Gamma M$ one deduces $d(\pi^*M) = m + d(M)$. (Compare 3.1.4.2 for $m = 1$.) This shows that π^*M is holonomic if M is.

<u>i^*</u> : Let N be an A_{m+n}-module. Then

$$i^*N = K[x]\otimes_{K[x,y]}N \cong K[x,y]/(y_1-p_1,\dots,y_n-p_n)\otimes_{K[x,y]}N,$$

where the A_m-structure on the right hand side is given by

$$x_j\cdot(q(x,y)\otimes u) = x_j q\otimes u\ ,$$

$$\partial_{x_j}\cdot(q(x,y)\otimes u) = \frac{\partial q}{\partial x_j}\otimes u + q\otimes\partial_{x_j}u + \sum_{i=1}^{n}\frac{\partial p_l}{\partial x_j}\partial_{y_l}u\ .$$

The automorphism of A_{m+n}

$$x_j' = x_j\ ,\ y_l' = y_l - p_l(x)\ ,\ (\partial_{x_j})' = \partial_{x_j} + \Sigma_l\frac{\partial p_l}{\partial x_j}\partial_{y_l}\ ,\ (\partial_{y_l})' = \partial_{y_l}$$

transforms this to $N/\Sigma_l y_l'N$ with the A_m-structure given just by left multiplication. This is holonomic, by (repeated application of) 3.1.7 . It remains to be shown that the automorphism does not affect the holonomicity. This follows from theorem 2.2.2 which gives a filtration free criterion of holonomicity.

Using 3.2.2 we proved

3.2.3 THEOREM. *If* M *is holonomic then* P^*M *is holonomic.*

Remark. In general P^*M might be not even finitely generated if M is . (E.g. : $M = A_1$, $P : K \to K$, $P(x) = x^2$.)

3.2.4 Define $D_{K^m\to K^n} = P^*A_n$. It carries the left A_m-structure defined above. There is also a right A_n-structure arising from the right A_n-module A_n . With this (A_m,A_n)-bimodule the inverse image P^*M can be written in a different way :

$$P^*M = K[x]\otimes_{K[y]}M \cong (K[x]\otimes_{K[y]}A_n)\otimes_{A_n}M = D_{K^m\to K^n}\otimes_{A_n}M\ .$$

We obtain the following formulas.

$$D_{K^{m+n}\xrightarrow{\pi}K^n} \cong A_{m+n}/\sum_{j=1}^{m}A_{m+n}\partial_{x_j}\ ,$$

$$D_{K^m\xrightarrow{i}K^{m+n}} \cong A_{m+n}/\sum_{i=1}^{n}(y_l-p_l(x))A_{m+n}\ ,$$

$$D_{K^m\xrightarrow{P}K^n} \cong A_{m+n}/\sum_{j=1}^{m}A_{m+n}\partial_{x_j} + \sum_{i=1}^{n}(y_l-p_l(x))A_{m+n}\ ,$$

with the bimodule structure as above.

3.3 *Direct images.* Again let $P : K^m \to K^n$ be a polynomial mapping, and M a *right* A_m-module. There is a natural definition of a direct image P_*M, this is going to be a right A_n-module, such that for every left A_n-module N there is a canonical isomorphism

$$P_*M \otimes_{A_n} N \cong M \otimes_{A_m} P^*N ,$$

namely $P_*M = M \otimes_{A_m} D_{K^m \to K^n}$. In order to obtain a definition for direct images of left A_n-modules we notice that the transposition

$$(\Sigma_\alpha a_\alpha(x)\partial^\alpha)^t = \Sigma_\alpha(-1)^{|\alpha|}\partial^\alpha a_\alpha(x) \ , \ a_\alpha(x) \in K[x] ,$$

defines an involutive anti-isomorphism of A_m. (I.e. $(D_1D_2)^t = D_2^tD_1^t$.) The application of the transposition to A_m turns a right (left) A_m-module into a left (right) A_m-module.

Remark. This definition of "turning left to right and vice versa" does not directly carry over to the more general case of D-modules on affine varieties. Then the "transposed" modules will in general also be different as modules (over the structure sheaf).

The transpositions of both the A_m- and the A_n-structures of $D_{K^m \to K^n}$ yields the (A_n,A_m)- bimodule $D_{K^n \leftarrow K^m}$ with A_n acting on the left and A_m on the right side.

Denote the transposition of a module M by M^t. If M is now a left A_m-module then its direct image under P is defined by

3.3.1 DEFINITION. $P_*M = D_{K^n \leftarrow K^m} \otimes_{A_m} M \cong (P_*M^t)^t$. ($P_*M^t$ was defined above .)

3.3.2 PROPOSITION. *If* $Q : K^\ell \to K^m$ *is another polynomial mapping, then there is a canonical isomorphism of bimodules.*

$$D_{K^n \leftarrow K^\ell} \cong D_{K^n \leftarrow K^m} \otimes_{A_m} D_{K^m \leftarrow K^l}$$

Proof. This follows from the isomorphism

$$D_{K^l \to K^n} \cong D_{K^l \to K^m} \otimes_{A_m} D_{K^m \to K^n} ,$$

which is just proposition 3.2.2 together with 3.2.4 .

3.3.3 PROPOSITION. *If* $P : K^m \to K^n$ *is a polynomial isomorphism, then there are canonical isomorphisms of bimodules*

$$D_{K^m \to K^n} \otimes_{A_n} D_{K^n \leftarrow K^m} \cong A_m ,$$

$$D_{K^n \leftarrow K^m} \otimes_{A_m} D_{K^m \to K^n} \cong A_n ,$$

i.e. P^* *and* P_* *are inverse to each other. (m = n of course, but different letters are used in order to avoid confusion.)*

Proof. Identify $D_{K^m \to K^n} = 1 \otimes A_n$ and $D_{K^n \leftarrow K^m}$ with A_n by $1 \otimes D \to D$ and $1 \otimes D \to D^t$, resp. The induced bimodule structures on A_n are

$$A_m \times A_n \times A_n \to A_n \qquad A_n \times A_n \times A_m \to A_n$$

and

$$(D_1,D,D_2) \to (P_*D_1)DD_2 \quad (D_1,D,D_2) \mapsto D_1D(P_*D_2) .$$

Here $P_* : A_m \to A_n$ is the isomorphism defined by

$$K[x] \ni f \mapsto foP^{-1} , \ \partial_{x_j} \mapsto \Sigma_j(\frac{\partial p_i}{\partial x_j} oP^{-1})\partial_{y_i} .$$

With this identifications the isomorphisms are

$$D_1 \otimes D_2 \mapsto P_*^{-1}(D_1D_2) \quad \text{and } D_1 \otimes D_2 \mapsto D_1D_2 .$$

Let $P : K^m \to K^n$ be a general polynomial mapping. We want to prove that P_* preserves the holonomicity of left A_m-modules. As in 3.2 we split P into an inclusion and a projection : $P = \pi \circ i$, and study two cases seperately (3.3.2).

$\underline{i_*}$: The inclusion is the composition

$$K^m \xrightarrow{j} K^{m+n} \xrightarrow{Q} K^{m+n} \quad , \; x \to (x,0), (x,y) \to (x,y+P(x)) \; .$$

As Q is an isomorphism, 3.3.3 implies $Q_* = (Q^{-1})^*$, so by 3.3.2 and 3.2.3 we only have to study the inclusion $j : x \mapsto (x,0)$. From the discussion of i^* we see that

$$D_{K^{m+n} \xleftarrow{i} K^m} \cong A_{m+n} / \sum_{i=1}^{n} A_{m+n}(y_i - p_i(x)) \; ,$$

$$D_{K^{m+n} \xleftarrow{j} K^m} \cong A_{m+n} / \sum_{i=1}^{n} A_{m+n} y_i \; .$$

If M is a holonomic left A_m-module, then

$$j_* M \cong A_{m+n} / \Sigma_i A_{m+n} y_i \otimes_{A_m} M$$

is a holonomic A_{m+n}-module by repeated application of 3.1.4.2 (use 3.3.2).

$\underline{\pi_*}$: $\pi : K^{m+n} \to K^n$, $(x,y) \mapsto y$. First compute $D_{\leftarrow}$:

$$D_{K^{m+n} \leftarrow K^n} \cong A_{m+n} / \Sigma_i A_{m+n} \partial_{x_i} \quad \text{implies}$$

$$D_{K^n \leftarrow K^{m+n}} \cong A_{m+n} / \sum_{i=1}^{m} \partial_{x_i} A_{m+n} \; .$$

If M is a holonomic left A_{m+n}-module, then we get

$$\pi_* M \cong M / \sum_{i=1}^{m} \partial_{x_i} M \; .$$

This is holonomic over A_n by successive application of 3.1.8 .

3.3.4 THEOREM. *If M is a holonomic A_m-module then $P_* M$ is a holonomic A_n-module.*

3.3.5 *Remark.*

$$D_{K^n \xleftarrow{P} K^m} \cong A_{m+n} / \sum_{i=1}^{m} \partial_{x_i} A_{m+n} + \sum_{i=1}^{n} A_{m+n}(y_i - p_i(x)) \; .$$

Here y_i , ∂_{y_i} operate on the left by multiplication with themselves; on the right x_i operates by multiplication with x_i , ∂_{x_i} by multiplication with $\partial_{x_i} + \Sigma_l \frac{\partial p_l}{\partial x_i} \partial_{y_l}$.

3.4 *Derived inverse images.* Denote by $\mu(A_n)$ the category of left A_n-modules and by $D^-(\mu(A_n))$ and $D^b(\mu(A_n))$ the corresponding derived categories of bounded above and bounded complexes, resp.

Let $P : K^m \to K^n$ be a polynomial mapping. As the functor $P^* : \mu(A_n) \to \mu(A_m)$ is given by the tensor product with the (A_m, A_n)-module $D_{K^m \to K^n}$, there exists a left derived functor $LP^* : D^-(\mu(A_n)) \to D^-(\mu(A_m))$. One way to construct LP^* is to look for a resolution $\tilde{D}^{\cdot}_{K^m \to K^n}$ by bimodules which are projective (or flat or free) as right A_n-modules. A standard free resolution is given by the Koszul complex:

$$\tilde{D}^{\cdot}_{K^m \to K^n} := K^{\cdot}_r(A_{m+n} / \Sigma_i A_{m+n} \partial_{x_i} \; ; y_1 - p_1(x), \dots, y_n - p_n(x)) \; .$$

Comments. 1) If M is a right R-module, R some (unitary) ring, and $a_1, \dots, a_n$ are commuting endomorphisms of M, then the right Koszul complex $K^{\cdot}_r(M; a_1, \dots, a_n)$ is defined by

$$\begin{array}{ccc} K_r^{k-n}(M; a_1, \dots, a_n) := M \otimes_{\mathbb{Z}} \Lambda^k(\mathbb{Z}^n) & \ni & u \otimes (v_1 \wedge \dots \wedge v_k) \\ \Big\downarrow d & & \Big\downarrow \\ M \otimes_{\mathbb{Z}} \Lambda^{k+1}(\mathbb{Z}^n) & \ni & \Sigma a_j(u) \otimes (e_j \wedge v_1 \wedge \dots \wedge v_k) \; . \end{array}$$

Here e_j is the j-th unit vector.
2) In the case at hand the $y_l - p_l(x)$ operate by multiplication from the left.
Since $[\partial_{x_i} + \Sigma_l \frac{\partial p_l}{\partial x_i}\partial_{y_l}, y_j - p_j(x)] = 0$, left multiplication with $y_j - p_j(x)$ commutes with the left A_m-action. Therefore, the differentials are (A_m,A_n)-linear. It is clear, that the module $A_{m+n}/\Sigma_{i=1}^m A_{m+n}\partial_{x_i}$ is right-A_n-free. The $y_j - p_j(x)$ are a regular sequence for the left $K[x,y]$-module $A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i}$. so the Koszul complex gives a resolution of

$$D_{K^m \to K^n} \cong A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i} + \Sigma_j (y_j - p_j(x))A_{m+n} .$$

If $M^\bullet$ is an object of $D^-(\mu(A_n))$ then $LP^*(M^\bullet)$ is given by

$$LP^*(M^\bullet) \cong \tilde{D}^\bullet_{K^m \to K^n} \otimes_{A_n} M^\bullet$$

(On the right hand side one has to take the single complex of a double complex.)

Since $\tilde{D}^\bullet_{K^m \to K^n}$ is bounded, this tensor product also defines a functor $D^b(\mu(A_n)) \to D^b(\mu(A_m))$ which is also denoted by LP^*.

We shall prove two properties of left derived images: First, LP^* depends functorially on P, and secondly, LP^* preserves the holonomicity of the cohomology of objects of $D^-(\mu(A_n))$.

Let $P : K^m \to K^n$ be as above and $Q : K^n \to K^p$ a second polynomial mapping.

3.4.1 PROPOSITION. *On* D^- *and* D^b, $(LP^*) \circ (LQ^*) \cong L(P^* \circ Q^*) (\cong L(Q \circ P)^*)$.

Proof. We have to show that the complexes of (A_m,A_p)-modules

$$\tilde{D}^\bullet_{K^m \to K^p} \text{ and } \tilde{D}^\bullet_{K^m \to K^n} \otimes_{A_n} \tilde{D}^\bullet_{K^n \to K^p}$$

are homology isomorphic. Writing $Q \circ P$ as the composition of the map $K^m \to K^{m+n+p}$ given by $x \mapsto (x, P(X), (Q \circ P)(x))$ and of the projection $(x,y,z) \mapsto z$ of K^{m+n+p} onto K^p, we obtain a new representation and resolution of $D_{K^m \to K^p}$:

$$\hat{D}^\bullet_{K^m \to K^p} := K^\bullet_r(A_{m+n+p}/\Sigma_i A_{m+n+p}\partial_{x_i} + \Sigma_j A_{m+n+p}\partial_{y_j}; y_1 - p_1(x), \ldots$$

$$\ldots, z_1 - q_1(P(x)), \ldots)$$

LEMMA. *The map*

$$[f(x,y)\partial_y^\alpha] \otimes [g(y,z)\partial_z^\beta] \mapsto [f(x,y)(\partial_y + \frac{\partial Q}{\partial y}\partial_z)^\alpha (g(y,z)\partial_z^\beta)].$$

defines an isomorphism of (A_m,A_p)*-modules*

$$(A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i}) \otimes_{A_n} (A_{n+p}/\Sigma_j A_{n+p}\partial_{y_j}) \xrightarrow{\sim} A_{m+n+p}/(\Sigma_i A_{m+n+p}\partial_{x_i} + \Sigma_j A_{m+n+p}\partial_{y_j}),$$

(Be aware of the structures of these three modules depending on P and Q as explained in 3.2 and 3.3.5 for the $D_\leftarrow$-case. On the right hand side, the right A_p-structure is just given by multiplication, as is the left action by x_i. The ∂_{x_i} operate by multiplication with

$$x_i + \Sigma_j \frac{\partial p_j}{\partial x_i}\partial_{y_j} + \Sigma_k \frac{\partial (Q \circ P)_k}{\partial x_i}\partial_{z_k} \quad .)$$

The proof of this lemma is similar to that of 3.2.2 , it is essentially the chain rule for $Q \circ P$.

This lemma yields an isomorphism of (A_m,A_p)-complexes

$$\tilde{D}^\bullet_{K^m \to K^n} \otimes_{A_n} \tilde{D}^\bullet_{K^n \to K^p} \cong \hat{D}^\bullet_{K^m \to K^p}$$

proving the proposition.

3.4.2 THEOREM. *Let P be as above and* $M^{\cdot}$ *an object of* $D^{-}(\mu(A_n))$ *or* $D^{b}(\mu(A_n))$ *with holonomic cohomology, i.e. all the* $H^{k}(M^{\cdot})$ *are holonomic. Then* $LP^{*}(M^{\cdot})$ *is also (cohomologically) holonomic.*

The proof of this theorem is similar to the analogous proof (of 3.5.2 below) for direct images in the next section, where a proof is given.

3.5 *Derived direct images.* Let $P : K^m \to K^n$ be a polynomial mapping. Then $P_* : \mu(A_m) \to \mu(A_n)$ is given by the tensor product with $D_{K^n \leftarrow K^m}$, so there is a left derived functor $LP_* : D^{-}(\mu(A_m)) \to D^{-}(\mu(A_n))$. A resolution $\tilde{D}^{\cdot}_{K^n \leftarrow K^m} \to D_{K^n \leftarrow K^m}$ of $D_{K^n \leftarrow K^m}$ (a formula of this is in 3.3.5) by (A_n, A_m)-modules which are free as A_m-modules is

$$\tilde{D}^{\cdot}_{K^n \leftarrow K^m} := K^{\cdot}_r(A_{m+n}/\textstyle\sum_j A_{m+n}(y_j - P_j(x)); \partial_{x_1}, \dots, \partial_{x_n}) .$$

This gives

$$LP_*(M^{\cdot}) = \tilde{D}^{\cdot}_{K^n \leftarrow K^m} \otimes_{A_m} M^{\cdot} \quad \text{for } M^{\cdot} D^{-}(\mu(A_m)) .$$

Since $D^{\cdot}_{K^n \leftarrow K^m}$ is bounded, we also have a derived functor

$$LP_* : D^{b}(\mu(A_m)) \to D^{b}(\mu(A_n)) .$$

Remark. In subsequent chapters we shall see that in the case of a morphism $P : X \to Y$ between non affine varieties there is a direct image functor between the corresponding categories of D-modules. That functor is the composition of first a left derived functor as here and a right derived functor. It will be denoted there RP_+ or P_+.

Let $Q : K^n \to K^p$ be a second polynomial mapping.

3.5.1 PROPOSITION. $(LQ_*) \circ (LP_*) \cong L(Q_* \circ P_*)$ $(\cong L(Q \circ P)_*)$.
The proposition can be obtained from 3.4.1 by transpositions.

3.5.2 THEOREM. *Let* $M^{\cdot}$ *be an object of* $D^{-}(\mu(A_m))$ *or* $D^{b}(\mu(A_m))$ *with holonomic cohomology. Then* $LP_*(M^{\cdot})$ *has holonomic cohomology.*

Proof. We write $P : K^m \to K^n$ as the composition

$$K^m \xrightarrow{i} K^{m+n} \xrightarrow{\alpha} K^{m+n} \xrightarrow{\pi} K^n$$

$$x \longmapsto (x,0) \qquad (x,y) \longmapsto y$$

$$(x,y) \longmapsto (x, y+P(x))$$

In view of 3.5.1 we can prove the theorem by proving it for i, α, π separately.

$\underline{i}$: $D_{K^{m+n} \xleftarrow{i} K^n} \cong A_{m+n}/\sum_j A_{m+n} y_j$ is free as a right A_m-module, so i_* is exact. Since i_* commutes with the cohomology functor, the result follows from 3.3.4 or 3.1.4.2 .

$\underline{\alpha}$: Since α is a isomorphism, α_* is exact. Then use 3.3.4 , for example.

$\underline{\pi}$: One can split π into a sequence of projections with one-dimensional fibres. By 3.5.1 is therefore enough to consider the case m = 1 . But first, let m be general. Let $N^{\cdot}$ be an object of $D^{-}(\mu(A_{m+n}))$. Then $L\pi_*(N^{\cdot})$ is represented by

$$K^{\cdot}_r(A_{m+n}; \partial_{x_1}, \dots, \partial_{x_n}) \otimes_{A_{m+n}} N^{\cdot} .$$

In order to prove the theorem we look at one of the spectral sequences of the double complex:

$$K^p_r(A_{m+n}; \partial_{x_1}, \dots, \partial_{x_n}) \otimes_{A_{m+n}} N^q =: E_0^{p,q} , \; d_0 : E_0^{p,q} \to E_0^{p,q+q} .$$

Then $E_1^{p,q} \cong K^p_r \otimes_{A_{m+n}} H^q(N^{\cdot})$. Now let m = 1 , then

$$E_2^{-1,q} \cong \mathrm{Ker}_{H^q(N^\cdot)}\,\partial_x \ , \quad E_2^{0,q} \cong H^q(N^\cdot)/\partial_x H^q(N^\cdot) \ ,$$

$$E_2^{p,q} = 0 \text{ for } p \neq -1,0 \ , \quad E_\infty = E_2 \ .$$

Let us assume the $H^q(N^\cdot)$ to be holonomic. Then $E_2^{-1,q}$ and $E_2^{0,q}$ are A_n-holonomic by 3.1.5 and 3.1.7. E_∞ being holonomic means, there is a filtration of the $H^k(L\pi_*(N^\cdot))$ with holonomic quotients. But then 1.11.1 implies that $H^k(L\pi_*(N^\cdot))$ are holonomic themselves. This proves the theorem.

3.5.3 *Remark.* The complex

$$DR^\cdot(N^\cdot;\partial_{x_i},\dots,\partial_{x_m}) = DR^\cdot(N^\cdot) = (K_r^\cdot \otimes_{A_{m+n}} N^\cdot)[-m] = N^\cdot \otimes_{\mathbb{Z}} \Lambda^\cdot \mathbb{Z}^m$$

with differential

$$d : \bigoplus_{p+q=k} (N^q \otimes \Lambda^p \mathbb{Z}^m) \to \bigoplus_{p+q=k+1} (N^q \otimes \Lambda^p \mathbb{Z}^m)$$

$$d(u \otimes v) = (-1)^p du \otimes v + \sum_{i=1}^{m} \partial_{x_i} u \otimes \ell_i \wedge v$$

$$(u \in N^q,\ v \in \Lambda^p \mathbb{Z}^m)$$

is called the de Rham complex of $N^\cdot$ relative to π or with respect to the ∂_{x_i}. The last step on the proof of the preceding theorem shows that the relative de Rham complex of an A_{m+n}-complex with holonomic cohomology has A_n-holonomic cohomology groups.

Remark. It has been shown by J. T. Stafford [St] that simple modules over A_n are not in general holonomic and do not share some of the finiteness properties proved here for holonomic ones.

BIBLIOGRAPHICAL NOTE

Most of the material of this chapter is contained in the two papers

[B1] BERNSTEIN, I.N., Modules over the rings of differential operators, a study of the fundamental solutions of equations with constant coefficients, Funct. anal. and Appl. 5,2 (1971), 1-16.

[B2] BERNSTEIN, I.N., The analytic continuation of generalized functions with respect to a parameter, Funct. Anal. and Appl. 6,4 (1972), 26-40.

The notions of §1 are all found in [B1], whereas the central result of §1, Bernstein's inequality, is proved in [B2]. The notions of inverse and direct images under polynomial mappings and the holonomicity theorems of §3 are also part of [B2]. The presentation of the homological algebra results of §2 follows:

[Bj] BJÖRK, J.E., Rings of differential operators, North-Holland Publishing Company, Amsterdam, 1979.

The main results about homological dimension are due to J.-E. Roos:

ROOS, J.E., Détermination de la dimension homologique globale des algèbres de Weyl, C.R. Acad. Sci. Paris **274** (1972), 23-26 .

For examples and properties of simple non-holonomic modules over A_n, see

[St] Stafford, J.T., *Non-holonomic modules over Weyl algebras and enveloping algebras*, Inv. Math. 79(1985), 619-638.

VI. OPERATIONS ON ALGEBRAIC $\mathcal{D}$ - MODULES

A. Borel

In the sequel, X and Y are smooth complex algebraic varieties of pure dimension and $\pi : Y \to X$ a morphism.

§ 1 GENERALITIES ON ALGEBRAIC $\mathcal{D}$ - MODULES

1.1 Let first X be affine. The ring $\mathcal{D}(X)$ of (algebraic) differential operators on X is the ring of operators on $\mathcal{O}(X)$, acting by multiplication, and by $\Theta_X(X) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(X), \mathcal{O}(X))$. We have therefore in $\mathcal{D}(X)$ the rule

$$(1) \qquad [\partial, f] = \partial f \qquad (f \in \mathcal{O}(X), \partial \in \Theta(X))$$

We shall soon deduce from V that $\mathcal{D}(X)$ is noetherian, coherent, of global homological dimension equal to dim X (see 1.9).
If Y is open affine, then $\Theta(Y)$ is generated over $\mathcal{O}(Y)$ be the restrictions of the elements in $\Theta(X)$, i.e. $\Theta(Y) = \mathcal{O}(Y) \underset{\mathcal{O}(X)}{\otimes} \Theta(X)$, whence also

$$(2) \qquad \mathcal{D}(Y) = \mathcal{O}(Y) \underset{\mathcal{O}(X)}{\otimes} \mathcal{D}(X)$$

There exists then a unique quasi-coherent $\mathcal{O}_X$-module on X, denoted $\mathcal{D}_X$, whose sections over Y open affine in X is $\mathcal{D}(Y)$.

1.2 We drop the assumption that X is affine. It follows from the last assertion in 1.1 that X carries a unique sheaf of rings $\mathcal{D}_X$ whose restriction to every open affine $Y \subset X$ is $\mathcal{D}_Y$. By definition, this is the sheaf of germs of algebraic differential operators on X, and its (regular) sections over X are the algebraic differential operators on X. It is quasi-coherent over $\mathcal{O}_X$. Its restriction to Y open in X is $\mathcal{D}_Y$.

1.3 If $X = \mathbb{A}^n$ is the affine n-space, then $\mathcal{D}(X)$ is the Weyl algebra A_n (cf V). In particular $\Theta(X)$ is spanned over $\mathcal{O}(X)$ by n commuting, everywhere linearly independent, vector fields. A similar statement is also valid around any point x (in the Zariski topology) of a general X. To see this, let $f_1,\dots,f_n$ ($n = \dim X$) be regular functions whose differentials are linearly independent at x. We may find an affine open neighborhood U of x such that the df_1 are linearly independent at every $u \in U$. There exist then uniquely determined $\partial_i \in \Theta(U)$ such that

$$\partial_i f_j = \langle \partial_i, df_j \rangle = [\partial_i, f_j] = \partial_{ij} \quad (1 \leq i,j \leq n).$$

The ∂_i commute on the subring of $\mathcal{O}(U)$ generated by the f_i's, hence also on $\mathcal{O}(U)$ since the latter is integral on the former. Then $\mathcal{D}(U)$ is free over $\mathcal{O}(U)$ with basis the $\partial^\alpha = \partial_1^{\alpha_1}, \dots \partial_n^{\alpha_n}$ ($\alpha_i \in \mathbb{N}$). The map $u \mapsto (f_1(u),\dots,f_n(u))$ is an étale morphism of U onto an open subset of $\mathbb{A}^n$ and ∂_i are the unique lifts of the standard ∂_i in A_n. We will refer loosly to such a system (f_i, ∂_j) as "local coordinates" around x. The $f_i - f_i(u)$ are indeed local coordinates, and the ∂_i the corresponding derivatives, in a neighborhood of any $u \in U$ in the *analytic* topology. However, in a Zariski neighborhood the f_i's do not necessarily separate the points.

1.4 *$\mathcal{D}$-modules.* A $\mathcal{D}$- module M on X is by definition a $\mathcal{D}_X$-module, i.e. a sheaf M on X such that $M(Y)$ is a $\mathcal{D}(Y)$-module for $Y \subset X$ open, with the usual compatibility with respect to restrictions. Let $\mathrm{Mod}(\mathcal{D}_X)$ be the category of all $\mathcal{D}_X$-modules. Most of the time, we shall consider only $\mathcal{D}$ -modules which are quasi-coherent as $\mathcal{O}_X$-modules. We let $\mu(\mathcal{D}_X)$ be the category of such $\mathcal{D}$-modules. A $\mathcal{D}$-module M is said to be coherent if it is locally so, i.e. if every $x \in X$ has an affine neighborhood Y on which $M(Y)$ is a coherent $\mathcal{D}(Y)$ module, which amounts to require the existence of an exact sequence:

(1) $$\mathcal{D}(Y)^q \to \mathcal{D}(Y)^p \to M(Y) \to 0$$

We let $\mathrm{coh}(\mathcal{D}_X)$ be the full subcategory of $\mu(\mathcal{D}_X)$ consisting of coherent $\mathcal{D}$-modules. We shall soon see that $\mathcal{D}_X$ is coherent and locally noetherian (1.9). Therefore M is coherent of and only if it is locally finitely generated.

1.5 *Relations with* PDE. Let X be affine. For simplicity, assume Θ_X is spanned by $n = \dim X$ commuting vector fields ∂_i. Consider a differential system of p equations in q unknown s on X

(1) $$\Sigma_{j=1}^{j=q} P_{ij} u_j = 0 \quad (1 \leq i \leq p)$$

where $P_{ij} \in \mathcal{D}(X)$, i.e. P_{ij} is a polynomial in the ∂^α, with coefficients in $\mathcal{O}(X)$. To (1) we associate a morphism of $\mathcal{D}(X)$-modules

$$\pi : \mathcal{D}(X)^p \to \mathcal{D}(X)^q$$

defined by

(2) $$(R_1,\dots,R_p) \longrightarrow (\Sigma_i R_i P_{i1}, \dots, \Sigma_i R_i P_{iq})$$

The image J of π is a left submodule. To (1) we associate the $\mathcal{D}(X)$-module $M = \mathcal{D}(X)^q/J$. We have then, by definition, an exact sequence

(3) $$\mathcal{D}(X)^p \to \mathcal{D}(X)^q \to M \to 0 .$$

In particular, M is coherent. Conversely, let M be a coherent $\mathcal{D}(X)$-module. There is then an exact sequence (3). If (u_i) is the canonical basis of $\mathcal{D}(X)^q$, then J is generated by p elements which can be written as the left-hand sides in (1) and then (1) is associated to M. As a consequence, there is a natural correspondance between coherent $\mathcal{D}(X)$-modules and systems of linear PDE.

To a q-tuple (f_j) of regular functions is associated a $\mathcal{D}(X)$-morphism $\mathcal{D}(X)^q$ into $\mathcal{O}(X)$ which maps u_i to f_i $(1 \leq i \leq q)$. This morphism can be factored through M if and only if the f_i are solutions of the system (1). The space of solutions in $\mathcal{O}(X)$ of (1) can be therefore identified with Sol $M = \mathrm{Hom}_{\mathcal{D}(X)}(M, \mathcal{O}(X))$. More generally, the space of solutions of (1) in any $\mathcal{D}(X)$-module F can be identified with $\mathrm{Hom}_{\mathcal{D}(X)}(M,F)$. These considerations go over to sheaves, and show that $\mathcal{H}om_{\mathcal{D}_X}(M,\mathcal{O}_X)$ is the sheaf of germs of solutions of the system of PDE which represents M locally as above. Sol M is a sheaf of $\mathbb{C}_X$-modules still with respect to the Zariski-topology. It is however more usual, and more useful, to pass to the analytic topology and consider the sheaf of germs of solutions in the latter. This passage, already discussed in IV, will be taken up again later when we set up the Riemann-Hilbert correspondence. More generally, one will consider the complex $Sol^{\cdot}M = R\mathcal{H}om^{\cdot}_{\mathcal{D}_X}(M,\mathcal{O}_X)$ of "generalized solutions" of M, again going over to the ordinary topology. We do not pursue that here, but mention it since it gives one of the motivations for passing from the system (1) to a $\mathcal{D}$-module.

1.6 *$\mathcal{D}$-modules and connections*. Let M be a quasi-coherent $\mathcal{O}_X$-module. A *connection* on M is an operation $\xi \mapsto \nabla_\xi$ of Θ_X on M which satisfies

(1) $$\nabla_{a\xi} + \nabla_{b\eta} = a\nabla_\xi + b\nabla_\eta \qquad (a,b \in \mathcal{O}_X; \xi,\eta \in \Theta_X)$$

(2) $$\nabla_\xi(f.m) = (\xi f).m + f\nabla_\xi m \ . \quad (\xi\in\Theta_X, f\in\mathcal{O}_X)), m\in M) \ :$$

the latter can also be written as

(3) $$[\xi,f]m = (\xi f).m$$

As usual, the curvature of a connection is given by $[\nabla_\xi,\nabla_\eta] - \nabla_{[\xi,\eta]}$ $(\xi,\eta\in\Theta_X)$. Therefore the connection extends to a $\mathcal{D}_X$-module structure if and only the curvature is zero i.e. if

(4) $$[\nabla_\xi,\nabla_\eta] = \nabla_{[\xi,\eta]} \qquad (\xi,\eta\in\Theta_X) \ .$$

In other words, a $\mathcal{D}_X$-module can also be defined as a $\mathcal{O}_X$-module endowed with a flat connection.

We are accustomed to that notion in case M is locally free of

finite rank, i.e. on case where M is the sheaf of germs of regular sections of an algebraic vector bundle. In fact, if suffices for this that M be $\mathcal{O}_X$-coherent:

1.7 PROPOSITION *Let M be a $\mathcal{D}_X$-module which is $\mathcal{O}_X$-coherent. Then it is locally free over $\mathcal{O}_X$.*

This is a local problem. Fix $x \in X$. Given a local section s of M around x, let $\bar{s}$ be its image in the vector space $\bar{M}_x = M_x/m_xM$ over $\mathcal{O}_{X,x}/m_x \cong \mathbb{C}$. We may find finitely many s_i $(1 \leq i \leq q)$ such that the $\bar{s}_i$ form a vector space basis of $\bar{M}_x$. By the Nakayama lemma, they generate M locally over $\mathcal{O}_X$. There remains to see the s_i are linearly independent over $\mathcal{O}_X$. Assume we have a relation

(1) $$\Sigma\phi_i.s_i = 0$$

Then we must have $\phi_i(x) = 0$, for $i = 1,\dots,q$. Let ν be the minimum of the orders of the zeroes of the ϕ_i at x. It is > 0. To arrive at a contradiction, an induction on ν shows that it suffices to construct a non-trivial relation with a strictly smaller ν .

Assume that ϕ_1 say, has order ν at 0. Using local coordinates, we can find $\partial \in \Theta_X$ such that $\partial\phi$, has a zero of order $> \nu$ at x. The relation $\Sigma\partial(\phi_is_i) = 0$ gives

$$\Sigma\partial\phi_i.s_i + \Sigma\phi_i.\partial s_i = 0$$

We may find regular functions a_{ij} around x such that

$$\partial s_i = \Sigma_j a_{ij}s_j \ .$$

We have then the relation

$$\Sigma_i s_i(\partial\phi_i + \Sigma_j a_{ij}\phi_j) = 0$$

where the coefficient of s_1 has at x a zero of order $> \nu$.

1.8 *The de Rham complex.* We let $\Omega^{\cdot}_X$ be the sheaf of germs of regular exterior differential forms on X. Since a $\mathcal{D}$-module may be viewed as a $\mathcal{O}_X$-module endowed with a flat connection, we may again form as usual (differential graded) sheaf of germs of exterior differential forms with coefficients in M,

$$\Omega^{\cdot}_X(M) = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} M$$

Locally, with a choice of lical coordinates $(x_i, {}_i)$, its differential can be expressed as

$$d(\omega \otimes m) = d\omega \otimes m + \Sigma_i dx_i \wedge \omega \otimes \partial_i m .$$

It is easily checked that it is independent of the choice of local coordinates and, using the flatness of the connection, that $d^2 = 0$.

1.9 *Local good filtrations, Characteristic variety.* On $\mathcal{D}_X$, we have the usual filtration by the degree, defined by induction on $m \in \mathbb{N}$ by:

(1) $$\mathcal{D}_X(0) = \mathcal{O}_X$$

(2) $$\mathcal{D}_X(m) = \{P \in \mathcal{D}_X \mid [P,\mathcal{O}_X] \subset \mathcal{D}_X(m-1)\} \quad (m \geqq 1)$$

More explicitely, in an affine neighborhood Y with local coordinates (x_i, ∂_i), $\mathcal{D}(Y)(m) = \mathcal{D}_X(m)(Y)$ consists of the P's of the form

(3) $$P = \Sigma_{|\alpha| \leqq m} f_\alpha \cdot \partial^\alpha, \text{ with } f_\alpha \in \mathcal{O}(Y)).$$

As in the case of the Weyl algebra, we see that

(4) $$\mathrm{Gr}\mathcal{D}(Y) = \mathcal{O}(Y)[\xi_1,\ldots,\xi_n] \quad (n=\dim Y),$$

where the ξ_i are commuting variables representing the ∂_i's. This is the direct image of the coordinate ring of the cotangent bundle $T^*(Y)$ of Y under the natural projection. Therefore

(5) $$\mathrm{Gr}\mathcal{D}_X = \tau_*(\mathcal{O}_{T^*(X)}),$$

where $\tau : T^*(X) \to X$ is the natural projection.

The ring $\mathcal{D}(Y)$ satisfies the assumptions (i),(ii) imposed on R in (V,2.2). As a consequence, the discussion there applies to $\mathcal{D}(Y)$, hence also, locally, to $\mathcal{D}_X$. In particular, the $\mathcal{D}_X$-module is locally finitely generated if and only if each $x \in X$ has an affine neighborhood Y such that $M(Y)$ has a good filtration Γ. with respect to $\mathcal{D}(Y)$. If so, we let again I be the annihilator of $\mathrm{Gr}M(Y)$ in $\mathcal{O}(T^*(Y))$ and J the radical of I. The variety in $T^*(Y)$ defined by J is the characteristic variety of $M(Y)$. Since J does not depend on the good filtration (V.2.2, lemma), this notion has an intrinsic character and allows one to define the *characteristic variety*, or *singular support* $SS(M)$ of any coherent $\mathcal{D}_X$-module M. It is conical, in the sense that

(6) $$\tau^{-1}(x) \cap SS(M) \neq x \Rightarrow \tau^{-1}(x) \subset SS(M), \quad (x \in X).$$

We let $d(M)$ be its dimension and, for $x \in X$, we let $d_x(M)$ be the dimension of $SS(M) \cap \tau^{-1}(Y)$ for sufficiently small neighborhoods Y of x. Let also

$$j_x(M) = \min\{j \mid \mathrm{Ext}^j_{\mathcal{D}_{X,x}}(M_x, \mathcal{D}_{X,x}) \neq 0\}$$

It is equal to

$$j(M(Y)) = \min\{j \mid \mathrm{Ext}^j_{\mathcal{D}(Y)}(M(Y), \mathcal{D}(Y)) \neq 0\}$$

for sufficiently small affine neighborhoods Y of x.

1.10 THEOREM (i) *The sheaf of rings $\mathcal{D}_X$ is locally noetherian and coherent.*

(ii) *The global homological dimension of $\mathcal{D}_{X,x}$ is equal to* $\dim X$ $(x \in X)$, *that of $\mathcal{D}_X$ is* $\leqq 2.\dim X$.

Let M be a $\mathcal{D}$-module:

(iii) $d_x(M) \geqq \dim X$ if $M_x \neq 0$

(iv) $d_x(M) + j_x(M) = 2.\dim X.$ $(x \in X)$

Proof. (i) follows from 1.9 as in the case of the Weyl algebra, and (iv) from 1.9 and V,2.2.2. Let M be a $\mathcal{D}$-module and fix $x \in X$. As recalled in 1.3 we may find an affine neighborhood Y of x, a hypersurface $T \subset Y$, an étale morphism $\sigma : Y - T \to \mathbb{A}^n$ ($n = \dim X$) wgose image is the complement of a hypersurface S. Let $U = Y - T$ and $V = \mathbb{A}^n - S$. The coordinate ring $\mathcal{O}(U)$ is finitely generated, integral over $\mathcal{O}(V)$. We may identify $\mathcal{D}(V)$ to the subring of $\mathcal{D}(U)$ generated over $\mathcal{O}(V)$ by the lifts of the canonical derivations $\partial_i \in A_n$, therefore

$$\mathcal{D}(U) = \mathcal{O}(U) \underset{\mathcal{O}(V)}{\otimes} \mathcal{D}(V) \tag{1}$$

The $\mathcal{D}(U)$-module $M(U)$ may then be viewed as a finitely generated $\mathcal{D}(V)$-module. According to Thm 1, p.17 in [Bu 2]

$$d_{\mathcal{D}(U)}(M(U)) = d_{\mathcal{D}(V)}(M(U)) \tag{2}$$

where $d_{\mathcal{D}(U)}$ (resp. $d_{\mathcal{D}(V)}$) refers to the dimension of the characteristic variety of $M(U)$, viewed as a $\mathcal{D}(U)$-module (resp. $\mathcal{D}(V)$)-module.

We have $\mathcal{O}(V) = \mathcal{O}(\mathbb{A}^n)[f^{-1}]$, where f generates the ideal of S, whence also $\mathcal{D}(V) = \mathcal{D}(\mathbb{A}^n)[f^{-1}] = A_n[f^{-1}]$. It is standard that

$$\mathrm{gl\ hd}(A_n[f^{-1}]) \leq \mathrm{gl\ hd}\ A_n = n \tag{3}$$

therefore

$$j_{\mathcal{D}(V)}(M(U)) \geq n . \tag{4}$$

and, by (iv),

$$d_{\mathcal{D}(V)}(M(U)) \geq n . \tag{5}$$

Together with (2), this yields (iii). Using (iv) again, we now have

$$\mathrm{Ext}^j_{\mathcal{D}(U)}(M(U),\mathcal{D}(U)) = 0 \text{ for } j > n \tag{6}$$

for every finitely generated $M(U)$. By V,2.3.3, the relation (6) remains true if the second argument is replaced by a finitely generated $\mathcal{D}(U)$-module and then, in view of §8, No.3 in [Bu 1], by a general $\mathcal{D}(U)$-module. It follows that we now have

$$(\mathcal{E}xt^j_{\mathcal{D}_X}(M,N))_x = 0 \text{ for } j > n \ (M,N \in \mu(\mathcal{D}_X);x \in X). \tag{7}$$

Similarly, we have

$$\mathrm{Ext}^j_{\mathcal{D}_{X,x}}(M_x,N_x) = 0 \text{ for } j > n , \tag{8}$$

which yields the first part of (ii). There exists a spectral sequence abutting to $\mathrm{Ext}^{\cdot}_{\mathcal{D}_X}(M,N)$ in which

$$E_2^{q,p} = H^p(X,\mathcal{E}xt^q_{\mathcal{D}_X}(M,N)) \tag{9}$$

[Go:II,7.3.3]. By (7), $E_2^{q,p}$ is zero if $q > n$. On the other hand, the cohomology of X with coefficients in a sheaf is zero above n by the Serre-Grothendieck vanishing theorem [Ha 2:III,2.7], therefore $\mathrm{Ext}^s_{\mathcal{D}_X}(M,N) = 0$ for $s > 2n$. This proves the second part of (ii).

1.11 If A is a category, then, as usual $D^b(A)$ (resp. $D^-(A)$, resp. $D^+(A)$) is the derived category of bounded (resp. bounded above, resp. bounded below) complexes in A. We shall however write $D^b(\mathcal{D}_X)$ for $D^b(\mu(\mathcal{D}_X))$ since this is the derived category of interest to us. Let us denote by $D^b_{qc}(\mathrm{Mod}(\mathcal{D}_X))$ the derived category of bounded complexes $\mathcal{F}^{\cdot}$ in $\mathrm{Mod}(\mathcal{D}_X)$ whose cohomology sheaves are quasi-coherent. If $\mathcal{F}^{\cdot}$ consists of q.c. $\mathcal{D}$-modules, then obviously $H^{\cdot}\mathcal{F}^{\cdot}$ is also q.c. The natural inclusion $\mu(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X)$ induces therefore a morphism of derived categories:

$$\alpha : D^b(\mathcal{D}_X) \to D^b_{qc}(\mathrm{Mod}(\mathcal{D}_X)) \tag{1}$$

J. Bernstein has recently shown that α is an equivalence of categories. His proof (or a more general result in fact) will be given at the end of Section 2.

We let $\mathrm{coh}(\mathcal{D}_X)$ be the category of coherent $\mathcal{D}_X$-modules and $D^b_{coh}(\mathcal{D}_X)$ the bounded derived category of q.c. $\mathcal{D}_X$-modules whose cohomology is coherent. A bounded complex of coherent $\mathcal{D}_X$-modules obviously belongs to it, whence again a natural morphism $D^b(\mathrm{coh}(\mathcal{D}_X)) \to D^b_{coh}(\mathcal{D}_X)$. It is also an equivalence of categories. Once it is known that any element of $\mu(\mathcal{D}_X)$ is an inductive limit of coherent submodules (2.3), this follows from an argument communicated to me by

Deligne (which yields more generally an equivalence between the bounded derived category of an abelien category A and the bounded derived category of the ind-objects of A with cohomology in A). For the sake of completeness, the proof in the case of interest here is given at the end of § 2 (2.11).

1.12 *Holonomic modules.* Let $M \in \mu(\mathcal{D}_X)$. Then M is *holonomic* if it is coherent and if $\dim_x SS(M) \leqq \dim X$ for all $x \in X$.

Note that if the inequality is strict, then $M_x = 0$ in view of 1.10(iii). We have again:

Let $M \in \mathrm{coh}(\mathcal{D}_X)$ *and* $x \in X$. *If* $M_x \neq 0$, *then* M *is holonomic around* x *if and only if* $Ext^j_{X,x}(M,\mathcal{D}_X) = 0$ *for* $j \neq \dim X$.

As in the case of the Weyl algebra. (V,1.11) we see that if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathcal{D}_X$-modules, then M is holonomic if and only if M' and M'' are so. The holonomic $\mathcal{D}_X$-modules form a full subcategory of $\mathrm{coh}(\mathcal{D}_X)$, to be denoted $\mathrm{hol}(\mathcal{D}_X)$. Also, 1.9 implies that M is holonomic if and only if $Ext^i_{\mathcal{D}_X}(M,\mathcal{D}_X) = 0$ for $i \neq \dim X$.

