Spherically symmetric quantum geometry: states and basic operators

Martin Bojowald

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1,
D–14476 Potsdam, Germany

E-mail: mabo@aei.mpg.de

Abstract
The kinematical setting of spherically symmetric quantum geometry, derived
from the full theory of loop quantum gravity, is developed. This extends
previous studies of homogeneous models to inhomogeneous ones where
interesting field theory aspects arise. A comparison between a reduced
quantization and a derivation of the model from the full theory is presented
in detail, with an emphasis on the resulting quantum representation. Similar
concepts for Einstein–Rosen waves are discussed briefly.

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1. Introduction

Since general relativity predicts singularities generically, and in particular in physically
interesting situations such as cosmology and black holes, it cannot be complete as a physical
theory. The situation improves when one quantizes general relativity in a background
independent manner, following loop quantum gravity [1–3]. The dynamics of the full theory
is not yet settled and is rather complicated, as expected for a full quantum theory of gravity.
Even classically one usually introduces symmetries for physical applications, which can also
be done in loop quantum gravity directly. This in fact led to the conclusion that isotropic
models in loop quantum cosmology [4] are non-singular [5] while at the same time they show
the usual classical behaviour at large scales [6].

One has to keep in mind, though, that symmetric models in a quantum theory play a
role different from symmetric classical solutions. While the latter are exact solutions of
the full theory, the former are obtained from the full theory by completely ignoring many
degrees of freedom which violates their uncertainty relations. One thus should weaken the
symmetry by looking at less symmetric models, and check if results obtained are robust. For

the isotropic results, this has been shown to be the case in a first step, reducing the isotropy by using anisotropic but still homogeneous models [7, 8]. This did not only show that the same mechanism for singularity freedom applied, and this in a more nontrivial way, but also led to new applications [9]. The latter allow tentative conclusions even for general, inhomogeneous singularities [10].

Nevertheless, one should go ahead and reduce the symmetry further. The next step must deal with inhomogeneous models, which for simplicity can first be taken to be 1+1 dimensional. This would also allow new physical applications concerning, e.g., spherically symmetric black holes and cylindrical gravitational waves. Furthermore, they allow additional tests of issues in the full theory which trivialize in homogeneous models, including field theory aspects, the constraint algebra and the role of anomalies, and specific constructions of semiclassical states using graphs. For (1+1)-dimensional models several alternative background independent quantization schemes have been applied, which can then be compared with loop results. The spherically symmetric model has been dealt with in the Dirac programme [11] as well as in a reduced phase space quantization [12, 13]. Einstein–Rosen waves can be mapped to a free field on flat spacetime allowing standard Fock quantization techniques [14], and there are several other interesting models with a two-dimensional Abelian symmetry group which have been quantized and studied extensively [15–27]. A wide class of models, which have finitely many physical degrees of freedom and also include the spherically symmetric model, are given by dilaton gravity in two dimensions [28] or, more generally, Poisson sigma models [29, 30]. These models have been quantized exactly in a background independent way with reduced phase space, Dirac or path integral methods.

The reason for the simplification in homogeneous models, which lead to explicit cosmological applications, is not just the finite number of degrees of freedom, but also a simplification of the volume operator (which at first sight is not always explicit [31]). In isotropic as well as diagonal homogeneous models the volume spectrum can be computed explicitly, which is not possible in the full theory. Since the volume operator plays a major role in defining the dynamics [32], also the evolution equation can be obtained and analysed in an explicit form. One can see that this is a consequence of either a nontrivial isotropy subgroup of the symmetry group, or of a diagonalization condition. Similar simplifications can be expected more generally, in particular in those inhomogeneous models which have a nontrivial isotropy group (spherical symmetry) or a diagonalization condition on the basic variables (polarized waves).

Nevertheless, the explicit reduction of spherically symmetric models done later in this paper, and also that of polarized cylindrical waves, shows that suitable canonical variables display a feature different from both the full theory and from homogeneous models: flux variables (canonical momenta of the connection) are not identical to the densitized triad which contains all information about spatial geometry. (A similar feature, though in a different manner, happens in the full theory when a scalar is coupled non-minimally [33].) Instead, the triad is a rather complicated function of the basic variables and in particular depends also on the connection. This seems to lead to an unexpected complication for the volume operator, and shows that 1+1 models are more complicated than homogeneous ones not just for the obvious reason of having infinitely many kinematical degrees of freedom, but also due to their canonical structure. The complicated expression for the triad could, *a priori*, even lead to a continuous volume spectrum, which would be difficult to reconcile with the full theory and homogeneous models. We will deal with the volume operator elsewhere [34], but already in this paper, where we introduce the kinematical set-up and discuss states and basic operators for connections and their momenta, we can see that this issue can have an influence on semiclassical properties.

We will start by recalling the definition of symmetric states in a quantum theory of connections, and then reduce the full phase space to that of the spherically symmetric sector. We introduce the states of the model by two procedures, first by loop quantizing the classically reduced phase space and then by reducing states of the full theory to be spherically symmetric. Both procedures lead to the same result, which is a 'mixed' quantization based on generalized connections, as in the full theory [35], as well as elements of the Bohr compactification of the real line, which is characteristic for homogeneous models [36]. Quantum numbers of the reduced quantization match with the spin labels obtained by restricting full states, and gauge invariant reduced states satisfy the reduced Gauss constraint. Basic operators on these states are given by holonomies and fluxes, which suggest conditions for the semiclassical regime. Finally, we will briefly discuss the model of Einstein–Rosen waves.

2. Symmetric states

Let Σ be a manifold carrying an action of a symmetry group S such that there is a dense subset of Σ where the group action has an isotropy subgroup isomorphic to $F < S$. In this case Σ, except for isolated points (symmetry axes or centres), can be decomposed as $\Sigma \cong B \times S/F$ with the reduced manifold $B = \Sigma/S$. On the symmetry orbits S/F there is a natural invariant metric which follows from the transitive group action, as well as preferred coordinates. On B, on the other hand, there is no natural metric and no preferred coordinates.

A given symmetry group S acting on a manifold Σ defines a class of inequivalent principal fibre bundles $P(\Sigma, G)$, for a given group G, which carry a lift of the action of S from Σ to P [37, 38]. For each such symmetric bundle there is a set of invariant connections having a $\mathcal{L}G$-valued 1-form on P satisfying $s^*\omega = \omega$ for each $s \in S$, giving rise to different embeddings $r_k \colon \mathcal{A}^{(k)}_{\text{inv}} \to \mathcal{A}$ in the full space of connections. Here, k is a label (topological charge) characterizing the type of symmetric bundle used. In a gravitational situation, where there is an additional condition that spaces of connections must allow non-degenerate dual vector fields, the would-be non-degenerate triads, usually only one value for the label k can be used such that we will suppress it later on.

An invariant connection then has the general form $A = A_B + A_{S/F}$ where A_B is a reduced connection over B, in general with a reduced gauge group, and $A_{S/F}$ contains additional fields in an associate bundle transforming as scalars taking values in a certain representation of the reduced structure group. The different forms r_k of embedding invariant connections into the full space of connections are classified by homomorphisms $\lambda_k \colon F \to G$ up to conjugacy in G. This map also determines the reduced structure group for the connections A_B as the centralizer $Z_G(\lambda_k(F))$ in G. The additional fields in $A_{S/F}$ are the components of a linear map $\phi \colon \mathcal{L}F_\perp \to \mathcal{L}G$ where the space S/F is assumed to be reductive, i.e. there is a decomposition $\mathcal{L}S = \mathcal{L}F \oplus \mathcal{L}F_\perp$ such that $\mathcal{L}F_\perp$ is fixed by the adjoint action of F. There are additional linear conditions ϕ has to satisfy when it comes from a full connection, namely

$$\phi(\text{Ad}_f X) = \text{Ad}_{\lambda_k(f)}\phi(X) \tag{1}$$

for all $X \in \mathcal{L}F_\perp$ and $f \in F$.

