$\mathrm{Diff}_+(S^1)/S^1$ AS A SPACE OF COMPLEX STRUCTURES ON LOOP SPACES OF COMPACT LIE GROUPS

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To Sergio Albeverio on behalf of his 60th birthday

ABSTRACT. The homogeneous Frechet space $\mathrm{Diff}_+(S^1)/S^1$ is realized as a space of complex structures on the loop space of a compact Lie group, compatible with the symplectic structure. The Kähler geometry of this manifold is analyzed. We define a symplectic twistor bundle over the loop space, having $\mathrm{Diff}_+(S^1)/S^1$ as a fibre, and present a scheme of twistor quantization of the loop space.

The study of the Kähler geometry of $\mathrm{Diff}_+(S^1)/S^1$ was initiated in [**6**] and [**10**]. In [**10**] several models for this space were proposed. In particular, it was realized as an orbit of the coadjoint representation of the Virasoro group, provided with the canonical symplectic form. This realization is closely related to two other models for $\mathrm{Diff}_+(S^1)/S^1$ as the moduli space of Hill's operators under the action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ and as the space of projective structures (cf. [**8**]). Another realization, which is important for the Kähler geometry of $\mathrm{Diff}_+(S^1)/S^1$, identifies $\mathrm{Diff}_+(S^1)/S^1$ as the space of univalent holomorphic functions on the unit disc, smooth up to the boundary (cf. [**9**]). We briefly recall these two realizations of $\mathrm{Diff}_+(S^1)/S^1$ in Sec.2.

In [**6**] still another model for $\mathrm{Diff}_+(S^1)/S^1$ was proposed. Namely, this space was realized as a space of complex structures on the space of loops in Minkowski space. This realization was later extended to loop spaces of compact Lie groups in [**11, 12**] (cf. also [**7**]). We have collected basic results on the Kähler geometry of this model in Sec.2.

The realization of $\mathrm{Diff}_+(S^1)/S^1$ as a space of complex structures, compatible with the symplectic structure, on loop spaces of compact Lie groups is closely related to the geometric quantization of these spaces. In Sec.3 we define a symplectic twistor bundle over the loop space of a compact Lie group, having $\mathrm{Diff}_+(S^1)/S^1$ as a fibre. This bundle, as in the conventional twistor theory, has a natural almost complex structure, which makes it possible to reformulate a geometric quantization problem in twistor terms. In Sec.3 a scheme of twistor quantization of loop spaces of compact Lie groups is presented, the details may be found in [**13, 7**].

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1. Loop spaces of compact Lie groups as Kähler Frechet manifolds

1.1. Loop spaces of compact Lie groups. Let G be a connected compact Lie group. Denote by $LG = \mathrm{Map}(S^1, G)$ the corresponding loop group, i.e. the space of smooth maps $S^1 \to G$. It is a Lie-Frechet group with the group law, given by the pointwise multiplication. The Frechet structure on $LG = \mathrm{Map}(S^1, G)$ is induced by the exponential map $\exp : \mathfrak{g} \to G$, where $\mathfrak{g}$ is the Lie algebra of G.

In more detail, the exponential exp defines a local diffeomorphism of some neighborhood $\mathfrak{u}$ of zero in the Lie algebra $\mathfrak{g}$ onto a neighborhood U of the identity e in the group G. This diffeomorphism generates a local chart of LG in a neighborhood of the identity in LG. Indeed, the identity in LG is given by the map $\mathbf{e} : S^1 \to e \in G$ and we take for a coordinate neighborhood of $\mathbf{e}$ the open subset $\mathbf{U} := \mathrm{Map}(S^1, U)$ in LG. The corresponding local chart is given by the bijection

$$\chi : \quad \mathfrak{U} = \mathrm{Map}(S^1, \mathfrak{u}) \longrightarrow \mathrm{Map}(S^1, U) = \mathbf{U} \ ,$$

induced by the exponential $\exp : \mathfrak{u} \to U$. Then the inverse map $\varphi_{\mathbf{e}} := \chi^{-1} : \mathbf{U} \to \mathfrak{U}$ is a bijection of $\mathbf{U}$ onto an open subset $\mathfrak{U}$ in the Frechet space $\mathrm{Map}(S^1, \mathfrak{g})$.

In the same way, a local chart in the neighborhood $\mathbf{U}_\gamma := \gamma \cdot \mathbf{U}$ of an arbitrary point $\gamma \in \mathrm{Map}(S^1, G)$ is given by the composition of the left shift $\gamma^{-1} : \mathbf{U}_\gamma \to \mathbf{U}$ and the map $\varphi_{\mathbf{e}} : \mathbf{U} \to \mathfrak{U} \subset \mathrm{Map}(S^1, \mathfrak{g})$:

$$\varphi_\gamma := \varphi_{\mathbf{e}} \circ \gamma^{-1} : \mathbf{U}_\gamma \to \mathfrak{U} \ .$$

The coordinate neighborhoods $\{\mathbf{U}_\gamma\}$ and local charts $\{\varphi_\gamma\}$, $\gamma \in \mathrm{Map}(S^1, G)$, define a Frechet manifold structure on LG with the local model, given by the Frechet space $\mathrm{Map}(S^1, \mathfrak{g})$.

We define the associated loop space ΩG to be the Frechet homogeneous space (of right conjugate classes) of the loop group LG of the form

$$\Omega G = LG/G \ ,$$

where G in the denominator is identified with the group of constant maps $S^1 \to g_0 \in G$.

The loop group LG acts on ΩG by left shifts. We denote by o the origin (neutral element) in ΩG, identified with the class of constant maps

$$o := [\mathbf{e}] = [G] \ .$$

The loop space ΩG (as a homogeneous space) can be identified with the subgroup $L_1 G$ of based maps in LG, sending $1 \in S^1$ to the identity of G: $\gamma(1) = e \in G$, i.e. with a closed Frechet submanifold (of codimension 1) of the Frechet manifold LG.

1.2. Symplectic structure. We shall define a symplectic structure on ΩG, which is invariant under the action of the loop group LG. Such a structure is uniquely determined by its value at the origin $o \in \Omega G$, i.e. by its restriction to the tangent space $T_o(\Omega G)$, which can be identified with the Frechet space $\Omega\mathfrak{g} := L\mathfrak{g}/\mathfrak{g}$. In terms of Fourier decompositions vectors from this space are given by the series

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k \quad , \ z = e^{i\theta} \ , \quad \xi_k \in \mathfrak{g}^{\mathbb{C}} \ ,$$

where $\xi_{-k} = \bar{\xi}_k$ and "bar" stands for the complex conjugation in the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. By dropping the reality condition, we obtain a description of the complexified tangent space $T_o^{\mathbb{C}}(\Omega G) = \Omega\mathfrak{g}^{\mathbb{C}} :== L\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}^{\mathbb{C}}$.