We also note that (V,1.16.1) implies that *any holonomic module has finite length*. It is enough to show that each point $x \in X$ has some affine neighborhood U in which this is the case. Assume $U,V = \mathbb{A}^n - S$ and f to be as in the proof of 1.10. Let M be a holonomic $\mathcal{D}(U)$-module It is then a $\mathcal{D}(V)$-module, which is holonomic by 1.10(2). We have $A_n \subset \mathcal{D}(V)$ and M is holonomic over A_n by (V,1.16.3), hence has finite length over A_n (V,1.16.1). Then $M = M[f^{-1}]$ has also finite length over $\mathcal{D}(V) = A_n[f^{-1}]$.

1.13 *Holonomic complexes.* A complex $M^\cdot \in D^b(\mathcal{D}_X)$ is said to be *holonomic* if the derived complex $H^\cdot M^\cdot$ is holonomic. We let $D^b_h(\mathcal{D}_X)$ be the category of bounded holonomic complexes.

Note it is *not* required that the M^i be holonomic. It would be more precise to call $M^\cdot$ a complex with holonomic cohomology, as in V. We shall follow the tradition by speaking of holonomic complexes. Of

course, it is the only sensible notion if we put ourselves squarely in the derived category, as we are supposed to do, since if $M^\cdot$ and $N^\cdot$ are h.i. complexes and one of them consists of holonomic $\mathcal{D}$-modules, the other one need not be so.

It follows from 1.12 that if $M^\cdot$ consists of holonomic $\mathcal{D}$-modules, then it is a holonomic complex. There is therefore a natural map

$$D^b(\mathrm{hol}(\mathcal{D}_X)) \to D^b_h(\mathcal{D}_X) .$$

Beilinson has recently proved that it is an equivalence of categories [Bi], but we shall not need this fact.

1.14 PROPOSITION. *If two vertices of an exact triangle in* $D^b(\mathcal{D}_X)$ *belong to* $D^b_h(\mathcal{D}_X)$ *, then so does the third one. The category* $D^b_h(\mathcal{D}_X)$ *is generated, as a triangulated category, by* $\mathrm{hol}(\mathcal{D}_X)$.

The first assertion follows, by using the long exact cohomology sequence, from the extension property stated in 1.12. This implies the second assertion.

1.15 PROPOSITION. *Let* $M^\cdot \in D^b(\mathcal{D}_X)$. *Assume there exists a spectral sequence* (E_r) *of* $\mathcal{D}_X$*-modules which converges to the graded module* $\mathrm{gr}\, H^\cdot M^\cdot$ *associated to a finite filtration of* $M^\cdot$ *and that* E_s *is holonomic for some* s . *Then* $M^\cdot$ *is holonomic.*

In fact, E_r is then holonomic for all $r \geqq s$, hence $\mathrm{gr}\, H^\cdot M^\cdot$ is holonomic too. Repeated use of 1.12 now shows that $M^\cdot$ is holonomic.

1.16 *Holonomic defect.* Let $M \in \mathrm{coh}(\mathcal{D}_X)$. The holonomic defect of M is, by definition, $e(M) = d(M) - \dim X$. By 1.10, it is $\geqq 0$, (unless $M = 0$).

1.17 *Support of a $\mathcal{D}$-module.* The support $\mathrm{Supp}(F)$ of $F \in \mu(\mathcal{D}_X)$ is, as usual, the set of $x \in X$ for which $F_x \neq 0$. If $F \in \mathrm{coh}(\mathcal{D}_X)$, it is closed, since the support of a section of a sheaf of abelian groups is always closed. Let $\sigma : T^*(X) \to X$ be the canonical projection. Then $\mathrm{Supp}\, F = \sigma(SS(F))$. In particular $\sigma(SS(F))$ is closed. This can also be seen directly: Let Z be an irreducible component of SS($\mathcal{F}$). If $Z \subset X$, then its image is clearly closed. If not, Z is conical (1.9(6)), hence the restriction of σ to Z factors through the projection onto X of the projectivised cotangent bundle, which is a proper map, and therefore sends Z onto a closed set.

§ 2 CONSTRUCTION OF RESOLUTIONS. TWO EQUIVALENCES OF CATEGORIES.

A. *Resolutions*

2.1 PROPOSITION. *Any $\mathcal{M} \in \mu(\mathcal{D}_X)$ may be embedded in an injective (quasi-coherent) $\mathcal{D}_X$-module $\mathcal{J}$.*

Let $(X_i)_{i\in I}$ be a finite open affine cover of X and $j_i : X_i \to X$ the inclusion. Let $\mathcal{M}_i = j_i^* \mathcal{M}_i$ be the restriction of $\mathcal{M}$ to X_i and $M_i = \mathcal{M}(X_i)$ $(i \in I)$. We can find an injective $\mathcal{D}(X_i)$-module J_i which contains M_i as a $\mathcal{D}(X_i)$-submodule. Let $\mathcal{J}_i$ be the associated ($\mathcal{O}_{X_i}$-quasi-coherent) $\mathcal{D}_{X_i}$-module. It is injective. We have an exact sequence

$$(1) \qquad 0 \to \mathcal{M}_i \to \mathcal{J}_i \qquad (i \in I).$$

The direct image j_{i*} in the $\mathcal{O}$-category preserves quasi-coherence and, since X_i is affine, is exact, whence an exact sequence

$$(2) \qquad 0 \to j_{i*}(\mathcal{M}_i) \xrightarrow{s_i} j_{i*}(\mathcal{J}_i) ,$$

These direct images are in an obvious way $\mathcal{D}_X$-modules: if $U \subset X$ is open then $\mathcal{D}(U \cap X_i)$ operates naturally on the space of sections over $U \cap X_i$. [In fact this is a special case of the notion of direct image of $\mathcal{D}$-modules, to be discussed later.] Since j_{i*} is left adjoint to the restriction j_i^* to X_i, which is exact, and $\mathcal{J}_i$ is injective, it follows that $j_{i*}(\mathcal{J}_i)$ is an injective $\mathcal{D}_X$-module. Let r_i be the canonical map

$$\mathcal{M}_i \to j_{i*} j_i^* \mathcal{M} = j_{i*} \mathcal{M}_i$$

and $\mathcal{J}$ be the direct sum of the $j_{i*}(\mathcal{J}_i)$. It is injective and the direct sum of the $s_i \circ r_i$ provides an embedding of $\mathcal{M}$ into $\mathcal{J}$.

2.2 THEOREM *Any $\mathcal{M} \in \mu(\mathcal{D}_X)$ has an injective resolution*

$$(1) \qquad 0 \to \mathcal{M} \to \mathcal{J}^0 \to \ldots \to \mathcal{J}^m \to 0$$

with $m \leq 2.\dim X + 1$.

By iterated use of 2.1, we can construct a resolution

(2) $$0 \to \mathcal{M} \to \mathcal{J}^0 \to \dots \to \mathcal{J}^{2n} \to \mathcal{N} \to 0$$

where $n = \dim X$ and the $\mathcal{J}^i$ are injective. It follows them from 1.9(ii) and the same (standard) arguments as in the proof of Prop.4 in [Bu 1: § 8, No 3] that $\mathcal{N}$ is injective.

2.3 LEMMA (i) *Let* $\mathcal{F} \in \mathrm{coh}(\mathcal{D}_X)$. *Then there exists a coherent* $\mathcal{O}_X$*-submodule* $\mathcal{G}$ *of* $\mathcal{F}$ *which generates* $\mathcal{F}$ *over* $\mathcal{D}_X$.

(ii) *Let* $U \subset X$ *be open,* $\mathcal{F} \in \mu(\mathcal{D}_X)$ *and assume that the restriction* $\mathcal{G}$ *of* $\mathcal{F}$ *to* U *belongs to* $\mathrm{coh}(\mathcal{D}_U)$. *Then there exists* $\tilde{\mathcal{G}} \in \mathrm{coh}(\mathcal{D}_X)$ *contained in* $\mathcal{F}$ *and such that* $\tilde{\mathcal{G}}|U = \mathcal{G}$.

(iii) *Any* $\mathcal{F} \in \mu(\mathcal{D}_Y)$ *is an inductive limit of* $\mathcal{D}_X$*-coherent submodules.*

(o) We remark first that (ii) is known if $\mathcal{D}_X$ is replaced by $\mathcal{O}_X$ [EGA I : 9.4.7]. It follows that any q.c. $\mathcal{O}_X$-module is inductive limit of its locally coherent $\mathcal{O}_X$-submodules [EGA I :9.4.9].

(i) There exists a finite open affine cover $(X_i)_{i\in I}$ of X such that the restriction of $\mathcal{F}$ to X_i is generated by finitely many sections s_j. By the result just recalled, there exists a $\mathcal{O}_X$-coherent submodule $\mathcal{G}_i$ of $\mathcal{F}$ whose restriction to X_i is the $\mathcal{O}_{X_i}$-submodule generated by the s_j's. Then we may take for $\mathcal{G}$ the sum of the $\mathcal{G}_i$'s.

(ii) By (i), applied to U, there exists a $\mathcal{O}_U$-coherent submodule $\mathcal{G}_o$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{D}_U.\mathcal{G}_o$. By our initial remark, we may extend $\mathcal{G}_o$ to a $\mathcal{O}_X$-coherent submodule $\tilde{\mathcal{G}}_o$ of $\mathcal{F}$. We let then $\mathcal{G} = \mathcal{D}_X.\tilde{\mathcal{G}}_o$.

(iii) now follows by applying (ii) to the members of a finite open affine cover.

2.4 PROPOSITION *Let* X *be quasi-projective and* $\mathcal{F} \in \mu(\mathcal{D}_X)$. *Then* $\mathcal{F}$ *is a quotient of a locally free* $\mathcal{D}_X$*-module.*

Proof. Let $i : X \to P = \mathbb{P}_N(\mathbb{C})$ be a locally closed projective embedding of X. It suffices to show that $i_*(\mathcal{F})$ is the quotient of a locally free $\mathcal{O}_P$-module Q because then $Q|X$ is locally free over $\mathcal{O}_X$ and $\mathcal{F}$ is the quotient of

$\mathcal{D}_X \otimes_{\mathcal{O}_X} (Q|X)$. But this is well-known : $i_*(\mathcal{F})$ is q.c. over $\mathcal{O}_P$, hence inductive limit of $\mathcal{O}_P$-coherent submodules $\mathcal{F}_i$ $(i\in I)$, (2.3(o)). For each i, there exists n_i such that $\mathcal{F}_i$ is quotient of a direct sum of twisted sheaves $\mathcal{O}_P(n_i)$ [Ha 2: II, 5.18], therefore $\mathcal{F}$ is the quotient of a direct sum $\mathcal{Q}$ of invertible sheaves $\mathcal{O}_P(n_s)$ (s S). Consider the open cover (P_j) of P where P_j is the set of points at which some homogeneous coordinate is non-zero. Then the restriction of $\mathcal{O}_P(n)$ $(n\in\mathbb{Z})$ to P_j is free, hence $Q|P_j$ is free and Q is locally free.

2.5 THEOREM *Assume* X *to be quasi-projective. Let* $\mathcal{F} \in \mu(\mathcal{D}_X)$ *and* $m\in\mathbb{N}$, $m \geqq \dim X$. *Then* $\mathcal{F}$ *admits a left resolution.*

(1) $$0 \to \mathcal{P}^{-m} \to \mathcal{P}^{-m+1} \to \dots \to \mathcal{P}^0 \to \mathcal{F} \to 0$$

in which $\mathcal{P}^{-m}$ *is locally projective and* $\mathcal{P}^{-i}$ *locally free for* $i \in [0,m)$.

By 2.4, we can find a resolution (1) in which the $\mathcal{P}^{-i}$ are locally free for $i \in [0,m)$. That $\mathcal{P}^{-m}$ is then locally projective follows by standard homological algebra, applied to the sections over an open affine subset, in view of the fact that, locally, the homological dimension of $\mathcal{D}_X$ is $\leqq \dim X$.

2.6 COROLLARY. *Let* $\mathcal{F}^\cdot \in D^b(\mathcal{D}_X)$.

(i) $\mathcal{F}^\cdot$ *admits a bounded right injective resolution and a bounded left flat resolution.*

(ii) *Let* X *be quasi-projective and* $m \in \mathbb{N}$. *Then* $\mathcal{F}^\cdot$ *admits a bounded locally projective left resolution which is locally free in degrees contained in* $[-m,m]$.

This follows from 2.5 by standard constructions.

2.7 *Resolution of* $\mathcal{O}_X$. We note first that, as a $\mathcal{D}_X$-module, $\mathcal{O}_X$ may be identified with $\mathcal{D}_X/\mathcal{D}_X.\Theta_X$. We consider the complex

(1) $$\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n\Theta_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{n-1}\Theta_X \to \dots \to \mathcal{D}_X,$$

where $\mathcal{D}_X \otimes \Lambda^i\Theta_X$ is given the degree $-i$, with differential

$$(2)\quad \begin{aligned}\partial(P\otimes\xi_1\wedge\ldots\wedge\xi_p) &= \Sigma(-1)^{i+1}P\xi_i \otimes \xi_1\wedge\ldots\wedge\widehat{\xi_i}\wedge\ldots\wedge\xi_p + \\ &+ \sum_{i<j}(-1)^{i+j}P[\xi_i,\xi_j]\otimes\xi_1\wedge\ldots\wedge\widehat{\xi_i}\wedge\ldots\wedge\widehat{\xi_j}\wedge\ldots\wedge\xi_p\end{aligned}$$

If (x_i,∂_i) is a local coordinate system around x and E is the complex vector space spanned by the ∂_i's, then we can write it as

$$(3)\quad \mathcal{D}_X \underset{\mathbb{C}}{\otimes} \Lambda^n E \to \mathcal{D}_X \underset{\mathbb{C}}{\otimes} \Lambda^{n-1}E \to \ldots \to \mathcal{D}_X$$

with differential

$$(4)\quad d(P\otimes\partial_{j_1}\wedge\ldots\wedge\partial_{j_p}) = \Sigma(-1)^{i+1}P\partial_{j_i} \otimes \partial_{j_1}\wedge\ldots\wedge\widehat{\partial_{j_i}}\wedge\ldots\wedge\partial_{j_p}\ .$$

This is a Koszul complex. Since the ∂_i form a regular sequence, it is acyclic, except in degree 0, where the cohomology is equal to $\mathcal{D}_X/\mathcal{D}_X.\Theta_X$. Moreover, left multiplication on $\mathcal{D}_X$ commutes with the differentials. Therefore (3), together with the augmentation $\mathcal{D}_X \to \mathcal{D}_X/\mathcal{D}_X.\Theta_X = \mathcal{O}_X$ provides a locally free resolution of $\mathcal{O}_X$ in $\mu(\mathcal{D}_X)$.

B. *Two equivalences of derived categories.*

2.9 We now turn to the theorem of Bernstein mentioned in 1.11. In this, X could be a quasi-compact and separated scheme. We assume given a sheaf of algebras A on X together with a sheaf morphism $\mathcal{O}_X \to A$ such that A is q.c. over $\mathcal{O}_X$. Let Mod(A) be the category of A-Modules and $\mu(A)$ the subcategory of q.c. A-Modules, D^b_{qc}(Mod A) the bounded derived category of complexes of A-Modules with q.c. cohomology.

2.10 THEOREM (J. Bernstein). *The natural morphism of derived categories* $\alpha : D^b(\mu(A)) \to D^b_{qc}(\text{Mod } A)$ *induced by the inclusion* $\mu(A) \to \text{Mod}(A)$ *is an equivalence.*

Proof. The category $D^b_{qc}(\text{Mod}(A))$ is generated, as a triangulated

category (I,§12) by its cohomology objects, i.e. by q.c. A-modules. In view of [I, 12.9] there remains to see that α is fully faithful, i.e. that the natural homomorphism

$$\text{Hom}^{\cdot}_{D(\mu(A))}(F^{\cdot},G^{\cdot}) \to \text{Hom}^{\cdot}_{D(\text{Mod}(A))}(F^{\cdot},G^{\cdot}) \quad (F^{\cdot},G^{\cdot} \in D^b(\mu(A))).$$

is an isomorphism. In fact, we prove it for R Hom. There are two steps to the proof:

a) *Reduction to the case where* X *is affine*. Let $U \subset X$ be open, affine and $i : U \to X$ the inclusion.Then $\mu(A|U)$ may be identified to the category mod(A(U)) of A(U)-modules. It has therefore enough injectives. If $J \in \text{mod}(A(U))$ is injective, then $i_*(\tilde{J})$, where $\tilde{J}$ is the associated q.c. $A|U$-module, is also injective, since i_* is adjoint to the restriction to U, which is exact. It follows (as in 2.1) that any $M \in \mu(A)$ can be embedded in a finite direct sum of modules of the type $i_*(\tilde{J})$, hence that $G^{\cdot}$ has a injective resolution of this type. It suffices therefore to prove that the natural map

$$\text{Hom}^{\cdot}_{D(\mu(A))}(F^{\cdot},i_*(\tilde{J})) \to \text{Hom}^{\cdot}_{D(\text{Mod } A)}(F^{\cdot},i_*(\tilde{J}))$$

is an isomorphism, for any $F^{\cdot}$ and any $i_*(\tilde{J})$ as above. But

$$\text{Hom}(F^{\cdot},i_*(\tilde{J})) = \text{Hom}(F^{\cdot}|U,\tilde{J})$$

in both categories. This reduces us to the second step:

b) X *is affine, and* $G^{\cdot}$ *is a single degree complex consisting of an injective q.c.* A*-module* J.

Let $A = A(X)$. Then $\mu(A)$ is isomorphic to the category mod(A) of A-modules. We may replace $F^{\cdot}$ by a bounded above locally free resolution. To prove that the RHom(F,J) in both categories are isomorphic, it suffices then to consider the case where $F^{\cdot}$is a free A-module. Since cohomology commutes with direct products, this reduces us to the case where $F^{\cdot} = A$. We have then on one hand

$$R^i\text{Hom}_{D(\mu(A))}(A,J) = R^i\text{Hom}_A(A,J(X)) = \begin{cases}0, & i \neq 0\\ J(X) & i = 0\ .\end{cases}$$

and on the other

$$R^i\mathrm{Hom}_{D(\mathrm{Mod}\ A)}(A,\mathcal{J}) = R^i\Gamma(X,\mathcal{J}) = \begin{cases} 0 & i \neq 0 \\ \Gamma(X,\mathcal{J}) = \mathcal{J}(X) & i = 0 \end{cases}$$

since $\mathcal{J}$ is quasi-coherent (Serre's theorem).

2.11 PROPOSITION *The natural morphism* $\mu : D^b(\mathrm{coh}(\mathcal{D}_X)) \to D^b_{\mathrm{coh}}(\mathcal{D}_X)$ *is an equivalence of categories.*

(a) We prove first the following assertion:

(*) *Let* $F^\cdot$ *be a bounded complex of q.c.*$\mathcal{D}_X$ *-modules with coherent cohomology. Then* $F^\cdot$ *is the inductive limit of coherent subcomplexes which are q.i. to* $F^\cdot$ *under the inclusion map.*

(2.3) implies obviously that $F^\cdot$ is an inductive limit of coherent subcomplexes. It suffices therefore to show that given a coherent subcomplex $K^\cdot$ of $F^\cdot$, there exists a coherent subcomplex $G^\cdot$ of $F^\cdot$ containing $K^\cdot$, which is h.i. to $F^\cdot$ under the inclusion map.

This is proved by a simple induction. Let $j \in \mathbb{Z}$. Assume that we have found a subcomplex $G^\cdot_j$ of $F_j = \oplus_{i \geq j} F^i$, containing $K_j = \oplus_{i \geq j} K^i$, such that $G^\cdot_j \to F_j$ is a h.i. in degrees $> j$ and that $G^j_j \cap Z(F^j)$ maps onto $H^j(F^\cdot)$ under the canonical projection. We can choose a coherent submodule A_{j-1} of F^{j-1} containing K^{j-1}, whose image under the differential is equal to $d(F^{j-1}) \cap A_j$. Moreover without altering these conditions, we may take its sum with a coherent submodule of cocycles so that the new module contains representatives of all the elements of $H^{j-1}(F^\cdot)$. Then for j small enough $G^\cdot_j$ is the sought for subcomplex.

(b) This implies that μ is surjective. There remains to show that μ is fully faithful, i.e.

$$\mathrm{RHom}^\cdot_{D^b(\mathrm{coh}(\mathcal{D}_X))}(F^\cdot,G^\cdot) = \mathrm{RHom}^\cdot_{D^b_{\mathrm{coh}}(\mathcal{D}_X)}(F^\cdot,G^\cdot) \quad (F^\cdot,G^\cdot \in D^b(\mathrm{coh}(\mathcal{D}_X))$$

We may assume $F^\cdot$ and $G^\cdot$ to be bounded and replace $F^\cdot$ by a bounded

locally projective left resolution, hence suppress the R. We prove first that μ is surjective. An element of $\mathrm{Hom}^\cdot_{D^b_{\mathrm{coh}}(\mathcal{D}_X)}(F^\cdot,G^\cdot)$ is represented by a diagram

$$\begin{array}{ccc} & & M^\cdot \\ F^\cdot & \overset{\alpha}{\nearrow} & \uparrow \beta \\ & & G^\cdot \end{array}$$

where β is a h.i. and $M^\cdot$ has coherent cohomology. Using truncation, we may assume $M^\cdot$ to be bounded. By (a), there exists a coherent subcomplex $N^\cdot$ of $M^\cdot$ containing $G^\cdot$ and h.i. to $M^\cdot$ under the inclusion. Then we may replace $M^\cdot$ by $N^\cdot$ and get a morphism in $D^b(\mathrm{coh}(\mathcal{D}_X))$. Let us prove now that μ is injective. Let $\alpha,\beta \in \mathrm{Hom}^\cdot_{D^b(\mathrm{coh}(\mathcal{D}_X))}(F^\cdot,G^\cdot)$ be homotopic in $D^b_{\mathrm{coh}}(\mathcal{D}_X)$. There exists then $L^\cdot \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$, a homology isomorphism $\gamma : G^\cdot \to L^\cdot$ and two homotopic morphisms $\alpha,\beta : F^\cdot \to L^\cdot$. The homotopy is given by a morphism $h : F^\cdot \to L^\cdot[1]$. By (a), we can find a coherent subcomplex $M^\cdot$ of $L^\cdot$ containing $h(F^\cdot)$ and $G^\cdot$, and h.i. to $L^\cdot$ under the inclusion. We can then replace $L^\cdot$ by $M^\cdot$, hence α and β are already equal in $D^b(\mathrm{coh}(\mathcal{D}_X))$.

§ 3 LEFT AND RIGHT $\mathcal{D}$-MODULES

3.1 Up to now, we have tacitly or explicitely assumed our $\mathcal{D}_X$-modules to be left modules. However, we shall have also to consider right $\mathcal{D}$-modules. For modules over the Weyl algebra, going over from one to the other was immediate, by use of the transposition isomorphism of A_n (see V, 3.3). In general however, a less obvious construction has to be used. We first give an example of a $\mathcal{O}_X$-module naturally endowed with a right $\mathcal{D}_X$-module structure.

3.2 *The right $\mathcal{D}_X$-module ω_X.* As usual, ω_X denotes the sheaf of germs of differential forms of top degree. It is locally free over $\mathcal{O}_X$, of rank one. A vector field $\xi \in \Theta_X$ operates on it naturally by means of the Lie derivative L_ξ . We obviously have

(1) $$[L_\xi, L_\eta] = L_{[\xi,\eta]} .$$

However, it is easily checked that

$$L_{f\xi}\omega = f.L_\xi\omega + \xi(f\omega) \qquad (\xi\in\omega_X, f\in\mathcal{O}_X, \xi\in\Theta_X) .$$

This action is therefore not $\mathcal{O}_X$-linear, hence is not a connection (cf.1.3). However, we can view it as a right action

$$L_{f.\xi}\omega = L_\xi(f\omega)$$

which, by (1), extends to a right $\mathcal{D}_X$-module structure, if ξ acts by $-\mathcal{D}_\xi$.

3.3 We can now exchange left and right $\mathcal{D}$-modules. Given a left $\mathcal{D}_X$-module F , let

(1) $$G = F \otimes_{\mathcal{O}_X} \omega_X$$

and define an action of Θ_X on G by:

(2) $$(f\otimes\omega)\xi = -\xi f \otimes \omega + f \otimes \omega.\xi$$

Easy computations show that it extends to a right $\mathcal{D}_X$-module structure. Let now G be a right $\mathcal{D}_X$-module. Then consider

(3) $$F = Hom_{\mathcal{O}_X}(\omega_X, G) = \omega_X^{-1} \otimes_{\mathcal{O}_X} G$$

and let Θ_X act by the rule

(4) $$\xi(\lambda)(\omega) = \lambda(\omega,\xi) - \lambda(\omega).\xi$$

Again, it is easily checked to yield a left $\mathcal{D}_X$-module structure. We shall keep the notation $\mu(\mathcal{D}_X)$ for the category of ($\mathcal{O}_X$-quasi-coherent) left $\mathcal{D}_X$-modules and let $\mu^r(\mathcal{D}_X)$ be the category of quasi-coherent right $\mathcal{D}_X$-modules. It is clear that the two previous operators define inverse equivalences of categories between $\mu(\mathcal{D}_X)$ and $\mu^r(\mathcal{D}_X)$, hence also between the corresponding bounded derived categories.

3.4 *Oda's rule* [O]. Let X be a complete curve of genus $g \neq 1$. A locally free $\mathcal{O}_X$-module M of rank one has a degree $d^\circ(M)$, which is in particular 0 if $M = \mathcal{O}_X$ and $2 - 2g$ if $M = \omega_X$. Moreover

(1) $$d^\circ(M\otimes N) = d^\circ(M) + d^\circ(N), \quad d^\circ(Hom_{\mathcal{O}_X}(M,N)) = d^\circ(N) - d^\circ(M) .$$

If M is a left $\mathcal{D}_X$-module, then it is endowed with a flat connection, hence $d^\circ(M) = 0$. It follows therefore from (1) and 3.3 that if N is a right $\mathcal{D}_X$-module, then $d^\circ(N) = 2 - 2g$.

Let M, M' be left $\mathcal{D}_X$-modules and N, N' be right $\mathcal{D}_X$-modules. Then, (1) implies that

(2) $$Hom_{\mathcal{O}_X}(N,M) \text{ and } N \otimes_{\mathcal{O}_X} N'$$

do not carry either a natural right or a left $\mathcal{D}_X$-module structure. On the other hand, for any x,

(3) $$M \otimes_{\mathcal{O}_X} M', Hom_{\mathcal{O}_X}(N,N')$$

are naturally left $\mathcal{D}_X$-modules and

(4) $$\mathcal{H}om_{\mathcal{O}_X}(M,N),\ M \otimes_{\mathcal{O}_X} N$$

are naturally right $\mathcal{D}_X$-modules. The actions of $\mathcal{D}_X$ on the second modules in (3) and (4) are given by 3.3(2) and (4) respectively. Those on the first modules in (3) and (4) are defined respectively by:

(5) $$\xi(m\otimes m') = \xi m \otimes m' + m \otimes \xi m' \quad (m\in M, m'\in M');$$

(6) $$\lambda.\xi(m) = \lambda(m).\xi - \lambda(\xi.m) \qquad (m\in M;\lambda\in \mathcal{H}om_{\mathcal{O}_X}(M,N)).$$

3.5 A resolution of ω_X in $\mu^r(\mathcal{D}_X)$. Consider the de Rham complex $\Omega^\cdot_X(\mathcal{D}_X) = \Omega^\cdot_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Its differential involves only left multiplication on $\mathcal{D}_X$. Therefore it commutes with the right $\mathcal{D}_X$-module structure defined by right multiplication on $\mathcal{D}_X$. Hence $\Omega^\cdot_X(\mathcal{D}_X)$ can be viewed as a complex in $\mu^r(\mathcal{D}_X)$. We claim that $\Omega^\cdot_X(\mathcal{D}_X)[d_X]$ is a resolution of ω_X. Choose local coordinates (x_i,∂_i). Let E be the $\mathbb{C}$-vector space spanned by the dx_i. Let $C^\cdot = \Omega^\cdot_X(\mathcal{D}_X)[d_X]$. We have

(1) $$C^{-j} = \Lambda^{n-j}E \otimes_{\mathbb{C}} \mathcal{D}_X$$

with differential

(2) $$d(\omega\otimes P) = (-1)^{n-j}\Sigma_i \omega\wedge dx_i \otimes \partial_i P \quad (\omega\in\Lambda^{n-j}E; P\in\mathcal{D}_X).$$

Therefore

(3) $$H^j(C^\cdot) = 0 \qquad (j\neq 0)$$

and we have an exact sequence, where $\omega_n = dx_1\wedge\ldots\wedge dx_n$:

(4) $$0 \to \omega_n \otimes_{\mathbb{C}} (\Sigma\partial_i.\mathcal{D}_X) \to \omega_n \otimes_{\mathbb{C}} \mathcal{D}_X \to \omega_n \otimes_{\mathbb{C}} \Theta_X\mathcal{D}_X\backslash\mathcal{D}_X \to 0$$

The last factor on the right may be identified with $\mathcal{O}_X$ and hence the last term is ω_X. Moreover, this identifiation carries the right $\mathcal{D}_X$-module structure defined by right multiplication on $\mathcal{D}_X$ onto the right $\mathcal{D}_X$-module structure considered above. Therefore

(5) $$\Omega^\cdot_X(\mathcal{D}_X)[d_X] \to \omega_X \to 0$$

is a locally free left resolution of ω_X in $\mu^r(\mathcal{D}_X)$.

3.6 *The dualizing $\mathcal{D}$-module.* We set

(1) $$\tilde{\mathcal{D}}_X = \mathcal{H}om_{\mathcal{D}_X}(\omega_X,\mathcal{D}^r_X) = \mathcal{D}^r_X \otimes_{\mathcal{O}_X} \omega_X^{-1},$$

where $\mathcal{D}^r_X$ refers to $\mathcal{D}_X$ viewed as a right $\mathcal{D}_X$-module via right multiplication. By 3.3 , it carries a natural left $\mathcal{D}_X$-module structure, call it a). In addition, it has a left $\mathcal{D}_X$-module structure stemming from the left multiplication on $\mathcal{D}_X$, call it b). They commute. In the sequel $\tilde{\mathcal{D}}_X$ is viewed as endowed with these two commuting left $\mathcal{D}_X$-modules structures. By definition the dualizing $\mathcal{D}$-module is $\tilde{\mathcal{D}}_X[d_X]$, i.e. $\tilde{\mathcal{D}}_X$ viewed as a single degree complex, non-zero only in degree $-d_X$. This is in analogy with the topological situation: since X is a $2d_X$-dimensional manifold, the dualizing sheaf underlying Verdier duality is $\mathbb{C}_X[2d_X]$. Again in analogy with the Verdier dual [IC:V,§7], given $M^\cdot \in D^b(\mathcal{D}_X)$, we define its dual differential graded $\mathcal{D}_X$-module by the formula

(2) $$\mathbb{D}_X M^\cdot = R\mathcal{H}om^\cdot_{\mathcal{D}_X}(M^\cdot,\tilde{\mathcal{D}}_X[d_X]) = R\mathcal{H}om^\cdot_{\mathcal{D}_X}(M^\cdot,\tilde{\mathcal{D}}_X)[d_X] .$$

It is understood that, in forming $\mathcal{H}om_{\mathcal{D}_X}(M,\tilde{\mathcal{D}}_X[d_X])$ we use on $\tilde{\mathcal{D}}_X$ the structure (a). Then structure (b) allows us to view $\mathbb{D}_X M^\cdot$ again as an element of $D^b(\mathcal{D}_X)$. We have

(3) $$R^i\mathbb{D}_X M^\cdot = \mathcal{E}xt^{i+d}_{\mathcal{D}_X}(M^\cdot,\tilde{\mathcal{D}}_X) \qquad (i\in\mathbb{Z})$$

3.7 THEOREM (i) *$M \in \mathrm{coh}(\mathcal{D}_X)$ is holonomic if and only if $R^i\mathbb{D}_X M = 0$ for $i \neq 0$. The map $M \mapsto M^* = R^0\mathbb{D}_X M$ is an equivalence of category on $\mathrm{hol}(\mathcal{D}_X)$, whose square is the identity.*

(ii) *$M^\cdot \mapsto M^* = \mathbb{D}_X M^\cdot$ is a equivalence of category on $D_h^b(\mathcal{D}_X)$, whose square is the identity.*

Proof. (i) The first assertion was already proved in a different notation, in 1.11 and the second one is derived as in V,2.4.3.

(ii) Let $M^\cdot \in D_h^b(\mathcal{D}_X)$. There is a spectral sequence in which

(1) $$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{H}^q M^\cdot, \mathcal{D}_X[d_X] \Rightarrow \mathcal{H}^{p-q}(\mathbb{D}_X M^\cdot).$$

By assumption, the $\mathcal{H}^q M^\cdot$ are holonomic, hence $E_2^{p,q} = 0$ for $p \neq 0$ by (i), whence

(2) $$H^q(\mathbb{D}_X M^\cdot) = \mathbb{D}_X \mathcal{H}^{-q} M^\cdot, \quad (q \in \mathbb{Z})$$

and (ii) follows from (i).

3.8 *Affine varieties.* Let X be affine. We set

(1) $$\omega(X) = \Gamma(X, \omega_X).$$

It is a projective $\mathcal{O}(X)$-module, locally of rank one and

$$\omega(X)^* = \Gamma(X, \omega_X^{-1}), \omega(X) \otimes_{\mathcal{O}(X)} \omega(X)^* \cong \mathcal{O}(X).$$

$\omega(X)$ is a right $\mathcal{D}(X)$-module and the exchange of left and right, in terms of $\mathcal{D}(X)$-modules, takes the following form:

If M is a left $\mathcal{D}(X)$-module, then $M \otimes_{\mathcal{O}(X)} \omega(X)$ is a right $\mathcal{D}(X)$-module.

If N is a right $\mathcal{D}(X)$-module, then

$$N \otimes_{\mathcal{O}(X)} \omega(X)^* = \mathrm{Hom}_{\mathcal{O}(X)}(\omega(X), M)$$

is a left $\mathcal{D}(X)$-module. The $\mathcal{D}(X)$-module structures are given by the analogues of 3.3.2 and 3.3.4. The dualizing module $\tilde{D}(X) = \Gamma(X, \tilde{\mathcal{D}}_X)$ is

then $\mathcal{D}(X) \otimes_{\mathcal{O}(X)} \omega(X)^*$ and we have

$$\mathbb{D}_X(M^\cdot) = \mathrm{RHom}^\cdot_{\mathcal{D}(X)}(M^\cdot, \tilde{\mathcal{D}}(X)[d_X]) \quad (M^\cdot \in D^b(\mathrm{mod}(\mathcal{D}(X))).$$

§ 4 INVERSE IMAGES

4.0 Our main objective in the next sections is to defines inverse image morphisms $D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ and direct image morphisms $D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ associated to π. The choice of notation will be in part dictated by the wish that the Riemann-Hilbert correspondence be as much as possible compatible with it. Let me recall briefly that in the topological case, say for pseudomanifolds, we have in the derived category of constructible sheaves a direct image π_*, a direct image with proper supports $\pi_!$, an inverse image π^* and an extraordinary inverse image $\pi^!$. The last one is defined from $\pi_!$ by an adjointness property; we have the relations

(1) $$\pi^! = \mathfrak{D}_Y \pi^* \mathfrak{D}_X \qquad \pi_! = \mathfrak{D}_X \pi_* \mathfrak{D}_Y .$$

where $\mathfrak{D}$ refers to Verdier duality (see e.g.[IC:V,§§7,8,10]).

In the discussion to come, I need to distinguish between the concepts of direct and inverse images in the categories of sheaves, of $\mathcal{O}$-modules and of $\mathcal{D}$-modules. I shall denote them respectively by $\pi, \pi^{-1}, \pi_*, \pi^*$ and π_+, π^+. It will turn out that the naturally defined functors are $\pi^!$ and π_+. The others will be obtained from them by use of duality, as in (1).

In the sequel all algebraic varieties are quasi-projective. If M *and* N *are algebraic varieties of pure dimension, we denote by* d_M or dim M *the dimension of* M *and set* $d_{M,N} = d_M - d_N$.

4.1 We want to associate to $\pi : Y \to X$ homomorphisms $\pi^o : \mu(\mathcal{D}_X) \to \mu(\mathcal{D}_X)$ and $\pi^! : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$. If M is $\mathcal{O}_X$-module, we have in the $\mathcal{O}$-category

(1) $$\pi^*(M) = \mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}M .$$

$\pi^*(M)$ is quasi-coherent if M is so. Assume now $M \in \mu(\mathcal{D}_X)$. We want to define a $\mathcal{D}_Y$-module structure on $\pi^*(M)$. We first define a connection locally. Let $y \in Y$, $x = \pi(y)$ and let (x_i, ∂_i) local coordinates around x. For $\xi \in \Theta(Y)$, we let

(2) $$\xi(f \otimes \pi^{-1}m) = \xi f \otimes \pi^{-1}m + \Sigma f\xi(x_i \circ \pi) \otimes \pi^{-1}(\partial_i m), (f \in \mathcal{O}_Y; m \in M)$$

This is independent of the choice of local coordinates. It is enough to check this in an arbitrary small neighborhood of y in the *analytic* topology. If (x'_j, ∂'_j) is another set of local coordinates around x then, sufficiently close to x in the ordinary topology, the x_i and the x'_j are two systems of analytic coordinates. We have

$$\Sigma_j f\xi(x'_j \circ \pi) \otimes \pi^{-1}(\partial'_j m) = \Sigma_{i,j,k} f\cdot\left(\frac{\partial x'_j}{\partial x_i}\circ\pi\right)\xi(x_i \circ \pi) \otimes \pi^{-1}\left(\frac{\partial x_k}{\partial x'_j}\cdot\partial_k m\right) =$$

$$= \Sigma_{i,k} f\xi(x_i \circ \pi)\circ\pi^{-1}(\partial_k m)\{\Sigma_j \frac{\partial x'_j}{\partial x_i}\cdot\frac{\partial x_k}{\partial x'_j}\circ\pi\} = \Sigma_{i,k} f\xi(x_i\circ\pi)\circ\pi^{-1}(\partial_k m)\delta_{ik} =$$

$$= \Sigma f\xi(x_i \circ \pi)\circ\pi^{-1}(\partial_i m) .$$

Furthermore, a simple computation, using the fact that the ∂_i's commute, shows that this connection is flat, whence the sought for $\mathcal{D}_Y$-module structure. Let $\pi^o M$ be the $\mathcal{D}_Y$-module so defined.

4.2 *The sheaf* $\mathcal{D}_{Y \to X}$. The above can be formulated a bit differently. We have just endowed $\pi^*(\mathcal{D}_X)$ with a structure of a left $\mathcal{D}_Y$-module. On the other hand, the right multiplication on $\mathcal{D}_X$ carries over to a right $\pi^{-1}\mathcal{D}_X$-module structure. We denote $\pi^*(\mathcal{D}_X)$, endowed with this structure of $\mathcal{D}_Y \times \pi^{-1}\mathcal{D}_X$ module, by $\mathcal{D}_{Y \to X}$. Since $\mathcal{D}_X$ is locally $\mathcal{O}_X$-free it follows that $\mathcal{D}_{Y \to X}$ is locally $\pi^{-1}\mathcal{O}_X$-free. We have

(1) $$\mathcal{D}_{Y \to X} \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}M = \mathcal{O}_Y \otimes_{\pi^{-1}(\mathcal{O}_X)} \pi^{-1}\mathcal{D}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}M = \pi^*(M)$$

hence

(2) $$\pi^o(M) = \mathcal{D}_{Y \to X} \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}M ,$$

where the $\otimes$ uses the right (resp. left) $\pi^{-1}\mathcal{D}_X$-module structure of

$\mathcal{D}_{Y\to X}$ (resp. $\pi^{-1}M$) and the left $\mathcal{D}_Y$-module structure is inherited from that of $\mathcal{D}_{Y\to X}$. This functor is only right exact in general and we can consider its left derived functor in $D^b(\mathcal{D}_Y)$, usually denoted:

(3) $$L\pi^{o}(M^{\cdot}) = \mathcal{D}_{Y\to X} \overset{L}{\otimes} \pi^{-1}M^{\cdot} \qquad (M^{\cdot} \in D^b(\mathcal{D}_X)) .$$

For reasons of convenience in dealing with the Kashiwara equivalence, adjointness and the Riemann-Hilbert correspondence, we wish to shift the degree and change the notation. We shall also view this as an element of the derived category and in general suppress the L. We now define

(4) $$\pi^{!}M^{\cdot} = \mathcal{D}_{Y\to X} \overset{L}{\underset{\pi^{-1}\mathcal{D}_X}{\otimes}} \pi^{-1}M[d_{Y,X}] \qquad (M^{\cdot} \in D^b(\mathcal{D}_X))$$

If $M \in \mu(\mathcal{D}_Y)$, we have therefore

(5) $$H^i\pi^{!}M = \mathrm{Tor}^{\pi^{-1}\mathcal{D}_X}_{i-d_{Y,X}}(\mathcal{D}_{Y\to X},\pi^{-1}M) \qquad (i\in\mathbb{Z})$$

If $M^{\cdot} \in D^b(\mathcal{D}_X)$ and $P^{\cdot} \to M^{\cdot}$ is a projective resolution, then

(6) $$\pi^{!}M^{\cdot} = \mathcal{D}_{Y\to X} \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}P^{\cdot}[d_{Y,X}] .$$

Since $\mathcal{D}_X$ is locally free over $\mathcal{O}_X$, the resolution $P^{\cdot}$ is also a projective resolution with respect to $\mathcal{O}_X$. Therefore we see from (1) that, as a $\mathcal{O}_Y$-module

(7) $$\pi^{!}M^{\cdot}[-d_{Y,X}] = \mathcal{O}_Y \overset{L}{\underset{\pi^{-1}\mathcal{O}_X}{\otimes}} \pi^{-1}M^{\cdot} = \mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}P^{\cdot} , \; (M^{\cdot} \in D^b(\mathcal{D}_X))$$

whence also

(8) $$H^i\pi^{!}M = \mathcal{T}or^{\pi^{-1}\mathcal{O}_X}_{i-d_{Y,X}}(\mathcal{O}_Y,\pi^{-1}M) \qquad (M \in \mu(\mathcal{D}_X)) .$$

At this point, the use of the notation $\pi^{!}$ is arbitrary, but we shall see that this functor behaves more like an extraordinary inverse image than an ordinary one. We then define the counterpart of the latter by

(9) $$\pi^{+} = \mathbb{D}_Y \circ \pi^{!} \circ \mathbb{D}_X .$$

Remark. If $M^{\cdot}$ is locally free over $\mathcal{O}_X$, or if π is flat, then all the $\mathcal{T}or_i(i\neq 0)$ are zero and (7) yields

(10) $$H^{i+d}\pi^{!}M^{\cdot} = \pi^{o}(H^iM^{\cdot}) \quad (M^{\cdot} \in D^b(\mathcal{D}_X)); (d=d_{Y,X})$$

This equality is also true if $H^{\cdot}M^{\cdot}$ is locally free over $\mathcal{O}_X$. In fact, in the spectral sequence abutting to $H^{\cdot}\pi^{!}M^{\cdot}$, in which $E_2^{p,q} = H^p\pi^{!}H^qM^{\cdot}$, the module $E_2^{p,q}$ is then 0 if $p \neq d_{Y,X}$, hence the spectral sequence degenerates and (10) follows. From this spectral sequence, we also see that the support of $\pi^{!}M^{\cdot}$ is contained in the inverse image of the support of $M^{\cdot}$.

4.3 PROPOSITION . *Let* Z *be a smooth variety and* $\sigma : Z \to Y$ *a morphism. Then*

(1) $$(\pi\circ\sigma)^{o} = \sigma^{o} \circ \pi^{o}.$$

(2) $$(\pi\circ\sigma)^{!} = \sigma^{!} \circ \pi^{!}.$$

Proof. As is known, we have in the $\mathcal{O}$- category the equality:

$$(\pi\circ\sigma)^{*}(M) = \sigma^{*}(\pi^{*}(M)) \qquad (M\in\mu(\mathcal{D}_X)) .$$

To check (1), it suffices to show that this equality is also valid for the connections defined in 4.1. This is a routine computation left to the reader.

To prove (2), we first remark that $d_{Z,Y} + d_{Y,X} = d_{Z,X}$ hence the shifts in degree compose well and we may ignore them. Let $M^{\cdot} \in D^b(\mathcal{D}_X)$ and $P^{\cdot} \to M^{\cdot}$ be a flat resolution. Up to the shift, $(\pi\circ\sigma)^{!}(M^{\cdot})$ is represented by $\sigma^*(\pi^*(P^\cdot))$. On the other hand, to get $\sigma^!(\pi^!(M^\cdot))$, we have to take a flat resolution $Q^\cdot$ of $\pi^*(P^\cdot)$ and consider $\sigma^*(Q^\cdot)$. For the latter to be h.i. to $(\pi\circ\sigma)^*(P^\cdot)$, if suffices that $\pi^*(P^\cdot)$ be flat over $\mathcal{O}_Y$, but this is clear from

$$\pi^*(P^\cdot) = \mathcal{O}_Y \underset{\pi^{-1}(\mathcal{O}_X)}{\otimes} \pi^{-1}(P^\cdot),$$

since flatness is preserved under base-change. q.e.d..

We now discuss inverse images in the special cases of open embeddings and projections.