2.1. Reduced loop quantization

Loop quantum gravity provides techniques to quantize theories of connections, possibly coupled to other fields, in a background independent manner. Following this procedure, the component A_B, which plays the role of the connection of the reduced theory, will be quantized by using its holonomies along curves in B as basic variables [35]. This leads to the space of

generalized reduced connections, $\bar{\mathcal{A}}_B$. Scalars like those in $A_{S/F}$ can be quantized according to [39, 40] with the result that the classical real values of the field are replaced by values in the Bohr compactification of the real line. In this way, the space $\bar{\mathcal{A}}_{B \times S/F}$ of generalized connections and scalar fields becomes a compact group which carries a Haar measure μ_0. The Hilbert space $L^2(\bar{\mathcal{A}}_{B \times S/F}, \mu_0)$ is then obtained by completing the space of continuous functions on this group with respect to the Haar measure.

Holonomies of the connection and analogous expressions for the scalar act as multiplication operators, while the momenta of the connection components, which can be written as fluxes, act as derivative operators. Both sets of basic operators are subsequently used to quantize more complicated, composite expressions.

2.2. Symmetric states from the full theory

Using the connection representation of states on the space of generalized connections, symmetric states can be defined in the full theory as distributional states supported only on invariant connections [41, 42] for a given symmetry. It is clear that such a state can also be represented as a function ψ on the space of reduced connections as before, but in addition it acquires an interpretation as a distribution in the full theory, i.e. as a linear functional on the space of cylindrical states depending only on finitely many holonomies and scalar values: for any cylindrical function f on the full space of connections,

$$\Psi[f] := \int_{\bar{\mathcal{A}}^{(k)}_{B \times S/F}} \mathrm{d}\mu_0(A) \overline{\psi(A)} \cdot r_k^* f(A) \tag{2}$$

defines a distribution in the full theory. Symmetric states thus form a subspace of the full distributional space which, using the measure on $\bar{\mathcal{A}}^{(k)}_{B \times S/F}$ can be equipped with an inner product.

Operators $\hat{O}$ of the full theory act on distributions Ψ via the dual action which defines $\hat{O}\Psi$ by

$$\hat{O}\Psi[f] = \Psi[\hat{O}^\dagger f] \qquad \text{for all} \quad f \in \text{Cyl}.$$

In general, however, $\hat{O}\Psi$ will not be a symmetric state even if Ψ is. The reason is that on states in the connection representation only the condition of having invariant A has been incorporated, but not the condition for invariant momenta E. Then, even classically the flow generated by a phase space function would in general not be tangential to the subspace given by invariant connections and arbitrary triads.

For general operators it is therefore necessary to implement the condition for invariant triads, which must be done by modifying the operators suitably. This is a complicated procedure which has not been developed in detail yet. Fortunately, one can use particular operators in the full theory whose dual action leaves the space of symmetric states invariant such that one can directly use them in the reduced model. Classical analogues of those operators generate a flow which is tangential to the subspace of invariant connections in phase space even if triads can be arbitrary. It is easy to see that such functions have to be linear in the triads (which is, however, not a sufficient condition). In fact, the reduced basic variables, holonomies and fluxes, are linear in the triads, and can be written such that they generate a flow parallel to invariant connections. Moreover, for the basic quantities, the classical Poisson *-algebra is represented faithfully on the Hilbert space such that the classical flow on phase space corresponds to a unitary transformation in the quantum theory. Thus, quantizations of basic variables will map symmetric states to symmetric states and can be used directly to derive the reduced operators. States as well as basic operators of a model are then defined directly in the full theory, and more complicated operators can be constructed from the basic

ones following the lines of the full theory. An advantage of relating the model to the full theory in this way is that there is a unique (under weak conditions) diffeomorphism invariant representation of the full holonomy/flux algebra [43] while within models one usually has several options.

The reduced theory is usually not a pure gauge theory even in the absence of matter since some components ϕ of the full connection play the role of scalar fields in the reduced model. Another difference to the full theory is that often the model has a reduced gauge group. States of the full theory and the model are then based on different groups. Nevertheless, representations of the reduced structure group automatically occur when the reduction from full states is done. For an explicit representation of states and operators, an Abelian gauge group is most helpful since all irreducible representations are then one dimensional and there are no complicated coupling coefficients between different representations. In fact, a nontrivial isotropy subgroup of the symmetry group often, as in the spherically symmetric case, implies an Abelian gauge group. Similarly, diagonalization conditions imposed on connections and triads can lead to Abelian gauge groups. On the other hand, a nontrivial isotropy group or additional diagonalization conditions lead to additional complications since the relations (1) have to be taken care of. Moreover, even though the reduced connection may be Abelian, its holonomies do in general not commute with expressions (point holonomies) representing scalars since this would be incompatible with a non-degenerate triad. Simplifications of an Abelian theory, such as spin network states with an Abelian group, then are not always obvious.

In many cases, the combined system of the reduced Abelian connection plus scalar fields ϕ can be simplified taking into account the special form of a given class of invariant connections. A model can be formulated as essentially Abelian if connection components along independent directions along B and in the orbits are perpendicular in the Lie algebra. For instance, in the (1+1)-dimensional case, a connection in general has the form

$$A = A_x(x)\Lambda_x(x)\,\mathrm{d}x + A_y(x)\Lambda_y(x)\,\mathrm{d}y + A_z(x)\Lambda_z(x)\,\mathrm{d}z + \text{field independent terms} \qquad (3)$$

where x is the inhomogeneous coordinate on B and $\Lambda_I(x) \in \mathcal{L}G$. (Depending on the symmetry, there can be additional terms not depending on fields A_I, as happens in the spherically symmetric case discussed later.) The fields A_y and A_z together with components of Λ_y and Λ_z comprise the field ϕ determining $A_{S/F}$ and are thus subject to (1). Simplifications occur if we have $\mathrm{tr}(\Lambda_x\Lambda_y) = \mathrm{tr}(\Lambda_x\Lambda_z) = \mathrm{tr}(\Lambda_y\Lambda_z) = 0$, as happens in the spherically symmetric case or for cylindrical gravitational waves with a polarization condition. Then, holonomies of invariant connections take a corresponding form with perpendicular internal directions, and thus obey special relations that would not hold true for holonomies of an arbitrary connection. The most important relation which will be used later is

Lemma 1. *Let* $g := \exp(A)$ *and* $h := \exp(B)$ *with* $A, B \in su(2)$ *such that* $\mathrm{tr}(AB) = 0$. *Then*

$$gh = hg + h^{-1}g + hg^{-1} - \mathrm{tr}(hg). \qquad (4)$$

Proof. Since equation (4) is invariant under conjugation of both g and h with the same $SU(2)$-element, we can first rotate A to equal $a\tau_1$ for some $a \in \mathbb{R}$. Then, $\mathrm{tr}(AB) = 0$ implies that $B = b_2\tau_2 + b_3\tau_3$ which can be rotated to $B = b\tau_2$ while keeping A fixed.

The proof proceeds by directly computing all products involved, using $\exp(a\tau_i) = \cos(a/2) + 2\tau_i \sin(a/2)$. $\qquad \square$

Thus, even though reduced holonomies do not all lie in an Abelian subgroup, they are almost commuting in the sense that products of two holonomies can always be expanded into terms where the order is reversed. It turns out that this is sufficient for a simplification of the

representation of states and basic operators, and in turn of other ones like the volume operator. This has been exploited in homogeneous models, where the special form of connections was a consequence of the nontrivial isotropy [4] or a diagonalization condition [7]. Similarly in (1+1)-dimensional models, a nontrivial isotropy group or a diagonalization condition can lead to connection components which are perpendicular for independent directions. As we will discuss in what follows, this leads to a similar simplification in the representation of states, but in inhomogeneous models there can be an additional complication for the volume operator.

3. Classical phase space

In the main part of this paper, we are interested in spherical symmetry where $S \cong SU(2)$ (in general, the action on P does not project to an $SO(3)$ action, even if it does so on Σ) and, outside symmetry centres, $F \cong U(1)$ such that $S/F \cong S^2$. The reduced (radial) manifold B is one dimensional. On the orbits we have an invariant metric which can be written as $d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2$ in angular coordinates which will be used from now on. A coordinate on B will be called x in what follows, but not fixed. The reduced phase space of this model has been studied in ADM variables [12] and complex Ashtekar variables [11, 13], which can be used for a reduced phase space or Dirac quantization. Many relations in complex Ashtekar variables also apply here, but one should be cautious since our notation is slightly different and in some places adapted to a loop quantization. The classical model in real Ashtekar variables and preliminary steps of a loop quantization have been described in [41, 42].

