ON TWISTOR QUANTIZATION OF LOOP SPACES AND KÄHLER MANIFOLDS

A.G.SERGEEV

Abstract. We present a twistor reformulation of the geometric quantization method for loop spaces of Lie groups and Kähler manifolds.

We consider a twistor approach to the geometric quantization of Kähler manifolds. The idea to use twistors in the geometric quantization of string theory was proposed in [3]. This idea was realized in [6,4] where the twistor quantization of string theory was considered. The main points of this approach are presented in Sec. 2 of this paper. In Sec. 3 we propose a definition of twistor quantization for Kähler manifolds satisfying to the quantization condition (the symplectic structure of a Kähler manifold should be integral).

1. Geometric quantization method

We recall first the geometric quantization method for finite-dimensional Kähler phase manifolds.

1.1. Classical systems.

We start from a `classical` (mechanical) `system` given by the pair $(M, \mathfrak{A})$ where M is a phase manifold and $\mathfrak{A}$ is an algebra of observables consisting of real-valued smooth functions (Hamiltonians) on M.

Mathematically, the `phase manifold` M is a symplectic manifold of dimension $2n$ with a symplectic form ω. Locally, M is isomorphic to the vector space $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega = \omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$. The coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ on $\mathbb{R}^{2n}$ for which the symplectic form has the above standard form are called canonical, q_i play the role of physical "coordinates" in the phase space while p_i – that of physical "momenta".

Apart from the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, the other standard examples of phase manifolds are provided by cotangent bundles and coadjoint orbits of Lie groups. In the first case, $M = T^*N$ where N is an n-dimensional smooth manifold (called the configuration space of the system). The canonical coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ where (q_i) are coordinates on N and (p_i) are coordinates in the fibre can be introduced now only locally but the form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ is still globally defined as a 2-form on M. Moreover, it is exact being the exterior derivative of the 1-form given in canonical coordinates by $\theta = \sum_{i=1}^n p_i dq_i$. In case

Research partly supported by the INTAS Association (grant 93-0047) and Russian Foundation for Fundamental Research (grant 95-011-00027)

of a coadjoint orbit of a Lie group the symplectic structure is given by the Kirillov form.

The `algebra of observables` $\mathfrak{A}$, mathematically, is a Lie algebra of real-valued smooth functions on the phase space M with respect to the Poisson bracket determined by the symplectic structure ω.

For the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ one can take for such an algebra the Heisenberg algebra $\text{heis}(\mathbb{R}^{2n})$ of $\mathbb{R}^{2n}$, i.e. a central extension of the Lie algebra generated by coordinate functions p_i, q_i, $i = 1, \ldots, n$. Another example arises when a Lie group of symplectic diffeomorphisms Γ acts on M so that its Lie algebra is a subalgebra of the Lie algebra of Hamiltonian vector fields on M. If M is simply connected, this Lie algebra may be dually realized as a Lie algebra of real smooth functions (Hamiltonians) on M. It is the Lie algebra $\text{Ham}(\Gamma)$ consisting of functions on M generating transformations from Γ.

1.2. Dirac quantization.

By the `quantization of a classical system` $(M, \mathfrak{A})$ we call an irreducible representation of the Lie algebra of observables

$$r : \mathfrak{A} \longrightarrow \text{End}^* H$$

in the Lie algebra of linear self-adjoint operators in a complex (separable) Hilbert space H with the Lie bracket given by the commutator of linear operators of the form $i(AB - BA)$. We also assume that $r(1) = \text{id}$. The Hilbert space H is called the `quantization space`. (If the above representation is reducible it is called the prequantization while H is called the prequantization space).

Note that the representation operators $r(f)$ for $f \in \mathfrak{A}$ are usually unbounded, hence defined only on a dense subspace in H. In this case we assume that all representation operators have a common dense definition domain in H.