We fix now an inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$, invariant under the adjoint action Ad of G on $\mathfrak{g}$, and define a 2-form ω_0 on $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$ by the formula

$$\omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(e^{i\theta}), \frac{d\eta(e^{i\theta})}{d\theta} > d\theta \tag{1}$$

for $\xi = \xi(e^{i\theta}), \eta = \eta(e^{i\theta}) \in L\mathfrak{g}$. It is a skew-symmetric bilinear form on $L\mathfrak{g}$, which vanishes if at least one of the maps ξ, η is constant. Because of the invariance of $< \cdot, \cdot >$, the form ω_0 is invariant under the adjoint action of the group G of constant loops on $L\mathfrak{g}$. By pushing ω_0 down to $\Omega\mathfrak{g}$, we get a non-degenerate form on $\Omega\mathfrak{g}$, which extends to an invariant non-degenerate 2-form ω on ΩG. The cocycle condition

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0 \ ,$$

which can be checked by a direct computation, implies that ω is closed on ΩG and thus determines an invariant symplectic structure on ΩG.

The choice of the formula (1) for the symplectic structure ω on ΩG looks ambiguous, however one can show that ω is uniquely determined by the invariant inner product $< \cdot, \cdot >$ on $\mathfrak{g}$ in the case of a semisimple group G. More precisely, we have the following

THEOREM 1.1 ([**14**]). *If the group G is semisimple, then any 2-form ω_0 on $L\mathfrak{g}$, satisfying the cocycle condition and invariant under the adjoint action of G on $L\mathfrak{g}$, is given by the formula* (1) *for some symmetric invariant bilinear form $< \cdot, \cdot >$ on $L\mathfrak{g}$.*

1.3. Complex structure. A complex structure on ΩG is provided by the "complex" representation of ΩG as a homogeneous space of a complex Lie-Frechet group. More precisely, we have the following equality

$$\Omega G = LG/G = LG^{\mathbb{C}} / \, L^+G^{\mathbb{C}} \ , \tag{2}$$

where $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is the subgroup of maps from $LG^{\mathbb{C}} = \mathrm{Map}(S^1, G^{\mathbb{C}})$, which extend to holomorphic (smooth up to the boundary) maps of the unit disc $\Delta \subset \mathbb{C}$ into the complexified Lie group $G^{\mathbb{C}}$.

For the unitary group $G = \mathrm{U}(n)$ the equality (2) means, in other words, that any smooth non-degenerate (i.e. with values in $\mathrm{GL}(n, \mathbb{C})$) matrix function $T(z)$ on the circle may be written as the product

$$T(z) = U(z) \cdot H_+(z) \ , \quad z = e^{i\theta} \ , \tag{3}$$

where $U(z)$ is a smooth unitary (i.e. with values in $\mathrm{U}(n)$) matrix function and $H_+(z)$ extends to a holomorphic (smooth up to the boundary) non-degenerate matrix function in the disc Δ. It is a parametric analogue of the standard representation of a non-degenerate matrix as the product of a unitary and an upper-triangular matrices.

The representation (3) is unique, if one fixes the value of U at 1 by assuming that $U \in L_1\mathrm{U}(n)$. Moreover, the product map $(U, H_+) \mapsto U \cdot H_+$ is a diffeomorphism $\Omega\mathrm{U}(n) \times L^+\mathrm{GL}(n, \mathbb{C}) \to L\mathrm{GL}(n, \mathbb{C})$. In the same way one should understand the equality (2) for an arbitrary (simply connected compact) Lie group G. Namely, we have the following

THEOREM 1.2 ([**14, 7**]). *The product map*

$$\Omega G \times L^+G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}}$$

is a diffeomorphism of Frechet manifolds.

The "complex" representation (2) yields a complex structure J^0 on ΩG. This complex structure is LG-invariant and its restriction J_o^0 to the tangent space $T_o^{\mathbb{C}}(\Omega G)$ may be described by an explicit formula. Namely, the value of J_o^0 on a vector $\xi = \sum_{k\neq 0} \xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}$ is given by the formula

$$J_o^0 \xi = -i \sum_{k>0} \xi_k z^k + i \sum_{k<0} \xi_k z^k \ .$$

The subspace $T_o^{1,0}(\Omega G)$ for this complex structure consists of vectors $\sum_{k<0} \xi_k z^k$ and $T_o^{0,1}(\Omega G)$ of vectors, having the form $\sum_{k>0} \xi_k z^k$.

In order to introduce local complex coordinates on ΩG, we use again the "complex" representation (2) and the Birkhoff theorem. Namely, denote by $L_1^- G^{\mathbb{C}}$ the closed subgroup of $LG^{\mathbb{C}}$, consisting of maps $\gamma \in LG^{\mathbb{C}}$, which extend to holomorphic, smooth up to the boundary, maps of the complement Δ_- of the unit disc $\overline{\Delta}$ on the Riemann sphere $\mathbb{CP}^1$, taking the value $e \in G^{\mathbb{C}}$ at infinity:

$$L_1^- G^{\mathbb{C}} = \{\gamma \in \mathrm{Hol}(\Delta_-, G^{\mathbb{C}}) : \gamma(\infty) = e\} \ .$$

It follows from the Birkhoff's theorem (cf. [**3, 4**]) that for all $\gamma \in LG^{\mathbb{C}}$ in a neighborhood of the identity $\mathbf{e} \in LG^{\mathbb{C}}$ the following representation holds

$$\gamma = \gamma_+ \cdot \gamma_- \ ,$$

where $\gamma_+ \in L^+ G^{\mathbb{C}}$, $\gamma_- \in L_1^- G^{\mathbb{C}}$. The maps $\gamma_\pm$ are uniquely determined by γ and the product map

$$L^+ G^{\mathbb{C}} \times L_1^- G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}}$$

is a local diffeomorphism. In particular, ΩG is locally diffeomorphic to the complex Lie–Frechet group $L_1^- G^{\mathbb{C}}$.

We note that the symplectic structure ω and the complex structure J^0 on ΩG are compatible in the sense that $\omega(J_o^0 \xi, J_o^0 \eta) = \omega(\xi, \eta)$ for any $\xi, \eta \in T_o(\Omega G)$ and the symmetric form

$$g_o^0(\xi, \eta) := \omega(\xi, J_o^0 \eta)$$

on $T_o(\Omega G) \times T_o(\Omega G)$ is positive definite. In other words, because of the invariance of ω and J^0, g_o^0 extends to an invariant Riemannian metric g^0 on ΩG. These two properties together mean that ΩG is a Kähler Frechet manifold, provided with a Kähler metric g^0.

1.4. Flag manifolds. The existence of two kinds of representations of the loop space ΩG as a homogeneous space of a real and a complex Lie–Frechet groups

$$\Omega G = LG/G = LG^{\mathbb{C}}/L^+ G^{\mathbb{C}}$$

is a characteristic property of ΩG. Finite-dimensional Kähler manifolds, having an analogous property, are called flag manifolds.

More precisely, a flag manifold F of a compact semisimple Lie group G can be represented in the form

$$F = G/G \cap P = G^{\mathbb{C}}/P$$

where $G^{\mathbb{C}}$ is the complexification of G and P is a parabolic (triangular) subgroup of $G^{\mathbb{C}}$. The complex representation $F = G^{\mathbb{C}}/P$ implies that F is a complex manifold, which has a G-invariant complex structure. The real representation implies that F is compact and Kähler. Note also that F is simply connected, if the group G has this property. One can show that the flag manifolds F exhaust all simply connected compact Kähler G-manifolds with a transitive action of a compact semisimple Lie group G (cf. [**5, 16**]).