4.4 *Open embeddings.* Assume π is an open embedding. Then π^{-1} is just the restriction to Y and in particular $\pi^{-1}\mathcal{D}_X = \mathcal{D}_Y$. Also $d_{Y,X} = 0$ Therefore $\pi^o(M) = M|Y$, viewed as $\mathcal{D}_Y = \mathcal{D}_X|Y$ module, hence π^o *is exact,* $H^i\pi^! = 0$ for $i \neq 0$ and $H^0\pi^! = \pi^o$. *The functor* $\pi^!$ *preserves coherence and holonomicity.*

4.5 *Projection.* Let $Y = Z \times X$ and π be the second projection. Let $\sigma : Y \to Z$ be the first projection. We have

(1) $$\mathcal{O}_Y = \sigma^{-1}(\mathcal{O}_Z) \underset{\mathbb{C}}{\otimes} \pi^{-1}(\mathcal{O}_X)$$

therefore

(2) $$\pi^*(M) = \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \pi^{-1}(M) = \sigma^{-1}(\mathcal{O}_Z) \underset{\mathbb{C}}{\otimes} \pi^{-1}(M) \qquad (M\in\mu(\mathcal{D}_X)).$$

We also have

(3) $$\Theta_Y = \sigma^{-1}(\Theta_Z) \underset{\mathbb{C}}{\otimes} \pi^{-1}(\Theta_X) = \sigma^*(\Theta_Z) \underset{\mathcal{O}_Z}{\oplus} \pi^*(\Theta_X)$$

(Using (1) to define the $\mathcal{O}_Y$-structure), and moreover

(4) $$[\sigma^{-1}(\Theta_Z), \pi^{-1}(\Theta_X)] = 0$$

It follows that

(5) $$\mathcal{D}_Y = \sigma^{-1}(\mathcal{D}_Z) \underset{\mathbb{C}}{\otimes} \pi^{-1}(\mathcal{D}_X)$$

(6) $$[\sigma^{-1}(\mathcal{D}_Z), \pi^{-1}(\mathcal{D}_X)] = 0$$

From (1) we get

(7) $$\pi^o(M) = \sigma^{-1}(\mathcal{O}_Z) \otimes M \qquad (M\in\mu(\mathcal{D}_X))$$

with the $\mathcal{D}_Y$-module structure defined via the decomposition (5). It follows that π^o *is exact*, hence, with $d = d_Z$,

(8) $$\pi^!(M^\cdot) = \pi^o(M^\cdot)[d] \qquad (M^\cdot\in D^b(\mathcal{D}_X)).$$

(9) $$H^i\pi^!M^\cdot = \pi^o(H^{i+d}(M^\cdot)) \qquad (i\in\mathbb{Z})$$

Also (7) shows that π^o *preserves coherence.* We claim that it also *preserves the holonomy defect.* In fact, it follows from the definitions that if we use the canonical isomorphism

(10) $$T^*(Y)_{(z,x)} = T^*(Z)_z \oplus T^*(X)_x \qquad (x\in X; z\in Z),$$

we can write

(11) $$SS(\pi^o M)_{(z,x)} = SS(M)_x \qquad (x\in X, z\in Z).$$

In particular, π^o and $\pi^!$ *preserve holonomicity.*

Closed embeddings will be discussed in §7.

4.6 *Remark.* The embedding of $\pi^*(\Theta_X)$ in Θ_Y given by 4.5(3) is defined by associating to $\xi \in \Theta_X$ its obvious lifting ξ' in Y which is tangent to the sections $\{z\} \times X$ $(z\in Z)$ of π. We also get from 4.5(3) a surjective morphism $\nu : \Theta_Y \to \pi^*(\Theta_X)$ with kernel $\sigma^*(\Theta_Z) = \Theta_{Y/X}$. For

$M \in \mu(\mathcal{D}_X)$, it is obvious that the left $\mathcal{D}_Y$-module structure of 4.1 can also be defined in the present case by

$$\text{(1)}\quad \xi(f\otimes\pi^{-1}m) = \xi f \otimes m + \Sigma_j' f.\xi(u_j) \otimes \pi^{-1}(\xi_j m) \quad (\xi\in\Theta_Y; f\in\mathcal{O}_Y; m\in M),$$

where

$$\text{(2)}\qquad \nu(\xi) = \Sigma_j' u_j \otimes \xi_j' \quad (u_j\in\mathcal{O}_Y, \xi_j\in\Theta_X).$$

4.7 *Affine varieties.* Assume X and Y to be affine. Then the $\mathcal{D}$-modules may be replaced by their modules of sections, which are modules over the rings of global algebraic differential operators. Accordingly, the inverse image may be viewed as a map $\mathrm{mod}(\mathcal{D}(Y)) \to \mathrm{mod}(\mathcal{D}(Y))$ and the derived inverse image as one $D^b(\mathrm{mod}(\mathcal{D}(Y))$ to $D^b(\mathrm{mod}(\mathcal{D}(X))$. We have

$$\text{(1)}\qquad \pi^o(M) = \mathcal{O}(Y) \underset{\mathcal{O}(X)}{\otimes} M \qquad (M\in\mathrm{mod}(\mathcal{D}(X)).$$

The $\mathcal{D}_Y$-module structure of the associated $\mathcal{D}_Y$-module $\pi^0(\tilde{M})$, where $\tilde{M}$ is the $\mathcal{D}_X$-module defined by M, gives a $\mathcal{D}(Y)$-module structure to the right hand side of (1). Let

$$\text{(2)}\qquad \mathcal{D}(Y\to X) = \mathcal{D}_{Y\to X}(Y) = \pi^o(\mathcal{D}(Y))$$

This is a $\mathcal{D}(Y) \times \mathcal{D}(X)$-module and we also have

$$\text{(3)}\qquad \pi^o(M) = \mathcal{D}(Y\to X) \underset{\mathcal{D}(X)}{\otimes} M$$

$$\text{(4)}\qquad \pi^!(M') = \mathcal{D}(Y\to X) \overset{L}{\underset{\mathcal{D}(X)}{\otimes}} M[d_{Y,X}] = O(Y) \overset{L}{\underset{\mathcal{O}(X)}{\otimes}} M''d_{Y,X}] .$$

For the case where $Y = \mathbb{A}^m$ and $X = \mathbb{A}^n$, see V,3.2.1, 3.2.4 .

4.8 PROPOSITION . *Assume π to be smooth. Then $H^i\pi^! = 0$ if $i \neq d_{Y,X}$ and $\pi^!$ preserves coherence and the holonomy defect, in particular holonomicity.*

Proof. Recall that, by definition, π is smooth if its differential $d\pi_y : T(Y)_y \to T(X)_{\pi(x)}$ is surjective for all y's. It is then dominant and $\mathcal{O}_Y$ is flat over $\mathcal{O}_{\pi(x)}$. The first assertion follows therefore from the remark to 4.4.

Locally, π can be factored as a product of an étale morphism α and a projection β. More precisely, given $y \in Y$, there exists an open neighborhood U of y, an integer N and an étale morphism $\alpha : U \to \mathbb{A}^N \times X$ such that $\pi|U$ is the product of α and of the second projection β. It is obvious that $\alpha^!$ preserves coherence and the holonomy defect, and $\beta^!$ does so by 4.5.

§ 5 DIRECT IMAGES

5.0 We want to associate to π a functor $\pi_+ : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$. The definition is more complicated than in the case of inverse images. π_+ will be a composition of a left derived and of a right derived functor from a map π_o defined at the module level. Unfortunately, it is not clear a priori that π_o preserves quasi-coherence. However, we shall see that π_+ maps $D^b(\mathcal{D}_Y)$ into $D^b_{qc}(\mathrm{Mod}\ \mathcal{D}_X)$. Since the latter is equivalent to $D^b_+(\mathcal{D}_X)$, by 2.10 this yields the desired homomorphism. As long as 2.10 was not available, the discussion was more involved. One way was to show that π_o preserves q.c. when π is an open or closed embedding or a projection and then either prove and use the composition law $(\pi o \sigma)_+ = \pi_+ \circ \sigma_+$, or define π_o, π_+ first for those elementary mappings and then as a composition, using the canonical factorisation of π into a closed embedding and a projection. However the latter makes the verification of the composition law more awkward, since one has to relate the canonical decomposition of $\pi \circ \sigma$ to those of π and σ.

5.1 The map π defines a morphism of ringed spaces $\pi_* : (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$, but not one of the ringed spaces $(Y,\mathcal{D}_Y)$ and $(X,\mathcal{D}_X)$, which is why some construction has to be found. It cannot be that obvious since there is no direct image of a linear system of PDE's.

But there is at least a canonical morphism of sheaves of rings

(1) $$\mathcal{D}_X \to \pi\pi^{-1}\mathcal{D}_X ,$$

hence, trivially, we can view π as a morphism of ringed spaces $(Y,\pi^{-1}\mathcal{D}_X) \to (X,\mathcal{D}_X)$. In that capacity, we denote it $\tilde{\pi}$. Since an injective $\pi^{-1}\mathcal{D}_X$-module is a flabby sheaf [Go:II,7.1], $R\tilde{\pi}$ agrees with the derived direct image for sheaves of abelian groups.

(a) The definition is most easily given for right $\mathcal{D}$-modules, and we start with it. Let $N \in \mu^r(\mathcal{D}_Y)$, and consider $N \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$. The tensor product uses up the right $\mathcal{D}_Y$-module structure of N and the left $\mathcal{D}_Y$-module structure of $\mathcal{D}_{Y\to X}$. There remains the right $\pi^{-1}\mathcal{D}_X$-module structure coming from that of $\mathcal{D}_{Y\to X}$. Therefore

$$\pi_o(N) = \tilde{\pi}(N \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$$

is a right $\mathcal{D}_X$-module. By definition, this is the direct image for right $\mathcal{D}$-modules. We then define π_+ for right $\mathcal{D}$-modules by

$$\pi_+N = R\tilde{\pi}(N \overset{L}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}) .$$

It is the composition of a left exact and a right exact functor. It would be difficult to make good sense of this without using the framework of derived categories. But there it is possible since both preserve exact triangles. Note however that $N \overset{L}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ does not carry in general a structure of $\mathcal{O}_Y$-module and we cannot view this as a derived direct image in the $\mathcal{O}$-category. We are therefore not sure at that point that the π_+N is q.c. We shall prove this first when Y and X are affine and then use 2.10 and a spectral sequence argument.

(b) Assume Y and X are affine. There exists a left free resolution $F^\cdot$ of N. Then

$$N \overset{L}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X} = F^\cdot \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$$

is a direct sum of copies of $\mathcal{D}_{Y\to X}$, hence is a q.c. $\mathcal{O}_Y$-module. We may compute the direct image in the $\mathcal{O}$-category hence the $R^i\tilde{\pi}$ are zero for $i > 0$, $\tilde{\pi}(F^\cdot \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$ is $\mathcal{O}_X$-q.c. and we have

$$H^i\pi_+(N) = H^i\pi_o(F^\cdot \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}), \quad (i\in\mathbf{Z})$$

in particular, for every affine open $U \subset X$

$$\Gamma(U;H^i\pi_+N) = \mathrm{Tor}^{-i}_{\mathcal{D}(V)}(N(V),\mathcal{D}_{Y\to X}(V)) \quad (i\in\mathbf{Z};V=f^{-1}(U))$$

hence

$$H^i\pi_+N = 0 \quad i \notin [-d_Y,0].$$

This shows in particular that π_+ maps $D^b(\mathcal{D}_Y)$ into $D^b(\mathcal{D}_X)$, (for right $\mathcal{D}$-modules).

(c) We go back to the general case. Since our assertion is local on X, we may assume X to be affine. Let $(Y_i)_{i\in I}$ be a finite affine open cover of Y. For $J \subset I$ let $Y_J = \bigcap_{i\in J} Y_i$ and π_J be the restriction of π to Y_J. There exists a spectral sequence abutting to $\pi_+ N$ in which E_1 is the Čech complex assigning $\pi_{J,+}(N|Y_J)$ to Y_J. By (b), E_1 is q.c. hence so is $H^\cdot \pi_+ N$. Thus $\pi_+ N$ has q.c. cohomology, hence is h.i. to a unique element of $D^b(\mathcal{D}_X)$ by 2.10. Since π_+ is a morphism of derived categories, it follows that it maps $D^b(\mathcal{D}_Y)$ into $D^b(\mathcal{D}_X)$, where right $\mathcal{D}$-modules are meant.

(d) We prefer however to work with left $\mathcal{D}$-modules. A simple way is to use the conversion between left and right of §3. Given $M \in \mu(\mathcal{D}_Y)$, we consider $N = M \otimes_{\mathcal{O}_Y} \omega_Y$. Then $\pi_+(N)$ is a right $\mathcal{D}_X$-module and we can go over to $\pi_+(N) \otimes \omega_X^{-1}$. We want to express this in a slightly different way, which will be more convenient to work with, using the $\mathcal{D}$-module $\tilde{\mathcal{D}}_X$ of 3.6. It has two commuting left $\mathcal{D}_X$-module structures, hence $\pi^{-1}(\tilde{\mathcal{D}}_X)$ has two commuting left $\pi^{-1}\mathcal{D}_X$-module structures. Therefore

$$\pi^o(\tilde{\mathcal{D}}_X) = \pi^*(\tilde{\mathcal{D}}_X) = \pi^*(\mathcal{D}_X) \otimes_{\mathcal{O}_Y} \pi^*(\omega_X)^{-1} \tag{2}$$

is on one hand a left $\mathcal{D}_Y$-module by means of structure (a) (cf.3.6) and the construction of the inverse image and on the other hand a left $\pi^{-1}(\mathcal{D}_X)$-module by means of structure (b). As a $\mathcal{O}_Y$-module, it can be written

$$\pi^*(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \pi^*(\omega_X)^{-1} \otimes_{\mathcal{O}_Y} \omega_Y = \pi^*(\mathcal{D}_X) \otimes_{\mathcal{O}_Y} \omega_{Y/X} . \tag{3}$$

We shall denote it by $\mathcal{D}_{X\leftarrow Y}$. Thus

$$\mathcal{D}_{X\leftarrow Y} = \pi^o(\mathcal{D}_X) \otimes_{\mathcal{O}_Y} \omega_{Y/X} = \mathcal{D}_{Y\to X} \otimes_{\mathcal{O}_Y} \omega_{Y/X} \tag{4}$$

is a $\pi^{-1}\mathcal{D}_X \times \mathcal{D}_Y$ module. As a consequence, if $M \in \mu(\mathcal{D}_X)$, then $\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} M$

is a left $\pi^{-1}\mathcal{D}_X$-module. We then define the direct image for left $\mathcal{D}$-modules by

$$\pi_o(M) = \tilde{\pi}(\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} M) \qquad (M\in\mu(\mathcal{D}_Y)) . \tag{5}$$

In the derived category, we put

$$\pi_+(M^\cdot) = R\tilde{\pi}(\mathcal{D}_{X\leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M^\cdot) \qquad (M^\cdot \in D^b(\mathcal{D}_Y)) . \tag{6}$$

To construct it, take a bounded left flat resolution $P^\cdot$ of $M^\cdot$. (It would also be possible to replace $\mathcal{D}_{X\leftarrow Y}$ by a bounded left $\mathcal{D}_Y$-locally free resolution in the category of $(\pi^{-1}(\mathcal{D}_X))\times\mathcal{D}_Y)$-modules, as will be done for projections.)
Then

$$\mathcal{D}_{X\leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M^\cdot = \mathcal{D}_{X\leftarrow Y} \otimes P^\cdot$$

is a left $\pi^{-1}\mathcal{D}_X$-module. Then choses an injective resolution $J^\cdot$ of the right-hand side in the category of $\pi^{-1}(\mathcal{D}_X)$-modules and consider $\tilde{\pi}(J^\cdot)$. Its cohomology sheaves are the $H^i\pi_+(M^\cdot)$. However, it may be unpleasant to deal with $J^\cdot$. Since the discussion of the direct image in case of morphisms of affine varieties is fairly direct (see 5.4), it is often more perspicuous to use the spectral sequence associated to a finite open affine covering, as we did above. A further simplification may be achieved by using the canonical factorisation of a map (see 5.3 for projections, 7.7, 7.8 for closed embeddings).

Remark. (1) There are various notations in the literature for the ith direct image, e.g. $R^i\pi_*$, $\int^i$ or $\int^i_{Y\to X}$.

(2) We have defined a direct image for left and right $\mathcal{D}$-modules, but an inverse image only for left $\mathcal{D}$-modules. To be more symmetrical, we could also introduce the inverse image for right $\mathcal{D}$-modules by the rule

$$\pi^{o}(N) = \pi^{-1}N \underset{\pi^{-1}(\mathcal{D}_X)}{\otimes} \mathcal{D}_{X\leftarrow Y}, \quad (N\in\mu^{r}(\mathcal{D}_X)) ,$$

where the right $\mathcal{D}_Y$-module structure stems from the one of $\mathcal{D}_{X\leftarrow Y}$, and its derived functors $L\pi^{o}$.

In this section, we shall discuss open embeddings and projections. Closed embeddings are postponed to §7.

5.2 *Open embeddings.* We assume here that Y is open in X. Then $\omega_{Y/X} = \mathcal{O}_Y$ and

(1) $$\mathcal{D}_{Y\to X} = \mathcal{D}_{X\leftarrow Y} = \mathcal{D}_Y, \text{ viewed as a } (\mathcal{D}_X\times\mathcal{D}_Y)\text{-module,}$$

so that π is a map of ringed spaces $(Y,\mathcal{D}_Y) \to (X,\mathcal{D}_X)$. We have

(2) $$\pi_o(M) = \tilde{\pi}(\mathcal{D}_Y \underset{\mathcal{D}_Y}{\otimes} M) = \tilde{\pi}(M)$$

and π_o may also be viewed as π_*. It is left exact. The sheaf $\mathcal{D}_Y$ is flat over $\mathcal{O}_Y$ hence if F is $\mathcal{D}_Y$-injective (resp. flat), it is also $\mathcal{O}_Y$-injective (resp. flat). Consequently

(3) $$H^{i}\pi_+(M) = R^{i}\pi_*M = \varinjlim_{x\in U} H^{i}(U\cap Y;M), \quad (M\in\mu(\mathcal{D}_Y))$$

(4) $$H^{i}\pi_+(M^{\cdot}) = R^{i}\pi_*(M^{\cdot}) = \varinjlim_{x\in U} \mathbb{H}^{i}(U\cap Y;M^{\cdot}) \quad (M^{\cdot}\in D^{b}(\mathcal{D}_Y)).$$

Here U runs through a fundamental set of affine open neighborhoods of x in X. Since we are in the $\mathcal{O}$-category, these modules are all quasi-coherent.

Assume now Y to be affine. Then $U \cap Y$ is also affine and $H^{i}(U\cap Y;M) = 0$ for $i \geq 1$ by Serre's theorem. Therefore

(5) $$H^{i}\pi_+(M) = 0 \text{ for } i \geq 1, \quad H^{0}\pi_+(M) = \pi_*(M), \quad (M\in\mu(\mathcal{D}_Y))$$

and $\pi_+ : D^{b}(\mathcal{D}_Y) \to D^{b}(\mathcal{D}_X)$ is an exact functor.

In general, π_o *does not preserve coherence.* To see this, assume that X is affine and Y is the complement of a hypersurface S with ideal generated by a regular function $f \in \mathcal{O}(X)$. Then $\mathcal{D}(Y) = \mathcal{D}(X)[f^{-1}]$ is not coherent over $\mathcal{D}(X)$, but $\pi_o(\mathcal{D}(Y))$ is $\mathcal{D}(Y)$ itself, viewed as a $\mathcal{D}(X)$-module. We shall see however that π_o preserves holonomicity (9.1). In the present example, with $X = \mathbb{A}^n$, this is just a reformulation of Bernstein's result (V,1.16.3).

5.3 *Projections.* We go back to the situation of 4.5. Thus π is a projection $Y = Z \times X \to X$ and σ denotes the first projection. We have

(1) $$\sigma^{*}(\omega_Z) = \omega_{Y/X},$$

(2) $$\Omega^{\cdot}_{Y/X} = \sigma^{*}(\Omega^{\cdot}_Z) = \sigma^{-1}(\Omega^{\cdot}_Z) \underset{\sigma^{-1}(\mathcal{O}_Z)}{\otimes} \mathcal{O}_Y = \sigma^{-1}(\Omega_Z) \underset{\mathbb{C}}{\otimes} \pi^{-1}(\mathcal{O}_X).$$

Given $M \in \mu(\mathcal{D}_Y)$, we let

(3) $$\Omega^{\cdot}_{Y/X}(M) = \Omega^{\cdot}_{Y/X} \underset{\mathcal{O}_Y}{\otimes} M$$

be the relative de Rham complex. It can also be written as

(4) $$\Omega^{\cdot}_{Y/X}(M) = \sigma^{-1}(\Omega^{\cdot}_Z) \underset{\sigma^{-1}(\mathcal{O}_Z)}{\otimes} M$$

with the differential given locally by

(5) $$d(\omega\otimes m) = d\omega \otimes m + \Sigma_i \sigma^{-1}(dz_i)\wedge\omega \otimes \partial_i m \quad (\omega\in\sigma^{-1}(\Omega^{\cdot}_Z); m\in M),$$

where (z_i,∂_i) is a system of "local coordinates" on Z (1.3). In view of 4.5(5), M can be viewed as a left $\pi^{-1}(\mathcal{D}_X)$-module, whence a left $\pi^{-1}(\mathcal{D}_X)$-module structure on $\Omega^{\cdot}_{Y/X}(M)$, which commutes with the $\sigma^{-1}(\mathcal{O}_Z)$ module structure. Therefore $\Omega^{\cdot}_{Y/X}(M)$ is also a $\mathcal{O}_Y$-module.

5.3.1 PROPOSITION (i) *We have the equalities*

(6) $$\mathcal{D}_{X\leftarrow Y} = \sigma^{-1}(\omega_Z) \otimes_{\mathbb{C}} \pi^{-1}(\mathcal{D}_X) = \sigma^{-1}(\omega_Z) \otimes_{\sigma^{-1}(\mathcal{D}_Z)} \mathcal{D}_Y$$

(7) $$\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} M = \sigma^{-1}(\omega_Z) \otimes_{\sigma^{-1}(\mathcal{O}_Z)} M \qquad (M\in\mu(\mathcal{D}_Y))$$

(ii) $\Omega^{\cdot}_{Y/X}(\mathcal{D}_Y)[d]$ *is a locally free left resolution of* $\mathcal{D}_{Y\leftarrow X}$ *in the category of* $\pi^{-1}(\mathcal{D}_X) \times \mathcal{D}_Y$ *modules and we have*

(8) $$\mathcal{D}_{X\leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M^{\cdot} = \Omega^{\cdot}_{Y/X}(M)[d] \qquad (M^{\cdot}\in D^b(\mathcal{D}_Y)),\ d=d_Z)$$

Proof. (i) follows from the definitions and 4.5(1), (5).
(ii) Using (4) and 4.5(5), we can write

$$\Omega^{\cdot}_{Y/X}(\mathcal{D}_Y) = C^{\cdot} \otimes_{\mathbb{C}} \pi^{-1}(\mathcal{D}_X),\ \text{where } C^{\cdot} = \sigma^{-1}(\Omega^{\cdot}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_Z) = \sigma^{-1}(\Omega^{\cdot}_Z(\mathcal{D}_Z)).$$

It follows from 3.5 that $C^{\cdot}$ is a locally free left resolution of $\sigma^{-1}(\omega_Z)$ in the category of right $\sigma^{-1}(\mathcal{D}_Z)$-modules. In view of (1) and 4.5(5), this shows that $\Omega^{\cdot}_{Y/X}(\mathcal{D}_Y)[d]$ is a locally free left resolution of $\mathcal{D}_{Y\leftarrow X}$ in $\mu^r(\mathcal{D}_Y)$. Moreover, left multiplication on $\pi^{-1}(\mathcal{D}_X)$ commutes with the differential, so that this complex can be viewed as a resolution in the category of $\pi^{-1}(\mathcal{D}_X) \times \mathcal{D}_Y$ modules. This proves the first assertion of (ii). We have now

$$\mathcal{D}_{X\leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M^{\cdot} = \Omega^{\cdot}_{Y/X}(\mathcal{D}_Y)[d] \otimes_{\mathcal{D}_Y} M^{\cdot} = \Omega^{\cdot}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} M^{\cdot}[d]$$

and (8) follows.

5.3.2 COROLLARY $\pi_+(M^{\cdot}) = R\pi_*(\Omega^{\cdot}_{Y/X}(M^{\cdot}))[d] \quad (M^{\cdot}\in D^b(\mathcal{D}_Y));d=d_Z)$.

This follows from 5.3.1, taking into account that we can replace $\tilde{\pi}$ by π_*, as was remarked at the beginning of 5.1. Note that this formula describes $\pi_+(M^{\cdot})$ as an $\mathcal{O}_X$-module only. Let

(9) $$L^iM = H^i(\Omega^{\cdot}_{Y/X}[M])[d] = H^{i-d}(\Omega^{\cdot}_{Y/X}(M)), \quad (M\in\mu(\mathcal{D}_Y);i\in\mathbb{Z})$$

Then

(10) $$L^0(M) = \omega_{Y/X} \otimes_{\mathcal{O}_Y} M/\Theta_Z M$$

(11) $$L^{-d}(M) = \cap_i \ker \partial_i$$

5.3.3 Assume Z to be affine. Then π is affine hence the $R^i\pi_*$ are zero for $i \geqq 1$; therefore π_+ is right exact and we have:

(12) $$H^i\pi_+ = H^i(\pi_*(\Omega^{\cdot}_{Y/X}(M^{\cdot})[d] \qquad (i\in\mathbb{Z})$$

which implies that $H^0\pi_+$ is right exact.

Assume moreover that X is affine. Then we have only to deal with global modules over coordinate rings. The direct image is just the base change $\mathcal{D}(X) \to \mathcal{D}(Y)$, and (10), (11) are valid for a $\mathcal{D}(Y)$-module M.

Assume now finally that Y is the affine n-space $\mathbb{A}^n$ with coordinates $x_1,\ldots,x_n$ and that Z is the line $x_1 = \ldots = x_{n-1} = 0$. Then (10), (11) yield

(13) $$H^{-1}\pi_+(M) = \ker_M \partial_n \quad , H^0\pi_+(M) = M/\partial_n M \quad ,$$

modules which were proved to be holonomic if M is (V, 3.1.8). More generally, in case of a polynomial mapping $\pi : \mathbb{A}^n \to \mathbb{A}^m$, the direct image is the one constructed in V, 3.5. It was shown there that π_+ preserves holonomicity (V, 3.5.2). This will be used in the next chapter in the case of a projection, which reduces easily to the above special case.

5.4 *Direct image for affine varieties.* Let X,Y be affine. As in 3.8 we want to express the previous constructions in terms of the modules of sections. We let

(1) $$\omega(Y/X) = \Gamma(Y,\omega_{Y/X}) = \omega(Y) \otimes_{\mathcal{O}(X)} \omega(X)^* ,$$

(2) $$\mathcal{D}(X\leftarrow Y) = \Gamma(Y,\mathcal{D}_{X\leftarrow Y}) = \mathcal{D}(Y\to X) \otimes_{\mathcal{O}(Y)} \omega(Y/X) .$$

Then

(3) $$\pi_o(N) = N \otimes_{\mathcal{D}(Y)} \mathcal{D}(Y\to X) \text{ , if } N \text{ is a right } \mathcal{D}(Y)\text{-module;}$$

(4) $$\pi_o(M) = \mathcal{D}(X\leftarrow Y) \otimes_{\mathcal{D}(Y)} M, \text{ if } M \text{ is a left } \mathcal{D}(Y)\text{-module.}$$

and π_+ is the left-derived functor

(5) $$\pi_+(M^\cdot) = \mathcal{D}(X\leftarrow Y) \overset{L}{\otimes}_{\mathcal{D}(Y)} M^\cdot \qquad (M^\cdot \in D^b(\mathrm{mod}(\mathcal{D}(Y))) .$$

For $N \in D^b(\mathrm{mod}^r(\mathcal{D}(Y))$, it would be

(6) $$\pi_+(N^\cdot) = N^\cdot \overset{L}{\otimes}_{\mathcal{D}(Y)} \mathcal{D}(Y\to X) .$$

These functors are right exact.

See V, 3.5 for the case $Y = \mathbb{A}^m$, $X = \mathbb{A}^n$.

§ 6 COMPOSITION OF DIRECT IMAGES

6.1 We recall first that on an algebraic variety Z the functor of sections in the Zariski topology commutes with direct sums of quasi-coherent sheaves. In fact, this is clear if Z is affine and therefore also in general, since any Z has a finite affine open cover. Similarly a sum of injective quasi-coherent $\mathcal{O}_Y$-modules is also injective. It follows that π_* and $R\pi_*$ commute with direct sums.

6.2 LEMMA *Let $A^\cdot$ a complex of right $\mathcal{D}_X$-modules and $B^\cdot$ a complex of left $\pi^{-1}(\mathcal{D}_X)$ modules. Then*

$$R\tilde{\pi}(\pi^{-1}A^\cdot \overset{L}{\otimes}_{\pi^{-1}(\mathcal{D}_X)} B^\cdot) = A^\cdot \overset{L}{\otimes}_{\mathcal{D}_X} R\tilde{\pi}B^\cdot .$$

Proof. We may replace $A^\cdot$ by a locally free resolution, hence assume $A^\cdot$ to be locally free. Then $\pi^{-1}A^\cdot$ is locally free over $\pi^{-1}\mathcal{D}_X$, and we can erase the L on the left-hand side. Moreover, if $B^\cdot$ is injective, then $\pi^{-1}A^\cdot \otimes B^\cdot$ is locally a direct sum of injective complexes hence is also injective. We may also erase the R and are reduced to prove

$$\pi(\pi^{-1}A^\cdot \otimes_{\pi^{-1}\mathcal{D}_X} B^\cdot) = A^\cdot \otimes_{\mathcal{D}_X} \tilde{\pi}(B^\cdot))$$

if $A^\cdot$ is locally free. This brings us back to 6.1.

We now consider a composite morphism

$$Z \xrightarrow{\sigma} Y \xrightarrow{\pi} X .$$

6.3 LEMMA *We have the equalities*

(1) $$\mathcal{D}_{Z\to X} = \mathcal{D}_{Z\to Y} \otimes_{\sigma^{-1}\mathcal{D}_Y} \sigma^{-1}(\mathcal{D}_{Y\to X}) = \mathcal{D}_{Z\to Y} \overset{L}{\otimes}_{\sigma^{-1}\mathcal{D}_Y} \sigma^{-1}(\mathcal{D}_{Y\to X})$$

(2) $$\mathcal{D}_{X\leftarrow Z} = \sigma^{-1}(\mathcal{D}_{X\leftarrow Y}) \underset{\sigma^{-1}\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\leftarrow Z} = \sigma^{-1}(\mathcal{D}_{X\leftarrow Y}) \overset{L}{\underset{\sigma^{-1}\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y\leftarrow Z}\ .$$

Proof. The first equality of (1) is just another way to write

(3) $$(\pi\circ\sigma)^{\circ}(\mathcal{D}_X) = \sigma^{\circ}(\pi^{\circ}(\mathcal{D}_X)),$$

which follows from 4.3. Together with 4.5(5) and

(4) $$\omega_{Z/X} = \omega_{Z/Y} \underset{\mathcal{O}_Z}{\otimes} \sigma^*(\omega_{Y/X}) = \omega_{Z/Y} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}(\omega_{Y/X}),$$

it implies the first equality of (2).

Since $\mathcal{D}_{Y\to X}$ is locally $\mathcal{O}_Y$-free (4.2), we have

$$\mathcal{D}_{Z\to X} = \sigma^{\circ}(\mathcal{D}_{Y\to X}) = \mathcal{O}_Z \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}(\mathcal{D}_{Y\to X}) = \mathcal{O}_Z \overset{L}{\underset{\sigma^{-1}\mathcal{O}_Y}{\otimes}} \sigma^{-1}(\mathcal{D}_{Y\to X}) = F^{\bullet} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}(\mathcal{D}_{Y\to X})$$

where $F^{\bullet}$ is a locally free left resolution of $\mathcal{O}_Z$ over $\sigma^{-1}(\mathcal{O}_Y)$. We have then also

$$\mathcal{D}_{Z\to X} = F^{\bullet} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}\mathcal{D}_Y \underset{\sigma^{-1}\mathcal{D}_Y}{\otimes} \sigma^{-1}(\mathcal{D}_{Y\to X})\ .$$

$F^{\bullet} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}\mathcal{D}_Y$ is locally a direct sum of copies of $\sigma^{-1}\mathcal{D}_Y$ hence

$$\mathcal{D}_{Z\to X} = (F^{\bullet} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}\mathcal{D}_Y) \overset{L}{\underset{\sigma^{-1}\mathcal{D}_Y}{\otimes}} \sigma^{-1}(\mathcal{D}_{Y\to X})$$

But

$$F^{\bullet} \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}\mathcal{D}_Y = \mathcal{O}_Z \overset{L}{\underset{\sigma^{-1}\mathcal{O}_Y}{\otimes}} \sigma^{-1}\mathcal{D}_Y = \mathcal{O}_Z \underset{\sigma^{-1}\mathcal{O}_Y}{\otimes} \sigma^{-1}\mathcal{D}_Y = \mathcal{D}_{Z\to Y}\ ,$$

and the second part of (1) follows. The second equality of (2) is proved similarly.

6.4 PROPOSITION *In the derived categories of bounded $\mathcal{D}$-modules, we have the equality* $(\pi\circ\sigma)_+ = \pi_+ \circ \sigma_+$.

Let $M^{\bullet} \in D^b(\mathcal{D}_Z)$. We have

$$\pi_+(\sigma_+(M^{\bullet})) = R\tilde{\pi}(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \sigma_+(M^{\bullet}))$$

$$\pi_+(\sigma_+(M^{\bullet})) = R\tilde{\pi}(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} R\tilde{\sigma}(\mathcal{D}_{Y\leftarrow Z} \overset{L}{\underset{\mathcal{D}_Z}{\otimes}} M^{\bullet}))\ .$$

We may replace $M^{\bullet}$ by a locally free resolution (2.4), hence suppress the second L on the right hand side. Locally on Z, $\mathcal{D}_{Y\leftarrow Z} \underset{\mathcal{D}_Z}{\otimes} M^{\bullet}$ is a direct sum of copies of $\mathcal{D}_{Y\leftarrow Z}$ hence is $\mathcal{O}_Z$-quasi-coherent, besides being a left $\sigma^{-1}\mathcal{D}_Y$-module. We can apply 6.2 to σ and get

$$\pi_+(\sigma_+(M)) = R\tilde{\pi}(R\tilde{\sigma}(\sigma^{-1}\mathcal{D}_{X\leftarrow Y} \underset{\sigma^{-1}\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\leftarrow Z} \underset{\mathcal{D}_Z}{\otimes} M^{\bullet})$$

We have then, using 6.3,

(1) $$\pi_+(\sigma_+(M^{\bullet})) = R\tilde{\pi}(R\tilde{\sigma}(\mathcal{D}_{X\leftarrow Z} \underset{\mathcal{D}_Z}{\otimes} M^{\bullet})\ .$$

On the other hand, for $M^{\bullet}$ locally free,

(2) $$(\pi\circ\sigma)_+(M^{\bullet}) = R(\pi\circ\sigma)(\mathcal{D}_{X\leftarrow Z} \underset{\mathcal{D}_Z}{\otimes} M^{\bullet})$$

It suffices then to know that $R(\pi\circ\sigma) = R\tilde{\pi} \circ R\tilde{\sigma}$, but this is true in the topological category, for sheaves of abelian groups, hence also in our situation, since both derived direct images agree (5.1).

Remark. In contrast with 6.4, it is not always true that we have $(\pi\circ\sigma)_{\circ} = \pi_{\circ} \circ \sigma_{\circ}$ at the $\mathcal{D}$-module level of 5.1(6).

§ 7 CLOSED EMBEDDINGS: INVERSE AND DIRECT IMAGES; THE FUNCTOR $\Gamma_{[Y]}$.

In this section, we assume that Y is a closed subvariety of X and that π is the inclusion map. We let $d = \mathrm{cod}_X Y = d_{X,Y}$ and denote by $\mathcal{J}$ the sheaf of ideals of Y. Let $\mu_Y(\mathcal{D}_X)$ be the set of $\mathcal{D}_X$-modules with support in Y.

7.1 We fix some notation and recall a few facts pertaining to this situation.

We note first that $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. We let $T^*_Y(X)$ denote the conormal cotangent bundle to Y. We have an exact sequence of bundles on Y:

(1) $$0 \to T^*_Y(X) \to T^*(X)|_Y \xrightarrow{T^*(\pi)} T^*(Y) \to 0 .$$

For the associated coherent sheaves, this can be written

(2) $$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^1_Y \to 0 .$$

Going over to top exterior powers, we get

(3) $$\Lambda^d \mathcal{J}/\mathcal{J}^2 \otimes_{\mathcal{O}_Y} \omega_Y = \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y ,$$

By definition

(4) $$\omega_{Y/X} = \omega_Y \otimes_{\mathcal{O}_Y} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y)^{-1}$$

whence

(5) $$\Lambda^d \mathcal{J}/\mathcal{J}^2 = \omega^{-1}_{Y/X} .$$

In order to study this situation locally around $y \in Y$, we choose a regular $\mathcal{O}_X$-sequence $(f_i)_{1 \leq i \leq d}$ of regular functions in some neighborhood

U of y in which the df_i are linearly independent at every point, and which generate $\mathcal{J}$. For $x \in U \cap Y$, the df_i at x provide a basis of $T^*_Y(X)$ and hence the df_i form a basis over $\mathcal{O}_Y$ of $\mathcal{J}/\mathcal{J}^2$. As a consequence $df_1 \wedge \dots \wedge df_d$ generates $\Lambda^d \mathcal{J}/\mathcal{J}^2$, hence

(6) $$\omega^{-1}_{Y/X} = \mathcal{O}_Y . df_1 \wedge \dots \wedge df_d .$$

7.2 In the discussion of the direct image we shall have to use the so-called *fundamental local isomorphism theorem:*

(1) $$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Y, M) = \mathcal{T}or^{\mathcal{O}_X}_{d-i}(\mathcal{O}_Y, M) \otimes_{\mathcal{O}_X} \omega_{Y/X} \quad (M \in \mu(\mathcal{D}_X); i \in \mathbf{Z})$$

This is of course "well known", although I have difficulty to give a reference for the statement exactly in that form.

The assertion to prove can also be written

(2) $$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, M)[d] = \mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} M \otimes_{\mathcal{O}_X} \omega_{Y/X} .$$

After having maybe shrunk U, we may assume there are commuting vector fields ∂_i on U such that $\partial_i f_j = \delta_{ij}$ $(1 \leq i, j \leq d)$. The df_i provide a basis of the *conormal sheaf* $\mathcal{J}/\mathcal{J}^2$ and the ∂_i a basis of the dual *normal sheaf* $\mathcal{T} = (\mathcal{J}/\mathcal{J}^2)^*$. We have

$$\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} M = \Lambda^{\cdot} \mathcal{J}/\mathcal{J}^2 \otimes_{\mathcal{O}_X} M[d] = \Lambda^{\cdot} E \otimes_{\mathbf{C}} M ,$$

where E is the $\mathbf{C}$-vector space spanned by the df_i, the space $\Lambda^i E \otimes_{\mathbf{C}} M$ has degree $-i$ and the differential is given by

$$d(df_{j_1} \wedge \dots \wedge df_{j_s} \otimes m) = \Sigma (-1)^{i+1} df_{j_1} \wedge \dots \wedge \widehat{df_{j_i}} \wedge \dots \wedge df_{j_s} \otimes f_{j_i} m$$

To compute the left-hand side we may use

$$\mathcal{H}om_{\mathcal{O}_X}(\Lambda^{\cdot}\mathcal{J}/\mathcal{J}^2, \mathcal{M}) = \Lambda^{\cdot}T \otimes_{\mathcal{O}_X} \mathcal{M} = \Lambda^{\cdot}E^* \otimes_{\mathbb{C}} \mathcal{M}$$

where E^* is the dual vector space to E, with basis the ∂_i. Ignoring the shift of degrees, we must deal with a complex $C^{\cdot}$ given by

$$C^i = \Lambda^{-i}E^* \otimes_{\mathbb{C}} \mathcal{M}$$

with differential $d : \Lambda^s E^* \otimes_{\mathbb{C}} \mathcal{M} \to \Lambda^{s+1}E^* \otimes_{\mathbb{C}} \mathcal{M}$ given by

$$\partial_{i_1} \wedge \ldots \wedge \partial_{i_s} \otimes m \mapsto \Sigma(-1)^{i+1}\partial_{i_1} \wedge \ldots \wedge \partial_{i_s} \wedge \partial_i \otimes f_i m$$

To establish the isomorphism of the theorem one defines a map

$$\Lambda^s E^* \otimes_{\mathbb{C}} \mathcal{M} \to \Lambda^{d-s}E \otimes_{\mathbb{C}} \mathcal{M} \text{ by}$$

$$df_{i_1} \wedge \ldots \wedge df_{i_s} \otimes m \mapsto (-1)^{\varepsilon(I,I')}\partial_{j_1} \wedge \ldots \wedge \partial_{j_{d-s}} \otimes m$$

where $I = \{i_1,\ldots,i_s\}$, $I' = \{j_1,\ldots,j_{d-s}\}$ is the strictly increasing complementary sequence to I in [1,d] and $\varepsilon(I,I')$ the number of inversions in the sequence (I,I') [EGA: III,1.1]. It commutes with d hence defines an isomorphism of the complexes (except for a change of degrees). This proves (1) locally, i.e. without the factor $\omega_{Y/X}$. However, it is easily seen that under a change of local coordinates,

$$\Lambda^{d-s}E \otimes \mathcal{M} \otimes \partial_1 \wedge \ldots \wedge \partial_d$$

behaves like a s-vector, hence (1) is the invariant way to write this isomorphism.

7.3 *Inverse image*. Since $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$, we have

(1) $$\pi^{\circ}(\mathcal{M}) = \mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{M}/\mathcal{J}\mathcal{M}$$

(2) $$\mathcal{D}_{Y\to X} = \mathcal{D}_X/\mathcal{J}\mathcal{D}_X .$$

By general construction, $\mathcal{D}_{Y\to X}$ is a $\mathcal{D}_Y \times \mathcal{D}_X$ module. The right $\mathcal{D}_X$-module structure is given in (2) by right multiplication on the right-hand side. The left $\mathcal{D}_Y$-module structure is not so obvious, because $\mathcal{D}_Y$ is not in general a subsheaf of $\mathcal{D}_X$. It would be if Y were a direct factor, as we saw in discussing projections, but in general it is only a subquotient. To describe this structure locally we use the previous (f_i,∂_i). Let $\Theta_{X,\underline{f}}$ be the $\mathcal{O}_X$-module of vector fields which annihilate the f_i's. Any vector field ξ can be written uniquely as

(3) $$\xi = \xi f_1.\partial_1 + \ldots + \xi f_d.\partial_d + \eta \quad \text{with } \eta \in \Theta_{X,\underline{f}} ,$$

i.e. there is a direct sum decomposition

(4) $$\Theta_X = \mathcal{O}_X\partial_1 \oplus \ldots \oplus \mathcal{O}_X\partial_d \oplus \Theta_{X,\underline{f}} .$$

Let $\mathcal{D}_{X,\underline{f}}$ be the subalgebra of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and $\Theta_{X,\underline{f}}$. Then we may view $\mathcal{D}_Y$ as the restriction of $\mathcal{D}_{X,\underline{f}}$ to Y:

(5) $$\mathcal{D}_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{D}_{X,\underline{f}} = \mathcal{D}_{X,\underline{f}}/\mathcal{J}\mathcal{D}_{X,\underline{f}} .$$

We can write $\mathcal{D}_X$ on U, as a vector space, in the form

(6) $$\mathcal{D}_X = \mathcal{D}_{X,\underline{f}} \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\ldots,\partial_d]$$

whence also on $U \cap Y$

(7) $$\mathcal{D}_{Y\to X} = \mathcal{D}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\ldots,\partial_d].$$

In this decomposition, the $\mathcal{D}_Y$-module structure is given by left multiplication on the first factor of the right hand side. We see from this that $\mathcal{D}_{Y\to X}$ is locally free over $\mathcal{D}_Y$ with basis the monomials in the (commuting) variables ∂_i . In particular it is not $\mathcal{D}_Y$-coherent, hence π^0 *does not preserve coherence*. To describe the right $\mathcal{D}_X$-module structure on the right hand side of (7), we can use the decomposition (6).

