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Chapter 0

Introduction

0.1 Exemples introductifs

Cette thèse est en premier lieu une contribution à l'automatisation du calcul infinitésimal. On vise la résolution de *systèmes asymptotiques*, c'est-à-dire de systèmes d'équations fonctionnelles, qui outre les opérations algébriques usuelles font intervenir des relations asymptotiques d'équivalence et d'inégalité.

Un exemple simple est l'équation

$$xe^x = y$$

en x, où x tend vers l'infini. Cette équation intervient par exemple en combinatoire dans l'étude asymptotique des nombres de Bell. En termes plus algébriques, elle est équivalente au système asymptotique

$$\begin{cases} xe^x = y\,; \\ 1 \prec\!\!\prec x. \end{cases}$$

Ici $\prec\!\!\prec$ désigne la relation de domination de Hardy. Les systèmes asymptotiques généralisent les systémes d'équations avec des perturbations, dont l'exemple

$$\begin{cases} t^3 + xt + y = 0\,; \\ x \prec\!\!\prec 1\,; \\ y \prec\!\!\prec 1, \end{cases}$$

intervient dans l'étude d'un point cuspidal en $x = y = 0$. Naturellement, ces exemples sont très simples et peuvent être traités à la main de façon relativement aisée. Mais la nécessité d'une démarche plus universelle devient plus apparente lorsque l'on considère un système asymptotique comme

$$\begin{cases} (f \ln f - x) + e^{-g} = 0\,; \\ (g^2 - f) - \frac{\ln x}{g} = 0\,; \\ x \succ\!\!\succ 1. \end{cases}$$

Dans cette thèse nous développons des outils suffisamment puissants pour résoudre un tel système de façon entièrement automatique. En particulier, nous pouvons affirmer que ce système n'admet qu'un nombre fini de solutions.

Plus généralement, on étudie des systèmes asymptotiques qui font intervenir la dérivation, voire même la composition fonctionnelle. Outre des équations linéaires du style

$$f''' - xe^x f' + e^{x^2} f = x^x \ (x \to \infty),$$

on envisage des équations non linéaires comme

$$W' + W^2 = e^{2x} \ (x \to \infty).$$

ou

$$f' f''' - f''^2 - e^{-e^x} f = e^{-x^2} \ (x \to \infty).$$

On peut également considérer des systèmes différentiels plus complexes avec paramètres, perturbations et/ou contraintes asymptotiques, dont voici un exemple :

$$\begin{cases} f^2 - e^{ax^2+bx} f' = g\,; \\ g^3 - ax^5 fg + b \log x f = e^x\,; \\ 1 \prec\!\!\prec x. \end{cases}$$

Pour les exemples ci-dessus, nos algorithmes permettent à nouveau de déterminer de façon entièrement automatique leurs ensembles de solutions.

Nous avouons avoir choisi des exemples un peu orientés : comme le lecteur le constatera, aucun d'entre eux ne fait intervenir des phénomènes oscillatoires. Nous dirons que nous nous sommes restreints à des systèmes asymptotiques *fortement monotones*. Bien que ceci soit un désavantage majeur de la théorie actuelle, nous avons tout de même franchi quelques premiers pas vers une théorie plus générale, qui incorpore ce genre de phénomènes. En particulier, nous verrons comment calculer les limites inférieures et supérieures à l'infini d'une fonction comme

$$\psi(x) = \frac{2 \sin x^2 - \sin(x^3/(x-1))}{3 + \sin ex^2 - \sin(ex^2 + 1)}.$$

Plus généralement, on est capable de déterminer automatiquement les comportements limites possibles d'une fonction comme

$$f(x) = \frac{e^{x^{2 \sin x} \sin x^2} - e^{x^{\sin 2x} \sin x^2}}{\Gamma(\log x \log \log x \sin 5x)},$$

pour $x \to \infty$ et sous la contrainte $\sin x \prec\!\!\prec 1$. Finalement, on sait trouver toutes les solutions de certains types d'équations différentielles linéaires comme

$$\log x e^{-x^2} f''' + (e^{2x} + e^x) f'' - 2e^x f' + (1 + x^{-\Gamma(x)}) f = \sin(\Gamma(e^{e^x})).$$

Enfin, il convient de noter que pour des raisons qui seront développées davantage plus bas, il est recommandable de passer par l'étude de systèmes asymptotiques fortement monotones, avant de s'attaquer au cas général : premièrement, le cas fortement monotone comporte déjà un grand nombre de difficultés qu'il faut résoudre de toute façon. Deuxièmement, la théorie générale s'appuiera vraisemblablement sur les méthodes développées pour le cas fortement monotone.

0.2 Historique et introduction générale

Après l'acceptation du formalisme rigoureux de l'analyse moderne, les aspects plutôt concrets du *calcul* infinitésimal n'ont pas connu un développement aussi explosif que l'analyse classique. Cependant, à l'époque de Newton, on concevait souvent le calcul infinitésimal comme un calcul plutôt concret sur des séries formelles. Aujourd'hui, ce point de vue est encore partagé par beaucoup « d'utilisateurs » de mathématiques, qui souvent ne voient pas en quoi la rigueur ε-δ leur est utile dans leurs calculs. Or, comme nous allons le montrer ci-dessous, derrière ce point de vue quelque peu naïf se cache une formidable théorie, dont certains aspects n'ont pas tellement évolué depuis l'ère de Newton. Cette théorie, initiée par Écalle, s'appelle la théorie des fonctions analysables (voir [Ec 92]). Retraçons maintenant les origines de cette théorie.

0.2.1 La théorie de resommation

La première source des fonctions analysables est la théorie de resommation, dont les débuts se situent dès l'époque d'Euler, qui étudia la série formelle

$$f(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1},$$

laquelle ne converge qu'en $x = 0$ au sens usuel. Par une intuition remarquable, il parvint à calculer des valeurs non triviales de cette série. Évidemment, ceci est absurde au sens moderne de la convergence ; mais la série f vérifie l'équation différentielle

$$x^2 f'(x) + f(x) = x, \tag{0.5}$$

et il fait sens de parler de solutions de cette équation. En privilégiant une solution particulière à cette équation, on peut donc évaluer f en des points différents de zéro.

À la fin du dix-neuvième et au début du vingtième siècle, on a beaucoup cherché à donner des sens plus précis à des séries divergentes comme f. L'introduction de la *méthode de Borel* a été particulièrement importante : elle consiste à appliquer d'abord l'opérateur de Borel formel $\sum f_n x^n \mapsto \sum f_n x^n / n!$, et puis l'opérateur de Laplace à la somme convergente du résultat. Ce procédé produit une fonction f qui vérifie l'équation différentielle (0.5). Ce procédé a la propriété importante de préserver l'équation différentielle vérifiée par f.

Après une période de silence relatif, la théorie de la resommation a recommencé à intéresser les mathématiciens vers la fin des années soixante-dix, avec l'arrivée des résultats de Ramis, Balser, Écalle, Braaksma, etc. (voir [Ram 93] et [Bal 94] pour des discussions). Un des aspects les plus importants des nouvelles méthodes de resommation et multisommation, est la stabilité de la classe des fonctions resommables par de nombreuses opérations algébriques, comme les opérations de corps, la dérivation, la composition, etc. En particulier, dans une théorie idéale on s'attend à ce que toutes les *fonctions ayant une source naturelle* (c'est à dire des solutions d'équations différentielles ou fonctionnelles, des transformées intégrales, etc.) soient resommables.

0.2.2 L'analyse asymptotique

Une deuxième source de la théorie des fonctions analysables est la théorie des développements asymptotiques. Dès son introduction par Du Bois-Raymond, Poincaré et Stieltjes, de nombreuses fonctions non développables dans l'échelle ordinaire apparurent. Pour remédier à ce défaut, Hardy étudie dans [Har 11] les *L-fonctions* (qui apparaissent par ailleurs déjá dans les travaux de Liouville (voir [Li 1837], [Li 1838])) comme étant des fonctions construites à partir de $\mathbb{R}$ et x par les opérations de corps, l'exponentielle, le logarithme et composition par des fonctions algébriques réelles. La plupart des fonctions ayant une source naturelle admettent des développements asymptotiques dans l'échelle de ces L-fonctions.

Après avoir introduit les L-fonctions, Hardy a démontré un théorème essentiel : les germes des L-fonctions à l'infini forment un corps différentiel totalement ordonné. En autres termes, l'ensemble des L-fonctions forme un corps stable par dérivation, et le signe de toute L-fonction $f(x)$ reste constant lorsque x tend vers l'infini. Bourbaki (voir [Bour 61]) a postulé cette propriété comme la base d'une nouvelle théorie : un corps de Hardy est un corps différentiel de germes de fonctions à l'infini. Cette définition donne en particulier une première formalisation au concept de comportement asymptotique fortement monotone, dont nous avons parlé dans la section précédente. Ensuite on a établi divers théorèmes de clôture pour les corps de Hardy (voir [Bour 61], [Rob 72], [Kho 83], [Ros 83a], [Ros 83b]). Par exemple, étant donnée une fonction f dans un corps de Hardy H, il existe un corps de Hardy H' qui contient H et $\int f$.

Malheureusement, la théorie classique des développements asymptotiques, même dans des échelles généralisées comme l'échelle des L-fonctions, admet deux inconvénients importants. Premièrement, cette théorie manque de propriétés de clôture sous différents types d'opérations : si f et g sont deux fonctions ayant le même développement asymptotique, la fonction $f - g$ n'admet pas nécessairement un développement asymptotique. Deuxièmement, en faisant le développement asymptotique d'une fonction, on a tendance à « perdre » de l'information : lorsque l'on développe la fonction

$$\frac{1}{1-x^{-1}} + e^{100} x^{-\log x} \approx 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$$

on a « perdu » le terme $e^{100}x^{-\log x}$. Ceci est d'autant plus nuisible, que ce terme détermine l'ordre de grandeur de la fonction tant que $x \ll e^{50}$.

La théorie des corps de Hardy admet quant à elle aussi plusieurs inconvénients. Premièrement, il peut être assez difficile de montrer qu'une fonction f donnée appartient à un corps de Hardy : pour cela, il faut vérifier que le signe de tout polynôme différentiel en f est constant à l'infini. Deuxièmement, puisque les fonctions appartenant à un corps de Hardy sont condamnées à ne jamais s'annuler au voisinage de l'infini, il est difficile de s'imaginer une théorie plus générale qui incorpore des fonctions oscillantes.

0.2.3 La genèse des transséries

Par les critiques formulées ci-dessus, on en vient à la troisième source de la théorie des fonctions analysables : le calcul sur des séries formelles. En effet, puisque les séries formelles sont des objets entièrement formels, elles ne souffrent d'aucun des inconvénients mentionnés ci-dessus. Le prix à payer est qu'elles n'ont pas toujours de signification analytique. Mais c'est là où intervient l'idée majeure de la théorie des fonctions analysables : on s'intéresse surtout aux séries formelles « ayant une source naturelle ». La théorie de la resommation permet alors de donner un sens analytique à ces objets formels. Cependant, puisque l'on veut étendre autant que possible cette classe de « fonctions ayant une source naturelle », on ne peut pas se contenter de travailler avec des séries formelles ordinaires. D'où la justification principale de l'introduction des *transséries* (voir [Dahn 84], [DG 86] et [Éc 92], la terminologie "transsérie" étant dû à Écalle), qui sont des expressions formées à l'aide de $\mathbb{R}$, x, les opérations de corps, l'exponentielle, le logarithme, et surtout un opérateur somme d'arité infinie. Les transséries généralisent en particulier les L-fonctions.

Étant donné que la puissance du calcul sur les séries formelles s'est montrée dans de nombreux domaines au fil des siècles, il est surprenant de constater qu'il a fallu attendre si longtemps l'introduction des transséries. Cependant, les transséries ont quelques précurseurs, que nous allons tenter de présenter brièvement.

Tout d'abord, pour la division et la résolution des équations algébriques, il a fallu autoriser des séries avec des puissances négatives et fractionnaires : les séries de Laurent et de Puiseux. Au début de ce siècle Hahn a considéré des exposants encore plus généraux : il montre dans [Hahn 07] qu'étant donné un groupe commutatif totalement ordonné X et un corps K, l'ensemble $K[[X]]$ des séries $f = \sum_{x \in G} f_x x$ *à support bien ordonné* est naturellement muni de la structure de corps totalement ordonné. Ce résultat a été étendu par Higman dans [Hig 52] au cas où G n'est plus que partiellement ordonné et non nécessairement commutatif, et où les séries ont des supports belordonnés. Pour l'étude de solutions formelles d'équations différentielles et à différences, on a également introduit la notion de séries formelles comportant des logarithmes (voir par exemple [LoRo 89]).

Écalle, qui utilise les fonctions analysables dans sa démonstration de la conjecture de Dulac, s'est davantage intéressé à la resommation des transséries qu'à leurs propriétés algébriques. De plus, Écalle se limite à l'étude de transséries fortement monotones en une variable, parce que la démonstration de la conjecture de Dulac repose précisément sur ce caractère fortement monotone. Dans la partie A de cette thèse, nous nous sommes alors proposé de donner une contribution à la théorie algébrique des transséries, d'étudier des transséries en plusieurs variables, ainsi que de préparer la route pour l'étude des transséries « faiblement oscillantes ».

Comme résultats principaux, il y a d'abord la mise en place d'un formalisme rigoureux pour l'étude de transséries à différents types de support. Utilisant ce formalisme nous apportons la réponse à une conjecture de Hardy (voir [Har 11]; nous notons que notre résultat a été démontré indépendamment dans [MMV *]). Ensuite, nous nous intéressons aux équations fonctionnelles, et nous proposons un algorithme théorique pour la résolution d'équations différentielles algébriques dans les transséries fortement monotones. Finalement, nous introduisons des transséries en plusieurs variables et des transséries faiblement oscillantes. L'idée majeure derrière cette introduction est que les transséries faiblement oscillantes sont en fait des transséries en plusieurs variables dans lesquelles on substitue aux variables des composants oscillants.

0.2.4 L'aspect effectif

Une dernière source majeure de la théorie des fonctions analysables est le constructivisme. En effet, les calculs sur les transséries, ainsi que les méthodes de resommation déployées pour leurs donner un sens analytique, sont toujours entièrement constructifs. Cependant, entre le constructivisme théorique et la mise en place concrète d'un logiciel de calcul formel, susceptible de faire ces calculs de façon entièrement automatique, il y a un grand écart, en particulier parce que les transséries sont des objets de nature infinie.

Les premiers algorithmes dans cette direction, qui dépassent la manipulation des « vulgaires séries de Taylor », sont dus à Shackell. Il a commencé par s'intéresser aux développements asymptotiques des fonctions exp-logs, qui sont des fonctions construites à partir de x et $\mathbb{R}$ par les opérations du corps, l'exponentielle et le logarithme. Travaillant dans le cadre des corps de Hardy, il obtient dans [Sh 90] le premier algorithme pour déterminer la limite d'une fonction exp-log à l'infini, modulo l'existence d'un oracle pour tester si une fonction exp-log est nulle au voisinage de l'infini. Pour ce faire, il utilise des *formes imbriquées* et plus tard des *développements imbriqués*, qui sont en quelque sorte des objets à mi-chemin entre des limites et des transséries complètes. Le principal problème que l'on rencontre dans ce domaine est celui de « l'annulation indéfinie » : un algorithme de développement « brutal » ne termine pas sur un exemple comme

$$\frac{1}{1-x^{-1}-e^{-x}}-\frac{1}{1-x^{-1}}.$$

Il est alors nécessaire de trouver la bonne échelle de développement $\{x^\alpha e^{\beta x} | \alpha, \beta \in \mathbb{R}\}$ et de développer d'abord par rapport à e^x, les coefficients de ce développement étant des transséries eux-mêmes, dont on garde des représentations exactes. Aussi, cet exemple illustre à nouveau l'intérêt de considérer des développements asymptotiques plus généraux que ceux qui sont employés classiquement.

Après ce premier algorithme de Shackell, il y a beaucoup d'autres résultats qui ont suivis : d'abord, Shackell met au point la technique « des ombres et des fantômes » pour donner un algorithme de développement complet (au sens des transséries) pour les fonctions exp-logs dans [Sh 96] (toujours modulo l'oracle mentionné plus haut). Des résultats allant dans ce sens ont été obtenus par Salvy dans sa thèse [Sal 91]. Il y obtient aussi des résultats concernant la clôture algébrique réelle et l'intégration. Ensuite, Shackell revient aux développements imbriqués pour lesquels il donne un procédé infini pour déterminer les types de développements imbriqués possibles des solutions à une équation différentielle algébrique donnée (voir [Sh 92]). En collaboration avec Salvy il obtient également un algorithme pour calculer des développements imbriqués d'inverses fonctionnels dans [SS 92] et des résultats en vue du traitement des fonctions implicites (voir [SS 96]). Ceci a notamment des applications dans le calcul d'intégrales (voir [Sal 91]).

Notre première contribution au calcul effectif sur des transséries est de simplifier l'algorithme de Shackell pour l'obtention du développement complet d'une fonction exp-log. Pour ceci, nous nous sommes basés sur les travaux de Gonnet et Gruntz sur les calculs de limites (voir [GoGr 92], [Gr 96]). Ensuite, nous avons raffiné cet algorithme de sorte que l'on n'a plus besoin que d'un oracle pour tester la nullité de constantes exp-logs. Par ailleurs, ce problème des constantes est très profond, et nous y reviendrons plus loin.

Nos principaux résultats concernent le développement de fonctions exp-logs avec paramètres, la résolution de systèmes asymptotiques, la résolution d'équations algébriques différentielles, et quelques premiers pas vers le traitement des transséries faiblement monotones. En simplifiant, nous démontrons dans le chapitre 11, qu'étant donné un oracle pour tester si un système d'équations et inégalités exp-logs admet une solution, il existe un algorithme pour trouver les développements asymptotiques génériques de transséries exp-logs en plusieurs variables, solutions d'un système asymptotique exp-log. Dans le chapitre 12, nous montrons comment résoudre dans le corps des transséries fortement monotones des équations différentielles algébriques dont les coefficients sont des fonctions exp-logs (par exemple). Dans le cas des équations différentielles linéaires, nos résultats généralisent des résultats plus classiques (voir [DDT 82] par exemple). Enfin, le chapitre 14 fournit un premier pas vers le traitement de transséries faiblement oscillantes.

0.2.5 Outils de calcul formel

Tous les algorithmes de développement asymptotique mentionnés plus haut présupposent la possibilité de faire des calculs exacts sur les transséries, en plus d'en extraire les coefficients. Or ceci a posé de nombreux problèmes en calcul formel : comment décider par exemple qu'une série fabriquée à partir de fonctions transcendantes usuelles est nulle ? Et même, comment décider si une constante exp-log est nulle ?

Ces deux problèmes ont fait l'objet de recherches intensives ces dernières années. Au départ, il y a les travaux [Li 1837] et [Li 1838] de Liouville sur les fonctions exp-logs. Risch est le premier à exploiter ces travaux d'un point de vue algorithmique et il démontre le théorème de structure de Risch (voir par exemple [Ris 75]). Ces travaux permettent en particulier de tester si deux fonctions exp-logs sont identiques au voisinage d'un point où elles sont toutes les deux définies (modulo un test à zéro pour les constantes exp-logs).

Dans [DL 89], Denef et Lipshitz montrent de façon plus générale le théorème suivant : soient $f_1, \cdots, f_n$ des séries formelles à coefficients dans un corps effectif, qui vérifient des équations différentielles algébriques « non singulières ». Alors il existe un algorithme pour décider si un polynôme P en $f_1, \cdots, f_n$ induit la série formelle nulle. Malheureusement, leur solution est basée sur la décomposition d'un idéal dans un anneau de polynômes en idéaux premiers, ce qui est un problème très couteux. Shackell propose d'autres approches dans [Sh 89] et [Sh 93b]. L'algorithme le plus prometteur dans ce domaine est dû à Péladan-Germa (voir [Pél 95]), et traite même de séries formelles en plusieurs variables. Dans l'annexe D, nous donnons encore d'autres approches, un des algorithmes ayant été obtenu en collaboration avec A. Péladan-Germa. On y étudie aussi les solutions de systèmes d'équations implicites.

Un problème beaucoup plus profond est de tester si des constantes exp-logs (voire des constantes transcendantes plus générales) sont zéro. Ceci n'est pas étonnant, dans la mesure que l'on ne sait même pas si des constantes « simples » comme $e + \pi$ sont transcendants. À ce titre nous mentionnons l'importante conjecture de Schanuel :

Conjecture 0.1. **(Schanuel)** *Si $\alpha_1, \cdots, \alpha_n$ sont des nombres complexes $\mathbb{Q}$-linearement indépendants, alors de degré de transcendance*

$$trdeg_{\mathbb{Q}}\left(\mathbb{Q}[\alpha_1, \cdots, \alpha_n, e^{\alpha_1}, \cdots, e^{\alpha_n}]\right)$$

est au moins égal à n.

Cette conjecture exprime entre autres l'idée (voir [CP 78], [Mac 91]) qu'il n'existe pas de relations « non triviales » entre les constantes exp-logs (c'est à dire, des relations qui ne se déduisent pas des lois habituelles sur l'exponentielle et le logarithme). Plus récemment, Richardson a donné un test à zéro pour des « constantes exp-logs implicites » modulo la conjecture de Schanuel (voir [Rich 95]) : soient $c_1, \cdots, c_n$

des constantes, qui sont les uniques solutions locales « non singulières » d'un système d'équations exp-logs, alors il existe un algorithme pour tester si la valeur d'un polynôme P à coefficients dans $\mathbb{Q}$ en $(c_1, \cdots, c_n)$ est nulle, modulo la conjecture de Schanuel. De plus, l'algorithme, s'il termine, rend toujours le résultat correct. En outre, en cas de non terminaison, les entrées fournissent « en quelque sorte » un contre-exemple concret à la conjecture de Schanuel.

Bien que les résultats précédents nous permettent en théorie de résoudre des problèmes très généraux, dans la pratique il est souvent nécessaire de recourir à des méthodes plus heuristiques pour des raisons d'efficacité (voir aussi la discussion dans la section D.4.4). Pour ces méthodes heuristiques, il est particulièrement important d'avoir des méthodes rapides d'évaluation de fonctions spéciales à de grandes précisions. À ce titre, nous avons inclus l'annexe C. Nous remarquons aussi, qu'au moins dans tous les exemples qui nous sont connus, les algorithmes heuristiques ne cessent de fonctionner que lorsque l'on recontre des phénomènes importantes d'annulations numériques. Un exemple notable est l'évaluation de

$$(e^{10^{-1000}} - 1)10^{1000}.$$

Il est décevant de constater que la plupart des systèmes de calcul formel renvoient la valeur zéro à cette évaluation. Une approche pour la résolution de ce problème a été proposée dans [VdH 95a]; voir aussi la conclusion de cette thèse.

0.3 Algèbre asymptotique

Dans la partie A de cette thèse nous étudions d'un point de vue théorique les aspects purement formels du calcul infinitésimal, et du calcul asymptotique. Ce sujet, que nous avons qualifié d'*algèbre asymptotique*, se fonde sur les transséries, que nous définissons d'une façon aussi générale que possible, et dont nous étudions les propriétés.

0.3.1 Les différentes transséries

Les transséries forment une généralisation des séries formelles, en autorisant l'intrusion récursive d'exponentielles et de logarithmes. En voici un exemple simple :

$$1!e^{e^x+e^x/x+e^x/x^2+\cdots} + 2!\log^{-1} x e^{e^x+e^x/x+e^x/x^2+\cdots} + 3!\log^{-2} x e^{e^x+e^x/x+e^x/x^2+\cdots} + \cdots,$$

où x tend vers l'infini. Les transséries se rencontrent naturellement lorsque l'on cherche des solutions formelles à des équations différentielles non linéaires dans le voisinage d'une singularité « explosive ». Un exemple simple est l'équation

$$ff'' = f'^2 + ff' \tag{0.6}$$

qui admet comme solutions toutes les *transséries finies* de la forme

$$f = ae^{be^x}.$$

En modifiant légèrement (0.6) :

$$ff'' = f'^2 + ff' + 1,$$

les solutions deviennent vite plus complexes ; en voici une :

$$f = e^{e^x} + \tfrac{1}{4}e^{-2x}e^{-e^x} + \cdots.$$

On peut également envisager des équations fonctionelles plus générales, comme

$$f(x) = \frac{1}{x} + f(x^2) + f(e^{\log^2 x}), \tag{0.7}$$

qui admet

$$\begin{aligned} f \quad = \quad & \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^4} + \cdots + \\ & \frac{1}{e^{\log^2 x}} + \frac{1}{e^{2\log^2 x}} + \frac{2}{e^{4\log^2 x}} + \frac{2}{e^{8\log^2 x}} + \cdots + \\ & \frac{1}{e^{\log^4 x}} + \frac{1}{e^{2\log^4 x}} + \cdots + \\ & \vdots \end{aligned}$$

comme solution particulière.

Comme on l'expliquera avec plus de détails ci-dessous, un problème majeur que l'on rencontre lors de la définition des transséries est que leurs « différentes origines naturelles » (équations différentielles, équations aux différences, transformations intégrales, etc.) induisent des types différents de transséries. Dans les chapitres 1 et 2 nous présentons une étude détaillé de ces différents types de transséries. Nos conclusions principales sont les suivantes :

- Il existe trois types principaux de transséries : les transséries réticulées, les transséries bien ordonnées, et les transséries imbriquées de force supérieure. Chaque type est caractérisé par des propriétés de clôture quant à la résolution d'équations fonctionnelles.

- Il n'existe pas de corps de transséries, qui soit à la fois stable par sommation infini et par exponentiation. En revanche, il *existe* un tel corps, si l'on se restreint a des sommations dénombrables.

- Les corps des transséries réticulées et bien ordonnées (de profondeur logarithmique fini) sont stables pour la « résolution fortement monotone » d'équations différentielles algébriques. Ce résultat semble se généraliser au cas des équations fonctionnelles générales dans le cadre des transséries imbriquées de force supérieure.

Séries formelles. Commençons par l'étude des généralisations classiques des séries formelles. Par analogie avec la définition des algèbres de polynômes sur un groupe quelconque, on souhaiterait définir des séries généralisées en prenant les monômes dans un groupe X quelconque. Le problème majeur ici est la définition du produit de deux séries. Pour que cela devienne possible, il faut supposer l'existence d'un (quasi-)ordre partiel sur X, et imposer des conditions sur les supports des séries. Higman a démontré dans [Hig 52] qu'il suffit de demander aux séries d'avoir des supports *belordonnés*, pour que l'on puisse définir les opérations habituelles sur les séries. Dans le cas où X est totalement ordonné, ceci revient à exiger des supports bien ordonnés ; par ailleurs, ce cas avait déjà été considéré par Hahn dans [Hahn 07].

Disposant des résultats généraux de Hahn et de Higman, on peut se demander si ce genre de supports se rencontre dans des problèmes concrets. Considérons pour cela les séries suivantes :

$$\begin{aligned} f &= 1 + x^{-1} + x^{-2} + x^{-e} + x^{-3} + x^{-e-1} + x^{-4} + x^{-e-2} + x^{-5} + x^{-2e} + \cdots ; \\ g &= 1 + x^{-\log 2} + x^{-\log 3} + x^{-\log 4} + x^{-\log 5} + \cdots ; \\ h &= 1 + x^{-1/2} + x^{-3/4} + x^{-7/8} + \cdots + x^{-1} + x^{-3/2} + x^{-7/8} + \cdots + x^{-2} + \cdots\cdots ; \end{aligned}$$

le groupe de monômes étant $X = x^{\mathbb{R}}$, c'est-à-dire le groupe formel de puissances réelles de x. Nous avons

$$f = \frac{1}{1 - x^{-1} - x^{-e}},$$

donc f est en particulier la solution d'une équation différentielle algébrique à coefficients dans $\mathbb{R}$. Nous avons

$$g(e^x) = \zeta(x),$$

où ζ est la fonction ζ de Riemann. Finalement $\psi = h(x-1)$ satisfait l'équation aux différences

$$\psi(x) = x + \psi(\sqrt{x}).$$

Bien que f ne soit pas une série de Puiseux, nous observons que le support de f est contenu dans le sous-groupe $\mathbb{Z} + e\mathbb{Z}$ finiment engendré de $\mathbb{R}$. Or Grigoriev et Singer (voir [GS 91]) ont démontré qu'il s'agit ici d'une propriété générale, résultant du fait que f vérifie une équation différentielle algébrique sur $\mathbb{R}$. Dans cette thèse, on montrera même un peu plus : le support d'une solution φ à une telle équation est toujours inclu dans un ensemble de la forme $a_1\mathbb{N} + \cdots + a_n\mathbb{N} + b$ $(a_1, \cdots, a_n > 0)$; nous disons que $\operatorname{supp} \varphi$ est *réticulé*. Nous notons qu'un sous-ensemble bien ordonné de $\mathbb{R}$, qui est inclus dans un sous-groupe finiment engendré de $\mathbb{R}$ n'est pas nécessairement réticulé. L'intérêt des séries réticulées est qu'elles se représentent comme des séries de Laurent en plusieurs variables, dans lesquelles on substitue des monômes dans X aux variables.

Les exposants en x^{-1} des éléments successifs dans le support de g tendent vers $+\infty$. En particulier le type d'ordre de $\operatorname{supp} g$ est ω, le plus petit ordinal dénombrable.

Plus généralement, considérons l'ensemble de séries dont le support n'admet pas de borne supérieure, à moins d'être fini (de telles séries apparurent pour la première fois dans [LC 1893] sous une forme légèrement différente). Comme l'ensemble des séries réticulées, cet ensemble forme également un corps. Les développements asymptotiques de certaines fonctions spéciales nécessitent ce genre de supports. De plus, ces séries peuvent encore être implantées de façon relativement aisée à l'aide de l'évaluation paresseuse (voir [Gr 96]), bien qu'une approche naïve puisse entraîner des augmentations dramatiques en complexité (voir la section 7.3).

Le dernier exemple montre que des supports bien plus généraux peuvent apparaître, dès que l'on considère des équations aux différences. En particulier, nous observons que le type d'ordre de $\operatorname{supp} h$ est ω^2. La théorie effective des séries avec des supports bien ordonnés généraux est bien plus complexe que celles des exemples précédents. Pour une approche, nous renvoyons vers [VdH 94a].

Venons en maintenant à l'introduction des transséries. Les transséries sont définies récursivement comme des séries généralisées, où le groupe totalement ordonné X des *transsmonômes* est un groupe d'exponentielles formelles de transséries « plus simples », les transmonômes « les plus simples » étant des logarithmes itérés. Comme on l'a montré plus haut, on peut envisager plusieurs types de supports pour ces séries généralisées, les choix les plus restrictifs ayant des avantages calculatoires, et les choix les moins restrictifs ayant des avantages au niveau des propriétés de clôture.

Transséries réticulées. Dans le chapitre 1, nous considérons des transséries dont les supports sont réticulés, c'est-à-dire inclus dans un ensemble de la forme $\text{ц}_1^{\mathbb{N}} \cdots \text{ц}_n^{\mathbb{N}} \text{щ}$, où $\text{ц}_1, \cdots, \text{ц}_n$ sont des transmonômes infinitésimaux, et щ un transmonôme arbitraire. Par exemple, tout germe à l'infini d'une fonction exp-log détermine une transsérie réticulée. En voici une

$$\frac{e^{x^2+e^{-x}}}{1-\log^{-1} x} = e^{x^2} + \frac{e^{x^2}}{\log x} + \frac{e^{x^2}}{\log^2 x} + \cdots + \frac{e^{x^2}}{e^x} + \frac{e^{x^2}}{\log x e^x} + \cdots + \cdots,$$

dont le support est contenu dans $\log^{-\mathbb{N}} x e^{-\mathbb{N}x} e^{x^2}$.

Une propriété importante de la définition des supports réticulés est la finitude de n. D'un point de vue asympotique elle affirme qu'il existe une échelle qui est générée par un nombre fini d'éléments dans laquelle on peut exprimer la transsérie. En fait, on peut même exiger de plus que les logarithmes de ces générateurs (sauf un) sont de nouveau exprimables par rapport à la même échelle : c'est *grosso modo* l'énoncé du théorème de structure démontré dans le chapitre 1. Dans la partie effective de cette thèse ce genre d'échelles joue un rôle très important.

Une autre conséquence importante des transséries réticulées est qu'elles se manipulent bien d'un point de vue effectif, à nouveau grâce aux conditions de finitude.

Pour cette raison, nous ne considérons que ce type de transséries dans la partie B de cette thèse.

Finalement, le corps des transséries réticulées admet d'excellentes propriétés de clôture. Notamment, il est stable par dérivation, intégration, composition et inversion fonctionnelle. En outre, il est stable pour la résolution d'équations différentielles algébriques, tant que l'on n'introduit pas des phénomènes oscillants. En conclusion, les transséries réticulées sont suffisamment générales pour couvrir la majorité des applications de l'algèbre asymptotique et, en plus, elles présentent des avantages calculatoires.

Transséries bien ordonnées. Cependant, les transséries réticulées ne suffisent plus, dès que l'on considère des équations fonctionnelles comportant la composition, ou des transformations intégrales complexes. Pour cette raison, nous introduisons dans le chapitre 2 des transséries aux supports bien ordonnés. Ceci ne se fait qu'au prix d'une technicité accrue : nous avons besoin de méthodes sophistiquées de la théorie du belordre (voir l'annexe A), rien que pour définir la dérivation et la composition fonctionnelle pour de telles transséries.

De surcroît, cette théorie plus générale introduit de nombreuses pathologies. Premièrement, il n'existe pas de corps de transséries qui est à la fois stable par l'exponentielle, le logarithme et la sommation infinie. Deuxièmement, des transséries convergentes comme

$$f = x^{-1} + e^{-x} + e^{-e^x} + e^{-e^{e^x}} + \cdots \tag{0.8}$$

ne sont pas analytiques, mais seulement quasi-analytiques (voir [Éc 92] pour plus de détails). Troisièmement, on n'a pas toujours la stabilité par l'intégration. Par exemple, l'intégration de

$$g = \frac{1}{x \log x \log \log x \cdots} = e^{-\log x - \log\log x - \log\log\log x - \cdots} \tag{0.9}$$

nécessite l'introduction d'un « itérateur » du logarithme (voir plus bas et [Éc 92]).

Néanmoins nous isolons dans le chapitre 2 le corps des transséries bien ordonnées de profondeurs exponentielle et logarithmique finies. Cette classe ne présente aucune des pathologies mentionnées ci-dessus, et on exclut en particulier des transséries comme dans (0.8) et (0.9). Nous montrons dans les chapitres 4 et 5 que cette classe est stable pour la résolution des équations différentielles algébriques.

Plus généralement, mais nous le démontrons pas dans cette thèse (voir aussi la conclusion), cette classe est stable pour la résolution d'équations différentielles aux différences algébriques, où les opérateurs aux différences sont des compositions à droite par des transséries *d'exponentialité zéro* : ce sont des transséries g telles qu'une contraction suffisamment itérée $\Delta^{-n}g$ de g est asymptotique à x. La contraction Δ^{-1} est définie par $f \mapsto \log \circ f \circ \exp$. En particulier, on peut prendre $g = x + 1$, $g = qx$, ou $g = x^M$, et traiter les équations aux différences les plus fréquentes.

Transséries imbriquées et de force supérieure. Comme nous l'avons souligné plus haut, l'intégration d'une transsérie g comme dans (0.9) nécessite l'introduction d'un itérateur du logarithme : un itérateur d'une transsérie $f \prec\!\!\prec x$ est une transséries f^* vérifiant

$$f^*(x) = f^*(f(x)) + 1.$$

Les itérateurs successifs $\log^*, \log^{**}, \cdots$ du logarithme ne sont pas des transséries au sens habituel. Néanmoins, nous discutons dans la section 2.7 la construction de corps de transséries de *force* supérieure, qui comprennent ce genre de transséries.

Une autre source d'instabilité des corps de transséries considérés plus haut provient de la résolution d'équations comme

$$f(x) = e^{x^2 + f(\log_2 x)},$$

où $\log_2$ désigne le logarithme itéré deux fois. Pour résoudre cette équation, il faut recourir à des transséries *imbriquées* du style

$$f = e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + \cdot^{\cdot^{\cdot}}}}}.$$

De telles transséries seront étudiées davantage dans la section 2.7.

Transséries en plusieurs variables et transséries faiblement oscillantes. Au delà des transséries fortement monones en une seule variable, on peut considérer des transséries en plusieurs variables, et les transséries « faiblement oscillantes ». L'idée « force » derrière l'introduction des transséries faiblement oscillantes est que ce sont des transséries fortement monotones en plusieurs variables, dans lesquelles on substitue des composantes oscillantes aux variables. De ce point de vue, on comprend pourquoi l'étude approfondie des transséries fortement monotones est essentielle pour aller plus loin. Ceci explique aussi la restriction principale de cette thèse, qui peut paraître arbitraire au premier abord.

Dans le chapitre 6 nous étudions les transséries en plusieurs variables et faiblement oscillantes d'un point de vue théorique. Nous donnons différentes voies de généralisations pour les différents types de transséries. Plutôt que d'établir une théorie profonde, nous étudions les premières conséquences des différents choix possibles. En particulier, on verra que les transséries Noethériennes en plusieurs variables (qui généralisent les transséries bien ordonnées) doivent être définies avec beaucoup de soin, si l'on veut que les dérivées partielles existent sur tout ouvert d'un espace affine. Nous montrons aussi qu'il y a un certain écart entre l'algébrique et l'analytique dans la définition des transséries faiblement oscillantes et complexes. Nous montrons finalement que les transséries réticulées et fortement monotones gardent toujours un sens dans un petit voisinage de l'axe réel à l'infini (dans le plan complexe des transséries).

Dans le chapitre 11, nous établissons une théorie effective pour les transséries réticulées en plusieurs variables, que nous discutons de façon plus détaillée plus bas. Ce chapitre peut être lu indépendamment du chapitre 11, qui n'a pas d'applications dans cette thèse.

0.3.2 La résolution d'équations fonctionnelles

Dans les chapitres 3, 4 et 5, nous considérons la résolution d'équations algébriques, différentielles linéaires et différentielles algébriques dans les transséries. Nous donnons des algorithmes théoriques (et non des procédés infinis) pour déterminer toutes leurs solutions (et non seulement les débuts potentiels de solutions).

Nous employons systématiquement deux techniques classiques : la méthode des polygones de Newton et la linéarisation. Dans le cas des équations différentielles algébriques, la deuxième méthode consiste à se ramener soit à des équations « quasi-linéaires », soit à des équations de Riccati d'ordre inférieur. À titre historique, nous remarquons que les polygones de Newton furent inventé par Newton dans [New 1671], mais il ne les utilisa pas pour donner toutes les solutions à une équation donnée. Pour cela, il a fallu attendre Puiseux (voir [Pui 1850]). Dans le cas des séries formelles, les polygones de Newton furent utilisés pour la première fois pour résoudre des équations différentielles dans [BB 1856] et [Fi 1889]. Plus récemment, ils sont réapparus dans [GS 91] et [Cano 93].

Nous traitons les deux techniques de façon détaillée, car leur application à la théorie présente est nouvelle, et comporte un certain nombre de subtilités qui ne sont pas toujours mentionnées dans des traités classiques :

L'approche de Smith et améliorations. D'abord, nous notons que la méthode classique des polygones de Newton est un processus *a priori* infini. Ceci présente des problèmes calculatoires lorsque l'on considère des équations du genre

$$f^2 - \frac{2}{1-x^{-1}} f + \frac{1}{(1-x^{-1})^2} = -e^{-x}.$$

En effet, en appliquant brutalement la méthode classique, on trouve successivement les termes $1, x^{-1}, x^{-2}, \cdots$ de f, sans s'aperçevoir que cette équation n'admet pas de solution dans les transséries !

Une approche pour ce problème fut donné par Smith (voir [Sm 1875]), et consiste à trouver d'abord les solutions de l'équation dérivée

$$2f - \frac{2}{1-x^{-1}} = 0.$$

Maintenant, au lieu de substituer $1+\tilde{f}$ à f, puis $1+x^{-1}+\tilde{\tilde{f}}$ à f, etc. dans la méthode classique de Newton, on substitue directement $\varphi+\tilde{f}$ à f, où $\varphi=(1-x^{-1})^{-1}$ est une solution de l'équation dérivée. Nous pensons que cette approche est aussi importante que la méthode de Newton elle-même et nous l'utilisons systématiquement pour garantir la terminaison de nos algorithmes. Dans le chapitre 3, nous l'améliorons même, en l'incluant en quelque sorte directement dans la méthode des polygones de Newton elle-même. Ceci entraîne d'importantes simplifications techniques pour les généralisations de la méthode de Newton considérées dans cette thèse.

Les supports des solutions. La méthode de Smith n'est pas seulement importante à titre calculatoire : elle intervient aussi pour démontrer des propriétés sur les supports des solutions. En effet, lorsque les coefficients d'une équation différentielle algébrique sont réticulés, bien ordonnés ou de profondeurs logarithmiques finis, il n'est pas évident *a priori* qu'il en est de même pour ses solutions. La méthode de Smith avec nos améliorations peut alors être utilisée pour réduire en un nombre fini d'étapes la résolution d'équations différentielles algébriques asymptotiques arbitraires à la résolution d'équations dites quasi-linéaires.

Un exemple d'une équation quasi-linéaire est

$$f = e^{-x} + e^{-x^2} f f' + f^5 \ (f \prec\!\!\prec 1).$$

L'idée est que les termes non linéaires sont asymptotiquement négligeables, lorsque l'on prend e^{-x} comme terme dominant pour f. Ensuite, la quasi-linéarité est préservée lorsque l'on retranche ce terme dominant de f et on résoud la nouvelle équation. En itérant, on obtient alors une solution de l'équation (comme on montrera dans le chapitre 5), qui est dite *distinguée*[1]. Bien que le développement de cette solution distinguée ne puisse être calculé que par un procédé infini en général, nous avons déjà atteint deux objectifs. Premièrement, on dispose d'une description précise d'une solution (au moins du point de vue théorique). Deuxièmement, en vue de ce qui précède, il suffit de démontrer que le support de cette solution possède les propriétés voulues, pour pouvoir conclure qu'il en est de même pour toute solution de l'équation différentielle algébrique originale.

La difficulté principale est donc de démontrer l'existence des solutions distinguées et que leurs supports vérifient les propriétés souhaitées. Dans le cas des équations différentielles algébriques, on tombe alors sur un problème difficile, qui est d'exclure la possibilité de l'intrusion de logarithmes itérés arbitrairement souvent. Pour cette raison, nous avons introduit un nouvel invariant discret, que nous appelons la *régularité de Newton*. Cet invariant est borné par l'ordre de l'équation, et augmente chaque fois qu'un nouveau logarithme apparaît lors du calcul des termes successifs de la solution distinguée.

Les polygones de Newton généralisés. Pour traiter des équations différentielles algébriques, il faut évidemment adapter la méthode classique des polygones de Newton. Une approches pour cela avait déjà été proposé dans [GrSi 91] lors

[1] Nous notons que de telles solutions était déjà calculées de façon systématique par Newton (voir [New 1671]).

de cas plus restreints. L'idée dans tous les cas est de décomposer le polynôme différentiel en composantes homogènes et de considérer les valuations de ces composantes. Malheureusement, ceci ne donne pas directement un polygone de Newton avec des propriétés adéquates dans notre cas. En effet, déjà dans le cas très simple de l'intégration :

$$f' - e^{e^x} = 0,$$

e^{e^x} n'est pas le premier terme d'une solution de l'équation. Par ailleurs, il existe des équations différentielles comme

$$f f'' - f'^2 = 0,$$

dont les monômes dominants potentiels de solutions ne correspondent pas à des pentes du polygone de Newton, mais seulement à des droites « admissibles » qui les coupent en des points extrémaux.

Pour cette raison, nous introduisons un nouvel artifice : la conjugaison multiplicative. Plus précisement, au lieu de définir un véritable polygone de Newton, nous considérons l'effet de la substitution $f = \text{ц}\tilde{f}$ dans l'équation différentielle algébrique $P(f) = 0$, où ц désigne un transmonôme. En fonction de l'équation $P(\text{ц}\tilde{f})$ ainsi obtenue, nous donnons alors de façon indirecte un critère, pour dire quand ц correspond à une pente du polygone de Newton ou à une droite admissible qui le coupe dans un point extrémal. Ces pentes sont toujours en nombre fini ; les droites admissibles correspondent à des (presques) solutions des équations de Riccati associées aux composantes homogènes de P.

En conclusion, il existe donc toujours quelque chose comme un polygone de Newton, mais on ne peut pas « le tracer » directement en regardant les monômes dominants des coefficients de P.

0.4 Asymptotique automatique

0.4.1 Les fonctions exp-logs

Le problème le plus naturel d'asymptotique automatique, qui dépasse substantiellement le cadre des « vulgaires développements en série de Taylor », est celui du développement des fonctions exp-logs. La résolution de ce problème est à la fois nécessaire et une source de nouvelles techniques pour traiter des cas plus complexes.

Le premier algorithme. Le premier algorithme pour manipuler automatiquement les (germes de) fonctions exp-logs à l'infini fut donné par Shackell dans [Sh 90], et est basé sur l'utilisation de « formes imbriquées ». Un exemple d'une forme imbriquée est donné par

$$f = \exp\exp(-\log x \exp(\log\log\log^3(3 + o(1)))).$$

Bien que ce premier algorithme de Shackell permette de contourner le problème de l'annulation indéfinie (voir la section 0.2.4), il se limite (grosso modo) au calcul de limites et est peu efficace dans la pratique (voir [Gr 96]).

Les développements complets. Dans [Sh 91], Shackell donne une deuxième approche, qui ne présente pas ces désavantages, et qui conduit à des développements complets au sens des transséries. Dans le cadre du calcul des limites, une variante de cette approche fut retrouvée indépendemment par Gonnet et Gruntz dans [GoGr 92], et implantée par Gruntz dans Maple V.3 (voir [Gr 96]). Nous avons adapté l'algorithme de Gonnet et Gruntz dans [VdH 94b] pour obtenir à nouveau des développements complets. Bien que plus ou moins équivalent à l'algorithme donné par Shackell dans [Sh 91], nous y introduisons la notion importante de « base normale » (voir aussi la section 7.2.2), qui joue un rôle clé dans la suite de la partie B de cette thèse.

Grosso modo, les bases normales sont des ensemble finis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ qui génèrent des échelles asymptotiques $S_B = \{\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} | \alpha_1, \cdots, \alpha_n \in \mathbb{R}\}$, satisfaisant à quelques conditions supplémentaires. Ces conditions facilitent les manipulations effectives de fonctions exp-logs développables dans cette échelle. De surcroît, les bases normales se laissent construire efficacement de façon dynamique (une idée qui remonte à [Sal 91]). Par conséquent, on obtient rapidement une échelle naturelle dans laquelle une fonction exp-log peut se développer à l'infini. La présentation de l'algorithme [VdH 94b] fut encore améliorée dans [RSSV 96], pour donner un algorithme réunissant simplicité, efficacité et la possibilité d'obtenir des développements complets.

Représentations cartésiennes. Cependant, deux problèmes importants ne furent pas traités dans [RSSV 96] : comment réduire le problème du développement des fonctions exp-logs au problème des constantes (voir aussi la section 0.2.5), et comment séparer des ordres de grandeurs très différents dans une même échelle. Le second problème est illustré par l'exemple suivant :

$$f = \frac{1}{1 - x^{-1} - x^{-N}} - \frac{1}{1 - x^{-1}}, \tag{0.10}$$

où N est très grand (disons $N = 10^{100}$). Bien que cette fonction se développe à l'infini par rapport à l'échelle $x^{-\mathbb{R}}$, l'algorithme dans [RSSV 96] prend un temps $O(N)$ pour calculer le premier terme de f.

Dans [VdH 96a] (voir aussi le chapitre 7), nous avons résolu les deux problèmes en introduisant un autre concept clé de la partie B de cette thèse: les *représentations cartésiennes*. Dans l'exemple précédant, ceci revient à représenter x^{-1} et x^{-N} par des variables formelles z_1 et z_2, et à développer f à la fois par rapport à z_1 et z_2. Voici une liste des avantages des représentations cartésiennes :

- Leur utilisation évite des annulations massives dans la même échelle.

- Elles permettent de réduire des questions sur des séries réticulées générales à des questions sur des séries formelles. En particulier des questions de tests à zéro.

- Elles ont des avantages calculatoires : puisqu'elles sont des séries de Laurent en plusieurs variables, les algorithmes efficaces de calcul avec des séries formelles s'appliquent (multiplication par dichotomie, par FFT, les algorithmes de Brent et Kung, etc.)

Fonctions exp-logs dépendants de paramètres. Le premier avantage des représentations cartésiennes mentionné dessus est d'autant plus important, quand on autorise les fonctions exp-logs à dépendre de paramètres. En effet, en considérant N comme un paramètre, l'utilisation des représentations cartésiennes garantit alors la terminaison du calcul du premier terme de f dans l'exemple (0.10), contrairement à l'algorithme présenté dans [RSSV 96].

Plus généralement, le développement de fonctions exp-logs dépendant de paramètres peut nécessiter la séparation de plusieurs cas : soit $f = e^{e^{\lambda x}}$ pour $x \to \infty$. Alors f se développe comme $f = e^{e^{\lambda x}}, f = e$ ou bien $f = 1 + e^{\lambda x} + \frac{1}{2} e^{2\lambda x} + \cdots$ suivant que $\lambda > 0, \lambda = 0$ ou $\lambda < 0$. Dans la section 8.3 (voir aussi [VdH 96a]), nous donnons un algorithme, pour développer des fonctions exp-logs f dépendant de paramètres, qui sépare ce genre de cas de façon automatique. Cet algorithme présuppose l'existence d'un oracle pour tester la consistence de systèmes d'équations exp-logs sur les constantes.

Néanmoins, il est important de noter qu'une partie importante de ce résultat subsiste sans cet oracle : l'algorithme donne alors une liste, toujours finie, de cas, et le développement de f pour chaque cas. Ces cas sont déterminés par des contraintes exp-logs sur les paramètres, et deux-à-deux exclusif. L'oracle est seulement utilisé pour déterminer quels cas peuvent réellement apparaître. Nous précisons aussi, que l'on n'a pas besoin d'un test à zéro pour les constantes exp-logs non plus.

Les algorithmes ci-dessus reposent sur une technique importante, et utilisée fréquemment dans cette thèse, de *séparation automatique des cas*. En termes d'informatique théorique ceci correspond à du non-détermisme ou de la programmation logique. Cette technique apparut pour la première fois en calcul formel dans [DDD 85], et est aussi connue sous le nom d'*évaluation dynamique*.

0.4.2 Systèmes de séries formelles en plusieurs variables

Dans le chapitre 11 nous montrons comment résoudre des systèmes de transséries en plusieurs variables. À cause de la complexité de notre méthode, nous considérons d'abord dans le chapitre 10 le cas plus simple de systèmes de séries formelles en plusieurs variables. Les techniques développées dans ce chapitre nous reserviront pour le traitement du cas général.

Comme toujours, l'ingrédient principal de notre algorithme de résolution est une généralisation de la méthode de Newton, avec les compléments discutés dans la section 0.3.2. Cependant, pour pouvoir appliquer cette méthode, il a fallu résoudre de nombreux problèmes techniques ; décrivons les brièvement :

La méthode de Newton lexicographique. Dans la méthode classique de Newton, nous considérons des séries formelles $f(x_1, x_2)$ en deux variables, que nous décomposons

$$f = \sum_{\alpha \in \mathbb{N}} f_\alpha(x_2) x_1^\alpha.$$

Chaque $f_\alpha(x_2)$ admet une valuation en x_2, et ces valuations déterminent le polygone de Newton. Dans le cas d'une série $f(x_1, \cdots, x_p)$ en plusieurs variables, nous pouvons toujours décomposer

$$f = \sum_{\alpha \in \mathbb{N}} f_\alpha(x_2, \cdots, x_p) x_1^\alpha,$$

mais en général, les $f_\alpha(x_2, \cdots, x_p)$ ne sont pas réguliers, donc elles n'admettent pas de valuations. Ici, une série est dite *régulière* si elle admet un terme dominant unique.

Modulo des « raffinements » (voir la section 0.3.2), il est néanmoins possible de distinguer un nombre toujours fini de « régions » sur lesquelles les $f_\alpha(x_2, \cdots, x_p)$ sont régulières. Par exemple la série $x_2^2 - x_3^3$ n'est pas régulière en général, mais elle l'est, si $x_2^2 \prec\!\!\prec x_3^3$ (c.à.d. que x_2^2 est négligeable devant x_3^3), ou $x_3^3 \prec\!\!\prec x_2^2$. En général il est nécessaire pour cela de résoudre de façon approchée les équations $f_\alpha(x_2, \cdots, x_p) = 0$. En effet, dans l'exemple de dessus, il faut entre autres traiter le cas ou x_2^2 est très voisin de x_3^3. Dans ce cas, il faut faire le changement de variables $x_2 = x_3^{3/2}(1 + x_2')$ avec $x_2' \prec\!\!\prec 1$ pour rendre $x_2^2 - x_3^3$ régulière.

Ainsi, on voit que pour résoudre l'équation $f(x_1, \cdots, x_p) = 0$ en x_1, il faut non seulements appliquer la méthode de Newton à f en x_1, mais aussi de façon lexicographique à ces coefficients itérés en $x_2, \cdots, x_{p-1}$. Il se trouve qu'un traitement uniforme pour cela est possible, ce qui est l'objet de la section 10.4.

Les systèmes de contraintes asymptotiques. On vient de voir qu'en général, les coordonnés $x_1, \cdots, x_p$ de la série f vérifient des contraintes comme $x_2^2 \prec\!\!\prec x_3^3$. Nous les appelons des contraintes asymptotiques et ce concept rentre naturellement dans le cadre des systèmes asymptotiques, dont on a parlé dans la section 0.1. Dans le chapitre 10, les contraintes asympotiques considérées sont toujours de la forme

$$\begin{cases} x_1^{\alpha_1} \cdots x_p^{\alpha_p} \prec\!\!\prec 1\,; \\ x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1, \end{cases} \tag{0.11}$$

et on parle de contraintes *expo-linéaires*. En fait, il s'agit de contraintes linéaires, mais écrites sous forme multiplicative. En particulier la théorie de la programmation linéaire permet de tester si un tel système de contraintes est consistant ou contradictoire.

Représentations cartésiennes. Car les séries qui interviennent lors de la résolution d'un systèmes de séries formelles en plusieurs variables ne sont pas nécessairement des séries formelles, nous avons besoin de les représenter par des représentations cartésiennes. Une série comme $(1 - x_1/x_2)^{-1}$, où $x_1 \lll x_2$ peut par exemple être représentée par $(1 - z)^{-1}$, où $z = x_1/x_2 \in S_{\{x_1,x_2\}}$.

Or pour appliquer la « méthode de Newton lexicographique », il faut pouvoir développer une série en fonction des vraies coordonnées $x_1, \cdots, x_p$ et non seulement en les variables $z_1, \cdots, z_k$ de sa représentation cartésienne. Nous n'avons pas d'algorithme pour faire cela en général, mais dans la section 9.4 nous introduisons les *pseudo-coefficients*, que nous pouvons calculer, et qui sont suffisamment proches des véritables coefficients pour que la méthode de Newton lexicographique s'applique encore.

Communautés locales effectives. Pour garantir que toutes les représentations cartésiennes qui interviennent dans les calculs intermédiaires puissent se faire de façon effective, il faut formuler les propriétés effectives de clôture que la classe des séries (en fait les représentations cartésiennes) considérées doit posséder. Pour pouvoir appliquer la méthode Smith, il est en particulier nécessaire que cette classe soit stable pour la résolution d'équations implicites.

Dans la section 9.3, nous introduisons les *communautés locales effectives* qui sont précisément les classes de séries qui possèdent les propriétés de clôture appropriées. Dans la section D.5.3, nous montrons que la classe des *séries D-algébriques* est une communauté locale effective (les séries D-algébriques étant *grosso modo* des solutions d'équations différentielles algébriques dans les séries). En particulier, cette classe contient les séries fabriquées à l'aide de la plupart des fonctions spéciales, et en particulier les séries exp-logs. La classe des séries convergentes sur $\mathbb{R}$ est un autre exemple d'une communauté locale (non effective). En particulier, la plus petite communauté locale qui contient les séries exp-logs ne contient que des séries convergentes. Dans le chapitre 11, ceci entraînera que les solutions de systèmes d'équations exp-logs sont toutes convergentes.

0.4.3 Systèmes de transséries en plusieurs variables

Lorsque l'on considère des transséries en plusieurs variables au lieu de séries formelles en plusieurs variables, les difficultés techniques se multiplient encore. En plus des méthodes exposées dans la section précédente, nous utilisons de façon systématique les bases normales. Décrivons brièvement les points essentiels :

Les systèmes de contraintes asymptotiques. Outre les contraintes de la forme (0.11), nous considérons aussi des contraintes de la forme

$$\left\{ \begin{array}{l} x_1^{\alpha_1} \cdots x_p^{\alpha_p} \prec\!\!\!\prec\!\!\!\prec x_1^{\beta_1} \cdots x_p^{\beta_p} ; \\ x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp\!\!\!\asymp x_1^{\beta_1} \cdots x_p^{\beta_p} \end{array} \right. \tag{0.12}$$

dans le chapitre 11. Ici $f \prec\!\!\!\prec\!\!\!\prec g$, si $\log |f| \prec\!\!\!\prec \log |g|$ et $f \asymp\!\!\!\asymp g$, si $\log |f| \asymp \log |g|$. Ces relations expriment que f est d'une « échelle plus faible » resp. « de la même échelle », ou dans les termes de Shackell que f a une « classe de comparabilité inférieure » à g, resp. f est « de la même classe de comparabilité » que g. Par exemple $x \asymp\!\!\!\asymp x^{10^{100}}$, mais $x \prec\!\!\!\prec\!\!\!\prec x^{\log \log x}$, pour $x \to \infty$. Dans la section 8.4, nous montrons que l'on peut toujours décider si ce genre de systèmes plus généraux de contraintes sont consistants.

Les bases normales. Au lieu de travailler avec une grande base normale B, nous travaillons plutôt avec p bases normales $B_1, \cdots, B_p$, une pour chaque coordonnée $x_1, \cdots, x_p$. Le jeu est alors d'introduire le moins possible de nouveaux éléments dans $B = B_1 \cup \cdots \cup B_p$ lors de la résolution d'un système de transséries. Plus précisément l'introduction d'un nouvel élément dans B_j doit toujours être compensée par l'élimination d'un élément dans B_i pour un $i < j$. Mais une telle élimination compense autant d'insertions dans $B_{j_1}, \cdots, B_{j_l}$ avec $j_1, \cdots, j_l > i$ que l'on souhaite.

Cependant, il faut de l'astuce pour arriver à cette fin. Une de ces astuces est la suivante : lorsque l'on veut résoudre une équation comme

$$x e^x = y, \tag{0.13}$$

avec $x, y \succ\!\!\!\succ 1$, il faut en particulier déterminer la classe d'équivalence de x (pour $\asymp$). Notons que $x e^x - y$ s'exprime par rapport à la base normale $B = B_1 \cup B_2 = \{x^{-1}, e^{-x}\} \cup \{y^{-1}\}$. La façon la plus simple de faire serait de prendre le logarithme de l'équation (0.13) :

$$x + \log x = \log y,$$

et de constater que $x \asymp \log y$. Cependant, le calcul de ce logarithme nécessite l'insertion de $\log^{-1} x$ et $\log^{-1} y$ dans B.

Pour contourner ce problème nous remarquons que $x \prec\!\!\!\prec\!\!\!\prec e^x$, et pour que $x e^x \asymp y$, il faut en particulier que $e^x \asymp\!\!\!\asymp y$, donc $x \asymp \log y$. Ceci permet à nouveau de déterminer la classe d'équivalence de x, mais cette fois-ci *sans* insérer $\log^{-1} x$ dans B. De plus, puisque $x e^x = y$, nous obtenons $e^x \asymp y / \log y$, et après plus de calculs, il devient possible de réécrire e^x en fonction de $y, \log y$ et $\log \log y$ seulement. On peut donc éliminer e^x de B_1 et le rajout des deux logarithmes $\log^{-1} y$ et $\log^{-1} \log y$ est compensé par cette élimination d'un élément de B_1.

Ultra-régularisation. Pour pouvoir (entre autres) extraire les pseudo-coefficients d'une transsérie en plusieurs variables, les variables de sa représentation cartésienne doivent être *ultra-régulières* : grosso modo, ceci veut dire que ces variables représentent des transmonômes dont les classes de comparabilité sont bien déterminées. Par conséquent, nous montrons dans le chapitre 11 comment « ultra-régulariser » des transséries, et comment conserver cette propriété lors des calculs.

Résultats principaux. Le résultat principal du chapitre 11 est un algorithme de résolution de n'importe quel système de transséries qui admettent des représentations cartésiennes dans une communauté locale effective fixée. Comme dans le cas de l'expansion de fonctions exp-logs paramétrées, on suppose l'existence d'un oracle pour tester la consistance de systèmes exp-logs sur les constantes. Néanmoins, en absense d'un tel oracle, nous avons quand même un algorithme presqu'aussi fort, comme dans le cas des fonctions exp-logs dépendant de paramètres.

De plus, on remarque que l'oracle ne concerne que la consistance de systèmes *exp-logs* sur les constantes, même si la communauté locale contient les développements asymptotiques d'autres fonctions spéciales que l'exponentielle et le logarithme. Cette observation justifie d'une autre manière la place spéciale accordée aux fonctions exp-logs dans l'asymptotique automatique, une chose qui est souvent surprenante pour les gens extérieurs du domaine. En effet, lorsque l'on calcule avec des formes closes il n'y a point d'intérêt à accorder une place si spéciale aux fonctions exp-logs.

En appliquant notre résultat principal à des communautés locales précises, nous obtenons divers théorèmes : l'algorithme peut s'utiliser en particulier pour résoudre des systèmes asymptotiqes d'équations exp-logs en plusieurs variables, mais on peut aussi traiter des systèmes de transséries bien plus générales en prenant la classe des séries D-algébriques comme communauté locale effective. De façon théorique, nous pouvons aussi considérer les communautés locales de toutes les séries convergentes ou de toutes les séries. Dans le premier cas, il s'ensuit que toute transsérie, solution d'un système de transséries convergentes, est convergente elle aussi. Dans le deuxième cas, nous avons une description théorique de l'ensemble de solutions d'un système de transséries. Enfin, on obtient la stabilité par composition et inversion fonctionnelle des différentes classes de transséries.

0.4.4 Les équations différentielles algébriques.

Dans le chapitre 12, nous rendons réellement effectif l'algorithme du chapitre 5 pour résoudre des équations différentielles algébriques à coefficients dans les transséries, dans le cadre des transséries réticulées. Comme les solutions d'une équation différentielle dépendent des conditions initiales, la solution asymptotique générique d'une telle équation doit faire intervenir des paramètres. La particularité de notre algorithme est qu'il détermine lui-même les paramètres qu'il convient d'introduire et qu'il calcule la solution générique en fonction de tels paramètres.

Le traitement effectif soulève, comme toujours, un certain nombre de difficultés supplémentaires. Discutons les brièvement ici.

L'oracle pour les constantes. Dans le cas des équations différentielles algébriques, nous avons besoin d'un oracle pour les constantes encore plus fort que dans les chapitres 8 et 11: nous supposons que nous pouvons tester la consistance de n'importe quel système du premier ordre formé à partir des rationnels (disons), les opérations du corps, l'exponentielle, le logarithme et la relation d'ordre. Dans [VdD 84], van den Dries montre que de tels systèmes ne se ramènent pas toujours à des systèmes exp-logs comme dans les chapitres 8 et 11. Un exemple de cette situation est donné par la formule

$$y > 0 \ \wedge \ \exists w \, (wy = x \ \wedge \ z = ye^w). \tag{0.14}$$

Néanmoins, nos résultats subsistent dans un sens plus faible sans l'oracle, comme expliqué dans la section 0.4.1 à propos des fonctions exp-logs paramétrées.

Théorèmes de stabilité. Dans le chapitre 5, nous avons montré comment résoudre des équations différentielles algébriques quand les coefficients sont des transséries bien ordonnées. Nous montrons dans la section 12.3 que si les coefficients sont réticulés, alors il en est de même pour les solutions.

Calculs effectifs avec les solutions distinguées. Le point le plus délicat du chapitre 12 est le calcul exact avec les solutions distinguées. Plus précisément, si les coefficients d'une équation quasi-linéaire vivent tous dans un corps différentiel effectif $\mathfrak{T}$ de transséries, on souhaiterait pouvoir étendre $\mathfrak{T}$ avec la solution distinguée f de l'équation. Bien que nous disposions d'une caractérisation de f, en regardant son support, cette caractérisation ne permet pas toujours de résoudre se problème.

Dans le chapitre 12, nous résolvons ce problème en introduisant des solutions *semi-distinguées*. Ces solutions sont définies de façon dynamique, et elles remplacent les solutions distinguées f dans les cas où l'on n'est pas capable d'effectuer un test à zéro dans l'extension de $\mathfrak{T}$ par f. Ce traitement est assez délicat et nous renvoyons vers le chapitre 12 pour plus de détails. Un aspect bizarre de cette théorie est que l'on sait calculer avec des solutions génériques d'équations différentielles algébriques, mais pas avec des solutions particulières !

0.4.5 Des comportements asymptotiques oscillants

Nous allons discuter ici les résultats du chapitre 14, qui forme une première contribution au calcul asymptotique général en présence de phénomènes oscillants.

L'algébrique versus l'analytique. Dans le cas des comportements fortement monotones, les propriétés analytiques des transséries coïncident — pour ce que l'on en sait — avec les propriétés algébriques. Pour la version la plus simple des transséries faiblement oscillantes, ceci n'est pas toujours le cas. Considérons par exemple la relation

$$2 - \sin x - \sin ex \geqslant_\infty \frac{1}{\Gamma(x+2)},$$

qui est analytiquement valide pour tout x suffisamment grand, à cause du développement en fraction continue de e. En revanche, dans les modèles les plus simples de calcul avec des transséries faiblement oscillantes, cette relation n'est pas toujours vraie, mais plutôt indécidable. On peut néanmoins *imposer* ce genre de relations dans le modèle algébrique, mais il faut des connaissances extérieures, non algébriques, pour cela ; ici, ce rôle est joué par la fraction continue de e.

Étant donné qu'il ne va pas de soi que l'algébrique et l'analytique coïncident, il est important d'étudier dans quels cas on a correspondance. Le premier résultat du chapitre 14 règle ce problème dans un cas non trivial. En effet, considérons des fonctions exp-logs (ou d'autres transséries réticulées ayant un sens analytique) $f_1(x), \cdots, f_p(x)$, positives à l'infini. Si φ est une fonction sympathique (disons algébrique) sur $[-1,1]^p$, on peut se demander quelles sont les limites supérieure et inférieure de $\varphi(\sin f_1(x), \cdots, \sin f_p(x))$. Il se trouve que la réponse à cette question peut être donnée de façon effective en n'utilisant que des calculs algébriques. Ceci se démontre en généralisant un théorème classique de Bohr, Sierpiński et Weyl sur la distribution uniforme, disons des progressions arithmétiques de pas irrationnel modulo un (voir [Kok 34], [KN 74]).

Développements de fonctions sin-exp-logs. Plus généralement, au lieu de ne calculer que des limites supérieures et inférieures, on peut se demander comment développer des « fonctions sin-exp-logs » à l'infini. Il faut, pour cela, utiliser les résultats ci-dessus et l'algorithme du chapitre 11 pour calculer les développements génériques de transséries en plusieurs variables. Bien qu'en toute généralité, ceci nécessite des oracles très puissants pour répondre à des questions d'approximation diophantienne, nous donnons dans la section 14.5.2 une approche qui pourrait permettre de traiter la plupart des cas que l'on rencontre dans la pratique, et ceci seulement à l'aide de l'oracle employé dans le chapitre 11.

Et au delà des fonctions sin-exp-logs ? Dans le chapitre 14 nous discutons enfin, mais pas en détail, l'extension de nos résultats à certains types d'équations différentielles, et nous soulèvons quelques problèmes qui restent à résoudre. *Grosso modo* nous savons assez bien traiter à présent le cas ou l'on considère des développements par rapport aux transmonômes fortement monotones, mais avec des coefficients qui sont des fonctions analytiques en des sinus de transséries fortement monotones.

Part A

Asymptotic algebra

Chapter 1

Grid-based transseries

1.1 Introduction

In this chapter we define grid-based transseries. Thereby, we lay the theoretical foundation for most algorithms of part B of this thesis, and provide the basics for the further development of "asymptotic algebra" as a subject on its own. Transseries were first introduced in [Dahn 84], [DG 86] and [Éc 92] in order to describe very general types of strongly monotonic asymptotic behaviour near infinity. This means that the functions we consider do not present any oscillatory phenomena at infinity. It can be seen as the algebraic counterpart of the theory of Hardy fields (see [Bour 61], [Har 10], [Har 11], [Ros 83]), which is also frequently used in the field of automatic asymptotics.

In fact, we think that the theory of transseries is more natural as a foundation for automatic asymptotics, because of its algebraic nature. Actually, the theory serves as an algebraic model for our computations. The advantage with respect to the former model of Hardy fields is that there is no need to establish any analytic lemmas in order to justify computations which are essentially algebraic. As a consequence, it is possible to solve certain types of functional equations, which had not previously been solved using the theory of Hardy fields. Moreover, the notion of a transseries is easier to generalize: in chapter 6 we will consider transseries in several variables and transseries with weakly oscillatory (in contrast to strongly monotonic) behaviour at infinity. Finally, resummation theory can be used to recover the analytic properties of transseries in many cases (see [Éc 92]), although we will not be concerned with this here.

Let us now come more specifically to the contents of this chapter. In section 1.2 we study ordered rings. In section 1.3 we introduce grid-based series, which generalize classical power series. Grid-based power series satisfy a strong finiteness condition on their supports. This condition is verified for many practical purposes (including the theory of algebraic differential equations), and simplifies a lot of computations. However, it is sometimes necessary to consider more general types of supports, which will be done in the next chapter. In section 1.4 we define the notion of an asymptotic scale in the context of grid-based series.

In section 1.5 we introduce grid-based transseries. The transseries form a totally ordered field, in which we have an exponentiation, a logarithm and an infinite summation operator. In section 1.6 we proceed with the study of some asymptotic properties of the field of transseries. In particular, we introduce the concept of normal bases and we prove a structure theorem. In the last section we define most of the common operations on transseries, that is the differentiation w.r.t. x, composition and functional inversion. We conclude this chapter by giving a natural solution to a conjecture of Hardy.

Throughout this chapter, we will frequently use order theoretic concepts. For definitions and elementary properties, we refer to appendix A.

1.2 Ordered rings

An **ordered semigroup** is a semigroup X, together with an ordering $\leqslant$, which is compatible with the multiplication — i.e. $x \leqslant y \ \wedge \ x' \leqslant y' \Rightarrow xx' \leqslant yy'$. An **ordered ring** is a ring A, together with an ordering $\leqslant$, which is compatible with the ring structure. This means that $x \leqslant y \ \wedge \ x' \leqslant y' \Rightarrow x + x' \leqslant y + y'$, $0 \leqslant 1$, and $0 \leqslant x \ \wedge \ 0 \leqslant y \Rightarrow 0 \leqslant xy$. Such an ordering is characterized by the set of positive elements. Let A be an ordered ring. An **ordered A-algebra** is a morphism of ordered rings $A \stackrel{\varphi}{\to} B$ — i.e. an increasing morphism of A into an other ordered ring B. In particular, A itself is an ordered $\mathbb{Z}$-algebra.

Let $A \stackrel{\varphi}{\to} B$ be a totally ordered A-algebra. The **absolute value** $|x|$ of $x \in B$ is defined by $|x| = x$ if $x \geqslant 0$, and $|x| = -x$ otherwise. We denote $x \prec\!\!\prec y$, if $|\lambda x| \leqslant |\mu y|$, for some $\mu \in A$ and all $\lambda \in A$, and we say that x is **negligible** w.r.t. y. Similarly, we denote $x \preccurlyeq\!\!\preccurlyeq y$, if $|x| \leqslant |\lambda y|$, for some $\lambda \in A$, and we say that x is **dominated** by y. Instead of **Hardy's notation**, one often uses **Landau's notation**, according to which we write $x = o(y)$ resp. $x = O(y)$ instead of $x \prec\!\!\prec y$ resp. $x \preccurlyeq\!\!\preccurlyeq y$.

If $x \prec\!\!\prec 1$, then we say that x is **infinitesimal**. If $1 \prec\!\!\prec x$, then we say that x is **infinitely large**. If $x \preccurlyeq\!\!\preccurlyeq 1$, then we say that x is **bounded** (and x is said to be **unbounded** if not). The sets of infinitesimal resp. bounded elements of B are denoted by B^o resp. B^O. We also define $x \asymp y \Leftrightarrow x \preccurlyeq\!\!\preccurlyeq y \ \wedge \ y \preccurlyeq\!\!\preccurlyeq x$, and $x \sim y \Leftrightarrow x - y \prec\!\!\prec x$. Both relations are equivalence relations. Elements x in A with $x \asymp 1$ are called **Archimedian**. If all non zero divisors in B are Archimedian, then we say that B is Archimedian. In particular, an ordered ring is said to be Archimedian, if it is Archimedian as a $\mathbb{Z}$-algebra.

For the definition of the last asymptotic relations, we need a preliminary. Any commutative group G can be seen as a $\mathbb{Z}$-module. If G is multiplicative, then $\mathbb{Z}$ acts on G by exponentiation, and we say that G is a group with $\mathbb{Z}$-powers. More generally, if R is any ring, then a **group with R-powers** G is an R-module, whose underlying group is multiplicative, and such that R acts on G by exponentiation. We also say that G is an **exponential R-module**. If G is ordered and R is an ordered ring, then we say that G is an **ordered group with R-powers**, if $1 \leqslant x \ \wedge\ 0 \leqslant \alpha \Rightarrow 1 \leqslant x^{\alpha}$. Similarly, a **totally ordered field with R-powers** is a totally ordered field K, whose multiplicative ordered group of strictly positive elements K_*^+ has R-powers.

Assume now that A and B are totally ordered fields with R-powers, such that $(xy)^{\alpha} = x^{\alpha} y^{\alpha}$, for all $x \in A_*^+$, $y \in B_*^+$ and $\alpha \in R$. We denote $\tilde{x} = |x|$ and $\tilde{x} = 1/|x|$, for $x \in B^*$ with $1 \leqslant |x|$ and $|x| \leqslant 1$ respectively. The **comparability class** over R of an element $x \in B^*$ is the set of $y \in B^*$, such that there exist $\alpha, \beta \in R_*^+$, with $\tilde{x} \leqslant \widetilde{y^{\alpha}}$ and $\tilde{y} \leqslant \widetilde{x^{\beta}}$. We write $x \asymp_R y$ if x and y have the same comparability class. We also write $x \prec\!\!\prec\!\!\prec_R y$, if $\widetilde{x^{\alpha}} \leqslant \widetilde{y^{\beta}}$, for some $\beta \in R_*^+$ and all $\alpha \in R_*^+$. If this is the case, then we say that x has a smaller comparability class than y over R (indeed, the relation $\prec\!\!\prec\!\!\prec$ determines an ordering on the set of comparability classes). If no confusion about R can arise then we will denote $\asymp_R$ and $\prec\!\!\prec\!\!\prec_R$ by $\asymp$ resp. $\prec\!\!\prec\!\!\prec$.

Remark 1.1. The above definitions of the asymptotic relations $\prec\!\!\prec$, $\preceq\!\!\preceq$, etc. are only valid in the case when B is totally ordered. However, as we will see in the next section, these asymptotic relations can often be introduced without having an ordering on B. Constructions such as completions and methods like the Newton polygon method can also be carried out independently from orderings. In fact, it is possible to introduce the concept of "asymptotic orderings" in a more axiomatic way, but this point of view will never be used in the rest of this thesis.

An **asymptotic ordering** on a ring A is a transitive relation $\ll$, such that for each $y \in A$, the set $\{x \in A | x \ll y\}$ is an additive subgroup of A, and such that $x \ll y \Rightarrow xz \ll yz$, for all x, y and z in A. If A is a field, then $\ll$ is determined by the additive subgroup $\{x \in A | x \ll 1\}$. The relations $\prec\!\!\prec$ and $\preceq\!\!\preceq$ from above are asymptotic orderings. However, we notice that an asymptotic ordering is not necessarily an ordering. Indeed, $\prec\!\!\prec$ is not a strict ordering, since $0 \prec\!\!\prec 0$. The relation $\preceq\!\!\preceq$ is only a quasi-ordering. Other examples of asymptotic orderings will be encountered in this thesis each time we extend the definitions for $\prec\!\!\prec$ and $\preceq\!\!\preceq$. Yet some other examples can be given, and we refer to [VdH 94a, p. 16].

1.3 Grid-based series

1.3.1 Algebras of grid-based series

Let X be an ordered commutative multiplicative semigroup. A subset $\Gamma \subseteq X$ is said to be **grid-based**, if we have

$$\Gamma \subseteq \text{ц}_1^{\mathbb{N}} \cdots \text{ц}_n^{\mathbb{N}} \text{Щ}, \tag{1.1}$$

for $\text{щ}_1 > 1, \cdots, \text{щ}_n > 1$ in X, and a finite subset Щ of X. If X is a group, which is generated by the set of elements which are strictly superior to 1, then we can take a singleton for Щ. Moreover, we have

$$\Gamma \subseteq ц_1^{(\mathbb{N}-p)} \cdots ц_n^{(\mathbb{N}-p)}, \tag{1.2}$$

for certain $ц_1 > 0, \cdots, ц_n > 0$ and $p \in \mathbb{Z}$ in this case. We remark that a grid-based subset of X is necessarily Noetherian by Dickson's lemma (see page 306).

Proposition 1.1. *Each finite subset of X is grid-based. If Γ and Γ' are grid-based subsets of X, then so are $\Gamma \cup \Gamma'$, $\Gamma\Gamma'$. Moreover, if $ц \geqslant 0$, for all $ц \in \Gamma$, then $\Gamma^{\diamond} \stackrel{def}{=} \{ц_1 \cdots ц_n | ц_1, \cdots, ц_n \in \Gamma\}$ is grid-based.*

Proof. All assertions except the last one are trivial. So assume that Γ is a positive grid-based subset of X, and let $ц_1, \cdots, ц_n, Щ$ be such that (1.1) is satisfied. We claim that Щ may be taken equal to $\{0\}$; from this we trivially deduce that $\Gamma^{\diamond}$ is grid-based. Let щ be in Щ. By Dickson's lemma, the final segment of $\mathbb{N}^n$ of those $(a_1, \cdots, a_n)$, such that $ц_1^{a_1} \cdots ц_n^{a_n} щ > 1$, is finitely generated. Let $ц_{n+1}, \cdots, ц_m$ be those elements of X of the form $ц_1^{a_1} \cdots ц_n^{a_n} щ$, where $(a_1, \cdots, a_n)$ is one of the generators mentioned above, and where щ runs over Щ. Then we have $\Gamma \subseteq ц_1^{\mathbb{N}} \cdots ц_n^{\mathbb{N}}$, which proves our claim. □

Now let C be a ring. We denote by $C[[X^{\leqslant}]]$ or by $C[[X]]$ the set of mappings from X to C with grid-based support, and we call it the set of **grid-based series** in C over X. Here the **support** of a mapping $\varphi : X \to C$ is the set $\operatorname{supp} \varphi = \{ц \in X | \varphi(ц) \neq 0\}$. Such mappings are also denoted by sums $\sum_{ц \in X} \varphi_ц ц$. More generally, we say that a family $(f_i)_{i \in I}$ of elements in $C[[X]]$ is a **grid-based**, if $\bigcup_{i \in I} \operatorname{supp} f_i$ is grid-based, and if $\{i \in I | ц \in \operatorname{supp} f_i\}$ is finite for each $ц \in X$. If $(f_i)_{i \in I}$ is such a family, then we define

$$\sum_{i \in I} f_i = \sum_{ц \in X} \left(\sum_{i \in I} f_{i,ц} \right) ц. \tag{1.3}$$

The elements of X are called **monomials** and the elements of C **coefficients**; X itself is also called a **monomial group** If $f \in A$ and $ц \in X$, then we say that $f_ц$ is the **coefficient** of ц in f and we say that $f_ц ц$ is a **term** occurring in f.

Let us show that $C[[X]]$ can naturally be given the structure of a C-algebra. First of all, C and X can be embedded canonically into $C[[X]]$, by $c \mapsto c \cdot 1$ resp. $ц \mapsto 1 \cdot ц$. Let f and g be in A. Then we define

$$f + g = \sum_{ц \in \operatorname{supp} f \cup \operatorname{supp} g} (f_ц + g_ц) ц,$$

Similarly, their product is defined by

$$fg = \sum_{(ц, щ) \in \operatorname{supp} f \times \operatorname{supp} g} f_ц g_щ ц щ.$$

To see that fg is well defined, we first observe that $\operatorname{supp} fg \subseteq (\operatorname{supp} f)(\operatorname{supp} g)$. Furthermore, each set of pairs $(\text{ц}, \text{щ})$ which give rise to the same monomial $\text{ц}\text{щ}$ forms an antichain for the product ordering on $\operatorname{supp} f \times \operatorname{supp} g$. Hence such sets are finite by proposition A.2(d). We leave it to the reader to verify that A is indeed a ring for these operations.

We denote by $C[\![X]\!]^{o}$ resp. $C[\![X]\!]^{O}$ (see below for a justification of these notations) the sets of series $f \in C[\![X]\!]$ such that $\text{ц} > 1$ resp. $\text{ц} \geqslant 1$ for all $\text{ц} \in \operatorname{supp} f$. Now let $\varphi \in C[[t]]$ be an ordinary power series. Then φ induces an application $\varphi \circ \cdot$ from $C[\![X]\!]^{o}$ into $C[\![X]\!]^{O}$, defined by

$$\varphi \circ f = \sum_{\text{ц}_1^{k_1} \cdots \text{ц}_n^{k_n} \in (\operatorname{supp} f)^{\diamond}} \varphi_{k_1+\cdots+k_n} f_{\text{ц}_1}^{k_1} \cdots f_{\text{ц}_n}^{k_n} \text{ц}_1^{k_1} \cdots \text{ц}_n^{k_n}. \tag{1.4}$$

Using proposition 1.1 and the corollary of Higman's theorem (see page 307), $\varphi \circ f$ is seen to be well defined as in the case of the multiplication. Moreover, the association $\varphi \mapsto \varphi \circ \cdot$ is a morphism of algebras. As an application, we observe that any element of $1 + C[\![X]\!]^{o}$ is invertible. In the case when C is a field and X a totally ordered group, we therefore conclude that $C[\![X]\!]$ is a field. Indeed, let м be the smallest element of the support of a non zero f. Then we can write $1/f = (f_{\text{м}}\text{м})^{-1}(f/f_{\text{м}}\text{м})^{-1}$. As another application, we remark (assuming that $C \supseteq \mathbb{Q}$) that we can take the exponential of any infinitesimal series, and the logarithm of any element in $1 + C[\![X]\!]^{o}$. Moreover, exponentiation and logarithm are inverse one to another.

Example 1.1. Let A be an ordered commutative group and z a formal infinitesimal variable. We denote by z^A the formal multiplicative group, which is isomorphic to A (via the isomorphism $a \mapsto z^a$). We call $C[\![z^A]\!]$ the ring of grid-based series over C in z along A. If no confusion about A can arise, then we also denote $C[\![z^A]\!]$ by $C[\![z]\!]$.

We can define grid-based series in several variables $z_1, \cdots, z_n$ along A in different ways. First, we can give $z_1^A \times \cdots \times z_n^A$ the natural product ordering $\leqslant_{prod}$. In this case

$$C[\![z_1, \cdots, z_n]\!] = C[\![(z_1^A \times \cdots \times z_n^A)^{\leqslant_{prod}}]\!]$$

is said to be the ring of grid-based series over C in $z_1, \cdots, z_n$ along A. Secondly, we can give $z_1^A \times \cdots \times z_n^A$ the lexicographical ordering $\leqslant_{lex}$. In this case

$$C[\![z_1; \cdots; z_n]\!] = C[\![(z_1^A \times \cdots \times z_n^A)^{\leqslant_{lex}}]\!]$$

is said to be the ring of lexicographical grid-based series over C in $z_1, \cdots, z_n$ along A. We have $C[\![z_1, \cdots, z_n]\!] \subseteq C[\![z_1; \cdots; z_n]\!]$, where the inclusion is strict if C is non trivial and $n > 1$:

$$1 + \frac{z_2}{z_1} + \frac{z_2^2}{z_1^2} + \cdots \in C[\![z_1; z_2]\!] \backslash C[\![z_1, z_2]\!].$$

If C is a field, and A is totally ordered, then $C[[z_1; \cdots; z_n]]$ is a field. Finally, we can consider the ring $C[[z_1]] \cdots [[z_n]]$, which contains $C[[z_1; \cdots; z_n]]$. Again, this inclusion is usually strict:

$$1 + \frac{z_2}{z_1} + \frac{z_2^2}{z_1^{2!}} + \frac{z_2^3}{z_1^{3!}} + \cdots \in C[[z_1]][[z_2]] \backslash C[[z_1; z_2]].$$

1.3.2 Asymptotic relations

The set of minimal elements of the support of a series f is said to be its set of **dominant monomials**. If this set consists of a singleton, the unique minimal element is said to be **the dominant monomial** of f, and we denote it by $\mathfrak{m}_f$. In this case f is said to be a **regular**, and we also define $c_f \stackrel{\text{def}}{=} f_{\mathfrak{m}_f}$ to be its **dominant coefficient** and $\tau_f = c_f \mathfrak{m}_f$ to be its **dominant term**. Each series f can be written as a finite sum of regular series. Indeed, let $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$ be the set of dominant monomials of f. Then we write

$$f = \sum_{i=1}^{n} \sum_{ц \in (\mathfrak{m}_1, \cdots, \mathfrak{m}_i) \backslash (\mathfrak{m}_1, \cdots, \mathfrak{m}_{i-1})} f_{ц} ц. \tag{1.5}$$

Here we recall that $(E) \subseteq X$ denotes the final segment generated by a subset E of X.

Let us now suppose that X is totally ordered. Then all series in $C[[X]]$ are regular. If C is an ordered ring, then we give $C[[X]]$ the structure of an ordered C-algebra, by setting $f > 0$ if $f \neq 0$ and $c_f > 0$. If C is a totally ordered ring, then so is $C[[X]]$.

Warning 1.1. It should be noticed that the ordering on the monomials in X is precisely the opposite from the ordering on X, considered as a subset of $C[[X]]$. For instance, if $C = \mathbb{Z}$ and $X = z^A$ in example 1.1, then $z <_X z^2$, although $z >_{C[[z^A]]} z^2$. In cases where confusion might arise, we will usually precise that the *monomial* ц is smaller than the *monomial* щ, if ц $\leqslant_X$ щ. Moreover, monomials are usually denoted by the Cyrillic characters ц and щ. Nevertheless, the reader should always be aware of this warning.

In the previous section we introduced the asymptotic relations $\prec\!\!\prec$, $\preceq\!\!\prec$, $\asymp$ and $\sim$ for totally ordered algebras. We will now give equivalent definitions in the case of $C[[X]]$, when both C and X are totally ordered. In fact, these alternative definitions can still be used when C is no longer ordered, and X only partially ordered. In particular, this will justify the above reintroduction of the notations $C[[X]]^o$ and $C[[X]]^O$.

We say that f is **infinitesimal**, if and only if $1 <$ supp f. Similarly, f is **bounded**, if and only if $1 \leqslant$ supp f. The relations $\prec\!\!\prec$ resp. $\preceq\!\!\prec$ are defined by $f \prec\!\!\prec g \Leftrightarrow f \in C[[X]]^o g$ resp. $f \in gC[[X]]^O$, and we respectively say that f is

negligible w.r.t. g and that f is **dominated** by g. We say that f is **asymptotic** to g, if $f \preccurlyeq g \preccurlyeq f$, and we write $f \asymp g$. We say that f is **equivalent** to g, if $f - g \prec f$, and we write $f \sim g$.

If X is totally ordered, then we still have some other characterizations of $\prec$, $\preccurlyeq$, $\asymp$ and $\sim$: $f \prec g \Leftrightarrow \mathfrak{m}_f > \mathfrak{m}_g$, $f \preccurlyeq g \Leftrightarrow \mathfrak{m}_f \geqslant \mathfrak{m}_g$, $f \asymp g \Leftrightarrow \mathfrak{m}_f = \mathfrak{m}_g$ and $f \sim g \Leftrightarrow c_f \mathfrak{m}_f = c_g \mathfrak{m}_g$. Here $\mathfrak{m}_0 = +\infty_X$, by convention. We can also make some more definitions. If $\mathfrak{m}_f = 1$, then we define its **limit** to be $\lim f = f_1$. If $\mathfrak{m}_f > 1$, we set $\lim f = 0$. If $\mathfrak{m}_f < 1$, we define $\lim f = \pm\infty_C$, depending on the sign of c_f. An unbounded series (which is necessarily infinitely large) is said to be **purely unbounded** , if there does not exist any $\text{ц} \in \operatorname{supp} f$ with $1 \leqslant \text{ц}$. The set of purely unbounded series together with 0 is denoted by $C[[X]]^{\uparrow}$. Then we have the canonical decomposition $C[[X]] = C[[X]]^{\uparrow} \oplus C \oplus C[[X]]^{\downarrow}$, and we denote by $f = f^{\uparrow} + f^{c} + f^{\downarrow}$ the corresponding decomposition of an element $f \in C[[X]]$. We also denote $f^{\updownarrow} = f^{\uparrow} + f^{c}$ and $f^{\updownarrow} = f^{c} + f^{\downarrow}$.

Assuming that X is a totally ordered group with R-powers, for some totally ordered ring R, we can also give some alternative definitions of $\lll_R$ and $\asymp_R$. Again, we denote $\tilde{\text{ц}} = \text{ц}$, if $1 \leqslant \text{ц}$, and $\tilde{\text{ц}} = \text{ц}^{-1}$ otherwise. Then the **comparability class** over R of an element $f \in C[[X]]$ is the set of $g \in B$, such that there exist $\alpha, \beta \in R_*^+$ with $\widetilde{\mathfrak{m}_f} \leqslant \widetilde{\mathfrak{m}_g^{\alpha}}$ and $\widetilde{\mathfrak{m}_g} \leqslant \widetilde{\mathfrak{m}_f^{\beta}}$. We denote $f \asymp_R g$, if f and g have the same comparability class over R. We also denote $f \lll_R g$, if for some $\beta \in R_*^+$ and all $\alpha \in R_*^+$ we have $\widetilde{\mathfrak{m}_f^{\alpha}} \leqslant \widetilde{\mathfrak{m}_g^{\beta}}$, and we say that f has a smaller comparability class than g over R. Indeed, it can be verified that $\lll_R$ induces an ordering on the compatibility classes over R. If no confusion about R can arise, then we denote $\asymp_R = \asymp$ and $\lll_R = \lll$.

1.3.3 Quasi-ordered monomial groups

A quasi-ordered group (with R-powers) is defined in a similar way as an ordered group (with R-powers), by replacing "ordering" by "quasi-ordering" (see page 304) in the definition. For certain purposes (see chapters 10 and 11), it is useful to extend the notion of grid-based series to the case when the monomial group X is only quasi-ordered. It is easily checked that all what has been said in the two previous sections generalizes to this case, when systematically replacing "ordered group (with R-powers)" by "quasi-ordered group (with R-powers)".

Let X be a quasi-ordered monomial group, and U its subgroup of elements ц with $\text{ц} \equiv 1$. Selecting a "natural right inverse" π^{-1} for the projection $\pi : X \to X/\equiv$, we obtain natural inclusions

$$\nu_1 : C[[X/\equiv]][U] \hookrightarrow C[[X]]$$

and

$$\nu_2 : C[[X]] \hookrightarrow C[U][[X/\equiv]],$$

by setting

$$\nu_1\left(\sum_{ц\in X/\equiv}\sum_{щ\in U} f_{ц,щ}цщ\right)=\sum_{ц\in X/\equiv}\sum_{щ\in U} f_{ц,щ}\pi^{-1}(ц)щ$$

and

$$\nu_2\left(\sum_{ц\in X/\equiv}\sum_{щ\in U} f_{ц,щ}\pi^{-1}(ц)щ\right)=\sum_{ц\in X/\equiv}\sum_{щ\in U} f_{ц,щ}цщ.$$

However, these inclusions are generally strict in both cases. For instance, let $X = z_1^{\mathbb{Z}} z_2^{\mathbb{Z}}$, with $1 < z_1$ and $1 < z_2$ but $z_1 \equiv z_2$. Then we have $U = (z_2/z_1)^{\mathbb{Z}}$ and $X/\equiv\cong \overline{z_1}^{\mathbb{Z}}$. Take $\pi^{-1}(\overline{z_1}^k) = z_1^k$ for all k. Now $1/(1 - z_1 - z_2)$ is in $C[[X]]$ but not in $\operatorname{im}\nu_1$, and $\sum_{k\in\mathbb{N}}(z_2/z_1)^{k!}\overline{z_1}^k$ is in $C[U][[X/\equiv]]$ but not in $\operatorname{im}\nu_2$.

1.4 Asymptotic scales

1.4.1 Invertible series

Let $R \supseteq \mathbb{Z}$ be a totally ordered ring and $C \supseteq R$ a field with R-powers. In this section we will only consider monomial groups with R-powers. Often, such groups X satisfy the condition

$$ц^{\alpha} \geqslant 1 \Rightarrow ц \geqslant 1,$$

for all $ц \in X$ and $\alpha \in R_*^+$. If this is the case, then we say that the ordering on X is **non degenerate** and the invertible elements of $C[[X]]$ can be characterized as follows:

Proposition 1.2. *Let X be a non degenerately ordered monomial group with R-powers. Then a series f in $C[[X]]$ is invertible, if and only if f is regular.*

Proof. First observe that if $\leqslant'\supseteq\leqslant$ is any other ordering such that $X^{\leqslant'}$ is a monomial group, then $C[[X^{\leqslant}]] \subseteq C[[X^{\leqslant'}]]$. Hence, if $f \in C[[X^{\leqslant}]]$ is invertible in $C[[X^{\leqslant'}]]$, then f is invertible in $C[[X^{\leqslant'}]]$, and both inverses coincide.

Now if $ц \in X$ is neither superior nor inferior to 1 for $\leqslant$, then $\leqslant$ can be extended into a total ordering $\leqslant'$ on X, for which $ц \geqslant 1$, and for which X remains a monomial group. We first observe that we can define an intermediate ordering $\leqslant_i$ by

$$щ \geqslant_i 1 \Leftrightarrow \exists \alpha, \beta \in R_*^+ \quad щ^{\alpha} \geqslant ц^{\beta},$$

for which X remains a monomial group, and for which $ц > 1$. This intermediate ordering extends into a total ordering by a classical argument, using Zorn's lemma.

Now assume that $f \neq 0$ is a series whose set of dominant monomials (for $\leqslant$) contains more than one element. By what precedes, we can construct extensions $\leqslant'$ and $\leqslant''$ of $\leqslant$ into total orderings (such that $X^{\leqslant'}$ and $X^{\leqslant''}$ are monomial groups), for which the dominant monomials of f are different. Hence the inverses of f in $C[\![X^{\leqslant'}]\!]$ and $C[\![X^{\leqslant''}]\!]$ are different, so that f can not be invertible in $C[\![X]\!]$. $\square$

Remark 1.2. The condition on X is necessary: if X is the multiplicative group $\{1, x\}$ with $x^2 = 1$, then $(\sqrt{2} + x)(\sqrt{2} - x) = 1$, although $\sqrt{2} + x$ is not regular.

1.4.2 Asymptotic scales and extension by strong linearity

Let X be any monomial group (with R-powers). If f is a regular series in $C[\![X]\!]$, then we can define f^α, for any $\alpha \in R$. Indeed, we write $f = c_f \text{м}_f(1+\varepsilon)$, with $\varepsilon \prec\!\!\prec 1$ and take $f^\alpha = c_f^\alpha \text{м}_f^\alpha p_\alpha \circ \varepsilon$, where $p_\alpha = 1 + \alpha z + \alpha(\alpha-1)z^2/2 + \cdots \in C[[z]]$. A multiplicative subgroup S of regular series with R-powers of $C[\![X]\!]$ is said to be an **asymptotic scale**, if the mapping $S \to X; \text{ц} \mapsto \text{м}_\text{ц}$ is injective. In this case, the ordering on X induces an ordering on S by $\text{ц} \leqslant_S \text{щ} \Leftrightarrow \text{м}_\text{ц} \leqslant_X \text{м}_\text{щ}$. The regularity condition is motivated by proposition 1.2, since the elements of an asymptotic scale are in particular invertible.

Proposition 1.3. *Let $S \subseteq C[\![X]\!]$ be an asymptotic scale. Then $C[\![S]\!]$ is naturally embedded in $C[\![X]\!]$.*

Proof. Let $f = \sum_{\text{щ} \in S} f_\text{щ} \text{щ}$ be an element of $C[\![S]\!]$. We have to prove that $\sum_{\text{щ} \in S} f_\text{щ} \text{щ}$ is also well defined as an element of $C[\![X]\!]$, i.e. that the family $(f_\text{щ} \text{щ})_{\text{щ} \in S}$ is grid-based. We have

$$\operatorname{supp}_S f \subseteq \text{щ}_1^{\mathbb{N}} \cdots \text{щ}_n^{\mathbb{N}} \{\text{щ}_{n+1}, \cdots, \text{щ}_N\},$$

for certain $\text{щ}_1, \cdots, \text{щ}_N$ in S, with $\text{щ}_1, \cdots, \text{щ}_n > 1$. We can write $\text{щ}_i = c_i \text{м}_i(1 + \varepsilon_i)$, for each i, with $c_i \in C$, $\text{м}_i = \text{м}_{\text{щ}_i}$ and $\varepsilon_i \prec\!\!\prec 1$. We have $\operatorname{supp}_X \varepsilon_i \subseteq \text{ц}_1^{\mathbb{N}} \cdots \text{ц}_k^{\mathbb{N}}$, for certain $\text{ц}_1, \cdots, \text{ц}_k > 1$ and all i. The sum $f = \sum_{\text{щ} \in S} f_\text{щ} \text{щ}$ can be rewritten as

$$f = \sum_{j=n+1}^{N} \sum_{\alpha_1, \cdots, \alpha_n \in \mathbb{N}} f_{\alpha_1, \cdots, \alpha_n, j} \text{щ}_1^{\alpha_1} \cdots \text{щ}_n^{\alpha_n} \text{щ}_j, \tag{1.6}$$

by choosing privileged $\alpha_1, \cdots, \alpha_n, j$ with $\text{щ} = \text{щ}_1^{\alpha_1} \cdots \text{щ}_n^{\alpha_n} \text{щ}_j$, for each $\text{щ} \in S$. Expanding $\text{щ}_i^{\alpha_i} = c_i^{\alpha_i} \text{м}_i^{\alpha_i} (1 + \varepsilon_i)^{\alpha_i}$, for each i, we deduce that

$$\operatorname{supp}_X f \subseteq \text{м}_1^{\mathbb{N}} \cdots \text{м}_n^{\mathbb{N}} \{\text{м}_{n+1}, \cdots, \text{м}_N\} \text{ц}_1^{\mathbb{N}} \cdots \text{ц}_k^{\mathbb{N}}.$$

Moreover, each element of $\operatorname{supp}_X f$ corresponds only to a finite number of terms in the sum (1.6) by Dickson's lemma (applied to $\mathbb{N}^n \times \mathbb{N}^{nk}$).

Finally, the set of dominant monomials D_X of $f \in C[\![S]\!]$ w.r.t. X is given by $D_X = \{\text{м}_\text{щ} | \text{щ} \in D_S\}$, where D_S is the set of dominant monomials of f w.r.t. S. In particular, $D_X = \phi$ if and only if $D_S = \phi$, since $S \overset{\text{м}}{\to} X; \text{щ} \mapsto \text{м}_\text{щ}$ is injective. Hence the natural mapping $C[\![S]\!] \to C[\![X]\!]$ is injective. $\square$

Remark 1.3. If the mapping $S \xrightarrow{\text{M}} X$ is not injective, then we still have a natural mapping from $C[\![S]\!] \to C[\![X]\!]$, but this mapping is not necessary an embedding. In this case it is also possible to take a family $(\text{ш}_s)_{s \in S}$ of elements in $C[\![X]\!]$, instead of a subset $S \subseteq C[\![X]\!]$. In each of these cases, we say that the resulting mapping $C[\![S]\!] \to C[\![X]\!]$ is obtained by **extension by strong linearity**. Notice finally that extension by strong linearity naturally carries over to the case when X is only a quasi-ordered monomial group.

If $C[\![S]\!] = C[\![X]\!]$, then the mapping $S \xrightarrow{\text{M}} X$ is bijective, and we call it a **scale change**. If B is a basis for X as an exponential R-module, then $\text{M}^{-1}(B)$ is a basis for S (such bases are also called **asymptotic bases**). Hence, scale changes are determined by **base changes** in this case. Inversely, if we have a basis B for S, then the embedding $C[\![S]\!] \subseteq C[\![X]\!]$ is entirely determined by its restriction to B (where we make take S as in the above remark).

1.4.3 Explicit base change formulae

We will now give some explicit formulae in the case when X admits a finite basis $B = \{z_1, \cdots, z_n\}$. These formulae are mainly useful in part B of this thesis and this section may temporarily be skipped by the reader. Let $g_1, \cdots, g_m$ be regular infinitesimal elements of $L = C[\![X]\!]$ and denote $M = C[\![g_1, \cdots, g_n]\!] = C[\![S]\!]$, where $S = R^m$ has the natural product ordering. We will now give an explicit formula for the natural mapping $M \to L$, which can be seen as a right **composition** with $(g_1, \cdots, g_m)$. It will be convenient to use **vector notation** for this. This means that we denote $\boldsymbol{z} = (z_1, \cdots, z_n)$ and $\boldsymbol{g} = (g_1, \cdots, g_m)$. We write $\boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, for $(\alpha_1, \cdots, \alpha_n) \in R^n$. Then the support of a series f in L can be seen as a set of vectors **supp** f. If f is regular, then we denote by $\boldsymbol{\mu}_f$ the minimal element in the support of f. This element is said to be the **dominant vector exponent** of f. We also denote by $\boldsymbol{\mu}_{\boldsymbol{g}}$ the matrix whose columns are $\boldsymbol{\mu}_{g_1}, \cdots, \boldsymbol{\mu}_{g_n}$. Matrix multiplication is denoted by $\cdot$.

We can write $g_i = c_i \boldsymbol{z}^{\boldsymbol{\mu}_{g_i}}(1 + \varepsilon_i)$, for each i, with $c_i \in C$ and $\varepsilon_i \prec\!\!\prec 1$. If $f \in M$ then its composition $f \circ \boldsymbol{g}$ with $\boldsymbol{g}$ is given by

$$f \circ \boldsymbol{g} = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \boldsymbol{c}^{\boldsymbol{\alpha}} z^{\boldsymbol{\mu}_{\boldsymbol{g}} \cdot \boldsymbol{\alpha}} (1 + \varepsilon_1)^{\alpha_1} \cdots (1 + \varepsilon_m)^{\alpha_m},$$

whence

$$f \circ \boldsymbol{g} = \sum_{\boldsymbol{\alpha}} \sum_{k_1, \cdots, k_m \in \mathbb{N}} \sum_{\boldsymbol{\beta}_{1,1}, \cdots, \boldsymbol{\beta}_{m,k_m}} C_{\boldsymbol{\alpha}, \boldsymbol{\beta}_{1,1}, \cdots, \boldsymbol{\beta}_{m,k_m}} \boldsymbol{z}^{\boldsymbol{\mu}_g \cdot \boldsymbol{\alpha} + \boldsymbol{\beta}_{1,1} + \cdots + \boldsymbol{\beta}_{m,k_m}}, \tag{1.7}$$

where

$$C_{\boldsymbol{\alpha}, \boldsymbol{\beta}_{1,1}, \cdots, \boldsymbol{\beta}_{m,k_m}} = f_{\boldsymbol{\alpha}} \boldsymbol{c}^{\boldsymbol{\alpha}} \binom{\alpha_1}{k_1} \cdots \binom{\alpha_m}{k_m} \varepsilon_{1, \boldsymbol{\beta}_{1,1}} \cdots \varepsilon_{m, \boldsymbol{\beta}_{m,k_m}}.$$

This formula often remains valid, if S is given a stronger ordering. It suffices that the matrix multiplication with $\boldsymbol{\mu_g}$ maps grid-based supports in M into grid-based supports in L. In particular, if $g_1 \lll \cdots \lll g_m$, then $M = C[\![g_1; \cdots; g_m]\!]$ embeds into $L = C[\![z_1; \cdots; z_n]\!]$.

Assume that $L = C[\![z_1; \cdots; z_n]\!]$ and $M = C[\![g_1; \cdots; g_n]\!]$, with

$$g_1 \lll \cdots \lll g_n.$$

For computational purposes it is sometimes useful to have explicit formulae for composition in terms of expansions w.r.t. z_n. Indeed, we can see L as a subset of $C[\![z_1; \cdots; z_{n-1}]\!][\![z_n]\!]$. The support of an element of $C[\![z_1; \cdots; z_{n-1}]\!][\![z_n]\!]$ is then considered as a subset of R. The minimal element of the support of a non-zero series f is denoted by μ_f, and we call it the **dominant exponent** or **valuation** of f. Right composition with () (for $n = 0$) is just the identity mapping. If $n > 0$, we have

$$f \circ (g_1, \cdots, g_n) = \sum_{\alpha_n} \sum_{\alpha_1, \cdots, \alpha_{n-1}} f_{\alpha_n, \cdots, \alpha_1} g_1^{\alpha_1} \cdots g_n^{\alpha_n}.$$

Now, let $1 \leqslant i \leqslant n-1$. If we put $g_i = g_{i,0} + g_i'$, we have

$$\begin{aligned} g_i^{\alpha_i} &= g_{i,0}^{\alpha_i} \left(1 + \frac{1}{g_{i,0}} \sum_{\beta_i > 0} g_{i,\beta_i} z_n^{\beta_i}\right)^{\alpha_i} \\ &= \sum_{k_i} \binom{\alpha_i}{k_i} g_{i,0}^{\alpha_i - k_i} \sum_{\beta_{i,1}, \cdots, \beta_{i,k_i} \in \operatorname{supp} g_i'} g'_{i,\beta_{i,1}} \cdots g'_{i,\beta_{i,k_i}} z_n^{\beta_{i,1} + \cdots + \beta_{i,k_i}}. \end{aligned}$$

Similarly, if we put $g_n = z^{\mu_{g_n}} (g_{n,\mu_{g_n}} + g_n')$,

$$g_n^{\alpha_n} = \sum_{k_n} \binom{\alpha_n}{k_n} g_{n,\mu_{g_n}}^{\alpha_n - k_n} z^{\mu_{g_n} \alpha_n} \sum_{\beta_{n,1}, \cdots, \beta_{n,k_n} \in \operatorname{supp} g_n'} g'_{n,\beta_{n,1}} \cdots g'_{n,\beta_{n,k_n}} z_n^{\beta_{n,1} + \cdots + \beta_{n,k_n}}.$$

Furthermore, for fixed α_n and k_i, we have

$$\sum_{\alpha_1, \cdots, \alpha_{n-1}} f_{\alpha_n, \cdots, \alpha_1} \prod_{i=1}^{n-1} \frac{\alpha_i!}{(\alpha_i - k_i)!} g_{i,0}^{\alpha_i - k_i} = \frac{\partial^{k_1 + \cdots + k_{n-1}} f_{\alpha_n}}{\partial z_1^{k_1} \cdots \partial z_{n-1}^{k_{n-1}}} \circ (g_{1,0}, \cdots, g_{n-1,0}).$$

Putting everything together, we obtain

$$f \circ (g_1, \cdots, g_n) = \sum_{\alpha_n} \sum_{\beta_{1,1}, \cdots, \beta_{n,k_n}} C_{\beta, \alpha_n} g_{n,\mu_{g_n}}^{\alpha_n - k_n} z_n^{\mu_{g_n} \alpha_n + \beta_{1,1} + \cdots + \beta_{n,k_n}},$$

where

$$C_{\beta, \alpha_n} = \frac{g'_{1,\beta_{1,1}} \cdots g'_{n,\beta_{n,k_n}}}{k_1! \cdots k_n!} \frac{\partial^{k_1 + \cdots + k_{n-1}} f_{\alpha_n}}{\partial z_1^{k_1} \cdots \partial z_{n-1}^{k_{n-1}}} \circ (g_{1,0}, \cdots, g_{n-1,0}).$$

If $z_1, \cdots, z_n$ and $g_1, \cdots, g_n$ correspond to normal bases (see section 1.6), then the above formulae may be simplified. In fact, in this case, we have $g_i = g_{i,0}$, for $1 \leqslant i \leqslant n-1$. Hence, the above formula reduces to

$$f \circ (g_1, \cdots, g_n) = \sum_{\alpha_n} \sum_{\beta_1, \cdots, \beta_k} C_{\beta,\alpha_n}\, g_{n,\mu_{gn}}^{\alpha_n - k_n}\, z_n^{\mu_{gn}\alpha_n + \beta_1 + \cdots + \beta_k}, \tag{1.8}$$

where

$$C_{\beta,\alpha_n} = \frac{g'_{n,\beta_1} \cdots g'_{n,\beta_k}}{k!}\, f_{\alpha_n} \circ (g_{1,0}, \cdots, g_{n-1,0}).$$

In particular, no partial derivatives are involved in this formula.

Still assume that $L = C\,[\![z_1; \cdots; z_n]\!]$ and $M = C\,[\![g_1; \cdots; g_n]\!]$, with

$$g_1 \lll \cdots \lll g_n.$$

We claim that right composition with $(g_1, \cdots, g_n)$ admits an inverse. In other words, the equation $f \circ (g_1, \cdots, g_n) = h$ admits a unique solution in L, for each $h \in L$. We give a method to compute f using vector notation. As before, we write $g_i = c_i \boldsymbol{z}^{\boldsymbol{\mu}_{g_i}}(1 + \varepsilon_i)$ for each i. Now let

$$\Gamma = \boldsymbol{\mu}_{\boldsymbol{g}}^{inv} \cdot (\mathbf{supp}\, h + (\mathbf{supp}\, \varepsilon_1 \cup \cdots \cup \mathbf{supp}\, \varepsilon_n)^{\diamond}).$$

Let us show that $f \circ \boldsymbol{g} = h$ admits a solution with supp $f \subseteq \Gamma$, by computing the coefficients of f by well-founded induction over $\boldsymbol{\alpha} \in \Gamma$. More precisely, our induction hypothesis assumes that we already computed the $f_{\boldsymbol{\beta}}$, for $\boldsymbol{\beta} < \boldsymbol{\alpha}$ and that the equation $(f \circ \boldsymbol{g})_{\boldsymbol{\gamma}} = h_{\boldsymbol{\gamma}}$ is verified for all $\boldsymbol{\gamma} < \boldsymbol{\mu}_{\boldsymbol{g}} \cdot \boldsymbol{\alpha}$, whatever we might take as coefficients $f_{\boldsymbol{\beta}}$, for $\boldsymbol{\beta} \geqslant \boldsymbol{\alpha}$.

We start with $\boldsymbol{\alpha} = \boldsymbol{\mu}_{\boldsymbol{g}}^{inv} \cdot \boldsymbol{\mu}_h$ for which the induction hypothesis is trivially verified. Next, let $\boldsymbol{\gamma} = \boldsymbol{\mu}_{\boldsymbol{g}} \cdot \boldsymbol{\alpha}$. We must have

$$(f \circ \boldsymbol{g})_{\boldsymbol{\gamma}} = \sum_{k_1, \cdots, k_m \in \mathbb{N}} \sum_{\boldsymbol{\beta}_{1,1}, \cdots, \boldsymbol{\beta}_{m,k_m}} C_{\boldsymbol{\alpha} - (\boldsymbol{\beta}_{1,1} + \cdots + \boldsymbol{\beta}_{m,k_m}), \boldsymbol{\beta}_{1,1}, \cdots, \boldsymbol{\beta}_{m,k_m}} = h_{\boldsymbol{\gamma}}.$$

By the induction hypothesis, $f_{\boldsymbol{\alpha}}$ is well defined by this formula. Next, we observe that the first $\boldsymbol{\alpha}' > \boldsymbol{\alpha}$, such that $(f \circ \boldsymbol{g})_{\boldsymbol{\mu}_{\boldsymbol{g}} \cdot \boldsymbol{\alpha}'}$ might be non zero, is the smallest upper bound of $\boldsymbol{\alpha}$ in Γ. Hence, the induction hypothesis is satisfied again, if we replace $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}'$, and we obtain a solution f by induction. Finally, as right composition by a fixed g is a field homomorphism, whose kernel is easily seen to be trivial, the solution f must be unique in L. Actually, it can easily be checked that there exists a $\boldsymbol{g}^{inv}$ with $f = h \circ \boldsymbol{g}^{inv}$ for all h.

1.5 Grid-based transseries

We will give purely algebraic variants to the definition of the field of grid-based transseries as given by Ecalle in [Éc 92]. Actually, we give two equivalent definitions, the first one of which corresponds to Ecalle's definition. In this chapter we restrict ourselves to transseries with grid-based support and we shortly call them grid-based transseries ; for more general definitions, we refer to the next chapter.

Let R be a ring. We say that R is a **partial exp-log ring**, if there exists a partially defined mapping $\exp : R \to R^*$, which satisfies $\exp 0 = 1$ and $\exp(x+y) = \exp x \exp y$, for all x and y, for which the identity make sense. Here we understand that whenever exp is defined for two values among x, y and $x+y$, then it is for the third. We remark that $\exp x \exp -x = 1$, for all x such that either $\exp x$ or $\exp -x$ is defined. Given a partial exp-log ring R, we define the logarithm to be the partial multivalued inverse of the exponentiation. If exp is totally defined, then we say that R is an **exp-log ring**. A **derivation** on R is a derivation ∂ on R as a ring, such that $\partial(\exp x) = (\partial x) \exp x$, for all $x \in \operatorname{dom} \exp$. An **ordered partial exp-log ring** is a partial exp-log ring with an ordering, such that $x \geqslant 0 \Leftrightarrow \exp x \geqslant 1$, for all $x \in \operatorname{dom} \exp$. In particular, log is univalued. If the ordering is total, then the image of exp is strictly positive. An **ordered exp-log ring** is an ordered partial exp-log ring R, for which exp is defined on R and log on R^+_*. We remark that a totally ordered exp-log ring is necessarily a field.

It will be convenient to adopt the notations $\log_k$ resp. $\exp_k$ for the k-th iterated logarithm resp. exponential. Here k can be taken in $\mathbb{Z}$, with the conventions $\log_0 x = \exp_0 x = x$ and $\log_{-k} = \exp_k$. More generally, let $\boldsymbol{p} = p_0 \cdots p_k$ be a word in C^*. Then we define the **logarithmic monomial** $\log_{\boldsymbol{p}} x$ by $\log_{\boldsymbol{p}} x = x^{p_0} \cdots \log_k^{p_k} x$. The multiplicative group of logarithmic monomials is denoted by $\log_{C^*} x$.

From now on we assume that we are given a fixed totally ordered exp-log constant field C and we will only consider totally ordered monomial groups with C-powers. Given such a group X, we remark that whenever we defined $\exp f$ for all elements f of a subgroup A of the additive group $C[\![X]\!]^{\uparrow}$, then we can canonically extend this definition to $A \oplus C \oplus C[\![X]\!]^{\downarrow}$ by $\exp f = \exp f^{\uparrow} \exp f^{c} \exp \circ f^{\downarrow}$, where $f = f^{\uparrow} + f^{c} + f^{\downarrow}$ is the canonical decomposition of f.

First construction. Let M denote the totally ordered monomial group $M = (1/x)^C$, with the same ordering as on C. Here x represents a variable which intuitively tends to infinity. The first step of the construction yields a sequence of monomial groups

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$$

each of which is naturally embedded in the next one. The construction of this sequence corresponds to the insertion of new exponentials and proceeds by induction (see also example 1.2 below).

We start with $E_0 = M$. Assume now that we have constructed our sequence up to E_k. Then we set

$$E_{k+1} = M \times \exp(C[\![E_k]\!]^{\uparrow}).$$

Here $\exp(C[\![E_k]\!]^{\uparrow})$ consists of the formal exponentials $\exp f$ for $f \in C[\![E_k]\!]^{\uparrow}$; hence, $\exp(C[\![E_k]\!]^{\uparrow})$ is a formal multiplicative monomial group which is isomorphic to the additive group $C[\![E_k]\!]^{\uparrow}$. Furthermore, $M \times \exp(C[\![E_k]\!])^{\uparrow}$ is ordered lexicographically: $1 \leqslant (\text{ц}, \text{щ})$, if either $1 < \text{щ}$, or $\text{щ} = 1$ and $1 \leqslant \text{ц}$. We finally have to check that E_k is naturally included in E_{k+1}. This is clear for $k = 0$. In general, $C[\![E_k]\!]$ is naturally included in $C[\![E_{k-1}]\!]$ by extension by strong linearity.

Example 1.2. The first monomial groups E_0, E_1 and E_2 are given by

$$\begin{aligned} E_0 &= \{x^\alpha | \alpha \in C\}; \\ E_1 &= \{x^\alpha e^{\Sigma_{\beta>0} f_\beta x^\beta} | \alpha \in C, f \in C[\![E_0]\!]^{\uparrow}\}; \\ E_2 &= \{x^\alpha e^f | \alpha \in C, f \in C[\![E_1]\!]^{\uparrow}\}. \end{aligned}$$

For instance, $e^{e^x + x^{-1}e^x + x^{-2}e^x + \cdots} \in E_2$.

Now consider the inductive limit of the sequence of embeddings of totally ordered partial exp-log fields $C[\![E_0]\!] \to C[\![E_1]\!] \to \cdots$. This limit is denoted by $C^{alog}[\![\![x]\!]\!]$, and we call it the **field of alogarithmic grid-based transseries**[1] over C in x. Alternatively, one can see $C^{alog}[\![\![x]\!]\!]$ as a field $C[\![E]\!]$ of grid-based series, by taking $E = \bigcup_{n\in\mathbb{N}} E_n$. The reason for this is that any grid-based subset of E is a grid-based subset of E_k for some k. We also remark that the exponentiation is totally defined in $C^{alog}[\![\![x]\!]\!]$. We claim that we can naturally embed $C^{alog}[\![\![x]\!]\!]$ into $C^{alog}[\![\![\log x]\!]\!]$. Roughly speaking, this embedding stems from the systematic replacement of x by $e^{\log x}$. Let us now give a more detailed description:

We already have a formal isomorphism between $C^{alog}[\![\![x]\!]\!]$ and $C^{alog}[\![\![\log x]\!]\!]$, by systematically replacing x by $\log x$. This isomorphism, which will be denoted by $\cdot\circ\log$, maps f to $f\circ\log$. Now the natural embedding of $C^{alog}[\![\![x]\!]\!]$ and $C^{alog}[\![\![\log x]\!]\!]$ maps $C[\![E_k]\!]$ into $C[\![E_{k+1} \circ \log]\!]$, for each k. For $k = 0$, we send monomials $x^c \in E_0$ to monomials $\exp(c \log x) \in E_1 \circ \log$ and extend by linearity. For $k > 0$, we use induction and send monomials $x^c \exp(f) \in M \times \exp(C[\![E_{k-1}]\!]^{\uparrow})$ to monomials $\exp(c \log x + f) \in \exp(C[\![E_k \circ \log]\!]^{\uparrow})$. Again, we extend by linearity. We observe that for each monomial in $C^{alog}[\![\![x]\!]\!]$, the logarithm of its image under the embedding is defined. Hence the logarithm of the image of any strictly positive element is defined.

We finally consider the inductive limit of the sequence

$$C^{alog}[\![\![x]\!]\!] \to C^{alog}[\![\![\log x]\!]\!] \to C^{alog}[\![\![\log_2 x]\!]\!] \to \cdots$$

of totally ordered exp-log fields. This limit is denoted by $C[\![\![x]\!]\!]$, and we call it the field of **grid-based transseries** over C in x. The **logarithmic depth** of a transseries f is the smallest k with $f \in C^{alog}[\![\![\log_k x]\!]\!]$. Often, when no confusion

[1] Alogarithmic means "without logarithms"

can arise, we will use the alternative notation $\mathbb{T} = C[[[x]]]$, and call it (abusively) *the* field of transseries. Again, $C[[[x]]]$ can be seen as a field of grid-based series, since $C[[[x]]] = C[[\text{Ц}]]$, with $\text{Ц} = E \cup E \circ \log \cup \cdots$, for the same reason as above. Elements of Ц are called **transmonomials**.

Second construction. Instead of constructing alogarithmic transseries first, we can also start with the construction of logarithmic transseries. More precisely, we call $[[L_0]]$ with $L_0 = \log_{C^*} x$ the set of **logarithmic transseries**. In a similar way as above, we define the L_k by $L_{k+1} = \exp(C[[L_k]]^\uparrow)$, using induction over k. Again, we can canonically embed L_k into L_{k+1} by sending ц to $\exp \log$ ц. Finally, we consider the inductive limit of the sequence

$$C[[L_0]] \to C[[L_1]] \to \cdots$$

which happens to be isomorphic to $C[[[x]]]$: the equivalence of both constructions is due to the finiteness condition in the definition of grid-based sets. In the next chapter we will see that both constructions are no longer equivalent for well-ordered supports.

The smallest number r such that $f \in C[[L_r]]$ is called the **exponential depth** of a transseries $f \in C[[[x]]]$. The set $C[[L_r]]$ is also denoted by $C_r[[[x]]]$. More generally, we denote by $C_r^k[[[x]]]$ the set of transseries whose logarithmic depth is bounded by k, and whose exponential depth is bounded by r.

Example 1.3. Let us give some explicit examples of transseries. The following transseries is in fact a series in $\mathbb{R}[[1/x]]$:

$$\frac{1}{1 - x^{-1} - x^{-e}} = 1 + x^{-1} + x^{-2} + x^{-e} + x^{-3} + x^{-e-1} + x^{-4} + x^{-e-2} + x^{-5} + x^{-2e} + \cdots.$$

The transseries $e^{x^2/(1-x^{-1})}$ can be rewritten as an infinite sum

$$e^{x^2/(1-x^{-1})} = e e^{x^2+x} + e x^{-1} e^{x^2+x} + \frac{3e}{2} x^{-2} e^{x^2+x} + \cdots$$

The transseries

$$e^{e^x/(1-x^{-1})} = e^{e^x + x^{-1} e^x + x^{-2} e^x + \cdots}$$

can not be rewritten in a similar way, because $e^x/(1-x^{-1})$ belongs to $\mathbb{T}^\uparrow$. In fact, $e^{e^x/(1-x^{-1})}$ is a transmonomial of exponential depth 2. The above transseries are all **finite transseries**, i.e. they all belong to the smallest exp-log subfield of $\mathbb{R}[[[x]]]$ which contains x and $\mathbb{R}$.

Example 1.4. The transseries

$$1 + (\log x) e^{-x} + 2!(\log^2 x) e^{-2x} + 3!(\log^3 x) e^{-3x} + \cdots$$

has logarithmic and exponential depths 1.

Example 1.5. The supports of

$$\begin{aligned} f_1 &= x^{-1} + x^{-\pi} + x^{-\pi^2} + x^{-\pi^3} + \cdots \\ f_2 &= x^{-1} + e^{-x} + e^{-e^x} + \cdots \\ f_3 &= 1 + xe^{-x} + x^{2!}e^{-2x} + x^{3!}e^{-3x} + \cdots \end{aligned}$$

are well-ordered, but not grid-based. Hence f_1, f_2 and f_3 are not transseries in the sense of this section, although f_1 and f_2 satisfy simple difference equations. In chapter 2.2, we will develop the theory of transseries with well-ordered supports. Finally,

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

is not a transseries at all, because its support is not well-founded (since $x \to \infty$).

1.6 Normal bases and the structure theorem

In this section $\mathbb{T}$ denotes the field of grid-based transseries in x over C. We state without proof the following easy characterizations of the asymptotic relations $\sim$, $\prec\!\!\prec$, $\preceq\!\!\!\prec$, $\asymp$, $\prec\!\!\prec\!\!\prec$ and $\asymp\!\!\!\!\asymp$:

1. $f \sim g \Leftrightarrow (\log|f|)^{\mapsup} = (\log|g|)^{\mapsup} \ \wedge \ |fg| = fg$.
2. $f \prec\!\!\prec g \Leftrightarrow (\log|f|)^{\uparrow} < (\log|g|)^{\uparrow}$.
3. $f \preceq\!\!\!\prec g \Leftrightarrow (\log|f|)^{\uparrow} \leqslant (\log|g|)^{\uparrow}$.
4. $f \asymp g \Leftrightarrow (\log|f|)^{\uparrow} = (\log|g|)^{\uparrow}$.
5. $f \prec\!\!\prec\!\!\prec g \Leftrightarrow \log|f| \prec\!\!\prec \log|g|$.
6. $f \asymp\!\!\!\!\asymp g \Leftrightarrow \log|f| \asymp \log|g|$.

In section 1.4 we discussed asymptotic scales. For computational purposes special types of asymptotic scales are particularly important when dealing with transseries. In fact, such asymptotic scales are given by asymptotic bases which satisfies some additional conditions. In section 1.4 we already met an application of this concept (see formula (1.8)), and in part B of this thesis normal bases will be of a crucial importance.

A linearly ordered set $B = \{\text{б}_1, \cdots, \text{б}_n\}$ of positive infinitesimal transseries is called a **normal basis**, if the following conditions are satisfied:

NB1. $\text{б}_1 \prec\!\!\prec\!\!\prec \cdots \prec\!\!\prec\!\!\prec \text{б}_n$.
NB2. $n > 0$ and $\text{б}_1 = \exp_l^{-1} x$ for some $l \in \mathbb{Z}$.
NB3. For all $i > 1$ we have $\log \text{б}_i \in C[\![\text{б}_1; \cdots; \text{б}_{i*}]\!]$, where $\log \text{б}_i \asymp\!\!\!\!\asymp \text{б}_{i*}$.

The integer l in condition **NB2** is called the **level** of the normal basis B. Given a normal basis B, we say that a transseries f can be expanded w.r.t. B if $f \in C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!]$. Equivalently, we say that B is a normal basis for f.

Example 1.6. The sets

$$B_1 = \{x^{-1}, e^{-x}, e^{-x^2}, e^{-x^3}\}$$

and

$$B_2 = \{\log^{-1} x, x^{-1}, e^{-\log^2 x}, e^{-x}, e^{-e^x/(1+x^{-1})}\}$$

are both normal bases. The set $B_3 = \{x^{-1}, e^{-x+e^{-x}}\}$ is not.

Remark 1.4. There are some slightly different alternatives for the definition of normal bases. Notably, the condition **NB2** can very well be omitted: we only use it to "standardize" our expansions with respect to the privileged transseries or "coordinate function" x. In higher dimensions, the condition **NB2** does not admit an analogue, unless we have a privileged system of coordinates (see chapter 6).

It is also possible to replace **NB3** by the slightly weaker condition that $\log \mathfrak{b}_i \in C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_{i-1}]\!]$ for all $i > 1$. Although this leads to less "canonical" expansions, for all our applications this weaker condition would also be sufficient. Inversely, we can make the extra requirement that $-\log \mathfrak{b}_i$ is a transmonomial for each i. Normal bases with this property are called **canonical bases**. Canonical bases do not admit higher dimensional analogues. This is due to the fact that the "coordinate function" x is heavily involved in the definition of transmonomials; actually, we should rather speak of transmonomials in x.

Having fixed a normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$, we will usually denote elements f of $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!]$ by series $f = \sum_{\alpha_1,\cdots,\alpha_n} f_{\alpha_n,\cdots,\alpha_1} \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$. Alternatively, we use vector notation instead and write $f = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \boldsymbol{\mathfrak{b}}^{\boldsymbol{\alpha}}$. Sometimes it is also useful to see f as a series in $\mathfrak{b}_n$ with coefficients in $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_{n-1}]\!]$, and we write $f = \sum_{\alpha_n} f_{\alpha_n} \mathfrak{b}_n^{\alpha_n}$. This last representation makes it often possible to solve problems by induction over n, while using **NB3**. Other interesting properties of normal bases are that they are stable under upward and downward movements (see section 1.7.2), and that $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_i]\!]$ is stable under differentiation (see section 1.7.1) for each i, if B has level zero or one. We have the following structure theorem:

Theorem 1.1. **(Structure theorem)** *Let f be a transseries and let B_0 be a normal basis. Then there exists a normal basis B for f which contains B_0.*

Proof. Let l be the level of B_0. We can write $f \in C^{alog}[\![\![\exp_{l'} x]\!]\!]$ for some $l' \in \mathbb{Z}$. If $l' < l$, then we insert $\exp_{l'}^{-1} x, \cdots, \exp_{l-1}^{-1} x$ into B_0. Therefore, we may assume without loss of generality that $l' = l$; hence, we can write $f \in C_k^{alog}[\![\![\exp_l x]\!]\!]$ for some k. We now prove the theorem by induction over k: if $k = 0$, then we have nothing to prove. Assume therefore that we proved the theorem up to $k \geqslant 0$.

Since the support of f is grid-based, we can write $\operatorname{supp} f \subseteq e^{\varphi_1 \mathbb{Z}} \cdots e^{\varphi_q \mathbb{Z}}$, for certain strictly negative $\varphi_1, \cdots, \varphi_q \in C_k [\![x]\!]^{\uparrow}$. By applying the induction hypothesis for $\varphi_1, \cdots, \varphi_q$, there exists a normal basis $B' = \{\mathfrak{b}'_1, \cdots, \mathfrak{b}'_{n'}\}$ for $\varphi_1, \cdots, \varphi_q$, which contains B_0. Now consider the following theoretical algorithm:

Algorithm $\mathtt{add}(g, B)$

INPUT: A negative infinitely large transseries g and a normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$, such that g can be expanded w.r.t. B.

OUTPUT: A normal overbasis of B for e^g.

STEP 1. **if** g is bounded, **then return** B.

STEP 2. **if** there exists $e^{\psi} \in B \backslash \{\mathfrak{b}_1\}$ such that $\psi \asymp g$,
then set $g' := g - \alpha\psi$, where $\alpha = \lim g/\psi$, and **return** $\mathtt{add}(g', B)$.

STEP 3. **otherwise** , let i^* be such that $g \asymp\!\!\!\asymp \mathfrak{b}_{i^*}$.
Set $g^+ := g_{0, \overset{n-i^* \text{ times}}{\cdots}, 0}$ and $g^- := g - g^+$.
return $B \cup e^{-|g^+|}$.

To prove the termination of $\mathtt{add}$, it suffices to observe that no infinite loops can arise from step 2, since ψ gets smaller and smaller for $\prec\!\!\!\prec$ during such a loop, while B remains fixed. Let us now prove the correctness of $\mathtt{add}$. The computation of the decompositions $g = g^+ + g^-$ guarantees that $B \cup e^{-|g^+|}$ is a normal basis at the end of the algorithm. Since g^- is necessarily bounded in such decompositions, $e^g = e^{g^+} e^{g^-}$ can indeed be expanded w.r.t. this normal basis. Whenever $g = \alpha\psi + g'$ in step 2, the same thing holds by induction.

Now we apply the algorithm for the φ_i, by executing $B := \mathtt{add}(g, B)$ for each $g \in \{\varphi_1, \cdots, \varphi_q\}$, starting with $B := B'$. This yields a normal overbasis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ for $\{\exp \varphi_1, \cdots, \exp \varphi_q\}$. We claim that f can be expanded w.r.t. B. Indeed, $e^{\varphi_1}, \cdots, e^{\varphi_q}$ are infinitesimal elements in $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!]$. Using the results from section 1.4, we therefore have a natural embedding of $C[\![e^{\varphi_1}, \cdots, e^{\varphi_q}]\!]$ into $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!]$. □

Remark 1.5. Intuitively speaking, g^+ and g^- correspond to the parts in the exponential which we do not resp. do expand in step 3 of $\mathtt{add}$. If we decompose $g = g^{\uparrow} + g^{\downarrow}$ instead of $g = g^+ + g^-$, then we obtain the analogue of the theorem where "normal basis" is replaced by "canonical basis".

1.7 Common operations on transseries

In this section $\mathbb{T}$ denotes the field of grid-based transseries in x over C. We show that we can define a natural derivation, composition and functional inversion on $\mathbb{T}$.