An important example of flag manifolds of G is given by the manifold of full flags, having a homogeneous space representation of the form

$$F_T = G/T = G^{\mathbb{C}}/B_+ \ ,$$

where T is a maximal torus in G, B_+ is the standard Borel subgroup in $G^{\mathbb{C}}$. The importance of this flag manifold is due to the fact that, by the Borel–Weil construction (cf., e.g., [**2**]), all irreducible representations of G can be realized in the spaces of holomorphic sections of complex line bundles $L_\lambda \to F_T$, associated with antidominant lowest weights λ of $\mathfrak{g}^{\mathbb{C}}$.

The loop space ΩG may be considered as a universal flag manifold of the group G in the sense that all flag manifolds of G can be canonically embedded in ΩG.

To show that, we note, first, that the real representation $F = G/L$ of a flag manifold F can be interpreted, otherwise, as the realization of F as an orbit of the adjoint action Ad of the group G on its Lie algebra $\mathfrak{g}$ (or as an orbit of the coadjoint action Ad^* of G on the dual Lie algebra $\mathfrak{g}^*$). Namely, the orbit of an element $\xi \in \mathfrak{g}$ under the adjoint action has the form

$$G/G(\xi) \ ,$$

where the isotropy subgroup $G(\xi)$ at ξ coincides with the centralizer of ξ, i.e. with $G(\xi) = \{g \in G : \ \mathrm{Ad}\, g(\xi) = \xi\}$. All such orbits are flag manifolds and, conversely, every flag manifold of a compact semisimple Lie group may be realized in this way.

Consider now the natural S^1-action of the circle group on the loop space ΩG (identified with the subgroup $L_1 G$ of LG), given by the rotation of loops:

$$\lambda \cdot \gamma(z) = \gamma(\lambda)^{-1}\gamma(\lambda z) \ , \quad \lambda \in S^1 \ ,$$

where $\gamma \in \Omega G$. A loop γ is the fixed point of S^1-action if and only if

$$\gamma(\lambda z) = \gamma(\lambda)\gamma(z) \quad \text{for all}\, \lambda, z \in S^1 \ ,$$

i.e. if γ is a homomorphism $S^1 \to G$. If $\gamma : S^1 \to G$ is a homomorphism, then so are all loops $\gamma_g = g\gamma g^{-1}$, $g \in G$, conjugated to γ. Hence, the set of all such loops (conjugacy classes of a homomorphism γ) is parametrized by points of the homogeneous space

$$F_\gamma = G/G(\gamma) \ ,$$

where $G(\gamma)$ is the centralizer of the 1-parameter subgroup $\gamma(S^1)$ in G. The homogeneous space F_γ can be identified, as we have just pointed out, with a flag manifold of the group G.

Hence, the set of fixed points of the S^1-action on ΩG is the disjoint union

$$\mathrm{Fix}(S^1) = \bigcup_\gamma F_\gamma$$

of flag manifolds F_γ, where γ runs over the set of conjugacy classes of homomorphisms $S^1 \to G$, and flag manifolds F_γ are embedded into ΩG as finite-dimensional Kähler submanifolds. This relation between flag manifolds and fixed points of the S^1-action on ΩG was pointed out in [**1**].

Along with ΩG, one can also define a "loop-space" analogue of the full flags manifold F_T. Namely, we fix again a maximal torus T in G and set

$$\Omega_T G = LG/T \ .$$

It is also a Kähler Frechet manifold. In order to define a symplectic structure on $\Omega_T G$, note that the loop group LG, as a manifold, is diffeomorphic to the direct product $G \times \Omega G$. This diffeomorphism maps a loop $\gamma \in LG$ into the pair $(\gamma(1), \gamma(1)^{-1}\gamma) \in G \times \Omega G$, where ΩG is identified with the subgroup $L_1 G = \{\gamma \in LG : \gamma(1) = e\}$ in LG. Hence, as a Frechet manifold, $\Omega_T G = LG/T = G/T \times \Omega G$. A symplectic structure on $\Omega_T G$ is generated by the symplectic structure on ΩG, defined above, and Kirillov symplectic structure on the manifold of full flags $F_T = G/T$. (Recall that flag manifolds of G can be considered as orbits of the coadjoint representation of G on the dual Lie algebra $\mathfrak{g}^*$, which have Kirillov symplectic structure).

A complex structure on $\Omega_T G$ is provided, as in the case of ΩG, by the "complex" representation of $\Omega_T G$ as a homogeneous space of the complexified loop group of the form

$$\Omega_T G = LG^{\mathbb{C}} / B_+ G^{\mathbb{C}} ,$$

where $B_+ G^{\mathbb{C}}$ is the subgroup in $L_+ G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$, consisting of maps $\gamma \in \mathrm{Map}(S^1, G^{\mathbb{C}})$, which can be extended to holomorphic (smooth up to the boundary) maps of the unit disc $\gamma : \Delta \to G^{\mathbb{C}}$, such that $\gamma(0) \in B_+$.

This complex structure on $\Omega_T G$ is compatible with the symplectic one and so defines the structure of a Kähler Frechet manifold on $\Omega_T G$.

2. $\mathrm{Diff}_+(S^1)/S^1$ as a Kähler Frechet manifold

We shall study now the action of the diffeomorphism group of the circle on loop spaces ΩG.

2.1. The diffeomorphism group of the circle. The group $\mathrm{Diff}(S^1)$ consists of diffeomorphisms of the circle S^1 with the group law, given by the composition of diffeomorphisms. This group is an open subset in the Frechet manifold $\mathrm{Map}(S^1, S^1)$ and is provided with the induced structure of a Frechet manifold, so that it is a Lie-Frechet group. $\mathrm{Diff}_+(S^1)$ has two connected components and we denote by $\mathrm{Diff}_+(S^1)$ the connected component of the identity, formed by the diffeomorphisms of S^1, preserving the orientation.

By pulling up to the universal covering $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \approx S^1$, we can associate with a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ a smooth function $h : \mathbb{R} \to \mathbb{R}$, which is defined up to a multiplicative constant $2\pi n$, $n \in \mathbb{Z}$, and satisfies the relation $h(x + 2\pi) = h(x) + 2\pi$, $x \in \mathbb{R}$. It follows that the derivative $h'(x)$, which is already uniquely determined by f, is a smooth 2π-periodic function on $\mathbb{R}$. Moreover, $h'(x) > 0$ and the average of $h'(x)$ over S^1 is equal to 1. Hence, we have associated with $f \in \mathrm{Diff}_+(S^1)$ a smooth 2π-periodic function h' on $\mathbb{R}$, which is strictly positive and has the average 1 over the period. We denote the set of such functions by C. It is a convex open subset in a closed affine subspace of codimension 1 in the Frechet space of smooth 2π-periodic functions on $\mathbb{R}$. The function h is uniquely determined by $h' \in C$ up to an additive constant $h(0) \in \mathbb{R}$ and also h is defined by $f \in \mathrm{Diff}_+(S^1)$ up to an additive constant $2\pi n$, $n \in \mathbb{Z}$. So f is uniquely determined by $h' \in C$ up to an element of $\mathbb{R}/2\pi\mathbb{Z} \approx S^1$. Hence, $\mathrm{Diff}_+(S^1)$ is diffeomorphic to $S^1 \times C$ and, in particular, homotopy equivalent to S^1.