7.4 THEOREM *We keep the previous assumptions. Let $M \in \mu(\mathcal{D}_X)$ and let ${}_J M = \{m\in M \mid Jm = 0\}$.*

(i) $H^i\pi^!(M) = 0$, for $i \notin [0,d]$;

(ii) $H^0\pi^!(M) = \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_Y} {}_J M$;

(iii) $H^d\pi^!(M) = M/JM$;

(iv) *if $M \in \mu_Y(\mathcal{D}_X)$, then $H^i\pi^!(M) = 0$ for $i \neq 0$.*

Proof. As $\mathcal{O}_Y$-modules, the $H^i\pi^!M$ are the cohomology sheaves of $\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} M[-d]$. To compute them we use a Koszul resolution. Let E be a d-dimensional complex vector space with basis the df_i's. We have a locally free resolution of $\mathcal{O}_Y$:

(1) $$0 \to C_{-d} \to C_{-d+1} \to \dots \to C_o \to \mathcal{O}_X/J \to 0$$

where

(2) $$C^{-i} = \Lambda^i E \otimes_{\mathbb{C}} \mathcal{O}_X$$

with $d : C_{-i} \to C_{-i+1}$ given by

(3) $$d(df_{j_1}\wedge\dots\wedge df_{j_i} \otimes f) = \Sigma(-1)^{s+1}(df_{j_1}\wedge\cdots\wedge\widehat{df_{j_s}}\wedge\dots\wedge df_{j_i} \otimes f_{j_s}.f)$$

Then $C^{\cdot} \otimes_{\mathcal{O}_X} M \to M \to 0$ is a locally free resolution of M, hence

(4) $$\pi^!M = C^{\cdot} \otimes_{\mathcal{O}_X} M[-d] .$$

This proves (i) and (iii). It also shows

(5) $$H^0\pi^!(M) = df_1\wedge \dots\wedge df_d \otimes_{\mathbb{C}} {}_J M.$$

whence (ii), after tensoring with $\mathcal{O}_Y$.

There remains to prove (iv). By composition (4.3), the proof is easily reduced to the case of a hypersurface. We have then d = 1 and write f,∂ instead of f_1,∂_1 . We now have to prove that $H^1\pi^+M = M/fM$ is zero, hence that

(6) $$M = f.M \qquad \text{if } M \in \mu_Y(\mathcal{D}_X) .$$

The relation $\partial f = [\partial,f] = 1$ gives

(7) $$\partial f^k = kf^{k-1} \qquad (k\in \mathbb{N}).$$

M is quasi-coherent over $\mathcal{O}_X$ with support in Y. This means that given $m \in M$, there exists $k \in \mathbb{N}$ such that $f^k m = 0$. To establish (6), it is then enough to show

(*) *Let $m \in M$. Assume that $f^k.m = 0$ for some $k \geq 1$, then $m \in f.M$.*

Let first k = 1. We have fm = 0, hence also

$$0 = \partial(fm) = m + f\partial m$$

$$m = -f\partial m \in fM .$$

let $k \geq 2$ and assume (*) to be true for k - 1. The relation $f^k m = 0$ gives

$$0 = \partial(f^k m) = kf^{k-1}m + f^k m$$

$$f^{k-1}(km+f.m) = 0$$

hence, by induction $km + fm \in f.M$ and $km \in f.M$.

7.5 COROLLARY *$M \mapsto H^0\pi^! M$ is an exact functor from $\mu_Y(\mathcal{D}_X)$ to $\mu(\mathcal{D}_Y)$.*

This follows from 7.4 (iv) by the long exact sequence in cohomology.

7.6 *Remarks* (1) In fact we shall soon see that this map is an equivalence of categories (Kashiwara's theorem).

(2) Implicit in (ii) is the fact that $\omega_{Y/X}^{-1} \otimes {}_J M$ is a $\mathcal{D}_Y$-module. To see this locally, note first that ${}_J M$ is stable under $\Theta_{X,\underline{f}}$: If $\xi \in \Theta_{X,\underline{f}}$ and $f \in J$, then $\xi.f = 0$ on Y, hence ξ and f commute on elements with support in Y. Consequently, ${}_J M$ is stable under $\mathcal{D}_{X,\underline{f}}$ and this operation yields one of $\mathcal{D}_{X,\underline{f}}/J\mathcal{D}_{X,\underline{f}} = \mathcal{D}_Y$ on ${}_J M$.

(3) Recall that in the topological setting. $\pi^!$ is the derived functor of the functor sections with support in the closed subvariety. Hence (ii) shows that $H^0\pi^!$ is a close analogue, whereas $H^d\pi^!$ is the usual inverse image. The decision to view our inverse image as extraordinary or ordinary hinges in a way on whether one concentrates on the lowest or the highest nontrivial derived functor. We will soon see that it is indeed the lowest one which matters most. One reason for the shift of degree in 4.2 (4) is precisely to have it in degree zero.

We now turn to the direct image.

7.7 PROPOSITION (i) *We have*

$$(1) \qquad \mathcal{D}_{X\leftarrow Y} = \mathcal{D}_X^r \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \omega_{Y/X} = \mathcal{D}_X/\mathcal{D}_X J \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

(ii) *$\mathcal{D}_{X\leftarrow Y}$ is locally free over $\mathcal{D}_Y$ and $H^i\pi_+(M) = 0$ for $i \geq 1$, $(M \in \mu(\mathcal{D}_Y))$.*

The functor π_+ preserves quasi-coherence.

Proof. By definition

$$(2) \qquad \mathcal{D}_{X\leftarrow Y} = \mathcal{D}_X^r \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \omega_Y = (\mathcal{D}_X^r \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_X^{-1} \otimes_{\mathcal{O}_X} \omega_Y .$$

The product of the two last factors is $\omega_{Y/X}$. Since $\mathcal{O}_Y = \mathcal{O}_X/J$ and we use right multiplication on $\mathcal{D}_X$, it is clear that the first tensor product is $\mathcal{D}_X/\mathcal{D}_X J$. This proves (1).

The discussion of 7.3 leading to 7.3(9), with left and right permuted, then gives

$$(3) \qquad \mathcal{D}_X/\mathcal{D}_X J = \mathcal{D}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\dots,\partial_d]$$

$$(4) \qquad \mathcal{D}_{X\leftarrow Y} = \mathcal{D}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\dots,\partial_d] \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

Hence $\mathcal{D}_{X\leftarrow Y}$ is locally free over $\mathcal{D}_Y$, and $\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y}$ is an exact functor. Moreover, π_o is the extension by zero, hence is exact and the assertion (ii) follows . Also $\mathcal{O}_X$ operates on $\pi_+(M)$ through $\mathcal{O}_Y$, therefore π_+ preserves quasi-coherence.

7.8 LEMMA *Let $N \in \mu(\mathcal{D}_Y)$.*

(i) *If N is finitely generated over $\mathcal{D}_Y$, then π_+N is finitely generated over $\mathcal{D}_X$.*

(ii) *$J\cdot\pi_+(N) = \pi_+(N)$ and ${}_J\pi_+(N) \cong N \otimes_{\mathcal{O}_Y} \omega_{Y/X}$.*

(iii) *$e(\pi_+(N)) = e(N)$; in particular, $\pi_+(N)$ is holonomic if and only if N is so.*

Proof. $\pi_+(N)$ is zero outside Y. It is enough to consider it around $y \in Y$. There it is isomorphic to $\mathbb{C}[\partial_1,\dots,\partial_d] \otimes_{\mathbb{C}} N$. The ∂_i act in the obvious way on the first factor, hence the latter is cyclic. (i) follows immediately from this.

f_i operates on $\mathbb{C}[\partial_1,\dots,\partial_d]$ by derivation, via

$$(1) \qquad [f_i,\partial_j^k] = -k\partial_j^{k-1} .$$

The action of J on $\mathbb{C}[\partial_1,\dots,\partial_d] \otimes N$ is on the first factor only. Then (1) implies (ii).

Let Γ' be a good filtration of N. The degree on $\mathbb{C}[\partial_1,\dots,\partial_d]$ and Γ' combine to a good filtration of π_+N and

$$\operatorname{gr}_{\Gamma}\pi_+(N) \cong \mathbb{C}[\partial_1,\dots,\partial_d] \underset{\mathbb{C}}{\otimes} \operatorname{gr}_{\Gamma'} N$$

hence

$$\operatorname{Ann} \Gamma_+(N) = \mathbb{C}[\partial_1,\dots,\partial_d] \underset{\mathbb{C}}{\otimes} \operatorname{Ann} N$$

and (iii) follows. This is also geometrically obvious. We have the natural projection

$$T^*(\pi) : T^*(X)|_Y \to T^*(Y) \to 0 \ .$$

and it is clear from the definition that

$$SS(\pi_+(N)) = T^*(\pi)^{-1}(SS(N)) \ .$$

7.9 *The functor* $\Gamma_{[Y]}$. For any ideal I of $\mathcal{O}_X$ and any $\mathcal{O}_X$-module M let

(1) $$\operatorname{Ann}_M I = {}_I M = \{m \in M, I\cdot m = 0\}.$$

By definition

(2) $$\Gamma_{[Y]}M = \cup_{k\geq 1}\, {}_{J^k}M \ .$$

Assume now M is a $\mathcal{D}_X$-module. For all $i,k \in N$ we have

$$J^{i+k}\mathcal{D}_X(k) \subset \mathcal{D}_X\cdot J^i$$

as follows by iterated use of the rule $[\partial,f] = \partial f \quad (\partial\in\Theta_X, f\in\mathcal{O}_X)$. Therefore, if $m \in M$ is annihilated by J^i, and $P \in \mathcal{D}_X(k)$, then $P\cdot m$ is annihilated by J^{i+k}; hence $\Gamma_{[Y]}M$ is a $\mathcal{D}_X$-module.

7.10 PROPOSITION *The map* $\mu : \mathcal{D}_X \times {}_J M \to M$ *defined by* $(P,m) \mapsto P\cdot m$ $(P\in\mathcal{D}_X, m\in {}_J M)$ *induces an isomorphism*

(1) $$\mathcal{D}_X/\mathcal{D}_X J \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X} \underset{\mathcal{D}_Y}{\otimes} \omega_{Y/X}^{-1} \underset{\mathcal{O}_Y}{\otimes} {}_J M \xrightarrow{\sim} \Gamma_{[Y]}M \ .$$

Proof. Since ${}_J M \subset \Gamma_{[Y]}M$ it is clear that $\operatorname{im}\mu \subset \Gamma_{[Y]}M$.

On the left-hand side of (1) we have used the $\mathcal{D}_Y$-module structures defined in 7.6(2) and 7.7. Locally, by 7.7(3) the left-hand side of (1) is given by

(2) $$\mathcal{D}_X/\mathcal{D}_X J \underset{\mathcal{D}_Y}{\otimes} {}_J M = \mathbb{C}[\partial_1,\dots,\partial_d] \underset{\mathbb{C}}{\otimes} {}_J M.$$

We have therefore to see that $\Gamma_{[Y]}M$ is the direct sum of the subspaces

(3) $$\partial^\alpha M \quad (\alpha\in\mathbb{N}^d)$$

where, as usual, we use the multiindex notation

(4) $$\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d} \quad (\alpha=(\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d)$$

(a) *These subspaces are linearly independent.* Assume we have a nontrivial linear relation

$$\Sigma c_\alpha \partial^\alpha m_\alpha = 0$$

and let c be the maximum of $|\alpha| = \Sigma\alpha_i$. If $|\alpha| = 0$, it reduces to one term. To argue by induction it suffices to manufacture a nontrivial relation with max. $|\alpha| < c$. Assume $c \geq 1$, let i be such that $\alpha_i \neq 0$ in a term of highest degree. Apply f_i to the left hand side; using 7.8(1), we get

(5) $$\Sigma - c_\alpha m_{\alpha_i} \partial^{\alpha'} m_\alpha = 0, \quad (\alpha'=(\alpha_1,\dots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\dots,\alpha_d)),$$

which is nontrivial, of degree c - 1.

(b) μ *is surjective.* It suffices to show that if $J^k m = 0$ for some $k \in \mathbb{N}$, then $m \in \operatorname{Im}\mu$. This is true ba definition if k = 1, so we proceed by induction, fix $k \geq 2$ and assume this to be true up to k - 1. Let $m \in M$ and assume $J^k\cdot m = 0$. We claim:

(6) $$\mathcal{J}^{k-1}((k-1)m + \Sigma\partial_i f_i m) = 0.$$

$\mathcal{J}^{k-1}$ is generated as an ideal monomials $u = f_1^{p_1}\dots f_d^{p_d}$ $(\Sigma p_i = k-1)$. Since $uf_i \in \mathcal{J}^k$, we have $uf_i m = 0$, hence $\partial_i(f_i u\cdot m) = 0$ and

(7) $$\partial_i(uf_i)\cdot m + uf_i\partial_i m = 0 \quad (i=1,\dots,d)\ .$$

But

$$\partial_i(uf_i) = (\partial_i u)f_i + u = (p_i+1)u,$$

(8) $$u((p_i+1)m + f_i\partial_i m) = 0$$

Summing over i, we get

(9) $$u((k+d-1)m + \Sigma_i f_i\partial_i m) = 0 \qquad (u\in\mathcal{J}^{k-1})$$

which proves (6). The induction assumption yields

(10) $$(k+d-1)m + \Sigma f_i\partial_i m \in \operatorname{Im}\mu$$

(11) $$f_i m \in \operatorname{Im}\mu,$$

whence also $\partial_i f_i m \in \operatorname{Im}\mu$; added to (10), this gives

$$(k+d-1)m + \Sigma_i[f_i,\partial_i]m \in \operatorname{Im}\mu.$$

But $\Sigma_i[f_i,\partial_i] = -d\cdot\mathrm{Id}$, whence $(k-1)m \in \operatorname{Im}\mu$.

Remark. This generalizes V, 3.1.2. The proof is essentially the same.

7.11 THEOREM (i) (Kashiwara) *$H^0\pi^!$ and π_+ are equivalences of categories between $\mu_Y(\mathcal{D}_X)$ and $\mu(\mathcal{D}_Y)$, inverse of one another. They preserve coherence holonomic defect and holonomicity.*

(ii) $\pi_+ H^0\pi^+(M) = \Gamma_{[Y]}M \quad (M\in\mu(\mathcal{D}_X))$.

Proof. By 7.4, we have $H^0\pi^!(M) = \mathcal{J}^M \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}^{-1}$, therefore, by 7.7

$$\pi_+ H^0\pi^+(M) = \mathcal{D}_X/\mathcal{D}_X\mathcal{J} \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X} \underset{\mathcal{D}_Y}{\otimes} \mathcal{J}^M \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}^{-1} = \mathcal{D}_X/\mathcal{D}_X\mathcal{J} \underset{\mathcal{D}_Y}{\otimes} \mathcal{J}^M\ .$$

and (ii) now follows from 7.10.

We have $\Gamma_{[Y]}M = M$ if and only if $M \in \mu_Y(\mathcal{D}_X)$, hence $\pi_+ \circ H^0\pi^! = \mathrm{Id}$. on $\mu_Y(\mathcal{D}_X)$.

Let now $N \in \mu(\mathcal{D}_Y)$. By 7.4(ii):

$$H^0\pi^!\pi_+(N) = \mathcal{J}^{\pi_+ N} \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}^{-1}\ .$$

Since $\mathcal{J}^{\pi_+ N} = N \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}$ (7.8(ii)), this shows that $H^0\pi^! \circ \pi_+ = \mathrm{Id}$ and prove the first part of (i). We already know that π_+ preserves coherence and defect (7.8). Hence so does its inverse. The second part of (i) follows.

Remarks. (1) We see in particular that a $\mathcal{D}_X$-module M with support in Y can be considered as a $\mathcal{D}_Y$-module. The analogous statement in the $\mathcal{O}$-category is well-known to be false: If a coherent $\mathcal{O}_X$-module is supported by Y, then it is annihilated by some power $\mathcal{J}^k$ of the ideal $\mathcal{J}$, hence may be viewed as a $\mathcal{O}_X/\mathcal{J}^k\mathcal{O}_X$-module. However, if M is not annihilated by $\mathcal{J}$ itself, it cannot be viewed as a $\mathcal{O}_Y$-module.

(2) Let $Y = \mathbf{A}^{n-1}$ be the hyperplane $x_n = 0$ in $X = \mathbf{A}^n$. We are therefore dealing with modules over the Weyl algebras A_{n-1} and A_n (cf.V) and 7.11 specializes to (V. 3.1.6). Let M be a module over A_n. Then by 7.4, the inverse images are $\ker_M x_n$ in dimension 0 and $M/x_n M$ in dimension one. If M is holonomic, 7.11 implies that $M/x_n M$ and $\ker_M x_n$ are holonomic A_{n-1}-modules, as was proved in V (see 3.1.7, 3.1.8).

7.12 We want now to state and prove the generalization of 7.11 to derived categories. If $M^\cdot \in D^b(\mathcal{D}_X)$, we define its support $\mathrm{Sup}(M^\cdot)$ by

(1) $$\mathrm{Sup}(M^\cdot) = \{x \in X \; H^\cdot M^\cdot_x \neq 0\}.$$

Let $D^b_Y(X) = \{M^\cdot \in D^b(\mathcal{D}_X),\ \mathrm{Sup}\ M^\cdot \subset Y\}$.

7.13 THEOREM. (i) (Kashiwara) π_+ *and* $\pi^!$ *define equivalences of categories between* $D^b_Y(\mathcal{D}_X)$ *and* $D^b(\mathcal{D}_Y)$*, which are inverse of one another and preserve coherence and holonomicity.*

(ii) We have $R\Gamma_{[Y]} = \pi_+ \circ \pi^!$.

The categories $\mu_Y(\mathcal{D}_X)$ and $\mu(\mathcal{D}_Y)$, viewed as single degree complexes in $D^b_Y(\mathcal{D}_X)$ and $D^b(\mathcal{D}_Y)$, generate those as triangulated categories [I, §12]. The first assertion then follows from 7.11(i) by induction on exact triangles. In the same way, $\mathrm{hol}_Y(\mathcal{D}_X) = \mathrm{hol}(\mathcal{D}_X) \cap \mu_Y(\mathcal{D}_X)$ and $\mathrm{hol}(\mathcal{D}_Y)$ generate $D^b_Y(\mathcal{D}_X) \cap D^b_h(\mathcal{D}_X)$ and $D^b_h(\mathcal{D}_Y)$. The preservation of holonomic complexes then follows in the same way from 7.11 and 1.13. Similarly 7.8, 7.11 imply the preservation of coherence.

(ii) follows from 7.11(ii) and the fact that π_+ is exact.

§ 8 SOME APPLICATIONS

A. *Affine morphisms and locally closed embeddings.*

8.0 Recall that π is said to be affine if X has a finite open affine cover (X_i) such that $\pi^{-1}(X_i)$ is affine for all i's. If so, the inverse image of any affine open set of X is affine [EGA: II,1.6]. If Y is affine, then π is always affine [EGA: I,5.5.10].

8.1 PROPOSITION *Assume that* π *is an affine morphism.*

(i) π_+ *is right exact.*

Assume moreover that π *is faithful and identify* Y *to its image.*

(ii) $H^i\pi_+ = 0$ *for* $i \geq 1$ *and* $H^0\pi_+$ *is injective, exact on* $\mu(\mathcal{D}_Y)$.

(iii) On $D^b_{\bar{Y}}(\mathcal{D}_X)$*, the functor* $\pi^!$ *has zero amplitude.*

Proof. These statements are local on X. We may therefore assume X to be affine and then Y is affine too. Therefore (i) follows from 5.4.

(ii) In view of (i), it suffices to show that π_+ is left exact. Let $S = \bar{Y} - Y$. We write π as the composition

(1) $$Y \xrightarrow{\alpha} X - S \xrightarrow{\beta} X$$

of a closed embedding α and an open embedding β . We know that α_+ is exact (7.11) and β_+ left exact (5.2). Hence π_+ is left exact.

(iii) To see this we use again the factorization (1). The image of $\beta^!$ is contained in $D^b_Y(\mathcal{D}_{X-S})$. We then use 7.11 for the restriction of $\alpha^!$ to the latter and 4.4 for $\beta^!$.

8.2 PROPOSITION *Let* π *be a locally closed embedding.*

(i) *Let* $F^\cdot \in D^b(\mathcal{D}_Y)$*. Then* $\pi_+F^\cdot$ *is supported by* $\bar{Y}$ *and* $\pi^! \circ \pi_+(F^\cdot) = F^\cdot$.

(ii) *Let* $G^\cdot \in D^b(\mathcal{D}_X)$*. Fix* $j \in \mathbb{Z}$ *and assume that* $H^iG^\cdot = 0$ *for* $i \neq j$. *Then* $H^i\pi^!G^\cdot = 0$ *for* $i \notin [j, j+d_{X,Y}]$.

Proof. Let $S = \bar{Y} - Y$. We decompose π again as in 8.1(1).

By 7.13, $\alpha_+(F^\cdot)$ has support on Y. Since β_+ is the derived functor of the usual direct image, the support of $\beta_+(M^\cdot)$ $(M^\cdot \in D^b(\mathcal{D}_{X-S}))$ is contained in the union of S and of the support of $M^\cdot$. The fist assertion of (i) follows.

It suffices to prove the second part of (i) for a module $F \in \mu(\mathcal{D}_Y)$. We have

(1) $$\pi^! \circ \pi_+ = \alpha^! \circ \beta^! \circ \beta_+ \circ \alpha_+ .$$

By 7.7, $\alpha_+(F)$ is a single degree complex. The $H^i\beta_+(\alpha_+F)$ have their support on S for $i \neq 0$, as follows from 5.2. Therefore

$$\beta^! . \beta_+ . \alpha_+(F) = \alpha_+(F) .$$

But $\alpha^! . \alpha_+(F) = F$ by 7.11.

(ii) $H^\cdot\beta^! G^\cdot$ is the restriction to X - S of $H^\cdot G^\cdot$, hence is zero in degrees $\neq j$. This reduces us to the case of a closed embedding, where our assertion follows from 7.4.

B. *An exact triangle.*

8.3 PROPOSITION *Let* Z *be a closed smooth subvariety of* X *and let us denote by* j *and* i *the inclusions* $Z \to X$ *and* $U = X - Z \to X$. *Let* $M^\cdot \in D^b(\mathcal{D}_X)$ *Then we have a distinguished triangle*

(1) $$\begin{tikzcd} j_+ j^!(M^\cdot) \arrow[rr] & & M^\cdot \arrow[dl] \\ & i_+(M^\cdot|U) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Proof. Recall that if $S^\cdot$ is a complex of sheaves on X, then we have an exact triangle

(2) $$\begin{tikzcd} j_! j^! S^\cdot \arrow[rr] & & S^\cdot \arrow[dl] \\ & Ri_* i^* S^\cdot \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $j^!$ is hypercohomology with support in Z. Moreover $j_* = j_!$ is extension by zero and $i^*S^\cdot = S^\cdot|U$. In the $\mathcal{O}$- category, if $F^\cdot$ is quasi-coherent, this can be written

(3) $$\begin{tikzcd} R\Gamma_{[Y]}F^\cdot \arrow[rr] & & F^\cdot \arrow[dl] \\ & Ri_* i^* F^\cdot \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Here i^* is just the restriction. Let now $M^\cdot \in D^b(\mathcal{D}_X)$. Then $Ri_* = i_+$ (5.2) and we have $j_+ \circ j^! = R\Gamma_{[Y]}$ by 7.13. This yields (1).

C. *Base change for locally closed embeddings.*

8.4 THEOREM *Let* $j : C \to X$ *be the inclusion of a locally closed smooth subvariety* C *and assume that* $\pi^{-1}C$ *is smooth in* Y. *Let* $\tilde{j} : \pi^{-1}C \to Y$ *be the inclusion and* π' *the restriction of* π *to* $\pi^{-1}C$. *On* $D^b(\mathcal{D}_Y)$ *we have*

(1) $$\pi'_+ \circ \tilde{j}^! = j^! \circ \pi_+ .$$

Proof. Assume first C to be open in X. Then $\pi^{-1}C$ is open in Y and $j, \tilde{j}$ are just the restrictions. Our assertion in this case means simply that the formation of the direct image is local on the base, which is clear from the definitions.

Let now C be closed in X. We consider the diagram

(2) $$\begin{tikzcd} \pi^{-1}C \arrow[r, "\tilde{j}"] \arrow[d, "\pi'"] & Y \arrow[d, "\pi"] & \tilde{U} = \pi^{-1}U \arrow[l, "i"'] \arrow[d, "\pi''"] \\ C \arrow[r, "j"] & X & U = X - C \arrow[l, "i"'] \end{tikzcd}$$

Let $M^\cdot \in D^b(\mathcal{D}_Y)$ and $N^\cdot = \pi_+ M^\cdot$. By the above:

(3) $$\pi''_+(M^\cdot|U) = (\pi_+ M^\cdot)|U = N^\cdot|U \ .$$

We have by 8.3 two exact triangles ,

(4) $$\tilde{j}_+\tilde{j}^!(M^\cdot) \to M^\cdot \longrightarrow \tilde{i}_+(M^\cdot|U) \xrightarrow{[1]}$$

on Y and

(5) $$j_+j^!(N^\cdot) \to N^\cdot \longrightarrow i_+(N^\cdot|U) \xrightarrow{[1]}$$

on X. Applying π_+ to (4), we get a second exact triangle on X:

(6) $$\pi_+ \circ \tilde{j}_+\tilde{j}^!(M^\cdot) \to N^\cdot \longrightarrow \pi_+(\tilde{i}_+(M^\cdot|\tilde{U})) \xrightarrow{[1]}$$

We have $\pi_+ \circ \tilde{i}_+ = i_+ \circ \pi''_+$ (6.4). Together with (2), this yields

(7) $$\pi_+(\tilde{i}_+(M^\cdot|\tilde{U})) = i_+(N^\cdot|U)$$

therefore (5) and (6) have a common edge. As a consequence the remaining vertices are homologically isomorphic; since $\pi_+ \circ \tilde{j}_+ = j_+ \circ \pi'_+$, (6.4) we get

(8) $$j_+\pi'_+j^!(M^\cdot) = j_+j^!\pi+(M^\cdot).$$

By Kashiwara's theorem (7.13), we may erase j_+ on both sides; this yields (1) when C is closed.

In the general case, C is open in its closure $\bar{C}$ by assumption. Let $S = \bar{C} - C$ and $T = \pi^{-1}S$. Let further $X' = X - S$ and $Y' = Y - T$. We have the commutative diagram of maps:

$$\begin{tikzcd}
\pi^{-1}C \arrow[r, "\tilde{k}"] \arrow[d, "\pi'"] & Y' \arrow[r, "\tilde{i}"] \arrow[d, "\pi''"] & Y \arrow[d, "\pi"] \\
C \arrow[r] & X' \arrow[r] & X
\end{tikzcd}$$

The base change is valid on the right square since X' is open and in the left square since C is closed in X'. We have then

$$\pi'_+ \circ \tilde{j}^! = \pi'_+ \circ \tilde{k}^! \circ \tilde{i}^! = k^! \circ \pi''_+ \circ \tilde{i}^! = k^! \circ i^! \circ \pi_+ = j^! \circ \pi_+ \ .$$

8.5 COROLLARY *Assume that* $\pi(Y) \cap C$ *is empty. Then* $j^! \circ \pi_+ = 0$.

This follows formally from 8.4 since in this case $\pi^{-1}(C)$ is empty. One may also argue more directly : we have now $\tilde{U} = Y$ hence $\pi = i \circ \pi''$, therefore

$$\pi_+(M^\cdot) = N^\cdot = i_+.\pi''_+(M^\cdot) \quad \text{and} \quad \pi''_+(M^\cdot) = N^\cdot|U \ .$$

The adjunction map $N^\cdot \longrightarrow i_+(N^\cdot|U)$ in the lower exact triangle 8.5(5) is therefore an isomorphism, hence $j_+j^!N^\cdot = 0$, and therefore $0 = j^!N^\cdot = j^!\pi_+M^\cdot$.

VII. COHERENT, HOLONOMIC AND REGULAR HOLONOMIC COMPLEXES

A. Borel

§ 9 COHERENT COMPLEXES

Notwithstanding our title, the first proposition is valid for general, not necessarily coherent, $\mathcal{D}_Y$-modules.

9.1 PROPOSITION *Let* X *be affine,* $P = \mathbb{P}_N(\mathbb{C})$ *for some* N *and* $Y = P \times X$.

(i) $F \to \Gamma(F)$ *is an exact functor on* $\mu(\mathcal{D}_Y)$.

(ii) *Any* $F \in \mu(\mathcal{D}_Y)$ *is generated by its cross sections.*

(iii) $\mathcal{D}_Y$ *is a projective generator of* $\mu(\mathcal{D}_Y)$

Let $A = \mathbb{A}^{N+1}$ and $A^* = A - \{0\}$. Let also $Z = A \times X$ and $Z^* = A^* \times X$. The complement of Z^* in Z is the subvariety $S = \{0\} \times X$. We let $j : Z^* \to Z$ be the inclusion and $\sigma : Z^* \to P \times X$ the natural projection. Note that σ is submersive, with fibre of dimension one, hence $\sigma^!$ is exact (4.8) and, if $M \in \mu(\mathcal{D}_Y)$, the inverse image $\sigma^! M$ is concentrated in degree -1 (see 4.5(8)). We denote by (a_i) the coordinates on A and also their inverse images on Z and by ∂_i the corresponding partial derivatives. The group $\mathbb{C}^*$ operates by homotheties on A^*, trivially on X, whence an operation on Z^*. The corresponding infinitesimal operator is $D = \Sigma a_i \partial_i$.

We have in $\mathcal{D}_Z$ the rules

$$(1) \quad D \circ a_i = a_i \circ D + a_i \quad D \circ \partial_i = \partial_i \circ D - \partial_i \qquad (i=1,\dots,n=N+1)$$

Let $F \in \mu(\mathcal{D}_Z)$. Assume D integrates to an algebraic action of $\mathbf{C}^*$ on F, e.g. $F = H^j j_+ \sigma^! M (M \in \mu(\mathcal{D}_Y); j \in \mathbb{Z})$. The action on $\Gamma(F)$ is then locally algebraic, semisimple. As a result, $\Gamma(F)$ is a semisimple $\mathbb{C}[D]$-module, with integral eigenvalues and we can write:

(2) $$\Gamma(F) = \oplus_{k\in\mathbb{Z}} \Gamma(F)^k ,$$

where

(3) $$\Gamma(F)^k = \{f\in F \mid D.f = k.f\} , \quad (k\in\mathbb{Z}).$$

The relations (1) show that

(4) $$\partial_i \Gamma(F)^k \subset \Gamma(F)^{k-1} \quad a_i.\Gamma(F)^k \subset \Gamma(F)^{k+1} \quad (k\in\mathbb{Z}).$$

We now prove:

(a) If $\Gamma(F)^k \neq 0$ for some $k > 0$, then $\Gamma(F)^0 \neq 0$.

In fact, if $s \in \Gamma(F)^k - \{0\}$ and $\partial_i s \neq 0$ for some i, then $\Gamma(F)^{k-1} \neq 0$. If $\partial_i s = 0$ for all i's, then $k = 0$. From this (a) follows by descending induction on k.

(b) If $\Gamma(F)^{-k}$ contains an element s with support not in S for some $k > 0$, then $\Gamma(F)^0$ contains an element with support not in S.

By assumption, there exists a point (a,x) $(a\in A^*, x\in X)$ at which s is not zero. Let i be such that $a_i(a) \neq 0$. Then $a_i^k.s$ belongs to $\Gamma(F)^0$ and is non-zero at (a,x).

(c) If $\mathrm{Supp}(F) \subset S$, then $\Gamma(F)^k \neq 0$ implies $k \leqq - n$.

By 7.11, F is the direct image of a $\mathcal{D}_S$-module G and we have by 7.10 an isomorphism

$$F = \mathbb{C}[\partial_i,\ldots,\partial_n] \otimes_{\mathbb{C}} G ,$$

As an $\mathcal{O}_Y$-module G is annihilated by the a_i's. Moreover we have

$$a_i(\partial_j^k) = - k\partial_j^{k-1}.\delta_{ij} .$$

The equality $[\partial_i, a_i] = 1$ implies

$$D = \Sigma\partial_i \circ a_i - n.I.$$

Therefore if $s \in G$, then $Ds = - n.s.$ and we get

$$D(\partial^\alpha s) = - |\alpha|\partial^\alpha s - n\partial^\alpha s,$$

which shows that $\partial^\alpha s$ belongs to the eigenvalue $- n - |\alpha|$.

(d) Let $M \in \mu(\mathcal{D}_Y)$. Clearly

(5) $$\Gamma(M) = \Gamma(Y;M) = \Gamma(Z^*, \sigma^! M[-1])^0 .$$

Let $N^\cdot = j_+\sigma^! M[-1]$. Since $H^0 j_+$ is the sheaf theoretic direct image (5.2), we have

(6) $$\Gamma(Z;H^0N^\cdot) = \Gamma(Z^*, \sigma^! M[-1]).$$

With (5), this yields a natural isomorphism

(7) $$\Gamma(Y;M) = \Gamma(Z, H^0N^\cdot)^0 , \quad (N^\cdot = j_+\sigma^! M[-1]).$$

We now show that $\Gamma(M) = 0$ implies $M = 0$. By (7) $\Gamma(Z;H^0N^\cdot)^0 = 0$. It follows from (a) that $\Gamma(Z;H^0N^\cdot)^k = 0$ if $k > 0$ and from (b) that any element of $\Gamma(Z;H^0N^\cdot)^k$ is supported by S if $k < 0$. Therefore, all elements of $\Gamma(Z,H^0N^\cdot)$ have support in S. But Z is affine, therefore $H^0N^\cdot$ is generated by its sections and we have $H^0N^\cdot|Z^* = 0$. Since $\sigma^! M[-1] = H^0N^\cdot|Z^*$, our assertion follows.

(e) We now prove (i). Let

(8) $$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of $\mathcal{D}_Y$-modules. Then

(9) $$0 \to \sigma^!(F')[-1] \to \sigma^!(F)[-1] \to \sigma^!(F'')[-1] \to 0$$

is also exact (4.8). Consider the long exact cohomology sequence obtained by applying Rj_+ to (9). The variety Z being affine, we still get an exact sequence if we apply $\Gamma(Z,.)$ to it. It is a long exact sequence of semisimple $\mathbb{C}[D]$-modules, therefore the 0-eigenspaces of D also form a long exact sequence. The derived functors $H^i j_+(\sigma^! G)[-1]$ are supported by S for $i \neq 0$, $(G\in\mu(\mathcal{D}_Y))$. By (c) the eigenvalues of D on their modules of sections are all $\leqq - n$. Therefore the exact sequence of the 0-eigenspaces reduces to the exact sequence

(10) $$0 \to \Gamma(Z,H^0j_+\sigma^! F'[-1])^0 \to \Gamma(Z,H^0j_+\sigma^!(F)[-1])^0 \to \Gamma(Z,H^0j_+\sigma^!(F'')[-1])^0 \to 0$$

By (7), this shows that

(11) $$0 \to \Gamma(Y,F') \to \Gamma(Y,F) \to \Gamma(Y,F'') \to 0$$

is exact, too.

(f) *Proof of* (ii). Let F' be the submodule of F generated by $\Gamma(F)$. We have the exact sequence

$$0 \to F' \to F \to F/F' \to 0$$

and, by construction, $\Gamma(F') \to \Gamma(F)$ is an isomorphism. By (i), it follows that $\Gamma(F/F') = 0$, hence that $F/F' = 0$ (see (d)).
This proves (ii).

(g) *Proof of* (iii). By (ii), $\mathcal{D}_Y$ is a generator of $\mu(\mathcal{D}_Y)$. For any $F \in \mu(\mathcal{D}_Y)$, we have $\Gamma(F) = \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y,F)$. Then (i) shows that $F \mapsto \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y,F)$ is exact, therefore $\mathcal{D}_Y$ is projective.

9.2 We keep the previous notation but assume X to be reduced to a point. We want to describe a generating set for the algebra $\mathcal{D}(P)$ of global differential operators on $P = \mathbb{P}_N(\mathbb{C})$.

The group $\mathbf{GL}_{n+1}(\mathbb{C})$ (resp. $\mathbf{PGL}_{n+1}(\mathbb{C})$ operates on A (resp. P) in the usual way, whence a representation of its Lie algebra $\mathfrak{gl}_{n+1}(\mathbb{C})$, (resp. $\mathfrak{pgl}_{n+1}(\mathbb{C}) \cong \mathfrak{sl}_{n+1}(\mathbb{C})$) in the Lie algebra of vector fields on A (resp. P), which is faithful. Let $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$) be its image.

PROPOSITION . *The algebra* $\mathcal{D}(P)$ *is generated by the constants and* $\mathfrak{g}$..

Proof. We go back to the proof of 9.1. Now Z = A, Z* = A*, and S is reduced to the origin. We claim that we have a natural isomorphism

(1) $$\mathcal{D}(P) \tilde{\to} \mathcal{D}(A)^{\mathbb{C}^*}/\mathcal{D}(A)^{\mathbb{C}^*}.\mathcal{D}$$

where the superscript $\mathbb{C}^*$ denotes the fixed point set under the natural action of $\mathbb{C}^*$ on $\mathcal{D}(A)$. The differential of the latter is $Q \mapsto [D,Q]$, therefore (1) implies easily that $\mathcal{D}(A)^{\mathbb{C}^*}$ is generated as an algebra by the constants and the operators $a_i\partial_j$ $(1 \le i,j \le n+1)$. Those make up $\tilde{g}$, hence $\mathcal{D}(P)$ is generated by $\mathbb{C}$ and $\tilde{g}/\mathbb{C}.D = g$.

Let $N^\cdot = j_+\sigma^!\mathcal{D}_P[-1]$. By 9.1(7)

(2) $$\mathcal{D}(P) = \Gamma(P,\mathcal{D}) \cong \Gamma(H^0N^\cdot)^0.$$

Let $M = \mathcal{D}_A/\mathcal{D}_A.D$. The left ideal generated by the Euler operator D is the ideal of differential operators which are zero on rational functions which are homogeneous of degree zero. Those supply local coordinates on P. This implies easily that the natural morphism $\mathcal{D}_A \to \sigma^!\mathcal{D}_P[-1]$ in $\mathcal{D}_A$ induces an isomorphism $(\mathcal{D}_A/\mathcal{D}_A.D)|A^* \tilde{\to} \sigma^!\mathcal{D}_P[-1]$. This yields a natural homomorphism $\alpha : M \to H^0N^\cdot$, which is an isomorphism on A*. As a consequence, $\ker\alpha$ and $\operatorname{coker}\alpha$ are concentrated at the origin. By 9.1(c) they are sums of eigenspaces of D corresponding to eigenvalues $\le -n$. Therefore α induces an isomorphism of $\Gamma(M)^0$ onto $\Gamma(H^0N^\cdot)^0$, i.e. onto $\mathcal{D}(P)$, in view of (2). But the action of $\mathbb{C}^*$ on $\mathcal{D}(A)$ is semisimple, therefore

$$(\mathcal{D}(A)/\mathcal{D}(A).D)^{\mathbb{C}^*} = \mathcal{D}(A)^{\mathbb{C}^*}/\mathcal{D}(A)^{\mathbb{C}^*}.D$$

and (1) follows.

9.3 LEMMA *Let* $F \in \mathrm{coh}(\mathcal{D}_X)$. *There exists* $Y \subset X$ *open dense on which* F *is* $\mathcal{O}_Y$*-locally free.*

Proof. This is a local question. We may assume X to be affine, irreducible and F to admit a global good filtration Γ. The graded module $\mathrm{Gr}^\Gamma F$ is finitely generated over $\mathcal{O}(T^*(X))$, which, in turn, is finitely generated over $\mathcal{O}(X)$. By [EGA IV:6.9.2] there exists a principal open subset $Y = X_f$ of X such that $\mathrm{gr}^\Gamma F|_Y$ is free over $\mathcal{O}(Y) = \mathcal{O}(X)[f^{-1}]$. [For another proof, without noetherian assumption in fact, see 2.6.3 in J. Dixmier, Enveloping algebras, North-Holland 1977.]
Since the direct sum of the $\Gamma^\gamma F/\Gamma^{\gamma-1}F$ is free, each $\Gamma^\gamma F/\Gamma^{\gamma-1}F$ is projective. It follows then by induction on j that $\Gamma^\gamma F$ is isomorphic to the direct sum of the $\Gamma^j F/\Gamma^{i-1}F$ for $i = 0,\dots,j$, hence that F is isomorphic to $\mathrm{Gr}^\Gamma F$ over $\mathcal{O}(Y)$; consequently it is free.

9.4 PROPOSITION *Assume* π *to be projective. Then* π_+ *maps* $D^b_{coh}(\mathcal{D}_X)$ *into* $D^b_{coh}(\mathcal{D}_X)$.

Proof. The category $D^b_{coh}(\mathcal{D}_Y)$ being generated by $coh(\mathcal{D}_Y)$, we need only consider the case of a single degree complex, i.e. of one coherent $\mathcal{D}_Y$-module. The maps π factors into a closed embedding $Y \to P \times X$ and the projection $P \times X \to X$, where P is a projective space. For closed embeddings, our assertion was proved in 7.11. There remains the case of projection with fibre P. The assertion being local on X, we may assume X to be affine. We prove first

$$(1) \qquad \pi_+(\mathcal{D}_Y) = \mathcal{D}_X[-d_P].$$

We have $\mathcal{D}_{X\leftarrow Y} = \pi^{-1}(\mathcal{D}_X) \otimes_{\mathbb{C}} \sigma^{-1}(\omega_P)$, where σ is the first projection $P \times X \to P$ (5.3.1(1)), therefore

$$\pi_+(\mathcal{D}_Y) = R\bar{\pi}(\pi^{-1}(\mathcal{D}_X) \otimes_{\mathbb{C}} \sigma^{-1}(\omega_P)).$$

By the (sheaf theoretic) projection formula we get

$$\pi_+(\mathcal{D}_Y) = \mathcal{D}_X \otimes_{\mathbb{C}} R\bar{\pi}(\sigma^{-1}(\omega_P)).$$

$$R^i\bar{\pi}(\sigma^{-1}(\omega_P)) = H^i(P;\omega_P) \qquad (i\in\mathbb{Z})$$

and $H^i(P;\omega_P)$ is equal to zero for $i \neq d_P$ and to $\mathbb{C}$ for $i = d_P$ [Ha;III,§5]. This proves (1) and our claim for $\mathcal{D}_Y$. Since $\mathcal{D}_Y$ is a generator for $coh(\mathcal{D}_Y)$ (9.2), the assertion for any coherent $\mathcal{D}_Y$-module follows by standard arguments. For the sake of completeness, we sketch them. Let $F \in coh(\mathcal{D}_Y)$. For any $N\in\mathbb{N}$, there exists a complex $\mathcal{Q}^\cdot$ of length N, graded by $[-N,0]$ whose elements are finite sums of copies of $\mathcal{D}_Y$, endowed with an augmentation $\mathcal{Q}^\cdot \to F \to 0$, which is a h.i. from dimension $-N+1$ on. By the above $\pi_+(\mathcal{Q}^\cdot)$ is coherent, hence belongs to $D^b_{coh}(\mathcal{D}_X)$). There is a spectral sequence abutting to $R\pi_+\mathcal{Q}^\cdot$ in which $E_2^{p,q} = R^p\pi_+H^q\mathcal{Q}^\cdot$. The corresponding spectral sequence for F degenerates at E_2. The augmentation induces a homomorphism of the E_2-terms which is an isomorphism in fibre degree $>-N$. The base degree is contained in $[0,2d_Y]$. If N is big enough, E_∞ will be isomorphic to $H^\cdot\pi_+F^\cdot$ in degrees $\leq 2d_Y$, which includes all the non-zero $H^i\pi_+F$.

9.5 PROPOSITION *The functor $\mathbb{D}_X$ defines an anti-automorphism of $D^b_{coh}(\mathcal{D}_X)$, whose square is the identity.*

Proof. Any coherent $\mathcal{D}_X$-module F has a left locally free resolution, finitely generated in each dimension, hence $\mathbb{D}_X F$ is coherent and in particular belongs to $D^b_{coh}(\mathcal{D}_X)$. Since $coh(\mathcal{D}_X)$ generates $D^b_{coh}(\mathcal{D}_X)$, it follows that $\mathbb{D}_X$ maps $D^b_{coh}(\mathcal{D}_X)$ into itself.

For $F \in coh(\mathcal{D}_X)$ we define a natural homomorphism

$$(1) \qquad \mu : F \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{D}_X}(F,\tilde{\mathcal{D}}_X[d_X]),\tilde{\mathcal{D}}_X[d_X]).$$

Fix $x \in X$. All $\mathcal{D}_X$-modules in sight being coherent, we have

$$(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{D}_X}(F,\tilde{\mathcal{D}}_X[d_X]),\tilde{\mathcal{D}}_X[d_X]))_x = \mathrm{Hom}_{\mathcal{D}_{X,x}}(\mathrm{Hom}_{\mathcal{D}_{X,x}}(F_x,\tilde{\mathcal{D}}_{X,x}[d_X])\tilde{\mathcal{D}}_{X,x}[d_X])$$

Let $f \in F_x$. Then $\mu_x(f)$ is the homomorphism

$$\mathrm{Hom}_{\mathcal{D}_{X,x}}(F_x,\tilde{\mathcal{D}}_{X,x}[d_X]) \to \tilde{\mathcal{D}}_{X,x}[d_X]$$

which to λ in the first term associates $\lambda(f)$. If $F_x = \mathcal{D}_{X,x}$, then the right hand side of (1) is naturally isomorphic to $\mathcal{D}_{X,x}$ in such a way that μ_x is the identity. It follows that if F is locally isomorphic to $\mathcal{D}_X$, then $\mu(F) = F$ and μ is the identity. Since any $F^\cdot \in D^b_{coh}(\mathcal{D}_X)$ has a locally free resolution, finitely generated in each dimension, we see that μ induces an isomorphism $F \xrightarrow{\sim} \mathbb{D}_X\mathbb{D}_X F$. That $\mathbb{D}_X \circ \mathbb{D}_X = \mathrm{Id}$ on $D^b_{coh}(\mathcal{D}_X)$ is now seen by using exact triangles.