Any invariant connection allowing a non-degenerate dual vector field can be written as

$$A = A_x(x)\Lambda_3 \, dr + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2) \, d\vartheta$$
$$+ (A_1(x)\Lambda_2 - A_2(x)\Lambda_1) \sin \vartheta \, d\varphi + \Lambda_3 \cos \vartheta \, d\varphi \qquad (5)$$

with three real functions A_x, A_1 and A_2 on B. The $su(2)$-matrices Λ_I are constant and are identical to $\tau_I = -\frac{i}{2}\sigma_I$ or a rigid rotation thereof. An invariant densitized triad has a dual form,

$$E = E^x(x)\Lambda_3 \sin \vartheta \frac{\partial}{\partial x} + (E^1(x)\Lambda_1 + E^2(x)\Lambda_2) \sin \vartheta \frac{\partial}{\partial \vartheta} + (E^1(x)\Lambda_2 - E^2(x)\Lambda_1) \frac{\partial}{\partial \varphi} \qquad (6)$$

such that the functions E^x, E^1 and E^2 on B are canonically conjugate to A_x, A_1 and A_2:

$$\Omega_B = \frac{1}{2\gamma G} \int_B dx (\delta A_x \wedge \delta E^x + 2\delta A_1 \wedge \delta E^1 + 2\delta A_2 \wedge \delta E^2) \qquad (7)$$

with the gravitational constant G and the Barbero–Immirzi parameter γ.

It will later be useful to keep in mind a peculiarity of one-dimensional models concerning the density weight of fields. As in the full theory, the connection has density weight zero, and the densitized triad is a vector field with density weight one. But in one dimension, the transformation properties with fixed orientation imply that a 1-form is equivalent to a scalar of density weight one, while a densitized vector field is equivalent to a scalar without density weight. Under a coordinate change $x \mapsto y(x)$, a densitized vector field, for instance, transforms as $E^a = \tilde{E}^b \partial x^a / \partial y^b |\det \partial y / \partial x|$, which implies $E^x = \tilde{E}^x \cdot |y'(x)|/y'(x) = \pm E^x$. Thus, E^x can be seen as the component of a densitized vector field on B or as a scalar, while E^1 and E^2 are densitized scalars (or 1-form components). Similarly, A_x is the component of a 1-form on B or a densitized scalar, while A_1 and A_2 are scalars (or densitized vector field components).

These variables are subject to constraints which are obtained by inserting the invariant forms into the full expressions. We have the Gauss constraint

$$G[\lambda] = \int_B dx \, \lambda (E^{x'} + 2A_1 E^2 - 2A_2 E^1) \approx 0 \qquad (8)$$

generating $U(1)$-gauge transformations, the diffeomorphism constraint

$$D[N_x] = \int_B dx \, N_x(2A_1' E^1 + 2A_2' E^2 - A_x E^{x'}) \tag{9}$$

and the Euclidean part of the Hamiltonian constraint

$$H[N] = 2 \int_B dx \, N(|E^x|((E^1)^2 + (E^2)^2))^{-1/2}(E^x(E^1 A_2' - E^2 A_1')$$
$$+ A_x E^x(A_1 E^1 + A_2 E^2) + (A_1^2 + A_2^2 - 1)((E^1)^2 + (E^2)^2)). \tag{10}$$

In what follows it will be more convenient to work with variables that are better adapted to the gauge transformations. We introduce the gauge invariant quantities

$$A_\varphi(x) := \sqrt{A_1(x)^2 + A_2(x)^2}, \tag{11}$$

$$E^\varphi(x) := \sqrt{E^1(x)^2 + E^2(x)^2} \tag{12}$$

and the internal directions

$$\Lambda_\varphi^A(x) := (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)/A_\varphi(x), \tag{13}$$

$$\Lambda_\varphi^E(x) := (E^1(x)\Lambda_2 - E^2(x)\Lambda_1)/E^\varphi(x) \tag{14}$$

in the Λ_1–Λ_2 plane. Furthermore, we parametrize $\Lambda_\varphi^A(x)$ and $\Lambda_\varphi^E(x)$, which in general are different from each other, by two angles $\alpha(x)$, $\beta(x)$:

$$\Lambda_\varphi^A(x) =: \Lambda_1 \cos\beta(x) + \Lambda_2 \sin\beta(x), \tag{15}$$

$$\Lambda_\varphi^E(x) =: \Lambda_1 \cos(\alpha(x) + \beta(x)) + \Lambda_2 \sin(\alpha(x) + \beta(x)). \tag{16}$$

Note that $\cos\alpha = \Lambda_\varphi^A \cdot \Lambda_\varphi^E$ is gauge invariant under $U(1)$-rotations, while the angle β is pure gauge.

In these new variables, the symplectic structure becomes

$$\Omega_B = \frac{1}{2\gamma G} \int_B dx(\delta A_x \wedge \delta E^x + 2\delta A_\varphi \wedge \delta(E^\varphi \cos\alpha) + 2\delta\beta \wedge \delta(A_\varphi E^\varphi \sin\alpha))$$

$$= \frac{1}{2\gamma G} \int_B dx(\delta A_x \wedge \delta E^x + \delta A_\varphi \wedge \delta P^\varphi + \delta\beta \wedge \delta P^\beta) \tag{17}$$

with new momenta

$$P^\varphi(x) := 2E^\varphi(x) \cos\alpha(x) \tag{18}$$

conjugate to A_φ and

$$P^\beta(x) := 2A_\varphi(x)E^\varphi(x) \sin\alpha(x) = A_\varphi(x)P^\varphi(x) \tan\alpha(x) \tag{19}$$

conjugate to β. The Gauss constraint then takes the form

$$G[\lambda] = \int_B dx \, \lambda(E^{x'} - P^\beta) \approx 0 \tag{20}$$

which is easily solved by $P^\beta = E^{x'}$, while the function $A_x + \beta'$ is manifestly gauge invariant.

Using these variables, the situation is different from that in the full theory in that the momentum conjugate to the connection component A_φ is not the triad component E^φ, which together with the momentum E^x would directly determine the geometry

$$ds^2 = E^x(x)^{-1} E^\varphi(x)^2 \, dx^2 + E^x(x)(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2). \tag{21}$$

Instead, the momentum P^φ is related to E^φ through the angle α. This angle is a rather complicated function of the variables, depending also on connection components: $\tan\alpha = (A_\varphi P^\varphi)^{-1} P^\beta$. This will complicate the quantum geometry since the fluxes P will be basic variables with simple quantizations, while geometric operators like the volume operator will be more complicated. In homogeneous models [7, 4], on the other hand, this complication does not appear since the Gauss constraint (20) with constant E^x implies $P^\beta = 0$ and thus $\alpha = 0$.

4. Kinematical Hilbert space

By definition, symmetric states can be described by restricting states of the full theory to invariant connections, which are of the form (5) in the spherically symmetric case (from a different point of view, focusing on coherent states, spherical symmetry has been considered in [44]). Using all states in the full theory, this leads to a complete, but not independent set of symmetric states, which then must be functionals of $A_x(x)$, $A_1(x)$ and $A_2(x)$. That such functionals have to be expected is also obvious from the reduced point of view where one just quantizes the classically reduced phase space. However, the class of functions obtained in this way depends on the quantization procedure, a loop quantization giving different results than, e.g., a Wheeler–DeWitt like quantization (as happens already in the isotropic case where both quantizations result in inequivalent representations [36]).

We first follow a reduced quantization point of view analogous to that followed in [36]. In constructing the quantum theory we perform the steps of the full loop quantization, thus obtaining a loop quantization of the reduced model. Thereafter, we will reduce states from the full theory and implement the reduction there, leading to the same results in particular for basic operators.

4.1. Reduced quantization

We start by choosing elementary functions on the classical phase space that will be promoted to basic operators of the quantum theory, acting on a suitable Hilbert space. The hallmark of loop quantizations is that those basic quantities are chosen to be holonomies of the connection and fluxes of the densitized triad. This choice incorporates a smearing of the classical fields along lines and surfaces, which is necessary for a well-defined representation, and does so in a background independent manner. From the reduced point of view, A_φ is a scalar for which there are analogous techniques [40, 39] which we will use below.