In case of a classical system given by the cotangent bundle $M = T^*N$ (with standard symplectic structure ω) and the "maximal" algebra of observables $\mathfrak{A} = C^\infty(M)$ one can easily construct its prequantization. Take for the Hilbert prequantization space H the space $H = L^2(M, \omega)$ of square integrable functions on M (with respect to the volume form on M given by the n-th power ω^n of symplectic form ω). A representation of the algebra of observables $C^\infty(M)$ in H is given by the formula

$$f \longmapsto r(f) = f - iX_f - \theta(X_f), \; f \in C^\infty(M) \tag{1}$$

where f in the right hand side is considered as the operator of multiplication by f in H, X_f is the Hamiltonian vector field associated with f, $\theta(X_f)$ is the value of the form θ on X_f (considered also as the multiplication operator).

Unfortunately, this representation is reducible (i.e. it does not yield a quantization). Physically, it is due to the fact that H contains functions depending both on p_j and q_j while, according to the Heisenberg uncertainty principle, the "physical" quantization space H should contain wave functions which cannot depend on some pair of variables (p_j, q_j) simultaneously.

1.3. Kostant–Souriau prequantization.

In case of a general symplectic manifold M of dimension $2n$, its symplectic form ω is exact only locally so that there exists an open covering $\{U_\alpha\}$ of M such that

$$\omega = d\theta_\alpha \quad \text{for} \quad U_\alpha$$

for some 1-forms θ_α. It turns out that the representation operators r_α in spaces $L^2(U_\alpha, \omega)$, defined by the formula (1), under some topological restrictions may be "glued together" into a one operator r, though acting not on functions but on sections of a certain complex line bundle over M.

This line bundle on M exists only if the manifold M satisfies to the following quantization condition: *the form $\omega/2\pi$ represents an integral cohomology class in $H^2(M,\mathbb{R})$*. Under this condition there exists a complex line bundle $L \to M$ with a linear connection ∇ on it having the curvature equal to ω. This line bundle is called the `prequantum bundle` on M.

We fix now a Hermitian structure $< \cdot, \cdot >$ on L and denote by $H = \mathcal{L}^2(M, L; \omega)$ the Hilbert space of square integrable sections of $L \to M$ with respect to the inner product define by

$$(s_1, s_2) = \int_M < s_1, s_2 > \omega^n \ .$$

Then the `Kostant-Souriau formula`

$$f \longmapsto r(f) = f - i\nabla_{X_f}$$

defines a representation of the Lie algebra $C^\infty(M)$ in $\mathcal{L}^2(M, L; \omega)$, i.e. a prequantization of $C^\infty(M)$ in H.

This representation is reducible by the same reasons as before. In order to make it irreducible one should restrict it, roughly speaking, to the "half" of the space H containing the functions from H which depend, say, only on momenta $\{p_j\}$ or only on coordinates $\{q_j\}$. This idea can be formalized using the notion of a polarization on the phase manifold M (cf., e.g., [5,8]). On a polarized phase manifold one should take for the "half" of H the subspace of polarized sections in H.

If M is Kähler then it's natural to take for the "half" of H the subspace $\mathcal{L}^2_{\mathcal{O}}(M, L; \omega)$ in $\mathcal{L}^2(M, L; \omega)$ consisting of square integrable holomorphic sections of L (such a polarization is called Kähler). If the Kostant-Souriau representation operators, acting in $\mathcal{L}^2(M, L; \omega)$, would leave the subspace $\mathcal{L}^2_{\mathcal{O}}(M, L; \omega)$ invariant, then their restrictions to $\mathcal{L}^2_{\mathcal{O}}(M, L; \omega)$ would provide an irreducible representation of the algebra $\mathfrak{A}$ in $\mathcal{L}^2_{\mathcal{O}}(M, L; \omega)$.