A morphism $\zeta : A \to B(A,B \in \mu(\mathcal{D}_X))$ induces canonically a morphism $\tilde{\zeta} : \mathrm{Hom}_{\mathcal{D}_X}(B,\tilde{\mathcal{D}}_X[d_X]) \to \mathrm{Hom}_{\mathcal{D}_X}(A,\tilde{\mathcal{D}}_X[d_X])$.

Let $A^\cdot,B^\cdot \in D^b_{coh}(\mathcal{D}_X)$ and $\zeta : A \to B$ a morphism. There are locally free left resolutions $F^\cdot,G^\cdot$ of $A^\cdot$ and $B^\cdot$ endowed with a morphism $F^\cdot \to G^\cdot$ sitting over ζ, which is a h.i. if ζ is one. From this we get a natural homomorphism

$$\mathbb{D}_X(\zeta) : \mathrm{RHom}_{\mathcal{D}_X}(A^\cdot,B^\cdot) \to \mathrm{RHom}_{\mathcal{D}_X}(\mathbb{D}_X(B^\cdot),\mathbb{D}_X(A^\cdot)),$$

and $\mathbb{D}_X\mathbb{D}_X(\zeta))$ is isomorphic to ζ. This proves the proposition.

9.6 PROPOSITION *Let π be proper. Then there is a natural isomorphism* $\mathbb{D}_X \circ \pi_+ \to \pi_+ \mathbb{D}_Y$ *of functors from* $D^b_{coh}(\mathcal{D}_Y)$ *to* $D^b_{coh}(\mathcal{D}_X)$.

The map π can be factored as a product of a closed embedding $Y \to P \times X$ and of a projection $P \times X \to X$, where P is a projective space. We need only consider these two cases. In each, the first point is to exhibit an isomorphism $\mu : \mathbb{D}_X\pi_+(\mathcal{D}_Y) \to \pi_+\mathbb{D}_Y(\mathcal{D}_Y)$, which is natural in the sense that it commutes with the endomorphisms of $\mathcal{D}_Y$ as a left $\mathcal{D}_Y$-module, i.e. with right multiplication by a section of $\mathcal{D}_Y$.

(a) *π is a closed embedding.* We use the notation of 7.7. We have

$$(1) \qquad \pi_+(\mathcal{D}_Y) = \mathcal{D}_{X\leftarrow Y} = \mathcal{D}_X/\mathcal{D}_X.J \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

$$(2) \qquad \mathbb{D}_Y(\mathcal{D}_Y) = \tilde{\mathcal{D}}_Y[d_Y].$$

Therefore

$$\pi_+\mathbb{D}_Y(\mathcal{D}_Y) = \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \tilde{\mathcal{D}}_Y[d_Y] = \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} (\mathcal{D}^r_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1})[d_Y] =$$

$$(3) \qquad = \mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_Y} \omega_{Y/X} \otimes_{\mathcal{D}_Y} (\mathcal{D}^r_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1})[d_Y] =$$

$$= \mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_Y} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y)^{-1} = \mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_X} \omega_X^{-1}[d_Y] \ ,$$

where $\mathcal{D}_X/\mathcal{D}_XJ$ is viewed as a $\mathcal{O}_X$-module via the action of $\mathcal{O}_Y \subset \mathcal{D}_Y$ on the right. On the other hand, (1) yields

$$(4) \qquad \mathbb{D}_X\pi_+(\mathcal{D}_Y) = R\mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_XJ \otimes \omega_{Y/X};\tilde{\mathcal{D}}_X[d_X]).$$

We have therefore to exhibit a natural isomorphism

$$(5) \qquad R\mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_Y} \omega_{Y/X}, \tilde{\mathcal{D}}_X[d_X]) \to \mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_X} \omega_X^{-1}[d_Y].$$

In the notation of 7.2, we have a locally free Koszul resolution $\mathcal{D}_X \otimes_{\mathbb{C}} \Lambda^{\cdot}E$ of $\mathcal{D}_X/\mathcal{D}_XJ$, with the understanding that Λ^iE has degree $-i$. The right hand side of (5) can therefore be written

$$(6) \qquad \mathrm{Hom}^{\cdot}_{\mathbb{C}_X}(\Lambda^{\cdot}E \otimes_{\mathbb{C}} \omega_{Y/X}, \tilde{\mathcal{D}}_X[d_X]),$$

hence also

$$(7) \qquad \Lambda^{\cdot}E^* \otimes_{\mathbb{C}} \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_X} \tilde{\mathcal{D}}_X[d_X].$$

Note that in (6) we are using the left $\mathcal{D}_X$-structure (a) of $\tilde{\mathcal{D}}_X$ (cf.3.6) i.e. the one coming from $\mathcal{D}^r_X \otimes_{\mathcal{O}_X} \omega_X^{-1}$. The space E^* has as a basis the ∂_i's ($1\leq i\leq d = d_{X,Y}$). The differential

$$\Lambda^iE^* \otimes_{\mathbb{C}} \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_X} \tilde{\mathcal{D}}_X[d_X] \to \Lambda^{i+1}E^* \otimes_{\mathbb{C}} \omega^{-1}_{Y/X} \otimes_{\mathcal{O}_X} \tilde{\mathcal{D}}_X[d_X]$$

can be written

$$d(w \otimes \omega^{-1}_{Y/X} \otimes P \otimes \omega_X^{-1}) = \Sigma\, dx_i\ w \otimes w^{-1}_{Y/X} \otimes (P.f_i \otimes \omega_X^{-1}).$$

We have

$$\mathcal{O}_Y.\partial_1\wedge\ldots\wedge\partial_d = \omega_{Y/X},$$

as follows from 7.1(6). Therefore, in the top dimension, which is $d_{X,Y} - d_X = -d_Y$, we have the exact sequence of left $\mathcal{D}_X$-modules:

$$0 \to \mathcal{D}_XJ \otimes_{\mathcal{O}_X} \omega_X^{-1} \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \to \mathcal{D}_X/\mathcal{D}_XJ \otimes_{\mathcal{O}_X} \omega_X^{-1} \ ,$$

which provides μ. Standard arguments using Koszul complexes show it does not depend on the choice of local coordinates (x_i) $(i=1,\ldots,n)$. If Y is given by $x_j = 0$ $(j>m)$, then μ commutes with right multiplication by ∂_i $(i\leq m)$ and by $f \in \mathcal{O}_Y$, (since f extends to an element of $\mathcal{O}_X$). Thus μ commutes with $\mathcal{D}_Y$ on the right. Both sides are single degree complexes concentrated in degree $-d_Y$.

From this we see that if S is a locally free $\mathcal{O}_Y$-module, then μ extends to an isomorphism

$$(8)\qquad \mu : \mathbb{D}_X\pi_+(\mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} S) \xrightarrow{\sim} \pi_+\mathbb{D}_Y(\mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} S)$$

and again, both sides are concentrated in degree $-d_Y$. Now every bounded coherent complex $F^\cdot$ admits a left resolution $G^\cdot$ by such $\mathcal{D}_Y$-modules, which is bounded above. We have

$$(9)\qquad \mathbb{D}_X\pi_+F^\cdot \xrightarrow{\sim} \mathcal{D}_X\pi_+G^\cdot \qquad\qquad \pi_+\mathbb{D}_YF^\cdot \cong \pi_+\mathbb{D}_YG^\cdot$$

Moreover, a spectral sequence argument (9.6.1 below) yields

$$(10)\qquad (\mathbb{D}_X\pi_+G^\cdot)^i = \mathbb{D}_X\pi_+G^{i+dy}$$

$$(11)\qquad (\pi_+\mathbb{D}_YG^\cdot)^i = \pi_+\mathbb{D}_YG^{i+dy} \qquad (i\in\mathbb{Z}).$$

These isomorphisms, combined with (8), provide the sought for isomorphism.

(b) *π is the projection* $Y = P \times X \to X$, *where* $P = \mathbb{P}_n(\mathbb{C})$ *for some* n. Again, we consider first $\mathcal{D}_Y$. We have

$$\mathbb{D}_Y(\mathcal{D}_Y) = \tilde{\mathcal{D}}_Y[-d_Y]$$

$$\pi_+(\mathbb{D}_Y(\mathcal{D}_Y)) = R\bar{\pi}(\mathcal{D}_{X\leftarrow Y} \underset{\mathcal{D}_Y}{\otimes} \omega_Y^{-1}))[-d_Y].$$

We use on $\mathcal{D}_Y^r \otimes \omega_Y^{-1}$ the $\mathcal{D}_Y$-structure defined by left multiplication. Therefore

$$\pi_+(\mathbb{D}_Y(\mathcal{D}_Y)) = R\bar{\pi}(\mathcal{D}_{X\leftarrow Y} \underset{\mathcal{O}_Y}{\otimes} \omega_Y^{-1})[-d_Y].$$

But

$$\mathcal{D}_{X\leftarrow Y} = \omega^{-1}\mathcal{D}_P \underset{\mathbb{C}}{\otimes} \pi^{-1}\mathcal{D}_X, \text{ and } \pi_Y^{-1} = \sigma^{-1}\omega_P{}^{-1} \underset{\mathbb{C}}{\otimes} \pi^{-1}\omega_X^{-1},$$

therefore

$$\mathcal{D}_{X\leftarrow Y} \underset{\mathcal{O}_Y}{\otimes} \omega_Y^{-1} = \pi^{-1}(\mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \pi^{-1}\omega_X^{-1} \underset{\mathbb{C}}{\otimes} \sigma^{-1}\mathcal{O}_P = \pi^{-1}(\tilde{\mathcal{D}}_X) \underset{\mathbb{C}}{\otimes} \sigma^{-1}\mathcal{O}_P.$$

By the projection formula

$$\pi_+(\mathbb{D}_Y(\mathcal{D}_Y)) = (\tilde{\mathcal{D}}_X \otimes R\pi_*(\sigma^{-1}\mathcal{O}_P))[-d_Y].$$

Since $R^i\pi_*(\sigma^{-1}\mathcal{O}_P) = H^i(P,\mathcal{O}_P)$, it is 0 for $i \neq 0$ and $\cong \mathbb{C}$ for $i = 0$.

Therefore

$$(12)\qquad \pi_+(\mathbb{D}_Y(\mathcal{D}_Y)) = \tilde{\mathcal{D}}_X[-d_Y].$$

On the other hand, by 9.4(1), $\pi_+(\mathcal{D}_Y) = \mathcal{D}_X[-d_P]$, hence

$$(13)\qquad \mathbb{D}_X(\pi_+(\mathcal{D}_Y)) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X[-d_P],\tilde{\mathcal{D}}_X[d_X]) = \tilde{\mathcal{D}}_X[-d_P-d_X] = \tilde{\mathcal{D}}_X[-d_Y].$$

(12) and (13) yield μ and show that the two complexes are concentrated in a single degree, namely d_Y. We have to check that this isomorphism commutes with the right multiplication by a section of $\mathcal{D}_Y$. We have

$$\Gamma(Y,\mathcal{D}_Y) = \mathcal{D}(Y) = \mathcal{D}(P) \underset{\mathbb{C}}{\otimes} \mathcal{D}(X)$$

The commutation with $\mathcal{D}(X)$ is clear. By 9.2, $\mathcal{D}(P)$ is generated by the constants and a Lie algebra g of vector fields isomorphic to $s\ell_{n+1}(\mathbb{C})$. The proof of 9.4(1) and the above argument show that g acts on $\mathbb{D}_X\pi_+(\mathcal{D}_Y)$ (resp. $\pi_+\mathbb{D}_Y(\mathcal{D}_Y)$) by the natural representation on $H^0(P,\mathcal{O}_P)$(resp. $H^n(P,\omega_P)$). Both spaces are one-dimensional and g is semisimple. Therefore these representations are trivial, hence $\mathcal{D}(P)$ acts trivially.

Let us call elementary a $\mathcal{D}_Y$-module of the form $\mathcal{D}_P \otimes_{\mathbf{C}} H$, where H is a locally free $\mathcal{D}_X$-module. Then we also get an isomorphism

(14) $$\mathbb{D}_X \pi_+ G \to \pi_+ \mathbb{D}_Y G, \quad \text{if } G \text{ is elementary.}$$

We now claim that any coherent $\mathcal{D}_Y$-module F is a quotient of an elementary one. Indeed, consider $H = \pi_*(F)$, viewed as a $\mathcal{D}_X$-module in the obvious way. Then, by 9.1, the natural morphism $\mathcal{D}_P \otimes_{\mathbf{C}} H \to F$ is surjective. Since H is a quotient of a direct sum of locally free modules, F is a quotient of a direct sum of elementary modules. By coherence, it is already the quotient of a finite such sum, hence of an elementary module. As a consequence, a coherent $\mathcal{D}_Y$-complex $F^\cdot$ admits a bounded above resolution by a complex $C^\cdot$ of elementary $\mathcal{D}_Y$-modules. Now, we can conclude as in (a). We have

(15) $$\mathbb{D}_X \pi_+ F^\cdot = \mathbb{D}_X \pi_+ C^\cdot \qquad \pi_+ \mathbb{D}_Y F^\cdot = \pi_+ \mathbb{D}_Y C^\cdot$$

and by 9.6.1

(16) $$(\mathbb{D}_X \pi_+ C^\cdot)^i = \mathbb{D}_X \pi_+ C^{i-d_Y} \qquad (\pi_+ \mathbb{D}_Y C^\cdot)^i = \pi_+ \mathbb{D}_Y C^{i-d_Y},$$

which, together with (14), prove 9.6 for projections.

There remains to prove (10),(11),(15) and (16). They follow from the following proposition :

9.6.1 PROPOSITION *Let* A *and* B *be abelian categories,* $F : A \to B$ *a functor which is either left (or right) exact and* $DF : D^- A \to D^- B$ *its derived functor. Let* R *be a class of elements in* A *with the following property: there exists a integer* d *such that for any* $C \in R$, $D(C)$ *is a single degree complex* $S[d]$ *concentrated in degree* $-d$. *Let* $C^\cdot$ *be a complex in* A *, consisting of elements of* R *, with degrees bounded above. Then* $DF(C^\cdot)$ *is represented by the complex* $B^\cdot = \{B^i[d]\}$ *, where* $B^i = DF(C^i)$ $(i \in \mathbf{Z})$.

Proof. To compute $DF(C^\cdot)$ we choose a resolution of $C^\cdot$ by a double complex $M^{\cdot\cdot} = (M^{ij})$ consisting of F-acyclic objects. Then $N^{\cdot\cdot} = (F(M^{ij}))$

is a double complex and the corresponding total complex $T(N^{\cdot\cdot})$ represents $DF(C^\cdot)$. For each column i, the complex $M^{i,\cdot} = (M^{ij})$, endowed with the second differential, represents $DF(C^i)$. By assumption

$$H^j(M^{i,\cdot}) = \begin{cases} B^i & \text{for } j = -d \\ 0 & \text{for } j \neq -d . \end{cases}$$

From this, we shall deduce that $T(N^{\cdot\cdot})$ is h.i. to $B^\cdot[d]$, which will prove the proposition. By shifting, we are reduced to the case where $d = 0$. Consider the truncated complex $\tau^{\geq 0}(N^{\cdot\cdot})$ defined by

$$\tau^{\geq 0}(N^{\cdot\cdot})^{ij} = \begin{cases} 0 & ,\text{if } j < 0 \\ N^{i,0}/\mathrm{Im}\, N^{i,-1} & ,\text{if } j = 0 \\ N^{i,j} & ,\text{if } j > 0 \end{cases}$$

This is a double complex, endowed with a morphism $\alpha : N^{\cdot\cdot} \to \tau^{\geq 0}(N^{\cdot\cdot})$, which is a h.i. in each column. Therefore α provides a h.i. of the underlying total complexes. We are now reduced to the case where $N^{ij} = 0$ for $j < 0$. We then use the truncation

$$\tau_{\leq 0}(N^{\cdot\cdot}) = \begin{cases} N^{ij} & , \text{if } j < 0 \\ \ker \ (N^{i,0} \to N^{i,1}) & , \text{if } j = 0 \\ 0 & , \text{if } j > 0 \end{cases}$$

The natural morphism $\tau_{\leq 0}(N^{\cdot\cdot}) \to N^{\cdot\cdot}$ is again a h.i.. $DF(C^\cdot)$ is now represented by a double complex $N^{\cdot\cdot}$ such that $N^{ij} = 0$ for $j \neq 0$, in which case our assertion is obvious.

9.6.2 The factorization of π used in 9.6 is not unique. To be complete, one should prove that the isomorphism $\mathbb{D}_X \circ \pi_+ \to \pi_+ \circ \mathbb{D}_Y$ obtained by composition does not depend on the factorization. Since this is not really needed in the sequel, we shall content ourselves with giving some indications on how to proceed to carry out such a check, without supplying all details, somewhat in the form of an exercise with hints.

If π is either a closed embedding or a projection $P \times X \to X$, we let μ_π be the isomorphism $\mathbb{D}_X \circ \pi_+ \to \pi_+ \circ \mathbb{D}_Y$ constructed in 9.6.

If $\pi = \tau \circ \sigma$, where $\sigma : Y \to Z$ and $\tau : Z \to X$ are each of one of these two types, we let $\mu_{\tau,\sigma}$ be the composition

$$\mathbb{D}_X \circ \pi_+ = \mathbb{D}_X \circ \tau_+ \circ \sigma_+ \to \tau_+ \circ \mathbb{D}_Z \circ \sigma_+ \to \tau_+ \circ \sigma_+ \circ \mathbb{D}_Y = \pi_+ \circ \mathbb{D}_Y$$

obtained from 9.6 and 6.4.

(a) We first remark that 9.1 remains valid if Y is a product of an affine variety U by k projective spaces. 9.1 is the case k = 1 and the proof proceeds from there by induction on k. Let P be one of the projective factors and let us now denote by X the product of the others by U. In the proof of 9.1, steps (a),(b) and (c) are valid for any X. The assumption X affine was used to be sure that : in (d), a $\mathcal{D}_Z$-module is generated by its cross sections and, in (e), the functor $\Gamma(Z,.)$ is exact. But Z is now a product of k − 1 projective spaces by the affine variety $A^{n+1} \times U$, and we may use induction assumption.

(b) We shall use the following lemma:

LEMMA *Assume that π is either a closed embedding or a projection with fibre P. Let $R \subset \mu(\mathcal{D}_Y)$ be a class of objects such that $\pi_+, \mathbb{D}_Y, \mathbb{D}_X\pi_+$, $\pi_+\mathbb{D}_Y$ map any $F \in R$ into a single degree complex, concentrated in a degree independent of F. Let $C^\cdot$ be a complex consisting of elements in R. Then the morphism $\mu_\pi(C^\cdot) : \mathbb{D}_X\pi_+C^\cdot \to \pi_+\mathbb{D}_YC^\cdot$ is completely determined by the morphisms $\mu_\pi(C^i)$.*

This is proved by the spectral sequence argument of 9.6.1.

(c) Let $\pi = \tau \circ \sigma$, where $\sigma : Y \to Z$ and $\tau : Z \to X$ are closed embeddings. We claim that $\mu_\pi = \mu_{\tau,\sigma}$. By (b), it suffices to prove this for a locally free $\mathcal{D}_Y$-module F. Since $\pi_+\mathbb{D}_YF$ is a module, it suffices to check this locally, i.e. we may assume that $F = \mathcal{D}_Y$ and that X is affine, with coordinates (x_i) $(1 \le i \le n)$, such that Z (resp. Y) is given by $x_j = 0$ with $j > k$ (resp. $j > m$). In this case, it is a direct computation.

(d) Let P_1, P_2 be projective spaces and $\pi : Y = P_1 \times P_2 \times X \to X$ the natural projection. It has two obvious factorizations in projections with fibres a projective space

$$Y \xrightarrow{q_1} P_1 \times X \xrightarrow{p_1} X \text{ and } Y \xrightarrow{q_2} P_2 \times X \xrightarrow{p_2} X$$

One has to prove that $\mu_{p_1,q_1} = \mu_{q_2,p_2}$. This is first checked on $\mathcal{D}_Y$, and then follows for elementary $\mathcal{D}_Y$-modules, defined as in 9.6(b). But, as there, (a) shows that any bounded complex of $\mathcal{D}_Y$-modules admits a resolution by bounded above complex of elementary $\mathcal{D}_Y$-modules and we can again use (b).

(e) Let π be a closed embedding and $Y \xrightarrow{i} P \times X \xrightarrow{p} X$ be a factorization of π. We want to prove that $\mu_\pi = \mu_{p,i}$. Using (b), we reduce this to the case of a locally free $\mathcal{D}_Y$-module F. Then $\mathbb{D}_X\pi_+F$ is a single degree complex, therefore it suffices to check the equality locally. As a consequence, we may assume Y,X to be affine and $F = \mathcal{D}_Y$. Any two isomorphisms $\mathbb{D}_X\pi_+F \to \pi_+\mathbb{D}_YF$ differ by an automorphism β of $\mathbb{D}_X\pi_+F$. Since $F = (\pi^!\mathbb{D}_X)(\mathbb{D}_X\pi_+F)$, the automorphism β is given by an automorphism α of F. The latter is the right multiplication by a section of $\mathcal{D}(Y)$, also denoted α. Since both μ_π and $\mu_{p,i}$ commute with the right action of $\mathcal{D}(Y)$ on F, the automorphism α also commutes with $\mathcal{D}(Y)$, hence lies in the center of $\mathcal{D}(Y)$, i.e. it is a constant. Therefore $\mu_\pi = \alpha.\mu_{p,i}$ with $\alpha \in \mathbf{C}^*$. Now we should check that our isomorphisms are normalized in such a way that $\alpha = 1$. It suffices to show that for the module $i_{y+}(\mathbf{C})$, where $i_y : y \to Y$ is the inclusion of a point. Using (c), we are reduced to the case where Y and X are points.

Now $d_X = 0$ and $\mathcal{D}_X \cong \tilde{\mathcal{D}}_X = \mathbf{C}$. In 9.6(a) μ_i is defined by identifying, by means of Koszul resolutions, $\mathbb{D}_P.i_+(\mathcal{D}_{\{x\}})$ and $i_+(\mathcal{D}_{\{x\}})$ to $\mathcal{D}_P/\mathcal{D}_PJ \otimes \omega_P^{-1}$, where J is the ideal of x. On the other hand, 9.4(1) is proved by first setting up a natural isomorphism

$$p_+(\mathbb{D}_P(\mathcal{D}_P)) = \mathcal{D}_{\{x\}} \otimes_{\mathbf{C}} H^0(P,\mathcal{O}_P)[-d] \cong H^0(P,\mathcal{O}_P)[-d] \qquad (d=d_P)$$

and the construction in 9.6(b) gives first an isomorphism

$$p_+(\mathcal{D}_P) = \mathcal{D}_{\{x\}} \otimes_{\mathbf{C}} H^d(P,\omega_P)$$

whence

$$\mathcal{D}_{\{x\}}(p_+(\mathcal{D}_P)) = H^d(P,\omega_P)^*.$$

Now duality on P provides an isomorphism $H^d(P,\omega_P) \simeq \mathbf{C}$ and a perfect pairing of $H^0(P,\mathcal{O}_P)$ and $H^d(P,\omega_P)$ to $H^d(P,\omega_P)$ [Ha: III, §7]. The task is then to make sure that the constructions of 9.6 are compatible with those. We omit the details.

(f) Let now

$$Y \xrightarrow{i_j} P_j \times X \xrightarrow{p_j} X \qquad (j=1,2)$$

be two factorizations of π, where i_j is a closed embedding, P_j a projective space and p_j the second projection ($j=1,2$). Our claim is that $\mu_{p_1,q_1} = \mu_{q_2,p_2}$. Consider the diagram of morphisms

$$\begin{tikzcd} & P_1 \times X \arrow[dr, "p_1"] & \\ Y \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & P_1 \times P_2 \times X \arrow[u, "q_1"] \arrow[d, "q_2"] \arrow[r, "p"] & X \\ & P_2 \times X \arrow[ur, "p_2"'] & \end{tikzcd}$$

where i is obtained from π and the mapping $Y \to P_j$ provided by i_j and q_j is the natural projection ($j=1,2$). It is commutative. By (c) $\mu_{i_1} = \mu_{q_1,i}$ and $\mu_{i_2} = \mu_{q_2,i}$. By (d) $\mu_{p_1,q_1} = \mu_{p_2,q_2}$. Our assertion now follows by simple diagram chasing.

9.7 PROPOSITION *Let* $F^\cdot \in D^b_{coh}(\mathcal{D}_X)$ *and* $G^\cdot \in D^b(\mathcal{D}_X)$ *Then*

(1) $$\omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \left(D_X F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot\right) = R\mathcal{H}om_{\mathcal{D}_X}(F^\cdot,G^\cdot)\ [d_X].$$

Proof. For $F \in coh(\mathcal{D}_X)$, let us define DF by

(2) $$DF = \mathcal{H}om_{\mathcal{D}_X}(F,\tilde{\mathcal{D}}_X)\ [d_X].$$

Therefore D_X is the right derived functor of $F \mapsto DF$. Given $F,G \in coh(\mathcal{D}_X)$ we first define a natural morphism

(3) $$\alpha : \omega_X \underset{\mathcal{D}_X}{\otimes} \left(DF \underset{\mathcal{O}_X}{\otimes} G\right) \to \mathcal{H}om_{\mathcal{D}_X}(F,G)\ [-d_X]$$

which is an isomorphism if $F = \mathcal{D}_X$. We have

(4) $$\omega_X \underset{\mathcal{D}_X}{\otimes} \left(DF \underset{\mathcal{O}_X}{\otimes} G\right) = \omega_X \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\mathcal{D}_X}(F,\mathcal{D}^r_X \otimes \omega_X^{-1}[d_X]) \otimes G.$$

Since F is coherent,

(5) $$(DF)_x = \mathcal{H}om_{\mathcal{D}_{X,x}}(F_x,\tilde{\mathcal{D}}_{X,x}[d_X]).$$

We define a homomorphism β of the left-hand side of (4) into

(6) $$\mathcal{H}om_{\mathcal{D}_X}(F,\omega_X \underset{\mathcal{D}_X}{\otimes} (\mathcal{D}^r_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}) \otimes G[d_X])$$

by assigning to

$$\omega \otimes \lambda \otimes \mu \qquad (\omega\in\omega_{X,x}, \lambda\in(DF)_x, \mu\in G_x)$$

the element $\phi_{\omega,\lambda,\mu}$ of the stalk at x of (6) defined by

$$\phi_{\omega,\lambda,\mu}(f) = \omega \otimes \lambda(f) \otimes \mu \qquad (f\in F).$$

Note that the $\mathcal{D}_X$ of $\mathcal{H}om_{\mathcal{D}_X}$ acts on $\mathcal{D}^r_X \otimes \omega_X^{-1}$ via the $\mathcal{D}_X$-structure (a) of 3.6, defined by using the right $\mathcal{D}_X$-module structure of $\mathcal{D}_X$, while the $\mathcal{D}_X$ under $\otimes$ acts via structure (b). They commute. Therefore the sheaf (6) is equal to

(7) $$\mathcal{H}om_{\mathcal{D}_X}(F,\mathcal{O}_X \underset{\mathcal{O}_X}{\otimes} G[d_X]) = \mathcal{H}om_{\mathcal{D}_X}(F,G)[d_X],$$

This yields α. If $F = \mathcal{D}_X$, then both sides of (3) are equal to G.

Given now $F^\cdot,G^\cdot$ as in the proposition, we replace them by bounded locally projective left resolutions $P^\cdot,Q^\cdot$, where we assume moreover $P^\cdot$ to be locally free from an (arbitrarily) low degree on. Then α provides a homomorphism of the complexes defining the two sides of (1), which is a h.i. from that degree on, and the proposition follows.

9.8 COROLLARY *Let* p *be the projection of* X *to a point. Then*

(1) $$p_+(D_X F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot) = R\mathcal{H}om^\cdot_{D^b(\mathcal{D}_X)}(F^\cdot,G^\cdot)[d_X].$$

In fact, it follows from the definitions that

$$\omega_X = \mathcal{D}_{pt\leftarrow X}\,, \tag{2}$$

therefore the module of global sections of the right-hand side of 3.7(1) is the direct image under p of $\mathbb{D}_X F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot$.

9.9 LEMMA *We have a natural isomorphism*

$$\pi_+(F^\cdot \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} \pi^!(G^\cdot)) = \pi_+F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot[d_{Y,X}] \qquad (F^\cdot \in D^b(\mathcal{D}_Y), G^\cdot \in D^b(\mathcal{D}_X)). \tag{1}$$

Proof. Using the definition of π_+ and $\pi^!$, we can write the left-hand side of (1) as

$$R\pi(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} F^\cdot \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} \mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}G^\cdot)[d_{Y,X}] = R\pi(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} F^\cdot \overset{L}{\underset{\pi^{-1}\mathcal{O}_X}{\otimes}} \pi^{-1}G^\cdot)[d_X]$$

By the sheaf theoretic projection formula, (cf. 6.2) this is equal to

$$(R\pi(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} F^\cdot) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot)[d_{Y,X}] = \pi_+F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot[d_{Y,X}].$$

9.10 PROPOSITION *Let π be proper. Then*

$$R\mathcal{H}om^\cdot_{\mathcal{D}_Y}(F^\cdot, \pi^!G^\cdot) = R\mathcal{H}om^\cdot_{\mathcal{D}_X}(\pi_+F^\cdot, G^\cdot) \qquad (F^\cdot \in D^b_{coh}(\mathcal{D}_Y), G^\cdot \in D^b_{coh}(\mathcal{D}_X)).$$

Proof. Let τ be the projection of X to a point. Then $\tau \circ \pi$ is the projection of Y to a point. By 9.8, we have

$$R\mathcal{H}om_{\mathcal{D}_Y}(F^\cdot, \pi^!G^\cdot) = (\tau\circ\pi)_+(\mathbb{D}_Y F^\cdot \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} \pi^!G^\cdot)[-d_Y].$$

Using the equality $(\tau\circ\pi)_+ = \tau_+ \circ \pi_+$ (see 6.4), 9.9 and then 9.6, we get

$$R\mathcal{H}om^\cdot_{\mathcal{D}_Y}(F^\cdot, \pi^!G^\cdot) = \tau_+(\pi_+\mathbb{D}_Y F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot)[-d_X] = \tau_+(\mathbb{D}_X\pi_+F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot)[-d_X].$$

But the last term is equal to $R\mathcal{H}om^\cdot_{\mathcal{D}_X}(\pi_+F^\cdot, G^\cdot)$ by 9.8, and the proposition follows.

9.11 We define on holonomic complexes

$$\pi^+ = \mathbb{D}_Y \circ \pi^! \circ \mathbb{D}_X \qquad \pi_! = \mathbb{D}_X \circ \pi_+ \circ \mathbb{D}_Y \tag{1}$$

π^+ (resp. $\pi_!$) extends to coherent complexes if π is smooth (resp. proper).

9.12 THEOREM *Let π be proper. Then, on $D^b_{coh}(\mathcal{D}_Y)$ and $D^b_{coh}(\mathcal{D}_X)$, we have $\pi_! = \pi_+$ and the functor $\pi_!$ (resp. π^+) is left adjoint to $\pi^!$ (resp. π_+).*

Proof. The first assertion follows from 9.5 and 9.6. Then 9.10 means that $\pi_!$ is left adjoint to $\pi^!$. Since duality is an anti-involution (9.5), it follows then formally that π^+ is left adjoint to π_+.

9.13 PROPOSITION. *Let π be smooth. Then there is a natural isomorphism of functors on $D^b_{coh}(\mathcal{D}_X)$.*

$$\mathbb{D}_Y \circ \pi^! = \pi^! \circ \mathbb{D}_X[-2d_{Y,X}]. \tag{1}$$

Proof. We have to show the existence of a natural isomorphism

$$R\mathcal{H}om^\cdot_{\mathcal{D}_Y}(\pi^!F^\cdot, \tilde{\mathcal{D}}_Y[d_Y 1]) = \pi^! R\mathcal{H}om^\cdot_{\mathcal{D}_X}(F^\cdot, \tilde{\mathcal{D}}_X[d_X])[-2d_{Y,X}] \qquad (F^\cdot \in D^b_{coh}(\mathcal{D}_X)). \tag{2}$$

We may assume $F^\cdot$ to consist of coherent $\mathcal{D}_X$-modules (2.11) hence to admit a locally free coherent left resolution. Moreover, $\pi^!$ is exact (4.8). This reduces us to showing the existence of a natural isomorphism

$$\mu : R\mathcal{H}om^\cdot_{\mathcal{D}_Y}(\pi^!\mathcal{D}_X, \tilde{\mathcal{D}}_Y[d_Y]) \tilde{\to} \pi^! R\mathcal{H}om^\cdot_{\mathcal{D}_X}(\mathcal{D}_X, \tilde{\mathcal{D}}_X[d_X])[-2d_{Y,X}]. \tag{3}$$

If π is an étale covering, $\pi^!\mathcal{D}_X = \mathcal{D}_Y$ and $\pi^{-1}\omega_X = \omega_Y$, so that our assertion is obvious. The morphism π is open and is locally the product of an étale covering and of a projection. This reduces us to the case where $Y = Z \times X$ and π is the second projection. Let $\sigma : Y \to Z$ be the first projection.

Recall that $\tilde{\mathcal{D}}_X = \mathcal{D}^r_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}$ is a double left $\mathcal{D}_X$-module and the $\mathcal{D}_X$-module structure involved in taking $R\mathcal{H}om^\cdot_{\mathcal{D}_X}$ is structure (a) of 3.6. Therefore

(4) $$R\mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{D}_X,\tilde{\mathcal{D}}_X[d_X]) = \mathcal{D}^r_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}[d_X],$$

where the $\mathcal{D}_X$-structure on the right-hand side is the obvious one, defined by the left multiplication on the first factor. We have

(5) $$\pi^!(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}[d_X]) = \pi^0(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}[d_X])[d_{Y,X}];$$

therefore the right-hand side RHS of (3) is given by

(6) $$\text{RHS} = \pi^0(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1})[d_Y].$$

The equalities $\pi^!\mathcal{D}_X = \pi^0\mathcal{D}_X[d_{Y,X}]$ and $d_Y - d_{Y,X} = d_X$ give

(7) $$R\mathcal{H}om^{\cdot}_{\mathcal{D}_Y}(\pi^!\mathcal{D}_X,\tilde{\mathcal{D}}_Y[d_Y]) = R\mathcal{H}om^{\cdot}_{\mathcal{D}_Y}(\pi^0\mathcal{D}_X,\tilde{\mathcal{D}}_Y)[d_X].$$

By 4.5(7)

(8) $$\pi^0\mathcal{D}_Y = \sigma^{-1}\mathcal{O}_Z \underset{\mathbb{C}}{\otimes} \pi^{-1}\mathcal{D}_X.$$

$\mathcal{O}_Z$ has the locally free left resolution $\Lambda^{\cdot}\theta_Z \underset{\mathcal{O}_Z}{\otimes} \mathcal{D}_Z$, where Λ^i is given the degree $-i$ (2.7), hence

(9) $$\Lambda^{\cdot}\sigma^{-1}(\theta_Z) \underset{\sigma^{-1}\mathcal{O}_Z}{\otimes} \sigma^{-1}\mathcal{D}_Z$$

is a locally free resolution of $\sigma^{-1}(\mathcal{O}_Z)$ over $\sigma^{-1}\mathcal{D}_Z$. We have $\mathcal{D}_Y = \sigma^{-1}\mathcal{D}_Z \underset{\mathbb{C}}{\otimes} \pi^{-1}\mathcal{D}_X$, therefore $\Lambda^{\cdot}\sigma^{-1}(\theta_Z) \underset{\mathbb{C}}{\otimes} \mathcal{D}_Y$ is a locally free $\mathcal{D}_Y$-resolution of $\pi^0\mathcal{D}_X$. Let (z_i) $(1\le i\le d_Z)$ be local coordinates in Z. Let E be the vector space spanned by the $\sigma^{-1}(\partial/\partial z_i)$. Its dual space E* is spanned by the $\sigma^{-1}(dz_i)$. We can write (9) as $\Lambda^{\cdot}E \underset{\mathbb{C}}{\otimes} \mathcal{D}_Y$. Therefore the left-hand side LHS of (3) is given by

(10) $$\text{LHS} = \mathcal{H}om^{\cdot}_{\mathcal{D}_Y}(\Lambda^{\cdot}E \underset{\mathbb{C}}{\otimes} \mathcal{D}_Y,\tilde{\mathcal{D}}_Y)[d_Y] = \mathcal{H}om^{\cdot}_{\mathbb{C}}(\Lambda^{\cdot}E,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y,\tilde{\mathcal{D}}_Y)[d_X]).$$

The remarks leading to (4) are also valid for $\tilde{\mathcal{D}}_Y$, therefore

(11) $$\text{LHS} = \Lambda^{\cdot}E^* \underset{\mathbb{C}}{\otimes} \mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} \omega_Y^{-1}[d_X],$$

where now Λ^iE^* has degree i. Using the standard tensor product decompositions of $\mathcal{D}_Y$ and ω_Y^{-1}, we can write this as

(12) $$\text{LHS} = \Lambda^{\cdot}E^* \underset{\mathbb{C}}{\otimes} \sigma^{-1}\mathcal{D}_Z \underset{\sigma^{-1}\mathcal{O}_Z}{\otimes} \sigma^{-1}\omega_Z^{-1} \underset{\mathbb{C}}{\otimes} \pi^{-1}\mathcal{D}_X \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}\omega_X^{-1}[d_X].$$

It follows from 3.5 that $\Lambda^{\cdot}E^* \underset{\mathbb{C}}{\otimes} \sigma^{-1}\mathcal{D}_Z[d_Z]$ is a free $\sigma^{-1}\mathcal{D}_Z$-resolution of $\sigma^{-1}\omega_Z$ (in the category of right $\sigma^{-1}\mathcal{D}_Z$-modules). The augmentation

$$\alpha : \Lambda^{\cdot}E^* \otimes \sigma^{-1}\mathcal{D}_Z[d_Z] \to \omega_Z$$

is a homology isomorphism. It yields a h.i.:

$$\beta : \text{LHS} \tilde{\to} \sigma^{-1}\omega_Z \underset{\sigma^{-1}\mathcal{O}_Z}{\otimes} \sigma^{-1}\omega_Z^{-1} \underset{\mathbb{C}}{\otimes} \pi^{-1}(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1})[d_X-d_Z].$$

The right-hand side can be written:

$$\sigma^{-1}\mathcal{O}_Z \underset{\mathbb{C}}{\otimes} {}^{-1}(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1})[d_X-d_Z] = \pi^0(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1})[d_X-d_Z]$$

whence a h.i.

(13) $$\beta : \text{LHS} \tilde{\to} \pi^0(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1})[d_X-d_{Y,X}].$$

Combined with (6), this provides the sought for h.i. μ of (3).

9.14 COROLLARY *If π is smooth, then $\pi^+ = \pi^![-2d_{Y,X}]$ on $D^b_{coh}(\mathcal{D}_X)$.*

In fact, $\pi^+ = \mathbb{D}_Y \circ \pi^! \circ \mathbb{D}_X$ by definition and $\mathbb{D}_X \circ \mathbb{D}_X = \text{Id}$ on $D^b_{coh}(\mathcal{D}_X)$ by 9.5.

§ 10 HOLONOMIC COMPLEXES

10.1 THEOREM *Holonomicity is preserved under* $\pi^!$, π_+ *and* $\mathbb{D}_X$.

Proof. (i) Preservation under duality has already been establishe (3.7).

(ii) *Preservation under direct images.* We distinguish several cases:

(a) π is a closed embedding. In this case, our assertion follows from 7.13. More precisely and more strongly, 7.13 shows that if $M^\cdot \in D^b(\mathcal{D}_Y)$, then $M^\cdot$ is holonomic if and only if $\pi_+M^\cdot$ is holonomic.

(b) $Y = \mathbb{A}^m$, $X = \mathbb{A}^n$ $(m > n)$ and π is the projection of $\mathbb{A}^m$ onto a direct factor $\mathbb{A}^n$. This case was handled in V,3.5.2.

(c) *Assume* Y *and* X *are affine.* Let $Y \overset{\beta}{\hookrightarrow} \mathbb{A}^N$ and $X \overset{\gamma}{\hookrightarrow} \mathbb{A}^M$ be closed embeddings. Consider the diagram

$$\begin{tikzcd}
Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & X \arrow[dd, "\gamma"] \\
F \subset Y \times X \arrow[d, "\beta\times\gamma"] & \\
\mathbb{A}^N \times \mathbb{A}^M \arrow[r, "\delta"] & \mathbb{A}^M
\end{tikzcd}$$

where α is the graph mapping : $y \mapsto (y,\pi(y))$ $(y\in Y)$. Let $M^\cdot \in D_h^b(\mathcal{D}_Y)$. We have to show that $\pi_+(M^\cdot) \in D_h^b(\mathcal{D}_X)$. By (a) this is equivalent to $\gamma_+\pi_+M^\cdot \in D_h^b(\mathcal{D}_{\mathbb{A}^M})$. But, by 6.4,

$$\gamma_+ \circ \pi_+ = \delta_+ \circ (\beta\times\gamma)_+ \circ \alpha_+ \ .$$

The mappings α and $\beta\times\gamma$ are closed embeddings and δ is the projection of an affine space onto a factor. The associated direct image functors preserve holonomicity by (a) and (b).

(d) π *is a projection* $Y = Z \times X \to X$. Let $\{Z_i\}_{i\in I}$ be a finite affine open cover of Z. We want to compute $\pi_+M^\cdot$ using the Čech complex associated to the cover $\{Z_i\times X\}$ of Y and a spectral sequence.

For $J = (i_1,\dots,i_s) \subset I$, let $Z_J = Z_{i_1}\cap\dots\cap Z_{i_s}$ and $Y_J = Z_J \times X$. Let π_J be the restriction of π to Y_J. There exists a spectral sequence abutting to $\pi_+M^\cdot$ in which E_1 is the Čech complex which associates $H^\cdot\pi_{J,+}(M^\cdot|Y_J)$ to Y_J, with the differential defined by the restrictions. By (c) the $H^j(\pi_{J,+}(M^\cdot|Y_J))$ are holonomic, hence E_1 is holonomic. $\pi_+M^\cdot$ is then also holonomic (1.13).

(e) That π_+ preserves holonomicity now follows from (b) and (d), applied to the canonical factorisation of π.

(iii) *Preservation under inverse images.* Let $M^\cdot \in D_h^b(\mathcal{D}_X)$. We have to show that $\pi^!M^\cdot \in D_h^b(\mathcal{D}_Y)$.

(f) By 4.4, 4.5, this is true if π is the inclusion of an open set or a projection.

(g) Let π be a closed embedding and $i : U = X - Y \to X$ the inclusion of the complement of Y. We have an exact triangle (8.3):

$$\pi_+\pi^!(M^\cdot) \to M^\cdot \to \pi_+(M^\cdot|U) \xrightarrow{[1]} \pi_+\pi^!(M^\cdot) \ .$$

By (ii) and (f), $i_+(M^\cdot|U)$ is holonomic. Therefore $\pi_+\pi^!M^\cdot$ is holonomic (1.12), but then $\pi^!M^\cdot$ is holonomic by (a).

10.2 THEOREM *On* $D_h^b(\mathcal{D}_X)$ *and* $D_h^b(\mathcal{D}_Y)$, *the morphism* $\pi_!$ *(resp.* π^+*) is left adjoint to* $\pi^!$ *(resp.* π_+*). There exists a natural morphism* $\pi_! \to \pi_+$ *of functors from* $D_h^b(\mathcal{D}_Y)$ *to* $D_h^b(\mathcal{D}_X)$ *which is an isomorphism extending to* $D_{coh}^b(\mathcal{D}_Y)$, $D_{coh}^b(\mathcal{D}_X)$ *if* π *is proper.*

Any morphism can be factored into proper maps and an open embedding. In view of 9.6, 9.12, there remains to consider the case of an open embedding. Then $\pi^!$ is the restriction and $\mathbb{D}_Y \circ \pi^! = \pi^! \circ \mathbb{D}_X$ follows from the definition, whence $\pi^+ = \pi^!$. Also, π_+ is the usual direct image, therefore π^+ is left adjoint to π_+. It follows formally, using 9.11, that $\pi_!$ is left adjoint to $\pi^!$. Let $F^\cdot \in D_h^b(\mathcal{D}_Y)$. Then

$$\pi^+\pi_!(F^\cdot) = \pi^+\mathbb{D}_X.\pi_+.\mathbb{D}_Y(F^\cdot) = \mathbb{D}_Y.\pi^+.\pi_+.\mathbb{D}_Y(F^\cdot) = F^\cdot$$

since $\pi^+ \circ \pi_+ = \mathrm{Id}.$ for an open embedding and $\mathbb{D}_Y \circ \mathbb{D}_Y = \mathrm{Id}.$, whence the existence of a natural morphism $\pi_!(F^\cdot) \to \pi_+(F^\cdot)$, or, equivalently, of a natural morphism $\mathbb{D}_X \circ \pi_+ \to \pi_+ \circ \mathbb{D}_Y$ of functors from $D_h^b(\mathcal{D}_Y)$ to $D_h^b(\mathcal{D}_X)$.

10.3 Let π be a locally closed embedding. We define a map $\pi_{!+} : \mathrm{hol}(\mathcal{D}_Y) \to \mathrm{hol}(\mathcal{D}_X)$ by

$$\pi_{!+}(F) = \mathrm{Im}(H^0(\pi_!(F)) \to H^0(\pi_+(F))) \quad (F \in \mathrm{hol}(\mathcal{D}_Y)).$$

If π is affine, in particular if Y is affine, then π_+ is exact (8.1), hence so is $\pi_!$ on $\mathrm{hol}(\mathcal{D}_Y)$. In that case we have therefore $\pi_{!+}(F) = \mathrm{Im}(\pi_!(F) \to \pi_+(F))$.