1.7.1 Differentiation and composition

To define the derivation, we proceed along the same lines as in the first construction of $\mathbb{T}$. For $f \in C[[E_0]]$, we define the derivative of f w.r.t. x (over C) by

$$f' = \left(\sum_{\alpha \in C} f_\alpha x^{-\alpha}\right)' = \frac{-1}{x} \sum_{\alpha \in C} \alpha f_\alpha x^{-\alpha}.$$

Assuming that we defined the derivation w.r.t. x on $C[[E_k]]$, we define it on $C[[E_{k+1}]]$ by

$$f' = \left(\sum_{ц \in x^{-C} \exp(C[[E_k]]^\uparrow)} f_ц ц\right)' = \sum_{ц \in x^{-C} \exp(C[[E_k]]^\uparrow)} f_ц (\log ц)' ц.$$

If the support of f is contained in $\Gamma = ц_1^{\mathbb{N}-p} \cdots ц_n^{\mathbb{N}-p}$, then the support of f' is contained in $(\operatorname{supp} \log ц_1 \cup \cdots \cup \operatorname{supp} \log ц_n)\Gamma$. Finally, we define $(f \circ \log x)' = (f' \circ \log x)/x$, for $f \in C^{alog}[[[\log_k x]]]$. The so defined mapping $\mathbb{T} \to \mathbb{T}; f \mapsto f'$ is a derivation over C, as is readily verified (see the next chapter for an explicit proof in a more general context).

The composition of $f \in \mathbb{T}$ with a positive infinitely large transseries $g \in \mathbb{T}^+_\infty$ is defined in a similar way. If $f \in C[[E_0]]$, we consider the natural embedding $C[[(1/g)^C]] \xrightarrow{\varphi} \mathbb{T}$ (see proposition 1.3) and the natural isomorphism $C[[(1/x)^C]] \xrightarrow{\psi} C[[(1/g)^C]]$, which corresponds to the substitution of x by g. Then we define $f \circ g = \varphi(\psi(f))$. Assume now that we defined $f \circ g$, for $f \in C[[E_k]]$. Then we define $(\exp h) \circ g = \exp(h \circ g)$, for $h \in C[[E_k]]^\uparrow$. Using induction over k, we observe that these $(\exp h) \circ g$ form an asymptotic scale S. Hence we have a natural embedding of $C[[S]]$ into $\mathbb{T}$. We also have a natural mapping from $C[[E_k]]$ into $C[[S]]$, obtained by substituting $ц \in E_{k+1}$ with $ц \circ g$. The composition of these two mappings determines the right composition by g on $C[[E_{k+1}]]$. Finally, we define $(f \circ \log) \circ g = f \circ (\log \circ g)$.

Let us mention some properties of $\circ$. First of all, $\circ_g = \cdot \circ g$ is a difference operator for any $g \in \mathbb{T}^+_\infty$. This follows directly from the fact that $\circ_g$ is defined as an inductive limit of ring homomorphisms. It is also readily verified that $\circ$ is associative, by using a double induction (see the next chapter for an explicit proof in a more general context). The above two properties are summarized by saying that $\circ$ is a **composition**. We also have $(f \circ g)' = (f' \circ g)g'$, and we say that $\circ$ is **compatible** with the derivation. We finally have $f \circ (g+h) = f \circ g + (f' \circ g)h + \cdots$, whenever both sides of the equation are well defined.

1.7.2 Upward and downward shiftings

It is interesting to study the action of the group of iterated logarithms and exponentials by left or/and right compositions on $\mathbb{T}$. Right compositions by exp resp. log are also referred to by **upward shiftings** (resp. **downward shiftings**) or **upward movements** (resp. **downward movements**). The upward (resp. downward) shift of $f \in \mathbb{T}$ is denoted by $f\uparrow$ (resp. $f\downarrow$). We observe that $\uparrow$ and $\downarrow$ are scale changes which preserve the set of transmonomials. In chapters 4 and 5, we will see that upward shiftings are particularly useful for differential calculus, where they allow to make transseries alogarithmic.

Other interesting related operations are the **dilatation** $\mathbb{T}_\infty^+ \xrightarrow{\Delta} \mathbb{T}_\infty^+; f \mapsto \exp \circ f \circ \log$ and the **contraction** $\mathbb{T}_\infty^+ \xrightarrow{\Delta^{-1}} \mathbb{T}_\infty^+; f \mapsto \log \circ f \circ \exp$. We claim that after a suitable number of contractions, any transseries $f > 0$ can be represented by $\Delta^{-k} f = \exp_l x + \varepsilon$, with $l \in \mathbb{Z}$ and $\varepsilon \prec\!\!\!\prec 1$. Indeed, it suffices to take the exponential depth plus one for k. The integer $l = \operatorname{expo} l$ is called the **exponentiality** of f. Hence, by using only left and right composition with exp and log we can recover $\mathbb{T}_\infty^+$ from the transseries of the form $x + \varepsilon$, where f is infinitesimal.

Let f and g be in $\mathbb{T}_\infty^+$. If $f^\uparrow < g^\uparrow$, then $\exp f \prec\!\!\!\prec \exp g$, and in particular $(\exp f)^\uparrow < (\exp g)^\uparrow$. Assume that the exponentiality of f is strictly smaller than g's. Then we have $(\log_k x \circ f \circ \exp_k x)^\uparrow < (\log_k x \circ g \circ \exp_k x)^\uparrow$, for k sufficiently large. Hence, $(f^\uparrow \circ \exp_k x)^\uparrow < (g^\uparrow \circ \exp_k x)^\uparrow$, by what precedes. Consequently, $f^\uparrow < g^\uparrow$, since downward shifting is a scale change of $\mathbb{T}$ which preserves the set of transmonomials. Similarly, we have $f \prec\!\!\!\prec g$ and $f \prec\!\!\!\prec\!\!\!\prec g$, using the characterizations of $\prec\!\!\!\prec$ and $\prec\!\!\!\prec\!\!\!\prec$ from the beginning of the previous section. In particular, we have $\mathfrak{m}_f \succ\!\!\!\succ \mathfrak{m}_{\log f}$, for any $f \in \mathbb{T}_\infty^+$.

1.7.3 Functional inversion

Proposition 1.4. *Any positive infinitely large transseries $g \in \mathbb{T}_\infty^+$ admits a functional* **inverse** *$g^{inv} \in \mathbb{T}_\infty^+$ for $\circ$.*

Proof. Modulo some left and right compositions with exp and log, it suffices to show this in the case when $g = x(1+\varepsilon)$ with $\varepsilon \prec\!\!\!\prec 1$ and g can be expanded w.r.t. a normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ of level zero. If $n = 1$, then we claim that g admits an inverse f in $C[[x^{-1}]]$. Indeed, its straightforward to check that the equation

$$x = \sum_{\alpha \in C} \frac{f_\alpha}{x^\alpha} \left(1 + \alpha\varepsilon + \binom{\alpha}{2}\varepsilon^2 + \cdots\right).$$

admits a solution f with $\operatorname{supp} f \subseteq x(\operatorname{supp} \varepsilon)^\diamond$, by using well-ordered induction.

The general case is proved by induction over n. Taking $n > 1$, our induction hypotheses are the following:

(a) There exists a functional inverse f_0 to g_0, when expanding g w.r.t. $\mathfrak{b}_n$.

(b) Denoting $\tilde{\mathfrak{b}}_1 = x, \cdots, \tilde{\mathfrak{b}}_{n-1} = \mathfrak{b}_{n-1} \circ f_{0,0}$, we have $f_0 \in C[[\tilde{\mathfrak{b}}_1; \cdots; \tilde{\mathfrak{b}}_{n-1}]]$.

(c) $C[[\tilde{\mathfrak{b}}_1; \cdots; \tilde{\mathfrak{b}}_{n-1}]]$ is stable under differentiation.

Writing $\tilde{\mathfrak{6}}_n = \mathfrak{6}_n \circ f_0$, we have for each $1 \leqslant i \leqslant n$:

$$\tilde{\mathfrak{6}}_i \circ g_0 = \mathfrak{6}_i + \mathfrak{6}_i'\xi + \frac{1}{2}\mathfrak{6}_i''\xi^2 + \cdots,$$

where $\xi = g_0 - x$ if $i = 0$, and otherwise $\xi = g_0 - g_{0,\overset{n+1-i \text{ times}}{\cdots},0}$. Hence, right composition with g_0 maps $C[[\tilde{\mathfrak{6}}_1; \cdots; \tilde{\mathfrak{6}}_n]]$ into $C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_n]]$. Similarly, denoting $\delta = g - g_0$ and $\eta = f_0 \circ (g_0 + \delta) - x$, we have $\eta \in C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_n]]$.

Now let $h = f \circ g_0$. We have $(f \circ g_0) \circ f_0 \circ (g_0 + \delta) = x$. Hence, we can write

$$h \circ (x + \eta) = \sum_{\alpha \in C} h_\alpha \mathfrak{6}_n^\alpha + (h_\alpha \mathfrak{6}_n^\alpha)'\eta + \cdots = x.$$

Again by well-ordered induction, this equation admits a solution h whose support in $\mathfrak{6}_n$ satisfies $\operatorname{supp} h \subseteq (\operatorname{supp} \eta)^\diamond$. In other words, $h \in C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_{n-1}]][[\mathfrak{6}_n]]$. To prove that h is in $C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_n]]$, we need to consider vector supports.

We first observe that $\mathbf{supp}\, \varphi' \subseteq \mathbf{supp}\, \varphi + \Delta$, for any φ in $C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_n]]$, where $\Delta = \mathbf{supp} \log \mathfrak{6}_1 \cup \cdots \cup \mathbf{supp} \log \mathfrak{6}_n \subseteq C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_{n-1}]]$. Again using well-ordered induction it can now be checked that

$$\mathbf{supp}\, h \subseteq \mathbf{supp}\, x + (\mathbf{supp}\, \eta + \Delta)^\diamond.$$

It follows that $f = h \circ f_0 \in C[[\tilde{\mathfrak{6}}_1; \cdots; \tilde{\mathfrak{6}}_n]]$. Finally, $C[[\mathfrak{6}_1; \cdots; \mathfrak{6}_n]]$ is stable under differentiation, because $\{\mathfrak{6}_1, \cdots, \mathfrak{6}_n\}$ is a normal basis of level 0. Hence, $C[[\tilde{\mathfrak{6}}_1; \cdots; \tilde{\mathfrak{6}}_n]]$ is stable under differentiation, since $\tilde{\mathfrak{6}}_n{}' = (\mathfrak{6}_n' \circ f_0) f_0'$. This completes the induction. □

Remark 1.6. It may be noticed that the "easier proof" of the above proposition in [Éc 92] (page 139) fails in general, because of the counterexample $g = x + 1/x + e^{-e^x}$.

1.7.4 On a conjecture of Hardy

To illustrate the formalism of transseries at work, we claim that the innocently looking property 1 from section 1.6 forms the key in the solution to a conjecture of Hardy (see [Har 11]), which states that the functional inverse of $\log x \log_2 x$ is not asymptotic to any L-function for $x \to \infty$. In other words, the conjecture states that there is no function, built up from $\mathbb{R}$ and x using the exp-log field operations and left composition with real algebraic functions, which is defined and asymptotic to the functional inverse of $\log x \log_2 x$ in a neighbourhood of infinity. Shackell proved in [Sh 93c] the slightly weaker assertion that the functional inverse of $\log_2 x \log_3 x$ is not equivalent to any L-function.

As our proof uses the theory of transseries, we will take for granted that the analytic properties of transseries coincide with their algebraic counterparts; this is indeed the case, because all the transseries we consider are convergent (see theorem 11.6). We will also use the following facts:

(1) $(x \log x)^{inv}$ is not an L-transseries.
(2) The unbounded part $f^{\uparrow}$ of an L-transseries is again an L-transseries.

Of course, an **L-transseries** is the transseries counterpart of an L-function. The first fact was already proved by Liouville in [Li 1838]. For the second fact, see proposition 9.10.

Theorem 1.2. *The functional inverse of* $\log x \log_2 x$ *is not asymptotic to an L-transseries.*

Proof. To prove our claim, we first observe that $\Delta^{-2} x \log x = x + \log(1 + x/e^x)$. By what precedes, $x + \log(1 + x/e^x)$ admits a functional inverse

$$f = x - \frac{x}{e^x} - \frac{x^2}{2e^{2x}} + \frac{x}{e^{2x}} \cdots \in \mathbb{R}[[x^{-1}; e^{-x}]].$$

Now

$$e^f = e^x \left(1 - \frac{x}{e^x} + \frac{x}{e^{2x}} + \cdots\right) \in \mathbb{R}[[x^{-1}; e^{-x}]]$$

and $e^{e^f} \in e^{e^x - x} \mathbb{R}[[x^{-1}; e^{-x}]]$. In particular, we have $(e^{e^f})^{\uparrow} = e^{e^f}$. Since

$$f = \left(\log \log e^x e^{e^x}\right)^{inv} = \left(e^x e^{e^x}\right)^{inv} \circ \exp \circ \exp$$

and since $(h \circ \exp)^{\uparrow} = h^{\uparrow} \circ \exp$ for all transseries h, we therefore obtain

$$(\log(\log x \log_2 x)^{inv})^{\uparrow} = (x \log x)^{inv}.$$

Hence, if $(\log x \log_2 x)^{inv}$ were equivalent to an L-transseries g, then $(x \log x)^{inv}$ would be equal to the unbounded part $(\log g)^{\uparrow}$ of $\log g$, which is an L-transseries by (2). This is impossible by (1). □

Actually, we have a more general theorem:

Theorem 1.3. *Let f be a positive infinitely large transseries, which is not an L-transseries. Let k resp. r be minimal, such that $f \in C_r^{alog}[[\log_k x]]$. Let $l = \operatorname{expo} f - k$ be the exponentiality of f minus k. Then for all $n \geqslant r - l + 2$, the n-th iterated dilatation $\Delta^n f$ of f is not asymptotic to an L-transseries.*

Proof. It suffices to prove that $(\Delta^{n-1} f)^{\uparrow} = \Delta^{n-1} f$. We prove this by induction over $r - l$. If $r = l$, then any transseries in $C_r^{alog}[[\log_k x]]$ is negligible with respect to e^f. In particular,

$$\Delta f = e^{f^{\uparrow} \circ \log} e^{f^{\downarrow} \circ \log} \in C_{r+1}^{alog}[[\log_k x]]^{\uparrow}.$$

If $r > l$, then $\operatorname{expo} \Delta f = \operatorname{expo} f$, $\Delta f \in C_r^{alog}[[\log_{k+1} x]]$ and the result follows by induction. □

Notice again that theorem 1.3 is purely algebraic; resummation techniques are generally needed to let the algebraic assertion coincide with its analytic counterpart. For instance, using these techniques, it can be shown that $e^{I(x)}$ is not asymptotic to any L-function, where $I'(x) = e^{x^2}$.

Notice also that the bound $n \geqslant r - l - 2$ in the theorem is sharp.

Remark 1.7. We have learned recently, that theorem 1.2 was proved independently by Macintyre, Marker and van den Dries in [MMV *]. They use a similar technique and the analogue of proposition 9.10 used in their proof has been established by Ressayre in [Res 93].

1.8 References

[Ber 1697] J. Bernoulli. Principia calculi exponentialium seu percurrentium. *Acta eruditorum (p. 125-133).*

[Bos 96] H.J.M. Bos. Johann Bernoulli on exponential curves ca. 1695. Innovation and habituation in the transition from explicit constructions to implicit functions. *Nieuw archief voor wiskunde serie 4, 14(1) (p. 1-19).*

[Bour 61] N. Bourbaki. *Éléments de mathématiques chap. V: Fonctions d'une variable réelle.* Hermann, 2-nd ed. (Appendice p 36-55).

[Br 58] N.G. de Bruijn. *Asymptotic methods in analysis.* North-Holland Publ. Co. Amsterdam.

[Dahn 84] B.I. Dahn. The limiting behaviour of exponential terms. *Fund. Math. 124 (p. 169-186).*

[DG 86] B.I. Dahn, P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae 127 (p. 45-50).*

[Ec 92] J. Ecalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Hermann, collection: Actualités mathématiques.

[Har 10] G.H. Hardy. *Orders of infinity.* Cambridge Univ. Press.

[Har 11] G.H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London mathematical society 10,2 (p. 54-90).*

[Li 1838] J. Liouville. Suite du mémoire sur la classification des transcendantes, et sur les racines de certaines équations en fonction finie explicite des coefficients. *J. Math. Pures et Appl. 3 (p. 523-546).*

[LoRo 89] D.E. Loeb, G.-C. Rota. Formal power series of logarithmic type. *Advances in mathematics 75(1), Academic Press.*

[MMV *] A. Macintyre, D. Marker, L. van den Dries. The elementary theory of restricted analytic fields with exponentiation. *To appear.*

[Res 93] J.P. Ressayre. Integer parts of real closed exponential fields. *Arithmetic, Proof Theory and Computational Complexity, P. Clote and J. Krajicek eds., Oxford Univ. Press.*

[Ros 83] M. Rosenlicht. Hardy fields. *Journal of Math. Analysis and Appl. 93 (p. 297-311).*

[Sh 93c] J. Shackell. Inverses of Hardy L-functions. *Bull. London Math. Soc. 25 (p. 150-156).*

[VdH 94b] J. van der Hoeven. General algorithms in asymptotics I, Gonnet and Gruntz's algorithm. *Research report LIX/RR/94/10, École polytechnique, France.*

Chapter 2

Well-ordered transseries

2.1 Introduction

In this chapter we introduce the concept of well-ordered transseries, thereby generalizing the grid-based transseries from chapter 1. Transseries of this type are naturally encountered as solutions to functional equations like

$$f(x) = \frac{1}{x} + f(x^2) + f(e^{\log^2 x}),$$

or as explicitly given expressions like

$$g(x) = \sum_{n=1}^{\infty} e^{-x^n}.$$

The price to be paid is that the classical operations, i.e. derivation, composition and inversion, are more technical to define. In fact, most of this chapter is devoted to a correct definition of well-ordered transseries and these classical operations.

In section 2.2, we introduce power series with Noetherian supports. In section 2.2 we define well-ordered transseries . However, we will see that there does not exist a totally ordered exp-log field of transseries, which is stable under infinite summation in the most general sense. However, we shall see that any exp-log field of transseries is included in a larger field of transseries which is stable under infinite summation, and vice versa.

In section 2.3, we prove a fixed point theorem for transseries.

In sections 2.4 and 2.5 we define the classical operations on transseries, namely differentiation, composition and inversion.

In section 2.6 we describe the compactification of the "transline" and establish a characterization for transseries intervals.

Finally, we will show in section 2.7 that some strongly monotonic solutions to very general systems of functional equations can not be represented by transseries in the sense of section 2.2 and we discuss further extensions of the concept of a transseries.

2.2 Well-ordered transseries

2.2.1 Algebras of Noetherian series

The definition of a well-ordered transseries is similar to the definition of a grid-based transseries, with this exception that the analogue of the identity $\mathbb{T} = C[\![Ц]\!]$ does not hold anymore. Indeed, we essentially needed the built-in finiteness condition in the definition of grid-based sets to prove this identity. As a consequence, larger and larger fields of transseries can be defined by transfinite induction. Apart from this major difference, the construction follows the same lines as in sections 1.3 and 1.5. We will therefore content ourselves to indicate the changes, and we leave the details to the reader. A review of Noetherian orderings is available in appendix A.

As in section 1.3, we first assume that C is a ring of constants, and X a commutative ordered semigroup. We denote by $C[[X^{\leqslant}]]$ or by $C[[X]]$ the set of mappings from X to C with Noetherian support, and we call it the set of **Noetherian series** in C over X. Such mappings are also denoted by sums $\sum_{ц \in X} \varphi_ц ц$. More generally, we say that a family $(f_i)_{i \in I}$ of elements in $C[[X]]$ is **Noetherian**, if $\bigcup_{i \in I} \operatorname{supp} f_i$ is Noetherian, and if $\{i \in I | ц \in \operatorname{supp} f_i\}$ is finite for each $ц \in X$. Given such a family, we define $\sum_{i \in I} f_i$ by (1.3).

As in section 1.3, we give $C[[X]]$ the structure of a C-algebra (instead of proposition 1.1, we use proposition A.2(c), proposition A.2(d). If X is totally ordered, then $C[[X]]$ is even a field, by using A.3 to Higman's theorem; in this case, elements of $C[[X]]$ are called **well-ordered series**. The asymptotic relations $\prec$, $\preceq$, $\asymp$, $\sim$, $\approx$ and $\lll$ are also introduced in a similar way as before.

Remark 2.1. It was first observed by Higman (see [Hig 52]) that $C[[X]]$ is a ring, and even a field if X is a totally ordered group and C a field. In fact, X does not need to be commutative, but for our purposes we can restrict ourselves to commutative semigroups.

Example 2.1. By analogy with example 1.1, we can consider the (product ordering, resp. lexicographical ordering) variants $C[[z_1^A, \cdots, z_n^A]]$ resp. $C[[z_1^A; \cdots; z_n^A]]$ of $C[\![z_1, \cdots, z_n]\!]$ and $C[\![z_1; \cdots; z_n]\!]$, for a fixed ordered commutative group A. This time, there exists an isomorphism between $C[[z_1^A; \cdots; z_n^A]]$ and $C[[z_1^A]] \cdots [[z_n^A]]$.

The definition of **asymptotic scales** needs some special care in our present context. This is because the proof of proposition 1.3 essentially used the finiteness condition in the definition of grid-based sets. However, if we add the condition that for each well-ordered subset W of S, the family $\{щ\}_{щ \in W}$ is Noetherian, then the analogue of proposition 1.3 does hold. Asymptotic scales which satisfy this additional condition are said to be of **well-ordered type**. Asymptotic scales in the former sense are said to be of **grid-based type**.

Let us finally remark that the ring $C[[X]]$ has been studied from an algebraic point of view by Ribenboim and others (e.g. see [Rib]). For instance, $C[[X]]$ is an entire ring, if and only C is entire and X is cancellative and torsion free. He also gives sufficient conditions for $C[[X]]$ to be a Noetherian ring. We believe that his results are also valid for $C[\![X]\!]$, but we have not checked it.

2.2.2 Well-ordered transseries

Assume now that we are given a totally ordered exp-log constant field C of characteristic zero, and a totally ordered monomial group X. The analogues of the two constructions from section 1.5 can again be carried out, but in both cases, the resulting field $\mathbb{T}$ is not complete ($\mathbb{T}$ is a **complete field of transseries**, if $\mathbb{T} = C[[\text{Ц}]]$, where Ц is the set of transmonomials in $\mathbb{T}$; complete fields are the only ones which are stable under well-ordered infinite summation). Moreover, both constructions are not equivalent anymore. Indeed, $f = x + \log x + \log_2 x + \cdots$ is not an element of $\mathbb{T}$, if we apply the first construction. However, $f \in \mathbb{T}$ if we apply the second construction, since $f \in C[[\log_{C^*} x]]$.

In fact, as we shall see below, it is not possible to construct a complete totally ordered exp-log field of transseries. However, we will show that how to construct fields which are closed under exponentiation, by analogy with the second construction from section 1.5. Then we obtain larger and larger fields of transseries, by alternating closure under exponentiation and closure under infinite summation. The transfinite induction we use for this generalizes the transfinite induction used by Conway to construct non standard numbers (see [Con 76]).

Closure under exponentiation and logarithm. Let X be a totally ordered monomial group. Assume that $C[[X]]$ is a totally ordered partial exp-log field. We say that $C[[X]]$ is **admissible**, if the logarithm is defined for all strictly positive elements, and if we have $\exp f = \exp f^{\uparrow} \exp f^{c} \exp f^{\downarrow}$, for all $f \in C[[X]]$ such that $\exp f$ or $\exp f^{\uparrow}$ is defined. Assuming that this is the case, let us show how to extend $C[[X]]$ into a totally ordered exp-log field.

The construction proceeds by induction, starting with $X_0 = X$. Next, let $k \geqslant 0$ be given and assume that we have given $C[[X_k]]$ the structure of an admissible totally ordered partial exp-log field. Then we define $X_{k+1} = \exp(C[[X_k]]^{\uparrow})$, and we embed X_k into X_{k+1} by sending ц to $\exp \log$ ц. Finally, we consider the inductive limit of the sequence $C[[X_0]] \to C[[X_1]] \to \cdots$, which satisfies our requirements.

Transfinite extensions. In general, the inductive limit of $C[\![X_0]\!] \to C[\![X_1]\!] \to \cdots$ is strictly included in $C[[X_\omega]]$, where X_ω is the inductive limit of $X_0 \to X_1 \to \cdots$ (we recall that ω stands for the smallest infinite ordinal number). Moreover, $C[[X_\omega]]$ is also an admissible totally ordered partial exp-log field. We can therefore repeat our construction with X_ω instead of X.

By using transfinite induction, we can even go much further. Indeed, we define $X_{\alpha+1} = \exp(C[[X_\alpha]]^{\uparrow})$, for any ordinal α. For limit ordinals, X_α is by definition the inductive limit of all X_β with $\beta < \alpha$. The corresponding inductive limit of the $C[[X_\beta]]$ has the structure of a totally ordered exp-log field.

Well-ordered transseries Now take the totally ordered group $\log_{C^*} x$ of logarithmic monomials in x for X in the previous construction. Then we denote $C[[X_\alpha]]$ by $C^\omega_\alpha[[[x]]]$, and call it the field of **well-ordered transseries of exponential depth** $\leqslant \alpha$. The **exponential depth** of a transseries f is the smallest ordinal α with $f \in C^\omega_\alpha[[[x]]]$. For limit ordinals α, the inductive limit of all fields $C[[X_\beta]]$, with $\beta < \alpha$ is denoted by $C^\omega_{<\alpha}[[[x]]]$, and is the totally ordered exp-log field of **transseries of exponential depth** $< \alpha$.

Alogarithmic transseries We define the **logarithmic depth** of a transseries to be the highest iterated logarithm "occurring" in its expansion: mimicking the first construction of grid-based transseries, we build the hierarchy of fields of alogarithmic transseries $C^{alog}_\alpha[[[x]]]$ of exponential depth $\leqslant \alpha$, for each ordinal α, and the inductive limits $C_\alpha[[[x]]]$ of $C^{alog}_\alpha[[[x]]] \to C^{alog}_\alpha[[[\log x]]] \to \cdots$. Then the logarithmic depth of f is the smallest k, with $f \in C^{alog}_\alpha[[[\log_k x]]]$. We extend the notations $C^k_\alpha[[[x]]] = C^{alog}_\alpha[[[\log_k x]]]$, $C^{<\omega}_\alpha[[[x]]] = C_\alpha[[[x]]]$, etc. Hence, the analogue of the first construction from section 1.5 yields $C[[[x]]] = C^{<\omega}_{<\omega}[[[x]]]$, and the second one yields $C^\omega[[[x]]] = C^\omega_{<\omega}[[[x]]]$.

Example 2.2. The following is a transseries of exponential depth ω:

$$e^{-x} + e^{-e^x} + e^{-e^{e^x}} + \cdots$$

The transseries

$$\exp_n\left(\sqrt{x} + e^{\sqrt{\log x}} + \exp_2\sqrt{\log_2 x} + \cdots\right)$$

has exponential depth $\omega + n$. The transseries

$$x + \log x + \log_2 x + \cdots$$

has infinite logarithmic depth.

Some results from chapter 1 can be adapted in a straightforward way to our present context. In particular, the properties 1-5 from section 1.6 still hold. The paragraph in section 1.7.1 about upward and downward movements, contractions and dilatations is also easily transposed. In particular, we still have $\mathfrak{M}_f \ggcurly \mathfrak{M}_{\log f}$, for positive infinitely large transseries f.

2.2.3 Closure properties

In what follows, by a field of transseries over C, we shall always mean a field of transseries $\mathbb{T}$ of the form $C^\alpha_\beta[[[x]]], C_\beta[[[x]]], C^\alpha_{<\beta}[[[x]]]$ or $C_{<\beta}[[[x]]]$. Here $\alpha \leqslant \omega$ and β are ordinals, and β is a limit ordinal in the last two cases.

Proposition 2.1. *Let f be a transseries of finite logarithmic depth. Then f has at most countable exponential depth.*

Proof. It suffices to consider the case when f is alogarithmic. Let us first prove by induction that each alogarithmic transmonomial ц of exponential depth d satisfies $\text{ц} \succ\!\!\!\succ\!\!\!\succ \exp_{d-1} x$. This is clear for $d = 0$. Assume that we have proved the assertion up to d (not included). Let м be the dominant monomial of $\log \text{ц}$ and assume that $\text{м} \preccurlyeq\!\!\!\prec \exp_{d-2} x$. Then $1 \prec\!\!\!\prec \text{щ} \preccurlyeq\!\!\!\prec \exp_{d-2} x$ for each $\text{щ} \in \operatorname{supp} \log \text{ц}$. Hence the exponential depth of each $\text{щ} \in \operatorname{supp} \log \text{ц}$ is bounded by $d-2$, by the induction hypothesis. But this means that the logarithmic depth of ц is bounded by $d-1$. This contradiction shows that we can not have $\text{м} \preccurlyeq\!\!\!\prec \exp_{d-2} x$. Therefore, $\log \text{ц} \asymp \text{м} \succ\!\!\!\succ \exp_{d-2} x$, whence $\text{ц} \succ\!\!\!\succ\!\!\!\succ \exp_{d-1} x$.

Now let Ц be the set of alogarithmic transmonomials of finite exponential depths. We must show that $C[[\text{Ц}]]$ is stable under exponentiation. Let $f \in C[[\text{Ц}]]$. Since $e^f = e^{f^\uparrow} e^{f^c} e^{f^\downarrow}$, we must show that $f^\uparrow$ has finite exponential depth, and we may assume without loss of generality that $f^\uparrow \neq 0$. Now let d be the exponential depth of the dominant monomial м of $f^\uparrow$. Then we have $\text{ц} \prec\!\!\!\prec\!\!\!\prec \exp_{d+1} x$, for all monomials $\text{ц} \in \operatorname{supp} f^\uparrow$. By what precedes, this means that the exponential depth of $f^\uparrow$ is bounded by $d+2$, whence the exponential depth of $e^{f^\uparrow}$ is bounded by $d+3$. □

As a consequence of the above proposition, the fields $C^\alpha_{<\beta}[[[x]]]$ and $C^\alpha_\beta[[[x]]]$ reduce to $C^\alpha_\omega[[[x]]]$, if $\alpha < \omega$ and $\beta > \omega$. Similarly, the fields $C_{<\beta}[[[x]]]$ and $C_\beta[[[x]]]$ reduce to $C_\omega[[[x]]]$, if $\beta > \omega$. The next proposition shows that no other such collapses take place:

Proposition 2.2. *The field $C^\omega_\alpha[[[x]]]$ is strictly contained in $C^\omega_\beta[[[x]]]$, for $\alpha < \beta$.*

Proof. Consider the sequence $(f_\alpha)_\alpha$, with $f_\alpha \in C^\omega_\alpha[[[x]]]$, defined by transfinite induction: we take $f_0 = -x^2$. For each ordinal α, we take $f_{\alpha+1} = f_\alpha - e^{f_\alpha \circ \log}$. For limit ordinals α, we let $f_\alpha = \operatorname{stat\,lim}_{\beta \to \alpha} f_\beta$ (see page 67 for the definition of stationary limits). It is easily verified that whenever $\beta < \alpha$, then $f_\alpha \notin C^\omega_\beta[[[x]]]$. Actually, the ordering on ordinals is even reproduced by the sequence.

Now we have classified the fields of transseries, their respective closure properties are listed in table 2.1 below.

Remark 2.2. Although the table shows that there are no fields of transseries which are stable under logarithm, exponentiation and well-ordered summation, there *do* exist fields of transseries which are stable under logarithm, exponentiation and *countable* well-ordered summation. Indeed, for each totally ordered monomial group, we may consider the field $C[[X]]_\omega$ of countable well-ordered linear combinations of monomials in X. Note that if the cardinal number of $C \cup X$ is at least 2^ω, then the cardinal numbers of $C \cup X$ and $C[[X]]_\omega$ coincide. Consequently, if we perform

	Infinite summation	Exponentiation	Logarithm
$C_d^l[[[x]]]$	✓		
$C^l[[[x]]]$		✓	
$C_\omega^l[[[x]]]$	✓	✓	
$C_d[[[x]]]$			✓
$C[[[x]]]$		✓	✓
$C_\omega[[[x]]]$		✓	✓
$C_\alpha^\omega[[[x]]]$	✓		✓
$C_{<\alpha}^\omega[[[x]]]$		✓	✓

Table 2.1: Closure properties of the different types of fields of transseries.

the construction from the previous section with fields of the form $C[[X]]_\omega$ instead of $C[[X]]$, the corresponding transfinite sequence of fields $C_\alpha^\omega[[[x]]]_\omega$ is stationary, since the cardinal numbers of these fields are all the same. The limit of the sequence is stable under logarithm, exponentiation and countable summation.

2.2.4 Tree representations of transseries

Let us show that each transseries f over C in x has a natural **tree representation** T_f. If f is an iterated logarithm, then T_f is a leaf labeled by f. In the other case, we write a transseries f as a well-ordered sum of transmonomials $f = \sum_{i \in I} c_i ц_i$. Let $ц = e^{\varphi_i}$, for some purely unbounded φ_i. Now assume that each φ_i admits a tree representation T_{φ_i}. Let U_i be the tree obtained by substituting the root of T_i by $ц_i$. Attaching the U_i together to a new root, labeled by f, and in the order determined by I, we obtain the tree representation of f. For an example, see figure 2.1.

Proposition 2.3. *Each well-ordered transseries f over C admits a tree representation T_f, which is well-founded.*

Proof. Let $f = \sum_{i \in I} c_i ц_i$ be as above, with $ц_i = e^{\varphi_i}$ for each i. If f is a finite linear combination of iterated logarithms, then T_{φ_i} admits a tree representation for each i, by definition. Hence, f admits a tree representation of depth $\leqslant 2$.

If f has exponential depth zero, then φ_i is a finite linear combination of iterated logarithms for each i. Hence, T_{φ_i} admits a tree representation for each i, by what precedes. Hence, f admits a tree representation of depth $\leqslant 3$.

Now assume that f has exponential depth α and that we have proved the proposition for all strictly smaller exponential depths. Then T_{φ_i} admits a well-founded tree representation for each I. Hence f admits a well-founded tree representation T_f. Indeed, any infinite path in T_f would yield an infinite path in one of the T_{φ_i}, which is impossible. This completes the proof, by transfinite induction. $\square$

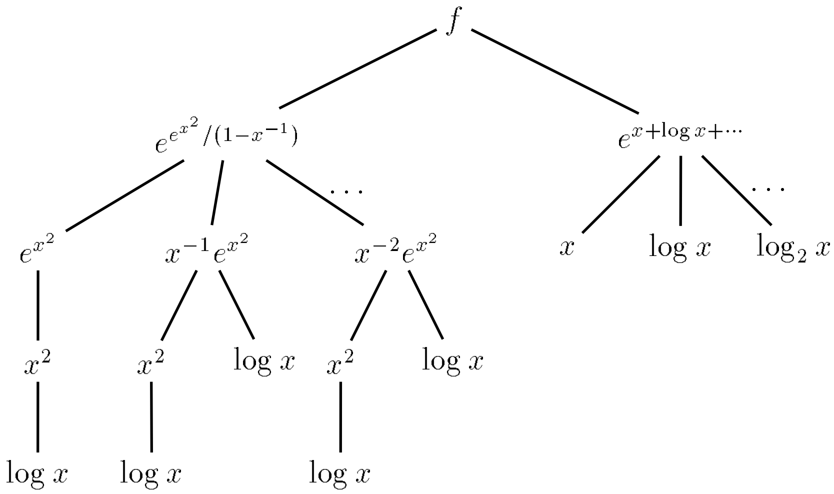


Figure 2.1: The tree representation of $f = e^{e^{x^2}/(1-1/x)} + e^{x+\log x+\cdots}$.

Tree representations provide good mental pictures for transseries, and will be useful for the combinatorial definitions of the derivative of a transseries and the composition of two transseries. Notice that the root plays a special rôle in a tree representation.

2.3 The fixed point theorem

2.3.1 Stationary limits

Let $\mathbb{T}$ be a field of transseries. We say that a transseries $f \in \mathbb{T}$ is a **truncation** of another transseries $g \in \mathbb{T}$, and we denote $f \trianglelefteq g$, if $f = g$ or $f = \sum_{\text{ц} < \text{ц}_0} g_{\text{ц}} \text{ц}$, for some transmonomial ц_0. This relation clearly determines an ordering on $\mathbb{T}$, which has the following properties:

(a) $0 \trianglelefteq f$;
(b) $f \trianglelefteq h \ \wedge \ g \trianglelefteq h \Rightarrow f \trianglelefteq g \ \vee \ g \trianglelefteq f$,

for all $f, g, h \in \mathbb{T}$.

Proposition 2.4. *Let $\mathbb{T}$ be a complete field of transseries. Let $(f_i)_{i \in I}$ be a family of transseries in $\mathbb{T}$, where I is totally ordered. Let T be the set of transseries f, such that for all $\text{ц} \in \operatorname{supp} f$ we have $f - f_i \prec\!\!\prec \text{ц}$ for all sufficiently large $i \in I$. Then T admits a unique maximal element for $\trianglelefteq$, which is called the* **stationary limit** *of $(f_i)_{i \in I}$ and denoted by $\operatorname{stat\,lim}_{i \in I} f_i$.*

Proof. Let $\tilde{f} \neq f$ be in T. The dominant monomial м of $\tilde{f} - f$ must be in $\operatorname{supp} f$, in $\operatorname{supp} \tilde{f}$, or in both. Assume that $\text{м} \in \operatorname{supp} f$. Then we have $f - f_i \prec\!\!\prec \text{м}$ for all sufficiently large i. If $\text{ц} \in \operatorname{supp} \tilde{f}$, then we also have $\tilde{f} - f_i = f - f_i + (\tilde{f} - f) \prec\!\!\prec \text{ц}$ for all sufficiently large i. Consequently, $\text{ц} \succ\!\!\succ \text{м}$, whence $\tilde{f} \lhd f$. Similarly, we find $f \lhd \tilde{f}$, if $\text{м} \in \operatorname{supp} \tilde{f}$. We infer that T is totally ordered for $\lhd$. In particular, a maximal element of T, if it exists, must be unique.

Now let $f, \tilde{f} \in T$ and let $\text{ц} \in \operatorname{supp} f, \tilde{\text{ц}} \in \operatorname{supp} \tilde{f}$ be monomials with $\text{ц} < \tilde{\text{ц}}$. If $f \trianglelefteq \tilde{f}$, then we clearly have $\text{ц} \in \operatorname{supp} \tilde{f}$. Otherwise, $\tilde{f} \lhd f$ and $f - \tilde{f} \prec\!\!\prec \tilde{\text{ц}}$, since $\tilde{\text{ц}} \in \operatorname{supp} \tilde{f}$. Hence, we again have $\text{ц} \in \operatorname{supp} \tilde{f}$. Now consider the set $S = \bigcup_{f \in T} \operatorname{supp} f$ is well-ordered. Let $\tilde{\text{ц}} \in \operatorname{supp} \tilde{f}$ for some $\tilde{f} \in T$. By what precedes, all $\{\text{ц} \in S | \text{ц} < \tilde{\text{ц}}\} \subset \operatorname{supp} \tilde{f}$. We deduce that S does not contain infinite decreasing sequences. Therefore, S is well-ordered.

Given $\text{ц} \in S$, there exists an $f \in T$ with $\text{ц} \in \operatorname{supp} f$, and the coefficient $c_{\text{ц}} = f_{\text{ц}}$ obviously does not depend on the choice of f. We claim that $l = \sum_{\text{ц} \in S} c_{\text{ц}} \text{ц}$ is a maximal element in T for $\trianglelefteq$. Let $f \in T$. Since $\operatorname{supp} f$ is an initial segment of S, and $c_{\text{ц}} = f_{\text{ц}}$ for all $\text{ц} \in \operatorname{supp} f$, we have $f \trianglelefteq l$. Hence, l is maximal for $\trianglelefteq$, if l is in T. Now given $\text{ц} \in \operatorname{supp} l$, there exists an $f \in T$ with $\text{ц} \in \operatorname{supp} f$. Then for all sufficiently large i, we have $l - f_i = (l - f) + (f_i - f) \prec\!\!\prec \text{ц}$, since $f \trianglelefteq l$. □

A subset U of $\mathbb{T}$, is said to be **complete**, if the stationary limit of any sequence with values in U lies again in U. A field of transseries is complete in the new sense, if and only if it is complete in the old sense: if $\mathbb{T} = C_{\beta}^{<\omega}[[[x]]]$ or $\mathbb{T} = C_{\beta}^{<\omega}[[[x]]]$, for some β, then $x, x + \log x, x + \log x + \log_2 x, \cdots$ is a sequence in $\mathbb{T}$ without stationary limit (in $\mathbb{T}$). If β is a limit ordinal, then the sequence $(f_\alpha)_{\alpha < \beta}$ from proposition 2.2 is an example of a sequence in $\mathbb{T} = C_{<\beta}^{\omega}$ without stationary limit.

We state without proof the following easy closure properties of complete sets:

Proposition 2.5. *Let U be a complete set. Then*

(a) $\{f \in U | f \prec\!\!\prec \varepsilon\}$ is complete, for all $\varepsilon \in \mathbb{T}^*$.
(b) $U\text{ц}$ is complete, for all transmonomials ц.
(c) If Ц is a set of transmonomials in $\mathbb{T}$, then $U \cap C[[\text{Ц}]]$ is complete.

2.3.2 The fixed point theorem

Let U be a complete subset of a field of transseries $\mathbb{T}$. Let Φ be a mapping from U into U. We say that Φ is a **contraction**, if

$$\forall f, g \in U \quad \Phi(g) - \Phi(f) \prec\!\!\prec g - f. \tag{2.1}$$

Then we have the following theorem:

Theorem 2.1. Fixed point theorem *Let U be a complete non empty subset of $\mathbb{T}$. Then any contraction $\Phi : U \to U$ admits a unique fixed point.*

Proof. Define a transfinite sequence $(f_\alpha)_\alpha$ of points in U as follows:

$$\begin{aligned} f_0 &= \text{An arbitrary point in } U; \\ f_{\alpha+1} &= \Phi(f_\alpha); \\ f_\alpha &= \operatorname*{stat\,lim}_{\beta\to\alpha} f_\beta \text{ for limit ordinals } \alpha. \end{aligned}$$

We claim that the stationary limit f of this sequence is a fixed point for Φ.

Actually, we will show by transfinite induction that the sequence $f_{\alpha+1} - f_\alpha$ is strictly decreasing for $\prec\!\!\prec$; this will imply that $f_{\alpha+1} = f_\alpha$ for all $\alpha \geqslant |\mathbb{T}|$. If $f_{\alpha+1} \neq f_\alpha$, then

$$f_{\alpha+2} - f_{\alpha+1} \prec\!\!\prec f_{\alpha+1} - f_\alpha \tag{2.2}$$

immediately follows from (2.1). Now assume that α is a limit ordinal and that we have proved the induction hypothesis for all $\beta < \alpha$. This implies in particular, that $f_\gamma - f_\beta \preceq\!\!\preceq f_{\beta+1} - f_\beta$ for all $\beta < \gamma < \alpha$ (by a second, straightforward transfinite induction). Consequently, $f_\alpha - f_\beta \preceq\!\!\preceq f_{\beta+1} - f_\beta$, whence $f_{\alpha+1} - f_{\beta+1} \prec\!\!\prec f_{\beta+1} - f_\beta$, by (2.1). Similarly, we have $f_\alpha - f_{\beta+1} \preceq\!\!\preceq f_{\beta+2} - f_{\beta+1} \prec\!\!\prec f_{\beta+1} - f_\beta$. Together this yields

$$f_{\alpha+1} - f_\alpha \prec\!\!\prec f_{\beta+1} - f_\beta, \tag{2.3}$$

as desired.

Now let $f' \neq f$ be a second point in U with $\Phi(f') = f'$. Then we have $f' - f \preceq\!\!\preceq f' - f \Rightarrow \Phi(f') - \Phi(f) = f' - f \prec\!\!\prec f' - f$. This contradiction proves the uniqueness of the fixed point. □

Remark 2.3. By analogy with classical fixed point theorems, one might expect the fixed point theorem to be very useful in the resolution of functional equations. Unfortunately, this is usually not the case, and even when the fixed point theorem can be applied, direct proofs are often shorter. Nevertheless, the transfinite approximation technique used in the proof is very powerful, and we will often use it in what follows.

2.4 Differentiation of transseries

Let us fix a field of transseries $\mathbb{T}$ over C. In this section we show that $\mathbb{T}$ can naturally be given a derivation over C. This derivation extends the derivation defined in section 1.7.1 for grid-based transseries. Given f in $\mathbb{T}$, we would like to define $f' = df/dx$ by transfinite induction, setting $(\log_k x)' = 1/x \cdots \log_{k-1} x$, and $(\sum_{g\in\mathbb{T}^\uparrow} c_g e^g)' = \sum_{g\in\mathbb{T}^\uparrow} c_g g' e^g$. Of course, elements of C are sent to zero. The hard thing to show is that f' has well-ordered support. Instead of showing this directly, we give an alternative combinatorial definition of f', based on the tree representation of f. Moreover, this combinatorial derivative has the properties mentioned above.

Consider a path P from a child of the root to a leaf in the tree representation T_f of f. Let $ц_{P,1}, \cdots, ц_{P,|P|}$ denote the labels of the consecutive nodes on this path, with $ц_{P,|P|} = \log_k x$. We denote

$$\begin{aligned} P &= [ц_{P,1}, \cdots, ц_{P,|P|}]; \\ c_P &= f_{ц_{P,1}} (\log ц_{P,1})_{ц_{P,2}} \cdots (\log ц_{P,|P|-1})_{ц_{P,|P|}}; \\ ц_P &= ц_{P,1} \cdots ц_{P,|P|}; \\ ц'_P &= ц_P / x \cdots \log_k x. \end{aligned}$$

Now we define f' by

$$f' = \sum_{P \in \text{path}(T_f)} c_P ц'_P, \tag{2.4}$$

where path(T_f) denotes the set of paths in T_f from a child of the root to a leaf.

Example 2.3. Let us illustrate the definition (2.4) on the example $f = e^{5e^{2x}+x^3} + e^x$. The paths from a child of the root to a leaf in T_f are

$$[e^{5e^{2x}+x^3}, e^{2x}, x], [e^{5e^{2x}+x^3}, x^3, \log x] \text{ and } [e^x, x].$$

The contributions of these paths to f' are respectively

$$10e^{5e^{2x}+x^3} e^{2x} x/x, 15e^{5e^{2x}+x^3} x^3 \log x / x \log x \text{ and } e^x x/x.$$

Indeed, $f' = 10e^{5e^{2x}+x^3} e^{2x} + 15e^{5e^{2x}+x^3} x^2 + e^x$.

In order to prove that f' is well defined by (2.4), we show that the ordering $\leqslant$ on paths defined by $P < Q \Leftrightarrow ц'_P < ц'_Q$ is Noetherian (remind warning 1.1: the ordering on transmonomials is opposite to the ordering on transseries). The main problem here is that it can happen that for paths P and Q with $ц_{P,1} < ц_{Q,1}$, we do not have $P < Q$. For instance, take $f = e^{e^x+\sqrt{x}} + e^{e^x}$, $P = [e^{e^x+\sqrt{x}}, \sqrt{x}, \log x]$ and $Q = [e^{e^x}, e^x, x]$. However, we have the following:

Lemma 2.1. *Let f be a transseries in $\mathbb{T}$ and let $P \nleqslant Q$ be paths in T_f with $ц_{P,1} < ц_{Q,1}$. Assume that $|Q| \geqslant 2$. Then $|P| \geqslant 2$ and $ц_{Q,2} \in \operatorname{supp} \log ц_{P,1}$.*

Proof. Let $\log_p x$ and $\log_q x$ be the labels of the ends of the paths P resp. Q. Let us first treat the case when $|P| \geqslant 2$ and $p = q$. We have

$$\frac{ц'_P}{ц'_Q} = \frac{ц_{P,1} \cdots ц_{P,|P|}}{ц_{Q,1} \cdots ц_{Q,|Q|}}.$$

Using that $P \nleqslant Q$ and the fact that $\text{ц}_{P,j} < 1$ for $j > 1$, we deduce

$$\text{ц}_{P,1} \geqslant \text{ц}_{Q,1} \cdots \text{ц}_{Q,|Q|}. \tag{2.5}$$

Assume now that $\text{ц}_{Q,2}$ were not in supp $\log \text{ц}_{P,1}$. Then $\text{ц}_{Q,2} \in$ supp $\log \text{ц}_{P,1}/\text{ц}_{Q,1}$, since

$$\text{supp } \log \text{ц}_{P,2} \subseteq \text{supp } \log \text{ц}_{P,1} \cup \text{supp } \log(\text{ц}_{P,2}/\text{ц}_{P,1}).$$

Recall that for any infinitely large transmonomial щ, we have $\text{щ} < \text{щ}'$ for each $\text{щ}' \in \text{supp } \log \text{щ}$. Hence,

$$\left(\frac{\text{ц}_{P,1}}{\text{ц}_{Q,1}}\right)^{1/(|Q|-1)} < \text{ц}_{Q,j},$$

for each $j \geqslant 2$, by induction over j. Consequently,

$$\frac{\text{ц}_{P,1}}{\text{ц}_{Q,1}} < \text{ц}_{Q,2} \cdots \text{ц}_{Q,|Q|},$$

which contradicts (2.5).

Assume now that $|P| \geqslant 2$, but $p \neq q$. If $p > q$, then we formally extend the path Q to

$$Q := [\text{ц}_{Q,1}, \cdots, \text{ц}_{Q,|Q|-1}, \log_q x, \cdots, \log_p x].$$

We observe that this extension does not alter $\text{ц}'_Q$ nor $\text{ц}_{Q,1}$. This ensures that the same arguments as in the case when $p = q$ can be applied. The case $p < q$ is treated similarly, by formally extending the path P.

Assume finally that $|P| = 1$ and let us come to a contradiction. Again, we formally extend the path P to the path

$$P := [\log_p x, \cdots, \log_q x].$$

Then the same arguments as before can be applied, to yield $\text{ц}_{Q,2} \in \log \text{ц}_{P,1}$. Hence $\text{ц}_{Q,2} = \log_{p+1} x$ and $|Q| = 2$. Consequently, $\text{ц}'_P/\text{ц}'_Q = \log_p x/\text{ц}_{Q,1} < 1$ leads to the desired contradiction. □

Theorem 2.2. *The ordering on the set of paths in the tree representation T_f of a transseries $f \in \mathbb{T}$ is Noetherian.*

Proof. Suppose that the conclusion of the theorem were false, and let $P_1, P_2, \cdots$ be a bad sequence of paths (see page 307). Using proposition A.1(d) (page 305), we may assume without loss of generality that $\text{ц}_{P_1,1} \leqslant \text{ц}_{P_2,1} \leqslant \cdots$. Clearly, we can not have $|P_i| = |P_j| = 1$ for two different indices $i < j$. Hence, for all sufficiently large $i \geqslant I > 1$, we have $|P_i| \geqslant 2$.

By lemma 2.1, we deduce that $|P_1| \geqslant 2$ and $\text{ц}_{P_i,2} \in$ supp $\log \text{ц}_{P_1,1}$, for each $i \geqslant I$. Hence, $Q_i = [\text{ц}_{P_i,2}, \cdots, \text{ц}_{P_i,|P_i|}]$ is a path in $\log \text{ц}_{P_1,1}$, for each such i. Moreover, we observe that $\mathfrak{m}'_{P_i} = \mathfrak{m}'_{Q_i}\mathfrak{m}_{P_i,1}$, for each $i \geqslant I$. Consequently, $Q_I, Q_{I+1}, \cdots$ is a bad sequence of paths in $T_{\log \mathfrak{m}_{P_1,1}}$, since $\mathfrak{m}_{P_I,1} \leqslant \mathfrak{m}_{P_{I+1},1} \leqslant \cdots$.

Repeating the argument, we can therefore construct an infinite sequence $f_1 = f, f_2, \cdots$, with $f_{i+1} \in \log \mathfrak{m}_i$ and $\mathfrak{m}_i \in \operatorname{supp} f_i$, for $i \geqslant 1$, and where none of the f_i is an iterated logarithm. The existence of such a sequence contradicts the well-foundedness of T_f. □

The theorem justifies our definition (2.4) of the derivative f' of a transseries $f \in \mathbb{T}$. Let us now show that the mapping $f \mapsto f'$ is indeed a derivation. Its linearity is clear. We also notice that $(e^f)' = f'e^f$ for all transmonomials e^f. Now by linearity, we only have to show that $(\mathfrak{m}_1\mathfrak{m}_2)' = m'_1\mathfrak{m}_2 + \mathfrak{m}_1 m'_2$ for transmonomials $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Writing $\mathfrak{m}_1 = e^{f_1}$ and $\mathfrak{m}_2 = e^{f_2}$, we have $(e^{f_1+f_2})' = (f_1 + f_2)'e^{f_1+f_2} = (e^{f_1})'e^{f_2} + e^{f_1}(e^{f_2})'$, again by linearity and the fact that $(e^f)' = f'e^f$, for each purely unbounded transseries f. The results of this section can be rephrased in

Theorem 2.3. *The relation (2.4) defines a derivation on the field of transseries.* □

2.5 Composition and inversion of transseries

In this section we introduce composition and functional inversion for grid-based transseries, thereby generalizing section 1.7.1.

2.5.1 Functional composition of transseries

In this section, we define the composition $f \circ g$ of a transseries f by a positive infinitely large transseries $g \in \mathbb{T}^{\infty}_{+}$, for some fixed field of transseries $\mathbb{T}$. In the next section, we give necessary and sufficient conditions for $\mathbb{T}$ to be stable under composition.

As in the case of the derivation, we would like to define $f \circ g$ by transfinite induction, setting $\log_k x \circ g = \log_k g$, and $(\sum_{h \in \mathbb{T}^\uparrow} c_h e^h) \circ g = \sum_{h \in \mathbb{T}^\uparrow} c_h e^{h \circ g}$. The case when $f \in C^\omega[[[x]]]$ will be treated in a direct way. However, to show that $f \circ g$ has well-ordered support in general, we will give an alternative combinatorial definition for the composition. Again, this definition will be equivalent to the definition by transfinite induction.

Let us first show that $f \circ g$ is well defined, if $f \in C^\omega_0[[[x]]]$. For each $k \geqslant 0$, we can write $\log_k g = c_k \mathfrak{M}_k (1 + \varepsilon_k)$, for $c_k \in C$, an infinitesimal ε_k and a transmonomial $\mathfrak{M}_k$. Then we have

$$\operatorname{supp} \log_{p_0 \cdots p_k} g \subseteq \mathfrak{M}_0^{p_0} \cdots \mathfrak{M}_k^{p_k} (\operatorname{supp} \varepsilon_0)^\diamond \cdots (\operatorname{supp} \varepsilon_k)^\diamond,$$

for each word $\boldsymbol{p} \in C^*$ over C. Moreover, there exist $k_0 \in \mathbb{N}$ and $l \in \mathbb{Z}$, such that $c_k \mathfrak{m}_k = \log_{k+l} x$, for $k \geqslant k_0$. Hence

$$\operatorname{supp} \varepsilon_{k+1} \subseteq \frac{(\operatorname{supp} \varepsilon_k)^\dagger}{\log_{k+l+1} x},$$

for $k \geqslant k_0$ (see page 307 for the notation $(\operatorname{supp} \varepsilon_k)^\dagger$). By induction we get

$$\operatorname{supp} \varepsilon_{k+1} \subseteq (\operatorname{supp} \varepsilon_{k_0})^\diamond \left\{ \frac{1}{\log_{k_0+l+1} x}, \cdots, \frac{1}{\log_{k+l} x} \right\}^\diamond \left\{ \frac{1}{\log_{k_0+l+1} x \cdots \log_{k+l+1} x} \right\}^\dagger.$$

Hence

$$\operatorname{supp} \varepsilon_k \subseteq S_{k_0} = (\operatorname{supp} \varepsilon_{k_0})^\diamond \left\{ \frac{1}{\log_{k_0+l+1} x}, \frac{1}{\log_{k_0+l+1} x \log_{k_0+l+2} x}, \cdots \right\}^\diamond,$$

for each $k \geqslant k_0$. We deduce that if $\operatorname{supp} f = \{\log_{p_{0,i} \cdots p_{n_i,i}} x | i \in I\}$, then

$$\operatorname{supp} f \circ g \subseteq S_0 = \{\mathfrak{m}_0^{p_{0,i}} \cdots \mathfrak{m}_{k_0-1}^{p_{k_0-1,i}} | i \in I\} (\operatorname{supp} \varepsilon_0)^\diamond \cdots (\operatorname{supp} \varepsilon_{k_0-1})^\diamond S_{k_0}.$$

Moreover, $\mathfrak{m}_0 \ggg \mathfrak{m}_1 \ggg \cdots$, so that the ordering on $\{\mathfrak{m}_0^{p_{0,i}} \cdots \mathfrak{m}_{k_0-1}^{p_{k_0-1,i}} | i \in I\}$ reproduces the ordering on $\{\log_{p_{0,i} \cdots p_{k_0-1,i}} x | i \in I\}$. Hence, $f \circ g$ has well-ordered support.

Consider the tree representation T_f of $C_\alpha^\omega[[[x]]]$. A **g-labeled tree** in T_f is a finite tree L, together with a mapping $\tau : L \to T_f$ and a labeling $a \mapsto \lambda_a$ of the leafs of L, with the following properties:

LT1. $\operatorname{pred}(\tau(\operatorname{root}(L))) = \operatorname{root}(T_f)$.
LT2. $\tau(\operatorname{pred}(a)) = \operatorname{pred}(\tau(a))$.
LT3. $a \in \operatorname{leaf}(L) \Rightarrow \tau(a) \in \operatorname{leaf}(T_f)$.
LT4. If $a \preceq_L b$ for the natural ordering (see page 308) on L, then $\tau(a) \preceq_{T_f} \tau(b)$.
LT5. If $\tau(a)$ is labeled by $\log_k x$ for $a \in \operatorname{leaf}(L)$, then $\lambda_a \in \operatorname{supp} \log_k g$.

Here $\operatorname{pred}(a)$ denoted the predecessor of a node a. We remark that τ is not necessarily injective. For each a in L, we denote by $\mathfrak{u}_a$ the label of $\tau(a)$ and by c_a its coefficient in $\log \operatorname{pred}(a)$ (resp. $\operatorname{pred}(a)$, if $a = \operatorname{root}(L)$).

Assume now that for some ordinal α we have shown that $f \circ g$ is well defined for all $f \in C_\beta^\omega[[[x]]]$, with $\beta < \alpha$. Then the mapping φ which associates the transmonomial $e^{(h \circ g)^\dagger}$ to a transmonomial e^h in $C_\alpha^\omega[[[x]]]$ is a morphism of ordered groups. Let L be g-labeled tree in T_f and L' a subtree of L. Let r be the root of L', and $L'_1, \cdots, L'_n$

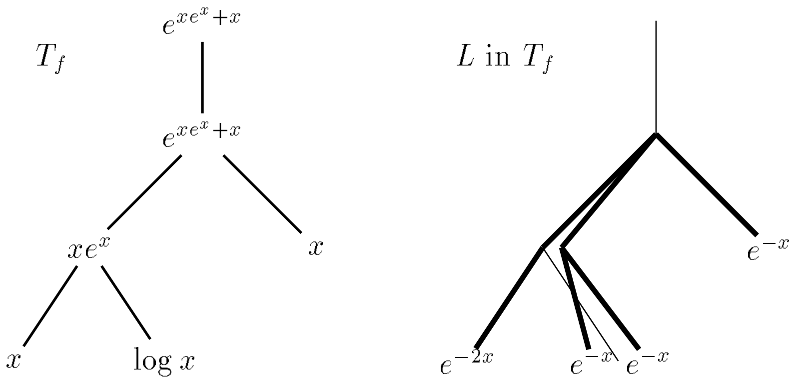


Figure 2.2: Illustration of a g-labeled tree in T_f (see example 2.4)

its children r. To L', we associate

$$\begin{aligned}
c_{L'} &= \begin{cases} c_r(\log_k g)_{\lambda_r}, & \text{if } n=0 \text{ and } ц_r = \log_k x; \\ \frac{1}{n!} c_r c_{L'_1} \cdots c_{L'_n}, & \text{if } n \neq 0; \end{cases} \\
ц_{L'} &= \prod_{a \in L' \setminus \operatorname{leaf}(L')} ц_a; \\
\lambda_{L'} &= \prod_{a \in \operatorname{leaf}(L')} \lambda_a; \\
ц^\circ_{L'} &= \varphi(ц_{L'})\lambda_{L'}.
\end{aligned}$$

We say that L is **admissible**, if $ц^\circ_{L'} \prec\!\!\prec 1$, for each strict subtree L' of L. We order the set $\operatorname{admt}_g(T_f)$ of admissible g-labeled trees in T_f by $L < L' \Leftrightarrow ц^\circ_L < ц^\circ_{L'}$. We define $f \circ g$ by

$$f \circ g = \sum_{L \in \operatorname{admt}_g(T_f)} c_L \varphi(ц_L)\lambda_L. \tag{2.6}$$

In order to prove that $f \circ g$ is well-defined, it suffices to show that the ordering $\leqslant$ on $\operatorname{admt}_g(T_f)$ is Noetherian.

Example 2.4. Let $f = e^{xe^x+x}$ and $g = x + \sqrt{x} + 3e^{-x} + 5e^{-2x}$. In figure 2.2, we have illustrated a sample admissible g-labeled tree L in T_f. We have

$$c_L = \frac{3^3 \cdot 5}{2 \cdot 3!} = \frac{45}{4}.$$

and

$$ц^\circ_L = (xe^{x+\sqrt{x}})^2 e^{(x+\sqrt{x})(e^{x+\sqrt{x}}+3e^{\sqrt{x}}+1)} e^{-5x}.$$

Lemma 2.2. *Let f be a transseries in $\mathbb{T}$ and let $L \not\leqslant M$ be admissible g-labeled trees in T_f, with $ц_{\text{root}(L)} < ц_{\text{root}(M)}$, and $\lambda_L \leqslant \lambda_M$. Assume that M has size $|M| \geqslant 2$. Then $|L| \geqslant 2$ and there exists a child M' of the root of M, such that $ц_{\text{root}(M')} \in$ supp $\log ц_{\text{root}(L)}$.*

Proof. Assume that the conclusion of the lemma were false. Then

$$ц_{\text{root}(M')} \in \text{supp} \log \frac{ц_{\text{root}(L)}}{ц_{\text{root}(M)}},$$

for each child M' of the root of M. Hence,

$$ц_{\text{root}(M')} > \left(\frac{ц_{\text{root}(L)}}{ц_{\text{root}(M)}}\right)^{1/(|M|-1)},$$

for each such M'. Using structural induction, we deduce that

$$ц_a > \left(\frac{ц_{\text{root}(L)}}{ц_{\text{root}(M)}}\right)^{1/(|M|-1)},$$

for any $a \in M \backslash \text{root}(M)$, whence

$$ц_M = ц_{\text{root}(M)} \prod_{a \in M \backslash \text{root}(M)} ц_a > ц_{\text{root}(L)} \geqslant ц_L.$$

This yields the desired contradiction, since $ц_M^\circ = \varphi(ц_M)\lambda_M > \varphi(ц_L)\lambda_L = ц_L^\circ$. $\square$

Theorem 2.4. *The ordering on the set of admissible g-labeled trees in the tree representation T_f of a transseries $f \in C_\alpha^\omega[[[x]]]$ is Noetherian.*

Proof. Suppose that the conclusion of the theorem were false, and let $L_1, L_2, \cdots$ be a bad sequence (see page 307) of admissible g-labeled trees in T_f. We say that such a bad sequence is admissible, if $ц_{\text{root}(L_1)} \leqslant ц_{\text{root}(L_2)} \leqslant \cdots$ and $\lambda_{L_1} \leqslant \lambda_{L_2} \leqslant \cdots$. Recall that λ_{L_i} is either in S_0 (if L_i consists of a leaf only), or in $\{ц \in S_0 | ц \prec\!\!\prec 1\}^\diamond$. By proposition A.1(d), we can extract from each bad sequence an admissible bad sequence. An admissible sequence is understood to be minimal, if for each i, and each admissible bad sequence $L_1, \cdots, L_i, L'_{i+1}, \cdots, L'_{i+2}, \cdots$ the number of children of the root of L'_{i+1} is strictly superior to the number of children of the root of L_{i+1}.

Without loss of generality, we may assume that $L_1, L_2, \cdots$ is a minimal admissible bad sequence. From the minimality hypothesis, we deduce that the number of the children of the root of L_i increases, as i increases. We claim that for all sufficiently large $i \geqslant I > 1$, the root of L_i has at least one child, whence $|L_i| \geqslant 2$. If this were not so, then all L_i would be roots whose images under τ are labeled by iterated logarithms. But this is impossible, since $\operatorname{supp} g \cup \operatorname{supp} \log g \cup \operatorname{supp} \log_2 g \cup \cdots$ is well ordered by the discussion at the beginning of this section.

Applying lemma 2.2, we deduce that $|L_1| \geqslant 2$ and for each $i \geqslant I$ there exists a child M_i of the root of L_i, such that $\text{ц}_{\mathrm{root}(M_i)} \in \operatorname{supp} \log \text{ц}_{\mathrm{root}(L_1)}$. Let L_i' be the admissible g-labeled tree obtained from L_i by deleting M_i. We claim that the induced ordering on $\{L_I', L_{I+1}', \cdots\}$ is Noetherian. Assuming the contrary, there exists an admissible bad sequence $L_{i_1}', L_{i_2}', \cdots$. Let k be such that i_k is minimal, and consider the sequence $L_1, \cdots, L_{i_k-1}, L_{i_k}', L_{i_k+1}', \cdots$. This sequence is also an admissible bad sequence, contradicting the minimality hypothesis. Indeed, we observe that $L_{i_j}' > L_{i_j}$ for each j, since $\text{ц}^\circ_{M_{i_j}} > 1$ and $\text{ц}^\circ_{L_{i_j}} = \text{ц}^\circ_{L_{i_j}'} \text{ц}^\circ_{M_{i_j}}$.

By proposition A.1(d), we can extract an increasing sequence $L_{i_1}', L_{i_2}', \cdots$, from $L_I', L_{I+1}', \cdots$. Using again that $\text{ц}^\circ_{L_{i_j}} = \text{ц}^\circ_{L_{i_j}'} \text{ц}^\circ_{M_{i_j}}$ for each j, we conclude that M_{i_1}, $M_{i_2}, \cdots$ is a bad sequence of admissible g-labeled trees in $T_{\log \text{ц}_{\mathrm{root}(L_1)}}$. Repeating the argument, we obtain a contradiction as in the proof of theorem 2.2. □

Corollary. *The composition $f \circ g$ is well defined by (2.6), for all transseries $f \in C_\alpha^\omega[[[x]]]$ and all ordinals α.*

2.5.2 Properties of functional composition

We now state some properties of the composition, leaving the proofs of (a), (c) and (f) as exercises for the reader.

Proposition 2.6. *Let $f \in C_\alpha^\beta[[[x]]]$ and $g \in (C_{\alpha'}^{\beta'}[[[x]]])_+^\infty$. Then*

(a) Right composition $\circ_g$ with g is a strong difference operator, i.e.

$$(\sum_{i \in I} f_i) \circ g = \sum_{i \in I} f_i \circ g,$$

for all Noetherian families $(f_i)_{i \in I}$.

(b) $(f \circ g) \circ h = f \circ (g \circ h)$ for all positive infinitely large h.

(c) $(f \circ g)' = g'(f \circ g)$.

(d) Let $\varepsilon \prec\!\!\!\prec 1$ such that $\varepsilon \prec\!\!\!\prec \text{ц}'/\text{ц}$, for all monomials $\text{ц} \neq 1$ in the support of f. Then

$$f \circ (x + \varepsilon) = f + f'\varepsilon + \frac{1}{2} f''\varepsilon^2 + \cdots. \tag{2.7}$$

Here $\varepsilon \prec\!\!\!\prec 1 \prec\!\!\!\prec \text{ц}'/\text{ц}$ holds in particular, if $\operatorname{expo} \text{ц} \leqslant 0$.

(e) $f \circ g \in C_{\alpha''}^{\beta''}[[[x]]]$, where $\alpha'' = \min(\alpha + \alpha', \max(\alpha, \alpha') + \omega)$ and $\beta'' = \min(\beta + \beta', \omega)$.

(f) $\operatorname{expo}(f \circ g) = \operatorname{expo} f + \operatorname{expo} g$.

Proof. We prove (b) using a double transfinite induction. We may assume without loss of generality that $\beta = \omega$, and that g and h are both in $\mathbb{T}_\infty^+$. Assume first that $f = \log_k x$. If $g = \log_l x$, then we clearly have associativity. In the other case, we write $g = c_g \text{м}_g (1+\varepsilon)$ and

$$
\begin{aligned}
\log(g \circ h) &= \log\left(c_g \text{м}_g \circ h \left(1 + \sum_{\text{ц}} \varepsilon_{\text{ц}} \text{ц} \circ h\right)\right) \\
&= \log g_{\text{м}} + \log \text{м} \circ h + \log\left(1 + \sum_{\text{ц}} \varepsilon_{\text{ц}} \text{ц} \circ h\right) \\
&= \log g_{\text{м}} + (\log \text{м}) \circ h + \log\left(1 + \sum_{\text{ц}} \varepsilon_{\text{ц}} \text{ц}\right) \circ h \\
&= \log\left(c_g \text{м}_g \left(1 + \sum_{\text{ц}} \varepsilon_{\text{ц}} \text{ц}\right)\right) \circ h = (\log g) \circ h.
\end{aligned}
$$

The induction hypothesis is used to write down the identity $\log(\text{м} \circ h) = (\log \text{м}) \circ h$. Iterating the obtained identity, we deduce that $\log_k(g \circ h) = (\log_k g) \circ h$, for all g and k. For more general f, we use a second transfinite induction and write

$$
\begin{aligned}
\left(\sum_{\text{ц}} f_{\text{ц}} \text{ц}\right) \circ (g \circ h) &= \sum_{\text{ц}} f_{\text{ц}} \text{ц} \circ (g \circ h) \\
&= \sum_{\text{ц}} f_{\text{ц}} (\text{ц} \circ g) \circ h \\
&= \left(\left(\sum_{\text{ц}} f_{\text{ц}} \text{ц}\right) \circ g\right) \circ h.
\end{aligned}
$$

This concludes the proof of (b).

The hard part of (d) is to prove that the right hand side of (2.7) is well defined. To do this, we shall use the concept of Noetherian operators from section A.4. Let X be the set of couples $(\text{ц}, n)$ where ц is a transmonomial with $\text{ц} = 1$ or $\varepsilon \prec\!\!\!\prec \text{ц}'/\text{ц}$ and $n \in \mathbb{N}$. We order X by $(\text{ц}, n) \leqslant (\text{щ}, m) \Leftrightarrow \text{ц}\varepsilon^n \leqslant \text{щ}\varepsilon^m$. Let ϑ be the strictly extensive choice operator on X, which sends $(\text{ц}, n)$ to the set of couples $(\text{щ}, n+1)$, with $\text{щ} \in \operatorname{supp} \text{ц}'$. From theorem 2.3, we deduce that the operator ϑ is actually Noetherian (apply the theorem to $\sum_{\text{ц} \in \text{Ц}} \text{ц}$ for each well-ordered set of transmonomials Ц). Hence, the operator ϑ^* is Noetherian, by theorem A.4.

We have a natural injection $\iota : \operatorname{supp} f \to X; \text{ц} \mapsto (\text{ц}, 0)$. Let $\varepsilon = c_\varepsilon \text{м}_\varepsilon (1+\delta)$. An element in $\vartheta^*(\iota(\operatorname{supp} f))(\operatorname{supp} \delta)^\diamond$ is represented by a chain $\text{ц}_0 \in \operatorname{supp} f, \text{ц}_1, \cdots, \text{ц}_n$ of monomials with $\text{ц}_{i+1} \in \operatorname{supp} \text{ц}_i'$ for all i and a chain $\text{щ}_1, \cdots, \text{щ}_m$ of monomials in the support of δ. If $n = m$, then we associate to such an element $(\mathbf{ц}, \mathbf{щ})$ a term

$$\tau_{(\text{ц},\text{щ})} = f_{\text{ц}_0} \text{ц}'_{0,\text{ц}_1} \cdots \text{ц}'_{n-1,\text{ц}_n} \text{ц}_n c_\varepsilon^n \text{м}_\varepsilon^n \delta_{\text{щ}_1} \cdots \delta_{\text{щ}_n}.$$

If $m \neq n$, then we take $\tau_{(ц,щ)} = 0$. Clearly, the mapping $(\mathbf{ц}, \mathbf{щ}) \mapsto \tau_{(ц,щ)}$ is increasing. Therefore, the sum $\sum_{(ц,щ)} \tau_{(ц,щ)}$ is well defined, since $\vartheta^*(\iota(\operatorname{supp} f))(\operatorname{supp} \delta)^{\diamond}$ is Noetherian. But this sum is nothing else than the right hand side of (2.7). We leave it as an exercise for the reader to prove the equality (2.7).

The only non trivial thing to prove in (e) is the bound $\alpha'' \leqslant \max(\alpha, \alpha') + \omega$. We may assume without loss of generality that $g = x + \varepsilon$, with $\varepsilon \prec\!\!\prec 1$, by means of a finite number of left and right compositions with exp and log. Let us prove by induction over l that if $\operatorname{supp} f$ contains only monomials with exponentiality $\leqslant l$, then we may take $\alpha'' \leqslant \max(\alpha, \alpha') + l$. If $l = 0$, then this results from (d). Otherwise, let $ц = e^{\varphi}$ be a monomial in $\operatorname{supp} f$. Then the exponentialities of the monomials in $\operatorname{supp} \varphi$ are all bounded by $l - 1$ by the induction hypothesis, whence the exponential depth of $e^{\varphi} \circ g$ is bounded by $\max(\alpha, \alpha') + l$. We conclude by the strong linearity of $\circ_g$.

In the general case, we decompose f as a sum $f = f_0 + f_1 + f_2 + \cdots$, where the monomials in $\operatorname{supp} f_0$ have exponentialities $\leqslant 0$, and the monomials in $\operatorname{supp} f_l$ have exponentialities l, for each $l > 0$. By what precedes, the exponential depth of $f_l \circ g$ is bounded by $\max(\alpha, \alpha') + l$, for each l. We now infer (e) from the strong linearity of $\circ_g$. $\square$

Using proposition 2.6(e), we can now characterize those fields of transseries $\mathbb{T}$ are stable under composition: $\mathbb{T} = C_\flat^\sharp[[[x]]]$ is stable under composition, if and only if $\sharp \in \{0, <\omega, \omega\}$, and $\flat \in \{0, <\omega\}$ or $\flat = <\alpha$ with $\alpha = \alpha + \omega$. In the cases when such a field contains $\log x$ and $\exp x$, then we say that $\mathbb{T}$ is a **stable** field of transseries. Summarizing, we have proved:

Theorem 2.5. *Let $\mathbb{T}$ be a stable field of transseries. Then the relation (2.6) defines a composition on $\mathbb{T}$, which is compatible with the derivation.* $\square$

2.5.3 Functional inversion of transseries

Theorem 2.6. *Let $\mathbb{T}$ be a stable field of transseries. Then each $g \in \mathbb{T}_+^{\infty}$ admits a functional inverse for $\circ$ in $\mathbb{T}$.*

Proof. Modulo a finite number of left and right compositions of g with exp or log, we may assume without generality that $g = x + \varepsilon$, for some $\varepsilon \prec\!\!\prec 1$. We will denote by κ resp. λ the exponential resp. logarithmic depth of g.

We first consider the case when all monomials in the support of ε have exponentiality $\leqslant 0$ or fixed exponentiality l. Then we define the sequence (f_α) by transfinite induction:

$$\begin{aligned} f_0 &= x; \\ f_{\alpha+1} &= f_\alpha - (f_\alpha \circ g - x) \circ f_\alpha; \\ f_\alpha &= \operatorname*{stat\,lim}_{\beta \to \alpha} f_\beta, \quad \text{for limit ordinals } \alpha. \end{aligned}$$

Using our hypothesis on ε, it can be verified by transfinite induction that the exponential and logarithmic depths of the f_α are bounded by κ resp. λ, so that the stationary limits are well defined. Indeed, this follows from proposition 2.6(d), by noticing that the exponentiality of $f_\alpha - x$ is either $\leqslant 0$ or equal to l for all $\alpha > 0$.

We claim that $f_\alpha \circ g = x$, for α sufficiently large (whence $g \circ f_\alpha = x$). It suffices to show that $f_\alpha \circ g - x \prec\!\!\prec f_\beta \circ g - x$, for all $\beta < \alpha$, with $f_\beta \circ g \neq x$. Indeed, this implies that $f_\alpha \circ g = x$ for $\alpha > |\mathbb{T}|$, where $|\mathbb{T}|$ denotes the cardinal of $\mathbb{T}$.

We first observe that for any infinitesimal δ, we have $\delta \sim \delta \circ (x + \delta)$. Hence

$$f_{\alpha+1} \circ g - x = f_\alpha \circ g - x - (f_\alpha \circ g - x) \circ (f_\alpha \circ g) \prec\!\!\prec f_\alpha \circ g - x.$$

For a limit ordinal α, let $\beta < \gamma < \alpha$. We have

$$(f_\beta \circ g - x) - (f_\gamma \circ g - x) \sim f_\beta \circ g - x \sim (f_\beta \circ g - x) \circ (f_\beta \circ g),$$

whence $f_\beta - f_\gamma \sim (f_\beta \circ g - x) \circ f_\beta$. Passing to the limit, we deduce $f_\beta - f_\alpha \sim (f_\beta \circ g) \circ f_\beta$. Hence

$$(f_\beta \circ g - x) - (f_\alpha \circ g - x) \sim f_\beta \circ g - x \sim (f_\beta \circ g - x) \circ (f_\beta \circ g),$$

so that $f_\alpha \circ g - x \prec\!\!\prec f_\beta \circ g - x$.

In general, we decompose $g = x + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \cdots$, where the exponentialities of the monomials in $\operatorname{supp} \varepsilon_0$ are bounded by 0 and the exponentialities of the monomials in ε_l are equal to l for each $l > 0$. By induction on l, we now have functional inverses $f_0, f_1, f_2, \cdots$ for $x + \varepsilon_0, x + \varepsilon_0 + \varepsilon_1, x + \varepsilon_0 + \varepsilon_1 + \varepsilon_2, \cdots$ respectively:

$$f_{l+1} = f_l \circ (x + \varepsilon_l \circ f_l)^{inv};$$

here $(x + \varepsilon_l \circ f_l)^{inv}$ is defined by what precedes. We also observe that the exponential resp. logarithmic depth of f_l is bounded by $\kappa + l$ resp. λ for each l. Hence $f = \operatorname{stat\,lim}_{l \to \infty} f_l$ exists and yields the desired functional inverse of g. Indeed, if $f \circ g \neq x$, then for any $l \in \mathbb{N}$, the exponentiality of $f \circ g - x \preceq f_l \circ g - x$ would be at least l. But this is impossible. □

2.6 Transseries intervals and compactification

Any totally ordered set E has a natural topology, called the **interval topology**, whose open sets are unions of open intervals. We recall that an **interval** is a subset I of E, such that for each $x < y < z$ with $x, z \in I$, we have $y \in I$. An interval is said to be **open**, if for each $x \in I$ we have: x is minimal resp. maximal, if and only if x is minimal resp. maximal in E. We observe that an increasing union of open intervals is an open interval. Hence, any open set U of E can be represented as the (generally infinite) disjoint union of intervals, by considering the largest open interval $I_x \subseteq U$ for each $x \in U$.

Now consider a field of transseries $\mathbb{T}$ with the interval topology. $\mathbb{T}$ is "very" non-Archimedian, whence disconnected and non compact. A natural question is how to characterize intervals in $\mathbb{T}$. Now open intervals in $\mathbb{R}$ are all of the form $]a, b[$, with $a \leqslant b$, and $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. Hence, extending $\mathbb{R}$ with $\{-\infty, \infty\}$ yields a simple description of the intervals of $\mathbb{R}$. Moreover, $\mathbb{R} \cup \{-\infty, \infty\}$ is a compactification of $\mathbb{R}$. We claim that an analogue of this holds for $\mathbb{T}$, but much more new values need to be inserted. We will first give an abstract construction in section 2.6.1 which works for any so called continuously totally ordered set. In section 2.6.2 we particularize the obtained results for the transline $\mathbb{T}$.

2.6.1 Compactification of continuous total orderings

Let E be any totally ordered set, and denote by $I(E)$ the set of its initial segments, ordered by inclusion. Let $\sim$ be the equivalence relation on $I(E)$ defined by $I \sim J \Leftrightarrow |I \bigtriangleup J| < \infty$, where $|I \bigtriangleup J|$ denotes the cardinal of $I \backslash J \cup J \backslash I$. This equivalence relation is compatible with the ordering on $I(E)$. Hence, we have a natural ordering on $\overline{E} \stackrel{\text{def}}{=} I(E)/{\sim}$. We also have a natural mapping $i : x \mapsto \overline{I_x}$ from E into $\overline{E}$, with $I_x = \{y \leqslant x\}$. We say that the ordering on E is **continuous**, if for each $x < y \in E$ there exists a $z \in E$, with $x < z < y$. This is always the case when E is a totally ordered field.

From now on we assume that $\leqslant_E$ is continuous. We first observe that $I_x \bigtriangleup I_y$ is infinite, for any $x \neq y$ in E, so that i is an embedding. Let us show that $\overline{\overline{E}} \cong \overline{E}$, i.e. that the natural mapping j from $\overline{E}$ into $\overline{\overline{E}}$ is an isomorphism. Let $\overline{I} < \overline{J}$ be in $\overline{E}$. Then there exists $x \in J \backslash I$. At least one of $I_x \backslash I$ and $J \backslash I_x$ must be non empty. If $y \in I_x \backslash I$, we have $I \leqslant I_y < I_z < I_x \leqslant J$ for any $y < z < x$. Similarly, if $y \in J \backslash I_x$, then $I < I_z < J$ for any $x < z < y$. We conclude that the ordering on $\overline{E}$ is continuous, whence j is injective. Next, let I be in $I(\overline{E})$. Consider $U = \bigcup_{\overline{V} \in I} V \in I(E)$. Then $I_{\overline{U}} \sim I$ and $\overline{I} = j(\overline{U})$.

Let us now show that $\overline{E}$ is connected. Assume the contrary. Then $\overline{E}$ is the disjoint union of two open sets, whence the disjoint union of at least two non empty open intervals. Let $I_1 < I_2$ be summands of this partition. Writing

$$\overline{E} = \cup_{x \in I_1}]-\infty_{\overline{E}}, x] \amalg \cap_{x \in I_1}]x, \infty_{\overline{E}}[,$$

we observe that we may assume without loss of generality that $\overline{E} = I_1 \amalg I_2$. Consider $U = \bigcup_{\overline{V} \in I_1} V \in I(E)$. We have either $\overline{U} \in I_1$ or $\overline{U} \in I_2$. In the first case, $\overline{U}$ would be a maximal element of I_1 different from $\overline{E}$. In the second case, it would be a minimal element of I_2 different from $\overline{\phi}$. This gives us the desired contradiction.

Let us finally show that $\overline{E}$ is compact. It suffices to show that from a covering $(I_\alpha)_{\alpha \in A}$ of $\overline{E}$ with open intervals we can extract a finite subcovering. This is done by the following procedure: let $x_0 = \overline{\phi}$ be the minimal element of $\overline{E}$. For each $k \geqslant 0$, we inductively define $x_{k+1} = \overline{\bigcup_{\overline{V} \in I_\alpha, x_k \in I_\alpha} V}$. We remark that we either have $x_{k+1} = \overline{E}$, or $I_\alpha < x_{k+1}$, for any α with $x_k \in I_\alpha$. We claim that $x_k = \overline{E}$ for k sufficiently large. Suppose the contrary and consider $x = \overline{\bigcup_{\overline{U_k} = x_k} U_k}$. There exists an α with $x \in I_\alpha$. Since I_α is open, there exists an $y < x$ in I_α. By the definition of x, there exists an n with $y \leqslant x_n$. But then x_n and $x_{n+1} < \overline{E}$ are both in I_α, which contradicts the fact that $x_{n+1} = \overline{E}$ or $I_\alpha < x_{n+1}$. Having proved the claim, we successively choose $\alpha_k, \cdots, \alpha_0$, such that $x_i \in I_{\alpha_i}$ $(0 \leqslant i \leqslant k)$, and $I_{\alpha_i} \cap I_{\alpha_{i+1}} \neq \phi$ $(0 \leqslant i < k)$. This is possible by the construction of the x_i, and we have $\overline{E} = I_{\alpha_0} \cup \cdots \cup I_{\alpha_k}$.

We have proved:

Proposition 2.7. *Let E be set with a continuous total ordering. Then*

(a) $\overline{\overline{E}} \cong \overline{E}$.
(b) $\overline{E}$ *is connected.*
(c) $\overline{E}$ *is compact.* □

2.6.2 Compactification of the transline

Let $\mathbb{T}$ be a field of transseries and $\amalg$ the subset of transmonomials. We will give a concrete description of $\overline{\mathbb{T}}$. Besides $\overline{C}$ and $\mathbb{T}$ which are naturally embedded into $\overline{\mathbb{T}}$, $\overline{\mathbb{T}}$ contains also two special elements $\infty_{\mathbb{T}} = \sup \mathbb{T}$ and $1/\infty_C = \sup\{f \in \mathbb{T} | f \prec\!\!\prec 1\}$. If $\mathbb{T}$ is not complete, then let $\hat{\mathbb{T}}$ denote the smallest complete subset containing $\mathbb{T}$ of a complete field of transseries $\mathbb{T}'$ which contains $\mathbb{T}$. $\hat{\mathbb{T}}$ is also natural embedded in $\overline{\mathbb{T}}$, and consists precisely of those elements f in $\mathbb{T}'$, such that for any $g \triangleleft f$ we have $g \in \mathbb{T}$.

Let us now show that the usual field operations, exponentiation and logarithm extend to $\overline{\mathbb{T}}$. Any increasing function $\varphi : I \to \mathbb{T}$ on an interval I of $\mathbb{T}$ naturally extends into an increasing function $\overline{\varphi} : \overline{I} \to \overline{\mathbb{T}}$ by $\overline{\varphi}(\sup J) = \sup \varphi(J)$, for all initial segments J of I. Indeed, this yields a function from $\overline{I}$ into $\overline{\mathbb{T}}$ since each element in $\overline{I} \backslash I$ can be represented uniquely in such a way. In particular, this shows that exponentiation and logarithm naturally extend to $\overline{\mathbb{T}}$. Right composition with a fixed infinitely large transseries is also defined on $\overline{\mathbb{T}}$.

In a similar way, decreasing functions $\varphi : I \to \mathbb{T}$ naturally extend into decreasing functions $\overline{\varphi} : \overline{I} \to \overline{\mathbb{T}}$ by $\overline{\varphi}(\sup J) = \inf \varphi(J)$. In particular, opposites $(-f)$ and inverses (f^{-1} for $f \neq 0$) are defined in $\overline{\mathbb{T}}$. Furthermore, the derivation is strictly decreasing on infinitesimal intervals, whence it can be extended to the compactifications of such intervals. Finally, functional inversion on $\mathbb{T}_\infty^+$ naturally extends to $\overline{\mathbb{T}_\infty^+}$.

Addition and multiplication also extend to $\overline{\mathbb{T}}$: addition is defined by $\sup I + \sup J = \sup I + J$ for all initial segments I and J of $\mathbb{T}$. Similarly, multiplication on $\overline{\mathbb{T}^+}$ is defined by $\sup I \sup J = \sup IJ$ for all initial segments I and J of $\mathbb{T}^+$. Multiplication is extended to $\overline{\mathbb{T}}$ by $(-x)y = x(-y) = -(xy)$ and $(-x)(-y) = xy$. We notice that $\overline{\mathbb{T}}$ does not possess much algebraic structure. For instance, $\infty_{\mathbb{T}} + \infty_{\mathbb{T}} = \infty_{\mathbb{T}}$, whence $\overline{\mathbb{T}}$ is not even a group.

To characterize $\overline{\mathbb{T}}$, only one type of elements can still not be constructed from $\hat{\mathbb{T}}, \overline{C}, \infty_{\mathbb{T}}$ and $1/\infty_C$ by using the above operations. Consider an expansion of the form

$$\varphi_0 + \lambda_0 e^{\varphi_1 + \lambda_1 e^{\varphi_2 + \lambda_2 e^{\cdot^{\cdot^{\cdot}}}}}, \tag{2.8}$$

where the φ_i are transseries in $\mathbb{T}$, and the λ_i non zero constants. If $\mathbb{T}$ is stable under exponentiation, then such expansions can be interpreted as elements in $\overline{\mathbb{T}}$ in the following way: for each i, let I_i be the initial segment of $\mathbb{T}$ with maximum

$$\varphi_0 + \lambda_0 e^{\varphi_i + \lambda_1 e^{\cdot^{\cdot^{\cdot^{\varphi_i}}}}}.$$

Then the expansion (2.8) can either be interpreted as the element $\sup \bigcap_{i \in \mathbb{N}} \bigcup_{j > i} I_i$ or $\sup \bigcup_{i \in \mathbb{N}} \bigcap_{j > i} I_i$ in $\overline{\mathbb{T}}$. In general, both interpretation yield different elements in $\overline{\mathbb{T}}$. Nevertheless, for what follows, we will only consider canonical expansions:

We say that an expansion of the form (2.8) is **canonical**, if the following conditions are satisfied:

(a) $\lambda_i = \pm 1$ for each i.

(b) φ_i is purely unbounded for each $i > 0$.

(c) for each i and each $\text{ц} \in \operatorname{supp} \varphi_i$, there exists a $j > i$, such that for all ψ with $\operatorname{supp} \varphi_j <_{\text{Ц}} \operatorname{supp} \psi$, we have

$$\text{ц} \succ\!\!\succ e^{\varphi_{i+1} + \lambda_{i+1} e^{\cdot^{\cdot^{\cdot^{\varphi_j + \psi}}}}}.$$

Given a canonical expansion of the form (2.8), both its interpretations as elements in $\overline{\mathbb{T}}$ coincide. Moreover, for each i and each $\text{ц} \in \operatorname{supp} \varphi_i$, we have

$$\text{ц} \succ\!\!\succ e^{\varphi_{i+1} + \lambda_{i+1} e^{\varphi_{i+1} + \lambda_{i+2} e^{\cdot^{\cdot^{\cdot}}}}}.$$

Here $f \prec\!\!\prec g$ for $f, g \in \overline{\mathbb{T}}$ if $\infty_C |f| < |g|$ or $f = g = 0$.

In fact, we extend the notion of canonical expansions to certain other elements in $\overline{\mathbb{T}}$: first, $\pm\infty_{\mathbb{T}}$, elements of $\hat{\mathbb{T}}$ and elements of the form $c\text{ц}$ ($c \in \overline{C}\setminus C, \text{ц} \in \text{Ц}$) and $(c \pm 1/\infty)\text{ц}$ ($c \in C, \text{ц} \in \text{Ц}$) are defined to be canonical expansions of themselves. Now assume that we are given $\varphi \in \mathbb{T}$ and a canonical expansion $\tilde{g}$ of $g \in \overline{\mathbb{T}}$, such that $\tilde{g} \prec\!\!\prec \text{ц}$ for each $\text{ц} \in \operatorname{supp} \varphi$. Then the expression $\varphi + \tilde{g}$ is by definition a canonical expansion of $\varphi + g$. If φ is purely unbounded and $g \succ\!\!\succ 1$, then the expression $\pm e^{\varphi+\tilde{g}}$ is by definition a canonical expansion of $\pm e^{\varphi+g}$.

Theorem 2.7. *Each $f \in \overline{\mathbb{T}}$ admits a unique canonical expansion of one of the following types:*

I. $f \in \mathbb{T}$.

II. $f = \pm\infty_{\mathbb{T}}$.

III. $f = \varphi_0 \pm e^{\varphi_1 \pm e^{\cdot^{\cdot^{\cdot^{\varphi_r + \lambda\text{ц}}}}}}$, $\lambda \in \overline{C}\setminus C$, $\text{ц} \in \text{Ц}$.

IV. $f = \varphi_0 \pm e^{\varphi_1 \pm e^{\cdot^{\cdot^{\cdot^{\varphi_r + (\lambda_r \pm 1/\infty_C)\text{ц}}}}}}$, $\lambda_r \in C$, $\text{ц} \in \text{Ц}$.

V. $f = \varphi_0 \pm e^{\varphi_1 \pm e^{\cdot^{\cdot^{\cdot^{\varphi_r}}}}}$, $\varphi_r \in \hat{\mathbb{T}}\setminus\mathbb{T}$.

VI. $f = \varphi_0 \pm e^{\varphi_1 \pm e^{\varphi_2 \pm e^{\cdot^{\cdot^{\cdot}}}}}$.

Proof. Let us describe a theoretical algorithm to compute the canonical expansion of f. If $f \in \hat{\mathbb{T}}$ or $f = \pm\infty_{\mathbb{T}}$, then we are done. In the remaining case, let I be the initial segment of $\mathbb{T}$ with $\sup I = f$ and let $h = \operatorname{stat\,lim}_{g \in I} g$. Since $f \notin \hat{\mathbb{T}}$, we have $h \in \mathbb{T}$. Distinguish the following cases:

1. The dominant monomial of $h - g$ is constant, for $g \in I$ sufficiently large.
2. We are not in case 1, and $I < h$.
3. We are not in case 1, and $g > h$ for some $g \in I$.

In case 1, we consider the set J of dominant coefficients of $g - h$, where we take $g \in I$ sufficiently large, so that its dominant monomial м is constant. Then J is an initial segment of C, which admits a lowest upper bound $\sup J \neq -\infty_C$ in $\overline{C}$. If $\sup J \in C$, then we have $f = h + (\sup J - 1/\infty)\text{м}$. If $\sup J \notin C$, then we have $\sup I = h + (\sup J)\text{м}$.

In case 2, $h - g$ is positive for all $g \in I$, so that the dominant monomial of $h - g$ increases (remind warning 1.1), when g approaches $\sup I$. Consider the set L of logarithms of dominant monomials of $h - g$, for $g \in I$. We recursively determine $\inf L$. We have $\inf L \neq -\infty_{\mathbb{T}}$, since $f \neq h \in \mathbb{T}$. If $\sup L$ has the form $\sup L = \varphi + \infty_C$, with $\varphi \in \mathbb{T}^{\uparrow}$, then $f = h - \infty_C e^{\varphi}$. In the other case, we have $f = h - e^{\inf L}$. Notice that the case $\sup L = \varphi + c$ with $\varphi \in \mathbb{T}^{\uparrow}$ and $c \in \overline{C}\setminus\{\infty_C\}$ is excluded.

In case 3, $h - g$ is negative for all sufficiently large $g \in I$, so that the dominant monomial of $g - h$ decreases, when g approaches $\sup I$. Let L be the set of logarithms of dominant monomials of $g-h$, for $g \in I$ with $g > h$. We recursively determine $\sup L$. If $\sup L$ has the form $\sup L = \varphi - \infty_C$, with $\varphi \in \mathbb{T}^\uparrow$, then $f = h + (1/\infty_C)e^\varphi$, In the other case, we have $f = h + e^{\sup L}$. Notice that the case $\sup L = \varphi + c$ with $\varphi \in \mathbb{T}^\uparrow$ and $c \in \overline{C} \backslash \{-\infty_C\}$ is excluded.

In the case when the theoretical algorithm terminates, we clearly obtain a canonical expansion for f of type I, II, III, IV or V. If the algorithm does not terminate, then we obtain an expression of type VI: the successive values of h in the algorithm determine $\varphi_0, \varphi_1, \cdots$. The signs are determined by the successive cases considered in the algorithm (case 2 versus case 3). The uniqueness of canonical expansions of types I, II, III, IV, V and VI is trivial. □

Example 2.5. The sequence $\log x, \log_2 x, \log_3 x, \cdots$ admits $\infty_C \in \overline{C}$ as a limit. The sequence $x, x^2, x^3, \cdots$ admits $e^{\infty \log x}$ as a limit. The sequence $x + x/\log x, x + x/\log_2 x, x + x/\log_3 x + \cdots$ admits $(1 + 1/\infty_C)x$ as a limit. The sequence $e^{x^2}, \exp(x^2 + e^{\log_2^2 x}), \exp(x^2 + \exp(\log_2^2 x + e^{\log_4^2 x})), \cdots$ admits

$$l = e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + \cdot^{\cdot^{\cdot}}}}}. \tag{2.9}$$

as a limit. The sequence $x, x + \log x, x + \log x + \log\log x, \cdots$ in $C[[[x]]]$ admits $x + \log x + \log\log x + \cdots$ in $\widehat{C[[[x]]]}$ as a limit.

The above theorem also yields a classification of transseries intervals:

Proposition 2.8. *Let I be a non empty interval of $\mathbb{T}$. Then there exist unique $f \leqslant g \in \overline{\mathbb{T}}$, such that I has one and only one of the following forms:*

I. $I =]f, g[\cap \mathbb{T}$.
II. $I = [f, g[\cap \mathbb{T}$, *with* $f \in \mathbb{T}$.
III. $I =]f, g] \cap \mathbb{T}$, *with* $g \in \mathbb{T}$.
IV. $I = [f, g] \cap \mathbb{T}$, *with* $f, g \in \mathbb{T}$. □

Our notation for elements in $\overline{\mathbb{T}}$ and transseries intervals might for example be useful in complexity theory: for a given problem, the set of complexities of algorithms (representable in $\mathbb{T}$) which solve this problem form a final segment F of $\mathbb{T}$. We define $\inf F$ to be the **complexity** of the problem **in** $\overline{\mathbb{T}}$. Similarly, one defines the **type** of a real number **in** $\overline{\mathbb{T}}$, which is studied in the field of Diophantine approximations.

2.6.3 Nested forms and nested expansions

In his automatic expansion algorithms, Shackell (see [Sh 93]) systematically uses nested forms and nested expansions: any non zero non Archimedian transseries f can be written in a unique way $f = \pm \exp_s^\varepsilon((\log_t^d x)g)$ $(s, t \in \mathbb{N}, \varepsilon = \pm 1,$

$d \in C^*$, $g > 0$), where $g \prec\!\!\prec \log_t x$, $d > 0$ unless $s = 0$, and $d \neq 1$ unless $s = 0$ or $t = 0$. Performing this operation recursively yields a **nested form** $f = \pm \exp_{s_1}^{\varepsilon_1}((\log_{t_1} x) \cdots \exp_{s_n}^{\varepsilon_n}((\log_{t_n} x) g))$, where g tends to a finite limit l. Repeating this procedure once again for $g - l$ yields a **nested expansion** for f.[1]

Nested forms are useful for estimating the order of growth of a transseries. In particular, the limit of a transseries can be deduced from its nested expansion, whence its usefulness for limit computations. However, much information is lost, even in the case of nested expansions. For instance, the nested expansions of $1+1/x+1/x^2+\cdots$ and $1+1/x+1/x^2+\cdots+e^{-x}$ are the same. For a similar reason, nested expansions do not yield equivalents: consider for example $f = \exp(e^x/(x-1)+x)$. Finally, a transseries like $x + e^{-x}$, whose "usual" expansion is finite, has an infinite nested expansion.

By what has been said in the previous section, we observe that nested expansions are special instances of expressions of the form (2.8). Consequently, nested expansions determine (and should be considered as?) elements in $\overline{\overline{\mathbb{T}}}$.

2.7 Complements

Certain, very general, functional equations admit solutions with a "strongly monotonic flavour" (Hardy field solutions, for instance), which are not representable by transseries in the sense of the previous sections. In this section we discuss possible extensions of the theory of transseries to cover such cases. Our presentation is informal; the generalized transseries we discuss only occur in quite "pathological" cases. Actually, their introduction is mainly motivated by the quest for a completely general theory.

2.7.1 Nested transseries

The first class of equations which does not admit solutions in any of the fields of transseries previously studied admits the following representative example:

$$f(x) = e^{x^2 + f(\log_2 x)}. \tag{2.10}$$

The natural transseries solution of this equation would be given by a "nested expansion"

$$f = e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + \cdots}}}. \tag{2.11}$$

Although such expressions do not belong to any field of transseries in the classical

[1] We note a slight difference between our definitions and Shackell's in the case when f is Archimedian: for us the nested form of f would just be f. Shackell requires one more term of the nested expansion for the nested form of f.

sense (see the previous section), it is plausible that fields of generalized transseries can be constructed, which contain f as a transmonomial.

Actually, one might be tempted to generalize the concept of a transseries by allowing tree representations which are not well-founded. However, to avoid paradoxes, an additional condition need be imposed on such tree representations. Indeed, consider the functional equation

$$g(x) = e^{x^2 + g(\log_2 x) + \log x}. \tag{2.12}$$

A "natural" transseries solution would be

$$g = e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + \cdot^{\cdot^{\cdot}} + \log_5 x} + \log_3 x} + \log x},$$

and one would like to think of g as a transmonomial. However, the above formula is misleading, because there exists a solution to (2.12) of the form

$$g = f(1 + \varepsilon).$$

Indeed,

$$\begin{aligned} \log g &= x^2 + g(\log_2 x) + \log x \\ &= x^2 + f(\log_2 x) + \varepsilon - \tfrac{1}{2}\varepsilon^2 + \cdots = \log f + \log(1 + \varepsilon), \end{aligned}$$

whence

$$f(\log_2 x)\varepsilon(\log_2 x) + \log x = \varepsilon - \tfrac{1}{2}\varepsilon^2 + \cdots$$

and we find a solution

$$\varepsilon(x) = -\frac{e^x}{f(x)} - \frac{e^{e^{e^x}}}{f(e^{e^x})} + \frac{e^{2e^{e^x}}}{2f(e^{e^x})^2} + \cdots.$$

The point is that we should forbid transseries whose tree representation contains an infinite path such that an infinite number of nodes on the path admit branches to the right hand side. Such "transseries" will be called **ill-founded**, while the remaining valid ones, like f, will be called **nested**.

Remark 2.4. It is interesting to notice that $\varepsilon < 0$, whence g is *smaller* than f. Actually, a similar phenomenon is encountered when considering transseries whose terms are ordered in the wrong way. For instance, let

$$\varphi = \cdots + x^2 + x + 1,$$

which seems to be positive. However, φ satisfies $\varphi = x\varphi + 1$, and the correct transseries solution to this equation is

$$\varphi = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots,$$

which is negative.

2.7.2 Transseries of positive strength

Another source of instability for the fields of transseries constructed in this chapter is revealed by considering iterators of positive infinitely large transseries of non zero exponentiality: consider the equations

$$\exp^*(x+1) = \exp(\exp^* x) \tag{2.13}$$

and

$$\log^*(x) = \log^*(\log x) + 1. \tag{2.14}$$

Any Hardy field solution to one of these equations has an order of growth which is superior to any iterated exponential resp. inferior to any iterated logarithm. In particular, such a solution can not be represented by a transseries in the sense of the previous sections, although it has a definite "strongly monotonic flavour".

Solutions $\exp^*$ and $\log^*$ are resp. called **iterators** of the exponential and the logarithm. The reason is that they enable us to define fractional iterates of the exponential and the logarithm:

$$\begin{aligned} \exp_\alpha x &= \exp^*(\log^* x + \alpha); \\ \log_\alpha x &= \exp^*(\log^* x - \alpha). \end{aligned}$$

More generally, even more violently increasing functions are obtained by repeatedly taking iterators of exp:

$$\begin{aligned} \exp^*(x+1) &= \exp(\exp^*(x)); \\ \exp^{**}(x+1) &= \exp^*(\exp^{**}(x)); \\ &\vdots \end{aligned}$$

Similarly, one defines $\log^*, \log^{**}, \cdots$. We also notice an alternative, formal way of introducing $\log^*$ and its repeated iterators, by means of integration:

$$\begin{aligned} \log^* x &= \int \frac{1}{x \log x \log_2 x \cdots}; \\ \log^{**} x &= \int \frac{1}{x \log x \cdots \log^* x \log\log^* x \cdots \log_2^* x \cdots}; \\ &\vdots \end{aligned}$$

A natural question is now to construct a field of transseries $C[[[[x]]]]$, which contains $\exp^* x, \log^* x, \exp^{**} x, \cdots$, and which is stable under derivation, composition, etc. Moreover, we want each transseries in $C[[[[x]]]]$ to be a well-ordered sum of transmonomials of bounded strength. Here the **strength** of a transseries f is defined to be the maximal number s, such that $\exp^{*^s}$ or $\log^{*^s}$ occurs in f.

The main difficulty we encounter here, is the characterization of transmonomials. Our criterion is as follows: if f is a positive infinitely large transseries in $C[[[[x]]]]$ and $s \in \mathbb{N}$, then $\exp^{*^s} f$ is a transmonomial, if and only if for each decomposition $f = f^+ + f^-$ with $0 \neq f^+ \lhd f$ we have

$$f^- \not\succ \frac{\exp^{*^s} f^+}{(\exp^{*^s})' f^+}.$$

The problem with this criterion is that the construction of $\exp^{*^s} f$ requires the preliminary construction of $\exp^{*^s} f^+, (\exp^{*^s})' f^+, \cdots$, for each strict truncation $0 \neq f^+ \lhd f$ of f. Nevertheless, we have checked the possibility of constructing $C[[[[x]]]]$ using our criterion, by alternating inductive closures and applications of Zorn's lemma. Because of its technical character, we plan to come back to this construction in a future paper.

Remark 2.5. As usual, we have only been concerned with the algebraic side of the story. Another interesting question is to *construct* solutions to equations like (2.13) and (2.14) at infinity. Hardy already constructed continuous solutions to similar equations, but his solutions are not even differentiable. Écalle constructed in [Éc 92] so called *quasi-analytic* solutions, which are in particular Hardy field solutions. Here we recall that quasi-analytic functions generalize analytic functions in the sense that they still admit derivatives up to any order and unique quasi-analytic continuations along the real axis. Unfortunately, no criterion is presently known to privilege particular quasi-analytic solutions to (2.13) and (2.14).[2] In any case, Écalle proved that such a criterion can not be entirely algebraic (i.e. involving $+, \cdot, d/dx, \circ$, etc.): this is his theorem of "indecernability". For more details about iterators, growth orders and "the Grand Cantor" we refer to [Éc 92].

2.7.3 Conclusion

Summarizing, the transseries from section 2.2 do not suffice to represent the strongly monotonic solutions to very general systems of differential difference equations. The source of troubles lies in the consideration of functional equations involving compositions with transseries of non zero exponentiality. We have shown two extensions of the concept of a transseries: nested transseries and transseries of positive strength. We conjecture that it is possible to construct fields of transseries which are stable under the strongly monotonic resolution of any system of functional equations, by combining these two extensions. In table 2.2 we have summarized the natural origins of different types of transseries in terms of the kind of functional equations which can give rise to it.

[2] We notice that given an iterator $\log^*$ of the logarithm, we have another iterator

$$\widetilde{\log^*} x = \log^* x + \varphi(\log^* x),$$

for any periodic function $\varphi \neq 0$ of period 1.

type	origin
Puiseux	algebraic equations
grid-based	algebraic differential equations
strength 0	functional equations, where only right compositions with transseries of exponentiality zero are allowed
positive strength	composition equations (see [Éc 92])
nested, positive strength	any functional equations

Table 2.2: Natural origins of transseries of a given type.

Let us finally notice that instead of considering more general transseries, it might also be interesting to study types of transseries between grid-based transseries and well-ordered transseries. Consider for instance those well-ordered transseries f with **recursively well-ordered support**: i.e. $\operatorname{supp} f \subseteq \mathfrak{T}$ for some set $\mathfrak{T}$ of transmonomials, such that for all $e^{\varphi} \in \mathfrak{T}$ we have $\operatorname{supp} \varphi \subseteq \mathfrak{T}$. The set of recursively well-ordered transseries is an exp-log field, stable under derivation, composition and the resolution of many systems of functional equations. However,

$$f = \sum_{i=1}^{\infty} e^{e^x x^{-i} + x^i}$$

is an explicit example of a transseries whose support is not recursively well-ordered.

2.8 References

[Can 1899] G. Cantor. Sur les fondements de la théorie des ensembles infinis. *Rééditions Jacques Gabay.*

[Con 76] J.H. Conway. *On numbers and games.* Academic Press.

[Dahn 84] B.I. Dahn. The limiting behaviour of exponential terms. *Fund. Math. 124 (p. 169-186).*

[DG 86] B.I. Dahn, P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae 127 (p. 45-50).*

[Éc 92] J. Ecalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Hermann, collection: Actualités mathématiques.

[Hig 52] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. 2 (p. 326-336).*

[Rib] P. Ribenboim. *Generalized power series rings.* Preprint of Queen's Univ. Kingston, Ontario, Canada.

[VdH 94a] J. van der Hoeven. Outils effectifs en asymptotique et applications. *Research report LIX/RR/94/09, Ecole polytechnique, France.*

Chapter 3

The Newton polygon method

3.1 Introduction

Almost all techniques for solving asymptotic systems of equations are explicitly or implicitly based on the Newton polygon method. In this section we explain this technique in the elementary case of algebraic equations over the ring of grid-based series $R = C[\![X]\!]$ or well-ordered series $R = C[[X]]$, where C is a constant field of characteristic zero and X a totally ordered monomial group with $\mathbb{Q}$-powers. In the subsequent sections, we will consider more and more general types of equations over the transseries. Notice also, that all "computations" we perform in this chapter are theoretical; for more details about the effective aspects of the Newton polygon method and its extensions, we refer to chapters 11 and 12.

3.2 The method illustrated by examples

In this section, we show how to apply the Newton polygon method to two specific examples. This will familiarize the reader with the basic concepts and some subtleties of the method. Let us first consider the equation

$$P(f) = \sum_i P_i z^i = z^3 f^6 + z^4 f^5 + f^4 - 2f^3 + f^2 + \frac{z}{1-z^2} f + \frac{z^3}{1-z} = 0. \tag{3.1}$$

Assume that the Puiseux series $f = c z^\mu + \cdots \in \mathbb{C}[\![z^{\mathbb{Q}}]\!]$ with $c \neq 0$ and μ as dominant exponent is a solution to this equation, and let

$$\alpha = \min\{3, \mu+1, 2\mu, 3\mu, 4\mu, 5\mu+4, 6\mu+3\}.$$

Since f is a solution, we in particular must have

$$[z^\alpha]\left(z^3 f^6 + z^4 f^5 + f^4 - 2f^3 + f^2 + \frac{z}{1-z^2} f + \frac{z^3}{1-z}\right) = 0. \tag{3.2}$$

By the choice of α, we have $[z^\alpha] P_i f^i = P_{i,\alpha-\mu i} c^i$, for each i, and $P_{i,\alpha-\mu i} \neq 0$ for some i.

In view of (3.2), this implies that $P_{i,\alpha-\mu i} \neq 0$ for at least two indices i. Consequently, α occurs at least twice among the numbers $3, \mu+1, 2\mu, 3\mu, 4\mu, 5\mu+4, 6\mu+3$, whence

$$\mu \in \{2, 1, 0, -\tfrac{3}{2}\}.$$

Graphically, these possible values of μ can be determined by considering the Newton polygon associated to (3.1), which is defined to be the convex hull of all points (f^i, z^ν) with $\nu \geqslant \mu_{P_i}$. We have illustrated the Newton polygon associated to (3.1) in figure 3.1. The diagonal slopes

$$\begin{array}{ll} (1, z^3) \to (f, z) & (\mu = 2); \\ (f, z) \to (f^2, 1) & (\mu = 1); \\ (f^2, 1) \to (f^4, 1) & (\mu = 0); \\ (f^4, 1) \to (f^6, z^3) & (\mu = -\tfrac{3}{2}). \end{array}$$

correspond to the possible values of μ. These values are also called the **potential dominant exponents** of f, where we consider f as an indeterminate solution to (3.1).

For given $\mu \in \mathbb{Q}$, the equation (3.2) is actually a non trivial polynomial equation in c over $\mathbb{C}$, which we call the **Newton equation** associated to μ. Hence, there are only a finite number of possible values for c, which are listed below as a function of μ:

$$\begin{array}{ll} \mu = 2, & c = -1; \\ \mu = 1, & c = -1; \\ \mu = 0, & c = 1 \text{ (double solution)}; \\ \mu = \tfrac{3}{2}, & c \in \{-i, i\}. \end{array}$$

The corresponding possible values for cz^μ are called the **potential dominant terms** of f.

For given $cz^\mu \in \mathbb{C}z^{\mathbb{Q}}$, we can now consider the equation $\tilde{P}(\tilde{f}) = 0$ which is obtained from (3.1), by substitution of f with $cz^\mu + \tilde{f}$, where $\tilde{f}$ satisfies the asymptotic constraint $\mu_{\tilde{f}} > \mu$. For instance, if $cz^\mu = 1z^0$, then we obtain:

$$\tilde{P}(\tilde{f}) = z^3\tilde{f}^6 + (6z^3)\tilde{f}^5 + (15z^3 + 5z^4 + 1)\tilde{f}^4 + (20z^3 + 10z^4 + 2)\tilde{f}^3 + (15z^3 + 10z^4 + 1)\tilde{f}^2 + (6z^3 + 5z^4 + z/(1-z^2))\tilde{f} + z^4 + z^3 + (z^4 + z^3 + z)/(1-z^2) = 0. \tag{3.3}$$

The potential dominant monomials of solutions $\tilde{f}$ to (3.3) are found by the same method, except that we now require the potential dominant exponents $\tilde{\mu}$ of $\tilde{f}$ to satisfy the additional condition $\tilde{\mu} > \mu$. The Newton polygon associated to (3.3) is illustrated in (3.2).

The above discussion illustrates that instead of studying polynomial equations $P(f) = 0$, it is more appropriate to study polynomial equations $P(f) = 0$, which satisfy additional constraints $\mu_f > \nu$ (the case of usual polynomial equations is

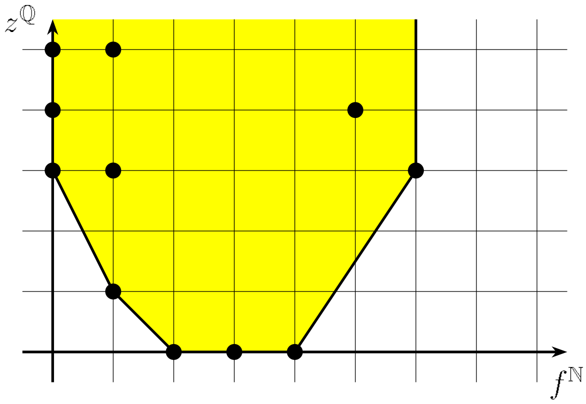


Figure 3.1: The Newton polygon associated to the equation

$$z^3 f^6 + z^4 f^5 + f^4 - 2f^3 + f^2 + zf/(1-z^2) + z^3/(1-z) = 0.$$

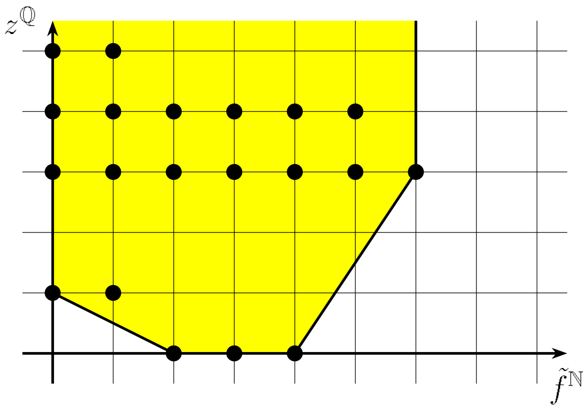


Figure 3.2: The Newton polygon associated to the equation

$$z^3 \tilde{f}^6 + (6z^3)\tilde{f}^5 + (15z^3 + 5z^4 + 1)\tilde{f}^4 + (20z^3 + 10z^4 + 2)\tilde{f}^3 +$$
$$(15z^3 + 10z^4 + 1)\tilde{f}^2 + (6z^3 + 5z^4 + z/(1-z^2))\tilde{f} + z^4 + z^3 + (z^4 + z^3 + z)/(1-z^2) = 0.$$

This equation is obtained from the equation in figure 3.1 when substituting $f = 1 + \tilde{f}$. The potential dominant monomial we chose corresponds to the horizontal slope.

recovered by allowing $\mu_f = -\infty$). Then the above method yields a way to transform such equations into new equations $\tilde{P}(\tilde{f}) = 0$, with $f = \varphi + \tilde{f}$, $\mu_\varphi > \mu$, $\mu_{\tilde{f}} > \tilde{\nu} \geqslant \mu_\varphi$. Such transformations are called **refinements**, which are said to be **admissible**, if $\tilde{P}$ is either divisible by $\tilde{f}$, or if there exists a potential dominant monomial relative to $\tilde{P}(\tilde{f}) = 0$.

Unfortunately, the process of computing potential dominant terms and their corresponding refinements is generally infinite and even transfinite. Hence, we only obtain necessary conditions for Puiseux series f to satisfy $P(f) = 0$ by this process. On the other hand, not all coefficients of a solution f to $P(f) = 0$ need to be determined in order to determine f itself: we merely want a suitable description for f. In our case, solutions to $P(f) = 0$ are represented by refinements as above, for which $\tilde{P}(\tilde{f}) = 0$ ($\mu_{\tilde{f}} > \tilde{\nu}$) has a unique solution. This leads to the question of finding a sufficient condition to guarantee this. It turns out (see the next section) that a sufficient condition is that $\tilde{P}(\tilde{f}) = 0$ ($\mu_{\tilde{f}} > \tilde{\nu}$) be quasi-linear. This means that either $\tilde{P}$ is divisible once by $\tilde{f}$, or the equation $\tilde{P}(\tilde{f}) = 0$ admits a unique potential dominant exponent, whose associated Newton equation has degree one.

Returning to our example equation (3.1), we observe that the refinements $f = -z^2 + \tilde{f}, f = -z + \tilde{f}, f = -iz^{-3/2} + \tilde{f}$, resp. $f = iz^{3/2} + \tilde{f}$ are quasi-linear (i.e. the corresponding equations in $\tilde{f}$ obtained from (3.1) are quasi-linear). The case $f = 1 + \tilde{f}$ ($\mu_{\tilde{f}} > 0$) necessitates one more step of the Newton polygon method: this yields the quasi-linear refinements $f = 1 - i\sqrt{z} + \tilde{\tilde{f}}$ resp. $f = 1 + i\sqrt{z} + \tilde{\tilde{f}}$ with $\mu_{\tilde{\tilde{f}}} > \frac{1}{2}$. Hence we obtain a complete description of the set of solutions to (3.1). The first terms of the expansions of the solutions are given by:

$$\begin{aligned}
f_I &= -z^2 + \cdots ; \\
f_{II} &= -z + \cdots ; \\
f_{III} &= 1 - iz^{1/2} + \cdots ; \\
f_{IV} &= 1 + iz^{1/2} + \cdots ; \\
f_V &= -iz^{-3/2} + \cdots ; \\
f_{VI} &= iz^{-3/2} + \cdots .
\end{aligned}$$

As explained in greater generality in chapter 8, the parallel process for computing these solutions can be represented conveniently by a labeled tree: the root of the tree is labeled by (3.1), its inner nodes by non quasi-linear refinements of (3.1), and the leaves by quasi-linear refinements of (3.1). This **computation tree** is illustrated in figure 3.3.

In some cases, quasi-linear refinements are harder to obtain. Consider for instance the equation

$$P(f) = 0$$

$$\begin{array}{ccccc} f = -z^2 + \tilde{f}, & \tilde{f} = -z + \tilde{f}, & f = 1 + \tilde{f}, & f = -iz^{-3/2} + \tilde{f}, & f = iz^{-3/2} + \tilde{f}, \\ \mu_{\tilde{f}} > 2 & \mu_{\tilde{f}} > 1 & \mu_{\tilde{f}} > 0 & \mu_{\tilde{f}} > -3/2 & \mu_{\tilde{f}} > -3/2 \end{array}$$

$$\begin{array}{cc} f = 1 - iz^{1/2} + \tilde{\tilde{f}}, & f = 1 + iz^{1/2} + \tilde{\tilde{f}}, \\ \mu_{\tilde{\tilde{f}}} > 1/2 & \mu_{\tilde{\tilde{f}}} > 1/2 \end{array}$$

Figure 3.3: Computation tree associated to (3.1).

$$\left(f - \sum_{k=1}^{\infty} z^{1-1/k}\right)^2 = z^2 \tag{3.4}$$

over $\mathbb{C}[[z^{\mathbb{Q}}]]$. In this case, iterated application of the ordinary Newton polygon method does not yield a quasi-linear equation after a finite number of steps. This is due to the fact that the respective Newton equations all have the same degree 2 and roots of maximal multiplicity 2. Therefore, an additional trick is applied: we take the first derivative

$$2f - \sum_{k=1}^{\infty} 2z^{1-1/k} = 0$$

of the equation (3.4) w.r.t. f, which is quasi-linear, and which has a unique solution

$$\varphi = \sum_{k=1}^{\infty} z^{1-1/k}.$$

Now, instead of performing the usual substitution $f = 1 + \tilde{f}$, we perform the substitution $f = \varphi + \tilde{f}$. This yields the equation $\tilde{f}^2 = z^2$ ($\mu_{\tilde{f}} > 0$). Applying one more step of the Newton polygon methods yields the admissible refinements $\tilde{f} = z + \tilde{\tilde{f}}$ ($\mu_{\tilde{\tilde{f}}} > 1$) and $\tilde{f} = -z + \tilde{\tilde{f}}$ ($\mu_{\tilde{\tilde{f}}} > 1$) for $\tilde{f}$. In both cases, we finally obtain a quasi-linear equation $2z\tilde{\tilde{f}} + \tilde{\tilde{f}}^2 = 0$ ($\mu_{\tilde{\tilde{f}}} > 1$) resp. equation $-2z\tilde{\tilde{f}} + \tilde{\tilde{f}}^2 = 0$ ($\mu_{\tilde{\tilde{f}}} > 1$) in $\tilde{\tilde{f}}$. In the next section, we will show that this trick generally applies, and that the resulting method always yields a complete description of the solution set after a finite number of steps.

Remark 3.1. The idea of using repeated differentiation in order to handle almost multiple solutions has been used for the first time in [Sm 1875]. The idea has also been used in computer algebra before (see [Ch 86] and [Gri 91]). Our contribution has been to incorporate it directly into the Newton polygon process, as will be shown in more detail in the next section.

3.3 The general method

Let C be a constant field of characteristic zero and X a totally ordered monomial group with $\mathbb{Q}$-powers. Having illustrated the Newton polygon method on some specific examples, we now turn to the general case of a polynomial equation

$$P_n f^n + \cdots + P_0 = 0 \ (f \prec\!\!\prec \text{ч}), \tag{3.5}$$

with coefficients in $C[[X]]$, subject to the constraint $f \prec\!\!\prec \text{ч}$ for some $\text{ч} \in X$. A **potential dominant monomial** of f relative to (3.5) is a monomial $\text{ц} \prec\!\!\prec \text{ч}$ in X, such that there exist $0 \leqslant i < j \leqslant n$ and $\text{щ} \in X$ with $P_i \text{ц}^i \asymp P_j \text{ц}^j \asymp \text{щ}$ and $P_k \text{ц}^k \underline{\prec\!\!\prec} \text{щ}$ for all other k. Graphically, potential dominant monomials correspond to diagonal slopes of the **Newton polygon** associated to (3.5), which is by definition the convex hull of all points $(i, \text{ц})$ with $\text{ц} \underline{\prec\!\!\prec} P_i$. To each potential dominant monomial ц we associate the equation

$$\mathrm{M}_{P,\text{ц}}(c) = P_{d,\text{щ}/\text{ц}^d} c^d + \cdots + P_{0,\text{щ}} = 0, \tag{3.6}$$

and $\mathrm{M}_{P,\text{ц}}$ is called the **Newton polynomial** (relative to (3.5) and ц). A **potential dominant term** of f relative to (3.5) is a term $c\text{ц}$, where ц is a potential dominant monomial of f relative to (3.5) and $c \in C^*$ a non zero root of the corresponding Newton polynomial. Notice that there are only a finite number of potential dominant terms relative to (3.5).

Proposition 3.1. *Let $f = c_f \mathrm{M}_f + \cdots$ be a non zero solution to (3.5), where c_f and M_f are the dominant coefficient and monomial of f. Then $c_f \mathrm{M}_f$ is a potential dominant term of f.* □

The **Newton degree** d of (3.5) is defined to be the largest degree of the Newton polynomial associated to a potential dominant monomial. By convention, the Newton degree is defined to be the valuation in f of P, if there exist no potential dominant monomials relative to (3.5). If $d = 1$, then we say that (3.5) is **quasi-linear**. The previous proposition implies that (3.5) does not admit any solution, if $d = 0$. The next proposition is a sort of implicit function theorem, which shows that quasi-linear equations admit unique solutions.

Lemma 3.1. *If (3.5) is quasi-linear, then it admits a unique solution f in $C[[X]]$. This solution satisfies* $\operatorname{supp} f \subseteq S$, *where*

$$S = \frac{\mathrm{M}(P_0)}{\mathrm{M}(P_1)} \left(\frac{\operatorname{supp} P_0}{\mathrm{M}(P_0)} \cup \frac{\operatorname{supp} P_1}{\mathrm{M}(P_1)} \cup \cdots \cup \frac{(\operatorname{supp} P_n)\mathrm{M}(P_0)^{n-1}}{\mathrm{M}(P_1)^n} \right)^{\diamond}.$$

Proof. The set $S \subseteq X$ is well-ordered, by Higman's theorem (see page 307). Let us now prove the existence of a solution to (3.5), by computing the coefficients $f_{\text{ц}}$. For this we use transfinite induction over $\text{ц} \in S$, and simultaneously prove that

$$P(\sum_{\text{ц}' \succcurlyeq\!\!\succ \text{ц}} f_{\text{ц}'}\text{ц}') \prec\!\!\prec \text{м}(P_1)\text{ц}.$$

The existence and uniqueness of $f_{\text{м}(P_0)/\text{м}(P_1)}$ is guaranteed by the quasi-linearity of (3.5). Assume that $\text{ц} \in S$ is given, and that the induction hypothesis is satisfied for all $\text{ц}' \succ\!\!\succ \text{ц}$ in S.

Let $\varphi = \sum_{\text{ц}' \succ\!\!\succ \text{ц}} f_{\text{ц}'}\text{ц}'$. By the choice of S, we have

$$\begin{aligned}
\operatorname{supp} P(\varphi) &\subseteq \bigcup_{i=0}^{n} (\operatorname{supp} P_i)(\operatorname{supp} \varphi)^i \\
&\subseteq \bigcup_{i=0}^{n} (\operatorname{supp} P_i) S^i \\
&\subseteq \bigcup_{i=0}^{n} \left(\frac{\text{м}(P_1)^{i+1}}{\text{м}(P_0)^i} S \right) S^i \\
&\subseteq \text{м}(P_1) S.
\end{aligned}$$

We claim that $P(\varphi) \preccurlyeq\!\!\prec \text{м}(P_1)\text{ц}$. Let $\text{ц}' \succ\!\!\succ \text{ц}$ be in S and denote $\psi = \sum_{\text{ц}'' \succcurlyeq\!\!\succ \text{ц}'} f_{\text{ц}''}\text{ц}''$. By the induction hypothesis, we have $P(\psi) \prec\!\!\prec \text{м}(P_1)\text{ц}'$. Furthermore, $\varphi - \psi \prec\!\!\prec \text{ц}'$, whence

$$\begin{aligned}
P(\varphi) &= P(\psi) + P'(\psi) o(\text{ц}') + \tfrac{1}{2} P''(\psi) o(\text{ц}'^2) + \cdots \\
&= P(\psi) + o(\text{м}(P_1)\text{ц}') + o(\text{м}(P_1)^2 \text{ц}'^2 / \text{м}(P_0)) + \cdots \\
&= P(\psi) + o(\text{м}(P_1)\text{ц}').
\end{aligned}$$

It follows that $P(\varphi) \prec\!\!\prec \text{м}(P_1)\text{ц}'$, for all $\text{ц}' \succ\!\!\succ \text{ц}$ in S. This proves our claim.

Taking $f_{\text{ц}} = (P(\varphi)/P'(\varphi))_{\text{ц}}$, the induction hypothesis is clearly satisfied for ц. For a similar reason as above, we have $P(f) \prec\!\!\prec \text{м}(P_1)\text{ц}'$, for all $\text{ц}'$ in S. Since $\operatorname{supp} P(f) \subseteq \text{м}(P_1)S$, by the choice of S, we therefore have $P(f) = 0$. Now consider the equation $Q(g) = 0$ with $g \prec\!\!\prec \text{ч}$, which is obtained by substitution of f by $f + g$ in (3.5). We have $Q_0 = 0$, and $Q_i \succcurlyeq\!\!\succ Q_1 \text{ч}^{1-i}$ for all i. Hence, $Q(g) = 0$ does not admit potential dominant monomials, whence $g = 0$ is its unique solution. □

A **refinement** is a change of variables $f = \varphi + \tilde{f}$ with $\varphi \prec\!\!\prec \text{ч}$, together with the imposition of a constraint $\varphi \succcurlyeq\!\!\succ \tilde{\text{ч}} \succ\!\!\succ \tilde{f}$ on $\tilde{f}$. Such a refinement transforms (3.5) into a polynomial equation in $\tilde{f}$:

$$\tilde{P}_n \tilde{f}^n + \cdots + \tilde{P}_0 = 0 \ (\tilde{f} \prec\!\!\prec \tilde{\text{ч}}), \tag{3.7}$$

where

$$\tilde{P}_i = \frac{P^{(i)}(\varphi)}{i!} = \sum_{k=i}^{n} \binom{k}{i} P_k \varphi^{k-i}, \tag{3.8}$$

for each i. The Newton degree of the refinement relative to (3.5) is by definition the Newton degree of (3.7), and the refinement is said to be **admissible** if its Newton degree is strictly positive.

Lemma 3.2. *Let $c\text{ц}$ be a potential dominant term of f relative to (3.5). Then the refinement $f = c\text{ц} + \tilde{f}$ ($\tilde{f} \prec\!\!\prec \text{ц}$) is admissible relative to (3.5), and its Newton degree is the multiplicity of c as a root of the Newton polynomial associated to* ц.

Proof. Let d be maximal such that $P_d \text{ц}^d$ is maximal for $\prec\!\!\prec$ and denote $\text{щ} = \text{м}(P_d)\text{ц}^d$. Then we have

$$\begin{aligned} \tilde{P}_i &= \frac{1}{i!} P^{(i)}(c\text{ц}) \\ &= \frac{1}{i!} \sum_{k=1}^{n} \binom{k}{i} P_k c^{k-i} \text{ц}^{k-i} \\ &= \frac{1}{i!} \sum_{k=1}^{n} \binom{k}{i} (P_{k,\text{щц}^{-k}} + o(1)) \text{щц}^{-k} c^{k-i} \text{ц}^{k-i} \\ &= \frac{1}{i!} \text{м}_{P,\text{ц}}^{(i)}(c) \text{щц}^{i} + o(\text{щц}^{i}), \end{aligned}$$

for all i. In particular, denoting the multiplicity of the root c by $\tilde{d}$, we have $\tilde{P}_{\tilde{d}} \asymp \text{щц}^{-\tilde{d}}$. Moreover, for all $i \geqslant \tilde{d}$, we have $\tilde{P}_i \preceq\!\!\preceq \text{щц}^{-i}$. Hence, for any $i > \tilde{d}$ and $\tilde{\text{ц}} \prec\!\!\prec \text{ц}$, we have $\tilde{P}_i \tilde{\text{ц}}^i \prec\!\!\prec \tilde{P}_{\tilde{d}} \tilde{\text{ц}}^{\tilde{d}}$. This shows already that the Newton degree of $\tilde{P}(\tilde{f}) = 0$ is at most $\tilde{d}$.

Let us now show that the Newton degree of $\tilde{P}(\tilde{f}) = 0$ is precisely $\tilde{d}$. If $c\text{ц}$ is a root of (3.5) of multiplicity $\tilde{d}$, then we have nothing to prove. Therefore, we may assume without loss of generality that $\tilde{P}_i \neq 0$, for a certain $i < \tilde{d}$. Take i such that $\tilde{\text{ц}} = \sqrt[\tilde{d}-i]{\text{м}(\tilde{P}_i(\tilde{f}))/\text{м}(\tilde{P}_{\tilde{d}}(\tilde{f}))}$ is maximal for $\prec\!\!\prec$. Then $\tilde{\text{ц}} \prec\!\!\prec \text{ц}$ is a potential dominant monomial for $\tilde{P}(\tilde{f}) = 0$, and the associated Newton polynomial has degree $\tilde{d}$. □

If one step of the Newton polygon method does not suffice to decrease the Newton degree, then two steps do, when applying the trick from the next lemma:

Lemma 3.3. *Let d be the Newton degree of (3.5) and let ц be a potential dominant monomial of f relative to (3.5). If $\text{м}_{P,\text{ц}}$ admits a unique root c of multiplicity d, then*

(a) $P^{(d-1)}(\varphi) = 0$ has a unique solution with $\varphi \prec\!\!\prec \text{ч}$.

(b) The refinement $f = \varphi + \tilde{f}$ ($\tilde{f} \prec\!\!\prec \text{м}_\varphi$) is admissible relative to (3.5).

(c) The Newton degree of any refinement $\tilde{f} = \tilde{\varphi} + \tilde{\tilde{f}}$ ($\tilde{\tilde{f}} \prec\!\!\prec \tilde{\text{ч}}$) relative to (3.7) is strictly inferior to d.

Proof. Notice first that the equation $P^{(d-1)}(\varphi)=0$ is quasi-linear, since taking the derivative of an equation corresponds to translating the associated Newton polygon by one place to the left. Hence, (a) follows from lemma 3.1. The proof of (b) is analogue to the proof of lemma 3.2. To prove (c), we first observe that $\tilde{P}_{d-1} = P^{(d-1)}(\varphi)=0$. It follows that if ц is a potential dominant monomial of $\tilde{f}$ relative to $\tilde{P}(\tilde{f})=0$, then $\mathrm{M}_{\tilde{P},ц,d-1}=0$. In particular, there do not exist $\alpha\neq 0,\beta\neq 0$ with $\mathrm{M}_{\tilde{P},ц}(\tilde{c})=\alpha(\tilde{c}-\beta)^d$. In other words, $\mathrm{M}_{\tilde{P},ц}$ does not admit roots of multiplicity d. We conclude by lemma 3.2. □

Theorem 3.1. *Let C be an algebraically closed field of characteristic zero and X a totally ordered monomial group with $\mathbb{Q}$-powers. Then both $C[[X]]$ and $C[\![X]\!]$ are algebraically closed.*

Proof. Consider the following theoretical algorithm:

Algorithm `polynomial_solve`
INPUT: An asymptotic polynomial equation (3.5).
OUTPUT: The solutions to (3.5).

STEP 1. Let d be the Newton degree of (3.5). If P is divisible by f^d, then separate the following two cases:
- A. Return 0.
- B. Proceed with step 2.

STEP 2. Compute the potential dominant terms $c_1ц_1,\cdots,c_\nu ц_\nu$ of f relative to (3.5).

STEP 3. If $\nu=1$ and c_1 is a root of multiplicity d of the Newton polynomial associated to $ц_1$, then proceed with step 5.

STEP 4. For each $1\leqslant i\leqslant \nu$, apply `polynomial_solve` to the equation obtained from (3.5), by refining $f=c_iц_i+\tilde{f}$ ($\tilde{f}\prec\!\!\prec ц_i$). Collect and return the so obtained solutions to (3.5)

STEP 5. Let φ be the unique solution to $P^{(d-1)}(\varphi)=0$ ($\varphi\prec\!\!\prec ч$). Apply `polynomial_solve` to the equation obtained from (3.5), by refining $f=\varphi+\tilde{f}$ ($\tilde{f}\prec\!\!\prec\varphi$). Return the so obtained solutions to (3.5).

The correctness of `polynomial_solve` is clear; its termination follows from lemma 3.2 and lemma 3.3(c). Since C is algebraically closed, all Newton polynomials considered in the algorithm split over C. Hence, `polynomial_solve` returns d solutions to (3.5) in $C[[X]]$. If ч is the formal monomial with $ц\prec\!\!\prec ч$ for all $ц\in X$, then we have $d=n$. Indeed, let i be such that $ц=\sqrt[n-i]{\mathrm{M}(P_i)/\mathrm{M}(P_n)}$ is maximal for $\prec\!\!\prec$. Then ц is an potential dominant monomial for (3.5) and its associated

Newton polynomial has degree n. We conclude that $C[[X]]$ is algebraically closed. Finally, if the coefficients of P are all in $C[\![X]\!]$, then all computations take place in $C[\![X]\!]$, because of the bound for $\operatorname{supp} f$ in lemma 3.1. We infer that $C[\![X]\!]$ is also algebraically closed. □

Corollary I. *Let C be a real algebraically closed field and X a totally ordered monomial group with $\mathbb{Q}$-powers. Then both $C[[X]]$ and $C[\![X]\!]$ are real algebraically closed.*

Proof. By the theorem, a polynomial equation of degree n over $C[i][[X]]$ admits n solutions (counting with multiplicities) in $C[i][[X]]$. Moreover, each root φ which lies in $C[i][[X]]\backslash C[[X]]$ is imaginary, because $C[[X]][\varphi] = C[i][[X]]$ for such φ. The proof is analogue for $C[\![X]\!]$. □

Corollary II. *The field $C[\![z^{\mathbb{Q}}]\!]$ of Puiseux series over an algebraically closed resp. real algebraically closed field C is algebraically closed resp. real algebraically closed.* □

3.4 A simple generalization

The restriction for P in (3.5) to be a polynomial is essentially superfluous for the method, and P can actually be replaced by any series in $C[[X, f^{\mathbb{Z}}]]$ or $C[\![X, f^{\mathbb{Z}}]\!]$. However, in this case we have to exclude those potential dominant monomials which lead to Newton polynomials of infinite degree. We notice that once we have chosen such a potential dominant monomial for the main equation (3.5), then this problem does not reappear for the refinements, because of lemma 3.2.

Let us also remark that, incorporating the above extension, there exist natural examples where the main equation has an infinity of potential dominant monomials. Such examples are constructed by considering certain types of difference equations. For instance, consider the system

$$\begin{cases} \varphi(x, f) = x + f\varphi(x^2, f) \\ \varphi(x, f) = 0, \end{cases}$$

where f is infinitesimal and x infinitely large. This system is equivalent to the equation

$$x + x^2 f + x^4 f^2 + x^8 f^3 + \cdots = 0.$$

This equation can be solved by the Newton polygon method and we find an infinite number $f_0, f_1, f_2, \cdots$ of solutions, with $f_i \sim -x^{-2^i}$.

Chapter 4

Linear differential equations

4.1 Introduction

Let C be a totally ordered exp-log field of constants. We will show how to solve linear differential equations

$$L(f) = g, \tag{4.1}$$

where L is a linear differential operator

$$L = L_0 + L_1\frac{\partial}{\partial x} + \cdots + L_r\frac{\partial^r}{\partial x^r} \qquad (L_r \neq 0) \tag{4.2}$$

with coefficients in the field $\mathbb{T} = C_\omega[[[x]]]$ of transseries of finite logarithmic and at most countable exponential depths.

We focus on the case when the coefficients of L are **purely exponential**, i.e. in $\mathbb{T}^{exp} = C_\omega^{alog}[[[\exp x]]]$. In this case, L maps the space $\mathbb{T}^{exp}[x]$ into itself, and we shall show that L admits a linear right inverse L^{-1}. The general case when the coefficients of L are in $\mathbb{T}$ will then be recovered by the use of upward shiftings. In sections 4.3 and 4.4, we prove the existence of a distinguished linear right inverse L^{-1}, and give a theoretical way to compute the distinguished solution $L^{-1}g$ to (4.1). Intuitively speaking, the distinguished solution to a linear equation is the simplest solution, which does not depend on the solutions to the homogeneous equation. For instance, in the case of integration ($L = \partial/\partial x$), the distinguished solution is precisely the one which corresponds to taking zero for the integration constant. We will also present a characterization for L^{-1} in section 4.4.

In section 4.5, we turn to the resolution of the homogeneous equation

$$Lh = 0. \tag{4.3}$$

The solutions of this equation form a finite dimensional totally ordered vector space H over C, which admits a basis $h_1 \prec\!\!\prec \cdots \prec\!\!\prec h_s$ (see lemma 8.1). Using the distinguished right inverse L^{-1}, it will then be sufficient to determine the dominant monomials of $h_1, \cdots, h_s$.

Finally, in section 4.6, we outline how the theory of this chapter generalizes to the case when we also allow oscillating solutions to (4.1). In this case, we show that the vector space of solutions to (4.3) has maximal dimension r.

When we restrict ourselves to linear differential equations with coefficients in rings of formal power series, then our results are more or less analogous to classical results: the introduction of formal local solutions for linear differential equations goes back to [Fab 1885]. The first algorithms are due to Della Dora, Tournier, and others (see [DDT 82], [Tour 87]). Their work is based on theoretical results of Malgrange and Ramis, who introduced Newton polygons in this context (see [Ram 78], [Mal 79]).

4.2 Preliminaries

For the purpose of differential calculus, we will need variants of the asymptotic relations $\prec$, $\preccurlyeq$ and $\asymp$ modulo $\lll h$ perturbations: let f, g and h be transseries with $h \neq 0$. We denote $f \prec_h g$ (or $f = o_h(g)$), if for all $\varphi \lll h$ we have $f \prec \varphi g$. Similarly, we denote $f \preccurlyeq_h g$ (or $f = O_h(g)$) resp. $f \asymp_h g$, if $f \preccurlyeq \varphi g$ resp. $f \asymp \varphi g$ for some $\varphi \lll h$. We state without proof the following easy properties:

Proposition 4.1. *Let $h \in \mathbb{T}^*$. Then*

(a) $\preccurlyeq_h$ is a quasi-ordering on $\mathbb{T}$ and $\asymp_h$ its associated equivalence relation.
(b) For fixed g, the set of f with $f \prec_h g$ resp. $f \preccurlyeq_h g$ forms an additive group.
(c) $f \prec_h g \Leftrightarrow f/g \prec_h 1$ for all $f, g \in \mathbb{T}$, and similarly for $\preccurlyeq_h$ and $\asymp_h$.

Let L be an arbitrary linear differential operator (4.2) with coefficients in $\mathbb{T}$. Then its **dominant monomial** $\mathrm{M}(L)$ is defined by

$$\mathrm{M}(L) = \max_{\prec}\{\mathrm{M}_{L_0}, \cdots, \mathrm{M}_{L_r}\}.$$

4.2.1 Multiplicative conjugation

Given a non zero transseries h, there exists a unique linear differential operator $L_{\times h}$ such that

$$L_{\times h}(f) = L(hf)$$

for all f. We call $L_{\times h}$ a **multiplicative conjugate** of L. The coefficients of $L_{\times h}$ are given explicitly by

$$L_{\times h, i} = \sum_{j=i}^{r} \binom{j}{i} L_j h^{(j-i)}. \tag{4.4}$$

Notice also that $L_{\times h_1 h_2} = L_{\times h_1, \times h_2}$ for all $h_1, h_2 \in \mathbb{T}^*$.

Proposition 4.2. *If $h \ggcurly x$, then*

$$\text{M}(L_{\times h}) \asymp_h h\text{M}(L).$$

Proof. By the hypothesis that $h \ggcurly x$, it follows that $h^{(i)} \asymp_h h$ for all i. In particular, (4.4) yields

$$L_{\times h,i} \preccurlyeq_h h\text{M}(L)$$

for all i, whence $\text{M}(L_{\times h}) \preccurlyeq_h \text{M}(l)$. Let m be the highest index for which $L_m \asymp_h \text{M}(L)$. Then (4.4) yields

$$L_{\times h,m} = hL_m + o_h(h\text{M}(L)).$$

In other words, $\text{M}(L_{\times h}) \succcurlyeq_h \text{M}(l)$. □

4.2.2 Upward shifting

To solve the equation $Lf = g$, it may be necessary to perform one or several upward shiftings. For this purpose, we define the **upward shifting** $L\uparrow$ of L by

$$(L\uparrow)(f\uparrow) = L(f)\uparrow,$$

for all f. In other words, solving the equation $Lf = g$ is equivalent to solving the equation $(L\uparrow)(f\uparrow) = g\uparrow$. The coefficients of $L\uparrow$ are given explicitly by

$$(L\uparrow)_i = \sum_{j=i}^{r} C_{i,j} e^{-jx} (L_j\uparrow), \tag{4.5}$$

where the $C_{i,j}$ are constants, determined by

$$(f(\log x))^{(j)} = \sum_{i=0}^{j} C_{i,j} x^{-j} f^{(i)}(\log x).$$

We have $C_{i,j} = s_{j,i}$, where the $s_{j,i}$ are the Stirling numbers of the first kind. Upward shifting is compatible with multiplicative conjugation, in the sense that

$$L_{\times h}\uparrow = (L\uparrow)_{\times h\uparrow}, \tag{4.6}$$

for all non zero transseries h.

Proposition 4.3. *For all $i, j \in \mathbb{N}$, we have*

$$\sum_{k=0}^{j} C_{k,j} (i+j)^k = \frac{(i+j)!}{i!}.$$

Proof. Consider the operator $L = \frac{\partial^j}{\partial x^j}$. The solution space of the homogeneous equation associated to the operator $L\uparrow_{\times e^{(i+j)x}}$ admits $e^{-(i+1)x}, \cdots, e^{-(i+j)x}$ as a basis. Hence, this operator factorizes

$$L\uparrow_{\times e^{(i+j)x}} = A\left(\frac{\partial}{\partial x} + i + j\right) \cdots \left(\frac{\partial}{\partial x} + i + 1\right),$$

and by looking at the coefficient of $\frac{\partial^j}{\partial x^j}$, we find that $A = e^{ix}$. It follows that

$$L\uparrow_{\times e^{(i+j)x},0} = \frac{(i+j)!}{i!}.$$

On the other hand, we have

$$L\uparrow_k = C_{k,j} e^{-jx},$$

for all k, by (4.4). Hence

$$L\uparrow_{\times e^{(i+j)x},0} = \sum_{k=0}^{j} C_{k,j}(i+j)^k,$$

by (4.5). □

4.3 Dominant monomials of distinguished solutions

Let L be a linear differential operator (4.2) with coefficients in $\mathbb{T}^{exp}$ and consider the equation (4.1) for $f, g \in \mathbb{T}^{exp}[x]$.

Theorem 4.1. *There exists a unique transmonomial* $ц \in \mathbb{T}^{exp}$, *such that* $\mathrm{M}_g/\mathrm{M}(L_{\times ц}) \in x^{\mathbb{N}}$.

Proof. The uniqueness of $ц$ trivially follows from proposition 4.2. Let us therefore prove its existence. Let $(ц_\alpha)_\alpha$ be the transfinite sequence of monomials, defined by

$$\begin{aligned} ц_0 &= 1; \\ ц_{\alpha+1} &= ц_\alpha \exp\left(\tau(\log(\mathrm{M}_g/\mathrm{M}(L_{\times ц_\alpha})))\right), \quad \text{if } \mathrm{M}_g/\mathrm{M}(L_{\times ц_\alpha}) \notin x^{\mathbb{N}}; \\ ц_\beta &= \exp\left(\operatorname{stat}\lim_{\alpha<\beta} \log ц_\alpha\right), \quad \text{for limit ordinals } \beta; \end{aligned}$$

here $\tau(\log \mathrm{M}_g/\mathrm{M}(L_{\times ц_\alpha}))$ denotes the dominant term of $\log \mathrm{M}_g/\mathrm{M}(L_{\times ц_\alpha})$. Intuitively speaking, this transfinite sequence corresponds to the computation of the successive terms of $\log ц$. We will show by transfinite induction that for all β:

– If $\alpha < \beta$, then $\log \text{ц}_\alpha \lhd \log \text{ц}_\beta$.

– $\forall \text{ч} \in \operatorname{supp} \log \text{ц}_\beta \quad \text{м}_g \asymp_{e^{\text{ч}}} \text{м}(L_{\times \text{ц}_\beta})$.

This will imply in particular that $\text{м}_g / \text{м}(L_{\times \text{ц}_\alpha}) \in x^{\mathbb{N}}$, for some ordinal α, since the length of the sequence $(\text{ц}_\alpha)_\alpha$ cannot exceed the cardinal number of $\mathbb{T}^{exp}[x]$, as a result of the first assertion. The induction hypothesis is clearly satisfied for $\beta = 0$. Assume now that for some $\beta > 0$, the induction hypothesis is satisfied for all smaller β.

Successor ordinals. Assume that $\beta = \alpha + 1$ and $\text{м}_g / \text{м}(L_{\times \text{ц}_\alpha}) \notin x^{\mathbb{N}}$. Let щ denote $\exp(\tau(\log(\text{м}_g / \text{м}(L_{\times \text{ц}_\alpha}))))$, so that $\text{м}_g \asymp_{\text{щ}} \text{щ}\text{м}(L_{\times \text{ц}_\alpha})$. Since $L_{\times \text{ц}_\beta} = L_{\times \text{ц}_\alpha, \times \text{щ}}$, we have

$$\text{м}(L_{\times \text{ц}_\beta}) \asymp_{\text{щ}} \text{щ}\text{м}(L_{\times \text{ц}_\alpha}),$$

by proposition 4.2. Hence,

$$\text{м}_g \asymp_{\text{щ}} \text{м}(L_{\times \text{ц}_\beta}).$$

More generally, $\text{м}_g \asymp_{e^{\text{ч}}} \text{м}(L_{\times \text{ц}_\beta})$ for all $\text{ч} \in \operatorname{supp} \log \text{ц}_\beta$, since $e^{\text{ч}} \underline{\ggg} \text{щ}$. Therefore, both induction hypotheses are satisfied at stage β.

Limit ordinals. Assume now that β is a limit ordinal. Given $\text{ч} \in \operatorname{supp} \log \text{ц}_\beta$, there exists an $\alpha < \beta$, with $\text{ч} \in \operatorname{supp} \log \text{ц}_\alpha$. By the induction hypothesis, we have

$$\text{м}_g \asymp_{e^{\text{ч}}} \text{м}(L_{\times \text{ц}_\alpha}).$$

We also have $\text{ц}_\beta / \text{ц}_\alpha \lll \text{ч}$, since $\text{ц}_\alpha \lhd \text{ц}_\beta$. Therefore,

$$\text{м}(L_{\times \text{ц}_\beta}) \asymp_{e^{\text{ч}}} \text{м}(L_{\times \text{ц}_\alpha}),$$

by proposition 4.2, whence the second induction hypothesis is again satisfied at stage β. The first one is trivially satisfied. □

With the notations from the theorem, let $\text{м}_g / \text{м}(L_{\times \text{ц}}) = x^i$, and let j be minimal, such that $\text{м}(L_{\times \text{ц}}) = \text{м}(L_{\times \text{ц}, j})$. Then $i + j$ is unique with the property that $\text{м}(L \text{ц} x^{i+j}) = \text{м}_g$. In the next section, we will show that there exists a "distinguished solution" f to (4.1) with $\text{м}_f = \text{ц} x^{i+j}$. Since this solution will be denoted by $L^{-1} g$, we denote $\text{м}_{L^{-1} g} = \text{ц} x^{i+j}$ by anticipation. Furthermore, the dominant coefficient of a solution to (4.1) with $\text{м}_f = \text{м}_{L^{-1} g}$ is necessarily given by $c_{L^{-1} g} \stackrel{\text{def}}{=} c_{g - L f_\alpha} / c_{L_{\times \text{ц}, 0}}$, i.e. the quotient of the dominant coefficients of $g - L f_\alpha$ and $L_{\times \text{ц}, 0}$. We will denote $\tau_{L^{-1} g} = c_{L^{-1} g} \text{м}_{L^{-1} g}$.

The mapping $\text{м}_g \mapsto \text{м}_{L^{-1} g}$ preserves $\prec\!\!\prec$:

Proposition 4.4. *If $g_1 \prec\!\!\prec g_2$, then $\text{м}_{L^{-1} g_1} \prec\!\!\prec \text{м}_{L^{-1} g_2}$.*

Proof. Modulo considering $g_1^{-1}L_{\times \mathfrak{m}_{L^{-1}g_1}}$ instead of L, we may assume without loss of generality that $\mathfrak{m}_{L^{-1}g_1} = g_1 = 1$. If $g_2 \asymp x^i$, then we are done by the definition of $\mathfrak{m}_{L^{-1}g_2}$. If $g_2 \ggg x$, then $\mathfrak{m}_{L^{-1}g_2} \asymp_{g_2} g_2$, by construction (indeed, $\text{ц}_1 = \exp(\tau(\log g_2))$ and $\text{ц}_1 \trianglelefteq \log \mathfrak{m}_{L^{-1}g_2}$ in the construction of $L^{-1}g$, whence $\mathfrak{m}_{L^{-1}g_2} \asymp \text{ц}_1 \asymp g_2$). □

We also have compatibility with upward shifting:

Proposition 4.5. *Let g be a non zero transseries. Then*

(a) $\mathfrak{m}_{(L\uparrow)^{-1}g\uparrow} = \mathfrak{m}_{L^{-1}g}\uparrow$.
(b) $\mathfrak{m}(L_{\times \mathfrak{m}_{L^{-1}g}}\uparrow) = \mathfrak{m}((L_{\times \mathfrak{m}_{L^{-1}g}}\uparrow)_0)$.

Proof. Let i be such that $\mathfrak{m}_g / x^i \in \mathbb{T}^{exp}$, and let j be such that $\mathfrak{m}_{L^{-1}g} = \text{ц} x^{i+j}$, with $\text{ц} \in \mathbb{T}^{exp}$. By construction, j is minimal such that $\mathfrak{m}(L_{\times \text{ц}}) = \mathfrak{m}(L_{\times \text{ц},j}) = \mathfrak{m}_g / x^i$. From (4.5), we deduce that

$$\begin{aligned} (L_{\times \text{ц}}\uparrow)_k &= e^{-jx} L_{\times \text{ц},j}\uparrow (C_{k,j} + o(1)), \text{ for } k \leqslant j; \\ (L_{\times \text{ц}}\uparrow)_k &\lll e^{-jx} L_{\times \text{ц},j}\uparrow, \text{ for } k > j. \end{aligned}$$

It follows that

$$\mathfrak{m}((L_{\times \text{ц}}\uparrow)_{\times e^{(i+j)x}}) = \mathfrak{m}_g\uparrow.$$

Using (4.6), we get

$$\mathfrak{m}(L_{\times \mathfrak{m}_{L^{-1}g}}\uparrow) = \mathfrak{m}_g\uparrow. \tag{4.7}$$

The above relations for the $(L_{\times \text{ц}}\uparrow)_k$ also imply that

$$\begin{aligned} (L_{\times \mathfrak{m}_{L^{-1}g}}\uparrow)_0 &= (L_{\times \text{ц}}\uparrow)_{\times e^{(i+j)x},0} \\ &= \sum_{k=0}^{r} (L_{\times \text{ц}}\uparrow)_k (i+j)^k e^{(i+j)x} \\ &= \left(\sum_{k=0}^{j} C_{k,j}(i+j)^k + o(1) \right) (L_{\times \text{ц}}\uparrow)_j e^{ix}. \end{aligned}$$

By proposition 4.3, we infer that 0 is the smallest index k, with $\mathfrak{m}(L_{\mathfrak{m}_{L^{-1}g}}\uparrow) = \mathfrak{m}((L_{\times \mathfrak{m}_{L^{-1}g}}\uparrow)_k)$. Together with (4.7), this implies that $\mathfrak{m}_{(L\uparrow)^{-1}g\uparrow} = \mathfrak{m}_{L^{-1}g}\uparrow$. □

It is finally natural to ask for which monomials $\mathfrak{m}$ there exists a $g \neq 0$ with $\mathfrak{m}_{L^{-1}g} = \mathfrak{m}$? Recalling that $h_1 \lll \cdots \lll h_s$ form a basis for the solution space H to the homogeneous equation (4.3), we have the following characterization of these monomials:

Proposition 4.6. *Let $\mathfrak{m}$ be a transmonomial in $\mathbb{T}^{exp}[x]$. Then a necessary and sufficient condition for the existence of a non zero transseries $g \in \mathbb{T}^{exp}[x]$ with $\mathfrak{m}_{L^{-1}g} = \mathfrak{m}$, is that $\mathfrak{m} \notin \{\mathfrak{m}_{h_1}, \cdots, \mathfrak{m}_{h_s}\}$.*

Proof. We will only prove the necessity of the condition here; the sufficiency will be proved in the next section. In view of the proof of the previous proposition, we may assume without loss of generality that $\text{м}(L_{\times\text{м}}) = \text{м}(L_{\times\text{м},0}) = \text{м}_g$, modulo one upward shifting. Then for any $h \in \mathbb{T}^{exp}[x]$ with $h \asymp \text{м}$, we have

$$Lh = L_{\times\text{м}}(h/\text{м}) = c_{L_{\times\text{м},0}} c_h \text{м}_g + o(\text{м}_g) \neq 0,$$

where $c_{L_{\times\text{м},0}}$ resp. c_h denote the dominant coefficients of $L_{\times\text{м},0}$ resp. h. Hence, h cannot be a solution to (4.3). □

4.4 Computation of distinguished solutions

Theorem 4.2. *There exists a right inverse for L.*

Proof. Let us show how to compute a solution to (4.1) for fixed g. Let $(f_\alpha)_\alpha$ be the transfinite sequence of transseries in $\mathbb{T}^{exp}[x]$, defined by:

$$\begin{aligned} f_0 &= 0; \\ f_{\alpha+1} &= f_\alpha + \tau_{L^{-1}(g-Lf_\alpha)}, \quad \text{if } Lf_\alpha \neq g; \\ f_\beta &= \operatorname*{stat\,lim}_{\alpha<\beta} f_\alpha \text{ for limit ordinals } \beta. \end{aligned}$$

We will show by transfinite induction that for all β:

- If $\alpha < \beta$, then $f_\alpha \lhd f_\beta$.
- $\forall \text{ч} \in \operatorname{supp} f_\beta \quad Lf_\beta - g \prec\!\!\prec L\text{ч}$.

This will in particular imply that $f_\alpha = g$ for some $\alpha \leqslant |\mathbb{T}^{exp}[x]|$, since the length of the sequence $(f_\alpha)_\alpha$ can not exceed the cardinal number of $\mathbb{T}^{exp}[x]$. The induction hypothesis is trivially satisfied for $\beta = 0$. Assume now that $\beta > 0$, and the induction hypothesis is satisfied for all smaller β.

Successor ordinals. Assume that $\beta = \alpha + 1$ and $Lf_\alpha \neq g$. Denote $\text{ц} = \tau_{L^{-1}(g-Lf_\alpha)}$. Since $c_{L^{-1}(g-Lf_\alpha)}$ is a constant, we have

$$L(c_{L^{-1}(g-Lf_\alpha)}\text{ц}) = L_{\times\text{ц}}(c_{L^{-1}(g-Lf_\alpha)}) = L_{\times\text{ц},0} c_{L^{-1}(g-Lf_\alpha)} = c_{g-Lf_\alpha} \text{м}_{g-Lf_\alpha}.$$

In particular, $L\text{ц} \asymp g - Lf_\alpha$, whence $\text{ц} \prec\!\!\prec \text{ч}$ for all $\text{ч} \in \operatorname{supp} f_\alpha$, by proposition 4.4 and the second induction hypothesis. Hence, the first induction hypothesis is satisfied at stage β. As to the second, let $\text{ч} \in \operatorname{supp} f_\beta$: then either $\text{ч} = \text{ц}$ or $\text{ч} \in \operatorname{supp} f_\alpha$ and $L\text{ц} \asymp g - Lf_\alpha \prec\!\!\prec L\text{ч}$. In both cases,

$$g - Lf_\beta = g - Lf_\alpha - L_{\times\text{ц},0} c_{L^{-1}(g-Lf_\alpha)} \prec\!\!\prec g - Lf_\alpha \preceq\!\!\preceq L\text{ч}.$$

Limit ordinals. Assume that β is a limit ordinal and let $\text{ч} \in \operatorname{supp} f_\beta$. Then there exists an $\alpha < \beta$ with $\text{ч} \in \operatorname{supp} f_\alpha$, and by the induction hypothesis, we have

$$Lf_\gamma - g \prec\!\!\prec L\text{ч},$$

for all $\alpha \leqslant \gamma < \beta$. Now we have

$$Lf_\beta - g = (Lf_\alpha - g) + \sum_{\alpha \leqslant \gamma < \beta} (Lf_{\gamma+1} - g) - (Lf_\gamma - g),$$

by the strong linearity of L and the fact that β is a limit ordinal. This implies the second induction hypothesis at stage β; the first one is trivially verified. □

In what follows, L^{-1} denotes the right inverse of L as constructed in the above proof. By construction, the dominant monomial of $L^{-1}g$ coincides with $\mathrm{M}_{L^{-1}g}$ as defined in the previous section. The operator L^{-1} satisfies the following characteristic property, for which we call it **distinguished**:

Proposition 4.7. *L^{-1} is the unique right inverse of L, such that* $\operatorname{supp} L^{-1}g \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi$ *for all* $g \in \mathbb{T}^{exp}[x]$.

Proof. By the necessity of the condition in proposition 4.6, we infer that we never have $\text{ц} \in \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\}$ during the construction of $f = L^{-1}g$. By a straightforward transfinite induction over α, it follows that $f_\alpha \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi$ for all α.

Assume now that $L\tilde{f} = Lf = g$, with $\operatorname{supp} \tilde{f} \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \operatorname{supp} f \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi$. Then $L(\tilde{f} - f) = 0$, whence $\tilde{f} - f$ is a linear combination of $h_1, \cdots, h_s$. Since, $\operatorname{supp} (\tilde{f} - f) \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi$, we get $\tilde{f} = f$, whence the uniqueness of L^{-1}. □

Corollary I. *The operator L^{-1} is linear.* □

Corollary II. *Let $\mathrm{M} \notin \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\}$ be a transmonomial in $\mathbb{T}^{exp}[x]$. Then there exists a transseries $g \in \mathbb{T}^{exp}[x]^*$ with $\mathrm{M}_{L^{-1}g} = \mathrm{M}$.*

Proof. Taking $g = L\mathrm{M}$, we have $\mathrm{M} - L^{-1}g \in \ker L$. If $\mathrm{M} = L^{-1}g$, then we are clearly done. Otherwise, we have $\mathrm{M} \not\succ \mathrm{M}_{\mathrm{M}-L^{-1}g} \in \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\}$. If $\mathrm{M} - L^{-1}g \prec\!\!\prec \mathrm{M}$, then $L^{-1}g \asymp \mathrm{M}$ and we are done. The other case is impossible, since we cannot have $\mathrm{M}_{L^{-1}g} \in \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\}$. □

It follows that $H = \ker L$ admits a supplement

$$\mathbb{T}^{exp}[x]/H \stackrel{\text{def}}{=} \{f \in \mathbb{T}^{exp}[x] | \operatorname{supp} f \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi\}$$

in $\mathbb{T}^{exp}[x]$, which is distinguished in the following sense:

Proposition 4.8. *The operator L determines an isomorphism*

$$\mathbb{T}^{exp}[x]/H \to \mathbb{T}^{exp}[x],$$

whose inverse is given by L^{-1}. □

Let us now consider upward shifting of linear differential operators and there right inverses:

Proposition 4.9. *For all linear differential operators (4.2) with coefficients in $\mathbb{T}^{exp}[x]$, we have $L^{-1}\!\uparrow = L\!\uparrow^{-1}$.*

Proof. This is a routine exercise, since upward shifting commutes with all operations used in the construction of L^{-1}, and in particular with the computation of the dominant monomials of solutions by proposition 4.5. □

From proposition 4.9, it follows that we can extend theorem 4.2 and proposition 4.8 to the case when L is an arbitrary linear differential operator with coefficients in a field of transseries $\mathbb{T}$ with finite logarithmic depth:

Theorem 4.3. *Let L be a linear differential operator (4.2) with coefficients in $\mathbb{T}$. Let $h_1 \prec\!\!\prec \cdots \prec\!\!\prec h_s$ be a basis for the vector space of solutions to (4.3). Then L determines an isomorphism*

$$\mathbb{T}/H \to \mathbb{T},$$

where

$$\mathbb{T}/H = \{f \in \mathbb{T} | \operatorname{supp} f \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \emptyset\}.$$

The inverse L^{-1} of L is linear.

Proof. Let $g \in \mathbb{T}$. For l large enough, $g\!\uparrow_l$ and the coefficients of $L\!\uparrow_l$ are in $\mathbb{T}^{exp}$, where $\uparrow_l$ denotes the l-th iteration of $\uparrow$. Then $f\!\uparrow_l = (L\!\uparrow_l)^{-1}(g\!\uparrow_l)$ is well defined, and we have $Lf = g$. Moreover, supp $f\!\uparrow_l \cap \{\mathrm{M}_{h_1}\!\uparrow_l, \cdots, \mathrm{M}_{h_s}\!\uparrow_l\} = \emptyset$ by proposition 4.8. Consequently, $f \in \mathbb{T}/H$. Finally, $f = L^{-1}g$ does not depend on the choice of l, by proposition 4.9. Consequently, L^{-1} is linear by theorem 4.2. □

Corollary. *There exists a unique integration operator $\int$ on $\mathbb{T}$, such that $(\int f)' = f$ and $(\int f)^c = 0$ for all $f \in \mathbb{T}$ (here we recall that $(\int f)^c$ denotes the constant part of $\int f$).* □

We have already shown that if $f \in \mathbb{T}^{exp}$, then $\int f \in \mathbb{T}^{exp}[x]$. Actually, a slightly stronger assertion holds:

Proposition 4.10. *If $f \in \mathbb{T}^{exp}$, then $\int f \in \mathbb{T}^{exp} + Cx$.*

Proof. Assume that $\int f \notin \mathbb{T}^{exp} + Cx$ and let ч be the biggest monomial in $\operatorname{supp} f$ for $\prec\!\!\prec$, with $(\int f)_{\text{ч}} \notin \mathbb{T}^{exp} \cup \{x\}$. Let $\varphi = f - \sum_{\text{ц} \succ\!\!\succ \text{ч}} f_{\text{ц}}$. Then we have $\text{м}_{\varphi} \in \mathbb{T}^{exp}$, while $\text{м}(\int \varphi) \notin \mathbb{T}^{exp} \cup \{x\}$.

If $\text{м}_{\varphi} = 1$, then we must have $\text{м}(\int \varphi) = x$ and we are done. Otherwise, there exists a unique transmonomial $\text{ц} \in \mathbb{T}^{exp}$ with $\text{м}((\partial/\partial x)_{\times \text{ц}}) = \text{м}_{\varphi}$ by theorem 4.1. Since $\text{м}_{\varphi} \succ\!\!\succ\!\!\succ x$, we have $\text{м}((\partial/\partial x)_{\times \text{ц}}) = \text{м}((\partial/\partial x)_{\times \text{ц},0})$, whence $\text{м}(\int \text{м}_{\varphi}) = \text{ц}$. Therefore $\text{м}(\int \varphi) = \text{ц}$ by construction: contradiction. □

4.5 Solving the homogeneous equation

Modulo upward shiftings, it suffices to show how to find the solutions to (4.3) in the case when L has coefficients in $\mathbb{T}^{exp}$. Moreover, in order to compute a basis $h_1 \prec\!\!\prec \cdots \prec\!\!\prec h_s$ it suffices to find the dominant monomials of $h_1, \cdots, h_s$. Indeed, given the dominant monomial м_{h_i} of h_i, we take

$$h_i = \text{м}_{h_i} - L^{-1} L \text{м}_{h_i}.$$

A special case in which dominant monomials of basis elements are easily determined is the following:

Proposition 4.11. *Assume that $L_0 \prec\!\!\prec \text{м}(L)$, and let i be minimal such that $L_i \asymp \text{м}(L)$. Then $1, \cdots, x^{i-1}$ are dominant monomials of solutions to (4.1).*

Proof. Let $\text{м} = x^j$ and $h = \text{м} - L^{-1} L\text{м}$, with $0 \leqslant j \leqslant i-1$. We claim that $\text{м}_h = \text{м}$. Since $j < i$, we have $L\text{м} \prec\!\!\prec \text{м}(L)$. Therefore, in the construction of $\text{м}_{L^{-1}L\text{м}}$, we have $\text{ц}_1 \prec\!\!\prec 1$. Since $\log \text{ц}_1 \vartriangleleft \log \text{ц}_{\alpha}$ for all $\alpha > 1$, it follows that $\text{м}_{L^{-1}L\text{м}} \prec\!\!\prec \text{м}$. Hence $h \sim \text{м}$, which proves our claim. □

The idea is now to determine those monomials $\text{ц} \in \mathbb{T}^{exp}$, such that $L_{\times \text{ц},0} \prec\!\!\prec \text{м}(L_{\times \text{ц}})$, and to prove that all м_{h_i} can be obtained from proposition 4.11, after multiplicative conjugation by such a ц. For this, we look at the **Riccati equation** associated to (4.3), which is an algebraic differential equation

$$L_0 R_0(\hat{h}) + L_1 R_1(\hat{h}) + L_2 R_2(\hat{h}) + \cdots + L_r R_r(\hat{h}) = 0 \tag{4.8}$$

in the logarithmic derivative $\hat{h}$ of h. Here the R_i are determined by $h^{(i)} = R_i(\hat{h})h$; for instance,

$$\begin{aligned} R_0(\hat{h}) &= 1; \\ R_1(\hat{h}) &= \hat{h}; \\ R_2(\hat{h}) &= \hat{h}^2 + \hat{h}'; \\ R_3(\hat{h}) &= \hat{h}^3 + 3\hat{h}'\hat{h} + \hat{h}''; \\ &\vdots \end{aligned}$$

In the case when $h \succ\!\!\succ\!\!\succ x$, we have $\hat{h}^{(i)} \asymp_{\hat{h}} \hat{h}$, whence $R_i(\hat{h}) = \hat{h}^i + o_{\hat{h}}(\hat{h}^i)$ for all i. Therefore, (4.8) becomes "quasi-algebraic":

$$L_0 + L_1\hat{h} + L_2\hat{h}^2(1+o(1)) + \cdots + L_r\hat{h}^r(1+o(1)).$$

In particular, the dominant monomial of any solution $\hat{h}$ is in $\mathbb{T}^{exp}$. On the other hand, if $h \preceq\!\!\!\prec\!\!\prec x$ is a solution to the homogeneous equation, then $\mathrm{M}(h)$ is determined via proposition 4.11:

Proposition 4.12. *If $\hat{h}$ is a solution to (4.8) with $h \preceq\!\!\!\prec\!\!\prec x$, then $\mathrm{M}_h = x^i$ for some $i \in \mathbb{N}$. Moreover, $L_j \prec\!\!\prec \mathrm{M}(L)$ for all $j \leqslant i$.*

Proof. Decompose $h = \varphi x^\lambda$, with $\varphi \prec\!\!\prec\!\!\prec x$, and assume that $h \notin x^{\mathbb{N}}$. If $\lambda \notin \mathbb{N}$, or $i \leqslant \lambda$, then we have

$$h^{(i)} \asymp \varphi x^{\lambda - i}.$$

Otherwise, we have $x\hat{\varphi} \prec\!\!\prec 1$, $x\hat{\varphi} \prec\!\!\prec\!\!\prec x$ and

$$h^{(i)} \asymp (x\hat{\varphi})\varphi x^{\lambda - i}.$$

In both cases, we thus have $h^{(i)} \asymp_x x^{\lambda - i}$ for all i. Since the coefficients of L are all purely exponential transseries, it follows that no cancelations can take place in (4.3): contradiction.

Therefore, we have $h = x^i$ for some $i \in \mathbb{N}$. Let j be the smallest index with $L_j \asymp \mathrm{M}(L)$. If j were smaller or equal to i, then we would have

$$L_0R_0(\hat{h}) + L_1R_1(\hat{h}) + L_2R_2(\hat{h}) + \cdots + L_rR_r(\hat{h}) \asymp L_jR_j(\hat{h}) \asymp \mathrm{M}(L)x^{-j},$$

since the coefficients of L are purely exponential transseries. Consequently, we must have $j > i$. $\square$

Putting together the above results, we get:

Proposition 4.13. *If h is a solution to (4.3), then $\mathrm{M}_h \in \mathbb{T}^{exp}x^{\mathbb{N}}$.*

Proof. Assume the contrary. Then $\hat{h} \notin \mathbb{T}^{exp}$; otherwise $\int \hat{h} \in \mathbb{T}^{exp} + Cx$, whence $h \in \mathbb{T}^{exp}$. Let ч be the biggest transmonomial in $\operatorname{supp}\hat{h}$ for $\prec\!\!\prec$, such that $\text{ч} \notin \mathbb{T}^{exp}$ and define $\varphi = \sum_{\text{ц} \succ\!\!\succ \text{ч}} \hat{h}_{\text{ц}}$.

Let us first assume that $\exp(\int \text{ч}) \succ\!\!\succ\!\!\succ x$. Then $\hat{h} - \varphi$ is a solution to the Riccati equation associated to the equation $L_{\times \exp(\int \varphi)}\tilde{h} = 0$. Since this Riccati equation is quasi-algebraic, the dominant monomial of $\hat{h} - \varphi$, which is ч, lies in $\mathbb{T}^{exp}$. Contradiction.

In the remaining case, the solution $h\exp(-\int \text{ц})$ to the equation $L_{\times \exp(\int \text{ц})}\tilde{h} = 0$ satisfies $\mathrm{M}(h\exp(-\int \text{ц})) = x^i$ for some $i \in \mathbb{N}$ by proposition 4.12. This leads again to the desired contradiction. $\square$

In view of the fact that the h_i has been chosen such that $h_i = \mathrm{M}_{h_i} - L^{-1} L \mathrm{M}_{h_i}$ for all i, the above proposition immediately implies the following theorem:

Theorem 4.4. *Assume that L has coefficients in $\mathbb{T}^{exp}$. Then any transseries solution h to (4.3) is in $\mathbb{T}^{exp}[x]$.* □

Although the above theorem describes the structure of the solutions to (4.3), it does not provide a theoretical way to construct the solutions. Therefore, we will now briefly describe how the algebraic Newton polygon method can be adapted to solve the Riccati equation (4.8). More details in a more general context will be given in sections 5.4 and 5.5.

First, we notice again that we only have to determine the dominant monomial of each $h \in \{h_1, \cdots, h_s\}$. Instead of (4.8), we will consider Riccati equations with an additional asymptotic constraint:

$$L_0 R_0(\hat{h}) + L_1 R_1(\hat{h}) + L_2 R_2(\hat{h}) + \cdots + L_r R_r(\hat{h}) = 0 \ (\hat{h} \prec\!\!\prec \text{ч}) \tag{4.9}$$

If $h \preceq\!\!\prec\!\!\prec x$, then h is given by proposition 4.11 and proposition 4.12. In the other case, the Riccati equation (4.9) is quasi-algebraic, and we define potential dominant monomials, Newton degree etc. as in the algebraic case, by neglecting the $o(1)$-terms. However, we restrict our attention to non infinitesimal potential dominant monomials, since we assumed $h \succ\!\!\succ\!\!\succ x$.

Now a refinement

$$\hat{h} = \varphi + \tilde{\hat{h}} \ (\tilde{\hat{h}} \prec\!\!\prec \varphi)$$

corresponds to a multiplicative conjugation

$$L(f) = L_{\times \exp(\int \varphi)}(\tilde{f}) \ (\tilde{f} \prec\!\!\prec\!\!\prec \exp(\int \varphi)).$$

Therefore $\tilde{\hat{h}}$ again satisfies an asymptotic Riccati equation of the form (4.9), which is solved by transfinite induction (in a similar way as in theorem 4.2). Actually, we can reduce the necessity of transfinite induction to the case when the asymptotic Riccati equation is quasi-linear, by using a generalization of the trick from lemma 3.3. For details, we refer to section 5.5.

4.6 Oscillating solutions

The theory of linear differential equations is the only one, which can be fairly easily generalized to include oscillating solutions. In this section, we briefly present the main differences that such a generalization introduces w.r.t. the theory from the previous sections. The main reason why the generalization is possible is that we allow oscillating behaviour in the space on which L operates, but not in the coefficients of L itself.

4.6.1 Notations

Let C be a real algebraically closed totally ordered exp-log field. Then $K = C + iC$ is algebraically closed, as well as $\hat{\mathbb{T}} = \mathbb{T} + i\mathbb{T}$ and $\hat{\mathbb{T}}^{exp} = \mathbb{T}^{exp} + i\mathbb{T}^{exp}$.

Given $\psi_1, \cdots, \psi_n \in \mathbb{T}^{\uparrow} \backslash \{0\}$, we may formally extend $\hat{\mathbb{T}}$ with $e^{i\psi_1}, \cdots, e^{i\psi_n}$. Then $R = \hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}]$ is a differential ring, when taking $(e^{i\psi_j})' = i\psi_j' e^{i\psi_j}$ for each j. In what follows we will always assume that we have either $\psi_{j'}/\psi_j \in C$ or $\psi_{j'} \not\asymp \psi_j$, for all $j' \neq j$; this can always be accomplished, by replacing the ψ_j by linear combinations of them. The elements in R can be decomposed in two ways:

Polynomial decomposition. Since elements in R are polynomials, we first have the following trivial decomposition:

$$f = \sum_{k_1, \cdots, k_n} f_{k_1, \cdots, k_n} e^{i(k_1\psi_1 + \cdots + k_n\psi_n)},$$

with coefficients $f_{k_1, \cdots, k_n}$ in $\hat{\mathbb{T}}$.

Asymptotic decomposition. Each element f in R can also be written uniquely as a sum

$$f = \sum_{\text{ц}} f_{\text{ц}} \text{ц},$$

where the ц ranges over transmonomials in $\mathbb{T}$ and the coefficients $f_{\text{ц}}$ are in

$$K[e^{i\psi_1}, \cdots, e^{i\psi_n}].$$

In particular, the asymptotic relations $\prec$, $\prec\!\!\prec$, etc. naturally extend to R.

4.6.2 Distinguished solutions

Consider the linear differential equation (4.1), where the coefficients of L are in $\hat{\mathbb{T}}^{exp}$, and $g \in \hat{\mathbb{T}}^{exp}[x]$. Let $h_1 \prec \cdots \prec h_s$ denote a basis for the solution space of the corresponding homogeneous equation (4.3) in $\hat{\mathbb{T}}^{exp}[x]$. Then we have the following straightforward generalization of theorem 4.2 and proposition 4.7:

Proposition 4.14. *There exists a unique strongly linear right inverse L^{-1} for L, such that* $\operatorname{supp} L^{-1}g \cap \{\mathrm{M}_{h_1}, \cdots, \mathrm{M}_{h_s}\} = \phi$ *for all* $g \in \hat{\mathbb{T}}^{exp}[x]$. □

This right inverse L^{-1} can be extended to $\hat{\mathbb{T}}^{exp}[x][e^{i\psi_1}, \cdots, e^{i\psi_n}]$ (with the notations from section 4.6.1) as follows: for each $g \in \mathbb{T}^{exp}[x]e^{i(k_1\psi_1 + \cdots + k_n\psi_n)}$ with $k_1, \cdots, k_n \in \mathbb{N}$, we take

$$L^{-1}g = (e^{-i(k_1\psi_1 + \cdots + k_n\psi_n)} L_{\times e^{i(k_1\psi_1 + \cdots + k_n\psi_n)}})^{-1}(e^{-i(k_1\psi_1 + \cdots + k_n\psi_n)} g).$$

Next, we use linearity

$$L^{-1}\left(\sum_{k_1,\cdots,k_n} g_{k_1,\cdots,k_n} e^{i(k_1\psi_1+\cdots+k_n\psi_n)}\right) = \sum_{k_1,\cdots,k_n} L^{-1}(g_{k_1,\cdots,k_n} e^{i(k_1\psi_1+\cdots+k_n\psi_n)})$$

to extend L^{-1} to $\hat{\mathbb{T}}^{exp}[x][e^{i\psi_1},\cdots,e^{i\psi_n}]$.

For elements f in $\hat{\mathbb{T}}[x]e^{i(k_1\psi_1+\cdots+k_n\psi_n)}$ it is convenient to define the oscillating dominant monomial

$$\tilde{\mathrm{M}}_f = e^{i(k_1\psi_1+\cdots+k_n\psi_n)}\mathrm{M}_f$$

for each $k_1,\cdots,k_n \in \mathbb{N}$. Then we have:

Proposition 4.15. *There exists a unique basis $h_1,\cdots,h_s$ for the vector space of solutions in $\hat{\mathbb{T}}^{exp}[x][e^{i\varphi_1},\cdots,e^{i\varphi_n}]$ to the homogeneous equation (4.3), which satisfies the following hypotheses:*

(a) For each j, there exist $k_1,\cdots,k_n \in \mathbb{N}$, such that $h_j \in \hat{\mathbb{T}}^{exp}[x]e^{i(k_1\psi_1+\cdots+k_n\psi_n)}$.
(b) With the notations from (a), we have

$$h_j = \tilde{\mathrm{M}}(h_j) + L^{-1}L\tilde{\mathrm{M}}(h_j).$$

Proof. Let $h \neq 0$ be a solution to (4.3). Then for each $k_1,\cdots,k_n \in \mathbb{N}$ with $h_{\mathrm{M}(h),k_1,\cdots,k_n} \neq 0$,

$$h[k_1,\cdots,k_n] = \tilde{\mathrm{M}}(h_{k_1,\cdots,k_n}) + L^{-1}L\tilde{\mathrm{M}}(h_{k_1,\cdots,k_n})$$

is a solution to (4.3). Moreover, we have

$$h' \stackrel{\text{def}}{=} h - \sum_{k_1,\cdots,k_n} c(h_{k_1,\cdots,k_n})h[k_1,\cdots,k_n] \prec\!\!\prec h.$$

Now the vector space H of solutions to (4.3) is finite dimensional. Hence, repeating the argument on h', we may write h as a linear combination of solutions to (4.3) in $\bigcup_{k_1,\cdots,k_n} \hat{\mathbb{T}}^{exp}[x]e^{i(k_1\psi_1+\cdots+k_n\psi_n)}$. In other words, we have

$$H = \bigoplus_{k_1,\cdots,k_n} H[k_1,\cdots,k_n],$$

where

$$H[k_1,\cdots,k_n] = H \cap \hat{\mathbb{T}}^{exp}[x]e^{i(k_1\psi_1+\cdots+k_n\psi_n)}$$

for each $h_1,\cdots,h_n$. By lemma 8.1, there exists a basis of pairwise incomparable (for $\asymp$) elements for each $H[k_1,\cdots,k_n]$. Replacing each such basis element h by $\tilde{\mathrm{M}}(h)+$

$L^{-1}L\tilde{\mathrm{M}}(h)$, we obtain a basis for H with the desired properties. The uniqueness of this basis is trivial. □

Let us now turn to the general equation (4.1), where the coefficients of L are in $\hat{\mathbb{T}}$ and the coefficients of g in $\hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}]$. Using upward shifting in a similar way as before, we obtain the following theorem:

Theorem 4.5. *Let L be a linear differential operator (4.2) with coefficients in $\hat{\mathbb{T}}$, and let $\psi_1, \cdots, \psi_n$ be in $\mathbb{T}^{\uparrow}\backslash\{0\}$. There exists a unique basis of the vector space H of solutions to (4.3) in $\hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}]$ which satisfies the following properties:*

(a) For each j, there exist $k_1, \cdots, k_n \in \mathbb{N}$, such that $h_j \in \hat{\mathbb{T}}e^{i(k_1\psi_1+\cdots+k_n\psi_n)}$.
(b) With the notations from (a), we have

$$h_j = \tilde{\mathrm{M}}(h_j) + L^{-1}L\tilde{\mathrm{M}}(h_j).$$

Furthermore, L determines an isomorphism

$$\hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}]/H \to \hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}],$$

where

$$\hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}]/H = \{f \in \hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_n}] | \operatorname{supp} f \cap \{\tilde{\mathrm{M}}_{h_1}, \cdots, \tilde{\mathrm{M}}_{h_s}\} = \emptyset\}.$$

The inverse L^{-1} of L is linear. □

4.6.3 Solving the homogeneous equation

Let us now consider the homogeneous equation (4.3), where we assume that the coefficients of L are in $\hat{\mathbb{T}}^{exp}$.

Proposition 4.16. *There exist transseries $\psi_1, \cdots, \psi_r \in (\mathbb{T}^{exp})^{\uparrow}\backslash\{0\}$, such the vector space of solutions to (4.3) in $\hat{\mathbb{T}}[x][e^{i\psi_1}, \cdots, e^{\psi_r}]$ has dimension r.* □

Proof. Consider the analogue of the theoretical transfinite algorithm at the end of section 4.5 to compute the set of dominant monomials of solutions to (4.3). Let m be minimal such that $\mathrm{M}(L) \asymp \mathrm{M}(L_m)$ in (4.9). If $\text{ч} \not\succeq 1$, then $\{1, \cdots, x^{m-1}\}$ is precisely the set of dominant monomials of a solution $h \preceq\!\!\!\prec x$ to (4.3), by suitable generalizations of proposition 4.11 and proposition 4.12.

On the other hand, since K is algebraically closed, there are exactly $d-m$ potential dominant terms $cц$ for $\hat{h}$ with $ц \not\succeq 1$, where d is the Newton degree of (4.9). For each such dominant term $cц$, the refinement

$$\hat{h} = cц + \tilde{\hat{h}} \ (\tilde{\hat{h}} \prec\!\!\prec ц)$$

leads to a new asymptotic Riccati equation associated to the equation

$$\exp(\int e^{-(\Im c)ц}) L_{\times \exp(\int e^{cц})} h = 0 \ (\tilde{h} \prec\!\!\prec\!\!\prec \exp(\int e^{cц})),$$

which has coefficients in $\hat{\mathbb{T}}^{exp}$. By a suitable generalization of lemma 3.2, the Newton degree of this asymptotic Riccati equation in $\tilde{\hat{h}}$ is precisely the multiplicity of c as a root of the Newton polynomial associated to $ц$.

By transfinite induction, it follows that there are precisely r monomials of the form $цe^{i\psi_k}$ with $ц \in \hat{\mathbb{T}}^{exp}[x]$, such that $\tilde{\mathrm{M}}_h = цe^{i\psi_k}$ for some $h \in H$. $\square$

Modulo upward shiftings, we have proved the following:

Theorem 4.6. *Let (4.3) be a homogeneous linear differential equation of order r with coefficients in $\hat{\mathbb{T}} = \mathbb{T} + i\mathbb{T}$, where $\mathbb{T}$ is a field of transseries of finite logarithmic depths over a real algebraically closed totally ordered exp-log field of constants. Then there exist transseries $\psi_1, \cdots, \psi_r \in \mathbb{T}^{\uparrow}\setminus\{0\}$, such that the vector space of solutions to (4.3) in $\hat{\mathbb{T}}[e^{i\psi_1}, \cdots, e^{i\psi_r}]$ has dimension r.* $\square$

4.7 References

[Bal 94] W. Balser. *From divergent power series to analytic functions.* Lect. Notes in Math. 1582, Springer-Verlag.

[DDT 82] J. Della Dora, C. Dicrescenzo, E. Tournier. An algorithm to obtain formal solutions of a linear homogeneous differential equation at an irregular singular point. *Eurocal '82, Lect. Notes in Comp. Sc. 174, Springer-Verlag (p. 273-280).*

[Fab 1885] M.E. Fabry. Sur les intégrales des équations différentielles linéaires à coefficients rationnels.. *Thèse à la faculté des sciences de Paris, Gauthier-Villars.*

[Mal 79] B. Malgrange. Sur la réduction formelle des équations différentielles à singularités irrégulières. *Manuscript.*

[Ram 78] J.P. Ramis. Dévissage Gevrey. *Astérisque 59/60 (p. 173-204).*

[Ram 93] J.P. Ramis. Séries divergentes et théories asymptotitques. *Suppl. au bulletin de la Soc. Math. de France 121.*

[Tour 87] É. Tournier. Solutions formelles d'équations différentielles. *Doctorat d'État, Univ. scientifique, technologique et médicale de Grenoble.*

[VH 96] M. van Hoeij. Factorization of linear differential operators. *PhD. Thesis, Univ. of Nijmegen, The Netherlands.*

Chapter 5

Algebraic differential equations

5.1 Introduction

Let C be a totally ordered exp-log field of constants and $\mathbb{T} = C_\omega[[[x]]]$. In this chapter we study the **asymptotic algebraic differential equation**

$$P(f) = \sum_{i_0,\cdots,i_r \in \mathbb{N}} P_{i_0,\cdots,i_r} f^{i_0} \cdots (f^{(r)})^{i_r} = 0 \ (f \prec\!\!\prec \text{ч}), \tag{5.1}$$

of order r, with coefficients in $\mathbb{T}$. P is also called a **differential polynomial**. We will give a theoretical resolution algorithm, and bounds for the logarithmic depths of solutions.

Questions of a similar nature were studied before by Strodt, Grigoriev, Singer and Shackell: in [Str 77], Strodt establishes a bound for the logarithmic depths of solutions (without exponentials) of certain first order differential equations. This work was based on earlier work in [SW 71]. Grigoriev and Singer proved in [GS 91] that the exponents of power series solutions to (5.1) are contained in a finitely generated group, if P has power series coefficients. In chapter 12, we will prove a similar theorem for transseries, using the techniques from this chapter. Finally, Shackell described in [Sh 92] an infinite process for obtaining the possible forms of nested expansions of solutions to (5.1).

The methods we use to solve (5.1) are classical: namely the Newton polygon method (see [New 1671], [Pui 1850], [BB 1856], [Fi 1889]) and linearization. However, several technical difficulties arise, which were absent in the purely algebraic case (see also the introduction). In section 5.3, we give present the differential version of the Newton polygon method. In section 5.4 we show how to compute distinguished solutions to "quasi-linear" equations. In particular, we show that at most $r+1$ upward shiftings are necessary to make the distinguished solutions purely exponential, if the coefficients of the quasi-linear equation are. Finally, we present the theoretical resolution algorithm in section 5.5 and obtain bounds for the logarithmic depths of solutions, using the bound mentioned above, and bounds for the size of the computation tree.

5.2 Preliminaries

5.2.1 Different decompositions of P

Let P be the differential polynomial from (5.1). We will usually adopt vector notation and denote

$$\begin{aligned} f^{(\boldsymbol{i})} &= f^{i_0} \cdots (f^{(r)})^{i_r}; \\ ||\boldsymbol{i}|| &= i_0 + \cdots + i_r, \end{aligned}$$

for all $\boldsymbol{i} = (i_0, \cdots, i_r)$. Then the decomposition

$$P(f) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{(\boldsymbol{i})} \tag{5.2}$$

of P is called the **natural decomposition** P. Another decomposition of P is given by

$$\begin{aligned} P(f) &= P_0(f) + \cdots + P_p(f); \\ P_i(f) &= \textstyle\sum_{||\boldsymbol{i}||=i} P_{\boldsymbol{i}}(f) \ (0 \leqslant i \leqslant p), \end{aligned} \tag{5.3}$$

where p is the degree of P. We call it the **decomposition** of P into **homogeneous parts**.

In some cases, it is more convenient to decompose P as a sum

$$P(f) = \sum_{\omega_1 \leqslant \cdots \leqslant \omega_{|\boldsymbol{\omega}|}} P_{\boldsymbol{i}(\boldsymbol{\omega})} f^{(\omega_1)} \cdots f^{(\omega_{|\boldsymbol{\omega}|})},$$

where $\boldsymbol{\omega} \in \{0, \cdots, r\}^*$ is a non commutative word of length $\leqslant p$ and

$$i(\boldsymbol{\omega})_j = |\{1 \leqslant k \leqslant |\boldsymbol{\omega}| \,|\, \omega_k = j\}|.$$

However, this formula presents the disadvantage of being asymmetric in $\omega_1, \cdots, \omega_{|\boldsymbol{\omega}|}$, therefore we usually prefer the following symmetric version:

$$P(f) = \sum_{\boldsymbol{\omega}} \binom{|\boldsymbol{\omega}|}{i(\boldsymbol{\omega})_0, \cdots, i(\boldsymbol{\omega})_{|\boldsymbol{\omega}|}}^{-1} P_{\boldsymbol{i}(\boldsymbol{\omega})} f^{(\omega_1)} \cdots f^{(\omega_{|\boldsymbol{\omega}|})}.$$

This latter formula is conveniently rewritten as

$$P(f) = \sum_{\boldsymbol{\omega}} P_{[\boldsymbol{\omega}]} f^{[\boldsymbol{\omega}]}, \tag{5.4}$$

where

$$\begin{aligned} P_{[\boldsymbol{\omega}]} &= \binom{|\boldsymbol{\omega}|}{i(\boldsymbol{\omega})_0, \cdots, i(\boldsymbol{\omega})_{|\boldsymbol{\omega}|}}^{-1} P_{\boldsymbol{i}(\boldsymbol{\omega})}; \\ f^{[\boldsymbol{\omega}]} &= f^{(\omega_1)} \cdots f^{(\omega_{|\boldsymbol{\omega}|})}. \end{aligned}$$

We say that (5.4) is the **decomposition** of P **along orders**. The **total order** of P is defined to be the maximal number ρ such that there exists a $\boldsymbol{\omega}$ with $||\boldsymbol{\omega}|| = \rho$ and $P_{[\boldsymbol{\omega}]} \neq 0$.

5.2.2 Additive conjugation

Given a transseries h, there exists a unique differential polynomial P_{+h} such that

$$P_{+h}(f) = P(h+f)$$

for all f. We call $P_{\times h}$ an **additive conjugate** of P. We have $P_{+(h_1+h_2)} = P_{+h_1,+h_2}$ for all $h_1, h_2 \in \mathbb{T}$.

Using a generalized Taylor series expansion, we obtain the following explicit formula for the coefficients of P_{+h}:

$$P_{+h,\boldsymbol{i}} = \frac{1}{\boldsymbol{i}!} P^{(\boldsymbol{i})}(h), \tag{5.5}$$

where

$$\begin{aligned} \boldsymbol{i}! &= i_0! \cdots i_r!; \\ P^{(\boldsymbol{i})} &= \frac{\partial^{\|\boldsymbol{i}\|} P}{\partial^{i_0} f \cdots \partial^{i_r} f^{(i_r)}}. \end{aligned}$$

In expanded form, this yields

$$P_{+h,\boldsymbol{i}} = \sum_{\boldsymbol{j} \geqslant \boldsymbol{i}} \binom{\boldsymbol{j}}{\boldsymbol{i}} h^{(\boldsymbol{j}-\boldsymbol{i})} P_{\boldsymbol{j}}. \tag{5.6}$$

Proposition 5.1.

(a) If $h \asymp 1$, then $\mathrm{M}(P_{+h}) = \mathrm{M}(P)$.
(b) If $h \asymp x^i$ for some $i \in \mathbb{Z}$, then $\mathrm{M}(P_{+h}) \asymp_{e^x} \mathrm{M}(P)$.

Proof. Assume that $h \asymp 1$. By (5.6), we clearly have $P_{+,\boldsymbol{i}} \preccurlyeq\!\!\!\prec \mathrm{M}(P)$ for all $\boldsymbol{i}$, whence $\mathrm{M}(P_{+h}) \preccurlyeq\!\!\!\prec \mathrm{M}(P)$. On the other hand, if $\boldsymbol{i}$ is chosen maximal for $\leqslant$ with $\mathrm{M}(P) = \mathrm{M}(P_{\boldsymbol{i}})$, then (5.6) implies that $\mathrm{M}(P_{+h,\boldsymbol{i}}) = P_{\boldsymbol{i}} + o(\mathrm{M}(P))$. This establishes (a); (b) is proven in a similar way. $\square$

5.2.3 Multiplicative conjugation

Given a non zero transseries h, there exists a unique differential polynomial $P_{\times h}$ such that

$$P_{\times h}(f) = P(hf)$$

for all f. We call $P_{\times h}$ a **multiplicative conjugate** of P. We have $P_{\times h_1 h_2} = P_{\times h_1, \times h_2}$ for all $h_1, h_2 \in \mathbb{T}^*$.

To get an explicit formula for the coefficients of $P_{\times h}$ we decompose P along orders:

$$\begin{aligned} P(hf) &= \sum_{\boldsymbol{\tau}} P_{[\boldsymbol{\tau}]}(hf)^{[\boldsymbol{\tau}]} \\ &= \sum_{\boldsymbol{\tau}} P_{[\boldsymbol{\tau}]} \sum_{\boldsymbol{\omega} \leqslant \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} h^{[\boldsymbol{\tau}-\boldsymbol{\omega}]} f^{[\boldsymbol{\omega}]}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\omega} \leqslant \boldsymbol{\tau} &\Leftrightarrow |\boldsymbol{\omega}| = |\boldsymbol{\tau}| = i \ \wedge\ \omega_1 \leqslant \tau_1 \ \wedge\ \cdots\ \wedge\ \omega_i \leqslant \boldsymbol{\tau}_i \\ \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} &= \binom{\tau_1}{\omega_1} \cdots \binom{\tau_i}{\omega_i}. \end{aligned}$$

It follows that

$$P_{\times h, [\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau} \geqslant \boldsymbol{\omega}} \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} h^{[\boldsymbol{\tau}-\boldsymbol{\omega}]} P_{[\boldsymbol{\tau}]}. \tag{5.7}$$

In particular, we notice that multiplicative conjugation acts by homogeneous parts, i.e.

$$P_{\times h, i} = P_{i, \times h},$$

for all i.

The **dominant monomial** $\mathrm{M}(P)$ of P is defined by

$$\mathrm{M}(P) = \max_{\preccurlyeq, i} \mathrm{M}_{P_i}.$$

In the purely exponential case, we have the following straightforward generalization of proposition 4.2 to the homogeneous algebraic case:

Proposition 5.2. *If $h \succ\!\!\succ x$ and P is homogeneous of degree i, then*

$$\mathrm{M}(P_{\times h}) \asymp_h h^i \mathrm{M}(P).$$

□

5.2.4 Upward shifting

The **upward shifting** $P{\uparrow}$ of P is defined by

$$(P{\uparrow})(f{\uparrow}) = P(f){\uparrow},$$

for all f. Upward shifting is compatible with multiplicative conjugation, in the sense that

$$P_{\times h}{\uparrow} = (P{\uparrow})_{\times h\uparrow}. \tag{5.8}$$

For all non zero transseries h.

The coefficients of $P\uparrow$ are again determined most conveniently, when decomposing P along orders:

$$\begin{aligned} P(f) = P(f\uparrow(\log x)) &= \sum_{\boldsymbol{\tau}} P_{[\boldsymbol{\tau}]}(f\uparrow(\log x))^{[\boldsymbol{\tau}]} \\ &= \sum_{\boldsymbol{\tau}} P_{[\boldsymbol{\tau}]} \sum_{\boldsymbol{\omega}\leqslant\boldsymbol{\tau}} C_{\boldsymbol{\omega},\boldsymbol{\tau}} x^{-\|\boldsymbol{\tau}\|}(f\uparrow)^{[\boldsymbol{\omega}]}, \end{aligned}$$

where

$$C_{\boldsymbol{\omega},\boldsymbol{\tau}} = C_{\omega_1,\tau_1}\cdots C_{\omega_{|\omega|},\tau_{|\tau|}},$$

and where we recall that the constants $C_{i,j}$ are determined by

$$(f(\log x))^{(j)} = \sum_{i=0}^{j} C_{i,j} x^{-j} f^{(i)}(\log x). \tag{5.9}$$

Actually, $C_{i,j} = s_{j,i}$, where the $s_{j,i}$ are the Stirling numbers of the first kind. It follows that

$$(P\uparrow)_{[\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau}\geqslant\boldsymbol{\omega}} C_{\boldsymbol{\omega},\boldsymbol{\tau}} e^{-\|\boldsymbol{\tau}\|x}(P_{[\boldsymbol{\tau}]}\uparrow), \tag{5.10}$$

for all $\boldsymbol{\omega}$. Notice also that upward shifting naturally commutes with additive and multiplicative conjugation.

Proposition 5.3. *For all $P \neq 0$, we have $\mathfrak{m}(P\uparrow)/\mathfrak{m}(P)\uparrow \preccurlyeq\!\!\prec e^x$.*

Proof. Since $\uparrow$ acts by homogeneous parts, it suffices to consider the case when P is homogeneous. Let $\boldsymbol{\tau}$ be maximal for $\leqslant$ such that $\mathfrak{m}(P_{[\boldsymbol{\tau}]}) = \mathfrak{m}(P)$. Then (5.10) yields

$$P\uparrow_{[\boldsymbol{\tau}]} = P_{[\boldsymbol{\tau}]} e^{-\|\boldsymbol{\tau}\|x}(1+o(1)).$$

(5.10) also yields

$$P\uparrow_{[\boldsymbol{\omega}]} = O(\mathfrak{m}(P)\uparrow),$$

for each $\boldsymbol{\omega}$. Hence $\mathfrak{m}(P)\uparrow e^{-\|\boldsymbol{\tau}\|x} \preccurlyeq \mathfrak{m}(P\uparrow) \preccurlyeq \mathfrak{m}(P)\uparrow$. □

5.3 The differential Newton polygon method

Let us come back to the equation (5.1). Except in the last paragraph, we assume in this section that it is purely exponential, i.e. ч and the coefficients of P are purely exponential. Let $ц \in \mathbb{T}^{exp}$ be a transmonomial. The **differential Newton polynomial** associated to ц (relative to (5.1)) defined by

$$\mathfrak{m}_{P,ц}(c) = \sum_{\boldsymbol{i}} P_{\times ц,\boldsymbol{i},\mathfrak{m}(P_{\times ц})} c^{\boldsymbol{i}}. \tag{5.11}$$

The **purely exponential Newton degree** of (5.1) is the highest degree d of the differential Newton polynomial associated to a monomial $\text{ц} \in \mathbb{T}^{exp}$. The **algebraic Newton polynomial** associated to ц is the algebraic part of $\text{м}_{P,\text{ц}}$, i.e.

$$\text{м}^{alg}_{P,\text{ц}}(c) = \sum_i P_{\times\text{ц},(i,0,\cdots,0),\text{м}(P_{\times\text{ц}})} c^i.$$

In what follows, by Newton polynomial, we mean differential Newton polynomial, unless stated otherwise. We notice that $\text{м}_{P,\text{ц}}(c) = 0 \Leftrightarrow \text{м}^{alg}_{P,\text{ц}}(c) = 0$, for $c \in C$.

We say that a monomial $\text{ц} \in \mathbb{T}^{exp}$ is a **potential dominant monomial** of f, if $\text{ц} \prec\!\!\prec \text{ч}$ and $\text{м}_{P,\text{ц}}$ admits a solution $c \in C^*$. In that case, $c\text{ц}$ is called a **potential dominant term** of f. Furthermore, ц (resp. $c\text{ц}$) is said to be **classical**, if the algebraic Newton polynomial $\text{м}^{alg}_{P,\text{ц}}$ is non trivial. The **multiplicity** of c as a root of (5.11) is the smallest integer i, such that there exists a $\boldsymbol{i}$ with $||\boldsymbol{i}|| = i$ and $\text{м}^{(\boldsymbol{i})}_{P,\text{ц}}(c) \neq 0$.

5.3.1 Classical potential dominant monomials

The classical potential dominant monomials are analogous to the potential dominant monomials encountered in the algebraic Newton polygon method. In particular, they are finite in number as follows from the following:

Theorem 5.1. *Assume that $P_i, P_j \neq 0$ with $i \neq j$. Then there exists a unique transmonomial $\text{ц} \in \mathbb{T}^{exp}$, such that $\text{м}(P_{\times\text{ц},i}) = \text{м}(P_{\times\text{ц},j})$.*

Instead of proving theorem 5.1, we will prove a slightly stronger assertion, which will be useful in section 5.4. Actually, the proof — which is very similar to the proof of theorem 4.1 — is merely given for convenience of the reader. The transfinite induction procedure we use also yields a theoretical way to compute the classical potential dominant monomials.

Theorem 5.2. *Assume that P has coefficients in $\mathbb{T}^{exp}[x]$. Let $P_i, P_j \neq 0$ be given with $i < j$. Then there exists a unique transmonomial $\text{ц} \in \mathbb{T}^{exp}$, such that $\text{м}(P_{\times\text{ц},i}) \asymp_{e^x} \text{м}(P_{\times\text{ц},j})$.*

Proof. The uniqueness of ц trivially follows from proposition 5.2. Let us therefore prove its existence. Let $(\text{ц}_\alpha)_\alpha$ be the transfinite sequence of monomials, defined by

$$\begin{aligned}
\text{ц}_0 &= 1; \\
\text{ц}_{\alpha+1} &= \text{ц}_\alpha \exp \tfrac{1}{j-i}(\tau(\log(\text{м}(P_{\times\text{ц}_\alpha,j})/\text{м}(P_{\times\text{ц}_\alpha,i})))), \quad \text{if } \text{м}(P_{\times\text{ц}_\alpha,j})/\text{м}(P_{\times\text{ц}_\alpha,i}) \not\asymp^x_e 1; \\
\text{ц}_\beta &= \exp\,(\operatorname*{stat\,lim}_{\alpha<\beta} \log \text{ц}_\alpha), \quad \text{for limit ordinals } \beta.
\end{aligned}$$

We recall that $\tau(f)$ stands for the dominant term of f. We will show by transfinite induction that for all β:

- If $\alpha < \beta$, then $\log \text{ц}_\alpha \vartriangleleft \log \text{ц}_\beta$.
- $\forall \text{ч} \in \operatorname{supp} \log \text{ц}_\beta \quad \text{м}(P_{\times \text{ц}_\beta, j}) \asymp_{e^\text{ч}} \text{м}(P_{\times \text{ц}_\beta, i})$.

This will imply in particular that the length of the sequence $(\text{ц}_\alpha)_\alpha$ cannot exceed the cardinal number of $\mathbb{T}^{exp}[x]$. The last term of this sequence satisfies the requirement of the theorem. The induction hypothesis is clearly satisfied for $\beta = 0$. Assume now that for some $\beta > 0$, the induction hypothesis is satisfied for all smaller β.

Successor ordinals. Assume that $\beta = \alpha + 1$ and $P_{\times \text{ц}_\alpha, j}/P_{\times \text{ц}_\alpha, i} \notin x^{\mathbb{N}}$. Let щ denote $\exp \frac{1}{j-i}(\tau(\log(P_{\times \text{ц}_\alpha, j}/P_{\times \text{ц}_\alpha, i})))$. Since $P_{\times \text{ц}_\beta} = P_{\times \text{ц}_\alpha, \times \text{щ}}$, we have

$$\frac{\text{м}(P_{\times \text{ц}_\beta, j})}{\text{м}(P_{\times \text{ц}_\beta, i})} \asymp_{\text{щ}} \text{щ}^{j-i} \frac{\text{м}(P_{\times \text{ц}_\alpha, j})}{\text{м}(P_{\times \text{ц}_\alpha, i})},$$

by proposition 5.2. Hence,

$$\text{м}(P_{\times \text{ц}_\beta, j}) \asymp_{\text{щ}} \text{м}(P_{\times \text{ц}_\beta, i}).$$

Furthermore, $\text{м}(P_{\times \text{ц}_\beta, j}) \asymp_{e^\text{ч}} \text{м}(P_{\times \text{ц}_\beta, i})$ for all $\text{ч} \in \operatorname{supp} \log \text{ц}_\beta$, since $e^\text{ч} \ggg\!\!\!\!\!-\; \text{щ}$. Hence, both induction hypotheses are satisfied at stage β.

Limit ordinals. Assume now that β is a limit ordinal. Given $\text{ч} \in \operatorname{supp} \log \text{ц}_\beta$, there exists an $\alpha < \beta$, with $\text{ч} \in \operatorname{supp} \log \text{ц}_\alpha$. By the induction hypothesis, we have

$$\text{м}(P_{\times \text{ц}_\alpha, j}) \asymp_{e^\text{ч}} \text{м}(P_{\times \text{ц}_\alpha, i}).$$

By proposition 4.2 and using that $\text{ц}_\beta/\text{ц}_\alpha \prec\!\!\!\lll \text{ч}$, we also have

$$\text{м}(P_{\times \text{ц}_\beta}) \asymp_{e^\text{ч}} \text{м}(P_{\times \text{ц}_\alpha}).$$

Therefore, the second induction hypothesis is again satisfied at stage β. The first one is trivially satisfied. $\square$

5.3.2 Non classical potential dominant monomials

There are two types of non classical potential dominant monomials of f: those for which the differential Newton polynomial is non homogeneous, and those for which it is. In the first case, we obtain a characterization by applying theorem 5.1. In order to characterize the non classical potential dominant monomials of f of the second type, we look at the Riccati equation associated to the corresponding homogeneous part of P.

More precisely, let ц be a transmonomial, whose associated Newton polynomial is homogeneous of degree i. Then we consider the equation

$$R_{P,i}(\hat{f}) = \sum_{|\boldsymbol{i}|=i} P_{\boldsymbol{i}} R_{\boldsymbol{i}}(\hat{f}) = 0 \tag{5.12}$$

in the logarithmic derivative $\hat{f}$ of f, where

$$R_{(i_0,\cdots,i_r)}(\hat{f}) = R_0^{i_0}(\hat{f}) \cdots R_r^{i_r}(\hat{f}),$$

with the notation from section 4.5. We call (5.12) the **Riccati equation** associated to (5.1) at degree i. We notice that the order and the total order of $R_{P,i}$ are both strictly smaller than those of P. Now we have:

Proposition 5.4. *Let $\text{ц} \in \mathbb{T}^{exp}$ with $\text{ц} \prec\!\!\prec \text{ч}$ be a monomial whose associated Newton polynomial is homogeneous of degree i. Then ц is a non classical potential dominant monomial of f, if and only if the equation*

$$R_{P,i,+\hat{\text{ц}}}(\tilde{\hat{f}}) = 0 \ (\tilde{\hat{f}} \prec\!\!\prec 1) \tag{5.13}$$

has strictly positive purely exponential Newton degree.

Proof. Assume first that ц is a potential dominant monomial of f. Since $\text{м}_{P,\times\text{ц}}(c) = P_{\times\text{ц},(i,0,\cdots,0),\text{м}(P_{\times\text{ц}})} c^i = 0$ for all $c \in C^*$, we must have $\text{м}(P_{\times\text{ц},(i,0,\cdots,0)}) \prec\!\!\prec \text{м}(P)$. Transferring this relation to the Riccati equation, we obtain

$$\text{м}(R_{P,i,+\hat{\text{ц}},0}) \prec\!\!\prec \text{м}(R_{P,i,+\hat{\text{ц}}}). \tag{5.14}$$

Let j be such that $\text{м}(R_{P,i,+\hat{\text{ц}},j}) = \text{м}(R_{P,i,+\hat{\text{ц}}})$. Unless $R_{P,i,+\hat{\text{ц}},0} = 0$ (in which case we are clearly done), there exists a unique transmonomial $\text{щ} \in \mathbb{T}^{exp}$ such that

$$\text{м}(R_{P,i,+\hat{\text{ц}},j,\times\text{щ}}) = \text{м}(R_{P,i,+\hat{\text{ц}},0,\times\text{щ}}),$$

by theorem 5.1. The degree of the Newton polynomial associated to щ relative to (5.13) is therefore strictly positive. Moreover, $\text{щ} \prec\!\!\prec 1$, by (5.14) and proposition 5.2.

On the other hand, if ц is not a potential dominant monomial of f, then we have $P_{\times\text{ц},(i,0,\cdots,0),\text{м}(P)} \neq 0$, whence

$$\text{м}(R_{P,i,+\hat{\text{ц}},0}) = \text{м}(R_{P,i,+\hat{\text{ц}}}).$$

Using proposition 5.2, it follows that

$$\text{м}(R_{P,i,+\hat{\text{ц}},\times\text{щ},0}) = \text{м}(R_{P,i,+\hat{\text{ц}},\times\text{щ}}),$$

for all $\text{щ} \prec\!\!\prec 1$ with $\text{щ} \in \mathbb{T}^{exp}$. Consequently, the purely exponential Newton degree of (5.13) is zero. □

5.3.3 Refinements

Now we know how to determine potential dominant terms, let us show how more terms of potential solutions are obtained. A **purely exponential refinement** (relative to (5.1)) is a change of variables together with an asymptotic constraint

$$f = \varphi + \tilde{f} \ (\tilde{f} \prec\!\!\prec \tilde{\text{ч}}), \tag{5.15}$$

where $\tilde{\text{ч}} \preccurlyeq\!\!\preccurlyeq \varphi \in \mathbb{T}^{exp}$. Such a refinement transforms (5.1) into

$$P_{+\varphi}(\tilde{f}) = 0 \ (\tilde{f} \prec\!\!\prec \tilde{\text{ч}}). \tag{5.16}$$

We say that the refinement (5.15) is **admissible**, if the purely exponential Newton degree of (5.16) is strictly positive. By convention, (5.15) is said to be admissible relative to the equation $0 = 0(f \prec\!\!\prec \text{ч})$. Clearly, if (5.16) admits a solution, then (5.15) must be admissible. The following generalizes lemma 3.2:

Lemma 5.1. *Let $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \varphi)$ be a purely exponential refinement relative to (5.1), and let $c\text{ц}$ be the dominant term of φ. Then the purely exponential Newton degree of (5.16) is equal to the multiplicity $\tilde{d}$ of c as a root of (5.11).*

Proof. Let $\boldsymbol{i}$ be an index with $||\boldsymbol{i}|| = \tilde{d}$, such that $\text{м}^{(\boldsymbol{i})}_{P,\text{ц}}(c) \neq 0$. From (5.6) it follows that:

$$\begin{array}{rcll}
\text{м}(P_{\times\text{ц},+\varphi/\text{ц},\boldsymbol{i}}) & = & \text{м}(P_{\times\text{ц}}); & \\
\text{м}(P_{\times\text{ц},+\varphi/\text{ц},\boldsymbol{j}}) & \preccurlyeq\!\!\preccurlyeq & \text{м}(P_{\times\text{ц}}), & \text{for all } \boldsymbol{j}; \\
\text{м}(P_{\times\text{ц},+\varphi/\text{ц},\boldsymbol{j}}) & \prec\!\!\prec & \text{м}(P_{\times\text{ц}}), & \text{for all } \boldsymbol{j} \text{ with } ||\boldsymbol{j}|| < \tilde{d}.
\end{array}$$

To see that the purely exponential Newton degree of (5.16) is bounded by $\tilde{d}$, let $\text{щ} \prec\!\!\prec \text{ц}$ be a monomial in $\mathbb{T}^{exp}$. Then

$$\begin{array}{rcl}
\text{м}(P_{+\varphi,\times\text{щ},j}) & = & \text{м}(P_{\times\text{ц},+\varphi/\text{ц},\times\text{щ}/\text{ц},j}) \\
& \asymp_{\text{щ}/\text{ц}} & (\text{м}/\text{ц})^{j}\,\text{м}(P_{\times\text{ц},+\varphi/\text{ц},j}) \\
& \prec\!\!\prec_{\text{щ}/\text{ц}} & (\text{м}/\text{ц})^{\tilde{d}}\,\text{м}(P_{\times\text{ц},+\varphi/\text{ц},j}) \\
& \preccurlyeq\!\!\preccurlyeq & (\text{м}/\text{ц})^{\tilde{d}}\,\text{м}(P_{\times\text{ц},+\varphi/\text{ц},\tilde{d}}) \\
& \asymp_{\text{щ}/\text{ц}} & \text{м}(P_{\times\text{ц},+\varphi/\text{ц},\times\text{щ}/\text{ц},\tilde{d}}) \\
& = & \text{м}(P_{+\varphi,\times\text{щ},\tilde{d}})
\end{array}$$

for all $j > \tilde{d}$.

On the other hand, for each $j < \tilde{d}$, there exists a unique transmonomial $\text{щ}_j \in \mathbb{T}^{exp}$, such that

$$\text{м}(P_{\times\text{ц},+\varphi/\text{ц},\times\text{щ}_j/\text{ц},j}) = \text{м}(P_{\times\text{ц},+\varphi/\text{ц},\times\text{щ}_j/\text{ц},\tilde{d}}).$$

Since $\text{м}(P_{\times\text{ц},+\varphi/\text{ц},j}) \prec\!\!\prec \text{м}(P_{\times\text{ц},+\varphi/\text{ц},\tilde{d}})$, it follows that $\text{щ}_j \prec\!\!\prec \text{ц}$ for each such j. Taking j such that щ_j is maximal for $\prec\!\!\prec$, the degree of $\text{м}_{P_{+\varphi},\text{щ}_j}$ is precisely $\tilde{d}$, as in the proof of lemma 3.2. □

Corollary I. *The purely exponential refinement $f = \varphi + \tilde{f}$ ($\tilde{f} \prec\!\!\prec \varphi$) is admissible, if and only if $c\text{ц}$ is a potential dominant term of f relative to (5.1).*

Proof. Immediate consequence of part (a). □

Corollary II. *For each $\boldsymbol{j}$ with $||\boldsymbol{j}|| < \tilde{d}$, the purely exponential refinement $f = \varphi + \tilde{f}$ ($\tilde{f} \prec\!\!\prec \varphi$) is admissible relative to the equation $P^{(\boldsymbol{j})}(f) = 0$ ($f \prec\!\!\prec \text{ч}$).*

Proof. Immediate consequence of the previous corollary, by observing that the Newton polynomial associated to ц, relative to the equation $P^{(\boldsymbol{j})}(f) = 0$ ($f \prec\!\!\prec \text{ч}$) is precisely $\text{м}_{P,\text{ц}}^{(\boldsymbol{j})}$. □

5.3.4 Upward shifting

Proposition 5.5. *The purely exponential Newton degree of (5.1) is bounded by the purely exponential Newton degree of the equation $P{\uparrow}(f{\uparrow}) = 0$ ($f{\uparrow} \prec\!\!\prec \text{ч}{\uparrow}$).*

Proof. Let $\text{ц} \in \mathbb{T}^{exp}$ be a monomial with $\text{ц} \prec\!\!\prec \text{ч}$, such that $\text{м}_{P,\text{ц}}$ has degree d. Then we claim that $\text{м}_{P{\uparrow},\times\text{ц}{\uparrow}e^{rx}}$ has degree d as well, whence the proposition.

We first deduce from (5.10) that

$$\text{м}(P_{\times\text{ц},i}{\uparrow}) = \text{м}(P_{\times\text{ц},i}){\uparrow}e^{-j_i x},$$

for each i, where j_i is minimal such that there exists an $\boldsymbol{\omega}$ with $|\boldsymbol{\omega}| = i$, $||\boldsymbol{\omega}|| = j_i$ and $\text{м}(P_{\times\text{ц}}) = \text{м}(P_{\times\text{ц},[\boldsymbol{\omega}]})$. Moreover, we have $||\boldsymbol{\omega}|| = j_i$ for each $\boldsymbol{\omega}$ such that $\text{м}(P_{\times\text{ц},[\boldsymbol{\omega}]}{\uparrow}) = \text{м}(P_{\times\text{ц},i}{\uparrow})$. Applying (5.7), we deduce that

$$\begin{aligned}
\text{м}((P_{\times\text{ц},i}{\uparrow})_{\times e^{rx}}) &\asymp \text{м}(P_{\times\text{ц}}){\uparrow}e^{(ir-j_i)x} \\
&\asymp \text{м}((P_{\times\text{ц},d}{\uparrow})_{\times e^{rx}})e^{((i-d)r+j_d-j_i)x} \\
&\preceq\!\!\preceq \text{м}((P_{\times\text{ц},d}{\uparrow})_{\times e^{rx}})
\end{aligned}$$

for each $i \leqslant d$ with $\text{м}(P_{\times\text{ц}}) = \text{м}(P_{\times\text{ц},i})$. Since $\text{м}(P_{\times\text{ц},i}{\uparrow}) \prec\!\!\prec_{e^{e^x}} \text{м}(P_{\times\text{ц}}{\uparrow})$ for all other i, we deduce that

$$(P{\uparrow})_{\times\text{ц}{\uparrow}e^{rx},i} \preceq\!\!\preceq (P{\uparrow})_{\times\text{ц}{\uparrow}e^{rx},d}$$

for all i. □

Proposition 5.6. *Let* $\mathfrak{m} \in \mathbb{T}^{exp}$. *Then*

$$\mathrm{M}_{P\uparrow,\mathfrak{m}\uparrow}^{alg} = \mathrm{M}_{P,\mathfrak{m}}^{alg}. \tag{5.17}$$

Proof. Decompose $\mathrm{M}_{P,\mathfrak{m}}$ along orders, and let i be minimal, such that there exists an $\boldsymbol{\omega}$ with $\mathrm{M}_{P,\mathfrak{m},[\boldsymbol{\omega}]} \neq 0$ and $\|\boldsymbol{\omega}\| = i$. By (5.10), we have $\mathrm{M}(P_{\times\mathfrak{m}}\uparrow) = \mathrm{M}(P_{\times\mathfrak{m}})\uparrow e^{-ix}$ and

$$\mathrm{M}_{P\uparrow,\mathfrak{m}\uparrow} = \sum_{\|\boldsymbol{\omega}\| \leqslant i} \sum_{\boldsymbol{\tau} \geqslant \boldsymbol{\omega}, \|\boldsymbol{\tau}\| = i} C_{\boldsymbol{\omega},\boldsymbol{\tau}} \mathrm{M}_{P,\mathfrak{m},[\boldsymbol{\tau}]} c^{[\boldsymbol{\omega}]}. \tag{5.18}$$

In the case when $i = 0$, this relation clearly yields (5.17). In the other case, we have $\mathrm{M}_{P,\mathfrak{m}}^{alg} = 0$. Furthermore, we observe from (5.9) that $C_{0,j} = 0$ for all $j > 0$, whence $C_{0,\boldsymbol{\tau}} = 0$ for all $\boldsymbol{\tau} > \mathbf{0}$. Consequently, $\mathrm{M}_{P\uparrow,\mathfrak{m}\uparrow}^{alg} = 0$ as desired. □

Corollary. *The transmonomial* $\mathfrak{m} \in \mathbb{T}^{exp}$ *(resp.* $c\mathfrak{m} \in \mathbb{T}^{exp}$*) is a potential dominant monomial or term of* f *relative to (5.1), if and only if* $\mathfrak{m}\uparrow$ *(resp.* $c\mathfrak{m}\uparrow$*) is a potential dominant monomial (resp. term) of* $f\uparrow$ *relative to the equation* $P\uparrow(f\uparrow) = 0$ $(f\uparrow \prec\!\!\prec \mathfrak{v}\uparrow)$. *Moreover,* $\mathfrak{m}$ *resp.* $c\mathfrak{m}$ *is classical if and only if* $\mathfrak{m}\uparrow$ *resp.* $c\mathfrak{m}\uparrow$ *is.* □

5.3.5 The general case

Assume now that the coefficients of P and $\mathfrak{v}$ are in $\mathbb{T}$. The results from the previous section enable us to extend our terminology for the purely exponential case to the general case. For instance, in view of proposition 5.5, the purely exponential Newton degree of

$$P\uparrow_l(f\uparrow_l) = 0 \ (f\uparrow_l \prec\!\!\prec \mathfrak{v}\uparrow_l) \tag{5.19}$$

is defined and remains constant for all sufficiently large l; this constant d is called the **Newton degree** of (5.1). Yet another example: given a monomial $\mathfrak{m} \in \mathbb{T}$, its l-th iterated upward shifting $\mathfrak{m}\uparrow_l$ is either always or never a potential dominant monomial of $f\uparrow_l$ relative to (5.19) for sufficient large l. In the first case, we say that $\mathfrak{m}$ is a **potential dominant monomial** of f (relative to (5.1)). In a similar fashion, we extend the notions of potential dominant terms, refinements, and so on. In this more general terminology, lemma 5.1 and its corollaries generalize to:

Proposition 5.7. *Let* $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \varphi)$ *be a refinement relative to (5.1), and let* $c\mathfrak{m}$ *be the dominant term of* φ. *Then* *(a)* *The Newton degree* $\tilde{d}$ *of (5.16) is equal to the multiplicity of* c *as a root of (5.11).*

(b) *The refinement* $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \varphi)$ *is admissible, if and only if* сц *is a potential dominant term of* f*.*

(c) *For each* $\boldsymbol{j}$ *with* $||\boldsymbol{j}|| < \tilde{d}$*, the refinement* $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \varphi)$ *is admissible relative to the equation* $P^{(\boldsymbol{j})}(f) = 0$ $(f \prec\!\!\prec$ ч$)$*.*

$\square$

5.4 Quasi-linear equations

The equation (5.1) is said to be quasi-linear, if its Newton degree is one. To emphasize quasi-linearity, we will write $L(f) = P(f) - P_0$ in this section. Putting $g = P_0$, (5.1) becomes

$$Lf = g \ (f \prec\!\!\prec \text{ч}). \tag{5.20}$$

Here we use the notation "L" instead of "L" in order to emphasize that L is not a linear, but merely a quasi-linear differential operator. We will show that there exists a quasi-linear analogue L^{-1} of the distinguished right inverse L^{-1} from section 4.4.

In order to generalize, we have to cope with one additional difficulty, which did not exist in the linear case: consider for example the quasi-linear equation

$$f' + e^{-x} f f'' = 1 + e^{-e^x} \ (f \prec\!\!\prec e^{x/2}).$$

The distinguished dominant term of f is x, and after the refinement $f = x + \tilde{f}(\tilde{f} \prec\!\!\prec x)$, we obtain

$$\tilde{f}' + x e^{-x} \tilde{f}'' + e^{-2x} \tilde{f} \tilde{f}'' = e^{-e^x}.$$

Now the distinguished dominant term of $\tilde{f}$ is $x^{-1} e^{-e^x - x}$, which is not in $\mathbb{T}^{exp}[x]$. Therefore, repeated appearance of this phenomenon might in principle necessitate the introduction of iterated logarithms of arbitrary strength. Nevertheless, we will associate a new invariant to quasi-linear equations: the Newton regularity. This invariant is bounded by the order of the quasi-linear equation, and strictly increases each time the above phenomenon occurs.

5.4.1 Notations

It is convenient to use notations based on the linear case: we denote L_{lin} for the homogeneous part of degree one of L, and $L_{rest} = L - L_{lin}$. We denote $L_i = L_{lin,i}$ instead of $L_{(i,0,\cdots,0)}$ for all i, where we notice that this notation invalidates the notation from (5.3). In case of additive conjugation, we remove the constant part, i.e.

$$L_{+\varphi} f = L(\varphi + f) - L\varphi.$$

Hence (5.20) becomes

$$L_{+\varphi}\tilde{f} = g - L\varphi \ (\tilde{f} \prec\!\!\prec \varphi),$$

after the refinement $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \varphi)$. Finally, it will be convenient to introduce the abbreviations

$$L_{\leqslant i}(f) = L_0 f + \cdots + L_i f^{(i)}$$

and

$$L_{>i}(f) = L_{i+1} f + \cdots + L_r f^{(r)},$$

for all $0 \leqslant i \leqslant r$.

5.4.2 Dominant terms of distinguished solutions

Lemma 5.2. *Assume that L and g have coefficients in $\mathbb{T}^{exp}[x]$, such that $\mathrm{M}(L)$ is purely exponential and $\mathrm{M}(L) \asymp_{e^x} \mathrm{M}_g$. Let j be minimal such that $\mathrm{M}(L) = \mathrm{M}(L_j)$ and let i be such that $\mathrm{M}_g/\mathrm{M}(L) \asymp x^i$. Then $(c_g i!/c_{L_j}(i+j)!)x^{i+j}$ is a classical potential dominant monomial for f relative to (5.20).*

Proof. We proceed in a similar way as in proposition 4.5. We first deduce from (5.10) that

$$\begin{aligned} (L\uparrow)_k &= e^{-jx} L_j\uparrow(C_{k,j} + o(1)), \text{ for } k \leqslant j; \\ \mathrm{M}((L - L_{\leqslant j})\uparrow) &\prec\!\!\prec e^{-jx} L_j\uparrow. \end{aligned}$$

It follows that

$$\mathrm{M}(L_{\times x^{i+j}}\uparrow) = \mathrm{M}_g\uparrow. \tag{5.21}$$

The above relations for the coefficients of $L\uparrow$ also imply that

$$\begin{aligned} (L_{\times x^{i+j}}\uparrow)_0 &= (L\uparrow)_{\times e^{(i+j)x},0} \\ &= \sum_{k=0}^{r} (L\uparrow)_k (i+j)^k e^{(i+j)x} \\ &= \left(\sum_{k=0}^{j} C_{k,j}(i+j)^k + o(1)\right)(L\uparrow)_j e^{ix}. \end{aligned}$$

By proposition 4.3 and (5.21), we infer that the algebraic Newton polynomial N associated to $e^{(i+j)x}$ and relative to the upward shifting of (5.20) is given by

$$N(\lambda) = \frac{(i+j)!}{i!} c_{L_j}\lambda - c_g.$$

Since this polynomial is not homogeneous, the potential dominant term $(c_g i!/(c_{L_j}(i+j)!))x^{i+j}$ of f corresponding to its unique solution is classical. □

Proposition 5.8. *If $g \neq 0$ then there exists a unique classical potential dominant term $\tau_{L^{-1}g} = c_{L^{-1}g}\mathrm{M}_{L^{-1}g}$ for f relative to (5.20).*

Proof. Modulo upward shifting, it suffices to consider the case when g and the coefficients of L are in $\mathbb{T}^{exp}$. Then by theorem 5.1, there exists a unique monomial $\text{ц} \in \mathbb{T}^{exp}$ with $\mathrm{M}(L_{\times\text{ц}}) = \mathrm{M}_g$, and the existence of $\tau_{L^{-1}g}$ follows by applying lemma 5.2 to $L_{\times\text{ц}}$ and g. The uniqueness of $\tau_{L^{-1}g}$ results from the corollary of theorem 5.1 and the fact that the Newton polynomial associated to $\mathrm{M}_{L^{-1}g}$ is linear. □

Proposition 5.9. *Assume that L and g have coefficients in $\mathbb{T}^{exp}[x]$. Let ц denote the unique purely exponential monomial, such that $\mathrm{M}(L_{\times\text{ц}}) \asymp_{e^x} \mathrm{M}(g)$ (see theorem 5.2). Then*

(a) $\mathrm{M}_{L^{-1}g} \in \text{ц}x^{\mathbb{Z}}\log^{\mathbb{N}} x$.
(b) If $\mathrm{M}(L_{\times\text{ц}})$ is purely exponential, then $\mathrm{M}_{L^{-1}g} \in \text{ц}x^{\mathbb{N}}$.
(c) If $\mathrm{M}(L_{\times\text{ц}})$ and g are purely exponential, and $\mathrm{M}(L_{\times\text{ц}}) = \mathrm{M}(L_{\times\text{ц},0})$, then $\mathrm{M}_{L^{-1}g} = 1$.

Proof. Applying lemma 5.2 to $L_{\times\text{ц}}$ and g, we have (b) and (c). In order to prove (a), we first observe that $\mathrm{M}(L_{\times\text{ц}}\uparrow)/\mathrm{M}_g\uparrow = e^{ix}$ for some $i \in \mathbb{Z}$. Next $\mathrm{M}(L_{\times\text{ц}}\uparrow_{\times e^{ix}}) = \mathrm{M}_g\uparrow$ by proposition 5.2, since $L_{\times\text{ц}}\uparrow$ is purely exponential. Now we have (a) by applying lemma 5.2 to $L_{\times\text{ц}x^i}\uparrow = L_{\times\text{ц}}\uparrow_{\times e^{ix}}$ and $g\uparrow$, and shifting downwards. □

5.4.3 Newton regularity

Assume that g, the coefficients of L and $\mathrm{M}_{L^{-1}g}$ are purely exponential. Then the **purely exponential Newton regularity** of (5.20) is defined to be the largest number χ, such that

$$\frac{\mathrm{M}(L_{\times\mathrm{M}_{L^{-1}g},\chi})}{\mathrm{M}(L_{\times\mathrm{M}_{L^{-1}g}})} \lll \frac{\mathrm{M}(L_{\times\mathrm{M}_{L^{-1}g},rest})}{\mathrm{M}(L_{\times\mathrm{M}_{L^{-1}g}})}. \tag{5.22}$$

We will show below that the Newton regularity of the upward shifting of (5.20) is equal to χ. This allows us to define the **Newton regularity** of (5.20) in the general case as being the purely exponential Newton regularity of a sufficiently large upward shifting of (5.20).

Proposition 5.10. *Assume that g, the coefficients of L and $\mathrm{M}_{L^{-1}g}$ are purely exponential. Then the purely exponential Newton regularities of (5.20) and the equation $L\uparrow(f\uparrow) = g\uparrow$ $(f\uparrow \prec\!\!\prec \text{ч}\uparrow)$ coincide.*

Proof. Modulo a multiplicative conjugation of L by $\mathrm{M}(L^{-1}g)$ and a division by $\mathrm{M}(L)$, it suffices to consider the case when $\mathrm{M}(L^{-1}g) = \mathrm{M}(L) = 1$. If χ denotes the purely exponential Newton regularity of (5.20), this yields $\mathrm{M}(L_\chi) \lll \mathrm{M}(L_{rest})$.

By proposition 5.3, we have $\mathfrak{m}(L_{rest}\uparrow)/\mathfrak{m}(L_{rest})\uparrow \underline{\lll} e^x$ and $\mathfrak{m}(L\uparrow_{>\chi})/\mathfrak{m}(L_{>\chi})\uparrow \underline{\lll} e^x$, since $L\uparrow_{>\chi} = L_{>\chi}\uparrow$. From (5.10), we also deduce that

$$L\uparrow_{\chi} = L_{\chi}\uparrow e^{-\chi x}(1 + o_{\mathfrak{m}(L_{rest})}) \lll \mathfrak{m}(L_{rest})\uparrow.$$

Therefore, the purely exponential Newton regularity of $L\uparrow(f\uparrow) = g\uparrow$ $(f\uparrow \prec\!\!\prec \mathfrak{v}\uparrow)$ equals χ, since $\mathfrak{m}((L\uparrow)^{-1}g\uparrow) = \mathfrak{m}(L^{-1}g)\uparrow = 1$. □

Contrary to the Newton degree, the Newton regularity increases during refinements:

Theorem 5.3. *Assume that the coefficients of L and g are purely exponential, and consider an admissible refinement*

$$L_{+\varphi}(\tilde{f}) = g - L\varphi \;\; (\tilde{f} \prec\!\!\prec \mathfrak{v}) \tag{5.23}$$

of (5.20) with $\varphi \in \mathbb{T}^{exp}[x]$ and $\tau_{\varphi} = \tau_{L^{-1}g}$. Let χ resp. $\tilde{\chi}$ be the Newton regularities of (5.20), and (5.23). Then $\tilde{\chi} \geqslant \chi$.

Proof. Modulo one upward shifting, we may assume without loss of generality that $\varphi \in \mathbb{T}^{exp}$. As in the proof of the previous proposition, we may also assume that $\mathfrak{m}(\varphi) = \mathfrak{m}(L) = 1$. Modulo one more upward shifting, we may finally assume that $\mathfrak{c} = \mathfrak{m}(L_{+\varphi}^{-1}(g - L\varphi)) \in \mathbb{T}^{exp}$. By proposition 5.1(a), we have $\mathfrak{w} \stackrel{\text{def}}{=} \mathfrak{m}(L_{rest}) = \mathfrak{m}(L_{+\varphi,rest})$.

Assume first that $\mathfrak{c} \lll \mathfrak{w}$. Then we have

$$L_{+\varphi,lin} = L_{lin} + o_{\mathfrak{w}}(1),$$

by (5.6), i.e. $L_{+\varphi,i} = L_i + o_{\mathfrak{w}}(1)$ for all i. Consequently,

$$L_{+\varphi,\times\mathfrak{c},\chi} = L_{+\varphi,\chi} + o_{\mathfrak{w}}(1) = L_{\chi} + o_{\mathfrak{w}}(1),$$

by (5.7). On the other hand,

$$\mathfrak{m}(L_{+\varphi,\mathfrak{c},rest}) = \mathfrak{m}(L_{+\varphi,rest,\mathfrak{c}}) \asymp_{\mathfrak{w}} \mathfrak{w},$$

by proposition 5.2. Therefore $\tilde{\chi} \geqslant \chi$.

Assume now that $\mathfrak{c} \underline{\ggg} \mathfrak{w}$. Let j be maximal, such that $\mathfrak{m}(L_{+\varphi,j}) \lll \mathfrak{c}$. Clearly, $j \geqslant \chi$. By (5.7) and proposition 5.2, we have

$$\mathfrak{m}(L_{+\varphi,\times\mathfrak{c},j}) \asymp_{\mathfrak{c}} \mathfrak{m}(L_{+\varphi,j})\mathfrak{c} \asymp_{\mathfrak{c}} \mathfrak{c},$$

and

$$\mathfrak{m}(L_{+\varphi,\times\mathfrak{c}}) \asymp_{\mathfrak{c}} \mathfrak{m}(L_{+\varphi})\mathfrak{c} = \mathfrak{c}.$$

Again by proposition 5.2, we also have

$$\mathfrak{m}(L_{+\varphi,\times\mathfrak{c},rest}) \preccurlyeq_{\mathfrak{c}} \mathfrak{m}(L_{+\varphi,rest})\mathfrak{c}^2.$$

Consequently, $\tilde{\chi} \geqslant j \geqslant \chi$. □

If the Newton regularity does not increase as the result of an admissible refinement, then no new iterated logarithms are needed to compute the next potential dominant term:

Theorem 5.4. *With the notations from theorem 5.3, assume that $\tilde{\chi} = \chi$. Then $\tau_{L_{+\varphi}^{-1}(g-L\varphi)} \in \mathbb{T}^{exp}[x]$. Moreover, if $\varphi \in \mathbb{T}^{exp}$ and $\tilde{\chi} = \chi = 0$, then $\tau_{L_{+\varphi}^{-1}(g-L\varphi)} \in \mathbb{T}^{exp}$.*

Proof. Modulo multiplicative conjugation of L by a purely exponential monomial, we may assume without loss of generality that $\textsc{m}(L) = \textsc{m}_g$ (by theorem 5.1). In particular, $\varphi \asymp x^i$ for some i, by proposition 5.1(b). Modulo division of L by $\textsc{m}(L)$ we may also assume that $\textsc{m}(L) = 1$. By proposition 5.9(a), $\textsc{m}(L_{+\varphi}^{-1}(g - L\varphi)) = \text{ц} x^\alpha \log^\beta x$ for some purely exponential monomial ц, $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{N}$. The key of the proof now lies in the following

Lemma 5.3. *$\textsc{m}(L_{+\varphi,\times\text{ц}})$ is purely exponential.*

Proof. We have $\text{щ} \stackrel{\text{def}}{=} \textsc{m}(L_{+\varphi,rest}) \asymp_{e^x} \textsc{m}(L_{rest})$, by proposition 5.1(b). Assume first that $\text{ц} \prec\!\!\prec \text{щ}$. Since $L = L_{\leqslant\chi} + o_{\text{щ}}(1)$, we have

$$L_{+\varphi} = L_{\leqslant\chi,+\varphi} + o_{\text{щ}}(1) = L_{\leqslant\chi} + o_{\text{щ}}(1),$$

by (5.6). Using proposition 5.2, we therefore have

$$L_{+\varphi,\times\text{ц}} = L_{\leqslant\chi,\times\text{ц}} + o_{\text{щ}}(1).$$

Since $L_{\leqslant\chi,\times\text{ц}} \asymp_{\text{ц}} \text{ц} \lll \text{щ}$ by proposition 5.2, it follows that $\textsc{m}(L_{+\varphi,\times\text{ц}})$ is purely exponential.

Assume now that $\text{ц} \underline{\ggg} \text{щ}$. By proposition 5.2, we have

$$\textsc{m}(L_{+\varphi,\times\text{ц},lin}) \asymp_{\text{ц}} \text{ц}\textsc{m}(L_{+\varphi,lin}) \asymp_{\text{ц}} \text{ц}$$

and

$$\textsc{m}(L_{+\varphi,\times\text{ц},rest}) \prec\!\!\prec_{\text{ц}} \text{ц}^2\textsc{m}(L_{+\varphi,rest}) \prec\!\!\prec_{\text{ц}} \text{ц}^2.$$

Since $\tilde{\chi} = \chi$ by assumption, we therefore must have $\textsc{m}(L_{+\varphi,\times\text{ц},>\chi})/\text{ц} \underline{\ggg} \text{ц}$. Using (5.7), it follows that

$$\textsc{m}(L_{+\varphi,>chi}) = \textsc{m}(L_{+\varphi,\times\text{ц},>\chi,\times\text{ц}^{-1}}) \asymp_{\text{ц}} \textsc{m}(L_{+\varphi,\times\text{ц},>\chi}) \underline{\ggg} \text{ц}.$$

Using (5.7) again, the other way around, we infer

$$\begin{aligned} L_{+\varphi,\times\text{ц},\leqslant\chi} &= L_{+\varphi,\leqslant\chi,\times\text{ц}} + L_{+\varphi,>\chi,\times\text{ц},\leqslant\chi} \\ &= L_{+\varphi,\leqslant\chi,\times\text{ц}} + o_{\text{ц}}(\text{ц}). \end{aligned}$$

Let as usual $\hat{\text{ц}}$ denote the logarithmic derivative of ц. We first consider the case when $\hat{\text{ц}} \prec\!\!\prec\!\!\prec \text{щ}$. Then

$$L_{+\varphi,\leqslant\chi} = L_{\leqslant\chi} + o_{\text{щ}}(1),$$

by (5.6) and using the fact that $\varphi^{(i)} \asymp_{\text{щ}} \varphi$ for all i. Consequently,

$$L_{+\varphi,\leqslant\chi,\times\text{ц}} = L_{\leqslant\chi,\times\text{ц}} + o_{\text{щ}}(\text{ц}),$$

by (5.7) and the fact mentioned above. On the other hand,

$$L_{\leqslant\chi,\times\text{ц},\chi} = \text{ц} L_\chi \succcurlyeq\!\!\succcurlyeq_{\text{щ}} \text{ц},$$

whence $\text{м}(L_{\leqslant\chi,\times\text{ц}}) \succcurlyeq\!\!\succcurlyeq_{\text{щ}} \text{ц}$. We conclude that $\text{м}(L_{+\varphi,\times\text{ц}}) = \text{м}(L_{+\varphi,\leqslant\chi,\times\text{ц}}) = \text{м}(L_{\leqslant\chi,\text{ц}})$ is purely exponential. Let us finally consider the case when $\hat{\text{ц}} \succcurlyeq\!\!\succcurlyeq\!\!\succcurlyeq \text{щ}$. Then (5.7) yields

$$L_{+\varphi,\leqslant\chi,\times\text{ц},i} \sim L_{+\varphi,\chi}\text{ц}^{(\chi-i)}$$

for all i. In particular,

$$\text{м}(L_{+\varphi,\leqslant\chi,\text{ц}}) = \text{м}(L_{+\varphi,\leqslant\chi,\text{ц},0}) = \text{м}(L_{+\varphi,\chi})\text{ц}\text{м}(\hat{\text{ц}})^\chi = \text{м}(L_\chi)\text{ц}\text{м}(\hat{\text{ц}})^\chi,$$

since $L_{+\varphi,\chi} = L_\chi + o_{\text{щ}}(1)$ and $L_\chi \prec\!\!\prec\!\!\prec \text{щ}$. We again conclude that $\text{м}(L_{+\varphi,\times\text{ц}}) = \text{м}(L_{+\varphi,\leqslant\chi,\times\text{ц}})$ is purely exponential. □

End of the proof of theorem 5.4. The lemma implies at once that $\alpha \in \mathbb{N}$ and $\beta = 0$, by proposition 5.9(b). Moreover, if $\varphi \in \mathbb{T}^{exp}$ and $\chi = \tilde{\chi} = 0$, then $g - Lf$ is purely exponential. Furthermore, for all $i > 0$,

$$\frac{\text{м}(L_{+\varphi,\times\text{ц}x^\alpha,i}\uparrow)}{\text{м}(L_{+\varphi,\times\text{ц}x^\alpha}\uparrow)} \succcurlyeq\!\!\succcurlyeq\!\!\succcurlyeq \frac{\text{м}(L_{+\varphi,\times\text{ц}x^\alpha,rest}\uparrow)}{\text{м}(L_{+\varphi,\times\text{ц}x^\alpha}\uparrow)} \succcurlyeq\!\!\succcurlyeq\!\!\succcurlyeq e^x\uparrow.$$

Here the first inequality results from the fact that $\tilde{\chi} = 0$; the second one follows from

$$\frac{\text{м}(L_{+\varphi,\times\text{ц},rest})}{\text{м}(L_{+\varphi,\times\text{ц}})} \succcurlyeq\!\!\succcurlyeq\!\!\succcurlyeq e^x$$

and

$$\begin{aligned} L_{+\varphi,\times\text{ц}x^\alpha,rest} &\asymp_{e^x} L_{+\varphi,\times\text{ц},rest}; \\ L_{+\varphi,\times\text{ц}x^\alpha} &\asymp_{e^x} L_{+\varphi,\times\text{ц}}. \end{aligned}$$

It follows that

$$\text{м}(L_{+\varphi,\times\text{ц},i}) \prec\!\!\prec_{e^x} \text{м}(L_{+\varphi,\times\text{ц}}),$$

for all $i > 0$, by proposition 5.2 and (5.7). Consequently, $\text{м}(L_{+\varphi,\times\text{ц}}) = \text{м}(L_{+\varphi,\times\text{ц},0})$, so that $\alpha = 0$ by proposition 5.9(c). □

5.4.4 Existence proof of distinguished solutions

Theorem 5.5. *The quasi-linear equation (5.20) of order r with coefficients in C^{exp} admits at least one solution in C^{exp} if $r = 0$, or in $C^{exp}[x]\downarrow_{r-1}$ if $r \geqslant 1$.*

Proof. Let $(f_\alpha)_\alpha$ be the transfinite sequence of transseries defined by:

$$\begin{aligned} f_0 &= 0; \\ f_{\alpha+1} &= f_\alpha + \tau_{L_{+f_\alpha}^{-1}(g - L f_\alpha)} \text{ if } L f_\alpha \neq g; \\ f_\beta &= \operatorname*{stat\,lim}_{\alpha<\beta} f_\alpha \text{ for limit ordinals } \beta. \end{aligned}$$

We will show by transfinite induction that for all β:

- If $\alpha < \beta$, then $f_\alpha \trianglelefteq f_\beta$.
- For all $\tilde{\text{ч}} \in \{\text{ч}\} \cup \operatorname{supp} f_\beta$, the equation
$$L_{+f_\beta}(\tilde{f}) = g - L f_\beta \ (\tilde{f} \prec\!\!\prec \tilde{\text{ч}})$$
has Newton degree one.
- Either $f_\beta \in \mathbb{T}^{exp}$, or $f_\beta \in \mathbb{T}^{exp}[x]\downarrow_l$ for some l, and the equation
$$L_{+f_\gamma}(\tilde{f}) = g \ (\tilde{f} \prec\!\!\prec \text{ч})$$
has Newton regularity $\geqslant l + 1$ for some $\gamma < \beta$ with $f_\gamma \in \mathbb{T}^{exp}\downarrow_l$.

The third hypothesis implies that $f_\alpha \in \mathbb{T}^{exp}\downarrow_{r-1}$ for all α (whence f_β is well-defined, if β is a limit ordinal). The first hypothesis implies that the sequence has a last term, which is necessarily a solution to (5.20). The induction hypothesis is trivially satisfied for $\beta = 0$. Assume now that $\beta > 0$, and the induction hypothesis is satisfied for all smaller β.

Successor ordinals. Assume that $\beta = \alpha + 1$ and $L f_\alpha \neq g$. Since

$$L_{+f_\alpha}(\tilde{f}) = g - L f_\alpha \ (\tilde{f} \prec\!\!\prec \tilde{\text{ч}}) \tag{5.24}$$

has Newton degree one for each $\tilde{\text{ч}} \in \operatorname{supp} f_\alpha$, we have $\tau_{L_{+f_\alpha}^{-1}(g - L f_\alpha)} \prec\!\!\prec \tilde{\text{ч}}$ for all $\tilde{\text{ч}} \in \operatorname{supp} f_\alpha$. Hence, the first induction hypothesis is satisfied at stage β.

Since $\tau_{L^{-1}(g - f_\alpha)}$ is an admissible dominant term for the equation (5.24), the equation

$$L_{+f_{\alpha+1}}(\tilde{\tilde{f}}) = g - L f_{\alpha+1} \ (\tilde{\tilde{f}} \prec\!\!\prec \tau_{L^{-1}(g - f_\alpha)}) \tag{5.25}$$

has Newton degree one. Therefore, the second induction hypothesis is satisfied at stage β, because $\text{ч} \prec\!\!\prec \tilde{\text{ч}} \prec\!\!\prec \tau_{L_{+f_\alpha}^{-1}(g - L f_\alpha)}$ for all $\tilde{\text{ч}} \in \operatorname{supp} f_\alpha$.

As to the third induction hypothesis, let us first consider the case when $f_\alpha \in \mathbb{T}^{exp}$. If $\beta = 1$, then we are done by proposition 5.9(b) and (c). If the Newton regularity χ of L_{+f_α} is strictly positive, then we are again done by proposition 5.9(b). Otherwise, we conclude by applying theorem 5.4 to the refinement $L_{+f_\beta}(\tilde{f}) = g - Lf_\alpha$ $(\tilde{f} \prec\!\!\prec \textsc{m}(f_1))$ of (5.20).

Assume now that $f_\alpha \notin \mathbb{T}^{exp}$. Choose l and $\gamma < \alpha$, such that $f_\alpha{\uparrow_l} \in \mathbb{T}^{exp}[x]$, $f_\gamma{\uparrow_l} \in \mathbb{T}^{exp}$ and the Newton regularity χ of

$$(L_{+f_\gamma}{\uparrow_l})(\tilde{\tilde{f}}{\uparrow_l}) = (g - Lf_\gamma){\uparrow_l} \ (\tilde{\tilde{f}}{\uparrow_l} \prec\!\!\prec \text{ч}{\uparrow_l})$$

satisfies $\chi \geqslant l + 1$. Let χ' be the Newton regularity of

$$(L_{+f_\alpha}{\uparrow_l})(\tilde{f}{\uparrow_l}) = (g - Lf_\alpha){\uparrow_l} \ (\tilde{f}{\uparrow_l} \prec\!\!\prec (f_\alpha - f_\gamma){\uparrow_l}).$$

If $\chi' > l + 1$, then the third induction hypothesis is clearly satisfied at stage β, by proposition 5.9(a). Otherwise, it is again, by theorem 5.4.

Limit ordinals. Assume that β is a limit ordinal and let $\tilde{\text{ч}} \in \operatorname{supp} f_\beta$. Then there exists an $\alpha < \beta$ with $\text{ч} \in \operatorname{supp} f_\alpha$, and by the second induction hypothesis, the equation (5.24) has Newton degree one. Since the dominant terms of $f_\beta - f_\alpha$ and $f_{\alpha+1} - f_\alpha$ coincide, the equations (5.25) and

$$L_{+f_\beta}(\tilde{\tilde{f}}) = g - Lf_\beta \ (\tilde{\tilde{f}} \prec\!\!\prec \tau_{L^{-1}_{+f_\alpha}(g - Lf_\alpha)})$$

are both admissible refinements of (5.24). This implies the second induction hypothesis at stage β, since $\text{ч} \prec\!\!\prec \tau_{L^{-1}_{+f_\alpha}(g - Lf_\alpha)}$. The first induction and third hypotheses are trivially satisfied. □

In what follows, $L^{-1}g$ denotes the solution f to (5.20) as constructed in the above proof, and we call it the **distinguished solution** to (5.20). Using upward shiftings, the definition of L naturally extends to the case when g and the coefficients of L are arbitrary elements of $\mathbb{T}$.

5.4.5 The "homogeneous" quasi-linear equation

In this section we prove the analogue of the fact that the dimension of the vector space of solutions to a homogeneous linear differential equation is bounded by the order of the equation.

Proposition 5.11. *Let $h_1 \succ\!\!\succ \cdots \succ\!\!\succ h_s \neq 0$ be transseries, such that*

$$L_{+h_1+\cdots+h_{i-1}} h_i = 0,$$

for all $1 \leqslant i \leqslant s$. Then $s \leqslant r$.

Proof. Let $\tilde{L} = L_{+h_1+\cdots+h_s}$. Since $Lh_1 + \cdots + h_i = 0$, we have

$$\tilde{L}(-h_{i+1} - \cdots - h_s) = 0,$$

for all $1 \leqslant i \leqslant s$. In particular, $\mathfrak{M}_{h_1}, \cdots, \mathfrak{M}_{h_s}$ are potential dominant monomials relative to the "homogeneous" quasi-linear equation

$$\tilde{L}\tilde{f} = 0 \ (\tilde{f} \prec\!\!\prec \mathfrak{v}).$$

By proposition 5.4, it follows that the asymptotic Riccati equation

$$R_{\tilde{L},1,+\widehat{\mathfrak{M}_{h_i}}}(\tilde{\tilde{f}}) = 0 \ (\tilde{\tilde{f}} \prec\!\!\prec 1)$$

has strictly positive purely exponential Newton degree for all $1 \leqslant i \leqslant s$. But this implies that $s \leqslant r$, since the dimension of the solution space of the *linear* differential equation $\tilde{L}_{lin} f = 0$ is bounded by r. □

5.5 Resolution of algebraic differential equations

5.5.1 Privileged refinements

In chapter 3, we have used refinements of the form $f = \varphi + \tilde{f}$ with $P^{(d-1)}(\varphi) = 0$ to make the Newton degree decrease at most every two steps in the Newton polygon method (see lemma 3.3). In this section, we will study a generalization of such refinements, which we qualify as being privileged. Intuitively speaking, privileged refinements can be thought of as the keystones of "asymptotic elimination" theory.

Let us first fix the total ordering $<^{tot}$ on $(r+1)$-tuples of indices, which is uniquely determined by the following two properties:

- If $||\boldsymbol{i}|| < ||\boldsymbol{j}||$, then $\boldsymbol{i} <^{tot} \boldsymbol{j}$.
- If $||\boldsymbol{i}|| = ||\boldsymbol{j}||$, $i_0 = j_0, \cdots, i_{k-1} = j_{k-1}$ and $i_k < j_k$, then $\boldsymbol{i} <^{tot} \boldsymbol{j}$.

Let $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \tilde{\mathfrak{v}})$ be a refinement relative to (5.1) with $\tilde{\mathfrak{v}} = \mathfrak{M}(\varphi)$. Denote by d resp. $\tilde{d}$ the Newton degrees of (5.1) resp. (5.16). The refinement $f = \varphi + \tilde{f}$ $(\tilde{f} \prec\!\!\prec \tilde{\mathfrak{v}})$ is said to be **privileged**, if either one of the following conditions is satisfied:

PR1. φ is a potential dominant term of f and $\tilde{d} < d$.

PR2. $\tau(\varphi)$ is a potential dominant term of f and $\tilde{d} = d$.
Moreover, $\varphi - \tau(\varphi)$ is the distinguished solution to the equation

$$P^{(\boldsymbol{i})}_{+\tau(\varphi)}(\tilde{f}) = 0 \ (\tilde{f} \prec\!\!\prec \tau(\varphi)),$$

where $\boldsymbol{i}$ is maximal for $<^{tot}$, $||\boldsymbol{i}|| = d-1$ and $\mathfrak{M}(P^{(\boldsymbol{i})}_{+\tau(\varphi)}) = \mathfrak{M}(P_{+\tau(\varphi)})$.

Proposition 5.12. *The length of a chain of privileged refinements of type* **PR2** *is bounded by* $(r+1)^d$.

Proof. Let

$$\begin{aligned} f_0 &= \varphi_1 + f_1 \ (f_1 \prec\!\!\prec \text{ч}_1); \\ &\vdots \\ f_{n-1} &= \varphi_n + f_n \ (f_n \prec\!\!\prec \text{ч}_n) \end{aligned}$$

be a chain of privileged refinements of type **PR2** with $f_0 = f$. For a fixed $\boldsymbol{i}$ with $||\boldsymbol{i}|| = d-1$ and $0 \leqslant i_1, \cdots, i_{d-1} \leqslant r$, let $j_1 < \cdots < j_m$ be those indices for which $\boldsymbol{i}$ coincides with the $\boldsymbol{i}$ in **PR2**. Let $k \leqslant m$. By the choice of φ_{j_k}, we have

$$P^{(\boldsymbol{i})}_{+\varphi_1+\cdots+\varphi_{j_k-1}}(\varphi_{j_k}) = 0.$$

If $j_k < n$, which is in particular the case if $k < m$, then we claim that $\mathrm{M}(\varphi_{j_k+1})$ is the dominant monomial of a solution to the "homogeneous" quasi-linear equation

$$P^{(\boldsymbol{i})}_{+\varphi_1+\cdots+\varphi_{j_k}}(h) = 0 \ (h \prec\!\!\prec \text{ч}_{j_{k+1}}).$$

Indeed, this follows by applying proposition 5.7(b) to the j_k-th refinement relative to the above equation, which is admissible by assumption. Our claim implies that $m-1 \leqslant r$ by proposition 5.11, because $\varphi_1 \succ\!\!\succ \cdots \succ\!\!\succ \varphi_n$. Since there are $(r+1)^{d-1}$ possible choices for $\boldsymbol{i}$, we conclude that $n \leqslant (r+1)^d$. □

Corollary. *The length of any chain of privileged refinements is bounded by* $(r+1)^{d+1}$. □

5.5.2 Theoretical resolution algorithm

In this section we describe a theoretical algorithm to determine all solutions to (5.1). The algorithm consists of five main parts:

- `ade_solve`: solves (5.1).
- `ade_mod_solve`: solves (5.1) modulo $o(1)$.
- `priviliged_refinements`: computes a privileged refinement of (5.1).

- `pdm`: computes a potential dominant monomials of f.
- `Newton_degree`: computes the Newton degree of (5.1).

Here we say that a transseries f is a solution modulo $o(1)$ to (5.1), if

$$P_{+f}(\tilde{f}) = 0 \ (\tilde{f} \prec\!\!\prec 1)$$

has strictly positive purely exponential Newton degree. The determination of the solutions modulo $o(1)$ to an asymptotic a.d.e. is used as a subalgorithm in order to find the non classical potential dominant monomials using proposition 5.4.

The algorithms are non deterministic (with the exception of `Newton_degree`). To implement the non determinism, we use the automatic case separation terminology from chapter 8. The main algorithm `ade_solve` has the property that each solution to (5.1) corresponds to exactly one branch in the computation tree.

Algorithm `ade_solve`

INPUT: An asymptotic algebraic differential equation (5.1).
OUTPUT: A solution to (5.1) (computed non deterministically, if a solution exists).

STEP 1. If $P_0 = 0$, then separate the following two cases:
  A. Return 0.
  B. Proceed with step 2.
STEP 2. Determine (non deterministically) a privileged refinement

$$f = \varphi + \tilde{f} \ (\tilde{f} \prec\!\!\prec \varphi) \tag{5.26}$$

to (5.1) by `priviliged_refinements`.
STEP 3. Recursively solve (5.26) by `ade_solve`.

Algorithm `ade_mod_solve`

INPUT: An asymptotic algebraic differential equation (5.1).
OUTPUT: A solution modulo $o(1)$ to (5.1) (computed non deterministically, if a solution exists).

STEP 1. Compute the Newton degree d of the equation $P(f) = 0 \ (f \prec\!\!\prec 1)$ by `Newton_degree`.
If $d > 0$, then separate the following two cases:
  A. Return 0.
  B. Proceed with step 2.
STEP 2. Determine (non deterministically) a privileged refinement (5.26) to (5.1).
Impose the constraint $\varphi \not\preceq 1$.
STEP 3. Recursively solve (5.26) modulo $o(1)$ by `ade_mod_solve`.

Algorithm `priviliged_refinements`

INPUT: An asymptotic algebraic differential equation (5.1).

OUTPUT: φ, such that (5.26) is a privileged refinement of (5.1); here φ is computed non deterministically, if such a φ exists.

STEP 1. Compute (non deterministically) a classical potential dominant monomial ц of f by `pdm`.
Choose (non deterministically) a non zero solution c to the algebraic Newton polynomial associated to ц.

STEP 2. Compute the Newton degree d of (5.1) by `Newton_degree`.

STEP 3. If c is a solution of multiplicity $< d$ to $P_{ц}$, then return $cц$.

STEP 4. Otherwise, let $\boldsymbol{i}$ be maximal for $<^{tot}$, with $||\boldsymbol{i}|| = d - 1$ and $\mathrm{M}(P_{+cц}^{(\boldsymbol{i})}) = \mathrm{M}(P_{+cц})$.
Compute the distinguished solution φ to

$$P_{+cц}^{(\boldsymbol{i})}(\tilde{f}) = 0 \ (\tilde{f} \prec\!\!\prec ц),$$

by theorem 5.5 and return $cц + \varphi$.

5.5.3 Computation of potential dominant monomials

Before stating the algorithms to compute the potential dominant monomials of f and the Newton degree of (5.1), let us introduce some more terminology. Assume that (5.1) is purely exponential. By proposition 5.3, we have $\mathrm{M}(P_i) \prec\!\!\prec \mathrm{M}(P) \Rightarrow \mathrm{M}(P_i\uparrow) \prec\!\!\prec \mathrm{M}(P\uparrow)$ for all i. More generally, if ц is a purely exponential monomial, then we have $\mathrm{M}(P_{\times ц,i}) \prec\!\!\prec \mathrm{M}(P_{\times ц}) \Rightarrow \mathrm{M}(P_{\times ц,i}\uparrow) \prec\!\!\prec \mathrm{M}(P_{\times ц}\uparrow)$ for all i. It follows that there exist unique indices $i \leqslant j$, such that for all sufficiently large l, we have

$$\mathrm{M}(P_{\times ц,i}\uparrow_l) = \mathrm{M}(P_{\times ц}\uparrow_l) = \mathrm{M}(P_{\times ц,j}\uparrow_l), \tag{5.27}$$

and

$$\mathrm{M}(P_{\times ц,k}\uparrow_l) \prec\!\!\prec \mathrm{M}(P_{\times ц}\uparrow_l) \tag{5.28}$$

for all $k < i$ and $k > j$. We will call ц an **(i,j)-monomial** (relative to (5.1)) if $i < j$ and an **i-monomial** (relative to (5.1)) if $j = i$. Modulo upward shifting, this terminology extends to the case when ц and P are arbitrary.

If ц is an potential dominant i-monomial, then ц is necessarily non classical. However, if ц is an potential dominant (i,j)-monomial, then ц may be non classical: for instance, 1 is a non classical potential dominant $(1,2)$-monomial for f relative to the equation $f' + ff' = 0$. By theorem 5.1, there is at most one (i,j)-monomial $ц_{i,j}$ for fixed $i < j$. The following algorithm computes this monomial if it exists.

Algorithm `ij_monomial`

INPUT: An asymptotic algebraic differential equation (5.1) and $i < j$.
OUTPUT: The unique (i, j)-monomial relative to (5.1), if it exists.

STEP 1. If $P_i = 0$ or $P_j = 0$ then return 'failed'.
STEP 2. Let $l \geqslant 0$ be minimal, such that the l-th upward shifting of (5.1) is purely exponential.
STEP 3. Compute the unique monomial ц $\in \mathbb{T}^{exp}\downarrow_l$ with $\text{м}(P_{\times \text{ц},i}\uparrow_l) = \text{м}(P_{\times \text{ц},j}\uparrow_l)$ by theorem 5.1.
STEP 4. Compute the unique monomial щ $\in \mathbb{T}^{exp}\downarrow_{l+1}$ with $\text{м}(P_{\times \text{щ},i}\uparrow_{l+1}) = \text{м}(P_{\times \text{щ},j}\uparrow_{l+1})$ by theorem 5.1.
STEP 5. If щ $\neq$ ц or the differential Newton polynomials $\text{м}_{P\uparrow_l,\text{ц}\uparrow_l}$ and $\text{м}_{P\uparrow_{l+1},\text{ц}\uparrow_{l+1}}$ do not coincide, then set $l := l + 1$ and go to step 3.
STEP 6. If ц satisfies (5.27) and (5.28), then return ц.
Otherwise, return 'failed'.

Lemma 5.4. *The algorithm* `ij_monomial` *is correct and terminates.*

Proof. In order to prove the correctness of `ij_monomial`, we have to show that if щ $=$ ц$\uparrow$, $\text{м}_{P\uparrow_l,\text{ц}\uparrow_l} = \text{м}_{P\uparrow_{l+1},\text{ц}\uparrow_{l+1}}$ and ц satisfies (5.27) and (5.28), then (5.27) and (5.28) are also satisfied for all larger values of l. Now from (5.18) it follows that for all i, $\text{м}(P\uparrow_{l+i+1}, \text{ц}\uparrow_{l+i+1})$ is determined as a function of $\text{м}(P\uparrow_{l+i}, \text{ц}\uparrow_{l+i})$ only. Consequently, $\text{м}(P\uparrow_{l+i}, \text{ц}\uparrow_{l+i})$ for all i, whence the result.

In order to prove the termination, we assume that the logarithmic depths of the solutions to the Riccati equation $R_{P,i}$ are bounded by a fixed constant; this will be proved in the next section. Now suppose that `ij_monomial` does not terminate. Then the logarithmic derivatives of the successive values of ц are all solutions to the Riccati equation $R_{P,i}$. By our assumption, it follows that ц is constant from a certain l on. Since $\text{м}(P\uparrow_l, \text{ц}\uparrow_l)$ changes at each iteration, we observe from (5.18) that the total order of $\text{м}(P\uparrow_l, \text{ц}\uparrow_l)$ strictly decreases at each iteration. But this impossible. □

Given a monomial ц, it is easy to check whether it is a potential dominant monomial of f: we shift upwards until ц and (5.1) become purely exponential, and check whether ц $\prec\!\!\prec$ ч and the algebraic Newton polynomial associated to ц admits a non zero solution. Now we know how to compute the (i, j)-monomials relative to (5.1) and we know how to compute recursively the non classical potential dominant i-monomials using the Riccati equation. This leads to the following:

Algorithm `pdm`

INPUT: An asymptotic algebraic differential equation (5.1).

OUTPUT: A potential dominant monomial of f relative to (5.1) (computed non deterministically, if it exists).

STEP 1. Separate two cases and respectively proceed with steps 2 and 4.

STEP 2. For each $0 \leqslant i < j \leqslant r$ separate a case and do the following:

A. Compute the (i, j)-monomial ц relative to (5.1) by `ij_monomial`. Kill the current process if ц does not exist.

B. If ц is a potential dominant monomial of f, then return ц. Otherwise, kill the current process.

STEP 3. For each $1 \leqslant i \leqslant r$, with $P_i \neq 0$, separate a case and do the following:

A. Solve the Riccati equation $R_{P,i}(\varphi) = 0$ modulo $o(1)$ by `ade_mod_solve` and set ц $:= \exp(\int \varphi)$.

B. If ц is not a potential dominant monomial of f, then kill the current process.

C. If ц occurs as an (i', j')-monomial for some $i' < j'$, then kill the current process.

D. Return ц.

Assume that ц is an (i, j)-monomial or a j monomial, and щ is an (i', j')-monomial or a i' monomial. If ц $\prec\!\!\prec$ щ, then applying proposition 5.2 after a sufficiently large number of upward shiftings, we get $j \leqslant i'$. This observation leads to the following algorithm to compute the Newton degree of (5.1):

Algorithm `Newton_degree`

INPUT: An asymptotic algebraic differential equation (5.1).

OUTPUT: The Newton degree of (5.1).

STEP 1. If ч is an (i, j)-monomial relative to (5.1) for some $i < j$, then return $j - 1$.

STEP 2. Return the unique i such that ч is an i-monomial relative to (5.1). Here i is computed as follows:

A. Let l be minimal, such that the l-th upward shifting of (5.1) is purely exponential.

B. While the Newton polynomial $\mathrm{M}_{P\uparrow_l, ч\uparrow_l}$ is not homogeneous, set $l := l + 1$.

C. Let i be the degree of $\mathrm{M}_{P\uparrow_l, ч\uparrow_l}$.

5.5.4 Correctness and termination proofs

We can now state and prove the main theorem of part A of the thesis.

Theorem 5.6. *Let (5.1) be an asymptotic algebraic differential equation of degree d and total order ρ, with coefficients in $\mathbb{T} = C_\omega[[[x]]]$. Then*

(a) The set of possible outcomes of the non deterministic theoretical algorithm `ade_solve` *is precisely the set of solutions to (5.1) in $\mathbb{T}$.*

(b) The algorithm `ade_solve` *terminates for all possible executions, and each solution to (5.1) in $\mathbb{T}$ corresponds to exactly one branch of the computation tree.*

(c) If the coefficients of (5.1) are in $\mathbb{T}^{exp}$, then the logarithmic depth of each solution to (5.1) in $\mathbb{T}$ is bounded by $(\rho+1)^{(\rho+2)^2/2+d}$.

Proof. We have already shown the partial correctness of `ade_solve` and of its subalgorithms in the previous sections. Therefore, it only remains to prove (b) and (c). We do this by induction over ρ, while proving simultaneously that the logarithmic depths of the possible outcomes of `ade_solve` and `ade_mod_solve` are bounded by $(\rho+1)^{(\rho+2)^2/2+d}$, if the coefficients of (5.1) are in $\mathbb{T}^{exp}$. We notice that if the coefficients of (5.1) are in $\mathbb{T}^{exp}\uparrow_l$ then one need add l to these bounds modulo l upward shiftings of the equation. For $\rho = 0$, the equation (5.1) is algebraic, and the statements (b) and (c) are trivially satisfied. Assume therefore that $\rho > 0$. As usual, we denote by r the order of (5.1).

By the induction hypothesis, all recursive invocations of `ade_mod_solve` in `pdm` for the Riccati equation terminate. The assumption made in the termination proof of `ij_monomial` in lemma 5.4 is nothing but the induction hypothesis concerning the bounds for the logarithmic depths of solutions. Finally, since a chain of privileged refinements is bounded by $(r+1)^{d+1}$ in view of proposition 5.12, we obtain the termination of `ade_solve` and `ade_mod_solve`. This proves the first part of (b).

Let us now examine more carefully the above argument, in order to obtain the explicit bound from (c), when the coefficients of (5.1) are in $\mathbb{T}^{exp}$ We have to examine how many "additional upward shiftings" each step in the algorithm may necessitate.

By the induction hypothesis, each invocation of `ade_mod_solve` in `pdm` necessitates at most $\rho^{(\rho+1)^2/2+\rho}$ additional upward shiftings, since the degrees of the Riccati equations $R_{P,i}$ are bounded by ρ. Furthermore, the computation of the (i,j)-monomials necessitates at most ρ additional upward shiftings, since $\mathrm{M}(P\uparrow_l, Ц\uparrow_l)$ needs to change at each iteration in `ij_monomial`, and this can happen at most ρ times. Consequently, the computation of a potential dominant monomial of f necessitates at most $\rho^{(\rho+1)^2/2+\rho} = \max(\rho, \rho^{(\rho+1)^2/2+\rho})$ additional upward shiftings.

The computation of a distinguished solution in step 4 of `privileged_refinements` necessitates at most r additional upward shiftings by theorem 5.5. Consequently, the computation of a privileged refinement necessitates at most $r + \rho^{(\rho+1)^2/2+\rho} \leqslant (\rho+1)^{(\rho+1)^2/2+\rho}$ additional upward shiftings. Finally, since the length of a chain of privileged refinements is bounded by $(r+1)^{d+1}$, the logarithmic depth of any possible outcome of `ade_solve` or `ade_mod_solve` is bounded by $(r+1)^{d+1}(\rho+1)^{(\rho+1)^2/2+\rho} \leqslant (\rho+1)^{(\rho+2)^2/2+d}$ as desired.

We finally have to prove that each solution to (5.1) in $\mathbb{T}$ corresponds to exactly one branch of the computation tree. Actually, we claim that each possible outcome of each of the subalgorithms correspond to exactly one branch of the computation tree. The only subalgorithm for which this is non trivial is `pdm`, in which the possible overlapping of branches is excluded by step 3d. □

Remark 5.1. Actually, part (a) of the theorem can be strengthened as follows: if $\mathbb{T}' \supseteq \mathbb{T}$ is any strongly monotonic field of transseries over C in the sense of chapter 2, then `ade_solve` even produces all solutions to (5.1) in $\mathbb{T}'$. Indeed, this follows immediately from our bounds for the logarithmic depths. In particular, no new solutions occur if we enlarge $\mathbb{T}$ by other strongly monotonic transseries.

Remark 5.2. For convenience of the reader, we have rather tried to keep the algorithms as simple as possible than to optimize the bound in (c). Let us now sketch how this bound can be improved. Instead of considering sequences of privileged refinements of type **PR2**, where the index $\boldsymbol{i}$ changes at each iteration, one can also consider similar sequences of refinements, but with the index $\boldsymbol{i}$ kept fixed. When doing so, the bound from proposition 5.12 improves to $r+1$ instead of $(r+1)^d$ and the bound in the corollary to $d(r+2)$ instead of $(r+1)^{d+1}$. Finally, the bound in part (c) of in the theorem improves to $(\rho+2)!^2 d$ instead of $(\rho+1)^{(\rho+2)^2/2+d}$.

In particular, for fixed total order, this improved bound is linear in d. This coincides with the result of Strodt in the case when $\rho = 1$, who proved in [Str 77] that we may even take $2d$ for the bound in this case. On the other hand, we have no evidence for the existence of linear bounds in ρ for the logarithmic depths of solutions, which would in particular imply Shackell's conjecture in [Sh 92].

5.6 References

[BB 1856] C.A. Briot, J.C. Bouquet. Propriétés des fonctions définies par des équations différentielles. *Journal de l'École Polytechnique 36 (p. 133-198).*

[Cano 93] J. Cano. An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms. *Ann. Institut Fourier de Grenoble 43(1) (p. 125-142).*

[New 1671] I. Newton. *De methodis serierum et Fluxionum.* Manuscript. French translation of 1740 reprinted by Librarie scientifique et technique Albert Blanchard.

[Fi 1889] H.B. Fine. On the functions defined by differential equations, with an extension of the Puiseux polygon construction to these equations. *Amer. Journ. of Math. 12 (p. 295-322).*

[GS 91] D. Grigoriev, M. Singer. Solving ordinary differential equations in terms of series with real exponents. *Trans. of the A.M.S. 327, Nr 1. (p 329-351).*

[Pui 1850] M.V. Puiseux. Recherches sur les fonctions algébriques. *J. Math. Pures et Appliquées 15 (p. 365-480).*

[Sh 92] J. SHACKELL. Growth orders occurring in expansions of Hardy-field solutions to algebraic differential equations. *Technical report, Univ. of Kent, Canterbury.*

[Str 77] W. STRODT. A differential algebraic study of the intrusion of logarithms into asymptotic expansions. *in "Contributions to algebra", H. Bass, P.J. Cassidy, J. Kovacic eds., Academic press (p. 355-375).*

[SW 71] W. STRODT, R.K. WRIGHT. Asymptotic behaviour of solutions and adjunction fields for nonlinear first order differential equations. *Mem. Amer. Math. Soc. 109.*

Chapter 6

Transvarieties

6.1 Introduction

The aim of this chapter is to present models for algebraic asymptotic calculus with transseries in several variables and transseries which are not longer strongly monotonic. Our approach is geometrical: we interpret transseries as elements of the function space on some "transvariety". Besides the ordered exp-log ring operations and relations, we also have partially defined infinitary summation operators on such function spaces. The difficulty is of course how and when to define these operators and here we use the abstract nonsense arguments from appendix B. The idea to use a pointwise definition of infinite summation: an infinite summation is defined if and only if it is defined in each point. Here a point is nothing else but a morphism of the function space into a field of transseries in the sense of chapter 1 or 2 for which infinite summation has already been defined.

It should be noted that this chapter should be seen as indicative for the development of a future, more complete theory (see also section 6.3.2). We restricted ourselves to transvarieties whose function spaces do not have nilpotent elements and we are mainly interested in transseries defined on open subsets of affine space. The advantage of this choice is that theory remains more natural, but it is probable that a more general approach will be necessary for some future applications.

Let us now come to a more detailed description of the contents of this chapter. In section 6.2 we introduce strong rings, which are rings with partially defined infinite summation operators, which satisfy certain conditions.

For the introduction of grid-based transvarieties, we also introduce more special, so called grid-based summation operators in section 6.3.1. These operators adequately reflect the finiteness properties of grid-based series, and we will establish a generalization of proposition 1.3. In section 6.4 we proceed with some examples and basic properties of grid-based transvarieties. In particular, transvarieties are given a natural topology, which generalizes the interval topology. It is next shown in section 6.5 that the partial derivations can be defined for open subvarieties of affine space.

In sections 6.6 we introduce Noetherian transseries as the multivariate generalization of well-ordered transseries. Using a careful definition, we show that the results from the previous sections can be generalized.

In section 6.7.1, we show that our algebraic geometry methods can also be applied to construct models for weakly oscillating transseries, i.e. transseries which are built up from the usual transseries operation plus the trigonometric functions. The idea is to see weakly oscillating transseries as multivariate transseries, in which the variables have been substituted by oscillating components. Contrary to the strongly monotonic case, the models for such transseries do not necessarily correspond to the analytic models, but a correspondence can often be forced. Using the notion of weakly oscillatory transseries, it is easy to introduce complex transseries by considering real and imaginary parts. This is done in section 6.7.2 and we prove a theorem which shows how far real transseries can be prolongated in the complex transplane.

6.2 Strongly linear algebra

In this section, we generalize linear algebra objects, replacing the usual addition by an infinitary summation operator. Implicitly, we already used such infinite summations in chapters 1 and 2, but it convenient to make their properties more precise. Two main problems are encountered here. First, infinite summation operator can only be partially defined. Secondly, it is not clear what the arity of such an operator should be. The first problem has been settled in section B.2. For the second problem we can either take a summation operator $\sum_I$ for each index set I, or for all subsets I of a fixed sufficiently large set. We choose for the first option and we remind the discussion at the end of section B.2. The prefix **strong** will be used to designate generalized linear algebra objects. We will now make this more precise.

Let A be a set with a zero element 0, and for each index set I a partially defined summation operator $\sum_I$. We will also denote $\sum = \sum_I$ if the index set I can be deduced from the context. Then A is said to be a **strong Abelian group**, if $\sum_I$ is totally defined for all finite I, and if for all I and $(x_i)_{i\in I} \in A^I$ we have

S1. $\sum(0)_{i\in I} = 0$;
S2. $\sum(x_i)_{i\in I} = \sum(x_{\sigma(i)})_{i\in I}$, for any permutation σ of I;
S3. $\sum(x_i)_{i\in I} = \sum(\sum(x_i)_{i\in I_j})_{j\in J}$, for any partition $I = \coprod_{j\in J} I_j$.

Here we use our convention by which each of the above equalities only holds if at least one of its sides is defined. We remark that A can be seen as an Abelian group by taking $x_1 + x_2 \stackrel{\text{def}}{=} \sum_{i\in\{1,2\}} x_i$, for all $x_1, x_2 \in A$. From **S2**, **S1** and **S3** it also follows that $\sum(x_i)_{i\in I} = \sum(x_{\varphi(i)})_{i\in J}$, for any bijection φ between two index sets I and J.

Assume now that R is a strong Abelian group with an associative multiplication $\cdot$ with unit 1. Then R is said to be a **strong ring** if

S4. $\sum(\lambda x_i)_{i\in I} = \lambda \sum(x_i)_{i\in I}$,

for all $(x_i)_{i\in I} \in R^I, \lambda \in R$ and index sets I, such that $\sum(x_i)_{i\in I}$ is defined. In particular, R can be seen as a ring. As before, we will only consider commutative rings. Let R be a strong ring and M a strong Abelian group, such that R operates on M by multiplication. We say that M is a **strong R-module**, if we have $\lambda(\mu x) = (\lambda\mu)x$, $1x = x$, **S4** and

$$\sum(\lambda_i x)_{i\in I} = \left(\sum(\lambda_i)_{i\in I}\right) x, \tag{6.1}$$

for all $\lambda, \mu \in R, (\lambda_i)_{i\in I} \in R^I, x \in M, (x_i)_{i\in I} \in M^I$ and index sets I. A mapping $\varphi : M \to N$ between two strong R-modules M and N is said to be **strongly linear** if

$$\varphi\left(\sum(\lambda_i x_i)_{i\in I}\right) = \sum(\lambda_i \varphi(x_i))_{i\in I}, \tag{6.2}$$

for all $(\lambda_i)_{i\in I} \in R^I, (x_i)_{i\in I} \in M^I$ and index sets I.

In a similar way many other linear algebra objects can be generalized, such as vector spaces, algebras, multilinear mappings, derivations, and so on. For abstract nonsense reasons (see chapter B), the categories of strong Abelian groups, rings, etc. have a lot of properties. For instance, tensor products, pullbacks, pushouts, direct limits and inverse limits exist in the category of strong rings by theorem B.2.

Example 6.1. Let $\mathbb{T}$ the field of grid-based transseries over C. Then $\sum(x_i)_{i\in I}$ is defined for any grid-based family $(x_i)_{i\in I} \in C^I$. Similarly, if $\mathbb{T}$ is a field of well-ordered transseries, then $\sum(x_i)_{i\in I}$ is defined for any Noetherian family $(x_i)_{i\in I} \in C^I$.

Example 6.2. If C is also a strong ring in the previous example, then we say that a family $(f_i)_{i\in I} \in \mathbb{T}^I$ is **weakly grid-based**, if $\cup_{i\in I}\operatorname{supp} f_i$ is grid-based and $\sum(f_{i,\text{ц}})_{i\in I}$ well defined for each transmonomial ц. Weakly Noetherian families are defined similarly. Now the previous example can be generalized using these new definitions.

Example 6.3. Strongly linear algebra can also be used in other mathematical topics, such as measure theory: $\mathbb{R}$ is a strong ring, by defining $\sum(x_i)_{i\in I}$ for all absolutely convergent families. A σ-algebra $\mathcal{B}$ on a set E can canonically be extended into a strong $\mathbb{R}$-module $M_{\mathcal{B}}$: we consider the free strong $\mathbb{R}$-module in $\mathcal{B}$ and then quotient by relations of the form

$$\sum(U_i)_{i\in I} = \amalg_{i\in I} U_i$$

for enumerable families $(U_i)_{i\in I} \in \mathcal{B}^I$, whose members are mutually disjoint. Finite measures can then be interpreted as strongly linear mappings from $M_{\mathcal{B}}$ into $\mathbb{R}$. It is also possible to replace $\mathbb{R}$ by a field of transseries $\mathbb{T}$, and we leave it as an exercise to the reader to generalize the Lebesgue measure (one has to be careful here: relatively few disjoint unions of intervals are measurable).

6.3 Grid-based transseries in several variables

6.3.1 Definition of grid-based transvarieties

In this chapter we fix a totally ordered exp-log field of constants C and we denote $\mathbb{T} = C[\![t]\!]$. In order to apply the abstract theory from the previous chapter, we have select a suitable signature T and a category P of point types. For each $n \in \mathbb{N}$ and each family $(c_{k_1,\cdots,k_n})_{k_1,\cdots,k_n\in\mathbb{N}^n}$ in $C^{\mathbb{N}^n}$, we define a **grid-based summation symbol** Σ_c^* of arity n. For infinitesimal $f_1, \cdots, f_n$ in $\mathbb{T}$, we define

$$\Sigma_c^*(f_1, \cdots, f_n) = \sum_{k_1,\cdots,k_n\in\mathbb{N}} c_{k_1,\cdots,k_n} f_1^{k_1} \cdots f_n^{k_n}. \tag{6.3}$$

We let T be the signature consisting of these symbols Σ_c^*, the exp-log field operations and the ordering relation $\leqslant$. We take $\operatorname{obj}(\mathsf{P}) = \{\mathbb{T}\}$.

In section B.5, we have constructed the category $\mathsf{Var}_{\mathsf{P}}$ of varieties relative to P. From now on such varieties are called **grid-based transvarieties** over C and their category is denoted by GTV_C. Then a **grid-based transseries in several variables** is just a partially (but somewhere) defined function on a transvariety. A transseries f in $x_1, \cdots, x_n$ can be seen as an expression in $x_1, \cdots, x_n$ and the function symbols from T, such that for all points P in a certain non empty subset of $\mathbb{T}^n$, the substitution $f(P)$ of $(x_1, \cdots, x_n)$ by P is well defined. The maximal subset with this property can be seen as the domain of f.

The ordering on the function space $\mathcal{F}(V)$ of a transvariety V is also determined pointwise, i.e. $f \geqslant g$ if and only if $f(P) \geqslant g(P)$ for all points $P \in V$. More generally, we have pointwise generalizations of the asymptotic relations $\prec\!\!\!\prec$, $\prec\!\!\!\prec\!\!\!\prec$, etc. (for instance, $f \prec\!\!\!\prec g \Leftrightarrow \forall P\in V \quad f(P) \prec\!\!\!\prec g(P)$). A consequence of the pointwise characterization of functions and relations on a transvariety is that all Horn clauses which are valid in $\mathbb{T}$ are valid in $\mathcal{F}(V)$ (see proposition B.4). In particular, $\mathcal{F}(V)$ is an ordered exp-log ring. However, $\mathcal{F}(V)$ is not necessarily totally ordered nor a field. Indeed, the axioms of a total ordering resp. a field are not all Horn clauses.

Example 6.4. The transseries $f(x, y) = \sum_{i,j,k} i!j!k!x^{-i}y^{-j}e^{-kxy}$ in x and y is well defined, for positive $x \succ\!\!\!\succ 1$ and $y \succ\!\!\!\succ 1$. Strictly speaking, f is given by $f = \Sigma_c^*(x^{-1}, y^{-1}, e^{-xy})$, and $c = (i!j!k!)_{(i,j,k)\in\mathbb{N}^3}$.

Let us show that the function space of a transvariety V has the structure of a strong ring. It will be convenient to extend the meaning of the grid-based summation symbols in the following way:

$$\Sigma_c^*(f_1, \cdots, f_n; g_1, \cdots, g_p) = \sum_{j=1}^{p} \Sigma_{c_j}^*(f_1, \cdots, f_n) g_j,$$

for all $c \in C^{\mathbb{N}_p\times\mathbb{N}^n}$, infinitesimal $f_1, \cdots, f_n \in \mathcal{F}(V)$ and arbitrary $g_1, \cdots, g_p \in \mathcal{F}(V)$. Now let $(h_i)_{i\in I}$ be a family of elements in $\mathcal{F}(V)$. Assume that there exist

$f_1, \cdots, f_n, g_1, \cdots, g_p$ and a family $({}^i c)_{i \in I} \in C^{\mathbb{N}_p \times \mathbb{N}^n}$ such that

$$\begin{cases} h_i = \Sigma^*_{{}^i c}(f_1, \cdots, f_n; g_1, \cdots, g_p), \text{ for all } i \in I; \\ \{i \in I | \, {}^i c_{j,k_1,\cdots,k_n}\} \text{ is finite for each } (j, k_1, \cdots, k_n) \in \mathbb{N}_p \times \mathbb{N}^n. \end{cases}$$

In this case, and only in this case we define

$$\sum (h_i)_{i \in I} = \Sigma^*_c(f_1, \cdots, f_n; g_1, \cdots, g_n), \tag{6.4}$$

where $c_{j,k_1,\cdots,k_n} = \sum_{i \in I} {}^i c_{j,k_1,\cdots,k_n}$, for all $j, k_1, \cdots, k_n$. It is not hard to check that $\sum_I$ is well defined by this formula for each I, and that $\mathcal{F}(V)$ is given the structure of a strong ring in this way.

Remark 6.1. One might be tempted to introduce grid-based transvarieties by taking the infinite summation symbols $\sum_I$ in the signature T instead of the grid-based summation symbols (see also example 6.1). However, consider the examples

$$\begin{aligned} f(x) &= x + x^2(x-1) + x^3(x-1)^2(x-2) + \cdots; \\ g(x,y) &= \frac{1}{y - x^2} + \frac{1}{y^2 - x^3} + \frac{1}{y^3 - x^4} + \cdots. \end{aligned}$$

Neither f nor g are grid-based transseries in our sense, but they would have been in the alternative sense, Indeed, $f(x)$ would have been defined for all x such that $x - n \prec\!\!\prec 1$ for some $n \in \mathbb{N}$. $g(x,y)$ would have been defined for all $x \succ\!\!\succ 1$ and $y \succ\!\!\succ 1$, such that $y^n \neq x^{n+1}$ for all $n \in \mathbb{N}$. In fact, our choice of T ensures that the finiteness condition in the definition of grid-based sets is adequately generalized.

6.3.2 Transrings of grid-based type

In this section, we study a purely algebraic generalization of the concept of transvarieties. Our approach is based on the structure of a transring, which can be modelized by using Horn clauses only, and in which is allowed to have nilpotent elements. The category of transrings has a lot of structure because of the abstract nonsense results from the previous chapter. However, we have not adopted this more general point of view in the rest of this chapter for simplicity reasons and because transvarieties have a strong geometrical appeal. Nevertheless, the theory of transvarieties will probably be replaced by a more algebraic theory in the future, using the ideas from this section and section B.6.2.

In order to define transrings, we start by examining the properties of the grid-based summation operators on the function space of a given transvariety. The following Horn-clause tells us when Σ^*_c is defined for a given $c \in C^{\mathbb{N}^n}$:

GS1. $\left(\bigwedge_{1 \leqslant i \leqslant n} \bigwedge_{\varepsilon \in C^+_*} -\varepsilon < f_i < \varepsilon \right) \Rightarrow \Sigma^*_c(f_1, \cdots, f_n)\#.$

In fact, we even have

GS1'. $\left(\bigwedge_{1\leqslant i\leqslant n}\bigwedge_{\varepsilon\in C_*^+} -\varepsilon < f_i < \varepsilon\right) \Leftrightarrow \Sigma_c^*(f_1,\cdots,f_n)\#.$

Whenever $c \in C^{\mathbb{N}^n}$ has finite support, we also have

GS2. $\Sigma_c^*(f_1,\cdots,f_n) = \sum_{k_1,\cdots,k_n\in\mathbb{N}} c_{k_1,\cdots,k_n} f_1^{k_1}\cdots f_n^{k_n}.$

This property ensures that the grid-based summation operators are compatible with the ordinary ring structure of $\mathcal{F}(V)$.

The last important property of the grid-based summation operators is analogue to **S3**. Let $({}^ic)_{i\in I}$ and $({}^ic')_{i\in I}$ be term by term sumable families in $C^{\mathbb{N}_p\times\mathbb{N}^n}$ and $C^{\mathbb{N}_{p'}\times\mathbb{N}^{n'}}$ respectively. Denote by c resp. c' their termwise sums. Then the following Horn-clause is satisfied:

GS3. $\left(\bigwedge_{i\in I}\Sigma_{{}^ic}^*(f_1,\cdots,f_n;g_1,\cdots,g_p) = \Sigma_{{}^ic'}^*(f'_1,\cdots,f'_{n'};g'_1,\cdots,g'_{p'})\right) \Rightarrow$
$\Sigma_c^*(f_1,\cdots,f_n;g_1,\cdots,g_p) = \Sigma_{c'}^*(f'_1,\cdots,f'_{n'};g'_1,\cdots,g'_{p'}).$

An ordered exp-log ring over C with grid-based summation operators which satisfy **GS1**, **GS2** and **GS3** is called a **grid-based transring** over C. Like in (6.4), grid-based transrings can be given the structures of strong rings. We finally remark that for some future purposes it might be necessary to replace **GS1** by the stronger condition **GS1'**, but we have not yet investigated this issue in detail.

6.3.3 Asymptotic scales of grid-based type

Let V be a grid-based transvariety. In contrast to the one variable case, there is no canonical way to write transseries in several variables as sums of coefficients times transmonomials. However, we will now show that certain subalgebras of $\mathcal{F}(V)$ can be seen as algebras of grid-based series. Let S be a subset of $\mathcal{F}(V)$. We interpret S as a generalized set of transmonomials, so that we partially order it by the opposite strict ordering $<$ of $\prec\!\!\!\prec$ (remind warning 1.1). We say that S is an **asymptotic scale** of grid-based type, if it satisfies the following conditions:

AS1. S is a multiplicative group with C-powers.

AS2. For any non trivial linear combination $f = \sum_{i=1}^n c_i щ_i$ with $щ_1,\cdots,щ_n \in S$, there exists a point $P \in V$ with $\mathrm{M}_{P(f)} = \min\{\mathrm{M}_{P(щ_1)},\cdots,\mathrm{M}_{P(щ_n)}\}$.

Let S be an asymptotic scale. The subset $S^c = \{щ \in S | щ \asymp 1\}$ of Archimedian elements in S forms a subgroup of S with C-powers. Elements in $C[\![S^c]\!]$ are polynomials in S^c over C. Because of **AS2**, these elements are also Archimedian.

Proposition 6.1. *If S is an asymptotic scale of grid-based type, then $C[\![S]\!]$ is naturally embedded in $\mathcal{F}(V)$.*

Proof. Let us first show that an element $f = \sum_{\text{щ} \in S} f_{\text{щ}} \text{щ}$ of $C[[S]]$ can naturally be interpreted as an element of $\mathcal{F}(V)$. We have

$$\operatorname{supp}_S f \subseteq \text{щ}_1^{\mathbb{N}} \cdots \text{щ}_n^{\mathbb{N}} F,$$

for certain infinitesimal $\text{щ}_1, \cdots, \text{щ}_n \in S$ and a certain finite subset $F = \{\text{щ}'_1, \cdots, \text{щ}'_p\}$ of S. Hence, we can write $f = \sum_{k_1, \cdots, k_n, j} c_{j,k_1,\cdots,k_n} \text{щ}_1^{k_1} \cdots \text{щ}_n^{k_n} \text{щ}'_j$, for some family c of constants. But this means that we can interpret f as the element of $\mathcal{F}(V)$:

$$f = \Sigma_c^*(\text{щ}_1, \cdots, \text{щ}_n; \text{щ}'_1, \cdots, \text{щ}'_p).$$

To prove that the natural mapping from $C[[S]]$ into $\mathcal{F}(V)$ is injective, we assume that $f = \sum_{\text{щ} \in S} c_{\text{щ}} \text{щ}$, with Noetherian support S' is in its kernel. Then the set F of minimal elements of S' is finite. Assume for contradiction that $F \neq \phi$. By **AS3** there exists a point $P \in V$, such that $\text{м}_{P(g)} = \min_{\text{щ} \in F} \text{м}_{P(\text{щ})} = \text{ц}$, where $g = \sum_{\text{щ} \in F} c_{\text{щ}} \text{щ}$. But $P(\sum_{\text{щ} \in S' \setminus F} c_{\text{щ}} \text{щ}) \prec\!\!\prec \text{ц}$ whence $\text{м}_{P(f)} = \text{ц}$ and $P(f) \neq 0$. □

Example 6.5. Take $V = (\mathbb{A}_\infty^+)^2 = \{(x,y) \in \mathbb{A}^2 | x \succ\!\!\succ 1 \wedge y \succ\!\!\succ 1 \wedge x > 0 \wedge y > 0\}$. Let Ц_x resp. Ц_y denote the sets of "transmonomials in x resp. y". Then $\text{Ц}_x \text{Ц}_y$ is an asymptotic scale. The following transseries is a sample element of $C[[\text{Ц}_x \text{Ц}_y]]$:

$$f = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{ij!k!!}{x^i y^j e^{kxy}}.$$

Example 6.6. Let $V = \{(x,y) \in \mathbb{A}^2 | 1 \preceq\!\!\preceq \lambda \preceq\!\!\preceq 1 \;\wedge\; z \prec\!\!\prec 1\}$. Then $\lambda^C \text{Ц}_{z^{-1}}$ is an asymptotic scale. For instance,

$$f = \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} (\lambda + 2^k \lambda^2 + \cdots + l^k \lambda^l) z^k e^{-l\lambda^2 z^{-1}}$$

is an element of $C[[\lambda^C \text{Ц}_{z^{-1}}]]$.

6.4 Examples and properties of transvarieties

The transline. Let us show that grid-based transseries in several variables indeed generalize transseries in one variable. To see this, we consider $\mathbb{T}$ as a transvariety. A point $\varphi : \mathbb{T} \to \mathbb{T}$ in $\mathcal{V}(\mathbb{T})$ must send t to a positive infinitely large transseries. Inversely, right composition with a positive infinitely large transseries determines a point in $\mathcal{V}(\mathbb{T})$. Hence, $\mathcal{V}(\mathbb{T})$ is isomorphic to the set $\mathbb{A}_\infty^+ = \{x \in \mathbb{A} | 0 \leqslant x \;\wedge\; x \succ\!\!\succ 1\}$ of positive infinitely large points on the transline.

Reparameterizations. The fact that right compositions in $\mathbb{T}$ correspond to morphisms of $\mathcal{V}(\mathbb{T})$ to $\mathcal{V}(\mathbb{T})$ suggests to interpret morphisms of transvarieties more generally as right compositions. Given a morphism φ between two transvarieties V and W and a function $f \in \mathcal{F}(W)$, we thus have a function $f \circ \varphi$ in $\mathcal{F}(V)$. In particular, for any positive infinitely large transseries $g \in \mathbb{T}$ and any point $P : \mathcal{F}(V) \to \mathbb{T}$ of a transvariety, we have a point $P \circ g$. We say that $P \circ g$ is a **reparameterization** of P and we denote $P \circeq Q$. Indeed, $\circeq$ is an equivalence relation.

Construction of transvarieties. We have shown in section B.5, that many categorical constructions can be carried out in GTV_C. For instance, we have the affine n-space $\mathbb{A}^n$, whose point space corresponds to $\mathbb{T}^n$. The direct product $V \times W$, and the direct sum $V \amalg W$ of two transvarieties V and W always exist, and correspond to the pointwise direct product and sum (i.e. the forgetfull functor which associates the underlying set of points to a variety preserves direct products and sums). We can also consider subvarieties of transvarieties, such as the circle $S^1 = \{(x, y) \in \mathbb{A}^2 | x^2 + y^2 = 1\}$. By proposition B.5 the circle indeed is a subvariety of $\mathbb{A}^2$.

Asymptotic systems. Other examples of subvarieties of $\mathbb{A}^2$ are the closed ball $\overline{B}^2 = \{(x, y) \in \mathbb{A}^2 | x^2 + y^2 \leqslant 1\}$ and $(\mathbb{A}^+_\infty)^2 = \{(x, y) \in \mathbb{A}^2 | x \succ\!\!\succ 1 \;\wedge\; y \succ\!\!\succ 1 \;\wedge\; x > 0 \;\wedge\; y > 0\}$. In fact, using the exp-log field operations, the ordering, and the asymptotic relations, one can construct very general **asymptotic systems** of equations and relations. For instance, one can consider relations like $x^2 + y^2 \prec\!\!\!\prec e^x$. The subvariety of points in $\mathbb{A}^2$ which satisfy this relations is shown in figure 6.1 (we recall that elements of $\mathbb{A}$ are considered as transseries in t).

Remark 6.2. Quotient transvarieties can not be constructed using the tools we have developed so far. However, following the discussion in section B.6 we can generalize the concept of transvarieties. Doing this, we can for example define **projective transvarieties** by $\mathbb{P}^n = \mathbb{A}^{n+1}/\!\sim$ with

$$(x_0, \cdots, x_n) \sim (y_0, \cdots, y_n) \Leftrightarrow \exists t \neq 0 \;\; y_0 = t x_0 \;\wedge\; \cdots \;\wedge\; y_n = t x_n.$$

Also less conventional quotient transvarieties, such as the **curve transvarieties** $V/\!\circeq$, where $\circeq$ is the reparameterization relation. An illustration of the plane curve transvariety $\mathbb{A}^2/\!\circeq$ is given in figure 6.2. In this case, $\circeq$ is given by

$$(x, y) \circeq (x', y') \Leftrightarrow \exists g \in \mathbb{A}^+_\infty \;\; x' = x \circ g \;\wedge\; y' = y \circ g.$$

Other examples of quotient transvarieties are obtained by quotienting $\mathbb{A}$ by $\sim, \asymp$ or $\overset{\asymp}{\smile}$.

Topologies on transvarieties. A transvariety V can be given different topologies. In section B.5 we defined the Zarisky topology on V. The interval topology on the transline can also be generalized to transvarieties: it is the smallest topology which makes all functions on the transvariety continuous, as well as all functions on open sets (recursively), when we give $\mathbb{T}$ the interval topology. This topology is called the

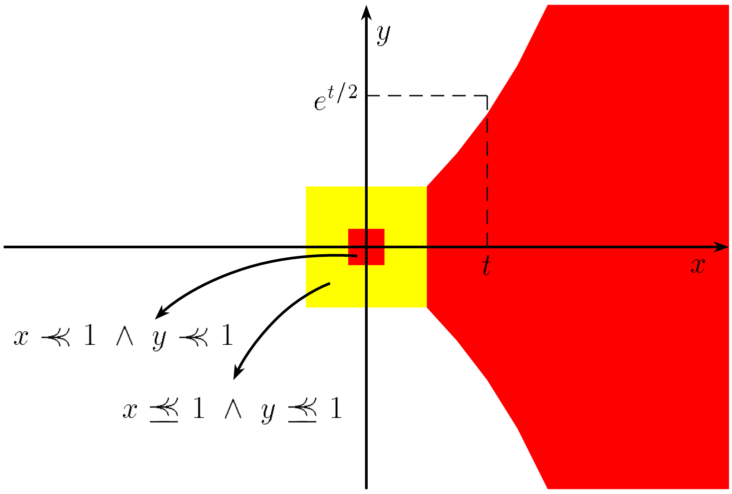


Figure 6.1: The set of points $(x, y) \in \mathbb{A}^2$, satisfying $x^2 + y^2 \prec\!\!\prec e^x$.

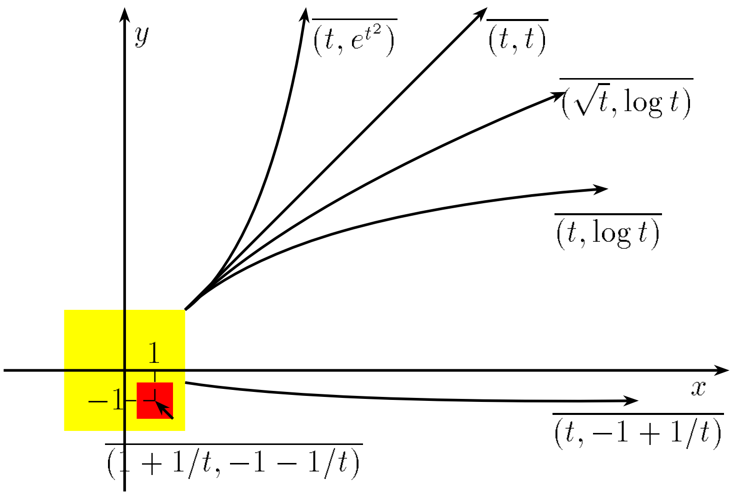


Figure 6.2: Some "points" in the plane curve transvariety.

natural topology on a transvariety. We claim that any map $V \xrightarrow{\varphi} W$ between transvarieties is continuous for the natural topology. Indeed, for an open set of the form $\psi^{-1}(I)$ (where $W \xrightarrow{\psi} \mathbb{T}$ and I is an interval), $(\psi \circ \varphi)^{-1}(I)$ is open by definition. The general case follows by the fact that ψ^{-1} preserves unions and intersections.

Dimension of a transvariety. The Zarisky topology can be used to define the **Zarisky dimension** of a transvariety as the dimension of the associated topological space (cf. [Har 77, p. 6]). Alternatively, we can define the **dimension** of a transvariety V to be the largest cardinal N, such that there exists a non empty open subset of $\mathbb{A}^N$ for the usual topology, which is isomorphic to an open subset of V. Both definitions do not coincide, but it can be shown that the Zarisky dimension is always superior or equal to the natural dimension.

6.5 Differential geometry on transvarieties

6.5.1 The tangent and cotangent bundles

Let $\mathcal{F}(V)$ be the function space of a transvariety. By **derivations** on $\mathcal{F}(V)$ we understand derivations on the strong exp-log ring $\mathcal{F}(V)$. More generally, let M be a strong $\mathcal{F}(V)$-module. Then a strong derivation on M is a strongly linear mapping $\mathcal{F}(V) \to M$, such that $d(fg) = f(dg) + g(df)$ and $de^f = (df)e^f$, for all $f, g \in \mathcal{F}(V)$. By theorem B.1, the category of strong $\mathcal{F}(V)$-modules M, together with a derivation $d : \mathcal{F}(V) \to M$ admits an initial object, which we denote by $\Omega_{\mathcal{F}(V)}$. In other words, $\Omega_{\mathcal{F}(V)}$ is a strong $\mathcal{F}(V)$-module together with a **universal derivation** $d : \mathcal{F}(V) \to \Omega_{\mathcal{F}(V)}$, which satisfies the universal property that for any other strong derivation $d' : \mathcal{F}(V) \to M$ into a strong $\mathcal{F}(V)$-module, there exists a unique strongly linear mapping $u : \Omega_{\mathcal{F}(V)} \to M$ with $d' = u \circ d$. We say that $\Omega_{\mathcal{F}(V)}$ is the **cotangent bundle** of V, and its dual $\Omega^*_{\mathcal{F}(V)}$ the **tangent bundle** of V. Elements of the tangent bundle can also be seen as strong derivations $\partial : \mathcal{F}(V) \to \mathcal{F}(V)$. In this section we will study the structure of the tangent bundle more closely and we start with affine varieties.

6.5.2 Partial differentiation on open domains of affine space

Let U be an open subset of affine X-space $\mathbb{A}^X$, for the natural topology. For each $f \in \mathcal{F}(U)$ and $x \in X$, we want to define the partial derivative $\partial_x f$ of f w.r.t. x. We proceed by structural induction: we set $\partial_x c = 0$, for $c \in C$, $\partial_x x = 1$ and $\partial_x y = 0$, for $y \in X \setminus \{x\}$. Next, $\partial_x(f \pm g) = \partial_x f \pm \partial_x g$, $\partial_x(fg) = f\partial_x g + \partial_x f g$, $\partial_x(1/f) = -\partial_x f / f^2$, $\partial_x e^f = \partial_x f e^f$ and $\partial_x \log f = \partial_x f / f$. Finally, $\partial_x \sum_{i \in I} f_i = \sum_{i \in I} \partial_x f_i$. Strictly speaking, we rather define

$$\partial_x \Sigma_c^*(g_1, \cdots, g_n) = \sum_{j=1}^{n} \partial_x g_j \Sigma_{jc}^*(g_1, \cdots, g_n)$$

in the last case, where ${}^j c_{k_1,\cdots,k_n} = (k_j + 1)c_{k_1,\cdots,k_j+1,\cdots,k_n}$, for all $k_1, \cdots, k_n, j$. In order to prove that ∂_x is well defined, we still need to verify that partial derivatives of equivalent expressions (i.e. representing the same function) are equivalent. We do this by establishing a pointwise characterization for ∂_x.

Let $P : \mathcal{F}(U) \to \mathbb{T}$ be a point of U. We denote by $S_{x,\varepsilon}P$ the point we obtain by sending x to $P(x) + \varepsilon$, and leaving the other coordinates unchanged. Since U is open, the points $S_{x,\varepsilon}P$ are in U for $|\varepsilon| > 0$ sufficiently small. We claim that

$$(S_{x,\varepsilon}P)(f) = P(f) + P(\partial_x f)\varepsilon + \frac{1}{2}P(\partial_x^2 f)\varepsilon^2 + \cdots, \tag{6.5}$$

for sufficiently small $|\varepsilon| > 0$. Indeed, this is easily proved by structural induction. For illustration, we give the proof in the case when $f = \Sigma_c^*(g_1, \cdots, g_n)$. By the induction hypothesis, there exists an $\varepsilon_0 > 0$, such that

$$(S_{x,\varepsilon}P)(g_i) = P(g_i) + P(\partial_x g_i)\varepsilon + \frac{1}{2}P(\partial_x^2 g_i)\varepsilon^2 + \cdots,$$

for sufficiently small $|\varepsilon| < \varepsilon_0$ and $1 \leqslant i \leqslant n$. Repeatedly using the generalized associativity property for strong rings, we deduce that (6.5) holds for all $|\varepsilon| < \varepsilon_0$.

The relation (6.5) gives us an intrinsic characterization of $\partial_x f$. Hence $\partial_x f$ is well defined for all $f \in \mathcal{F}(U)$. In fact, the tangent bundle of U is generated by the partial derivatives, in the sense that each strong derivation ∂ of the exp-log ring $\mathcal{F}(U)$ is of the form $\partial = \sum_{x \in X} a_x \partial_x$. Here the sum is formal and does not necessarily have finite support. Indeed, each element of $\mathcal{F}(U)$ is can be written as a finite tree with leafs in X and nodes in T, since the arities of all symbols in T are finite. The derivative of such an expression is uniquely determined by the images of the elements of X. Consequently, $\Omega_{\mathcal{F}(U)}$ is isomorphic to the free strong $\mathcal{F}(U)$-module over X.

6.5.3 Regular algebraic extensions

For non open subvarieties of affine space, the structure of $\Omega_{\mathcal{F}(U)}$ may become more complicated. We do not think that the classical exactness properties for Kähler differentials hold (see for example [Mat 70, Th 57,58, p. 186,187]). This is due to the pointwise nature of transvarieties; in the more algebraic theory of transrings, these properties might actually hold. Nevertheless, we are often interested in transvarieties which are locally isomorphic to open affine sets with a potentially extended function space. This is for example so for any non singular algebraic curve. Let us investigate more closely the effect of function space extensions with respect to differential calculus. Assume therefore that we are given a projection $V \to U$ of another transvariety onto U, such that each point in U has precisely one preimage. We want a sufficient condition for $\Omega_{\mathcal{F}(V)}$ to be isomorphic to the free strong $\mathcal{F}(V)$-module over X.

Let $Y = \{y_1, \cdots, y_p\} \subseteq \mathcal{F}(V)$. There exists a natural mapping φ from V into $U \times \mathbb{A}^Y$, which maps each $y_i \in \mathcal{F}(U \times \mathbb{A}^Y)$ to y_i. Now assume that there exists an open subset W of $U \times \mathbb{A}^Y$ and functions $f_1, \cdots, f_p \in \mathcal{F}(W)$, such that φ factors through the common zero set Z of $f_1, \cdots, f_p$. Assume also that the Jacobian of $f_1, \cdots, f_p$ has maximal rank on Z — i.e. the vectors $((\partial_{W,v} f_i)(P))_{v \in X \amalg Y}$ with $1 \leqslant i \leqslant p$ are linearly independent, for each $P \in Z$. Since only a finite number of variables from X occur in $f_1, \cdots, f_p$ we may assume without loss of generality that $X = \{x_1, \cdots, x_n\}$ is finite.

For given $(x_1, \cdots, x_n) \in U$, there exists precisely one corresponding point in Z. Hence the operators S_{x_i, ε_i} are still well defined on Z. Now we can formally expand

$$(S_{x_1,\varepsilon_1} \cdots S_{x_n,\varepsilon_n} P)(y) = \sum_{k_1,\cdots,k_n} \frac{1}{k_1! \cdots k_n!} P(\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} y) \varepsilon_1^{k_1} \cdots \varepsilon_n^{k_n} \qquad (6.6)$$

for all $y \in Y$. The successive partial derivatives of y are formally computed by the implicit function theorem. We observe that the right hand side of (6.6) is actually well defined for sufficiently small ε_i. Since $y_1, \cdots, y_p$ are determined uniquely as a function of $x_1, \cdots, x_n$ this means that (6.6) is a genuine expansion for sufficiently small ε_i. Hence, the partial derivatives of the y_j are well defined on $\mathcal{F}(Z)$ and $\Omega_{\mathcal{F}(Z)}$ is isomorphic to the free strong $\mathcal{F}(Z)$-module over X.

More generally, if for each $g \in \mathcal{F}(V)$ there exist $y_1, \cdots, y_n$ with the above properties and such that $g \in \mathcal{F}(Z)$, then the above discussion also shows that $\Omega_{\mathcal{F}(V)}$ is isomorphic to the free strong $\mathcal{F}(V)$-module over X. In that case, we say that $\mathcal{F}(V)$ is a **regular algebraic extension** of $\mathcal{F}(U)$ and elements $\mathcal{F}(V)$ are said to be **regular algebraic** over $\mathcal{F}(U)$.

6.6 Noetherian transseries

6.6.1 Definition of Noetherian transvarieties

If we are considering well-ordered transseries instead of grid-based transseries, we do not have to take into account finiteness conditions. Consequently, the grid-based summation symbols Σ_c^* are replaced by the infinite summation symbols $\sum_I$ for all index sets I. However, we can not content ourselves to take one of the fields of transseries as constructed in section 2.2 for $\mathbb{T}$ in this case. The reason is that the partial derivations would not be well defined on open subsets of affine space. Consider for instance the transseries

$$f = \frac{\varepsilon}{t(\varepsilon - 1/t)^2 + 1/t} + \frac{\varepsilon}{t^2(\varepsilon - 1/e^t)^2 + 1/e^t} + \cdots,$$

where t is infinitely large, and ε infinitely small. This transseries is defined for all $\varepsilon \prec\!\!\prec 1$, with $\varepsilon \in C_\beta^\alpha[[[t]]]$.

The solution to this dilemma is to consider also fields of transseries of the type $C_{<\alpha}^\sharp[[[t_1; \cdots; t_n]]]$ with $\sharp =< \omega$ or $\sharp = \omega$. These are defined by induction:

$$C_{<\alpha}^\sharp[[[t_1; \cdots; t_n]]] = C_{<\alpha}^\sharp[[[t_1; \cdots; t_{n-1}]]]_{<\alpha}^\sharp[[[t_n]]].$$

The transseries f mentioned above is then undefined for $\varepsilon \in C[[[u]]]^* \subseteq C[[[u;t]]]$. Hence, f is not defined on an open subset of $C[[[u;t]]]$. We also write $f \lll g$ for non zero transseries f and g, if $\exp_p \tilde{f} < \tilde{g}$, for all $p \in \mathbb{N}$. Here $\tilde{f} = |f|$ if $|f| \succcurlyeq 1$, and $\tilde{f} = 1/|f|$ otherwise. Thus, f is defined for all $\varepsilon \prec 1$, except for $0 \neq \varepsilon \lll t$.

So let P consist of those fields of transseries $C^\sharp_{<\alpha}[[[t_1; \cdots; t_n]]]$, where α is a stable limit ordinal. Objects of Var_P are called **Noetherian transvarieties** of finite resp. infinite logarithmic depth. Elements of their function spaces are called **Noetherian transseries**. Alternatively, we can restrict the α from above to be bounded by some fixed stable limit ordinal β. In that case we obtain transvarieties of exponential depth $< \beta$. The category of transvarieties of finite exponential and logarithmic depths will be denoted by NTV_C.

Most of the theory from the previous sections can easily be adapted to the context of Noetherian transvarieties. We now give some more details. We start by showing how the transline can be given the interval topology, thus permitting the definition of the natural topology on Noetherian transvarieties. Observe first that the elements of P can all be embedded in the inductive limit $C^\sharp_\beta[[[t_1; \cdots; t_{<\omega}]]]$ of

$$C \to C^\sharp_\beta[[[t_1]]] \to C^\sharp_\beta[[[t_1; t_2]]] \to \cdots$$

Here β is taken to be the limit meta-ordinal of all small ordinals (for instance). Then the intervals of this inductive limit induce a topology on the transline. We also remark that the transvarieties associated to the $C^\sharp_\beta[[[t_1; \cdots; t_n]]]$ are given up to isomorphism by

$$\mathcal{V}(C^\sharp_\beta[[[t_1; \cdots; t_n]]]) \cong \{(x_1, \cdots, x_n) \in \mathbb{A}^n | x_1 \lll \cdots \lll x_n\}.$$

The intrinsic definition of ∂_x by (6.5) also remains valid: let $P : \mathcal{F}(V) \to \mathbb{T}$ be a point in U and let $\delta \in \mathbb{T}^+_*$ be such that $S_{x,\varepsilon}P \in U$ for all $|\varepsilon| < \delta$. In particular, $S_{x,\varepsilon}P \in U$, for all $0 < \varepsilon \lll \mathbb{T}^+_*$ (and in particular $1/u \in \mathbb{T}^\sharp_\alpha[[[u]]]$). It is easily verified by structural induction that (6.5) holds for $0 < \varepsilon \lll \mathbb{T}^+_*$. For illustration, let us do this in the case when $f = \sum (f_i)_{i \in I}$. The set $S = \text{Щ}\varepsilon^C$ forms an asymptotic scale, where Щ is the set of transmonomials in $\mathbb{T}$. By the structural induction hypothesis, we have $(S_{x,\varepsilon}P)(f_i) \in C[[S]]$ for each i. Since $\sum (S_{x,\varepsilon}P)(f_i)_{i \in I}$ is well defined, the number of elements $\text{щ} \in S$ in the support of $(S_{x,\varepsilon}P)(f_i)$ w.r.t. S is finite. In particular, the number of indices in support of $P(\partial^j_x f_i)$ is finite for each $j \in \mathbb{N}$. It follows that $\sum_N P(\partial^j_x f_i)_{i \in N}$ is well defined for all $j \in \mathbb{N}$. The relation (6.5) holds by strong linearity.

It is also true that $\Omega_{\mathcal{F}(U)}$ is isomorphic to the free strong $\mathcal{F}(U)$-module in V. The key point here is that transseries which are defined on an open subset of affine space only depend on a finite number of variables. Indeed, a relation like (6.5) still holds if we simultaneously change all elements in X: $x \mapsto x + \varepsilon_x$. If f depended on an infinite number of variables, then we would obtain a contradiction by selecting a suitable family $(\varepsilon_x)_{x \in X}$ (exercise).

Remark 6.3. One has to be a bit careful with the definition of Noetherian transvarieties. Consider for instance the category $\mathsf{P}' = \{C^\sharp_\beta[[[t_1; \cdots; t_{<\omega}]]]\}$ of point types. At the first sight, the categories $\mathsf{Var}_{\mathsf{P}}$ and $\mathsf{Var}_{\mathsf{P}'}$ coincide. However, consider

$$V = \{(x_1, x_2, \cdots) \in \mathbb{T}^{\mathbb{N}}_{\mathsf{P}'} | x_1 \lll x_2 \lll \cdots\}.$$

Then V is a nonempty variety relative to P', although V has not a single P-point. In fact, the subcategories of $\mathsf{Var}_{\mathsf{P}}$ and $\mathsf{Var}_{\mathsf{P}'}$ only coincide for those varieties whose function spaces are finitely generated. Here a partial T-algebra F is said to be finitely generated, if $F \subseteq I_X$ for some finite set X.

6.6.2 Asymptotic scales of Noetherian type

It is also possible to generalize asymptotic scales to the context of Noetherian transseries in several variables: we say that a subset S of the function space $\mathcal{F}(V)$ of some Noetherian transvariety is an **asymptotic scale of Noetherian type**, if it satisfies **AS1**, **AS2** and

AS3. For each point $P \in V$ and each well-ordered subset T of S, the set $\bigcup_{f \in T} \operatorname{supp} f(P)$ is well-ordered.

Proposition 6.2. *If S is an asymptotic scale of Noetherian type, then $C[[S]]$ is naturally embedded in $\mathcal{F}(V)$.*

Proof. Assume for contradiction that not all elements of $C[[S]]$ naturally induce elements in $\mathcal{F}(V)$, and let $f = \sum_{щ \in S} c_щ щ$ be a sum with Noetherian ordered support T (w.r.t. $\succ\!\!\succ$) which is not defined in $\mathcal{F}(V)$. Then there is a point $P \in V$, such that $\bigcup_{щ \in T} \operatorname{supp} щ(P)$ is not Noetherian. Let $щ_1, щ_2, \cdots$ be a sequence of elements in T, such that there exists a bad sequence $ц_1, ц_2, \cdots$, with $ц_i \in \operatorname{supp} щ_i(P)$, for each i. Since $\operatorname{supp} f$ is Noetherian, we may assume without loss of generality (modulo extracting a subsequence using proposition A.1(d)), that the sequence $щ_1, щ_2, \cdots$ is increasing. Hence $\bigcup_i \operatorname{supp} щ_i(P)$ is well-ordered by **AS3**, which contradicts the existence of the bad sequence $ц_1, ц_2, \cdots$. Finally, the mapping from $C[[S]]$ into $\mathcal{F}(V)$ is injective for the same reason as in the proof of proposition 6.1. □

Example 6.7. Take $V = \mathbb{A}^2_\infty = \{(x, y) \in \mathbb{A}^2 | x \succ\!\!\succ 1 \ \wedge \ y \succ\!\!\succ 1\}$. Let $Ц_x$ resp. $Ц_y$ denote the sets of "transmonomials in x resp. y". Then $Ц_x Ц_y$ is an asymptotic scale and

$$f = \frac{1}{e} + \frac{1}{xy} + \frac{1}{x^2 e^{\log^2 y}} + \frac{1}{x^3 e^{\log^3 y}} + \cdots$$

is a transseries in $C[[Ц_x Ц_y]]$.

6.7 Weakly oscillating and complex transseries

A major limitation of the theory we have developed so far is that we only consider strongly monotonic asymptotic behaviour. In general, we also want to consider trigonometric (and sometimes even fractal) functions. In this section we show that the trigonometric functions and their inverses can be incorporated in a generalized theory of so called weakly oscillatory transseries. The construction heavily relies on the algebraic geometry methods we have introduced. Considering real and imaginary parts, we will also show how this theory can be applied to construct complex transseries.

Unfortunately, the semantics of the usual asymptotic relations on $\mathbb{R}$ and the asymptotic relations that will be introduced below do not necessarily coincide. A typical example is the asymptotic relation

$$\frac{1}{\Gamma(x+2)} \prec\!\!\prec 2 - \cos x - \cos ex, \tag{6.7}$$

for $1 \succ\!\!\succ x$. Classically, this relation can easily be deduced from the expansion $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$. In our most primitive model for asymptotic calculus with trigonometric functions, this relation does not hold. Intuitively speaking, this is due to the possible existence of non standard reals x for which both $\cos x$ and $\cos ex$ are very close to 1. Nevertheless, we will show that many relations like (6.7) can be forced. We also remark that the above example is rather pathological, in the sense that it heavily relies on the number theoretical properties of e. We refer to chapter 14 for a further discussion of these issues.

6.7.1 Weakly oscillating transseries

In this section, we assume that C is a totally ordered elementary function field. This means that C is a totally ordered exp-log field, on which we have the additional functions sin and arctan, which satisfy the following conditions for all $x, y \in C$:

EF1. $\sin -x = -\sin x$.
EF2. $\sin(x+y) = \sin x \cos y + \sin y \cos x$.
EF3. $\tan \arctan x = x$.
EF4. arctan is strictly increasing and $\operatorname{im} \arctan =]-\pi/2, \pi/2[$.

Here $\pi = 4 \arctan 1$, $\cos x = \sin(x + \pi/2)$ and $\tan x = \sin x / \cos x$. In a similar fashion, one defines (ordered) **elementary function rings** to be (ordered) exp-log rings with totally defined functions sin and arctan, which satisfy the conditions **EF1**, **EF2** and **EF3** (and **EF4**).

It is not hard to deduce the classical trigonometric relations from the axioms **EF1** until **EF4**. The condition **EF1** might actually be superfluous, but we have not yet checked this (if $C = \mathbb{R}$, then **EF1** follows by continuity). The inverse trigonometric functions are defined by $\arcsin x = \arctan x/\sqrt{1-x^2}$ and $\arccos x = \frac{\pi}{2} - \arcsin x$ for

$-1 < x < 1$. Here $\sqrt{x} = e^{(\log x)/2}$ for $x > 0$. Finally, a **derivation** on an elementary function ring R is a derivation ∂ on R, considered as an exp-log ring, which satisfies $\partial \tan f = \partial f/(1+f^2)$ and $\partial \sin f = \partial f \cos f$ for all f.

For the definition of weakly oscillatory transseries, we need to add sin and arctan to the signature T. Now arctan is totally defined on any field $\mathbb{T}$ of grid-based or well-ordered transseries, by using the Taylor expansions of arctan either at points in C or at $\pm\infty_C$. It can be checked that all theory of the previous sections remains valid, if we replace T by $T \cup \{\arctan\}$ and from now on we assume that we have done this. To incorporate the sine function, two different constructions can be used. The first one uses abstract nonsense: we repeatedly insert all sines of elements in the function space of a given transvariety V. The sine of a bounded function is given by its Taylor series. The other sines are inserted freely under the constraints **EF1** until **EF4**. The transvariety $\tilde{V}$ we finally obtain is called the free weakly oscillatory completion of V. The second construction gives us more insight in the structure of $\tilde{V}$ and this is the one we now give in more detail.

Let $V = V_0$ be a transvariety and F_0 its function space. Since $\mathbb{Q} \subseteq C$, we can see F_0 as a vector space over $\mathbb{Q}$. The bounded elements of F_0 form a sub-vector space $F_0^{\top}$. Using Zorn's lemma, we can select a basis B_0 for some supplement of $F_0^{\top}$. Now consider the extension $F_0\langle S_0|R_0\rangle$ of F_0 with the symbols in S_0 subject to the relations in R_0, where

$$\begin{aligned} S_0 &= \{\cos(f/n) | f \in B_0 \ \wedge \ n \in \mathbb{N}^*\} \cup \{\sin(f/n) | f \in B_0 \ \wedge \ n \in \mathbb{N}^*\}; \\ R_0 &= \{\cos^2(f/n) + \sin^2(f/n) = 1 | f \in B_0 \ \wedge \ n \in \mathbb{N}^*\} \cup \\ & \quad \{\cos f = \Phi_n(\cos(f/n)) | f \in B_0 \ \wedge \ n \in \mathbb{N}^*\}. \end{aligned}$$

Here the generating function $\Phi(t,u) = \sum_{n \geqslant 0} \Phi_n(t) u^n$ of the polynomials Φ_n is given by $\Phi(t,u) = (1-ut)/(1-2tu+u^2)$. On $F_0^{\top}$, the sine and the cosine are naturally defined by their Taylor series expansions. Using the relations **EF1** and **EF2**, we can therefore extend the mappings $\sin, \cos : B_0/\mathbb{N}^* \to F_0\langle S_0|R_0\rangle$ to F_0.

Now let V_1 be the transvariety associated to $C \to F_0\langle S_0|R_0\rangle$. Identifying varieties with their point spaces, the natural morphism $V_1 \to V_0$ is surjective: to any point in V_0 we associate a point in V_1 by sending the symbols $\cos(f/n)$ to one and the symbols $\sin(f/n)$ to zero. Dually, this means that the mapping $\mathcal{F}(V_0) \to \mathcal{F}(V_1)$ is injective. Repeating the above construction (with $B_0 \subseteq B_1 \subseteq \cdots$), we can consider the direct limit

$$\tilde{F} = \varinjlim \mathcal{F}(V_0) \to \mathcal{F}(V_1) \to \mathcal{F}(V_2) \to \cdots$$

The corresponding transvariety $\tilde{V} = C \to \tilde{F}$ is the[1] **free weakly oscillatory completion** of V. Points of $\tilde{V}$ correspond to sequences $(\mathcal{F}(V_n) \stackrel{P_n}{\to} \mathbb{T})_{n \in \mathbb{N}}$ of morphisms, such that P_j extends P_i whenever $i \leqslant j$. A transseries in $f \in \tilde{F}$ is said to be a

[1]In fact it is easy to check the independence of the construction of $\tilde{V}$ on the successive choices of the supplements and the $\mathbb{Q}$-bases: exercise.

weakly oscillatory transseries. Its **weakly oscillatory depth** is the smallest integer $n \in \mathbb{N}$, such that $f \in \mathcal{F}(V_n)$. By construction, $\tilde{F}$ is an ordered elementary function ring.

As we already mentioned in the introduction to this section, the semantics of the usual asymptotic relations on $\mathbb{R}$ and the asymptotic relations that are verified by the free weakly oscillatory completion of a transvariety over the reals do not coincide. For instance, consider the free weakly oscillatory completion of $\widetilde{\mathbb{A}^+_\infty}$, where $\mathbb{A}^+_\infty = \{x \in \mathbb{A} | x > 0 \ \wedge \ x \succ\!\!\succ 1\}$. There exists a point of $\widetilde{\mathbb{A}^+_\infty}$, which sends x to $t \in \mathbb{T}$ (recall that $t \succ\!\!\succ 1$) and both $\sin x$ and $\sin ex$ to 1. Consequently, (6.7) is not verified by $\widetilde{\mathbb{A}^+_\infty}$.

Nevertheless, in the construction of $\widetilde{\mathbb{A}^+_\infty}$, we may replace V_1 by the subvariety of V_1, for which (6.7) is valid (see proposition B.5): the natural projection of this subvariety onto $\mathbb{A}^+_\infty$ is still surjective. More generally, at each step of the construction, we can replace V_i by any subvariety of V_i, as long as the natural projections $V_{i+1} \to V_i$ remain surjective. Doing this, we say that the resulting $\tilde{V}$ is *a* **weakly oscillatory completion** of V. Now (6.7) is an example of a relation which can be forced in such a completion. Actually, many natural relations can be forced simultaneously, although this point needs further investigation.

Now assume that we have selected a category $\tilde{\mathsf{P}}$, whose objects are weakly oscillating completions of objects in P. We consider the objects of $\tilde{\mathsf{P}}$ as partial $(T \cup \{\sin\})$-algebras. Varieties relative to $\tilde{P}$ are defined to be **weakly oscillating transvarieties** (relative to $\tilde{\mathsf{P}}$) For instance, $\tilde{\mathsf{P}}$ can be taken to be the category of all free weakly oscillating completions of objects of P. Again, it is possible to define a natural topology on the weakly oscillating transline $\tilde{\mathbb{A}}$: we take a basis of open sets of the form $B(P, \varepsilon)$ with $P \in \tilde{\mathbb{T}} \subseteq \mathcal{P}(\tilde{\mathbb{A}})$ and $0 < \varepsilon \in \mathbb{T}$. Here $C \to \tilde{\mathbb{T}} \in \mathrm{obj}(\tilde{\mathsf{P}})$ is a completion of $C \to \mathbb{T} \in \mathrm{obj}(\mathsf{P})$ and $B(P, \varepsilon)$ consists of all points $Q \in \widetilde{\mathbb{T}'}$, such that there exist a commutative diagram

$$\begin{tikzcd} \mathcal{F}(\tilde{\mathbb{A}}) \arrow[r, "P"] \arrow[d, "Q"] & \tilde{\mathbb{T}} \arrow[d, "\varphi"] \\ \widetilde{\mathbb{T}'} \arrow[r, "\psi"] & \widetilde{\mathbb{T}''} \end{tikzcd},$$

where $\varphi, \psi \in \hom(\tilde{\mathsf{P}}, \tilde{\mathsf{P}})$ and such that $|\varphi(P(x)) - \psi(Q(x))| < \varepsilon$. Of course, x denotes the coordinate function in $\mathcal{F}(\tilde{\mathbb{A}})$. As before, the topology on $\tilde{\mathbb{T}}$ induces a natural topology on all weakly oscillating transvarieties. It can also be checked that the partial derivatives can again be defined on the function spaces of open subvarieties of weakly oscillating affine space. Here we need a precaution in the case of Noetherian transvarieties: for each $C \to \tilde{\mathbb{T}} \in \tilde{\mathsf{P}}$ there should exist a prolongation $C \to \widetilde{\mathbb{T}[[[u]]]} \in \tilde{\mathsf{P}}$.

6.7.2 Complex transseries

Let $C \to \tilde{\mathbb{T}}$ be a weakly oscillating completion of a point $C \to \mathbb{T}$ in P. Then $C + Ci = C^2$ has the natural structure of a strong exp-log field and $\tilde{\mathbb{T}} + \tilde{\mathbb{T}}i = \tilde{\mathbb{T}}^2$ has the natural structure of a strong exp-log ring over $C + Ci$. Indeed, all laws on $C + Ci$ resp. $\tilde{\mathbb{T}} + \tilde{\mathbb{T}}i$ are defined in analogy with their definitions for the complex numbers. For instance, $e^{f+gi} = (\cos g)e^f + (\sin g)e^f i$. By convention, we make log univariate by defining $\log(f + gi) = \log f + \arctan(g/f)i$ only for $f > 0$.

Assume now that we have selected a category $\tilde{\mathsf{P}}$ like above and let C denote the category whose objects are of the form $C + iC \to \tilde{\mathbb{T}} + \tilde{\mathbb{T}}i$, with $C \to \tilde{\mathbb{T}} \in \operatorname{obj}(\tilde{\mathsf{P}})$. Then a **complex transvariety** (relative to $\tilde{\mathsf{P}}$) is a variety relative to C. The complex transplane $\mathbb{A}_{\mathsf{C}}$ can be given a natural topology in a similar way as the weakly oscillating transline. Again, this allows us to define the complex partial derivations on the function spaces of open subvarieties of complex affine space. It also allows us to generalize transmanifolds to the complex case. The following theorem indicates how far real grid-based transseries extend into complex transseries.

Theorem 6.1. *Let $f \in C_r^{alog}[\![x]\!]$ be an alogarithmic grid-based transseries of exponential depth r. Then there exists a normal basis $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ for f, such that $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!] \subseteq \mathcal{F}(U)$ with*

$$U = \begin{cases} \{z \in \mathbb{A}_{\mathsf{C}} | \Re z \succ\!\!\succ 1\}, & \textit{if } r = 0; \\ \{z \in \mathbb{A}_{\mathsf{C}} | \Re z \succ\!\!\succ 1 \ \wedge\ \Im z \prec\!\!\prec \Re z\}, & \textit{if } r = 1; \\ \{z \in \mathbb{A}_{\mathsf{C}} | \Re z \succ\!\!\succ 1 \ \wedge\ \Im z \prec\!\!\prec \exp_{r-2}^{-1}(\Re z)^{\lambda}\} & \\ \qquad \textit{for some suitable } \lambda > 0, & \textit{if } r > 1. \end{cases}$$

In particular, f is defined on U.

Proof. By the structure theorem, there exists a normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ for f. Following the construction in the proof, we may assume without loss of generality that the elements of B are all in $C_r^{alog}[\![x]\!]$ and that $\mathfrak{b}_1 = x$. If $r > 1$, then we take λ such that $\log_{r-2} \mathfrak{b}_i(x) \prec\!\!\prec x^{-\lambda-\varepsilon}$ for all i and some $\varepsilon > 0$. We claim that $\log \mathfrak{b}_i(z) \sim \log \mathfrak{b}_i(\Re z)$ for all i. From this it follows that for g of the form $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$ we have $g(x) \prec\!\!\prec 1 \Rightarrow \log f(x) \prec\!\!\prec 1 \Rightarrow \log f(z) \prec\!\!\prec 1 \Rightarrow |f(z)| \prec\!\!\prec 1$. In other words, the ordering on the monomials in $\mathfrak{b}_1, \cdots, \mathfrak{b}_n$ is preserved in the extension from the real to the complex case. Hence, the claim implies the theorem.

If $r = 0$, then our claim trivially holds. In the case $r = 1$ it follows from the fact that $\log z^{\alpha} = \alpha \log(\Re z) + \alpha \arctan(\Im z / \Re z) \sim \log(\Re z)^{\alpha}$, for all $\alpha \in C$. If $r > 1$, then we expand $g_i = \log_2 \mathfrak{b}_i(z)$ for all i:

$$g_i(z) = g_i(\Re z) + g_i'(\Re z) i \Im z - \frac{1}{2} g_i''(\Re z)(\Im z)^2 + \cdots$$

The choice of λ ensures the validity of this expansion and the fact that $g_i(z) - g_i(\Re z) \prec\!\!\prec 1$ for each i. Consequently, $\log \mathfrak{b}_i(z) = e^{g_i(z)} \sim \log \mathfrak{b}_i(\Re z)$ for each i. $\square$

Corollary. *Let* $f \in C_r^{alog}[\![\log_l x]\!]$ *with* $l > 0$*. Then the previous theorem still holds if we take*

$$U = \begin{cases} \{z \in \mathbb{A}_{\mathbb{C}} | \Re z \succ\!\!\succ 1\}, & \text{if } r \leqslant 1; \\ \{z \in \mathbb{A}_{\mathbb{C}} | \Re z \succ\!\!\succ 1 \wedge \Im(\log_l z) \prec\!\!\prec \exp_{r-2}^{-1}(\Re(\log_l z))^{\lambda}\} \\ \qquad \text{for some suitable } \lambda > 0, & \text{if } r > 1. \end{cases}$$

□

Remark 6.4. Similar types of theorems were previously proved in the context of Hardy fields (see [Gokh 93a] and [Gokh 93b]). Although our theorem does not give any information about convergence, its scope is far more general, and its proof more natural.

6.7.3 Extensions

Although the introduction of weakly oscillating resp. complex transseries as in the previous section permits us to solve an increasing number of functional equations, there are still many equations which can not be treated by the theory from sections 6.7.1 resp. 6.7.2. A typical example is

$$f'(x) = e^{\sin x}.$$

The point here is that a solution to this equation is available in the form of a Fourier expansion, and such a Fourier expansion involves an infinite number of sines with different arguments.

In the case when a differential ring of periodic functions Φ from C into C is given, such that each $\varphi \in \Phi$ admits a Fourier expansion

$$\varphi = \sum_{k \in \mathbb{Z}} \varphi_k e^{ikx},$$

the definition of weakly oscillating transseries can be generalized in a natural way in order to include the functions in Φ. Indeed it suffices to extend the signature T with Φ and to mimic the construction from section 6.7.1.

6.8 References

[Gokh 93a] D. Gokhman. Regular growth of solutions to the Riccati equation $W' + W^2 = e^{2z}$ in the complex plane. *Tech. Rep. of the Univ. of Texas at San Antonio.*

[Gokh 93b] D. Gokhman. Limits in differential fields of holomorphic germs. *Tech. Rep. of the Univ. of Texas at San Antonio.*

[Mat 70] H. Matsumara. *Commutative algebra.* W.A. Benjamin Co., New York.

Part B

Automatic asymptotics

Chapter 7

Asymptotic expansions of exp-log functions

7.1 Introduction

An exp-log function is a function built up from x and the rational numbers $\mathbb{Q}$ by the field operations, exponentiation and logarithm. These functions were introduced by Hardy (see [Har 11]), and he showed that their germs at infinity form a totally ordered field. But how to decide whether a given exp-log function is asymptotically superior to another one in a neighbourhood of infinity? More generally, is it possible to compute an asymptotic expansion of a given exp-log function in a natural asymptotic scale?

The first attempt to solve these problems was made by Geddes and Gonnet (see [GeGo 88]). Shackell is the first to give an algorithm in [Sh 90] for computing the limit of an exp-log function at infinity, under the assumption that an oracle is given to decide whether an exp-log function vanishes in a neighbourhood of infinity. His technique is based on so called nested expansions, by which one can find the order of growth of exp-log functions at infinity, but which do not allow to derive complete asymptotic expansions. This drawback is removed in [Sh 91], where Shackell gives a complete and natural asymptotic expansion algorithm. A weaker version of this algorithm, which only computes limits of exp-log functions was discovered independently in [GoGr 92], and is currently incorporated in MAPLE V.3. The author generalized this limit computation algorithm and obtained variants of Shackell's algorithm in [VdH 94a] and [VdH 94b]. In particular, [VdH 94b] contains a compact version of Shackell's algorithm. This algorithm has been further improved in [RSSV 96].

However, several related problems were overlooked up till now. First, can we reduce the problem of deciding whether an exp-log function is zero at infinity to the corresponding problem for exp-log constants? Although several algorithms exist for deciding whether a given exp-log function f is locally zero in the neighbourhood of a point of analyticity (see [Ris 75], [DL 89], [Sh 89], [Pél 95]), no one considered the problem of deciding whether the germ of f at infinity is zero. The second problem concerns the improvement of the dramatic complexity of Shackell's algorithm. Indeed, we will see that the complexity is worse than any iterated exponential, even for simple examples and if we assume that all computations with constants are done in unit time. Moreover, for the generic version of the expansion algorithm (see section 8.3), termination of the basic algorithm is not guaranteed any longer.

These two problems are dealt with in this chapter, our approach being based on [VdH 96a]. In section 7.2 we recall the basic expansion algorithm from [Sh 91], with the improvements from [VdH 94b] and [RSSV 96].

In section 7.3 we show that the germ at infinity of an exp-log function can be represented by a Laurent series in other, but simpler, germs of exp-log functions at infinity. Such representations are called Cartesian representations and they allow to detect efficiently cancelations of large numbers of terms, thus speeding up the algorithm from section 7.2. We also show that Cartesian representations can be used to decide whether a germ of an exp-log function is zero, modulo an oracle for determining the sign of a given exp-log constant.

Modulo Schanuel's conjecture (see the introduction), Richardson gave an algorithm to decide whether an exp-log constant is zero. His algorithm has the property that whenever it produces an answer, then this answer is correct. Moreover, if we can prove that the algorithm does not terminate on a given input, then we can construct a counterexample to Schanuel's conjecture from this input. In principle, Richardson's algorithm also yields a method to compute the sign of an exp-log constant: it suffices to perform a floating point evaluation at a sufficient precision. In practice, this method is intractable and a more efficient algorithm for sign computations was proposed in [VdH 95a].

For convenience of the reader, we have tried to formalize the expansion algorithm in a language which is as comprehensive as possible. In particular, in sections 7.2 and 7.3 we recall the necessary concepts from chapter 1 in a simplified and effective context. Hence, these sections can be read independently from part A , although it is easily verified that all reintroduced concepts are indeed special instances of the theory from chapter 1. In chapter 9, we will reinterpret the algorithm in the richer context of transseries from chapter 1, and discuss some variants of the expansion algorithm.

7.2 The basic algorithm

Let $\mathfrak{T}$ denote the field of germs at infinity of exp-log functions and $\mathfrak{C}$ the subfield of exp-log constants. Elements of $\mathfrak{T}$ can be represented by exp-log expressions — i.e. finite trees whose internal nodes are labeled by $+, -, \cdot, /, \exp$ or log, and whose leafs are labeled by x or rational numbers. The set of exp-log expressions which can be evaluated in a neighbourhood of infinity is denoted by $\mathfrak{T}^{expr}$. We have a natural projection $f \mapsto \overline{f}$ from $\mathfrak{T}^{expr}$ onto $\mathfrak{T}$. Until section 7.4, we make the assumption that we have an oracle which decides whether a given exp-log expression in $\mathfrak{T}^{expr}$ is zero in a neighbourhood of infinity. In this section we recall the classical asymptotic expansion algorithm for exp-log functions at infinity from [Sh 91], using the presentation from [VdH 94b].

7.2.1 Grid-based series

Let us first recall some basic concepts. An **effective asymptotic basis** is an ordered finite set $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ of positive infinitesimal exp-log expressions in $\mathfrak{T}^{expr}$, such that $\log \mathfrak{b}_i = o(\log \mathfrak{b}_{i+1})$ for $1 \leqslant i \leqslant n-1$. For instance, the set $B = \{\log^{-1} x, x^{-1}, e^{-x^2}\}$ is an effective asymptotic basis. An effective asymptotic basis B generates an **effective asymptotic scale**, namely the set S_B of all products $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$ of powers of the $\mathfrak{b}_i$, with the α_i in $\mathfrak{C}$. Elements of S_B are also called **monomials**.

Given an effective asymptotic basis B, let $\mathfrak{G}_B^{expr}$ denote the set of expressions which are built up from $\mathfrak{C}, S_B, +, -, \cdot, /$ and the operations $\varepsilon \mapsto \exp \varepsilon$ resp. $\varepsilon \mapsto \log(1+\varepsilon)$ for infinitesimal ε. We observe that each exp-log expression $f \in \mathfrak{G}_B^{expr}$ has a series expansion of the form

$$f = \sum_{(\alpha_1, \cdots, \alpha_n) \in \mathfrak{C}^n} f_{\alpha_1, \cdots, \alpha_n} \mathfrak{b}_1^{\alpha_n} \cdots \mathfrak{b}_n^{\alpha_n}. \tag{7.1}$$

Alternatively, we can expand f as a series in $\mathfrak{b}_n$ with coefficients in $\mathfrak{G}^{expr}_{\{\mathfrak{b}_1, \cdots, \mathfrak{b}_{n-1}\}}$. These coefficients can recursively be expanded in $\mathfrak{b}_{n-1}, \cdots, \mathfrak{b}_1$:

$$\begin{aligned} f &= \sum_{\alpha_n \in \mathfrak{C}} f_{\alpha_n} \mathfrak{b}_n^{\alpha_n} \\ &\vdots \\ f_{\alpha_n, \cdots, \alpha_2} &= \sum_{\alpha_1 \in \mathfrak{C}} f_{\alpha_n, \cdots, \alpha_1} \mathfrak{b}_1^{\alpha_1}. \end{aligned}$$

The exp-log expressions of the form $f_{\alpha_n, \cdots, \alpha_i}$ are called **iterated coefficients** of f. In particular, the iterated coefficients of the form $f_{\alpha_n, \cdots, \alpha_1}$ are exp-log constants.

The above expansions of f have an important property (see chapter 1): the support of f as a series in $\mathfrak{b}_n$ (resp. $\mathfrak{b}_1, \cdots, \mathfrak{b}_n$) is included in a set of the form $\lambda_1 \mathbb{N} + \cdots + \lambda_p \mathbb{N} + \nu$ — we say that f is a **grid-based series**. Here the λ_i and ν are constants in $\mathfrak{C}$ (resp. vectors in $\mathfrak{C}^n$). From this property, it follows that the support of f is well-ordered. If f is non zero, then the first term of its expansion is called the **dominant term** of f. The corresponding monomial in S_B and its coefficient are called the **dominant monomial** and **dominant coefficient** of f respectively.

Another important property of the expansion of f in $\mathfrak{b}_n$ and the expansions of its iterated coefficients is that they can be computed automatically. By this we mean that for each integer i, we can compute the first i terms of the expansion of f and so can we for its iterated coefficients. In particular, we can compute the sign of f, test whether f is infinitesimal, etc.

For the computation of the expansions of f in $\mathfrak{б}_n$, we use the usual Taylor series formulas. In the case of division $1/f$, we compute the first term $f_\mu \mathfrak{б}_n^\mu$ of f and then use the formula $1/f = (1/f_\mu)\mathfrak{б}_n^{-\mu}(1/(1+\varepsilon))$, where $\varepsilon = (f/f_\mu \mathfrak{б}_n^\mu) - 1$. The only problem when applying these formulas is that we have to avoid indefinite cancelation: note that indefinite cancelation only occurs if after having computed the first i terms of the expansion, f is actually equal to the sum of these terms. But this can be tested using the oracle, and we stop the expansion in this case.

7.2.2 Automatic expansions of exp-log expressions

The asymptotic expansion algorithm takes an exp-log expression $f \in \mathfrak{T}^{expr}$ on input, computes a suitable effective asymptotic basis B and rewrites f into an element of $\mathfrak{G}_B^{expr}$. The main idea of the algorithm lies in the idea to impose some suitable conditions on B: we say that a linearly ordered set $B = \{\mathfrak{б}_1, \cdots, \mathfrak{б}_n\}$ is an **effective normal basis** if

NB1. B is an effective asymptotic basis.

NB2. $\mathfrak{б}_1 = \log_l^{-1} x$ for some $l \in \mathbb{N}$, where $\log_l x \stackrel{\text{def}}{=} \log \overset{l \text{ times}}{\cdots} \log x$.

NB3. $\log \mathfrak{б}_i \in \mathfrak{G}_{\{\mathfrak{б}_1, \cdots, \mathfrak{б}_{i*}\}}^{expr}$ for all $i > 1$, where $\log\log \mathfrak{б}_i^{-1} \asymp \log \mathfrak{б}_{i*}$.

Such a basis is constructed gradually during the algorithm — i.e. B is a global variable in which we insert new elements during the execution of the algorithm, while maintaining the property that B is an effective normal basis. We also say that B is a **dynamic effective normal basis**. We initialize B with $B := \{x^{-1}\}$. Let us now explicitly give the algorithm, using a PASCAL-like notation:

Algorithm `expand`(f).
INPUT: An exp-log expression $f \in \mathfrak{T}^{expr}$.
OUTPUT: A grid-based series φ in $\mathfrak{G}_B^{expr}$ with $\overline{\varphi} = \overline{f}$.

Case $f \in \mathbb{Q}$: **return** f

Case $f = x$: **return** $(x^{-1})^{-1}$

Case $f = g \top h$, $\top \in \{+, -, \cdot, /\}$:
  if $\top = /$ **and** $\overline{h} = 0$ **then error** "division by zero"
  return `expand`$(g) \top$ `expand`(h)

Case $f = \log g$:
  $g :=$ `expand`(g)
  • Denote $B = \{\mathfrak{б}_1 = \log_l^{-1} x, \mathfrak{б}_2, \cdots, \mathfrak{б}_n\}$.
  if $g \leqslant 0$ **then error** "invalid logarithm"
  • Rewrite $g = c\mathfrak{б}_1^{\alpha_1} \cdots \mathfrak{б}_n^{\alpha_n}(1+\varepsilon)$, with infinitesimal ε in $\mathfrak{G}_B^{expr}$.
  if $\alpha_1 \neq 0$ **then** $B := B \cup \{\log_{l+1}^{-1} x\}$
  return $\log c + \alpha_1 \log \mathfrak{б}_1 + \cdots + \alpha_n \log \mathfrak{б}_n + \log(1+\varepsilon)$

Case $f = e^g$:

$g := \mathtt{expand}(g)$

• Denote $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$.

if $g = O(1)$ **then return** $e^c e^{g-c}$, where $c := g_{0, \overset{n \text{ times}}{\cdots}, 0}$

if $\exists 1 < i \leqslant n \;\; g \asymp \log \mathfrak{b}_i$ **then**

$\alpha := \lim g / \log \mathfrak{b}_i$

return $\mathfrak{b}_i^\alpha \mathtt{expand}(e^{g - \alpha \log \mathfrak{b}_i})$

• Let i^* be such that $\log |g| \asymp \log \mathfrak{b}_{i^*}$.

$g^+ := g_{0, \overset{n-i^* \text{ times}}{\cdots}, 0}$

$g^- := g - g^+$

$B := B \cup \{e^{-|g^+|}\}$

return $(e^{-|g^+|})^{-\operatorname{sign} g^+} e^{g^-}$

Let us comment the algorithm. The first three cases do not need explanation. In the case $f = \log g$, the fact that B is an effective normal basis is used at the end: $\alpha_1 \log \mathfrak{b}_1 + \cdots + \alpha_n \log \mathfrak{b}_n$ is indeed an expression in $\mathfrak{G}_B^{expr}$. The expansion of the exponential of a bounded series g is done by a straightforward Taylor series expansion. If g is unbounded, then we test whether g is asymptotic to the logarithm of an element in B — i.e. we test whether $\alpha := \lim g / \log \mathfrak{b}_i$ is a non zero finite number for some i. If this is so, then $f = \mathfrak{b}_i^\alpha e^{g - \alpha \log \mathfrak{b}_i}$ and $e^{g - \alpha \log \mathfrak{b}_i}$ is expanded recursively. We remark that no infinite loops can arise from this, because successive values of g in such a loop would be asymptotic to the logarithms of smaller and smaller elements of B, while B remains unchanged. Finally, if g is not asymptotic to the logarithm of an element in B, then B has to be extended with an element of the order of growth of f. The decomposition $g = g^+ + g^-$ is computed in order to ensure that B remains an effective normal basis.

7.2.3 A detailed example

Let us exemplify our algorithm on the exp-log expression

$$f = \log \log (x e^{x e^x} + 1) - \exp \exp (\log \log x + \frac{1}{x})$$

from [RSSV 96]. Initially, the effective normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ is $\{x^{-1}\}$. We start with the innermost subexpression e^x of the first part of f. The argument x of the exponential is not asymptotic to any $\log \mathfrak{b}_i$, with $i > 1$. Hence e^{-x} is inserted at the end of B and e^x is rewritten as $\mathfrak{b}_2^{-1}$. Next, for the expansion of e^{xe^x}, the argument $xe^x = \mathfrak{b}_1^{-1} \mathfrak{b}_2^{-1}$ is compared to $-x = \log \mathfrak{b}_2 = -\mathfrak{b}_1^{-1}$. We deduce that $-x = o(xe^x)$, whence e^{-xe^x} is inserted at the end of B.

At this stage, $B = \{x^{-1}, e^{-x}, e^{-xe^x}\}$. The next expression we consider is $\log(xe^{xe^x} + 1)$, where $xe^{xe^x} + 1$ is represented as $\mathfrak{b}_1^{-1} \mathfrak{b}_3^{-1} + 1$. The exponent of

$б_1$ is not zero, therefore $\log^{-1} x$ is inserted at the beginning of B. We now have $B = \{\log^{-1} x, x^{-1}, e^{-x}, e^{-xe^x}\}$ and

$$\log(xe^{xe^x} + 1) = б_2^{-1} б_3^{-1} + б_1^{-1} + \log(1 + б_2 б_4).$$

The next logarithm is treated similarly: the normal basis need not be extended and we rewrite

$$\log\log(xe^{xe^x} + 1) = б_2^{-1} + б_1^{-1} + \log[1 + б_2 б_3 [б_1^{-1} + \log(1 + б_2 б_4)]].$$

We now consider the second part of f: $\log x$ is obviously rewritten as $б_1^{-1}$. Taking its logarithm, we insert $\log\log x$ at the beginning of B. This yields $B = \{\log^{-1}\log x, \log^{-1} x, x^{-1}, e^{-x}, e^{-xe^x}\}$. The argument of the innermost exponential is $\log\log x + x^{-1} = б_1^{-1} + б_3$, which tends to infinity. This is found to be asymptotic to the logarithm of $б_2$ and we rewrite

$$\exp(\log\log x + \tfrac{1}{x}) = б_2^{-1} e^{б_3},$$

where the argument of the new exponential tends to zero. Now $б_2^{-1} e^{б_3}$ is asymptotic to $\log б_3$, whence the next exponential $\exp\exp(\log\log x + x^{-1})$ is rewritten as

$$\exp\exp(\log\log x + \tfrac{1}{x}) = б_3^{-1} \exp[б_2^{-1} \exp б_3 - б_2^{-1}].$$

The argument of the outermost exponential of the right hand side tends to zero, so that no further rewriting is necessary.

At this stage, we have constructed an effective normal basis

$$B = \{\log^{-1}\log x, \log^{-1} x, x^{-1}, e^{-x}, e^{-xe^x}\},$$

with respect to which we can expand

$$f = б_3^{-1} + б_2^{-1} + \log[1 + б_3 б_4 [б_2^{-1} + \log(1 + б_3 б_5)]] - б_3^{-1} \exp[б_2^{-1} \exp б_3 - б_2^{-1}].$$

and its subexpressions as grid-based series in $\mathfrak{G}_B^{expr}$. For instance, we now detail the computation of an equivalent of f. The first step consists in computing the dominant term with respect to $б_5$. We illustrate the algorithm on the expansion of the subexpression $б_2^{-1} + \log(1 + б_3 б_5)$.

First, the argument $б_3 б_5$ of the special function $\log(1 + z)$ expands to itself and the dominant term of the logarithm is $б_3 б_5$. Next, $б_2^{-1}$ is seen to be the dominant term of the sum. The rest of the computation of the dominant term of f w.r.t. $б_5$ is straightforward and yields

$$f_0 = б_3^{-1} + б_2^{-1} + \log(1 + б_2^{-1} б_3 б_4) - б_3^{-1} \exp[б_2^{-1} \exp б_3 - б_2^{-1}].$$

We then proceed with the computation of the dominant term of this expression w.r.t. $\mathfrak{6}_4$. A similar computation leads to

$$f_{0,0} = \mathfrak{6}_3^{-1} + \mathfrak{6}_2^{-1} - \mathfrak{6}_3^{-1} \exp[\mathfrak{6}_2^{-1} \exp \mathfrak{6}_3 - \mathfrak{6}_2^{-1}].$$

Next, we compute the dominant term of this expression w.r.t. $\mathfrak{6}_3$. The computation of the dominant term of the argument g of the outermost exponential leads to the cancelation $\mathfrak{6}_2^{-1} - \mathfrak{6}_2^{-1} = 0$, which is recognized by the oracle, whereas the function g itself is not zero. By computing the next term of its expansion, we obtain the dominant term $\mathfrak{6}_2^{-1}\mathfrak{6}_3$ of g. The dominant term of $\mathfrak{6}_3^{-1}e^g$ is $\mathfrak{6}_3^{-1}$, whence a new cancelation $\mathfrak{6}_3^{-1} - \mathfrak{6}_3^{-1} = 0$ occurs in the computation of the expansion of $f_{0,0}$. Computing the next term of the expansion leads to another cancelation $\mathfrak{6}_2^{-1} - \mathfrak{6}_2^{-1} = 0$. One more term is necessary before arriving at the conclusion that the dominant term of $f_{0,0}$ w.r.t. $\mathfrak{6}_3$ is

$$f_{0,0,1} = -\tfrac{1}{2}(\mathfrak{6}_2^{-2} + \mathfrak{6}_2^{-1})\mathfrak{6}_3.$$

Computing the dominant monomial of this expression w.r.t. $\mathfrak{6}_2$ yields the desired equivalent for f:

$$f \sim -\frac{1}{2}\mathfrak{6}_2^{-2}\mathfrak{6}_3 = -\frac{\log^2 x}{2x}.$$

In particular, we see that f is infinitesimal and ultimately negative.

7.3 Cartesian representations

In practice it is not always efficient to perform the expansions of elements in $\mathfrak{G}_B$ by applying the classical formulas for Taylor series expansions in a direct way. Consider for example the expression

$$f(x) = \frac{1}{1 - x^{-1}} - \frac{1}{1 - x^{-1}} + x^{-N},$$

or, alternatively,

$$f(x) = \frac{1}{1 - x^{-1}} - \frac{1}{1 - x^{-1} - x^{-N}},$$

where N is very large (say $N = 10^{10^{100}}$) and x tends to infinity. Determining the first term of this series using a straightforward expansion would need a time proportional to N. The point here is that, in order to detect the cancelation $1/(1 - x^{-1}) - 1/(1 - x^{-1}) = 0$, we need to represent f as a Laurent series in two variables, namely x^{-1} and x^{-N}. This is possible by the fact that f is a grid-based series in x^{-1}. In this section we show that any exp-log expression f can be represented in such a way and how to exploit this in order to improve the algorithm `expand` from section 7.2.2.

7.3.1 Cartesian representations

A Laurent series u in several variables $z_1, \cdots, z_k$ is a series in $z_1, \cdots, z_k$ whose support is included in $(\mathbb{N}+p_1) \times \cdots \times (\mathbb{N}+p_k)$ for certain $p_1, \cdots, p_k \in \mathbb{Z}$. We say that u is infinitesimal if its support is included in $\mathbb{N}^k \backslash (0, \cdots, 0)$. The α-th coefficient of u in z_i is denoted by $[z_i^\alpha]u$. We abbreviate $[z_{i_1}^{\alpha_1}] \cdots [z_{i_j}^{\alpha_j}]u$ by $[z_{i_1}^{\alpha_1} \cdots z_{i_j}^{\alpha_j}]u$. We notice that $z_1, \cdots, z_k$ should be interpreted as variables which tend to zero.[1]

Let B be an effective asymptotic basis and let $Z = \{z_1, \cdots, z_k\}$ be a finite set of infinitesimal monomials in S_B. For convenience, we order Z by $z_1 \leqslant_B \cdots \leqslant_B z_k$. We denote by $\mathfrak{L}_L^{expr}$ the set of expressions built up from $\mathfrak{C}, z_1, z_1^{-1}, \cdots, z_k$ and z_k^{-1} by $+, -, \cdot$ and the operations $\varepsilon \mapsto e^\varepsilon, \varepsilon \mapsto \log(1+\varepsilon)$ and $\varepsilon \mapsto 1/(1+\varepsilon)$ for infinitesimal ε. Given such a Laurent series $u \in \mathfrak{L}_Z^{expr}$, its expansion

$$u = \sum_{\alpha=p_i}^{\infty} ([z_i^\alpha]u)\, z_i^\alpha$$

in any of the z_i can be computed automatically. Moreover, the coefficients $[z_i^\alpha]u$ of such an expansion are also expressions in $\mathfrak{L}_Z^{expr}$, so that they can recursively be expanded — we say that u is an **automatic Laurent series**. In what follows, we will only consider automatic Laurent series which are in $\mathfrak{L}_Z^{expr}$ for some Z.

Remark 7.1. We notice that all classical efficient expansion algorithms for formal Laurent series in $\mathfrak{L}_Z^{expr}$ can be used, such as Karatsuba's algorithm for multiplication [Kn 81] and Brent and Kung's algorithms for composition [BK 75], [BK 78]. We also remark that we systematically store all coefficients of all expansions we compute, in order to perform these computations only once (i.e. we use a MAPLE-like remember option).

We denote by $\overline{u}$ the germ at infinity of the exp-log function represented by a Laurent series u in $\mathfrak{L}_Z^{expr}$. We call u a **Cartesian representation** of $\overline{u}$. Let an expression $f \in \mathfrak{G}_B^{expr}$ be given. The aim of the rest of this section is to compute a Cartesian representation $u \in \mathfrak{L}_Z^{expr}$ of $\overline{f}$ for some suitable subset Z of S_B. Furthermore, we will show how to compute the expansion of f from the knowledge of u only. Clearly, this will enable us to replace all computations with elements in $\mathfrak{G}_B^{expr}$ by computations with Cartesian representations in `expand`.

[1] The fact that x tends to infinity and $z_1, \cdots, z_k$ to zero might confuse the reader. This apparently illogical choice stems from the potentially different asymptotic behaviours of an exp-log function $f(x)$, if x tends to zero from below or from above.

Warning 7.1. One should carefully distinguish Cartesian representation from the germs at infinity they represent. For instance, if $B = \{x^{-1}, e^{-x}\}$, $z_1 = x^{-1}$ and $z_2 = z^{-x}$, then $\overline{z_1^{-1} z_2}$ is infinitesimal, while $z_1^{-1} z_2$ is not. In order to avoid confusion, we therefore scrupulously distinguish u from $\overline{u}$ by means of the upper bar. Moreover, we will use the prefix "C-" to emphasize that we are referring to properties of Cartesian representations. For instance, infinitesimal Cartesian representations will be called C-infinitesimal.

7.3.2 Restrictions of Cartesian representations

Let $Z = \{z_1, \cdots, z_k\}$ and let $S_Z = \{z_1^{\alpha_1} \cdots z_k^{\alpha_k} | \alpha_1, \cdots, \alpha_k \in \mathbb{Z}\}$ be the set of monomials in $z_1, \cdots, z_k$. We have a natural partial ordering on S_Z:

$$z_1^{\alpha_1} \cdots z_k^{\alpha_k} \leqslant_Z z_1^{\beta_1} \cdots z_k^{\beta_k} \Leftrightarrow \alpha_1 \leqslant \beta_1 \ \wedge \ \cdots \ \wedge \ \alpha_k \leqslant \beta_k.$$

Let u be a Laurent series in $z_1, \cdots, z_k$ and let Ц be a subset of S_Z. We denote by

$$[u|\text{Ц}] = \sum_{z_1^{\alpha_1} \cdots z_k^{\alpha_k} \in \text{Ц}} ([z_1^{\alpha_1} \cdots z_k^{\alpha_k}]u)\; z_1^{\alpha_1} \cdots z_k^{\alpha_k}$$

the **restriction** of u w.r.t. Ц. For singletons $\text{Ц} = \{\text{ц}\}$ we also write $[u|\text{ц}] = [u|\{\text{ц}\}]$. We finally define $(\text{Ц}) = \{\text{щ} | \exists \text{ц} \in \text{Ц} \ \ \text{ц} \leqslant_Z \text{щ}\}$ to be the final segment generated by Ц. Here we recall that a **final segment** of S_Z is a subset $F \subseteq S_Z$ such that $\text{ц} \in F \ \wedge \ \text{ц} \leqslant \text{щ} \Rightarrow \text{щ} \in F$ for all $\text{ц}, \text{щ} \in S_Z$.

Proposition 7.1. *Let u be a Laurent series in $\mathfrak{L}_Z^{expr}$. There exists an algorithm to compute the restriction $[u|(\text{Ц})]$ of u w.r.t. any final segment (Ц) for finite Ц.*

Proof. Let Ц_α denote the subset of Ц of monomials whose exponents in z_k are inferior or equal to α and let λ be the smallest integer with $\text{Ц} = \text{Ц}_\lambda$. Let Ш_α denote the set of monomials щ in $z_1, \cdots, z_{k-1}$, such that $\text{щ} z_k^\beta$ is in Ц_α, for some $\beta \leqslant \alpha$. Now expand u up to order λ in z_k, say $u = u_{p_k} z_k^{p_k} + \cdots + u_{\lambda-1} z_k^{\lambda-1} + \tilde{u}$. Then we have

$$[u|\text{Ц}] = \sum_{i=p_k}^{\lambda-1} [u_j|(\text{Ш}_\alpha)]\; z_k^\alpha + [\tilde{u}|(z_k^\lambda \text{Ш}_\lambda)].$$

The right hand side of this equation is evaluated by expanding each of the terms in $z_{k-1}, \cdots, z_1$ using the same method. □

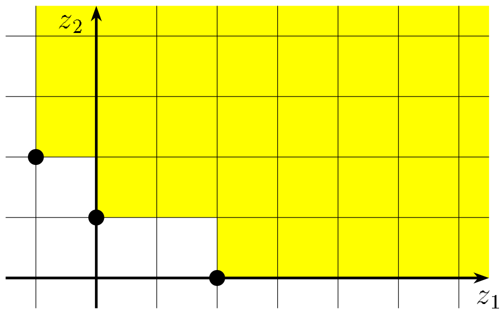


Figure 7.1: Dominant monomials for $u = \frac{1}{z_1(1-z_1-z_2)} - z_1^{-1} - 1 - z_1 - 2z_2 - z_1^{-1}z_2$.

7.3.3 Intermediary dominant monomials

Let $u \in \mathfrak{L}_Z^{expr}$ be a Cartesian representation. A set of **intermediary dominant monomials** of u is a finite subset G of S_Z, such that $\overline{u} = \overline{[u|(G)]}$, and such that the dominant monomial of f is equal to the minimal monomial in G. Most of the time, but not always, G is unique and we say that G is the set of intermediary dominant monomials.

In figure 7.1, we have represented the dominant monomials of $u = z_1^{-1}(1 - z_1 - z_2)^{-1} - z_1^{-1} - 1 - z_1 - 2z_2 - z_1^{-1}z_2$; these are by definition the minimal elements in the support of u w.r.t. $\leqslant_Z$. If $\overline{z_1} \neq \overline{z_2}$, then $\{z_1^{-1}z_2^2, z_2, z_1^2\}$ is also the set of intermediary dominant monomials of u. If $\overline{z_1} = \overline{z_2}$, then $\{z_1^{-1}z_2^3, z_2^2, z_1z_2, z_1^2\}$ is the set of intermediary dominant monomials of u, because of the cancelation $z_1^{-1}z_2^2 - z_2 = 0$.

In order to compute intermediary dominant monomials, we first need to introduce some more orderings. First, we have a total ordering $\leqslant_B$ on S_B, which is analogous to $\leqslant_Z$ on S_Z:

$$\text{б}_1^{\alpha_1} \cdots \text{б}_n^{\alpha_n} \leqslant_B \text{б}_1^{\beta_1} \cdots \text{б}_n^{\beta_n} \Leftrightarrow \text{б}_1^{\beta_1} \cdots \text{б}_n^{\beta_n} = O(\text{б}_1^{\alpha_1} \cdots \text{б}_n^{\alpha_n}).$$

Via the natural (not necessarily injective) mapping $\nu : S_Z \to S_B$, the ordering $\leqslant_B$ induces a quasi-ordering $\preccurlyeq_B$ on S_Z: $\text{ц} \preccurlyeq_B \text{щ} \Leftrightarrow \nu(\text{ц}) \leqslant_B \nu(\text{щ})$ for all $\text{ц}, \text{щ} \in S_Z$. The reader should not confuse this quasi-ordering with $\leqslant_Z$, nor with the usual asymptotic ordering on germs of exp-log functions (which is actually opposite to $\preccurlyeq_B$ on S_Z). Now consider the following algorithm:

Algorithm `idm`(u).
INPUT: A Cartesian representation $u \in \mathfrak{L}_Z^{expr}$ with $\overline{u} \neq 0$.
OUTPUT: A set of intermediary dominant monomials for u.

• Let $z_i^{p_i}$ be the dominant monomial of u in z_i, for $1 \leqslant i \leqslant k$.
$G := \{z_1^{p_1} \cdots z_k^{p_k}\}$
while true
 $M := \{ц \in G | \forall щ \in G \;\; ц \preccurlyeq_B щ\}$
 if $\overline{\sum_{ц \in M} u_ц} \neq 0$ **then return** G
 • Denote $G \backslash M = \{ц_1, \cdots, ц_q\}$, with $ц_1 \preccurlyeq_B \cdots \preccurlyeq_B ц_q$.
 if $\exists 0 \leqslant i \leqslant q \;\; \overline{u - [u|(ц_i, \cdots, ц_q)]} = 0$
  then $G := \{ц_i, \cdots, ц_q\}$ (with i chosen minimal)
  else $G := (G \backslash M) \cup M\{z_1, \cdots, z_k\}$
 • Eliminate non minimal elements from G.

Remark 7.2. We recall the existence of an oracle to decide whether a given exp-log expression in $\mathfrak{T}^{expr}$ is zero in a neighbourhood of infinity. Hence, the test $\exists 0 \leqslant i \leqslant q \;\; \overline{u - [u|(ц_i, \cdots, ц_q)]} = 0$ is indeed effective, by proposition 7.1.

Proposition 7.2. *The algorithm* `idm` *is correct and terminates.*

Proof. Let $G_1, G_2, \cdots$ be the successive values of G at the beginning of the main loop. By induction, we observe that $\overline{u} = \overline{[u|(G_j)]}$ for all j. This proves the correctness of `idm`. Suppose that the algorithm does not terminate. Let $F = \bigcap_{j \geqslant 1}(G_j)$. We have $\overline{u} = \overline{u_F}$. By Dickson's lemma, F is finitely generated, say by Γ. There are only a finite number of monomials $ц \geqslant_Z z_1^{p_1} \cdots z_k^{p_k}$ with $ц <_Z щ$ for some $щ \in \Gamma$. For sufficiently large j, none of these monomials belongs to G_j. We have $\Gamma \subseteq G_j$, since $\Gamma \subseteq (G_j)$. There do not exist $ц \in G_j \backslash \Gamma$ and $щ \in \Gamma$, with $щ <_B ц$: indeed, such a $ц$ would belong to $G_{j'}$ for all $j' \geqslant j$, although $ц \notin F = (\Gamma)$. We deduce that $\Gamma = \{ц_i, \cdots, ц_q\}$ at the j-th iteration of the main loop for some q. But this means that $G_{j+1} = \Gamma$ and $\Gamma \nsubseteq (G_{j+2})$. This contradiction proves the termination of `idm`. □

Remark 7.3. We observe that the dominant term τ_f of f is given by $\tau_f = \sum_{ц \in M} u_ц ц$ at the end of the algorithm. More terms of the expansion of f can be obtained by rerunning the algorithm recursively on $u - \tau_f$.

7.3.4 On the computation of Cartesian representations

Lemma 7.1. *There exists an algorithm, which given a Cartesian representation $u \in \mathfrak{L}_Z^{expr}$ of an infinitesimal germ $\overline{u}$ at infinity computes $Z' = \{z'_1, \cdots, z'_{k'}\}$ and a C-infinitesimal Cartesian representation $u' \in \mathfrak{L}_{Z'}^{expr}$ for $\overline{u}$.*

Proof. We first compute a set of intermediary dominant monomials $G = \{\text{ц}_1, \cdots, \text{ц}_m\}$ for u. If all monomials in G are strictly superior to 1, then we can take $Z' = Z$ and $u' = [u|(1)]$. More generally, we can write $u = v_1 + \cdots + v_m$, with $v_i = [u|(\text{ц}_1, \cdots, \text{ц}_i)] - [u|(\text{ц}_1, \cdots, \text{ц}_{i-1})]$, for $1 \leqslant i \leqslant m$. Putting $v_i = \text{ц}_i h_i$, each h_i belongs to $\mathfrak{L}_Z^{expr}$ and its support is included in $\mathbb{N}^k$. Now $u' = \text{ц}_1 h_1 + \cdots + \text{ц}_m h_m \in \mathfrak{L}_{Z'}^{expr}$, with $Z' = \{z_1, \cdots, z_k, \text{ц}_1, \cdots, \text{ц}_m\}$ satisfies the requirements of the lemma. □

Remark 7.4. Actually, we can take $k' \leqslant k$, as will easily follow from lemma 7.3 below.

Example 7.1. Assume that u' is the Laurent series from figure 7.1. We can take $\text{ц}_1 = z_1^{-1} z_2^2, \text{ц}_2 = z_2$ and $\text{ц}_3 = z_1^2$. Then we get $v_1 = z_1^{-1} z_2^2 (1/(1 - z_2))$, $v_2 = z_2((1 + z_1)/(1 - z_2))$ and $v_3 = z_1^2(1/(1 - z_1 - z_2))$. We observe that $\overline{u}$ is infinitesimal if $\overline{z_1^{-1} z_2^2}$ is. In that case, an expression like e^u can be expanded in z_1, z_2 and $z_1^{-1} z_2^2$, by using the identity $e^u = \exp(v_1 + v_2 + v_3)$.

Theorem 7.1. *Let B be an effective normal basis. Then there exists an algorithm which given an expression f in $\mathfrak{G}_B^{expr}$ computes a finite set Z of infinitesimals in S_B, and a Cartesian representation $u \in \mathfrak{L}_Z^{expr}$ for $\overline{f}$.*

Proof. Constants are by definition Cartesian representations of themselves. If $f \in S_B \backslash \{1\}$, then either $f \in \mathfrak{L}_{\{f\}}^{expr}$ or $f \in \mathfrak{L}_{\{1/f\}}^{expr}$. Now assume that u' and u'' are Laurent series in $\mathfrak{L}_{Z'}^{expr}$ and $\mathfrak{L}_{Z''}^{expr}$ respectively, with $Z' = \{z_1', \cdots, z_{k'}'\}$ and $Z'' = \{z_1'', \cdots, z_{k''}''\}$. Then $u' + u''$, $u' - u''$ and $u'u''$ are Cartesian representations for $\overline{u'} + \overline{u''}$, $\overline{u'} - \overline{u''}$ resp. $\overline{u'u''}$ in $\mathfrak{L}_{Z' \cup Z''}^{expr}$. If $\overline{u'}$ is infinitesimal, then we may assume without loss of generality that u' is C-infinitesimal by lemma 7.1. Hence, we have straightforward Cartesian representations for $1/(1 + \overline{u'}), \log(1 + \overline{u'})$ and $\exp \overline{u'}$ in $\mathfrak{L}_{Z'}^{expr}$. □

7.3.5 Asymptotic expansions via Cartesian representations

Having computed a Cartesian representation u for $\overline{f}$ by theorem 7.1, we would like to take advantage of u to compute the asymptotic expansion of f.

Lemma 7.2. *There exists an algorithm, which given a Laurent series u in $\mathfrak{L}_Z^{expr}$ with $\overline{u} \neq 0$ computes $Z' = \{z_1', \cdots, z_{k'}'\}$ and $u' \in \mathfrak{L}_{Z'}^{expr}$ with $\overline{u'} = \overline{u}$, such that u' has only one dominant monomial.*

Proof. Let $\{\text{ц}_1, \cdots, \text{ц}_m\}$ be a set of intermediary dominant monomials for u. Let $c = u_{\text{ц}_1} + \cdots + u_{\text{ц}_m}$ and $\varepsilon = (u - u_{\text{ц}_1} \text{ц}_1 - \cdots - u_{\text{ц}_m} \text{ц}_m)/\text{ц}_1$. By lemma 7.1 we can compute a C-infinitesimal Cartesian representation $\varepsilon' \in \mathfrak{L}_{Z'}^{expr}$ for $\overline{\varepsilon}$, where $Z' = \{z_1', \cdots, z_{k'}'\}$. Now we take $u' = (c + \varepsilon')\text{ц}_1$. □

Modulo this lemma, we may assume without loss of generality, that u has a unique dominant monomial. The following proposition gives us the first term of the expansion of f w.r.t. $\mathfrak{G}_n$:

Proposition 7.3. *Let $f \in \mathfrak{G}_B^{expr}$ and let $u \in \mathfrak{L}_Z^{expr}$ be a Cartesian representation of $\overline{f}$ with a unique dominant monomial $z_1^{\mu_1} \cdots z_k^{\mu_k}$ and $Z \subseteq S_B$. Let $z_1, \cdots, z_l$ those elements among $z_1, \cdots, z_k$ which depend on $\mathfrak{G}_n$, say $z_i = z_i' \mathfrak{G}_n^{\alpha_i}$ for $1 \leqslant i \leqslant l$, with z_i' free from $\mathfrak{G}_n$ and $\alpha_i > 0$. Then the dominant exponent of f w.r.t. $\mathfrak{G}_n$ equals*

$$\mu_f = \mu_1 \alpha_1 + \cdots + \mu_l \alpha_l$$

and

$$([z_1^{\mu_1} \cdots z_l^{\mu_l}]u) z_1'^{\mu_1} \cdots z_l'^{\mu_l}$$

is a Cartesian representation for $\overline{[\mathfrak{G}_n^{\mu_f}]f}$. □

Clearly, this proposition enables us to extract the first term of the expansion of f w.r.t. $\mathfrak{G}_n$. More terms can be obtained by subtracting the first term from f and iterating the process. Similarly, we can iterate the process on the coefficients of this expansion in order to obtain the iterated coefficients of f. In particular, this yields an algorithm to compute the iterated coefficients $g_{0, \overset{n-i^* \text{ times}}{\cdots}, 0}$ of g involved in the exponential case $f = e^g$ in `expand`.

7.4 An asymptotic zero test for exp-log functions

In this section we no longer assume that we have an oracle for deciding whether an exp-log function is zero in a neighbourhood of infinity. Instead, we assume that we dispose of an oracle which can decide whether an exp-log constant is zero. Such an oracle is in fact an algorithm under the assumption that Schanuel's conjecture holds (see the introduction). Now a zero test for Laurent series in $\mathfrak{L}_Z^{expr}$, which depends on the oracle, is given in [Pél 95] (see also appendix D).

Lemma 7.3. *Let B be an effective normal basis and let $z_1, \cdots, z_{k+1}$ be infinitesimals in S_B. Assume that $z_{k+1} = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ with $\alpha_1, \cdots, \alpha_k \in \mathbb{Z}$. There is an algorithm which computes $z_1', \cdots, z_k' \in S_B$ and a matrix $M = (\beta_{i,j})$ with $i \in \{1, \cdots, k+1\}$, $j \in \{1, \cdots, k\}$ and coefficients in $\mathbb{N}$, such that*

$$z_i = z_1'^{\beta_{i,1}} \cdots z_k'^{\beta_{i,k}}$$

for all $1 \leqslant i \leqslant k+1$.

Proof. Let us describe a recursive method to compute such z_j' and $\beta_{i,j}$. Since one of the α_i must be strictly positive, we may assume without loss of generality that $\alpha_k > 0$ by permuting variables. Now $\overline{z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-1}}}$ is either infinitesimal, equal to 1, or infinitely large.

In the first case, we recursively compute $z'_1, \cdots, z'_{k-1} \in S_B$ and $\gamma_{i,j} \in \mathbb{N}$ for $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, k\}$, such that

$$z_i = {z'_1}^{\gamma_{i,1}} \cdots {z'_{k-1}}^{\gamma_{i,k-1}}$$

for all $1 \leqslant i \leqslant k-1$ and

$$z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-1}} = {z'_1}^{\gamma_{k,1}} \cdots {z'_{k-1}}^{\gamma_{k,k-1}}.$$

Now we can take $z'_k = z_k$ and

$$M = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k-1} & 0 \\ \vdots & & \vdots & \vdots \\ \gamma_{k-1,1} & \cdots & \gamma_{k-1,k-1} & 0 \\ 0 & \cdots & 0 & 1 \\ \gamma_{k,1} & \cdots & \gamma_{k,k-1} & 1 \end{pmatrix}.$$

The second case is trivial, since $z_{k+1} = z_k$.

In the last case, we recursively compute $z'_1, \cdots, z'_{k-1} \in S_B$ and $\gamma_{i,j} \in \mathbb{N}$ for $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, k\}$, such that

$$z_i = {z'_1}^{\alpha_k \gamma_{i,1}} \cdots {z'_{k-1}}^{\alpha_k \gamma_{i,k-1}}$$

for all $1 \leqslant i \leqslant k-1$ and

$$z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-1}} = {z'_1}^{-\alpha_k \gamma_{k,1}} \cdots {z'_{k-1}}^{-\alpha_k \gamma_{k,k-1}}.$$

Now take $z'_k = z_k {z'_1}^{\alpha_1} \cdots {z'_{k-1}}^{\alpha_{k-1}}$ and

$$M = \begin{pmatrix} \alpha_n \gamma_{1,1} & \cdots & \alpha_n \gamma_{1,k-1} & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_n \gamma_{k-1,1} & \cdots & \alpha_n \gamma_{k-1,k-1} & 0 \\ \gamma_{k,1} & \cdots & \gamma_{k,k-1} & 1 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix}.$$

□

The following is an easy consequence of the lemma:

Lemma 7.4. *Let B be an effective normal basis and let $z_1, \cdots, z_{k+1}$ be infinitesimals in S_B. Assume that $z_1^{\alpha_1} \cdots z_k^{\alpha_k} = 1$, for $\alpha_1, \cdots, \alpha_k \in \mathbb{Z}$ not all zero. There is an algorithm which computes $z'_1, \cdots, z'_{k-1} \in S_B$ and a matrix $M = (\beta_{i,j})$ with $i \in \{1, \cdots, k\}$, $j \in \{1, \cdots, k-1\}$ and coefficients in $\mathbb{N}$, such that*

$$z_i = {z'_1}^{\beta_{i,1}} \cdots {z'_{k-1}}^{\beta_{i,k-1}}$$

for all $1 \leqslant i \leqslant k$. □

Theorem 7.2. *Assuming Schanuel's conjecture, there exists an algorithm which given an exp-log expression $f \in \mathfrak{T}^{expr}$:*

(a) computes an effective normal basis B for f.
(b) computes an asymptotic expansion for f w.r.t. B at any order.
(c) determines the sign of f.
(d) determines whether f is infinitesimal.

Proof. In view of what precedes, we only have to show how to decide whether $\overline{u} = 0$ for a given $u \in \mathfrak{L}_Z^{expr}$, with the notations from the previous section. To do this, we slightly modify `idm`:

Algorithm `zero_test`(u).
INPUT: A Cartesian representation $u \in \mathfrak{L}_Z^{expr}$ for some Z.
OUTPUT: Result of the test $\overline{u} = 0$.

if $u = 0$ **then return true**
• Let $z_i^{p_i}$ be the dominant monomial of u in z_i, for $1 \leqslant i \leqslant k$.
$G := \{z_1^{p_1} \cdots z_k^{p_k}\}$
while true
    $M := \{ц \in G | \forall щ \in G \;\; ц \preccurlyeq_B щ\}$
    if $|M| > 1$ **then return** `zero_test(simplify(`u, M`))`
    if $\sum_{ц \in M} u_ц \neq 0$ **then return false**
    • Denote $G \backslash M = \{ц_1, \cdots, ц_q\}$, with $ц_1 \preccurlyeq_B \cdots \preccurlyeq_B ц_q$.
    if $\exists 0 \leqslant i \leqslant q \;\; u - [u | (ц_i, \cdots, ц_q)] = 0$ **then** $G := \{ц_i, \cdots, ц_q\}$
    else $G := (G \backslash M) \cup M\{z_1, \cdots, z_k\}$
    • Eliminate non minimal elements from G.

Let us comment this algorithm. All zero tests we perform are zero tests for Laurent series. If the cardinal $|M|$ of M never exceeds 1, then the usual termination proof of `idm` remains valid and we are done. In the other case, the function `simplify` is invoked, which undertakes the following action:

STEP 1. Determine a non trivial relation of the form $z_1^{\alpha_1} \cdots z_k^{\alpha_k} = 1$ in S_B, with $\alpha_1, \cdots, \alpha_k \in \mathbb{Z}$.
STEP 2. Apply lemma 7.4 to find infinitesimals $z'_1, \cdots, z'_{k-1} \in S_B$ and positive integers $\beta_{i,j}$ with $z_i = {z'_1}^{\beta_{i,1}} \cdots {z'_{k-1}}^{\beta_{i,k-1}}$ for each i.
STEP 3. Return u after having replaced each z_i by ${z'_1}^{\beta_{i,1}} \cdots {z'_{k-1}}^{\beta_{i,k-1}}$.

The recursive call of `zero_test` terminates, since $Z' = \{z'_1, \cdots, z'_{k-1}\}$ has one element less than Z. □

Remark 7.5. A heuristic zero test for Laurent series u in $\mathfrak{L}_Z^{expr}$ often suffices for practical purposes: we perform a floating point evaluation of u in a random point $(\zeta_1, \cdots, \zeta_k)$ with reasonably small ζ_i. Instead of rewriting u in the above algorithm, whenever we find a dependency $z_1^{i_1} \cdots z_k^{i_k} = 1$, we use these dependencies to impose a posteriori additional conditions on the ζ_i.

7.5 References

[BK 75] R.P. Brent, H.T. Kung. $O((n \log n)^{3/2})$ algorithms for composition and reversion of power series. *In Analytic Computational Complexity, J.F. Traub Ed. Proc. of a symposium on analytic computational complexity held by Carnegie-Mellon University.*

[BK 78] R.P. Brent, H.T. Kung. Fast algorithms for manipulating formal power series. *Journal of the ACM 25 (p. 581-595).*

[DL 89] J. Denef, L. Lipshitz. Decision problems for differential equations. *The Journ. of Symb. Logic 55(3) (p. 941-950).*

[GeGo 88] K.O. Geddes, G.H. Gonnet. A new algorithm for computing symbolic limits using hierarchical series. *Proc. ISSAC 1988, Lecture notes in computer science 358, (p. 490-495).*

[GoGr 92] G.H. Gonnet, D. Gruntz. Limit computation in computer algebra. *Report technique 187 du ETH, Zürich.*

[Gr 96] D. Gruntz. On computing limits in a symbolic manipulation system. *PhD. Thesis, E.T.H. Zürich, Switzerland.*

[Har 10] G.H. Hardy. *Orders of infinity.* Cambridge Univ. Press.

[Har 11] G.H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London mathematical society 10,2 (p. 54-90).*

[Kn 81] D.E. Knuth. *The art of computer programming.* Vol 2: Seminumerical algorithms. Addison-Wesley (2-nd ed.).

[Pél 95] A. Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948 (p. 393-407).*

[PV 96] A. Péladan-Germa, J. van der Hoeven. Un algorithme de Buchberger local. *Submitted to C.R.A.S.*

[Rich 92] D. Richardson. The elementary constant problem. *Proc. ISSAC 92 (p. 108-116).*

[Rich 94] D. Richardson. How to recognize zero. *Preprint, University of Bath, England.*

[Ris 75] R.H. Risch. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math. 4, 101 (p. 743-759).*

[RSSV 96] D. Richardson, B. Salvy, J. Shackell, J. van der Hoeven. Asymptotic expansions of exp-log functions. *Submitted to ISSAC '96.*

[Sal 91] B. Salvy. Asymptotique automatique et fonctions génératrices. *PhD. thesis, Ecole Polytechnique, France.*

[Sh 89] J. Shackell. A differential-equations approach to functional equivalence. *Proc. ISSAC 89, Portland, Oregon, A.C.M., New York, (p. 7-10).*

[Sh 90] J. Shackell. Growth estimates for exp-log functions. *Journal of symbolic computation 10 (p. 611-632).*

[Sh 91] J. Shackell. Limits of Liouvillian functions. *Technical report, Univ. of Kent, Canterbury. Appeared in 1996 in the Proc. of the London Math. Soc. 72 (p. 124-156).*

[VdH 94a] J. van der Hoeven. Outils effectifs en asymptotique et applications. *Research report LIX/RR/94/09, Ecole Polytechnique, France.*

[VdH 94b] J. van der Hoeven. General algorithms in asymptotics I, Gonnet and Gruntz's algorithm. *Research report LIX/RR/94/10, Ecole Polytechnique, France.*

[VdH 95a] J. van der Hoeven. Automatic numerical expansions. *Local proc. of the conference "Real numbers and computers", St-Etienne, April 4-6, France (p. 261-274).*

[VdH 96a] J. van der Hoeven. Generic asymptotic expansions. *Local Proc. of the Rhine workshop on computer algebra, Saint-Louis, April 1-3, France, Submitted to AAECC..*

Chapter 8

Automatic case separation

8.1 Introduction

This chapter constitutes both the informal description of a general method (automatic case separation) that will be used throughout the rest of part B of this thesis, and two applications of this method to the parameterized exp-log expansion problem and linear programming. The main application of the automatic case separation strategy is to solve problems involving parameters. For instance, consider the asymptotical problem of expanding $e^{e^{\lambda x}}$: if $\lambda < 0$, then $e^{e^{\lambda x}} = 1 + e^{\lambda x} + e^{2\lambda x}/2 + \cdots$. If $\lambda > 0$, then $e^{e^{\lambda x}}$ forms its own expansions. The same situation is encountered when solving differential equations, due to the presence of initial conditions.

In section 8.2 we give an informal description of the strategy of **automatic case separation**. This technique is the analogue of constraint logical programming, although the resolution techniques used here are different from the classical ones.

In section 8.3, we give an expansion algorithm for parameterized exp-log functions, using automatic case separation. In the example mentioned above, the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ are distinguished automatically while computing the expansion of $e^{e^{\lambda x}}$. Next, the expansions in these different cases are found in the usual way, by interpreting the parameters as constants.

In section 8.4, we briefly review a variant of the simplex method in linear programming, based on automatic case separation. Next, we extend the usual constraint checkers for linear equalities and inequalities with the possibility to impose asymptotic linear constraints.

Let us finally notice that automatic case separation is historically referred to as **dynamic evaluation**, but we think that our nomenclature is more suggestive. The first appearance of this technique in computer algebra goes back to [DDD 85]. Until now, it has mainly be used for computations in parameterized algebraic number fields (see also [GoDi 94]). But actually, it underlies many algorithms in computer algebra, such as the Boulier-Seidenberg-Ritt algorithm (see [Seid 56], [Boul 94], [BLOP 95], [VdH 96c]), many asymptotic expansion algorithms, interval analysis, etc. In particular, we will make extensive use of it in the following chapters.

8.2 Automatic case separation

8.2.1 The strategy

Let us be given an arbitrary program P which takes p arguments $\lambda_1, \cdots, \lambda_p$ of types $T_1, \cdots, T_p$ on input and produces an output of type T'. We are interested in determining all possible outcomes of the program. More precisely, we see the arguments $\lambda_1, \cdots, \lambda_p$ as formal parameters and we want to determine the outcome of the algorithm in a symbolic way as a function of these parameters. Of course, different cases need to be distinguished in general: by a **generic output** of P relative to a subdivision

$$T_1 \times \cdots \times T_p = R_1 \amalg \cdots \amalg R_r \tag{8.1}$$

of $T_1 \times \cdots \times T_p$ into **regions** $R_1, \cdots, R_r$, we mean a list

$$P_1(\lambda_1, \cdots, \lambda_p), \cdots, P_r(\lambda_1, \cdots, \lambda_p)$$

of symbolic formulas, such that each $P_i(\lambda_1, \cdots, \lambda_p)$ is the output of the algorithm in the case $(\lambda_1, \cdots, \lambda_p) \in R_i$. We say that $P_i(\lambda_1, \cdots, \lambda_p)$ is the generic output of P relative to the region R_i.

Example 8.1. Let P be the program which takes on input an exp-log constant λ and returns the result of the expansion algorithm from section 7.2 applied on $e^{e^{\lambda x}}$. Then a generic output of P is

$$\begin{cases} \exp^{-1}(-\exp^{-1}(-\lambda(x^{-1})^{-1})), & \text{if } \lambda > 0; \\ e, & \text{if } \lambda = 0; \\ \exp(\exp(\lambda(x^{-1})^{-1})), & \text{if } \lambda < 0. \end{cases}$$

In this case, $P_1(\lambda), P_2(\lambda)$ and $P_3(\lambda)$ are exp-log expressions in x and λ, such that we obtain the result of P applied to a particular exp-log constant c by replacing λ by c in the appropriate P_i.

Let us make the notion of symbolic formula more precise. We assume that the program P is built up from a certain number of elementary functions $f_1, f_2, \cdots$ and relations $\rho_1, \rho_2, \cdots$ by the usual constructs of some imperative programming language. In the generic version P^{gen} of P, the types of all variables, arguments to subprograms, etc. which depend directly or indirectly on $\lambda_1, \cdots, \lambda_p$ are replaced by the symbolic type G of expressions in $\lambda_1, \cdots, \lambda_p, f_1, f_2, \cdots$. The only other difference between P^{gen} and P is that we have to respecify the elementary functions and relations.

Now f_i applied on arguments $e_1, \cdots, e_j$ of type G simply yields $f_i(e_1, \cdots, e_j)$. Each test which depends on the value of $\rho_i(e_1, \cdots, e_j)$ for some $e_1, \cdots, e_j$ of type G leads to the separate consideration of the cases $\rho_i(e_1, \cdots, e_j)$ and $\neg\rho(e_1, \cdots, e_j)$. It is convenient to adopt a **parallel computation model** for this purpose: both cases induce two distinct processes in which the relations $\rho_i(e_1, \cdots, e_j)$ and $\neg\rho_i(e_1, \cdots, e_j)$ are respectively imposed as constraints. This is done by introducing a global variable which contains the successive constraints we impose. Of course, we need to know whether there exist actual substitutions for the λ_i which verify a given set of constraints. A program which can check this is called a **constraint checker**. Assuming that we dispose of a constraint checker, all processes in which the set of the constraints admits no solutions are eliminated.

Example 8.2. In the case of expansions of parameterized exp-log expressions, the elementary functions are the exp-log field operations and for each constant c the zero-ary function yielding c. The elementary relations are $=$ and $<$. The constraint checker should be able to decide whether a given system of exp-log equalities and inequalities admits a solution.

The parallel computation process can be represented by a binary tree, which is called the **computation tree**: the outgoing edges of each internal node of this tree are labeled by $\rho_i(e_1, \cdots, e_j)$ and $\neg\rho_i(e_1, \cdots, e_j)$ for some $i, e_1, \cdots, e_j$. Each leaf l of the tree are labeled by the generic output of the algorithm relative to the region determined by the constraints on the path from the root to l. By Königs lemma on trees [Kön 50], the computation tree is finite if and only if their are no infinite branches. This means that the generic program P^{gen} terminates, if each of its parallel processes terminates. Usually, this is due to a Noetherian property for the regions which are determined by the successive constraints.

8.2.2 Examples

In this section we give a list of classical and new examples where the technique of automatic case separation can be applied, and briefly discuss these examples. Of course, our list is not exhaustive and merely included to give the reader an impression of the scope of the technique of automatic case separation.

Linear constraints. The problem of checking a finite system of linear equalities and inequalities in finite dimension for consistency is an important problem in mathematics, with many applications to economics: the theory of **linear programming** and **linear optimization** has known a big development since the fifties. We will discuss this theory in more detail in section 8.4, where we also extend the classical theory in order to deal with linear **asymptotic** constraints, which turn out to be important in chapter 11.

Algebraic constraints. The problem of determining the consistency of a finite system of polynomial equalities and inequations over an effective algebraically closed field in finite dimension can either be solved by Groebner basis techniques or the Ritt-Wu's algorithm (see [Wu 87]). In particular, the Ritt-Wu algorithm can be seen as an application of the automatic case separation strategy to the classical g.c.d. algorithm.

Although the complexities of algorithms in this area are always very bad, several interesting problems can nevertheless be treated. Notably, the technique of automatic case separation has nice applications in classical geometry, because different geometrical configurations can be distinguished automatically. We refer to [GoDi 94] for more details.

Real algebraic constraints. The problem of determining the consistency of a finite system of polynomial equalities and inequalities over an effective real algebraically closed field in finite dimension is an even more difficult problem. Nevertheless, algorithms exist for this (see for instance [Col 75]).

Real exp-log constraints. A crucial problem in the field of automatic asymptotics in higher dimensions is to determine the consistency of a finite system of real exp-log equalities and inequalities in a finite number of variables over $\mathbb{Q}$. We refer to the conclusion of this thesis for a discussion of this problem.

Arithmetic constraints. A well-known theorem of computability theory is that there exists no algorithm to determine the consistency of a finite system of polynomial equalities and inequalities in a finite number of variables over the integers (see [Mat 70]). This clearly marks a limit to the capability of constraint checkers. However, there are algorithms to determine the consistency of special types of Diophantine equations, such as linear equations over $\mathbb{Z}$.

Algebraic differential constraints. In [Seid 56] and [Boul 94] (see also [BLOP 95] and [VdH 96c]), it is shown that there exist constraint checkers for certain systems of partial algebraic differential equations and inequations. The Boulier-Seidenberg-Ritt algorithm can actually be seen as an application of the theory of case separation to Ritt's reduction theory. In [VdH 96c], we have generalized this algorithm to more general mixed differential-difference equations.

Constraints in free algebras. The elementary theory of **logical programming** can be interpreted as constraint checking in free algebras of a certain signature. Of course, many extensions exist in the literature and real logic programming languages such as PROLOG are far from theory. Currently, much research is done in the area of **constraint logical programming languages** and we hope that computer algebra will soon benefit from this research. See [Gal 87] for more information about this area.

8.2.3 Remarks

Many remarks can be made about the strategy of automatic case separation. In this section we list the principal ones.

Initial constraints. By default, the set of constraints is empty at initialization; taking another initial value, we can impose additional constraints on the parameters.

Error treatment. The strategy favorites a very flexible error treatment: we can localize regions where errors occur and eliminate the corresponding processes, or return error messages, etc.

Partial constraint checking. A very important feature of the strategy of automatic case separation is that we do not necessarily need a complete constraint checker: in cases where it is very expensive or impossible to check the consistency of the constraints, we can temporarily or permanently allow inconsistent sets of constraints. Indeed, even the virtual answer on a potentially empty region might interest the user. Moreover, inconsistent results can often be refuted by the user on the base of physical or other considerations. Generic outputs, for which the underlying partition does not contains empty regions are called **consistent**; non consistent generic outputs are called **virtual**.

Another advantage of partial constraint checkers is that it often increases the efficiency of the strategy. Indeed, whenever the constraint checker is written in a recursive style, constraint checking itself may lead to the separation of many cases. This is one of the reasons for which we will make extensive use of partial constraint checking in part B of this thesis. Moreover, we remark that the consistency of constraints can always be checked a posteriori, if we also have a complete constraint checker.

Let us finally notice that algorithms which are based on partial constraint checkers necessitate termination proofs which depend on the nature of these constraint checkers. Indeed, termination also has to be guaranteed on empty but not eliminated regions.

Parallel computation model. Let us comment the parallel computational model we use. To our knowledge, no computer algebra systems support parallel constructs yet. Nevertheless, parallelism can be simulated, by replacing all variables by lists in which each item corresponds to a set of constraints plus the corresponding value of the variable. Such lists are called **generic variables**. Another way to simulate parallelism is to rerun the program several times, by choosing each time another branch of the computation tree. This strategy has the advantage that no code has to be rewritten; the price to be paid is that the same computations are often performed several times.

The most elegant and efficient way to implement automatic case separation is to use the second strategy mentioned above, but by remembering the information which is common to all processes. However, this necessitates the extension of conventional programming languages such as C with parallel data types and control structures. At present such extensions are only in an experimental stage.

8.3 Expansions of parameterized exp-log functions

Let K be a totally ordered exp-log field and Λ a set of formal parameters and C a subset of K. An **exp-log expression** f over C in Λ is an expression built up from C, Λ by $+, -, \cdot, /, \exp$ and $\log$. The **domain** of f is the subset $\operatorname{dom} f$ of K^Λ, consisting of those substitutions $\varphi : \Lambda \to K$, such that $\varphi(f)$ is naturally defined. A **system of exp-log equalities and inequalities** is a pair $\Sigma = (\Sigma_e, \Sigma_i)$ of finite sets of exp-log expressions over C in Λ. The **domain** of Σ is defined by $\operatorname{dom} \Sigma = \bigcap_{f \in \Sigma_e \cup \Sigma_i} \operatorname{dom} f$. We say that a substitution $\Lambda \to K$ in $\operatorname{dom} \Sigma$ is a **solution** to Σ, if $\varphi(f) = 0$, for each $f \in \Sigma_e$, and $\varphi(f) > 0$, for each $f \in \Sigma_i$.

Assume now that $\mathfrak{C}$ is an effective ordered field of constants and a subset of K. Let $\hat{\mathfrak{C}}$ be the smallest subset of K, such that any system of exp-log equalities over $\mathfrak{C}$ in some finite Λ which admits a solution in K^Λ admits a solution in $\hat{\mathfrak{C}}^\Lambda$. We call $\hat{\mathfrak{C}}$ the **exp-log closure** of $\mathfrak{C}$ in K and we observe that $\hat{\mathfrak{C}}$ is an exp-log field. If $\mathfrak{C} = \hat{\mathfrak{C}}$, then we say that $\mathfrak{C}$ is **exp-log closed**.

8.3.1 The algorithm

In this section we present two generic expansion algorithms for exp-log functions over $\mathfrak{C}$ depending on a finite number of parameters in K. In practice, we usually have $K = \mathbb{R}$, and $\mathfrak{C} = \mathbb{Q}$ or $\mathfrak{C} = \hat{\mathbb{Q}}$. For the first algorithm, no additional hypothesis need to be made, but the constraint checker being only a partial one, the algorithm may yield virtual generic expansions. For the second algorithm, we make the hypothesis that $\mathfrak{C}$ is an **effective totally ordered exp-log closed constant field**; i.e. we have an algorithm or oracle which can check the consistency of any given system of exp-log equalities and inequalities over $\mathfrak{C}$ in any finite set of parameters Λ. In this case, the computed generic expansions are always consistent.

Theorem 8.1. **(Parameterized expansion theorem, weak form)** *Let $\Lambda = \{\lambda_1, \cdots, \lambda_p\}$ be a finite set of parameters. There exists an algorithm which takes an exp-log expression in $x, \lambda_1, \cdots, \lambda_p$ over $\mathfrak{C}$ on input and computes*

(a) A partition $\mathfrak{C}^\Lambda = R_1 \amalg \cdots \amalg R_r$ of $\mathfrak{C}^\Lambda$, which we denote by P;
(b) A generic effective normal basis B relative to P;
(c) An algorithm which computes the generic asymptotic expansion of f w.r.t. B relative to P at any order.

Each possibly empty region R_i is represented as the solution set to a system Σ_i of exp-log equalities and inequalities.

Proof. We apply the strategy of automatic case separation to the expansion algorithm `expand` from chapter 7, incorporating the optimizations from sections 7.3 and 7.4. We first remark that modulo the introduction of a finite number of new parameters, a finite number of elements in the exp-log field $\mathfrak{C}$ may be introduced in the algorithm. For instance, if $\mathfrak{C} = \mathbb{Q}$ and $K = \mathbb{R}$, then e can be represented by λ_{p+1}, where λ_{p+1} satisfies the constraint $\lambda - e^1 = 0$. Representing elements in $\mathfrak{C}$ in this way, all computations on constants are done in the ring $\mathfrak{C}[\lambda_1, \cdots, \lambda_q]$, where $q \geqslant p$ may increase during the algorithm. In particular, the constraints we impose on constants are only checked for their *algebraic* consistency, for instance by using cylindrical decompositions (see [Col 75]).

Let us verify that all parallel processes terminate. The only loops in the expansion algorithm come from the recursive application of `expand` in the exponential case and the main loops in `idm` and `zero_test`. For a similar reason as in the previous chapter, no infinite loops can arise from `expand`.

Assume that the main loop in `idm` or `zero_test` does not terminate on a given input. Now u is a Laurent series with coefficients in $\mathfrak{C}[\lambda_1, \cdots, \lambda_q]$ in these algorithms, since q can not increase during such a loop. Let $c_1, c_2, \cdots$ denote the successive values of $\Sigma_{\mathfrak{n} \in M} u_{\mathfrak{n}}$ during the loop. Since $\mathfrak{C}[\lambda_1, \cdots, \lambda_q]$ is Noetherian, the chain of ideals $(c_1), (c_1, c_2), \cdots$ is stationary. In particular, $c_i = 0$ can be deduced from the constraints, for sufficiently large i, and the usual termination argument is used to obtain a contradiction. □

Remark 8.1. We notice that the zero test from section 7.4 can indeed be applied: in section D.4.1 (see remark D.5), we have given a zero test for parameterized Laurent series. We remark that some new but finite branching may occur during the execution of such a generic zero test.

Theorem 8.2. **(Parameterized expansion theorem, strong form)** *Let $\Lambda = \{\lambda_1, \cdots, \lambda_p\}$ be a finite set of parameters. Assume that we have an oracle which decides whether a given system of exp-log equalities and inequalities over $\mathfrak{C}$ in any finite set of parameters admits a solution. Then there exists an algorithm which takes an exp-log expression in $x, \lambda_1, \cdots, \lambda_p$ over $\mathfrak{C}$ on input and computes*

(a) A partition $\mathfrak{C}^\Lambda = R_1 \amalg \cdots \amalg R_r$ of $\mathfrak{C}^\Lambda$, which we denote by P;
(b) A generic effective normal basis B relative to P;
(c) An algorithm which computes the generic asymptotic expansion of f w.r.t. B relative to P at any order.

Each non empty region R_i is represented as the solution set to a system Σ_i of exp-log equalities and inequalities.

Proof. The present theorem is a trivial corollary of the previous one, since the oracle can be used to check the regions for non emptiness. □

8.3.2 An example

Let us consider the expansion of the exp-log function

$$f(x) = e^{1/x + e^{\lambda x}} - e^{1/x},$$

depending on one formal parameter λ. The expansions of $1/x$ and λx are straightforward. For the expansion of $e^{\lambda x}$, one needs to compute the sign of λx and thus of λ. This leads to a branching into three processes, corresponding to the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. The first case leads to the expansion

$$f = \text{б}_2 + \text{б}_1\text{б}_2 + \frac{1}{2}\text{б}_1^2\text{б}_2 + \cdots + \frac{1}{2}\text{б}_2^2 + \frac{1}{2}\text{б}_1\text{б}_2^2 + \cdots,$$

with effective normal basis $B = \{\text{б}_1 = x^{-1}, \text{б}_2 = e^{\lambda x}\}$. The second case leads to the expansion

$$f = (e-1) + (e-1)\text{б}_1 + \frac{e-1}{2}\text{б}_1^2 + \cdots,$$

where $B = \{\text{б}_1 = x^{-1}\}$. Finally, the case $\lambda > 0$ leads to the expansion

$$f = \text{б}_3 + \text{б}_1\text{б}_3 + \frac{1}{2}\text{б}_1^2\text{б}_3 + \cdots - 1 - \text{б}_1 - \cdots,$$

with $B = \{\text{б}_1 = x^{-1}, \text{б}_2 = e^{-\lambda x}, \text{б}_3 = e^{-e^{\lambda x}}\}$.

8.4 Linear constraint checkers

In this section we give algorithms for determining the consistency for systems of linear inequalities, equations, and asymptotic inequalities. In terms of asymptotics, this means that we can check the consistency of systems of relations involving $\prec$, $\preccurlyeq$, $\asymp$, $\prec\!\!\prec$, $\preccurlyeq\!\!\preccurlyeq$ and $\asymp\!\!\asymp$ in certain monomial groups.

The topic of finding the maxima of linear functionals on convex sets determined by linear inequalities is known as **linear programming**. In particular, the consistency of systems of linear inequalities can be determined in this way. The most frequently used method in linear programming is the simplex method. Roughly speaking, this method proceeds by following a path on the edges of the simplicial solution set of a system of linear inequalities. However, the simplex method has an exponential worst case complexity. This has lead to the introduction of the ellipsoid method in [Kat 79], which is a polynomial time algorithm. Unfortunately, the complexity involves a large constant factor, which makes the Katchian's algorithm little useful in practice. More recently, this drawback has been removed by Karmakar (see [Kar 84]), whose algorithm is currently the fastest.

In sections 8.4.1 and 8.4.2 we respectively study systems of strict linear inequalities and mixed systems of linear inequalities and equalities. The consistency of such systems is determined by computing the simplicial set of solutions. We give a geometrical approach, which is basically equivalent to the simplex method. However, the fact that we actually compute the simplicial solutions sets implies that we can quickly check whether a given linear equation or inequality is or can be verified on such sets. This is useful, if these tests are performed with a rate which is very superior to rate of imposition of new constraints.

In sections 8.4.3 and 8.4.4 we reduce the problems of determining the consistency of certain systems of asymptotic linear constraints to the problem of determining the consistency of systems of linear inequalities and equalities. Hence, the algorithms from sections 8.4.1 and 8.4.2 can be applied, as well as the ellipsoid method, and Karmakar's optimization.

8.4.1 Linear inequalities

Let X be a totally ordered non zero vector space over an effective totally ordered field $\mathfrak{K}$. Let $z_1, \cdots, z_k$ be a finite number of positive parameters in X. In this section, we give an algorithm to check whether a finite system of constraints of the form

$$a_1 z_1 + \cdots a_k z_k > 0 \tag{8.2}$$

is consistent (where $a_1, \cdots, a_k \in \mathfrak{K}$). In particular, if X is a monomial group with $\mathfrak{K}$-powers, this yields a method to check whether a finite system of constraints of the form

$$z_1^{a_1} \cdots z_k^{a_k} \prec\!\!\prec 1 \tag{8.3}$$

is consistent (where $z_1, \cdots, z_k \prec\!\!\prec 1$). Inequalities of the form (8.3) are also called **expo-linear inequalities**.

Since the z_i are assumed to be positive, and since constraints of the form (8.2) are homogeneous, we may assume without loss of generality that $X = \mathfrak{K}$ and impose the additional constraint

$$z_1 + \cdots + z_k = 1. \tag{8.4}$$

The set of solutions to (8.4) and a finite number of linear inequalities like (8.2) is a bounded convex subset of $\mathfrak{K}^k$. Our algorithm is based on the representation of such convex sets C by their sets of vertices V_C, edges E_C and hypersurfaces H_C. For each edge $e \in E_C$, we store its two endpoints $e_{1,C}, e_{2,C} \in V_C$. For each vertex $v \in V_C$, we store its coordinates and the set $H_{v,C}$ of hypersurfaces to which it belongs. Each hypersurface is characterized by a linear constraint of the form

$$a_1 z_1 + \cdots + a_k z_k = 0.$$

By convention, we represent the empty set by $V_\phi = E_\phi = H_\phi = \phi$.

In absence of constraints, C is the simplex with k vertices $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ which are pairwise connected by edges. C has k hypersurfaces, which are characterized by the equations $z_1 = 0, \cdots, z_k = 0$. Each point $(0, \overset{i-1 \text{ times}}{\cdots}, 0, 1, 0, \cdots, 0)$ belongs to all hypersurfaces, except the one which is determined by $z_i = 0$.

Assume now that we have the representation of the convex solution set C of a given system of constraints. Imposing the additional constraint (8.2), we wish to compute the corresponding convex solution set C'. To do this, we first label each vertex $(v_1, \cdots, v_k) \in V_C$ by the sign of $a_1 v_1 + \cdots + a_k v_k$. Let us denote by V^+, V^0 and V^- the sets of vertices which are labeled by $+, 0$, resp. $-$. If $V^+ = \phi$, then we have $C' = \phi$. If $V^- = \phi$, then we have $C' = C$.

In the remaining case, we determine all edges whose endpoints are labeled by opposite signs and we compute the intersections of these edges with the hypersurface H^{new} determined by $a_1 z_1 + \cdots + a_k z_k = 0$. Let I be the set of these intersections. We have $V_{C'} = I \amalg V^0 \amalg V^+$. $H_{C'}$ is given by H^{new} and those hypersurfaces H in H_C, with $H \in H_{v,C}$ for some $v \in V^+$. We have $H_{v,C'} = H_{v,C}$ for $v \in V^+$. We have $H_{v,C'} = \{H\} \cup (H_{e_{1,C},C} \cap H_{e_{2,C},C})$ for each $v \in I$ which lies on the edge $e \in E_C$. Finally, we have $H_{v,C'} = (\{H\} \cup H_{v,C}) \cap H_{C'}$. All edges in E_C between vertices with non negative labels are conserved in $E_{C'}$. Each edge $e \in E_C$, on which lies a vertex v in I, is replaced by the edge from v to the endpoint of e with a positive label. Finally, two vertices v and w in $V^0 \amalg I$ are connected by an edge in $E_{C'}$ if and only if the intersection of all hyperplanes in $H_{v,C'} \cap H_{v,C}$ forms a line (linear algebra methods are used to check this). This completes the algorithm and we have proved:

Theorem 8.3. *Let $\mathfrak{K}$ be an effective totally ordered constant field and X a totally ordered vector space over $\mathfrak{K}$. Then there exists a constraint checker for systems of constraints of the form (8.2).* □

Remark 8.2. Given a convex set C represented in the above way, the maximum of $a \cdot v = a_1 v_1 + \cdots + a_k v_k$, for $v \in V_C$ and given $a_1, \cdots, a_k \in \mathfrak{K}$ can be determined as follows: we first randomly choose a vertex $v \in V_C$. Next, we follow those edges which increase $a \cdot v$, until we have reached the maximum. This algorithm can be speeded up by choosing v among an appropriate number of random vertices in such a way that $a \cdot v$ is maximal.

Remark 8.3. Let C_1 and C_2 be arbitrary bounded convex sets, which are represented in the above way. Then the method can be generalized in order to compute the intersection of C_1 and C_2; in this case, V^+, V^0 and V^- are the sets of vertices of V_{C_1} which are inside, on the boundary of, resp. outside C_2.

8.4.2 Linear equalities

In this section, we consider systems of constraints of the type (8.2) or

$$a_1 z_1 + \cdots + a_k z_k = 0. \tag{8.5}$$

The latter constraints take the form

$$z_1^{a_1} \cdots z_k^{a_k} \asymp 1 \tag{8.6}$$

if X is a totally ordered monomial group with $\mathfrak{K}$-powers. Equalities of the form (8.6) are also called **expo-linear equalities**.

The solution set of (8.4) and a system of constraints of the form (8.2) or (8.5) is an open bounded convex subset C of some subvector space of $\mathfrak{K}^k$. This subvector space is said to be the underlying vector space of C. We represent C as before, except that we work in the underlying vector space W_C of C instead of the hyperplane $z_1 + \cdots + z_k = 0$. This means that the hypersurfaces of C are represented as intersections of W_C with hyperplanes $H \not\supseteq W$ determined by equations like (8.5).

Assume now that we are given such a convex set C with underlying vector space W_C. The algorithm from the previous section to compute the intersection of C with the half space determined by (8.2), remains valid modulo a minor change: we first have to check by linear algebra that W_C is not contained in H^{new}, in which case C' is empty.

If we want to compute the intersection C' of C with the hyperplane H^{new} determined by (8.5), we proceed in a similar way: if H^{new} contains W_C, then $C' = C$. If V^+ or V^- then so is C', with similar notations as before. In the remaining case, we compute the intersection $W_{C'}$ of W_C and H^{new} by linear algebra. Finally, $V_{C'}$, $E_{C'}$ and $H_{C'}$ are computed as in the previous section, but we only conserve those vertices and edges which lay in H^{new} and those hypersurfaces on which lies at least one vertex in V^+ and one vertex in V^-. Summarizing, we have

Theorem 8.4. *Let $\mathfrak{K}$ be an effective totally ordered constant field and X a totally ordered vector space over $\mathfrak{K}$. Then there exists a constraint checker for systems of constraints of the form (8.2) or (8.5).* □

An immediate corollary of this theorem is that there exists a constraint checker for systems of constraints of the form (8.2), (8.5) or

$$a_1 z_1 + \cdots + a_k z_k \geqslant 0. \tag{8.7}$$

Indeed, it suffices to split up each constraint (8.7) in the constraints (8.2) and (8.5), by using the automatic case separation strategy.

Alternatively, one can describe the solution sets C to such systems of equations in the above way, with this difference that we associate a flag to each hypersurface in H_C which indicates whether $C \cap H_C$ is empty or not. It is not hard to modify our algorithms in order to compute with such representations of solution sets.

Let us finally notice that if X is an totally ordered monomial group, then the constraint (8.7) takes the form

$$z_1^{a_1} \cdots z_k^{a_k} \preccurlyeq 1.$$

8.4.3 Asymptotic linear inequalities

The totally ordered $\mathfrak{K}$-vector space X is said to be **complete**, if for each $x, y \in K$ with $x \asymp y$ there exists a $\lambda \in \mathfrak{K}^*$ with $x \sim \lambda y$. In this section, we assume that X is a complete infinite dimensional vector space. Under this assumption, we will show how the constraint checker from the previous section can be extended in order to handle constraints of the form

$$|a_1 z_1 + \cdots + a_k z_k| < \mathfrak{K}_*^+ |b_1 z_1 + \cdots + b_k z_k|. \tag{8.8}$$

If X is a totally ordered monomial group with $\mathfrak{K}$-powers, then such constraints take the form

$$z_1^{a_1} \cdots z_k^{a_k} \lll z_1^{b_1} \cdots z_k^{b_k}.$$

We notice that in order to impose a constraint

$$|a_1 z_1 + \cdots + a_k z_k| < \mathfrak{K}_*^+ |b_1 z_1 + \cdots + b_k z_k|,$$

it suffices to consider the case in which

$$\begin{cases} a_1 z_1 + \cdots + a_k z_k > 0; \\ b_1 z_1 + \cdots + b_k z_k > 0, \end{cases} \tag{8.9}$$

modulo the imposition of some additional linear constraints like (8.2) or (8.5) and some case separation.

Theorem 8.5. *Let $\mathfrak{K}$ be an effective totally ordered field and X a complete infinite dimensional totally ordered $\mathfrak{K}$-vector space. Then there exists a constraint checker for systems of constraints of the form (8.2), (8.5), (8.7) or (8.8).*

Proof. The idea of the proof is to work in the effective totally ordered $\mathfrak{K}$-algebra $\mathfrak{K}(\varepsilon)$ instead of $\mathfrak{K}$, where ε is a positive infinitesimal. Then we replace each constraint of the form

$$a_1 z_1 + \cdots + a_k z_k < \mathfrak{K}_*^+ (b_1 z_1 + \cdots + b_k z_k) \tag{8.10}$$

under the assumption (8.9) by

$$a_1 \zeta_1 + \cdots + a_k \zeta_k < \varepsilon (b_1 \zeta_1 + \cdots + b_k \zeta_k). \tag{8.11}$$

We have to show that a system Σ of constraints of the form (8.2), (8.5) or (8.10) admits a solution in X^k if and only if the corresponding system Σ' of constraints of the form (8.2), (8.5) admits a solution in $\mathfrak{K}(\varepsilon)^k$. This will prove the theorem in view of theorem 8.4.

Lemma 8.1. *Each finite dimensional $\mathfrak{K}$-subvector space Y of X admits a basis of positive pairwise comparable elements for $\lll$.*

Proof. We use induction over the dimension of Y. If $\dim Y = 0$ then there is nothing to prove. Assume that $\dim Y > 0$ and let Y' be a hyperplane in Y. By the induction hypothesis, we can find a basis of Y' of positive elements $\text{б}_1 \prec\!\!\prec \cdots \prec\!\!\prec \text{б}_n$. Let x be an arbitrary positive element in $Y \backslash Y'$. As long as x is asymptotic to one of the б_i, we replace it by $|x - \lambda \text{б}_i|$, where $x \sim \lambda \text{б}_i$ (the existence of such a $\lambda \in \mathfrak{K}^*$ is guaranteed by the completeness of X). This process yields a positive element x which is not asymptotic to any of the б_i after at most n steps, and $\{\text{б}_1, \cdots, \text{б}_n, x\}$ yields the desired basis for Y. □

Remark 8.4. We notice that the lemma is a weaker analogue of the structure theorem for transseries from page 53.

End of the proof of the theorem. Let $(z_1, \cdots, z_k)$ be a solution to Σ. By the lemma, there exists a basis $\text{б}_1 \prec\!\!\prec \cdots \prec\!\!\prec \text{б}_n$ of positive elements for the vector space Y spanned by $z_1, \cdots, z_k$. Let E be the vector space spanned by $1, \cdots, \varepsilon^{2(n-1)}$ and consider the linear transformation which sends each б_i to $\varepsilon^{2(n-i)}$. This transformation actually determines an isomorphism φ of ordered $\mathfrak{K}$-vector spaces. Moreover, we have $x \prec\!\!\prec y \Leftrightarrow x < \varepsilon y$ for positive elements x, y in E. Hence $(\varphi(z_1), \cdots, \varphi(z_k))$ is a solution to Σ'.

Inversely, let $(\zeta_1, \cdots, \zeta_k)$ be a solution to Σ'. By the lemma, there exists a basis $\text{б}_1 \prec\!\!\prec \cdots \prec\!\!\prec \text{б}_n$ of positive elements for the vector space V spanned by $\zeta_1, \cdots, \zeta_n$. Since the ζ_i are Laurent series in ε, we may assume without loss of generality that $\text{б}_n \asymp 1$, by multiplying the ζ_i by a suitable power of ε. Let p be such that $\text{б}_1 \asymp \varepsilon^p$. Let E be the vector space spanned by $1, \varepsilon, \cdots, \varepsilon^p$.

The truncation of power series at order p determines mapping φ from V into E. We claim that this mapping is an embedding of ordered $\mathfrak{K}$-vector spaces. It suffices to check that strictly positive elements in V are mapped to strictly positive elements in E. Moreover, given $x > 0$ in V, we can write $x \sim \lambda \text{б}_i$ for some $\lambda > 0$ and i. Now $\text{б}_i \sim \mu \varepsilon^j$ for some $\mu > 0$ and $j \leqslant p$. Therefore, $x \sim \lambda \mu \varepsilon^j$ and $\varphi(x) > 0$.

By lemma 8.1 and the fact that X has infinite dimension, there exist elements $\varepsilon_0 \succ\!\!\succ \cdots \succ\!\!\succ \varepsilon_p$ in X. Now there exists a natural isomorphism ψ between the ordered $\mathfrak{K}$-vector spaces E and $\mathfrak{K}\varepsilon_0 \oplus \cdots \oplus \mathfrak{K}\varepsilon_p$, which sends each ε^i to ε_i. Then $(\psi(\varphi(\zeta_1)), \cdots, \psi(\varphi(\zeta_k)))$ is a solution to Σ'. □

8.4.4 Asymptotic linear equalities

In this section, we still assume that X is a complete infinite dimensional vector space. We will consider asymptotic linear constraints of the form

$$\left\{ \begin{array}{ll} \exists M \in \mathfrak{K}_*^+ & |a_1 z_1 + \cdots + a_k z_k| \leqslant M |b_1 z_1 + \cdots + b_k z_k| \wedge \\ \exists N \in \mathfrak{K}_*^+ & |b_1 z_1 + \cdots + b_k z_k| \leqslant N |a_1 z_1 + \cdots + a_k z_k|. \end{array} \right. \tag{8.12}$$

resp.

$$\exists M \in \mathfrak{K}_*^+ \quad |a_1 z_1 + \cdots + a_k z_k| \leqslant M |b_1 z_1 + \cdots + b_k z_k|. \tag{8.13}$$

If X is a totally ordered monomial group with $\mathfrak{K}$-powers, then such constraints take the form

$$z_1^{a_1} \cdots z_k^{a_k} \asymp z_1^{b_1} \cdots z_k^{b_k}$$

resp.

$$z_1^{a_1} \cdots z_k^{a_k} \underline{\prec\!\!\prec} z_1^{b_1} \cdots z_k^{b_k}.$$

Clearly, it suffices to consider constraints of the form (8.13) and without loss of generality we may assume (8.9) modulo the imposition of some additional linear constraints and some case separation.

Theorem 8.6. *Let $\mathfrak{K}$ be an effective totally ordered field and X a complete infinite dimensional totally ordered $\mathfrak{K}$-vector space. Then there exists a constraint checker for systems of constraints of the form (8.2), (8.5), (8.7), (8.8), (8.12) or (8.13).*

Proof. Let $\mathfrak{K}(\omega)$ be a totally ordered field, where ω is positive and infinitely large over $\mathfrak{K}$. Let X' be any complete infinite dimensional totally ordered vector space over $\mathfrak{K}(\omega)$. Given a system Σ of constraints of the form (8.2), (8.5), (8.8) or (8.13), let Σ' be the system of constraints of the form (8.2), (8.5) or (8.8), which is obtained by replacing constraints of the form

$$\exists M \in \mathfrak{K}_*^+ \quad a_1 z_1 + \cdots + a_k z_k < M(b_1 z_1 + \cdots + b_k z_k), \tag{8.14}$$

under the assumption (8.9) by

$$a_1 \zeta_1 + \cdots + a_k \zeta_k < \omega(b_1 \zeta_1 + \cdots + b_k \zeta_k). \tag{8.15}$$

Let us show that Σ admits a solution in X^k if and only if Σ' admits a solution in ${X'}^k$. This will prove the theorem in view of theorem 8.5 and the introductory remarks.

Let $(z_1, \cdots, z_k)$ be a solution to Σ. By lemma 8.1, we construct a basis $\mathfrak{b}_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}_n$ for the $\mathfrak{K}$-vector space V generated by $z_1, \cdots, z_k$. Let V' be a subvector space of X' of dimension n. Again by lemma 8.1, we construct a basis $\mathfrak{b}'_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}'_n$ for V'. Now let φ be the $\mathfrak{K}$-linear mapping from V to V' which sends each $\mathfrak{b}_i$ to $\mathfrak{b}'_i$. Then $(\varphi(z_1), \cdots, \varphi(z_k))$ is a solution to Σ'.

Inversely, let $(\zeta_1, \cdots, \zeta_k)$ be a solution to Σ' and construct a basis $\mathfrak{b}'_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}'_n$ for the $\mathfrak{K}$-vector space V' generated by $\zeta_1, \cdots, \zeta_k$. Let V be a subvector space of X of dimension n, with basis $\mathfrak{b}_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}_n$. Let φ_M be the partial mapping which maps elements $f_1 \mathfrak{b}'_1 + \cdots + f_k \mathfrak{b}'_k$ in V' to elements $f_1(M)\mathfrak{b}_1 + \cdots + f_k(M)\mathfrak{b}_k$ in V. We claim that for sufficiently large M, $(\varphi_M(\zeta_1), \cdots, \varphi_M(\zeta_k))$ is well defined and a solution to Σ.

Consider a constraint of the form (8.15) in Σ'. We can write $a_1\zeta_1 + \cdots + a_k\zeta_k = f_1\mathfrak{b}'_1 + \cdots + f_p\mathfrak{b}'_p$ for certain $f_1, \cdots, f_p \in \mathfrak{K}(\lambda)$ and $f_p > 0$. Similarly, we can write $b_1\zeta_1 + \cdots + b_q\zeta_q = g_1\mathfrak{b}'_1 + \cdots + g_q\mathfrak{b}'_q$, with $g_q > 0$ and $q \geqslant p$. Since $f_q < (\omega + 1)g_q$ (with $f_q = 0$ if $q > p$), we have $f_q(M) < (M + 1)g_q(M)$ and $0 < g_q(M)$ for all sufficiently large M. Hence, taking $z_i = \varphi_M(\zeta_i)$ for each i, the constraint (8.14) of Σ is satisfied for sufficiently large M. A similar argument applies to constraints of the form (8.2), (8.5) resp. (8.11). This proves our claim, because Σ' contains only finitely many constraints. $\square$

8.5 References

[BLOP 95] F. Boulier, D. Lazard, F. Ollivier, M. Petitot. Représentation du radical d'un idéal différentiel de type fini. *Proc. ISSAC 1995, Montreal.*

[Boul 94] F. Boulier. Étude et implantation de quelques algorithmes en algèbre différentielle. *PhD. Thesis, University of Lille I, France.*

[Col 75] G.E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Proc. 2-nd conf. on automata theory and formal languages, Springer Lect. Notes in Comp. Sc. 33 (p. 134-183).*

[DDD 85] J. Della Dora, C. Dicrescenzo, D. Duval. A new method for computing in algebraic number fields. *Eurocal'85 (2), Springer Lect. Notes in Comp. Sc. 358, ed. G. Goos, J. Hartmanis (p. 321-326).*

[DuR 94] D. Duval, J-.C. Reynaud. Sketches and computation I: basic definitions and static evaluation, II: dynamic evaluation and applications. *Math. Struct. in Comp. Sc. 4 (p. 185-238, 239-271).*

[Gal 87] H. Gallaire. Boosting logic programming. *In Proc. of the fourth Intern. Conf. on Logic Programming, Melbourne, Australia (ed. J.-L. Lassez), MIT. Press, Cambridge (p. 928-962).*

[GoDi 94] T. Gomez-Diaz. Quelques applications de l'évaluation dynamique. *PhD. Thesis, Univ. of Limoges, France.*

[Kar 84] N. Karmakar. A new polynomial-time algoritm for linear programming. *Combinatorica 4(4) (p. 373-395).*

[Kat 79] L.G. Katchian. A polynomial algoritm for linear programming. *Doklady Akad. Nauk USSR 244(5) (p. 1093-1096). Soviet Math. Doklady 20 (p. 191-194).*

[Kön 50] D. König. *Theorie der endlichen und unendlichen Graphen.* Chelsea publ. Comp. New York.

[Matij 70] Yu. Matijasevic. Enumerable sets are Diophantine. *Dokl. Akad. Nauk SSSR 191 (p. 279-282). Soviet Math. Dokl. 11 (p. 354-357).*

[Pél 95] A. Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948.*

[Rich 92] D. Richardson. The elementary constant problem. *Proc. ISSAC 92 (p. 108-116).*

[Rich 94] D. Richardson. How to recognize zero. *Preprint, University of Bath, England.*

[Ris 75] R.H. Risch. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math. 4, 101 (p. 743-759).*

[Seid 56] A. Seidenberg. An elimination theorem for differential algebra. *Univ. California Publ. Math. (N.S.) (p 31-38).*

[VdH 96a] J. VAN DER HOEVEN. Generic asymptotic expansions. *Local Proc. of the Rhine workshop on computer algebra, Saint-Louis, April 1-3, France, Submitted to AAECC..*

[VdH 96c] J. VAN DER HOEVEN. Differential and mixed differential difference equations from the effective viewpoint. *Research report LIX/RR/96/11, Ecole polytechnique, France.*

[Wu 87] W.T. WU. A Zero structure theorem for polynomial equation solving and its applications. *Proceedings of ISSAC'88, Roma, Springer Verlag.*

Chapter 9

Basics for automatic asymptotics

9.1 Introduction

In chapter 7 we have given an expansion algorithm for germs of exp-log functions at infinity. Actually, we have limited ourselves to such germs for pedagogical reasons, and we have opted to represent them by exp-log expressions. However, as we will show in this chapter, most of the methods apply in a far more general context. Our approach is based on the theory of grid-based transseries from chapter 1.

In section 9.2, we introduce some very general terminology. Section 9.2.1 concerns the concept of "effective algebraic structures", which provides a useful approach to computer algebra. In sections 9.2.2 and 9.2.3, we give general definitions of automatic power series and automatic Laurent series. In section 9.2.4 we introduce the concept of automatic transseries and prove an effective version of the structure theorem from chapter 1. Let us also notice that the above concepts will be refined further in chapter 12.

In section 9.3, we generalize the concept of Cartesian representations and some of the corresponding algorithms from chapter 7.

In section 9.5.1, we generalize the asymptotic expansion algorithm from chapter 7 to a far more general class of functions, using the concepts introduced in section 9.2, and the algorithms from section 9.3. We also give an alternative canonical expansion algorithm.

In section 9.6.3, we introduce the concept of local communities: a theoretical and effective tool for computations with implicitly defined series. An application is given in section 9.6.3, where we show how certain asymptotic implicit equations can be solved automatically.

Finally, section 9.7 deals with Newton polytopes — a classical concept, which will be needed in chapters 10 and 11.

9.2 A formalism for effective asymptotic algebra

9.2.1 Effective algebraic structures

An **effective algebraic structure** is an algebraic structure, represented by some data type, together with algorithms for performing a certain number of operations and/or constructions effectively. The data type is not necessarily required to represent the elements of the algebraic structure in a one-to-one manner, although we do require the algorithms to be compatible for this representation. For instance, elements of $\mathfrak{T}$ are non uniquely represented by exp-log expressions in $\mathfrak{T}^{expr}$ in chapter 7. The exp-log field operations $0, 1, +, -, \cdot, /, \exp, \log$ can clearly be performed by algorithm and we do have compatibility: for example, the exponentials of two expressions which represent the same germ in $\mathfrak{T}$ also represent the same germ. Moreover, assuming Schanuel's conjecture, we have shown how to test whether a given exp-log expression in $\mathfrak{T}^{expr}$ represents the zero germ at infinity; we say that $\mathfrak{T}$ is an **effective exp-log field**.

Remark 9.1. In general, the exact meaning of computable can be made easily be made precise, for example in the language of Turing machines. Sometimes, a more general notion of computability is needed. For instance, in section 7.3 we have shown that $\mathfrak{T}$ is an effective exp-log field, if $\mathfrak{C}$ is an effective field of constants. This relative notion of computability is usually formalized using oracles (which we actually used already).

In particular, for theoretical purposes it is sometimes useful to assume that all algebraic structures are effective. Algorithms for these structures are then called **theoretical algorithms**. Although they can not be executed in practice, their correctness proofs may yield interesting theoretical results. For the same reason, definitions made in an effective context do also make sense in a theoretical context.

We notice that for many mathematical structures the definitions of their effective counterparts are clear. Most importantly, for a given signature Σ (see section B.2), we define an **effective Σ-structure** to be a Σ-structure, with algorithms for all functions and relations in Σ and for the equality test. In particular, this definition covers the notions of effective groups, effective rings, effective exp-log fields, and so on. More generally, any axiom imposed on the algebraic structure should be satisfied in a suitable effective way. For instance, an effective algebraically closed field is an effective field $\mathfrak{K}$, which is algebraically closed, and such that we have an algorithm which computes the solution set to any polynomial equation over $\mathfrak{K}$.

Example 9.1. Adopting the effective algebraic spirit of thinking we get theorems like: if $\mathfrak{K}$ is an effective field, then we can compute its effective algebraic closure $\hat{\mathfrak{K}}$. Indeed, this is a classical exercise which we leave to the reader. We notice the precise statement of this theorem: we do not only announce that the algebraic closure is effective, but there exists an algorithm which computes $\hat{\mathfrak{K}}$ as a function of $\mathfrak{K}$.

9.2.2 Automatic power series

Special care needs to be paid to effective algebraic structures whose elements are of an infinite nature; since such elements can not be stored as a whole into a machine, they are represented by programs which extract suitable finite information about them. For instance, an infinite power series can not be stored completely into a machine, but we may have algorithms for extracting the first n elements of it for any n. Actually, we use a double representation for series: we both compute with series as elements of an abstract effective series algebra and with their truncations up till a finite order. This strategy is classically called **lazy evaluation**.

Univariate power series. Assume that we are given an effective constant field $\mathfrak{C}$ and an effective $\mathfrak{C}$-algebra $\mathfrak{R}$ of power series over $\mathfrak{C}$ in z, which contains z. A subset $\mathfrak{A}$ of $\mathfrak{R}$ is said to be **automatic**, if there exists an algorithm which takes $u \in \mathfrak{A}$ and $\alpha \in \mathbb{N}$ on input, and which computes $[z^\alpha]u$. A power series u in $\mathfrak{R}$ is said to be **automatic**, if u is an element of an automatic subset $\mathfrak{A}$ of $\mathfrak{R}$. Consequently, automatic power series can automatically be expanded up till any order and in particular, we can compute their valuations.

Multivariate power series. More generally, assume that we have an effective $\mathfrak{C}$-algebra $\mathfrak{R}$ of power series in $z_1, \cdots, z_k$, which contains $z_1, \cdots, z_k$. A subset $\mathfrak{A}$ of $\mathfrak{R}$ is said to be **automatic**, if there exists an algorithm which takes $u \in \mathfrak{A}$, $1 \leqslant i \leqslant k$ and $\alpha_i \in \mathbb{Z}$ on input and which computes $[z_i^{\alpha_i}]u$; moreover, we require $[z_i^{\alpha_i}]u$ to be in $\mathfrak{A}$. If $\mathfrak{A}$ is an automatic subset of $\mathfrak{R}$, then so is the subalgebra of $\mathfrak{R}$ which is effectively generated by $\mathfrak{A}$. A power series u in $\mathfrak{R}$ is said to be **automatic**, if u is an element of an automatic subset of $\mathfrak{R}$. Such series can automatically be expanded in each z_i up to any order.

Proposition 9.1. *If $\mathfrak{A}$ is an automatic subset of $\mathfrak{R}$, then so is the subalgebra of $\mathfrak{R}$ which is effectively generated by $\mathfrak{A}$.*

Proof. We first detail that elements of the effective subalgebra of $\mathfrak{R}$ are represented by trees whose inner nodes are labeled by $+, -$ and $\cdot$ and whose leaves are labeled by elements in $\mathfrak{A}$ or $\mathfrak{C}$. Such trees represent series in $\mathfrak{R}$, which can be expanded automatically w.r.t. each z_i, by using the usual rules for the expansions of sums differences and products. The global zero-test for such series is given by the zero-test in $\mathfrak{R}$. □

Power series in infinitely many variables. For some purposes it is convenient to allow k to be infinite. In that case, $\mathfrak{R}$ is an effective $\mathfrak{C}$-subalgebra of the direct limit $\mathfrak{C}[[z_1, z_2, \cdots]]$ of $\mathfrak{C}, \mathfrak{C}[[z_1]], \mathfrak{C}[[z_1, z_2]], \cdots$, which contains $z_1, z_2, \cdots$. Series in $\mathfrak{R}$ only depend on a finite number of variables, although this number is not bounded a priori.

If $Z = \{z_{i_1}, \cdots, z_{i_j}\}$ is a finite set of variables, then we denote by $\mathfrak{R}_Z = \mathfrak{R}_{z_{i_1},\cdots,z_{i_j}}$ the subalgebra of $\mathfrak{R}$ of power series in Z.

Effective Cartesian algebras. For the purpose of Cartesian representations, it is convenient to introduce the concept of Cartesian algebras of power series over $\mathfrak{C}$. These are subalgebras $\mathfrak{R}$ of $\mathfrak{C}[[z_1, z_2, \cdots]]$, which are stable under substitutions which send a finite number of z_i to power products $z_1^{\alpha_{i,1}} \cdots, z_{k_i}^{\alpha_{i,k_i}}$, with $\alpha_{i,1}, \cdots, \alpha_{i,k_i} \in \mathbb{N}$, while leaving the others invariant.

If these substitutions are effective, and $\mathfrak{R}$ is effective and automatic, then we say that $\mathfrak{R}$ is an **effective Cartesian algebra** of power series over $\mathfrak{C}$. In particular, $\mathfrak{R}_Z$ is unique up to isomorphism, if Z is a finite subset of $\{z_1, z_2, \cdots\}$ of fixed the cardinality, since $\mathfrak{R}$ is stable under finite permutations of variables. Therefore, we may extend the notation $\mathfrak{R}_Z$ to the case when Z is any finite set of formal variables.

9.2.3 Automatic Laurent series.

Automatic Laurent series. Let $\mathfrak{L}$ be an effective $\mathfrak{C}$-algebra of Laurent series in $z_1, \cdots, z_k$, which contains $z_1, z_1^{-1}, \cdots, z_k$ and z_k^{-1}. A subset $\mathfrak{A}$ of $\mathfrak{L}$ is said to be **automatic**, there exists algorithms which take $u \in \mathfrak{A}$, $1 \leqslant i \leqslant k$ and $\alpha_i \in \mathbb{Z}$ on input and compute the valuation of u in z_i and its α_i-th coefficient; moreover, we require $[z_i^{\alpha_i}]u$ to be in $\mathfrak{A}$. A Laurent series u in $\mathfrak{L}$ is said to be **automatic**, if u is an element of an automatic subset of $\mathfrak{L}$. We can compute the valuations of such a series u in each z_i, as well as its expansion up to any order.

Effective Cartesian algebras. As in the previous section, we may allow k to be infinite in the above definition. Then effective Cartesian algebras of Laurent series over $\mathfrak{C}$ are defined in a similar manner as in the previous section.

From power series to Laurent series. Assume that $\mathfrak{R}$ is an effective $\mathfrak{C}$-algebra of power series in $z_1, \cdots, z_k$, which contains $z_1, \cdots, z_k$. Then we can naturally associate an effective $\mathfrak{C}$-algebra $\mathfrak{L}$ of Laurent series in $z_1, \cdots, z_k$ to $\mathfrak{R}$ by

$$\mathfrak{L} = \{z_1^{p_1} \cdots z_k^{p_k} u | p_1, \cdots, p_k \in \mathbb{Z}, u \in \mathfrak{R}\}.$$

Laurent series in $\mathfrak{L}$ are redundantly represented as pairs $(z_1^{p_1} \cdots z_k^{p_k}, u)$, whence we always have *a priori* bounds for their valuations in $z_1, \cdots, z_k$. Therefore, if $\mathfrak{R}$ is automatic, then so is $\mathfrak{L}$. Allowing k to be infinite, we deduce that effective Cartesian algebras of power series naturally induce effective Cartesian algebras of Laurent series.

From Laurent series to power series. Assume that $\mathfrak{L}$ is an effective $\mathfrak{C}$-algebra of Laurent series in $z_1, \cdots, z_k$, which contains $z_1, z_1^{-1}, \cdots, z_k, z_k^{-1}$. Then we can naturally associate the effective $\mathfrak{C}$-algebra $\mathfrak{R} = \mathfrak{L} \cap \mathfrak{C}[[z_1, \cdots, z_k]]$ of power series to $\mathfrak{L}$, which contains $z_1, \cdots, z_k$. We notice that if $\mathfrak{L}$ is automatic, then so is $\mathfrak{R}$, and we have an algorithm for testing whether a series $u \in \mathfrak{L}$ belongs to $\mathfrak{R}$.

Remark 9.2. Let $\mathfrak{R}$ be an effective algebra of power series, $\mathfrak{L}$ the natural effective algebra of Laurent series associated to $\mathfrak{R}$, and $\overline{\mathfrak{R}}$ the natural effective algebra of power series associated to $\mathfrak{L}$. In general, we have $\overline{\mathfrak{R}} \supseteq \mathfrak{R}$, but not $\overline{\mathfrak{R}} = \mathfrak{R}$. This is because $z_i u \in \mathfrak{R}$ does not necessarily imply $u \in \mathfrak{R}$ for $u \in \mathfrak{C}[[z_1, \cdots, z_k]]$. On the other hand, we do have $\overline{\mathfrak{L}} = \mathfrak{L}$, when starting with an effective algebra $\mathfrak{L}$ of Laurent series instead.

9.2.4 Automatic transseries

Automatic transseries. Let $\mathfrak{C}$ be any effective totally ordered exp-log field of constants and $\mathbb{T} = \mathfrak{C}[[x]]$ the field of grid-based transseries in x over $\mathfrak{C}$, as defined in chapter 1. Let $\mathfrak{T}$ be an effective totally ordered exp-log subfield of $\mathbb{T}$, which contains x and $\mathfrak{C}$. A subset $\mathfrak{A}$ of $\mathfrak{T}$ is said to be **automatic**, if there exists an algorithm which takes $f \in \mathfrak{A}$ on input and which returns a finite labeled tree T with the following properties:

- **AT1.** Each node of T is labeled by a couple (g, P), where $g \in \mathfrak{A}$ and P is an algorithm.
- **AT2.** The root is labeled by a couple of the form (f, P).
- **AT3.** The label of each leaf has the form $(-\exp_l x, P)$, where P is a 'dummy' algorithm.
- **AT4.** For each interior node labeled by (g, P), whose successor nodes are labeled by $(\varphi_1, Q_1), \cdots, (\varphi_k, Q_k)$, the transseries g is an automatic Laurent series in $e^{\varphi_1}, \cdots, e^{\varphi_k}$ and P an expansion algorithm for g.[1]

The tree T is called an **automatic expansion tree** for f. A transseries in $\mathfrak{T}$ is said to be **automatic**, if it is an element of some automatic subset of $\mathfrak{T}$.

[1]Here we understand that the underlying automatic set of Laurent series for this expansion algorithm is a suitable set of Laurent series in $\mathfrak{C}[[e^{\varphi_1}, \cdots, e^{\varphi_k}]] \cap \mathfrak{A}$.

Effective normal bases and the structure theorem. An asymptotic expansion tree relative to a normal basis $B \subseteq \mathfrak{T}$ (as defined on page 52) is an asymptotic expansion tree T, whose labels except the root's one have the form $(\alpha \log \mathfrak{b}, P)$, with $\alpha \in \mathfrak{C}$ and $\mathfrak{b} \in B$. An **effective normal basis** is a normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_i\}$, such that we have an asymptotic expansion tree for each $\mathfrak{b}_i$ with $i > 1$ relative to $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_{i*}\}$, where $\log \mathfrak{b}_i \asymp \mathfrak{b}_{i*}$.[2]

Theorem 9.1. **(Effective structure theorem)** *Let* $\mathfrak{A}$ *be an automatic subset of* $\mathfrak{T}$ *and* $B_0 \subseteq \mathfrak{T}$ *an effective normal basis. Then there exists an algorithm which takes* $f \in \mathfrak{A}$ *on input and which computes an effective normal overbasis* $B \subseteq \mathfrak{T}$ *of* B_0 *and an automatic expansion tree for* f *relative to* B.

Proof. We only give a sketch of the proof, which is analogous to the proof of the theoretical structure theorem on page 53. Given $f \in \mathfrak{A}$, we first compute an automatic expansion tree T of f. If T is a leaf, whence $f = \exp_{l_0}^{-1} x$, then we compare l with the level l_0 of B_0, and either insert $\log_l^{-1} x, \cdots, \log_{l_0-1}^{-1}$ into B_0, in which case we are done, or replace f by $e^{\log f}$ and recursively apply the algorithm.

If T is not a leaf, then let $(\varphi_1, Q_1), \cdots, (\varphi_k, Q_k)$ denote the labels of the successors of the root. We recursively compute an effective normal overbasis $B' \subseteq \mathfrak{T}$ of B_0 and automatic expansion trees for $\varphi_1, \cdots, \varphi_k$ relative to B'. Now we compute the normal basis B as in the proof of the theoretical structure theorem, using the algorithm `add`. In our case, B is an effective normal basis and the image of f by the embedding of $C[\![e^{\varphi_1}, \cdots, e^{\varphi_q}]\!]$ into $C[\![\mathfrak{b}_1; \cdots; \mathfrak{b}_n]\!]$ is computed by using the formulas from the end of section 1.4.3. □

The following corollary of the structure theorem is proved by mimicking the algorithm `expand` from chapter 7. As the result will not be used in what follows, the details of the proof are left as an exercise to the reader.

Corollary. *If* $\mathfrak{A}$ *is an automatic subset of* $\mathfrak{T}$, *then so is the exp-log subfield of* $\mathfrak{T}$ *which is effectively generated by* $\mathfrak{A}$. □

Automatic multivariate transseries. Automatic multivariate transseries can be defined in a similar way as above, by labeling the leaves by pairs $(-\exp_l x_i, P)$ in **AT3**, for some x_i in a finite set $X = \{x_1, \cdots, x_p\}$. For more details about multivariate transseries from the effective point of view, we refer to chapter 11.

[2]We remark that this definition implies that we have automatic Cartesian representations for the $\log \mathfrak{b}_i$ with $i > 1$. Although this property could not be derived directly from the definition on page 168, it resulted indirectly from theorem 7.1, because of the restriction to exp-log functions.

9.3 Cartesian representations

Let $\mathfrak{C}[[\mathfrak{X}]]$ be the ring of grid-based series over an effective field $\mathfrak{C}$ in an effective quasi-ordered monomial group $\mathfrak{X}$. Let $z_1, \cdots, z_k$ be formal variables, which represent infinitesimal monomials $\overline{z_1}, \cdots, \overline{z_k}$ in $\mathfrak{X}$. Then we have a natural mapping $\mathfrak{C}[[z_1, \cdots, z_k]] \to \mathfrak{C}[[\mathfrak{X}]]; u \mapsto \overline{u}$ by proposition 1.3 and remark 1.3. We say that a Laurent series $u \in \mathfrak{C}[[z_1, \cdots, z_k]]$ is a **Cartesian representation** for $\overline{u}$. In this section, we will generalize the algorithms from section 7.3 for Cartesian representations which belong to an effective Cartesian algebra $\mathfrak{L}$ of Laurent series.

9.3.1 Intermediary dominant monomials

Let $u \in \mathfrak{L}_Z$ be a Cartesian representation of a series in $\mathfrak{C}[[\mathfrak{X}]]$ and let $\mathrm{M}_1, \cdots, \mathrm{M}_d$ be the dominant monomials of $\overline{u}$. A set of **intermediary dominant monomials** of u is a finite set G of monomials in S_Z with $\overline{u} = \overline{[u|(G)]}$, such that each of the M_i is represented by one of the elements in G and such that each element in G represents a monomial in $(\mathrm{M}_1, \cdots, \mathrm{M}_d)$. If $\overline{u} = 0$ then such a set is necessarily empty.

In this section we will present a generalization of the algorithm `idm` which incorporates an asymptotic zero-test at the same time. As in section 7.4, this implies the need for an algorithm `simplify` in order to eliminate redundant elements in Z. If $\mathfrak{X}$ is an effective totally ordered monomial group with $\mathbb{Q}$-powers, such an algorithm is obtained by a straightforward generalization of the algorithm `simplify` from section 7.4. In the other case, we assume that $\mathfrak{X}$ is generated by a finite number of monomials, on which asymptotic constraints of the forms (8.3) and (8.6) may be imposed. Then the algorithm `simplify` from section 7.4 again generalizes: the cases when $\text{ц} = \overline{z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-1}}}$ are infinitesimal, Archimedian, resp. infinitely large in lemma 7.3 are now distinguished using automatic case separation, by imposing the constraints $\text{ц} \prec\!\!\prec 1, \text{ц} \asymp 1$ resp. $\text{ц} \succ\!\!\succ 1$.

Whenever a redundant element in Z is eliminated by `simplify`, the Cartesian representation of u needs to be rewritten as an element in $\mathfrak{L}_{z'_1, \cdots, z'_{k'}}$, where the z'_j represent new monomials in $\mathfrak{X}$ and the $\overline{z_i}$ are positive integer power products of the $\overline{z'_i}$. Sets like Z and Z' are called **Cartesian coordinates**, changes of coordinates of the above type **Cartesian changes of coordinates**, and Z' is said to be **wider** than Z. We notice that the rewriting of u can be done automatically, by the properties of effective Cartesian algebras.

As before, we will denote by $\leqslant_Z$ the componentwise ordering on $S_Z = z_1^{\mathbb{Z}} \cdots z_k^{\mathbb{Z}}$. Restrictions of Cartesian representations in $\mathfrak{L}_Z$ are computed in a similar fashion as in proposition 7.1. We will also denote by $\preccurlyeq_{\mathfrak{X}}$ the quasi-ordering on S_Z induced by $\leqslant_{\mathfrak{X}}$, where we warn the reader not to confuse these different quasi-orderings and that the quasi-ordering $\leqslant_{\mathfrak{X}}$ on $\mathfrak{X}$ is opposite to the asymptotic ordering $\prec\!\!\prec$. We can now specify the generalization of `idm`; its termination and correctness proofs are analogous to those of `idm` from page 175 and `zero_test` from page 179.

Algorithm `idm`(u).
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series $\overline{u} \in \mathfrak{C}\llbracket\mathfrak{X}\rrbracket$.
OUTPUT: A set of intermediary dominant monomials for u.

• Let $z_i^{p_i}$ be the dominant monomial of u in z_i, for $1 \leqslant i \leqslant k$.
$G := \{z_1^{p_1} \cdots z_k^{p_k}\}$
while true
  if $u = 0$ **then return** ϕ
  $M := \{\text{ц} \in G | \forall \text{щ} \in G \quad \text{ц} \not\succ_{\mathfrak{X}} \text{щ}\}$
  if M contains $\text{щ} \neq \text{ц}$ with $\overline{\text{щ}} = \overline{\text{ц}}$ **then return** `idm`(`simplify`(u, M))
  if $\sum_{\text{ц} \in M} u_{\text{ц}} \neq 0$ **then return** G
  if there exists a final segment Ц of $G \backslash M$ for $\preccurlyeq_{\mathfrak{X}}$ with $u - [u|(\text{Ц})] = 0$
    then choose Ц maximal with this property, and set $G := \text{Ц}$.
    else $G := (G \backslash M) \cup M\{z_1, \cdots, z_k\}$
  • Eliminate non minimal elements from G.

9.3.2 Infinitesimalization and regularization

Let u be a Cartesian representation of an infinitesimal series in $\mathfrak{C}\llbracket\mathfrak{X}\rrbracket$. Although u is not necessarily C-infinitesimal itself, it is always possible to compute another Cartesian representation u' of $\overline{u}$ which is C-infinitesimal. The replacement of u by such a u' is called **infinitesimalization** of u.

Proposition 9.2. *There exists an algorithm, which given a Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ of an infinitesimal series $\overline{u}$ computes a C-infinitesimal Cartesian representation $u' \in \mathfrak{L}_{z'_1,\cdots,z'_{k'}}$ for $\overline{u}$, with $k' \leqslant k$.*

Proof. The algorithm is analogous to the algorithm from the proof of lemma 7.1. Here we apply `simplify` each time a new Cartesian coordinate is introduced, in order to force $k' \leqslant k$. □

Let u be a Cartesian representation of a regular series in $\mathfrak{C}\llbracket\mathfrak{X}\rrbracket$. Again, u is not necessarily C-regular, although it is possible to compute a C-regular Cartesian representation u' for u. Replacement of u by u' is called **regularization** of u.

More generally, a Cartesian representation $u \in \mathfrak{C}\llbracket z_1, \cdots, z_k\rrbracket$ for $\overline{u}$ is said to be **faithful**, if the natural mapping $z_1^{\mathbb{Z}} \cdots z_k^{\mathbb{Z}} \to \mathfrak{X}$ induces a bijection between the set of dominant monomials of the Laurent series u and the set of dominant monomials of $\overline{u}$. The following proposition generalizes and improves lemma 7.2:

Proposition 9.3. *There exists an algorithm, which given a Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series $\overline{u}$ computes a faithful Cartesian representation $u' \in \mathfrak{L}_{z'_1,\cdots,z'_{k'}}$ in $\overline{u}$, where $k' \leqslant k$.*

Proof. We first compute a set of intermediary dominant monomials $G = \{\text{ц}_1, \cdots, \text{ц}_m\}$ for u by `idm`. We notice that there do not exist $\text{ц}_i \neq \text{ц}_j$ in G such that $\overline{\text{ц}_i} = \overline{\text{ц}_j}$ is a dominant monomial of $\overline{u}$, because this would have been detected in

$$\textbf{if } |M| > 1 \textbf{ then return } \texttt{idm}(\texttt{simplify}(u, M)).$$

Now as long as G contains monomials $\text{ц}_i <_Z \text{ц}_j$, we do the following: we introduce the new Cartesian coordinate $\zeta = \text{ц}_j / \text{ц}_i$ and rewrite

$$u := \frac{[u|(\text{ц}_j)]\text{ц}_i\zeta}{\text{ц}_j} + (u - [u|\text{ц}_j]).$$

In these new coordinates, u admits $G := \{\text{ц}_1, \cdots, \text{ц}_{j-1}, \text{ц}_{j+1}, \cdots, \text{ц}_m\}$ as set of intermediary dominant monomials. We finally eliminate one of the new coordinates, using `simplify` and the multiplicative relation $\zeta\text{ц}_i = \text{ц}_j$.

After a finite number of steps, our procedure yields the desired faithful Cartesian representation for $\overline{u}$. □

9.3.3 A variant of `idm`

The algorithm `idm` from section 9.3.1 has the property that the Cartesian coordinates may be simplified during the execution. However, for the purposes of chapters 10 and 11 where additional assumptions are made on the Cartesian coordinates, such simplifications may be undesirable. In this section, we show how an admissible set of dominant monomials w.r.t. the original coordinates can be found, if such a set is known w.r.t. wider coordinates.

Lattice subalgorithms. Before stating the algorithm, we first need a preliminary. Let Z and Z' be sets of Cartesian coordinates, such that Z' is wider than Z. Then we have a natural mapping $\varphi : S_Z \to S_{Z'}$, which is neither injective nor surjective in general. Interpreting φ as a linear mapping from $\mathbb{Z}^k$ into $\mathbb{Z}^{k'}$ via natural isomorphisms, we recall that there exists algorithms to compute generators for the kernel of φ and to compute a preimage of any element in the image of φ. For instance, the LLL-algorithm can be used for this (see [LLL 82]).

Algorithm `idm`(u).
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series $\overline{u} \in \mathfrak{C}[\![\mathfrak{X}]\!]$.
OUTPUT: A set of intermediary dominant monomials for u w.r.t. the Cartesian coordinates $z_1, \cdots, z_k$.

$G := \phi$
• Let $\Gamma = z_1^{\alpha_1+\mathbb{N}} \cdots z_k^{\alpha_k+\mathbb{N}}$ be a superset of $\operatorname{supp} u$.
repeat
 • Let H be a set of intermediary dominant monomials for $u - [u|(G)]$ w.r.t. a wider set of Cartesian coordinates $z_1', \cdots, z_{k'}'$.
 • Compute $\varphi^{-1}(H) \cap \Gamma$, where φ is the natural mapping from S_Z into $S_{Z'}$.
 $G := G \cup (\varphi^{-1}(H) \cap \Gamma)$.
until $H = \phi$
return G

Proposition 9.4. *The above algorithm* `idm` *is correct and terminates.*

Proof. Let $G_1, G_2, \cdots$ resp. $H_1, H_2, \cdots$ denote the successive values of G resp. H during the execution. Since $(G_1) \subset (G_2) \subset \cdots$ forms a strictly increasing sequence of final segments of Γ, `idm` terminates by Dickson's lemma.

At the end of the algorithm, we have $\overline{[u|(G)]} = \overline{u}$, since $H = \phi$. Furthermore, since H_i is chosen to be a set of intermediary dominant monomials for $u - [u|(G_{i-1})]$ for all i, we have

$$\overline{u} \succ\!\!\succ \overline{u - [u|(G_1)]} \succ\!\!\succ \overline{u - [u|(G_2)]} \succ\!\!\succ \cdots.$$

This implies in turn that the maximal elements in each $\overline{G_i}$ for $\prec\!\!\prec$ were already in $\overline{G_1}$. Since there exists an element $\mathfrak{q} \in H_1$ with $\overline{\mathfrak{q}} \asymp \overline{u}$ (if $\overline{u} \neq 0$), there exists also such an element $\mathfrak{q}' \in G_1$. Therefore, G is an admissible set of dominant monomials. □

9.4 Extraction of coefficients

Let $\mathfrak{X}$ be an effective quasi-ordered asymptotic scale, which admits a basis B, and let $\mathfrak{L}$ be an effective Cartesian algebra of Laurent series. Given a Cartesian representation $u \in \mathfrak{L}_Z$ of an element $\overline{u} \in \mathfrak{C}[\![\mathfrak{X}]\!]$ and $\mathfrak{b}$ in B, a natural question is how to compute the asymptotic expansion of $\overline{u}$ in $\mathfrak{b}$. Modulo the insertion of new elements into Z, we may assume without loss of generality that for all $z \in Z$ we have $\overline{z} = \overline{z^{free}\mathfrak{b}^{\alpha_z}}$, for some $z^{free} \in Z^{\mathbb{Z}}$ free from $\mathfrak{b}$. If $\alpha_z \geqslant 0$ for all $z \in Z$, then the first terms of the asymptotic expansion of $\overline{u}$ can be computed by a natural generalization of proposition 7.3 (see section 9.4.1). The other case is far more intricate and we do not know of a general algorithm to compute the expansion of $\overline{u}$ in $\mathfrak{b}$.

To see where the difficulties lie, consider the following example: let $B = \{\mathfrak{b}_1, \mathfrak{b}_2\}$ and $Z = \{z_1, z_2\} = \{\underline{\mathfrak{b}_1}, \underline{\mathfrak{b}_2/\mathfrak{b}_1}\}$, where $\mathfrak{b}_1 \succ\!\!\succ \mathfrak{b}_2$.[3] Let u be a Cartesian representation in z_1, z_2. First of all, we observe that the expansion of $\overline{u}$ w.r.t. $\mathfrak{b}_1$ is not grid-based in general, although the coefficients are grid-based. For instance, if $u = 1/(1 - z_1 - z_2)$, then

$$\overline{u} = \sum_{i \in \mathbb{Z}_-^*} \frac{\mathfrak{b}_2^{-i}}{1 - \mathfrak{b}_2} \mathfrak{b}_1^i + \sum_{i \in \mathbb{N}} \frac{1}{1 - \mathfrak{b}_2} \mathfrak{b}_1^i.$$

A second, more serious problem is that we need to compute the **diagonal**

$$[u|\{z_1^\alpha z_2^\alpha | \alpha \in \mathbb{Z}\}],$$

in order to get a Cartesian representation for $[\mathfrak{b}_1^0]\overline{u}$. Unfortunately, "the largest class", as far as we know, in which diagonals of the above kind can be computed automatically, is the class of holonomic functions (see [Lip 89], for instance). In particular, there is no reason for a Cartesian algebra like $\mathfrak{L}^{expr}$ from chapter 7 to be stable under taking diagonals.

Fortunately, the exact computation of coefficients of the form $[\mathfrak{b}^\alpha]\overline{u}$ is not needed for the applications we have in mind. Instead, we will only need suitable approximations of them, which we call pseudo-coefficients. Before introducing these in section 9.4.4, we first show how to test whether u is equal to its restriction w.r.t. a generalized diagonal (see section 9.4.2) and whether $\overline{u}$ depends on $\mathfrak{b}$ (see 9.4.3). Actually, we will even consider the extraction of pseudo-coefficients w.r.t. several variables in B.

9.4.1 The "easy" case

Assume that for all $z \in Z$ we can write $\overline{z} = \overline{z'}\mathfrak{b}^\alpha$, where $z' \in Z$ is free from $\mathfrak{b}$ and $\alpha \geqslant 0$. Then `idm`, proposition 9.3 and the following straightforward generalization of proposition 7.3 can be used to compute the dominant term of the expansion of $\overline{u}$ in $\mathfrak{b}$ (whence the first n terms for any $n \in \mathbb{N}$):

Proposition 9.5. *Let u be a Cartesian representation in $\mathfrak{L}_Z$ of a series f in $\mathfrak{C}[\![\mathfrak{X}]\!]$, where $\mathfrak{X}$ admits a basis B. Assume that u has a unique dominant monomial $\underline{z_1^{\mu_1} \cdots z_k^{\mu_k}}$. Let $\mathfrak{b} \in B$ be such that z_i does not depend on $\mathfrak{b}$ for $l < i \leqslant k$ and $\overline{z_i} = \overline{z_i^{free}}\mathfrak{b}^{\alpha_i}$ for $1 \leqslant i \leqslant l$, with z_i^{free} free from $\mathfrak{b}$ and $\alpha_i > 0$. Then the dominant exponent of f w.r.t. $\mathfrak{b}$ equals*

$$\mu_f = \mu_1\alpha_1 + \cdots + \mu_l\alpha_l$$

and

$$([z_1^{\mu_1} \cdots z_l^{\mu_l}]u)\,(z_1^{free})^{\mu_1} \cdots (z_l^{free})^{\mu_l}$$

is a Cartesian representation for $[\mathfrak{b}^{\mu_f}]f$. □

[3] The notation $\underline{f}$ is used in order to refer to a "natural Cartesian reprentation" of a transseries (or series f). In the present case, we have $\underline{\mathfrak{b}_1} = z_1$ and $\underline{\mathfrak{b}_2/\mathfrak{b}_1} = z_2$ by definition.

9.4.2 Diagonal tests

Let $u(z_1, z_2)$ be a Laurent series in two variables in an effective Cartesian algebra $\mathfrak{L}$. Then $u = [u|(z_1z_2)^{\mathbb{Z}}]$, if and only if $u(z_1z_2, z_1z_2) = u(z_1^2, z_2^2)$. In particular, we have an algorithm to test this, by the properties of effective Cartesian algebras. More generally, let $Z = \{z_1, \cdots, z_k\}$ and let Δ be a **generalized diagonal** of $Z^{\mathbb{Z}}$, i.e. Δ is a subgroup of $Z^{\mathbb{Z}}$, which is generated by the intersection of a subspace of $Z^{\mathbb{Q}}$ with $Z^{\mathbb{N}}$. We will show how to decide whether $u = [u|\Delta]$ for $u \in \mathfrak{L}_{z_1,\cdots,z_k}$. This problem is essentially a problem of "discrete ordered linear algebra":

Proposition 9.6. *Let A and M be $k \times l$ resp. $k \times k$ matrices with entries in $\mathbb{N}$ resp. $\mathbb{N}^*$. Assume that $l \leqslant k$ and the rank of $B = MA$ is l. Then there exists an invertible matrix U with entries in $\mathbb{Q}_+^*$, such that $B = UA$.*

Proof. By classical linear algebra, there exists an invertible matrix V with coefficients in $\mathbb{Q}$, such that $B = VA$. Now consider the matrix $U_\lambda = \lambda V + (1-\lambda)M$. We have $B = U_\lambda A$ for all $\lambda \in \mathbb{Q}$. V being invertible the polynomial $\det U_\lambda$ in λ is non zero. Hence, $\det U_\lambda \neq 0$ for all $\lambda \neq 0$ sufficiently close to zero. Since the entries of M are in $\mathbb{N}^*$, the matrix U_λ has coefficients in $\mathbb{Q}_+^*$ for λ sufficiently small. Hence, $U = U_\lambda$ fulfills our hypothesis for some sufficiently small λ. □

Proposition 9.7. *Let A be an $k \times l$ matrix of rank $l \leqslant k$ with entries in $\mathbb{N}$. Then there exist matrices M and U with entries in $\mathbb{N}^*$, and respective ranks l and k, such that $MA = UA$ and MA has rank l.*

Proof. Let M_1 be an arbitrary matrix of rank l and with entries in $\mathbb{N}^*$, such that M_1A has rank l. By the previous proposition, there exists an invertible matrix U_1 with entries in $\mathbb{Q}_+^*$, such that $U_1A = M_1A$. Let $p \in \mathbb{N}^*$ be the greatest common divisor of the denominators of the entries in U_1. Then $A = pA_1$ and $U = pU_1$ obviously fulfill the requirements. □

Clearly, the above proofs actually provide an algorithm to construct M and U. Let us now come back to our initial problem, and let $\delta_1, \cdots, \delta_l \in Z^{\mathbb{N}}$ be expo-linearly independent generators for the group $\Delta^{\mathbb{Q}}$ with $\mathbb{Q}$-powers. Each δ_i corresponds to a column of a $k \times l$ matrix A with entries in $\mathbb{N}$, by decomposing δ_i w.r.t. Z. Now let M and U be as in proposition 9.7. Then $u = [u|\Delta]$, if and only if $M \star u = U \star u$, where the matrix action $\star$ is defined as follows:

$$M \star u = \sum_{V \in \mathbb{Z}^k} u_{V_1,\cdots,V_k} z_1^{M_{1,\cdot} V} \cdots z_k^{M_{k,\cdot} V}.$$

Remark 9.3. To check $M \star u = U \star u$ in a reasonably efficient way, it is important to choose M and U, such that their entries are as small as possible in proposition 9.7. We have not yet undertaken a detailed study of this issue.

9.4.3 Dependency on basis elements

Let B' be a subset of B. We now present an algorithm to test whether the class $\overline{u}$ of a Cartesian representation $u \in \mathfrak{L}_Z$ depends on B'. That is, we test whether $\overline{u} = [\prod_{\mathfrak{б} \in B'} \mathfrak{б}^0]\overline{u}$.

Algorithm `depends`(u, B').
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series $\overline{u} \in \mathfrak{C}[[\mathfrak{X}]]$.
An subset B' of the basis B for $\mathfrak{X}$.
OUTPUT: The result of the test whether $\overline{u}$ depends on B'.

$\Delta := \phi$
while true
  if $u = 0$ **then return true**
  $G :=$ `idm`(u)
  if $\mathfrak{ц}$ depends on B' for some $\mathfrak{ц} \in G$ **then return false**
  $\Delta := \Delta \cup G$
  • Choose $\mathfrak{ц}$ minimal in G for $\preccurlyeq_{\mathfrak{X}}$.
  if $[u|\mathfrak{ц} I^{\mathbb{N}}] = [u|\mathfrak{ц}(I^{\mathbb{N}} \cap \Delta^{\mathbb{Q}})]$ for some initial segment I of G for $\preccurlyeq_{\mathfrak{X}}$ **then**
    • Choose I maximal with this property, and set $u := u - [u|\mathfrak{ц} I^{\mathbb{N}}]$.

Remark 9.4. We notice that the problem of computing generators for $I^{\mathbb{N}} \cap \Delta^{\mathbb{Q}}$ is an easy application of the LLL-algorithm.

Remark 9.5. We assume that redundant elements in Z are eliminated during the successive applications of `idm`, whenever possible.

Proposition 9.8. *The algorithm* `depends` *is correct and terminates.*

Proof. The correctness of the algorithm is clear, since the part of $\overline{u}$ which depends on B' does not change throughout the algorithm. Assume that `depends` does not terminate on some input. The dimension of $\Delta^{\mathbb{Q}}$ being bounded, $\Delta^{\mathbb{Q}}$ and $\Delta^{\mathbb{N}}$ are constant after a sufficiently large number of iterations. Similarly, Z remains constant after a sufficiently large number of iterations.

By Dickson's lemma, the sequence of successive choices for $\mathfrak{ц}$ admits an increasing subsequence for $\leqslant_Z$, say $\mathfrak{ц}_1, \mathfrak{ц}_2, \cdots$. Without loss of generality, we may assume that $\Delta^{\mathbb{Q}}$ and Z are constant from the choice of $\mathfrak{ц}_1$ on. Let I be the smallest initial segment of Z for $\preccurlyeq_{\mathfrak{X}}$, such that there exists an n, for which $\mathfrak{ц}_n, \mathfrak{ц}_{n+1}, \cdots$ are all contained in $\mathfrak{ц}_n I^{\mathbb{N}}$. We claim that for each $i \in \mathbb{N}$, there exists an $m \geqslant n$ with $\mathfrak{ц}_m \in \mathfrak{ц}_n(\prod I)^i$. Indeed, otherwise there would exist a $z \in I$, such that the exponent of $\mathfrak{ц}_m$ in z remains bounded by i for all $m \geqslant n$, whence this exponent tends to a finite limit. But this would imply that $\mathfrak{ц}_{n'}, \mathfrak{ц}_{n'+1}, \cdots$ are all contained in $\mathfrak{ц}_{n'}(I - \{z\})^{\mathbb{N}}$ for some $n' \geqslant n$, which contradicts the minimality hypothesis on I.

Let $v = [u|\mathfrak{ц}_n I^{\mathbb{N}}] - [u|\mathfrak{ц}_n(I^{\mathbb{N}} \cap \Delta^{\mathbb{Q}})]$, just after the choice of $\mathfrak{ц}_n$. Let $\mathfrak{m} = \mathfrak{ц}_n z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ be a dominant monomial for v. Such a monomial exists, since $v \neq 0$, and depends on B'. Let $\alpha = \max(\alpha_1, \cdots, \alpha_k)$. By our claim, there exists an $m \geqslant n$ with $\text{ц}_m \in \text{ц}_n(\prod I)^{\alpha+1}$. Since u remains unaltered except for terms which do not depend on B', and since $\text{м} \prec\!\!\prec \text{м}$, the term $u_{\text{м}}\text{м}$ of u must have canceled out against another term in the call of `idm` which precedes the choice of ц_m. But this contradicts the constancy assumption on Z. □

9.4.4 Pseudo-coefficients

Let B' be a subset of B and let $\text{ч} = \prod_{\text{б} \in B'} \text{б}^{\alpha_{\text{б}}}$ be a monomial in $S_{B'}$. In this section we will show how to compute the **pseudo-coefficient** $\langle \text{ч} \rangle f$ of f in ч, where f resp. ч are represented by u resp. ц. The idea behind pseudo-coefficients is the following: if $f/\text{ч}$ is a series which is free from $B' \subseteq B$, then $\langle \text{ч} \rangle f$ should coincide with $[\text{ч}]f$. Otherwise, they should coincide at least up till terms τ, with $\tau \not\succcurlyeq \text{м}$ for each dominant monomial м of $f/\text{ч} - [\text{ч}]f$. Intuitively speaking, a pseudo-coefficient is just sufficiently close to the real coefficient to make the algorithms which use them work. Pseudo-coefficients are computed by slightly modifying the algorithm `depends` from the previous section:

Algorithm `pseudo_coefficient`$(u, \text{ч})$.
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series $f \in \mathfrak{C}[[\mathfrak{X}]]$.
A monomial ч in $S_{B'}$, where $B' \subseteq B$.
OUTPUT: A Cartesian representation for $\langle \text{ч} \rangle f$.

$u := u/\underline{\text{ч}}$
$u_0 := u$
$\Delta := \phi$
while true
  if $u = 0$ **then return** $u_0 - u$
  $G :=$ `idm`(u)
  if ц depends on B' for all $\text{ц} \in G$ **then return** $u_0 - u$
  $\Delta := \Delta \cup G$
  • Choose ц minimal in G for $\preccurlyeq_{\mathfrak{X}}$, such that ц does not depend on B'.
  if $[u|\text{ц}I^{\mathbb{N}}] = [u|\text{ц}(I^{\mathbb{N}} \cap \Delta^{\mathbb{Q}})]$ for some initial segment I of Z for $\preccurlyeq_{\mathfrak{X}}$ **then**
    • Choose I maximal with this property, and set $u := u - [u|\text{ц}I^{\mathbb{N}}]$.

Remark 9.6. In principle, we may have to introduce some new Cartesian coordinates in order to compute the Cartesian representation $\underline{\text{ч}}$. However, in practice such a Cartesian representation is often known already, in which case we may use this one instead.

The termination of this algorithm is proved in a similar way as the termination of `depends`. Although pseudo-coefficients are not defined canonically by the algorithm `pseudo_coefficient`, they always do satisfy the following property:

Proposition 9.9. *For any possible outcome of* `pseudo_coefficient`*, each dominant monomial of* $f/\text{ч} - \langle\text{ч}\rangle f$ *depends on* B'. □

9.5 Automatic expansion of $\mathfrak{L}$-finite transseries

Let $\mathfrak{L}$ be an effective Cartesian algebra of Laurent series over an effective ordered exp-log field $\mathfrak{C}$. We say that $\mathfrak{L}$ is an **effective exp-log Cartesian algebra**, if $\mathfrak{L}$ is effectively stable for left composition of infinitesimal Laurent series with $1/(1+z), \log(1+z)$ and e^z. Let $\mathfrak{T}$ be the smallest subfield of $\mathfrak{C}[[[x]]]$, containing x, which is stable under left composition of infinitesimal transseries with Laurent series in $\mathfrak{L}$. Transseries in $\mathfrak{T}$ are called **$\mathfrak{L}$-finite**. They are represented by $\mathfrak{L}$-exp-log expressions, which are expressions built up from x by the exp-log field operations and left compositions with elements in $\mathfrak{L}$. We notice that $\mathfrak{L}$ contains in particular the constants in $\mathfrak{C}$. In this section, we will show that $\mathfrak{T}$ is an automatic effective exp-log field.

9.5.1 The basic expansion algorithm

The following expansion algorithm is a straightforward generalization of `expand` from chapter 7. As before, $B = \{\text{б}_1, \cdots, \text{б}_n\}$ is a dynamic effective normal basis, which is initialized by $B := \{x^{-1}\}$.

Algorithm `expand`(f).
INPUT: An $\mathfrak{L}$-exp-log expression f in $\mathfrak{T}$.
OUTPUT: A Cartesian representation $\underline{f}$ for f.

Case $f = x$: Return $(\underline{x^{-1}})^{-1}$.

Case $f = u(g_1, \cdots, g_i) \in \mathfrak{L}_{g_1,\cdots,g_i}$:
STEP 1. Compute Cartesian representations $\underline{g_1}, \cdots, \underline{g_i}$ for $g_1, \cdots, g_i$ by `expand`.
Check whether $g_1, \cdots, g_i$ are infinitesimal.
Infinitesimalize $\underline{g_1}, \cdots, \underline{g_i}$.
STEP 2. Return $u(\underline{g_1}, \cdots, \underline{g_i})$.

Case $f = g_1 \square g_2$, $\square \in \{+, -, \cdot, /\}$:
STEP 1. Compute Cartesian representations $\underline{g_1}, \underline{g_2}$ for g_1 and g_2 by `expand`.
If $\square = /$, then check whether $g_2 \neq 0$ and regularize $\underline{g_2}$.
STEP 2. Return $\underline{g_1} \square \underline{g_2}$.

Case $f = \log g$:
STEP 1. Compute a Cartesian representation $\underline{g}$ for g by `expand`.
Compute the dominant monomial of g via `idm`.
Regularize $\underline{g}$.

STEP 2. Compute $\underline{\varepsilon}$ and $\alpha_1, \cdots, \alpha_n \in \mathfrak{C}$ with $\varepsilon \prec\!\!\prec 1$ and $\underline{g} = c_g \underline{\mathfrak{б}_1^{\alpha_1} \cdots \mathfrak{б}_n^{\alpha_n}}(1 + \underline{\varepsilon})$. If $\alpha_1 \neq 0$, then insert $\log^{-1} \mathfrak{б}_1^{-1}$ into B.

STEP 3. Return $\log c_g + \alpha_1 \underline{\log \mathfrak{б}_1} + \cdots + \alpha_n \underline{\log \mathfrak{б}_n} + \log(1 + \underline{\varepsilon})$.

Case $f = \exp g$:

STEP 1. Compute a Cartesian representation $\underline{g}$ for g by `expand`.

STEP 2. If g is bounded, then do the following:

A. Compute $c_g = g_{0, \overset{n \text{ times}}{\cdots}, 0}$ and $\underline{\varepsilon} = \underline{g} - c$.

B. Infinitesimalize $\underline{\varepsilon}$.

C. Return $e^c e^{\underline{\varepsilon}}$.

STEP 3. If there exists an $1 < i \leqslant n$ with $g \asymp \log \mathfrak{б}_i$, then do the following:

A. Compute the limit λ of $g / \log \mathfrak{б}_i$.

B. Return $\underline{\mathfrak{б}_i^{\lambda}}$`expand`$(e^{g - \lambda \log \mathfrak{б}_i})$.

STEP 4. In the remaining case, do the following:

A. Compute i^* with $\log |g| \asymp \log \mathfrak{б}_{i^*}$.

B. Decompose $g = g^+ + g^-$, with $g^+ = g_{0, \overset{n-i^* \text{ times}}{\cdots}, 0}$.

C. Insert $e^{-|g^+|}$ into B.

D. Return $(\underline{e^{-|g^+|}})^{-\operatorname{sign} g^+} e^{\underline{g^-}}$.

Theorem 9.2. *Let $\mathfrak{L}$ be an effective exp-log Cartesian algebra of Laurent series. Then the exp-log field $\mathfrak{T}$ of $\mathfrak{L}$-finite transseries in x is effective and automatic.* □

Example 9.2. The above theorem seriously enlarges the class of functions for which we can compute automatic asymptotic expansions. For instance, modulo Schanuel's conjecture, we may include the power series of the trigonometric functions at zero into $\mathfrak{L}$. Then the expansion algorithm may be applied to expressions like

$$e^{x^5 \sin \sin(x^{-1} + e^{-x})} - \frac{1}{\sin \sin e^{-x^4}}$$

and many others. When we allow heuristic zero-tests for Laurent series, then we may also include the Laurent series

$$\log \Gamma(x) = x \log x - \frac{1}{2} \log x + \frac{1}{2} \log(2\pi) + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

in $\log^{-1} x$ and x^{-1} into $\mathfrak{L}_{\log^{-1} x, x^{-1}}$. Hence, we can automatically expand expressions like

$$\frac{\Gamma(x + 1/\Gamma(x)) - \Gamma(x)}{\log x}.$$

In a similar fashion, many other expressions involving special functions can be expanded automatically.

9.5.2 A canonical expansion algorithm

In section 1.6 we have introduced the concept of canonical bases. If instead of computing decompositions $g = g^+ + g^-$, we compute decompositions $g = g^\uparrow + g^c + g^\downarrow$ in step 4b of the exponential case of `expand`, then the normal basis B actually remains a canonical basis. Such decompositions are computed using the formula

$$g^\uparrow = \sum_{\alpha_n < 0} [\text{б}_n^{\alpha_n}] g + \sum_{\alpha_{n-1} < 0} [\text{б}_n^0 \text{б}_{n-1}^{\alpha_{n-1}}] g + \cdots + \sum_{\alpha_1 < 0} [\text{б}_n^0 \cdots \text{б}_2^0 \text{б}_1^{\alpha_1}] g.$$

Expansions of exp-log expressions with respect to canonical bases are interesting because they do not depend on the order in which the expansions of the subexpressions are computed. However, we do not recommend the use of canonical bases for practical purposes for two reasons: first, the complexity of algorithms based on canonical may be dramatic, as shows the following example:

$$f(x) = \exp \frac{x^N}{1 - x^{-1}} = e e^{x^N + \cdots + x} + e x^{-1} e^{x^N + \cdots + x} + \cdots,$$

where N is a very large integer. Indeed, in this example the computation of $x^N + \cdots + x$ takes a time proportional to N. A second reason not to use canonical bases is that they essentially depend on the choice of x as "coordinate function". Therefore, canonical bases admit no natural generalization to higher dimensions.

Nevertheless, canonical bases are interesting for certain theoretical purposes, because the above algorithm shows that the purely unbounded part $f^\uparrow$ of a transseries $f \in \mathfrak{T}$ is actually in $\mathfrak{T}$. If we take for $\mathfrak{L}$ the smallest Cartesian algebra of Laurent series over $\mathfrak{C}$, such that its subset of infinitesimal elements is stable under left composition with $\exp z - 1, \log(1 + z)$ and infinitesimal real algebraic power series, then we have in particular:

Proposition 9.10. *If f is an L-transseries, then so are $f^\uparrow, f^{↥}, f^{↧}$ and $f^\downarrow$.* □

9.6 Computations with implicit functions

9.6.1 Local communities

A **local community** of power series over C is a Cartesian algebra R of power series over C, which satisfies the following additional conditions:

- **LC1.** $z_1 \in R_{z_1}$.
- **LC2.** R is stable under the partial derivation ∂_{z_1}.
- **LC3.** For each infinitesimal $P \in R_{z_1, \cdots, z_{k+1}}$ with $[z_1^0 \cdots z_k^0 z_{k+1}^1] P \neq 0$ we have a homomorphism $R_{z_1, \cdots, z_{k+1}} \xrightarrow{\imath_P} R_{z_1, \cdots, z_k}$, which corresponds to the implicit definition of $z_{k+1} \in C[[z_1, \cdots, z_k]]$ by $P = 0$.

The condition **LC3** means that local communities are stable under the resolution of regular systems of functional equations. In particular, R is stable under composition and functional inversion, whence local communities are the natural local structures in which all classical operations are defined:

Proposition 9.11. *Let R be a local community of power series over C. Then*

(a) R is stable under composition.
(b) R is stable under regular functional inversion.
(c) R is stable under extraction of coefficients.

Proof. Let $u \in R_{v_1,\cdots,v_l}$, where $v_1, \cdots, v_l$ are infinitesimal power series $\overline{v_1}, \cdots, \overline{v_l}$ in $R_{z_1,\cdots,z_k}$. Applying k times **LC3** to the equations $v_1 - \overline{v_1}, \cdots, v_l - \overline{v_l}$ we obtain a natural homomorphism of $R_{v_1,\cdots,v_l,z_1,\cdots,z_k}$ into $R_{z_1,\cdots,z_k}$, which sends u to the composition of u with $(v_1, \cdots, v_l)$. This proves (a).

Next, let $v_1, \cdots, v_k$ represent infinitesimal power series $\overline{v_1}, \cdots, \overline{v_k} \in R_{z_1,\cdots,z_k}$. If the linear parts of $v_1, \cdots, v_k$ determine an isomorphism of C^k, then applying k times **LC3** to the equations $v_1 - \overline{v_1}, \cdots, v_k - \overline{v_k}$, we obtain a natural homomorphism of $L_{v_1,\cdots,v_k,z_1,\cdots,z_k}$ into $L_{v_1,\cdots,v_k}$. Left composition of this homomorphism with the natural inclusion $L_{z_1,\cdots,z_k} \to L_{v_1,\cdots,v_k,z_1,\cdots,z_k}$ yields a natural isomorphism between $L_{z_1,\cdots,z_k}$ and $L_{v_1,\cdots,v_k}$. This proves (b).

Finally, let $u \in L_{z_1,\cdots,z_k}$ be given. Since $[z_1^0]u$ is precisely the substitution of z_1 by 0 in u, we have $[z_1^0]u \in L_{z_2,\cdots,z_k}$ by (a). More generally, for each $i \geqslant 0$, we have

$$[z_1^i]u = \frac{1}{i!}[z_1^0]\left(\frac{\partial^i u}{\partial z_1^i}\right),$$

whence $[z_1^i]u \in L_{z_2,\cdots,z_k}$ for each i. This proves (c). □

A Cartesian algebra L of Laurent series over C is said to be a **local community**, if its associated Cartesian algebra of power series is. By what has been said in section 9.2.3, we can naturally associate local communities of power series to local communities of Laurent series and vice versa. However, we usually start with a local community of Laurent series because of remark 9.2.

Example 9.3. The intersection of an arbitrary family of local communities is a local community. In particular, there exists a smallest local community over C. This local community contains all algebraic power series.

Example 9.4. Let $C[[z_1, z_2, \cdots]]^{conv}$ denote the set of power series $u(z_1, \cdots, z_k)$ which converge in an open neighbourhood of $(0, \cdots, 0)$. and let $C\llbracket z_1, z_2, \cdots\rrbracket^{conv}$ denote the natural C-algebra of Laurent series associated to $C[[z_1, z_2, \cdots]]^{conv}$. By the implicit function theorem, $C[[z_1, z_2, \cdots]]^{conv}$ and $C\llbracket z_1, z_2, \cdots\rrbracket^{conv}$ are local communities.

Example 9.5. The subalgebras of all regular D-algebraic power series resp. Laurent series of $C[[z_1, z_2, \cdots]]$ resp. $C[\![z_1, z_2, \cdots]\!]$ are local communities (see section D.5.3). More generally, the regular D-algebraic closure of a local community is also a local community.

9.6.2 Effective local communities

Let $\mathfrak{C}$ be an effective field of constants of characteristic zero. An **effective local community** over $\mathfrak{C}$, is an effective Cartesian algebra over $\mathfrak{C}$, which satisfies **LC1**, **LC2** and **LC3** effectively. I.e. the partial derivations are effective, as well as the natural mappings $\imath_P$ from **LC3**. Moreover, these mappings $\imath_P$ can be computed as a function of P. The following proposition is the effective counterpart of proposition 9.11:

Proposition 9.12. *Let $\mathfrak{R}$ be an effective local community of power series over C. Then*

(a) $\mathfrak{R}$ is effectively stable under composition.
(b) $\mathfrak{R}$ is effectively stable under regular functional inversion.
(c) $\mathfrak{R}$ is automatic. □

Let $\mathfrak{C}$ be an effective totally ordered exp-log field of constants. The above proposition implies that $\mathfrak{R}$ is an effective exp-log Cartesian algebra, as soon as C is a totally ordered exp-log field with $\log(1+z), e^z \in R_z$. In this case we say that $\mathfrak{R}$ is an **effective exp-log local community**.

Example 9.6. In section D.5.3 we have shown that the regular D-algebraic series over $\mathfrak{C}$ form an effective local community. In fact, local communities of this type are sufficiently general for the applications to asymptotic expansion algorithms which will be considered in this thesis. However, there is no need to restrict our attention to this particular type of local communities.

Example 9.7. There exists a smallest local community $\mathfrak{R}$ of power series over $\mathfrak{C}$, which contains $\exp z_1$ and $\log(1+z_1)$. Since $\exp z_1$ and $\log(1+z_1)$ are regular D-algebraic, $\mathfrak{R}$ is effective. This local community will be sufficiently rich for the applications in chapter 11. We notice that the series in $\mathfrak{R}$ are convergent, if $\mathfrak{C}$ is a subfield of $\mathbb{C}$.

9.6.3 Automatic expansions of implicit functions

In section 3.1, we have described the Newton polygon method from a theoretical point of view. In this section, we show how this method can be made effective in the setting of local communities. Although the results of this section will not be applied in what follows, the section can be seen as an introduction to section 10.4.1.

Let $\mathfrak{C}$ be an effective field of constants of characteristic zero, such that the solutions in $\mathfrak{C}$ to any polynomial equation over $\mathfrak{C}$ can be computed by algorithm. Let $\mathfrak{X}$ be an effective totally ordered monomial group with $\mathbb{Q}$-powers. Let $\mathfrak{L}$ be a local community of Laurent series, and $\mathfrak{R}$ the associated local community of power series. Our aim is to compute the solutions to asymptotic equations in $\mathfrak{C}[\![\mathfrak{X}]\!]$ of the form

$$P_0 + P_1 f + P_2 f^2 + \cdots = 0 \quad (f \prec\!\!\prec ч \in \mathfrak{X}), \tag{9.1}$$

where there exists an $i \in \mathbb{N}$, with $P_i \succcurlyeq P_j ч^{i-j}$ for all $j \in \mathbb{N}$. To do this, we assume that the series $P_0 + P_1 f + \cdots$ is given by a Cartesian representation $u \in \mathfrak{R}_{z_1,\cdots,z_k,f/ч}$, for some finite set $Z = \{z_1, \cdots, z_k\}$ of infinitesimals in $\mathfrak{X}$. We also assume that $ч \in S_Z$.

Remark 9.7. If $P(f) = P_0 + P_1 f + \cdots$ is a polynomial, then we obtain all solutions to (9.1) by taking ч small enough. Equations of the form $P_\mu f^\mu + P_{\mu+1} f^{\mu+1} + \cdots = 0$ with $\mu \in \mathbb{Z}$ can be reduced to (9.1) by multiplication with $f^{-\mu}$.

Theorem 9.3. *There exists an algorithm, which given P and ч satisfying the above assumptions computes the set of all solutions to (9.1) in $\mathfrak{C}[\![\mathfrak{X}]\!]$.*

Proof. We have to show how to make the different steps of `polynomial_solve` in section 3.3 effective, taking into account the extension from section 3.4.

Let us first show how to compute an a priori bound d_0 for the Newton degree of (9.1): we substitute f by $f/ц$ in (9.1), thus reducing the general case to the case when $ц = 1$. Next we apply the algorithm `idm` to find the dominant monomial of P when considered as a series in $\mathfrak{C}[[f]][\![\mathfrak{X}]\!]$. Then d_0 is just the valuation of the corresponding dominant coefficient as a series in f.

Let us now show how to perform steps 1 and 2, if we have an a priori bound d_0 for the Newton degree of (9.1): we start by computing $\mathrm{M}_{P_0}, \cdots, \mathrm{M}_{P_{d_0}}$, using `idm`. Then the potential dominant monomials $ц > ч$ relative to (9.1) are necessarily of the form $ц = \sqrt[j-i]{\mathrm{M}_{P_i}/\mathrm{M}_{P_j}}$ with $0 \leqslant i < j \leqslant d_0$. To decide whether such a ц is indeed an potential dominant monomial, it suffices to check that $\mathrm{M}_{P_l} ц^l \preccurlyeq \mathrm{M}_{P_i} ц^i$ for all $0 \leqslant l \leqslant d_0$. In particular, this yields an algorithm to compute d. The Newton polynomials associated to the potential dominant monomials can be computed by the hypothesis on $\mathfrak{C}$. Hence, we have an algorithm to compute the potential dominant terms of f.

The only non trivial thing which remains to be shown is how to compute the unique solution to $P^{(d-1)}(\varphi) = 0$ in step 5. Clearly, it suffices to show this for $d = 1$. We first reduce the problem to the case when $ч = 1$, $u_1 = [f^1]u = 1$ and $u \in \mathfrak{R}_{z_1,\cdots,z_k,f}$. This is done by substituting f by $f/ц$ in (9.1), dividing u by u_1 and regularizing u as a Laurent series in $\mathfrak{C}[[f]][\![\mathfrak{X}]\!]$. Now the effective counterpart of **LC3** yields the unique infinitesimal solution to (9.1) in $\mathfrak{R}_{z_1,\cdots,z_k}$. □

Remark 9.8. The substitution of f by $\varphi + \tilde{f}$ in step 5 of `polynomial_solve` may be unnecessarily expensive from a complexity point of view. Consider for instance the example

$$\left(f - \frac{1}{1-x^{-1}}\right)^2 - e^{-x^2} f = e^{-x} \ (x \to \infty).$$

Application of the above algorithm leads to the refinement

$$f = \frac{1}{1-x^{-1}} + \frac{e^{x^2}}{2} + \tilde{f}, \ (\tilde{f} \prec\!\!\prec 1).$$

However, the refinement

$$f = \frac{1}{1-x^{-1}} + \tilde{f}, \ (\tilde{f} \prec\!\!\prec 1)$$

would clearly avoid infinite loops as well and leads to simpler formulas. The reason is that we did not exploit the information that $x^{-1} \prec\!\!\prec\!\!\prec e^{-x} \prec\!\!\prec\!\!\prec e^{-x^2}$.

Let us now sketch an alternative approach, based on the observation that we can often efficiently check whether $P(\varphi) = 0$: for instance, a heuristic test is usually sufficient. In the case when P is a polynomial, we can check whether $P(\varphi) = 0$, by considering the g.c.d. of $P, \partial P/\partial f, \cdots, \partial^{d-1} P/\partial f^{d-1}$. As in the example above, we will replace the substitution of f by $\varphi + \tilde{f}$ by a less expensive substitution if $P(\varphi) \neq 0$. We denote $u_i = [f^i]u$ for each i. We assume that $\mathfrak{v} = 1$ and that we have ordered $z_1 \preccurlyeq_{\mathfrak{X}} \cdots \preccurlyeq_{\mathfrak{X}} z_k$.

Let $\mathfrak{u} = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ and let $z_1^{\nu_1} \cdots z_k^{\nu_k}$ be the dominant monomial of P_0. Without loss of generality, we may assume that u_i has been regularized, and that its dominant monomial is $z_1^{\nu_1 - i\alpha_1} \cdots z_k^{\nu_k - i\alpha_k}$ for $0 \leqslant i \leqslant d$: otherwise, we eliminate one of the z_i using `simplify`. Now consider the Newton polynomial in z_k:

$$[z_k^{\nu_k}]u_0 + ([z_k^{\nu_k - \alpha_k}]u_1) f_* + \cdots + ([z_k^{\nu_k - d\alpha_k}]u_d) f_*^d.$$

If this polynomial has a root f_* of multiplicity d, then we substitute f by $f_* z_k^{\alpha_k} + \tilde{f}$. Otherwise, we consider the Newton polynomial in z_k and z_{k-1}:

$$[z_{k-1}^{\nu_{k-1}} z_k^{\nu_k}]u_0 + ([z_{k-1}^{\nu_{k-1} - \alpha_{k-1}} [z_k^{\nu_k - \alpha_k}]u_1) f_* + \cdots + ([z_{k-1}^{\nu_{k-1} d\alpha_{k-1}} [z_k^{\nu_k - d\alpha_k}]u_d) f_*^d.$$

We repeat this procedure until such a Newton polynomial admits a root of multiplicity d. Of course, this is ultimately the case, since the Newton polynomial in $z_1, \cdots, z_k$ is nothing but the usual Newton polynomial.

9.7 Newton polytopes

Let $\mathfrak{C}$ and $\mathfrak{K}$ be effective totally ordered fields and $\mathfrak{X}$ be an effective monomial group with $\mathfrak{K}$-powers. We assume that $\mathfrak{X}$ is **admissible**, i.e. given $z_1, \cdots, z_k \in \mathfrak{X}$, we can compute a finite system of expo-linear constraints Σ on $z_1^{\mathfrak{K}} \cdots z_k^{\mathfrak{K}}$, which determines precisely the quasi-ordering induced by $\mathfrak{X}$.

Let f be a non zero series in $\mathfrak{C}[[\mathfrak{X}]]$ and M its set of dominant monomials. The convex envelope C (when considering $\mathfrak{X}$ as a vector space over $\mathfrak{K}$) of the final segment generated by M is called the **Newton polytope** associated to f. The intersection $N = C \cap M$ is said to be the **combinatorial Newton polytope** associated to f. If H is a hyperplane with $H \cap C \neq \phi$ and such that C is contained in a halfspace with border H, then $H \cap C$ is said to be a **facet** of the Newton polytope C and $H \cap N$ a facet of the combinatorial Newton polytope.

Proposition 9.13. *Let $\mathfrak{L}$ be an effective Cartesian algebra of Laurent series over $\mathfrak{C}$. There exists an algorithm, which given a Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ for a series f in $\mathfrak{C}[[\mathfrak{X}]]$ computes the combinatorial Newton polytope associated to f.*

Proof. Compute a set G of intermediary dominant monomials for u. Now assume that the ordering on $\mathfrak{X}$ induces on $z_1^{\mathfrak{K}} \cdots z_k^{\mathfrak{K}}$ an ordering which is determined by a set of constraints Σ as above. Let F be a subset of G and $\text{ц} \in F$, such that $\overline{\text{ц}}$ is a dominant monomial of $\overline{u}$. Let Σ' be the set of constraints $\text{ц} \asymp \text{щ}$ for $\text{щ} \in F \backslash \{\text{ц}\}$ and $\text{ц} \not\succ \text{щ}$ for $\text{щ} \in G \backslash F$. Then a subset F represents a facet of the combinatorial Newton polytope associated to u if and only if $\Sigma \cup \Sigma'$ is consistent. The consistency of $\Sigma \cup \Sigma'$ is checked using theorem 8.4. $\square$

Remark 9.9. Let $\mathfrak{Y}$ be a totally ordered monomial groups with $\mathfrak{K}$-powers. Let $\mathfrak{X} \to \mathfrak{C}[[\mathfrak{Y}]]$ be a morphism of multiplicative ordered groups. Such a morphism extends by linearity into a mapping $\mathfrak{C}[[\mathfrak{X}]] \xrightarrow{\varphi} \mathfrak{C}[[\mathfrak{Y}]]$. Then the set F of those ц in M which minimize $\mathrm{M}_{\varphi(\text{ц})}$ is a facet of N. Usually, $\mathrm{M}_{\varphi(f)} = \mathrm{M}_{\text{ц}}$ for all $\text{ц} \in F$: this is called the regular case. The case when $\mathrm{M}_{\varphi(f)} > \mathrm{M}_{\text{ц}}$ for all $\text{ц} \in F$ is called the singular case. See section 10.2 for more details.

In the case when we want to solve systems of equations $f_1 = \cdots = f_n = 0$ with $f_1, \cdots, f_n \in \mathfrak{C}[[\mathfrak{X}]]$, we need to consider the Newton polytopes associated to $f_1, \cdots, f_n$ simultaneously. Then the analogue of a facet of the Newton polytope in the case $n = 1$ is a compatible set of facets of these Newton polytopes. Let us now make this concept more precise.

Let $C_1, \cdots, C_n$ resp. $N_1, \cdots, N_n$ be the Newton polytopes resp. combinatorial Newton polytopes associated to $f_1, \cdots, f_n$. Let $B_1, \cdots, B_n$ be facets of $C_1, \cdots, C_n$ respectively. Considering $\mathfrak{X}$ as an ordered $\mathfrak{K}$-vector space, there exist by definition linear functionals $\varphi_1, \cdots, \varphi_n : \mathfrak{X} \to \mathfrak{K}$, and numbers $c_1, \cdots, c_n \in \mathfrak{K}$ such that $C_i \subseteq \varphi_i^{-1}([c_i, \infty[)$ and $B_i = C_i \cap \varphi_i^{-1}(c_i)$ for all i. If we can choose $\varphi_1 = \cdots = \varphi_n$, then we say that the facets $B_1, \cdots, B_n$ are **compatible**. In this case, the corresponding facets $F_1 = B_1 \cap N_1, \cdots, F_n = B_n \cap N_n$ of the combinatorial Newton polytopes $N_1, \cdots, N_n$ are said to be compatible as well. In the effective context from above, compatible facets of the combinatorial Newton polytopes can be computed by checking the consistency of systems $\Sigma \cup \Sigma_1' \cup \cdots \cup \Sigma_n'$, with obvious notations.

Remark 9.10. Let φ be as in the previous remark and let $M_1, \cdots, M_n$ be the respective sets of dominant monomials of $f_1, \cdots, f_n$. Let F_i be the set of those ц in M_i which minimize $\text{м}_{\varphi(\text{ц})}$ for each i. Then $F_1, \cdots, F_n$ are compatible.

Chapter 10

Multivariate series

10.1 Introduction

Note. For convenience, we exceptionally use the letters x, y, etc. to denote positive infinitesimal parameters in this and the following chapter. In order to avoid confusion with the usual convention, we distinguish between x and x by using a different font.

Let K be a field of characteristic zero. Let C be a totally ordered real algebraically closed field with K-powers. Let G be a non trivial totally ordered monomial group with K-powers. Let $X = \{\mathsf{x}_1, \cdots, \mathsf{x}_p\}$ be a finite set of strictly positive infinitesimal parameters in $C[\![G]\!]^{\downarrow}_{+}$, the set of positive infinitesimal grid-based series in G. We recall that a series $f \in C[\![S_X]\!]$ is **regular**, if f admits a unique dominant monomial (see page 42). Regular series are important because they are the only ones which are invertible in $C[\![S_X]\!]$ by proposition 1.2, whence the only ones which can be composed on the left with other univariate series.

Although series f are not regular in general, they may be regular on certain regions R of $(C[\![G]\!]^{\downarrow}_{+})^k$. For instance, $f = \mathsf{x}_1^2 - \mathsf{x}_2^3$ is not regular, but on the region $R = \{(\mathsf{x}_1, \mathsf{x}_2) \in (C[\![G]\!]^{\downarrow}_{+})^2 | \mathsf{x}_1^2 \prec\!\!\prec \mathsf{x}_2^3\}$, f is a regular series in $C[\![S_X]\!]$. Here $S_X = \mathsf{x}_1^{\mathbb{Z}} \cdots \mathsf{x}_p^{\mathbb{Z}}$ carries the natural[1] ordering determined by the asymptotic relation $\prec\!\!\prec$ on R.

Unfortunately, given an arbitrary series f on a region R, it is not always possible to split up R in a finite number of regions, each on which f is regular: in general, a sequence of changes of coordinates, or refinements, needs to be made, whose inverses are determined by Puiseux series. For instance, in the case of $f = \mathsf{x}_1^2 - \mathsf{x}_2^3$, f is regular on the regions $R = \{(\mathsf{x}_1, \mathsf{x}_2) \in (C[\![G]\!]^{\downarrow}_{+})^2 | \mathsf{x}_1^2 \prec\!\!\prec \mathsf{x}_2^3\}$ and $R = \{(\mathsf{x}_1, \mathsf{x}_2) \in (C[\![G]\!]^{\downarrow}_{+})^2 | \mathsf{x}_1^2 \succ\!\!\succ \mathsf{x}_2^3\}$, but some additional action needs to be undertaken on the region $R = \{(\mathsf{x}_1, \mathsf{x}_2) \in (C[\![G]\!]^{\downarrow}_{+})^2 | \mathsf{x}_1^2 \asymp \mathsf{x}_2^3\}$.

[1] In terms of section 1.3, this means that $\preccurlyeq_R$ is defined by $\mathsf{x}_1^{\alpha_1} \cdots \mathsf{x}_p^{\alpha_p} <_R \mathsf{x}_1^{\beta_1} \cdots \mathsf{x}_p^{\beta_p}$, iff $\overline{\mathsf{x}_1}^{\alpha_1} \cdots \overline{\mathsf{x}_p}^{\alpha_p} \succ\!\!\succ \overline{\mathsf{x}_1}^{\beta_1} \cdots \overline{\mathsf{x}_p}^{\beta_p}$ for all $(\overline{\mathsf{x}_1}, \cdots, \overline{\mathsf{x}_p}) \in R$. Here we remind warning 1.1.

In this chapter, we give an algorithm, which automatically performs the necessary refinements in order to compute the generic dominant monomial of f. In other words, we are able to decompose R in a finite number of regions, each on which f has a well determined asymptotic magnitude. In section 10.2, we introduce coordinate systems and refinements from a theoretical point of view. In section 10.3, we consider the effective counterpart of section 10.2. In section 10.4 we describe a generic generalization of the Newton polygon process. Finally, the algorithm to compute generic dominant monomials is given in section 10.5.

Our algorithm has several applications: it provides a uniform way to compute with multivariate series, but it can also be used to solve systems of asymptotic equations. The algorithm can finally be used to desingularize singularities determined by multivariate series, although this topic will not be studied here.

Let us discuss the relation between the results in this chapter and the classical desingularization problem in algebraic geometry. Geometrically speaking, the main difference between our approach and the more classical napproach is that we do not search for a global non singular projective variety which parameterizes the original singular variety: instead, we cut the singular variety into pieces, such that each piece is non singular. Moreover, the resulting pieces can be described and parameterized effectively. However, the different pieces are not determined uniquely (the way we cut in particular depends on an elimination order on the variables), and our algorithms do not provide information about how the pieces glue together.

In other words, we consider a somewhat easier problem in a less general setting. Nevertheless, our way of describing singularities also has an important advantage: instead of embedding the variety in a higher dimensional one, the dimensions of the pieces are all bounded by the dimension of the original variety. Moreover, our parameterization is more natural in some respects, because each piece is described in a fairly easy manner, while the description of the non singular projective variety may involve several complicated relations.

It is also in order to compare our results to what is known when the Laurent series we consider are polynomials. In that case, Mora's tangent cone algorithm applies (see [Mora 82], resp. [MPT 92]), and this algorithm can be used (see [AMR 89]) to compute standard bases in the sense of Hironaka (see [Hir 64]). In particular, the desingularization process from algebraic geometry can be carried out effectively.

From the complexity point of view, one might expect that the more general types of orderings, which are used in the tangent cone algorithm are more efficient, for a similar reason that Groebner basis computations are usually more efficient than the computation of Ritt-Wu bases. Whether this analogy holds is not clear at present: our main technical ingredient is a generalization of the Newton polygon method, which has no counterpart in the Ritt-Wu method. Furthermore, since our algorithm has not been implemented, no practical evidence is available yet.

10.2 Terminology

Linguistic convention. The following convention will be useful in what follows: the set X is called a set of **variables**. Whenever the values of the x_i are restricted to belong to a region R, then X is called a set of **coordinates** (for R). Hence coordinates are really determined by pairs (X, R), although we will often abusively write X instead of (X, R).

Abstract definition of the function space $\mathcal{F}(R)$. We now describe a function space consisting of formal expressions build up from X, the field operations and taking power series in other infinitesimal expressions. Hidden infinitesimal parameters, which are needed to describe singularities are directly available as expressions in this function space.

Denote by $\mathcal{F}(R, C[\![G]\!])$ the set of mappings from R into $C[\![G]\!]$. This set is an ordered ring, on which we define the asymptotic relations $\prec\!\!\!\prec$, $\preceq\!\!\!\preceq$, $\asymp$, $\prec\!\!\!\prec\!\!\!\prec$ and $\asymp\!\!\!\asymp$ componentwise: for instance, $f \prec\!\!\!\prec g$ if and only if $f(P) \prec\!\!\!\prec g(P)$ for all $P \in R$. There exists a natural mapping ν_X of X into $\mathcal{F}(R, C[\![G]\!])$, which maps x_i to the projection mapping

$$(\hat{x}_1, \cdots, \hat{x}_p) \in R \mapsto \hat{x}_i.$$

Now let $f_1, \cdots, f_p$ be positive infinitesimal elements in $\mathcal{F}(R, C[\![G]\!])$ and let us give $S_{f_1,\cdots,f_p}$ the natural ordering determined by $\prec\!\!\!\prec$. Then we have a natural mapping $\nu_{f_1,\cdots,f_p}$ from $C[\![S_{f_1,\cdots,f_p}]\!]$ into $\mathcal{F}(R, C[\![G]\!])$, which maps

$$\sum_{\alpha_1,\cdots,\alpha_n} c_{\alpha_1,\cdots,\alpha_n} f_1^{\alpha_1} \cdots f_p^{\alpha_n} \in C[\![S_{f_1,\cdots,f_p}]\!]$$

to the mapping

$$P \in R \mapsto \sum_{\alpha_1,\cdots,\alpha_n} c_{\alpha_1,\cdots,\alpha_n} f_1(P)^{\alpha_1} \cdots f_p(P)^{\alpha_n}.$$

We define $\mathcal{F}(R)$ to be the smallest subring of $\mathcal{F}(R, C[\![G]\!])$, which contains the image of ν_X, and such that $\operatorname{im} \nu_{f_1,\cdots,f_p}$ is contained in $\mathcal{F}(R)$, for all $f_1, \cdots, f_p$ in $\mathcal{F}(R)^{\downarrow}_{+} = \{f \in \mathcal{F}(R) | f \prec\!\!\!\prec 1 \wedge f > 0\}$. In particular, we have a natural mapping ν_R of $C[\![S_X]\!]$ into $\mathcal{F}(R)$, where $S_X = x_1^{\mathbb{Z}} \cdots x_p^{\mathbb{Z}}$ carries the natural ordering determined by $\prec\!\!\!\prec$ on R (notice that we should actually write S_X^R instead of S_X). Whenever convenient, we (abusively) identify elements with their images through natural mappings.

Changes of coordinates. Let $X' = \{x'_1, \cdots, x'_{k'}\}$ be a second set of strictly positive infinitesimal variables. Let ξ be a mapping of X' into $\mathcal{F}(R)^{\downarrow}_{+}$. Such a mapping induces a region R_ξ of $(C[\![G]\!]^{\downarrow}_{+})^{k'}$ by

$$R_\xi = \{(\xi(x'_1)(\hat{x}_1, \cdots, \hat{x}_p), \cdots, \xi(x'_{k'})(\hat{x}_1, \cdots, \hat{x}_p)) | (\hat{x}_1, \cdots, \hat{x}_p) \in R\}.$$

Then ξ naturally extends into a mapping from $C[\![S_{X'}]\!]$ into $\mathcal{F}(R)$, and from now on we assume that we have done this. Moreover, there exists a natural mapping $\tilde{\xi}$ such that $\xi = \tilde{\xi} \circ \nu_{R_\xi}$.

If $\tilde{\xi}$ is an isomorphism and if there exists a morphism[2] $\chi : C\llbracket S_X \rrbracket \to C\llbracket S_{X'} \rrbracket$, with $\nu_R = \xi \circ \chi$, then we say that ξ is a **change of variables** relative to R or a **change of coordinates**. Assuming that ξ is such, we have the following commutative diagram:

$$\begin{tikzcd}
C\llbracket S_{X'} \rrbracket \arrow[r, "\nu_{R_\xi}"] \arrow[dr, "\xi"] & \mathcal{F}(R_\xi) \arrow[d, "\tilde{\xi}"] \\
C\llbracket S_X \rrbracket \arrow[u, "\chi"] \arrow[r, "\nu_R"] & \mathcal{F}(R)
\end{tikzcd}$$

We notice that χ preserves regular series. We also remark that coordinate changes can be composed: if $\xi' : C\llbracket S_{X''} \rrbracket \to \mathcal{F}(R_\xi)$ is a second change of coordinates, then $\tilde{\xi} \circ \xi'$ is also a change of coordinates, as illustrates the following diagram:

$$\begin{tikzcd}
C\llbracket S_{X''} \rrbracket \arrow[r, "\nu_{R_{\xi,\xi'}}"] \arrow[dr, "\xi'"] & \mathcal{F}(R_{\xi,\xi'}) \arrow[d, "\tilde{\xi'}"] \\
C\llbracket S_{X'} \rrbracket \arrow[u, "\chi'"] \arrow[r, "\nu_{R_\xi}"] \arrow[dr, "\xi"] & \mathcal{F}(R_\xi) \arrow[d, "\tilde{\xi}"] \\
C\llbracket S_X \rrbracket \arrow[u, "\chi"] \arrow[r, "\nu_R"] & \mathcal{F}(R)
\end{tikzcd}$$

A **restriction** of the coordinate system (X, R) is a coordinate system (X, R') with $R' \subseteq R$. Such a restriction induces a natural morphism $\mathcal{F}(R) \xrightarrow{\nu} \mathcal{F}(R')$. More generally, a **refinement** is a change of coordinates $\xi : C\llbracket S_{X'} \rrbracket \to \mathcal{F}(R')$ with $R' \subseteq R$; the coordinate system (X', R'_ξ) is said to refine (X, R). Again, refinements can be composed: if $\xi' : C\llbracket S_{X''} \rrbracket \to \mathcal{F}(R'')$ is a second refinement, then $\varphi \circ \xi'$ is also a refinement, where φ is the natural isomorphism between $\mathcal{F}(R'')$ and $\mathcal{F}(R''_{\xi^{-1}\circ\nu_{R'}})$. This situation is illustrated by the following diagram:

$$\begin{tikzcd}
 & & C\llbracket S_{X''} \rrbracket \arrow[r, "\nu_{R''_{\xi'}}"] \arrow[dr, "\xi'"] & \mathcal{F}(R''_{\xi'}) \arrow[d, "\tilde{\xi'}"] \\
 & C\llbracket S_{X'} \rrbracket \arrow[ur, "\chi'"] \arrow[r, "\nu_{R'_\xi}"] \arrow[dr, "\xi"] & \mathcal{F}(R'_\xi) \arrow[r] \arrow[d, "\tilde{\xi}"] & \mathcal{F}(R'') \arrow[d, "\varphi"] \\
C\llbracket S_X \rrbracket \arrow[ur, "\chi"] \arrow[r, "\nu_R"] & \mathcal{F}(R) \arrow[r] & \mathcal{F}(R') \arrow[r] & \mathcal{F}(R''_{\xi^{-1}\circ\nu_{R'}})
\end{tikzcd}$$

Let f be a Laurent series in $C\llbracket S_X \rrbracket$. A **desingularization** of f relative to a region $R' \subseteq R$ is a refinement $\xi : C\llbracket S_{X'} \rrbracket \to \mathcal{F}(R')$, such that $\nu_{R'_\xi}$ is injective

[2]By morphism we mean here a morphism of strong C-algebras. I.e. χ is a C-algebra homomorphism which preserves infinite summation.

and $\chi(f)$ is a regular series in $C[\![S_{X'}]\!]$. Here χ is a mapping as mentioned above, which is actually uniquely determined by the requirement that ξ be a desingularization. It can be checked that the composition of a desingularization of f with another desingularization (in the sense of composition of refinements) yields again a desingularization of f.

Example 10.1. Let $f = x_1^2 - x_2^3$. If we have $x_1^2 \prec\!\!\prec x_2^3$ or $x_2^3 \prec\!\!\prec x_1^2$, then we say that we are in the **regular case** and ν_R yields a trivial desingularization of f. In particular, we can expand

$$\frac{1}{f} = -\frac{1}{x_2^3} - \frac{x_1^2}{x_2^6} - \frac{x_1^4}{x_2^9} + \cdots$$

on the region where $x_1^2 \prec\!\!\prec x_2^3$ and

$$\frac{1}{f} = \frac{1}{x_1^2} + \frac{x_2^3}{x_1^4} + \frac{x_2^6}{x_1^6} + \cdots$$

on the region where $x_2^3 \prec\!\!\prec x_1^2$. If $x_1^2 \asymp x_2^3$, we either have $f \asymp x_1^2$ or $f \prec\!\!\prec x_1^2$. In the first case, the **critical case**, we can write $x_2 = x_1^{2/3}(\lambda+\varepsilon)$ with $\varepsilon \prec\!\!\prec 1$ and $0 < \lambda \neq 1$. We then distinguish the three cases $\varepsilon > 0$, $\varepsilon = 0$ and $\varepsilon < 0$, for which the following refinements respectively desingularize f:

$$\begin{cases} x_1 = {x_1'}^3,\ x_2 = {x_1'}^2(\lambda + x_2') & (x_2' \prec\!\!\prec 1); \\ x_1 = {x_1'}^3,\ x_2 = {x_1'}^2(\lambda - x_2') & (x_2' \prec\!\!\prec 1); \\ x_1 = {x_1'}^3,\ x_2 = \lambda {x_1'}^2. \end{cases}$$

They respectively lead to the following expansions of $1/f$:

$$\frac{1}{f} = \begin{cases} \frac{1}{(1-\lambda^3){x_1'}^6} + \frac{3\lambda^2 x_2'}{(1-\lambda^3)^2 {x_1'}^6} + \cdots; \\ \frac{1}{(1-\lambda^3){x_1'}^6} - \frac{3\lambda^2 x_2'}{(1-\lambda^3)^2 {x_1'}^6} + \cdots; \\ \frac{1}{(1-\lambda^3){x_1'}^6}. \end{cases}$$

Finally, let us consider the **singular case** $x_1^2 \asymp x_2^3$ and $f \prec\!\!\prec x_1^2$. Writing $x_1^2 - x_2^3 = x_1^2\varepsilon$, with $\varepsilon \prec\!\!\prec 1$, we have either $\varepsilon > 0$, $\varepsilon = 0$ or $\varepsilon < 0$. These three cases respectively lead to the desingularizations

$$\begin{cases} x_1 = {x_1'}^3,\ x_2 = {x_1'}^2(1 + x_2') & (x_2' \prec\!\!\prec 1); \\ x_1 = {x_1'}^3,\ x_2 = {x_1'}^2(1 - x_2') & (x_2' \prec\!\!\prec 1); \\ x_1 = {x_1'}^3,\ x_2 = {x_1'}^2. \end{cases}$$

of f and to the following expansions of $1/f$:

$$\frac{1}{f} = \begin{cases} -\frac{1}{3{x_1'}^6 x_2'} + \frac{1}{3{x_1'}^6} + \cdots; \\ -\frac{1}{3{x_1'}^6 x_2'} - \frac{1}{3{x_1'}^6} + \cdots; \\ \text{Error}. \end{cases}$$

10.3 Effective refinements

In this section, we describe how the concept of coordinate systems and refinements can be made effective, thereby establishing the framework for the algorithms in the next sections.

10.3.1 Effective assumptions

In the rest of this chapter, we make the following effective assumptions:

A0. $\mathfrak{K}$ is an effective field of characteristic zero.
A1. $\mathfrak{C}$ is an effective totally ordered constant field with $\mathfrak{K}$-powers.
A2. $\breve{\mathfrak{C}} = \mathfrak{C}(\lambda_1, \lambda_2, \cdots)$ is the effective parameterized constant field over $\mathfrak{C}$.
A3. $\mathfrak{L}$ is an effective local community of Laurent series over $\breve{\mathfrak{C}}$.

Let us detail condition **E2**: any element in $\breve{\mathfrak{C}}$ is a rational fraction in a finite number of parameters over $\mathfrak{C}$. In our algorithms, we allow the dynamic imposition of polynomial constraints on these parameters (either equations, inequations or inequalities). The consistency of such systems can be checked by classical algorithms from effective real algebraic geometry (see for instance [Col 75]). In practice, only a finite number of parameters $\lambda_1, \cdots, \lambda_i$ are used at each instant, and new parameters $\lambda_{i+1}, \lambda_{i+2}, \cdots$ are introduced whenever necessary.

10.3.2 The coordinates

Coordinates. The coordinates are determined by couples (X, Σ), where

– $X = \{\mathsf{x}_1, \cdots, \mathsf{x}_p\}$ is a set of formal variables.

– Σ is a set of asymptotic constraints which determines a region R.

As was the normal basis B in the expansion algorithm, our coordinate system is determined dynamically. This means that X and Σ are global variables, which may change during the execution due to refinements. We always work with respect to the current coordinate system, which is determined by the last refinement.

The set of variables X. We assume the existence of an elimination ordering

$$\mathsf{x}_1 >^{elim} \cdots >^{elim} \mathsf{x}_p$$

on the variables. Intuitively speaking, this means that x_1 will be eliminated before x_2, when necessary.

The set of constraints Σ. The set Σ is a consistent set of constraints of one of the following forms:

$$\left\{ \begin{array}{l} x_1^{\alpha_1} \cdots x_p^{\alpha_p} \prec\!\!\prec 1; \\ x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1, \end{array} \right. \tag{10.3}$$

where $\alpha_1, \cdots, \alpha_p \in \mathbb{Z}$. In section 8.4.2, we have shown how the consistency of Σ can be checked by algorithm[3].

Initialization. At initialization, Σ contains the constraints $x_i \prec\!\!\prec 1$ for all $1 \leqslant i \leqslant k$, and optionally some additional constraints.

10.3.3 Cartesian representations

Dependency on coordinates. Let u be a Cartesian representation in $z_1, \cdots, z_k$ of a series f in $\breve{\mathfrak{C}}\llbracket S_X \rrbracket$. We say that u depends on the coordinate x_i, if $\overline{z_j}$ depends on x_i for one of the underlying Cartesian coordinates. Here we notice that this not imply $\overline{u}$ to depend on x_i: consider for example the Cartesian representation $z_1 z_2$ of y, with $\overline{z_1} = xy^{-1}, \overline{z_2} = y$.

Intermediary dominant monomials. For the computation of intermediary dominant monomials, we use the algorithm `idm` from section 9.3.3. We recall that no changes of the Cartesian coordinates may result from the application of this version of `idm`.

10.3.4 Effective refinements

Refinements. In our algorithms, we only consider refinements of the form

R1. $x_q = \overline{\text{ц}}(\overline{v} + x_q')\ (x_q' \prec\!\!\prec 1)$;
R2. $x_q = \overline{\text{ц}}(\overline{v} - x_q')\ (x_q' \prec\!\!\prec 1)$;
R3. $x_q = \overline{\text{ц} v}$.

Here ц is a C-infinitesimal monomial and $v \asymp 1$ a C-regular Cartesian representation, which do not depend on $x_1, \cdots, x_{q-1}$. Notice that we have eliminated x_q in the last case, so that $x_{q+1} = x_q', \cdots, x_p = x_{p-1}'$.

Automatic updating of Cartesian representations. Assume that we perform a sequence of refinements of the above forms. Then the old coordinates can always be expressed effectively in the new ones and vice versa on the current region. Refinements resp. desingularizations with this property are called **effective**. By proposition 9.12, we can rewrite any series in the old coordinates as a series in the new coordinates.

To perform these rewritings automatically when necessary, each Cartesian representation in memory contains a field, specifying its Cartesian coordinates. Each

[3]Indeed, it suffices to consider the x_i as parameters in G; this is equivalent from an asymptotic point of view, since any series in $\mathfrak{C}\llbracket G \rrbracket_{+}^{\downarrow}$ is asymptotic to its dominant monomial.

time we attempt to access a Cartesian representation whose coordinates are not the current ones, then we perform the necessary rewritings. We call this strategy the **automatic updating strategy**.

Automatic updating of Cartesian monomials. Let щ be a Cartesian monomial. Then after a refinement of one of the form **R1**, **R2** or **R3**, or a sequence of such refinements, it is straightforward to compute the dominant monomial м of щ w.r.t. the new coordinates. Indeed, in the case of one refinement it suffices to replace x_q by $\overline{\text{ц}}$ in щ. Using the automatic updating strategy, we assume that automatically replace щ by м whenever necessary.

Notice that we made a small abuse of language: the automatic updating of a Cartesian monomial as a monomial and as a Cartesian representation do not coincide in general. Nevertheless, 'Cartesian monomial' and 'Cartesian representation' should rather be considered as data types, thereby eliminating the risk of confusion.

Automatic updating of Σ. We finally use the automatic updating strategy in order to update the constraints in Σ: each time we perform a refinement, we replace each monomial occurring in Σ by its dominant monomial.

10.3.5 Imposition of constraints

The default way of imposing a constraint of the form (10.3), is to insert it into Σ. However, in case of constraints of the form

$$x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1,$$

the dominant monomial of $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ w.r.t. the new coordinates is not necessarily equal to 1, although is equivalent to 1. For this reason, we sometimes need the following alternative algorithm to impose such constraints:

Algorithm `constraint`($\text{ц} \asymp 1$).

Input: A monomial ц in S_X.

Action: Restricts and cuts the current region into parts, such that the dominant monomial of ц is 1 on each of these parts.

Step 1. Let $\text{ц} = x_q^{\alpha_q} \cdots x_p^{\alpha_p}$, with $\alpha_q \neq 0$ (if $l = k$, then we have nothing to do).
If $\alpha_q < 0$ then set $\alpha_j := -\alpha_j$ for all $j \geqslant q$.
Let щ be a Cartesian monomial with $\overline{\text{щ}} = \sqrt[\alpha_q]{x_{q+1}{}^{\alpha_{q+1}} \cdots x_p{}^{\alpha_p}}$.
Infinitesimalize щ.

Step 2. Introduce a new parameter $\lambda > 0$ in $\breve{\mathfrak{C}}$.
Separate three cases and respectively refine:

$$\begin{cases} x_q = \overline{\text{щ}}(\lambda + x_q') \quad (x_q' \prec\!\!\prec 1); \\ x_q = \overline{\text{щ}}(\lambda - x_q') \quad (x_q' \prec\!\!\prec 1); \\ x_q = \lambda \overline{\text{щ}}. \end{cases} \tag{10.4}$$

10.4 The generic Newton polygon method

The main piece of the algorithm to compute generic dominant monomials, is a suitable generalization of the Newton polygon method. The idea is to consider a series in $\mathsf{x}_1, \cdots, \mathsf{x}_p$ as a series in x_1 with generic coefficients. The main difficulty is that these coefficients are not regular in general, and we need to compute their generic dominant monomials recursively, modulo refinements. Such recursive computations lead to applications of the Newton polygon method on series in $\mathsf{x}_q, \cdots, \mathsf{x}_p$, with $q \geqslant 1$.

The above discussion motivates the following definition: let f be a series in $\mathsf{x}_1, \cdots, \mathsf{x}_p$ and let ч be a monomial in $\mathsf{x}_1, \cdots, \mathsf{x}_{q-1}$ for some $q \geqslant 1$. We say that the coefficient $[\text{ч}]f$ is **Newton prepared**, if $[\text{ч}]f$ is a power series in x_q and the dominant monomials of $[\text{ч}]f$ as a series in $\mathsf{x}_q, \cdots, x_p$ are of the form $\text{щ}(\mathsf{x}_q/\text{ц})^\alpha$ for fixed ц and щ in $\mathsf{x}_{q+1}, \cdots, \mathsf{x}_p$. The corresponding **Newton polynomial** of $[\text{ч}]f$ is defined by

$$P(\lambda) = \sum_{\alpha \in \mathbb{N}} [\text{чщ}(z_l/\text{ц})^\alpha] f \lambda^\alpha.$$

The degree of P is called the **Newton degree** of $[\text{ч}]f$.

10.4.1 The algorithm

For concrete computations, we first need an analogue of Newton prepared series for Cartesian representations. This introduces a technical difficulty, since no Cartesian representation for $[\text{ч}]f$ is available in general (see section 9.4). To overcome this difficulty, we use pseudo-coefficients and we restrict ourselves to the case when f admits a dominant monomial in $\text{ч}S_{\mathsf{x}_q, \cdots, \mathsf{x}_p}$.

More precisely, let u be a Cartesian representation of f in $z_1, \cdots, z_k$. We say that x_q is an **ordinary** variable in u, if u is a power series in z_l, for some l with $\overline{z_l} = \mathsf{x}_q$, and such that $\overline{z_i}$ does not depend on x_q for $i \leqslant l$. Let F be a set of intermediary set of dominant monomials of u. We denote

$$F_{\text{ч}} = \{\text{м} \in F | \overline{\text{м}} \in \text{ч}S_{\mathsf{x}_q, \cdots, \mathsf{x}_p}\}.$$

Finally, let M the subset of F of monomials which represent a dominant monomial of f.

We say that u is **Newton prepared** relative to ч and F, if x_q is ordinary in u, $M \cap F_{\text{ч}} \neq \phi$ and the elements in $M \cap F_{\text{ч}}$ are of the form $\text{м} = \text{щ}(z_l/\text{ц})^\alpha$ for fixed ц and щ with $\overline{\text{ц}} \in S_{\mathsf{x}_{q+1}, \cdots, \mathsf{x}_p}$ and $\overline{\text{щ}} \in \text{ч}S_{\mathsf{x}_{q+1}, \cdots, \mathsf{x}_p}$. In this case, the coefficient $[\text{ч}]f$ is clearly Newton prepared as well, and its Newton polynomial is given by

$$P(\lambda) = \sum_{\text{щ}(z_l/\text{ц})^\alpha \in M \cap F_{\text{ч}}} u_{\text{щ}(z_l/\text{ц})^\alpha} \lambda^\alpha.$$

We can now state the algorithm which performs one step of the Newton polygon method:

Algorithm `Newton_step`$(u, \text{ч}, F)$.

INPUT: A Newton prepared Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ relative to a monomial $\text{ч} \in S_{\mathsf{x}_1,\cdots,\mathsf{x}_{q-1}}$ and a set F of intermediary dominant monomials for u.

ACTION: The algorithm refines $\mathsf{x}_q = \overline{\text{ц}}(\overline{v} \pm \mathsf{x}_q')$ $(\mathsf{x}_q' \prec\!\!\prec 1)$ or eliminates $\mathsf{x}_q = \overline{\text{ц}v}$, where $\overline{\text{ц}v}$ is a first approximation to a solution of $\overline{u}\text{ч} = 0$ in x_q, and $\overline{v} \asymp 1$.

STEP 1. Adopting the notations from above, test whether the Newton polynomial P admits a root of multiplicity $\deg P$. If so, then proceed with step 3.

STEP 2. Introduce the formal parameter $\lambda > 0$ in $\breve{\mathfrak{C}}$.
Impose the constraint $P(\lambda) = 0$.
Infinitesimalize ц.
Separate three cases and respectively refine:

$$\left\{\begin{array}{l} \mathsf{x}_q = \overline{\text{ц}}(\lambda + \mathsf{x}_q') \quad (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = \overline{\text{ц}}(\lambda - \mathsf{x}_q') \quad (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = \lambda\overline{\text{ц}}. \end{array}\right. \tag{10.5}$$

Return.

STEP 3. Compute $w :=$ `pseudo_coefficient` $\left(\dfrac{\partial^{\deg P - 1} u}{\partial z_l^{\deg P - 1}}, \text{ч}\right)$.
Divide w by $[z_l^1]w$.
Regularize and infinitesimalize $[z_l{}^0]w$ and $w - [z_l{}^0]w$.
Compute the unique C-infinitesimal solution $\text{ц}v$ to the equation $w = 0$ in z_l by **LC3**, where $v \asymp 1$.
Regularize v.

STEP 4. Impose the constraint $c > 0$ on the dominant coefficient c of v.
Separate three cases and respectively refine:

$$\left\{\begin{array}{l} \mathsf{x}_q = \overline{\text{ц}}(\overline{v} + \mathsf{x}_q') \quad (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = \overline{\text{ц}}(\overline{v} - \mathsf{x}_q') \quad (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = \overline{\text{ц}v}. \end{array}\right. \tag{10.6}$$

10.4.2 Termination lemmas

Let (X', R') and (X, R) be coordinate systems, such that (X', R') refines (X, R). We say that X' **q-refines** X, $\mathsf{x}_1' = \mathsf{x}_1, \cdots, \mathsf{x}_{q-1}' = \mathsf{x}_{q-1}$ and $\mathsf{x}_q = \boldsymbol{g} \pm \boldsymbol{h}\mathsf{x}_q'$ $(\mathsf{x}_q' \prec\!\!\prec 1)$, where $\boldsymbol{g} \not\succeq \boldsymbol{h}$ are regular infinitesimal series in $\mathsf{x}_{q+1}, \cdots, \mathsf{x}_p$.

For termination purposes, we now need to find suitable analogues for lemma 3.2 and lemma 3.3 from section 3.3. These analogues will establish that in a sequence of q-refinements, resulting from repeated applications of `Newton_step`, the successive Newton degrees in x_q decrease by one at most every two steps. Truly, it happens that this phenomenon is independent of the first $q-1$ exponents in the corresponding dominant monomials.

Lemma 10.1. *Let f be a series in $x_1, \cdots, x_q$ and assume that X' is a coordinate system which q-refines X via $x_q = g \pm h x'_q$ ($x'_q \prec\!\!\prec 1$). Assume that*

1. $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ *and* $x_1'^{\alpha'_1} \cdots x_{p'}'^{\alpha'_{k'}}$ *are dominant monomials of f w.r.t. X resp. X'.*
2. $[x_1^{\alpha_1} \cdots x_{q-1}^{\alpha_{q-1}}]f$ *is Newton prepared, of Newton degree α_q.*

Then $\alpha'_q \leqslant \alpha_q$.

Proof. We prove the lemma in the case when $x_q = g + h x'_q$; the case $x_q = g - h x'_q$ is treated similarly. Denote $\text{ч} = x_1^{\alpha_1} \cdots x_{q-1}^{\alpha_{q-1}}$ and $\text{ч}' = x_1'^{\alpha'_1} \cdots x_{q-1}'^{\alpha'_{q-1}}$. Let P be the Newton polynomial associated to $\mathrm{M}(g)$ and relative to $[\text{ч}]f = 0$, and let ν denote the multiplicity of c_g as a root of P. We have

$$[\text{ч} x_q'^{\nu}]f = \sum_{\gamma \geqslant 0} \binom{\nu+\gamma}{\gamma} ([\text{ч} x_q^{\nu+\gamma}]f)\, g^{\gamma} h^{\nu},$$

by expanding $x_q = g + h x'_q$ and

$$[\text{ч} x_q'^{\nu}]f = \mathrm{M}(([\text{ч} x_q^{\alpha_q}]f)\, g^{\alpha_q - \nu}) h^{\nu} (\tfrac{1}{\nu!} P^{(\nu)}(\lambda) + o(1)),$$

by splitting the dominant part from the rest. Since $g \sim x_q$ on R', we obtain

$$[\text{ч} x_q'^{\nu}]f \asymp ([\text{ч} x_q^{\alpha_q}]f)\, x_q^{\alpha_q - \nu} h^{\nu}. \tag{10.7}$$

More generally, we have on R'

$$[\text{ч}' x_q'^{\alpha'_q}]f = \sum_{\gamma \geqslant 0} \binom{\alpha'_q + \gamma}{\gamma} ([\text{ч}' x_q^{\alpha'_q + \gamma}]f)\, g^{\gamma} h^{\alpha'_q}.$$

Since $[\text{ч} x_q^{\alpha_q}]f$ is regular, we also have

$$([\text{ч} x_q^{\alpha_q}]f)\, \text{ч} x_q^{\alpha_q} \asymp x_1^{\alpha_1} \cdots x_p^{\alpha_p}.$$

Since $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ is a dominant monomial of f, we therefore obtain

$$([\text{ч}' x_q^{\alpha'_q + \gamma}]f)\, \text{ч}' x_q^{\alpha'_q + \gamma} \preccurlyeq\!\!\preccurlyeq ([\text{ч} x_q^{\alpha_q}]f)\, \text{ч} x_q^{\alpha_q},$$

for all $\gamma \geqslant 0$. Hence,

$$\begin{array}{rcl} ([\text{ч}' x_q'^{\alpha'_q}]f)\, \text{ч}' x_q'^{\alpha'_q} & \preccurlyeq\!\!\preccurlyeq & ([\text{ч} x_q^{\alpha_q}]f)\, \text{ч} x_q^{\alpha_q - \alpha'_q} x_q'^{\alpha'_q} h^{\alpha'_q} \\ & \asymp & ([\text{ч} x_q'^{\nu}]f)\, \text{ч} x_q'^{\nu} \left(\frac{h x'_q}{x_q}\right)^{\alpha'_q - \nu} \\ & \preccurlyeq\!\!\preccurlyeq & ([\text{ч}' x_q'^{\alpha'_q}]f)\, \text{ч}' x_q'^{\alpha'_q} \left(\frac{h x'_q}{x_q}\right)^{\alpha'_q - \nu}, \end{array} \tag{10.8}$$

using (10.7) and the fact that $x_1'^{\alpha_1'} \cdots x_p'^{\alpha_p'}$ is a dominant monomial of f. We conclude that $\alpha_q' \leqslant \nu \leqslant \alpha_q$, since $h x_q' \prec\!\!\prec x_q$. $\square$

Let us now establish a generalization of lemma 3.3, which says that if one step of the Newton polygon method does not suffice to decrease the Newton degree, then two steps do. We start by the ideal case, in which we know how to extract the coefficient $[\text{ч}]f$ of f.

Lemma 10.2. *With the notations and assumptions from lemma 10.1, let X'' be a coordinate system which q-refines X'. Assume that*

1. *$\alpha_q' = \alpha_q$ and g is the unique solution to the equation in x_q:*

$$\langle x_1^{\alpha_1} \cdots x_{q-1}^{\alpha_{q-1}} \rangle \frac{\partial^{\alpha_q - 1} g}{\partial x_q^{\alpha_q - 1}} = 0.$$

2. *$x_1''^{\alpha_1''} \cdots x_{p''}''^{\alpha_{k''}''}$ is a dominant monomial of f w.r.t. X''.*
3. *$[x_1'^{\alpha_1'} \cdots x_{q-1}'^{\alpha_{q-1}'}]f$ is Newton prepared.*

Then $\alpha_q'' < \alpha_q$.

Proof. It suffices to consider the case when $x_q = g + h x_q'$ and $x_q' = g' + h' x_q''$; the other three cases are treated similarly. Assume that $\alpha_q'' = \alpha_q$ for contradiction and let

$$l = \frac{\partial^{\alpha_q - 1} g}{\partial x_q^{\alpha_q - 1}}.$$

We first treat the "ideal case", when $\langle \text{ч} \rangle l = [\text{ч}] l$. In this case, we have $[\text{ч} x_q'^{\alpha_q - 1}] f = 0$. It follows that on R'':

$$[\text{ч} x_q''^{\alpha_q - 1}] f \asymp ([\text{ч} x_q'^{\alpha_q}] f)\, g' h'^{\alpha_q - 1}. \tag{10.9}$$

Since $x_1'^{\alpha_1'} \cdots x_p'^{\alpha_p'}$ is a dominant monomial of f and $\alpha_q' = \nu = \alpha_q$, (10.8) becomes

$$\begin{aligned} ([\text{ч}' x_q'^{\alpha_q}] f)\, \text{ч}' x_q'^{\alpha_q} &\asymp ([\text{ч} x_q^{\alpha_q}] f)\, \text{ч} x_q'^{\alpha_q} h^{\alpha_q} \\ &\asymp ([\text{ч} x_q'^{\alpha_q}] f)\, \text{ч} x_q'^{\alpha_q}. \end{aligned} \tag{10.10}$$

Similarly (and with obvious notations), we have

$$\begin{aligned} ([\text{ч}'' x_q''^{\alpha_q}] f)\, \text{ч}'' x_q''^{\alpha_q} &\asymp ([\text{ч}' x_q'^{\alpha_q}] f)\, \text{ч}' x_q''^{\alpha_q} h'^{\alpha_q} \\ &\asymp ([\text{ч}' x_q''^{\alpha_q}] f)\, \text{ч}' x_q''^{\alpha_q}. \end{aligned} \tag{10.11}$$

for the second q-refinement. Putting things together, we obtain

$$\begin{aligned} ([\text{ч}'' x_q''^{\alpha_q}] f)\, \text{ч}'' x_q''^{\alpha_q} &\asymp ([\text{ч}' x_q'^{\alpha_q}] f)\, \text{ч}' x_q''^{\alpha_q} h'^{\alpha_q} && \text{(by (10.11))} \\ &\asymp ([\text{ч} x_q'^{\alpha_q}] f)\, \text{ч} x_q''^{\alpha_q} h'^{\alpha_q} && \text{(by (10.10))} \\ &\asymp ([\text{ч} x_q''^{\alpha_q - 1}] f)\, \text{ч} x_q''^{\alpha_q - 1} \left(\frac{x_q'' h'}{g'}\right) && \text{(by (10.9))} \\ &\prec\!\!\prec ([\text{ч} x_q''^{\alpha_q - 1}] f)\, \text{ч} x_q''^{\alpha_q - 1} \end{aligned}$$

This yields the desired contradiction.

Let us now consider the other case, when $\langle \text{ч} \rangle l \neq [\text{ч}] l$. Let $\cdot \circ_{x_q} g$ denote the substitution of x_q by g. Since $(\langle \text{ч} \rangle l) \circ_{x_q} g = 0$, we have

$$l \circ_{x_q} g = (\text{ч}(l/\text{ч} - \langle \text{ч} \rangle l)) \circ_{x_q} g.$$

By proposition 9.9, each dominant monomial of $l/\text{ч} - \langle \text{ч} \rangle l$ depends on $x_1, \cdots, x_{q-2}$ or x_{q-1}. Since

$$\begin{aligned} [x_q'^{\alpha_q - 1}] f &= (l \circ_{x_q} g) h^{\alpha_q - 1} \\ &= ((l/\text{ч} - \langle \text{ч} \rangle l) \circ_{x_q} g) \text{ч} h^{\alpha_q - 1}, \end{aligned}$$

we infer:

$$\text{Each dominant monomial of } \text{ч}^{-1} [x_q'^{\alpha_q - 1}] f \text{ depends on } x_1, \cdots, x_{q-2} \text{ or } x_{q-1}. \tag{10.12}$$

Now expand

$$[x_q''^{\alpha_q - 1}] f = ([x_q'^{\alpha_q - 1}] f + (\alpha_q - 1)([x_q'^{\alpha_q}] f) \, g' + \mathrm{Rest}) h'^{\alpha_q - 1}.$$

Since $\mathrm{M}([\text{ч} x_q'^{\alpha_q}] f) \text{ч} x_q'^{\alpha_q}$ is a dominant monomial of f on R' by (10.10), each dominant monomial of $[x_q'^{\alpha_q}] f$ is bounded by $([\text{ч} x_q'^{\alpha_q}] f) \, \text{ч}$ for $\preccurlyeq$, and $\mathrm{Rest} \prec\!\!\prec ([\text{ч} x_q'^{\alpha_q}] f) \, \text{ч} g'$. We claim that any dominant monomial M of $[x_q''^{\alpha_q - 1}] f$ satisfies

$$\mathrm{M} \succcurlyeq ([\text{ч} x_q'^{\alpha_q}] f) \, \text{ч} g' h'^{\alpha_q - 1}.$$

This follows from (10.12), if each dominant monomial of $[x_q'^{\alpha_q - 1}] f$ is bounded by $([\text{ч} x_q'^{\alpha_q}] f) \, \text{ч} g'$ for $\preccurlyeq$, and is trivial in the other case. From our claim we deduce in a similar way as before that

$$([\text{ч}'' x_q''^{\alpha_q}] f) \, \text{ч}'' x_q''^{\alpha_q} \prec\!\!\prec \mathrm{M} x_q''^{\alpha_q - 1}.$$

But this contradicts the fact that $x_1''^{\alpha_1''} \cdots x_p''^{\alpha_p''}$ is a dominant monomial of f on R''. $\square$

10.5 Computation of generic dominant monomials

10.5.1 The algorithm `dom_mon`

In this section, we present the algorithm `dom_mon`, which computes generic dominant monomials of a series f, whose variables are infinitesimal parameters in some non trivial totally ordered algebra of grid-based series with $\mathfrak{K}$-powers. Hence, the algorithm splits up the region on which f in a finite number of regions, each on which f is regular, and on which the corresponding dominant monomial of f is computed.

Algorithm `dom_mon`(u).
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ of a series f in $x_1, \cdots, x_p$.
OUTPUT: The generic dominant monomial of f.
By convention, we return 0 for regions on which $f = 0$.

STEP 1. Repeat the following until м $\neq$ 'Recommence':
A. Compute a set M of intermediary dominant monomials of u on R by `idm`.
B. If $M = \phi$ then return 0.
C. Separate a case for each facet $F \subseteq M$, and do the following:
Select an arbitrary ц $\in F$.
Impose the constraint $\overline{щ} \prec\!\!\prec \overline{ц}$ for each щ $\in M \backslash F$.
Impose the constraint $\overline{щ} \asymp \overline{ц}$ for each щ $\in F \backslash \{ц\}$.
Set м := `dom_sub`$(u, 1, F)$.
STEP 2. Return м.

Let us shortly explain the algorithm: we repeatedly refine the coordinates until f is regular. To compute the refinements, we select a combinatorial Newton polytope, whose vertices are maximal for $\preccurlyeq\!\!\!\prec$ among the dominant monomials, and then apply a subalgorithm `dom_sub`. This subalgorithm either directly returns the dominant monomial of f, or, in the case when more refinements are necessary, it returns the symbolic value 'Recommence'.

Subalgorithm `dom_sub`$(u, ч, F)$.
INPUT: A Cartesian representation $u \in \mathfrak{L}_{z_1,\cdots,z_k}$ of a series f in $x_1, \cdots, x_p$.
A monomial ч in $S_{x_1,\cdots,x_{q-1}}$.
A set of monomials F, such that $F_ч \neq \phi$.
OUTPUT: Either the dominant monomial м of $[ч]f$ or 'Recommence'.
In the first case, $\overline{F_ч} = \{м/ч\}$ at the end of the algorithm.

STEP 1. If x_q is ordinary in u, then separate the non singular from the singular case and respectively proceed with step 3 or step 4.
STEP 2. If $\overline{ц/щ}$ depends on x_q for some ц, щ $\in F_ч$, then fix ц $\in F_ч$, execute `constraint`$(\overline{щ} \asymp \overline{ц})$ for all щ $\in F_ч \backslash \{ц\}$ and return 'Recommence'.
Otherwise, let x_q^α be the unique monomial such that $F_{ч x_q^\alpha} \neq 0$, and return `dom_sub`$(u, ч x_q^\alpha, F) x_q^\alpha$ (or 'Recommence', if `dom_sub` returns 'Recommence').
STEP 3. Let ц be an arbitrary element in $F_ч$.
Execute `constraint`$(\overline{щ} \asymp \overline{ц})$ for all щ $\in F_ч \backslash \{ц\}$.
Let м be the unique element of $F_ч$ and impose the constraint $[м]u \neq 0$.
Return $\overline{м}$.

STEP 4. For each α such that $F_{\text{ч}\mathsf{x}_q^\alpha} \neq \phi$, execute `dom_sub`$(u, \text{ч}\mathsf{x}_q^\alpha, F)$.
Let n be the number of times that `dom_sub` does not return 'Recommence'.
If $n = 0$, then return 'Recommence'.
If $n = 1$, then kill the current process.
Otherwise, choose $\text{ц}, \text{ц}' \in F_{\text{ч}}$, with $\overline{\text{ц}} = \text{ч}\mathsf{x}_q^\alpha \text{щ}$ and $\overline{\text{ц}'} = \text{ч}\mathsf{x}_q^{\alpha'} \text{щ}'$, such that $\alpha' \neq \alpha$. Next, execute `constraint`$(\text{щ}^{\alpha''-\alpha'} \text{щ}'^{\alpha-\alpha''} \text{щ}''^{\alpha'-\alpha} \asymp 1)$ for each $\text{ц}'' \in F_{\text{ч}}$, with $\overline{\text{ц}} = \text{ч}\mathsf{x}_q^{\alpha''} \text{щ}''$.

STEP 5. Execute `Newton_step`$(u, \text{ч}, F)$ and return 'Recommence'.

Let us again detail the computations. Steps 1 and 2 reduce the general case to the case when x_q is ordinary in u. Then we distinguish the non singular case from the singular one: the **non singular** case is when the monomials in the selected combinatorial Newton polytope are equivalent to f in the refined coordinates; this case corresponds to step 3 in the algorithm and directly yields the desired dominant monomial of f. The remaining, **singular case** corresponds to steps 4 and 5 in the algorithm and is essentially treated by the algorithm `Newton_step` described in the previous section. Step 4 serves to Newton prepare u.

10.5.2 Termination proof of `dom_mon`

In this section we prove the termination of `dom_mon`; the correctness is easily verified step by step, by checking the specification of each subalgorithm.

Theorem 10.1. *Let $X = \{\mathsf{x}_1, \cdots, \mathsf{x}_p\}$ a coordinate system, whose underlying region is determined by a finite set of constraints of the form $\mathsf{x}_1^{\alpha_1} \cdots \mathsf{x}_p^{\alpha_p} \prec\!\!\prec 1$. Then there exists an algorithm which takes a Laurent series $u \in \mathfrak{L}_{\mathsf{x}_1,\cdots,\mathsf{x}_p}$ on input and computes a generic dominant monomial of it via a suitable refinement.*

Proof. Assume that `dom_mon` does not terminate on some input u. Let q be minimal such that x_q is refined infinitely often. Replacing u by its value after a large number of iterations of step 2 of `dom_mon`, we may assume without loss of generality that $\mathsf{x}_1, \cdots, \mathsf{x}_{q-1}$ are constant during the execution. Moreover, modulo one refinement of x_q in step 2 of `dom_sub`, we may also assume that $\overline{z_l} = \mathsf{x}_q$ is ordinary in u throughout the execution.

Lemma 10.3. *Ultimately, all refinements of x_q exclusively occur in* `Newton_step`.

Proof. We first observe that only the subalgorithm `dom_sub` may lead to refinements during the execution of `dom_mon`, since `idm` does not affect the system of Cartesian coordinates. Now each call of `dom_sub`, for which the present q and the q in the algorithm correspond, falls into the singular case: otherwise, either one of the coordinates $x_1, \cdots, x_{q-1}$ would change or the algorithm would terminate. By what has been said above this lemma, x_q does not change in step 2 of `dom_sub` either. Consequently, during the execution of the algorithm, x_q can only be refined in `Newton_step`. □

By lemma 10.3, we may assume without loss of generality, that all refinements of x_q occur in `Newton_step`. By lemma 10.1 and lemma 10.2, an infinite sequence of such refinements does not exist. □

10.6 Comments and extensions

Case separations. In practice, it is necessary to limit as much as possible the number of case separations. In particular, instead of splitting up into three processes in (10.4),(10.5) and (10.6), the sign determinations can be postponed until we need them: only in order to compute negative powers of monomials, we should check them for being zero, and only in order to compute fractional powers, we should check them for positivity. In example 10.1 this leads to the distinction of only 5 cases instead of 8.

Algebraically closed constant fields. For convenience, we have limited ourselves to the case of a real algebraically closed constant field. Actually, the algorithms can be adapted to the case when $\mathfrak{C}$ is an algebraically closed field: in this case, $x_1, \cdots, x_p$ are not required to be positive, but we assume the existence of canonical i-th roots in $\mathfrak{C}$ for all i.

Solving implicit equations. The algorithm can also be used to solve the equation $f = 0$ in x_1. Indeed, it suffices to choose the branches of the computation tree in which x_1 is eliminated from the equation $f = 0$. It is also possible to solve a system of equations $f_1, \cdots, f_n = 0$ in $x_1, \cdots, x_n$, by successively eliminating x_1 from $f_1 = 0$, x_2 from $f_2 = 0$, etc. However, our algorithm can be optimized in order to eliminate $x_1, \cdots, x_n$ simultaneously in this latter case. This is interesting because it strongly reduces the number of cases to be separated during the execution.

The idea behind simultaneous elimination of variables is to replace `dom_mon` by a routine which simultaneously computes the generic dominant monomials of $u_1, \cdots, u_n$. In the main loop, we now consider tuples $(F_1, \cdots, F_n)$ of compatible facets of the combinatorial Newton polytopes associated to $u_1, \cdots, u_n$. The subalgorithm `dom_sub` is applied to each $(u_i, 1, F_i)$ in the same way as before.

Normalization of Σ. The presence of constraints of the form $x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1$ is quite uncommon, since the resulting scales S_X are only quasi-ordered. We say that Σ is a **normal** system of constraints, if it contains only constraints of the form $x_1^{\alpha_1} \cdots x_p^{\alpha_p} \prec\!\!\prec 1$. An arbitrary system Σ of constraints of the form (10.3) can be normalized by executing $\texttt{constraint}(x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1)$ for each $x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1$ in Σ, starting with those constraints which depend on x_1, next those on x_2, and so on.

It can be shown that `dom_mon` followed by the above normalization of Σ actually computes a generic desingularization of f. Since this fact will not be needed in what follows, we will not prove it here.

10.7 References

[AMR 89] M.E. Alonso, T. Mora, M. Raimondo. Computing with algebraic series. *Proc. of ISSAC '89, A.C.M. (p. 101-111).*

[Col 75] G.E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Proc. 2-nd conf. on automata theory and formal languages, Springer Lect. Notes in Comp. Sc. 33 (p. 134-183).*

[Hir 64] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Annals of mathematics 79 (p. 109-326).*

[MMV 94] A. Macintyre, D. Marker, L. van den Dries. The elementary theory of restricted analyic fields with exponentiation. *Annals of Math. 140 (p. 183-205).*

[Mora 82] T. Mora. An algorithm to compute the equations of tangent cones. *Proc. of EUROCAL '83, Lect. Notes in Comp. Science 144 (p. 158-165).*

[MPT 92] T. Mora, G. Pfister, C. Traverso. An introduction to the tangent cone algorithm. *Advances in Comp. Research, Vol. 6 (p. 199-270).*

[Pél 95] A. Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948.*

[Tar 31] A. Tarski. *Sur les ensembles définissables de nombres réels.* Fund. Math. 17 (p. 210-239).

Chapter 11

Multivariate transseries

11.1 Introduction

In this chapter, we generalize the generic expansion algorithm from chapter 10 to an expansion algorithm for multivariate transseries. The hard core of the algorithm is the same as in the previous section, and consists of the algorithms `dom_mon`, `dom_sub` and `idm`. However, to handle exponentials, many additional problems arise. We will now give a brief overview of our approach to handle these.

The theoretical framework. In section 11.2 we define multivariate transseries. Such transseries were already introduced in chapter 6, but we will recall a more restricted setting which is sufficient for our applications. Consequently, all what follows can be read independently from chapter 6, although the general theory may provide better insight. We also define refinements and desingularizations of transseries, without searching for maximal generality.

The effective framework. In section 11.3, we describe the effective framework which we use in this chapter. We introduce normal bases, as being the lexicographical multivariate counterpart of the previously defined normal bases: let $\mathsf{x}_1, \cdots, \mathsf{x}_p$ be the coordinates. Then a normal basis B is a disjoint reunion $B = B_1 \amalg \cdots \amalg B_p$, where each B_q is the set of basis elements which depend on x_q, but not on $\mathsf{x}_1, \cdots, \mathsf{x}_{q-1}$. Each B_q corresponds to a normal basis in the old sense.

Normal basis can be well-quasi-ordered, by ordering first on the number of elements in B_1, then the number of elements in B_2, etc. This quasi-ordering underlies most of the termination proofs in this chapter. The main difficulty we will encounter is to avoid as much as possible the insertion of new logarithms into B. Nevertheless, in view of the above quasi-ordering, the elimination of a single element in B_q compensates the insertion of any number of logarithms and/or exponentials into $B_{q+1} \amalg \cdots \amalg B_p$.

In section 11.3.3, we define admissible Cartesian representations as being Cartesian representations so that of the classes of its underlying coordinates are ultra-regular. Roughly speaking, this means that the comparability classes of the coordinates are well determined. For technical reasons, we will exclusively work with admissible Cartesian representations in this chapter.

In section 11.3 we also add two new types of refinements to the ones we considered in chapter 10: upward shiftings $x_q = \exp^{-1} x_q'^{-1}$ and split-offs $x_q = \overline{\text{ц}} x_q'^{\pm 1}$ ($x_q' \lll x_q$). We finally introduce the concept of exponential rewritings, which is the analogue of refinements for exponential basis elements in B.

The main algorithms. The sections 11.4, 11.5, 11.6 and 11.7 are highly interdependent, although we have tried to keep them as understandable as possible when read linearly.

In section 11.4, we give the expansion algorithm for multivariate transseries; actually, the algorithm can easily be derived from the one on page 212 when consistently applying the automatic case separation strategy.

Since we require all Cartesian representations to be admissible, the infinitesimalization and regularization algorithms from chapter 9 can not be used, without showing how new Cartesian coordinates can be introduced while preserving admissibility. In section 11.5, we present the algorithm `ultra_regularize` for this purpose.

Section 11.6 is devoted to the analogue of the algorithm `constraint` from section 10.3.5, for the imposition of asymptotic constraints of the form $\text{ц} \asymp 1$. The treatment is far more complicated than the one from chapter 10, because we need to avoid as much as possible the insertion of logarithms into the normal basis.

Finally, in section 11.7 we give the overall termination proof of our algorithms, the algorithms `Newton_step`, `dom_mon` and `dom_sub` being identical to those from chapter 10 (with some obvious changes).

Complements. The last two sections are complements.

Our algorithms are based on automatic case separation using a partial constraint checker. In section 11.8, we provide a complete constraint checker, modulo an oracle to decide the consistency of finite exp-log systems over the constants.

Finally, in section 11.9 we give an application of the main theorem of this section to classical analysis. We show that the field of convergent transseries is a Hardy field which is stable under the resolution of consistent zero-dimensional systems of exp-log equations. In particular, this field is stable under functional composition and inversion. As an application, the functional inverse of $\log x \log_2 x$ admits a convergent transseries expansion (see also section 1.7.4).

11.2 Multivariate transseries and normal sets

Let us fix a totally ordered exp-log constant field C and a finite set of transseries parameters $X = \{x_1, \cdots, x_p\}$. We denote $\mathbb{T} = C[[[t]]]$. Let R be a region of $\mathbb{T}^p$.

Abstract definition of the function space $\mathcal{F}(R)$. Let $\mathcal{F}(R, \mathbb{T})$ be the set of mappings $R \to \mathbb{T}; P \mapsto f(P)$. Then $\mathcal{F}(R, \mathbb{T})$ has the componentwise structure of an ordered partial exp-log ring: the logarithm $\log f$ of $f \in \mathcal{F}(R, \mathbb{T})$ is defined if and only if $\log f(P)$ is defined for all $P \in R$. As in section 10.2, the asymptotic relations $\prec, \preceq, \asymp, \prec\!\!\prec, \succcurlyeq$ are naturally defined on $\mathcal{F}(R, \mathbb{T})$. We also have a natural mapping ν_X of X into $\mathcal{F}(R, \mathbb{T})$, and if $f_1, \cdots, f_n$ are positive infinitesimal elements in $\mathcal{F}(R, \mathbb{T})$ then we have a natural mapping $\nu_{f_1, \cdots, f_n}$ of $C[\![S_{f_1, \cdots, f_n}]\!]$ into $\mathcal{F}(R, \mathbb{T})$, where $S_{f_1, \cdots, f_n} = f_1^C \cdots f_n^C$. We define $\mathcal{F}(R)$ to be the smallest partial exp-log subring of $\mathcal{F}(R, C[\![X]\!])$, which contains the image of ν_X, and such that $\operatorname{im} \nu_{f_1, \cdots, f_p}$ is contained in $\mathcal{F}(R)$, for all $f_1, \cdots, f_p$ in $\mathcal{F}(R)$. We also say that $\mathcal{F}(R)$ is **generated** by $\operatorname{im} \nu_X$. Whenever convenient, we abusively identify elements with their images through natural mappings.

Normal sets. In order to generalize refinements, we need to generalize normal bases. However, for technical reasons, it is more convenient to work with a slightly weaker concept at this point, namely the concept of normal sets. Let B be a finite set of positive infinitesimal elements in $\mathcal{F}(R)$, which contains $X = \{x_1, \cdots, x_p\}$. We say that B is a **normal set** relative to R and some total elimination ordering $<^{elim}$ on B, if the following conditions are satisfied:

NS1. $X \subseteq B$.

NS2. The logarithm of each element $\mathfrak{b} \in B \backslash X$ is a regular transseries in $C[\![S_{<^{elim}\mathfrak{b}}]\!]$, where

$$S_{<^{elim}\mathfrak{b}} = \{(\mathfrak{b}'_1)^{c_1} \cdots (\mathfrak{b}'_k)^{c_k} | \mathfrak{b}'_1, \cdots, \mathfrak{b}'_k \prec \mathfrak{b} \ \wedge \ c_1, \cdots, c_k \in C\}$$

is given the natural[1]ordering determined by $\prec\!\!\prec$.

Usually, B satisfies some additional conditions; see for instance the next section. If B is a normal set, then we denote by $S_B = B^C$ the multiplicative group with C-powers generated by B.

Refinements. Assume that B is a normal set relative to R and let ξ be a mapping of a set $X' = \{x'_1, \cdots, x'_{p'}\}$ into $\mathcal{F}(R)$. Then ξ naturally determines a region of $(\mathbb{T}^+_\infty)^{p'}$:

$$R_\xi = \{(\xi(x'_1)(\hat{x}_1, \cdots, \hat{x}_p), \cdots, \xi(x'_{p'})(\hat{x}_1, \cdots, \hat{x}_p)) | (\hat{x}_1, \cdots, \hat{x}_p) \in R\}.$$

Given a normal set B' relative to R_ξ, ξ can naturally be extended into a mapping from $C[\![S_{B'}]\!]$ into $\mathcal{F}(R)$ and from now on we assume that we have done this. Moreover, there exists a natural mapping $\mathcal{F}(R_\xi) \xrightarrow{\tilde{\xi}} \mathcal{F}(R)$ with $\xi = \tilde{\xi} \circ \nu_{R_\xi}$. If $\tilde{\xi}$ is bijective and there exists a morphism[2] $\chi : C[\![S_B]\!] \to C[\![S_{B'}]\!]$ with $\nu_R = \xi \circ \chi$, then

[1] In view of section 1.3, this ordering is the opposite ordering induced by $\prec\!\!\prec$. Here we remind warning 1.1.

[2] By morphism we mean here a morphism of strong exp-log C-algebras. I.e. χ preserves the exp-log C-algebra structure as well as infinite summation.

we say that ξ is a **change of coordinates**. Refinements and desingularizations are defined in a similar way as in section 10.2 and they have similar properties as before.

11.3 The effective framework

11.3.1 Basic assumptions

In the rest of this chapter, we make the following effective assumptions:

A1. $\mathfrak{C}$ is an effective totally ordered constant field, contained in some totally ordered exp-log field.

A2. $\breve{\mathfrak{C}}$ is the effective parameterized exp-log constant field over $\mathfrak{C}$.

A3. $\mathfrak{L}$ is an effective exp-log local community of Laurent series over $\breve{\mathfrak{C}}$.

Let us detail condition **A2**: any element in $\breve{\mathfrak{C}}$ is an exp-log expression in a finite number of parameters over $\mathfrak{C}$. As in section 10.3, we allow the dynamic imposition of polynomial constraints on these parameters (either equations, inequations or inequalities). However, we only check the real algebraic consistency of such systems, although we will sometimes assume the existence of an oracle to check the exp-log consistency.

We will denote by $\mathfrak{L}$ the $\breve{\mathfrak{C}}$-algebra of Laurent series associated to $\mathfrak{C}$. Furthermore, $\mathfrak{T}$ denotes the set of parameterized transseries over $\mathfrak{L}$; i.e. $\mathfrak{T}$ is the smallest set of expressions, which contains a countable set of transseries parameters, which is stable under the exp-log field operations, and which contains $\mathfrak{L}_{g_1,\cdots,g_i}$ for any $g_1, \cdots, g_i \in \mathfrak{T}$. In particular, $\mathfrak{T}$ contains the set of all exp-log expressions in the countable set of parameters. Notice also that expressions in $\mathfrak{T}$ may very well be defined nowhere (example: $\log x + \log(-x)$); nevertheless, we will be able to detect this modulo an oracle for checking the exp-log consistency of exp-log systems over $\mathfrak{C}$.

11.3.2 The coordinates

Coordinates. From an effective point of view, the coordinates are determined by triples (X, Σ, B), where

– $X = \{\mathsf{x}_1, \cdots, \mathsf{x}_p\}$ is a set of formal variables.

– Σ is a system of asymptotic constraints on X, which determines a region R.

– $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ is a normal basis of positive infinitesimal multivariate transseries on R.

In all our algorithms, X, B and Σ are all global variables.

The set of variables X. As in chapter 10, we assume the existence of an elimination ordering on X:

$$x_1 >^{elim} \cdots >^{elim} x_p.$$

Moreover, X is required to be a subset of the countable set of transseries parameters mentioned in section 11.3.1.

The normal basis B. At this point, it suffices to assume that B is a normal set relative to a suitable elimination ordering which extends the elimination ordering on X. Actually, we assume that B is an effective normal basis, but this assumption will only be detailed in the next section.

Roughly speaking, if B_q denotes the set of elements in B which depend on x_q, but not on $x_1, \cdots, x_{q-1}$, then we require that B_q is a normal basis of level 0 for each q. As a consequence, we have a natural extension of the elimination ordering on X: if for each q we write

$$B_q = \{\mathfrak{b}_{q,1}, \cdots, \mathfrak{b}_{q,n_q}\},$$

with $x_q = \mathfrak{b}_{q,1} \lll \cdots \lll \mathfrak{b}_{q,n_q}$, then we order the elements in B by

$$\mathfrak{b}_{1,n_1} >^{elim} \cdots >^{elim} \mathfrak{b}_{1,1} >^{elim} \cdots >^{elim} \mathfrak{b}_{p,n_p} >^{elim} \cdots >^{elim} \mathfrak{b}_{p,1}.$$

This will precisely be the elimination ordering we mentioned above.

The set Σ of asymptotic constraints. Each of the constraints in Σ has one of the following forms:

$$\left\{\begin{array}{l} \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \prec 1; \\ \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \asymp 1; \\ \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \lll \mathfrak{b}_1^{\beta_1} \cdots \mathfrak{b}_n^{\beta_n}; \\ \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \underline{\lll} \mathfrak{b}_1^{\beta_1} \cdots \mathfrak{b}_n^{\beta_n}, \end{array}\right. \tag{11.1}$$

with $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \breve{\mathfrak{C}}$. Modulo case separations, we notice that we can also insert constraints of the forms $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \preceq 1$ and $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n} \not\asymp 1$ into Σ.

Partial constraint checking of Σ. We only check the constraints (11.1) in Σ for their expo-linear consistency by 8.4.4.[3] Consequently, the asymptotic relations $\prec$, $\lll$, etc. do not necessarily coincide with the usual asymptotic relations determined by R. Indeed, expo-linear consistency of Σ does not imply overall consistency, although a complete constraint checker will be given in section 11.8.

Constraint saturation. To reduce the number of case separations, we will always saturate the set Σ in the following way: for each $\mathfrak{b} \in B \backslash X$, we assume that $\log \mathfrak{b}$

[3]We notice that it suffices to consider the $\mathfrak{b}_i$ as transmonomial parameters in order to apply 8.4.4. This is equivalent from an asymptotic point of view, since any transseries in $\mathbb{T}$ is asymptotic to its dominant monomial.

is regular, and we impose the constraint $\text{м}(\log \text{б}) \prec\!\!\prec\!\!\prec \text{б}$. Furthermore, whenever we have $\text{м}(\log \text{б}) \prec\!\!\prec \text{м}(\log \text{б}')$ for $\text{б}, \text{б}' \in B \backslash X$, then we impose the constraint $\text{б} \prec\!\!\prec\!\!\prec \text{б}'$ and vice versa. Similarly, whenever we have $\text{м}(\log \text{б}) \underline{\prec\!\!\prec} \text{м}(\log \text{б}')$ for $\text{б}, \text{б}' \in B \backslash X$, then we impose the constraint $\text{б} \underline{\prec\!\!\prec\!\!\prec} \text{б}'$ and vice versa.

Initialization. The triple (X, Σ, B) determined by $\Sigma = \{x_1 \prec\!\!\prec 1, \cdots, x_p \prec\!\!\prec 1\}$ and $B = \{x_1, \cdots, x_p\}$ satisfy our hypothesis. These are the coordinates we use to initialize our algorithms; additional constraints may be imposed by the user afterwards.

11.3.3 Effective representations

Asymptotic scales. Let

$$S_B = B^{\check{\mathfrak{C}}}.$$

In the remainder of this chapter, we will only consider asymptotic expansions w.r.t. S_B, i.e. we work with transseries in $\check{\mathfrak{C}}[[S_B]]$. We abusively call S_B an **asymptotic scale**. Indeed, S_B is not an asymptotic scale in the sense of chapter 6 in general: a counterexample is given by $B = \{x, e^{-y^{-1}-yx^{-1}}, y, e^{-y^{-1}}\}$, where $x \prec\!\!\prec\!\!\prec y$.

Admissible Cartesian representations. A monomial $\text{ц} \in S_B$ is said to be **ultra-regular**, if it has the form

$$\text{ц} = \text{б}^{\alpha} \text{щ},$$

where $\text{б} \in B$, $\alpha > 0$, $\text{щ} \in S_{B, \prec\!\!\prec\!\!\prec \text{б}}$ and

$$S_{B, \prec\!\!\prec\!\!\prec \text{б}} = \{\text{б} \in B | \text{б} \prec\!\!\prec\!\!\prec z\}^{\check{\mathfrak{C}}}.$$

By convention, 1 is said to be ultra-regular too. An **ultra-regular transseries** is a regular transseries whose dominant monomial is ultra-regular. Ultra-regular monomials and transseries are interesting, because their comparability classes are well determined.

A Cartesian representation $u \in \mathfrak{L}_Z$ is said to be **admissible**, if each Cartesian coordinate in Z represents an ultra-regular monomial. From now on we will assume without further mention that all Cartesian representations are admissible.

Example 11.1. Let $X = \{x, y\}$. The Cartesian coordinate z with $\overline{z} = y/x$ is admissible, if and only if $x \prec\!\!\prec\!\!\prec y$.

Dependence on coordinates. Let f be a transseries represented by an admissible Cartesian representation $\underline{f} \in \mathfrak{L}_{z_1, \cdots, z_k}$. We will now define when $\underline{f}$ depends on a coordinate x in X.

Each z_i has the form $z_i = \text{б}_1^{\alpha_{i,1}} \cdots \text{б}_n^{\alpha_{i,n}}$. We say that z_i **depends** on б_j in B, if $\alpha_{i,j} \neq 0$. Let $B_{z_1, \cdots, z_k}$ be the union of these elements б_j in B, when i ranges over

$1, \cdots, k$. Then we say that $\underline{f}$ **depends** on $x \in X$, if $x \in B_{z_1,\cdots,z_k}$, or if there exists an $б_j \in B_{z_1,\cdots,z_k} \backslash X$ such that $б_j$ recursively depends on x. Here $б_j$ is (abusively: see the warning below) said to depend on x if the natural Cartesian representation $\underline{\log б}$ of $\log б$ does.

Warning 11.1. It may happen that $\underline{f}$ depends on a certain coordinate, while f does not. For instance, taking $X = \{x, y\}$, the transseries $f = y$ may very well be represented by $\underline{f} = \underline{y} + (\underline{e^{2x}})^3 - (\underline{e^{3x}})^2$, which depends on x. For a similar reason, a basis element $б$ in B may depend on x in the above syntactical sense, while the transseries $б$ does not depend on x in the usual sense.

Normal bases. For each $1 \leqslant q \leqslant p+1$, let $B_{\geqslant q}$ denote the set of those $б$ in B which do not depend on $x_1, \cdots, x_{q-1}$. We also abbreviate $B_q = B_{\geqslant q} \backslash B_{\geqslant q+1}$, $B_{>q} = B_{\geqslant q+1}$, etc. We say that B is an **effective normal basis**, if for each $1 \leqslant q \leqslant p$:

NB1. $B_q = \{б_{q,1}, \cdots, б_{q,n_q}\}$ is linearly ordered w.r.t. $\prec\!\!\!\prec\!\!\!\prec$.

NB2. $б_{q,1} = x_q$.

NB3. Each $б = б_{q,i} \in B_q$ with $i > 1$ has the form $б = e^{\overline{u}}$, where $u = \underline{\log б}$ is a C-regular Cartesian representation and $\overline{u} \in \check{\mathfrak{C}}[[S]]$ with

$$S = \{б \in B_{\geqslant q} | б \in B_{>q} \ \vee \ б \ \underline{\prec\!\!\!\prec\!\!\!\prec} \ \text{м}(\log б)\}^{\check{\mathfrak{C}}}.$$

We notice that the above hypotheses indeed imply that B is a normal set relative to the elimination ordering mentioned in the previous section. From now on, we will always assume that B is an effective normal basis.

11.3.4 Refinements and exponential rewritings

In this chapter, we will exclusively consider effective refinements of one of the following forms:

R1. $x_q = \exp^{-1} x'_q$;

R2. $x_q = \overline{ц} x'_q \ (x'_q \prec\!\!\!\prec\!\!\!\prec \overline{ц})$;

R3. $x_q = \overline{ц} x'_q \ (x'_q \prec\!\!\!\prec\!\!\!\prec \overline{ц})$;

R4. $x_q = \overline{ц}(\overline{u} + x'_q) \ (x'_q \prec\!\!\!\prec 1)$;

R5. $x_q = \overline{ц}(\overline{u} - x'_q) \ (x'_q \prec\!\!\!\prec 1)$;

R6. $x_q = \overline{ц u}$.

Here $ц$ is a C-infinitesimal C-regular Cartesian monomial such that $\overline{ц}$ is ultra-regular, and u a C-regular Cartesian representation of a transseries with $\overline{u} \asymp 1$. Moreover, neither u nor $ц$ depend on $x_1, \cdots, x_q$.

Refinements of the form **R1** are called **upward shiftings** and need to be used with care, because they insert new elements into B. Refinements of the forms **R2** and **R3** are called **split-offs**. Refinements of the forms **R4**, **R5** and **R6** are called **ordinary** refinements.

Refinements and constraints in Σ. Whenever we perform a refinement of one of the above types, we rewrite all constraints in Σ w.r.t. the new coordinates. This process is straightforward: for instance, in the ordinary case, each occurrence of x_q is replaced by $\overline{\text{ц}}$.

Automatic updating of Cartesian representations. As in chapter 10, we will always automatically update Cartesian representations when necessary, to make them available w.r.t. the current coordinates. The rewritings involved in this process are straightforward, because x_q admits a C-regular Cartesian representation w.r.t. the new coordinates for any refinement of the above types, while the other elements in B remain unchanged. Furthermore, admissibility is preserved, since $\overline{\text{ц}}$ is required to be ultra-regular. We also notice that C-regularity and C-infinitesimality are preserved under these rewritings.

Updating of monomials in S_B. In our algorithms, monomials ц in S_B are *not* updated automatically, and we explicitly say that we "update ц", if ц has to be replaced by its dominant monomial w.r.t. the current coordinates.

Besides rewriting coordinates in X, we sometimes also need to rewrite elements in $B\backslash X$: an **exponential rewriting** of $\text{б} \in B_q\backslash\{x_q\}$ is a rewriting of the form:

$$\text{б} = \overline{\text{ц}}\, \text{б}'^{\varepsilon}\, \overline{u} \tag{11.2}$$

Here $\overline{\text{ц}}\text{б}'^{\varepsilon} \in S_{B'_{\succcurlyeq q}\backslash\{x_q\}}$ is ultra-regular, $\varepsilon \in \{-1,0,1\}$ and u is a C-regular Cartesian representation free from $x_1, \cdots, x_{q-1}$ with $\overline{u} \asymp 1$. Moreover, at least one of the following two conditions holds:

- **E1.** ц does not depend on x_q.
- **E2.** u does not depend on basis elements in B_q which are strictly larger than $\text{м}(\log \text{б})$ for $\lll$.

As a result of the exponential rewriting, б is removed from B, but $\text{б}' \lll \text{б}_{i_j}$ is a new element in B', if $\varepsilon \neq 0$.

Exponential rewritings and constraints in Σ. Whenever we perform an exponential rewriting, all constraints in Σ are modified accordingly, by replacing each occurrence of б by $\overline{\text{ц}}\text{б}'^{\varepsilon}$.

Exponential rewritings and Cartesian representations. The process of automatic updating of Cartesian representations after exponential rewritings is straightforward. This is again due to the fact that б admits a C-regular C-infinitesimal Cartesian representation w.r.t. the new coordinates. Furthermore, C-regularity and C-infinitesimality are preserved under exponential rewritings.

11.3.5 Renormalization of B

In general, the basis B need not remain normal, when we perform an ordinary refinement; therefore, some additional action need be undertaken each time we do.

Example 11.2. Let $B = \{x, e^{-x^{-1}}, y\}$ and assume that we perform the refinement $x = y(1 + x')$, with $x' \ggg y$. Then $B = \{x', e^{-y^{-1}/(1+x')}, y\}$ after the refinement. Although B is no longer normal, B can be renormalized by means of the exponential rewriting

$$e^{-y^{-1}/(1+x')} = e^{-y^{-1}} e^{-y^{-1}x'/(1+x')}.$$

More generally, the renormalization process relies on the algorithm `exponentiate`, which will only be specified in the next section. This algorithm is invoked in order to recompute some of the exponential basis elements in B_q, after a refinement of x_q.

Algorithm `renormalize`.

ACTION: Renormalizes B after an ordinary refinement.

STEP 1. Let $\mathfrak{b}_{q,1} \lll \cdots \lll \mathfrak{b}_{q,n_q}$ be the elements of B_q before the refinement.
In case of a refinement of type **R6**, let $i = n_q$ and go to step 2.
Separate n_q cases and respectively impose the constraints:

$$\begin{cases} x'_q \underline{\lll} \mathrm{M}(\log \mathfrak{b}_{q,2}) \ (i = 1); \\ \mathrm{M}(\log \mathfrak{b}_{q,i}) \lll x'_q \underline{\lll} \mathrm{M}(\log \mathfrak{b}_{q,i+1}) \ (2 \leqslant i \leqslant n_q - 1); \\ \mathrm{M}(\log \mathfrak{b}_{q,n_q}) \lll x'_q \ (i = n_q). \end{cases}$$

STEP 2. Recompute $\exp(\log \mathfrak{b}_{q,2}), \cdots, \exp(\log \mathfrak{b}_{q,i})$ by `exponentiate`;
This yields expressions of the form (11.2) for $\mathfrak{b}_{q,2}, \cdots, \mathfrak{b}_{q,i}$ satisfying **E1**, and we perform the corresponding exponential rewritings.

Proposition 11.1. *The algorithm* `renormalize` *is correct. Moreover, the the size of B_q does not increase*[4] *as a result of the renormalization, and the exponential elements in B_q remain unaltered.*

Proof. In case of refinements of type **R6**, the algorithm is obviously correct. Assume therefore that the refinement has type **R4** or **R5**.

If $i > 1$, then the fact that $x'_q \ggg \mathfrak{b}_{q,i}$ ensures $\underline{g^+}$ to be free from x'_q in the decomposition $g = g^+ + g^-$ involved in the re-exponentiation of each $\log \mathfrak{b}_{q,2}$ with $2 \leqslant j \leqslant i$ (see the next section). Hence, the resulting expressions for $\mathfrak{b}_{q,2}, \cdots, \mathfrak{b}_{q,i}$ indeed have the desired form and their dominant monomials in the new coordinates are free from x_q.

On the other hand, if $i < n_q$, the fact that $x'_q \underline{\lll} \log \mathfrak{b}_{q,i+1}$ ensures that $\{x'_q, \mathfrak{b}_{q,i+1}, \cdots, \mathfrak{b}_{q,n_q}\}$ is again a normal basis after the refinement. □

[4]By convention, we understand that the size of B_q strictly decreases if x_q is eliminated from X.

In principle, renormalization of B might also be necessary after exponential rewritings. However, exponential rewritings of type **E1** are only performed in `renormalize` in this chapter, whence no additional renormalization is needed in this case. Furthermore, we claim that renormalization is never necessary after exponential rewritings of type **E2**. Indeed, the only exponential basis elements $\tilde{\mathfrak{b}}$ in B_q, such that $\underline{\log \tilde{\mathfrak{b}}}$ depends on $\mathfrak{b}$ are strictly larger than $\mathfrak{b}$ for $\prec\!\!\!\prec$. Since u does not depend on elements in B_q which are larger than $\log \mathfrak{b}$, it follows that no exponential rewritings are necessary for $\tilde{\mathfrak{b}}$.

11.4 The expansion algorithm

In this section we present an algorithm to compute asymptotic expansions of multivariate transseries in $\mathfrak{T}$. The algorithm relies on a certain number of subalgorithms which will be specified in the next sections.

Algorithm `expand`(f).
INPUT: An $\mathfrak{L}$-exp-log expression f in $\mathfrak{T}$.
OUTPUT: A generic Cartesian representation $\underline{f} \in \mathfrak{L}_Z$ for f.

Case when $f = x_q$ is a new transseries parameter:
Separate five cases, respectively refine

$$\begin{cases} x_q = x_q'^{-1} \ (x_q' \prec\!\!\prec 1); \\ x_q = -x_q'^{-1} \ (x_q' \prec\!\!\prec 1); \\ x_q = \lambda_q + x_q' \ (x_q' \prec\!\!\prec 1); \\ x_q = \lambda_q - x_q' \ (x_q' \prec\!\!\prec 1); \\ x_q = \lambda_q, \end{cases}$$

and return x_q. In the last three cases, λ_q denotes a new parameter in $\breve{\mathfrak{C}}$.

Case $f = u(g_1, \cdots, g_i) \in \mathfrak{L}_{g_1,\cdots,g_i}$:
STEP 1. Compute Cartesian representations $\underline{g_1}, \cdots, \underline{g_i}$ for $g_1, \cdots, g_i$ by `expand`. Compute their respective dominant monomials $\mathfrak{m}_1, \cdots, \mathfrak{m}_i$ by `dom_mon`. Regularize $\underline{g_1}, \cdots, \underline{g_i}$.
STEP 2. Impose the constraints $\mathfrak{m}_1 \prec\!\!\prec 1, \cdots, \mathfrak{m}_i \prec\!\!\prec 1$. Infinitesimalize $\underline{g_1}, \cdots, \underline{g_i}$.
STEP 3. Return $u(\underline{g_1}, \cdots, \underline{g_i})$.

Case $f = g_1 \star g_2$, $\star \in \{+, -, \cdot, /\}$:
STEP 1. Compute Cartesian representations $\underline{g_1}, \underline{g_2}$ for g_1 and g_2 by `expand`.
STEP 2. If $\star = /$, then compute the dominant monomial $\mathfrak{m}$ of g_2 by `dom_mon`, kill the current process if $\mathfrak{m} = 0$, and regularize $\underline{g_2}$.
STEP 3. Return $\underline{g_1} \star \underline{g_2}$.

Case $\boldsymbol{f} = \log \boldsymbol{g}$:

STEP 1. Compute a Cartesian representation $\underline{\boldsymbol{g}}$ for $\boldsymbol{g}$ by `expand`.
Compute the dominant monomial of $\boldsymbol{g}$ by `dom_mon`.
Regularize $\underline{\boldsymbol{g}}$.

STEP 2. Compute $\underline{\varepsilon}$ and $\alpha_1, \cdots, \alpha_n \in \breve{\mathfrak{C}}$ with $\underline{\boldsymbol{g}} = c_{\boldsymbol{g}} \underline{\text{б}_1^{\alpha_1} \cdots \text{б}_n^{\alpha_n}(1+\underline{\varepsilon})}$.
For each i with $\alpha_i \neq 0$, perform the upward shifting $\mathsf{x}_i = \exp^{-1} \mathsf{x}_i'^{-1}$.

STEP 3. Return $\log c_{\boldsymbol{g}} + \alpha_1 \underline{\log \text{б}_1} + \cdots + \alpha_n \underline{\log \text{б}_n} + \log(1+\varepsilon)$.

Case $\boldsymbol{f} = \exp \boldsymbol{g}$:

STEP 1. Compute a Cartesian representation $\underline{\boldsymbol{g}}$ for $\boldsymbol{g}$ by `expand`.
Compute the dominant monomial $\mathrm{M}(\boldsymbol{g})$ of $\boldsymbol{g}$ by `dom_mon`.
Separate two cases, and respectively proceed with steps 2 and 3.

STEP 2. Separate two cases and respectively proceed with a and b

A. Impose the constraint $\mathrm{M}(\boldsymbol{g}) \prec\!\!\prec 1$.
Regularize and infinitesimalize $\underline{\boldsymbol{g}}$ and return $\exp \underline{\boldsymbol{g}}$.

B. Execute `constraint`$(\mathrm{M}(\boldsymbol{g}) \asymp 1)$.
Regularize $\underline{\boldsymbol{g}}$.
Let $c = \underline{\boldsymbol{g}}_{0,\cdots,0}$.
Return $\exp c \exp(\underline{\boldsymbol{g}} - c)$.

STEP 3. Impose the constraint $\mathrm{M}(\boldsymbol{g}) \succ\!\!\succ 1$.
Let q be minimal such that $\underline{\boldsymbol{g}}$ depends on x_q.
Let $\mathsf{x}_q = \text{б}_{q,1} \lll \cdots \lll \text{б}_{q,n_q}$ be the elements in B_q.
Separate two cases and respectively proceed with steps 4 and 5.

STEP 4. For each $2 \leqslant i \leqslant n_q$ separate a case and impose the constraint $\mathrm{M}(\boldsymbol{g}) \asymp \mathrm{M}(\log \text{б}_{q,i})$.
Compute the limit $\lambda = \lim \boldsymbol{g} / \log \text{б}_{q,i}$.
Return $\text{б}_{q,i}^{\lambda}$`expand`$(e^{\boldsymbol{g} - \lambda \log \text{б}_{q,i}})$.

STEP 5. Separate two cases and respectively impose the constraints

$$\begin{cases} \mathrm{M}(\boldsymbol{g}) \lll \mathsf{x}_q; \\ \mathrm{M}(\boldsymbol{g}) \underline{\ggg} \mathsf{x}_q. \end{cases}$$

Separate $n_q + 1$ cases and respectively impose the constraints

$$\begin{cases} \mathrm{M}(\boldsymbol{g}) \prec\!\!\prec \mathrm{M}(\log \text{б}_{q,2}); \\ \mathrm{M}(\log \text{б}_{q,i}) \prec\!\!\prec \mathrm{M}(\boldsymbol{g}) \prec\!\!\prec \mathrm{M}(\log \text{б}_{q,i+1}) \ (2 \leqslant i \leqslant n_q); \\ \mathrm{M}(\log \text{б}_{q,n_q}) \prec\!\!\prec \mathrm{M}(\boldsymbol{g}). \end{cases} \tag{11.6}$$

STEP 6. Let $\text{б}_{i_1}, \cdots, \text{б}_{i_j}$ be those elements б in B, with $\text{б} \ggg \mathrm{M}(\boldsymbol{g})$.
Decompose $\boldsymbol{g} = \boldsymbol{g}^+ + \boldsymbol{g}^-$, with $\boldsymbol{g}^+ = [\text{б}_{i_1}^0 \cdots \text{б}_{i_j}^0]\boldsymbol{g}$.
If $\underline{\boldsymbol{g}^+}$ depends on x_q, then insert $\text{б} = e^{-|\operatorname{sign} \boldsymbol{g}^+| \boldsymbol{g}^+}$ into B.
Return `expand`$(e^{\boldsymbol{g}^+})$`expand`$(e^{\boldsymbol{g}^-})$.

Remark 11.1. In step 6 of the exponential case, the decomposition $g = g^+ + g^-$ is computed by repeated applications of proposition 9.5, where we start by extracting the coefficients of the basis-elements which are maximal for $\lll$. Notice that we may indeed apply proposition 9.5, since all Cartesian coordinates are admissible.

Let us briefly explain the different cases of the algorithm, which is fairly similar to the algorithm on page 212.

The case when f is a new transseries parameter x_q is reduced to the case when x_q is positive and infinitesimal, by a separation into five cases.

In step 5 of the exponential case, we make sure by means of case separations that $\mathrm{M}(g)$ and x_q are comparable for $\lll$ and we determine the place in B where e^{g^+} might need be inserted. In step 6, $\underline{g^+}$ either depends on x_q, in which case insertion takes place, or $\underline{g^+}$ does not depend on x_q, in which case the part g^- of g which depends on x_q is bounded so that we "split it off".

Obviously, the exponential case algorithm can also be applied to transseries g for which we already have a Cartesian representation $\underline{g}$, by skipping the very first line of step 1. This algorithm, called `exponentiate`, has the following obvious property:

Proposition 11.2. *Exponentials computed by* `exponentiate` *are always ultra-regular.* □

11.5 Ultra-regularization

Since not all Cartesian monomials are admissible, we can not merely introduce new infinitesimal Cartesian coordinates in the infinitesimalization and regularization algorithms from section 9.3.2. For this purpose, we will show in this section how to make an arbitrary monomial $\mathfrak{u}$ in S_B ultra-regular modulo a sequence of refinements.

11.5.1 The subalgorithm `rewrite`

If the basis elements in B are linearly ordered w.r.t. $\lll$, then all infinitesimal monomials in S_B are represented by Cartesian coordinates. Therefore, a major ingredient of the ultra-regularization process is a partial linearization of B by means of the following algorithm:

Algorithm `rewrite`(б, щ).

INPUT: A basis element $б \in B_q$ and an infinitesimal ultra-regular monomial $(щ \asymp\!\!\!\!\!\!\times б) \in S_{B_{\geqslant q \setminus \{x_q\}}}$.

ACTION: б is either rewritten in terms of щ and smaller basis elements for $\lll$, or eliminated[5] from B_q.

Case $б = x_q$: Separate three cases, and respectively refine

$$\begin{cases} x_q = щ^\lambda x_q' & (x_q' \lll щ); \\ x_q = щ^\lambda x_q'^{-1} & (x_q' \lll щ), \end{cases}$$

or use `constraint` in order to impose the constraint

$$x_q \asymp щ^\lambda,$$

where $\lambda > 0$ is a new parameter in $\breve{\mathfrak{C}}$.

Case $б \in B_q \setminus \{x_q\}$:

STEP 1. Execute `constraint`(м(log б) $\asymp$ м(log щ)).

STEP 2. Compute the limit λ of log б/ log щ.
Compute the exponential of $\varphi = \log б - (\log щ)/\lambda$ by `exponentiate`.
If б has not been eliminated from B_q,
then perform the exponential rewriting $б = щ^\lambda e^\varphi$ of type **E2**.

We notice that in the last step of the case $б \in B_q \setminus \{x_q\}$, б may indeed have been eliminated from B_q. Indeed, the preceding steps may lead to refinements of x_q. If, as a result of these, б is eliminated from B_q, then the exponential rewriting $б = щ^\lambda e^\varphi$ becomes either invalid or superfluous.

We also notice that by the fact that щ does not depend on the basis element x_q, the computation of φ does not necessitate the insertion of new logarithms into B_q.

Let us exemplify the use of `rewrite` in the ultra-regularization process:

Example 11.3. Let $B = \{x, e^{-y^{-1}(1+x)}, y, e^{-y^{-1}}\}$ with $x \lll y$. Assume that we want to ultra-regularize $e^{-y^{-1}(1+x)}/e^{-y^{-1}}$. Since $e^{-y^{-1}(1+x)}/e^{-y^{-1}}$ we first re-express $e^{-y^{-1}(1+x)}$ in terms of $e^{-y^{-1}}$ times smaller terms for $\lll$, using `rewrite`:

$$e^{-y^{-1}(1+x)} = e^{-y^{-1}} e^{-y^{-1}x}.$$

After this exponential rewriting, $e^{-y^{-1}(1+x)}/e^{-y^{-1}} = e^{-y^{-1}x}$ is ultra-regular.

[5]By convention, we say that б has been eliminated from B_q, if x_q is eliminated from X.

11.5.2 The algorithm `ultra_regularize`

In general, the following algorithm is both used in order to infinitesimalize a Cartesian monomial and to ultra-regularize the transmonomial it represents:

Algorithm `ultra_regularize`(ц).

INPUT: An infinitesimal monomial ц in S_B.

ACTION: After the execution, the dominant monomial of ц w.r.t. the current coordinates is ultra-regular.

STEP 1. Let M be the set of maximal basis elements occurring in ц.
Select a non empty subset $S \subseteq M$, using $2^{|M|} - 1$ case separations.
Impose the constraint $\text{б} \lll \text{б}'$ for all $\text{б} \in M \backslash S$ and $\text{б}' \in S$.
If S contains only one element, then return ц.

STEP 2. Let q be minimal such that $B_q \cap S \neq \phi$.
Let б be the unique element in $B_q \cap S$.
Let α be the exponent of б in ц.
Let Q be the set of basis elements in $B_{\geqslant q} \backslash \{\text{б}\}$ which occur in ц.
Decompose $\text{ц} = \text{ч}\text{б}^{\alpha}\text{щ}$, with $\text{ч} \in S_{<q}$ and $\text{щ} \in S_Q$.

STEP 3. Separate three cases, and respectively proceed with a,b or c:

- A. Impose the constraint $\text{щ} \prec\!\!\prec 1$ and execute `ultra_regularize`(щ).
- B. Execute `constraint`($\text{щ} \asymp 1$) and return.
- C. Impose the constraint $\text{щ} \succ\!\!\succ 1$ and execute `ultra_regularize`(щ^{-1}).

STEP 4. If б has been eliminated[5] from B_q, then update ц and return to step 1.
Update щ, separate two cases and respectively proceed with a or b:

- A. Impose the constraint $\text{щ} \lll \text{б}$ and return.
- B. Impose the constraint $\text{щ} \asymp \text{б}$.

STEP 5. Let β be the exponent of x_q in щ and set $\text{щ} := \text{щ}/\mathsf{x}_q^{\beta}$.
Execute `rewrite`(б, щ).
Update ц and re-execute `ultra_regularize`(ц).

Proposition 11.3. *The algorithm* `ultra_regularize` *is correct and terminates.*

Proof. The correctness of the algorithm is obvious. Before proving its termination, let us first introduce the following invariant: let ц be an element in $S_{B_{\geqslant q}}$, and let $\text{б}_{q,i}$ be maximal for $\lll$ while occurring in ц. If ц does not depend on x_q, then we define $\chi(\text{ц}) = i$. Otherwise, we define $\chi(\text{ц}) = i + \frac{1}{2}$.

Assume for contradiction that the recursive invocations of `ultra_regularize` (or the jumps to step 1 in step 4) provoke an infinite loop. Let $q_1, q_2, \cdots$ be the successive values of q in these recursive invocations. Let q be minimal, such that $q = q_i$ for infinitely many i. Without loss of generality, we may assume that $q_i \geqslant q$ for all i. We now restrict our attention to calls of `ultra_regularize` for which $q_i = q$. Assume that all such calls for which $\chi(\text{ц}) < d$ terminate. This is clearly so for $d = 1$. Now consider a call of `ultra_regularize`, for which $\chi(\text{ц}) = d$. We will prove that this call terminates. In particular, this yields the desired contradiction, by induction.

First, the recursive invocations of `ultra_regularize` in step 3 always terminate, since $\chi(\text{щ}) < \chi(\text{ц})$. Similarly, if we return to step 1 in step 4, then we have termination by the fact that the new ц does not depend on x_q any more. Now consider the monomial ц just before a recursive invocation of `ultra_regularize` in step 5b. Decompose $\text{ц} = \text{ц}_1 \text{ц}_2$, where $\text{ц}_1 \in S_{B_{<q}}$ and $\text{ц}_2 \in S_{B_{\geqslant q}}$. We will show that ц_2 is either ultra-regular, or $\chi(\text{ц}_2) < d$; the termination of `ultra_regularize` is clear in both cases. We will denote by ${}_*x_q$ resp. ${}^*x_q$ the respective values of x_q before and after the execution of `rewrite`. Assume first that $\text{б} = {}_*x_q$. Then $\text{ц}_2 = {}_*x_q^{\alpha}\, \text{щ}$, whence after the execution of `rewrite` we have

$$\text{м}(\text{ц}_2) = \text{м}(\text{щ}^{\lambda\alpha+1}\, {}^*x_q^{\varepsilon}) \ (\varepsilon \in \{-1, 0, 1\}),$$

where λ is as in `rewrite`. Since щ is ultra-regular and ${}^*x_q \lll \text{щ}$, we conclude that ц_2 is ultra-regular.

Assume now that $\text{б} \neq {}_*x_q$. If б is eliminated in `rewrite`, then the dominant monomial of ц does not depend on x_q, after the execution of `rewrite`, and we are done. Otherwise, we have

$$\text{ц}_2 = \text{б}^{\alpha}\, {}_*x_q^{\beta}\, \text{щ} = {}_*x_q^{\beta}\, \text{щ}^{\lambda\alpha+1} e^{\varphi}$$

after the execution and with the notations of `rewrite`. Since $e^{\varphi} \lll \text{щ}$ and ${}_*x_q^{\beta} \lll \text{щ}$, we are done in particular, if $\lambda\alpha+1 \neq 0$. Assume therefore that $\lambda\alpha+1 = 0$. By proposition 11.2, e^{φ} is ultra-regular. Hence, if $\beta = 0$ or ${}_*x_q \not\asymp e^{\varphi}$, then we are done. Furthermore, if $\beta \neq 0$ and x_q has undergone an ordinary refinement, then the exponent of ${}^*x_q$ in $\text{м}(\text{ц}_2)$ vanishes, and we are done again.

The only case which remains be treated is when x_q has only undergone split-offs and ${}_*x_q \asymp e^{\varphi}$. We claim that these split-offs did not occur during the computation of the dominant monomial (see section 11.7) of φ in `exponentiate`, unless an element of $B_{q, \lll \text{б}}$ is eliminated from B_q. Assume first that we never execute `constraint(`$\text{ч} \asymp 1$`)` in `dom_sub`, for a monomial ч which depends on x_q. Since x_q is the largest variable occurring in φ, x_q is not refined at all in this case. In the other case, the imposition of the constraint either leads to an ordinary refinement of x_q or the elimination of an element in B_q (see section 11.6), which is necessarily in $B_{q, \lll \text{б}}$, since φ only depends on elements in B_q, which are in $B_{q, \lll \text{б}}$.

Now if we eliminated an element in $B_{q, \lll \text{ч}}$, then we clearly have $\chi(\text{ц}) < d$ in the recursive call of `ultra_regularize`. Assume therefore that no split-offs occur during the computation of the dominant monomial of φ. Then $x_q = {}_*x_q$, when we compute the decomposition $\varphi = g^+ + g^-$ in `exponentiate`. It follows that the principal part g^+ does not depend on x_q, since $\varphi \asymp \log x_q \lll x_q$. Consequently, $\chi(\text{ц}) \leqslant d - \frac{1}{2}$ in the recursive call of `ultra_regularize`. $\square$

11.6 Imposition of constraints

In this section, we describe the analogue of the algorithm `constraint` from section 10.3.5 in the present context. Any monomial $\text{ц} = \text{б}_1^{\alpha_1} \cdots \text{б}_n^{\alpha_n}$ in S_B can canonically be decomposed as follows:

$$\text{ц} = \mathsf{x}_q^{\alpha} \text{ц}^{expo} \text{ц}^{free}, \tag{11.8}$$

where q is maximal such that ц does not depend on $\mathsf{x}_1, \cdots, \mathsf{x}_{q-1}$, $\alpha \in \breve{\mathfrak{C}}$, $\text{ц}^{expo} \in S_{B_q \setminus \{\mathsf{x}_q\}}$ and $\text{ц}^{free} \in S_{B_{>q}}$. We say that ц is a **$q$-monomial**. Now three cases are distinguished:

1. The ground case $\text{ц}^{expo} = 1$.
2. The exponential case $\alpha = 0, \text{ц}^{expo} \neq 1$.
3. The mixed ground-exponential $\alpha \neq 0, \text{ц}^{expo} \neq 1$.

We will respectively qualify ц as a **ground**, **exponential** and **mixed monomial** in these cases. In sections 11.6.1, 11.6.2 and 11.6.3 we consider the imposition of the constraint $\text{ц} \asymp 1$ for these three cases.

11.6.1 The ground case

The algorithm in the ground case is analogous to the algorithm from section 10.3, and needs no further explanation:

Algorithm `constraint`($\text{ц} \asymp 1$). (ground case)

INPUT: A ground q-monomial ц in S_B.

ACTION: Restricts and cuts the current region into parts, such that the dominant monomial of ц is 1 on each of these parts.

STEP 1. If $\alpha < 0$, then set $\alpha := -\alpha$ and $\text{ц}^{free} = (\text{ц}^{free})^{-1}$.
Impose the constraint $\text{ц}^{free} \not\succ\!\!\succ 1$.
Ultra-regularize $(\text{ц}^{free})^{-1/\alpha}$ and update ц^{free}.

STEP 2. Let λ be a new parameter in $\breve{\mathfrak{C}}$.
Impose the constraint $\lambda > 0$.
Separate three cases and respectively refine:

$$\begin{cases} \mathsf{x}_q = (\text{ц}^{free})^{-1/\alpha}(\lambda + \mathsf{x}_q') \ (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = (\text{ц}^{free})^{-1/\alpha}(\lambda - \mathsf{x}_q') \ (\mathsf{x}_q' \prec\!\!\prec 1); \\ \mathsf{x}_q = \lambda(\text{ц}^{free})^{-1/\alpha}. \end{cases}$$

11.6.2 The exponential case

If we are not in the ground case, the constraint $\text{ц} \asymp 1$ may be imposed by computing the dominant monomial $\textsc{m}$ of $\log \text{ц}$ and imposing the constraint $\textsc{m} \underline{\prec\!\!\prec} 1$. If $\alpha = 0$, we hereby benefit from the fact that a Cartesian representation for $\log \text{ц}^{expo}$ is already beforehand, therefore $\log^{-1} x_q^{-1}$ needs not be inserted into B. On the other hand, the computation of $\log \text{ц}^{free}$ may necessitate the insertion of new logarithms into B, which are free from x_q. To compensate these insertions, the relation $\text{ц} \asymp 1$ is used to eliminate at least one element from B_q.

Algorithm `constraint`($\text{ц} \asymp 1$). (exponential case)

INPUT: An exponential q-monomial ц in S_B.

ACTION: Restricts and cuts the current region into parts, such that the dominant monomial of ц is 1 on each of these parts.

STEP 1. Let $\text{б} \in B_q$ be maximal for $\lll$, while occurring in ц^{expo}.
Let λ be the power of б in ц^{expo}.
Ultra-regularize $(\text{ц}^{free})^{-1/\lambda}$ and update ц^{free}.

STEP 2. Let $\varphi = \underline{\log \text{ц}^{expo} + \log \text{ц}^{free}}$.
Compute the dominant monomial м of $\overline{\varphi}$ by `dom_mon`.
Regularize φ and separate the following two cases:

A. Impose the constraint $\overline{\text{м}} \prec\!\!\prec 1$ and infinitesimalize φ.

B. Execute `constraint`($\text{м} \asymp 1$).

STEP 3. If б has not been eliminated[5] from B_q,
then perform the exponential rewriting of type **E2**

$$\text{б} = ((\text{ц}^{expo}/\text{б}^{\lambda})\text{ц}^{free})^{-1/\lambda}\overline{e^{\varphi/\lambda}}.$$

STEP 4. Otherwise, update ц and impose the constraint $\text{ц} \asymp 1$.

Proposition 11.4. *The above algorithm* `constraint` *is correct and terminates. Moreover, at each invocation, $|B_q|$ strictly decreases*[4]*, while $B_{<q}$ remains unchanged.*

Proof. In step 1, we reduce the general case to the case when $(\text{ц}^{free})^{-1/\lambda}$ is ultra-regular (this is needed in step 3, in order to guarantee that the rewriting of б is indeed an exponential rewriting). Step 2 is equivalent to the imposition of the desired constraint. In step 3, we check whether the coordinate x_q has been eliminated from б as a result of step 2. If not, we perform an exponential rewriting to eliminate б from B_q. Otherwise, `constraint` must be re-applied to the dominant monomial of ц in the current coordinates, which is free from x_q. □

11.6.3 The mixed ground-exponential case

In general, $\alpha \neq 0$ and $\text{ц}^{expo} \neq 1$ Consider for example the constraint

$$x e^{-x^{-1}} \asymp y, \tag{11.10}$$

with $X = \{x, y\}$. In order to avoid the insertion of $\log^{-1} x^{-1}$ into B, we first compute a monomial щ, asymptotic to x, but independent of x. This is achieved by exploiting the fact that (11.10) implies $e^{x^{-1}} \succcurlyeq y$, whence $x^{-1} \asymp \log y^{-1}$. Consequently, (11.10) transforms into

$$e^{-x^{-1}} \asymp y \log y^{-1},$$

and we have reduced the present case to the exponential case.

For more general monomials ц, the idea of the algorithm is to compute a decomposition

$$ц = x_q^{\alpha} e^{f} e^{g},$$

where f is an ultra-regular transseries whose dominant monomial depends on x_q, and g is a transseries which is free from x_q. If f is bounded, then have reduced our problem to the ground case. Otherwise, since $f \underline{\succ\!\!\succ\!\!\succ} x_q$, we must have $f \asymp g$. Then, by induction, the recursive imposition of the constraint $f \asymp g$ either leads to the elimination of a basis element in B_q or an ordinary refinement of x_q (in which case we have the desired equivalent щ for x_q).

However, we have to cope with one additional difficulty: how to compute the part of a transseries which does not depend on x_q? By what has said in section 9.4, we do not have a general algorithm to do this. For this reason, we will specify an algorithm `dep_dom_mon` in section 11.7, which given a Cartesian representation u, simultaneously computes a decomposition $u = \varphi + \psi$, where ψ is free from x_q, and a generic dominant monomial м of $\overline{\varphi}$ which depends on x_q. Furthermore, this algorithm aborts whenever an ordinary refinement for x_q occurs. Anyway, we also obtain the desired equivalent щ for x_q in this case.

Algorithm `constraint`(ц $\asymp 1$). (mixed ground-exponential case)

INPUT: A mixed q-monomial ц in S_B.

ACTION: Restricts and cuts the current region into parts, such that the dominant monomial of ц is 1 on each of these parts.

STEP 1. $u := \underline{\log ц^{expo} + \log ц^{free}}$.
и $:= x_q$.
Let $(\varphi, \psi, м) :=$ `dep_dom_mon`(u).

STEP 2. If the dominant monomial of и w.r.t. the current coordinates is free from x_q, then update ц, execute `constraint`(ц $\asymp 1$), and return.

STEP 3. Separate the following two cases:

A. Impose the constraint м $\succ\!\!\succ 1$.

B. Impose the constraint м $\underline{\prec\!\!\prec} 1$.
Execute `constraint`(x_q^{α}`exponentiate`$(\overline{\psi}) \asymp 1$)
Re-execute step 2.

STEP 4. Ultra-regularize м.
Re-execute step 2.
Let ч $\in S_{B_{>q}}$ be such that и $\asymp$ ч$x_q^{\pm 1}$.
Set $\psi := \psi + \underline{\log ч}$.

STEP 5. Compute the dominant monomial м$'$ of ψ by `dom_mon`.
Update м and execute `constraint`(м$'$/м $\asymp 1$).
Update ц and execute `constraint`(ц $\asymp 1$).

Proposition 11.5. *The above algorithm* `constraint` *is correct and terminates. Moreover, at each invocation, $|B_q|$ strictly decreases*[4]*, while $B_{<q}$ remains unchanged.*

Proof. The correctness and termination of the algorithm are clear, if we return in step 2: if $ц$ does not depend on x_q any more after the updating, then all basis elements in B_q which were present in the original $ц$ must have been eliminated. Otherwise, $ц$ is an exponential q-monomial after the updating, and we are done by proposition 11.4.

In step 4, we indeed have $и \asymp ч x_q^{\pm 1}$, for some $ч \in S_{B_{>q}}$, since all refinements of x_q must have been split-offs. Furthermore, the dominant monomial of $м$ w.r.t. the current coordinates depends on x_q and is ultra-regular. Now we must have

$$x_q^{\pm\alpha} e^{\overline{\varphi}} e^{\overline{\psi}} \asymp 1, \tag{11.11}$$

just before the imposition of the constraints in step 5. Since the dominant monomial of $\overline{\psi}$ is ultra-regular, and depends on x_q, we have $x_q \underline{\lll} \overline{\psi}$ and $x_q \lll e^{\overline{\psi}}$. Consequently, (11.11) admits no solutions if $\overline{\varphi} \not\asymp \overline{\psi}$. Therefore, it is legitimate to impose the constraint $м'/м \asymp 1$ in step 5 (and we have termination by the fact that $м'/м$ only depends on elements in B_q which are strictly smaller for $\lll$ than the largest element in B_q occurring in $ц^{expo}$). Now the imposition of this constraint either provokes an ordinary refinement of x_q, or the elimination of an element in B_q. In the first case, the recursive application of the constraint $ц \asymp 1$ falls into the exponential case. In the second case, the termination follows from finiteness of B_q. □

11.7 Computation of generic dominant monomials

The implementation of the algorithms `dom_mon`, `dom_sub` and `Newton_step` is the same as in chapter 10 with the obvious changes:

- We compute with Cartesian representations of multivariate transseries instead of multivariate Laurent series.
- The monomial $ч \in S_{x_1,\cdots,x_{q-1}}$ in `Newton_step` and `dom_sub` is now a monomial in $S_{B_{\leqslant q} \setminus \{x_q\}}$.
- At the very beginning of Newton step, we ultra-regularize and update $\overline{ц}$.
- In step 1 of `dom_mon`, we first test whether there exists a monomial $и \in S_{B_q \setminus \{x_q\}}$ such that each monomial in $F_ч$ is in $F_{чи}$. If so, then $ч$ is replaced by $чи$. Otherwise, we proceed with step 2.
- We use the algorithms from the previous sections for the imposition of constraints and computations with Cartesian representations.

Modulo these changes, we again have:

Theorem 11.1. *The algorithm* `dom_mon` *and its subalgorithms are correct and terminate.*

Proof. The correctness' of `dom_mon` and its subalgorithms are clear from the comments made in the text. Assume that `dom_mon` does not terminate on a given input. Without loss of generality, we may assume that p and all basis elements in $B_{<q}$ remain fixed during the computations, while B_q is altered infinitely many times.

Furthermore, by proposition 11.4 and proposition 11.5, among all calls of `constraint`($ц \asymp 1$), there are only a finite number such that ц depends on an element in $B_q \backslash \{x_q\}$. Hence, using proposition 11.1, we may also assume that the elements in $B_q \backslash \{x_q\}$ remain unaltered throughout the execution.

The remainder of the proof is analogous to the proof of theorem 10.1: after one call of `constraint`($ц \asymp 1$), such that ц depends on x_q, the variable x_q becomes and remains ordinary in the Cartesian representation u. After this, all refinements of x_q are ordinary, and they exclusively occur in `Newton_step`. We finally obtain a contradiction by the analogues of lemma 10.1 and lemma 10.2. □

In section 11.6.3, we assumed the existence of a variant `dep_dom_mon` of the algorithm to compute generic dominant monomials. Let us now state this algorithm:

Algorithm `dep_dom_mon`(u).

INPUT: A Cartesian representation $u \in \mathfrak{L}_Z$ of a transseries f in $x_1, \cdots, x_q$.

OUTPUT: Whenever x_q is refined, the algorithm aborts and returns $(0,0,0)$.
Otherwise, the algorithm returns $(\varphi, \psi, \text{м})$, where $u = \varphi + \psi$ of u, with ψ free from x_q, and м is a generic dominant monomial of $\overline{\varphi}$.
By convention, we return $\text{м} = 0$ on regions where $\overline{\varphi} = 0$.

STEP 1. Let $\varphi := u$ and $\psi := 0$.

STEP 2. Repeat the following until м $\neq$ 'Recommence':

A. Compute $\xi :=$ `pseudo_coefficient`$(\varphi, б_{q,1}^0 \cdots б_{q,n_q}^0)$.
Set $\varphi := \varphi - \xi$ and $\psi := \psi + \xi$.

B. Compute a set M of intermediary dominant monomials of φ on R by `idm`.

C. If $M = \phi$ then return $(\varphi, \psi, 0)$.

D. Separate a case for each facet $F \subseteq M$, and do the following:
Select an arbitrary $ц \in F$.
Impose the constraint $\overline{щ} \prec\!\!\prec \overline{ц}$ for each $щ \in M \backslash F$.
Impose the constraint $\overline{щ} \asymp \overline{ц}$ for each $щ \in F \backslash \{ц\}$.
Set м := `dom_sub`$(\varphi, 1, F)$.

STEP 3. Return $(\varphi, \psi, \text{м})$.

Theorem 11.2. *The algorithm* `dep_dom_mon` *is correct and terminates.*

Proof. By proposition 9.9, the class of each of element in a facet F in step 3d depends on x_q. Consequently, whenever we perform a refinement during the execution, each dominant monomial of $\overline{\varphi}$ is also a dominant monomial of $f - [\mathfrak{S}^0_{q,1} \cdots \mathfrak{S}^0_{q,n_q}]f$. The remainder of the termination proof is now analogous to the termination proof of `dom_mon`, by applying the analogues of lemma 10.1 and lemma 10.2 to $f - [\mathfrak{S}^0_{q,1} \cdots \mathfrak{S}^0_{q,n_q}]f$ instead of f. □

Putting all pieces together, we have proved:

Theorem 11.3. **(Main theorem, weak form)** *Under the assumptions* **A1**, **A2** *and* **A3**, *there exists an algorithm which takes an $\mathfrak{L}$-exp-log expression in $x_1, \cdots, x_p$ over $\mathfrak{C}$ on input and which computes*

(a) A partition $\mathbb{T}^p = R_1 \amalg \cdots \amalg R_r$ of $\mathbb{T}^p$, which we denote by P;
(b) A generic effective normal basis B relative to P;
(c) An algorithm which computes the generic asymptotic expansion of f w.r.t. B relative to P at any order.

The regions $R_1, \cdots, R_r$ may be empty. □

11.8 Constraint checking

In this section, we assume the existence of an oracle to test the exp-log consistency of systems of constraints imposed on $\breve{\mathfrak{C}}$. Under this assumption, we will design a complete constraint checker for the asymptotic constraints in Σ. This in particular reduces the asymptotic expansion problem of multivariate exp-log functions to the correspondent constant problem. More precisely, we will prove the following theorem:

Theorem 11.4. **(Main theorem, strong form)** *Assume* **A1**, **A2**, **A3**, *and that the exp-log consistency of finite systems of exp-log constraints on $\mathfrak{C}$ can be checked by algorithm. Then there exists an algorithm which takes an $\mathfrak{L}$-exp-log expression in $x_1, \cdots, x_p$ over $\mathfrak{C}$ on input and which computes*

(a) A partition $\mathbb{T}^p = R_1 \amalg \cdots \amalg R_r$ of $\mathbb{T}^p$, which we denote by P;
(b) A generic effective normal basis B relative to P;
(c) An algorithm which computes the generic asymptotic expansion of f w.r.t. B relative to P at any order.

Each region R_i is non empty and represented as the solution set to a system Σ_i of exp-log equalities, inequalities and asymptotic relations.

Proof. An asymptotic constraint of the form (11.1) is said to be a q-**constraint** if it depends on x_q, but not on $x_1, \cdots, x_{q-1}$. A **normal constraint** is a constraint of one of the following forms:

$$\begin{cases} x_q \prec\!\!\!\prec ц; \\ x_q \succ\!\!\!\succ ц; \\ x_q \lll ц; \\ x_q \ggg ц; \\ x_q \underline{\lll} ц; \\ x_q \underline{\ggg} ц, \end{cases}$$

where $ц \in S_{>q}$. We say that Σ is **normal**, if all its constraints can be deduced from a subset Σ^{norm} of normal constraints; i.e. each $\sigma \in \Sigma$ is an expo-linear consequence of the saturation of Σ^{norm} (see section 11.3.2).

In section 11.8.1, we shall show how Σ can be normalized. In section 11.8.2, we show that if Σ is normal, then Σ is consistent if and only if it is expo-linearly consistent. This will clearly enable us to check the consistency of general systems Σ. □

11.8.1 Normalization of systems of asymptotic constraints

We introduce the following elimination ordering on constraints of the form (11.1): we write $\sigma <^{elim} \sigma'$ whenever the highest monomial (for $<^{elim}$ on monomials) occurring in σ is strictly smaller than the highest monomial occurring in σ'. Then we have the following normalization algorithm for Σ:

Algorithm `normalize`(Σ).
INPUT: A set of constraints Σ of the form (11.1).
ACTION: The algorithm normalizes Σ.

STEP 1. While Σ is not normal, let σ be a maximal non normal constraint in Σ for $<^{elim}$, and do the following:

- A. If σ has the form $ц \asymp 1$, then execute `constraint`($ц \asymp 1$).
- B. If σ has the form $ц \prec\!\!\!\prec 1$, then execute step 2.
- C. If σ has the form $ц \lll ц'$ resp. $ц \underline{\lll} ц'$, then execute step 3.

STEP 2. Ultra-regularize $ц$.
Let $б \in B$ be the maximal element for $\lll$ occurring in $ц$.
Let λ be the power of $б$ in $ц$ and impose the constraint $\lambda > 0$.

STEP 3. Ultra-regularize $ц$ and $щ$.
Replace $ц$ by a basis element $б$ with $ц \asymp\!\!\!\asymp б$.
Replace $ц'$ by a basis element $б'$ with $ц' \asymp\!\!\!\asymp б'$.
If the constraint $ц \lll ц'$ resp. $ц \underline{\lll} ц'$ is not normal, then impose the constraint $м(\log ц) \prec\!\!\!\prec м(\log ц')$ resp. $м(\log ц) \preceq\!\!\!\preceq м(\log ц')$.

Proposition 11.6. *Then above algorithm* `normalize` *is correct and terminates.*

Proof. The correctness of the algorithm is obvious. In order to prove its termination, we first observe that during the treatment of a q-constraint σ (in steps 1a, 1b, 1c, 2 and 3), the number $|B_q|$ does not increase. Moreover, after the treatment, σ can be deduced from constraints in Σ which are strictly smaller than σ for $<^{elim}$. These two properties clearly imply the termination of `normalize`. $\square$

11.8.2 Consistency of normal systems of constraints

Theorem 11.5. *Let Σ be a normal system of constraints, which is expo-linearly consistent. Then $R \neq \phi$.*

Proof. We prove the theorem by induction over p. For $p \geqslant 1$, the theorem obviously holds. Assume now that the system Σ' of all constraints in Σ which do not depend on x_1 is consistent. Then the region R' associated to Σ' is non empty, whence there exists a point $P' = (\hat{x}_2, \cdots, \hat{x}_p)$ in R'.

Let $\Sigma^{norm} \subset \Sigma$ be the subset of Σ of normal constraints, so that the constraints in Σ are expo-linear consequences of the saturation of Σ^{norm}. The comparability class constraints in $\Sigma^{norm} \backslash \Sigma'$ are each of one of the following forms:

$$\begin{cases} x_1 \prec\!\!\!\prec\!\!\!\prec \text{ц}; \\ x_1 \preceq\!\!\!\preceq\!\!\!\preceq \text{ц}; \\ \text{ц} \prec\!\!\!\prec\!\!\!\prec x_1; \\ \text{ц} \preceq\!\!\!\preceq\!\!\!\preceq x_1, \end{cases}$$

where ц does not depend on x_1. Among the ц occurring in such constraints, let ц^{max} resp. ц^{min} be the ones for which $\text{ц}(P')$ is maximal resp. minimal for $\prec\!\!\!\prec\!\!\!\prec$. By convention, we may have $\text{ц}^{max} = 1/\infty_{\mathbb{T}}$ resp. $\text{ц}^{max} = 1/\infty_{\mathfrak{C}}$.

The remaining constraints in $\Sigma^{norm} \backslash \Sigma'$ are of one of the following forms:

$$\begin{cases} x_1 \prec\!\!\!\prec \text{щ}; \\ \text{щ} \prec\!\!\!\prec x_1, \end{cases}$$

where щ does not depend on x_1. Among the щ occurring in such constraints, let щ^{max} resp. щ^{min} be the ones for which $\text{щ}(P')$ is maximal resp. minimal for $\prec\!\!\!\prec$. By convention, we may have $\text{щ}^{max} = \infty_{\mathfrak{T}}$ resp. $\text{щ}^{max} = 1/\infty_{\mathbb{T}}$.

Since Σ is expo-linearly consistent, we must have

$$\text{ц}^{min}(P') \preceq\!\!\!\preceq\!\!\!\preceq \text{щ}^{min}(P') \prec\!\!\!\prec \text{щ}^{max}(P') \preceq\!\!\!\preceq\!\!\!\preceq \text{ц}^{max}(P').$$

Hence, there exists a $\hat{x}_1 \in \mathbb{T}$ with $\text{щ}^{min} \prec\!\!\!\prec \hat{x}_1 \prec\!\!\!\prec \text{щ}^{max}$. Then $(\hat{x}_1, \cdots, \hat{x}_p)$ is a point in R, as desired. $\square$

11.9 Applications

A **Hardy field** is a field of germs of functions at infinity, which is stable under derivation. Hardy fields are the classical analytical analogues for fields of transseries. The analogues of many stability theorems for transseries also hold for Hardy fields and there is an extensive literature on this subject (see [Bour 61], [Rob 72], [Ros 83a], [Ros 83b], [Ros 87]). In this section, we give an example of how theorem 11.4 can be used to transfer such theorems directly from the transseries setting to the Hardy field setting.

Let $\mathbb{T}^{conv}$ be the field of convergent transseries, i.e. the field of $\mathfrak{C}[\![z_1, z_2, \cdots]\!]$-finite transseries. Clearly, $\mathbb{T}^{conv}$ is stable under differentiation, since $\mathfrak{C}[\![z_1, z_2, \cdots]\!]$ is stable under the partial derivations. We also notice that transseries in $\mathbb{T}^{conv}$ naturally converge in a neighbourhood of infinity, whence we may consider them as germs of functions at infinity.

Theorem 11.6. *$\mathbb{T}^{conv}$ is a Hardy field which is stable under composition, inversion and resolution of consistent zero-dimensional exp-log systems of equations.*

Proof. We may consider $\mathfrak{C}[\![z_1, z_2, \cdots]\!]$ as a theoretical effective exp-log local community of Laurent series. Therefore, the main theorems of this chapter apply if we take $\mathfrak{L} = \breve{\mathfrak{C}}[\![z_1, z_2, \cdots]\!]$.

Now let f and g be two convergent transseries in $\mathbb{T}^{conv}$, where g is positive and infinitely large. Expressing z as a function of x, by applying theorem 11.4 to eliminate of y and z from the equations $z = g(y)$ and $y = f(x)$ yields the composition of f by g.

Similarly, the functional inverse of g is obtained by expressing y as a function of x after elimination of y from the equation $f(y) = x$.

More generally, if a system of transseries equations in $\mathfrak{T}$ admits a finite number of solutions, then the solution set can theoretically be computed by theorem 11.4, whence all solutions must be convergent. □

Remark 11.2. Actually, more direct algorithms can be given for the computation of functional compositions and inverses (see [VdH 94c]), based on the formula's from sections 1.7.1 and 1.7.3. However, it is quite cumbersome to verify the preservation of convergence in a direct manner.

Although we have restricted ourselves here to convergent transseries, similar transfer theorems hold whenever we identify suitable exp-log local communities. For instance, if one is able to generalize the concept of multisummable Laurent series to several dimensions, and to prove that the set of such series forms an exp-log local community $\mathfrak{L}$, then the analogue of the above theorem would hold for $\mathfrak{L}$-finite transseries in several variables. If one is also able to generalize Braaksma's theorem (see [Br 92]), then this analogue of theorem 11.6 would encapsulate virtually all known stability theorems for Hardy fields (for germs with a natural origin *d'après* Écalle).

However, the above project can probably only be carried after some suitable modifications: in the next chapter, we shall see that the current definition of effective local communities is not general enough for the systematic treatment of differential equations. Nevertheless, we shall indicate how to generalize this concept in order to incorporate solutions to algebraic differential equations.

11.10 References

[Bour 61] N. Bourbaki. *Éléments de mathématiques chap. V: Fonctions d'une variable réelle.* Hermann, 2-nd ed. (Appendice p 36-55).

[Br 92] B.L.J. Braaksma. Multisummability of formal power series solutions of non linear meromorphic differential equations. *Ann. Inst. Fourier, Grenoble 42 (p. 517-540).*

[Mar 96] D. Marker. Model theory and exponentiation. *Notices of the A.M.S. 43(7) (p. 753-759).*

[Pél 95] A. Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948.*

[Rich 92] D. Richardson. The elementary constant problem. *Proc. ISSAC '92 (p. 108-116).*

[Rich 94] D. Richardson. How to recognize zero. *Preprint, University of Bath, England.*

[Ris 75] R.H. Risch. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math. 4, 101 (p. 743-759).*

[Rob 72] A. Robinson. On the real closure of a Hardy field. *In "Theory of sets and topology", G. Asser et al. ed., Berlin, Deut. Verlag. Wissenschaften.*

[Ros 83a] M. Rosenlicht. Hardy fields. *Journal of Math. Analysis and Appl. 93 (p. 297-311).*

[Ros 83b] M. Rosenlicht. The rank of a Hardy field. *Trans. Amer. Math. Soc. 280 (2) (p. 659-671).*

[Ros 87] M. Rosenlicht. Growth properties of functions in Hardy fields. *Trans. Amer. Math. Soc. 299 (1) (p. 261-272).*

[SS 96] B. Salvy, J. Shackell. Symbolic asymptotics: functions of two variables, implicit functions. *Tech. Rep. 2883, INRIA, France.*

[Tar 31] A. Tarski. *Sur les ensembles définissables de nombres réels.* Fund. Math. 17 (p. 210-239).

[VdH 94c] J. van der Hoeven. General algorithms in asymptotics II, Common operations. *Research report LIX/RR/94/10, École polytechnique, France.*

[VdH 96a] J. van der Hoeven. Generic asymptotic expansions. *Local Proc. of the Rhine workshop on computer algebra, Saint-Louis, April 1-3, France, Submitted to AAECC..*

Chapter 12

Algebraic differential equations

12.1 Introduction

This chapter establishes the effective counterpart of chapter 5. We show how to compute the generic solution to an asymptotic algebraic differential equation with grid-based transseries coefficients

$$P(f) = 0 \ (f \prec\!\!\!\prec \text{ч}), \tag{12.1}$$

under suitable effective hypothesis. Our algorithm has the particularity that we automatically introduce the necessary parameters, which correspond to the integration constants. The chapter is divided into three main parts: in section 12.2, we introduce a new sort of "lexicographically automatic Cartesian representations". In section 12.3, we show that the transseries solutions to the above equation are indeed grid-based. In section 12.4, we give the effective hypothesis and the algorithm.

In view of the undecidability results of Denef and Lipshitz (see [DL 89]) and Grigoriev and Singer (see [GS 91]), the existence of our algorithm might seem surprising. The fact that we can actually give an algorithm relies on two observations:

- The field of grid-based transseries is better behaved for the resolution of algebraic differential equations than the field of grid-based power series.
- Since we search for generic solutions, we may decide ourselves how we want to represent them.

Let us detail these issues.

In [GS 91], Grigoriev and Singer consider the following system Σ of differential equations in grid-based power series[1]:

$$\begin{cases} y' x = \beta y; \\ \beta' = 0; \\ z' y x + z'' x^2 = y + x. \end{cases}$$

[1]Actually, Grigoriev and Singer work with well-ordered power series whose supports are included in a finitely generated subgroup of $x^{\mathbb{R}}$.

They show that Σ admits a solution if and only if $\beta = n^{-1}$ for some $n \in \mathbb{N}$. Based on Σ, they construct a system T_φ of algebraic differential equations for any Diophantine equation φ, such that the existence of solutions to T_φ is equivalent to the existence of solutions to φ. The latter problem is known to be undecidable (see [Matij 70]), whence so is the former. What saves us, is that Σ always admits a natural transseries solution, although this transseries is not necessarily a grid-based series!

Another important issue is how to decide whether $Q(f) = 0$, where Q is another differential polynomial and f is a well determined solution to (12.1)? To see why this problem is difficult, let

$$f = \int g$$

be the primitive of some transseries g, by taking 0 for the integration constant (i.e. if $B = \{\mathfrak{6}_1, \cdots, \mathfrak{6}_n\}$ denotes the normal basis w.r.t. which we expand g, then $[\mathfrak{6}_1^0 \cdots \mathfrak{6}_n^0]g = 0$; stated differently, f is the distinguished solution to $f' = g$). Assume that we want to test whether $Q(f) = 0$. Replacing the derivatives of f by g, this question reduces to the case when Q is a polynomial.

Now the point is that although we can compute the roots of Q, we are not able in general to test whether the constant parts of these roots vanish. In other words, whenever we find a primitive $\tilde{f}$ of g among the roots, we can not test whether $\tilde{f} = f$! Nevertheless, we notice that this problem does not arise in the case when Q does not depend on parameters, since in this case, we can compute the canonical expansions of the roots of Q as shown in section 9.5.2. Unfortunately, this algorithm does not work any more in presence of parameters.

What saves us in this case, is that in order to compute the generic solution to $f' = g$, we may choose the integration constant ourselves: initially, we choose the primitive f_0 of g with integration constant 0 and represent the generic solution to $f' = g$ by $f_0 + \lambda$. Now if we need to test whether $Q(f) = 0$, somewhere later during the computations, and if the polynomial Q admits a root f_1 with $f_1' = g$, then we replace the previous generic solution $f_0 + \lambda$ to $f' = g$ by the generic solution $f_1 + \mu$. In other words, the knowledge that there exists a primitive for g, which is also a solution to an algebraic equation, enables us to represent the generic solution in a simpler way than in the general case. However, this knowledge may become available, only when we explicitly test whether one of the roots of some polynomial has derivative g! This issue will be treated in detail in section 12.4.4.

12.2 Cartesian representations reviewed

In this section, we introduce "lexicographically automatic" Laurent series and Cartesian representations. These are different from the previously defined, "symmetrically automatic" Laurent series in this respect that they can only be expanded automatically w.r.t. a single variable, the coefficients of this expansion being recursively expandable in the same way. Nevertheless, we *do* know that the series in question is a Laurent series (and not merely a series in some ring like $\mathfrak{C}((z_1))\cdots((z_k)))$, and we demand explicit bounds for the valuations in each variable. Hence, we do conserve some of the properties of the symmetrically automatic Laurent series, and this will enable us to generalize the algorithm `idm`.

So why the need for lexicographically automatic Laurent series? Consider for instance the primitive F of

$$f(x) = \frac{1}{(1-x^{-2})(1-e^{-x^2})}.$$

Clearly, F admits an asymptotic expansion w.r.t. the normal basis $\{x^{-1}, e^{-x^2}\}$, say $F(x) = u(z,\zeta) = u(x^{-1}, e^{-x^2})$. We have

$$F' = -z^2 u_z - 2z^{-1}\zeta u_\zeta = \frac{1}{(1-z^2)(1-\zeta)},$$

together with the initial condition

$$-z^2 u_z(z,0) = \frac{1}{1-z^2}.$$

The problem with this kind of equations is that although the expansion of u w.r.t. ζ is easily derived, the same does not hold for the expansion w.r.t. z. Even though this problem might still be feasible in this special case, we do not know of any general method in order to obtain such expansions. For this reason, we will now consider lexicographically automatic series.

12.2.1 Lexicographically automatic series

Lexicographically automatic power series. Let $\mathfrak{C}$ be an effective constant field and $\mathfrak{R}$ an effective $\mathfrak{C}$-algebra of series in $\mathfrak{C}[[z_1,\cdots,z_k]]$. We say that a subset $\mathfrak{A}$ of $\mathfrak{R}$ is **lexicographically automatic**, if there exists an algorithm, which takes $u \in \mathfrak{A}$ and $\alpha_l, \cdots, \alpha_k \in \mathbb{N}$ $(1 \leqslant l \leqslant k)$ on input, and which computes $[z_l^{\alpha_l}\cdots z_k^{\alpha_k}]u$; moreover, this coefficient is required to be in $\mathfrak{A}$. As before, if $\mathfrak{A}$ is a lexicographically automatic subset of $\mathfrak{R}$, then so is the $\mathfrak{C}$-algebra which is effectively generated by $\mathfrak{A}$. A series $u \in \mathfrak{R}$ is said to be **lexicographically automatic**, if it is contained in a lexicographically automatic subset of $\mathfrak{R}$.

Lexicographically automatic Laurent series. Let $\mathfrak{L}$ now be an effective $\mathfrak{C}$-algebra of Laurent series in $z_1,\cdots,z_k$, which contains $z_1, z_1^{-1},\cdots,z_k,z_k^{-1}$. We say that a subset $\mathfrak{A}$ of $\mathfrak{L}$ is **lexicographically automatic**, if

- There exists an algorithm, which takes $u \in \mathfrak{A}$ on input, and which computes lower bounds $\mu_1,\cdots,\mu_k$ for the valuations of u in $z_1,\cdots,z_k$ respectively.

– There exists an algorithm, which takes $u \in \mathfrak{A}$ and $\alpha_l, \cdots, \alpha_k \in \mathbb{N}$ ($1 \leqslant l \leqslant k$) on input, and which computes $[z_l^{\alpha_l} \cdots z_k^{\alpha_k}]u$; moreover, this coefficient is required to be in $\mathfrak{A}$.

Again, if $\mathfrak{A}$ is a lexicographically automatic subset of $\mathfrak{L}$, then so is the $\mathfrak{C}$-algebra which is effectively generated by $\mathfrak{A}$. A Laurent series $u \in \mathfrak{L}$ is said to be **lexicographically automatic**, if it is contained in a lexicographically automatic subset of $\mathfrak{L}$.

Lexicographically automatic transseries. We define lexicographically automatic transseries in a similar way as symmetrically automatic transseries in section 9.2.4, by requiring g to be a lexicographically automatic Laurent series instead of a symmetrically automatic Laurent series in **AT4**. It may be checked that the effective stability theorems from section 9.2.4 extend to the present context.

12.2.2 Lexicographical Cartesian representations

Let $\mathfrak{C}$ be an effective field of constants and $\mathfrak{X}$ an effective (quasi-ordered) monomial group. We assume that $\mathfrak{X}$ is generated by a finite number of monomials, on which expo-linear constraints are imposed. Contrary to what we did in section 9.3, in this chapter we will work with Cartesian representations of series in an effective $\mathfrak{C}$-subalgebra $\mathfrak{S}$ of $\mathfrak{C}[[\mathfrak{X}]]$. In other words, we do not assume the existence of a zero-test for the Cartesian representations, but for the series they represent. In particular, for the computation of intermediary dominant monomials, there is no need for an analogue of the algorithm `simplify`.

A **lexicographical Cartesian representation** of a series f in $\mathfrak{S}$ is a Laurent series u in $z_1, \cdots, z_k$, such that the Cartesian coordinates verify $z_1 \preccurlyeq_{\mathfrak{X}} \cdots \preccurlyeq_{\mathfrak{X}} z_k$. Here we notice that we may always order the Cartesian coordinates in such a way, by separating at most 2^k cases. We say that u is a **lexicographically automatic** Cartesian representation, if we have bounds $p_1, \cdots, p_k \in \mathbb{Z}$ for the valuations of u in $z_1, \cdots, z_k$, and an algorithm which given $\alpha_l, \cdots, \alpha_k \in \mathbb{Z}$ ($1 \leqslant l \leqslant k$) computes $[z_l^{\alpha_l} \cdots z_k^{\alpha_k}]u$, where we assume this series to be in $\mathfrak{S}$. By "formal nonsense", it follows that the coefficients $[z_l^{\alpha_l} \cdots z_k^{\alpha_k}]u$ are lexicographically automatic Cartesian representations as well.

In the remainder of this chapter, *Cartesian representations are always understood to be lexicographically automatic Cartesian representations.*

12.2.3 Computation of intermediary dominant monomials

We now have the following algorithm to compute intermediary dominant monomials in the context of lexicographically automatic Cartesian representations:

Algorithm `idm`(u).
INPUT: A Cartesian representation $u \in \mathfrak{C}[\![z_1, \cdots, z_k]\!]$ of a series $\overline{u} \in \mathfrak{S}$.
OUTPUT: A set of intermediary dominant monomials for u.

if $\overline{u} = 0$ **then return** ϕ
• Let $z_i^{p_i}$ be the dominant monomial of u in z_i, for $1 \leqslant i \leqslant k$.
$H := \{1\}$
while true
  $\alpha := p_k$
  $G := \phi$
  repeat
    $G := G \cup \mathtt{idm}([z_k^\alpha]u)$
    $\alpha := \alpha + 1$
  until $G \neq \phi$ and $\text{ц} \preccurlyeq_{\mathfrak{x}} z_1^{p_1} \cdots z_{k-1}^{p_{k-1}} z_k^\alpha$ for all $\text{ц} \in G$
  $G := G \cup \{z_1^{p_1} \cdots z_{k-1}^{p_{k-1}} z_k^\alpha\}$
  if G is an intermediary set of dominant monomials of u **then return** GH
  else
    • Let $\text{ц} = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ and $\text{ц}' = z_1^{\alpha_1'} \cdots z_k^{\alpha_k'}$ be in G, such that $\overline{\text{ц}} = \overline{\text{ц}'}$ is minimal for $\leqslant_{\mathfrak{x}}$, $\alpha_k < \alpha_k'$, and where α_k' is chosen minimal with these properties.
    $u := u + (\text{ц}' - \text{ц})([z_k^{\alpha_k}]u)$
    $H := H \cup \{\text{ц}/\text{ц}'\}H$

Proposition 12.1. *The above algorithm* `idm` *is correct and terminates.*

Proof. The correctness of `idm` is immediate from the following two observations: first, ц and ц′ with the required properties indeed exist in the before last step: for each ц_1 chosen maximal in G for $\preccurlyeq_{\mathfrak{x}}$, the $\text{ц}_2, \cdots, \text{ц}_i \in G$ with $\overline{\text{ц}_2} = \cdots = \overline{\text{ц}_i} = \overline{\text{ц}_1}$ satisfy $[\text{ц}_1]u + \cdots + [\text{ц}_i]u = 0$. Secondly, at the end of the algorithm, GH is indeed an intermediary set of dominant monomials for the original value of u; G is only an intermediary set of monomials for the final value of u.

Let us prove by induction on k that `idm` terminates. This is clear for $k = 0$. Let $k > 0$ and assume that we have proved the assertion for all smaller k. No infinite loops can occur in the repeat-until loop, since we assumed that $z_i \preccurlyeq_{\mathfrak{x}} z_k$ for all $i < k$. Furthermore, the step $u := u + (\text{ц}' - \text{ц})([z_k^{\alpha_k}]u)$ can be executed only a finite number of times, since it has the effect of increasing the valuation of u in z_k, while $\overline{u} \neq 0$ remains invariant. □

Remark 12.1. Instead of returning GH at the end of the algorithm, we may first eliminate its redundant elements. Notice also, that if we are allowed to change the Cartesian representation u during the execution (while preserving $\overline{u}$), then the set H has no use, and we may return G instead of GH.

Remark 12.2. As before, we also have algorithms for the infinitesimalization and regularization of Cartesian representations, based on `idm`.

12.3 Stability theorems

In chapter 5, we gave a theoretical algorithm to solve asymptotic algebraic differential equations in the case of well-ordered transseries. In this section, we show that if the coefficients of such an equation were actually grid-based transseries, then so are its solutions. Moreover, we describe some changes in the theoretical algorithm, which will allow a purely effective treatment in the next section. Because of its technical character, this section may be skipped without much harm at a first reading.

In view of the theoretical algorithm to solve asymptotic algebraic differential equations, it suffices to show that the distinguished solutions of quasi-linear equations with grid-based coefficients are also grid-based. In section 12.3.1, we give several ways to compute distinguished solutions. In section 12.3.2, we prove that the distinguished solutions of linear differential equations with grid-based coefficients are grid-based. In section 12.3.3, we treat the general case.

Since this section is purely theoretical, no effective assumptions need be made, and C denotes the real closed exp-log field of constants we are working over.

12.3.1 Distinguished solutions w.r.t. normal bases

When dealing with grid-based transseries, it is convenient to compute w.r.t. normal bases, especially for the effective purposes of the next section. The construction by transfinite induction of distinguished solutions to linear or quasi-linear differential equations corresponds to the case when the normal basis is actually a canonical basis. Indeed, if B is a canonical basis, then the elements of the asymptotic scale S_B generated by B are transmonomials. In general, distinguished solutions are constructed in a similar way as in chapters 4 and 5, but they depend on the choice of B.

In this section, we briefly explain the construction of distinguished solutions w.r.t. general normal bases $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$. We will also present an alternative, lexicographical construction: this yields the expansion in $\mathfrak{b}_n$, and, recursively, the full expansion w.r.t. B. By analogy, distinguished solutions w.r.t. general normal bases share all properties of the distinguished solutions as introduced in chapters 4 and 5. In principle, distinguished solutions w.r.t. B are transseries in $\mathfrak{C}[[[S_B]]]$. In sections 12.3.2 and 12.3.3, we will show that they are actually grid-based.

Classical construction of distinguished solutions. Before coming to the construction of distinguished solutions, we first notice that, more generally, potential dominant monomials and terms of solutions to asymptotic algebraic differential equations are easier to compute in the grid-based case than in the context from chapter 5. This is because the transfinite induction procedure in theorem 5.2 may now be replaced by a finite one: with the notations from chapter 5, we first compute the unique constant $\alpha_n \in \mathfrak{C}$ such that

$$\mathrm{M}(P_{\times \mathfrak{b}_n^{\alpha_n}, i}) \asymp_{\mathfrak{b}_n} \mathrm{M}(P_{\times \mathfrak{b}_n^{\alpha_n}, j}),$$

next the unique constant $\alpha_{n-1} \in \mathfrak{C}$ such that

$$\mathrm{M}(P_{\times \mathfrak{b}_{n-1}^{\alpha_{n-1}} \mathfrak{b}_n^{\alpha_n}, i}) \asymp_{\mathfrak{b}_n} \mathrm{M}(P_{\times \mathfrak{b}_{n-1}^{\alpha_{n-1}} \mathfrak{b}_n^{\alpha_n}, j}),$$

and so on for $\alpha_{n-2}, \cdots, \alpha_1$. In the linear case, this yields a finite procedure to compute $\mathrm{M}_{L^{-1}g}$ in section 4.3. In the quasi-linear case, this yields a finite procedure to compute $\mathrm{M}_{L^{-1}g}$, which is in particular needed for the construction of $L^{-1}g$.

Let us now come to the construction of distinguished solutions, which is analogous to the one given in the proof of theorem 5.5. This time, g and the coefficients of L are in $C[[[S_B]]]$, and we are looking for solutions in $C[\log_r x][[[S_{\{x^{-1}, \cdots \log_{r-1} x\} \cup B}]]]$. The main differences with respect to before are the following:

- The definition of stationary limits in $C[[[S_B]]]$ is analogous to the transseries case, except that we now write elements in $C[[[S_B]]]$ as strong linear combinations of monomials in S_B, instead of strong linear combinations of transmonomials.
- Using upward shiftings, we maintain the normal basis B of level 1 during the "transfinite computation": each time when x^{-1} need be inserted into B, we perform an upward shifting. In view of the Newton regularity, this may happen at most $r+1$ times. Alternatively, we shift $r+1$ times upwards at the beginning of the computation, thereby guaranteeing *a priori* that the distinguished solution will be expandable w.r.t. B.

Taking into account these two differences, we define the transfinite sequence (f_α) as before, and the last term of the sequence is the distinguished solution $L^{-1}g$ to $Lf = g$.

Lexicographical construction of distinguished solutions. Instead of computing the distinguished solution f term by term, one might also want to compute the successive terms of f, when expanded in $\mathfrak{b}_n$, the coefficients of this expansion being recursively expandable in a similar manner. To do so, we will construct a transfinite sequence (f_α^*) in a similar manner as before; but now $f_{\alpha+1}^*$ is obtained from f_α^* by adding a term τ_α in $C[[[\mathfrak{b}_1; \cdots; \mathfrak{b}_{n-1}]]]\mathfrak{b}_n^C$ instead of a term in CS_B. Actually, this sequence will be a subsequence of (f_α).

Before we show how to construct the sequence (f_α^*), let us first introduce some more notations. For any f, we denote by $\mu(f)$ or μ_f the valuation of f in $\mathfrak{b}_n$. We will also consider expansions of quasi-linear operators L w.r.t. $\mathfrak{b}_n$:

$$L = \sum_{\boldsymbol{i}} \sum_{\alpha} L_{\boldsymbol{i},\alpha} \mathfrak{b}_n^\alpha = \sum_\alpha ([\mathfrak{b}_n^\alpha] L)\, \mathfrak{b}_n^\alpha,$$

and we denote by $\mu(L)$ or μ_L the valuation of L in $\mathfrak{б}_n$. The coefficients $[\mathfrak{б}_n^\alpha]L$ of L w.r.t. B are themselves operators with coefficients in $C[[[S_{B\setminus\{\mathfrak{б}_n\}}]]]$. Moreover, the operator $[\mathfrak{б}_n^{\mu_L}]L$ is quasi-linear.

Now we define (f_α^*) by

$$\begin{aligned} f_0^* &= 0; \\ f_{\alpha+1}^* &= f_\alpha^* + \tau_\alpha; \\ f_\alpha^* &= \operatorname*{stat\,lim}_{\beta<\alpha} f_\beta^* \text{ (if } \alpha \text{ is a limit ordinal)}, \end{aligned}$$

where

$$\begin{aligned} \tau_\alpha &= c_\alpha \mathfrak{б}_n^{\mu_\alpha}; \\ c_\alpha &= ([\mathfrak{б}_n^{\mu(L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})^{-1}\,([\mathfrak{б}_n^{\mu_{g-Lf_\alpha^*}}](g-Lf_\alpha^*)); \\ \mu_\alpha &= \mu(L_{+f_\alpha^*}^{-1}(g-Lf_\alpha^*)) = \mu(g-Lf_\alpha^*) - \mu(L_{+f_\alpha^*,lin}). \end{aligned}$$

The fact that $[\mathfrak{б}_n^{\mu(L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}}$ is a quasi-linear operator with coefficients in $C[[[S_{B\setminus\{\mathfrak{б}_n\}}]]]$ guarantees the existence of a distinguished right-inverse for it.

Proposition 12.2.

(a) The sequence (f_α^*) is a subsequence of (f_α).

(b) The lexicographical and the classical constructions of distinguished solutions are equivalent.

Proof. It is straightforward to verify (a) and (b) follows directly from (a). □

12.3.2 Distinguished solutions: the linear case

Let $L = L_0 + \cdots + L_r\partial^r/\partial x^r$ be a linear differential operator with coefficients in $\mathfrak{C}[\![\mathfrak{б}_1;\cdots;\mathfrak{б}_n]\!]$. We study the action of L on $\mathfrak{C}[x][\![\mathfrak{б}_1;\cdots;\mathfrak{б}_n]\!]$, where we recall that B is purely exponential of level 1. The **usual support** $\operatorname{supp} L$ of L is defined to be the union of the supports of its coefficients $L_0,\cdots,L_r$. The **operator support** of L is defined to be the smallest set $\operatorname{supp}_* L$, such that

$$\operatorname{supp} Lg \subseteq (\operatorname{supp}_* L)(\operatorname{supp} g),$$

for all g. The operator L is actually a **grid-based linear operator**, in the sense that $\operatorname{supp}_* L$ is grid-based. Indeed,

$$\operatorname{supp}_* L \subseteq \operatorname{supp} L_0 \cup (\operatorname{supp} L_1)(\operatorname{supp}_* \partial/\partial x) \cup \cdots \cup (\operatorname{supp} L_r)(\operatorname{supp}_* \partial/\partial x)^r$$

and

$$\operatorname{supp}_* \partial/\partial x \subseteq \{x^{-1}\} \cup \operatorname{supp} \hat{\text{б}}_1 \cup \cdots \cup \operatorname{supp} \hat{\text{б}}_n,$$

where we recall that $\hat{f}$ stands for the logarithmic derivative of f.

In this section, we will show that the distinguished right inverse L^{-1} of L w.r.t. B as defined in the previous section is also a grid-based linear operator. This implies in particular that the distinguished solution to the equation $L^{-1} f = g$ is grid-based, for all g.

Lemma 12.1. *Let*

$$\begin{aligned}
\Phi &= \left\{ \frac{\text{ц}}{\text{м}(L_{\times \text{ц}})} \middle| \text{ц} \in S_B \right\};\\
\Gamma &= \bigcup_{\text{ц} \in S_B} \frac{\operatorname{supp} L_{\times \text{ц}}}{\text{м}(L_{\times \text{ц}})};\\
\hat{\Phi} &= x^{r - \mathbb{N}} \Phi;\\
\hat{\Gamma} &= x^{-\mathbb{N}} \cup x^{r - \mathbb{N}} (\Gamma \backslash \{1\}).
\end{aligned}$$

Then

$$\operatorname{supp}_* L^{-1} \subseteq \hat{\Phi} \hat{\Gamma}^{\diamond}.$$

Proof. We must show that for all g, the distinguished solution f to the equation $Lf = g$ satisfies $\operatorname{supp} f \subseteq \hat{\Phi} \hat{\Gamma}^{\diamond} \operatorname{supp} g$. Let the f_α be as in the classical construction of $L^{-1} g$ in the previous section. We will prove by transfinite induction over α that:

- $\operatorname{supp} (g - L f_\alpha) \subseteq \hat{\Gamma}^{\diamond} \operatorname{supp} g$.
- $\operatorname{supp} f_\alpha \subseteq \hat{\Phi} \hat{\Gamma}^{\diamond} \operatorname{supp} g$.

This is clear for $\alpha = 0$.

Assume that $\alpha = \beta + 1$ is a successor ordinal. Let $\text{щ} x^j$ be the dominant monomial of $g - L f_\beta$, where щ is purely exponential. By the induction hypothesis, $\text{щ} x^j \in \hat{\Gamma}^{\diamond} \operatorname{supp} g$. Let

$$\text{ц} x^i = \text{м}_{L^{-1}(g - L f_\beta)} = \text{м}_{L^{-1} \text{м}},$$

where ц is purely exponential. Then we have

$$\text{щ} = \text{м}(L_{\times \text{ц}}). \tag{12.3}$$

Now $f_\alpha = f_\beta + c \text{ц} x^i$ for some constant c. Hence,

$$\text{ц} = \text{щ} \frac{\text{ц}}{\text{м}(L_{\times \text{ц}})} \in \text{щ} \Phi$$

and $цx^i \in щx^j\hat{\Phi} \subseteq \hat{\Phi}\hat{\Gamma}^{\diamond}\operatorname{supp} g$. This proves the second induction hypothesis. As to the first, we have $g - Lf_\alpha = g - Lf_\beta - L(cцx^i)$. Now

$$L(cцx^i) = cL_{\times ц}x^i = cL_{\times ц,0}x^i + \cdots + ci!L_{\times ц,i},$$

whence

$$\operatorname{supp} L(cцx^i) \subseteq x^{i-\mathbb{N}}\operatorname{supp} L_{\times ц}.$$

Using (12.3), we get

$$\operatorname{supp} L(cцx^i) \subseteq щx^j x^{i-j-\mathbb{N}}\frac{\operatorname{supp} L_{\times ц}}{м(L_{\times ц})}$$

Since $i - j \leqslant r$ and $L(cцx^i) \sim щx^j$, we conclude that $\operatorname{supp} L(cцx^i) \subseteq щx^j\hat{\Gamma} \subseteq \hat{\Gamma}^{\diamond}\operatorname{supp} g$, whence the first induction hypothesis.

If α is a limit ordinal, then the induction hypothesis are trivially satisfied (in the case of the first hypothesis, we use the linearity of L). □

Lemma 12.2. *The sets $\hat{\Phi}$ and $\hat{\Gamma}$ from the previous lemma are grid-based.*

Proof. Each $ц \in S_B$ has the form $ц = б_1^{\alpha_1}\cdots б_n^{\alpha_n}$, whence we may consider $ц^{-1}L_{\times ц}$ as a linear differential operator with coefficients in $\mathfrak{C}[\alpha_1, \cdots, \alpha_n][\![б_1; \cdots; б_n]\!]$, whose support is a grid-based set Δ. Indeed, the coefficients of $ц^{-1}L_{\times ц}$ are given by

$$(ц^{-1}L_{\times ц})_i = \sum_{j=i}^{r} L_j ц^{(j-i)}/ц,$$

and the quotients $ц^{(j-i)}/ц$ may be rewritten as differential polynomials with constant coefficients in the logarithmic derivative $\hat{ц}$ of $ц$.

Since the ring $\mathfrak{C}[\alpha_1, \cdots, \alpha_n]$ is Noetherian, it follows that $м(ц^{-1}L_{\times ц})$ can only take a finite number of values, when varying $\alpha_1, \cdots, \alpha_n$. For each semi-algebraic subset of $(\alpha_1, \cdots, \alpha_n)$ on which $м(ц^{-1}L_{\times ц})$ is constant, we also have a uniform bound

$$\frac{\operatorname{supp} L_{\times ц}}{м(L_{\times ц})} = \frac{ц^{-1}\operatorname{supp} L_{\times ц}}{м(ц^{-1}L_{\times ц})} \subseteq \frac{\{щ \in \Delta | щ \preccurlyeq м(ц^{-1}L_{\times ц})\}}{м(ц^{-1}L_{\times ц})},$$

for $\operatorname{supp} L_{\times ц}/м(L_{\times ц})$. Hence, the sets Φ, Γ, $\hat{\Phi}$ and $\hat{\Gamma}$ are indeed grid-based. □

Combining both lemmas, we have proved:

Theorem 12.1. *The distinguished linear right inverse L^{-1} of L is grid-based.* □

12.3.3 Distinguished solutions: the quasi-linear case

We now turn to the case of a quasi-linear operator L of order r with coefficients in $\mathfrak{C}[[\mathfrak{б}_1;\cdots;\mathfrak{б}_n]]$. Given $g\in\mathfrak{C}[[\mathfrak{б}_1;\cdots;\mathfrak{б}_n]]$, we study the distinguished solution $L^{-1}g$ to the equation $Lf=g$, as defined in section 5.4.4. Modulo a suitable number of upward shiftings, we may assume without loss of generality that B has level 1, and that $L^{-1}g$ is expandable w.r.t. B.

Consider the lexicographical construction of $L^{-1}g$. The dominant term f_1^* of $L^{-1}g$ w.r.t. $\mathfrak{б}_n$ is the distinguished solution to the equation

$$([\mathfrak{б}_n^{\mu(L_{\times\mathfrak{б}_n^{\mu_0}})}]L_{\times\mathfrak{б}_n^{\mu_0}})\, c_0=([\mathfrak{б}_n^{\mu_{g-Lf_\alpha^*}}](g-Lf_\alpha^*)),$$

which is quasi-linear in general. The remaining coefficients are solutions of the equations

$$([\mathfrak{б}_n^{\mu(L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})^{-1}\ c_\alpha([\mathfrak{б}_n^{\mu_{g-Lf_\alpha^*}}](g-Lf_\alpha^*)),$$

with $\alpha>0$. We will now show that these equations are actually linear and not merely quasi-linear. Moreover, there exists a linear differential operator L, such that

$$c_\alpha=L^{-1}_{\times\mathfrak{б}_n^{\mu_\alpha}}\ ([\mathfrak{б}_n^{\mu_{g-Lf_\alpha^*}}](g-Lf_\alpha^*)),$$

for all $\alpha>0$.

Lemma 12.3.

(a) $[\mathfrak{б}_n^{\mu(L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}}$ *is a linear differential operator for all* $\alpha>0$.

(b) The linear differential operator

$$L=([\mathfrak{б}_n^{\mu(L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_\alpha}})_{\times\mathfrak{б}_n^{-\mu_\alpha}} \qquad (12.4)$$

is independent of α*, for* $\alpha>0$.

Proof. For $\alpha>0$, we have

$$\mu(L_{+f_\alpha^*,\times MB_n^{\mu_0}})=\mu(L_{+f_\alpha^*,\times MB_n^{\mu_0},lin}).$$

Since $\mu_\alpha>\mu_0$, it follows by (5.6) that

$$\mu(L_{+f_\alpha^*,\times MB_n^{\mu_1}})=\mu(L_{+f_\alpha^*,\times MB_n^{\mu_1},lin})>\mu(L_{+f_\alpha^*,\times MB_n^{\mu_1},rest}).$$

This proves (a).

For $\alpha>1$, we have $f_\alpha^*-f_1^*\asymp\mathfrak{б}_n$, whence

$$L_{+f_\alpha^*,\times\mathfrak{б}_n^{\mu_1}}=L_{+f_1^*,\times\mathfrak{б}_n^{\mu_1}}+o_{\mathfrak{б}_n}(\mathfrak{б}_n^{\mu(L_{+f_1^*,\times\mathfrak{б}_n^{\mu_1}})}),$$

by (5.6). Hence,

$$L_{+f_\alpha^*,\times б_n^{\mu_\alpha}} = L_{+f_1^*,\times б_n^{\mu_\alpha}} + o_{б_n}(б_n^{\mu(L_{+f_\alpha^*,\times б_n^{\mu_\alpha}})}),$$

by (5.7). Hence,

$$([б_n^{\mu(L_{+f_\alpha^*,\times б_n^{\mu_\alpha}})}]L_{+f_\alpha^*,\times б_n^{\mu_\alpha}})_{\times б_n^{-\mu_\alpha}} = ([б_n^{\mu(L_{+f_1^*,\times б_n^{\mu_1}})}]L_{+f_1^*,\times б_n^{\mu_1}})_{\times б_n^{-\mu_1}},$$

for all $\alpha > 1$, because of (a). This proves (b). □

From both lemmas and theorem 12.1, we now deduce the following:

Lemma 12.4. *The distinguished solution to $Lf = g$ is grid-based.*

Proof. We prove the theorem by induction over n. For $n = 0$, we have nothing to prove. Assume therefore that $n > 0$. By the induction hypothesis, f_1^* is grid-based. Modulo the replacement of g by $g - Lf_1^*$, we may assume without loss of generality that (12.4) holds for all α. Let

$$\begin{aligned} L &= L + R; \\ \Delta &= \operatorname{supp} \widehat{б_1} \cup \cdots \cup \operatorname{supp} \widehat{б_n}, \end{aligned}$$

and define

$$\begin{aligned} \Phi &= (\operatorname{supp}_* L^{-1})(\operatorname{supp} g); \\ \Gamma &= \bigcup_{\boldsymbol{i}} (\operatorname{supp}_* L^{-1})(\operatorname{supp} R_{\boldsymbol{i}})\Phi^{|\boldsymbol{i}|-1}\Delta^{\|\boldsymbol{i}\|}. \end{aligned}$$

Let us show by transfinite induction that

$$\operatorname{supp} f_\alpha \subseteq \Phi\Gamma^\diamond$$

for all α. This is clear for $\alpha = 0$ and for limit ordinals α.

Assume that $\alpha = \beta + 1$ is a successor ordinal. We claim that $\operatorname{supp}(g - Lf_\beta^* - R(f_\beta^*)) \subseteq \operatorname{supp} g \cup \operatorname{supp} R(f_\beta^*)$. Indeed, by the construction of f_β^*, all terms of Lf_β^* cancel out against terms of $g - R(f_\beta^*)$. It follows that

$$\begin{aligned} \operatorname{supp} f_\alpha^* &\subseteq \operatorname{supp} f_\beta^* \cup \operatorname{supp} \tau(L^{-1}(g - Lf_\beta^* - R(f_\beta^*))) \\ &\subseteq \operatorname{supp} f_\beta^* \cup (\operatorname{supp}_* L^{-1})(\operatorname{supp} g \cup \operatorname{supp} R(f_\beta^*)) \\ &\subseteq \Phi\Gamma^\diamond \cup \left(\bigcup_{\boldsymbol{i}} (\operatorname{supp}_* L^{-1})(\operatorname{supp} R_{\boldsymbol{i}})(\operatorname{supp}(f_\beta^*)^{(\boldsymbol{i})})\right) \\ &\subseteq \Phi\Gamma^\diamond \left(\bigcup_{\boldsymbol{i}} (\operatorname{supp}_* L^{-1})(\operatorname{supp} R_{\boldsymbol{i}})\Phi^{|\boldsymbol{i}|-1}\Delta^{\|\boldsymbol{i}\|}\right) \\ &\subseteq \Phi\Gamma^\diamond. \end{aligned}$$

This completes the proof of the theorem. □

In view of this lemma and the theoretical resolutions algorithm, we have proved the main theorem of this section:

Theorem 12.2. *Any transseries solution to an algebraic differential equation with grid-based transseries coefficients is itself grid-based.* □

12.4 Effective asymptotic resolution of a.d.e.'s

In this section, we show how to compute the generic solution to the asymptotic algebraic differential equation (12.1) under suitable effective hypothesis. For this purpose, we use the theoretical algorithm from section 5.5, with the following changes:

- The distinguished solutions are now computed w.r.t. a dynamic normal basis B, as explained in section 12.3.1 and below.
- Instead of computing with distinguished solutions, we will sometimes compute with semi-distinguished solutions (see section 12.4.4).

Of course, we also have to fix an effective context and to show how to render the theoretical algorithm effective in this context. This is the object of section 12.4.1. The sections 12.4.2, 12.4.3 and 12.4.4 deal with effective extensions of the effective field of transseries coefficients we compute in.

12.4.1 Basics for the effective treatment

Basic assumptions. In the rest of this chapter, we make the following effective hypotheses:

A1. $\mathfrak{C}$ is an effective totally ordered constant field.
A2. $\mathfrak{T}$ is an effective differential field of transseries over $\mathfrak{C}$.

In what follows, we allow ourselves to enlarge $\mathfrak{T}$ with real parameters, logarithms, (certain: see below) exponentials and distinguished solutions of quasi-linear equations. In the next sections, we will give effective zero-tests[2] in such extensions fields.

As in previous chapters, we allow the imposition of constraints on the real parameters by which we extend $\mathfrak{T}$. Contrary to before, we allow the imposition of any

[2]Actually, we will not really give a full zero-test in the case of extensions by distinguished solutions. Nevertheless, we will present an approach which is equivalent to giving such a zero-test for the purposes we have in mind.

first order formula, using constants in $\mathfrak{C}$, the exp-log field operations and the ordering operation. The regions defined by such constraints do not coincide with the regions defined by the exp-log systems considered in chapters 7 and 11; see [VdD 84] and (0.14). When necessary, we assume the existence of an oracle to check the consistency of systems of such constraints. As usual, such an oracle is used to eliminate empty regions, but it suffices to check for real algebraic consistency in order to guarantee termination. This can for instance be done using cylindrical decomposition (see [Col 75]).

The normal basis B**.** In the remainder of this chapter, we will always work w.r.t. the effective dynamic normal basis $B = \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ of level 1. Initially, the coefficients of (12.1) can be effectively expanded w.r.t. B, after having them made purely exponential by means of upward shiftings (i.e. the coefficients of P admit automatic Cartesian representations relative to $\mathfrak{T}$, where the Cartesian coordinates are in $\underline{S_B}$). On the other hand, we will always assume that the elements in $\mathfrak{T}$ can all be expanded w.r.t. B.

Automatic upward shiftings. In order to keep the normal basis purely exponential of level 1, we use upward shiftings and the automatic updating strategy: we introduce a global level l, which is increased each time an upward shifting is necessary. To each transseries f (resp. differential polynomial with transseries coefficients, etc.) we consider during the computations, we also associate a level. If this level is different from the global level l, when we access the transseries, then the necessary upward shiftings of f are made. This may lead to the insertion of new (inverses of) logarithms into B and the extension of $\mathfrak{T}$ with new logarithms.

Insertion of new exponentials into B**.** It may happen that a monomial $\mathfrak{q}$ is a potential dominant monomial w.r.t. a normal overbasis of B, but not w.r.t. B itself. We need be able to detect this situation, and insert the necessary elements into B when it occurs. Now in order to find the potential dominant (i,j)-monomials of f (see section 5.5.3), it suffices to perform at most one additional upward shifting, whence no new exponentials need be inserted into B.

In the case of potential dominant i-monomials, we first solve the corresponding Riccati equation at degree i. If φ is a solution modulo $o(1)$, then we compute e^{φ} by the usual exponentiation algorithm. The dominant monomial of e^{φ} is now a potential dominant i-monomial. Here we notice that both the computations of φ and e^{φ} may provoke the insertion of new elements into B. We also notice that the field $\mathfrak{T}$ is extended by each new exponential which is inserted into B; the exponential of “the infinitesimal part” of φ is not needed for further computations.

Cartesian representations of distinguished solutions. The lexicographical construction in section 12.3.1 shows in principle how to compute the expansion of the distinguished solution to quasi-linear equation $Lf = g$ term by term. However, since we represent all our transseries by Cartesian representations, we also need a Cartesian representation for the distinguished solution. We will now explain how to compute such a Cartesian representation.

We start by regularizing all coefficients of the quasi-linear operator, and extract their dominant coefficients w.r.t. $\mathfrak{b}_n$. This allows us to compute (recursively) the first term τ of the expansion of f w.r.t. $\mathfrak{b}_n$. We then reduce the general case to the case when (12.4) holds for all α, by replacing g with $g - L\tau$ and searching a solution to $Lf = g$.

Next, we compute a set $Z = \{z_1, \cdots, z_k\}$ of Cartesian coordinates with $z_1 \succ\!\!\succ \cdots \succ\!\!\succ z_k$, w.r.t. which f can be expanded, using the effective bounds for $\operatorname{supp} f$ given in sections 12.3.2 and 12.3.3. We take this set Z sufficiently large, so that it contains all Cartesian coordinates of the coefficients of L. We now let $z_l \succ\!\!\succ \cdots \succ\!\!\succ z_k$ be those Cartesian coordinates in Z which depend on $\mathfrak{b}_n$. The expansion of the Cartesian representation of f w.r.t. Z is now computed lexicographically, by expanding first in z_k, then in z_{k-1} and so on. To do this, we proceed in a similar manner as in the lexicographical construction of distinguished solutions, where the elements of Z now play the roles of the normal basis elements.

We must finally show that we indeed compute a Cartesian representation of the distinguished solution in this way. Let L be the linear operator from lemma 12.3. By proposition 4.7, the distinguished solution is unique with the property that $\operatorname{supp} f \cap \{\mathfrak{m}_{h_1}, \cdots, \mathfrak{m}_{h_s}\} = \phi$, where $h_1 \prec\!\!\prec \cdots \prec\!\!\prec h_s$ form a basis for the solutions space of the homogeneous equation $Lh = 0$. But the transseries represented by the Cartesian representation computed by the above construction clearly satisfies this property.

12.4.2 Effective extensions of $\mathfrak{T}$ by logarithms and exponentials

In the previous section, we have seen that we sometimes need to extend $\mathfrak{T}$ by logarithms or exponentials during the computation of privileged refinements. We have also observed that we only need to extend $\mathfrak{T}$ by (inverses of) logarithms and exponentials $\mathfrak{b}$, which are also inserted into B.

Let B be the normal basis before we insert $\mathfrak{b}$. We recall that we assumed that all elements in $\mathfrak{T}$ can be expanded w.r.t. B. We claim that a polynomial $R(\mathfrak{b})$ in $\mathfrak{b}$ with coefficients in $\mathfrak{T}$ vanishes, if and only if $R = 0$. Indeed, from our hypothesis on $\mathfrak{T}$ it follows that the supports of the different terms of R are pairwise disjoint. Consequently, we have a straightforward zero-test in $\mathfrak{T}(\mathfrak{b})$. We also notice that the elements in $\mathfrak{T}(\mathfrak{B})$ can again be expanded w.r.t. $B \cup \{\mathfrak{b}\}$. Hence, we may recursively extend $\mathfrak{T}$ by new logarithms and exponentials.

12.4.3 Recall of some results from differential algebra

In this section P denotes an arbitrary differential polynomial, and not necessarily the P from (12.1).

Let f be a solution to an algebraic differential equation $P(f) = 0$. Assume that Q is another differential polynomial and suppose that we want to test whether $Q(f) = 0$. Often this question can be answered by using only algebraic considerations. By analogy with classical algebra, this is done by computing some kind of greatest common divisor of P and Q. If this greatest common divisor is a scalar, then we know that $Q(f) \neq 0$. If the greatest common divisor is P, then we know that $Q(f) = 0$. In the remaining case, we need more information to decide, but we still gained something. Namely, a simpler equation then $P(f) = 0$ must be satisfied by f, in order $Q(f) = 0$ to hold true.

Actually, there is no straightforward analogue of the g.c.d. of two polynomials in the differential setting. However, something like the above still holds in this case. To show this, we will use Ritt reduction (see [Ritt 50], or [Kap 76]), and an algorithm which resembles a lot the Boulier-Seidenberg-Ritt algorithm for testing whether a differential polynomial belongs to a given perfect ideal (see [Boul 94], [BLOP 95], [VdH 96c]).

We will say that a differential polynomial $P(f)$ is **simpler** than another differential polynomial $Q(f)$, if the order of P is strictly inferior to the order of Q, or the orders r of P and Q coincide and the degree of P in $f^{(r)}$ is strictly inferior of the degree of Q in $f^{(r)}$; we write $P \prec Q$. Let $P(f)$ be a non zero differential polynomial of order r, and write $P = P_d(f^{(r)})^d + \cdots P_0$ as a polynomial in $f^{(r)}$. The initial of P is defined by $I_P = P_d$ and the separant of P by $S_P = \partial P / \partial f^{(r)}$. The differential polynomial $H_P = I_P S_P$ is always simpler then P.

If Q is another differential polynomial, and the the coefficients of P and Q live in an effective differential field, then Ritt gave an algorithm to compute $i \in \mathbb{N}$, a linear combination $\boldsymbol{\omega} P$ of $P, P', P'', \cdots$ and a differential polynomial R which is simpler then P, such that

$$H_P^i Q = \boldsymbol{\omega} P + R.$$

In particular, f is a simultaneous solution to P and Q, if and only if $H_P(f) \neq 0$ and $P(f) = R(f) = 0$, or $H_P(f) = P(f) = Q(f) = R(f) = 0$.

More generally, we may consider pairs (Σ, T), where $\Sigma = \{P_1, \cdots, P_s\}$ is a non empty set of non zero differential polynomials, and T a single differential polynomial. Such a pair corresponds to the **system** $P_1(f) = \cdots = P_s(f) = 0$ and $T(f) \neq 0$ of differential equations and inequations. We say that (Σ, T) is **elementary**, if Σ is a singleton. We say that a system (Σ, T) is **simpler** than a system (Σ', T'), if (Σ, T) is elementary, or there exists a $P \in \Sigma$, which is simpler than all elements in Σ'.

Let (Σ, T) be a non elementary system and write $\Sigma = \{P, Q_1, \cdots, Q_s\}$, where P is simplest among the elements in Σ. Now compute $\alpha_i, \boldsymbol{\omega}_i$ and R_i as above, such that

$$H_P Q_i = \boldsymbol{\omega}_i P + R_i,$$

for each $1 \leqslant i \leqslant s$. Let $\Sigma' = \{R_i | R_i \neq 0\}$. Then the systems $(\Sigma' \cup \{P\}, T H_P)$ and $(\Sigma' \cup \{H_P\} \cup \Sigma, T)$ are both simpler than (Σ, T), their solutions spaces are disjoint, and f is a solution to (Σ, T), if and only if f is a solution to one of the these systems.

In general, a system (Σ, T) is said to be **equivalent** to a list L of systems whose solution spaces are mutually disjoint, if f is a solution to (Σ, T) if and only if f is a solution to one of the systems in L. Then the above generalization of Ritt-reduction yields a procedure, which given a system (Σ, T) with $|\Sigma| \geqslant 2$ computes an equivalent list of simpler systems. Repeating this procedure on each of the systems in the obtained list, we eventually obtain an equivalent list of elementary systems. Indeed, there exists no infinite chain of simpler and simpler systems. We have proved:

Theorem 12.3. *There exists an algorithm, which given a system (Σ, T) computes an equivalent list of elementary systems.* □

In what follows, we will apply this theorem in the case when $(\Sigma, T) = (\{P, Q\}, 1)$. In this case, we obtain an equivalent list of elementary systems, which describes the set of common solutions to P and Q.

12.4.4 Effective extensions of $\mathfrak{T}$ by distinguished solutions

Assume that we want to adjoin the distinguished solution f to a quasi-linear equation $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$[3] to $\mathfrak{T}$, where P has coefficients in $\mathfrak{T}$. We have to design an algorithm in order to test whether $Q(f) = 0$ for arbitrary differential polynomials Q with coefficients in $\mathfrak{T}$. If Λ_0 denotes the set of parameters in $\mathfrak{T}$, we may formulate the problem in a different way: determine the region R of $\mathfrak{C}^{\Lambda_0}$ on which $Q(f) = 0$. We will assume by induction that we know how to solve this problem for all distinguished solutions of simpler differential polynomials than P. We know how to treat the base step of this induction, namely the case of linear equations.

The first step of our algorithm consists of computing an equivalent list of simpler elementary systems $(\{A_1\}, T_1), \cdots, (\{A_l\}, T_l)$ for the system $(\{P, Q\}, 1)$, by the algorithm from the previous section. Then the region R can be written as a disjoint union

$$R = R_1 \amalg \cdots \amalg R_l,$$

where R_i is the region on which $A_i(f) = 0$ and $T_i(f) \neq 0$ for all i.

[3] For the purposes of this section, the notation $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$ is more convenient than $Lf = g$.

The simple case. Let us first consider the case when (for all possible values of the parameters in Λ_0) the distinguished solution is the *unique* solution to the quasi-linear equation $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$. In this case, whenever for some i, $\tilde{f}$ is a solution to the asymptotic differential equation

$$A_i(\tilde{f}) = 0 \ (\tilde{f} \prec\!\!\prec 1) \tag{12.5}$$

with the extra condition

$$T_i(\tilde{f}) \neq 0, \tag{12.6}$$

then we *must* have $\tilde{f} = f$. On the other hand, we must have $A_i(f) = 0$ $(f \prec\!\!\prec 1)$ with $T_i(f) \neq 0$ for some i, if f is a solution to $Q(f) = 0$. Hence, we have reduced our problem to determining whether for a fixed index i, there exists a solution to (12.5) with the extra condition (12.6).

By the induction hypothesis, we can solve the equation (12.5).[4] Again by the induction hypothesis, we can determine the common solutions to (12.5) and $T_i(\tilde{f}) = 0$. More precisely, each generic solution $\tilde{f}$ to (12.5) leads to the introduction of a new set of parameters Λ_1. For each such solution, we then obtain the region S of $\mathfrak{C}^{\Lambda_0 \amalg \Lambda_1}$ on which $T_i(\tilde{f})$ vanishes. Then $\mathfrak{C}^{\Lambda_0 \amalg \Lambda_1} \setminus S$ is the region on which $T_i(\tilde{f})$ does not vanish. The projections of these regions on $\mathfrak{C}^{\Lambda_0}$ (when taking into account all generic solutions to (12.5)) yield the desired region on which there exists a solution to (12.5) under the condition (12.6). It is in order to carry out these projections, that we need an oracle for determining the consistency of first order formulas in the theory of exp-log fields, and not merely the consistency of exp-log systems in the sense of chapter 11.

The general case. In general, f is not the unique solution to $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$, so we can not guarantee that a given solution to (12.5) and (12.6) is the same one as f. Although a test whether this is the case can be designed in most of the cases encountered in practice (by using the properties of the supports of distinguished solutions), we were not able to design a fully general test.

Nevertheless, we will now present two "dirty tricks", which provide the last piece of our algorithm to solve algebraic differential equations. Actually, our trick consists of working with *dynamically* determined **semi-distinguished solutions** instead of distinguished solutions. By default, the semi-distinguished solution to a quasi-linear equation $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$ is the distinguished solution. However, when we find *a posteriori* a solution to $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$, which is also a solution to a simpler algebraic differential equation, then we will use this solution instead as the semi-distinguished solution.

Coming back to our original problem, such replacements *a posteriori* of distinguished solutions by semi-distinguished solutions in privileged refinements occur

[4]We notice that it suffices to solve this equation w.r.t. the current normal basis B; i.e. any solutions which necessitate the insertion of new exponentials are discarded.

precisely then, when we can not decide whether a solution to (12.5) under the condition (12.6) is the distinguished solution to $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$. More precisely, this situation amounts to the separation of $l + 1$ cases: on the complement of the region R, the result to the test is negative and no additional action is undertaken. On each region R_i, we compute a solution $\tilde{f}$ to (12.5) under the condition (12.6). Next, all computations which depend on f are done over, where f is substituted by $\tilde{f}$.

Termination is guaranteed by the fact that at each replacement of a distinguished solution by a semi-distinguished solution, the semi-distinguished solution satisfies simpler and simpler algebraic differential equations.

Remark 12.3. We qualified our trick as "dirty", because we may have to do over part of the computations. This amounts to quite complicated and unconventional control structures. To handle this problem, we propose the following approach, based on a variant of the automatic case separation strategy: at each time we introduce a distinguished solution, we separate a usual case, and a virtual case. Whenever the distinguished solution need be replaced by another solution, we kill the current process, and activate a virtual process in which we work with the new solution instead of the distinguished solution.

Remark 12.4. We notice that distinguished solutions are needed at two places: in privileged refinements, and recursively during the computation of the expansion of a distinguished solution. In particular, the replacement of an iterated coefficient c of a distinguished solution f to some quasi-linear equation $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$ by an iterated coefficient $\tilde{c}$ which is no longer a distinguished solution to some equation, yields a new solution $\tilde{f}$ to $P(f) = 0$ $(f \prec\!\!\prec \text{ч})$, which is only semi-distinguished! In particular, suitable new Cartesian coordinates for $\underline{\tilde{f}}$ are computed in this case.

12.5 Conclusion

Putting together the results from this chapter, we have proved

Theorem 12.4. *Let $\mathfrak{C}$ be an effective exp-log field. Then there exists an algorithm which takes on input*

(a) An effective differential field $\mathfrak{T}$ of transseries over $\mathfrak{C}$;
(b) An effective normal basis $B \subseteq \mathfrak{T}$;
(c) An asymptotic algebraic differential equation (12.1), whose coefficients can effectively be expanded w.r.t. B,

and which computes

(a) A finite family $(\Lambda_i, \Sigma_i)_{1 \leqslant i \leqslant l}$ of pairs of finite sets of real parameters and first order systems of exp-log constraints;
(b) For each $1 \leqslant i \leqslant l$, an expansion algorithm for a virtual generic transseries solution f_i to (12.1), parameterized by Λ_i, where the parameters satisfy Σ_i,

such that $f_1, \cdots, f_l$ yield all solutions to (12.1) in a non redundant way.

Modulo an oracle to determine the consistency of first order systems of exp-log constraints, we may eliminate those indices i, for which Σ_i has no solutions. In this case, the f_i are generic solutions and not merely virtual generic solutions to (12.1). □

Remark 12.5. Although the algorithm is finite, the actual expansion of the trans-series solutions to (12.1) may involve a potentially unbounded number of case separations. For instance, consider the differential equation

$$f' = \frac{x^\lambda}{1 - x^{-1}},$$

where λ is a real parameter and $x \to \infty$. The computed generic solution to this equation is

$$f = \int \frac{x^\lambda}{1 - x^{-1}} + \mu,$$

where the integration constant is zero. However, expansion of this solution up to the first $n \in \mathbb{N}^*$ terms yields the following result:

$$\begin{array}{rcl}
\lambda < -1 & : & f = \mu + \frac{1}{\lambda+1} x^{\lambda+1} + \cdots + \frac{1}{\lambda-n+3} x^{\lambda-n+3} + O(x^{\lambda-n+2}). \\
\lambda = -1 & : & f = \mu + \log x - x^{-1} - \cdots - \frac{1}{n-2} x^{-n+2} + O(x^{-n+1}); \\
-1 < \lambda < 0 & : & f = \frac{1}{\lambda+1} x^{\lambda+1} + \mu + \cdots + \frac{1}{\lambda-n+3} x^{\lambda-n+3} + O(x^{\lambda-n+2}). \\
\lambda = 0 & : & f = x + \log x + \mu - x^{-1} - \cdots - \frac{1}{n-3} x^{-n+3} + O(x^{-n+3}); \\
 & \vdots & \\
\lambda = n-3 & : & f = \frac{1}{n-2} x^{n-2} + \cdots + x + \log x + \mu + O(x^{-1}); \\
\lambda > n-3 & : & f = \frac{1}{\lambda+1} x^{\lambda+1} + \cdots + \frac{1}{\lambda-n+3} x^{\lambda-n+2} + O(x^{\lambda-n+1}).
\end{array}$$

Here we assumed that $\mu \neq 0$; a similar list is returned for the other case.

Remark 12.6. If we want to compute the first n terms of the expansion of a generic solution to the differential equation, the computations may induce substitutions of auxiliary distinguished solutions by semi-distinguished solutions. Such substitutions may actually alter the expansions which have already been computed! Nevertheless, if we allow terms in the expansion to be zero, then no zero-tests are needed during the expansion, and this problem does not occur. Furthermore, if all parameters are substituted by constants, then we are able to perform all necessary zero-tests effectively, by computing canonical expansions (see section 9.5.2).

Remark 12.7. The generic solutions to algebraic differential equations may be used themselves to extend $\mathfrak{T}$. Hence, we may recursively solve algebraic differential equations whose coefficients are solutions to other algebraic equations. However, it should be noted that such computations may alter the representations of the coefficients of the equation.

Remark 12.8. Since arbitrary D-algebraic[5] systems of ordinary algebraic differential equations may be reduced to equivalent lists of elementary systems (see section 12.4.3) by a straightforward generalization of the reduction algorithm, our results may in principle be used to solve such systems. However, from the efficiency point of view, a simultaneous resolution algorithm for systems of asymptotic algebraic differential equations would be preferable. We have not studied this issue in detail, but we think that our methods generalize to this case without to many difficulties.

12.6 References

[Col 75] G.E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Proc. 2-nd conf. on automata theory and formal languages, Springer Lect. Notes in Comp. Sc. 33 (p. 134-183).*

[DL 89] J. Denef, L. Lipshitz. Decision problems for differential equations. *The Journ. of Symb. Logic 55(3) (p. 941-950).*

[GS 91] D. Grigoriev, M. Singer. Solving ordinary differential equations in terms of series with real exponents. *Trans. of the A.M.S. 327(1) (p 329-351).*

[Kap 76] I. Kaplanski. *An introduction to differential algebra.* Hermann (2-nd ed.).

[Kol 73] E.R. Kolchin. *Differential algebra and algebraic groups.* Academic Press, New-York.

[Matij 70] Yu. Matijasevic. Enumerable sets are Diophantine. *Dokl. Akad. Nauk SSSR 191 (p. 279-282). Soviet Math. Dokl. 11 (p. 354-357).*

[Ritt 50] J.F. Ritt. *Differential algebra.* Amer. Math. Soc, New-York.

[Sh 91] J. Shackell. Limits of Liouvillian functions. *Technical report, Univ. of Kent, Canterbury. Appeared in 1996 in the Proc. of the London Math. Soc. 72 (p. 124-156).*

[VdD 84] L. van den Dries. Remarks on Tarski's problem concerning $(\mathbb{R}, +, \cdot, \exp)$. *Logic Colloquium '82, G. Longi, G. Longo and A. Marcja eds., North-Holland.*

[5]This means that the "differential dimension" of the variety of solutions is zero; see also [Kol 73].

Chapter 14

Oscillatory asymptotic behaviour

14.1 Introduction

In the previous chapters of part B of this thesis, we have been concerned with automatic asymptotic expansions of "strongly monotonic" transseries. In this chapter, we make a first step towards the automatic treatment of functions involving oscillatory behaviour. We notice that Grigoriev obtained some very interesting related results in [Gri 94] and [Gri 95], although his point of view is different yet complementary to ours. The results of section 14.2, 14.3 and 14.4 should soon appear in [VdH *].

The structure of this chapter is as follows: in section 14.2, we recall a classical density theorem for linear curves on the n-dimensional torus (see for example [Kok 37] or [KN 74]). In section 14.3, this theorem is generalized to more general classes of curves on the torus.

In section 14.4, we are given an algebraic function φ defined on $[-1,1]^q$, and exp-log functions at infinity $F_1(x), \cdots, F_q(x)$ in x. We show how to compute

$$\limsup_{x\to\infty} \varphi(\sin(F_1(x)), \cdots, \sin(F_q(x))).$$

In section 14.4, we will assume the existence of an oracle for checking the $\mathbb{Q}$-linear dependence of exp-log constants. Actually, Richardson's algorithm (see [Rich 95]) can easily be adapted to yield an algorithm for doing this modulo Schanuel's conjecture.

Section 14.5 contains extensions of the obtained results. For simplicity, we have based our exposition in section 14.4 on the case of exp-log functions. In view of the algorithms from the previous chapter, the reader will notice that the results can be easily extended to more general classes of transseries. This issue will be discussed in section 14.5.1. In section 14.5.2, we sketch an approach for computing complete asymptotic expansions of so called sin-exp-log functions of trigonometric depth one (sines may not be nested). We finally discuss further extensions of our results to the resolution of differential equations, in section 14.5.3.

14.2 A density theorem on the n-dimensional torus

Let $\lambda_1, \cdots, \lambda_n$ be $\mathbb{Q}$-linearly independent numbers: we will use vector notation, and denote the vector $(\lambda_1, \cdots, \lambda_n)$ by $\boldsymbol{\lambda}$. In this section, we prove that the image of $x \mapsto \boldsymbol{\lambda}x$, from $\mathbb{R}$ into the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is dense. Notice that we use the same notation for $\boldsymbol{\lambda}x$ and its class modulo $\mathbb{Z}^n$. Moreover, we show that the "density" of the image is uniform is a sense that will be made precise. The following theorem is classical:

Theorem 14.1. **(Kronecker)** *Let $\lambda_1, \cdots, \lambda_n$ be $\mathbb{Q}$-linearly independent real numbers. Let $e_1, \cdots, e_n$ be the canonical base of $\mathbb{R}^n$. Then $\lambda_1 e_1\mathbb{Z} + \cdots + \lambda_n e_n\mathbb{Z} + \mathbb{R}(e_1 + \cdots + e_n)$ is dense in $\mathbb{R}^n$.*

Proof. Let $G = \lambda_1 e_1\mathbb{Z} + \cdots + \lambda_n e_n\mathbb{Z}$, $u = e_1 + \cdots + e_n$, and let A be the closure of $G + \mathbb{R}u$. By the classification theorem of closed abelian subgroups of $\mathbb{R}^n$, we can decompose $A = V \oplus D$, where V is a subvector space of $\mathbb{R}^n$ and where D is a discrete subgroup of $\mathbb{R}^n$, with $V \cap \operatorname{vect} D = \{0\}$ (here we recall that $\operatorname{vect} D$ stands for the vector space spanned by D). Then we have projections $\pi : G \to G \cap V$ and $\pi' : G \to D$, with $Id = \pi + \pi'$. Now let $\{v_1, \cdots, v_d\} \subseteq G$ be an $\mathbb{R}$-base of G and complete this base into an $\mathbb{R}$-base $\{v_1, \cdots, v_n\} \subseteq G$ of $\mathbb{R}^n$. Then $\pi(v_{d+1}), \cdots, \pi(v_n)$ form an $\mathbb{R}$-base of V. Since $\mathbb{R}u \subseteq V$, u is an $\mathbb{R}$-linear combination of $\mu(v_{d+1}), \cdots, \mu(v_n)$. This can be written in matrix form

$$\Lambda M X = U,$$

where Λ is the diagonal matrix with entries $\lambda_1, \cdots, \lambda_n$, M an integer matrix, X some column matrix, and U the column matrix with 1-entries. If d were strictly superior to 0, then we would obtain a $\mathbb{Q}$-linear relation between the λ_i, by doing row operations on M. This completes the proof. □

Now let X be a measurable subset of T^n, and let I be some interval of $\mathbb{R}$. Denoting the Lebesgue measure by μ, we define

$$\rho(I, X) = \frac{\mu(\{x \in I | \boldsymbol{\lambda}x \in X\})}{\mu(I)}. \tag{14.1}$$

Let us also denote by d the Euclidean distance on T^n. Let S_d, resp. $S_{\boldsymbol{d}}$ denote the shift operator on $\mathbb{R}$, resp. $\mathbb{R}^n$ or T^n: $S_d(x) = x + d$ and $S_{\boldsymbol{d}}(\boldsymbol{x}) = S_{(d_1, \cdots, d_n)}(x_1, \cdots, x_n) = (x_1 + d_1, \cdots, x_n + d_n) = \boldsymbol{x} + \boldsymbol{d}$. The following are immediate consequences of the definition of ρ:

Proposition 14.1. *We have*

(a) $\rho(I, X) = \sum_{i \in \mathbb{N}} \rho(I, X_i)$, *if* $X = \coprod_{i \in \mathbb{N}} X_i$.
(b) $|\rho(I, X) - \rho(S_d I, X)| \leqslant |d|/\mu(I)$, *for all* d.
(c) $\rho(I, X) = \rho(S_{-d} I, S_{\boldsymbol{\lambda}d} X)$, *for all* d. □

It will be convenient to adopt some conventions for intervals $I = [a, b]$ (resp. $I = [a, b[, I =]a, b]$ or $I =]a, b[$) whose lengths $b - a$ tend to infinity: we say that a property P holds uniformly in I, if the property holds uniformly in a:

$$\exists l_0 \forall a \forall l > l_0 \quad P([a, a + l]).$$

We say that P holds for all I sufficiently close to infinity, if P holds for all sufficiently large a.

The next theorem is also classical, but for convenience of the reader we present a proof, since similar techniques will be used in the next section:

Theorem 14.2. **(Bohr, Sierpiński, Weyl)** *Let $\lambda_1, \cdots, \lambda_n$ be $\mathbb{Q}$-linearly independent real numbers and let ρ be given by (14.1). Let*

$$X = [a_1, b_1[\times \cdots \times [a_n, b_n[\subseteq T^n$$

be an n-dimensional block, with $0 \leqslant a_i \leqslant b_i \leqslant 1$ for all i). Then

$$\lim_{\mu(I) \to \infty} \rho(I, X) = \mu(X),$$

uniformly in I.

Proof. The theorem trivially holds, if $a_i = 0$ and $b_i = 1$, for all but one $1 \leqslant i \leqslant n$. Hence, it suffices to prove the theorem, when the a_i and the b_i are rational numbers. Indeed, let $a'_1, b'_1, \cdots, a'_n, b'_n$ be rational numbers with $|a'_1 - a_1| + |b'_1 - b_1| + \cdots + |a'_n - a_n| + |b'_n - b_n| \leqslant \delta$, and denote $X' = [a'_1, b'_1[\times \cdots \times [a'_n, b'_n[$. Then $|\rho(I, X') - \rho(I, X)| \leqslant 2\delta$ for $\mu(I)$ sufficiently large, uniformly in I.

Because of proposition 14.1(b), it therefore suffices to prove the theorem for fixed $\boldsymbol{p} = (p_1, \cdots, p_n) \in (\mathbb{N}^*)^n$ and for all

$$X = X_{\boldsymbol{k}} = \left[\frac{k_1}{p_1}, \frac{k_1 + 1}{p_1}\right[\times \cdots \times \left[\frac{k_n}{p_n}, \frac{k_n + 1}{p_n}\right[,$$

with $0 \leqslant k_1 < p_1, \cdots, 0 \leqslant k_n < p_n$. We remark that $[0, 1[^n = \coprod_{\boldsymbol{k}} X_{\boldsymbol{k}}$, so that $\sum_{\boldsymbol{k}} \rho(I, X_{\boldsymbol{k}}) = 1$.

Now let $\varepsilon > 0$. For each $\boldsymbol{k}$, we can find $x_{\boldsymbol{k}}$, with $\mathrm{d}(\boldsymbol{\lambda} x_{\boldsymbol{k}}, \boldsymbol{k}) < \varepsilon/n$, by theorem 14.1. Consequently, we have $\mu(S_{-\boldsymbol{\lambda} x_{\boldsymbol{k}}} X_{\boldsymbol{k}} \bigtriangleup X_{\boldsymbol{0}}) < \varepsilon$, where $A \bigtriangleup B$ denotes the symmetric difference of A and B. Hence, $\mu(X_{\boldsymbol{l}} \bigtriangleup S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}} - x_{\boldsymbol{k}})} X_{\boldsymbol{k}}) < 2\varepsilon$, for each $\boldsymbol{l}$ with $l_1 < p_1, \cdots, l_n < p_n$. Using proposition 14.1, we can now estimate

$$\begin{aligned} |\rho(I, X_{\boldsymbol{l}}) - \rho(I, X_{\boldsymbol{k}})| &\leqslant |\rho(I, S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}} - x_{\boldsymbol{k}})}) - \rho(I, X_k)| + \\ &\quad\ \mu(X_{\boldsymbol{l}} \bigtriangleup S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}} - x_{\boldsymbol{k}})} X_{\boldsymbol{k}}) \\ &\leqslant |\rho(S_{x_{\boldsymbol{k}} - x_{\boldsymbol{l}}} I, X_{\boldsymbol{k}}) - \rho(I, X_{\boldsymbol{k}})| + 2\varepsilon \\ &\leqslant \frac{|x_{\boldsymbol{k}} - x_{\boldsymbol{l}}|}{\mu(I)} + 2\varepsilon. \end{aligned}$$

Taking $\mu(I) > |x_{\boldsymbol{k}} - x_{\boldsymbol{l}}|/\varepsilon$, for any $\boldsymbol{k}$ and $\boldsymbol{l}$, we get

$$\left|\rho(I, X_{\boldsymbol{k}}) - \frac{1}{p_1 \cdots p_n}\right| \leqslant \frac{1}{p_1 \cdots p_n} \sum_{\boldsymbol{k}} |\rho(I, X_{\boldsymbol{k}}) - \rho(I, X_{\boldsymbol{l}})| < 3\varepsilon.$$

Hence $|\rho(I, X_{\boldsymbol{k}}) - \mu(X_{\boldsymbol{k}})| < 3\varepsilon$, for sufficiently large $\mu(I)$, uniformly in I. This completes our proof. □

Remark 14.1. The theorem does not longer hold, if we replace X by an arbitrary measurable subset of the torus. Nevertheless, it can be shown that it does hold for any measurable X, whose boundary is a finite union of differentiable hypersurfaces.

14.3 A more general density theorem

In this section we will obtain a more general uniform density theorem on the torus, when the application $x \mapsto \boldsymbol{\lambda} x$ from section 14.2 is replaced by a non linear mapping, which satisfies suitable regularity conditions. Before coming to this generalization, we will need some definitions and lemmas. We say that a function f defined in a neighbourhood of infinity is **steadily dominated** by x, if f has a continuous second derivative, f tends to infinity, f' decreases strictly towards zero, and f''/f' tends to zero. We remark that such functions f admit functional inverses f^{inv} in a neighbourhood of infinity.

More generally, we say that if f and g are functions in a neighbourhood of infinity, such that g is invertible, then f is steadily dominated by g, if $f \circ g^{inv}$ is steadily dominated by x. In this case, we write $f \prec\!\!\prec^s g$. It is easily verified that if $f \prec\!\!\prec^s x$ and $g \prec\!\!\prec^s x$, then $f \circ g \prec\!\!\prec^s x$, so that $\prec\!\!\prec^s$ is transitive. We also remark that if $f \prec\!\!\prec^s g$ and if h is a function, which has a continuous second derivative and tends to infinity, then $f \circ h \prec\!\!\prec^s g \circ h$. We finally have the following property of steady domination:

Lemma 14.1. *Let h be steadily dominated by x and let $l > 0$ and $\varepsilon > 0$ be given. Then for all sufficiently large x we have $|h'(x+d) - h'(x)| < \varepsilon h'(x)$, for all d with $|d| < l$.*

Proof. Let x_0 be such that $|h''(x)/h'(x)| < \varepsilon h'(x)$, for all $x \geqslant x_0 - l$. We have $|h'(x+d) - h'(x)| \leqslant |dh''(\xi)| < \varepsilon h'(\xi)$, for some ξ between x and $x+d$. If d is positive, then $h'(\xi) \leqslant h'(x)$, and we are done. In the other case, we have $|h'(x+d) - h'(x)| \leqslant \varepsilon h'(x) - \varepsilon |h'(x+d) - h'(x)|$, whence $|h'(x+d) - h'(x)| < (\varepsilon/(1-\varepsilon)) h'(x)$. □

Now let X be a measurable subset of $\mathbb{R}$. For each interval I, we define:

$$\rho(I, X) = \frac{\mu(I \cap X)}{\mu(I)}.$$

We say that X admits an **asymptotic density** $\rho(X)$ if

$$\lim_{\mu(I)\to\infty} \rho(I,X) = \rho(X),$$

uniformly in I, for I sufficiently close to infinity.

Lemma 14.2. *Let X be a measurable subset of $\mathbb{R}$ and let h be steadily dominated by x. If $\rho(X)$ exists, then so does $\rho(h(X))$ and we have $\rho(h(X)) = \rho(X)$.*

Proof. Let $\varepsilon > 0$. Let $l \in \mathbb{R}$ be such that $|\rho(I,X) - \rho| < \varepsilon$, whenever $\mu(I) > l$. Taking $I = [\alpha, \beta[$, we subdivide $h^{inv}(I)$ in $q = \lfloor (h^{inv}(\beta) - h^{inv}(\alpha))/l \rfloor$ parts of equal length $l' > l$

$$[(h^{inv}(\alpha), h^{inv}(\beta))[= [a_1, b_1[\amalg \cdots \amalg [a_q, b_q[,$$

with $b_i = a_{i+1}$ for $1 \leqslant i < q$. Then we have

$$\begin{gathered}(\rho - \varepsilon)\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(b_i)dx \;\leqslant\; \sum_{i=1}^{q}\int_{a_i}^{b_i} \chi_X(x)h'(b_i)dx \leqslant \\ \mu(h^{inv}(X)\cap I) \leqslant \\ \sum_{i=1}^{q}\int_{a_i}^{b_i} \chi_X(x)h'(a_i)dx \;\leqslant\; (\rho+\varepsilon)\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(a_i)dx.\end{gathered}$$

By lemma 14.1, for all sufficiently large x, we have $|h'(x+d) - h'(x)| \leqslant \varepsilon h'(x)$, for all d with $|d| \leqslant l'$. Hence,

$$\begin{aligned}&\left|\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(x)dx - \sum_{i=1}^{q}\int_{a_i}^{b_i} h'(b_i)dx\right| \leqslant \\ &\sum_{i=1}^{q}\int_{a_i}^{b_i} |h'(x) - h'(b_i)|dx \leqslant \\ &\varepsilon\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(x)dx = \varepsilon\mu(I),\end{aligned}$$

and we have a similar estimation, when replacing b_i by a_i. Consequently,

$$(\rho - \varepsilon)(1-\varepsilon)\mu(I) \leqslant \mu(h(X)\cap I) \leqslant (\rho+\varepsilon)(1+\varepsilon)\mu(I).$$

This completes our proof. □

Let $f_1 \prec\!\!\prec^s \cdots \prec\!\!\prec^s f_p$ be continuous functions defined in a neighbourhood of infinity, which strictly increase towards infinity. Let $\lambda_{i,j} > 0$ $(1 \leqslant j \leqslant n_i)$ be such that $\lambda_{i,1}, \cdots, \lambda_{i,n_i}$ are $\mathbb{Q}$-linearly independent for each i. Now consider the curve

$$g(x) = (f_1(\lambda_{1,1}x), \cdots, f_1(\lambda_{1,n_1}x), \cdots, f_p(\lambda_{p,1}x), \cdots, f_p(\lambda_{p,n_p}x))$$

on T^n ($n = n_1 + \cdots + n_p$), which is defined for sufficiently large x. By analogy with the preceding section, we define

$$\rho_{f,g}(I, X) = \frac{\mu(\{x \in I | g(f_1^{inv}(x)) \in X\})}{\mu(I)}, \tag{14.2}$$

for intervals I sufficiently close to infinity, and measurable subsets X of T^n.

Theorem 14.3. *Let $f_1, \cdots, f_p, g$ and $\rho_{f,g}$ be given as above and let*

$$X = [a_1, b_1[\times \cdots \times [a_n, b_n[\subseteq T^n$$

be an n-dimensional block. Then

$$\lim_{\mu(I) \to \infty} \rho_{f,g}(I, X) = \mu(X),$$

uniformly, for intervals sufficiently close to infinity.

Proof. We proceed by induction over p. If $p = 0$, we have nothing to prove. Otherwise, we decompose $X = X_1 \times \tilde{X}$, with $X_1 \subseteq T^{n_1}$ and $\tilde{X} \subseteq T^{\tilde{n}}$, where $\tilde{n} = n_2 + \cdots + n_p$. We denote by $g_1(x)$ resp. $\tilde{g}(x)$ the projections of $g(x)$ on T^{n_1} resp. $T^{\tilde{n}}$, when considering T^n as the product of T^{n_1} and $T^{\tilde{n}}$. Without loss of generality, we may assume that $f_1 = x$.

Given a subset A of $\mathbb{R}$ or T^n and its frontier ∂A, we denote for any $\varepsilon > 0$

$$\Omega_\varepsilon A = \{x \in A | \mathrm{d}(x, \partial A) > \varepsilon\}.$$

Let $\varepsilon > 0$. If $g_1(x) \in \Omega_\varepsilon X_1$, then $g_1(x + d) \in X_1$ for all d with $|d| < l$, where $l = \max(1/\lambda_{1,1}, \cdots, 1/\lambda_{1,n_1})\varepsilon$. Hence, for I sufficiently close to infinity,

$$I \cap g_1^{inv}(\Omega_\varepsilon X_1) \ \subseteq \ (I \cap \Omega_l g_1^{inv}(X_1)) +]-l, l[\subseteq \ I \cap g_1^{inv}(X_1).$$

Therefore, theorem 14.2 implies that for I sufficiently close to infinity

$$\left| \frac{\mu(I \cap g_1^{inv}(X_1))}{\mu(I)} - \mu(X_1) \right| < \varepsilon \tag{14.3}$$

and (using that $\mu(\Omega_\varepsilon X_1 \bigtriangleup X_1) < 2n_1$)

$$\left| \frac{\mu((I \cap \Omega_l g_1^{inv}(X_1)) +]-l, l[)}{\mu(I)} - \frac{\mu(I \cap g_1^{inv}(X_1))}{\mu(I)} \right| \leqslant (2n_1 + 1)\varepsilon. \tag{14.4}$$

Now $(I \cap \Omega_l g_1^{inv}(X_1)) +]-l, l[$ is a finite union of intervals, say

$$I \cap \Omega_l g_1^{inv}(X_1) \cap]-l, l[= I_0 \amalg \cdots \amalg I_{q+1},$$

where $I_1, \cdots, I_q$ have length at least $2l$, and where I_0 and I_{q+1} have length at most $2l$.

By the induction hypothesis, we have

$$\lim_{\mu(J)\to\infty} \frac{\mu(J \cap f_2(\tilde{g}^{inv}(\tilde{X})))}{\mu(J)} = \mu(\tilde{X}),$$

uniformly, for J sufficiently close to infinity. Using lemma 14.2, this gives us

$$\lim_{\mu(f_2(J))\to\infty} \frac{\mu(J \cap \tilde{g}^{inv}(\tilde{X}))}{\mu(J)} = \mu(\tilde{X}),$$

uniformly, for J sufficiently close to infinity. In particular, we have

$$\left|\frac{\mu(J \cap \tilde{g}^{inv}(\tilde{X}))}{\mu(J)} - \mu(\tilde{X})\right| < \varepsilon,$$

for all J sufficiently close to infinity, with $\mu(J) > l$. Thus, choosing I sufficiently close to infinity, we have

$$\left|\frac{\mu(I_i \cap \tilde{g}^{inv}(\tilde{X}))}{\mu(I_i)} - \mu(\tilde{X})\right| < \varepsilon,$$

for all $1 \leqslant i \leqslant q$.
Taking $\mu(I) > 2l/\varepsilon$, and using (14.3) and (14.4), this gives us

$$\begin{aligned}
|\rho_{f,g}(I,X) - \mu(X)| &\leqslant \left|\frac{\mu(\coprod_{i=0}^{q+1} I_i \cap \tilde{g}^{inv}(\tilde{X}))}{\mu(I)} - \mu(X)\right| + \\
&\quad \left|\frac{\mu(I \cap g^{inv}(X))}{\mu(I)} - \frac{\mu(\coprod_{i=0}^{q+1} I_i \cap \tilde{g}^{inv}(\tilde{X}))}{\mu(I)}\right| \\
&\leqslant \left|\sum_{i=0}^{q+1} \frac{\mu(I_i \cap \tilde{g}^{inv}(\tilde{X})) - \mu(\tilde{X})\mu(I_i)}{\mu(I)}\right| + \\
&\quad \left|\frac{\mu(\tilde{X})\mu(\coprod_{i=0}^{q+1} I_i)}{\mu(I)} - \mu(X)\right| + (2n_1+1)\varepsilon \\
&\leqslant \sum_{i=1}^{q} \frac{\mu(I_i)\varepsilon}{\mu(I)} + (4n_1+5)\varepsilon \leqslant (4n_1+6)\varepsilon.
\end{aligned}$$

This completes the proof. □

14.4 On the automatic computation of limsups

In this section we show how theorem 14.3 can be applied to compute limsups (or liminfs) of certain bounded functions, involving trigonometric functions. The idea is based on the following consequence of theorem 14.3.

Theorem 14.4. *Let $1 \prec\!\!\prec f_1 \prec\!\!\prec \cdots \prec\!\!\prec f_p$ be exp-log functions at infinity. Let $\lambda_{i,j} > 0$ ($1 \leqslant j \leqslant n_i$) be such that $\lambda_{i,1}, \cdots, \lambda_{i,n_i}$ are $\mathbb{Q}$-linearly independent for each i. Denote $U = \{x + \sqrt{-1}\, y \in \mathbb{C} | x^2 + y^2 = 1\}$ and $n = n_1 + \cdots + n_p$. Let φ be a continuous function from U^n into $\mathbb{R}$ and let*

$$\psi(x) = \varphi(e^{\sqrt{-1}\,\lambda_{1,1} f_1(x)}, \cdots, e^{\sqrt{-1}\,\lambda_{p,n_p} f_p(x)}).$$

Then

$$\limsup_{x \to \infty} \psi(x) = \sup_{\boldsymbol{x} \in U^n} \varphi(\boldsymbol{x}).$$

Proof. We first notice that we will be able to apply theorem 14.3 on our input data: by a well known theorem, which goes back to Hardy (see [Har 11]), the germs at infinity of $f_1, \cdots, f_p$ lie in a common Hardy field. Consequently, $f_1 \prec\!\!\prec^s \cdots \prec\!\!\prec^s f_p$, and $f_1, \cdots, f_p$ are strictly increasing in a suitable neighbourhood of infinity.

The mapping ψ is defined in a neighbourhood V of infinity, and can be factored $V \overset{\psi}{\to} \mathbb{R} = V \overset{\psi_1}{\to} T^n \overset{\psi_2}{\to} \mathbb{R}$, with

$$\psi_1(x) = \left(\frac{\lambda_{1,1} f_1(x)}{2\pi}, \cdots, \frac{\lambda_{p,n_p} f_p(x)}{2\pi} \right)$$

and

$$\psi_2(x_1, \cdots, x_n) = \varphi(e^{2\pi\sqrt{-1}\,x_1}, \cdots, e^{2\pi\sqrt{-1}\,x_n}),$$

where ψ_1 and ψ_2 are both continuous. Since T^n is compact, there exists a point $\boldsymbol{x}$ in which ψ_2 attains its maximum. Let $\varepsilon > 0$. There exists a neighbourhood V of $\boldsymbol{x}$, such that $|\psi_2(\boldsymbol{y}) - \psi_2(\boldsymbol{x})| < \varepsilon$, for any $\boldsymbol{y}$ in V. By theorem 14.3, there exist x, with $\psi_1(x) \in V$ as close to infinity as we wish. For such x, we have $|\psi(x) - \sup_{\boldsymbol{x} \in U^n} \varphi(\boldsymbol{x})| < \varepsilon$. □

We now turn to the computation of this limit.

Theorem 14.5. *Let $F_1, \cdots, F_q$ be exp-log functions at infinity. Let $\varphi : U^q \to \mathbb{R}$ a real algebraic function, where we consider U^q as a real algebraic variety. Assume that we have an oracle to test the $\mathbb{Q}$-linear dependence of exp-log constants. Then there exists an algorithm to compute the limsup of $\psi(x) = \varphi(e^{\sqrt{-1}\,F_1(x)}, \cdots, e^{\sqrt{-1}\,F_q(x)})$.*

Proof. Using the identity $e^{-x} = 1/e^x$, we may always assume without loss of generality, that the F_i are all positive. Now the algorithm consists of the following steps:

Step 1. Compute a common effective normal basis for $F_1, \cdots, F_p$, using the algorithm from section 7.2. Order the F_i w.r.t. $\prec\!\!\prec$; that is, $F_i \asymp F_j$ or $F_i \prec\!\!\prec F_j$, whenever $i < j$.

Step 2. Simultaneously modify the F_i and the algebraic function φ in the $e^{\sqrt{-1}\,F_i}$, until we either have $F_i \prec\!\!\prec F_j$, or $F_i = \lambda F_j$, for some λ, whenever $i < j$. As long as this is not the case, we take j maximal, such that the above does not hold, and do the following:

First compute the limit λ of F_i/F_j. Next insert $F_i' := F_i - \lambda F_j$ and $F_j' := \lambda F_j$ into the set of F_i and remove F_i. The new expression for φ is obtained by replacing each $e^{\sqrt{-1}\,F_i}$ by $e^{\sqrt{-1}\,F_i'}e^{\sqrt{-1}\,F_j'}$.

Step 3. Compute exp-log functions $f_1 \prec\!\!\prec \cdots \prec\!\!\prec f_p$, and constants $\lambda_{i,j}$ $(1 \leqslant j \leqslant n_i)$, such that each F_l can be written as $F_l = \lambda_{i,j} f_i$, for some i and j. Replacing $e^{\sqrt{-1}\,F_i}$ by its limit for each bounded F_i, we reduce the general case to the case when $1 \prec\!\!\prec f_1$.

Step 4. This step consists in making the $\lambda_{i,j}$ $\mathbb{Q}$-linearly independent for each fixed i. Whenever there exists a non trivial $\mathbb{Q}$-linear relation between the $\lambda_{i,j}$ (for fixed i), we may assume without loss of generality that this relation is given by

$$a_{n_i}\lambda_{i,n_i} = a_1\lambda_{i,1} + \cdots + a_{n_i-1}\lambda_{i,n_i-1},$$

for $a_1, \cdots, a_{n_i}$ in $\mathbb{Z}$ and $a_{n_i} > 0$. As long as we can find such a relation, we do the following:

For all $j < n_i$, replace $\lambda_{i,j}$ by $\lambda_{i,j}' := \lambda_{i,j}/a_{n_i}$ and $e^{\sqrt{-1}\,\lambda_{i,j}f_i}$ by $(e^{\sqrt{-1}\,\lambda_{i,j}'f_i})^{a_{n_i}}$ in the expression for φ. Next, replace $e^{\sqrt{-1}\,\lambda_{i,n_i}f_i}$ by $(e^{\sqrt{-1}\,\lambda_{i,1}'f_i})^{a_1}\cdots(e^{\sqrt{-1}\,\lambda_{i,n_i-1}'f_i})^{a_{n_i-1}}$ in the expression for φ.

Step 5. By theorem 14.4, the limsup of ψ is the maximum of φ on U^n, where $n = n_1 + \cdots + n_p$. To compute this maximum, we determine the set of zeros of the gradient of φ on U^n. Then φ is constant on each connected component and the maximum of these constant values yields $\max_{U^n}\varphi$. To compute the zero set of the gradient of φ and its connected components, one may for instance use cylindrical decomposition (see [Col 75]). Of course, other algorithms from effective real algebraic geometry can be used instead.

The correctness of our algorithm is clear. The termination of the loop in step 2 follows from the fact that the new F_i' is asymptotically smaller then F_j, so that either the $\asymp$-class of F_j strictly decreases, or the number of i with $F_i \asymp F_j$, but not $F_i = \lambda F_j$ for some λ. The number of $\asymp$-classes which can be attained is bounded by the initial value of q. □

Corollary. *Let $F_1, \cdots, F_q$ be exp-log functions at infinity and φ be an algebraic function in q variables, defined on $[-1,1]^q$. Assume that we have an oracle to test the $\mathbb{Q}$-linear dependence of exp-log constants. Then there exists an algorithm to compute the limsup of $\psi(x) = \varphi(\sin(F_1(x)), \cdots, \sin(F_q(x)))$.* □

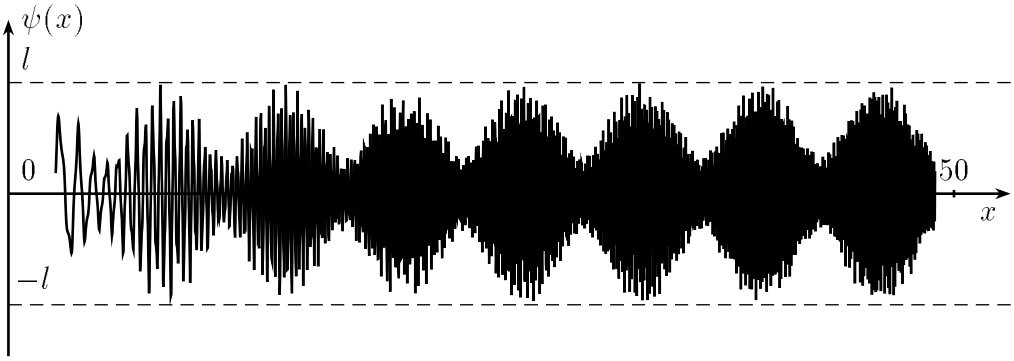


Figure 14.1: Plot of the function ψ from example 14.1.

Example 14.1. Consider the function

$$\psi(x) = \frac{2\sin x^2 - \sin(x^3/(x-1))}{3 + \sin ex^2 - \sin(ex^2+1)}.$$

The first step consists in expanding $x^2 = x^2$, $x^3/(x-1) = x^2 + x + \cdots$, $ex^2 = ex^2$ and $ex^2+1 = ex^2+1$. All these functions have the same $\asymp$-class, but they are not all homothetic. Therefore, some rewriting needs to be done. First, $x^3/(x-1) = x^2 + x^2/(x-1)$, and we rewrite

$$e^{\sqrt{-1}\, x^3/(x-1)} = e^{\sqrt{-1}\, x^2} e^{\sqrt{-1}\, x^2/(x-1)},$$

which corresponds to the rewriting

$$\sin\frac{x^3}{x-1} = \sin x^2 \cos\frac{x^2}{x-1} + \sin\frac{x^2}{x-1}\cos x^2,$$

if we consider real and imaginary parts. Similarly, we rewrite

$$e^{\sqrt{-1}\,(ex^2+1)} = e^{\sqrt{-1}\, ex^2} e^{\sqrt{-1}},$$

which corresponds to the rewriting

$$\sin(ex^2+1) = \sin ex^2 \cos 1 + \sin 1 \cos ex^2.$$

In step 4, no $\mathbb{Q}$-linear relations are found, so that we have to determine the maximal value of

$$\varphi(a,\hat{a},b,\hat{b},c,\hat{c}) = \frac{2a - a\hat{c} - c\hat{a}}{3 + b - b\cos 1 - \hat{b}\sin 1} \tag{14.5}$$

on U^3. Here we have abbreviated $a = \sin x^2, \hat{a} = \cos x^2, b = \sin ex^2, \hat{b} = \cos ex^2, c =$

$\sin(x^2/(x-1)), \hat{c} = \cos(x^2/(x-1))$ (hence U^3 is the set of points with $a^2 + \hat{a}^2 = b^2 + \hat{b}^2 = c^2 + \hat{c}^2 = 1$). The maximum of φ is attained for $a = 1, \hat{a} = 0, b = -1/2, \hat{b} = \sqrt{3}/2, c = 0, \hat{c} = -1$. We deduce that

$$\limsup_{x\to\infty} \psi(x) = \frac{6}{5 + \cos 1 - \sqrt{3}\ \sin 1} = l.$$

Similarly, exploiting the symmetry of (14.5), we have

$$\liminf_{x\to\infty} \psi(x) = \frac{-6}{5 + \cos 1 - \sqrt{3}\ \sin 1} = -l.$$

14.5 Extensions

We have shown how to compute limsups of certain functions involving trigonometric functions, exponentiation and logarithm. Actually, the techniques we have used are far more general than theorem 14.5 might suggest. Let us now discuss the possible extensions of our theorem.

14.5.1 More general classes

More general $F_1, \cdots, F_q$. In theorem 14.5, the crucial property of the functions $F_1, \cdots, F_q$ is that they are strongly monotonic and that we have an asymptotic expansion algorithm for them. Consequently, functions more general than exp-log functions can be taken instead, especially classes of (germs of) functions, which can be represented by transseries for which the asymptotic expansion algorithms from the previous chapters apply. Now convergent transseries naturally represent germs of functions at infinity. More generally, resummation techniques are needed in order to associate germs of functions at infinity to transseries.

Resummation theories. So what natural axioms should a resummation theory actually satisfy? We propose to define a **real resummation theory** as being a mapping Σ from a differential subfield K of the field of transseries to the ring of germs of real functions at infinity, which satisfies the following properties:

- **RT0.** K contains all convergent transseries and $\Sigma(f)$ is defined in the natural way for such transseries f.
- **RT1.** Σ preserves the field operations.
- **RT2.** Σ preserves differentiation.
- **RT3.** Σ preserves the ordering.
- **RT4.** Σ preserves infinitesimals.
- **RT5.** Σ preserves composition (when defined).

The image of a resummation theory Σ is a Hardy field. Via Σ, the results of this chapter generalize to the case when $F_1, \cdots, F_q$ are in an automatic subfield of K.

There do exist non trivial resummation theories. For instance, one may extend the differential field of convergent transseries with the Gamma function. Similarly, we may use the classical closure properties of Hardy fields in order to construct non trivial resummation theories. However, no resummation theories with very general closure properties (stability for resolution of algebraic differential equations and composition) are known at the moment. Indeed, this is a very important open question, and especially the preservation of realness is very hard to achieve (see [Ec 92] and [Men 96] for an approch to this problem).

More general functions φ. We may also take φ in a more general class than the class of algebraic functions defined on U^q or $[-1,1]^q$. The interesting property of the class of algebraic functions is that there exists a cylindrical decomposition algorithm for it. In particular, modulo suitable oracles, one may consider the class of solutions to real exp-log systems in several variables (see also the conclusion).

If one drops the effectiveness condition, one may also consider the class C of all real analytic functions defined on U^q. We claim that there exists a cylindrical decomposition theorem for C. To show this, we have to prove that the zero-set of any finite system of functions in C can be decomposed in a finite set of connected real analytic subvarieties. Now the theory from chapter 10 implies that such decompositions exist locally, by taking the set of all convergent power series for the local community. By the compactness of the zero-set, we need only a finite number of such local pieces in order to recover the entire zero-set.

From the effective point of view, it is attractive to restrict ones attention to an effective differential ring R of effective real analytic functions on U^q. Effective real numbers are numbers which can be approximated automatically to any desired precision by rationals. Effective real analytic functions are effective real functions f, such that for any effective point x_0 in the domain of f, we can compute $f(x_0)$, a small disk around x_0 on which f is analytic *and* bounds for f on any closed subdisk. In this context, it is not hard to show that there exists an algorithm to compute the maximum of a real analytic function φ on the torus U^q. On the other hand, exact zero-tests for constants in R are usually very hard to design, if they exist at all. We also refer to appendix C for computations with effective reals and complex numbers.

14.5.2 Complete expansions

The approach. In principle, our techniques can also be used to compute automatic asymptotic expansions of sin-exp-log functions ψ at infinity of trigonometric depth one (i.e. without nested sines). This is done as follows: let $\sin f_1, \cdots, \sin f_p$ be the sines occurring in ψ. Using the rewriting algorithm from section 14.4, we first reduce the general case to the case in which the f_i are $\mathbb{Q}$-linearly independent, and homothetic whenever equivalent for $\asymp$. Next, we consider $g_1 = \sin f_1, \cdots, g_p = \sin f_p$ as formal transseries parameters, which satisfy the constraints $-1 \leqslant \sin f_i \leqslant 1$ for all i. At this point we can apply the generic expansion algorithm for multivariate transseries from chapter 11.

Simplifying a bit (see the final remarks at the end of this section), the result is a list of regions, determined by asymptotic constraints on the g_i, together with a generic expansion algorithm on each region. Of course, some regions may not correspond to actual asymptotic behaviours of ψ. Besides checking the constraints for exp-log consistency, which can be done modulo the oracle used in chapter 11, we need check whether the constraints (and the extra constraints induced by the final refinements) can actually be realized when we know that the formal parameters were originally sines of exp-log functions. This issue has still to be studied in detail. Let us briefly discuss some aspects of this problem.

Diophantine problems. In its full generality the problem may lead to very difficult number theoretical phenomena, as the following example illustrates:

$$2 - \sin x - \sin ex \geqslant_\infty \frac{1}{\Gamma(x+2)}.$$

This asymptotic inequality follows from the number theoretical properties of e. But what about the positive infinitesimal exp-log functions f for which

$$2 - \sin x - \sin ax \geqslant f(x) \tag{14.1}$$

for all sufficiently large x, where a is an arbitrary exp-log constant? Clearly, this is a problem of Diophantine approximation which is very hard to solve in general, if solvable at all (see [Lang 71] for a nice survey on Diophantine approximation). Yet, it is one of the simplest situations which can arise, since in general, we want to study far more general systems of constraints.

Nevertheless, we notice that (14.1) has been chosen in a very special way: 2 is precisely the $\limsup$ of $\sin x + \sin ax$; we say that (14.1) is "degenerate". If 2 is replaced by any other real number, then the problem becomes "non degenerate" and trivial. Therefore, our example is quite pathological. Furthermore, we notice that a is usually not an arbitrary real number, but say an algebraic number, or a parameter. In the first case, the theory of Diophantine approximation may give us *some* information, which we are able to exploit effectively, using the theory from chapter 11. In the second case, one can keep f fixed, and study the measure of the set of a for which (14.1) holds (by analogy with Kolmogorov-Arnold-Moser theory), which is again a simpler problem.

A pragmatic attitude. Finally, let us explain why from a numerical point of view, it is not really necessary to decide whether complicated asymptotic systems involving trigonometric functions are consistent. Indeed, assume for instance that during the computation of the expansion of a sin-exp-log function ψ of trigonometric depth one, we need determine the asymptotic sign of $g(x) = 2 - \sin x - \sin ax - f(x)$

for some reason (with a and f as in (14.1)). Then this asymptotic sign does not really matter for numerical applications: if we want to evaluate $\varphi(x_0)$ for a large value of x_0, then we just compute the sign of g in x_0. If this sign is positive resp. negative, then we compute $\varphi(x_0)$ using the asymptotic estimate obtained for it under the assumption that $g(x)$ is asymptotically positive resp. negative. Clearly, the fact that the generic asymptotic expansion algorithm from section 11 returns always a finite list of regions is very important for this mechanism to work in general.

On the other hand, certain questions which are, logically speaking, decidable, are completely undecidable from the practical point of view. A good example is to determine the sign of

$$c = \sin 10^{10^{10^{10}}},$$

which is needed in order to compute the asymptotic expansion of

$$\exp\exp(\sin 10^{10^{10^{10}}} x)$$

at infinity.

Final remarks. We have "cheated" a bit in the application of our generic expansion algorithm from chapter 11. Indeed, the algorithm may introduce some real parameters and impose exp-log constraints on them. In particular, when introducing a sine $g_i = \sin f_i \preccurlyeq 1$ as a formal parameter, we refine

$$g_i = \lambda \pm \varepsilon \ (\varepsilon \prec\!\!\prec 1)$$

or

$$g_i = \lambda.$$

However, we do not wish to interpret g_i as the sum of a real parameter plus an infinitesimal expression, since $\sin f_i(x)$ does not necessarily tend to a constant as x tends to infinity in the region which interests us.

Let us briefly discuss an approach to handle this problem, which we intend to develop in a future paper. Instead of introducing real parameters in the generic expansion algorithm, an alternative approach would be to divide the set of transseries parameters in two subsets: the infinitesimal and bounded parameters. Here the status of a parameter may be changed dynamically, while maintaining the elimination ordering. From the constraint point of view, the bounded parameters behave in a similar way as the real parameters before. However, if the limit behaviour of a bounded parameter x_i can be expressed as a function of other bounded parameters x_j with $j > i$, then we separate three cases, as we would have done in the usual algorithm.

Example 14.2. Assume that we want to expand

$$\varphi = \frac{1}{\sin x - \sin e^x}.$$

Initially, $\sin x <^{elim} \sin e^x$ correspond to bounded parameters. In order to invert $\sin x - \sin e^x$ we have to separate the cases

$$\sin x - \sin e^x \not\succ 1$$

and

$$\sin x - \sin e^x \prec\!\!\prec 1.$$

In the first case, φ is bounded, and we are done[1]. In the second case, we have expressed the limit behaviour of $\sin x$ as a function of the limit behaviour of $\sin e^x$, and we separate three cases:

$$\begin{cases} \sin x = \sin e^x + \varepsilon \ (\varepsilon \prec\!\!\prec 1); \\ \sin x = \sin e^x - \varepsilon \ (\varepsilon \prec\!\!\prec 1); \\ \sin x = \sin e^x. \end{cases}$$

The last case is eliminated, since $\sin x - \sin e^x$ should be invertible. In the first two cases, the bounded parameter $\sin x$ is rewritten in terms of an infinitesimal parameter ε (change of status). The constraints $\sin x = \sin e^x + \varepsilon$ $(\varepsilon \prec\!\!\prec 1)$ resp. $\sin x = \sin e^x - \varepsilon$ $(\varepsilon \prec\!\!\prec 1)$ induced by these refinements are both consistent.

14.5.3 Differential equations

In section 4.6, we have given a theoretical algorithm to determine all solutions to linear differential equations with transseries coefficients, even those solutions which involve oscillatory behaviour. The method becomes really effective, if the transseries live in an effective field of transseries, which satisfies the hypothesis from chapter 12. Indeed, we use similar algorithms as in the previous chapter.

More generally, one may consider transseries whose transmonomials are strongly monotonic, but the coefficients are analytic functions in a finite number of exponentials $e^{iF_1}, \cdots, e^{iF_p}$ of purely imaginary strongly monotonic transseries. Of course, the analytic functions are restricted to belong to some effective class, as discussed before. In this context, we have already some results about the computation of distinghuished solutions to linear and quasi-linear equations, but these results still require a full development.

14.6 References

[Col 75] G.E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Proc. 2-nd conf. on automata theory and formal languages, Springer lect. notes in comp. sc. 33 (p. 134-183).*

[Ec 92] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Hermann, collection: Actualités mathématiques.

[1] From the constraint point of view, $\sin x - \sin e^x \not\succ 1$ is handled as an inequality $\sin x \neq \sin e^x$ between real parameters.

[GoGr 92] G.H. Gonnet, D. Gruntz. Limit computation in computer algebra. *Technical report 187, ETH. Zürich.*

[Gri 94] D. Grigoriev. Deviation theorems for solutions to differential equations and applications to lower bounds on parallel complexity of sigmoids. *Theoretical Comp. Sc. 133(1) (p. 23-33).*

[Gri 95] D. Grigoriev. Deviation theorems for solutions of linear ordinary differential equations and applications to parallel complexity of sigmoids. *St.Petersburg Math. Jorn. 6(1) (p. 89-106).*

[Har 11] G.H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London mathematical society 10,2 (p 54-90).*

[HW 38] G.H. Hardy, E.M. Wright. *An introduction to the theory of numbers (chapter XXIII).* Oxford science publications.

[KN 74] L. Kuipers, H. Niederreiter. *Uniform distribution sequences.* Wiley (New York).

[Kok 34] J.F. Koksma. Diophantische Approximationen. *Ergebnisse der Mathematik, Band IV, Heft 4, Springer, Berlin.*

[Kr 1884] Kronecker. Werke iii(i). *Berliner Sitzungsberichte (p. 47-110).*

[Lang 71] S. Lang. Transcendental numbers and Diophantine approximation. *Bulletin of the A.M.S. 77(5) (p. 635-677).*

[Men 96] F. Menous. Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle. *PhD. thesis, Univ. d'Orsay, France.*

[Rich 95] D. Richardson. A simplified method for recognizing zero among elementary constants. *Proc. ISSAC '95 (p. 104-109).*

[RSSV 96] D. Richardson, B. Salvy, J. Shackell, J. van der Hoeven. Asymptotic expansions of exp-log functions. *Accepted for ISSAC '96, E.T.H. Zürich, july 24-26, Switzerland.*

[Sal 91] B. Salvy. Asymptotique automatique et fonctions génératrices. *PhD. thesis, Ecole Polytechnique, France.*

[VdH 96a] J. van der Hoeven. Generic asymptotic expansions. *Local proc. of the Rhine workshop on computer algebra, ed. A. Carrière, L.R. Oudin, Saint-Louis, April 1-3, France (p. 17.1-17.17), submitted to AAECC.*

[VdH *] J. van der Hoeven. On the automatic computation of limsups. *Available at* `http://www.win.tue.nl/win/mega96/papers`. *To appear in a special issue of the Journal of Pure and Applied Algebra (eds. A.M. Cohen, M.F. Roy), Elsevier.*

Appendices

Appendix A

Noetherian orderings

A.1 Introduction

It should not be surprising that in a general theory of asymptotic expansions the usual theory of orderings plays an important role. In fact, it turns out that asymptotics is closely related to the subject of **Noetherian orderings**. Because of its richness, we decided to devote this appendix to it. One of its origins clearly comes from commutative algebra, as developed by Noether and others in the twenties and thirties. In section 2, we translate the basic properties concerning Noetherian rings in the language of orderings; in fact, we mainly recall this very classical material to make the reader familiar with our nomenclature.

The study of Noetherian orderings for their own right would not have been justified without the appearance of a certain number of non trivial theorems in the fifties and the sixties, most importantly Higman's theorem (see [Hig 52]). Various motivations led to these theorems: a conjecture of Vázsonyi and a question from Erdős (see [Er 49]), order theoretic reasons (see [Krus 60], [NaWi 63]), etc. It is interesting to notice, that Higman's original motivation was the same as ours: the construction of algebras of generalized power series. The idea is to fix a ring, a partially ordered monomial group, and to consider those series, whose supports are Noetherian (that is, the induced ordering is Noetherian). We shall prove Higman's theorem and some of its generalizations in section 3.

The interest of theorems like Higman's theorem is that they permit us to prove the existence of solutions to certain types of equations over generalized power series and transseries, and in many cases to explicitly find a solution. The idea is that these theorems permit us to construct Noetherian orderings from others by means of **elementary constructions**. In terms of asymptotics, this allows us to confirm that the supports of candidate solutions are indeed Noetherian. However, for several reasons the classical theorems are insufficient to handle very general functional equations. Therefore, we introduce in section 4 the concept of Noetherian operators, which will allow us to generalize some classical theorems about Noetherian orderings. In section 5, we will show that these theorems are in fact effective, in a sense which will be made precise.

Let us finally notice that some other interesting constructions can be performed on Noetherian orderings. First, a Noetherian ordering determines a topology, whose closed sets are the final segments. Then it is interesting to study the Boolean algebras or σ-algebras generated by this topology. The elements of this boolean algebra are called constructible sets (by analogy with algebraic geometry). Often it is possible to define a natural measure on the σ-algebra on a Noetherian ordering, and often the measures of constructible sets can be computed effectively. This gives an final justification of the use of Noetherian orderings for doing asymptotics, although the degree of generality obtained here hides some other problems, which are discussed more fully in part B of this thesis. For more details, we refer to [VdH 94a].

A.2 Definitions and basic properties

Let E be a set. We recall that an **ordering** on E is a reflexive transitive and antisymmetric relation on E. One also defines **strict orderings** on E to be anti-reflexive transitive and antisymmetric relations on E. Of course, a strict ordering naturally determines an ordering and vice versa. If an ordering $\leqslant$ is given on E, we (abusively) say that E is an **ordered set**, and that $\leqslant$ is the underlying ordering. Two elements x, y of E are said to be **comparable**, if either $x \leqslant y$ or $y < x$. The ordering is **total**, if any two elements are comparable (to emphasize that an ordering is not total, we will sometimes say that it is **partial**). A subset of E is said to be a **chain**, if it is totally ordered by the induced ordering. A subset of E is said to be an **antichain**, if no two of its elements are comparable. The ordering is said to be **well founded**, if there is no infinite strictly decreasing sequence of elements in E. Equivalently we say that the **descending chain condition** holds. Dually, we say that the **ascending chain condition** holds, if there is not infinite strictly increasing sequence of elements in E. The ordering is said to be **Noetherian**, if it is well founded, and if there are no infinite antichains. Finally, a total well founded ordering is called a **well-ordering**. A well-ordering is in particular Noetherian.

Remark A.1. Many other names appear in the literature for the concept of Noetherian orderings. It is also possible to define Noetherian quasi-orderings, in which case the name **well-quasi-ordering** is used most commonly instead. Here a **quasi-ordering** on a set E is a reflexive transitive relation and the definitions of well-foundedness and antichain are the same. Given a quasi-ordering $\leqslant$, one can define an equivalence relation $\equiv$ by $x \equiv y \Leftrightarrow x \leqslant y \ \wedge \ y \leqslant x$. Then "dividing out" $\leqslant$ with respect to $\equiv$, one obtains an ordering. This ordering is Noetherian, if $\leqslant$ is a well-quasi-ordering.

We will need some more definitions. Let $\leqslant$ still be an ordering on E. A **final segment** for $\leqslant$ is a subset F of E, such that $x \in F \ \wedge \ x \leqslant y \Rightarrow y \in F$. We denote by $(A) = \{y \in E | \exists x \in A \ \ x \leqslant y\}$ the final segment **generated** by $A \subseteq E$. Dually, we say that I is an **initial segment** for $\leqslant$, if $y \in I \ \wedge \ x \leqslant y \Rightarrow x \in I$. We will also refer to final segments as **closed** sets and to initial segments as **open** sets. Indeed, it is not hard to verify that the open sets form a topology on E; they are even stable under infinite intersections. We have the following classical equivalent conditions for an ordering to be Noetherian:

Proposition A.1. *Let $\leqslant$ be an ordering on E. Then the following are equivalent:*

(a) The ordering $\leqslant$ is Noetherian.
(b) Any final segment of E is finitely generated.
(c) The ascending chain condition w.r.t. inclusion holds for final segments of E.
(d) One can extract an increasing sequence from any sequence $x_1, x_2, \cdots \in E$.
(e) Any extension of E into a total ordering yields a well-ordering.

Proof. Let F be a final segment of E and $G \subseteq F$ the subset of minimal elements of F. G is an antichain, whence finite. Moreover, G generates F, since $\leqslant$ is well-founded. Inversely, if $x_1, x_2, \cdots$ is an infinite antichain or an infinite strictly decreasing sequence, the final segment generated by $\{x_1, x_2, \cdots\}$ is not finitely generated. This proves (a) $\Leftrightarrow$ (b).

Now let $F_1 \subseteq F_2 \subseteq \cdots$ be an ascending chain of final segments. If the final segment $F = \bigcup_n F_n$ is finitely generated, say by G, then we must have $G \subseteq F_n$, for some n. This shows that (b) $\Rightarrow$ (c). Inversely, let G be the set of minimal elements of a final segment F. If $x_1, x_2, \cdots$ are pairwise distinct elements of G, then $(x_1) \subseteq (x_1, x_2) \subseteq \cdots$ forms an infinite strictly ascending chain of final segments.

Now consider a sequence $x_1, x_2, \cdots$ of elements in E, and assume that $\leqslant$ is Noetherian. We extract an increasing sequence $x_{i_1}, x_{i_2}, \cdots$ from it by the following procedure: Let F_n be the final segment generated by the x_k, with $k > i_n$ and $x_k \geqslant x_{i_n}$ ($F_0 = E$ by convention) and assume by induction that the subsequence of $x_1, x_2, \cdots$ of those x_j which are in F_n is infinite. Since F_n is finitely generated by (b), we can select a generator $x_{i_{n+1}}$, with $i_{n+1} > i_n$ and such that the subsequence of $x_1, x_2, \cdots$ of those x_j which are in F_{n+1} is infinite. On the other hand, it is clear that it is not possible to extract an increasing sequence from an infinite strictly decreasing sequence or from a sequence, whose image forms an infinite antichain.

Let us finally prove (a)$\Leftrightarrow$(e). An ordering containing an infinite antichain or an infinite strictly decreasing sequence can always be extended as to contain a copy of $-\mathbb{N}$, by a straightforward application of Zorn's lemma. Inversely, any extension of a Noetherian ordering is Noetherian. $\square$

In order to state some applications of proposition A.1, we need some more definitions. Let E and F be ordered sets. Then we define the natural ordering on the disjoint union $E \amalg F$ of E and F, by taking the induced ordering on each of the summands, and by taking E and F mutually incomparable. Similarly, we define the product ordering on $E \times F$ by $(x, y) \leqslant_{E\times F} (x', y') \Leftrightarrow x \leqslant_E x' \ \wedge \ y \leqslant_F y'$.

An increasing mapping φ between from E into F is a mapping such that $x \leqslant_E y \Rightarrow \varphi(x) \leqslant_F \varphi(y)$. We remark that if this is the case, then $\varphi(S)$ is a final segment of E for any final segment S of F. Now assume that $\sim$ is an equivalence relation on E. We say that $\sim$ is **compatible** with $\leqslant_E$, if $x \leqslant y \wedge x \sim x' \Rightarrow \exists y' \sim y \ \ x' \leqslant y'$. In this case, we have a natural ordering defined on $E/{\sim}$, and the projection $E \to E/{\sim}$ is an increasing mapping. We state without prove the following easy proposition, where (c), (d) and (e) follow from proposition A.1:

Proposition A.2.

(a) Any ordering on a finite set is Noetherian.
(b) The usual ordering on $\mathbb{N}$ is Noetherian.
(c) If E and F are Noetherian ordered sets, then so is $E \amalg F$.
(d) If E and F are Noetherian ordered sets, then so is $E \times F$.
(e) Let E be a Noetherian ordered set and let φ be a surjective increasing mapping from E onto an ordered set F. Then F is Noetherian.
(f) If E is a Noetherian ordered set, then so is any subset of E for the induced ordering.
(g) If E is a Noetherian ordered set, and $\sim$ an equivalence relation, compatible with $\leqslant_E$, then $E/{\sim}$ is a Noetherian. □

Corollary. (Dickson's lemma) *$\mathbb{N}^n$ is a Noetherian ordered set for each n.* □

To finish this section, let us state the so called **Noetherian induction principle**, which generalizes the classical induction principle over $\mathbb{N}$ as well as transfinite induction:

Proposition A.3. *Let P be some property concerning ordered sets, such that $P(E)$ is true whenever $P(I)$ is true for every proper initial segment I of E. Then $P(E)$ holds for all Noetherian ordered sets.*

Proof. Assume that $P(E)$ is false for some Noetherian ordered set, then we can construct a strictly decreasing sequence of initial segments $I_1 \supset I_2 \supset \cdots$ of E, such that $P(I_n)$ is false for each n. But then $E \backslash I_1 \subset E \backslash I_2 \subset \cdots$ would be an infinite strictly increasing sequence of final segments. □

A.3 Classical theorems on Noetherian orderings

In this section we will generalize proposition A.2 in order to provide more examples of Noetherian orderings. Let us be given an ordered set E. We will denote by E^* resp. $E^\diamond$ the sets of **non commutative words** and **commutative words** over E. The sets of non empty words resp. commutative words are denoted by E^+ resp. $E^\dagger$. We denote words either by products of letters $x_1 \cdots x_n$, or by n-tuples $[x_1, \cdots, x_n]$, in the case when confusion might arise. Elements of $E^\diamond$ are also denoted by words, although it is understood that the letters commute. We remark that an element of $E^\diamond$ can always be represented by a word $x_1 \cdots x_n \in E^*$, with $x_i < x_j \Rightarrow i < j$, for all i, j. If the ordering on E is total, then this representation is canonical.

The sets of commutative and non commutative words can be given "natural" orderings in the following way: we define $x_1 \cdots x_n \leqslant_{E^*} y_1 \cdots y_m$, if and only if there exists a strictly increasing mapping $\varphi : \{1, \cdots, n\} \to \{1, \cdots, m\}$, such that $x_i \leqslant_E y_{\varphi(i)}$, for all $1 \leqslant i \leqslant n$. It is not hard to verify that the equivalence relation determined by the permutation of letters is compatible with $\leqslant_{E^*}$. Hence, we also have a natural ordering defined on $\leqslant_{E^\diamond}$. For instance, if $E = \mathbb{N}$, then we have

$$\begin{array}{lll} [2,31,15,7] & \leqslant_{\mathbb{N}^*} & [2,8,35,17,3,7,1] \\ [2,31,15,7] & \not\leqslant_{\mathbb{N}^*} & [2,8,35,17,3,2,1] \\ [2,31,15,7] & \leqslant_{\mathbb{N}^\diamond} & [2,8,35,17,3,2,1] \end{array}$$

Theorem A.1. (Higman) *If E is a Noetherian ordered set, then so is E^*.*

Proof. We will give a proof due to Nash-Williams (see [NaWi 65]), using his technique of minimal bad sequences. If $\leqslant$ denotes any ordering, then we say that $(x_1, x_2, \cdots)$ is a **bad sequence**, if there do not exist $i < j$ with $x_i \leqslant x_j$. An ordering is Noetherian, if and only if there are no bad sequences. Now assume for contradiction that $s = (w_1, w_2, \cdots)$ is a bad sequence for $\leqslant_{E^*}$. Without loss of generality, we may assume that each w_i was chosen such that the length (as a word) of w_i were minimal, under the condition that w_i be in $E^* \backslash (w_1, \cdots, w_{i-1})$. We say that $(w_1, w_2, \cdots)$ is a **minimal bad sequence**.

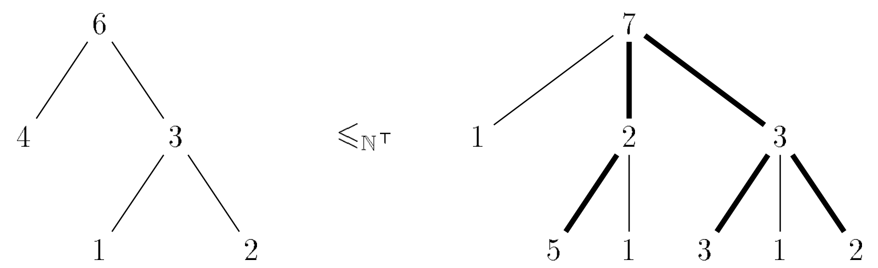
Now none of the w_i can be the empty word, so we can factorize $w_i = x_i w_i'$, where x_i is the first letter of w_i. By proposition A.1(d), we can extract an increasing sequence $x_{i_1}, x_{i_2}, \cdots$ from $x_1, x_2, \cdots$. Now consider the sequence $s' = (w_1, \cdots, w_{i_1 - 1}, w_{i_1}', w_{i_2}', \cdots)$. By the minimality of s, this sequence is good. Hence, there exist $j < k$ with $w_{i_j}' \leqslant_{E^*} w_{i_k}'$. But then, $w_{i_j} \leqslant_{E^*} w_{i_k}$, which contradicts the badness of s. □

Corollary. *If E is a Noetherian ordered set, then so is $E^\diamond$.*

More generally, one can consider the set $E^\top$ of **finite trees**, whose nodes are labeled by elements of E. We recall that a finite E-labeled tree T is recursively defined as being an element x of E, together with an n-tuple $T_1, \cdots, T_n$ of E-labeled trees ($n = 0$ being allowed). We write $T = x[T_1, \cdots, T_n]$. Equivalently, we can see an E-labeled tree as a finite set T of points, called **nodes**, labeled by elements of E. One of the nodes $\mathrm{root}(T)$ of T is said to be the **root** of T. To each other node is associated a unique distinct node, which is said to be its **predecessor** or **parent**. Finally, a total ordering is given on the successors of each node.

A node which is not the predecessor of any other node is called a **leaf**. The set of leafs of T is denoted by $\mathrm{leaf}(T)$. The transitive closure $\preceq_T$ of the predecessor relation determines a partial ordering on T; we say that a is an **ancestor** of b, if $a \preceq_T b$. For each node $a \in T$, the descendants of a form a **subtree** of T, of which a is the root. The subtrees determined by the successors of a node are said to be its **children**. Any two nodes a, b of T admit a lowest common upper bound w.r.t. $\preceq_T$, which we denote by $a \vee b$.

The partial ordering $\preceq_T$ can canonically be extended into a total ordering $\leqslant_T$ by imposing that the children of each node a are ordered by the corresponding ordering on the successors of a and that $a' \leqslant_T b'$ for any $a' \preceq_T a$, $b' \preceq_T b$ such that $a \leqslant_T b$. The "natural" embeddability ordering on $E^\top$ is given by $T \leqslant_{E^\top} T'$, if and only if there exists an injective strictly increasing (for the total orderings) mapping $\varphi : T \to T'$, such that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$, and $l(a) \leqslant_E l(\varphi(a))$, for all $a, b \in T$. The following is an example of a tree which embeds into another tree, if $E = \mathbb{N}$:



Theorem A.2. (Kruskal) *If E is a Noetherian ordered set, then so is $E^\top$.*

Proof. Assume that there exists a minimal bad sequence $(T_1, T_2, \cdots)$ in the sense that the cardinal of T_i is minimal, for fixed $T_1, \cdots, T_{i-1}$. We can write $T_i = x_i[T_{i,1}, \cdots, T_{i,n_i}]$ for each i. We claim that the induced ordering on $S = \{T_{i,j} | j \leqslant n_i\}$ is Noetherian.

Indeed, suppose for contradiction that the claim is false, and let

$$(T_{i_1,j_1}, T_{i_2,j_2}, \cdots)$$

be a bad sequence. Let k be such that i_k is minimal. Then the sequence

$$(T_1, \cdots, T_{i_k - 1}, T_{i_k, j_k}, T_{i_{k+1}, j_{k+1}}, \cdots)$$

is also bad, which contradicts the minimality of $(T_1, T_2, \cdots)$.

Finally, we know that $E \times S^*$ is a Noetherian ordered set by Higman's theorem and proposition A.2(d). But each tree T_i can be interpreted as an element of this set. Hence, we obtain the desired contradiction. □

Remark A.2. In the case when we restrict ourselves to trees of bounded arity, the above theorem was already due to Higman. The general theorem was first conjectured by Vázsonyi. Let us also notice that much research has been done in order to extend Kruskal's theorem to trees with infinite arities. This led to the concept of **better-quasi-ordering**, which lies between well-quasi-ordering and well-ordering. We refer to [Mil 85] and [Pouz 85] for surveys. It would be interesting to know whether this related theory can be given an interpretation in our context. Another very deep result (see [RS]) is that the **graph minor ordering** on the set of finite labeled graphs is Noetherian. That is, $G \leqslant G'$, if G can be obtained from G' by deleting and contracting edges and decreasing labels. Again we do not have an interpretation in our context.

By playing some combinatorial games with the encoding of finite trees, one can obtain many variants of Kruskal's theorem. We will mention one of them now. Let X be any ordered set and let Ω be an ordered set of operations on X (that is, each $f \in \Omega$ is an n_f-ary operation $f : X^{n_f} \to X$). We say that the operations in Ω are **extensive**, and that the ordering on M is **compatible** with the ordering on X, if the following two conditions are verified respectively:

O1. $x_i \leqslant_X f(x_1, \cdots, x_{n_f})$ for any $f \in \Omega$, and $1 \leqslant i \leqslant n_f$;

O2. Let $f \leqslant_\Omega g$ be in Ω. Then $f(x_1, \cdots, x_{n_f}) \leqslant_X g(y_1, \cdots, y_{g_n})$, whenever there exists a strictly increasing mapping $\varphi : \{1, \cdots, n_f\} \to \{1, \cdots, n_g\}$, such that $x_i \leqslant_X y_{\varphi(i)}$ for each i.

Now let G be a subset of X. The smallest subset of X which contains G and which is stable under Ω is said to be the subset of X **generated** by G w.r.t. Ω, and will be denoted by $(G)_\Omega$. Then we have

Theorem A.3. *Let X be an ordered set and Ω a Noetherian ordered set of operations on X verifying the above conditions. Then if $(G)_\Omega$ is a Noetherian subset of X, whenever G is.*

Proof. Any element of $(G)_\Omega$ can be represented as a finite tree labeled by elements of $G \amalg \Omega$. We conclude by proposition A.2(c) and Kruskal's theorem.

We remark that Kruskal's theorem can also be derived from the present theorem. To do this, one takes $G = \emptyset$, $\Omega = E \times \mathbb{N}$ (with $n_{(x,k)} = k$) and $X = E^\top$. Then the smallest ordering relation on X verifying conditions **O1** and **O2** is the embeddability ordering. □

A.4 Noetherian choice operators

The results of this section find their origin in the theory of transseries (see chapters 1 and 2). They can be used to establish implicit function theorems of a very general type. Roughly speaking, we prove a generalization of Kruskal's theorem, where the elements of Ω are not operations, but rather mappings f from X^{n_f} into the *power set* $\mathcal{P}(X)$ of X. The reason why we need this, is that the support of the derivative of a transmonomial is generally not a singleton (think of $\partial e^{e^x/(x-1)}/\partial x = (x^{-1} - x^{-3} - 2x^{-4} - \cdots)e^x e^{e^x/(x-1)}$); we do not have this difficulty in the case of ordinary power series, where $\partial x^k/\partial x = kx^{k-1}$.

Let X denote an ordered set and let Σ be a set of X**-labeled structures**. This means that to each $\sigma \in \Sigma$ we can associate a set I_σ and a mapping $l_\sigma : I_\sigma \to X$ (note that this association need not to be injective). If $Y \subseteq X$, then we denote by $\Sigma_Y = \{\sigma \in \Sigma | \operatorname{im} l_\sigma \subseteq Y\}$ the subset of Σ of Y-labeled structures. We order couples in $\Sigma \times X$ by $(\sigma, x) < (\sigma', x') \Leftrightarrow x < x'$.

A mapping $\vartheta : \Sigma \to \mathcal{P}(X)$ is called a **choice operator**. We say that ϑ is **Noetherian**, if for any Noetherian subset Y of X, the subset $\{(\sigma, x) | \sigma \in \Sigma_Y \wedge x \in \vartheta(\sigma)\}$ of $\Sigma \times X$ is Noetherian. We say that ϑ is **extensive**, if for each $\sigma \in \Sigma$, $a \in \operatorname{im} l_\sigma$ and $b \in \vartheta(\sigma)$, we have $a \leqslant b$. We say that ϑ is **strictly extensive**, if for each $\sigma \in \Sigma$, $a \in \operatorname{im} l_\sigma$ and $b \in \vartheta(\sigma)$, we have $a < b$.

Remark A.3. Let us comment why we insist the subset $S = \{(\sigma, x) | \sigma \in \Sigma_Y \wedge x \in \vartheta(\sigma)\}$ of $\Sigma \times X$ to be Noetherian in our definition, and not simply the set $S' = \bigcup_{\sigma \in \sigma_Y} \vartheta(\sigma)$. Indeed, this condition guarantees that for a given x, we only have a finite number of σ with $(\sigma, x) \in S$ — a property which will prove to be important for future purposes.

Example A.1. Let f be an extensive n-ary operation, and let $\Sigma = X^n$, with

$$\begin{array}{rccc} l_{(x_1, \cdots, x_n)} : & \{1, \cdots, n\} & \to & X \\ & i & \mapsto & x_i. \end{array}$$

Then

$$\begin{array}{rccc} \vartheta : & \Sigma & \to & X \\ & (x_1, \cdots, x_n) & \mapsto & \{f(x_1, \cdots, x_n)\} \end{array}$$

is an extensive Noetherian choice operator.

As in the previous section, we now want to build trees. In fact, it suffices to show how to build words, because of the strength of the formalism. Let $\vartheta : \Sigma \to \mathcal{P}(X)$ be a choice operator and let us inductively define the choice operator ϑ^*. First we set $T_0 = X$, with $l_x : \{1\} \to X; 1 \mapsto x$ and $\vartheta^*(x) = \{x\}$. Next, assume that we defined ϑ^* on the domain $T_0 \amalg \cdots \amalg T_k$. Let $\sigma \in \Sigma$ be given together with a family $(\tau_i)_{i \in I_\sigma}$ of elements of $T_0 \amalg \cdots \amalg T_k$. Assume that $\tau_i \in T_k$, for at least one i (we say that τ_i has **depth** k), and that $l_\sigma(i) \in \vartheta^*(t_i)$, for each i. Then these data determine a unique structure $\tau = \sigma[\tau_i]_{i \in I_\sigma}$ in T_{k+1}. We take $I_\tau = \amalg_{i \in I_\sigma} I_{\tau_i}$, with $l_\tau(j) = l_{\tau_i}(j)$, for $j \in I_{\tau_i}$, and we define $\vartheta^*(\tau) = \vartheta(\sigma)$. This construction inductively determines a choice operator ϑ^* with domain $T_0 \amalg T_1 \amalg \cdots$. We take the choice operator ϑ^+ to be the restriction of ϑ^* to the domain $T_1 \amalg T_2 \amalg \cdots$.

Theorem A.4. *Let ϑ be a strictly extensive Noetherian choice operator. Then ϑ^+ is a strictly extensive Noetherian choice operator.*

Proof. The tree operator ϑ^+ is clearly strictly extensive, by induction. Let Y be a Noetherian subset of X. Assume that there exists a minimal bad sequence

$$((\tau_1, x_1), (\tau_2, x_2), \cdots),$$

with $x_i \in \vartheta^+(\tau_i)$ and $\operatorname{im} l_{\tau_i} \subseteq Y$ for each i, in the sense that the depth of τ_i is minimal, for fixed $x_1, \cdots, x_{i-1}$. Write $\tau_i = \sigma_i[\tau_{i,j}]_{j \in I_{\tau_i}}$ for each i. We claim that the induced ordering on $S = \{(\vartheta^+(\tau_{i,j}), l_{\sigma_i}(j)) | j \in I_{\tau_i}\}$ is Noetherian.

Indeed, suppose for contradiction that the claim is false, and let

$$((\vartheta^+(\tau_{i_1,j_1}), l_{\sigma_{i_1}}(j_1)), (\vartheta^+(\tau_{i_2,j_2}), l_{\sigma_{i_2}}(j_2)), \cdots)$$

be a bad sequence. Let k be such that i_k is minimal. Then the sequence

$$((\tau_1, x_1), \cdots, (\tau_{i_k-1}, x_{i_k-1}), (\vartheta^+(\tau_{i_k,j_k}), l_{\sigma_{i_k}}(j_k)), (\vartheta^+(\tau_{i_{k+1},j_{k+1}}), l_{\sigma_{i_{k+1}}}(j_{k+1})), \cdots)$$

is also bad, since ϑ is strictly extensive. But this contradicts our minimality hypothesis. Having proved the claim, the desired contradiction is obtained by using the Noetherianity of ϑ for S. □

A.5 References

[Can 1899] G. Cantor. Sur les fondéments de la théorie des ensembles infinis. *Rééditions Jacques Gabay.*

[Erd 49] P. Erdős. Problem 4358. *Amer. Math. Monthly 56 (p. 480).*

[Hahn 07] H. Hahn. Über die nichtarchimedischen Grössensysteme. *Sitz. Akad. Wiss. Wien 116 (p. 601-655).*

[Haus 08] F. Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Math. Ann. 65 (p. 435-505).*

[Hig 52] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. 2 (p. 326-336).*

[Krus 60] J.B. Kruskal. Well-quasi-ordering, the tree theorem and Vázsonyi's conjecture. *Trans. Amer. Math. Soc. 95 (p. 210-225).*

[Mal 48] A.I. Malcev. On the embedding of group algebras in division algebras. *Daklady Akad. Nauk. SSSR. 60 (p. 1409-1501).*

[Mil 85] E.C. Milner. Basic wqo- and bqo-theory. *In Rival (ed.) Graphs and orders. D. Reidel Publ. Comp. (p. 487-502).*

[NaWi 63] C. St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Philos. Soc 59 (p. 833-835).*

[Pouz 85] M. Pouzet. Applications of well quasi-ordering and better quasi-ordering. *In Rival (ed.) Graphs and orders. D. Reidel Publ. Comp. (p. 503-519).*

[RS] N. Robertson, P. Seymour. Graph minors — a survey. *Manuscript.*

[VdH 94a] J. van der Hoeven. Outils effectifs en asymptotique et applications. *Rapport de recherche LIX/RR/94/09 de L'Ecole polytechnique, France.*

Appendix B

Partial algebras and their geometry

B.1 Introduction

The aim of this appendix is to introduce algebraic geometry methods for structures which are different from rings, by adopting the language of category theory. In chapter 6, this theory is applied to define and manipulate transseries in several variables, but this quite technical appendix can be skipped without much harm, at least at a first reading of 6. To facilitate the reading of this appendix, let us now comment its motivation and main results.

Just as rings are used in algebraic geometry to study algebraic equations, we want to construct generalized transseries in order to study very general singular equations. Following this parallel, elements of a generalized transseries ring need an interpretation as functions on some space (just as $k[x, y]$ is the ring of polynomial functions on the plane). Different problems arise at this point. First, contrary to the ring operations, the logarithm can only be defined for positive elements (or non zero elements, if one considers $\log |x|$). Secondly, we need to have an interpretation for the ordering relation. Thirdly, we need to interpret infinite summation. Fourthly, we should make the concept of "the space" associated to a ring of generalized transseries more precise. Fifthly, we would like to incorporate in our theory the most common spaces, such as the line, the plane, etc. And we can go on.

Now the first problem also arises in algebraic geometry, if we want to incorporate the division. This is precisely what motivates the systematic use of localization. We might therefore borrow some of these ideas for our treatment, and consider spaces with sheaves of functions defined on it. Alternatively, we can consider spaces with partially defined functions on it. The second problem is not a classical one. Nevertheless, it has a strong analogy with the first one: we can consider the definability of a partially defined function as a relation. In other words, the first problem can be seen as a special case of the second one.

The algebraic translation of what precedes is that we will consider categories of sets with partially defined functions and relations, so called partial Σ-algebras.

Next, we impose axioms on such structures, which are of a particular type, namely Horn clauses. Ordered rings can for example be defined using such axioms. It can be shown that the category of partial Σ-algebras which satisfy certain Horn-clauses has a very rich structure, i.e. many abstract nonsense constructions can be carried out. Moreover, we can solve the third problem by allowing functions and relations to have infinite arities; infinite summation can then be seen as a partially defined function. Partial Σ-algebras will be studied in sections B.2, B.3 and B.4.

Geometrically, partial Σ-algebras correspond to functions on a space. The next step is to introduce the spaces themselves. As in algebraic geometry we will work "over" a fixed partial Σ-algebra A. For instance, a $\mathbb{Z}$-algebra is a morphism from $\mathbb{Z}$ into a ring R. Now a B-point of a partial Σ-algebra F over A is a morphism from F into B, where B is another partial Σ-algebra over A. For example, an $\mathbb{R}$-point (x, y) in the "plane" $\mathbb{Z} \to \mathbb{Z}[x, y]$ naturally associates a value $P(x, y) \in \mathbb{R}$ to each polynomial in $\mathbb{Z}[x, y]$. Instead of considering B-points for all possible B, we may restrict our attention to B-points with $B \in \mathrm{obj}(\mathsf{P})$, where P is a subcategory of the category of partial Σ-algebras over C. For instance, we may restrict B to be a field in the case of $\mathbb{Z}$-algebras; in terms of systems of algebraic equations, this means that we are only looking for solutions in a field.

Assume that we have fixed P, so that the points of a partial Σ-algebra over A are B-points, with $B \in \mathrm{obj}(\mathsf{P})$. Not all partial Σ-algebras F have a "pointwise" geometrical interpretation. In particular, it is not always possible to interpret elements of F as unique functions from the point space of F to a Σ-algebra B over A in P (the union of all these B corresponds to the affine line). For instance, in the case of $\mathbb{Z}$-algebras, where P is taken to be the category of fields, this is due to the possibility of nilpotent elements: for every morphism of $\mathbb{R}[x]/(x^2)$ into a field, 0 and x have the same image, yet they are not equal. Now varieties are partial Σ-algebras F over A for which a suitable pointwise interpretation of the elements of F is possible. In section B.5 we will show that there is a canonical way of associating a variety $\hat{F}$ to any partial Σ-algebra F over A. In the case of $\mathbb{Z}$-algebras, we just quotient by the ideal of all elements x, which are sent to zero by every point in the above sense. The canonical nature of the association $F \mapsto \hat{F}$ implies that we can carry out many abstract nonsense constructions. In particular, we can construct many familiar spaces such as the line, the plane, etc.

A consequence of the pointwise nature of a variety is that the function space of a variety V shares a lot of properties with A. In particular, any Horn clause verified by A is verified by the function space of V. The interesting point is that this resemblance is preserved for a lot of other intrinsic properties of A, which can not — or not easily — be modelized by Horn clauses. This it what makes the construction useful in the case of transseries. In the last section we discuss some possible extensions of our theory. More precisely, we discuss a trick of Lawvere based on the construction of affine varieties from section B.5, which permits us to incorporate (the analogue of) nilpotent elements in the function spaces of varieties.

We also discuss another extension of the notion of a point, which allows us to recover the analogues of projective varieties and others.

In this appendix, we assume the reader is familiar with category theory, although a certain number of definitions will be recalled. We refer to [ML 71] for a more extensive treatment. A very basic knowledge of commutative algebra, universal algebra, and algebraic geometry might also ease the reading. We respectively refer to [AtMac 69] or [Lang 84], [Cohn 65] and [Har 77]. Let us finally remark that some of the nomenclature may not correspond to the classical nomenclature. For instance, varieties may be reducible (which is not the case in [Har 77]), and are always understood in the algebraic geometry sense (in [Cohn 65] a variety is not at all the same thing).

Most of the results of this appendix are not new, perhaps with the exception of the last two sections. However, our way of exposition is not very standard. The classical theory of Σ-algebras does not treat partially defined functions nor relations, and the arities are classically restricted to be finite. In particular, the device of subquotient objects seems to be new. However, we think that all our results about partial Σ-algebras can be obtained by reformulation of related theories. For simplicity reasons, we have chosen a quite classical approach, but it should be noticed that many generalizations can be obtained by a more extensive use of category theory. For instance, it is possible to study topological Σ-algebras and so on.

B.2 Partial algebras

A **signature** Σ is a set of **function symbols** Σ_f and **relation symbols** Σ_R , together with there associated arities. That is, to every $f \in \Sigma_f$, resp. $R \in \Sigma_r$ corresponds a set N_f resp. N_R , which is called the **arity** of f resp. R. For our purposes, we may assume that all arities are at most countable.

From now on we will fix a signature Σ. A **Σ-algebra** is a set E on which these function and relation symbols correspond to functions resp. relations. That is, each $f \in \Sigma_f$ corresponds to a function f (or f_E, whenever confusion might arise) from E^{N_f} into E and similarly for relations. A Σ-algebra E is said to be **full**, if the relation symbols of Σ_R are maximal: $R_E = E^{N_R}$, for each $R \in \Sigma_R$. A **partial Σ-algebra** is defined like a Σ-algebra, with the exception that the function symbols need not to be defined totally. We will note by $\operatorname{dom} f$ the domain of a function f.

A **morphism** of Σ-algebras $E \to F$ is a mapping φ, such that $\varphi(f(x)) = f(\varphi(x))$ resp. $R(x) \Rightarrow R(\varphi(x))$, for all $f \in \Sigma_f$ resp. $R \in \Sigma_R$ and $x \in E^{N_f}$ resp. $x \in E^{N_R}$. Of course, $f(x)$ is defined by $(f(x))_i = f(x_i)$, for each I and $x \in E^I$. A **morphism** of partial Σ-algebras is defined similarly, with the exception that $\varphi(f(x)) = f(\varphi(x))$ only needs to hold, when $\varphi(f(x))$ is defined (hence, $x \in \operatorname{dom} f_E$ implies $\varphi(x) \in \operatorname{dom} f_F$). Similarly, one defines **partial morphisms** of partial Σ-

algebras, by demanding $\varphi(f(x)) = f(\varphi(x))$ resp. $R(x) \Rightarrow R(\varphi(x))$ to hold when defined only. Morphisms of partial Σ-algebras will often be called **Σ-morphisms**. It is straightforward to verify that Σ-algebras and partial Σ-algebras form categories and we will note them respectively by Σ-Alg resp. Σ-PAlg.

Let X be any set and consider the set I_X of trees, labeled by elements of $\Sigma_f \amalg X$. Here we assume that the arities of each node of the tree correspond to the arity of each label (the arities of elements of X being zero). The set I_X has a natural Σ_f-algebra structure. Taking $R_{I_X} = I_X^{N_R}$, for each $R \in \Sigma_R$, we give I_X the structure of a full Σ-algebra which satisfies the following universal property: if ξ is any mapping from X into a full Σ-algebra E, then there exists a unique Σ-morphism $I_X \stackrel{\varphi}{\to} E$, such that $\xi = \varphi \circ \iota_X$, where ι_X is the natural inclusion from X into I_X. We say that I_X is the **free full Σ-algebra** over X.

Usually, one is interested in Σ-algebras or partial Σ-algebras which verify certain axioms. Let us make this more precise. Let W denote a set of **variable symbols** (which is always assumed to be sufficiently large), and let $T = I_W$ denote the set of **ground terms** w.r.t. Σ. If E is a partial Σ-algebra, then by an **assignment**, we mean a mapping $\nu : W \to E$. An assignment ν can recursively be extended to a subset of T, by putting $\nu(f(x)) = f(\nu(x))$, for all $f \in \Sigma_f$ and $x \in T_k^{Nf}$, with $\nu(x) \in \operatorname{dom} f$. If $\nu(t)$ is defined for every assignment ν, then we say that t is **totally defined**, or **defined** on E.

A **ground property** w.r.t. Σ is either an expression of the form $t = t'$, where $t, t' \in T$, or an expression of the form $R(x)$, where $R \in \Sigma_R$ and $x \in T^{N_R}$. A ground property $t = t'$ resp. $R(x)$ is **valid** for an assignment ν, if t and t' resp. the x_i are defined and if $\nu(t) = \nu(t')$ resp. $\nu(R(t(x))$ holds. If a ground property is valid for all assignments into E, then we say that it is valid on E. A ground property of the form $t = t$, for some ground term t, is valid for ν if and only if $\nu(t)$ is defined. We will write $t\#$ instead of $t = t$ and read "t is defined".

A **Horn clause** (w.r.t. Σ) is a pair $((P_i)_{i\in I}, Q)$, where the P_i and Q are ground properties. We say that $((P_i)_{i\in I}, t)$ is **valid** in a partial Σ-algebra E, if $\nu(Q)$ holds for all assignments ν, such that every $\nu(P_i)$ holds. We write $E \models \bigwedge_{i\in I} P_i \Rightarrow Q$, if this is the case. We remark that ground properties can be interpreted as Horn clauses, by taking $I = \phi$. We denote by C_Σ the set of Horn clauses w.r.t. Σ. If E is a partial Σ-algebra, then we denote by C_E the set of Horn clauses which are valid in E. Inversely, let C be a set of clauses or **axioms**. A **model for** C is a partial Σ-algebra E with $C \subseteq C_E$. These models form a subcategory A-$\mathsf{Mod} = (\Sigma, C)$-$\mathsf{PAlg}$ of Σ-PAlg, which is called the category of partial Σ-algebras verifying C.

Let us now discuss some matters concerning the foundations of set theory and category theory. The objects of a category C do not form a set in general, and neither do the morphisms of C. However, by enlarging our model of set theory, we can interpret the objects resp. morphisms of most categories C as sets $\mathrm{obj}(\mathsf{C})$, resp. $\hom(\mathsf{C}, \mathsf{C})$. More precisely, assume that we have a model M of the Zermelo-Fraenkel axioms, such that there exists a privileged set U, which is also a model of the Zermelo-Fraenkel axioms for the induced relation $\in$.

To avoid confusion between sets w.r.t. M and U, sets w.r.t. M are called **meta-sets** (alternatively, sets w.r.t. U are sometimes called **small sets**). Using this formalism, we can for example speak about the meta-set of all groups. We also have $U = \operatorname{obj}(\mathsf{Set})$, where Set denotes the category of sets. The formalism is also useful when considering mathematical structures where we do not have any a priori bounds for the arities of the operations. This arises for example in the case of infinite summation of well-ordered transseries, where the arity depends on the exponential depths of the transseries we are considering. This problem is solved by taking meta-sets of function and relation symbols, instead of classical sets.

We finally remark that our formalism can still be strengthened by allowing meta-meta-sets, and so on. Indeed, this is done by assuming that we have an even bigger model $\hat{M}$ of the Zermelo-Fraenkel axioms, for which M is a set. Nevertheless, for most of the practical applications, the consideration of meta-sets (and sometimes meta-meta-sets) is sufficient.

B.3 The lattice of subquotient objects

In this section, Σ denotes a fixed signature and C a set of axioms w.r.t. Σ. Let E be a partial Σ-algebra. A **subalgebra** F of E is a subset of E on which we have a partial Σ-algebra structure, such that the canonical injection is a Σ-morphism. Assume now that E is a full Σ-algebra. Then a **quotient algebra** F of E is a quotient set of E on which we have a Σ-algebra structure, such that the canonical surjection is a Σ-morphism. A **subquotient algebra** F of E is a partial subalgebra of a quotient algebra of E. These definitions can be extended to the case in which F models C. In that case, we say that F is a **(Σ, C)-subalgebra** resp. **quotient (Σ, C)-algebra** resp. **subquotient (Σ, C)-algebra**.

To a subalgebra of a partial Σ-algebra E corresponds a monomorphism $F \xrightarrow{\iota} E$, which is said to be a **subobject** of E. We quasi-order subobjects by

$$F \xrightarrow{\iota} I \leqslant F' \xrightarrow{\iota'} E \Leftrightarrow \exists F \xrightarrow{\varphi} F' \quad \iota = \varphi \circ \iota'.$$

Similarly, if E is a full Σ-algebra, then a quotient algebra of E gives rise to an epimorphism $E \xrightarrow{\pi} F$, which is said to be a **quotient object** of E. We quasi-order quotient objects by:

$$E \xrightarrow{\pi} F \leqslant E \xrightarrow{\pi'} F' \Leftrightarrow \exists F \xrightarrow{\varphi} F' \quad \pi' = \varphi \circ \pi.$$

In the literature, the opposite quasi ordering is usually taken. The ordering on subquotient objects justifies our inversion: a **subquotient object** of E is a pair of morphisms $F \xrightarrow{\iota} Q \xleftarrow{\pi} E$, where π is a quotient object of E, and ι a subobject of Q. We quasi-order subquotient objects of E by

$$F \xrightarrow{\iota} Q \xleftarrow{\pi} E \leqslant F' \xrightarrow{\iota'} Q' \xleftarrow{\pi'} E,$$

if and only if there exist $F \xrightarrow{\varphi} F'$ and $Q \xrightarrow{\psi} Q'$, such that

$$\begin{tikzcd}
F \arrow[r, "\iota"] \arrow[d, "\varphi"] & Q \arrow[d, "\psi"] & \\
 & & E \arrow[ul, "\pi"'] \arrow[dl, "\pi'"] \\
F' \arrow[r, "\iota'"] & Q' &
\end{tikzcd}$$

commutes. We remark that it can be shown that the above definitions of subobjects and quotient objects coincide with the usual definitions (as monomorphism $F \to E$ resp. epimorphisms $E \to F$) up to isomorphism. In particular, one can generalize and define quotient objects of partial Σ-algebras. However, this leads to several complications, and our restricted definition will suffice for what follows.

Proposition B.1. *Let E be a full Σ-algebra. The set of subquotient (Σ, C)-objects of E forms a complete lattice for $\leqslant$.*

Proof. $S \xrightarrow{\iota} E/\sim \xleftarrow{\pi} E$ is maximal for $\leqslant$, if we take $\sim = E^2$, $S = E/\sim$ and $\iota = Id_{E/\sim}$. Next, let $(S_k \xrightarrow{\iota_k} E/\sim_k \xleftarrow{\pi_k} E)_{k\in K}$ be a family of (Σ, C)-subquotient objects. Let $\sim = \bigcap_{k\in K} \sim_k$. The natural surjection $E \xrightarrow{\pi} E/\sim$ naturally induces a Σ_f-algebra on $E/\sim$. Let $S = \pi(\bigcap_{k\in K} \pi_k^{-1}(S_k))$. We give S the structure of a partial Σ-algebra, by taking $\operatorname{dom} f_S = \pi(\bigcap_{k\in K} \pi_k^{-1}(\operatorname{dom} f_{S_k}))$ and $R_S = \pi(\bigcap_{k\in K} \pi_k^{-1}(R_{S_k}))$, for each $f \in \Sigma_f$ resp. $R \in \Sigma_R$. It is clear from the definition that $S \xrightarrow{\iota} E/\sim \xleftarrow{\pi} E$ must be the infimum of the $(S_k \xrightarrow{\iota_k} E/\sim_k \xleftarrow{\pi_k} E)$, if S models C. Let us show that this is indeed the case.

Remark first that the mapping $\xi_k = \pi_k \circ (\pi^{-1})$ is well defined for $k \in K$ and as $\operatorname{im} \xi_k \circ \iota \subseteq \operatorname{im} \iota_k$, there exists a unique mapping $S \xrightarrow{\chi_k} S_k$ with $\iota_k \circ \chi_k = \xi_k \circ \iota$. Let us now show by structural induction, that if t is a ground term and ν a substitution into S, then $\nu(t)$ is defined, iff $(\chi_k \circ \nu)(t)$ is defined for every $k \in K$. Surely, this is the case, if $t \in W$. Suppose now that $t = f(x)$, where $\nu(x)$ and every $(\chi_k \circ \nu)(x)$ is defined. Then $\nu(t)\# \Leftrightarrow \nu(x) \in \operatorname{dom} f_S \Leftrightarrow \forall k \in K \quad \chi_k(\nu(x)) \in \operatorname{dom} f_{S_K} \Leftrightarrow \forall k \in K \quad (\chi_k \circ \nu)(t)\#$. Similarly $\nu(P)$ is valid, for a ground property P, iff $(\chi_k \circ \nu)(P)$ is valid for every $k \in K$. Finally, let $((P_i)_{i\in I}, Q)$ be a Horn clause in C and ν a substitution into S, such that $\nu(P_i)$ holds for every $i \in I$. Then $(\chi_k \circ \nu)(P_i)$ holds for every $i \in I$ and $k \in K$. Therefore $(\chi_k \circ \nu)(Q)$ is holds for every $k \in K$, so that $\nu(t)$ holds. $\square$

Let E be a partial Σ-algebra E and let $\overline{E}$ be the infimum of all quotient algebras $I_E/\sim$ of I_E, such that $E \to I_E \to I_E/\sim$ is a Σ-morphism. This infimum is defined by proposition B.1, noticing that any quotient object $I_E/\sim \leftarrow E$ can be interpreted as a subquotient object $I_E/\sim \xrightarrow{Id} I_E/\sim \leftarrow E$. We say that $\overline{E}$ is the **Σ-closure** of E, and we have

Proposition B.2. *Let E be a partial Σ-algebra. Then*

(a) *For each Σ-morphism $E \xrightarrow{\varphi} F$ into a full Σ-algebra, there exists a unique Σ-morphism $\overline{E} \xrightarrow{\psi} F$, such that $\varphi = E \to \overline{E} \xrightarrow{\psi} F$.*
(b) *$E \to \overline{E} \leftarrow I_E$ is a subquotient algebra of I_E.*

Proof. Let φ be given. There exists a unique Σ-morphism $I_E \xrightarrow{\xi} F$, such that $\varphi = E \to I_E \xrightarrow{\xi} F$. This ξ factorizes uniquely into the composition of a surjection and an injection $\xi = I_E \to \operatorname{im}\xi \to F$. There exists a unique Σ-morphism $\overline{E} \to \operatorname{im}\xi$, which makes all triangles commutative, by the minimality of $\overline{E}$. This yields the desired Σ-morphism $\overline{E} \to F$ which is easily seen to be unique by decomposing it as the composition of a surjection and an injection.

To prove (b), we consider any extension $\hat{E}$ of the partial Σ-algebra structure on E into a full Σ-algebra structure. By (a), there exists a Σ-morphism $\overline{E} \to \hat{E}$, such that $E \xrightarrow{Id_E} \hat{E} = E \to \overline{E} \to \hat{E}$. The injectivity of $E \to \overline{E}$ follows from the injectivity of Id_E. $\square$

Proposition B.3. *Let E be a partial Σ-algebra. The set of (Σ, C)-subobjects of E forms a complete lattice for $\leqslant$.*

Proof. This follows easily by abstract nonsense from proposition B.1, when representing (Σ, C)-subobjects of E as (Σ, C)-subquotient objects of I_E, using proposition B.2(b). $\square$

Let us finally remark that subquotient algebras can also be considered as "quotientsub algebras" and vice versa (under some restrictions). More precisely, let $S \xrightarrow{\iota} Q \xleftarrow{\pi} E$ be a quotient object of E. Denoting $S' = \pi^{-1}(S)$, we have a canonical injection $S' \xrightarrow{\iota'} E$, a canonical surjection $S' \xrightarrow{\pi'} S$, and the following diagram commutes:

$$\begin{tikzcd} Q & E \arrow[l, "\pi"'] \\ S \arrow[u, "\iota"] & S' \arrow[l, "\pi'"'] \arrow[u, "\iota'"'] \end{tikzcd}$$

Inversely, assume that we are given a partial subobject $S \xrightarrow{\iota} E$ of E and an equivalence relation $\sim$ on S, such that $S \xrightarrow{\pi} S/\sim$ is a morphism of partial Σ-algebras. Then we can consider the smallest relation $\overline{\sim}$ containing $\sim$ on E, such that $Q = E/\overline{\sim}$ is a Σ-algebra. In general, $S/\sim$ is not a subalgebra of Q. Nevertheless, a sufficient condition is that S is an initial segment for the smallest ordering on E such that $x_i \leqslant f(x)$, for any $f \in \Sigma_f$, $x \in E^{N_f}$ and $i \in N_f$. In the proof of theorem B.3 we will see a natural example of the use of quotientsub algebras.

B.4 Existence theorems for adjoints

In this section we will show that a lot of so called universal constructions can be carried out in (Σ, C)-PAlg. We first recall the necessary language. If C is a category, its **opposite** category C^o is obtained by reversing all arrows. If $C' \subseteq C$ and $\Sigma' \subseteq \Sigma$, then each partial (Σ, C)-algebra is in particular a partial (Σ', C')-algebra. The natural functor from (Σ, C)-PAlg into (Σ', C')-PAlg is called the **forgetfull functor**.

Let H be any functor $\mathsf{C}^o \times \mathsf{K}$ from into the category Set of sets. A **universal object** associated to an object x in C is an object $F(x)$ in K, together with a mapping $\varphi_x \in H(x, F(x))$, such that for any other object y in K, together with a mapping $\psi \in H(x, y)$, there exists a unique morphism $F(x) \xrightarrow{\xi} y$, with $\psi = \xi \circ \varphi_x$. If such a couple $(F(x), \varphi_x)$ exists for each object x, then it is a classical exercise to verify that F is a functor, called **universal functor**. Dually, we have **couniversal objects** and **couniversal functors**. I.e. a couniversal object associated to $y \in \mathsf{K}$ is an object $G(y)$, together with a mapping $\varphi_y \in H(G(y), y)$, such that the natural universal property holds.

Usually, $H(x, y)$ is the set of homomorphisms from x to $G(y)$, where G is some functor from K to C. In that case, a universal functor is called a **left-adjoint** for G. Similarly, a couniversal functor is called a **right-adjoint**. It is easy to verify that if F is a left-adjoint for G, then G is a right adjoint for F and the following relation holds:

$$\hom_{\mathsf{C}}(x, G(y)) \cong \hom_{\mathsf{K}}(F(x), y).$$

Theorem B.1. *Let U be the forgetfull functor from (Σ, C)-PAlg to (Σ', C')-PAlg, where $\Sigma' \subseteq \Sigma$ and $C' \subseteq C$. Then U admits a left adjoint.*

Proof. Let A be in (Σ', C')-PAlg and let I_A be the free full Σ-algebra over A. Consider the family $\Omega = (S_i \to I_A/{\sim_i} \leftarrow I_A)_{i \in I}$ of subquotient (Σ, C)-objects, such that each $A \to I_A/{\sim_i}$ is of the from $A \xrightarrow{\varphi_i} S_i \to I_A/{\sim_i}$, where φ_i is a Σ'-morphism. The maximal subquotient object is in Ω, whence $\Omega \neq \phi$. By proposition B.1 we may consider the partial (Σ, A)-algebra U, which is the infimum of all elements in Ω. Then $A \to I_A/\sim$ factors through U by mapping x to $\bigcap_{i \in I} \varphi_i(x)$. Here $\bigcap_{i \in I} \varphi_i(x)$ is well defined in view of the proof of proposition B.1. Moreover, $A \to U$ is a Σ'-morphism, so that $U \in \Omega$. We claim that U satisfies the universal property.

Let $A \xrightarrow{\varphi} B$ be a Σ'-morphism into a partial (Σ, C)-structure B. Then φ factors uniquely through the smallest partial (Σ, C)-subalgebra $(\operatorname{im} \varphi)$ of B containing $\operatorname{im} \varphi$.

Let us show that $(\mathrm{im}\,\varphi)$ is a subquotient object of $I_{\mathrm{im}\,\varphi}$ and therefore of I_A. First, we have a natural maximally defined partial Σ-morphism ρ from $I_{\mathrm{im}\,\varphi}$ into B. It is straightforward to verify that $\mathrm{im}\,\rho$ is a (Σ, C)-algebra, whence $(\mathrm{im}\,\varphi) \subseteq \mathrm{im}\,\rho$. Therefore, we can canonically extend the injection of $\mathrm{im}\,\varphi$ into $\overline{\mathrm{im}\,\varphi}$ into a mapping ι of $(\mathrm{im}\,\varphi)$ into $\overline{\mathrm{im}\,\varphi}$, by sending each $\rho(x)$ in $(\mathrm{im}\,\varphi)$ to $\overline{x}$. Then ι is injective, since $(\mathrm{im}\,\varphi) \to \overline{(\mathrm{im}\,\varphi)}$ is. In fact, we have $\overline{(\mathrm{im}\,\varphi)} \cong \overline{\mathrm{im}\,\varphi}$.

We now observe that the resulting subquotient object $(\mathrm{im}\,\varphi) \to \overline{\mathrm{im}\,\varphi} \leftarrow I_A$ is in Ω. Hence, there exist unique Σ-morphisms $U \to (\mathrm{im}\,\varphi)$ and $I_A/{\sim} \to I_A/{\sim'}$ such that

$$\begin{tikzcd}
{} & U \arrow[r] \arrow[d] & I_A/{\sim} \arrow[d] & {} \\
A \arrow[ur] \arrow[r] \arrow[dr, "\varphi"'] & (\mathrm{im}\,\varphi) \arrow[r] \arrow[d] & \overline{\mathrm{im}\,\varphi} & I_A \arrow[ul] \arrow[l] \\
{} & B & {} & {}
\end{tikzcd}$$

commutes. Now let $U \overset{\psi}{\to} B$ be another Σ-morphism with $\varphi = A \to U \overset{\psi}{\to} B$. We decompose $\psi = U \overset{\psi_1}{\to} (\mathrm{im}\,\psi) \overset{\psi_2}{\to} B$, and observe that $(\mathrm{im}\,\varphi) \subseteq (\mathrm{im}\,\psi)$. Again, the Σ-morphism from $\mathrm{im}\,\varphi$ into $\overline{\mathrm{im}\,\varphi}$ naturally extends to an injective Σ-morphism from $(\mathrm{im}\,\psi)$ into $\overline{\mathrm{im}\,\varphi}$. Thus, $(\mathrm{im}\,\psi)$ is a subquotient algebra of I_A in Ω, and we have a unique morphism from U into $(\mathrm{im}\,\psi)$ which makes all relevant triangles commutative. We deduce that $\psi_1 = U \to (\mathrm{im}\,\varphi) \to (\mathrm{im}\,\psi)$, whence $(\mathrm{im}\,\varphi) = (\mathrm{im}\,\psi)$. $\square$

Let G be a graph, where we allow multiple edges and let C be a category. Then the category C^G is defined in the following way: objects are pairs (o, m), where $o : V(G) \to \mathrm{obj}(\mathsf{C})$ is a labeling of the vertices of G with objects in C and $m : E(G) \to \mathrm{hom}(\mathsf{C}, \mathsf{C})$ is a labeling of the edges with morphisms in C, such that the obvious commutation rules hold. A morphism from (o, m) to (o', m') is a family of morphisms from $o(a)$ to $o'(a)$, where a runs over $V(G)$, such that the obvious commutation rules hold. The category C^g is called a **C-pointed category**. We have the so called **diagonal functor** from C into C^G, such that $\Delta(x) = (G, V(G) \mapsto x, E(G) \mapsto Id_x)$.

Theorem B.2. *Let G be a graph. Then there exists a left adjoint and a right adjoint to the diagonal functor (Σ, C)-$\mathsf{PAlg} \overset{\Delta}{\to} ((\Sigma, C)$-$\mathsf{PAlg})^G$.*

Proof. The existence proof of the left adjoint runs along the same lines as the proof of theorem B.1. This time, assuming that $O = (o, m) \in ((\Sigma, C)$-$\mathsf{PAlg})^G$ and denoting $X = \coprod_{v \in V(G)} o(v)$, we take Ω to be the set of subquotient (Σ, C)-objects $S \to I_X/{\sim} \leftarrow I_X$ of $I_X/{\sim}$, such that each $o(v) \to I_X/{\sim}$ factors through S and such that for each $vw \in E$ we have $o(v) \to U = o(v) \overset{m(vw)}{\longrightarrow} o(w) \to U$. We leave it as an exercise to the reader to carry out the remaining details.

Let us now treat the right adjoint. Suppose that $O = (o, m)$ is in $((\Sigma, C)\text{-}\mathsf{PAlg})^G$. Let $P = \prod_{v \in V(G)} o(v)$ and denote by π_v the natural projection from P onto $o(v)$, for each $v \in V(G)$. We give P the natural partial (Σ, C)-algebra structure, by setting $\pi_{o(v)}(f_P(x)) = f_{o(v)}(\pi_v(x))$, and $R_P(x)$ iff $\forall v \in V(G) \ \ R_{o(v)}(\pi_v(x))$. Then the projections π_v are Σ-morphisms. Let U be the set of points $x \in P$, such that $m(vw)(\pi_v(x)) = \pi_w(x)$, for all $vw \in E(G)$. For each $f \in \Sigma_f$ and $x \in \operatorname{dom} f \cap U^{N_f}$, we have $f(x) \in U$. From this, it is straightforward to verify that U is a partial (Σ, C)-subalgebra of P. We claim that $\Delta U \to O$ satisfies the universal property.

Let $\Delta B \to O$ be a morphism. Then there exists a unique mapping $B \xrightarrow{\varphi} P$ such that $\Delta B \to O = \Delta B \xrightarrow{\Delta\varphi} \Delta P \xrightarrow{(\varphi_v)_{v \in V(G)}} O$ as a family of mappings. This mapping factors uniquely $\varphi = B \to (\operatorname{im}\varphi) \to P$. We observe that $\operatorname{im}\varphi \subseteq U$, whence $(\operatorname{im}\varphi) \subseteq U$. Denoting the corresponding inclusion by ι, we deduce that

$$\begin{tikzcd}
 & \Delta B \arrow[d] \arrow[dr] & \\
 & \Delta(\operatorname{im}\varphi) \arrow[r] \arrow[dl, "\Delta\iota"'] \arrow[d] & O \\
\Delta U \arrow[r] & \Delta P \arrow[ur] &
\end{tikzcd}$$

commutes. We thus get a morphism $B \xrightarrow{\psi} U$ which is unique with the property that $\Delta B \to O = \Delta B \xrightarrow{\Delta\psi} \Delta U \to O$. Indeed, if ψ' is another such morphism, then $B \xrightarrow{\psi'} U \to P = B \xrightarrow{\psi} U \to P$, because of the uniqueness of φ and its factorization. Hence $\psi' = \psi$ since $U \to P$ is injective. $\square$

Many universal results about the category (Σ, C)-PAlg can be deduced immediately from the two above theorems. Taking the category of sets Set for (Σ', C')-Palg in theorem B.1, we prove the existence of the free partial (Σ, C)-algebra $\langle X \rangle$ on X for any set X. In particular, (Σ, C)-PAlg has an initial object, by taking $X = \emptyset$. (Σ, C)-PAlg also has the trivial full singleton Σ-algebra as a terminal object. Taking a suitable C, (Σ, C)-PAlg is the category of full Σ-algebras. Hence, we can reinterpret the Σ-closure for partial Σ-algebras as the left adjoint of the forgetfull functor from (Σ, C)-PAlg to $(\Sigma, \emptyset)$-PAlg.

From theorem B.2 it follows that direct sums, direct products, pushouts, pullbacks, direct and inverse limits exist in (Σ, C)-PAlg. Taking the direct sum $A\langle X \rangle = A \amalg \langle X \rangle$ of A and $\langle X \rangle$ gives the free extension of a partial (Σ, C)-algebra by a set X. It can also be proved (exercise; this does not follow from theorem B.2) that for each partial (Σ, C)-algebra B the functor $A \mapsto A \times B$ admits a left adjoint. Together with the existence of a terminal object and the existence of direct products, this implies that (Σ, C)-PAlg is **Cartesian closed**.

B.5 Varieties

B.5.1 Definitions and the right adjoint functor theorem

In algebraic geometry, it is a classical wish to interpret elements of partial (Σ, C)-algebras as functions on a variety. In general, partial (Σ, C)-algebras may contain functions without any direct pointwise geometric interpretation, although such an interpretation is sometimes possible. By definition, varieties are partial (Σ, C)-algebras F in which such a suitable pointwise interpretation of the elements of F is possible. In this section we will see that the notion of a point is in fact variable, but once we made it precise, an appropriate theory of varieties can be developed. As is customary in algebraic geometry, we will work over a fixed partial Σ-algebra A.

The category of (Σ, C)-**prevarieties** over A (or shortly prevarieties, if no confusion is possible) is defined to be $(\Sigma, C)\text{-}\mathsf{PVar}_A = \hom(A, (\Sigma, C)\text{-}\mathsf{PAlg})^o$, which we also denote by PVar_A, if $C = \phi$. That is, a prevariety is a Σ-morphism from A into a partial (Σ, C)-algebra. A morphism between two prevarieties $A \to F$ and $A \to F'$ is a Σ-morphism $F' \to F$ so that $A \to F = A \to F' \to F$. Given a variety $V = A \to F$, we say that $\mathcal{F}(V) \stackrel{\text{def}}{=} F$ is the **function space** of V. A morphism between two prevarieties $(A \to F) \to (A \to F')$ maps functions in F' to functions in F. Partial (Σ, C)-algebras $A \to F$ are often abusively denoted by F. To avoid confusion, we will denote $A \to F$ considered as a prevariety by $\mathcal{V}(F)$.

A natural question is how to define the points of a prevariety. Using a simplistic point of view, the point-prevariety is defined by $\star = A \stackrel{Id_A}{\to} A$. Then a **point** of a prevariety V is just a morphism from $\star$ to V. We denote the set of these points by $\mathcal{P}(V)$, and call it the **point space** of V. We observe that a morphism $V \stackrel{\varphi}{\to} W$ from one variety to another induces a mapping $\mathcal{P}(\varphi)$ from $\mathcal{P}(V)$ to $\mathcal{P}(W)$ and that this association is functorial. Pursuing this geometric point of view, we would like to be able to interpret functions on V as actual morphisms between prevarieties. This can be done by considering the line-prevariety $L = \mathcal{V}(A\langle x\rangle)$, whose points correspond precisely to the elements of A. We have a natural bijection between the elements of F and morphisms from V into L, by mapping $f \in F$ to the morphism $A\langle x\rangle \stackrel{\varphi(f)}{\to} F$, with $\varphi(f)(x) = f$.

More generally, the point-prevariety does not need to be the identity morphism Id_A. In fact, we may allow different types of points: let P be a subcategory of $(\Sigma, C)\text{-}\mathsf{PVar}_A$, whose objects are called **point types**. Given a variety V, a **point** of type $B \in \mathrm{obj}(\mathsf{P})$ in V is a morphism $B \to V$. Such points are also called B-**points** and the set of B-points is denoted by $\mathcal{P}_B(V)$. The point space of V is by definition the meta-set $\mathcal{P}(V) = \bigcup_{B \in \mathrm{obj}(\mathsf{P})} \mathcal{P}_B(V)$. The point space of the line-prevariety $\mathcal{V}(A\langle x\rangle)$ from above is isomorphic to the disjoint union of all B in $\mathrm{obj}(\mathsf{P})$.

Now consider the mapping $\chi : (V \stackrel{\varphi}{\to} L) \mapsto (\mathcal{P}(V) \stackrel{\mathcal{P}(\varphi)}{\to} \mathcal{P}(L))$. This mapping is not necessarily injective, as shows the example $V = \mathbb{Z} \stackrel{\pi}{\to} \mathbb{Z}/2\mathbb{Z}$ from the category of rings (with $\mathrm{obj}(\mathsf{P}) = \{\mathbb{Z} \overset{Id_{\mathbb{Z}}}{\to} \mathbb{Z})$. A prevariety for which χ is injective is said to be **reduced**. This is equivalent to the condition that

$$(\forall F \overset{\varphi}{\to} B \in \mathcal{P}(V)) \;\Rightarrow\; f = g \quad \varphi(f) = \varphi(g), \tag{B.1}$$

for all $f, g \in F$. Dually, χ generally is not surjective. Nevertheless, the relations and the domains of partially defined functions on F can often be extended in a natural way. If this is not the case, then V is said to be **full**. More precisely, this means that

$$(\forall F \overset{\varphi}{\to} B \in \mathcal{P}(V) \quad (\varphi \circ \nu)(t)\#) \;\Rightarrow\; \nu(t)\# \tag{B.2}$$

$$(\forall F \overset{\varphi}{\to} B \in \mathcal{P}(V) \quad (\varphi \circ \nu)(P)) \;\Rightarrow\; \nu(P). \tag{B.3}$$

for all ground terms t, ground properties P and assignments ν into F. A prevariety which is both reduced and full is said to be a **variety**[1] relative to P (or shortly a variety, if no confusion about P is possible).

The idea behind the definition of varieties is that the properties of the different types of points should be reflected in each point of a variety. As a consequence, if B models a certain Horn clause H, for each $B \in \mathrm{obj}(\mathsf{P})$, then any Σ-variety relative to P automatically models H too (see the proposition below). In particular, the category $\mathsf{Var}_{\mathsf{P}}$ of varieties relative to P is a subcategory of (Σ, C)-PVar_A.

Proposition B.4. *Let $V = A \to F$ be a variety. Then $\bigcup_{B \in \mathrm{obj}(\mathsf{P})} C_B$ is included in C_F.*

Proof. Let $((P_i)_{i \in I}, Q)$ be a Horn clause in $\bigcup_{B \in \mathrm{obj}(\mathsf{P})} C_B$. Let ν be an assignment into F, such that all $\nu(P_i)$ are valid. Let $F \overset{\varphi}{\to} B$ be a point of V. Then all $(\varphi \circ \nu)(P_i)$ are valid in B. Therefore, $(\varphi \circ \nu)(Q)$ is valid. Consequently, $\nu(Q)$ is valid. $\square$

Theorem B.3. *The forgetfull functor from $\mathsf{Var}_{\mathsf{P}}$ into PVar_A admits a right adjoint $\hat{}$. The points of any prevariety V and the* **variety $\hat{V}$ associated** *to V are in a natural one-to-one correspondence.*

Proof. Let $V = \mathcal{V}(F)$ be a prevariety, and let ι be the canonical inclusion from F into I_F. Let S be the subset of I_F given by

$$(\iota \circ \nu)(t) \in S \Leftrightarrow \forall F \overset{\varphi}{\to} B \in \mathcal{P}(V) \quad (\varphi \circ \nu)(t)\#, \tag{B.4}$$

for all ground terms t and assignments ν into F. Notice that $\iota(F) \subseteq S$. We define the relations $R \in \Sigma_R$ on S by

$$(\iota \circ \nu)(R(t)) \Leftrightarrow \forall F \overset{\varphi}{\to} B \in \mathcal{P}(V) \quad (\varphi \circ \nu)(R(t)), \tag{B.5}$$

[1] We notice that varieties are not necessarily irreducible, when using this definition. In algebraic geometry it is often assumed that varieties are irreducible.

for all families t of ground terms and assignments ν into F. Finally, let $\sim$ be the equivalence relation on S defined by

$$(\iota \circ \nu)(t) \sim (\iota \circ \nu)(t') \Leftrightarrow \forall F \xrightarrow{\varphi} B \in \mathcal{P}(V) \quad (\varphi \circ \nu)(t) = (\varphi \circ \nu)(t'), \tag{B.6}$$

for all ground terms t and t' and assignments ν into F. The equivalence relation is compatible with the partial Σ-algebra structure on S, whence $\hat{V} \stackrel{\text{def}}{=} \mathcal{V}(S/\sim)$ is a partial Σ-algebra. By construction, the points of V and $\hat{V}$ are in one-to-one correspondence and $\hat{V}$ is a variety. We claim that $\hat{V}$ satisfies the desired universal property.

Let $F \xrightarrow{\xi} G$ be a morphism of a variety $W = \mathcal{V}(G)$ into V. There is a natural maximally defined partial Σ-morphism $\tilde{\xi}$ from I_F into G. Now let t be a ground term and ν an assignment into F, such that $(\iota \circ \nu)(t) \in S$. Then $(\varphi \circ \nu)(t)$ is defined for all points $\varphi \in \mathcal{P}(V)$. In particular, $(\varphi \circ \xi \circ \nu)(t)$ is defined for points $\varphi \in \mathcal{P}(W)$. Hence $(\xi \circ \nu)(t)$ is defined in G, since W is a variety, so that $\tilde{\xi}$ is totally defined on S. Similarly, it can be shown that $\tilde{\xi}$ preserves the relations $R \in \Sigma_R$, whence is a Σ-morphism. Finally, again using the same type of arguments, $\tilde{\xi}$ maps equivalent elements in S to the same elements in G. In other words, $\tilde{\xi}$ factors uniquely through $S/\sim$: $\tilde{\xi} = \hat{\xi} \circ \pi$. Then $\hat{\xi}$ is the desired Σ-morphism. □

Proposition B.5. *Let V be a variety, let P be a ground property and let ν be an assignment into $\mathcal{F}(V)$. Then there exists a natural subvariety of V, whose point space is in one-to-one correspondence with those points $\varphi : \mathcal{F}(V) \to B$ in V, such that $(\varphi \circ \nu)(P)$ is valid.*

Proof. Consider the smallest partial Σ-subalgebra G of $\overline{\mathcal{F}(V)}$, which contains $\mathcal{F}(V)$ and in which $\nu(P)$ is valid. The point space of the prevariety $\mathcal{V}(G)$ is in a natural one-to-one correspondence with those points $\varphi : \mathcal{F}(V) \to B$ in V, such that $(\varphi \circ \nu)(P)$ is valid. We now apply theorem B.3 to this prevariety. The terminology "subvariety" is justified below. □

B.5.2 Elementary properties

Let us now investigate the properties of the categories of prevarieties over A and varieties relative to P. First, for each set X we have the free prevariety $\mathcal{V}(A\langle X\rangle)$ over A in X. The variety $\mathbb{A}^X_{\mathsf{P}}$ relative to P associated to this prevariety is called the **affine X-space relative to** P Next, we observe that theorem B.2 still holds, if we replace (Σ, C)-PAlg by PVar_A. Indeed the right adjoint version is obtained by joining a new vertex to the graph with edges to all other vertices and applying the theorem for left adjoints. What concerns the left adjoint: suppose that $O = (o, m)$ is in $(\mathsf{PVar}_A)^G$. We can interpret O as a morphism from ΔA to $O' = (o', m')$ for some object O' in $((\Sigma, C)$-$\mathsf{PAlg})^G$. Applying the right adjoint to the diagonal functor to

O', we get partial (Σ, C)-algebra U and a morphism from A to U, by the universal property. Composing with the functor ^ we deduce that the analogue of theorem B.2 also holds for varieties.

From what precedes it follows that direct sums and products, pushouts and pullbacks, direct limits and inverse limits, etc. exist in $\mathsf{Var}_{\mathsf{P}}$. In the case of direct products, pullbacks, inverse limits, etc. the corresponding right adjoint functor commutes with the functor which associates to each variety its point space. This is easily proved by abstract nonsense. For instance, $\mathcal{P}(V \times W)$ is naturally isomorphic to $\mathcal{P}(V) \times \mathcal{P}(W)$. A similar statement does not hold for direct sums, pushouts, etc. Nevertheless, in the case of entire rings, $\mathcal{P}(V \amalg W)$ is naturally isomorphic to $\mathcal{P}(V) \amalg \mathcal{P}(W)$ (i.e. Σ contains the ring operations and $B \in \mathrm{obj}(\mathsf{P})$ is an entire ring). Indeed, if φ is a ring homomorphism from $R \times S$ into an entire ring B, then $\varphi(1,0)\varphi(0,1) = 0$. Hence φ is either of the form $\varphi(x,y) = \psi(x)$ or $\varphi(x,y) = \psi(y)$, where ψ is a ring homomorphism from R resp. S into B.

From now on, it will be convenient to identify varieties V with their point spaces $\mathcal{P}(V)$, Let X be any meta-subset of points in V. Replacing $\mathcal{P}(V)$ by X in (B.4), (B.5) and (B.6), we construct a variety $V_{|X} = \mathcal{V}(F_{|X})$ instead of X. Each point $F_{|X} \to A$ in $V_{|X}$ determines a unique point $F \to F_{|X} \to A$ in V, and we will identify both from now on. We say that $V_{|X}$ is the smallest **subvariety** of V containing X. In particular, a subvariety of V is a meta-subset X of V, such that $V_{|X} = X$. The meta-set of subvarieties of V is stable under arbitrary intersections. This can be shown by considering the pullback of the inclusion morphisms associated to a family $(U_i)_{i \in I}$ of subvarieties of V, and observing that this pullback satisfies the same universal property as the smallest subvariety which contains the intersection of the U_i.

It should be remarked that a variety often has a lot of subvarieties. For instance, assume that A is a field and that $\mathrm{obj}(\mathsf{P}) = \{Id_A\}$. Then $\mathbb{A}_A \backslash \{a\}$ is a subvariety of the affine line $\mathbb{A}_A$, for any $a \in \mathbb{A}_{\mathsf{P}} \cong A$, whose function space is isomorphic to $A[x, 1/(x-a)]$. Hence, all subsets of $\mathbb{A}_{\mathsf{P}}$ are subvarieties. A subvariety W of V is said to be **Zarisky closed**, if $\mathcal{F}(W) = \widehat{\mathcal{F}(V)/\sim}$ for some quotient relation $\sim$ on $\mathcal{F}(V)$. By abstract nonsense, the intersection of an arbitrary family of closed subvarieties is closed. Hence, the Zarisky closed subvarieties determine a closure operator, the **Zarisky closure**. Usually, the Zarisky closed subvarieties of V do not determine a topology. Nevertheless, in the case of entire rings, the Zarisky closed subsets do determine a topology, because the zero-set of fg is then given by the union of the zero-sets of f and g. This topology generalizes the usual **Zarisky topology**.

B.5.3 Partial functions on a variety

A **partial function** on a variety V is an element of the function space $\mathcal{O}_V(U) = \mathcal{F}(U)$ of a subvariety U of V. In fact, $\mathcal{O}_V$ is a functor which associates $\mathcal{O}_V(U)$ to each subvariety U of V, and a Σ-morphism $\mathcal{O}_V(U_2) \stackrel{\rho_{U_2 U_1}}{\to} \mathcal{O}_V(U_1)$ to each inclusion morphism $U_1 \to U_2$. This functor determines a presheaf: the **presheaf of partial functions** on V. We recall that a presheaf is defined by the following conditions:

P1. $\mathcal{O}_V(\phi)$ is the trivial full Σ-structure $\{0\}$.
P2. We have $\rho_{UU} = Id_{\mathcal{O}_V(U)}$ for each U.
P3. We have $\rho_{U_3U_1} = \rho_{U_3U_2} \circ \rho_{U_2U_1}$, for $U_1 \subseteq U_2 \subseteq U3$.

Let us finally investigate what are the analogues of stalks and function fields in our context. Restricting the presheaf $\mathcal{O}_V$ to the open subvarieties of V for some topology or closure operator, we can define the stalk of $\mathcal{O}_V$ at $P \in V$ to be the direct limit of the $\mathcal{O}_V(U)$ with open $U \ni P$ via the restriction morphisms. Here the concept of functions defined in a neighbourhood of P is not always the best reflected by taking the Zarisky closure. For instance in the case of the line transvariety (see chapter 6), any series in x should clearly be a function in the neighbourhood of $x = 0$. This will not be the case if we take the Zarisky topology.

Similarly, if V is the variety associated to an integral domain, then its function field is the direct limit of all $\mathcal{O}_V(U)$ with non empty Zarisky open U, via the restriction morphisms. In our context, the Zarisky topology is again replaced by any other topology on V. We remark that for this definition the function "field" does not necessarily extend $\mathcal{F}(V)$: in the classical case $\mathcal{F}(V)$ needs to be an integral domain for this. Nevertheless, the direct limit along all monomorphisms $\mathcal{O}_V(U_1) \to \mathcal{O}_V(U_2)$, for open subvarieties U_1 and U_2 of V always extends $\mathcal{F}(V)$. In the commutative algebra setting, this corresponds to inverting all non zero divisors.

B.6 Complements

B.6.1 P-structures

The concept of varieties as introduced in the preceding section has sometimes the disadvantage that the analogues of nilpotent elements in the category of rings do not exist. An obvious solution would be to work in the category of partial (Σ, C)-algebras itself. However, this is not possible if the properties of the objects we want to manipulate can not adequately — or easily — be modelized by Horn clauses. This happens for example in the case of transseries. The solution to this dilemma is to generalize a trick, which was invented by Lawvere (see [MoRe 91]) in order to define C^∞-rings.

Let us explain this trick. Let B be a ring. Then every map $\mathbb{Z}^n \xrightarrow{p} \mathbb{Z}^m$ given by an m-tuple of polynomials, naturally induces a map $B^n \xrightarrow{\Phi(p)} B^m$, in such a way that projections, composition and identity are preserved by Φ. Actually, this can be taken as an equivalent definition for rings. Now a **C^∞-ring** is defined analogously, except that we now ask that each smooth map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ induces a map $B^n \xrightarrow{\Phi(f)} B^m$ (i.e. can be **interpreted** in B). We remark that any smooth map $\mathbb{R}^n \to \mathbb{R}^m$ can be seen as an m-tuple of smooth maps $\mathbb{R}^n \to \mathbb{R}$. Hence, it suffices to have interpretations for these latter maps.

In our case, we are given a partial Σ-structure A and a category P of point types. In order to apply Lawvere's trick, we have to consider those partial Σ-algebras F, in which we can interpret all partially defined functions and relations on all affine spaces $\mathbb{A}_{\mathsf{P}}^N$. Let us reformulate this by introducing a signature $\Sigma(\mathsf{P})$ and axioms $C(\mathsf{P})$, taking into account the discussion at the end of section B.2. The function symbols of arity N of $\Sigma(\mathsf{P})$ are partially defined functions on $\mathbb{A}_{\mathsf{P}}^N$ and the relation symbols of arity N are the subvarieties of $\mathbb{A}_{\mathsf{P}}^N$. Now consider Horn clauses of the form $R(x) \Rightarrow P(x)$, where R is a relation symbol, $(x_i)_{i\in N_R}$ a family of variable symbols, and $P(x)$ a ground property in the x_i. Such a Horn clause is in $C(\mathsf{P})$, if $P(x)$ is well defined and valid for all points x in R (considered as a subvariety of $\mathbb{A}_{\mathsf{P}}^{N_R}$).

We define a **P-structure** to be a partial $(\Sigma(\mathsf{P}), C(\mathsf{P}))$-algebra. Let us now assume that each $B \in \mathrm{obj}(\mathsf{P})$ is a partial (Σ, C)-algebra and let F be an P-structure. Then each function symbol $f \in \Sigma_f$ determines a function in $\mathbb{A}_{\mathsf{P}}^{N_f}$ and hence a function on F. Similarly, the relation symbols determine relations on F. Consequently, we can see each partial $(\Sigma(\mathsf{P}), C(\mathsf{P}))$-algebra as a partial Σ-algebra. Now all axioms in C which are equations are also satisfied by F. However, this is not the case for general Horn clauses: for instance, if we take $A = \mathbb{Z}$ and $\mathrm{obj}(\mathsf{P}) = \{Id_{\mathbb{Z}}\}$, then we obtain nothing but the category of rings. Now $\mathbb{Z}$ is a ring without nilpotent elements (which can be modelized using Horn clauses), while every ring is not.

Example B.1. Using the notion of partial $(\Sigma(\mathsf{P}), C(\mathsf{P}))$-algebras, we can define a **transring** to be a P-structure, where P is one of the point type categories given in chapter 6. Transrings can have nilpotent elements, as in the case of $\mathbb{R}\langle x|x^2 = 0\rangle$. In fact, $\mathbb{R}\langle x|x^2 = 0\rangle \cong \mathbb{R}^2$, where x corresponds to $(0, 1)$. We have $\exp(a + bx) = (\exp a)(1 + bx)$ and $\log(a + bx) = \log a + bx/a$. Finally, $\sum_{i\in I} a_i + b_i x$ is defined if and only if $\sum_{i\in I} a_i + (\sum_{i\in I} b_i)x$ is, in which case they are equal.

B.6.2 Generalized varieties

Another disadvantage of the theory of varieties from the previous section is that we do not have the analogues of projective varieties (and more generally of schemes). In fact, following [MoRe 91], the concept of a point can still be extended. In the previous section, we had for each $B \in \mathrm{obj}(\mathsf{P})$ the set $\mathcal{P}_B(V) = \hom(\mathcal{F}(V), B)$ of B-points for a given variety V. The association $B \mapsto \mathcal{P}_B(V)$ is functorial, and this is precisely the property we retain in order to generalize. More precisely, we define a **generalized variety** to be a contravariant functor from P into Set. The category of generalized varieties is denoted by $(\mathsf{Set}^{\mathsf{P}})^o$, morphisms of generalized varieties being natural transformations. The category of varieties relative to CP can be embedded into $(\mathsf{Set}^{\mathsf{P}})^o$ by the Yoneda embedding Y:

$$Y(B) : \mathsf{P} \to \mathsf{Set};\ B \mapsto \hom(F, B).$$

In fact, the category $\mathsf{Set}^{\mathsf{P}}$ can be seen as an enriched set-theoretic universe, in which we can perform algebraic as well as set-theoretic constructions, using a non standard language. We will not go any further into this, but we remark that properties of a variety V in the usual sense may no longer hold if we consider V as a generalized variety, using non-standard logic. However, the preservation of properties can often be forced by considering certain subcategories of $(\mathsf{Set}^{\mathsf{P}})^o$. We refer to [MoRe 91] for such a study in the case of C^∞-rings.

Example B.2. Take $A = \mathbb{Z}$ and let $\mathrm{obj}(\mathsf{P})$ be the category of rings. Then the plane is the generalized set $Y(\mathbb{Z}[x, y])$. The circle can be considered as the generalized subset of the plane which associates to each B the set of those ring homomorphisms $\mathbb{Z}[x, y] \xrightarrow{\varphi} B$, such that $\varphi(x)^2 + \varphi(y)^2 = 1$. The projective line associates to each ring B the set of those ring homomorphisms $\mathbb{Z}[x, y] \xrightarrow{\varphi} B$, quotiented by the equivalence relation $\sim$ defined by

$$\varphi \sim \psi \Leftrightarrow \exists u \in B^* \quad (\varphi(x), \varphi(y)) = u(\psi(x), \psi(y)).$$

Here B^* denotes the set of invertible elements of B.

B.7 References

[AtMac 69] M.F. Atiyah, I.G. Mac Donald. *Introduction to commutative algebra.* Addison-Wesley.

[Cohn 65] P.M. Cohn. *Universal algebra.* M.I.A. 6, Reidel, Dordrecht.

[Dav 77] M. Davis. *Applied nonstandard analysis.* J. Wiley, New York.

[MoRe 91] I. Moerdijk, G.E. Reyes. *Models for smooth infinitesimal analysis.* Springer-Verlag, New-York.

[ML 71] S. Mac Lane. *Categories for the working mathematician.* Springer-Verlag, New-York.

[Har 77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New-York.

[Lang 84] S. Lang. *Algebra.* Addison-Wesley.

Appendix C

Fast evaluation of holonomic functions

C.1 Introduction

A holonomic function is an analytic function f, which satisfies a linear differential equation over $\mathbb{C}[z]$. In other words,

$$P_p(z)f^{(p)}(z) + \cdots + P_0(z)f(z) = 0, \tag{C.1}$$

for some polynomials $P_0, \cdots, P_p \in \mathbb{C}[z]$, with $P_p \neq 0$. Many classical special functions like the exponential, the logarithm, trigonometric functions, Bessel functions, hypergeometric functions, etc. are holonomic. It is a well known fact, that a function f which satisfies (C.1) can be analytically continued in all points where P_p does not vanish. Hence, denoting the zeros of P_p by $w_1, \cdots, w_\nu$, f is defined on a Riemann surface $\mathcal{X}$ over $U = \mathbb{C}\backslash\{w_1, \cdots, w_\nu\}$. We denote by π the natural projection $\mathcal{X} \to U$.

For each point ξ on $\mathcal{X}$, we denote by $F(\xi)$ the vector

$$F(\xi) = \begin{pmatrix} f(\xi) \\ \vdots \\ f^{(p-1)}(\xi) \end{pmatrix}.$$

If $\xi \rightsquigarrow \xi'$ is a path[1] on $\mathcal{X}$, from ξ to another point $\xi' \in \mathcal{X}$, then $f(\xi')$ is uniquely determined by $F(\xi)$ and $\xi \rightsquigarrow \xi'$, by integrating (C.1). The aim of this article is to compute $f(\xi')$ as a function of $F(\xi)$ and $\xi \rightsquigarrow \xi'$. The effective point of view requires to make two additional assumptions on f: first, $P_0, \cdots, P_p$ should be polynomials over some algebraic number field K. Secondly, $F(\xi)$ and $\xi \rightsquigarrow \xi'$ need to be effective, in a sense that will be made precise in section C.2.

[1]In all what follows, we assume that paths are determined up to homotopy; i.e. a path on $\mathcal{X}$ is really the homotopy class of a continuous mapping $[0, 1] \to \mathcal{X}$.

In section C.3, we treat the simplest case when $\xi \rightsquigarrow \xi'$ is a straight line path between two sufficiently close points ξ and ξ' above $\mathbb{Z}[\frac{1}{2}, i]$ on the Riemann surface $\mathcal{X}$. In section C.4 we extend this result to compute so called truncated transition matrices between ξ and ξ'. In section C.5, such matrices are used to perform analytic continuations and to tackle the general case. We will show that the asymptotic cost of the computation of the first n digits of $f(\xi')$ is $O(n \log^3 n \operatorname{llog}^2 n + T(n + O(1)))$, if the first n digits of $\pi(\xi), \pi(\xi')$ and $F(\xi)$ can be computed in time $O(T(n))$.

Our algorithms involve bounds for f and its iterated derivatives on certain compact subsets of $\mathcal{X}$. Although the existence of such bounds is guaranteed theoretically, we show how to compute them in section C.6.

C.2 Preliminaries

In this section, we establish an effective framework for dealing with complex numbers and points on Riemann surfaces over open subsets of $\mathbb{C}$. Throughout this article, we will use the notation $\mathtt{size}(O)$ for the natural size of an object O. For instance, the size of an integer is its length in bits, the size of a matrix is the sum of the sizes of its entries, etc. We will also denote $\operatorname{llog} x = \log \log x$.

A complex number $z \in \mathbb{C}$ is said to be **effective**, if there exists an **approximation algorithm** for z, which given $n \in \mathbb{N}$ computes a $\tilde{z} \in \mathbb{Z}[\frac{1}{2}, i]$, such that $|\tilde{z} - z| < 2^{-n}$. The set of these numbers is denoted by $\mathbb{C}^{\mathit{eff}}$, and numbers in $\mathbb{C}^{\mathit{eff}}$ are redundantly represented by approximation algorithms for them. A number $z \in \mathbb{C}^{\mathit{eff}}$ is said to have **time complexity** $O(T(n))$ if there exists an $O(T(n))$ approximation algorithm it; i.e. the computation of an approximation $\tilde{z}$ of z with $|\tilde{z} - z| < 2^{-n}$ is performed in time $O(T(n))$. The following theorem is classical (see [SS 71], [Kn 81]):

Theorem C.1. *Let $z_1, z_2 \in \mathbb{C}^{\mathit{eff}}$ have time complexities $O(T(n))$. Then $z_1 + z_2, z_1 - z_2, z_1 z_2$ and z_1 / z_2 (provided $z_2 \neq 0$) admit approximation algorithms with time complexities $O(T(n)), O(T(n)), O(T(n) + n \log n \operatorname{llog} n)$ and $O(T(n) + n \log n \operatorname{llog} n)$ respectively.* □

Using Newton's method for the approximation of roots, the following theorem is also classical:

Theorem C.2. *Let $z \in \hat{\mathbb{Q}}$ be an algebraic number. Then z has an $O(n \log n \operatorname{llog} n)$ approximation algorithm.* □

Let $\mathcal{X}$ be a connected Riemann surface over an open subset U of $\mathbb{C}$ and let $\pi : \mathcal{X} \to U$ denote the natural projection. For each $z \in U$, let $d(z, \partial U)$ denote the distance between z and the frontier ∂U of U. For $\zeta_0 \in \mathcal{X}$ and $|u| < \delta(\zeta_0)$, there is a unique point $\zeta_1 = \zeta_0 + u \in \mathcal{X}$ at distance $|u|$ of ζ_0, with $\pi(\zeta_1) = \pi(\zeta_0) + u$. Such points ζ_0, ζ_1 are said to be **close**, if $|\zeta_1 - \zeta_0| < \frac{1}{2}\max\{d(\pi(\zeta_0), \partial U), d(\pi(\zeta_1, \partial U))\}$.

Paths $\xi \rightsquigarrow \xi'$ on $\mathcal{X}$ can always be written as compositions

$$\xi \rightsquigarrow \xi' = (\xi = \zeta_0 \to \zeta_1 \to \cdots \to \zeta_\kappa = \xi')$$

of straight line paths $\zeta_0 \to \zeta_1, \cdots, \zeta_{\kappa-1} \to \zeta_\kappa$, where ζ_{i+1} and ζ_i are close for each $0 \leqslant i < \kappa$. If $\pi(\xi)$ and $\pi(\xi')$ are effective, then we say that the path $\xi \rightsquigarrow \xi'$ is **effective**. Such paths are represented as above, where we require $\pi(\zeta_1), \cdots, \pi(\zeta_{\kappa-1})$ to be in $\mathbb{Z}[\frac{1}{2}, i]$. Points ξ' on $\mathcal{X}$ are represented by paths $\xi \rightsquigarrow \xi'$ from a fixed point ξ to ξ'. If such a path is effective, then so is the point ξ'.

We recall that an algebraic number field K is a field of characteristic zero of finite dimension over the rationals $\mathbb{Q}$. During intermediate computations, we will frequently approximate complex numbers by elements of a fixed algebraic number field $K \ni i$. For most practical applications, we may assume that $K = \mathbb{Q}[i]$.

Let us now detail how arithmetic in K is performed. We assume that K is given through a subring Z of K, which is a free $\mathbb{Z}$-module with $K = \mathbb{Q}Z$. Then we have a basis for K resp. Z, such that each $b_i b_j$ is a $\mathbb{Z}$-linear combination of the b_k. We represent elements in Z by d-tuples of integers: $(a_1, \cdots, a_d)$ represents $a_1 b_1 + \cdots + a_d b_d$. Then FFT-multiplication naturally extends to Z and has the usual asymptotic complexity $O(n \log n \operatorname{llog} n)$. Elements of K are represented as elements of Z divided by a strictly positive integer, where common factors in the numerators and the denominator are allowed. Then the naive addition, and multiplication algorithms in K, based on those in Z, have complexities $O(n \log n \operatorname{llog} n)$.

A **truncation** of an element $z \in K$ at precision $0 < \varepsilon \in \mathbb{Z}[\frac{1}{2}]$ is by definition an approximation $\tilde{z} = (a + bi)/2^m \in \mathbb{Z}[\frac{1}{2}, i]$ of z, with $|\tilde{z} - z| < \varepsilon$ and $2^{2-m} > \varepsilon$. Hence, the asymptotic size of a truncation at precision $\varepsilon = 2^{-n}$ is $O(n)$. By theorem C.2, a truncation of $z \in K$ at precision $\varepsilon = 2^{-n}$ can be computed in time $O(s \log s \operatorname{llog} s)$, where $s = \max\{\mathtt{size}(z), n\}$.

Remark C.1. For some applications, we may restrict ourselves to effective real numbers instead of effective complex numbers. In this case, numbers are approximated by elements in $\mathbb{Z}[\frac{1}{2}]$, and we do not need require $i \in K$. The theory of this article is easily adapted to this case, but no analytic continuation around singularities is possible.

C.3 Evaluations near a non singular point above $\mathbb{Z}[\frac{1}{2}, i]$

Let $\mathcal{D}$ be a compact disk on $\mathcal{X}$ with center above $\mathbb{Z}[\frac{1}{2}, i]$ and radius in $\mathbb{Z}[\frac{1}{2}]$. Let ζ_0 and ζ_1 be points in $\mathcal{D}$, such that the compact disk with center ζ_0 and radius $2|\zeta_1 - \zeta_0|$

is contained in the interior of $\mathcal{D}$. We denote by $\delta(\zeta_0)$ the distance between ζ_0 and ∂B. Let f be the function, which satisfies (C.1) and such that the entries of $F(\zeta_0)$ are effective, of time complexities $O(T(n))$. Let $B_0 \in \mathbb{Z}[\frac{1}{2}]$ be a bound for $|f|$ on $\mathcal{D}$. In what follows, we will denote by $|M|$, resp. $|M|_1$ the L^∞ resp. L^1-norm of a matrix (or vector) M.

Let $f(\zeta_0 + u) = f_0 + f_1 u + f_2 u^2 + \cdots$ be the power series expansion of f at ζ_0. Using the rewriting rules

$$\begin{cases} [u^k]ug(u) = [u^{k-1}]g(u); \\ [u^k]g'(u) = (k+1)[u^{k+1}]g(u) \end{cases}$$

for the extraction of k-th coefficients in power series, the relation (C.1) transforms into a linear difference relation for $f_0, f_1, \cdots$ over $K[k]$. Substitution of f_k by $(f_k(\zeta_1 - \zeta_0)^k)(\zeta_1 - \zeta_0)^{-k}$ and division by a suitable power of k yields a linear difference relation

$$f_{k+q}(\zeta_1 - \zeta_0)^{k+q} = Q_{k+q-1}(k) f_{k+q-1}(\zeta_1 - \zeta_0)^{k+q-1} + \cdots + Q_0(k) f_k(\zeta_1 - \zeta_0)^k, \quad \text{(C.2)}$$

with $Q_0, \cdots, Q_{q-1} \in K[1/k]$. We notice that the polynomials $Q_0, \cdots, Q_{q-1}$ depend on ζ_0 and ζ_1. Furthermore, $q \neq p$, in general, but q does not depend on ζ_0 nor ζ_1.

Example C.1. Let us perform the above rewritings, if we take a simple Bessel differential equation

$$f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{1}{z^2}\right) f(z) = 0 \quad \text{(C.3)}$$

for (C.1) and $\zeta_0 = 3, \zeta_1 = 5$. Writing $z = 3 + u$, (C.3) becomes

$$(u^2 + 6u + 9) f''(3 + u) + (u + 3) f'(3 + u) + (u^2 + 6u + 8) f(3 + u) = 0.$$

Using the rewriting rules, this leads to the following recurrence relation for the f_k:

$$(9k^2 + 63k + 108) f_{k+4} + (6k^2 + 31k + 39) f_{k+3} + (k^2 + 8k + 14) f_{k+2} + 6 f_{k+1} + f_k = 0.$$

Substitution of f_k by $2^{-k}(f_k 2^k)$ in this equation and division by k^2 finally yields

$$\begin{array}{l} \frac{1}{16}(9 + 63\frac{1}{k} + 108\frac{1}{k^2})(f_{k+4}2^{k+4}) + \frac{1}{8}(6 + 31\frac{1}{k} + 39\frac{1}{k^2})(f_{k+3}2^{k+3}) + \\ \frac{1}{4}(1 + 8\frac{1}{k} + 14\frac{1}{k^2})(f_{k+2}2^{k+2}) + 3(f_{k+1}2^{k+1}) + (f_k 2^k) = 0. \end{array}$$

This is the desired equation (C.2).

Denote by Φ_k the column vectors formed by $f_k(\zeta_1 - \zeta_0)^k, \cdots, f_{k+q}(\zeta_1 - \zeta_0)^{k+q}$. Repeated differentiation of (C.1) and substitution of z by $\pi(\zeta_0)$ yields expressions for any finite number of coefficients f_k as linear combinations of $f(\zeta_0), \cdots, f^{(p-1)}(\zeta_0)$. In particular, we have a matrix $A(\zeta_0)$ with $\Phi_0 = A(\zeta_0) F(\zeta_0)$.

Let $\sigma_{k;\alpha} = f_k(\zeta_1 - \zeta_0)^k + \cdots + f_{k+\alpha-1}(\zeta_1 - \zeta_0)^{k+\alpha-1}$ for all $\alpha \in \mathbb{N}$ and let $\Sigma_{k;\alpha}$ denote the column vectors with entries $\sigma_{k;\alpha}, \cdots, \sigma_{k+q;\alpha}$. We claim that if α is a power of two, then there exist matrix equations with coefficients in K of the following form:

$$\begin{array}{rcl} \Sigma_{k+\alpha;\alpha} & = & M_{k;\alpha}\Phi_k; \\ \Phi_{k+\alpha} & = & N_{k;\alpha}\Phi_k. \end{array} \tag{C.4}$$

If $\alpha = 1$, this follows directly from (C.2). Now assume that we have proved the claim for a certain α. Then the claim also holds for 2α, by taking

$$\begin{array}{rcl} M_{k;2\alpha} & = & M_{k;\alpha} + M_{k+\alpha;\alpha}N_{k;\alpha}; \\ N_{k;2\alpha} & = & N_{k+\alpha;\alpha}N_{k;\alpha}. \end{array} \tag{C.5}$$

From an effective point of view, the matrices $M_{k;\alpha}$ and $N_{k;\alpha}$ are represented as matrices $M'_{k;\alpha}$ and $N'_{k;\alpha}$ with entries in Z, divided by a common denominator $q_{k;\alpha}$. In this representation, (C.5) becomes:

$$\begin{array}{rcl} q_{k;2\alpha} & = & q_{k+\alpha;\alpha}q_{k;\alpha}; \\ M'_{k;2\alpha} & = & q_{k+\alpha;\alpha}M'_{k;\alpha} + M'_{k+\alpha;\alpha}N'_{k;\alpha}; \\ N'_{k;2\alpha} & = & N'_{k+\alpha;\alpha}N'_{k;\alpha}. \end{array} \tag{C.6}$$

We now have the following approximation algorithm for $f(\zeta)$:

Algorithm `approximate1` (path length 1, begin and end points in $\mathbb{Z}[\frac{1}{2}, i]$).
INPUT: A precision $0 < \varepsilon \in \mathbb{Z}[\frac{1}{2}]$.
OUTPUT: An approximation of $f(\zeta_1)$ with error $< \varepsilon$.

STEP 1. Compute the difference equation (C.2) from (C.1) as a function of ζ_0 and ζ_1.

STEP 2. Compute the smallest $\beta = 2^l$ for which

$$2B_0 \left(\frac{|\zeta_1 - \zeta_0|}{\delta(\zeta_0)}\right)^\beta \leqslant \frac{\varepsilon}{2}.$$

STEP 3. For $\alpha = 1, 2, 4, \cdots, \beta$ compute $M_{0;\alpha}, \cdots, M_{\beta-\alpha;\alpha}$ and $N_{0;\alpha}, \cdots, N_{\beta-\alpha;\alpha}$, using (C.2) and (C.6).

STEP 4. Compute $A(\zeta_0)$ by repeated differentiation of (C.1), and the first line $L_{;\beta}$ of the product $M_{0;\beta}A(\zeta_0)$.

STEP 5. Compute an approximation $\tilde{F}(\zeta_0)$ of $F(\zeta_0)$, with $r|L_{;\beta}|_1|\tilde{F}(\zeta_0) - F(\zeta_0)| < \varepsilon/2$. Return $L_{;\beta}\tilde{F}(\zeta_0)$.

Proposition C.1. *The algorithm* `approximate1` *is correct and has asymptotic complexity $O(n \log^3 n \operatorname{llog} n + T(n + O(1)))$, for $\varepsilon = 2^{-n}$.*

Proof. Let us prove the correctness of `approximate1`. By Cauchy's formula:

$$|f_k| = \left|\frac{1}{2\pi i}\int_{|u|=\delta(\zeta_0)} \frac{f(\zeta_0 + u)}{u^{k+1}}du\right| \leqslant \frac{B_0}{\delta(\zeta_0)^k},$$

for all k. In particular,

$$|f_\beta(\zeta_1-\zeta_0)^\beta + f_{\beta+1}(\zeta_1-\zeta_0)^{\beta+1}+\cdots| \leqslant 2B_0\left(\frac{|\zeta_1-\zeta_0|}{\delta(\zeta_0)}\right)^\beta < \frac{\varepsilon}{2}.$$

Hence,

$$|L_{;\beta}\tilde{F}(\zeta_0) - f(\zeta_1)| \leqslant |L_{;\beta}\tilde{F}(\zeta_0) - L_{;\beta}F(\zeta_0)| + |L_{;\beta}F(\zeta_0) - f(\zeta_1)| < \varepsilon.$$

Let us now estimate the complexity of `approximate1`: step 1 is a precomputation of cost $O(1)$. In step 2, we have $\beta = O(n)$.

In step 3, we have $\mathtt{size}(M_{k;1}) + \mathtt{size}(N_{k;1}) = O(\log k)$, since the Q_i are polynomials in $K[1/k]$. By induction, it follows that $\mathtt{size}(M_{k;\alpha}) + \mathtt{size}(N_{k;\alpha}) = O(\alpha(\log k))$. Hence, the computations of $M_{0;\alpha}, \cdots, M_{\beta-\alpha;\alpha}$ require β/α operations of costs $O(\alpha \log^2 n \operatorname{llog} n)$ for fixed α. Thus, the computation of $M_{0;\beta}$ requires $O(n \log^3 n \operatorname{llog} n)$ operations.

In step 4, the computation of $A(\zeta_0)$ is actually another precomputation of cost $O(1)$. The matrix multiplication $M_{0;\beta}A(\zeta_0)$ takes a time $O(n\log^2 n \operatorname{llog} n)$.

In step 5, $L_{;\beta}$ tends to the first line in the transition matrix between ζ_0 and ζ_1 (see the next section) for $\beta \to \infty$. Hence, $\lim_{\beta\to\infty} |L_{;\beta}|$ is finite and $\tilde{F}(\zeta_0)$ is computed in time $O(T(n+O(1)))$. Finally, the multiplication $L_{;\beta}\tilde{F}(\zeta_0)$ takes a time $O(n \log^2 n \operatorname{llog} n)$. □

C.4 Transition matrices

In this section we introduce the main tool for performing analytic continuations: transition matrices and their truncations. Let $\zeta_0 \in \mathcal{X}$ and let I be a column vector with p entries. Let $f[\zeta_0 : I]$ denote the unique function f, which satisfies (C.1) and such that $F(\zeta_0) = I$. Denote $f[\zeta_0 : I]^{(i)} = f^{(i)}[\zeta_0 : I]$ for each i. We notice that $f^{(i)}[\zeta_0 : I + J] = f^{(i)}[\zeta_0 : I] + f^{(i)}[\zeta_0 : J]$ for all i, I and J, by linearity. If $\zeta_0 \rightsquigarrow \zeta_1$ is a path on $\mathcal{X}$ from ζ_0 to ζ_1, then $F(\zeta_1)$ depends linearly on I, by integrating (C.1). The matrix $\Delta_{\zeta_0 \rightsquigarrow \zeta_1}$ such that

$$F(\zeta_1) = \Delta_{\zeta_0\rightsquigarrow\zeta_1} F(\zeta_0)$$

for all $F(\zeta_0)$ is called the **transition matrix** associated to $\zeta_0 \rightsquigarrow \zeta_1$ (and of course to the equation (C.1)). Obviously,

$$\Delta_{\zeta_0\rightsquigarrow\zeta_1\rightsquigarrow\zeta_2} = \Delta_{\zeta_1\rightsquigarrow\zeta_2}\Delta_{\zeta_0\rightsquigarrow\zeta_1},$$

for all compositions $\zeta_0 \rightsquigarrow \zeta_1 \rightsquigarrow \zeta_2$ of $\zeta_0 \rightsquigarrow \zeta_1$ with a path $\zeta_1 \rightsquigarrow \zeta_2$ on $\mathcal{X}$.

Assume now that $\mathcal{D}, \zeta_0, \zeta_1, \delta(\zeta_0)$ are as in the previous section. Let

$$f^{(i)}[\zeta_0 : I](\zeta_0 + u) = f_0^{(i)}[\zeta_0 : I] + f_1^{(i)}[\zeta_0 : I]u + \cdots$$

denote the power series of $f^{(i)}[\zeta_0 : I]$ at ζ_0 for each i. For $\beta \geqslant p$, let $F_{;\beta}(\zeta_1)$ be the column vector

$$F_{;\beta}(\zeta_1) = \begin{pmatrix} f_0 + \cdots + f_{\beta-1}(\zeta_1 - \zeta_0)^{\beta-1} \\ \vdots \\ f_0^{(r-1)} + \cdots + f_{\beta-p}^{(r-p)}(\zeta_1 - \zeta_0)^{\beta-p} \end{pmatrix}$$

The matrix $\Delta_{\zeta_0 \to \zeta_1;\beta}$ with

$$F_{;\beta}(\zeta_1) = \Delta_{\zeta_0 \to \zeta_1;\beta} F(\zeta_0)$$

for all $F(\zeta_0)$ is called **truncated transition matrix** at order β associated to $\zeta_0 \to \zeta_1$. The following technical lemma gives us precise information about the computational complexity of truncated transition matrices:

Lemma C.1. *Let* $s = \texttt{size}(\pi(\zeta_0)) + \texttt{size}(\pi(\zeta_1))$. *There exists an algorithm which computes* $\Delta_{\zeta_0 \to \zeta_1;\beta}$ *in time*

$$O(\beta(s + \log \beta) \log^2 \beta \operatorname{llog} \beta + s \log s \operatorname{llog} s),$$

uniformly in $\zeta_0, \zeta_1 \in \mathcal{D}$, *provided that* $|\zeta_1 - \zeta_0| < \frac{1}{2}\delta(\zeta_0)$.

Proof. The computation of $\Delta_{\zeta_0 \to \zeta_1;\beta}$ is done by steps 1, 3 and 4 of `approximate1` with the following modifications: instead of working with coefficients in K, we work with coefficients in $K[\eta]/(\eta^p)$. Instead of starting from the difference relation (C.2), we now start from the difference equation satisfied by the sequence $f_0, f_1(\zeta_1 + \eta - \zeta_0), f_2(\zeta_1 + \eta - \zeta_0)^2, \cdots$. Modulo these changes, the line $L_{;\beta}$ computed in step 4 has the form $L_{;\beta} = L_{;\beta,0} + L_{;\beta,1}\eta + \cdots + L_{;\beta,p-1}\eta^{p-1}$, where $L_{;\beta,0}, \cdots, L_{;\beta,p-1}$ are precisely the lines of $\Delta_{\zeta_0 \to \zeta_1;\beta}$. Intuitively speaking, η is a formal infinitesimal variable, which enables us to compute formal expansions at ζ_1 up to the order p.

Let us now bound the complexity: step 1 is a precomputation of cost $O(s \log s \operatorname{llog} s)$.

In step 3, we have $\texttt{size}(M_{k;1}) + \texttt{size}(N_{k;1}) = O(s + \log k)$, since the Q_i are polynomials in $K[1/k]$ with coefficients of sizes $O(s)$. By induction, it follows that $\texttt{size}(M_{k;\alpha}) + \texttt{size}(N_{k;\alpha}) = O(\alpha(s + \log k))$. Hence, the computations of $M_{0;\alpha}, \cdots, M_{\beta-\alpha;\alpha}$ require β/α operations of costs $O(\alpha(s+\log \beta) \log \beta \operatorname{llog} \beta)$ for fixed α. Thus, the computation of $M_{0;\beta}$ requires $O(\beta(s + \log \beta) \log^2 \beta \operatorname{llog} \beta)$ operations.

In step 4, the (pre-)computation of $A(\zeta_0)$ has cost $O(s \log s \operatorname{llog} s)$. The matrix multiplication $M_{0;\beta}A(\zeta_0)$ takes a time $O(\beta(s + \log \beta) \log \beta \operatorname{llog} \beta)$. □

The remainder of this section is devoted to the obtaining of theoretical bounds concerning transition matrices.

Forward transitions. Let $E_0, \cdots, E_{p-1}$ be the columns of the identity matrix. Then the entry $\Delta_{\zeta_0 \to \zeta_1, i, j}$ equals $f^{(i)}[\zeta_0 : E_j](\zeta_1)$. By Cauchy's formula, we have:

$$|f_k^{(i)}[\zeta_0 : E_j]| = \left| \frac{1}{2\pi\sqrt{-1}} \int_{|u|=\delta(\zeta_0)} \frac{f^{(i)}[\zeta_0 : E_j](\zeta_0 + u)}{u^{k+1}} du \right| \leqslant \frac{\sup_{\mathcal{D}} f^{(i)}[\zeta_0 : E_j]}{\delta(\zeta_0)^k}.$$

Let $\beta \geqslant p$. Summing over $k \geqslant \beta$, we have for each ζ_1 with $|\zeta_1 - \zeta_0| < \frac{1}{2}\delta(\zeta_0)$:

$$|\Delta_{\zeta_0 \to \zeta_1; \beta, i, j} - \Delta_{\zeta_0 \to \zeta_1, i, j}| \leqslant 2 \sup_{\mathcal{D}} f^{(i)}[\zeta_0 : E_j] \left(\frac{|\zeta_1 - \zeta_0|}{\delta(\zeta_0)} \right)^{\beta - i}. \tag{C.7}$$

Hence, denoting

$$S(\zeta_0) = \max_{0 \leqslant i,j \leqslant p-1} \sup_{\mathcal{D}} f^{(i)}[\zeta_0 : E_j],$$

we have

$$|\Delta_{\zeta_0 \to \zeta_1; \beta} - \Delta_{\zeta_0 \to \zeta_1}| \leqslant 2S(\zeta_0) \left(\frac{|\zeta_1 - \zeta_0|}{\delta(\zeta_0)} \right)^{\beta + 1 - p}. \tag{C.8}$$

Notice that this proves in particular our claim in the proof of proposition C.1 that $L_{;\beta}$ tends to the first line of $\Delta_{\zeta_0 \to \zeta_1}$ for $\beta \to \infty$.

Backward transitions. Let C_0 be a constant with $C_0 > \sup_{\zeta_0, \zeta_1 \in \mathcal{D}} p |\Delta_{\zeta_1 \to \zeta_0}|_1$. Let $\Xi(\zeta_0)$ resp. $\Xi(\zeta_1)$ be the matrix, with entries $f^{(i)}[\zeta_0 : E_j]$ resp. $f^{(i)}[\zeta_1 : E_j]$. We observe that

$$\Xi(\zeta_1) = \Delta_{\zeta_1 \to \zeta_0} \Xi(\zeta_0).$$

In particular, $S(\zeta_1) \leqslant C_0 S(\zeta_0)$. Let C_1 be a constant with $C_1 > C_0 S(\zeta_0)$. Then (C.8) implies

$$|\Delta_{\zeta_0 \to \zeta_1; \beta} - \Delta_{\zeta_0 \to \zeta_1}| \leqslant C_1 \left(\frac{|\zeta_1 - \zeta_0|}{\delta(\zeta_0)} \right)^{\beta + 1 - p}, \tag{C.9}$$

and this relation is valid for all $\zeta_0, \zeta_1 \in \mathcal{D}$ with $|\zeta_1 - \zeta_0| < \frac{1}{2}\delta(\zeta_0)$. In particular, for $C_2 > p|F(\zeta_0)|C_0$, we get

$$|\Delta_{\zeta_0 \to \zeta_1; \beta} F(\zeta_0) - F(\zeta_1)| \leqslant C_2 \left(\frac{|\zeta_1 - \zeta_0|}{\delta(\zeta_0)} \right)^{\beta + 1 - p}, \tag{C.10}$$

where the bound is again valid for all $\zeta_0, \zeta_1 \in \mathcal{D}$ with $|\zeta_1 - \zeta_0| < \frac{1}{2}\delta(\zeta_0)$.

Infinitesimal transitions. Let ρ be the radius of $\mathcal{D}$. Then substitution of β by p in (C.9) yields

$$|\Delta_{\zeta_0 \to \zeta_1; p} - Id| \leqslant (e^\rho - 1)|\zeta_1 - \zeta_0|.$$

Applying (C.9) once, we get

$$|\Delta_{\zeta_0 \to \zeta_1} - Id| \leqslant (C_1 + e^\rho - 1)|\zeta_1 - \zeta_0|.$$

Let $C_3 > (e^\rho - 1) + 2C_1$. Applying (C.9) once more, we deduce that

$$|\Delta_{\zeta_0 \to \zeta_1;\beta} - Id| \leqslant C_3|\zeta_1 - \zeta_0|,$$

whence

$$p|\Delta_{\zeta_0 \to \zeta_1;\beta}|_1 \leqslant 1 + C_3|\zeta_1 - \zeta_0|, \tag{C.11}$$

for all $\beta \geqslant p$.

C.5 Analytic continuation

In this section, we give the complete evaluation algorithm for holonomic functions. Let $\mathcal{C}$ be a connected finite union of compact disks in $\mathcal{X}$ with centers above $\mathbb{Z}[\frac{1}{2}, i]$ and radii in $\mathbb{Z}[\frac{1}{2}]$. For each $\xi \in \mathcal{C}$, denote by $\delta(\xi)$ the distance between ξ and $\partial\mathcal{C}$. Let $B_1 \in \mathbb{Z}[\frac{1}{2}]$ be such that

$$|\Delta_{\xi \to \xi';\beta} F(\xi) - F(\xi')| \leqslant B_1 \left(\frac{|\xi' - \xi|}{\delta(\xi)}\right)^{\beta+1-p},$$

for all $\beta \geqslant p$ and $\xi, \xi' \in \mathcal{C}$, such that $|\xi' - \xi| < \frac{1}{2}\delta(\xi)$. Such a constant B_1 exists by (C.10). Let $1 \leqslant B_2 \in \mathbb{Z}[\frac{1}{2}]$ be a constant, such that for all $\xi, \xi' \in \mathcal{D}_i$ with $|\xi' - \xi| < \frac{1}{2}\delta(\xi')$, we have

$$p|\Delta_{\xi \to \xi';\beta}|_1 \leqslant 1 + B_2|\xi' - \xi|. \tag{C.12}$$

Such a constant B_2 exists, by (C.11).

C.5.1 Begin and endpoints above $\mathbb{Z}[\frac{1}{2}, i]$

We first consider the case of an effective path

$$\xi \rightsquigarrow \xi' = \zeta_0 \to \cdots \to \zeta_\kappa,$$

with $\pi(\zeta_0), \pi(\zeta_\kappa) \in \mathbb{Z}[\frac{1}{2}, i]$ and such that $|\zeta_{i+1} - \zeta_i| < \frac{1}{2}\delta(\zeta_i)$ for all $0 \leqslant i < \kappa$. For each $0 \leqslant i < \kappa$, let

$$\Pi_i = (1 + B_2|\zeta_{i+1} - \zeta_i|) \cdots (1 + B_2|\zeta_\kappa - \zeta_{\kappa-1}|).$$

Let $B_3 \in \mathbb{Z}[\frac{1}{2}]$ be a constant for which $B_3 \geqslant \Pi_0$.

Algorithm `approximate2` (begin and end points in $\mathbb{Z}[\frac{1}{2}, i]$).
INPUT: A precision $0 < \varepsilon \in \mathbb{Z}[\frac{1}{2}]$.
OUTPUT: An approximation of $f(\zeta_\kappa)$ with error $< \varepsilon$.

STEP 1. For $i = 0, \cdots, \kappa$ do the following:

A. Compute the smallest $\beta_i = 2^l \geqslant p$ for which

$$B_1 B_3 \left(\frac{|\zeta_{i+1} - \zeta_i|}{\delta(\zeta_i)}\right)^{\beta_i+1-p} \leqslant \varepsilon/4\kappa.$$

B. Compute $\Delta_{\zeta_i \to \zeta_{i+1};\beta_i}$, by the algorithm from lemma C.1.

STEP 2. Compute an approximation $\tilde{F}(\zeta_0)$ of $F(\zeta_0)$, with $B_3|\tilde{F}(\zeta_0) - F(\zeta_0)| < \varepsilon/4$.

STEP 3. For each $i = 0, \cdots, \kappa-1$, compute a truncation $\tilde{F}(\zeta_{i+1})$ of $\Delta_{\zeta_i \to \zeta_{i+1};\beta_i}\tilde{F}(\zeta_i)$ at precision $\varepsilon/4\kappa B_3$.

STEP 4. Return the first element of $\tilde{F}(\zeta_k)$.

Remark C.2. The need of truncating the $\tilde{F}(\zeta_i)$ will become apparent in the next algorithm, which uses `approximate2` as a subalgorithm.

Proposition C.2. *The algorithm* `approximate2` *is correct and has asymptotic complexity $O(n \log^3 n \operatorname{llog} n)$, for $\varepsilon = 2^{-n}$.*

Proof. For each $0 \leqslant i < \kappa$, we have:

$$\begin{aligned}\Pi_{i+1}|\tilde{F}(\zeta_{i+1}) - F(\zeta_{i+1})| &\leqslant \Pi_{i+1}|\tilde{F}(\zeta_{i+1}) - \Delta_{\zeta_i \to \zeta_{i+1};\beta_i}\tilde{F}(\zeta_i)| + \\ &\quad\ \Pi_i|\tilde{F}(\zeta_i) - F(\zeta_i)| + \\ &\quad\ \Pi_{i+1}|\Delta_{\zeta_i \to \zeta_{i+1};\beta_i}F(\zeta_i)\Delta_{\zeta_i \to \zeta_{i+1}}F(\zeta_i)| \\ &\leqslant \varepsilon/2\kappa + \Pi_i|\tilde{F}(\zeta_i) - F(\zeta_i)|.\end{aligned}$$

By induction, we get

$$|\tilde{F}(\zeta_\kappa) - F(\zeta_\kappa)| \leqslant \varepsilon/2 + \Pi_0|\tilde{F}(\zeta_0) - F(\zeta_0)|,$$

whence the correctness of `approximate2`, since $\Pi_0|\tilde{F}(\zeta_0) - F(\zeta_0)| < \varepsilon/4$. The complexity bound is proved in a similar way as in proposition C.1, using lemma C.1. □

C.5.2 Arbitrary paths

Now let $\xi \rightsquigarrow \xi'$ be an arbitrary effective path on $\mathcal{D}$, represented by $\xi \to \xi_0 \to \cdots \to \xi_\lambda \to \xi'$. We assume that $d = |\xi_0 - \xi| < \frac{1}{4}\delta(\xi)$, $d' = |\xi' - \xi_\lambda| < \frac{1}{4}\delta(\xi')$ and $|\xi_{i+1} - \xi_i| < \frac{1}{2}\delta\xi_i$, for all $0 \leqslant i < \lambda$. Clearly, any effective path can be represented in this way, by subdividing abnormally long segments.

To approximate $f(\xi')$, we need two special paths in order to connect good approximations ξ_μ and $\xi'_{\mu'}$ above $\mathbb{Z}[\frac{1}{2}, i]$ of ξ resp. ξ' to their rough approximations ξ_0

and ξ_λ. Then the effective path to compute $f(\xi')$ has the form

$$\xi \approx \chi_\mu \to \cdots \to \chi_0 \to \xi_0 \to \cdots \to \xi_\lambda \to \chi'_0 \to \cdots \to \chi'_{\mu'} \approx \xi'. \tag{C.13}$$

Given $0 < \varepsilon \in \mathbb{Z}[\frac{1}{2}]$, such a path is called an **ε-refinement** of $\xi \rightsquigarrow \xi'$, when computed as follows: first compute approximations $\tilde{\xi}$ and $\tilde{\xi}'$ of ξ resp. ξ' with error $< \varepsilon$. Next, truncate $\tilde{\xi}$ at precisions $2^{-2^0}d, 2^{-2^1}d, \cdots, 2^{-2^\mu}d$, until $2^{-2^\mu}d < \varepsilon$; this yields $\chi_0, \cdots, \chi_\mu$. Similarly, we truncate $\tilde{\xi}'$ at precisions $2^{-2^0}d', \cdots, 2^{-2^{\mu'}}d'$ until $2^{-2^{\mu'}}d' < \varepsilon$ in order to yield $\chi'_0, \cdots, \chi'_{\mu'}$. Of course, the starting point of the refinement is $\xi + (\chi_\mu - \pi(\xi))$.

In order to apply `approximate2` we have to find a suitable constant B_3, which works for all possible ε-refinements (C.13). We observe that

$$\prod_{i=0}^{\infty} \left(1 + \frac{B_2 d}{2^{2^i}}\right) \leqslant C_3 = 2(1 + B_2 d)^{\lceil \log(\log \max\{B_2, 2\}/\log 2)/\log 2 \rceil} \tag{C.14}$$

and

$$\prod_{i=0}^{\infty} \left(1 + \frac{B_2 d'}{2^{2^i}}\right) \leqslant C_4 = 2(1 + B_2 d')^{\lceil \log(\log \max\{B_2, 2\}/\log 2)/\log 2 \rceil}. \tag{C.15}$$

Now for B_3 we take a constant in $\mathbb{Z}[\frac{1}{2}]$, such that

$$C_3 C_4 (1 + B_2 d)(1 + B_2 d') \prod_{i=0}^{\lambda-1} (1 + B_2 |\xi_{i+1} - \xi_i|) \leqslant B_3. \tag{C.16}$$

Hence $B_3 \geqslant \Pi_0$. Finally, let $B_4 \in \mathbb{Z}[\frac{1}{2}]$ be a constant, with $\sup_{\mathcal{C}} |f^{(i)}| \leqslant B_4$ for all $1 \leqslant i \leqslant p$.

Algorithm `approximate3` (general case).
INPUT: A precision $0 < \varepsilon \in \mathbb{Z}[\frac{1}{2}]$.
OUTPUT: An approximation of $f(\xi')$ with error $< \varepsilon$.

STEP 1. Compute an $(\varepsilon/8B_3B_4)$-refinement of $\xi \rightsquigarrow \xi'$, and let $\zeta_0 \to \cdots \to \zeta_\kappa$ be its representation.

STEP 2. Apply `approximate2` to the path $\zeta_0 \to \cdots \to \zeta_\kappa$ with initial conditions $F(\xi)$ at ζ_0.

Theorem C.3. *Let f be a holonomic function satisfying (C.1), such that $P_0, \cdots, P_p \in K[z]$ for an algebraic number field K. Let $\mathcal{X}$ be the Riemann surface of f over $\mathbb{C}\backslash\{w_1, \cdots, w_\nu\}$, where $w_1, \cdots, w_\nu$ are the zeros of P_p. Let $\xi \rightsquigarrow \xi'$ be an effective path on $\mathcal{X}$, such that the initial conditions $f(\xi), \cdots, f^{(p-1)}(\xi)$ are effective. Assume that $\pi(\xi), \pi(\xi'), f(\xi), \cdots, f^{(p-1)}(\xi)$ have time complexities $O(T(n))$. Then there exists an $O(n \log^3 n \operatorname{llog}^2 n + T(n + O(1)))$ expansion algorithm for $f(\xi')$.*

Proof. Let us first prove the correctness of `approximate3`. Since $|\zeta_0 - \xi| \leqslant \varepsilon/8B_3B_4$, we have $|F(\zeta_0) - F(\xi)| \leqslant \varepsilon/8B_3 \leqslant \varepsilon/8\Pi_0$. Hence, $|\tilde{F}(\zeta_0) - F(\zeta_0)| \leqslant 3\varepsilon/8$. As in the proof of proposition C.2, we deduce that $|\tilde{F}(\zeta_\kappa) - F(\zeta_\kappa)| \leqslant 7\varepsilon/8$. Finally, $|F(\xi') - F(\zeta_\kappa)| \leqslant \varepsilon/8$, since $|\xi' - \zeta_\kappa| \leqslant \varepsilon/8B_3B_4$ and $B_3 \geqslant 1$. It follows that $|F(\xi') - \tilde{F}(\zeta_\kappa)| \leqslant \varepsilon$, which completes the correctness proof.

Let us now prove the complexity bound. First we notice that $|\zeta_{i+1} - \zeta_i| < \frac{1}{2}\delta(\zeta_i)$ for all $0 \leqslant i < \kappa$, so that lemma C.1 applies. Now, using similar notations as above, let $\zeta_0 \to \cdots \to \zeta_\kappa$ be the path (C.13) and let $\varepsilon = 2^{-n}$. By the definition of ε-refinements, we have $\kappa = O(\log n)$, $\mathtt{size}(\pi(\chi_i)) = O(2^i)$, $\mathtt{size}(\pi(\chi_i')) = O(2^i)$, $\log|\chi_{i+1} - \chi_i| = O(2^i)$ and $\log|\chi_{i+1}' - \chi_i'| = O(2^i)$. Furthermore, if γ_i resp. γ_i' denotes the β_j which corresponds to χ_i resp. χ_i', then

$$\gamma_i = O\left(\frac{\log\varepsilon - \log 4\kappa B_1 B_3}{\log|\chi_i - \chi_{i-1}| - \log\delta(\xi + (\chi_{i-1} - \pi(\xi)))}\right) = O\left(\frac{n}{2^i}\right)$$

resp.

$$\gamma_i' = O\left(\frac{n}{2^i}\right).$$

Hence, by lemma C.1, the execution time of `approximate3` is bounded by

$$O\left(n\log^3 n \operatorname{llog} n + \sum_{i=0}^{O(\log n)} \frac{n}{2^i}(2^i + \log n)\log^2 n \operatorname{llog} n\right).$$

Now

$$\begin{aligned}\sum_{i=0}^{O(\log n)} \frac{n}{2^i}(2^i + \log n)\log^2 n \operatorname{llog} n &= \\ \left(\sum_{i=0}^{\lfloor\log\log n\rfloor} + \sum_{i=\lfloor\log\log n\rfloor+1}^{O(\log n)}\right)\frac{n}{2^i}(2^i + \log n)\log^2 n \operatorname{llog} n &= O(n\log^3 n \operatorname{llog}^2 n),\end{aligned}$$

whence the desired complexity bound. □

C.6 Computation of the constants B_i

In this section, we show how the constants B_1, B_2, B_3, B_4 can be computed effectively. We first show how to compute bounds for $|f|, |f'|, \cdots$ on a fixed compact disk $\mathcal{D}$ on $\mathcal{X}$, which center ζ_0 above $\mathbb{Z}[\frac{1}{2}, i]$ and radius $\rho \in \mathbb{Z}[\frac{1}{2}]$, where we assume that $F(\zeta_0)$ is effective. Let $f(\zeta_0 + u) = f_0 + f_1 u + \cdots$ be the power series expansion of f at ζ_0. Let $\zeta_1 = \zeta_0 + \rho$, and let the matrices $N_{k;\alpha}$ be defined as in section C.3. We start by showing how to compute suitable bounds for the f_k.

For each $\alpha = 2^l$ there exists a matrix $N_{;\alpha}$ with coefficients in $K(k)$, such that $N_{k';\alpha}$ is obtained by replacing k by k' for all k'. Indeed, such matrices are obtained in a similar way as the $N_{k;\alpha}$ from (C.2) and (C.5), by considering k as a formal parameter. Hence, we even have an algorithm to compute the $N_{;\alpha}$.

Substitution of k by infinity in $N_{;\alpha}$ yields a matrix $N_{\infty;\alpha}$, which is an $O(1/k)$-perturbation of $N_{\infty;\alpha}$ (i.e. the entries of $N_{\infty;\alpha} - N_{;\alpha}$ are $O(1/k)$). Intuitively speaking, taking $k = \infty$ amounts to neglecting the contribution of the terms stemming from $P_{p-1}f^{(p-1)}, \cdots, P_0 f$ in the recurrence relation (C.2). Therefore, $N_{\infty;\alpha}$ reflects the linear recurrence relation satisfied by the coefficients of the Taylor series expansion of $1/P_p(\zeta_0 + u\rho)$ in u. In particular, the non zero eigenvalues of $N_{\infty;\alpha}$ are $(\rho/(w_1 - \zeta_0))^\alpha, \cdots, (\rho/(w_n - \zeta_0))^\alpha$.

Now consider the following algorithm:

Algorithm `bound`.

INPUT: $\zeta_0, \zeta_1 \in \mathbb{Z}[\frac{1}{2}]$.

OUTPUT: Constants C_5 and $|\tau| \leqslant 1$, such that $|f_k(\zeta_1 - \zeta_0)^k| \leqslant C_5 \tau^k$ for all k.

STEP 1. Compute the smallest $\alpha_0 = 2^l$, such that $p|N_{\infty;\alpha_0}|_1 < 1$.

STEP 2. Compute the matrix $N_{;\alpha_0} - N_{\infty;\alpha_0}$ with $O(1/k)$ entries in $K(k)$.

STEP 3. Compute a k_0 and $\eta < 1 - p|N_{\infty;\alpha_0}|_1$, such that $p|N_{k;\alpha_0} - N_{\infty;\alpha_0}|_1 \leqslant \eta$, for all $k \geqslant k_0$.

STEP 4. Let $\tau := (p|N_{\infty;\alpha_0}|_1 + \eta)^{1/\alpha_0}$, and $C_5 := \max\{|f|, \cdots, |f_{k_0-1}((\zeta_1 - \zeta_0)/\tau)^{k_0-1}|\}$

In particular, the algorithm yields the desired upper bounds

$$\sup_{\mathcal{D}} |f^{(i)}| \leqslant C_5 \sum_{k=0}^{\infty} \frac{(k+i)!\tau^k}{k!|\zeta_1 - \zeta_0|^i} \leqslant \frac{C_5 \tau^i i!}{(1-\tau)^{i+1}|\zeta_1 - \zeta_0|^i}. \tag{C.17}$$

In section C.3, this yields an algorithm to compute B_0, if we take ζ_0 to be the center of $\mathcal{D}$. Adopting the notations from section C.4, we also obtain an algorithm to compute upper bounds for $\sup_D |f^{(i)}[\zeta_0 : E_j]|$, for all i and j.

Correctness and termination proof of `bound`. In step 1, the power of two α_0 indeed exists, because the eigenvalues of $N_{\infty;\alpha}$ are all strictly inferior to 1, and $N_{\infty;2\alpha} = N^2_{\infty;\alpha}$, for all powers of two α, by (C.5). The validity and termination of the remainder of `bound` is trivial. □

For the computation of the constants B_1, B_2, B_3, B_4, we have to show how to compute C_0 in section C.4. We start with the following preliminary: let $\mathcal{D}$ and ζ_0 be as above, and let ζ_1 be such that $|\zeta_1 - \zeta_0| < \frac{1}{2}\rho$. Applying (C.8) twice, we deduce as we did in section C.4 that

$$|\Delta_{\zeta_0 \to \zeta_1} - Id| \leqslant (2S(\zeta_0) + e^\rho - 1)|\zeta_1 - \zeta_0|.$$

Using (C.17), we can compute a constant $C_6(\zeta_0) \in \mathbb{Z}[\frac{1}{2}]$, such that $C_6(\zeta_0) > 2S(\zeta_0) + e^\rho - 1$. We have

$$|\Delta_{\zeta_1 \to \zeta_0} - Id| \leqslant \frac{C_6(\zeta_0)|\zeta_1 - \zeta_0|}{1 - C_6(\zeta_0)|\zeta_1 - \zeta_0|},$$

for $|\zeta_1-\zeta_0|<C_6(\zeta_1)^{-1}$. In particular, this yields the bound

$$p|\Delta_{\zeta_1'\to\zeta_0'}|_1 \leqslant 4 \tag{C.18}$$

for all $\zeta_0',\zeta_1'\in\mathcal{D}$, with $|\zeta_0'-\zeta_0|<1/2C_6(\zeta_0)$ and $|\zeta_1'-\zeta_0|<1/2C_6(\zeta_0)$. Hence we may take $C_0=4$, whenever ζ_1 is sufficiently close to ζ_0.

Let us now treat the general case. We construct a sequence $\mathcal{U}_1,\mathcal{U}_2,\cdots$ of open disks on $\mathcal{X}$ as follows: assuming that $\mathcal{U}_1,\cdots,\mathcal{U}_n$ have been constructed for $n\geqslant 0$, the center ξ_{n+1} of $\mathcal{D}_{n+1}$ is taken to be any point above $\mathbb{Z}[\frac{1}{2},i]$ in $\mathcal{D}\backslash(\mathcal{U}_1\cup\cdots\cup\mathcal{U}_n)$; if such a point does not exist, the construction is terminated. Next, we let $\mathcal{D}_{n+1}$ be any compact disk on $\mathcal{X}$, with center ξ_{n+1} and radius in $\mathbb{Z}[\frac{1}{2}]$. We next compute a number $\rho_{n+1}\in\mathbb{Z}[\frac{1}{2}]$ in the way described above, such that (C.18) holds for all $\zeta_0',\zeta_1'\in\mathcal{D}_{n+1}$, with $|\zeta_0'-\xi_{n+1}|\leqslant\rho_{n+1}$ and $|\zeta_1'-\xi_{n+1}|\leqslant\rho_{n+1}$. Termination of this procedure is guaranteed by the compactness of $\mathcal{D}$. Ultimately, we take $C_0:=4^n$.

Proposition C.3. *There is an algorithm to compute the bounds B_1,B_2,B_3,B_4 involved in* `approximate2` *and* `approximate3`*.*

Proof. Using the algorithms above, we compute the constants C_1,C_2,C_3,C_4 by the formulas from section C.4. To compute the constants B_1,B_2 and B_4, we use the fact that $\mathcal{C}$ is a finite union of compact disks, each on which the preceding algorithms apply. In the case of B_4, we use (C.10) and (C.1) (for $\sup_{\mathcal{C}}|f^{(p)}|$). Finally, B_3 is computed using (C.16). □

Remark C.3. For simplicity of the presentation, the constants B_1,B_2,B_3,B_4 are "relative to $\mathcal{C}$ as a whole". In practice, we compute such constants on each compact disk, which constitutes $\mathcal{C}$. This allows us to avoid the computation of the sequence $\mathcal{U}_1,\mathcal{U}_2,\cdots$ by precomputing a suitable refinement of the path $\xi\rightsquigarrow\xi'$ in `approximate3`.

C.7 Conclusion and extensions

We have presented an algorithm to evaluate certain holonomic functions. Although our algorithm has a good asymptotic complexity, it also has several disadvantages:

- Bad complexity as a function of q (= order of (C.2)).
- The initial conditions need to be specified in a point different from $w_1,\cdots,w_\nu$.
- Bad behaviour near singular points.
- Not clear how to choose the path for analytic prolongation.

On the other hand, several generalizations of the algorithm are possible:

- Holonomic functions in several variables.

– Small perturbations of (C.1).

In this section, we briefly discuss these issues.

Dependence on q. Our algorithm has a quite dramatic complexity as a function of q, namely the complexity of q by q matrix multiplications (which is $O(q^{2.81})$, see [Str 69]).

Nevertheless, we are currently investigating an optimization of the algorithm, where the matrices $M_{k;\alpha}$, $N_{k;\alpha}$ are not computed up to order β in `approximate1`, but up till a slightly smaller power of two; the last steps of the evaluation are performed by a more classical Horner-like method. This should lead to a general purpose algorithm, which reduces to the standard way of evaluating power series for small precisions and which partially uses our optimized algorithm for large precisions. Moreover, this approach should extend to the case when the coefficients are no longer algebraic numbers, although the complexity drops down to $O(n^{\frac{3}{2}} \log^{\alpha} n \operatorname{llog} n)$ in this case, at a first sight.

Let us finally notice that for large values of q, FFT-multiplication becomes profitable for smaller precisions, since we can FFT-transform the entire matrices.

Initial conditions. Sometimes, f may be analytically continued above some of the points $w_1, \cdots, w_\nu$. For many classical special functions, the initial conditions are even specified in such "fake singularities": as an example, we mention the sine-integral function

$$\operatorname{Si} x = \int_0^x t^{-1} \sin t dt,$$

which satisfies the equation

$$z \operatorname{Si}''' z + 2 \operatorname{Si}'' z + z \operatorname{Si}' z = 0,$$

with initial conditions $\operatorname{Si}(0) = 0, \operatorname{Si}'(0) = 1, \operatorname{Si}''(0) = 0$. Using the recurrence relation

$$\operatorname{Si}_{k+2} + \frac{2}{k} \operatorname{Si}_{k+1} + \frac{1}{(k+1)(k+2)} \operatorname{Si}_k = 0,$$

`approximate1` still applies in this case. Moreover, modulo some precautions, this is a general situation: it suffices that the power series expansion be convergent and that K contains $\pi(\zeta_0)$. Then taking $\zeta_1 - \zeta_0 \in \mathbb{Z}[\frac{1}{2}, i]$, a suitable adaptation of lemma C.1 applies. This reduces the problem to the case when the initial conditions are specified in ζ_1. The bounds involved in these computations are computed in a similar way as in section C.6.

Singularities. When the point ξ' in which we want to evaluate f is very near to a singularity, the bounds B_i and the complexity of the algorithm as a function of ξ' may become very bad. No straightforward numerical methods can be applied to solve this problem, and resummation techniques are essentially needed to handle this situation (see [Th 95], for instance). Here we notice that the Borel and Laplace transforms preserve holonomy. Therefore, our algorithm can theoretically be used in the resummation process.

Analytic continuation path. Clearly, the complexity of the algorithm heavily depends on the choice of the path used for analytic continuation. At present, we have not studied this point in detail. We expect several similarities with the way paths are chosen for the computation of Laplace transforms in resummation algorithms.

Multivariate holonomic functions. A multivariate function $f(z_1, \cdots, z_k)$ is said to be holonomic, if f is holonomic in each of its variables. It is classical that the restriction of a multivariate holonomic function to a straight line segment is a holonomic function in one variable only. Moreover, the differential equation satisfied by this restriction can be computed in a generic way, i.e. for a generic straight line segment. Consequently, our algorithms generalize in a straightforward way to the multivariate case.

Small perturbations. In the proof of lemma C.1, we used a trick to compute $f(\zeta_1), \cdots, f^{(p-1)}(\zeta_1)$ simultaneously, by introducing the infinitesimal variable η. If instead of working in the ring $K[\eta]/(\eta^p)$, we work in the ring $K[\eta]/(\eta^r)$, the same method yields approximations for the first r terms of the power series expansion of f in ζ_1. Moreover, the cost is just multiplied by a factor $O(r)$ in this case. More generally, we may allow the polynomials $P_0, \cdots, P_p$ themselves to depend on η. In this case, we compute the effect of such a small perturbation in ζ_1, up to a precision of r terms.

Divergent series. In priciple, the techniques of this appendix can also be used to effeciently evaluate holonomic functions in the neighbourhood of points where the series expansion diverges, by summing "up to the smallest term". Of course, we only get approximations of the value of the holonomic function in this way, but it is well known that these approximations have exponential accuracy, when we approach the singularity.

C.8 References

[Br 76] R.P. Brent. Fast multiple-precision evaluation of elementary functions. *Journal of the Assoc. for Comp. Machinery 23(2) (p. 242-251).*

[Kn 81] D.E. Knuth. *The art of computer programming.* Vol 2: Seminumerical algorithms. Addison-Wesley (2-nd ed.).

[Lip 89] L. Lipshitz. D-finite power series. *Journal of algebra 122 (p 353-373).*

[Nae 95] F. Naegele. Autour de quelques équations fonctionnelles analytiques. *PhD. Thesis, l'I.N.P.G., Grenoble, France.*

[Rich 94] D. Richardson. How to recognize zero. *Preprint, University of Bath, England.*

[SS 71] A. Schönhage, V. Strassen. Schnelle Multiplikation grosser Zahlen. *Computing 7 (p. 281-292).*

[Str 69] V. Strassen. Gaussian elimination is not optimal. *Numer. Math. 13 (p. 352-356).*

[Th 95] J. Thomann. Procédés formels et numériques de sommation de séries d'équations différentielles. *Expositiones Math. 13 (p. 223-246).*

[VdH 96b] J. van der Hoeven. Fast evaluation of holonomic functions. *Research report LIX/RR/96/05, Ecole polytechnique, France.*

[Zeil 90] D. Zeilberger. A holonomic systems approach to special functions identities. *Journal of computational and applied mathematics 32, p 321-368.*

Appendix D

Computations with special functions

D.1 Introduction

Transcendental functions like $\exp, \log, \sin, \wp$, etc. have been studied since a long time. In our age of symbolic computation it is natural to ask whether computations with such functions can be done automatically. Essentially, this question can be reduced to the following one: given an expression built up from the rationals, a finite number of indeterminates and a given set of elementary functions, can we decide whether the expression is zero? Since the expressions are not necessarily canonically determined (they usually admit non trivial Riemann surfaces), the problem should be specified further: can we decide whether the expression is locally zero at a given point on the Riemann surface? We also have to specify what we mean by elementary functions: in this chapter, we will consider a very large class of elementary functions, namely those which can be entirely specified by a finite number of algebraic partial differential equations with initial conditions. In what follows, such functions will be called D-algebraic functions.

Let us first briefly discuss some of the history of the above problem. Initially, most of the research has been centered around finding canonical ways for representing expressions of the above type, based on our experience with polynomials. The study of functions built up from algebraic functions, exponentiation and logarithm was started by Liouville (see [Li 1837] and [Li 1838]) and culminated one and a half century later in the Risch structure theorem (see [Ris 75]). These techniques were extended to include a few other transcendental functions such as the error function by Cherry and Caviness (see [Ch 83], [CC 85]). However, for many other special functions, the desire of having canonical representations seems to be to ambitious.

The emergence of holonomic functions has provided a new way of looking at the question. Holonomic functions (in one variable) are functions which satisfy a non trivial linear differential equation over the polynomials with rational coefficients. They are represented, although not uniquely, by such a differential equation and a number of initial conditions. The basic idea is now to compute with these representations, without searching for canonical ones. Denef and Lipshitz, followed by others have generalized the holonomic function approach to D-algebraic functions (see [DL 89], [SH 89]). At this moment, the most promising algorithm for computations with D-algebraic functions is due to Péladan-Germa (see [Pél 95]). However, no implementation of this algorithm is available yet.

We finally mention that in the above discussion, we implicitly assumed the existence of an oracle, to perform the necessary computations with constants. Actually, this is a very strong hypothesis since computations with transcendental numbers turn out to be even harder than computations with transcendental functions (modulo a suitable oracle for the constants). Although it is often easy to decide whether a constant is zero (it suffices to perform a floating point evaluation at a sufficient precision), it can be very hard to prove that a constant is zero. Nevertheless, in the case of constants determined by exp-log equations, an algebraic zero-test does exist modulo Schanuel's conjecture and we refer to the introduction for more details.

Let us now come more particularly to the contents of this chapter. We have chosen the differential algebra with initial conditions setting to study local functions. This has the disadvantage of restricting the class of functions which can be studied, but the advantage of being suitable for effective computations by its algebraic character.

In section D.2, we introduce the formalism of D-rings. This formalism is due to Nichols and Weisfeiler (see [NiWe 82], [Bu 92]) and provides an algebraic setting for studying p.d.e.'s on curved geometrical objects. Its originality with respect to the classical theory of differential algebra (as developed by Riquier, Janet, Ritt, Raudenbush, Seidenberg, Kolchin, etc.; see [Riq 10], [Jan 20], [Ritt 50], [Kol 73], [Kap 76]) is that the derivations do not necessarily commute. Consequently, p.d.e.'s on non affine objects such as spheres can be considered, even though no essentially new functions are introduced by this. Actually, the formalism of D-rings mainly allows us to place ourselves in the coordinates, which correspond to the underlying geometry of the problem. Moreover, in [VdH 96c] it is shown that the main results from classical differential algebra can be generalized without much effort. In section D.3, we introduce D-rings with initial conditions. We will mainly consider initial conditions in a point, which correspond to (non differential) maximal ideals of the D-ring.

In section D.4, we establish the main algorithms for computations with D-algebraic functions. We start with a generalization of an algorithm due to Shackell and an optimization of this algorithm using a local pseudo-Buchberger algorithm. This work was carried out jointly with A. Péladan-Germa in [PV 96]. For the local pseudo-Buchberger algorithm, we refer to section D.6. We proceed with a zero-equivalence algorithm which is particularly useful when the point in which the zero-test is performed may be chosen randomly: in that case, virtually all functions which evaluate to zero are zero, and this property is exploited in the algorithm.

In section D.5 we consider some other computations which can be done with D-algebraic functions. Most importantly, we obtain an implicit function theorem, which permits to solve effectively certain systems of implicit equations determined by D-algebraic functions. This is a crucial result on which many algorithms in part B of this thesis rely.

D.2 Basic concepts

D.2.1 Definition of a D-ring and examples

A **D-ring** is a couple (A, D) satisfying

DR1. A is a commutative ring.

DR2. D is an A-module of derivations on A satisfying

$$\begin{aligned} 0_D\, a &= 0; \\ (bd)a &= b(da); \\ (d_1 + d_2)a &= d_1 a + d_2 a, \end{aligned}$$

for all $d, d_1, d_2 \in D$ and $a, b \in A$.

DR3. D has the structure of a Lie algebra and

$$\begin{aligned} [d_1, d_2]a &= d_1 d_2 a - d_2 d_1 a; \\ [d_1, a d_2] &= (d_1 a) d_2 + a[d_1, d_2], \end{aligned}$$

for all $d_1, d_2 \in D$ and $a \in A$.

For simplicity, we often write A instead of (A, D). In practice, (A, D) is **finite dimensional**, i.e. D is a finitely generated A-module. We notice that D-ring theory generalizes ring theory, by taking $D = 0$ for the set of derivations.

Example D.1. If k is a field, then $(k[x, y], (d_x, d_y))$ is a D-ring. Here d_x and d_y denote the partial derivatives with respect to x resp. y and $D = (d_x, d_y)$ the $k[x, y]$-module generated by d_x and d_y. D has a natural Lie algebra structure, given by

$$\begin{aligned} [A d_x + B d_y, A' d_x + B' d_y] &= (A A'_x + B A'_y - A_x A' - A_y B') d_x + \\ &\quad\; (A B'_x + B B'_y - B_x A' - B_y B') d_y. \end{aligned}$$

The D-ring $(k[x, y], (d_x, d_y))$ corresponds to the plane (over k). In a similar fashion, one defines affine n-space $(k[x_1, \cdots, x_n], (d_{x_1}, \cdots, d_{x_n}))$.

Example D.2. If k is a field, then $(k[x, y]/(x^2 + y^2 - 1), (y d_x - x d_y))$ is a D-ring. This object corresponds to the circle with its natural derivations. Similarly,

$(k[x,y,z]/(x^2+y^2+z^2-1),(d_1,d_2,d_3))$ is a D-ring, where $d_1 = yd_x - xd_y, d_2 = zd_y - yd_z$ and $d_3 = xd_z - zd_x$. We have $[d_1,d_2] = d_3, [d_2,d_3] = d_1$ and $[d_3,d_1] = d_2$. Finally, $(k[x,y]/(xy),(xd_x,yd_y))$ is a non entire D-ring, which corresponds to the union of two lines.

Example D.3. Assume that (A,D) is a D-ring and that I is a usual ideal of A. Then $A_{|I} = A/I$ can naturally be given the structure of a D-ring by taking $D_{|I} = \{\overline{d} \in D/ID | dI \subseteq I\}$ for the derivations. Indeed, we have a natural induced Lie bracket on $D_{|I}$, since $dI \subseteq I$ and $d'I \subseteq I$ imply $[d,d']I \subseteq I$, for all $d,d' \in D$. The D-ring $(A_{|I}, D_{|I})$ is called the **restriction of domain** of (A,D) by I. If A is Noetherian and finite dimensional, then so is $A_{|I}$. The D-rings of example D.2 are obtained as restrictions of domain of $k[x,y]$ by x^2+y^2-1, of $k[x,y,z]$ by $x^2+y^2+z^2-1$ and of $k[x,y]$ by xy.

Example D.4. Let A be an algebra over R. Denote by $Der_R(A)$ the set of R-derivations on A (i.e. the set of derivations $d : A \to A$ with $dR = 0$). Then $(A, Der_R(A))$ is a D-ring. If A is finitely generated, then this D-ring is finite dimensional.

D.2.2 Morphisms of D-rings

Let us now show how familiar concepts in differential algebra generalize to the context of D-rings. A **morphism of D-rings** or **D-morphism** $(A,D) \xrightarrow{\varphi,\psi} (A',D')$ is a pair of mappings $A \xrightarrow{\varphi} A'$ and $D \xrightarrow{\psi} D'$, preserving all D-ring operations. Clearly, D-rings with their morphisms form a category. Let us show that each morphism $(A,D) \xrightarrow{\varphi,\psi} (A',D')$ can be factored canonically through $(A', A' \otimes_A D)$, where we consider A' as an A-algebra, by $\lambda a = \varphi(\lambda)a$, for $\lambda \in A$ and $a \in A'$. Roughly speaking, this means that we can decompose a morphism in a part which preserves the structure of the module of derivations, and in a part which preserves the structure of the ring.

As we have a A-bilinear mapping $\mu : A' \times D \to D', (a,d) \mapsto a\psi(d)$, there exists a unique A-linear mapping $A' \otimes_A D \xrightarrow{\xi} D'$, such that $\mu = \xi \circ (1 \otimes Id)$. This mapping induces a canonical operation of $A' \otimes_A D$ on A' by $da = \xi(d)a$. This makes it possible two define a Lie bracket on $A' \otimes_A D$ by $[a \otimes d, a' \otimes d'] = aa' \otimes [d,d'] + a(da') \otimes d' - a'(d'a) \otimes d$. Then we have the desired factorization

$$(A,D) \xrightarrow{\varphi,1\otimes Id} (A', A' \otimes_A D) \xrightarrow{Id,\xi} (A',D').$$

A D-morphism is said to be **pure**, if $\xi = Id$ in the above decomposition. By the transitivity of base change, D-rings with pure D-morphisms form a category.

Remark D.1. Consider the D-ring $(k[x,y],(d_x,d_y))$. Then interchanging x and y resp. d_x and d_y gives a D-automorphism φ of $k[x,y]$. We remark that this would not be the case in differential algebra, because the derivations d_x and d_y are restricted to remain fixed. Nevertheless, φ is not a $k[x,y]$-morphism of D-algebras (see below).

D.2.3 D-ideals, D-A-modules and D-A-algebras

A **D-ideal** of (A,D) is an ideal, stable under D. We denote by $[\Sigma]$ the D-ideal generated by a subset Σ of A. If I is such a D-ideal, then A/I has a canonical **quotient D-ring** structure. If S is a multiplicatively stable subset, we each derivation $d \in D$ naturally gives rise to a derivation on $S^{-1}A$ by $d(a/s) = (da/s) - (a/s^2)$. Therefore, $S^{-1}A$ has a canonical D-ring structure and is called a **local D-ring** of A. We recall that $A \to S^{-1}A$ is injective if and only if S contains no zero divisors. The **total D-ring of fractions** is the D-ring $Q(A) = S^{-1}A$, where S is the set of non zero-divisors. In particular, $Q(A)$ is the **quotient D-field**, if A is entire.

A **D-A-module** or **D-module** over A is an A-module M, such that each derivation $d \in D$ gives rise to a derivation $\hat{d}$ on M, satisfying $\hat{d}(ax) = (da)x + a\hat{d}x$ and $\widehat{[d_1,d_2]}x = \hat{d}_1\hat{d}_2x - \hat{d}_2\hat{d}_1x$, for $a \in A$, $d, d_1, d_2 \in D$ and $x \in M$. A **morphism of D-modules** over A is an A-linear mapping, which commutes with the derivations of D.

A **D-A-algebra** or **D-algebra** over A is a D-A-module, which is an A-algebra B, such that $\hat{d}(xy) = x\hat{d}y + (\hat{d}x)y$, for each $x, y \in B$. We remark that (B, D_B) is a D-ring in this case (assuming that B has a unit), where $D_B = B \otimes_A D_A$ acts naturally on B by $(x \otimes d)y = xdy$. We have a canonical D-morphism of (A, D_A) into (B, D_B). Inversely, given such a morphism, we can consider B as a D-A-algebra in a natural way. A **morphism of D-A-algebras** is a morphism of A-algebras, which commutes with the derivations of D.

D.2.4 D-operator algebras

Let (A,D) be a finite dimensional Ritt D-ring. One can naturally associate the **free linear D-operator algebra** $\Omega = A[D]$ to (A,D): this is the free associative A-algebra, generated by A and D, subject to the relations

$$\begin{aligned} a \cdot_\Omega d &= ad; \\ d \cdot_\Omega a &= da; \\ d_1 \cdot_\Omega d_2 - d_2 \cdot_\Omega d_1 &= [d_1, d_2]. \end{aligned}$$

We also define $\Omega_0 = A$ and $\Omega_{r+1} = \Omega_r \cup D\Omega_r$, for each $r \in \mathbb{N}$. These sets are subsets of Ω, with $\Omega = \bigcup_{r\in\mathbb{N}} \Omega_r$. If $\omega \in \Omega$, we define its **order** to be the smallest r, with $\omega \in \Omega_r$.

Proposition D.1. *Let $d_1, \cdots, d_r$ be in D. Then $d_{\sigma(1)} \cdots d_{\sigma(r)} - d_1 \cdots d_r$ has order strictly inferior to r, for any permutation σ.*

Proof. It suffices to prove this, in the case when σ is a transposition of two subsequent indices i and $i+1$. In that case, we have

$$d_1 \cdots d_{i+1} d_i \cdots d_n - d_1 \cdots d_i d_{i+1} \cdots d_n = d_1 \cdots [d_{i+1}, d_i] \cdots d_n,$$

which has order at most $n-1$. □

Operators of the form $d_1 \cdots d_r$ are called **words**. The word operator $d_{\sigma(1)} \cdots d_{\sigma(r)}$ is said to be a **shuffle** of the word operator $d_1 \cdots d_r$. Suppose that we have fixed generators or a basis $d_1, \cdots, d_n$ for D. Then we denote $\Theta = \{d_1^{\alpha_1} \cdots d_n^{\alpha_n} | \alpha_1, \cdots, \alpha_n \in \mathbb{N}\}$ and $\Theta_r = \{d_1^{\alpha_1} \cdots d_n^{\alpha_n} | \alpha_1 + \cdots + \alpha_n \leqslant r\}$, for each r. Then we have

Proposition D.2. *The set Θ (resp. Θ_r) generates Ω (resp. Ω_r) as an A-module. It even forms a basis, if $d_1, \cdots, d_n$ form a basis of D.*

Proof. Let us show by induction over r that Θ_r generates Ω_r as an A-module. This is clear for $r = 0$. Assume that Θ_{r-1} generates Ω_{r-1}. By linearity, it suffices to show that $d_i d_1^{\alpha_1} \cdots d_n^{\alpha_n} \in (\Theta_r)$, for each i and $d_1^{\alpha_1} \cdots d_n^{\alpha_n} \in \Theta_{r-1}$. By the previous proposition, we have $d_i d_1^{\alpha_1} \cdots d_i^{\alpha_i} \cdots d_n^{\alpha_n} - d_1^{\alpha_1} \cdots d_i^{\alpha_i+1} \cdots d_n^{\alpha_n} + \omega$, with $\omega \in \Omega_{r-1}$. This completes the induction. As $\Omega = \bigcup_{r \in \mathbb{N}} \Omega_r$ and $\Theta = \bigcup_{r \in \mathbb{N}} \Theta_r$, this implies that Ω is generated by Θ.

Suppose now that $d_1, \cdots, d_n$ form a basis for D. The free A-module Ω' generated by Θ, can naturally be given the structure of an associative A-algebra, and it is easily checked that this algebra satisfies the universal property of Ω. Hence, Ω' is isomorphic to Ω. Therefore, Θ is linearly independent over A, and so is Θ_r, for each r. □

D.2.5 Geometric interpretation of D-rings

The concept of D-rings has a strong geometric appeal: we can interpret A as the space of functions on a manifold and D as its tangent bundle. In order to let things correspond properly, assume that A is entire and that D finitely generated by $d_1, \cdots, d_n$. Then we remark that D is locally trivial. Indeed, whenever we have a relation $a_1 d_1 + \cdots + a_i d_i = 0$, with $a_i \neq 0$, then D is generated by $\{d_1, \cdots, d_n\} \backslash \{d_i\}$, when localizing with respect to the multiplicative subgroup generated by a_i. After a finite number of such localizations, the tangent bundle becomes trivial.

Now the analogy can be carried out further. Finitely generated A-modules (which are locally trivial, by the above argument) correspond to vector bundles. For example, we have the cotangent space $D^* = Lin_A(D, A)$, the tensor bundles

$$D \otimes_A \overset{n \text{ times}}{\cdots} \otimes_A D \otimes_A D^* \otimes_A \overset{m \text{ times}}{\cdots} \otimes_A D^*,$$

etc. Other geometric structures can be imposed on A such as metrics (which are just elements of $D^* \otimes_A D^*$), connections (which are $\mathbb{Z}$-bilinear maps from $D \times D$ into D, such that

$$\begin{aligned} \nabla_{ad} d' &= a \nabla_d d'; \\ \nabla_d(ad') &= (da)d' + a\nabla_d d', \end{aligned}$$

and, optionally, $\nabla_d d' - \nabla_{d'} d = [d, d']$), etc.

Many differential geometric properties admit straightforward algebraic analogues. This observation, combined with the results of subsequent sections, makes it possible to perform many differential geometrical computations automatically.

D.3 D-rings with initial conditions

In this section we will algebrize the notion of a system of partial differential equations with boundary conditions. In section D.3.1, we first give a very general definition, with arbitrary partial differential equations and partially specified boundary conditions. In section D.3.2, and all what follows, we will restrict ourselves to initial conditions in a point.

D.3.1 D-boundary value problems

A **D-boundary value problem** is a chain of triplets $(A_n, J_n, I_n), \cdots, (A_1, J_1, I_1)$, where the J_i are D-ideals of the D-rings A_i, where the I_i are ideals of A_i/J_i and where $A_{i-1} = (A_i/J_i)_{|I_i}$, for each $2 \leqslant i \leqslant n$. Denote $A_0 = (A_1/J_1)_{|I_1}$. We have canonical mappings

$$A_n \to A_n/J_n \to A_{n-1} \to \cdots \to A_1 \to A_1/J_1 \to A_0.$$

The composite of these mappings is denoted by ε and it is called the **evaluation mapping**. We define an equivalence relation $\sim$ on A_n by

$$a \sim b \Leftrightarrow \forall \theta \in \Theta_{A_n} \quad \varepsilon(\theta(a)) = \varepsilon(\theta(b)),$$

for all a and b in A_n.

Remark D.2. This definition of equivalence may appear non natural at a first time, because of the example $f = e^{-1/x^2}$. However, f can not be specified in $x = 0$, because $1/x^2$ would not be defined. In fact, the theory of D-rings with initial conditions somehow generalizes complex function theory, where a function is also determined by the values of its iterated derivatives in a point.

The zero-equivalent elements form an ideal, which is easily checked to be a D-ideal. If this ideal is non zero, we say that the D-boundary value problem is

non reduced. In that case we can transform the problem into a **reduced** D-boundary value problem $(A'_n, J'_n, I'_n), \cdots, (A'_1, J'_1, I'_1)$, with $A'_i = A_i/\sim, J'_i = J_i \otimes A'_i$ and $I'_i = I_i \otimes (A'_i/J'_i)$, for all i. If I_1 is a maximal ideal, then A_0 is a field and the D-boundary value problem is said to be **completely specified**.

Example D.5. Suppose that we wish to represent $f = e^{x+y}$ as a function which is equal to e^y, for $x = 0$, and which satisfies the differential equation $f_x = f$. We take $A_2 = k[x, y]\{f\}$, $J_2 = [f_x - f]$ and $I_2 = (x)$. Then $A_1 \cong k[y]\{f_{|x}\}$ and we take $J_1 = [(f_{|x})_y - f_{|x}]$ and $I_1 = (x, f_{|x} - 1)$. We remark that f can also be specified by two partial differential equations and initial conditions in a point (see the next example).

D.3.2 D-systems

In the rest of this chapter, we will restrict our attention to D-boundary value problems, with $n = 1$, $J_1 = 0$ and where I_1 is maximal. This leads to the notion of a **D-system**, which is a pair $((A, D), \mathfrak{m})$, where (A, D) is a D-ring and $\mathfrak{m}$ a maximal ideal of A. Again, we often write A instead of $((A, D), \mathfrak{m})$. D-systems correspond to D-rings with initial conditions in a point. We have an evaluation mapping $A \to A/\mathfrak{m}$. A morphism of a D-system $((A, D_A), \mathfrak{m}_A)$ into a D-system $((B, D_B), \mathfrak{m}_B)$ is a morphism of D-rings $(A, D_A) \to (B, D_B)$, which commutes with the evaluation mappings. This means that $\mathfrak{m}_A$ is the inverse image of $\mathfrak{m}_B$.

Example D.6. A D-system in which we can represent the function $f = e^{x+y}$ is

$$(k[x, y]\{f\}/[f_x - f, f_y - f], (x, y, f - 1)),$$

with the usual partial derivations d_x and d_y on $k[x, y]$. Indeed, f is determined by the equations $f_x = f_y = f$ and the initial condition $f(0, 0) = 1$. To represent $g = e^{xe^{x+y}}$, we build a tower on this D-system. Indeed, it suffices to consider the D-supersystem

$$(k[x, y]\{f, g\}/[f_x - f, f_y - f, g_x - f - xf_x, g_y - xf_y], (x, y, f - 1, g - 1)).$$

Example D.7. An example of a non reduced system is $k[x]\{f, g\}/[f_x - f, g_x - g], (x, f - 1, g - 1)$. Indeed, $f \neq g$ are formally different in $k[x]\{f, g\}/[f_x - f, g_x - g]$, but they both represent the function e^x, so that $f \sim g$.

Example D.8. Consider the D-system $((k[x, y]/(xy), (xd_x, yd_y)), (x - 1, y))$. A polynomial $P(x, y) = c + xP_1(x) + yP_2(y)$ is zero-equivalent, iff $\varepsilon(\theta(P)) = 0$, for any linear differential operator θ. Now $\varepsilon(yd_yQ) = 0$, for any Q, so that $P \sim 0 \Leftrightarrow x = P_1 = 0$. This means that the behaviour of P on the y-axis is irrelevant for its zero-equivalence. This should not surprise, since the initial point does not lie on the y-axis.

More strikingly, if we took (x, y) as our initial condition, then all polynomials vanishing in 0 would even have been zero-equivalent. This comes from the fact that 0 is a singular point. The same holds true, if we consider $((k[x, y]/(x^2 - y^3), (3y^2 d_x + 2x d_y)), (x, y))$.

Proposition D.3. *Let $((A, D), \mathfrak{m})$ be a D-system, such that $A/\mathfrak{m}$ has characteristic zero. Then $A/\sim$ is an entire ring.*

Proof. Suppose that $xy \sim 0$, but $x \not\sim 0$ and $y \not\sim 0$. Let θ and θ' be linear differential operators, of minimal orders k and l, such that $\varepsilon(\theta x) \neq 0$ and $\varepsilon(\theta' y) \neq 0$. Thus, for any $\xi \in \Theta_{k-1}$ and $\xi' \in \Theta_{l-1}$, we have $\varepsilon(\xi x) = \varepsilon(\xi' y) = 0$. As $d_1 \cdots d_k - d_{\sigma(1)} \cdots d_{\sigma(k)}$ has order $< k$, for any derivations $d_1, \cdots, d_k$ and any permutation σ, we have $\varepsilon(d_1 \cdots d_k x) = \varepsilon(d_{\sigma(1)} \cdots d_{\sigma(k)} x)$. Similarly, $\varepsilon(d_1 \cdots d_l y) = \varepsilon(d_{\sigma(1)} \cdots d_{\sigma(l)} y)$.

Let us fix some well ordering $\leqslant$ on D. This ordering induces well orderings on the $\mathcal{M}_p(D)$, the sets of multisets of p elements of D. More precisely, we order the elements of multisets in increasing order and take the lexicographical orderings. We also have a well ordering on $\mathcal{M}(D) = \amalg_{p \in \mathbb{N}} \mathcal{M}_p(D)$, by ordering first on size and then using the above ordering on each component. We remark that the union operation is compatible with this ordering, so that $\mathcal{M}(D)$ is an ordered commutative monoid.

Now take $\{d_1, \cdots, d_k\} \in \mathcal{M}_k(D)$ and $\{d_{k+1}, \cdots, d_{k+l}\} \in \mathcal{M}_l(D)$ minimal, such that $\varepsilon(d_1 \cdots d_k x) \neq 0$ and $\varepsilon(d_{k+1} \cdots d_{k+l} y) \neq 0$. Then

$$\varepsilon(d_1 \cdots d_{k+l}(xy)) = \sum_{\{i_1, \cdots, i_k\} \amalg \{j_1, \cdots, j_l\} = \{1, \cdots, k+l\}} \varepsilon(d_{i_1} \cdots d_{i_k} x) \varepsilon(d_{j_1} \cdots d_{j_l} y) = 0.$$

Now if $\{d_{i_1}, \cdots, d_{i_k}\} \neq \{d_1, \cdots, d_k\}$ as a multiset, then either $\{d_{i_1}, \cdots, d_{i_k}\} < \{d_1, \cdots, d_k\}$, or $\{d_{j_1}, \cdots, d_{j_l}\} < \{d_{k+1}, \cdots, d_{k+l}\}$, because of the compatibility of the union with the ordering. Therefore, either $\varepsilon(d_{i_1} \cdots d_{i_k} x) = 0$ or $\varepsilon(d_{j_1} \cdots d_{j_l} y) = 0$ from the minimality hypothesis. We conclude that

$$\varepsilon(d_1 \cdots d_{k+l}(xy)) = m \varepsilon(d_1 \cdots d_k x) \varepsilon(d_{k+1} \cdots d_{k+l} y) = 0,$$

for some integer $m > 0$. This yields a contradiction, since $m \neq 0$ in $A/\mathfrak{m}$. □

We can perform different constructions on a D-system $((A, D), \mathfrak{m})$. First, we can naturally associate the **reduced D-system** $((A/\sim, D/\sim), \mathfrak{m}/\sim)$ to it, where, denoting by $\mathfrak{z}$ the D-ideal of zero-equivalent elements, $D/\sim = D/\mathfrak{z}D$ and $\mathfrak{m}/\sim = \mathfrak{m}/\mathfrak{z}\mathfrak{m}$, with $A/\mathfrak{z} \cong A/\sim$.

Secondly, we can associate the **local D-system** $((A_\mathfrak{m}, D_\mathfrak{m}), \mathfrak{m}_\mathfrak{m})$ to it. Here $M_\mathfrak{m}$ denotes the localization of any A-module or ideal M w.r.t. $\mathfrak{m}$. We say that $A_\mathfrak{m}/\sim$ is the local ring of **functions at** $\mathfrak{m}$.

Finally, if $I \subseteq \mathfrak{m}$, we have the **restriction of domain** $((A_{|I}, D_{|I}), \mathfrak{m}_{|I})$ of $((A, D), \mathfrak{m})$ w.r.t. I, where $\mathfrak{m}_{|I} = \mathfrak{m}/I\mathfrak{m}$. The next propositions show how these constructions are related.

Proposition D.4. *Let $((A, D), \mathfrak{m})$ be a D-system. Then $A_{\mathfrak{m}}/\sim \cong (A/\sim)_{\mathfrak{m}/\sim}$ as D-systems.*

Proof. We claim that the mapping from $A_{\mathfrak{m}}/\sim$ into $(A/\sim)_{\mathfrak{m}/\sim}$ defined by $\overline{a/s} \mapsto \overline{a}/\overline{s}$ is well defined and bijective. Indeed, $\overline{a/s} = 0$ is equivalent to saying that $\varepsilon(\theta(a/s)) = 0$, for any $\theta \in \Theta_{A_{\mathfrak{m}}}$. By induction over the order of θ, this is equivalent to $\varepsilon(\theta(a)) = 0$ for each $\theta \in \Theta_{A_{\mathfrak{m}}}$, since $\varepsilon(s) \neq 0$. Hence, $\overline{a/s} = 0 \Leftrightarrow \overline{a} = 0$. Next, $\overline{a}/\overline{s} = 0$ is equivalent to the existence of an s', with $\overline{s'a} = 0$. By a similar argument, one shows that this is also equivalent to $\overline{a} = 0$. □

Proposition D.5. *Let $I \subseteq \mathfrak{m}$ be a finitely generated ideal of a D-system $((A, D), \mathfrak{m})$. Then $(A_{\mathfrak{m}})_{|I_{\mathfrak{m}}} \cong (A_{|I})_{\mathfrak{m}_{|I}}$ as D-systems.*

Proof. Let M be an A-module. Then we have a natural isomorphism between $M_{\mathfrak{m}}/I_{\mathfrak{m}}M_{\mathfrak{m}}$ and $(M/IM)_{\mathfrak{m}_{|I}}$, which sends $\overline{x/s}$ to $\overline{x}/\overline{s}$. Therefore, it suffices to check that $\{\overline{d}/\overline{s} | dI \subseteq I\} = \{\overline{d/s} | (d/s)I_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}\}$, when identifying $D_{\mathfrak{m}}/I_{\mathfrak{m}}D_{\mathfrak{m}}$ with $(D/ID)_{\mathfrak{m}_{|I}}$. If $dI \subseteq I$, then clearly $(d/s)I_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$. Inversely, suppose that $(d/s)I_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$. If $a_1, \cdots, a_n$ are generators for I, then we have $s_i((d/s)a_i) \in I$, for certain s_i and all i. This means that $d'I \subseteq I$, where $d' = s_1 \cdots s_n d$, and $d/s = d'/(ss_1 \cdots s_n)$. □

Example D.9. The restriction of domain operator does not satisfy any simple commutation rule with the equivalence operator: take $A = K[x, y]$, $D = (d_x)$, $\mathfrak{m} = (x, y)$ and $I = (xy)$. Then $y \sim 0$, so that $A/\sim \cong K[x]$ and $I/\sim = 0$. However, $Pd_x(xy) = Py$, so that the set of derivations leaving invariant I is generated by xd_x. Thus, all elements of $A_{|I}$ are zero-equivalent.

Example D.10. The restriction of domain operator does not necessarily satisfy $A_{|I|J_{|I}} \cong A_{|J}$ for $I \subseteq J$. A counterexample is given by $A = K[x, y]$, $D_A = (d_x, d_y)$, $I = (xy)$ and $J = (x)$. Similarly, we do not necessarily have $((A/\sim)_{|I/\sim}/\sim)_{|(J/\sim)_{|I/\sim}/\sim} \cong (A/\sim)_{|J/\sim}$.

D.4 Zero-equivalence algorithms

In this and the next section, we shall borrow without further mention some concepts of the theory of Groebner bases (see for instance [CLO 92]). The sections D.4.1 and D.4.2 are the result of a collaboration between A. Péladan-Germa and the author (see also [PV 96]).

Let $\mathfrak{C}$ be an effective field — i.e. we have algorithms for performing the field operations of $\mathfrak{C}$ and we have an effective zero-test (see also section 9.2.1 for this terminology). A **simple effective D-system** over $\mathfrak{C}$ is a couple $((\mathfrak{A}, \mathfrak{D}), \mathfrak{m})$ which satisfies the following conditions:

ES1. $\mathfrak{A} = \mathfrak{C}[f_1, \cdots, f_k]/\mathfrak{i}$ and we have a Groebner basis $G_\mathfrak{A}$ for the ideal $\mathfrak{i}$.

ES2. $\mathfrak{D}$ is a free $\mathfrak{A}$-module with basis $d_1, \cdots, d_n$.

ES3. $(\mathfrak{A}, \mathfrak{D})$ is an effective D-ring — i.e. $\mathfrak{A}$, the action of $\mathfrak{D}$ on $\mathfrak{A}$ and the Lie bracket on $\mathfrak{D}$ are effective.

ES4. $\mathfrak{m}$ is a maximal ideal of $\mathfrak{A}$, such that $\mathfrak{A}/\mathfrak{m} \cong \mathfrak{C}$, and the evaluation mapping $\varepsilon : \mathfrak{A} \to \mathfrak{C}$ is effective.

In the remainder of this section, $((\mathfrak{A}, \mathfrak{D}), \mathfrak{m})$ is a D-system which satisfies the above requirements.

The aim of this section is to show how to compute with special functions in $\mathfrak{A}_\mathfrak{m}/\sim$. Such functions are redundantly represented by rational fractions in $\mathfrak{C}[f_1, \cdots, f_k]$, whose denominators do not evaluate to zero, whence the ring operations in $\mathfrak{A}/\sim$ can be implemented in a straightforward way. However, for the equality test, we need a zero-equivalence test in $\mathfrak{A}$. In this section, we shall provide several of such zero-equivalence tests.

D.4.1 A naive zero-equivalence algorithm

In what follows, `Groebner-basis` stands for an algorithm to compute Groebner basis in $\mathfrak{C}[f_1, \cdots, f_k]$. Given a polynomial $P \in \mathfrak{C}[f_1, \cdots, f_k]$, we will abusively denote its natural projection on $\mathfrak{A}$ by P as well. The following zero-equivalence algorithm generalizes the first algorithm from [Sh 89] to test zero-equivalence in the context of ordinary differential equations over $\mathbb{Q}$:

Algorithm `zero_equivalence_1`(P).
INPUT: A polynomial $P \in \mathfrak{C}[f_1, \cdots, f_k]$.
OUTPUT: The result of the zero-equivalence test for P.

if $\varepsilon(P) \neq 0$ **then return false**
$G :=$ `Groebner-basis`$(G_\mathfrak{A} \cup \{P\})$
while $\exists i\ \exists Q \in G \quad d_i Q \mod G \neq 0$ **do**
  if $\varepsilon(d_i Q) \neq 0$ **then return false**
  $G :=$ `Groebner-basis`$(G \cup \{d_i Q\})$
return true

Proposition D.6. *The algorithm* `zero_equivalence_1` *is correct and terminates.*

Proof. Let us first prove the correctness. It is clear that if the algorithm returns false, then P is not zero-equivalent. If the algorithm returns true, then let G be the Groebner basis at the end of the algorithm. We have $(d_i Q) \in (G)$, for each $1 \leqslant i \leqslant k, Q \in G$. Hence, (G) is stable by Δ, and $\varepsilon(P) = 0$ for each $P \in (G)$. Consequently, all elements of (G) — which contains P — are zero-equivalent.

As to the termination of `zero_equivalence_1`, the heads (see also section D.6 for this terminology) of the polynomials in the successive values of G form a strictly increasing chain of ideals. Now the termination follows from the Noetherianity of polynomial rings. □

Remark D.3. A slight modification of the algorithm allows us to exploit previous computations: since we are interested in $\mathfrak{A}/\sim$ rather than $\mathfrak{A}$ itself, we may turn $G_{\mathfrak{A}}$ into a global variable. Then setting $G^{glob} := G$ just before we return true in `zero_equivalence_1` has the effect of remembering all non trivial relations we find between the f_i in $\mathfrak{A}/\sim$.

Remark D.4. It is also possible to test several polynomials $P_1, \cdots, P_p$ for zero-equivalence at the same time. This is done by checking first whether they evaluate to zero and then replacing the line $G :=$ `Groebner-basis`$(G_{\mathfrak{A}} \cup \{P\})$ by $G :=$ `Groebner-basis`$(G_{\mathfrak{A}} \cup \{P_1, \cdots, P_p\})$.

Remark D.5. The algorithm naturally extends to the case when the initial conditions depend on parameters via the automatic case separation strategy (see chapter 8). More precisely, we may take $\mathfrak{C}$ to be a parameterized constant field $\mathfrak{C} = \mathfrak{K}(\lambda_1, \cdots, \lambda_p)$ over an effective field $\mathfrak{K}$. This means that the elements in $\mathfrak{C}$ are rational fractions in a finite number of parameters $\lambda_1, \cdots, \lambda_p$. These parameters are subject to polynomial constraints, which are either equations or inequations. The consistency of such systems of constraints can be checked by Groebner basis techniques. Moreover, no infinite loops can arise from the parameterized Groebner basis computations in `zero_equivalence_1` (see [GoDi 94], for instance).

D.4.2 An optimized zero-equivalence algorithm

In the naive zero-equivalence algorithm, we do not exploit the local character of our problem from an algebraic point of view. Now in section D.6, we show that Buchberger's algorithm can be generalized to local rings, although the computed pseudo-Groebner bases do not possess all of the nice properties of usual Groebner bases. Nevertheless, this local pseudo-Buchberger algorithm can be used instead of the usual one in `zero_equivalence_1`, yielding the following optimized zero-equivalence test:

Algorithm `zero_equivalence_2`(P).
INPUT: A polynomial $P \in \mathfrak{C}[f_1, \cdots, f_k]$.
OUTPUT: The result of the zero-equivalence test for P.

if $\varepsilon(P) \neq 0$ **then return false**
$G :=$ `Pseudo-Groebner-basis`$(G_{\mathfrak{A}} \cup \{P\})$
while $\exists i\ \exists Q \in G \quad Red(d_i Q, G) \neq 0$ **do**
  if $\varepsilon(d_i Q) \neq 0$ **then return false**
  $G :=$ `Pseudo-Groebner-basis`$(G \cup \{d_i Q\})$
return true

Proposition D.7. *The algorithm* `zero_equivalence_2` *is correct and terminates.*

Proof. The termination is proved in a similar way as before. As to the correctness, it is again clear that if the algorithm returns false, then P is not zero-equivalent. If the algorithm returns true, then let G be the pseudo-Groebner basis at the end of the algorithm. We have $Red(d_i Q, G) = 0$, for each $1 \leqslant i \leqslant k, Q \in G$. In particular, $\Delta G \subseteq (G)_{\mathfrak{A}/\mathfrak{S}}$, where $(G)_{\mathfrak{A}/\mathfrak{S}}$ denotes the ideal in $\mathfrak{A}/\mathfrak{S}$ generated by G. This implies that $(G)_{\mathfrak{A}/\mathfrak{S}}$ is stable by Δ. Since all elements of G evaluate to zero, so do all elements of $(G)_{\mathfrak{A}/\mathfrak{S}}$. Hence all elements of $(G)_{\mathfrak{A}/\mathfrak{S}}$ — which contains P — are zero-equivalent. □

The interest of this local pseudo-Buchberger algorithm is illustrated on the following example, proposed by Shackell: let $\mathfrak{A} = \mathfrak{C}[f_1, f_2, f_3, f_4]$, $\mathfrak{D} = \mathfrak{A} d_x$, $d_x f_1 = 1$, $d_x f_2 = f_2$, $d_x f_3 = 2 f_1 f_3$, $d_x f_4 = 2 f_1 f_4$, $\varepsilon(f_1) = 0$, $\varepsilon(f_2) = \varepsilon(f_3) = \varepsilon(f_4) = 1$. Then the polynomial $P = (f_1^M + f_2)(f_3 - f_4)$ is zero-equivalent since $f_3 - f_4$ is. However, the naive algorithm needs $O(M)$ steps to conclude this, whereas the new one terminates after one step: $d_x P$ is pseudo-reduced to zero by P.

D.4.3 A randomized zero-equivalence algorithm

Often, if we want to determine whether some special function — such as an exp-log function — is zero, then the initial point may be chosen randomly, provided that we avoid singularities. Now the set of points in which a non zero function vanishes, has measure zero. In this section, we show how this observation can be used to speed up the zero-equivalence algorithm, if the initial point may be chosen by the algorithm.

The idea of the algorithm is the following: an initial point is said to be **good**, if all polynomials $P \in \mathfrak{m}$ considered during the computations are actually zero-equivalent. Otherwise, the initial point is said to be **bad**. Under the hypothesis that an initial point is good, we can insert any polynomial which vanishes under evaluation into the Groebner basis $G_{\mathfrak{A}}$. Whenever 1 is in the ideal generated by the Groebner basis G, this means that the initial point is bad, and an exception is raised. This leads to the following algorithm:

Algorithm `zero_equivalence_3`(P).
INPUT: A polynomial $P \in \mathfrak{C}[f_1, \cdots, f_k]$.
OUTPUT: The result of the zero-equivalence test for P. The algorithm aborts whenever a bad initial point was chosen.

if $\varepsilon(P) \neq 0$ **then return false**
$G_{\mathfrak{A}} :=$ `Groebner-basis`$(G_{\mathfrak{A}} \cup \{P\})$
while $\exists i\ \exists Q \in G_{\mathfrak{A}}\quad d_i Q \mod G_{\mathfrak{A}} \neq 0$ **do**
  if $\varepsilon(d_i Q) \neq 0$ **then raise** "bad initial point"
  $G :=$ `Groebner-basis`$(G_{\mathfrak{A}} \cup \{d_i Q\})$
return true

Remark D.6. The Groebner basis computations may also be speeded up by inserting each polynomial $P \in \mathfrak{M}$ we encounter during these computations into $G_{\mathfrak{A}}$.

Let us now sketch in which circumstance the above algorithm applies. Assume that we are given an analytic function f defined on some Riemann surface. Assume also that we are given a sequence of points $z_1, z_2, \cdots$ in which f is defined, such that $\{z_1, z_2, \cdots\}$ is dense in some open set U. Assume finally that to each initial point z_i corresponds a simple effective D-system $((\mathfrak{A}, \mathfrak{D}), \mathfrak{m}_i)$, which specifies f in z_i (notice that $\mathfrak{A}$ and $\mathfrak{D}$ do not depend on i). Then we claim that we can test the zero-equivalence of f by the above algorithm, by running it successively in $z_1, z_2, \cdots$ until we have found a good initial point.

First, the zero-equivalence algorithm can be aborted due to the vanishing of only a finite number of non zero functions at the initial point. Now at least one of the parts of a finite partition of U is also dense in some open subset (the measure of the closure of one of the parts has to be non zero). Therefore, if there were no good initial point in the sequence $z_1, z_2, \cdots$, there would exist an open subset on which a non zero function would vanish. This is not possible.

D.4.4 Other algorithms and conclusion

A very nice zero-equivalence algorithm — quite different in spirit from those considered in the previous sections — has been given by Péladan-Germa in [Pél 95] in the context of commutating derivations $d_1, \cdots, d_k$. In a nutshell, the idea is to consider both the initial points and the initial conditions to be *variable*. Then algebraic conditions on the initial point and the initial conditions are given under which a *fixed* function in $\mathfrak{A}$ is zero-equivalent. These algebraic conditions are obtained via Ritt's classical differential elimination theory.

Another advantage of Péladan-Germa's approach is that her algorithm partially generalizes to the case of more general boundary value problems, where the initial conditions are no longer specified in a point (see [Pél 96]). However, in its full generality, this algorithm crucially depends on Kolchin's problem (see [Kol 73] and also [VdH 96c]). Nevertheless, the algorithm can be applied in several non trivial and interesting cases.

It should be noticed that certain more general boundary value problems can also be treated by the approach of this section. This is for instance the case if the quotient field of $\mathfrak{A}/\sim$ is taken as the constant field w.r.t. a new derivation. We also notice that the algorithms from this section apply in characteristic p, while Péladan-Germa's approach fails in this case.

Another question which can be raised is the following: since the zero-equivalence elements in $\mathfrak{A}$ form an ideal, there exists an ideal $\mathfrak{z}$ with $\mathfrak{A}/\sim = \mathfrak{C}[f_1, \cdots, f_k]/\mathfrak{z}$. Now can we compute a Groebner basis for $\mathfrak{z}$? This question is very hard in general, and algorithms are only known in the case of exp-log functions, using the Risch structure theorem (see [Ris 75]), and in a few other cases (see [Ch 93], [CC 85]).

After all the theoretical considerations made up till here, the reader might wonder how to implement an efficient zero-equivalence algorithm. For this purpose, several remarks of a more heuristic nature should be made.

1. In the zero-equivalence problem the hard thing is to *prove* that a function is zero-equivalent, whenever this is the case. On the contrary, it is usually easy to prove that a function is not zero-equivalent, either by evaluating some terms of the power series expansion, or by choosing a suitable initial point (when we are allowed to do so).

2. Following the previous remark, two types of zero-equivalence problems should be distinguished: those in which the initial point is fixed, and those in which the initial point may be chosen by the algorithm. In the first case, only power series expansions can be used to prove that a function is not zero-equivalent — and many terms may need be evaluated. In the second case, we would rather search for a point in which the function does not vanish; such a point is chosen at random with probability 1.

3. Many different (partial) methods may be used to prove or disprove a function to be zero-equivalent. A good final algorithm should start with cheap tests for zero-equivalence and non zero-equivalence and proceed with the more expensive ones, whenever these tests fail to decide. In particular, the time spent on tries to prove zero-equivalence should be proportional to the time spent on tries to disprove zero-equivalence.

4. In some circumstances, it is not reasonable to demand an immediate answer to a zero-equivalence quest, but we rather postpone a decision to a later moment and temporarily perform a case separation (see chapter 8).

5. In relation to 4. it should be noticed that the efficiency of successive zero-equivalence tests may crucially depend on the order in which we perform them (when applying remark D.3).

D.5 Implicit functions

D.5.1 Inversion of regular matrices

Let $(\mathfrak{A}, \mathfrak{m})$ be a simple effective D-system and let $(M_{i,j})$ be a matrix with $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant n$ and $p \leqslant n$. We say that M is **regular matrix**, if its evaluation

$$\varepsilon(M) = \begin{pmatrix} \varepsilon(M_{1,1}) & \cdots & \varepsilon(M_{1,n}) \\ \vdots & & \vdots \\ \varepsilon(M_{p,1}) & \cdots & \varepsilon(M_{p,n}). \end{pmatrix}$$

has rank p. Given such a matrix, we will now show how to compute an invertible square matrix U with entries in $\mathfrak{A}_{\mathfrak{m}}$, such that

$$MU = I_{n,p} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \dots & 0 \\ \vdots & 1 & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \dots & 0 \end{pmatrix} \tag{D.1}$$

in $\mathfrak{A}_{\mathfrak{m}}/\sim$. The algorithm proceeds by swapping rows and columns in a straightforward manner:

Algorithm `invert`(M)
INPUT: A regular n by p matrix M with entries in $\mathfrak{A}$.
OUTPUT: An invertible n by n matrix U with entries in $\mathfrak{A}_{\mathfrak{m}}$ satisfying (D.1).

$U := Id$
for $i := 1$ **to** p **do**
  let $j \geqslant i$ **be** such that $\varepsilon(M_{i,j}) \neq 0$
  `swap`($M_{i,\cdot}, M_{j,\cdot}$)
  `swap`($U_{i,\cdot}, U_{j,\cdot}$)
  $M_{i,\cdot} := (d_i g_i)^{-1} M_{i,\cdot}$
  $U_{i,\cdot} := (d_i g_i)^{-1} U_{i,\cdot}$
  for $j \in \{1, \cdots, n\} \setminus \{i\}$ **do**
    $M_{\cdot,j} := M_{\cdot,j} - M_{i,j} M_{\cdot,i}$
    $U_{\cdot,j} := U_{\cdot,j} - M_{i,j} U_{\cdot,i}$

D.5.2 Restriction of domain and resolution of implicit equations

Let $(\mathfrak{A}, \mathfrak{m})$ be a simple effective D-system of characteristic zero, such that $\mathfrak{D}_{\mathfrak{A}}$ admits $d_1, \cdots, d_n$ as a basis. Let $\mathfrak{j} = (g_1, \cdots, g_p)$ be a finitely generated ideal of $\mathfrak{A}$, such that $\varepsilon(g_1) = \cdots = \varepsilon(g_p) = 0$. The **Jacobian matrix** of $g_1, \cdots, g_p$ is defined by

$$J = \begin{pmatrix} d_1 g_1 & \cdots & d_n g_1 \\ \vdots & & \vdots \\ d_1 g_p & \cdots & d_n g_p \end{pmatrix}.$$

We say that the ideal $\mathfrak{j}$ is **regular**, if $\operatorname{rank}(\varepsilon(J)) = p$. Under this assumption, we will now show how to compute a simple effective D-system $(\mathfrak{B}, \mathfrak{n})$, such that

$$\mathfrak{B}_{\mathfrak{n}}/{\sim} \cong (\mathfrak{A}_{\mathfrak{m}}/{\sim})_{|\mathfrak{j}_{\mathfrak{m}}\not\sim}/{\sim}\ .$$

We take $\mathfrak{B} = \mathfrak{C}[f_1, \cdots, f_k]/(\mathfrak{i}, \mathfrak{j})$, so that we start by computing a Groebner basis for $(\mathfrak{i}, \mathfrak{j})$. In order to compute a basis for $\mathfrak{D}_{\mathfrak{B}}$, we first compute a matrix U with $JU = I_{n,p}$ by `invert`. Performing the base change

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} := {}^tU \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix},$$

we then reduce the general case to the case when $J = I_{n,p}$. In this case, $d_{p+1}, \cdots, d_n$ leave $\mathfrak{j}$ invariant and it is easily seen that their natural images in $(\mathfrak{A}_{\mathfrak{m}}/{\sim})_{|\mathfrak{j}_{\mathfrak{m}}\not\sim}/{\sim}$ form a basis for $(\mathfrak{A}_{\mathfrak{m}}/{\sim})_{|\mathfrak{j}_{\mathfrak{m}}\not\sim}/{\sim}$.

In practice, when we solve the equations $g_1 = \cdots = g_p$, we often want to express the solutions w.r.t. given coordinates $g_{p+1}, \cdots, g_n \in \mathfrak{A}$. In order to make this possible, we need assume that the evaluation

$$\varepsilon(J) = \begin{pmatrix} \varepsilon(d_1 g_1) & \cdots & \varepsilon(d_n g_1) \\ \vdots & & \vdots \\ \varepsilon(d_1 g_n) & \cdots & \varepsilon(d_n g_n). \end{pmatrix}$$

of the Jacobian matrix of $g_1, \cdots, g_n$ is invertible. Now compute a matrix U with $JU = Id$ by `invert`. We again reduce the general case to the case when $J = 1$ via the base change

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} := {}^tU \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

Then the natural images of $d_{p+1}, \cdots, d_n$ in $\mathfrak{B}_{\mathfrak{n}}/{\sim}$ have the desired property that

$$\begin{pmatrix} d_{p+1} g_{p+1} & \cdots & d_n g_{p+1} \\ \vdots & & \vdots \\ d_{p+1} g_n & \cdots & d_n g_n. \end{pmatrix} = Id.$$

Remark D.7. As in remark D.5, the above computations generalize in a straightforward way to the case when the initial conditions depend on parameters, using the automatic case separation strategy (see chapter 8).

D.5.3 D-algebraic power series

In this section, all D-systems $(A, \mathfrak{m})$ we consider have characteristic zero; i.e. $A/\mathfrak{m}$ has characteristic zero.

Let $((A, D), \mathfrak{m})$ be a reduced D-system, such that A is a finitely generated algebra over $C = A/\mathfrak{m}$, and D is a free A-module, which is finitely generated by pairwise commuting derivations $\partial/\partial z_1, \cdots, \partial/\partial z_n$. Then A admits a natural differential embedding ν into the ring $C[[z_1, \cdots, z_n]]$ of formal power series by

$$f \mapsto \nu(f) = \sum_{i_1,\cdots,i_n} \frac{1}{i_1! \cdots i_n!} \varepsilon \left(\frac{\partial^{i_1+\cdots+i_n}}{\partial^{i_1} z_1 \cdots \partial^{i_n} z_n} \right) z_1^{i_1} \cdots z_n^{i_n}.$$

A power series of the form $\nu(f)$ (for some D-system $((A, D), \mathfrak{m})$ which satisfies the above hypotheses) is called a **regular D-algebraic power series**.

Remark D.8. In characteristic $p > 0$, the above embedding does not exist. Actually, we may interpret elements in A as formal Borel transforms of power series in this case.

From our definition, it follows immediately that the regular D-algebraic power series form a local ring, which is stable under the partial derivations, and permutation of coordinates. Moreover, if we are given a regular D-algebraic power series $f \in C[[z_1, \cdots, z_{n+1}]]$, such that $f(0, \cdots, 0) = 0$ and $(\partial f/\partial z_{n+1})(0, \cdots, 0) \neq 0$, then by what has been said in the previous section, there exist derivations $d_1, \cdots, d_{n+1}$, such that the Jacobian matrix of $z_1, \cdots, z_n, f$ is the identity (assuming that $z_1, \cdots, z_n \in A$). It is easily checked that $d_1, \cdots, d_n$ commute for the Lie bracket, whence we have the natural embedding

$$\overline{g} \mapsto \sum_{i_1,\cdots,i_n} \frac{1}{i_1! \cdots i_n!} \varepsilon(d^{i_1} z_1 \cdots d^{i_n} z_n g) z_1^{i_1} \cdots z_n^{i_n}.$$

from $A_{|(f)}/\sim$ into $C[[z_1, \cdots, z_n]]$. This mapping sends f to zero and fixes $z_1, \cdots, z_n$. In other words, the mapping corresponds to the implicit definition of z_{n+1} by $f = 0$. Consequently, the regular D-algebraic power series form a local community (see page 9.6.1).

If $(\mathfrak{A}, \mathfrak{m})$ is a simple effective D-system with a basis of pairwise commuting derivations, the above passage from functions in $\mathfrak{A}_{\mathfrak{m}}/\sim$ to power series yields an effective way to compute with regular D-algebraic power series over $\mathfrak{C} = \mathfrak{A}/\mathfrak{m}$. In view of the algorithm from the previous section to solve implicit equation, it follows that the set of regular D-algebraic power series over $\mathfrak{C}$ forms an effective local community.

A **regular D-algebraic Laurent series** is a Laurent series f, such that $z_1^{\alpha_1} \cdots z_n^{\alpha_n} f$ is a D-algebraic power series for suitable $\alpha_1, \cdots, \alpha_n \in \mathbb{N}$. Unfortunately, we did not solve the following problem: prove or disprove that if z_i is a power series in $z_1, \cdots, z_n$ and $z_1 f$ is D-algebraic, then so is f. Consequently, we have no proof that the set of D-algebraic Laurent series forms a local community. Nevertheless, we will now define D-algebraic power and Laurent series, which do have the desired property.

Let A be as in the beginning of this section and denote by $\tilde{A}$ the set of those fractions f/s in $Q(A)$, such that there exists a power series g with $\nu(f) = g\nu(s)$. We extend the evaluation mapping on A to $\tilde{A}$ by $\varepsilon(f/s) = g(0, \cdots, 0)$, where $\nu(f) = g\nu(s)$. Clearly, $\tilde{A}$ forms a reduced local D-system over C. The natural inclusion of A into $C[[z_1, \cdots, z_n]]$ also extends to $\tilde{A}$ by $\nu(f/s) = g$, where $\nu(f) = g\nu(s)$. A power series g of the form $\nu(f/s)$ (for some A) is said to be **D-algebraic**. A Laurent series f is said to be **D-algebraic**, if $f z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is a D-algebraic power series for certain $\alpha_1, \cdots, \alpha_n \in \mathbb{N}$.

Now let $(\mathfrak{A}, \mathfrak{m})$ again be a simple effective D-system with a basis of pairwise commuting derivations. The set $\tilde{\mathfrak{A}} = \widetilde{\mathfrak{A}_{\mathfrak{m}}/\sim}$ is clearly an effective D-algebra, since it is a subfield of the field of fractions of $\mathfrak{A}_{\mathfrak{m}}/\sim$. Notice however, that we do not claim that we can test whether a given fraction $f/s \in \mathfrak{A}_{\mathfrak{m}}/\sim$ is in $\tilde{\mathfrak{A}}$ (actually, we think that such a test can be designed using the theory from chapter 10, but we have not studied this issue in detail).

Given an element $f/s \in \tilde{\mathfrak{A}}$, we can also compute its evaluation: we first compute a dominant monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\nu(s)$ by `idm` (see section 9.3.1). Then $\varepsilon(f/s) = [z_1^{\alpha_1} \cdots z_n^{\alpha_n}]f/[z_1^{\alpha_1} \cdots z_n^{\alpha_n}]s$.

Having shown that all D-system operations in $\tilde{\mathfrak{A}}$ can be carried out effectively, the algorithm to solve implicit equations from section D.5.2 naturally generalizes, if we replace $\mathfrak{A}_{\mathfrak{m}}/\sim$ by $\tilde{\mathfrak{A}}$. In particular, the sets of D-algebraic power series resp. Laurent series over $\mathfrak{C}$ are both effective local communities.

D.6 A local pseudo-Buchberger algorithm

This section is the result of a collaboration between A. Péladan-Germa and the author (see also [PV 96]).

Let $\mathfrak{A} = \mathfrak{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates over an effective field $\mathfrak{C}$ of constants, and $\mathfrak{S}$ be an effective multiplicative subset of $\mathfrak{A}$ — that is, provided with an effective membership test. We present here a method to test whether a given polynomial $P \in \mathfrak{A}$ belongs to the ideal generated by polynomials $Q_1, \ldots, Q_s$ in the quotient ring $\mathfrak{A}/\mathfrak{S}$. We only give a weak membership test in the sense that $P \in (Q_1, \ldots, Q_s)_{\mathfrak{A}/\mathfrak{S}}$, whenever the algorithm returns true. However, in the case of a negative response, P might still be in $(Q_1, \ldots, Q_s)_{\mathfrak{A}/\mathfrak{S}}$. Nevertheless, for the application in section D.4.2 such a weak membership test is sufficient.

Actually, our algorithm is based on the heuristic idea that the exploitation of local information should accelerate Buchberger's algorithm. Unfortunately, the pseudo-Groebner bases we compute does not have all the theoretical properties of classical Groebner bases. However, up to our knowledge, no complete membership test has been given yet in the case of a general effective multiplicative set $\mathfrak{S}$. