SOME REMARKABLE PROPERTIES OF SINC AND RELATED INTEGRALS

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Abstract. Using Fourier transform techniques, we establish inequalities for integrals of the form

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{a_k x}\, dx.$$

We then give quite striking closed form evaluations of such integrals and finish by discussing various extensions and applications.

1. Introduction. Motivated by questions about the integral[1]

$$\mu := \int_0^\infty \prod_{k=1}^\infty \cos\left(\frac{x}{k}\right)\, dx, \tag{1}$$

we study the behaviour of integrals of the form

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{a_k x}\, dx.$$

In Section 2 we use Fourier transform theory to establish monotonicity properties of these integrals as functions of n. In Section 3, by direct methods, we give closed forms for these integrals and for similar integrals also incorporating cosine terms. In Section 4, we provide a very different proof of one of these results following an idea in an 1885 paper of Störmer [2]. Finally, in Section 5 we return to the study of (1).

2. Fourier cosine transforms and sinc integrals. Define

$$\operatorname{sinc}(x) := \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

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[1] Through J. Selfridge and R. Crandall we learned that B. Mares discovered that $\mu < \dfrac{\pi}{4}$.

and, for $a > 0$,

$$\chi_a(x) := \begin{cases} 1 & \text{if } |x| < a \\ \frac{1}{2} & \text{if } |x| = a \\ 0 & \text{if } |x| > a. \end{cases}$$

We first state some standard results about the Fourier cosine transform (FCT) which may be found in texts such as [4, ch. 13].

The FCT of a function $f \in L_2(-\infty,\infty)$ is defined to be the function $\hat{f}$ that is the L_2-limit as $y \to \infty$ of

$$c_y(x) := \frac{1}{\sqrt{2\pi}} \int_{-y}^{y} f(t)\cos(xt)\,dt, \text{ i.e. } \int_{-\infty}^{\infty} |c_y(x) - \hat{f}(x)|^2\,dx \to 0 \text{ as } y \to \infty.$$

This function $\hat{f}$ exists, is unique apart from sets of zero Lebesgue measure, and $\hat{f} \in L_2(-\infty,\infty)$; and if f is even so also is $\hat{f}$ and f is the FCT of $\hat{f}$. If, in addition, f is continuous on $(-\alpha,\alpha)$ for some $\alpha > 0$ and $\hat{f} \in L_1(-\infty,\infty)$, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x)\cos(xt)\,dx = f(t) \text{ for } -\alpha < t < \alpha,$$

since the left-hand term is also continuous on $(-\alpha,\alpha)$ by dominated convergence.

Note that, for $a > 0$, the FCT of χ_a is $a\sqrt{\frac{2}{\pi}}\operatorname{sinc}(ax)$, so that the FCT of $a\sqrt{\frac{2}{\pi}}\operatorname{sinc}(ax)$ is equivalent to χ_a. In fact, it can easily be shown that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a\sqrt{\frac{2}{\pi}}\operatorname{sinc}(ax)\cos(xt)\,dx = \chi_a(t) \text{ for all } t \in (-\infty,\infty),$$

either directly or by appeal to a standard result about inverse Fourier transforms of functions of local bounded variation.

Note also that if $\hat{f}_1, \hat{f}_2$ are FCTs of even functions $f_1, f_2 \in L_2(-\infty,\infty)$, then $\hat{f}_1\hat{f}_2$ is the FCT of $\frac{1}{\sqrt{2\pi}} f_1 * f_2$, where

$$f_1 * f_2(x) := \int_{-\infty}^{\infty} f_1(x-t) f_2(t)\,dt \text{ for all } x \in (-\infty,\infty).$$

In addition, we have the following version of Parseval's theorem for such even functions:

$$\int_0^{\infty} f_1(x) f_2(x)\,dx = \int_0^{\infty} \hat{f}_1(x)\hat{f}_2(x)\,dx,$$

provided at least one of the functions f_1, f_2 is real.

We are now in a position to prove:

Theorem 1. *Suppose that $\{a_n\}$ is a sequence of positive numbers. Let $s_n := \sum_{k=1}^{n} a_k$ and*

$$\tau_n := \int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx.$$

(i) *Then*

$$0 < \tau_n \leq \frac{1}{a_0}\frac{\pi}{2},$$

with equality if $n = 0$, or if $a_0 \geq s_n$ when $n \geq 1$.

(ii) *If $a_{n+1} \leq a_0 < s_n$ with $n \geq 1$, then*

$$0 < \tau_{n+1} \leq \tau_n < \frac{1}{a_0}\frac{\pi}{2}.$$

(iii) *If $a_0 < s_{n_0}$ with $n_0 \geq 1$, and $\sum_{k=0}^{\infty} a_k^2 < \infty$, then there is an integer $n_1 \geq n_0$ such that*

$$\tau_n \geq \int_0^\infty \prod_{k=0}^{\infty} \operatorname{sinc}(a_k x)\, dx \geq \int_0^\infty \prod_{k=0}^{\infty} \operatorname{sinc}^2(a_k x)\, dx > 0 \text{ for all } n \geq n_1.$$

Observe that applying Theorem 1 to different permutations of the parameters will in general yield different inequalities.