The Lie algebra of $\mathrm{Diff}(S^1)$ concides with the algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle S^1. In other words, $\mathrm{Vect}(S^1)$ concides with the Frechet space of smooth sections of the tangent bundle of the circle $TS^1 \to S^1$. Elements $v \in$

$\mathrm{Vect}(S^1)$ are written down in the form $v = v(\theta)\frac{d}{d\theta}$, where $v(\theta)$ is a smooth 2π-periodic function of θ. The bracket of two vector fields $v_1, v_2 \in \mathrm{Vect}(S^1)$ is given by the standard formula

$$\left[v_1(\theta)\frac{d}{d\theta}, v_2(\theta)\frac{d}{d\theta}\right] = \{v_1(\theta)v_2'(\theta) - v_1'(\theta)v_2(\theta)\}\frac{d}{d\theta} ,$$

so that $\mathrm{Vect}(S^1)$ is a Lie–Frechet algebra.

In terms of Fourier decompositions, the coefficients $v(\theta)$ of vector fields $v = v(\theta)\frac{d}{d\theta}$ are given by the series

$$v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta} , \quad v_n \in \mathbb{C} ,$$

subject to the relation: $v_{-n} = \bar{v}_n$.

2.2. The action of $\mathrm{Diff}_+(S^1)$ on loop spaces ΩG. The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts naturally on ΩG by the reparametrization of loops:

$$f_*\gamma(\theta) := \gamma\left(f(\theta)\right) \quad \text{for } \gamma \in LG,\ f \in \mathrm{Diff}_+(S^1) .$$

It follows from the definition of symplectic structure ω on ΩG, associated with the form

$$\omega_0(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta$$

on $L\mathfrak{g}$, that this action preserves ω, i.e. diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$ generate symplectomorphisms of ΩG.

Consider next the action of $\mathrm{Diff}_+(S^1)$ on the complex structure J^0 on ΩG. A diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ transforms it into a complex structure

$$J := f_*^{-1} \circ J^0 \circ f_* , \tag{4}$$

where f_* is the differential of f.

THEOREM 2.1 (cf. [**7**]). *The complex structure J coincides with J^0 if and only if the diffeomorphism $f \in \mathit{Diff}_+(S^1)$ is a rotation, i.e. $f \in S^1 \subset \mathit{Diff}_+(S^1)$.*

Indeed, if $f \in \mathrm{Diff}_+(S^1)$ does not change the complex structure, it also preserves the "complex" representation (2) of ΩG. This implies that f leaves the subspaces $L^+G^{\mathbb{C}}$ and $L_1^-G^{\mathbb{C}}$ in $LG^{\mathbb{C}}$ invariant, which is possible only if $f \in S^1$.

Recall that the complex structure J^0 on ΩG is invariant under left LG-shifts on ΩG and compatible with the symplectic structure ω on ΩG. The invariance of ω with respect to the action of $\mathrm{Diff}_+(S^1)$ implies that the complex structures J, obtained from J^0 by formula (4), are also invariant under left LG-shifts and compatible with ω. In particular, any such J generates an invariant Kähler metric g on ΩG, given by the formula

$$g_\gamma(\xi,\eta) := \omega_\gamma(\xi, J_\gamma\eta) , \quad \gamma \in \Omega G,$$

for any $\xi, \eta \in T_\gamma(\Omega G)$.

We shall call the complex structures J on ΩG, obtained from J^0 by the action of $\mathrm{Diff}_+(S^1)$ according to the formula (4), the admissible complex structures on ΩG.

According to Theorem 2.1, admissible complex structures on ΩG are parametrized by points of the manifold $\mathrm{Diff}_+(S^1)/S^1$, which we shall study next.

2.3. Symplectic structure of $\mathrm{Diff}_+(S^1)/S^1$. The space $\mathbf{S} := \mathrm{Diff}_+(S^1)/S^1$ is a homogeneous space of the group $\mathrm{Diff}_+(S^1)$, which acts on $\mathbf{S}$ by left shifts. It can be identified (as a homogeneous space) with the subgroup of $\mathrm{Diff}_+(S^1)$, consisting of all diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$, fixing the point $1 \in S^1$: $f(1) = 1$. This provides $\mathbf{S}$ with a Frechet manifold structure.

As in the case of ΩG, invariant geometric structures on ΩG are uniquely determined by their values at the origin $o \in \mathrm{Diff}_+(S^1)/S^1$.

The tangent space $T_o\mathbf{S}$ at the origin is identified with the space $\mathrm{Vect}_0(S^1)$, consisting of smooth 2π-periodic functions $\xi(\theta)$ with zero average: $\frac{1}{2\pi}\int_0^{2\pi}\xi(\theta)d\theta = 0$. In terms of Fourier decompositions tangent vectors $\xi \in T_o\mathbf{S}$ are given by the series

$$\xi = \sum_{n\neq 0} \xi_n e_n \ , \quad \xi_n \in \mathbb{C} \ ,$$

subject to the condition $\bar{\xi}_n = -\xi_n$, where $e_n = ie^{in\theta}\frac{d}{d\theta}$, $n = 0, \pm 1, \pm 2, \ldots$, is the standard base of the Lie algebra $\mathrm{Vect}(S^1)$.

In particular, an invariant symplectic structure on $\mathbf{S}$ is uniquely determined by its restriction to $T_o\mathbf{S}$, i.e. by a 2-cocycle w on the Lie algebra $\mathrm{Vect}(S^1)$, invariant under rotations. It is well known (cf., e.g., [**14**]), that all such cocycles are parametrized by two real numbers α and β, their values on the base elements $\{e_n\}$ are given by the formula

$$w(e_n, e_m) = (\alpha n^3 + \beta n)\delta_{n,-m} \ .$$

The form $w_{\alpha,\beta}$, corresponding to parameters α, β, is non-degenerate on $\mathrm{Vect}_0(S^1)$ if and only if $\alpha n^3 + \beta n \neq 0$ for all natural n. The latter condition is satisfied if either $\alpha = 0$, $\beta \neq 0$ or $-\beta/\alpha$ is not the square of a natural number. In the first case the form $w_{\alpha,\beta}$, as it's easy to see, is exact, we take the second possibility.

In a more invariant way the above cocycle $w_{\alpha,\beta}$ may be written in the form

$$\omega_{\alpha,\beta}(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \xi(\theta)\left(\beta\eta'(\theta) - \alpha\eta'''(\theta)\right) d\theta \ ,$$

where $\xi, \eta \in \mathrm{Vect}_0(S^1)$.

The family of symplectic structures $w_{\alpha,\beta}$, determined by these cocycles, has a natural interpretation in terms of coadjoint orbits of the group $\mathrm{Diff}_+(S^1)$ on the Virasoro algebra vir^*. Recall that the Virasoro algebra vir is the central extension of the Lie algebra $\mathrm{Vect}(S^1)$, determined by the cocycle $w_{1,0}$ on $\mathrm{Vect}(S^1)$, generating the cohomology group $H^2(\mathrm{Vect}(S^1), \mathbb{R})$. The dual algebra $\mathrm{Vect}^*(S^1)$ may be identified with the space of quadratic differentials with distributional coefficients on S^1, so the regular part of $\mathrm{Vect}^*(S^1)$ may be identified with the space $Q(S^1)$ of quadratic differentials on S^1 of the form $q = q(\theta)(d\theta)^2$, where $q(\theta)$ is a smooth 2π-periodic function of θ. The coadjoint action of $\mathrm{Diff}_+(S^1)$ on the regular part of the dual Virasoro algebra, identified with $Q(S^1) \oplus \mathbb{R}$, is given by the formula (cf. [**8**])

$$f \in \mathrm{Diff}_+(S^1) \longmapsto K(f)(q,t) = (g^*(q) + tS(g)d\theta^2, t) \ ,$$

where $g = f^{-1}$ and $g^*(q)$ denotes the natural action of a diffeomorphism g on a quadratic differential q. The functional $S(\varphi)$ coincides with the Schwarz derivative of a diffeomorphism φ, i.e.

$$S(\varphi) = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 \ .$$

One can describe explicitly the orbits (under the action of $\mathrm{Diff}_+(S^1)$) of the elements $(q,t) = (qd\theta^2, t)$, where q and t are real numbers. If $2q/t$ is not the square of a natural number, then the orbit of $(qd\theta^2, t)$ coincides with $\mathbf{S}$. If $2q/t = n^2$, then the orbit of $(qd\theta^2, t)$ coincides with the homogeneous space

$$\mathrm{Diff}_+(S^1)/\,\mathrm{PSL}^{(n)}(2,\mathbb{R})\ ,$$

where $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ is the n-fold covering of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ of fractional-linear automorphisms of the unit disc Δ.

Recall now that the orbits of the coadjoint representation have a canonical symplectic structure, given by the Kirillov form. Identifying the homogeneous space $\mathbf{S}$ with the orbits of elements $(qd\theta^2, t)$ for the different choice of (q,t) with $2q/t \neq n^2$ and providing these orbits with the corresponding Kirillov forms, we reproduce again the above two-parameter family of symplectic structures on $\mathbf{S}$.

2.4. Complex structure of $\mathbf{Diff}_+(S^1)/S^1$. We switch now to the invariant complex structure on $\mathbf{S}$. Its restriction to $T_o\mathbf{S} = \mathrm{Vect}_0(S^1)$ is given by the Hilbert transform, associating with $\xi \in \mathrm{Vect}_0(S^1)$ the element

$$(I_o\xi)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ctg}\,\frac{\theta-\psi}{2}\cdot[\xi(\theta)-\xi(\psi)]\,d\psi\ , \quad 0 \le \theta \le 2\pi\ .$$

In terms of Fourier decompositions $\xi = \sum_{n\neq 0}\xi_n e_n \in \mathrm{Vect}_0(S^1)$ we obtain

$$I_o\xi = -i\sum_{n>0}\xi_n e_n + i\sum_{n<0}\xi_n e_n\ ,$$

so that the subspace $T_o^{0,1}(\mathbf{S})$ consists of the vectors of the form $\sum_{n>0}\xi_n e_n$ and $T_o^{1,0}(\mathbf{S})$ — of the vectors $\sum_{n<0}\xi_n e_n$. The complex structure I_o on $T_o\mathbf{S}$ uniquely determines a $\mathrm{Diff}_+(S^1)$-invariant almost complex structure I on $\mathbf{S}$ (obtained from I_o by left translations). It's easy to show that I is integrable, i.e. the Lie bracket of two $(1,0)$-vector fields is again a $(1,0)$-vector field. Moreover, I is the only integrable $\mathrm{Diff}_+(S^1)$-invariant almost complex structure on $\mathbf{S}$ (cf. [**15**]).

Moreover, one can introduce local complex coordinates on $\mathbf{S}$ (i.e. realize $\mathbf{S}$ as a complex Frechet manifold), using a realization of $\mathbf{S}$ as a space of holomorphic univalent functions in the disc from [**9**].

Consider the Frechet space $\mathbf{A}$ of all smooth complex-valued functions on the closure of the unit disc Δ, which are holomorphic inside Δ and vanish at the origin. Let $\mathbf{A}_0$ be the open subset of $\mathbf{A}$, consisting of all $f \in \mathbf{A}$ which are univalent on Δ and map diffeomorphically $S^1 = \partial\Delta$ onto its image. We denote by $\mathfrak{S}$ the set of functions $f \in \mathbf{A}_0$, such that $f'(0) = 1$. It is an open subset of the affine subspace of $\mathbf{A}$, consisting of $f \in \mathbf{A}$, subject to $f'(0) = 1$. Functions $f \in \mathfrak{S}$ are holomorphic and univalent in Δ, they map diffeomorphically $\overline{\Delta} \to \overline{f(\Delta)}$ and satisfy the normalization conditions $f(0) = 0$, $f'(0) = 1$.

Functions $f \in \mathfrak{S}$ have the power series decompositions of the form

$$f(z) = z + c_2z^2 + c_3z^3 + \dots\ .$$

Hence, $\mathfrak{S}$ is a complex Frechet manifold with the complex coordinates, given by the coefficients $(c_2, c_3, \dots)$.

THEOREM 2.2 (cf. [**9**]). *The space $\mathfrak{S}$ of smooth univalent functions on Δ is diffeomorphic to the space $\mathbf{S}$ of admissible complex structures on ΩG.*

This provides $\mathbf{S}$ with the structure of a complex Frechet manifold.

2.5. Kähler structure of $\mathrm{Diff}_+(S^1)/S^1$. The complex structure I on $\mathbf{S}$ is compatible with all symplectic structures $w_{\alpha,\beta}$ and so generates a family of Kähler metrics $g_{\alpha,\beta}(\xi,\eta) := w_{\alpha,\beta}(\xi, I\eta)$ on $\mathbf{S}$, given at the origin by the formula

$$g_{\alpha,\beta,o}(\xi,\eta) = \sum_{n\geq 1}(\alpha n^3 + \beta n)\xi_n \bar{\eta}_n \ ,$$

where $\xi = \sum_{n\neq 0} \xi_n e_n$, $\eta = \sum_{n\neq 0} \eta_n e_n \in T_o\mathbf{S}$. Thus, $\mathbf{S}$ is a Kähler Frechet manifold with a two-parameter family of Kähler metrics $g_{\alpha,\beta}$.