10.4 PROPOSITION (i) *Let $M \in \mathrm{coh}(\mathcal{D}_X)$. Then M is $\mathcal{O}_X$-coherent if and only if* $SS(M) \subset X$.

(ii) *Let $M \in \mathrm{hol}(\mathcal{D}_X)$. Let Y be the smooth part of $supp(M)$ and $i : Y \to X$ the inclusion. Then $i^! M$ is a $\mathcal{O}$-coherent connection on a dense Zariski-open subset of Y. The module M is $\mathcal{O}$-coherent on a Zariski open dense subset U of X.*

Proof.

(i) This is a local assertion. We may assume X to be affine. Choose a good filtration on M and consider the $\mathrm{gr}\, D(X)$-module $\mathrm{gr}\, M$. Clearly M is $\mathcal{O}(X)$-coherent if and only if $\mathrm{gr}\, M$ is $\mathcal{O}(X)$-coherent and the latter condition is equivalent to $\mathrm{gr}\, M$ being annihilated by some power of the ideal

$$J = \sum_{i>0} \mathrm{gr}^i D(X),$$

a condition which, in turn, is equivalent to $SS(M) \subset X$.

(ii) We may assume X to be irreducible. It suffices to prove the first assertion for each irreducible component of Y. We may therefore also assume Y to be irreducible.

Let $N = i^! M$. It is holonomic (10.1). Let $\sigma : T^*Y \to Y$ be the canonical projection. It maps every irreducible component Z of SSN onto a closed algebraic set. This is clear if $Z \subset Y$ (viewed, as usual, as the zero-section of T^*Y); if not, since it is conical (see 1.9(6)), the restriction of σ to Z can be factored through the projection on Y of the projective cotangent bundle, which is proper. The irreducible components of SSN have dimension $\le d_Y$, since N is holonomic. Let W be the union of the $\sigma(Z)$ of dimension $< d_Y$. Then

the characteristic variety of $N|(Y-W)$ is contained in $Y - W$. Therefore N is $\mathcal{O}$-coherent on $Y - W$ by (i), and is a connection by 1.7.

This also proves the last assertion if Y is dense in X. If not, then $\mathrm{supp}\, M$ is contained in a proper subvariety and M is zero on its complement.

10.5 PROPOSITION *Let $i : Y \to X$ be an affine locally closed embedding and E an irreducible holonomic $\mathcal{D}_Y$-module. Then*

(i) i_+E and $i_!E$ are holonomic.

(ii) $i_{!+}E$ is the unique irreducible submodule of i_+E (resp. quotient of $i_!E$).

(iii) For any open affine $V \subset X$, either $i_+E|V = 0$ or $i_{!+}E|V$ is the only irreducible submodule of $i_+E|V$.

The first assertion follows from 10.1. In particular i_+E and $i_!E$ have finite length (1.10), hence i_+E has an irreducible submodule, say F. Its support is contained in $\bar{Y}$. In the present case i_+E reduces to $H^0 i_+E$, which is just the usual direct image of $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} E$ see (5.2). Therefore no section of i_+E has a support entirely outside Y. As a consequence, $i^! F \neq 0$. By the exactness of $i^!$ on $\mu_{\bar{Y}}(\mathcal{D}_X)$) (see 8.1(iii)), we have $i^! F \subset i^! i_+ E = E$, hence $i^! F = E$, since E is irreducible. It also shows then that

$$i^!(i_+(E)/F) = i^! i_+ E / i^! F = E/E = 0$$

i.e. $\mathrm{Supp}(i_+E/F) \subset \bar{Y} - Y$. Since no non-zero submodule of i_+E is supported in $\bar{Y} - Y$, this implies that F is the unique irreducible submodule of i_+E. In particular $F \subset H = i_{!+}E$. Consider now the exact sequence

$$0 \to F \to H \to H/F \to 0.$$

By the above, $\mathrm{Supp}(H/F) \subset \bar{Y} - Y$. Going over to the duals, we see that $\mathbb{D}_X(H/F) \subset \mathbb{D}_X(H)$. On the other hand, the surjection $\mathbb{D}_X . i_+ . \mathbb{D}_X E \to H \to 0$ also shows, by duality, that $\mathbb{D}_X H \subset i_+ \mathbb{D}_X E$. It follows that $\mathbb{D}_X(H/F) \subset i_+ \mathbb{D}_X E$. Since the support of H/F is contained in $\bar{Y} - Y$. This yields a contradiction, unless $H/F = 0$. Thus $F = H$. In the same way,

$\mathbb{D}_X F$ is the unique irreducible submodule of $i_+\mathbb{D}_X E$, hence F is the unique irreducible quotient of i_+E. This proves (ii).

(iii) We let $H = i_{!+}E$ and denote by j the inclusion $V \to X$. If $Y = X$, our assertion amounts to saying that if H is an irreducible holonomic $\mathcal{D}_X$-module then $j^!H$ is irreducible or zero. We prove this first.

Suppose $j^!H$ is not zero and not irreducible. There exists then $F \in \mu(\mathcal{D}_V)$ and a non-zero morphism $\alpha : j^!H \to F$ with non-zero kernel. From this we get a non-zero morphism $\alpha' : H \to j_+F$. We have $\ker\alpha_{|V} = \ker\alpha \neq 0$, therefore $\ker\alpha' \neq 0$, contradicting the fact that H is irreducible.

We now prove (iii) in general. We assume that $i_+(E)|V \neq 0$, otherwise there is nothing to prove. If $A \subset B$ are two smooth locally closed subvarieties of X, we denote by $i_{(A\to B)+}$ the direct image from A to B. We have the equality

$$i_{(Y\to X)+}(E)|V = i_{(Y\cap V\to V)+}(E|Y\cap V).$$

(Note that Y is affinely embedded by assumption, hence $Y \cap V$ is affine, and we are reduced to the usual direct image (8.1). Therefore $E|Y \cap V \neq 0$. It is then irreducible by our initial remark, hence the right-hand side contains a unique irreducible submodule by (ii). But $H|V \neq 0$ since $\operatorname{supp}(i_+E/H) \subset \bar{Y}-Y$, hence $H|V$ is irreducible for the same reason and we get the equality $H|V = i_{(Y\cap V\to V)+!}(E|Y\cap V)$.

10.6 THEOREM *Let M be a holonomic $\mathcal{D}_X$-module.*

(i) Assume M to be irreducible. Then there exists a locally closed smooth affine subvariety Y of X and an irreducible ($\mathcal{O}_Y$-coherent) connection E on Y such that $M = i_{!+}E$.

(ii) M is completely reducible if and only if it is locally completely reducible.

Proof. (i) Let Y be an irreducible open smooth subset of $\operatorname{Supp} M$ and $i : Y \to X$ the inclusion. Replacing Y by an open subset, we may assume that N is $\mathcal{O}$-coherent (10.4) and Y is affine. Let U be an open affine subset of X whose intersection with $\operatorname{Supp} M$ is Y. Then $N = i^!(M|U)$ and, by Kashiwara's theorem (7.11), $M|U = i_{(Y\to U)+}(N)$. In particular $N \neq 0$.

Let E be an irreducible quotient of N. Then the non-zero morphism

$$M|U = i_{(Y\to U)+}(N) \to i_{(Y\to U)+}(E)$$

gives rise to a morphism

$$M \to (i_{U\to X}\circ i_{Y\to U})_+(E) = i_+(E).$$

By 10.5, $M = i_{!+}(E)$. This also implies that N is equal to E and is irreducible.

(ii) We saw in the proof of 10.5 (iii) that if H is an irreducible holonomic $\mathcal{D}_X$-module, then the restriction of H to any affine open subset is either 0 or irreducible. Therefore, if M is a direct sum of irreducible submodules, then so is its restriction to any affine open subset.

Assume now conversely that there is a finite open affine cover $(X_j)_{j\in I}$ of X such that $M|X_j$ is completely reducible for every $j \in I$. We want to prove that M is completely reducible. Let i_j be the inclusion of X_j into X. We can write $M|X_j = \oplus_j N_{jk}$ with N_{jk} holonomic, irreducible, and have a natural map

$$\zeta_j : M \to \oplus_k\, i_{j+}(N_{jk}) \qquad (j\in I).$$

Let H_{jk} be the unique irreducible submodule of $i_{j+}(N_{jk})$ $(j,k\in I)$, (10.5(ii)). It follows from 10.5(iii) that for any $\ell \in I$, the $H_{jk}|U_\ell$ are the unique irreducible submodules of $i_{j+}(i_j^!M)$. Since $M|U_\ell$ is completely reducible, we see that $\zeta_j(M)|U_\ell$ is contained in $\oplus H_{jk}|U_\ell$ for any $\ell \in I$, hence

$$\zeta_j(M) \subset \oplus_k H_{jk} \qquad (j\in I).$$

But the direct sum of the ζ_j defines an embedding of M into the direct sum of the $i_{j+}(N_{jk})$. Its image is then contained in the direct sum of the H_{jk}, hence is completely reducible.

10.7 PROPOSITION *Let $F^\cdot \in D^b(\mathcal{D}_X)$. Then the two following conditions are equivalent:*

(i) *$F^\cdot$ is holonomic.*

(ii) $H^{\cdot}F^{\cdot}$ *is coherent and for every* $x \in X$, $H^{\cdot}i_x^{!}F^{\cdot}$ *is finite dimensional, where* i_x *is the inclusion of* x *in* X.

Proof. (i) $\Longrightarrow$ (ii). $H^{\cdot}F^{\cdot}$ being holonomic, is coherent. Since $i_x^{!}$ preserves holonomicity (10.1), the modules $H^i i_x^{!}F^{\cdot}$ are therefore coherent over $\mathcal{D}_{\{x\}} \cong \mathbb{C}$, hence are finite dimensional vector spaces.

(ii) $\Longrightarrow$ (i) Let S be the support of $F^{\cdot}$. The proof is by induction on dim S. If dim $S = 0$, then we have $F^{\cdot} = i_+ i^+ F^{\cdot}$ by 7.13, and our assertion follows from 7.13. Assume now dim $S > 0$. There exists $Y \subset S$, smooth, open affine and dense, on which $G^{\cdot} = i^{!}_{Y\to X}F^{\cdot}$ has locally $\mathcal{O}_Y$-free cohomology (9.3). We claim that $H^{\cdot}G^{\cdot}$ is $\mathcal{O}_Y$-coherent.

We have $i^{!}_{y\to Y}G^{\cdot} = i^{!}_{y\to X}F^{\cdot}$ $(y \in Y)$, therefore $H^{\cdot}i^{!}_{y\to Y}G^{\cdot}$ is finite dimensional. The complex $G^{\cdot}$ is $i^{!}_{y\to Y}$-acyclic, therefore

$$H^j i^{!}_{y\to Y}G^{\cdot} = i^{!}_{y\to Y}H^j G^{\cdot} \qquad (j \in \mathbb{Z}) \tag{1}$$

and $H^{\cdot}i^{!}_{y\to Y}G^{\cdot}$ is a direct sum of copies of $i_y^{!}(\mathcal{O}_Y) = \mathcal{O}_{Y,y}$, in number equal to the rank of $H^{\cdot}G^{\cdot}$ in a neighbourhood of y. Since the left-hand side of (1) is finite dimensional, this proves our assertion. As a consequence, $G^{\cdot}$ is a $\mathcal{O}_Y$-coherent integrable connection, hence is holonomic and then so it $i_+G^{\cdot} = i_+ i^{!}F^{\cdot}$, where i now stands for the inclusion $Y \to X$.

Consider the natural morphism $u : F^{\cdot} \to i_+ i^{!}F^{\cdot}$ and let $A^{\cdot}$ be the cone over it. We have just seen that $i_+ i^{!}F^{\cdot}$ is holonomic. It suffices therefore to show that $A^{\cdot}$ is holonomic. The long exact cohomology sequence associated to the exact triangle

$$0 \to F^{\cdot} \to i_+ i^{!}F^{\cdot} \to A^{\cdot} \to \dots \tag{2}$$

shows that $H^{\cdot}A^{\cdot}$ is coherent. Let $x \in X$. By applying $i_x^{!}$ to (2), we get an exact triangle, whence again a long exact cohomology sequence. There $H^{\cdot}i_x^{!}F^{\cdot}$ is finite dimensional by assumption and $H^{\cdot}i_x^{!}(i_+ i^{!}F^{\cdot})$ is finite dimensional by (i). Therefore $H^{\cdot}i_x^{!}A^{\cdot}$ is finite dimensional. The induction assumption then implies that $A^{\cdot}$ is holonomic.

10.8 *Addendum to* 10.6. D. Milicic has pointed out to me that 10.6(ii) is in fact valid for coherent $\mathcal{D}_X$-modules. We outline his argument, assuming, as we may, that X is irreducible.

(a) Let $F \in \mu(\mathcal{D}_X)$. Given an open subset U of X and a submodule G of $F|U$ we let $\bar{G}$ be the canonical extension in F of G. By definition, for V open in X, $\Gamma(V,\bar{G})$ is the set of sections $s \in \Gamma(V,F)$ such that $s|U \cap V \in \Gamma(U\cap V, F)$. This is the largest subsheaf of F whose restriction to U is G. It is obviously a $\mathcal{D}_X$-submodule of F. By [EGA: I, 6.9.2], it is quasi-coherent over $\mathcal{O}_X$, hence $G \in \mu(\mathcal{D}_X)$. As a consequence, if F is irreducible, then $F|U$ is either irreducible or zero. This also implies that any $\mathcal{D}_X$-module of finite length is coherent and that if F is completely reducible, then its restriction to any open subset is completely reducible, too.

(b) Let $\mathfrak{U} = \bigcup_{j\in J} U_j$ be a finite open cover of X and $F \in \mu(\mathcal{D}_X)$ such that $F|U_j$ has finite length $(i \in I)$. Then F has finite length. In fact, being coherent (by (a)), it has a composition series F_i $(i \leq 0, F_0 = F)$ such that F_i/F_{i+1} is irreducible. The restriction of this series to U_j is stationary for all $j \in J$, hence the series itself is stationary.

(c) Assume now that $F \in \mu(\mathcal{D}_X)$ is such that $F|U_j$ is irreducible for all j's. We claim that F is a direct sum of finitely many irreducible submodules with disjoint supports. By (b), F has finite length, hence is coherent, and therefore has closed support (1.10). We now argue by induction on the length $\ell(F)$ of F: If $\ell(F) \neq 1$, let G be a non-trivial submodule of F. For $x \in \mathrm{Supp}(F)$, we have an exact sequence

$$0 \to G_x \to F_x \to (F/G)_x \to 0.$$

Since x lies in some U_i, the fibre F_x is irreducible, therefore either $G_x = F_x$ or $G_x = 0$; hence Supp F is the disjoint union of Supp G and Supp (F/G).

(d) We can now prove that if $F|U_j$ is completely reducible for all j's, then F is completely reducible. We use induction on $\ell(F)$. Let G be a maximal $\mathcal{D}_X$-submodule of F. By induction, it is completely reducible. If the supports of G and F/G are disjoint, then F is the direct sum of G and F/G, hence is completely reducible. If not, choose j such that $G|U_j$ and $(F/G)|U_j$ are non-zero. By our assumption, $F|U_j$ is the direct sum of $(F/G)|U_j$ and of some irreducible submodule M. The canonical extension $\bar{M}$ of M (see (a)) is $\neq F$, hence is completely reducible by induction assumption. Moreover, $G + \bar{M}$ is a $\mathcal{D}_X$-submodule containing G and different from

G, hence equal to F. Therefore F is a quotient of the completely reducible $\mathcal{D}_X$-module $G \oplus \bar{M}$, and is completely reducible, too.

Remark. We recall that irreducible $\mathcal{D}_X$-modules are not necessarily holonomic, as shown by J. T. Stafford (see p. 204).

§ 11 REGULAR HOLONOMIC COMPLEXES

11.0 We shall soon have to shift between algebraic and analytic objects in the usual way, as was already done in IV, and we collect here some of the standard notation and facts. However to simplify the typography, we shall use the superscript "a" rather than "an" to refer to analytic objects or maps.

Let M be a smooth algebraic variety. As usual, we let M^a be the underlying complex analytic manifold and $\mathcal{O}_{M^a}$ its structural sheaf. If Θ_{M^a} is the sheaf of germs of holomorphic vector fields, then $\Theta_{M^a} = \mathcal{O}_{M^a} \otimes_{\mathcal{O}_M} \Theta_M$. The ring of sheaf of germs of holomorphic differential operators $\mathcal{D}_{M^a}$ is the sheaf of germs of operators on $\mathcal{O}_{M^a}$ generated by $\mathcal{O}_{M^a}$ and Θ_{M^a}. We have $\mathcal{D}_{M^a} = \mathcal{O}_{M^a} \otimes_{\mathcal{O}_M} \mathcal{D}_M$. If F is $\mathcal{D}_M$-module, then $F^a = \mathcal{O}_{M^a} \otimes_{\mathcal{O}_M} F$ is a $\mathcal{D}_{M^a}$-module. The map $F \mapsto F^a$ induces a functor $D^b(\mathcal{D}_M) \to D^b(\mathcal{D}_{M^a})$.

If $\sigma : U \to V$ is a morphism of algebraic varieties, then σ^a denotes the corresponding morphism $U^a \to V^a$.

11.1 Let C be a smooth curve and $C \to \bar{C}$ a completion. Let F be a connection on C. Then i_+F is a meromorphic extension on $\bar{C}$. Therefore F is regular if and only if $(i_+F)^a$ has regular singularities around the points of $\bar{C} - C$. If C' is the complement of a divisor in C, then F is regular if and only if $F|C'$ is so.

The embedding $C \to \bar{C}$ is affine, hence i_+ is exact. Therefore if

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of connections on C, then

$$0 \to i_+F' \to i_+F \to i_+F'' \to 0$$

and

$$0 \to (i_+F')^a \to (i_+F)^a \to (i_+F'')^a \to 0$$

are also exact. It follows then from (III,2.4.4) that F is regular if and only F' and F'' are regular.

We also deduce from III,2.4.5 and 2.4.6 that if $\pi : C \to C'$ is an unramified covering and F (resp. G) is a connection on C (resp. C') then F (resp. G) is regular if and only if $\pi_+ F$ (resp. $\pi^o G$) is regular.

11.2 Let more generally F be a holonomic module on C. There is a Zariski open dense set C' on which it is a connection. We say that F is regular holonomic (r.h.) if $F|C'$ is a regular connection. In particular, any holonomic module with support of dimension zero is r.h.. From this definition it follows that if

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of $\mathcal{D}$-modules, then F is regular holonomic if and only if F' and F'' are so.

A complex $F^\cdot \in D(\mathcal{D}_C)$ is said to be regular holonomic if $H^\cdot F^\cdot$ consists of r.h. modules.

We let $D^b_{rh}(\mathcal{D}_C)$ be the bounded derived category of complexes of r.h. $\mathcal{D}_C$-modules.

From the previous remarks, we see that : (1) any subquotient of a r.h. module is r.h.; (2) if two vertices of an exact triangle in $D^b(\mathcal{D}_C)$ are r.h., then so is the third one ; (3) let $C \to C'$ be a dominant morphism and $F^\cdot \in D^b(\mathcal{D}_C)$ (resp. $G^\cdot \in D^b(\mathcal{D}_C)$. Then $F^\cdot$ (resp. $G^\cdot$) is r.h. if and only if $\pi_+ F^\cdot$ (resp. $\pi^! G^\cdot$) is r.h.

In particular $D^b_{rh}(\mathcal{D}_C)$ is triangulated.

11.3 DEFINITION An element $F^\cdot \in D(\mathcal{D}_X)$ is regular holonomic (r.h.) if :

(i) it is holonomic ;

(ii) for any morphism $\phi : C \to X$ of a smooth irreducible curve C into X, the complex $\phi^! F^\cdot$ is r.h. on C.

This definition applies in particular to a single degree complex. Thus a $\mathcal{D}_X$-module F is r.h. if it is holonomic and if for every ϕ as in (ii), the $\mathcal{D}_C$-modules $H^i \phi^! F$ are r.h. ($i \in \mathbb{Z}$)

We let $D^b_{rh}(\mathcal{D}_X)$ be the bounded derived category of r.h. complexes.

11.4 We now make a number of remarks on this notion.

(a) It does not imply a priori that $H^\cdot F^\cdot$ consists of r.h. modules, because $\phi^!$ is not exact (and therefore $\phi^! H^i F^\cdot \neq H^i \phi^! F^\cdot$ in general). We shall prove it later, however; more generally we shall see that any subquotient of $H^\cdot F^\cdot$ is r.h. (12.8).

(b) As already pointed out in the foreword, this definition was proposed orally by Bernstein in 1984 and is not the one of [Be]. In fact the property discussed in (a) was built in the definition there.

(c) Since inverse images compose (4.3) and preserve holonomicity (10.1) it is clear that $\pi^!$ maps $D^b_{rh}(\mathcal{D}_X)$ into $D^b_{rh}(\mathcal{D}_Y)$.

(d) Let $F^\cdot \to G^\cdot \to H^\cdot \to F^\cdot[1]$ be an exact triangle in $D^b(\mathcal{D}_X)$. If two vertices are r.h. then so is the third one. In fact, it is first of all holonomic (1.12). If ϕ is as in 11.3(ii), then

$$\phi^! F^\cdot \to \phi^! G^\cdot \to \phi^! H^\cdot \to \phi^! F^\cdot[1]$$

is again an exact triangle and we may use 11.2.

(e) If ϕ is a constant map and $F^\cdot$ is holonomic, then $\phi^! F^\cdot$ is $\mathcal{O}_C$-free and coherent, hence regular.

Proof. Let $x = \phi(C)$. Then ϕ is the composition $i_x \circ p$ if the projection p of C onto x and of the inclusion of x in X, hence $\phi^! = p^! \circ i_x^!$ (4.3). By 10.7, $H^\cdot i_x^! F^\cdot$ is a finite dimensional complex vector space. Since p is constant, $p^!(i_x^! F^\cdot)$ is a direct sum of copies of $\mathcal{O}_C$ indexed by a basis of $H^\cdot i^! F^\cdot$.

(f) If $F^\cdot$ is holonomic with support of dimension 0 and $\phi : C \to X$ is not constant, then the support of $\phi^! F^\cdot$ has dimension zero (4.4, remark), therefore $\phi^! F^\cdot$ is r.h.. Together with (e) this shows that $F^\cdot \in D^b_h(\mathcal{D}_X)$ with support of dimension 0 is r.h..

(g) We now want to show that it is enough to check (ii) for embeddings. By (e), we need only consider non-constant maps. There exists then $C' \subset C$, Zariski open, such that $\phi(C')$ is smooth and $\phi' : C' \to \phi(C')$ is an unramified covering. Let $i : C' \to C$ and $j : \phi(C') \to X$ be the inclusion maps. The regularity of $\phi^! F$ is equivalent to that of $(i^! o \phi^!) F^\cdot$. But $\phi \circ i = j \circ \phi'$. If $j^! F^\cdot$ is regular, then $(\phi'^! . j^! F^\cdot)$ is also regular (11.1).

(h) Assume that F is a ($\mathcal{O}_X$-coherent)-connection. Then (4.2, Remark) $\phi^! F$ is a single degree complex and its non (necessarily) zero component is the inverse image of the connection F, as defined in IV, §3. Therefore the present definition of regularity coincides with that of IV,7.1 in that case. It follows then from IV, 3.2 that if

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of connections, then F is regular if and only if F' and F'' are so.

(i) If X is a curve, we have *a priori* two definitions of regular holonomicity. An "internal" one given by 11.2 and an external one given by 11.3 : They are of course equivalent. In fact, if $F^{\cdot}$ is r.h. in the sense of 11.3 then, by applying 11.3(ii) to the identity map, we see that $F^{\cdot}$ is r.h. in the sense of 11.2. Assume now the latter and let $\phi : C \to X$ be a morphism. If it is constant, then $\phi^! F^{\cdot}$ is r.h. by (e). If it is not, then it is dominant (assuming, as we may, that C is irreducible), and $\phi^! F$ is regular holonomic by 11.2(c).

11.5 The smooth variety X admits a smooth completion $\bar{X}$ which is a projective variety, such that $Z = \bar{X} - X$ is a smooth divisor with normal crossings. We shall call such an $\bar{X}$ a *regular completion* of X.

Let $i : X \to \bar{X}$ be the inclusion. Since Z is of pure codimension one, its intersection with any open affine subset of X is affine, hence i is an affine embedding and i_+ is exact (8.1).

We let $\mathcal{D}_{\bar{X},Z}$ be the sheaf of germs of differential operators generated by $\mathcal{O}_{\bar{X}}$ and the $\mathcal{O}_{\bar{X}}$-module of vector fields $\Theta_{\bar{X},Z}$ which are tangent to Z on Z. Of course $\mathcal{D}_{\bar{X},Z}|X = \mathcal{D}_X$. As in IV, let $\Theta_{\bar{X}^a,Z^a}$ be the sheaf of germs of holomorphic vector fields on $\bar{X}^a$ which are tangent to Z^a on Z^a . Then

(1) $$\Theta_{\bar{X}^a,Z^a} = \mathcal{O}_{\bar{X}^a} \underset{\mathcal{O}_{\bar{X}}}{\otimes} \Theta_{\bar{X},Z} \ ; \ \mathcal{D}_{\bar{X}^a,Z^a} = \mathcal{O}_{\bar{X}^a} \underset{\mathcal{O}_{\bar{X}}}{\otimes} \mathcal{D}_{\bar{X},Z} \ .$$

We now want to prove a variant of IV,5.10 in the algebraic context.

11.6 PROPOSITION *We keep the notation of* 11.5.*Let* E *be a holonomic module on* X *with support* X *and* $F = i_+E$. Then E is a regular connection if and only if F is a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$-modules.

Proof. (a) Let E be a regular connection. F^a is a meromorphic extension of E^a . Therefore it is the unique meromorphic extension of E^a which is regular along Z (IV;§5). By (IV,5.10) it is an increasing union of $\mathcal{O}_{\bar{X}}$ -coherent and $\mathcal{D}_{\bar{X}^a,Z^a}$-submodules, G_j ($j \in \mathbb{N}$). By GAGA there exists for each j a unique $\mathcal{O}_{\bar{X}}$-coherent module $G_{j,o}$ such that $G_j = \mathcal{O}_{\bar{X}^a} \underset{\mathcal{O}_{\bar{X}}}{\otimes} G_{j,o}$. By uniqueness, $G_{j,o}$ is $\mathcal{D}_{\bar{X},Z}$-stable. On the other hand, F is an increasing union of coherent $\mathcal{O}_{\bar{X}}$ -submodules F_k ($k \in \mathbb{N}$). Given k, there exists then j and k' such that $F^a_{k'} \supset G_j \supset F^a_k$ hence such that $F_{k'} \supset G_{j,o} \supset F_k$. Therefore F is the union of the $G_{j,o}$.

(b) Assume F to be a union of $\mathcal{O}_{\bar{X}}$ -coherent $\mathcal{D}_{\bar{X},Z}$-modules F_j ($j \in \mathbb{N}$). Then E is the union of the restrictions E_j to X of the F_j's, which are $\mathcal{O}_X$-coherent and $\mathcal{D}_X$-stable. Being $\mathcal{D}_X$-coherent, E is then equal to E_j for some j, hence is $\mathcal{O}_X$-coherent as well. Since its support is X, it is a connection on X (1.6). F^a is a meromorphic extension of E^a and is the union of the F^a_j , which are $\mathcal{O}_{\bar{X}^a}$-coherent. In view of 11.5(1) they are also $\mathcal{D}_{\bar{X}^a,Z^a}$-stable. Therefore F^a is regular (IV,5.10). Then F is regular (11.4(h)).

11.7 PROPOSITION *We keep the notation of* 11.5. *Let* F *be a holonomic module on* $\bar{X}$ *and assume that* F *is a union of* $\mathcal{O}_{\bar{X}}$*-coherent* $\mathcal{D}_{\bar{X},Z}$ *-modules . Then* F *is r.h. and its restriction to* X *is a regular connection. Every* $\mathcal{D}_{\bar{X}}$ *-subquotient of* F *is r.h.*

Proof. We show first that the restriction E of F to X is a regular connection. It is holonomic, hence coherent, and a union of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-submodules. Those are all connections (1.6) and, by coherence, E is equal to one of them. We claim that $G = i_+E$ is also a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$-submodules. Let $F' \subset F$ be a $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$-submodule whose restriction to X is E and F'' be the image of F' under the canonical map $F' \to i_+E$. It is the quotient of F' by its biggest submodule with support Z, which is also $\mathcal{D}_{\bar{X},Z}$-stable. F'' is consequently a $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$-module. Now the sheaf $\mathcal{O}_{\bar{X}}[Z]$ of germs of rational functions with poles on Z is also a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$submodules, hence $\mathcal{O}_{\bar{X}}[Z] \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{F}"$ has the same property. But it is equal to i_+E. By 11.6, E is regular.

To prove the first assertion, we have to test that $\phi\ \mathcal{F}$ is regular for any embedding $\phi : C \to X$ of a smooth curve in X and we may always replace C by the complement of a divisor.

Assume first that $C \cap X$ is dense in C. Then, replacing C by $C \cap X$, we may assume $C \subset X$. Let $j : C \to X$ be the inclusion ; then $\phi^! \mathcal{F} = j^! i^! \mathcal{F} = j^! E$ is regular since E is a regular connection.

Assume now that $C \subset Z$. Let W be the intersection of the components of Z containing a dense set of C and T the union of the intersections of W with the other components of Z. Let $W' = W - T$. Then $C \cap W'$ is open dense in C and we may replace C by $C \cap W'$, hence assume $C \subset W'$. Note that W is a regular completion of W'. We have then inclusions

$$C \xrightarrow{\alpha} W' \xrightarrow{\beta} W \xrightarrow{\gamma} X$$

We now claim :

(*) *$H^{\cdot}\gamma^! \mathcal{F}^{\cdot}$ is a union of $\mathcal{O}_W$-coherent $\mathcal{D}_{W,T}$-modules.*

Assume this provisorily. Then, by the first part of the proof $\beta^!(H^{\cdot}\gamma^!\mathcal{F}^{\cdot}) = H^{\cdot}\beta^!\gamma^!\mathcal{F}^{\cdot}$ is a regular connection on W'. Hence $\alpha^! H^{\cdot}\beta^!\gamma^!\mathcal{F}^{\cdot}$ is r.h.. But (4.2, Remark)

$$\alpha^!(H^{\cdot}\beta^!\gamma^!\mathcal{F}^{\cdot}) = H^{\cdot}\alpha^!\beta^!\gamma^!\mathcal{F}^{\cdot} = H^{\cdot}\phi^!\mathcal{F}^{\cdot}[d_{W,X}],$$

hence $\phi^! \mathcal{F}^{\cdot}$ is r.h. on C.

There remains to prove (*). Let Z_1 be one component of Z containing W and J its sheaf of ideals. Let $j_1 : Z_1 \to X$ be the inclusion. By 7.4, we have

(1) $\quad H^1 j_1^! \mathcal{F} = \mathcal{F}/J\mathcal{F};\ H^0 j_1 \mathcal{F} = {}_J\mathcal{F} \otimes \omega_{\bar{X}/Z_1};\ H^i j_1^! \mathcal{F} = 0 \quad (i \neq 0,1).$

Let Y_1 be the union of the intersections of Z_1 with the other components of Z. Then $\mathcal{D}_{Z_1,Y_1}$ is a subquotient of $\mathcal{D}_{\bar{X},Z}$. Since $\mathcal{O}_{Z_1} = \mathcal{O}_{\bar{X}}/J$, the assumption on F implies that the $\mathcal{D}_{Z_1}$-modules in (1) are union of $\mathcal{O}_{Z_1}$-

coherent $\mathcal{D}_{Z_1,Y_1}$-modules. If $W = Z_1$ we are finished. If not we argue by an easy induction on the number of components containing W, replacing X by Z_1. This then proves (*) and concludes the proof that $\mathcal{F}$ is r.h.. Since every $\mathcal{D}_{\bar{X}}$-submodule of $\mathcal{F}$ satisfies the conditions imposed on $\mathcal{F}$, the last assertion follows, too.

11.8 COROLLARY *Let E be a regular connection on X . Then every $\mathcal{D}_{\bar{X}}$-submodule M of i_+E is r.h..*

M is holonomic (1.11). By 11.6, M is a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},Z}$-submodules. It is then r.h. by 11.7.

§ 12 PRESERVATION OF REGULAR HOLONOMICITY.

We already noticed that regular holonomicity is preserved under $\pi^!$ (11.4(c)). Our goals here are to show that r.h. is also invariant under direct images and duality, goes over to the subquotients of $H^\cdot F^\cdot$, and to give a generating class of modules for $D^b_{rh}(\mathcal{D}_X)$.

12.1 *Standard $\mathcal{D}_X$-modules.* We shall call standard a $\mathcal{D}_X$-module of the form $i_+(E)$ where $i : Y \to X$ is the inclusion of a smooth, irreducible affine locally closed subvariety and E is an irreducible regular connection on Y.

We note that, Y being affine, $i_+(E)$ reduces indeed to one $\mathcal{D}_X$-module (8.1).

12.2 THEOREM (i) *Regular holonomicity is preserved under π_+.*
(ii) *The standard modules are r.h. and form a generating class for $D^b_{rh}(\mathcal{D}_X)$.*

The proof is contained in the next three sections. Let $F^\cdot \in D^b_{rh}(\mathcal{D}_Y)$. Then $\pi_+(F^\cdot)$ is holonomic (10.1). In order to establish (i) it suffices to check the condition (ii) of 11.3, and we may restrict ourselves to embedded curves by 11.4(g). Let then $\phi : C \to X$ be an embedding of a smooth irreducible curve. We have to prove that $\phi^!\pi_+(F^\cdot)$ is r.h..

We note first that (i) is easy to prove if Y and X are curves: In that case either $\pi(Y)$ has dimension zero and we may apply 11.4(f) or π is dominant and we may find smooth embedded curves $D \xrightarrow{i} Y$ and $\pi(D) \xrightarrow{j} X$ such that $\pi' = \pi|D$ is an unramified covering. Then $j^!F^\cdot$ is r.h., and $\pi'_+j^!F^\cdot$ is r.h. by 11.2(3). Since $\pi'_+ \circ j^! = i^! \circ \pi_+$ (8.4), this implies that $i^!\pi_+F^\cdot$ is r.h.. But this is equivalent to $\pi_+F^\cdot$ being r.h..

12.3 *Proof of* (i) *for π injective.* (a) Assume first that $C' = C \cap Y$ is open dense in C. Let $j : C' \to C$ be the inclusion. It suffices to show that $j^!.\phi^!.\pi_+F^\cdot$ is regular holonomic. For this, we note that, in the diagram

$$\begin{array}{ccc} C' & \xrightarrow{\psi} & Y \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\phi} & X \end{array}$$

we have, by 4.3, 8.2,

$$j^! \circ \phi^! \circ \pi_+ = \psi^! \circ \pi^! \circ \pi_+ = \psi^! .$$

(b) We are now reduced to the case where $C \cap Y$ is finite. We may replace C by $C - (C\cap Y)$ hence assume that $C \cap Y = \emptyset$. The curve C is necessarily locally closed in X. Therefore $\phi^! \circ \pi_+ = 0$ by 8.5.

12.4 *Proof of* (ii). We note first that, by 12.3, the standard modules are r.h.. Before proving the second assertion of (ii), we make two simple remarks.

(a) Let $F^\cdot \in D^b_{rh}(\mathcal{D}_X)$ and assume that $H^\cdot F^\cdot$ is $\mathcal{O}_X$-coherent. Then $H^\cdot F^\cdot$ is a regular connection.

In fact, $H^\cdot F^\cdot$ is locally free over $\mathcal{O}_X$ (1.6). If $\phi : C \to X$ is as in 11.3(ii), then (4.2, Remark), we have $\phi^! H^q F^\cdot = H^q \phi^! F^\cdot$, which is regular on C by definition.

(b) The irreducible regular connections form a generating class for the bounded derived category B of r.h. complexes whose cohomology is $\mathcal{O}_X$-coherent.

We have just seen that $H^\cdot F^\cdot$ is a regular connection. The last remark of 11.4(h) implies that $H^\cdot F^\cdot$ belongs to the category generated by the irreducible regular connections. But then B is generated by its cohomology objects [I:§12].

(c) Let A be the subcategory of $D^b_{rh}(\mathcal{D}_X)$ generated by the standard modules. We prove that a r.h. bounded complex $F^\cdot$ belongs to A by induction on the dimension of the support S of $F^\cdot$.

Assume first that dim S = 0 and let $i : S \to X$ be the inclusion. By 7.13, we have $F^\cdot = i_+i^!F^\cdot$. The module $i^!F^\cdot$ is the direct sum of the $i_x^!F^\cdot$ $(x \in S)$. Each $H^\cdot i_x^! F^\cdot$ is a finite dimensional vector space, hence $i_+i^!F^\cdot$ is a finite direct sum of standard modules $i_{x,+}(\mathbb{C})$ $(x \in S)$.

Let now $k \geq 1$ and assume our assertion proved for dim S < k. Let S be of dimension k. We shall argue by induction on the number of irreducible components of S of dimension k. We can find an affine irreducible smooth subvariety Y of X which is Zariski open in an irreducible component of dimension k of S, on which $i^! F^\cdot$, where $i : Y \to X$ is the inclusion, has $\mathcal{O}_Y$-coherent cohomology (10.4). By (b), $i^! F^\cdot$ belongs to the category generated by the irreducible regular connections on Y. Since i_+ is exact (Y being affine) it follows that $i_+ i^! F^\cdot$ belongs to A. Let $G^\cdot$ be the cone on the canonical morphism $u : F^\cdot \to i_+ i^! F^\cdot$. It is r.h. (11.4(d)). Both $F^\cdot$ and $i_+ i^! F^\cdot$ have support in S. The morphism u is an isomorphism on Y. Therefore the support of $G^\cdot$ is contained in S - Y, hence has (at least) one less irreducible component of dimension k than S. By induction assumption $G^\cdot \in A$. Since we already have $i_+ i^! F^\cdot \in A^\cdot$, this concludes the proof of (ii).

12.5 *Proof of* (i) *for projections.* This is the main case. It contains in particular the regularity of the Gauss-Manin connection. In fact, thanks to the results proved so far, the proof is reduced rather formally to the consideration of a regular connection in a special case, rather analogous to the one to which Deligne reduces the proof of the regularity of the Gauss-Manin connection [De:§7], but even more special in that we may moreover assume the total space to have dimension 2. Indeed , the formalism of $\mathcal{D}$-modules will allow us to reduce the proof to the following crucial case:

12.5.1 π *is the projection* $Y = \mathbb{A}^1 \times X \to X$, *where* X *is an affine irreducible smooth curve,*

By 12.2, we only need consider a standard module $F = j_+(E)$, where E is an irreducible regular connection on a smooth irreducible affine locally closed subvariety Z of Y and $j : Z \to Y$ is the inclusion. If Z is itself a curve, then $\pi \circ j$ is a morphism of curves and our assertion was proved at the end of 12.2. From now on, we assume that Z *is open in* Y. Its image in X is open dense. Recall that to check regularity on a curve, we may always leave out an arbitrary finite set of points. If π' is the restriction of π to Z and k the inclusion of $X' = \pi'(Z)$ into X, we have $\pi_+ i_+(E) = k_+(\pi'_+(E|Z))$. By 12.3, it suffices to see that $\pi'_+(E|Z)$ is regular. There again, we may restrict to an open dense set of X' and to its inverse image in Z.

We can find a regular completion $\bar{Z}$ of Z mapping onto the regular completion $\bar{X}$ of X, such that the inverse image of $S = \bar{X} - X$ is contained in $T = \bar{Z} - Z$ and that S contains the image of the singular set of T. By adding some points of X to S, and their inverse images to T, if needed, we may further arrange that any component of T which maps onto X is an unramified covering outside its intersection with the singular set of T (cf also [De:7.9.2]). We now change the notation slightly, write again π for π', let $\bar{\pi}$ be its extension to $\bar{Z}$ and $i : Z \to \bar{Z}$, $j : X \to \bar{X}$ the inclusions. By construction $\bar{\pi}$ is a submersion outside $\bar{\pi}^{-1}(S)$.

We have to prove that $\pi_+ E$ is regular. By 11.6, it suffices to show that $j_+ \pi_+ E$ is a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},S}$-submodules. But $j_+ \pi_+ E = \bar{\pi}_+ i_+ E$ and similarly, $i_+ E$ is a union of $\mathcal{O}_{\bar{Z}}$-coherent $\mathcal{D}_{\bar{Z},T}$- modules. The idea is then to define a notion of direct image from $\mathcal{D}_{\bar{Z},T}$-modules to $\mathcal{D}_{\bar{X},S}$-modules, which will preserve $\mathcal{O}$-coherence (because $\bar{\pi}$ is proper) and give the same underlying $\mathcal{D}_{\bar{X},S}$-modules as $\bar{\pi}_+$ when applied to $i_+ E$. We explain this for right modules, since this simplifies slightly the formalism (as in §5) .

At each point $z \in \bar{Z}$ the differential $d\bar{\pi}$ maps $T(Z)_z$ into $T(X)_x$, where $x = \bar{\pi}(z)$. From this we get a canonical morphism of $\mathcal{O}_{\bar{Z}}$-modules

$$\text{(1)} \qquad \nu : \Theta_{\bar{Z},T} \to \pi^* \Theta_{\bar{X},S}.$$

(and also of $\Theta_{\bar{Z}}$ into $\pi^*\Theta_{\bar{X}}$, but this is not needed).
We claim that it is part of an *exact* sequence:

$$\text{(2)} \qquad 0 \to \Theta^{\rho}_{\bar{Z}/\bar{X}} \to \Theta_{\bar{Z},T} \xrightarrow{\nu} \pi^* \Theta_{\bar{X},S} \to 0,$$

where $\Theta^{\rho}_{\bar{Z}/\bar{X}}$ is the module of germs of vector fields tangent to the fibres of π . That the latter is $\ker\nu$ is obvious. The main point is to see that ν is surjective. This means that locally, one can find liftings of elements in $\Theta_{\bar{X},S}$ which are contained in $\Theta_{\bar{Z},T}$. This is a local assertion on $\bar{Z}$. Let $z \in \bar{Z}$. If $z \in \bar{\pi}^{-1}(X-S)$, then $d\bar{\pi}$ is submersive at z and our assertion is clear. Assume that $\bar{\pi}(z) \in S$. We can choose a local coordinate x on $\bar{X}$ with origin $\pi(z)$ and local coordinates u,v around z, with origin z, such that $x \circ \bar{\pi} = a(u,v)u^m v^n$, where a(u,v) is regular and invertible around z and $m > 0$, $n > 0$. Around $\pi(z)$, the $\mathcal{O}_{\bar{X}}$-module $\Theta_{\bar{X},S}$ is spanned by $x.\partial_x$. It suffices to "lift" this vector field in $\Theta_{\bar{Z},T}$, i.e. to find $\xi \in \Theta_{\bar{Z},T}$, defined around z, such that

$\xi(x\circ\pi) = x \circ \pi$. Note that, around z, the $\mathcal{O}_Z$-module $\Theta_{\bar{Z},T}$ contains $u.\partial_u$ [and is spanned by it and $v.\partial_v$ if $n > 0$, ∂_v if $n = 0$]. It is then readily seen that

(3) $$\xi = (m+a^{-1}.u.\partial_u a)^{-1}.u.\partial_u$$

satisfies our condition.