4.1.1. Cylindrical states. A loop quantization in the connection representation is based on cylindrical functions which depend on the connection only via holonomies. If we just consider the space $\mathcal{A}_B$ of reduced $U(1)$-connections given by $A_x(x)$ on B, cylindrical functions are continuous functions on the space of generalized connections $\bar{\mathcal{A}}_B$. As in the full theory, $\bar{\mathcal{A}}_B$ can be written as a projective limit over graphs in B, which in the one dimensional case are simply characterized by a disjoint union of non-overlapping edges, $g = \dot{\bigcup}_i e_i$, whose vertex set $V(g)$ is the union of all endpoints of the e_i. Choosing an orientation of B, we fix the orientation of all edges to be compatible with that of B. Holonomies then define spaces $\mathcal{A}_B^g$ of maps from the set of edges of a given graph of n edges to $SU(2)^n$, which for classical connections reduces to $A_B: g \to U(1), e \mapsto h^{(e)} := \exp \frac{1}{2}\mathrm{i} \int_e A_x(x)$ (the factor 1/2 comes from taking matrix elements of Λ_3-holonomies). The space of generalized connections is obtained as the projective limit

$$\bar{\mathcal{A}}_B = \mathrm{proj}\lim_{g \subset B} \mathcal{A}_B^g \tag{22}$$

with the usual projections $p_{gg'}(\bar{A}_B^g) := \bar{A}_B^g\big|_{g'}$ for $g' \subset g$.

Since A_φ transforms as a scalar, its holonomies with respect to edges in B would not be well defined. Instead, following [39, 40] one considers 'point holonomies' $\exp(\mathrm{i}\mu A_\varphi(x))$ such that, for a fixed point $x \in B$, the relevant space of states is the space $C(\bar{\mathbb{R}}_{\mathrm{Bohr}})$ of continuous (almost periodic) functions on the Bohr compactification of the real line $\mathbb{R} \ni A_\varphi(x)$. The remaining independent scalar function in the connection, $\beta(x)$, takes values in the

circle S^1 which is already compact. Corresponding point holonomies are simply exponentials $\exp(\mathrm{i}\beta(x)) \in U(1)$.

Since all points are independent, the space of generalized fields $\bar{\mathcal{A}}_{S^2}$ is again a projective limit, this time over sets of points $\{x_i\}_{i=1...m} \subset B$, which can be taken as the vertex set $V(g)$ of a graph g. For a fixed graph, we obtain the space $\mathcal{A}_{S^2}^g$ of maps from the set $V(g)$ of m vertices to $(\bar{\mathbb{R}}_{\mathrm{Bohr}} \times U(1))^m$, which for classical fields is $A_{S^2}: V(g) \to \bar{\mathbb{R}}_{\mathrm{Bohr}} \times U(1)$, $v \mapsto (A_\varphi(v), \mathrm{e}^{\mathrm{i}\beta(v)})$. The space of generalized fields is

$$\bar{\mathcal{A}}_{S^2} = \mathrm{proj\,lim}_{g \subset B} \mathcal{A}_{S^2}^g$$

which can easily be combined with $\bar{\mathcal{A}}_B$ to obtain the space of generalized spherically symmetric connections

$$\bar{\mathcal{A}}_{B \times S^2} = \mathrm{proj\,lim}_{g \subset B} \mathcal{A}_B^g \otimes \mathcal{A}_{S^2}^g. \tag{23}$$

Since $\bar{\mathcal{A}}_{B \times S^2}$ is the projective limit of tensor products of compact groups, $U(1)$ and $\bar{\mathbb{R}}_{\mathrm{Bohr}}$, it carries a normalized Haar measure which is analogous to the Ashtekar–Lewandowski measure in the full theory and will be called μ_0. The kinematical Hilbert space is then obtained by completing the space of cylindrical functions on $\bar{\mathcal{A}}_{B \times S^2}$ with respect to μ_0. Holonomies defined above act by multiplication on this space.

As in the full theory, one can use spin network states as a convenient basis, which in the connection representation become functionals of A_x, A_φ and β. They are cylindrical states based on a given graph g whose edges e are labelled by irreducible $U(1)$-representations $k_e \in \mathbb{Z}$, and whose vertices v are labelled by irreducible $\bar{\mathbb{R}}_{\mathrm{Bohr}}$-representations $\mu_v \in \mathbb{R}$ as well as irreducible S^1-representation $k_v \in \mathbb{Z}$. The value of such a spin network state in a given (generalized) spherically symmetric connection A then is

$$T_{g,k,\mu}(A) = \prod_{e \in g} k_e(h^{(e)}) \prod_{v \in V(g)} \mu_v(A_\varphi(v)) k_v(\beta(v))$$

$$= \prod_{e \in g} \exp\left(\frac{1}{2}\mathrm{i}k_e \int_e A_x(x)\,\mathrm{d}x\right) \prod_{v \in V(g)} \exp(\mathrm{i}\mu_v A_\varphi(v)) \exp(\mathrm{i}k_v \beta(v)). \tag{24}$$

Since A_φ and β are scalars on B, they are not integrated over in the states. On the other hand, A_x as a connection component is integrated to appear only via holonomies. Alternatively, as discussed before, we can view the one-dimensional connection component A_x as a density-valued scalar. Also from this perspective it would have to appear integrated along regions in the above form. Since A_φ is by definition non-negative, we will restrict the states to only those values.

4.1.2. Flux operators. For the flux of E^x it is also helpful to view it in the unconventional way as a scalar. At a given point x, $E^x(x)$ will then simply be quantized to a single derivative operator without integration:

$$\hat{E}^x(x)f(h) = -\mathrm{i}\frac{\gamma \ell_{\mathrm{P}}^2}{4\pi} \sum_e \frac{\partial f}{\partial h^{(e)}} \frac{\delta h^{(e)}}{\delta A_x(x)} = \frac{\gamma \ell_{\mathrm{P}}^2}{8\pi} \cdot \frac{1}{2} \sum_{e \ni x} h^{(e)} \frac{\partial f}{\partial h^{(e)}}, \tag{25}$$

where f is a cylindrical function depending on the holonomies $h^{(e)} = \exp\left(\frac{1}{2}\mathrm{i}\int_e A_x\,\mathrm{d}x\right)$. To simplify the notation, we assumed that x lies only at boundary points of edges, which can always be achieved by subdivision, and which contributes the additional $\frac{1}{2}$. The other flux components, P^φ and P^β, are density-valued scalars and thus will be turned to well-defined

operators after integrating over regions $\mathcal{I} \subset B$. We obtain

$$\int_{\mathcal{I}} \hat{P}^\varphi f(h) = -\mathrm{i}\frac{\gamma \ell_\mathrm{P}^2}{4\pi} \int_{\mathcal{I}} \frac{\delta}{\delta A_\varphi(x)} \, \mathrm{d}x \, f(h) = -\mathrm{i}\frac{\gamma \ell_\mathrm{P}^2}{4\pi} \int_{\mathcal{I}} \mathrm{d}x \sum_v \frac{\partial f}{\partial h^{(v)}} \frac{\delta h^{(v)}}{\delta A_\varphi(x)}$$

$$= -\mathrm{i}\frac{\gamma \ell_\mathrm{P}^2}{4\pi} \sum_v \int_{\mathcal{I}} \mathrm{d}x \, \frac{\partial f}{\partial h^{(v)}} \delta(v, x) = -\mathrm{i}\frac{\gamma \ell_\mathrm{P}^2}{4\pi} \sum_{v \in \mathcal{I}} \frac{\partial}{\partial A_\varphi(v)} f(h) \tag{26}$$

with $h^{(v)} := A_\varphi(v)$, and similarly

$$\int_{\mathcal{I}} \hat{P}^\beta f(h) = -\mathrm{i}\frac{\gamma \ell_\mathrm{P}^2}{4\pi} \sum_{v \in \mathcal{I}} \frac{\partial}{\partial \beta(v)} f(h). \tag{27}$$