The Kähler metrics $g_{\alpha,\beta}$, being $\mathrm{Diff}_+(S^1)$-invariant, are uniquely determined by their restriction to the tangent space $T_o\mathbf{S}$ at the origin. The Ricci curvature of $g_{\alpha,\beta}$ in the standard base $\{e_n\}$ of $T_o\mathbf{S}$ was computed in [**6**]. Surprisingly, it is finite and given by the formula

$$\mathrm{Ric}_{-n,m} := \mathrm{Ric}(e_{-n}, e_m) = \left(-\frac{13}{3}n^3 + \frac{1}{3}n\right)\delta_{n,m} \ .$$

The coefficient of n in this formula is not essential, since it can be changed by modifying the symplectic structure $w_{\alpha,\beta}$ inside its cohomology class in $H^2(\mathrm{Vect}(S^1), \mathbb{R})$ (or, in physical terms, by changing the normal ordering). The coefficient of n^3 does not alter inside the given cohomology class of $w_{\alpha,\beta}$ and so is more interesting. First of all, note that it does not depend on the values of parameters α and β (assuming, that $\alpha \neq 0$). Secondly, one can consider the Lie algebra $\mathrm{Vect}(S^1)$ as an infinite-dimensional analogue of the semisimple Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and, accordingly, the space $\mathrm{Diff}_+(S^1)/S^1$ as a Frechet analogue of the unit disc $\mathrm{SL}(2,\mathbb{R})/S^1$. In this way, one should think of the $\mathrm{Diff}_+(S^1)$-invariant Kähler metrics $g_{\alpha,\beta}$ on $\mathrm{Diff}_+(S^1)/S^1$ as infinite-dimensional analogues of the $\mathrm{SL}(2,\mathbb{R})$-invariant Poincaré metric on the disc. They have the Ricci curvature, equal (essentially) to $-\frac{13}{3}n^3$, which should be compared to the curvature $-\frac{1}{4}$ of the Poincaré metric on the disc.

3. A symplectic twistor bundle over ΩG and the geometric quantization of loop spaces

3.1. A twistor bundle over ΩG. Consider the bundle $\pi : Z \to \Omega G$ of admissible complex structures on ΩG with the fibre $\pi^{-1}(\gamma)$ at $\gamma \in \Omega G$, given by the restrictions J_γ of admissible complex structures J on ΩG to the tangent space $T_\gamma\Omega G$. Points $z \in Z$ are the pairs $z = (\gamma, J_\gamma)$, where $\gamma \in \Omega G$, J_γ is an admissible complex structure on $T_\gamma\Omega G$.

Admissible complex structures are invariant under the LG-action on ΩG, so there is a natural action of LG on Z and the quotient of this action coincides with the manifold $\mathbf{S} = \mathrm{Diff}_+(S^1)/S^1$. Thus we have a double fibration

$$\begin{array}{ccc} Z & \xrightarrow{\ p\ } & \mathbf{S} \\ {\scriptstyle\pi}\big\downarrow & & \\ \Omega G & & \end{array} \tag{5}$$

where p is the natural projection of Z onto $\mathbf{S}$. The fibre $p^{-1}(J)$ at $J \in \mathbf{S}$ can be identified with the complex manifold $(\Omega G, J)$, i.e. with the space ΩG, provided with the complex structure, corresponding to $J \in \mathbf{S}$ (we denote a point $J \in \mathbf{S}$ and the corresponding complex structure on ΩG by the same letter J).

As in the conventional twistor theory, the space Z of admissible complex structures on ΩG has a natural almost complex structure. Namely, the tangent bundle

TZ of Z can be split into the direct sum $TZ = V \oplus H$ of vertical and horizontal subbundles. The vertical subbundle $V \subset Z$ coincides with the kernel of the differential π_* of projection $\pi : Z \to \Omega G$. Respectively, the horizontal subbundle $H \subset Z$ is given by the kernel of the differential p_* of projection $p : Z \to \mathbf{S}$. The fibre V_z at $z \in Z$ is identified by p_* with the tangent space $T_J\mathbf{S}$ at $J = p(z) \in \mathbf{S}$, and so has a natural complex structure $\mathbf{J}^v$, induced by the Kähler complex structure of $\mathbf{S}$. We define now an almost complex structure $\mathbf{J}$ on Z in terms of the splitting $TZ = V \oplus H$ as

$$\mathbf{J} = \mathbf{J}^v \oplus \mathbf{J}^h \ ,$$

where the complex structure $\mathbf{J}^h$ at $z \in Z$ is equal to the complex structure J_γ on $T_zH \approx T_{\pi(z)}\Omega G$, determined by the point $z = (\gamma, J_\gamma)$. Note that the projection $p : Z \to \mathbf{S}$ is holomorphic with respect to $\mathbf{J}$ (while π is not!).

The double fibration (5) is somewhat similar to the twistor fibration over hyperkähler manifolds. Recall that these are the Riemannian manifolds M, provided with three covariantly constant complex structures I, J, K, satisfying the usual quaternion relations. The fibre of the twistor bundle $\pi\colon Z \to M$ of admissible (compatible with the Riemannian metric) complex structures on M is identified with the sphere $S^2 \cong \mathbb{CP}^1$, consisting of complex structures of the form $aI + bJ + cK$, where (a, b, c) is a point of the unit sphere in $\mathbb{R}^3$. The twistor space Z has a natural complex structure and the natural projection $p\colon Z \to \mathbb{CP}^1$ is a holomorphic mapping. Hence, we have the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow[p]{} & \mathbb{CP}^1 \\ {\scriptstyle\pi}\downarrow & & \\ M & & \end{array} \tag{6}$$

The main difference between this twistor diagram and the double diagram (5) in the case of loop spaces ΩG is that in (6) we consider complex structures, compatible with the Riemannian metric, while in (5) the involved complex structures are compatible with the symplectic structure. This fact implies, in particular, that the fibre of $\pi : Z \to M$ in (6) coincides with the Riemann sphere and so has the positive curvature and a non-trivial topology, while in (5) it is identified with the "Frechet disc" $\mathrm{Diff}_+(S^1)/S^1$ and so is contractible and, in the above sense, negatively curved.

3.2. Twistor quantization of the loop space of a vector space. The space $\mathbf{S}$ plays a central role in the twistor approach to the geometric quantization of loop spaces. We recall here briefly the twistor quantization construction, details may be found in [**7**]. In this Subsection we present the main ideas for the case of the loop space of the d-dimensional vector space $\mathbb{R}^d$, where the role of the group G is played by the group $\mathbb{R}^d$ of translations of $\mathbb{R}^d$, and in the next Subsection we generalize them to the loop spaces of compact Lie groups.

We consider again the twistor double fibration (5) with $G = \mathbb{R}^d$. The twistor quantization of $\Omega\mathbb{R}^d$ in terms of (5) means (cf. [**14, 7**]) the construction of a holomorphic (Fock) Hilbert bundle over the space $\mathbf{S}$ of admissible complex structures on $\Omega\mathbb{R}^d$ and a flat unitary connection on it.

We start from the construction of a Fock bundle over $\mathbf{S}$. For that we should enlarge first our Frechet loop space $\Omega\mathbb{R}^d$ to a Hilbert space. The natural choice is the Sobolev space $V^d := H_0^{1/2}(S^1, \mathbb{R}^d)$, consisting of functions $v \in L^2(S^1, \mathbb{R}^d)$, which have the zero mean value over the circle and generalized derivative of order

$1/2$ in $L^2(S^1, \mathbb{R}^d)$. (In some sense, it is the largest Sobolev space, for which our construction is still valid). In other words, the elements of V^d are the functions $v \in L^2(S^1, \mathbb{R}^d)$, having Fourier series

$$v(\theta) := v(e^{i\theta}) = \sum_{k \neq 0} v_k e^{ik\theta} \ , \quad v_k = \bar{v}_{-k}$$

with finite Sobolev norm

$$||v||^2_{1/2} = \sum_{k \neq 0} |k| ||v_k||^2 \ .$$

(In physical applications $\mathbb{R}^d$ is provided with the Lorentz metric of signature $(d-1, 1)$).

