# On Renormalization

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**Abstract**

The basic ideas of renormalization are presented with the help of scalar $\phi^4$-theory as the simplest example. It is shown that — in the first instance — the renormalization procedure is universal and has *nothing* to do with divergencies.

## Free theory

The Lagrangian of free $\phi^4$-theory reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \quad . \tag{1}$$

The physical meaning of all quantities is clear:

- $\phi$ creates/annihilates a free particle
- $m$ is the (physical) mass of the free particle

The Green's function is

$$G^{(2)}(x,y) := < 0|\mathrm{T}\phi(x)\phi(y)|0 > \tag{2}$$

$$G^{(2)}(k^2) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{3}$$

which (in momentum space) has a pole at $k^2 = m^2$. We can also *define* the mass in that way: The mass squared $m^2$ is defined as the pole of the Green's function.

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## Theory with interaction

With interaction the Lagrangian reads

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{\lambda}{4!}\phi^4 \quad . \tag{4}$$

But now we must be very careful: We no longer know what the exact physical meaning of $\phi$, $m$, $\lambda$ is!

Therefore we use a suffix "$B$" to emphasize that these quantities are merely (bare) parameters of the theory *without any direct physical meaning:*

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_B)^2 - \tfrac{1}{2}m_B^2\phi_B^2 - \tfrac{\lambda_B}{4!}\phi_B^4 \tag{5}$$

However, a physical theory should deal with *physical*, i.e. observable quantities. Therefore in order to make our theory sensible we have to relate $\phi_B$, $m_B$, $\lambda_B$ to physical quantities. To do so, we must define *measuring processes* for physical quantities.

## Some general remarks on perturbation theory

Let us start from the Lagrangian (4). (We omit the suffix "$B$" here.) Then the Green's functions are given by

$$G^{(2)}(x,y) := <0|\mathrm{T}\phi(x)\phi(y)|0> \tag{6}$$

$$G^{(n)}(x_1,x_2\dots x_n) := <0|\mathrm{T}\phi(x_1)\dots\phi(x_n)|0> \quad . \tag{7}$$

With interaction, the *full* propagator is

where the hatched bubbles denote *1-particle irreducible* self energy insertions. (1-particle irreducible means that the graph cannot be split into two separate parts by cutting just one line.) If in momentum space the self energy is given by $\frac{1}{i}\Sigma(k^2)$ we obtain the mathematical expression

$$\begin{aligned} G^{(2)}(k^2) &= \frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}\frac{1}{i}\Sigma(k^2)\frac{i}{k^2-m^2} + \dots \\ &= \frac{i}{k^2-m^2-\Sigma(k^2)} \quad . \end{aligned} \tag{8}$$

The vertex functions change, too, e.g.

such that $G^{(4)} \sim \Gamma^{(4)} \circ G^{(2)} \circ G^{(2)} \circ G^{(2)} \circ G^{(2)}$.

### Defining physical quantities

Now we proceed with our Lagrangian (5) which is written in terms of bare quantities. Consequently, we get a *bare* Green's function

$$G_B^{(2)}(k^2) = \frac{i}{k^2 - m_B^2 - \Sigma_B(k^2)} \quad . \tag{9}$$

We postulate in analogy to the free theory: The *physical mass* $m_F$ is the pole of the Green's function, i.e.

$$m_F^2 - m_B^2 - \Sigma_B(m_F^2) \stackrel{!}{=} 0 \quad . \tag{10}$$

We assume that we can develop $\Sigma_B(k^2)$ into a series about $k^2 = m_F^2$. (This assumption simplifies our calculation, but it is not essential; all the important results can be derived without using it.) Hence

$$\Sigma_B(k^2) = \Sigma_0 + \Sigma_1 \cdot (k^2 - m_F^2) + \Sigma_2 \cdot (k^2 - m_F^2)^2 + \dots \quad , \tag{11}$$

and the (bare) Green's function can be written as

$$G_B^{(2)}(k^2) = \frac{iZ_3}{k^2 - m_F^2 - \Sigma_F(k^2)} \tag{12}$$

where

$$m_F^2 = m_B^2 + \Sigma_0 \tag{13}$$

$$Z_3 := \frac{1}{1 - \Sigma_1} \tag{14}$$

$$\Sigma_F(k^2) = Z_3 \cdot [\Sigma_2 \cdot (k^2 - m_F^2)^2 + \dots] \tag{15}$$

such that

$$\Sigma_F(m_F^2) = 0 \qquad ; \qquad \frac{\partial}{\partial k^2} \Sigma_F(k^2) \Big|_{k^2 = m_F^2} = 0 \quad . \tag{16}$$

Thus in a whole *neighbourhood* of $k^2 = m_F^2$

$$G_B^{(2)}(k^2) \stackrel{k^2 \to m_F^2}{\approx} \frac{iZ_3}{k^2 - m_F^2} \quad . \tag{17}$$

However, in analogy to the free theory, the *physical* Green's function should have the form

$$G_F^{(2)}(k^2) \overset{k^2 \to m_F^2}{\approx} \frac{i}{k^2 - m_F^2} \quad . \tag{18}$$

Hence

$$G_F^{(2)} = \frac{1}{Z_3} G_B^{(2)} \quad . \tag{19}$$

Since $G_{B,F}^{(2)} \sim < 0|\mathrm{T}\phi_{B,F}\phi_{B,F}|0 >$, we define the *physical field* $\phi_F$:

$$\phi_F := \frac{1}{\sqrt{Z_3}} \phi_B \tag{20}$$

Furthermore, we define

$$Z_0 := Z_3 \frac{m_B^2}{m_F^2} \quad . \tag{21}$$

Now let us consider the *vertex functions*. Remember

$$G_{B,F}^{(4)} \sim \Gamma_{B,F}^{(4)} \circ G_{B,F}^{(2)} \circ G_{B,F}^{(2)} \circ G_{B,F}^{(2)} \circ G_{B,F}^{(2)} \quad . \tag{22}$$