Proof. Part (i). That $\tau_0 = \frac{1}{a_0}\frac{\pi}{2}$ is a standard result (proven e.g., by contour integration in [1, p. 157] and by Fourier analysis in [3, p. 563]) with the integral in question being improper (i.e. not absolutely convergent—the integrals in the other cases are absolutely convergent). Assume therefore that $n \geq 1$, and let

$$F_0 := \frac{1}{a_0}\sqrt{\frac{\pi}{2}}\chi_{a_0}, \quad F_n := (\sqrt{2\pi})^{1-n} f_1 * f_2 * \cdots * f_n, \text{ where } f_n := \frac{1}{a_n}\sqrt{\frac{\pi}{2}}\chi_{a_n}.$$

Then it is readily verified by induction that, for $n \geq 1, F_n(x)$ is an even function which vanishes on $(-\infty, -s_n) \cup (s_n, \infty)$ and is positive on $(-s_n, s_n)$. Moreover, $F_{n+1} = \frac{1}{\sqrt{2\pi}} F_n * f_{n+1}$, so that

$$F_{n+1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_n(x-t) f_{n+1}(t)\, dt = \frac{1}{2a_{n+1}} \int_{x-a_{n+1}}^{x+a_{n+1}} F_n(u)\, du.$$

Hence $F_{n+1}(x)$ is absolutely continuous on $(-\infty, \infty)$ and, for almost all $x \in (-\infty, \infty)$,

$$2a_{n+1} F'_{n+1}(x) = F_n(x + a_{n+1}) - F_n(x - a_{n+1}) = F_n(x + a_{n+1}) - F_n(a_{n+1} - x).$$

Since $(x+a_{n+1}) \geq \max\{(x-a_{n+1}),(a_{n+1}-x)\} \geq 0$ when $x>0$, it follows that if $F_n(x)$ is monotone non-increasing on $(0,\infty)$, then $F'_{n+1}(x) \leq 0$ for a.a. $x \in (0,\infty)$, and so $F_{n+1}(x)$ is monotone non-increasing on $(0,\infty)$. This monotonicity property of F_n on $(0,\infty)$ is therefore established by induction for all $n \geq 1$. Also

$$F_n \text{ is the FCT of } \sigma_n(x) := \prod_{k=1}^{n} \operatorname{sinc}(a_k x), \text{ and } \sigma_n \text{ is the FCT of } F_n.$$

Thus, all our functions and transforms are even and are in $L_1(0,\infty)\cap L_2(0,\infty)$. Hence, by the above version of Parseval's theorem,

$$\tau_n = \int_0^\infty F_n(x)F_0(x)\,dx = \frac{1}{a_0}\sqrt{\frac{\pi}{2}}\int_0^{\min(s_n,a_0)} F_n(x)\,dx. \tag{2}$$

When $a_0 \geq s_n$, the final term is equal to $\frac{1}{a_0}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}}\sigma_n(0) = \frac{1}{a_0}\frac{\pi}{2}$ since $\sigma_n(x)$ is continuous on $(-\infty,\infty)$; and when $a_0 < s_n$, the term is positive and less than $\frac{1}{a_0}\frac{\pi}{2}$ since $F_n(x)$ is positive and continuous for $0<x<s_n$. This establishes part (i).

Part (ii). Observe again that $F_{n+1} = \frac{1}{\sqrt{2\pi}} F_n * f_{n+1}$, and hence that, for $y>0$,

$$\begin{aligned}
\int_0^y F_{n+1}(x)\,dx &= \frac{1}{\sqrt{2\pi}}\int_0^y dx \int_{-\infty}^{\infty} F_n(x-t)f_{n+1}(t)\,dt = \frac{1}{2a_{n+1}}\int_0^y dx\int_{-a_{n+1}}^{a_{n+1}} F_n(x-t)\,dt\\
&= \frac{1}{2a_{n+1}}\int_{-a_{n+1}}^{a_{n+1}} dt \int_0^y F_n(x-t)\,dx = \frac{1}{2a_{n+1}}\int_{-a_{n+1}}^{a_{n+1}} dt\int_{-t}^{y-t} F_n(u)\,du\\
&= \int_0^y F_n(u)\,du + \frac{1}{2a_{n+1}}(I_1+I_2),
\end{aligned}$$

where

$$I_1 := \int_{-a_{n+1}}^{a_{n+1}} dt \int_{-t}^{0} F_n(u)\,du \quad \text{and} \quad I_2 := \int_{-a_{n+1}}^{a_{n+1}} dt\int_y^{y-t} F_n(u)\,du.$$