The definitions of the complex structure J^0 and symplectic structure ω, given in Sec.1 for the space ΩG, are easily extended to the space V^d so that V^d becomes a Kähler Hilbert space. Its complexification

$$V^d_{\mathbb{C}} = H^{1/2}_0(S^1, \mathbb{C}^d)$$

is a complex Hilbert space with the Hermitian inner product, defined by the Kähler metric on V^d. The symplectic form ω and complex structure operator J^0 extend complex linearly to $V^d_{\mathbb{C}}$ and determine a decomposition of $V^d_{\mathbb{C}}$ into the direct orthogonal sum

$$V^d_{\mathbb{C}} = W^0_+ \oplus W^0_-$$

of $(\pm i)$-eigenspaces of J^0 on $V^d_{\mathbb{C}}$, which are isotropic with respect to the symplectic form ω, i.e. $\omega(x, y) = 0$ if $x, y \in W^0_+$ or $x, y \in W^0_-$.

The action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the loop spaces ΩG by the reparametrization, defined in Sec.2.2, naturally extends to its action on the space V^d by symplectic transformations. Under the action of $f \in \mathrm{Diff}_+(S^1)$ the original complex structure J^0 on V^d transforms into another complex structure J, which coincides with J^0 iff $f \in S^1$. This complex structure J is also compatible with the symplectic form ω and so generates a decomposition

$$V^d_{\mathbb{C}} = W_+ \oplus W_- \tag{7}$$

into the direct sum of isotropic eigenspaces $W_\pm$ of the operator J on $V^d_{\mathbb{C}}$. The complex structure J, being compatible with ω, defines a Kähler metric on V^d, which extends to a Hermitian inner product on $V^d_{\mathbb{C}}$ and the decomposition (7) is orthogonal with respect to this inner product. We call again the complex structures J on V^d, obtained from J^0 by the action of $\mathrm{Diff}_+(S^1)$, admissible.

With any admissible complex structure on V^d, generating a decomposition (7), we associate a Fock space F_J of holomorphic functions on $V^d_{\mathbb{C}}$ (with respect to J), defined in the following way. We consider the algebra $\mathfrak{S}(W_+)$ of symmetric polynomials in variables $z \in W_+$ and provide it with a natural inner product, generated by the inner product on $V^d_{\mathbb{C}}$. The completion of $\mathfrak{S}(W_+)$ with respect to this inner product is called the Fock space $F_J = F(V^d_{\mathbb{C}}, J)$ over $V^d_{\mathbb{C}}$ with respect to the admissible complex structure J.

We introduce next the Heisenberg Lie algebra $\mathrm{heis}(V^d)$, which is the central extension of the Abelian Lie algebra V^d. In other words, it coincides with the vector space

$$\mathrm{heis}(V^d) = V^d \oplus \mathbb{R} \ ,$$

provided with the Lie bracket

$$[(x,s),(y,t)] := (0,\omega(x,y)) \ , \quad x,y \in V^d, \ s,t,\in \mathbb{R} \ .$$

This algebra has a standard (Heisenberg) irreducible representation r_J in the Fock space F_J, acting on vectors $f \in F_J$ by the formula

$$v \in V^d_{\mathbb{C}} \longmapsto r_J(v)f(z) = -\partial_v f(z) + < z, v > f(z) \ ,$$

where ∂_v is the derivative in the direction of v, and $< z, v >$ is the multiplication by the function $< z, v >$.

According to a theorem of Shale, all representations r_J are unitary equivalent. In other words, for any admissible complex structure J on V^d there exists a unitary operator $T_J : F_0 \to F_J$, intertwining the representation r_J in F_J with the representation r_0 in F_0, corresponding to the original complex structure J^0. Moreover, the map $\mathrm{Diff}_+(S^1) \ni f \longmapsto T_J : F_0 \to F_J$, where $J = J_f := f_* J$, defines a projective unitary action of $\mathrm{Diff}_+(S^1)$ (or a unitary action of the Virasoro group) on the union F of Fock spaces F_J over all admissible J. The infinitesimal version of this action yields a projective representation ρ_J of the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}_+(S^1)$ (or a representation of the Virasoro algebra vir) in the Fock space F_J. It is given explicitly in terms of the Virasoro operators L_n (cf. [**6, 14, 7**]).

This Fock space construction admits a natural interpretation in terms of the twistor bundle $\pi : Z \to V^d$ of admissible complex structures on V^d with the fibre at $v \in V^d$, formed by the restrictions of admissible complex structures J to the tangent space $T_v V^d \cong V^d$. We have again the twistor diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & \mathbf{S} \\ {\scriptstyle\pi}\downarrow & & \\ V^d & & \end{array}$$

where the fibre $p^{-1}(J)$ over $J \in \mathbf{S}$ is identified with the space $V_J = (V, J)$, provided with the complex structure J. We define an almost complex structure $\mathbf{J}$ on Z, as in Sec.3.1. Then the Fock space F_J for an admissible complex structure $J \in \mathbf{S}$ is identified with the Fock space $F(p^{-1}(J))$ of holomorphic functions on the fibre $p^{-1}(J)$ (with respect to the complex structure $\mathbf{J}$). The union of Fock spaces

$$F = \bigcup_{J \in \mathbf{S}} F_J$$

is a holomorphic Hilbert bundle over $\mathbf{S}$, which is trivial on $\mathbf{S}$ (since $\mathbf{S}$ is contractible). The holomorphic trivialization of H is given explicitly by the above map

$$T_J : F_0 \longrightarrow F_J \ .$$

Note that this trivialization is equivariant with respect to the action of the Virasoro group.

Recall that the infinitesimalization of this $\mathrm{Diff}_+(S^1)$-action yields a projective representation ρ_0 of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0. Having such a representation, we can construct a connection A on the Fock bundle F, whose curvature is equal to the cocycle of ρ_0 (cf. [**7**]). More explicitly, we identify $\mathbf{S} = \mathrm{Diff}_+(S^1)/S^1$ with the subgroup $\mathrm{Diff}_1(S^1)$, which consists of diffeomorphisms in $\mathrm{Diff}_+(S^1)$ fixing the point $1 \in S^1$. Then the covariant derivative of A at $f \in \mathrm{Diff}_1(S^1)$ in the direction $\xi \in T_f\mathbf{S}$ is equal to

$$\nabla_{A,\xi} v = \rho_0(L_{f^{-1},*}\xi)v \ , \quad v \in F_0 \ ,$$

where $L_{f^{-1}}$ is the left translation on $\mathrm{Diff}_1(S^1)$ by the element f^{-1}. This connection is invariant under the Virasoro group action.