Unfortunately, this idea works only for a "minimal" algebra of observables, the Heisenberg algebra heis($\mathbb{R}^{2n}$) of the vector space $\mathbb{R}^{2n}$ (and for its analogue – the translation algebra – in case of a homogeneous phase manifold M). In case of a general phase manifold the Kostant–Souriau representation operators, in general, do not preserve the space $\mathcal{L}^2_{\mathcal{O}}(M, L; \omega)$. The reason for that is that symplectic diffeomorphisms generated by the elements of the algebra $\mathfrak{A}$ (considered as a subalgebra of the algebra of Hamiltonian vector fields on M), in general, do not preserve the original complex structure of the Kähler manifold M. Otherwise speaking, the complex structure, contrary to its symplectic counterpart, is not a "physical quantity" attributed to the quantum system associated with $(M, \mathfrak{A})$ and we have no right to fix it.

In this situation we propose, instead of fixing a complex structure on M, to consider all possible complex structures obtained from the original one by the action of symplectic diffeomorphisms corresponding to Hamiltonian vector fields from $\mathfrak{A}$. Such structures can be identified with sections of a certain twistor bundle $Z \to M$ over the phase manifold M which has (as in conventional twistor theory) a canonical almost complex structure. The idea of twistor quantization is then to reformulate the original quantization problem in terms of this twistor bundle provided with its canonical almost complex structure. Such twistor quantization will not depend on the choice of a particular complex structure on M and, having solved the twistor problem, we may look for conditions under which this solution can be "pushed down" from the twistor bundle Z to its base manifold M. The latter problem may be treated then using methods of the conventional twistor theory.

2. Twistor quantization of loop spaces

The idea of twistor quantization proposed at the end of last section was realized in [6,4] in case of bosonic string theory.

2.1. Classical system.

For strings in the d-dimensional vector space $\mathbb{R}^d$ the `phase space` of this theory is identified with the `loop space` $\Omega^d := \Omega\mathbb{R}^d$, i.e. with the Frechet vector space $\mathrm{Map}_0(S^1, \mathbb{R}^d)$ of smooth maps $S^1 \to \mathbb{R}^d$ having the zero average over S^1.

The `algebra of observables` $\mathfrak{A}$ has two components. The first one is the Heisenberg algebra of Ω^d, i.e. a central extension of the Lie algebra generated by coordinate functions on Ω^d. The second one is the Lie algebra $\mathrm{Ham}(\Gamma)$ where $\Gamma = \mathrm{Diff}_+(S^1)$ is the group of orientation preserving diffeomorphisms of S^1 (or, physically speaking, the group of string reparametrizations). This algebra is identified with the Lie algebra $\mathrm{Vect}(S^1)$ of smooth tangent vector fields on S^1. The algebra of observables $\mathfrak{A}$ is then the semi-direct product of these two components, i.e. of algebras $\mathrm{heis}(\Omega^d)$ and $\mathrm{Ham}(\mathrm{Diff}_+(S^1))$. Similar to the Poincaré group of special relativity, the first component $\mathrm{heis}(\Omega^d)$ (a "linear" part) is the group of translations of Ω^d and the second component $\mathrm{Ham}(\mathrm{Diff}_+(S^1))$ (a "quadratic" part) is a subgroup of "symplectic rotations" group of Ω^d.

For strings in a compact (simply connected) Lie group G the `phase space` is identified with the loop space $\Omega_T = \mathrm{Map}(S^1, G)/T$, i.e. with the quotient of the mapping group $\mathrm{Map}(S^1, G)$ modulo the subgroup consisting of constant maps $S^1 \to t_0 \in T$ into a fixed maximal torus T of G. The `algebra of observables` $\mathfrak{A}$ has again two components. The first one, analogous to the Heisenberg algebra above, is given by a central extension $\widetilde{L\mathfrak{g}}$ of the mapping Lie algebra $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. The second component is again the Lie algebra $\mathrm{Vect}(S^1)$ of smooth tangent vector fields on S^1 so that $\mathfrak{A}$ is the semi-direct product of these two components.

2.2. Twistor bundle for flat space loops.

The geometric quantization of the classical system, we have just defined, means the construction of an irreducible representation of the algebra of observables $\mathfrak{A}$ in the Lie algebra End^*H of linear self-adjoint operators in a complex Hilbert space H. We reformulate this problem in twistor terms.