Now $I_1 = 0$ since $\int_{-t}^0 F_n(u)\,du$ is an odd function of t, and for $y \geq a_{n+1}$,

$$\begin{aligned}
I_2 &= \int_0^{a_{n+1}} dt\int_y^{y-t} F_n(u)\,du + \int_{-a_{n+1}}^0 dt\int_y^{y-t} F_n(u)\,du\\
&= -\int_0^{a_{n+1}} dt\int_{y-t}^{y} F_n(u)\,du + \int_0^{a_{n+1}} dt\int_y^{y+t} F_n(u)\,du\\
&= \int_0^{a_{n+1}} dt\int_{y-t}^{y} \big(F_n(u+t) - F_n(u)\big)\,du \leq 0
\end{aligned}$$

since $F_n(u)$ is monotonic non-increasing for $u \geq y - t \geq y - a_{n+1} \geq 0$. Hence

$$\int_0^y F_{n+1}(x)\,dx \leq \int_0^y F_n(x)\,dx \text{ when } a_{n+1} \leq y < s_n. \tag{3}$$

It follows from (2), and (3) with $y = a_0$, that $0 < \tau_{n+1} \leq \tau_n$ if $a_{n+1} \leq a_0 < s_n$, and this completes part (ii).

Part (iii). Let $\rho(x) := \lim_{n\to\infty} \sigma_n^2(x) = \prod_{k=1}^{\infty} \operatorname{sinc}^2(a_k x)$ for $x > 0$. Observe that the limit exists since $0 \leq \operatorname{sinc}^2(a_k x) < 1$, and that there is a set A differing from $(0,\infty)$ by a countable set such that $0 < \operatorname{sinc}^2(a_k x) < 1$ whenever $x \in A$ and $k = 1, 2, \ldots$. Now

$$\operatorname{sinc}(a_k x) = 1 - \delta_k, \text{ where } 0 \leq \frac{\delta_k}{a_k^2} \to \frac{x^2}{3} \text{ as } k \to \infty,$$

so that $\sum_{k=1}^{\infty} \delta_k < \infty$, and hence, by standard theory of infinite products, $\sigma(x) := \lim_{n\to\infty} \sigma_n(x)$ exists and $\sigma^2(x) = \rho(x) > 0$ for $x \in A$. It follows, by part (ii), that

$$\tau_n \geq \int_0^\infty \sigma_n^2(x)\,dx \geq \int_0^\infty \rho(x)\,dx > 0$$

for all $n \geq n_1$, where $n_1 \geq n_0$ is an integer such that $a_{n+1} \leq a_0$ for all $n \geq n_1$. In addition, by dominated convergence,

$$\lim_{n\to\infty} \tau_n = \int_0^\infty \sigma(x)\,dx \geq \int_0^\infty \rho(x)\,dx,$$

and this completes the proof of part (iii). $\square$

3. Some elementary identities. In this section we prove some identities involving products of sines and cosines by straightforward methods not involving Fourier transform theory. We adopt the usual convention that empty sums have the value 0 and empty products have the value 1, and we define

$$\operatorname{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Theorem 2. *Let $a_0, a_1, \ldots, a_n$ be complex numbers with $n \geq 1$. For each of the 2^n ordered n-tuples $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \{-1, 1\}^n$ define*

$$b_\gamma := a_0 + \sum_{k=1}^{n} \gamma_k a_k, \qquad \epsilon_\gamma := \prod_{k=1}^{n} \gamma_k.$$

(i) *Then*

$$\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^r = \begin{cases} 0 & \text{for } r = 1, 2, \ldots, n-1 \\ 2^n n! \prod_{k=1}^n a_k & \text{for } r = n, \end{cases}$$

and

$$\prod_{k=0}^n \sin(a_k x) = \frac{1}{2^n} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma \cos\left(b_\gamma x - \frac{\pi}{2}(n+1)\right).$$

(ii) *If* $a_0, a_1, \ldots, a_n$ *are real, then*

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \frac{1}{2^n n!} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

If, in addition,

$$a_0 \geq \sum_{k=1}^n |a_k|,$$

then

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^n a_k.$$

Proof. Observe that

$$e^{a_0 t} \prod_{k=1}^n (e^{a_k t} - e^{-a_k t}) = \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma e^{b_\gamma t}.$$

Since $e^{a_k t} - e^{-a_k t} = 2a_k t + O(t^2)$ as $t \to 0$, the first summation formula in part (i) follows on equating coefficients of t^r in the above identity. Note that the formula also holds for $r = 0$ if we define $b_\gamma^0 = 1$ even when $b_\gamma = 0$. Similarly

$$\begin{aligned} \prod_{k=0}^n \sin(a_k x) &= \frac{1}{(2i)^{n+1}} (e^{ia_0 x} - e^{-ia_0 x}) \prod_{k=1}^n (e^{ia_k x} - e^{-ia_k x}) \\ &= \frac{1}{(2i)^{n+1}} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma \left(e^{ib_\gamma x} - (-1)^n e^{-ib_\gamma x}\right) \\ &= \frac{1}{2^n} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma \cos\left(b_\gamma x - \frac{\pi}{2}(n+1)\right), \end{aligned}$$

and this completes the proof of part (i).