If M is a left $\mathcal{D}_{\bar{X},S}$-module, then, in analogy with 4.6, we define on $\bar{\pi}^*(M)$ a structure of $\mathcal{D}_{\bar{Z},T}$-module by setting

(4) $$\xi(f\otimes m) = \xi f \otimes m + \Sigma_i f.u_i \otimes \pi^{-1}(\xi_i m) \quad (\xi\in\Theta_{\bar{Z},T}),$$

where

(5) $$\nu(\xi) = \Sigma_i u_i \otimes \pi^{-1}\xi_i \quad (u_i\in\mathcal{O}_{\bar{Z}};\xi_i \in \Theta_{\bar{X},S}).$$

It follows from 4.6 that on Z, this is indeed the $\mathcal{D}_Z$-module structure of $\pi^o(M)$ defined in 4.1. We let $\mathcal{D}_{\bar{Z},T\to\bar{X},S}$ be $\pi^*(\mathcal{D}_{\bar{X},S})$ endowed with this left $\mathcal{D}_{\bar{Z},T}$-module structure, and the natural right $\pi^{-1}\mathcal{D}_{\bar{X},S}$-module structure, stemming from right multiplication. We have then

(6) $$\mathcal{D}_{\bar{Z},T\to\bar{X},S}|Z = \mathcal{D}_{Z\to X}\,.$$

We now define the relative direct image $\bar{\pi}_+^{\phi}N$ of a right $\mathcal{D}_{\bar{Z},T}$-module N by

(7) $$\bar{\pi}_+^{\phi}N = R\bar{\pi}_*(N \overset{L}{\underset{\mathcal{D}_{\bar{Z},T}}{\otimes}} \mathcal{D}_{\bar{Z},T\to\bar{X},S}).$$

The $R\bar{\pi}_*$ refers here to a morphism of ringed spaces $(\bar{Z},\pi^{-1}\mathcal{D}_{\bar{X},S}) \to (\bar{X},\mathcal{D}_{\bar{X},S})$

From (6), we see that

(8) $$\bar{\pi}_+^{\phi}|Z = \pi_+.$$

Since i,j are affine, we get from (8) and the definitions

(9) $$\bar{\pi}_+^{\phi} \circ i_+ = j_+ \circ (\bar{\pi}_+^{\phi}|Z) = j_+\circ \pi_+\,.$$

In order to prove that $j_+(\pi_+E)$ is a union of $\mathcal{O}_{\bar{X}}$-coherent $\mathcal{D}_{\bar{X},S}$-submodules it is then sufficient to show that $\bar{\pi}_+^{\phi}$ preserves $\mathcal{O}$-coherence. The module $\mathcal{D}_{\bar{Z},T\to\bar{X},S}$ admits a locally free left resolution

(10) $$0 \to \Theta^{\rho}_{\bar{Z}/\bar{X}} \underset{\mathbb{C}}{\otimes} \mathcal{D}_{\bar{Z},T} \to \mathcal{D}_{\bar{Z},T} \to \mathcal{D}_{\bar{Z},T\to\bar{X},S} \to 0$$

obtained from (2). Therefore

(11) $$N \overset{L}{\underset{\mathcal{D}_{\bar{Z},T}}{\otimes}} \mathcal{D}_{\bar{Z},T\to\bar{X},S} = \{N \underset{\mathbb{C}}{\otimes} \Theta^{\rho}_{\bar{Z}/\bar{X}} \to N\}$$

where the differential on the right hand side is the obvious one:

$$u\otimes\xi \mapsto v.\xi \qquad (u\in N;\xi\in\Theta^{\rho}_{\bar{Z}/\bar{X}})$$

it consists of $\mathcal{O}_{\bar{Z}}$-coherent modules. Since $\bar{\pi}$ is proper, the direct image of this complex is $\mathcal{O}_{\bar{X}}$-coherent.

12.5.2 Let now π be the projection $\mathbb{A}^1 \times X \to X$ for a general X. Let $\psi : \pi^{-1}C = \mathbb{A}^1 \times C \to \mathbb{A}^1 \times X$ be the inclusion, and π' the restriction of π to $\pi^{-1}C$. The curve C being locally closed in X, we have by base change (8.6) :

$$\phi^! \circ \pi_+ = \pi'_+ \circ \psi^!$$

Since $\psi^!F^{\cdot}$ is r.h., $\pi'_+.\psi^!(F^{\cdot})$ is r.h. by 12.5.1, hence $\phi^!.\pi_+(F^{\cdot})$ is r.h..

12.5.3 *The general case.* It suffices to prove our assertion for standard modules. Using composition, we are reduced to the case where Y is affine, irreducible and $F^{\cdot} = E$ is an irreducible regular connection on Y. Let $j : Y\to \mathbb{A}^N$ be a closed embedding. We can factor π as

$$Y \xrightarrow{\alpha} Y\times X \xrightarrow{\beta} \mathbb{A}^N \times X \xrightarrow{\gamma} X$$

where α is the isomorphism of Y onto the graph of π, $\beta = j \times Id$. and

γ is the second projection. By 12.3, α and β preserve r.h.; but γ is a composition of projections with fibre $\mathbb{A}^1$, hence preserves r.h. by 12.5.2.

12.6. DEFINITION Let $F^\cdot \in D^b(\mathcal{D}_X)$ and assume that $H^\cdot F^\cdot$ has finite length. Then the contents $\mathrm{cont}(F^\cdot)$ of $F^\cdot$ is the set of irreducible subquotients of $H^\cdot F^\cdot$.

If $F^\cdot, G^\cdot, H^\cdot$ are the vertices of an exact triangle and have all cohomology of finite length, then, obviously,

$$\mathrm{cont}(F^\cdot) \subset \mathrm{cont}(G^\cdot) \cup \mathrm{cont}(H^\cdot) .$$

12.7 THEOREM *Let* $F^\cdot \in D^b_{rh}(\mathcal{D}_X)$. *Then*

(i) $\mathbb{D}_X F^\cdot$ *is r.h..*

(ii) $\mathrm{cont}\ F^\cdot$ *consists of r.h. modules.*

Proof of (i). We shall use induction on the dimension of the support S of $F^\cdot$. Fix $k \geq 0$. We assume (i) proved whenever $\dim S < k$ and consider now complexes with support of dimension k. It suffices to prove (i) for a generating class, hence we may assume that F is a standard module. Thus there exists an irreducible smooth affine subvariety Y of dimension k and a regular irreducible connection on Y such that $F = i_+(E)$, where $i : Y \to X$ is the inclusion. By Hironaka, there exists a regular completion $\alpha : Y \to \tilde{Y}$ and a proper map $\beta : \tilde{Y} \to X$ such that $i = \beta\circ\alpha$.

We have to prove that $\mathbb{D}_X i_+ E$ is r.h.. We have $i_+E = \beta_+ \circ \alpha_+(E)$. Since β is proper, $\beta_+ \circ \mathbb{D}_{\tilde{Y}} = \mathbb{D}_X \circ \beta_+$ (9.6), therefore

$$\mathbb{D}_X i_+ E = \beta_+ \mathbb{D}_{\tilde{Y}} \alpha_+ E.$$

Since β_+ preserves r.h. (12.2), it suffices to prove that $\mathbb{D}_{\tilde{Y}}\alpha_+ E$ is r.h.

By 10.5, $\alpha_+ E$ has a unique irreducible submodule $G = \alpha_{!+}E$, the image of the natural morphism $\alpha_! E \to \alpha_+ E$. It follows from 11.8 that it is r.h.. The quotient $H = \alpha_+ E/G$ is then also r.h. However its support is contained in $\bar{Y} - Y$, hence has dimension $< k$. By induction $\mathbb{D}_Y H$ is r.h.. The exact sequence

(1) $$0 \to G \to \alpha_+ E \to H \to 0$$

gives by duality an exact sequence

(2) $$0 \to \mathbb{D}_{\tilde{Y}} H \to \mathbb{D}_{\tilde{Y}} \alpha_+ E \to \mathbb{D}_{\tilde{Y}} G \to 0.$$

By definition, G is a quotient of $\alpha_! E = (\mathbb{D}_{\tilde{Y}} . \alpha_+ . \mathbb{D}_Y)(E)$. By duality, $\mathbb{D}_{\tilde{Y}} G$ is contained in $\alpha_+ \mathbb{D}_Y E$. But $\mathbb{D}_Y E = E^*$ is a regular connection hence $\mathbb{D}_{\tilde{Y}} G$ is again r.h. by 11.8. Since $\mathbb{D}_{\tilde{Y}} H$ is already known to be r.h., (2) shows that $\mathbb{D}_{\tilde{Y}} \alpha_+ E$ is r.h..

Proof of (ii). The proof also proceeds by induction on the dimension of the support S of $F^\cdot$. Fix $k \geq 0$ and assume (ii) proved if $\dim S < k$. The remark following the definition of $\mathrm{cont}\ F^\cdot$ (12.6) shows that it suffices to prove our assertion for elements in a generating class, hence it is enough to consider the case of a standard module $F = i_+(E)$, with Y, i and S as before. By 12.2 and (i) we now know that $i_! E$ is also r.h.. Our next goal is to prove that

(3) $$G = i_{!+} E = \mathrm{Im}(i_! E \xrightarrow{u} i_+ E)$$

is also r.h.. We consider the cone $A^\cdot$ over u. It is r.h. since $i_! E$ and $i_+ E$ are r.h., and its support has dimension $< k$. Therefore $\mathrm{cont}\ A^\cdot$ is r.h. by induction assumption. The long exact sequence

(4) $$0 \to H^{-1}A^\cdot \to i_! E \to i_+ E \to H^0 A^\cdot \to 0$$

associated to the exact triangle $i_! E \to i_+ E \to A^\cdot \to$ gives

(5) $$H^{-1}A^\cdot = \ker u \qquad H^0 A^\cdot = \mathrm{coker}\ u.$$

Thus cont (coker u) is r.h.. The exact sequence

(6) $$0 \to G \to i_+ E \to \mathrm{coker}\ u \to 0$$

then shows that G is r.h.. Moreover, since G is irreducible, (10.5) we have

(7) $$\mathrm{cont}\ i_+E \subset \mathrm{cont}(\mathrm{coker}\ u) \cup \{G\}$$

therefore cont i_+E consists or r.h. modules, too.

12.8 COROLLARY *Let $F^\cdot \in D^b(\mathcal{D}_X)$.*

(i) *If $F^\cdot$ is r.h., then all subquotients of $H^\cdot F^\cdot$ are r.h..*

(ii) *Let* (E_r) be a *convergent spectral sequence abutting to* $\mathrm{Gr} H^\cdot F^\cdot$ *for a bounded filtration. If E_s is r.h. for some s, then $H^\cdot F^\cdot$ is r.h..*

(iii) *$F^\cdot$ is r.h. if and only if $H^\cdot F^\cdot$ is r.h..*

Proof. (i) 12.7(ii) implies, by induction on the length, that every submodule of $H^\cdot F^\cdot$ is r.h., hence every subquotient is r.h. too.

(ii) By (i) E_r is r.h. for every $r \geq s$. Hence Gr $H^\cdot F^\cdot$ is r.h.. Then all its subquotients are r.h., hence so is $H^\cdot F^\cdot$.

(iii) If $F^\cdot$ is r.h., then $H^\cdot F^\cdot$ is so by (i). Assume now $H^\cdot F^\cdot$ to be r.h.. Then all its subquotients are r.h. by (i). That $F^\cdot$ is r.h. can now be deduced by induction on the cohomological length (cf I, 12.5) of $F^\cdot$, using (I, 12.6).

We end up this section with a digression, to prove a result which is basic for the applications of $\mathcal{D}$-modules to representations of reductive groups, due to Beilinson-Bernstein.

12.9 PROPOSITION *Assume that π is smooth and surjective. Let $F^\cdot \in D^b_{coh}(\mathcal{D}_X)$ be such that $\pi^! F^\cdot$ is r.h.. Then $F^\cdot$ is r.h..*

Proof. The assumption implies that $\pi^! F^\cdot$ is holonomic. Since $\pi^!$ preserves the holonomy defect (4.8), $F^\cdot$ is also holonomic. There remains to check 11.3(ii) and we may restrict ourselves to embedded curves (11.4(g)). Then $\pi^{-1}C$ is smooth. Let $\alpha : \pi^{-1}C \to Y$ be the inclusion. The complex $\alpha^! \pi^! F^\cdot$ is r.h. (11.4(c)). We have $\pi^! \phi^! F^\cdot = \alpha^! \pi^! F^\cdot$ and we must show that $\phi^! F^\cdot$ is r.h.. This reduces us to the case where X is a curve, and it is enough to check r.h.. outside a given finite

set of points of X. We can always find an embedded smooth curve $\beta : D \to Y$ such that $\pi|\beta(D) = \pi'$ is dominant. Leaving out a few points, we may assume $\pi' = \pi|D$ to be smooth and surjective. We know that $\beta^! \pi^! F^\cdot$ is r.h.. But $\pi' = \pi \circ \beta$. Hence $\pi'^! F^\cdot$ is r.h. and $F^\cdot$ is r.h. by 11.2.

12.10 Let G be a complex linear algebraic group. We assume that X is an algebraic G-space, i.e. there is an action of G on X defined by means of a morphism $\nu : G \times X \to X$ satisfying the usual properties:

$$\nu(g,h,x) = \nu(g,\nu(h,x)) \quad \nu(1,x) = x \quad (g,h\in G, x\in X).$$

As is customary, we write g.x for $\nu(g,x)$. Let ϕ_g $(g\in G)$ be the automorphism $x \mapsto g.x$ of X. The group G operates via the ϕ's on $\mu(\mathcal{D}_X)$, $\mathrm{coh}(\mathcal{D}_X)$ and on the various categories or derived categories of $\mathcal{D}_X$-modules we have considered. The transform g^*F of an element F in one of those categories can be viewed as the inverse image $\phi_g^!$. Let $i_g : X \to G \times X$ be the closed embedding $x \mapsto (g.x)$ and $p : G \times X \to X$ be the projection. Then

(1) $$\nu \circ i_g = \phi_g \ , \ p \circ i_g = \mathrm{Id}.,$$

Hence

(2) $$g^* = i_g^! . \nu^!(F) \quad i_g^! \circ p^! = \mathrm{Id}.$$

The $\mathcal{D}_X$-module F is said to be G-*equivariant* if $\nu^! F$ is isomorphic to $p^! F$. If so, then (2) shows that g^*F is isomorphic to F for all $g \in G$.

We let $D^b_G(\mathcal{D}_X)$ (resp. $D^b_{c,G}(\mathcal{D}_X)$) be the bounded derived category of G-equivariant bounded (resp. and coherent) complexes of $\mathcal{D}_X$-modules.

We shall not develop all the generalities about G-equivariante. In particular we take for granted that if Y is also an algebraic G-space and π is G-equivariant, then $\pi^!$ maps $D^b_G(\mathcal{D}_X)$ into $D^b_G(\mathcal{D}_Y)$.

12.11 THEOREM *We keep the notation and assumption of 12.10 and assume moreover that X is the union of finitely many orbits of G. Then every $F^\cdot \in D^b_{c,G}(\mathcal{D}_X)$ is r.h..*

Proof. The connected component of the identity G^o of G has finite index in G and a G-equivariant $\mathcal{D}_X$-module is of course G^o-equivariant. Our assumptions are therefore also satisfied by G^o and we may assume G to be connected. The orbits are then irreducible smooth varieties.

(a) Assume first that $X = G$, with G operating on itself by left translations. We claim that $\mathcal{H}^\cdot\mathcal{F}^\cdot$ is a direct sum of a finite number of copies of $\mathcal{O}_G$. To see this, consider the diagram

$$G \xrightarrow{u} G \times G \overset{\nu}{\underset{p}{\rightrightarrows}} G$$

where $u(g) = (g^{-1}, g)$ and ν is now the product in G. Then $\nu \circ u$ is the projection on 1 and $p \circ u$ is the identity. Therefore $\mathcal{F}^\cdot = u^! \circ p^! \mathcal{F}^\cdot$ while $\mathcal{H}^\cdot u^!.\nu^!(\mathcal{F}^\cdot) = (u^!.\nu^!)\mathcal{H}^\cdot\mathcal{F}^\cdot$ is the free $\mathcal{O}_G$ module with fibre $(\mathcal{H}^\cdot\mathcal{F}^\cdot)_1$. Since $p^!\mathcal{F}^\cdot = \nu^!\mathcal{F}^\cdot$ by assumption, our assertion follows. In view of 12.8, this shows that $\mathcal{F}^\cdot$ is r.h..

(b) Assume next that X is homogeneous. Fix $x \in X$. The orbit map $h_x : g \mapsto g.x$ is a smooth surjective morphism, therefore $h_x^!.\mathcal{F}^\cdot$ is coherent (4.8). Being also G-equivariant, it is r.h. by (a). Then $\mathcal{F}^\cdot$ is r.h. by 12.9.

(c) In the general case we shall argue by an induction.

By standard facts about algebraic transformation spaces, each orbit Z is locally closed and $\bar{Z}$ is the union of Z and of orbits of strictly smaller dimension. For $i \in \mathbb{N}$, let X_i be the union of orbits of dimension $\geqq i$. It is then open in X. We have then inclusions

$$X = X_0 \supset X_1 \supset \dots \supset X_d \supset \emptyset\ , \text{ where } d = d_X\ ,$$

and $Z_k = X_k - X_{k+1}$ is the disjoint union of the (finitely many) orbits of dimension k. It is therefore a closed smooth subvariety of X_k. We now prove by descending induction on k that the restriction $\mathcal{F}_k^\cdot$ of $\mathcal{F}^\cdot$ to X_k is r.h.. Let first $k = d$. Then X_d is open in X, and is the union of finitely many open orbits. The restriction to any of them is first of all coherent, and then r.h. by (b).

Fix $k < d$ and assume that $\mathcal{F}_{k+1}$ is r.h.. Let $j : Z_k \to X_k$ and $i : X_{k+1} \to X_k$ be the inclusions. Since Z_k is closed, smooth, we have the exact triangle

(1) $$j_+j^!\mathcal{F}_k^\cdot \to \mathcal{F}_k^\cdot \to i_+\mathcal{F}_{k+1}^\cdot \to j_+j^!\mathcal{F}_k^\cdot[1]$$

(8.3). $\mathcal{F}_k^\cdot$ is coherent. By the induction assumption and 12.2, $i_+(\mathcal{F}_{k+1}^\cdot)$ is r.h. and in particular is coherent. From (1), we see that $j_+j^!\mathcal{F}_k^\cdot$ is coherent. Then $j^!\mathcal{F}_k^\cdot$ is also coherent (7.13). By (b) again $j^!\mathcal{F}_k$ is r.h., therefore $j_+j^!\mathcal{F}_k^\cdot$ is r.h. (12.3). We now see from (1) and 11.4(d) that $\mathcal{F}_k^\cdot$ is r.h..

VIII. THE RIEMANN-HILBERT CORRESPONDENCE

A. BOREL

§ 13 THE DE RHAM FUNCTOR

13.1 Let $F \in \mu(\mathcal{D}_X)$. We consider the complex $\Omega_X^{\cdot}(F)^a$ of analytic objects associated to the de Rham complex $\Omega_X^{\cdot}(F)$ (see 1.8), i.e.

$$(1) \qquad F^a \to (\Omega_X^1 \underset{\mathcal{O}_X}{\otimes} F)^a \to (\Omega_X^2 \underset{\mathcal{O}_X}{\otimes} F)^a \to \dots \to (\Omega_X^{d_X} \underset{\mathcal{O}_X}{\otimes} F)^a$$

or, equivalently

$$(2) \qquad F^a \to \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} F^a \to \Omega_X^2 \underset{\mathcal{O}_X}{\otimes} F^a \to \dots \to \Omega_X^{d_X} \underset{\mathcal{O}_X}{\otimes} F^a,$$

with a differential given locally as in 1.8. If $F = \mathcal{O}_X$, we write $\Omega_X^{\cdot a}$. The complex $\Omega_X^{\cdot}(\mathcal{D}_X)^a$ is a complex of right $\mathcal{D}_{X^a}$-modules. It follows by similar computations to those of 3.5 or by using the flatness of $\mathcal{O}_{X^a}$ over $\mathcal{O}_X$, that it is a locally free left resolution of $\omega_X^a[-d_X] = \omega_{X^a}[-d_X]$, therefore

$$\Omega_X^{\cdot}(F)^a[d_X] = \Omega_X^{\cdot}(\mathcal{D}_X)^a \underset{\mathcal{D}_X^a}{\otimes} F^a[d_X] = \omega_X^a \overset{L}{\underset{\mathcal{D}_X^a}{\otimes}} F^a.$$

Similarly for $F^{\cdot} \in D^b(\mathcal{D}_X)$

$$(3) \qquad \Omega_X^{\cdot}(F^{\cdot})^a[d_X] = \Omega_X^{\cdot}(\mathcal{D}_X)^a \underset{\mathcal{D}_{X^a}}{\otimes} F^{\cdot a}[d_X] = \omega_X^a \overset{L}{\underset{\mathcal{D}_{X^a}}{\otimes}} F^{\cdot a}.$$

It is the right hand side of (3) which we take as "de Rham functor" to go from $D^b(\mathcal{D}_X)$ to the bounded derived category $D^b(X^a)$ of sheaves on X^a. By definition then:

(4) $$\mathrm{DR}_X F^\cdot = \mathrm{DR}(F^\cdot) = \omega_X^a \overset{L}{\underset{\mathcal{D}_{X^a}}{\otimes}} F^\cdot a = \Omega_X^\cdot(F^\cdot)^a[d_X].$$

This notation is different from the one of IV, or of 1.7, where DR or DR_X denote the de Rham complex itself, without shift in degree. To avoid any confusion, we shall only use $\Omega_X^\cdot(F^\cdot)^a$ or $\Omega_X^{\cdot a} \underset{\mathcal{O}_{X^a}}{\otimes} F^{\cdot a}$ for the latter.

Let $\{x_i\}$ be local coordinates around a point of X^a and $\{\partial_i\}$ the associated derivatives. Around x, $\Omega_X^\cdot(F) = \oplus_{i_1<\ldots<i_k} dx_{i_1}\ldots dx_{i_k}.F$ $(1\leq i_1,\ldots,i_k\leq n=d_X)$, so that $\mathrm{DR}(F)$ is the Koszul complex $K^\cdot(\{\partial_i\},F)$ with respect to the operators ∂_i.

13.2 In the analytic topology, we dispose of the holomorphic Poincaré lemma. Therefore, if $F \in \mu(\mathcal{D}_X)$ is $\mathcal{O}_X$-coherent, hence $\mathcal{O}_X$-locally free (1.6), then

$$H^0(\Omega_X^\cdot(F)^a) = \tilde{F}^a, \quad H^i(\Omega_X^\cdot(F)^a) = 0 \quad (i\neq 0)$$

where $\tilde{F}^a$ is the (locally constant) sheaf of horizontal sections of F^a. In other words, $\Omega_X^\cdot(F)^a$ is a locally free bounded right resolution of $\tilde{F}^a$. In particular, $\Omega_X^\cdot(\mathcal{O}_X)^a$ is such a resolution of $\mathbf{C}_{X^a}$.

13.3 PROPOSITION. *We have an isomorphism.*

(1) $$\mathrm{DR}(\mathbb{D}_X F^\cdot \underset{\mathcal{O}_X}{\otimes} G^\cdot) = R\mathcal{H}om_{\mathcal{D}_{X^a}}(F^{\cdot a}, G^{\cdot a})[d_X], \quad (F^\cdot \in D^b_{coh}(\mathcal{D}_X), G^\cdot \in D^b(\mathcal{D}_X)).$$

Proof. 9.7 implies a similar isomorphism for the corresponding analytic objects and then the left-hand side is the one of (1) by definition of the de Rham functor.

13.4 *The solution complex.* In the case where $G^\cdot$ reduces to $\mathcal{O}_X$ in degree 0, 13.3 yields

(1) $$\mathrm{DR}(\mathbb{D}_X F^\cdot) = R\mathcal{H}om^\cdot_{\mathcal{D}_{X^a}}(F^{\cdot a}, \mathcal{O}_{X^a})[d_X], \quad (F^\cdot \in D^b_{coh}(\mathcal{D}_X)).$$

The right hand side, without the shift $[d_X]$, is, by definition, the solution complex Sol $F^{\cdot a}$ of $F^{\cdot a}$. To see the origin of this terminology, let us go back to the situation of 1.5 and assume that $F^\cdot = M$ is the

$\mathcal{D}_X$-module associated to the system of linear equations 1.5(1). Let $(u_1,\ldots,u_q)$ be free generators of $\mathcal{D}_X^q$. Any assignment $u \mapsto f_i$ $(f_i \in \mathcal{O}_X)$ defines a homomorphism of $\mathcal{D}_X^q$ into $\mathcal{O}_X$. This homomorphism factors through M if and only if the f_i's are solutions of 1.5.1. Going over to the analytic topology, we see that $\mathcal{H}om_{\mathcal{D}_{X^a}}(M^a, \mathcal{O}_{X^a})$ is the *sheaf of germs of solutions of the system* 1.5(1). Thus Sol $F^\cdot$ is the derived category counterpart of the sheaf of germs of solutions, (in the analytic topology).

§ 14 STATEMENT OF THE MAIN RESULTS. SOME CONSEQUENCES.

14.1 The space X^a is a complex d_X-dimensional manifold, hence also a $2d_X$-dimensional real manifold. The dualizing sheaf for sheaves of complex vector spaces is therefore the constant sheaf $\mathbf{C}_{X^a}$ in degree $-2d_X$. We shall use the notation $\mathbf{D}_{Ve,M}$ or simply $\mathbf{D}_{Ve}$ for Verdier duality on the space M:

(1) $$\mathbf{D}_{Ve,X^a}(S^\cdot) = R\mathcal{H}om^\cdot(S^\cdot,\mathbf{C}_{X^a})[2d_X]. \qquad (S^\cdot \in D^b(X^a))$$

(cf. [IC: V, §7]). For direct and inverse images in sheaf theory, we follow the usual notation, as in [IC].

14.2 A stratification $\mathfrak{X}$ of X is a finite partition $X = \cup_i X_i$ of X into locally closed smooth algebraic subvarieties, called the *strata*, such that the closure of any stratum is a union of strata. A complex of sheaves $S^\cdot \in D^b(X^a)$ is said to be *constructible* if there exists a stratification $\mathfrak{X}$ such that $H^\cdot S^\cdot$ is locally constant (in the analytic topology) on each stratum, with finite dimensional stalks. If this needs to be specified, we shall say that $S^\cdot$ is $\mathfrak{X}$-constructible.

This was called $\mathfrak{X}$-cohomological constructibility ($\mathfrak{X}$-cc) in [IC: V], except for the fact that the requirements on $\mathfrak{X}$ were more stringent: it was supposed to satisfy certain local triviality conditions [IC: I, 1.1]. However, by [V], any stratification in the present sense admits a refinement which is a Whitney stratification, hence fulfills these local triviality conditions. As a consequence, a constructible $S^\cdot$ is also constructible in the sense of [IC: V] and we may use the results proved there for cohomologically constructible complexes of sheaves.

14.3 As usual, $\boxtimes$ will refer to the exterior tensor product. When it occurs, it is understood that we deal with a product, say $X = U \times V$ and we let, for $F^\cdot \in D^b(\mathcal{D}_U)$, $G^\cdot \in D^b(\mathcal{D}_V)$, $F^\cdot \boxtimes G^\cdot$ be the quasi-coherent sheaf which, on a product $A \times B$ of open affine subsets of U and V respectively, is the quasi-coherent complex of sheaves over $\mathcal{O}_X(A\times B) = \mathcal{O}_U(A) \otimes_{\mathbf{C}} \mathcal{O}_V(B)$ associated to $F^\cdot(A) \otimes_{\mathbf{C}} G^\cdot(B)$. In particular, 4.5(5) can be written as $\mathcal{D}_X = \mathcal{D}_U \boxtimes \mathcal{D}_V$, hence $F^\cdot \boxtimes G^\cdot$ is canonically a $\mathcal{D}_X$-module.

Similarly, in sheaf theory over X^a, we shall consider the exterior tensor product

(2) $$S^\cdot \boxed{X}^a T^\cdot = p^{a*}S^\cdot \otimes_{\mathbf{C}} q^{a*}T^\cdot \qquad (S^\cdot \in D^b(U^a), T^\cdot \in D^b(V^a)).$$

It follows immediately from the definitions that if $Y = U' \times V'$ is also a product and π is a product of morphisms $\mu : U' \to U$ and $\nu : V' \to V$, then $\pi^!(E \boxed{X} F) = \mu^! E \boxed{X} \nu^! F$.

This operation preserves coherence, holonomicity and regular holonomicity. This is clear for coherence. The cotangent bundle $T^*(X)$ is the exterior direct sum of $T^*(U)$ and $T^*(V)$ and it follows from the definitions that $SS(E \boxed{X} F)$ is the direct product of $SS(E)$ and $SS(F)$ $(E \in \mathrm{coh}(\mathcal{D}_U), F \in \mathrm{coh}(\mathcal{D}_V))$, whence the preservation of holonomicity (more generally, the holonomicity defects add up). Let now $\bar{U}$ and $\bar{V}$ be smooth completions of U and V. Then $\bar{U} \times \bar{V}$ is a smooth completion of X. If E and F are regular connections, then 11.6, 11.7 show that $E \boxed{X} F$ is regular. From this the preservation of r.h. follows if U and V have dimension one. Let now F, G be r.h. and $\varphi : C \to X$ a smooth embedded smooth irreducible curve. If $\varphi(C)$ is contained in a fibre of one of the projections (up to finitely many points), then it is obvious that $\varphi^!(E \boxed{X} F)$ is r.h.. If not, taking out finitely many points, we may assume that $C' = p(\varphi(C))$ and $C'' = q(\varphi(C))$ are smooth curves. Let $\mu : C' \to U$, $\nu : C'' \to V$ and $\eta : \varphi(C) \to C' \times C''$ be the inclusions. Then $\varphi^!(E \boxed{X} F) = \eta^!(\mu\times\nu)^!(E \boxed{X} F) = \eta^!(\mu^! E \boxed{X} \nu^! F)$. It follows from the above (and 11.4(c)) that $\varphi^!(E \boxed{X} F)$ is r.h..

14.4 THEOREM (i) *The de Rham functor* DR *establishes an equivalence of categories between* $D^b_{rh}(\mathcal{D}_X)$ *and* $D^b_{cs}(X^a)$ *which commutes with direct images, inverse images,* $\boxed{X}$ *and duality.*

(ii) DR *defines an equivalence between the categories of regular holonomic modules and of perverse sheaves.*

The definition of perverse sheaves will be recalled in §22.

14.5 The proof is divided into a number of steps, some of which are valid more generally for holonomic complexes. More precisely, we shall prove the following assertions, where the argument belongs to $D^b_{rh}(\mathcal{D}_X)$, unless otherwise stated.

(1) $$\mathrm{DR} \circ \pi_+ = \pi^a_* \circ \mathrm{DR}$$

(2) $$\mathrm{DR}(F^\cdot \boxtimes G^\cdot) = \mathrm{DR}\, F^\cdot \boxtimes^a \mathrm{DR}\, G^\cdot, \quad (F^\cdot \text{ holonomic, } G^\cdot \text{ coherent}).$$

(3) $$\mathrm{DR}(F^\cdot) \in D^b_{cs}(X^a) \text{ if } F^\cdot \text{ is holonomic.}$$

(4) $$\mathrm{DR} \circ \pi^! = \pi^{a!} \circ \mathrm{DR}.$$

(5) $$\mathrm{DR} \circ \mathbb{D}_X = \mathbb{D}_{Ve,X^a} \circ \mathrm{DR} \text{ on } D^b_h(\mathcal{D}_X) .$$

(6) DR is fully faithful.

(7) DR is essentially surjective.

(8) $H^\cdot F^\cdot$ is concentrated in degree 0 if and only if DR $F^\cdot$ is perverse.

14.6 *Remark.* (a) In view of (5), (1) and (4) also imply similar commutations for π^+ and π^{a*} (resp. $\pi_!$ and $(\pi^a)_!$).

(b) Since (3) is valid for holonomic complexes, it follows from these results that given $F^\cdot$ holonomic, there exists a unique regular holonomic $G^\cdot$ such that DR $F^\cdot$ = DR $G^\cdot$, hence also such that Sol $F^\cdot$ = Sol $G^\cdot$.

(c) The functor $\mathbb{D}_X$ being a antiequivalence of $D^b_{rh}(\mathcal{D}_X)$ with itself, we could also use the functor $\mathrm{Sol}^\cdot = \mathrm{DR} \circ \mathbb{D}_X$ (cf. 13.4) to establish the Riemann-Hilbert correspondence. 14.4 shows that it defines an antiequivalence between $D^b_{rh}(\mathcal{D}_X)$ and $D^b_{cs}(X^a)$.

(d) For the reader who wants to compare this version of the Riemann-Hilbert correspondence and the holomorphic one, we should point out three differences in degree conventions which may occur:

In the holomorphic case, it is either Sol or the de Rham complex, i.e. our DR$[-d_X]$, which is used. Second, there is no shift by $d_{Y,X}$ in the definition of the derived inverse image (4.1). Third, the dualizing sheaf on the topological side is sometimes taken to be $\mathbb{C}_{X^a}$ in degree 0, rather than in degree $-2d_X$, as is done here. These differences cancel out so that the statement of 14.4 in the holomorphic case is formally the same.

(e) In view of 13.4, the relation 14.5(5) is obviously equivalent to

(1) $$\mathbb{D}_{Ve}\mathrm{DR}\, F^\cdot[-d_X] = \mathrm{Sol}\, F^\cdot \qquad (F^\cdot \in D^b_h(\mathcal{D}_X)).$$

In other words, the de Rham complex $\Omega^\cdot_{X^a}(F^\cdot)$ and Sol $F^\cdot$ are Verdier duals of one another.

14.7 In the analytic case, there are several equivalent definitions of the notion of regular holonomic system (see e.g. [0]). We shall show that the one used here implies one of these other definitions. Fix $x \in X$. As usual, let $\hat{\mathcal{O}}_{X^a,x}$ be the completion of $\mathcal{O}_{X^a,x}$ with respect to the maximal ideal of x. In analogy with 13.4, $\mathrm{Hom}_{\mathcal{D}_{X^a,x}}(F^a_x, \hat{\mathcal{O}}_{X^a,x})$ may be viewed as the space of germs of formal solutions of a system of equations giving rise to F_x and $\mathrm{RHom}^\cdot_{\mathcal{D}_{X^a,x}}(F^a_x, \hat{\mathcal{O}}_{X^a,x})$ as the formal solution complex at x. The inclusion of $\mathcal{O}_{X^a,x}$ into $\hat{\mathcal{O}}_{X^a,x}$ induces a natural morphism

(1) $$\nu : R\mathcal{H}om_{\mathcal{D}_{X^a}}(F^{\cdot a}, \mathcal{O}_{X^a})_x \to \mathrm{RHom}_{\mathcal{D}_{X^a,x}}(F^{\cdot a}_x, \hat{\mathcal{O}}_{X^a,x}) \qquad (x \in X).$$

In the one-dimensional case, it was shown by Malgrange that the Fuchs condition (cf III,1.1.1) is equivalent to :: ν is an isomorphism. This was extended to regular connections and then taken as one of the possible definitions of regular holonomicity. We show here that it is implied by regular holonomicity in the sense of § 13.

14.8 PROPOSITION *Assume that* $F^\cdot \in D^b_{rh}(\mathcal{D}_X)$. *Then the morphism of* 14.7(1) *is an isomorphism for all* $x \in X$.

Proof. By 13.3(1), the left hand side is $\mathrm{DR}\,\mathbb{D}_X F^\cdot[d_X]$. Let $x \in X$. We have

(1) $$R\mathcal{H}om_{\mathcal{D}_{X^a}}(F^{\cdot a}, \mathcal{O}_{X^a})_x = i^{a*}_x(\mathrm{DR}\,\mathbb{D}_X F^\cdot[d_X]).$$

By 14.6(a) (it is here, and only here, that we use r.h.) and 9.13, we have

(2) $$i_x^{a*} \mathrm{DR}\, \mathbb{D}_X F^\cdot[d_X] = \mathrm{DR}\, i_x^! \mathbb{D}_X F^\cdot[d_X] = \mathrm{DR}\, \mathbb{D}_{\{x\}} i_x^! F^\cdot[d_X].$$

Over a point, DR is the identity, and $\mathbb{D}_{\{x\}}$ is just the usual duality of vector spaces. Therefore

(3) $$R\mathcal{H}om^\cdot_{\mathcal{D}_{X^a}}(F^{\cdot a}, \mathcal{O}_{X^a}) = \mathrm{Hom}^\cdot(i_x^! F^\cdot[d_X], \mathbb{C}).$$

We have then to show that

(4) $$\mathrm{Hom}(i_x^! F^\cdot[d_X], \mathbb{C}) = R\mathrm{Hom}_{\mathcal{D}_{X^a,x}}(F_x^{\cdot a}, \hat{\mathcal{O}}_{X^a,x}).$$

The complex $F^\cdot$ admits a locally free resolution. It suffices therefore to prove:

(5) $$\mathrm{Hom}(i_x^! F[d_X], \mathbb{C}) = \mathrm{Hom}_{\mathcal{D}_{X^a,x}}(F_x^a, \hat{\mathcal{O}}_{X^a,x}) \quad \text{for} \quad F = \mathcal{D}_X .$$

In this case, the right hand side is equal to $\hat{\mathcal{O}}_{X^a,x}$. Since $\mathcal{D}_X$ is locally free over $\mathcal{O}_X$ it follows from 7.4 that

(6) $$H^j i_x F = 0 \ (j \neq d_X) \text{ and } H^j i_x F = F/\mathfrak{m}_x F \quad \text{if } j = d_X,$$

where $\mathfrak{m}_x$ is the maximal ideal of x. Therefore the left hand side of (5) is equal to $\mathrm{Hom}(\mathcal{D}_X/\mathfrak{m}_x\mathcal{D}_X, \mathbb{C})$. Let (x_i) be local analytic coordinates on X^a around x and $\partial^i = \partial/\partial x_i$. Then $\mathcal{D}_X/\mathfrak{m}_x\mathcal{D}_X$ may be viewed as the space of polynomials in the ∂^i's, with constant coefficients. Its dual is then canonically isomorphic to the space of formal power series in these variables, i.e. to $\hat{\mathcal{O}}_{X^a,x}$.

14.9 The R.H. correspondence was established first in the holomorphic setting by Kashiwara [K 2] and Z. Mebkhout [M1, M2, M3] using the functor Sol. Following this, Beilinson and Bernstein developed a theory of algebraic $\mathcal{D}$-modules, including a version of the R.H. correspondence, which is at the basis of these Notes, as already mentioned in the introduction. A number of steps in the proof of 14.4 are analogous to the corresponding ones in the holomorphic case, often simpler because of lesser analytical difficulties. Specific to the algebraic case however is the method of §§19, 20, which relies on the fact that an increasing sequence of Zariski open subsets is stationary.

In a more recent paper, Kashiwara has introduced an inverse functor to DR (or Sol), but this requires more analysis [K3]. In the earlier treatments, there is already a substitute for an inverse functor: it goes from constructible complexes to holonomic complexes of possible infinite order, rather than to r.h. ones, so that there is in fact an equivalence between three categories.

Z. Mebkhout has pointed out to me that his forthcoming presentation of the holomorphic theory [M4] will also include a proof of the R.H. correspondence in the algebraic case.

§ 15 PROOF OF 14.5(1) : $DR \circ \pi_+ = \pi^a_* \circ DR$.

15.1 We have to use here the analogue in the analytic context of the direct image, which we shall denote π^a_+. However, this is mainly an intermediary and we need not study its properties, which are not as simple as in the algebraic case. First of all, in the same way as in 5.1, we define a $\pi^{a,-1}\mathcal{D}_{X^a} \times \mathcal{D}_{Y^a}$ module $\mathcal{D}_{X^a \leftarrow Y^a}$ by

(1) $$\mathcal{D}_{X^a \leftarrow Y^a} = \omega_{Y^a} \underset{\mathcal{O}_{Y^a}}{\otimes} \pi^{a,-1}\mathcal{D}_{X^a} \underset{\pi^{a,-1}\mathcal{O}_{X^a}}{\otimes} \omega^{-1}_{X^a} .$$

We have therefore

(2) $$\mathcal{D}_{X^a \leftarrow Y^a} = (\mathcal{D}_{X \leftarrow Y})^a .$$

The direct image is then defined by

(3) $$\pi^a_+ F^\cdot = R\tilde{\pi}^a(\mathcal{D}_{X^a \leftarrow Y^a} \overset{L}{\underset{\mathcal{D}_{Y^a}}{\otimes}} F^\cdot) \quad (F^\cdot \in D^b(\mathcal{D}_{Y^a})).$$

Here $\tilde{\pi}^a$ refers to the morphism $(Y^a,(\pi^a)^{-1}\mathcal{D}_{X^a}) \to (X^a,\mathcal{D}_{X^a})$.

We need the :

LEMMA *There is a natural functor* $\mu : (\pi_+)^a \to \pi^a_+$ *, which is an isomorphism if* π *is proper.*

In fact, in view of (2), we have

(4) $$(\mathcal{D}_{X \leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} F^\cdot)^a = \mathcal{D}_{X^a \leftarrow Y^a} \overset{L}{\underset{\mathcal{D}_{Y^a}}{\otimes}} F^{\cdot a}, \quad (F^\cdot \in D^b(\mathcal{D}_Y)).$$

Since the Zariski topology is coarser than the analytic one, there is further a natural map $R\tilde{\pi} \to R\tilde{\pi}^a$, whence the morphism $\mu : (\pi_+ F^\cdot)^a \to \pi^a_+(F^{\cdot a})$. The complex $\mathcal{D}_{X \leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} F^\cdot$ consists of $\mathcal{O}_Y$-quasi-coherent modules, since $\mathcal{D}_{X \leftarrow Y}$ is so. Let now π be proper. Then $(R\tilde{\pi})^a \to R\tilde{\pi}^a$ is an isomorphism on quasi-coherent complexes by GAGA; therefore μ is an isomorphism.

15.2 PROPOSITION *There exists a natural morphism of functors* $DR \circ \pi_+ \to \pi^a_* \circ DR$ *from* $D^b(\mathcal{D}_Y)$ *to* $D^b(X^a)$, *which is an isomorphism if* π *is proper.*

In view of 15.1, it is enough to show the existence of a functorial map

(1) $$\mu : DR\ \pi^a_+ F^{\cdot a} \to \pi^a_* DR\ F^\cdot \quad (F^\cdot \in D^b(\mathcal{D}_Y)) ,$$

which is an isomorphism if π is proper. We have

(2) $$DR\ \pi^a_+ (F^{\cdot a}) = \omega_{X^a} \overset{L}{\underset{\mathcal{D}_{X^a}}{\otimes}} R\tilde{\pi}^a(\mathcal{D}_{X^a \leftarrow Y^a} \overset{L}{\underset{\mathcal{D}_{Y^a}}{\otimes}} F^{\cdot a}).$$

Though the projection formula is not true in general, there is still always a functorial map ν of the right hand side of (2) to

(3) $$R\tilde{\pi}^a(\pi^a)^{-1}(\omega_{X^a}) \overset{L}{\underset{(\pi^a)^{-1}\mathcal{D}_{X^a}}{\otimes}} \mathcal{D}_{X^a \leftarrow Y^a} \overset{L}{\underset{\mathcal{D}_{Y^a}}{\otimes}} F^{\cdot a}) ,$$

which is an isomorphism if π is proper. By 15.1(1) $\mathcal{D}_{X^a \leftarrow Y^a}$ is locally free over $\pi^{a,-1}\mathcal{D}_{X^a}$, hence the first L can be erased, and we see moreover that (3) reduces to

(4) $$R\tilde{\pi}^a(\omega_{Y^a} \overset{L}{\underset{\mathcal{D}_{Y^a}}{\otimes}} F^{\cdot a}) = R\tilde{\pi}^a(DR\ F^\cdot) .$$

Since $R\tilde{\pi}^a$ is the sheaf theoretic derived direct image for the underlying topological sheaves, this is equal to $\pi^a_*\ DR(F^\cdot)$

15.3 *Proof of* 14.5(1). We consider first two special cases.

(a) π is the inclusion of Y into a smooth completion X and F is a regular connection on Y. The assertion to prove can also be written:

(1) $$\Omega^\cdot_{X^a}(\pi_+ F) = \pi^a_*(\Omega^\cdot_{Y^a}(F)).$$

(cf 13.1(4)). The module $(\pi_+F)^a$ is an extension of F^a which is regular, meromorphic along $X^a - Y^a$, in the terminology of IV. On the other hand

$$(2)\qquad \Omega_X^{\cdot}(\pi_+F)^a\big|_{Y^a} = \Omega_Y^{\cdot}(F)^a,$$

so that the right-hand side of (1) can be written

$$\pi_*^a(\pi^{a*}(\Omega_X^{\cdot}(\pi_+F))^a).$$

Our assertion now reduces to the Grothendieck-Deligne theorem (IV,6.2).

(b) F is a regular connection. By Hironaka desingularisation and Nagata [Na] we can find a smooth completion $j : Y \to Z$ of Y endowed with a proper map $p : Z \to X$ such that $\pi = p \circ j$. We have $DR\ j_+F = j_*^a\ DR\ F$ by (a). Together with 15.2, this yields

$$DR\ \pi_+F = DR\ p_+ \cdot j_+F = p_*^a \cdot j_*^a\ DR\ F = \pi_*^a\ DR\ F.$$

(c) The general case of 14.5(1) is now an easy consequence. In view of the existence of a morphism of the left hand side into the right hand side (15.2), it suffices to prove our assertion on a generating class for $D^b_{rh}(\mathcal{D}_Y)$. We take of course the standard modules (12.1) and are reduced to the case $F = i_+(E)$, where $i : Z \to Y$ is the inclusion of a locally closed smooth irreducible affine subvariety and E is a regular connection on Z. Using (b) and 6.4, we get

$$DR\ \pi_+F = DR\ \pi_+ \cdot i_+E = DR\ (\pi o i)_+E = (\pi o i)_*^a\ DR\ E.$$

We have then, applying (b) again

$$DR\ \pi_+F = \pi_*^a \circ i_*^a\ DR\ E = \pi_*^a\ DR\ i_+E = \pi_*^a\ DR\ F.$$

§ 16 PROOF OF $DR\ F^{\cdot} \boxed{x}^a DR\ G^{\cdot} = DR(F^{\cdot} \boxed{x} G^{\cdot})$ FOR $F^{\cdot}$ AND $G^{\cdot}$ R.H.

16.1 We assume here that $X = M \times N$ is a product and let p,q be the two projections. The natural map

$$(1)\qquad \mu : \Omega_M^{\cdot a} \boxed{x}^a \Omega_N^{\cdot a} \to \Omega_X^{\cdot a}$$

induces a morphism

$$(2)\qquad \tilde{\mu} : \Omega_M^{\cdot}(F^{\cdot})^a \boxed{x}^a \Omega_V^{\cdot}(G^{\cdot})^a \to \Omega_X^{\cdot}(F^{\cdot} \boxed{x} G^{\cdot})^a$$

and we want to prove that it is an isomorphism. Since $d_X = d_U + d_V$ this will imply 14.5(2) under the present assumptions.