We define first a twistor bundle of complex structures on the phase manifold of our theory in case of Ω^d. Consider all complex structures J on Ω^d obtained from the standard one J^0 by the natural action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on Ω^d. We call these complex structures `admissible`. These complex structures on Ω^d are compatible with the symplectic structure of Ω^d and invariant under translations of Ω^d. We consider next the `twistor bundle` $\pi : Z \to \Omega^d$ of admissible complex structures on Ω^d having the fibre Z_x at $x \in \Omega^d$ formed by restrictions J_x of admissible complex structures J to the tangent space $T_x\Omega^d \cong \Omega^d$. This fibre may be identified with the Frechet Kähler manifold $\mathcal{S} := \mathrm{Diff}_+(S^1)/(S^1)$. The bundle $\pi : Z \to \Omega^d$ is trivial and admissible complex structures on Ω^d are translation-invariant sections of this bundle. It follows that there exists a natural projection $p : Z \to \mathcal{S}$ assigning to a point $z = (x, J_x) \in Z$ the translation-invariant complex structure $J = J_x$ on Ω^d. The fibre $p^{-1}(J)$ of this projection is identified with the space $\Omega^d_J = (\Omega^d, J)$ provided with the complex structure J.

The twistor space Z has a natural `complex structure`. Indeed, we consider the decomposition of the tangent bundle TZ into the direct sum of subbundles

$$TZ = \mathcal{V} \oplus \mathcal{H}$$

where the vertical subbundle $\mathcal{V}$ identifies with the tangent bundle to the fibres of π and the horizontal subbundle $\mathcal{H}$ — with the tangent bundle to the fibres of p. The complex structure $\mathcal{I}$ on Z is, by definition, the direct sum

$$\mathcal{I}_z = \mathcal{I}_z^v \oplus \mathcal{I}_z^h$$

of the natural complex structure $\mathcal{I}_z^v$ on the vertical space $\mathcal{V}_z$ identified (by p_*) with the tangent space $T_{p(z)}\mathcal{S}$ to the Kähler manifold $\mathcal{S}$ and the complex structure $\mathcal{I}_z^h = J_{\pi(z)}$ on the horizontal space $\mathcal{H}_z$ identified (by π_*) with the tangent space $T_{\pi(z)}\Omega^d$. (Note that p is holomorphic with respect to the introduced complex structure while π not!).

The Heisenberg group (the Lie group of the Heisenberg algebra $\mathrm{heis}(\Omega^d)$) acts on Z horizontally, preserving the fibres of the projection p and the complex structure of Ω^d. The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts on Z vertically and this action is induced by the action of $\mathrm{Diff}_+(S^1)$ on $\mathcal{S}$ by left translations.

2.3. Fock bundle.

We shall reformulate the original quantization problem in terms of the twistor bundle $\pi : Z \to \Omega^d$. The quantization space H will correspond in twistor terms to a quantization bundle $\mathcal{H} \to \mathcal{S}$ and the representation of the algebra of observables $\mathfrak{A}$ will correspond to a flat connection on $\mathcal{H} \to \mathcal{S}$.

The quantization bundle $\mathcal{H} \to \mathcal{S}$ will be introduced later on as an "extended" Fock bundle of half-densities on Ω^d. We start with the construction of a Fock bundle to be considered as a "first approximation" to our quantization bundle.

Let $V := H_0^{1/2}(S^1, \mathbb{R}^d)$ be the Sobolev space completion of the phase space Ω^d (subindex "0" means that we take the functions from the Sobolev space $H^{1/2}(S^1, \mathbb{R}^d)$ having the zero average over the circle). Consider the Fock space F_J of V provided with an admissible complex structure J (extended by continuity to V) which is defined as follows. The complex structure J generates a decomposition of the complexified vector space $V^{\mathbb{C}}$ into the direct sum $V^{\mathbb{C}} = W \oplus \bar{W}$ of $(\pm i)$-eigenspaces of J on $V^{\mathbb{C}}$. In terms of this decomposition the Fock space F_J is identified with the completion of the algebra of symmetric polynomials on W with respect to a natural inner product on this algebra induced by the inner product on $H^{1/2}(S^1, \mathbb{R}^d)$ (cf. [1] for details).

The union of Fock spaces

$$F = \bigcup_{J \in \mathcal{S}} F_J$$

forms a Hermitian holomorphic Hilbert bundle, `Fock bundle`, over $\mathcal{S}$ which is trivial on $\mathcal{S}$ (note that $\mathcal{S}$ is contractible).

We define next a natural connection on this bundle. Note first that there is a `canonical (Fock) representation` of the Heisenberg algebra heis(Ω^d) in every F_J (cf. [1]). This representation is equivalent to the one given by the Kostant–Souriau construction in this case. On the other hand, if we apply the Kostant–Souriau construction to another component of the algebra of observables, the Lie algebra Vect(S^1), then the representation operators given by the Kostant–Souriau formula will not preserve, in general, Fock spaces F_J.

We make this statement more precise. Denote by X_f the Hamiltonian vector field on Ω^d corresponding to an observable $f \in$ Vect(S^1). This field generates a (local) 1-parameter group of diffeomorphisms $\varphi_t = \exp(2\pi i t X_f)$ of Ω^d. Then the operators $\exp(2\pi i t r(f))$, where the operator $r(f)$ is given by the Kostant–Souriau formula, transform the Fock space F_J into the Fock space F_{J_t} which is defined with respect to the complex structure J_t obtained from J by the action of diffeomorphism φ_t.

So, in order to construct a representation of the algebra Vect(S^1) in the Fock space F_J using the Kostant–Souriau formula, one should find a canonical way of identification of spaces F_J for different complex structures J. It can be done by use of a flat connection on the bundle $F \to \mathcal{S}$.

Instead of the action of diffeomorphisms given by exponentials $\exp(2\pi i t r(f))$ on the Fock bundle one can construct, infinitesimally, a <u>projective</u> representation of the algebra Vect(S^1) in any fixed Fock space F_J, in particular in the Fock space F_o associated with the standard complex structure J^0 on Ω^d corresponding to the origin o of $\mathcal{S}$. This projective representation of Vect(S^1) generates, as usual, a representation of a central extension of Vect(S^1) called the Virasoro algebra.

From the projective representation of the Lie algebra Vect(S^1) in the Fock space F_o one can construct in a natural way a `linear connection` on the Fock bundle F, invariant under the projective action of the group Diff$_+(S^1)$ (associated with the projective representation of Vect(S^1)) and having the curvature at the origin equal to the cocycle of this projective representation. In particular, this connection is never flat and cannot be used for the identification of spaces F_J for different complex structures J.

2.4. Twistor quantization of flat space loops.

The same problem arises also in the geometric quantization of finite-dimensional phase manifolds. It is well known that, in order to overcome this difficulty, one should use instead of Fock spaces F_J the "extended" Fock spaces of half-densities. We exploit an analogous idea in our case. Namely, we introduce an `anticanonical bundle` $K^{-1} \to \mathcal{S}$ and consider its `square root` $K^{-1/2} \to \mathcal{S}$ (which exists since $\mathcal{S}$ is contractible). The Kähler metric on $\mathcal{S}$ generates a `natural connection` on $K^{-1/2}$ which is invariant under a natural projective action of $\mathrm{Diff}_+(S^1)$ on $K^{-1/2}$.

Now we define the `quantization bundle` $\mathcal{H} \to \mathcal{S}$ to be the tensor product

$$\mathcal{H} := F \otimes K^{-1/2} \longrightarrow \mathcal{S}$$

and provide it with a connection which is the tensor product of connections on F and $K^{-1/2}$ described above. It turns out that this connection is flat precisely when $d = 26$.