To prove part (ii) of the theorem, observe that

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\, dx = \frac{1}{2^n} \int_0^\infty x^{-n-1} C_n(x)\, dx, \tag{4}$$

where $C_n(x) := \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma \cos\left(b_\gamma x - \frac{\pi}{2}(n+1)\right)$. Because $C_n(x)$ is an entire function, bounded for all real x, with a zero of order $n+1$ at $x=0$, we can integrate the right-hand side of (4) by parts n times to get

$$\begin{aligned}\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\,dx &= \frac{1}{2^n n!}\int_0^\infty \frac{dx}{x} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \sin(b_\gamma x)\\ &= \frac{1}{2^n n!} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \int_0^\infty \frac{\sin(b_\gamma x)}{x}\,dx\\ &= \frac{\pi}{2}\frac{1}{2^n n!} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).\end{aligned}$$

Since the additional hypothesis implies that $b_\gamma \geq 0$ for all $\gamma \in \{-1,1\}^n$, the final formula in the theorem follows from part (i). ☐

Corollary 1. *If $2a_k \geq a_n > 0$ for $k = 0, 1, \ldots, n-1$ and*

$$\sum_{k=1}^n a_k > a_0 \geq \sum_{k=1}^{n-1} a_k,$$

then

$$\int_0^\infty \prod_{k=0}^r \frac{\sin(a_k x)}{x}\,dx = \frac{\pi}{2}\prod_{k=1}^r a_k \text{ for } r = 0, 1, \ldots, n-1,$$

while

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\,dx = \frac{\pi}{2}\left\{\prod_{k=1}^n a_k - \frac{(a_1+a_2+\cdots+a_n-a_0)^n}{2^{n-1}n!}\right\}.$$

Proof. Let $\gamma' := (-1,-1,\ldots,-1) \in \{-1,1\}^n$. Observe that $b_{\gamma'} := a_0 - a_1 - \cdots - a_n < 0$, that $b_\gamma \geq 0$ for every other $\gamma \in \{-1,1\}^n$, and that $\epsilon_{\gamma'} = (-1)^n$. It follows that

$$\begin{aligned}\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\,dx &= \frac{\pi}{2}\frac{1}{2^n n!} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma)\\ &= \frac{\pi}{2}\frac{1}{2^n n!}\left(\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n + \epsilon_{\gamma'} b_{\gamma'}^n \left(\operatorname{sign}(b_{\gamma'}) - 1\right)\right)\\ &= \frac{\pi}{2}\left\{\prod_{k=1}^n a_k - \frac{2(-b_{\gamma'})^n}{2^n n!}\right\},\end{aligned}$$

as desired. ☐

Remarks 1. (a) If $a_0, a_1, \ldots, a_n$ are real and non-zero, then, by Theorem 2(ii),

$$\begin{aligned}\tau_n :&= \int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{a_k x}\,dx = \frac{\pi}{2}\frac{1}{2^n n! a_0 a_1 \cdots a_n} \sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma)\\ &= \frac{\pi}{2}\frac{1}{2^n n! a_0 a_1 \cdots a_n}\left(\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n + \sum_{b_\gamma<0} \epsilon_\gamma b_\gamma^n \bigl(\operatorname{sign}(b_\gamma)-1\bigr)\right)\\ &= \frac{\pi}{2a_0}\left\{1 - \frac{1}{2^{n-1} n! a_1 a_2 \cdots a_n} \sum_{b_\gamma<0} \epsilon_\gamma b_\gamma^n\right\}.\end{aligned}$$

(b) Suppose further that $a_k > 0$ for $k = 0, 1, \ldots, n$. Consider the polyhedra

$$\begin{aligned}P_n =&P_n(a_0, a_1, \cdots, a_n)\\ &:= \{(x_1, x_2, \cdots, x_n)\bigl| -a_0 \le \sum_{k=1}^n x_k \le a_0, -a_k \le x_k \le a_k \text{ for } k = 1, 2, \ldots, n\},\\ Q_n =&Q_n(a_0, a_1, \cdots, a_n)\\ &:= \{(x_1, x_2, \cdots, x_n)\bigl| -a_0 \le \sum_{k=1}^n a_k x_k \le a_0, -1 \le x_k \le 1 \text{ for } k = 1, 2, \ldots, n\},\\ &H_n := \{(x_1, x_2, \cdots, x_n)\bigl| -1 \le x_k \le 1 \text{ for } k = 1, 2, \ldots, n\}.\end{aligned}$$