Acting on spin network states (24), we obtain

$$\hat{E}^x(x) T_{g,k,\mu} = \frac{\gamma \ell_\mathrm{P}^2}{8\pi} \frac{k_{e^+(x)} + k_{e^-(x)}}{2} T_{g,k,\mu} \tag{28}$$

$$\int_{\mathcal{I}} \hat{P}^\varphi T_{g,k,\mu} = \frac{\gamma \ell_\mathrm{P}^2}{4\pi} \sum_{v \in \mathcal{I}} \mu_v T_{g,k,\mu} \tag{29}$$

$$\int_{\mathcal{I}} \hat{P}^\beta T_{g,k,\mu} = \frac{\gamma \ell_\mathrm{P}^2}{4\pi} \sum_{v \in \mathcal{I}} k_v T_{g,k,\mu} \tag{30}$$

where $e^\pm(x)$ are the two edges (or two parts of a single edge) meeting in x. Thus, spin networks are eigenstates of all flux operators and all flux operators have discrete spectra (normalizable eigenstates). Note in particular that $\int_{\mathcal{I}} \hat{P}^\varphi$ is self-adjoint even though the range of A_φ is restricted to non-negative values. In a Schrödinger representation the corresponding derivative operator would not have a self-adjoint extension, which is the case here on the restricted Bohr Hilbert space.

Knowing the flux operators allows us to quantize and solve the Gauss constraint (20). Restricting attention for simplicity to piecewise constant λ, it suffices to quantize the integrated density $E^{x\prime}$, which can be done easily by using $\int_{x_-}^{x_+} E^{x\prime} \, \mathrm{d}x = E^x(x_+) - E^x(x_-)$. Thus,

$$\hat{G}[\lambda] T_{g,k,\mu} = \frac{\gamma \ell_\mathrm{P}^2}{8\pi} \sum_v \lambda(v)(k_{e^+(v)} - k_{e^-(v)} - 2k_v) T_{g,k,\mu} = 0 \tag{31}$$

which is solved by

$$k_v = \tfrac{1}{2}(k_{e^+(v)} - k_{e^-(v)}) \tag{32}$$

for all vertices v of a given spin network. Since the k_v must be integer, all differences $k_{e^+(v)} - k_{e^-(v)}$ must be even, restricting the allowed values. The labels k_v are then determined completely by the edge labels k_e and can be dropped, while the other vertex labels, μ_v, are not restricted by the Gauss constraint. In fact, when (32) is satisfied, the spin network (24) takes the form

$$T_{g,k,\mu} = \prod_e \exp\left(\frac{1}{2}\mathrm{i}k_e \int_e (A_x + \beta') \, \mathrm{d}x\right) \prod_v \exp(\mathrm{i}\mu_v A_{\varphi(v)}) \tag{33}$$

which depends only on manifestly gauge invariant quantities.

The diffeomorphism constraint only generates transformations along B, which can be dealt with by group averaging in complete analogy with the full theory (see also [41, 42] for more details).

4.2. Spherically symmetric states from the full theory

Since symmetric states are by definition full states supported on invariant connections, one can, for a given symmetry action, find the form of symmetric states by restricting a complete set of full states. In a second step, one can then select an independent set of the resulting functions on $\bar{\mathcal{A}}_{B \times S^2}$

Inserting these holonomies is most easily done for an alternative form of the states rather than the spin network basis. All states can be obtained (in an overcomplete way) as products of Wilson loops which in our case are composed of radial or orbital holonomies. (If B has a boundary, there can be open ends of the loop where gauge transformations are frozen.) Each such state is a superposition of matrix elements of the form

$$h_x^{(e_1)} \cdot h_\vartheta^{(v_1,\mu_{\vartheta,1}^1)} h_\varphi^{(v_1,\mu_{\varphi,1}^1)} \cdots h_\vartheta^{(v_1,\mu_{\vartheta,n_1}^1)} h_\varphi^{(v_1,\mu_{\varphi,n_1}^1)}$$

$$\cdot h_x^{(e_2)} \cdot h_\vartheta^{(v_2,\mu_{\vartheta,1}^2)} h_\varphi^{(v_2,\mu_{\varphi,1}^2)} \cdots h_\vartheta^{(v_2,\mu_{\vartheta,n_2}^2)} h_\varphi^{(v_2,\mu_{\varphi,n_2}^2)}$$

$$\cdot h_x^{(e_3)} \cdots$$

with radial edges $e_1, e_2, e_3, \ldots$, and vertices $v_1, v_2, \ldots$, where v_1 is the endpoint of e_1 and the starting point of e_2. (A given edge e can appear several times in such an expression since it can be traversed back and forth with running through orbital edges in between.) The parameters $\mu_{\vartheta/\varphi,j}^i$ are the parameter lengths of orbital edges and can take any real value.

To simplify the general expressions evaluated in spherically symmetric connections we assume that the states are gauge invariant under gauge transformations around Λ_3, constant on the orbits (which still allows also open graphs). We can then gauge the angle $\beta(x)$ to be constant (with a local gauge transformation $\exp(-\beta(x)\Lambda_3)$, possibly up to a global gauge transformation if there is a boundary). Then also Λ_φ^A is constant and we can apply lemma 1 to (almost) commute the holonomies. In particular, we can order the radial holonomies according to coordinate values x of their starting points, also re-orienting them if necessary such that they run in the positive orientation of B. Between different edges there are vertices which can have the following forms:

$$\cdots \left(h_x^{(e_-)}\right)^{k_-} \cdot h_\varphi^{(v,\mu)} \cdot \left(h_x^{(e_+)}\right)^{k_+} \cdots,$$

$$\cdots \left(h_x^{(e_-)}\right)^{k_-} \Lambda_3 \cdot h_\varphi^{(v,\mu)} \cdot \left(h_x^{(e_+)}\right)^{k_+} \cdots \qquad \text{or} \qquad (37)$$

$$\cdots \left(h_x^{(e_-)}\right)^{k_-} \cdot h_\varphi^{(v,\mu)} \cdot \Lambda_3\left(h_x^{(e_+)}\right)^{k_+} \cdots$$

where possible factors of Λ_3 come from $\exp\left(\frac{1}{2}\pi\Lambda_3\right)$ in ϑ-holonomies (35).

Matrix elements of the resulting products of holonomies can easily be seen to be superpositions of states of the form (33), keeping in mind that we chose the gauge such that β is constant along B. It is also possible, though more tedious, to follow this procedure without fixing the gauge. We just mention the example of a single 'rectangular' loop made of one radial edge with holonomy h_x and two orbital ones along the equator at $x_\pm$ with holonomies $h_\pm$. The corresponding Wilson loop is

$$\text{tr}(h_x h_+ h_x^{-1} h_-^{-1}) = \text{tr}\left(\left(\cos\frac{1}{2}\int A_x + 2\Lambda_3\sin\frac{1}{2}\int A_x\right)\right.$$

$$\times \left(\cos\frac{1}{2}A_\varphi(x_+) + 2\Lambda_\varphi^A(x_+)\sin\frac{1}{2}A_\varphi(x_+)\right)$$

$$\times \left(\cos\frac{1}{2}\int A_x - 2\Lambda_3\sin\frac{1}{2}\int A_x\right)$$

$$\times \left.\left(\cos\frac{1}{2}A_\varphi(x_-) - 2\Lambda_\varphi^A(x_-)\sin\frac{1}{2}A_\varphi(x_-)\right)\right)$$

$$= 2\cos\frac{1}{2}A_\varphi(x_+)\cos\frac{1}{2}A_\varphi(x_-)$$

$$+ 2\cos\left(\int A_x + \beta(x_+) - \beta(x_-)\right)\sin\frac{1}{2}A_\varphi(x_+)\sin\frac{1}{2}A_\varphi(x_-) \qquad (38)$$

where we used $\mathrm{tr}\left(\Lambda_\varphi^A(x_+)\Lambda_\varphi^A(x_-)\right) = -\frac{1}{2}\cos(\beta(x_+) - \beta(x_-))$ and $\mathrm{tr}\left(\Lambda_3\Lambda_\varphi^A(x_+)\Lambda_\varphi^A(x_-)\right) = \frac{1}{4}\sin(\beta(x_+) - \beta(x_-))$. This state can clearly be written as a superposition of states (33).