Hence, the *twistor quantization of the loop space* Ω^d*, i.e. the construction of a flat (Hermitian) connection in the quantization bundle* $\mathcal{H} = F \otimes K^{-1/2} \to \mathcal{S}$*, invariant under the projective action of* $\mathrm{Diff}_+(S^1)$ *on* $\mathcal{H}$*, is possible for* $d = 26$.

In order to derive from the twistor quantization a solution of the original quantization problem, i.e. a representation of the algebra of observables $\mathfrak{A}$ in the extended Fock space $\mathcal{H}_o = F_o \otimes K_o^{-1/2}$, identified with the fibre of the quantization bundle at the origin $o \in \mathcal{S}$, one should proceed in the following way. The representation of the Heisenberg algebra in fibres of the Fock bundle F extends trivially to its representation in fibres of the quantization bundle $\mathcal{H}$. The group $\mathrm{Diff}_+(S^1)$ acts projectively on the bundle $\mathcal{H}$ and this action intertwines with the representation of the Heisenberg algebra. As we know, the Kostant–Souriau operators, associated with observables $f \in \mathfrak{A}$ do not preserve, in general, the spaces F_o and $\mathcal{H}_o$ because the symplectic diffeomorphisms φ^t, generated by the Hamiltonian vector fields X_f, transform F_o and $\mathcal{H}_o$ into the spaces F_t and $\mathcal{H}_t$ associated with the complex structure $J^t = \varphi^t_*(J^0)$. However, using an invariant flat Hermitian connection on the quantization bundle $\mathcal{H}$, one can, by integrating it, construct a unitary operator U_t identifying $\mathcal{H}_t$ again with $\mathcal{H}_o$. The composition of operators U_t with Kostant–Souriau representation operators acts already in $\mathcal{H}_o$, and, after the differentiation, provides a representation of the algebra $\mathfrak{A}$ in $\mathcal{H}_o$ we are looking for.

2.5. Twistor quantization of loop group spaces.

In case of the loop space Ω_T we proceed much in the same way as in case of Ω^d. Instead of Fock spaces with the associated Fock representations of the Heisenberg algebra we have now the lowest weight representations $H_J = H_J^\lambda$ associated with the lowest weights λ of the Lie algebra $\widetilde{L\mathfrak{g}}$.

Consider again the `twistor bundle` $\pi : Z \to \Omega_T$ of admissible complex structures on Ω_T with fibre $\mathcal{S}$. Admissible complex structures on Ω_T are identified now with the sections of the twistor bundle invariant under the action of the mapping group $\mathrm{Map}(S^1, G)$ on Ω_T by left translations. So we have again a natural projection $p : Z \to \mathcal{S}$. Provide Z with an almost complex structure in the same way as before so that p becomes a holomorphic map. The group $\mathrm{Map}(S^1, G)$ acts on Z horizontally, preserving the fibres of p without changing the complex structure on Ω_T, and the group $\mathrm{Diff}_+(S^1)$ acts on Z vertically preserving the fibres of π, this action being induced by the action of $\mathrm{Diff}_+(S^1)$ on $\mathcal{S}$.

The union of lowest weight representation spaces H_J

$$H := \bigcup_{J \in \mathcal{S}} H_J$$

forms again a **Hermitian holomorphic Hilbert bundle** which is trivial. The group $\mathrm{Diff}_+(S^1)$ acts projectively on H. An infinitesimal analogue of this action is a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the lowest weight λ in the fibre H_o at the origin.

From the projective representation of $\mathrm{Vect}(S^1)$ we construct a natural **linear connection** on H which is invariant under the projective action of the group $\mathrm{Diff}_+(S^1)$ and has the curvature at the origin equal to the cocycle of the projective representation of $\mathrm{Vect}(S^1)$.