(i) If we return to equation (2) we observe that

$$\tau_n = \frac{\pi}{a_0}\frac{1}{2^n a_1 a_2 \cdots a_n}\int_0^{\min(s_n, a_0)} \chi_{a_1} * \chi_{a_2} * \cdots * \chi_{a_n}\,dx = \frac{\pi}{2a_0}\frac{\operatorname{Vol}(P_n)}{2^n a_1 a_2 \cdots a_n} = \frac{\pi}{2a_0}\frac{\operatorname{Vol}(Q_n)}{\operatorname{Vol}(H_n)}.$$

Moreover, we now explain the behaviour of τ_n when we note that the value drops precisely when the constraint $-a_0 \le \sum_{k=1}^n a_k x_k \le a_0$ becomes active and bites into the hypercube H_n.

(ii) We sketch a probabilistic interpretation. From (i) it follows that $p_n := 2a_0\,\tau_n/\pi$ may be regarded as the probability that independent random variables $\{x_k, k = 1, 2, \ldots\}$ identically distributed in $[-1, 1]$ satisfy $|\sum_{k=1}^n a_k x_k| \le a_0$. Correspondingly

$$p_\infty := \frac{2a_0}{\pi}\int_0^\infty \prod_{k=1}^\infty \operatorname{sinc}(a_k x)\,dx$$

is the probability that the constraint $|\sum_{k=1}^\infty a_k x_k| \le a_0$ is met. We have also shown that p_n decreases monotonically to p_∞.

(c) Consider now the special case

$$\mu_n := \tau_{n-1} = \int_0^\infty \operatorname{sinc}^n(x)\,dx.$$

In this case we have $a_k = 1$ for $k = 0, 1, \ldots, n-1$, and it is straightforward to verify that

$$\sum_{\gamma \in \{-1,1\}^{n-1},\, b_\gamma < 0} \epsilon_\gamma b_\gamma^{n-1} = \sum_{1 \le r \le \frac{n}{2}} (-1)^{r+1} \binom{n-1}{r-1} (n-2r)^{n-1},$$

and hence that

$$\begin{aligned}
\mu_n =& \frac{\pi}{2} \left\{ 1 - \frac{2}{2^{n-1}(n-1)!} \sum_{1 \le r \le \frac{n}{2}} (-1)^{r+1} \binom{n-1}{r-1} (n-2r)^{n-1} \right\} \\
=& \frac{\pi}{2} \left\{ 1 + \frac{1}{2^{n-2}} \sum_{1 \le r \le \frac{n}{2}} \frac{(-1)^r}{(r-1)!} \frac{(n-2r)^{n-1}}{(n-r)!} \right\}.
\end{aligned}$$

□

The following formula for μ_n appears as an exercise in [5, p. 123]:

$$\mu_n = \frac{\pi}{2^n (n-1)!} \sum_{0 \le r \le \frac{n}{2}} (-1)^r \binom{n}{r} (n-2r)^{n-1}.$$

To show that this formula for μ_n is equivalent to the one derived above, it clearly suffices to prove that

$$\sum_{0 \le r \le \frac{n}{2}} (-1)^r \binom{n}{r} (n-2r)^{n-1} = 2^{n-1}(n-1)! + \sum_{1 \le r \le \frac{n}{2}} (-1)^{r+1} \binom{n-1}{r-1} (n-2r)^{n-1}.$$

Since

$$\binom{n}{r} - 2\binom{n-1}{r-1} = \frac{1}{n}\binom{n}{r}(n-2r),$$

this is equivalent to proving that

$$\sum_{0 \le r \le \frac{n}{2}} (-1)^r \binom{n}{r} (n-2r)^n = 2^{n-1} n!,$$

which, by symmetry, is equivalent to proving that

$$\frac{1}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-2r)^n = 2^n.$$

But the left-hand side of this latter identity is the coefficient of t^n in

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}e^{(n-2r)t} = e^{nt}(1-e^{-2t})^n = (2\sinh t)^n.$$

Since $2\sinh t = 2t + O(t^2)$ as $t \to 0$, the coefficient is indeed 2^n, and the desired equivalence of the formulae for μ_n is proved.

The next theorem extends Theorem 2 by adjoining cosines to the product of sines.