Thus, the states obtained before from a loop quantization of the classically reduced phase space also emerge as symmetric states in the full theory. This is true, however, only with a slight restriction since full gauge invariant spin network states evaluated in spherically symmetric connections satisfy an additional condition: the gauge transformation $\exp\left(\frac{1}{2}\pi\Lambda_3\right)$ changes the sign of A_1 and A_2 everywhere, which means that all those states will be even under changing the sign of all $A_\varphi(x)$, as e.g. (38). This can easily be imposed as an additional condition on the reduced states, and it will be respected by operators coming from full ones.

4.2.2. Flux operators. In general, the dual action of operators of the full theory applied to distributional symmetric states will not lead to another symmetric state. The reason is that symmetric states only incorporate the condition for the connection to be invariant, but if full operators are used there is no condition for an invariant triad. In such a case, even classically the Hamiltonian flow generated by an arbitrary function on the phase space would in general leave the subspace of invariant connections with arbitrary triads (while the flow would always stay inside the subspace of invariant connections and invariant triads if the symmetric model is well defined). There are, however, notable exceptions which allow us to obtain all operators for the basic variables directly from the full theory. This is true for holonomies of A_x and A_φ which commute with connections, anyway. But we can also find special fluxes whose classical expressions generate a flow that stays in the subspace of invariant connections. For instance, for the Λ_3-component of a full flux for a symmetry orbit S^2, $F_{S^2}^3(x) := \int_{S^2}\Lambda_3 \cdot (i_{E(x)}\,\mathrm{d}x)\,\mathrm{d}^2y$, we have

$$\left\{A_a^i(x), F_{S^2}^3\right\}|_{\mathcal{A}_{\mathrm{inv}}\times\mathcal{E}} = \gamma\kappa\Lambda_3^i\delta_a^x \int_{S^2}\delta(x, y)\,\mathrm{d}^2y$$

which defines a distributional vector field on the phase space parallel to the subspace $\mathcal{A}_{\mathrm{inv}} \times \mathcal{E}$ of invariant connections (parallel to A_x). If we would look at any other internal component, e.g., $F_{S^2}^2$ using Λ_2, on the other hand, the Poisson bracket would be proportional to $\Lambda_2^i\delta_a^x$, which is not parallel to the subspace of invariant connections. Similarly, one can see that the flux

$$F_{\mathcal{I}\times S^1} := \int_{\mathcal{I}\times S^1}\left(\Lambda_\varphi^A(x) \cdot (i_{E(x)}\,\mathrm{d}\varphi)\,\mathrm{d}x\,\mathrm{d}\vartheta + \Lambda_\vartheta^A(x) \cdot (i_{E(x)}\,\mathrm{d}\vartheta)\,\mathrm{d}x\,\mathrm{d}\varphi\right)$$

for a cylindrical surface along an interval $\mathcal{I} \subset B$ generates a flow parallel to A_φ, which leaves the space of invariant connections invariant.

These two fluxes are sufficient for the basic momenta since

$$\int_{S^2}\Lambda_3 \cdot (i_{E(x)}\,\mathrm{d}x)\,\mathrm{d}^2y = 4\pi E^x(x)$$

and

$$\int_{\mathcal{I}\times S^1}\left(\Lambda_\varphi^A(x) \cdot (i_{E(x)}\,\mathrm{d}\varphi)\,\mathrm{d}x\,\mathrm{d}\vartheta + \Lambda_\vartheta^A(x) \cdot (i_{E(x)}\,\mathrm{d}\vartheta)\,\mathrm{d}x\,\mathrm{d}\varphi\right) = 4\pi\int_{\mathcal{I}}P^\varphi(x)\,\mathrm{d}x$$

whose quantizations can thus be obtained directly from the full theory. Note that this would not be possible for E^φ, for instance, since its corresponding flux would generate a transformation changing the invariant form of A (since any full expression reducing to E^φ upon reduction would involve the nonlinear function Λ_E^φ depending on the triad components).

For the flow $F_{S^2}^3(x)$ we obtain the Λ_3-component of an invariant vector field associated with the edges containing x. The pull back to invariant connections in (2) ensures that the dual action on the distribution Ψ can be expressed by an invariant vector field on the representation

ψ where only radial holonomies $h_x^{(e)}$ of the form (34) appear. For the explicit expression we again assume that x is an endpoint of two edges, $e^+(x)$ and $e^-(x)$ which can be achieved by appropriate subdivision, and obtain

$$\hat{F}_{S^2}^3(x) = \frac{1}{2} i \gamma \ell_P^2 \left(\text{tr}\left(\Lambda_3 h_x^{(e^+(x))}\right)^T \frac{\partial}{\partial h_x^{(e^+(x))}} + \text{tr}\left(\Lambda_3 h_x^{(e^-(x))}\right)^T \frac{\partial}{\partial h_x^{(e^-(x))}} \right). \tag{39}$$

Since Λ_3 commutes with radial holonomies $h_x^{(e)}$, we do not need to distinguish between left and right invariant vector field operators. According to the derivation of states above, they can be seen as polynomials in the radial holonomies. The action of a derivative operator $\text{tr}(\Lambda_3 h_x^{(e)})^T \partial / \partial h_x^{(e)}$ with respect to $h_x^{(e)}$ then amounts to replacing $(h_x^{(e)})^k$ by $k \Lambda_3 (h_x^{(e)})^k$ and $\Lambda_3 (h_x^{(e)})^k$ by $-\frac{1}{4} k (h_x^{(e)})^k$ (note that insertions of Λ_3 appear automatically as in (37); in any case, they would occur when considering a more general class of gauge non-invariant states in the full theory which are invariant from the reduced point of view). Eigenstates of the derivative can then be obtained by forming linear combinations such that only the combinations $(h_x^{(e)})^k \pm 2i\Lambda_3 (h_x^{(e)})^k$ appear, which are mapped to $\mp \frac{1}{2} ik((h_x^{(e)})^k \pm 2i\Lambda_3 (h_x^{(e)})^k)$. In this way, one obtains eigenstates of $\hat{E}^x(x) = (4\pi)^{-1} \hat{F}_{S^2}^3(x)$ and a spectrum identical to (28).

The operator $\hat{E}^x$ also appears in the Gauss constraint. A gauge invariant state in the full theory in particular satisfies $\Lambda_3 \cdot \left(J_L(h_x^{(e^+(x))}) - J_R(h_x^{(e^-(x))}) + J_L(h_\varphi^{(x)}) - J_R(h_\varphi^{(x)}) \right) = 0$ in any vertex x where we can assume the form (37) for the spin network (a general vertex would just be a superposition of those vertices). The operators $\Lambda_3 \cdot J(h_x^{(e^\pm(x))})$ simply give operators $\hat{E}^x$ where we do not need to distinguish between right and left invariant ones, while for the derivative operators with respect to φ-holonomies we have

$$\Lambda_3 \cdot \left(J_L(h_\varphi^{(x)}) - J_R(h_\varphi^{(x)}) \right) = -i \left(\text{tr}(h_\varphi^{(x)} \Lambda_3)^T \frac{\partial}{\partial h_\varphi^{(x)}} - \text{tr}(\Lambda_3 h_\varphi^{(x)})^T \frac{\partial}{\partial h_\varphi^{(x)}} \right)$$

$$= i \, \text{tr}[\Lambda_3, h_\varphi^{(x)}]^T \frac{\partial}{\partial h_\varphi^{(x)}} = i \, \text{tr} \frac{\partial h_\varphi^{(x)}}{\partial \beta(x)} \frac{\partial}{\partial h_\varphi^{(x)}}$$

$$= i \frac{\partial}{\partial \beta(x)}$$

using $[\Lambda_3, h_\varphi^{(x)}] = \partial h_\varphi^{(x)} / \partial \beta(x)$ with $\Lambda_\varphi^A(x) = \cos \beta(x) \Lambda_1 + \sin \beta(x) \Lambda_2$. The right-hand side is then simply proportional to $\hat{P}^\beta(x)$ and we see that the Λ_3-component of the full quantum Gauss constraint is identical to the reduced Gauss constraint. (The remaining components of the full Gauss constraint would not fix the space of symmetric states and thus cannot be dealt with in this way. They would have to be satisfied identically.) In what follows we use the above equation to define the operator $\hat{P}^\beta$ which then has the spectrum (30) as in the reduced case.