Using (7) and (20) we obtain

$$G_B^{(n)} = Z_3^{n/2} \cdot G_F^{(n)} \tag{23}$$

and hence

$$\Gamma_F^{(4)} = Z_3^2 \cdot \Gamma_B^{(4)} \quad . \tag{24}$$

We define the *physical coupling constant* $\lambda_F$ to be the value of $\Gamma_F^{(4)}$ at a given reference point (renormalization point) $\mu$:

$$\lambda_F := \Gamma_F^{(4)}\Big|_\mu \tag{25}$$

From the Lagrangian one can calculate (in perturbation theory) $\Gamma_B^{(4)}\Big|_\mu$. To lowest order $\Gamma_B^{(4)}\Big|_\mu = \lambda_B$, but in general $\Gamma_B^{(4)}\Big|_\mu \neq \lambda_B$. With

$$Z_1 := \frac{\lambda_B}{\Gamma_B^{(4)}\Big|_\mu} \tag{26}$$

and (24) we get

$$\lambda_F = \frac{Z_3^2}{Z_1} \lambda_B \quad . \tag{27}$$

### The renormalized Lagrangian

Expressed only in terms of *physical* quantities (i.e. inserting (20,21,27) into the "bare" Lagrangian (5)), the Lagrangian now reads:

$$\begin{aligned}\mathcal{L} = &\ \tfrac{1}{2}\ (\partial_\mu \phi_F)^2 - \tfrac{1}{2} m_F^2 \phi_F^2 - \tfrac{\lambda_F}{4!}\phi_F^4 + \\ + &\ \underbrace{\tfrac{1}{2}(Z_3-1)(\partial_\mu\phi_F)^2 - \tfrac{1}{2}(Z_0-1)m_F^2\phi_F^2 - (Z_1-1)\tfrac{\lambda_F}{4!}\phi_F^4}_{\text{"Counterterms"}}\end{aligned} \tag{28}$$

The 3 constants $Z_3$, $Z_0$, $Z_1$ are *uniquely determined* by the 3 *renormalization conditions*

$$\Sigma_F(m_F^2) = 0 \tag{29}$$

$$\frac{\partial}{\partial k^2}\Sigma_F(k^2)\bigg|_{k^2=m_F^2} = 0 \tag{30}$$

$$\Gamma_F^{(4)}\Big|_\mu = \lambda_F \quad . \tag{31}$$

$Z_3$, $Z_0$, $Z_1$ are not observable quantities.

Now we can use this "renormalized" Lagrangian (28) as the basis for a perturbation expansion, but now with the *physical* quantities $m_F$ and $\lambda_F$ as expansion parameters (rather than $m_B$, $\lambda_B$).

The first part of the Lagrangian has the original form, but now with the bare parameters replaced by physical ones, and hence yields the same perturbation expansion. But there are corrections coming from the counterterms.

**Note:** *So far we have with no word mentioned divergencies. This renormalization procedure (i.e. relating parameters of the Lagrangian to physical quantities) is universal and must be done in* all *quantum field theories with interaction, whether finite or infinite!*

This explains our statement at the beginning of this work when we claimed that in the first instance renormalization had nothing to do with divergencies. Rather *it turns out to be merely a by-product that in gauge theories the counterterms cancel all divergencies.*

Remember that only *interacting* theories must be renormalized.

A theory is called *renormalizable* if, to all orders of perturbation theory, all divergencies are cancelled by the counterterms.

Every sensible theory must give finite results for observable quantities. Thus *a theory is sensible only if it is renormalizable.* Our example, the $\phi^4$-theory, is indeed renormalizable.

In fact, the condition of renormalizability is a strong restriction. For example, requiring renormalizability of the electroweak interaction leads *uniquely* to the Glashow-Salam-Weinberg Lagrangian!

### How to handle divergencies

Although they will be cancelled in the final result by the counterterms, divergencies may occur at an *intermediate stage* of calculations. (In principle, by including all counterterms from the beginning, one could avoid any divergencies.)

To handle divergent integrals one has to use some *regularization*. The final result does *not* depend on *which* regularization is used. However, some methods are more convenient than others. The most commonly used are

- dimensional regularization (widely used; advantage: preserves gauge invariance)
- cut-off (now old-fashioned)
- Pauli-Villars (now old-fashioned)

$\vdots$

## How to perform a concrete calculation

1. Choose order $n$ in perturbation theory. Stick to this order throughout the calculation.
2. Choose method of regularization. Stick to this method throughout the calculation.
3. Take Lagrangian with *bare* quantities and calculate $\Sigma_B$, $\Gamma_B^{(4)}$ to $n$-th order p.t.; regularize divergent integrals.
4. Use the 3 renormalization conditions to determine $Z_3$, $Z_0$, $Z_1$. (*Note:* For different orders $n$ get different $Z_i$'s.)
5. Write down the *renormalized* Lagrangian.
6. Use the first part of the renormalized Lagrangian to calculate the process to $n$-th order p.t.; regularize divergent integrals.
7. Now calculate all corrections using the counterterms.
8. Add all contributions and obtain the final result (which should be finite if the theory is renormalizable).

## References

Here we give some general references where a detailed introduction to renormalization theory can be found:
S. Pokorski, *Gauge Field Theories*, Cambridge Univ. Press; J. Collins, *Renormalization*, Cambridge Univ. Press; L. H. Ryder, *Quantum Field Theory*, Cambridge Univ. Press; R. Kleiss, Lecture Notes, ICTP Trieste 1989.