We define the **quantization bundle**, as before, to be the tensor product $\mathcal{H} := H \otimes K^{-1/2} \to \mathcal{S}$ and provide it with the tensor product of connections on H and $K^{-1/2}$ constructed above. The curvature of this tensor product connection is zero when the following condition is satisfied

$$c(\mathfrak{g}) = \frac{h \dim \mathfrak{g}}{h + \kappa(\mathfrak{g})} = 26 \; . \tag{2}$$

Here h is the level of the lowest weight representation and $\kappa(\mathfrak{g})$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$.

Hence, *the twistor quantization of the theory determined by the phase manifold $\Omega_T G$ and the lowest weight λ is possible if the condition* (2) *is satisfied.*

3. Twistor quantization of general Kähler manifolds

We turn now to the twistor quantization of general Kähler phase manifolds. In twistor quantization of loop spaces above we have used the holomorphic projection $p : Z \to \mathcal{S}$. The existence of this projection was due to the fact that loop spaces are homogeneous with respect to left translations and we have identified $\mathcal{S}$ with the space of invariant complex structures. For a general Kähler manifold there is no hope to find a twistor bundle with a holomorphic projection of the above type. Moreover, the twistor bundle Z, in general, has no natural almost complex structure. We take, instead, another way and consider the space $\mathcal{Z}$ of all Kähler structures on M given by sections of the twistor bundle $Z \to M$ with vanishing Nijenhuis tensor. (This space, as we shall see, has always a canonical almost complex structure). We define the Fock bundle to be a bundle over $\mathcal{Z}$. This bundle has a natural almost complex structure and the twistor quantization problem for M may be formulated as before: construct a flat Hermitian connection on the Fock bundle.

3.1. Twistor bundle.

We start with the definition of a twistor bundle. For a finite-dimensional Kähler manifold of dimension $2n$ we define $\pi : Z \to M$ to be the bundle of Kähler structures on M, i.e. complex structures compatible with the symplectic structure. It is a vector bundle which is associated with the $\mathrm{Sp}(2n, \mathbb{R})$-bundle of symplectic frames on M and has the fibre at $x \in M$ given by the space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ of Kähler structures on $T_x M$.

If M is a Kähler manifold modelled on a separable complex Hilbert space, we define $\pi : Z \to M$ to be the bundle of restricted Kähler structures on M. Its fibre at $x \in M$ is identified with the space $\mathrm{Sp}_{res}(T_xM)/\mathrm{U}(T_xM)$ of restricted Kähler structures on T_xM where $\mathrm{Sp}_{res}(T_xM)$ is the space of linear symplectic transformations of T_xM differing from a unitary one by a Hilbert–Schmidt operator (cf. [7] for details).

The twistor space Z for a general Kähler manifold M has no distinguished complex (or almost complex) structure. In order to define it one needs a vertical-horizontal decomposition $TZ = V \oplus H$ of the tangent bundle TZ. Such a decomposition can be defined using a symplectic connection on M but, contrary to the Riemannian case, we have no canonical choice of a symplectic connection on a general Kähler manifold.

3.2. Twistor quantization.

We consider now the `space` $\mathcal{Z}$ `of Kähler structures` on M which is identified with the space $\Gamma(M, Z)$ of smooth sections J of $\pi : Z \to M$ satisfying to the Nijenhuis integrability condition (cf., e.g., [2]). A tangent vector X to $\mathcal{Z}$ at J is represented by a tangent vector field $X = \{X_z \in T_zZ : z \in J(M)\}$ to Z along the image $J(M)$ of $J : M \to Z$. Any such vector X has a vertical-horizontal decomposition $X = X^v \oplus X^h$ where X_z^v, $z \in J(M)$, is tangent to the fibre of $\pi : Z \to M$ at z and X_z^h is tangent to $J(M)$ at z.

We define an almost complex structure $\mathcal{J}$ on $\mathcal{Z}$ by setting

$$\mathcal{J}_J X = \{\mathcal{J}_z^v X_z^v \oplus z X_z^h : z \in J(M)\}$$

at $J \in \mathcal{Z}$. Here $\mathcal{J}_z^v$ is the Kähler structure on the Kähler manifold $\pi^{-1}(\pi(z))$ and z in the second summand is considered as the complex structure (defined by z) on $T_zJ(M)$ identified with $T_{\pi(z)}M$ by π_*.