It remains to look at the full quantization of

$$\int_{\mathcal{I}} P^\varphi \, dx = \frac{1}{4\pi} \int_{\mathcal{I} \times S^1} \left(\Lambda_\varphi^A \cdot (i_E \, d\varphi) \, dx \, d\vartheta + \Lambda_\vartheta^A \cdot (i_E \, d\vartheta) \, dx \, d\varphi \right)$$

acting on symmetric states. Since they are now Λ_φ^A-components of derivative operators with respect to h_φ, the end result is again a simple derivative operator acting on powers h_φ^μ, which gives a spectrum proportional to that in (29). It is only proportional since we chose the surface for this flux using the great circle S^1, which contributes a factor 2π. Choosing other circles on the orbits would change the factor which, anyway, can always be absorbed by a unitary transformation. (Such a rescaling is unitary for this operator, as in the isotropic case [36], since the range of eigenvalues μ is the real line.)

4.3. Reduced states from the full point of view

The derivations presented above demonstrate that the states obtained from the purely reduced point of view are also necessary in this form when viewing them as being obtained by restricting full states. Also basic flux operators obtained in the reduced quantization and from the full theory via the dual action on distributional symmetric states agree. Thus, the model can be seen as a symmetric sector of the full theory, associated with a subspace of the distribution space Cyl*.

Full symmetric states which satisfy the 3-component of the full Gauss constraint are also gauge invariant from the reduced point of view. In fact, we can directly take the dual action of the 3-component of the Gauss constraint to obtain the reduced constraint (the other components do not map the space of symmetric states to itself). In the reduced model, this has as a consequence that the difference of labels associated with neighbouring radial edges has to be even. The analogue in the full theory can be seen by considering vertices where radial and orbital edges meet. After inserting invariant connections, a state with such a vertex is equivalent to a superposition of states with a vertex having one incoming radial edge, a composition of several orbital edges and an outgoing radial edge. From this point of view, this gauge invariant vertex is a $2(n + 1)$-vertex with the ingoing and outgoing radial edges with representations j_- and j_+, respectively, as well as n-closed orbital edges which each contribute one incoming and one outgoing part with spin j_i. For an intertwiner we can first construct the tensor product of the orbital representations, $\bigotimes_{i=1}^{n} j_i \otimes j_i = \bigoplus_i l_i$ where only integer l_i occur in the decomposition. The vertex intertwiner maps this representation to the tensor product $j_+ \otimes j_-$, which for integer l_i is possible in a nontrivial way only if j_+ and j_- are either both integer or both $1/2$ times an integer. Thus, $j_+ - j_- \in \mathbb{Z}$, which is equivalent to the fact that the difference of charges $k_+ - k_-$ must be even.

Similarly, the reduced diffeomorphism constraint can be obtained directly from the dual action of the full diffeomorphism constraint for a radial shift vector. For other shifts, the dual action would not fix the space of symmetric states. For a radial shift, then, the constraint generates transformations which move vertices along the radial manifold, which is the same as the action generated by the reduced diffeomorphism constraint. Thus, also the reduced diffeomorphism constraint can be obtained via the dual action on symmetric states which leads to the same results as quantizing the classically reduced constraint. The Hamiltonian constraint, on the other hand, is nonlinear in the triads such that its dual action cannot be used.

We thus have seen how states and basic operators of the reduced model can be obtained from the full theory. Composite operators can then be built from the basic ones within the model. An analogous derivation of composite-reduced operators from those in the full theory is more complicated since a direct application of the dual action would not fix the space of symmetric states.

It follows from these considerations that the representation of the reduced model is determined by that of the full theory. Since the diffeomorphism covariant holonomy/flux representation of full loop quantum gravity is unique under certain weak conditions [43], a representation for the model is selected naturally. Starting from the classically reduced model, on the other hand, would have left open the choice of representation. In such a case, the representation is usually selected in such a way that explicit calculations are possible, which does not say anything about physical correctness. Even working in the framework of this paper and using the same variables, there would be other possibilities. For instance, viewing the scalar density P^φ as a 1-form, which in one dimension has the same transformation properties if the orientation is preserved, suggests to quantize it via holonomies. In this case, A_φ rather than P^φ would become discrete. Similarly, we could use point holonomies for the densitized

vector field component E^x, which can also be viewed as a scalar. This would give a discrete A_x. All these alternatives are possible only in the reduced model due to the special behaviour under coordinate transformations. But they are not possible in the full theory and thus cannot be obtained when the link between the model and the full quantization is taken into account. Studying these representations further can shed light on physical properties and effects that are unique to the loop representation of the full theory.

4.4. Semiclassical geometry

The flux eigenvalues allow us to find conditions for states which would be expected in regimes where the spatial geometry is almost classical. Comparing (21) with the Schwarzschild solution at large radius, or general asymptotic conditions, shows that E^x (corresponding to r^2 for Schwarzschild) should be large together with $E^{x\prime} = P^\beta$. This implies that the edge labels k_e and the differences $k_{e^+(v)} - k_{e^-(v)} = 2k_v$ have to be large compared to one since eigenvalues of $\hat{E}^x$ and $\hat{P}^\beta$ are directly given by the labels without summing over vertices. This is analogous to the homogeneous case where for a semiclassical geometry all labels have to be large.

The situation is different for the other triad components. From the metric we see that also E^φ has to be large which, for generic α implies that P^φ must be large. However, the quantization of the density P^φ is well defined only if it is first integrated over an interval in B, which means that the relevant eigenvalues are given by a vertex sum $\sum_v \mu_v$, which needs to be large. This can be realized by large individual μ_v, or by a dense distribution of vertices such that many small μ_v add up to a large value. This situation is analogous to that in the full theory where geometric operators are always given by vertex sums. It is then expected that states with many small labels are relevant for a semiclassical geometry since they dominate the counting of states.

At this point, the difference between P^φ and E^φ suggests possible consequences for semiclassical physics. If we fix a μ_φ (which happens, e.g., if we consider the dynamics [36]) and restrict the operators to a separable subspace of our Hilbert space generated by $e^{i\mu_\varphi A_\varphi}$, we would have a discrete set $\mu_\varphi n$ of eigenvalues with integer n. Then, the sum $\sum_v \mu_v = \mu_\varphi \sum_v n_v$ would still be of the same form and not become denser at large eigenvalues (as would happen in the full theory for, e.g., the area operator). The triad component E^φ, which appears in the metric (21), however, is a more complicated function of the basic variables and thus is likely to have a more complicated vertex contribution which could lead to crowded eigenvalues [34].

5. Other 1+1 models

There are many other models in 1+1 dimensions which have infinitely many physical degrees of freedom, but are integrable [14, 45], and which would be interesting to compare with a loop quantization. The general form of an invariant connection in those cases is (3) where the $\Lambda_I(x) \in su(2)$, $\mathrm{tr}(\Lambda_I(x)^2) = -\frac{1}{2}$ can be restricted further depending on the symmetry action. In general, however, they do not satisfy $\mathrm{tr}(\Lambda_I \Lambda_J) = -\frac{1}{2}\delta_{IJ}$, which was the case in the spherically symmetric model with its nontrivial isotropy group and was responsible for the simplified structure of states and basic operators.

In cylindrically symmetric models with a space manifold $\Sigma = \mathbb{R} \times (S^1 \times \mathbb{R})$, for instance, the symmetry group $S = S^1 \times \mathbb{R}$ acts freely, and invariant connections and triads have the form

$$A = A_x(x)\Lambda_3 \, dx + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2) \, dz + (A_3(x)\Lambda_1 + A_4(x)\Lambda_2) \, d\varphi \qquad (40)$$

$$E = E^x(x)\Lambda_3 \frac{\partial}{\partial x} + (E^1(x)\Lambda_1 + E^2(x)\Lambda_2)\frac{\partial}{\partial z} + (E^3(x)\Lambda_1 + E^4(x)\Lambda_2)\frac{\partial}{\partial \varphi} \tag{41}$$

such that $\text{tr}(\Lambda_3\Lambda_z) = 0 = \text{tr}(\Lambda_3\Lambda_\varphi)$, but in general $\text{tr}(\Lambda_z\Lambda_\varphi) \neq 0$.