16.2 In this section, we consider the case where $F^{\cdot} = E$ is a $\mathcal{O}_M$-coherent $\mathcal{D}_M$-module and $G^{\cdot} \in D^b_{coh}(\mathcal{D}_N)$.

Since we have a global morphism, it suffices to check our assertion for the stalks. Given $x = b \times c$ ($b \in M, c \in N$) in X, we may always replace M and N by arbitrary small open neighborhoods of b and c, in the analytic topology, and a fortiori in the Zariski topology. Since E is locally free over $\mathcal{O}_M$ (1.6), we may then assume it to be free, which reduces us to the case where $E = \mathcal{O}_N$. On the other hand, we may replace $G^{\cdot}$ by a bounded locally projective resolution, which is locally free in a given range of degrees (VI, 2.6). This reduces us to the case where $G^{\cdot} = \mathcal{D}_N$.

Fix $x \in M^a$ (resp. $y \in N^a$) and a sufficiently small neighborhood U of x (resp. V of y). Let $\{x_i\}$, $i=1,\dots,d_M=m$ (resp. $\{y_j\}$, $j=1,\dots,d_N=n$) be local coordinates on U (resp. V) and $\{\partial_i\}$ (resp. $\{\partial_j'\}$) the associated derivatives. From 13.1, we have

$$DR_X(F \boxed{x} G) = K^{\cdot}(\{\partial_i\},\{\partial_j'\};\ (F \boxed{x} G)^a).$$

This can be computed in two stages:

$$DR_X(F^{\cdot} \boxed{x} G) = K^{\cdot}(\{\partial_i\}; K^{\cdot}(\{\partial_j'\}; (E \boxed{x} G)^a)).$$

This is true without special assumption on F and G. But we are now assuming that F is locally free over $\mathcal{O}_M$ and that $G = \mathcal{D}_N$. Thus, on V, $G = \mathbf{C}[\partial_1',\dots,\partial_n'] \otimes_{\mathbf{C}} \mathcal{O}_N^a$. This implies that $K^{\cdot}(\{\partial_j'\}; (F \boxed{x} G)^a)$

has zero cohomology except in degree 0, where it is equal to $(F \boxtimes \mathcal{O}_N)^a$. Assuming, as we may, that $F = \mathcal{O}_M$, we have $(F \boxtimes \mathcal{O}_N)^a = \mathcal{O}_X^a$ and $DR_X(E \boxtimes G) = K^\cdot(\{\partial_i\}; \mathcal{O}_X^a)$.

By the holomorphic Poincaré lemma on A, this complex has zero homology except in degree $-m$, where it reduces to the subsheaf of elements in $\mathcal{O}_X^a$ which are annihilated by the ∂_i's, i.e. to $\mathbf{C}_U{}^a \boxtimes^a \mathcal{O}_V^a$. But we saw in 13.2 that $DR_U(\mathcal{O}_U) = \mathbf{C}_U[d_U]$. Altogether we get

$$DR_X(F \boxtimes G) = \mathbf{C}_M[d_M] \boxtimes^a \mathcal{O}_N^a = DR_M F \boxtimes^a DR_N G,$$

as claimed. This proves that $\tilde{\mu}$ is an isomorphism when F is $\mathcal{O}_M$-coherent and $G \in D^b_{coh}(\mathcal{D}_X)$.

16.3 Fix $G^\cdot \in D^b_{rh}(\mathcal{D}_N)$. We have two functors $A : F^\cdot \to DR\, F^\cdot \boxtimes^a DR\, G^\cdot$ and $B : F^\cdot \to DR(F^\cdot \boxtimes G^\cdot)$ from $D^b(\mathcal{D}_X)$ to $D^b(X^a)$, and $\tilde{\mu}$ defines a morphism of functors $A \to B$. If $F^\cdot = E$ is a connection, then μ is an isomorphism of $A(F^\cdot)$ onto $B(F^\cdot)$. Let now $F = i_+(E)$ be a standard module, where E is a regular connection on a locally closed smooth affine subvariety Z. Using 14.5(1), we get

$$DR\, i_+ E \boxtimes^a DR\, G^\cdot = i_*^a(DR\, E) \boxtimes^a DR\, G^\cdot = (i^a \times Id)_*(DR\, E \boxtimes^a DR\, G^\cdot)$$

$$= DR(i \times Id)_+(E \boxtimes G^\cdot) = DR\,(F^\cdot \boxtimes G^\cdot).$$

Since the standard modules form a generating class for $D^b_{rh}(\mathcal{D}_M)$, it follows that $\tilde{\mu}$ is an isomorphism of $A(F^\cdot)$ onto $B(F^\cdot)$ for $F^\cdot \in D^b_{rh}(\mathcal{D}_M)$ and $G^\cdot \in D^b_{rh}(\mathcal{D}_N)$.

§ 17 PROOF OF DR $F^\cdot \in D^b_{cs}(X^a)$ IF $F^\cdot$ IS REGULAR HOLONOMIC.

It suffices to prove it for standard modules. Let then $i : Z \to X$ be a locally closed embedding of a smooth subvariety, E a regular connection on Z and $F = i_+(E)$. We have $DR\, i_+(E) = i_*^a(DR\, E)$ by 14.5(1). On the other hand $DR\, E$ is a local system in degree d_Y (13.2), hence is constructible. The map i can be viewed as a stratified algebraic map, since there always exists a stratification of X by locally closed algebraic sets of which Z is one stratum. Then i_*^a preserves constructibility (see e.g. [IC:V,10.16]).

§ 18 PROOF OF $DR \circ \pi^! = \pi^{a!} \circ DR$ FOR $F^\cdot$ REGULAR HOLONOMIC

18.1 Let π be an open embedding. Then $\pi^!$ and $\pi^{a!}$ are just the restriction to Y and it is obvious that DR commutes with restriction.

18.2 Let $Y = Z \times X$ and π be the second projection. The relation 4.5(8) can be written.

(1) $$\pi^! F^\cdot = \mathcal{O}_Z \boxtimes F^\cdot[d_Z].$$

By 14.5(2) :

(2) $$DR\ \pi^! F^\cdot = DR\ \mathcal{O}_Z \boxtimes^a DR\ F^\cdot[d_Z],$$

But $DR\ \mathcal{O}_Z[d_Z]$ is a resolution of $\mathbb{C}_{Z^a}$ (13.2), therefore

(3) $$DR\ \pi^! F^\cdot = \mathbb{C}_{Z^a} \boxtimes^a DR\ F^\cdot[2d_Z].$$

If $S^\cdot$ is a complex of sheaves on X^a, then

$$\mathbb{C}_Z \boxtimes^a S^\cdot = \pi^{a*}(S^\cdot),$$

therefore we have

$$DR\ \pi^! F^\cdot = \pi^{a*}(DR\ F^\cdot)[2d_Z].$$

Our assertion in the present case now follows from the relation

(4) $$\pi^{a!} = \pi^{a*}[2d_Z]$$

on $D^b_{cs}(X^a)$ (see 18.5).

18.3 There remains to consider the case of a closed embedding. We first construct in this section a natural morphism of functors

(1) $$\mu : DR \circ \pi_+ \circ \pi^! \to \pi^a_* \circ \pi^{a!} \circ DR.$$

In view of 7.13 and of its topological counterpart [IC: V, 1.8], this can also be written, in the notation used there

$$DR(R\Gamma_{[Y]}(F^\cdot)) \to R\gamma_Y DR(F^\cdot).$$

The natural morphism $R\Gamma_{[Y]}F^\cdot \to F^\cdot$ (8.3) gives rise to

$$\nu : DR(R\Gamma_{[Y]}F^\cdot) \to DR\ F^\cdot$$

But $DR(R\Gamma_{[Y]}F^\cdot)$ is necessarily concentrated on Y. Therefore, ν factors through $R\gamma_Y DR\ F^\cdot$, providing μ.

18.4 Let π be a closed embedding and j the inclusion of X − Y into X. We have an exact triangle

(1) $$\pi_+\pi^! F^\cdot \to F^\cdot \to j_+ j^! F^\cdot \to \dots$$

(8.3). By applying DR, we get the exact triangle

(2) $$DR\ \pi_+\pi^! F^\cdot \to DR\ F^\cdot \to DR\ j_+ j^! F^\cdot \to \dots$$

On the topological side, there is the exact triangle

(3) $$\pi^a_*\pi^{a!} DR\ F^\cdot \to DR\ F^\cdot \to j^a_* j^{a*} DR\ F^\cdot \to \dots$$

(see [IC:V,5.14]). By 14.5(1) and 18.1 there is a natural isomorphism

(4) $$\nu : DR\ j_+ . j^! F^\cdot \to j^a_* j^{a*} DR\ F^\cdot.$$

We leave it to the reader to check that μ, the identity on $DR\ F^\cdot$ and ν define a morphism of (2) to (3). Since it is an isomorphism on two vertices, it is also one for the third one, hence

$$\mu : DR\ \pi_+\pi^! F^\cdot \to \pi^a_*\pi^{a!} DR\ F^\cdot$$

is an isomorphism. By 14.5(1), $DR \circ \pi_+ = \pi^a_* \circ DR$. Since π^a_* is an extension by zero, the equality

$$DR\ \pi^! F^\cdot = \pi a^! DR\ F^\cdot$$

follows.

18.5 We still have to prove 18.2(4). This is a topological assertion, which fits into the framework of [IC:V]. Let $C = A \times B$ be a product of two locally compact spaces satisfying the general assumptions of [IC:V] and p,q the two projections. Assume that A is a manifold We claim

(1) $$\mathbb{D}_{Ve,C} \circ q^* = q^* \circ \mathbb{D}_{Ve,B}[\dim A] \quad \text{on } D^b(B).$$

We let $\mathcal{D}_{V,Z}$ be the dualizing sheaf on a space Z. By definition

(2) $$\mathbb{D}_{Ve,C}\, q^* S^\cdot = RHom^\cdot(q^* S^\cdot, \mathcal{D}_{V,C}) \quad (S^\cdot \in D^b(B)).$$

By [IC:V,10.26] :

(3) $$\mathcal{D}_{V,C} = p^* \mathcal{D}_{V,A} \otimes q^* \mathcal{D}_{V,B}.$$

Since A is a manifold, $\mathcal{D}_{V,A} = \mathbb{C}_A[\dim A]$, therefore

(4) $$\mathcal{D}_{V,C} = p^* \mathbb{C}_A \otimes q^* \mathcal{D}_{V,B}[\dim A] = q^*(\mathcal{D}_{V,B})[\dim A].$$

Using 10.21 in *loc.cit.* we get then

(5) $$\mathcal{D}_{Ve,C}\, q^* S^\cdot = RHom^\cdot(q^* S^\cdot, q^* \mathcal{D}_{V,B}) = q^* RHom^\cdot(S^\cdot, \mathcal{D}_{V,B})[\dim B] =$$

$$= q^* \mathcal{D}_{Ve,B}(S^\cdot)[\dim A].$$

On $D^b_{cs}(C)$ we have Verdier biduality [IC:V,§8]. Besides

$$q^! = \mathbb{D}_{Ve,C} \circ q^* \circ \mathbb{D}_{Ve,B}$$

[IC:V,10.11]. The relation 18.2(4) now follows from (5).

§ 19 PROOF OF 14.5(5) : $DR \circ \mathbb{D}_X = \mathbb{D}_{Ve} \circ DR$ ON $D^b_h(\mathcal{D}_X)$

19.1 PROPOSITION *There exists a natural morphism* $\mu : DR \circ \mathbb{D}_X \to \mathbb{D}_{Ve} \circ DR$ *, which is an isomorphism on complexes with* $\mathcal{O}_X$ *- coherent cohomology.*

Proof. Let $\mathcal{O}_{X^a} \to J^\cdot$ be an injective resolution of $\mathcal{O}_{X^a}$ in $D^b(\mathcal{D}_{X^a})$ and $\mathbb{C}_X[2d_X] \to K^\cdot$ an injective resolution in $D^b(X^a)$ of the dualizing sheaf. By 13.4(1), we may write

(1) $$DR\, \mathbb{D}_X F^\cdot = Hom^\cdot_{\mathcal{D}_{X^a}}(F^\cdot a, J^\cdot)[d_X] \qquad (F^\cdot \in D^b_{coh}(\mathcal{D}_X)).$$

Let α be the obvious map

(2) $$\alpha : Hom_{\mathcal{D}_{X^a}}(F^\cdot a, J^\cdot) \underset{\mathbb{C}_{X^a}}{\otimes} \Omega^\cdot_{X^a} \underset{\mathcal{O}_{X^a}}{\otimes} F^\cdot a \to \Omega^\cdot_{X^a} \underset{\mathcal{O}_{X^a}}{\otimes} J^\cdot$$

defined stalkwise, using the fact that

(3) $$Hom_{\mathcal{D}_{X^a}}(F^\cdot a, J^\cdot)_x = Hom_{\mathcal{D}_{X^a,x}}((F^\cdot a)_x, J^\cdot_x) ,$$

since $F^\cdot$ may be assumed to be coherent. We have, using 13.1, 13.2

(4) $$DR(J^\cdot) = DR(\mathcal{O}_X) = \mathbb{C}_{X^a}[d_X].$$

Combining α and (4), we get a morphism

$$DR\, \mathbb{D}_X F^\cdot[-d_X] \underset{\mathbb{C}}{\otimes} DR\ (F^\cdot)[-d_X] \to DR(J^\cdot)[-d_X] = \mathbb{C}_{X^a} .$$

Composed with the resolution $\mathbb{C}_{X^a} \to K^\cdot[-2d_X]$, this yields a morphism

$$DR \circ \mathbb{D}_X F^\cdot[-d_X] \to Hom(DR\ F^\cdot[-d_X], K^\cdot[-2d_X]).$$

The right hand side being equal to $Hom(DR\ F^\cdot, K^\cdot)[-d_X]$, we now have a morphism

$$DR \circ \mathbb{D}_X F^\cdot \to Hom(DR\ F^\cdot, K^\cdot).$$

Since the right hand side is equal to $\mathbb{D}_{Ve}\mathrm{DR}\ \mathcal{F}^{\bullet}$ by definition, this provides μ.

Assume now that $\mathcal{F}$ is a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. We have to prove that μ is an isomorphism. This is a local statement. The module $\mathcal{F}$ being locally $\mathcal{O}_X$-free (1.6), we may assume $\mathcal{F} = \mathcal{O}_X$. Then $\mathbb{D}_X\mathcal{F} = \mathcal{O}_X$ and by 13.1, 13.2

$$\mathrm{DR}\ \mathcal{F} = \mathbb{C}_X a[d_X].$$

On the other hand $\mathcal{H}om(\mathrm{DR}\ \mathcal{F}, K^{\bullet}) = \mathcal{H}om(\mathbb{C}_X a[d_X], K^{\bullet}) = K^{\bullet}[-d_X] = \mathbb{C}_X a[d_X]$.

Assume now that $\mathcal{F}^{\bullet}$ has coherent cohomology. The sheaf complexes $\mathrm{DR}.\mathbb{D}_X\mathcal{F}^{\bullet}$ and $\mathbb{D}_{Ve}.\mathrm{DR}\ \mathcal{F}^{\bullet}$ are the respective abutments of spectral sequences in which the E_2 terms are $\mathrm{DR}\ \mathbb{D}_X\ \mathcal{H}^{\bullet}\mathcal{F}^{\bullet}$ and $\mathbb{D}_{Ve}\mathrm{DR}\ \mathcal{H}^{\bullet}\mathcal{F}^{\bullet}$ respectively. By what has been proved, μ induces an isomorphism of one onto the other, hence μ is an isomorphism also in this case.

19.2 We now assume $\mathcal{F}^{\bullet}$ to be holonomic, let $\mathrm{Err}(\mathcal{F}^{\bullet})$ or $\mathrm{Err}_X\mathcal{F}^{\bullet}$ be the cone on $\mu : \mathrm{DR}.\mathbb{D}_X\mathcal{F}^{\bullet} \to \mathbb{D}_{Ve}\mathrm{DR}\ \mathcal{F}^{\bullet}$. We have to prove that $\mathrm{Err}(\mathcal{F}^{\bullet}) = 0$. This functor obviously commutes with the restriction to an open subset. By 10.4, there exists an open dense set $U \subset X$ on which $\mathcal{H}^{\bullet}\mathcal{F}^{\bullet}$ is $\mathcal{O}_X$-coherent. By 19.1, $\mathrm{Err}\ \mathcal{F}^{\bullet} = 0$ on U. Since an increasing sequence of dense Zariski open subsets of an algebraic variety is stationary, it suffices to prove the following statement:

(*) *Let* U *be a Zariski open dense subset of* X *on which* $\mathrm{Err}\ \mathcal{F}^{\bullet} = 0$. *If* $U \neq X$, *there exists a smooth non-empty open subvariety* W *of* $X - U$ *on which* $\mathrm{Err}\ \mathcal{F}^{\bullet} = 0$.

19.3 LEMMA (i) *If* π *is proper, then* $\pi_*^a \circ \mathrm{Err}_Y = \mathrm{Err}_X \circ \pi_+$.

(ii) *If* π *is a locally closed embedding, then* $\mathrm{Err}_X\pi_+\mathcal{F}^{\bullet}|Y = \mathrm{Err}_Y\mathcal{F}^{\bullet}$ $(\mathcal{F}^{\bullet} \in D_h^b(\mathcal{D}_Y))$.

(iii) *If* π *is an étale covering, then* $\pi^{a*} \circ \mathrm{Err}_X = \mathrm{Err}_Y \circ \pi^!$.

Proof. (i) By 9.6 and 15.2, we have

(1) $$\mathrm{DR}.\mathbb{D}_X(\pi_+\mathcal{F}^{\bullet}) = \mathrm{DR}.\pi_+(\mathbb{D}_Y\mathcal{F}^{\bullet}) = \pi_*^a\mathrm{DR}\ (\mathbb{D}_Y\mathcal{F}^{\bullet}),$$

and also, since Verdier duality commutes with proper maps [IC:V,10.11]:

(2) $$\mathbb{D}_{Ve,X}.\mathrm{DR}.\pi_+\mathcal{F}^{\bullet} = \mathbb{D}_{Ve,X}.\pi_*^a.\mathrm{DR}\ \mathcal{F}^{\bullet} = \pi_*^a\mathbb{D}_{Ve,Y}\mathrm{DR}\ \mathcal{F}^{\bullet}$$

and (i) follows.

(ii) Let $Z = \bar{Y} - Y$, and $X' = X - Z$. Then $\pi = \beta \circ \alpha$, where $\alpha : Y \to X'$ is a closed embedding and $\beta : X' \to X$ an open one. Let $\mathcal{F}^{\bullet} \in D_{coh}^b(\mathcal{D}_Y)$.
We have first

(3) $$\mathrm{Err}_{X'}\alpha_+\mathcal{F}^{\bullet}|Y = \mathrm{Err}_Y\mathcal{F}^{\bullet}.$$

In fact, $\mathrm{Err}_{X'}\alpha_+\mathcal{F}^{\bullet} = \alpha_*^a\mathrm{Err}_Y\mathcal{F}^{\bullet}$ by (i); since α_* is the extension by zero, the restriction of the right hand term to Y is indeed $\mathrm{Err}_Y\mathcal{F}^{\bullet}$. Moreover Err commutes with the restriction to an open set, therefore

(4) $$\mathrm{Err}_X\pi_+\mathcal{F}^{\bullet}|_Y = (\mathrm{Err}_X\beta_+(\alpha_+\mathcal{F}^{\bullet})|X')|_Y = (\mathrm{Err}_{X'}\alpha_+\mathcal{F}^{\bullet})|_Y = \mathrm{Err}_Y\mathcal{F}^{\bullet}.$$

(iii) is obvious: If π is étale, then $\pi^!$ commutes with DR and duality, and π^{a*} commutes with Verdier duality.

19.4 LEMMA *Let* U *be as in* 19.2(*) *and* W *be an open non-empty subvariety in* $X - U$. *Assume there exists a proper morphism* $p : X \to W$ *which is the identity on* W. *Then* $\mathrm{Err}_X\mathcal{F}^{\bullet} = 0$ *on some Zariski open dense subset of* W.

Proof. By 19.3(i) we have

(1) $$p_*^a\mathrm{Err}_X\mathcal{F}^{\bullet} = \mathrm{Err}_W p_+\mathcal{F}^{\bullet}$$

Since $p_+\mathcal{F}^{\bullet}$ is holonomic (10.1), we see as in 19.2 that there exists a Zariski dense open subset W' of W on which $\mathrm{Err}_W p_+\mathcal{F}^{\bullet} = 0$. Replacing W by W' and X by $\pi^{-1}W'$, we may assume that $p_*^a\mathrm{Err}_X\mathcal{F}^{\bullet} = 0$, (using 19.1(i)).

Let $Z = X - U$ and $j : Z \to X$ be the inclusion. By assumption, $\mathrm{Err}_X\mathcal{F}^{\bullet}$ is supported by Z, therefore

(2) $$j^{a!}\mathrm{Err}_X\mathcal{F}^{\bullet} = \mathrm{Err}_X\mathcal{F}^{\bullet}|Z = j^{a*}\mathrm{Err}_X\mathcal{F}^{\bullet}.$$

(3) $$j_{a*}j^{a!}\mathrm{Err}_X F^\cdot = \mathrm{Err}_X F^\cdot .$$

The composition p o j is the restriction of p to Z. We now get

(4) $$p_*^a(\mathrm{Err}_X F^\cdot|Z) = 0.$$

Let k be the embedding of W into Z. Then p o k is the identity. Therefore $\mathrm{Err}_X F^\cdot|W$ is a direct summand of $p_*^a(\mathrm{Err}\ F^\cdot|Z)$, and is 0 by (4). (Note that W is open and closed in Z.)

19.5 The rest of the proof is basically a geometric construction which shows how one can reduce the proof of 19.2(*) to the case just treated.

We first choose an open affine subvariety V of X such that $V \cap W$ is open dense in W and closed in V. Shrinking V further if necessary, we may assume there are on V "local coordinates" x_i $(1 \leq i \leq n = d_X)$ such that W is defined by $x_n = \dots = x_{n-k} = 0$ (k=dim W). These coordinates define an étale map $\alpha : V \to \mathbb{A}^n$ which sends W into the coordinate plane $\mathbb{A}^k$ of the first k coordinates. We now restrict F to V, which we call again X. It is then still true that Err $F^\cdot$ is zero on a dense open subset, and that W is open in its complement. Composing α with the projection of A^n onto A^k, we have now a morphism $\phi : X \to \mathbb{A}^k$, whose restriction to W is étale. Let $\psi : X \to P$ be a locally closed embedding of X into some projective space. Let X' be the image of X into $P \times \mathbb{A}^k$ under the morphism $\eta : x \to (\phi(x),\psi(x))$; let W' be the image of W and $F'^\cdot = \eta_+ F^\cdot$. The complex $\eta_+ F^\cdot$ is holonomic (10.1), zero on the complement $C(\bar{X}')$ of $\bar{X}'$, therefore Err $F^\cdot$ is zero on the open set $C(\bar{X}') \cup \eta(U)$, and $W' = \eta(W)$ is open in the complement of the latter. η is the composition of an isomorphism and of a locally closed embedding, therefore Err $F' = \eta_*^a$ Err $F^\cdot$ by 19.3(ii). The map η is an isomorphism of X onto X' and of W onto W', therefore η_*^a defines an isomorphism of Err $F^\cdot|W$ onto Err $F'^\cdot|W'$. It suffices to show that Err $F'^\cdot$ is zero on some dense open subset of W'. Changing the notation again, we are reduced to proving 19.2(*) in the case where $X = \mathbb{A}^k \times P$ and where the restriction μ of the first projection is étale on W. Consider now the diagram

$$\begin{array}{ccc} X' = W \times P & \xrightarrow{\mu \times \mathrm{Id.}} & \mathbb{A}^k \times P = X \\ \downarrow q & & \downarrow \\ W & \xrightarrow{\mu} & \mathbb{A}^k \end{array}$$

where the vertical maps are the first projections. Let $W' = \{(w,w)\,|\,(w \in W)\} \subset X'$ and r be the restriction of q to W'. It is an isomorphism. We let $p = r^{-1} \circ q$. Then p is a proper morphism of X' on W' which is the identity on W'. The morphism $\nu = \mu \times \mathrm{Id.}$ is an étale covering. Let $G^\cdot = \nu^!(F^\cdot)$. By 19.3(iii), $\mathrm{Err}_{X'}G^\cdot = \nu^{a*}\mathrm{Err}_X F^\cdot$, hence $\mathrm{Err}_{X'}G^\cdot$ is zero on the open set $U' = \nu^{-1}(U)$. The morphism ν defines an isomorphism of W' onto W, therefore ν^{a*} yields an isomorphism of $\mathrm{Err}_X F^\cdot|W$ onto $\nu^{a*}(\mathrm{Err}_X F^\cdot)|W'$. But $\mathrm{Err}_{X'}G^\cdot = \nu^{a*}(\mathrm{Err}_X F^\cdot)$ by 19.3(iii). We are now reduced to the situation of 19.4, (with $X,U,W,F^\cdot$ replaced by X',U',W' and $G^\cdot$), hence 19.4 proves 19.2(*) and the theorem.

Remark. The argument used here is a variation on one of P. Deligne's in [De 1].

§ 20 PROOF OF 14.5(2), (3) FOR HOLONOMIC COMPLEXES

So far, we proved 14.5(2) and the constructibility of DR $\mathcal{F}^\cdot$ for $\mathcal{F}^\cdot$ regular holonomic. This is sufficient to establish the Riemann-Hilbert correspondence. However, the previous argument also allows one to prove them for $\mathcal{F}^\cdot$ holonomic. We now briefly indicate how.

20.1 The proof in § 19 shows in fact the following: Let for every smooth quasi-projective variety X be given a functor A_X from $D^b_h(\mathcal{D}_X)$ to $D^b(X^a)$ which satisfies the following properties:

(A) it commutes with the restriction to an open set. Given $\mathcal{F}^\cdot \in D^b_h(\mathcal{D}_X)$ there is a dense Zariski open subset V of X on which $A_X(\mathcal{F}^\cdot) = 0$.

(B) A_X satisfies the conditions of Lemma 19.3.

Then $A_X(\mathcal{F}^\cdot) = 0$ for every $\mathcal{F}^\cdot \in D^b_h(\mathcal{D}_X)$.

We want to apply this to 14.5(2). We fix V and $\mathcal{G}^\cdot \in D^b_{coh}(\mathcal{D}_V)$. For any U, we define a functor A_U which associates to $\mathcal{F}^\cdot \in D^b_h(\mathcal{D}_U)$ the cone $A_U(\mathcal{F}^\cdot)$ over DR $\mathcal{F}^\cdot \boxtimes$ DR $\mathcal{G}^\cdot \to$ DR $(\mathcal{F}^\cdot \boxtimes \mathcal{G}^\cdot)$. It obviously commutes with the restriction to an open subset. In 16.2, we proved that $A_U(\mathcal{F}^\cdot) = 0$ if $\mathcal{F}^\cdot$ is a $\mathcal{O}_U$-coherent connection. By the usual spectral sequence argument, this is then also true if $\mathcal{F}^\cdot$ has $\mathcal{O}_U$-coherent cohomology. Since $\mathcal{F}^\cdot$ is holonomic, $\mathcal{H}^\cdot\mathcal{F}^\cdot$ is $\mathcal{O}$-coherent on some dense open subset (10.4), therefore A_U satisfies (A). That it satisfies (B) is easily seen: it follows from the fact that DR commutes with proper mappings and the inverse image under an étale covering. This then yields 14.5(2) for $\mathcal{F}^\cdot$ holonomic.

20.2 Let us now replace (A) by :

(A') A_X commutes with the restriction to an open subset and, given $\mathcal{F}^\cdot \in D^b_h(\mathcal{D}_X)$, there exists a dense open subset U of X such that $A_X(\mathcal{F}^\cdot)|U$ belongs to $C^b_{cs}(U^a)$.

We claim that if (A') and (B) hold, then $A_X(\mathcal{F}^\cdot) \in D^b_{cs}(X^a)$ for $\mathcal{F}^\cdot$ holonomic.

In fact, the argument of § 19 yields a partition $X = \cup_{i\in I} X_i$ of X into locally closed smooth subvarieties such that $\mathcal{H}^\cdot$ DR $\mathcal{F}^\cdot$ is locally constant on each X_j^a. We have only to make sure that the X_i may be so chosen as to define a stratification of X (cf 14.2). But this is easy

to arrange: To see that note first that the condition of 14.2 could also be stated in the following way: If $X_i \cap \bar{X}_j \neq \emptyset$, then $X_i \subset \bar{X}_j$ $(i,j\in I)$. Now replace 19.2(*) by

(**) Let U be a Zariski open dense subset of X admitting a stratification $\mathfrak{U} = (U_i)_{i\in I}$ such that $A_U(\mathcal{F}^\cdot)$ is $\mathfrak{U}$ -constructible. Then there exists a smooth non-empty subvariety W of X - U on which $\mathcal{H}^\cdot A_X\mathcal{F}^\cdot$ is locally constant.

Again, this is what follows from (A') and (B), as in 19.4, 19.5 . Assume, as we may, that W is irreducible. Let $J \subset I$ be the set of indices for which $W \cap \bar{U}_i$ is non-empty and open in W. Let $W' = \cap_{i\in J}(W\cap\bar{U}_i)$. Then W' is still open, non empty, in X - U and $U' = U \cup W$ is open in X. Moreover, W'and $\mathfrak{U}$ define a stratification of U' with respect to which $A_X\mathcal{F}^\cdot|U'$ is constructible. This induction procedure yields therefore a stratification $\mathfrak{X}$ of X such that $A_X(\mathcal{F}^\cdot)$ is $\mathfrak{X}$ -constructible.

Take in particular for A the de Rham functor itself. If $\mathcal{F}^\cdot$ is a $\mathcal{O}_X$-coherent connection, then DR $\mathcal{F}^\cdot$ is a locally constant system (13.2) of finite rank, hence is constructible. Then again 10.4 shows that $\mathcal{H}^\cdot$ DR $\mathcal{F}^\cdot$ is locally constant on a dense open subset of X. Thus (A') holds. That (B) holds for DR has already been used repeatedly. Therefore DR $\mathcal{F}^\cdot$ is constructible if $\mathcal{F}^\cdot$ is holonomic.

§ 21 DR : $D^b_{rh}(\mathcal{D}_X) \to D^b_{cs}(X^a)$ IS AN EQUIVALENCE OF CATEGORIES

21.1 Let $F^\cdot, G^\cdot \in D^b_{coh}(\mathcal{D}_X)$. By 9.7, 9.8 we have

(1) $$R\mathcal{H}om^\cdot_{\mathcal{D}_X}(F^\cdot, G^\cdot) = \omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathbb{D}_X F^\cdot \overset{L}{\underset{\mathcal{D}_X}{\otimes}} G^\cdot[-d_X].$$

(2) $$\mathrm{RHom}^\cdot_{D^b(\mathcal{D}_X)}(F^\cdot, G^\cdot) = p_+(\mathbb{D}_X F^\cdot \overset{L}{\otimes} G^\cdot)[-d_X],$$

where p is the projection of X to a point. We want first to transform slightly the right hand side of these formulas. Let

(3) $$X \xrightarrow{\delta} X \times X \begin{array}{l} \xrightarrow{\pi_1} X \\ \xrightarrow{\pi_2} X \end{array}$$

where δ is the diagonal map and π_1, π_2 are the projections. We claim that we have

(4) $$F^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} G^\cdot = \delta^!(F^\cdot \boxtimes G^\cdot)[d_X].$$

Let first $F, G \in \mu(\mathcal{D}_X)$. By definition (see 4.1)

(5) $$\delta^o(F \boxed{\times} G) = \mathcal{O}_X \underset{\delta^{-1}\mathcal{O}_{X\times X}}{\otimes} \delta^{-1}(\pi_1^{-1}F \underset{\mathbf{C}}{\otimes} \pi_2^{-1}G).$$

It is easily seen that the map defined by

$$f \otimes \delta^{-1}(\pi_1^{-1}u \otimes \pi_2^{-1}v) \mapsto fu \otimes v \quad (u \in F, v \in G, f \in \mathcal{O}_X)$$

provides an isomorphism of $\mathcal{O}_X$-modules of the right hand side of (5) onto $F \underset{\mathcal{O}_X}{\otimes} G$, whence an isomorphism

$$\mu : \delta^o(F \boxtimes G) \tilde{\to} F \underset{\mathcal{O}_X}{\otimes} F;$$

at first, it is one of $\mathcal{O}_X$-modules, but it follows immediately from the definition of the $\mathcal{D}_X$-module structure on the inverse image (4.1(2)) that it is also one of $\mathcal{D}_X$-modules. Let now $F^\cdot, G^\cdot$ be as above. By definition

$$\delta^!(F^\cdot \boxed{\times} G^\cdot) = \mathcal{O}_X \overset{L}{\underset{\delta^{-1}\mathcal{O}_{X\times X}}{\otimes}} \delta^{-1}(\pi_1^{-1}F^\cdot \underset{\mathbf{C}}{\otimes} \pi_2^{-1}G^\cdot)[-d_X].$$

$F^\cdot$ and $G^\cdot$ have locally free resolutions. It suffices therefore to prove (4) when $F^\cdot$ and $G^\cdot$ are locally free. Since

$$\mathcal{O}_{X\times X} = \mathcal{O}_X \boxtimes \mathcal{O}_X$$

$F^\cdot \boxed{\times} G^\cdot$ is locally free over $\mathcal{O}_{X\times X}$ hence $\delta^{-1}(F^\cdot \boxed{\times} G^\cdot)$ is locally free over $\delta^{-1}(\mathcal{O}_{X\times X})$ and the right-hand side of (4) is equal to

(6) $$\mathcal{O}_X \underset{\delta^{-1}\mathcal{O}_{X\times X}}{\otimes} \delta^{-1}(\pi_1^{-1}F^\cdot \underset{\mathbf{C}}{\otimes} \pi_2^{-1}G^\cdot).$$

But the above μ provides an isomorphism of (6) onto $F^\cdot \underset{\mathcal{O}_X}{\otimes} G^\cdot[-d_X]$, i.e. onto the left-hand side of (4).
Therefore (1) and (2) give

(7) $$R\mathcal{H}om^\cdot_{\mathcal{D}_X}(F^\cdot, G^\cdot) = \omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \delta^!(\mathbb{D}_X F^\cdot \boxtimes G^\cdot)$$

(8) $$\mathrm{RHom}_{D^b(\mathcal{D}_X)}(F^\cdot, G^\cdot) = p_+(\delta^!(\mathbb{D}_X F^\cdot \boxtimes G^\cdot)).$$

where $F^\cdot, G^\cdot \in D^b_{coh}(\mathcal{D}_X)$ and p is the projection of X onto a point.

21.2 We now prove that DR is fully faithful on $D^b_{rh}(\mathcal{D}_X)$. We have to show

(1) $$\mathrm{RHom}^\cdot_{D^b_{rh}(\mathcal{D}_X)}(F^\cdot, G^\cdot) = \mathrm{RHom}^\cdot_{D^b_{cs}(X^a)}(\mathrm{DR}\,F^\cdot, \mathrm{DR}\,G^\cdot) \qquad (F^\cdot, G^\cdot \in D^b_{rh}(\mathcal{D}_X)).$$

We note first that, in view of 14.3(2), the relation 10.27(1), p. 181 in [IC: V] can be written, in our present notation

(2) $$R\mathcal{H}om^{\cdot}_{D^b(X^a)}(A^{\cdot},B^{\cdot}) = \delta^{a!}(D_{Ve}A^{\cdot}\boxtimes B^{\cdot}) \qquad (A^{\cdot},B^{\cdot}\in D^b_{cs}(X^a)),$$

which implies the following topological counterpart to 21.1(8):

(3) $$\mathrm{RHom}_{D^b(X^a)}(A^{\cdot},B^{\cdot}) = p^a_*\delta^{a!}(D_{Ve}A^{\cdot}\boxtimes B^{\cdot}) \qquad (A^{\cdot},B^{\cdot}\in D^b_{cs}(X^a)).$$

The de Rham functor for a point is just the identity. Therefore we can write 21.1(8) as

(4) $$\mathrm{RHom}^{\cdot}_{D^b_{rh}(\mathcal{D}_X)}(F^{\cdot},G^{\cdot}) = \mathrm{DR}\ p_+(\delta^!\mathbb{D}_X F^{\cdot}\boxtimes G^{\cdot})).$$

By 14.5(1), DR o $p_+ = Rp^a_*$ o DR, therefore

(5) $$\mathrm{RHom}^{\cdot}_{D^b_{rh}(\mathcal{D}_X)}(F^{\cdot},G^{\cdot}) = Rp^a_*.\mathrm{DR}\ (\delta^!\mathbb{D}_X F^{\cdot}\boxtimes G^{\cdot})).$$

This is the basic step; it expresses the left hand side by means of analytic objects. Using successively (1),(2) and (5) of 14.5, we get

$$\mathrm{DR}(\delta^!(\mathbb{D}_X F^{\cdot}\boxtimes G^{\cdot})) = \delta^{a!}(\mathrm{DR}(\mathbb{D}_X F^{\cdot}\boxtimes G^{\cdot})) = \delta^{a!}(\mathrm{DR}\ \mathbb{D}_X F^{\cdot}\boxtimes^a \mathrm{DR}\ G^{\cdot}) =$$
$$= \delta^{a!}(\mathbb{D}_{Ve}\mathrm{DR}\ F^{\cdot}\boxtimes^a \mathrm{DR}\ G^{\cdot}).$$

Together with (3), for $A^{\cdot} = \mathrm{DR}\ F^{\cdot}$, $B^{\cdot} = \mathrm{DR}\ G^{\cdot}$, and (5), this proves (1).

21.3 To finish the proof that DR is an equivalence of categories, it suffices to show that the image of DR contains a generating class of $D^b_{cs}(X^a)$ [I: 12.9]. It is known and easily seen that the direct images of locally constant sheaves of finite rank on locally closed smooth varieties form such a class. Let then $j : Z \to X$ be the inclusion of a locally closed smooth subvariety and S a locally constant system of finite dimensional vector spaces on Z^a. By Deligne's solution to

the Riemann-Hilbert problem for regular connections (IV: 1.1, 7.2.1), there exists a regular connection E on Z such that S is the local system of horizontal sections of E, i.e. (13.2) such that $\mathrm{DR}_Z\ E[-d_Z] = S$. We have then by 14.5(1):

$$\mathrm{DR}_X j_+(E)[-d_Z] = j^a_*\mathrm{DR}_Z(E)[-d_Z] = j^a_*(S)\ .$$

§ 22 REGULAR HOLONOMIC MODULES AND PERVERSE SHEAVES

22.1 Consider first the bounded derived category $D^b(A)$ of some abelian category (cf. I). Let I be a set of integers. Then $F^{\cdot} \subset D^b(A)$ is said to be concentrated in degrees belonging to I if $H^iF^{\cdot} = 0$ for $i \notin I$. We let in particular $D^b(A)^{\geq 0}$ (resp. $D^b(A)^{\leq 0}$) be the subcategory of elements concentrated in $[0,\infty]$ (resp. $(-\infty,0]$) and $D^b(A)^0 = D^b(A)^{\geq 0} \cap D^b(A)^{\leq 0}$ be the subcategory of elements with cohomology concentrated in degree 0. It is equivalent to A.

22.2 Let $S^{\cdot} \in D^b_{cs}(X^a)$. As recalled in 14.2, there is Whitney stratification such that $\mathcal{H}^{\cdot}S^{\cdot}$ is locally constant, with finite dimensional stalks, on the strata of X. By [IC:V,3.7] the assignment $x \mapsto H^{\cdot}i_x^{a!}S^{\cdot}$ is locally constant in each stratum. It follows therefore that if Z is a locally closed smooth subvariety of X, then Z has an open dense Zariski open subset Z' where $\mathcal{H}^{\cdot}S^{\cdot}$ and $H^{\cdot}i_Z^{a!}S^{\cdot}$ are locally constant. We consider the condition:

(p) *Any locally closed smooth subvariety* Z *of* X *contains a dense Zariski open subset on which* $i_Z^!S^{\cdot}$ *is locally constant and concentrated in degrees* $\geq -d_Z$.

By (one) definition, $S^{\cdot}$ is perverse if and only if $S^{\cdot}$ and $D_{Ve}S^{\cdot}$ satisfy (p) [BBD].
If $F^{\cdot} \in D^b_h(\mathcal{D}_X)$, we have $\mathcal{H}^q\mathbb{D}_XF^{\cdot} = \mathbb{D}_X\mathcal{H}^{-q}F^{\cdot}$ $(q \in \mathbb{Z})$ by 3.7(2), therefore

(1) $$F^{\cdot} \in D^b_h(\mathcal{D}_X)^{\geq 0} \Longleftrightarrow D_XF^{\cdot} \in D^b_h(\mathcal{D}_X)^{\leq 0}, \quad (F^{\cdot} \in D^b_h(\mathcal{D}_X)).$$

22.3 LEMMA *Let* $F^{\cdot}$ *be holonomic. Then* $F^{\cdot} \in D^b_h(\mathcal{D}_X)^{\geq 0}$ *if and only if the following condition is fulfilled:*

(*) Any locally closed smooth subvariety Z *of* X *contains a dense Zariski open subset on which* $i_Z^!(F^{\cdot})$ *has* $\mathcal{O}_Z$*-coherent cohomology, concentrated in degrees* ≥ 0.

Proof. Recall first that if Z is locally closed, smooth in X, there is a spectral sequence abutting to $\mathcal{H}^{\cdot}i_Z^!F^{\cdot}$ in which $E_2 = i_Z^!\mathcal{H}^{\cdot}F^{\cdot}$.

Let $F^{\cdot} \in D^b_h(\mathcal{D}_X)^{\geq 0}$, and Z a locally closed smooth subvariety. It follows from 8.2(ii) that $i_Z^!\mathcal{H}^{\cdot}F^{\cdot}$ is concentrated in positive degrees, hence so is $i_Z^!F^{\cdot}$. Moreover $\mathcal{H}^{\cdot}i_Z^!F^{\cdot}$ is $\mathcal{O}$-coherent on a dense Zariski open subset by 10.4. Assume now (*). Let j be the smallest integer such that $\mathcal{H}^jF^{\cdot} \neq 0$. We can find Z open in the support Y of $\mathcal{H}^jF^{\cdot}$ on which $\mathcal{H}^jF^{\cdot}$ is $\mathcal{O}_Z$-coherent (10.4). We have to show that $j \geq 0$. In view of (*), it suffices to prove that $\mathcal{H}^ji_Z^!F^{\cdot} \neq 0$. We can find an open dense set U in X such that $Z \cap Y$ is closed in U and equal to Z. By 7.4, $i_Z^!\mathcal{H}^jF^{\cdot}$ is concentrated in degree zero and by 7.11, $i_{Z\to U,+}(i_Z^!\mathcal{H}^jF^{\cdot}) = \mathcal{H}^jF^{\cdot}|U$, therefore $\mathcal{H}^0i_Z^!\mathcal{H}^jF^{\cdot} \neq 0$. In the spectral sequence mentioned above we have $E_2^{p,q} = \mathcal{H}^pi_Z^!\mathcal{H}^qF^{\cdot}$. Here $p \in [0,d_{X,Z}]$ and $q \geq j$. It follows that $E_\infty^{0,j} \neq 0$, whence our assertion.

22.4 THEOREM *Let* $F^{\cdot} \in D^b_{rh}(\mathcal{D}_X)$. *Then* $DR\,F^{\cdot}$ *is perverse if and only if* $F^{\cdot} \in D^b_{rh}(\mathcal{D}_X)^0$.

Proof. It follows from 22.2(1) and 22.3 that

(1) $$F^{\cdot} \in D^b_h(\mathcal{D}_X)^0 \Longleftrightarrow F^{\cdot} \text{ and } \mathbb{D}_XF^{\cdot} \text{ satisfy } 22.3(*).$$

Let Z be a locally closed smooth subvariety of X. Since $DR\,F^{\cdot}$ is constructible (14.5(3)) and $i_Z^!F^{\cdot}$ is holonomic (12.1), there exists a dense Zariski open subset Z' of Z on which $\mathcal{H}^{\cdot}i_Z^!F^{\cdot}$, $\mathcal{H}^{\cdot}\,DR\,F^{\cdot}$ and $\mathcal{H}^{\cdot}i_Z^{a!}DR\,F^{\cdot}$ are locally constant in the analytic topology (10.4 and 22.2). Therefore $i_Z^!F^{\cdot}$ is concentrated in degrees ≥ 0 if and only if $DR\,i_Z^!F^{\cdot}$ is concentrated in degrees $\geq -d_Z$. But $F^{\cdot}$ is r.h., hence $DR\,i_Z^!F^{\cdot} = i_Z^{a!}DR\,F^{\cdot}$ by 14.5(4). Consequently, (*) for $F^{\cdot}$ is equivalent to (p) for $DR\,F^{\cdot}$. Similarly, (*) for $\mathbb{D}_XF^{\cdot}$ is equivalent to (p) for $DR\,\mathbb{D}_XF^{\cdot}$. But DR commutes with duality (14.5(5)), therefore (*) for $\mathbb{D}_XF^{\cdot}$ is equivalent to (p) for $D_{Ve}DR\,F^{\cdot}$. The theorem now follows from 22.3 and the definition of a perverse sheaf.

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