Theorem 3. *Let $a_0, a_1, \ldots, a_{n+m}$ be complex numbers with $n \geq 1$ and $m \geq 0$. For each of the 2^{n+m} ordered $(n+m)$-tuples $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_{n+m}) \in \{-1,1\}^{n+m}$ define*

$$b_\gamma := a_0 + \sum_{k=1}^{n+m}\gamma_k a_k, \qquad \epsilon_\gamma := \prod_{k=1}^{n}\gamma_k.$$

(i) *Then*

$$\sum_{\gamma\in\{-1,1\}^{n+m}}\epsilon_\gamma b_\gamma^r = \begin{cases} 0 & \text{for } r = 1,2,\ldots,n-1 \\ 2^{n+m}n!\prod_{k=1}^{n}a_k & \text{for } r = n, \end{cases}$$

and

$$\left(\prod_{k=0}^{n}\sin(a_k x)\right)\left(\prod_{k=n+1}^{n+m}\cos(a_k x)\right) = \frac{1}{2^{n+m}}\sum_{\gamma\in\{-1,1\}^{n+m}}\epsilon_\gamma\cos\left(b_\gamma x - \frac{\pi}{2}(n+1)\right).$$

(ii) *If $a_0, a_1, \ldots, a_{n+m}$ are real, then*

$$\int_0^\infty\left(\prod_{k=0}^{n}\sin(a_k x)\right)\left(\prod_{k=n+1}^{n+m}\cos(a_k x)\right)dx = \frac{\pi}{2}\frac{1}{2^{n+m}n!}\sum_{\gamma\in\{-1,1\}^{n+m}}\epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

If, in addition,

$$a_0 \geq \sum_{k=1}^{n+m}|a_k|,$$

then

$$\int_0^\infty\left(\prod_{k=0}^{n}\sin(a_k x)\right)\left(\prod_{k=n+1}^{n+m}\cos(a_k x)\right)dx = \frac{\pi}{2}\prod_{k=1}^{n}a_k.$$

Proof. By Theorem 2 we have that

$$\prod_{k=1}^{n+m}\sin(a_k x) = \frac{1}{2^{n+m}}\sum_{\gamma\in\{-1,1\}^{n+m}}\epsilon'_\gamma\cos\left(b_\gamma x - \frac{\pi}{2}(n+m+1)\right),$$

where, for each $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{n+m}) \in \{-1, 1\}^{n+m}$,

$$b_\gamma = a_0 + \sum_{k=1}^{n+m} \gamma_k a_k, \quad \epsilon'_\gamma = \prod_{k=1}^{n+m} \gamma_k = \pm 1,$$

and

$$\sum_{\gamma \in \{-1,1\}^{n+m}} \epsilon'_\gamma b_\gamma^r = \begin{cases} 0 & \text{for } r = 1, 2, \dots, n+m-1 \\ 2^{n+m}(n+m)! \prod_{k=1}^{n+m} a_k & \text{for } r = n+m. \end{cases}$$

Differentiating these expressions partially with respect to $a_{n+1}, a_{n+2}, \dots, a_{n+m}$ yields part (i) of Theorem 3 with

$$\epsilon_\gamma = \epsilon'_\gamma \prod_{k=1}^{m} \gamma_{n+k} = \left(\prod_{k=1}^{n} \gamma_k \right) \prod_{k=1}^{m} \gamma_{n+k}^2 = \prod_{k=1}^{n} \gamma_k.$$

To deal with part (ii) of Theorem 3 we observe that, by Theorem 2, if $a_0, a_1, \dots, a_{n+m}$ are real, then

$$\int_0^\infty \prod_{k=0}^{n+m} \frac{\sin(a_k x)}{x} \, dx = \frac{\pi}{2} \frac{1}{2^{n+m}(n+m)!} \sum_{\gamma \in \{-1,1\}^{n+m}} \epsilon'_\gamma b_\gamma^{n+m} \operatorname{sign}(b_\gamma).$$

Differentiating partially with respect to $a_{n+1}, a_{n+2}, \dots, a_{n+m}$, we get

$$\int_0^\infty \left(\prod_{k=0}^{n} \frac{\sin(a_k x)}{x} \right) \left(\prod_{k=n+1}^{n+m} \cos(a_k x) \right) dx$$
$$= \frac{\pi}{2} \frac{1}{2^{n+m} n!} \sum_{\gamma \in \{-1,1\}^{n+m}} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

If, in addition,

$$a_1 \geq \sum_{k=2}^{n+m} |a_k|,$$

then, by Theorem 2,

$$\int_0^\infty \prod_{k=0}^{n+m} \frac{\sin(a_k x)}{x} \, dx = \frac{\pi}{2} \prod_{k=1}^{n+m} a_k.$$

Differentiating partially with respect to $a_{n+1}, a_{n+2}, \dots, a_{n+m}$, we get

$$\int_0^\infty \left(\prod_{k=0}^{n} \frac{\sin(a_k x)}{x} \right) \left(\prod_{k=n+1}^{n+m} \cos(a_k x) \right) dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \qquad \square$$