The corresponding metric is

$$ds^2 = (E^x)^{-1}(E^1E^4 - E^2E^3)\,dx^2 + E^x(E^1E^4 - E^2E^3)^{-1}(((E^3)^2 + (E^4)^2)\,dz^2$$
$$- (E^2E^4 + E^1E^3)\,dz\,d\varphi + ((E^1)^2 + (E^2)^2)\,d\varphi^2) \tag{42}$$

which is not diagonal. To simplify the model further, one often requires that the metric is diagonal, which physically corresponds to selecting a particular polarization of Einstein–Rosen waves. This is achieved by imposing the additional condition $E^2E^4 + E^1E^3 = 0$ which, in order to yield a non-degenerate symplectic structure, has to be accompanied by $A_2A_4 + A_1A_3 = 0$ for the connection components. Thus, polarized cylindrical waves of this form also have perpendicular internal directions since now $\text{tr}(\Lambda_z\Lambda_\varphi) = 0$ for both A and E, and similar simplifications as in the spherically symmetric case can be expected.

The form of the metric now is

$$ds^2 = (E^x)^{-1}E^zE^\varphi\,dx^2 + E^x(E^\varphi/E^z\,dz^2 + E^z/E^\varphi\,d\varphi^2) \tag{43}$$

with

$$E^z := \sqrt{(E^1)^2 + (E^2)^2}, \qquad E^\varphi := \sqrt{(E^3)^2 + (E^4)^2}. \tag{44}$$

Einstein–Rosen waves are usually represented in the form

$$ds^2 = e^{2(\gamma - \psi)}\,dr^2 + e^{2\psi}\,dz^2 + e^{-2\psi}r^2\,d\varphi^2 \tag{45}$$

with only two free functions γ and ψ. Thus, compared with (43) one function has been eliminated by gauge fixing the diffeomorphism constraint.

In fact, this form can be obtained from the more general (43) by a field-dependent coordinate change [46]: the symmetry reduction leads to a spacetime metric $ds^2 = e^\Lambda\,dU\,dV + W(e^{-\Psi}\,dx^2 + e^\Psi\,dy^2)$ which indeed has a spatial part as in (43) with three independent functions Λ, $W = E^x$ and $\Psi = \log(E^\varphi/E^z)$. One then introduces $t := \frac{1}{2}(V - U)$ and $\rho := \frac{1}{2}(V + U)$ such that $ds^2 = e^\Lambda(-dt^2 + d\rho^2) + \rho(e^{-\Psi}\,dx^2 + e^\Psi\,dy^2)$. Finally, defining $\Lambda = 2(\gamma - \psi)$, $e^{-2\psi}\rho := e^{-\Psi}$ and renaming $x =: \varphi$, $y =: z$ leads to the metric (45). Einstein's field equations then imply that ψ behaves as a free scalar on a flat spacetime, which can be quantized with standard Fock techniques [14]. However, to arrive at this form of the metric, several coordinate transformations have been performed which mix coordinates with the physical fields. This potentially eliminates any contact the model may have with a full theory and indicates that results may be very particular to this kind of model. From the point of view taken here, where the quantum representation comes directly from the full theory, a subsequent transformation in such a way is impossible, which also means that the quantization of the model will be more complicated. We will see soon that the transversal geometry given by (43) becomes discrete after loop quantizing, which ψ when treated as an ordinary scalar will not be. Thus, the quantum geometries obtained from both representations differ from each other, which can also lead to differing physical results (as in the homogeneous case, where loop properties are extremely different from Wheeler–DeWitt results concerning the issue of singularities and also phenomenology). This may in particular be of interest in view of large quantum gravity effects derived from wave models [17–19]. On the other hand, a direct comparison between different quantizations is made more complicated by the coordinate-dependent field transformation.

We now present the initial steps of a loop quantization along the lines followed in the spherically symmetric model. This will allow us to see some properties of the quantum

geometry. Most of the steps to arrive at the kinematical Hilbert space and basic operators can be done almost identically to those followed before.

In analogy with the spherically symmetric model, we now introduce

$$A_z := \sqrt{A_1^2 + A_2^2}, \qquad A_\varphi := \sqrt{A_3^2 + A_4^2} \tag{46}$$

$$\Lambda_z^A := \frac{A_1\Lambda_1 + A_2\Lambda_2}{A_z}, \qquad \Lambda_\varphi^A := \frac{A_3\Lambda_1 + A_4\Lambda_2}{A_\varphi} \tag{47}$$

and analogously E^z, E^φ, Λ_E^z and Λ_E^φ. Furthermore, we write

$$\Lambda_\varphi^A = \cos\beta\Lambda_1 + \sin\beta\Lambda_2, \tag{48}$$

$$\Lambda_E^\varphi = \cos(\alpha + \beta)\Lambda_1 + \sin(\alpha + \beta)\Lambda_2. \tag{49}$$

With the polarization condition, this implies

$$\Lambda_z^A = -\sin\beta\Lambda_1 + \cos\beta\Lambda_2 \qquad \Lambda_E^z = -\sin(\alpha + \beta)\Lambda_1 + \cos(\alpha + \beta)\Lambda_2$$

such that we have only two angles, β which is pure gauge and α as in the spherically symmetric model.

The symplectic structure tells us that momenta of A_z and A_φ are not given by triad components directly, but by $P^z := E^z\cos\alpha$ and $P^\varphi := E^\varphi\cos\alpha$. The momentum of β is $P^\beta := (A_z E^z + A_\varphi E^\varphi)\sin\alpha$, which is related to E^x by the same Gauss constraint as in the spherically symmetric case.

The adaptation of the construction of states and operators to this model is now straightforward, the only difference being that we have one additional degree of freedom per point on B, given by A_z for which we have additional holonomies $\exp(i\mu_z A_z)$ in vertices of spin network states. As before, flux operators do not give us direct information about the geometry since fluxes are related to the triad in a more complicated way. Still, the orbital components of the metric in the z and φ directions are easily accessible since $E^z/E^\varphi = P^z/P^\varphi$ thanks to a cancellation of $\cos\alpha$. Thus, the spectrum of the orbital geometry can easily be computed, after using techniques as in [47] to quantize the inverse momenta. The radial geometry, however, and thus the volume are more complicated, similarly to the spherically symmetric volume.

6. Conclusions

As discussed in this paper, states and basic operators for symmetric models can be obtained from full loop quantum gravity in a direct way and lead to considerable simplifications even in inhomogeneous models. Hopefully, this will eventually lead to explicit investigations of important problems in the full theory, such as general field theory aspects (in particular relating loop to standard field theory techniques, e.g., [27, 48–50]), issues of the constraint algebra [32, 51–54] and the master constraint [55], as well as explicit constructions of semiclassical states [56–59]. Even though, compared to homogeneous models, the system is much more complicated with infinitely many kinematical degrees of freedom, the situation is simpler than in the full theory.

In addition to the structure of states and basic operators discussed before, an advantage is that fluxes commute with each other such that there exists a flux representation. Transforming to such a representation from the connection representation has been of significant advantage in homogeneous models, but is not possible in the full theory with its non-commuting fluxes [60]. Even so, the Hamiltonian constraint equation will turn into a functional difference

equation with infinitely many independent variables for which most likely new techniques would have to be developed.

An unexpected complication can arise in inhomogeneous models since momenta conjugate to the connection may not be identical to triad components. Thus, even though basic operators are easy to deal with explicitly, this does not necessarily translate to direct access to the quantum geometry, most importantly the volume operator. Since the volume operator also plays an important role in defining the dynamics [32] and other interesting operators, a complicated volume operator whose spectrum is not known explicitly would probably render calculations in the model almost as hard as those in the full theory. It turns out that the spherically symmetric model still allows us to diagonalize the volume operator explicitly, and to develop an explicit calculus rather similar to that in homogeneous models [34]. This fact opens up the possibility of new conceptual investigations and applications to the physics of black holes.

Acknowledgments

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