The curvature of A at the origin $o \in \mathbf{S}$ in the standard base $e_n = ie^{in\theta}\frac{d}{d\theta} \in \mathrm{Vect}_0(S^1)$ of $T_o\mathbf{S}$ is given by (cf. [**6, 12**])

$$R_A(e_{-n}, e_m) = -\frac{d}{12}(n^3 - n)\delta_{n,m} .$$

As we see from this formula, this curvature never vanishes.

In order to get a flat connection, one should tensorially multiply F by, what physisists call, the "ghost bundle". This bundle in mathematical terms is equal to the square root $K^{-1/2} \to \mathbf{S}$ of the anticanonical bundle $K^{-1} \to \mathbf{S}$ of the space $\mathbf{S}$. The bundle $K^{-1/2} \to \mathbf{S}$ is a holomorphic line bundle over $\mathbf{S}$ with an action of the Virasoro group, covering the action of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Diff}_+(S^1)/S^1$ by left shifts. It can be provided with a canonical (spinor) connection B, invariant under this action. The curvature of B is equal to $-\frac{1}{2}\mathrm{Ric}$, where Ric is the Ricci curvature of $\mathbf{S}$, computed at the origin of $\mathbf{S}$ in Sec.2. In other words, the curvature of B in the standard base $\{e_n\}$ of $T_o\mathbf{S}$ is equal to

$$R_B(e_{-n}, e_m) = \left(\frac{13}{6}n^3 - \frac{1}{6}n\right)\delta_{n,m} .$$

Now we define the quantization bundle $\mathbf{H}$ as the tensor product of the Fock bundle $F \to \mathbf{S}$ and the ghost bundle $K^{-1/2} \to \mathbf{S}$:

$$\mathbf{H} := F \otimes K^{-1/2} \to \mathbf{S}$$

and provide it with the Vir-invariant tensor product connection

$$\mathbf{D} := A \otimes 1 + 1 \otimes B .$$

Then the curvature of $\mathbf{D}$ in the base $\{e_n\}$ of $T_o\mathbf{S}$ is equal to

$$R_{\mathbf{D}}(e_{-n}, e_m) = \left(\frac{26-d}{12}n^3 + \frac{d-2}{12}n\right)\delta_{n,m} .$$

As we have remarked in Sec.2, the coefficient of n may be made arbitrary by changing the symplectic structure inside its cohomology class (or modifying the normal ordering), so we can assert that the curvature $R_{\mathbf{D}}$ vanishes for $d = 26$. Hence, the *twistor quantization of the loop space $\Omega\mathbb{R}^d$, i.e. the construction of a Vir-invariant flat (Hermitian) connection on the quantization bundle $\mathbf{H} = F \otimes K^{-1/2} \to \mathbf{S}$ is possible for $d = 26$.*

3.3. Twistor quantization of loop spaces of compact Lie groups. In the case of the loop space of a compact Lie group G, we take for a twistor bundle the bundle $\pi : Z \to \Omega_T G$ of admissible complex structures on $\Omega_T G$ with the fibre $\mathbf{S} = \mathrm{Diff}_+(S^1)/S^1$ and associated double fibration

$$\begin{array}{ccc} Z & \xrightarrow{p} & \mathbf{S} \\ {\scriptstyle\pi}\downarrow & & \\ \Omega_T G & & \end{array} \tag{8}$$

We provide Z with the canonical almost complex structure as above, so that p becomes a holomorphic map. The group LG acts on Z horizontally, preserving the fibres of p without changing the complex structure, and the group $\mathrm{Diff}_+(S^1)$ acts on Z vertically, preserving the fibres of π, this action being induced by the action of $\mathrm{Diff}_+(S^1)$ on $\mathbf{S}$.

Let $\widetilde{L\mathfrak{g}}$ be the central extension of the loop algebra $L\mathfrak{g}$ and λ be an antidominant lowest weight of $\widetilde{L\mathfrak{g}}$. The algebra $\widetilde{L\mathfrak{g}}$ plays the role of the Heisenberg algebra (cf. Sec.3.2), whereas its lowest weight representation is an analogue of the Heisenberg representation.

The twistor quantization of $\Omega_T G$ is based on a construction, similar to that of Borel–Weil (cf. [**2**]). We consider the complex line bundle $L^\lambda \to \Omega_T G$ over the loop space $\Omega_T G$, associated with the lowest weight λ. The pull-back $\widetilde{L} = \pi^* L$ of this bundle to the twistor space Z has a natural almost complex structure.

Recall that any fibre $p^{-1}(J)$, $J \in \mathbf{S}$, of the projection $p : Z \to \mathbf{S}$ is identified with the loop space $\Omega_T G$, provided with the complex structure J. Accordingly, for any $J \in \mathbf{S}$ the irreducible lowest weight representation of $\widetilde{L\mathfrak{g}}$ with the weight λ may be realized in a Hilbert space H_J^λ of holomorphic sections of $\widetilde{L}_\lambda$ along the fibre $p^{-1}(J)$. The union H^λ of the Hilbert spaces H_J^λ over all $J \in \mathbf{S}$ forms a holomorphic Hilbert bundle $H^\lambda \to \mathbf{S}$ over $\mathbf{S}$, which is trivial since $\mathbf{S}$ is contractible. The action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on $\mathbf{S}$, as in Sec.3.2, is covered by a projective action of $\mathrm{Diff}_+(S^1)$ (or a Virasoro group action) on H^λ, intertwining the lowest weight representations of $\widetilde{L\mathfrak{g}}$ in H_J^λ for different $J \in \mathbf{S}$.

Using this action, we define, as in Sec.3.2, a connection A on H^λ, invariant under the Virasoro group action. The curvature of this connection at the origin of $\mathbf{S}$, computed in the base $e_n = ie^{in\theta}\frac{d}{d\theta}$ of $T_o\mathbf{S}$, has the form (cf. [**11, 12**])

$$R_A(e_{-n}, e_m) = \frac{c(\mathfrak{g})}{12}(-n^3 + n)\delta_{n,m} \ .$$

Here the number $c(\mathfrak{g})$ is determined by the formula

$$c(\mathfrak{g}) = \frac{h \dim \mathfrak{g}}{h + \kappa(\mathfrak{g})} \ ,$$

where h is the level of the lowest weight representation and $\kappa(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$.

Following the scheme of Sec.3.2, we take the tensor product $\mathbf{H}$ of the bundle $H^\lambda \to \mathbf{S}$ with the ghost bundle $K^{-1/2} \to \mathbf{S}$:

$$\mathbf{H} := H^\lambda \otimes K^{-1/2} \to \mathbf{S}$$

and provide it with the Vir-invariant tensor product connection

$$\mathbf{D} := A \otimes 1 + 1 \otimes B \ .$$

Then the curvature of $\mathbf{D}$ in the base $\{e_n\}$ of $T_o\mathbf{S}$ is equal to

$$R_{\mathbf{D}}(e_{-n}, e_m) = \left(\frac{26 - c(\mathfrak{g})}{12}n^3 + \frac{c(\mathfrak{g}) - 2}{12}n\right)\delta_{n,m} \ .$$

Having this formula, we can assert that *the twistor quantization of the loop space $\Omega_T G$ is possible if*

$$c(\mathfrak{g}) = \frac{h \dim \mathfrak{g}}{h + \kappa(\mathfrak{g})} = 26 \ .$$

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