Corollary 2. *If* $2a_k \geq a_{n+m} > 0$ *for* $k = 0, 1, \ldots, n+m-1$ *and*

$$\sum_{k=1}^{n+m} a_k > a_0 \geq \sum_{k=1}^{n+m-1} a_k,$$

then

$$\int_0^\infty \left(\prod_{k=0}^{r} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=r+1}^{r+m} \cos(a_k x)\right) dx = \frac{\pi}{2}\prod_{k=1}^{n} a_k \text{ for } r = 1, 2, \ldots, n-1,$$

while

$$\int_0^\infty \left(\prod_{k=0}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=n+1}^{n+m} \cos(a_k x)\right) dx$$
$$= \frac{\pi}{2}\left\{\prod_{k=1}^{n} a_k - \frac{(a_1 + a_2 + \cdots + a_{n+m} - a_0)^n}{2^{n+m-1} n!}\right\}.$$

Proof. The first part follows immediately from Theorem 3, and the second part can be derived from Corollary 1 with $n+m$ in place of n by differentiating partially with respect to $a_{n+1}, a_{n+2}, \ldots, a_{n+m}$, as above. □

4. An alternative proof. The next theorem is a restatement of the last part of Theorem 3 restricted to real numbers. It appears as an example without proof in [5, p. 122] where it is ascribed to Carl Störmer [2]. Störmer's article does not contain the integral in question, but his proof for the series identity

$$\sum_{r=1}^{\infty}(-1)^{r+1}\left(\prod_{k=1}^{n} \frac{\sin(r a_k)}{r}\right)\left(\prod_{j=1}^{m} \cos(r c_j)\right) = \frac{1}{2}\prod_{k=1}^{n} a_k,$$

$$\text{provided } \sum_{k=1}^{n} |a_k| + \sum_{j=1}^{m} |c_j| < \pi,$$

is readily adapted to yield a proof of the theorem which is radically different from the proof of Theorem 3.

Theorem 4. *If* $a, a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m$, *are real numbers with* $a > 0$ *and*

$$a \geq \sum_{k=1}^{n} |a_k| + \sum_{j=1}^{m} |c_j|,$$

then

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2}\prod_{k=1}^{n} a_k. \tag{5}$$

Proof. We prove the theorem by induction. Applying as before the convention that empty sums have the value 0 and empty products have the value 1, we observe that formula (5) for the case $n = m = 0$ reduces to the standard result

$$\int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}.$$

Formula (5) also holds for the case $n = 1, m = 0$, by the case $n = 1$ of Theorem 1 (which can easily be proved directly).

Assume that the theorem holds for certain integers $n \geq 1$ and $m \geq 0$. First suppose that

$$a \geq \sum_{k=1}^{n} |a_k| + \sum_{j=1}^{m+1} |c_j|.$$

Then

$$a \geq |a_1 \pm c_{m+1}| + \sum_{k=2}^{n} |a_k| + \sum_{j=1}^{m} |c_j|,$$

and hence

$$\begin{aligned}\int_0^\infty \frac{\sin(a_1 \pm c_{m+1})}{x} \left(\prod_{k=2}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\,dx \\ = \frac{\pi}{2}(a_1 \pm c_{m+1}) \prod_{k=2}^{n} a_k.\end{aligned} \tag{6}$$

Adding the two identities in (6), we immediately obtain

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m+1} \cos(c_j x)\right) \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \tag{7}$$

Next suppose that

$$a \geq \sum_{k=1}^{n+1} |a_k| + \sum_{j=1}^{m} |c_j|,$$

and let t lie between 0 and a_{n+1}. Then, by (7), we have

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \cos(tx) \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \tag{8}$$

Now integrate (8) with respect to t from 0 to a_{n+1} to get

$$\int_0^\infty \left(\prod_{k=1}^{n+1} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2} \prod_{k=1}^{n+1} a_k. \tag{9}$$

Identities (7) and (9) show that if the theorem holds for a pair of integers n, m with $n \geq 1, m \geq 0$, then it also holds for the pairs $n, m+1$ and $n+1, m$. Since it holds for $n = 1, m = 0$, the proof is completed by induction. $\square$

Remarks 2. Parts of our previous theorems do, of course, overlap with Theorem 4, but this latter theorem does not deal with cases where the identity in (4) fails, whereas the other theorems do. Thus, for example,

$$\int_0^\infty \operatorname{sinc}(x)\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\, dx = \frac{\pi}{2},$$

$$\cdots$$

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{13}\right)\, dx = \frac{\pi}{2},$$

yet

$$\begin{aligned}
&\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right)\, dx \\
&\quad = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi,
\end{aligned} \tag{10}$$

and this fraction in (10), in accord with Corollary 1, is approximately equal to 0.499999999992646. When this fact was recently verified by a researcher using a computer algebra package, he concluded that there must be a "bug" in the software. Not so. In the above example, $\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$, but with the addition of $\frac{1}{15}$, the sum exceeds 1 and the identity no longer holds. This is a somewhat cautionary example for too enthusiastically inferring patterns from symbolic or numerical computation. $\square$