Suppose now that the symplectic space (M, ω) is quantizable (cf. Sec. 1.3), i.e. the 2-form $\omega/2\pi$ is integral. Then there exists a prequantum line bundle L over M with a Hermitian connection ∇ having the curvature $F_\nabla = \omega$. Denote by $\tilde{L} \to Z$ the pull-back of L to Z provided with the pulled-back connection $\tilde{\nabla}$.

Define a `prequantum bundle` $\mathcal{L} \to \mathcal{Z}$ over $\mathcal{Z}$ with the fibre at $J \in \mathcal{Z}$ given by $\mathcal{L}_J = \Gamma(J(M), \tilde{L})$. A local section σ of $\mathcal{L}$ at J is a collection $\sigma = \{\sigma_z : z \in J(M)\}$ where σ_z is a local section of $\tilde{L} \to Z$ at z. The prequantum bundle $\mathcal{L}$ can be provided with a natural structure of a `holomorphic vector bundle` over $\mathcal{Z}$. This structure is determined by a $\bar{\partial}$-operator acting on local sections of $\mathcal{L} \to \mathcal{Z}$ according to the following rule. If $\sigma = \{\sigma_z : z \in J(M)\}$ is a section of $\mathcal{L}$ at J then

$$\bar{\partial}_J \sigma = \{\tilde{\nabla}_z^{(0,1)} \sigma_z : z \in J(M)\}$$

where $\tilde{\nabla}_z^{(0,1)}$ is the pull-back to Z of the $(0,1)$-component of $\nabla_{\pi(z)}$ with respect to the complex structure on $T_{\pi(z)}M$ given by z.

We define a `Fock bundle` $\mathcal{H} \to \mathcal{Z}$ over $\mathcal{Z}$ with the fibre at $J \in \mathcal{Z}$ given by the Fock space $F(J(M), \tilde{L})$ of holomorphic sections σ of $\mathcal{L}$ at J. They satisfy to the equation

$$\tilde{\nabla}_z^{(0,1)} \sigma_z = 0 \quad \text{for } z \in J(M).$$

With this definition, the twistor quantization problem for a Kähler phase manifold M may be formulated as before: *construct a (projectively) flat Hermitian connection on the Fock bundle* $\mathcal{H} \to \mathcal{Z}$.

References

1. F.A.Berezin, *The Method of Second Quantization.*– Academic Press, N.Y.-London, 1966
2. A.L.Besse, *Einstein manifolds.*– Springer Verlag: Berlin, Heidelberg, N.Y., 1987.
3. M.J.Bowick, S.G.Rajeev, *The holomorphic geometry of closed bosonic string theory and* $\mathit{Diff}\, S^1/S^1$.– Nucl. Phys., B293, 1987, 348–384
4. J.Davidov, A.G.Sergeev, *Twistor quantization of loop spaces.*– Steklov Inst. Proc., 217, 1997
5. A.A.Kirillov, *Geometric Quantization.* In: Dynamical Systems, IV. Encycl. of Math. Sciences, vol. 4.– Springer Verlag, Berlin-Heidelberg-N.Y., 1990; 137–172
6. A.D.Popov, A.G.Sergeev, *Symplectic twistors and geometric quantization of strings.* In: Algebraic Geom. and its Appl.; Aspects of Math., vol. E25 (eds. A.Tikhomirov, A.Tyurin).– Vieweg Verlag, Wiesbaden, 1994; 137–157
7. A.Pressley, G.Segal, *Loop groups.*– Clarendon Press, Oxford, 1986.
8. N.M.J.Woodhouse, *Geometric Quantization, 2nd ed.*.– Clarendon Press, Oxford, 1992

Steklov Mathematical Institute, Gubkina 8, 117966, GSP-1, Moscow, RUSSIA
E-mail address: armen@sergeev.mian.su