5. An infinite product of cosines. We return to the integral, which we denote by μ, in (1). Let

$$C(x) := \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right).$$

This product is absolutely convergent, since $\cos\left(\frac{x}{n}\right) = 1 - \frac{x^2}{2n^2} + O\left(\frac{1}{n^4}\right)$ as $n \to \infty$. Here and elswhere in this section we ignore the countable set of points on which individual terms of such an infinite product vanish. Recall the absolutely convergent Weierstrass products [4, p. 144]

$$\operatorname{sinc}(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right), \quad \cos(x) = \prod_{k=0}^{\infty}\left(1 - \frac{4x^2}{\pi^2(2k+1)^2}\right),$$

from which it follows that

$$C(x) = \prod_{n=1}^{\infty}\prod_{k=0}^{\infty}\left(1 - \frac{4x^2}{\pi^2 n^2 (2k+1)^2}\right) = \prod_{k=0}^{\infty}\prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{\pi^2 n^2 (2k+1)^2}\right) = \prod_{k=0}^{\infty}\operatorname{sinc}\left(\frac{2x}{2k+1}\right). \tag{11}$$

It is interesting to note that the alternative absolutely convergent product expansion of $C(x)$ afforded by (11) can also be derived from the Weierstrass expansion of $\operatorname{sinc}(x)$ together with Vieta's formula [3, p. 419] in the form

$$\operatorname{sinc}(2x) = \prod_{n=0}^{\infty} \cos\left(\frac{x}{2^n}\right),$$

since every positive integer is uniquely expressible as an odd integer times a power of 2.

Now apply Theorem 1 and (11) to obtain

$$0 < \mu = \int_0^\infty C(x)\,dx = \lim_{N\to\infty} \int_0^\infty \prod_{k=1}^{N} \operatorname{sinc}\left(\frac{2x}{2k-1}\right)\,dx < \frac{\pi}{4}.$$

These sinc integrals are essentially those of the previous Remarks. Note that all parts of Theorem 1 apply since $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} < \infty = \sum_{k=1}^{\infty} \frac{1}{2k-1}$.

We observe that Theorem 1 allows for reasonable lower bounds on μ. Indeed, as $\cos^2 x > 1 - x^2 > 0$ for $0 < x < 1$, we see — using the product form for sinc — that $C^2(x) > \operatorname{sinc}(\pi x)$ on the same range. Hence, by Theorem 1(iii),

$$\frac{\pi}{4} > \mu > \int_0^\infty C^2(x)\,dx > \frac{1}{\pi}\int_0^\pi \operatorname{sinc}(x)\,dx \approx .5894898722.$$

We could produce a better lower bound, and indeed lower bounds for our more general sinc integrals in the same way.

In fact

$$\int_0^\infty C(x)\,dx \approx 0.78538055729863287349258301146733252476 1$$

while $\frac{\pi}{4} \approx .785398$ only differs in the fifth significant place. We note that high precision numerical evaluation of these highly oscillatory integrals is by no means straightforward.

If $C(x)$ is replaced by

$$C^*(x) := \cos(2x)C(x) = \cos(2x)\prod_{n=1}^{\infty}\cos\left(\frac{x}{n}\right),$$

we similarly obtain

$$C^*(x) = \operatorname{sinc}(4x)\prod_{n=1}^{\infty}\operatorname{sinc}\left(\frac{2x}{2n+1}\right). \tag{12}$$

It now takes 55 terms before $\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} > 2$, so that the corresponding integrals drop below $\frac{\pi}{8}$. Indeed, lengthy numerical computation shows that

$$0 < \frac{\pi}{8} - \int_0^\infty C^*(x)\,dx < \frac{1}{10^{41}}.$$

We finish by recording without details that (11) allows us to obtain the Maclaurin series for $\log C(x)$. It is

$$\log C(x) = -\sum_{k=1}^{\infty} \frac{4^k - 1}{k} \frac{\zeta^2(2k)}{\pi^{2k}} x^{2k},$$

with radius of convergence $\frac{1}{2}\pi$. This in turn shows that the coefficient of x^{2n} in the Maclaurin series for $C(x)$, say c_n, is a rational multiple of π^{2k} and is explicitly given by the recursion

$$c_0 := 1,\ c_n := -\frac{1}{n} \sum_{k=1}^{n} (4^k - 1) \frac{\zeta^2(2k)}{\pi^{2k}} c_{n-k} \text{ for } n > 0.$$

Thus

$$C(x) = 1 - \frac{1}{12}\,\pi^2 x^2 + \frac{11}{4320}\,\pi^4 x^4 - \frac{233}{5443200}\,\pi^6 x^6 + \frac{1429}{3048192000}\,\pi^8 x^8 + O\left(x^9\right).$$

Incidentally, as pointed out by David Bradley, the Maclaurin series of $\log C(x)$ can be obtained without appeal to (11) via the Weierstrass product for $\cos(x)$.

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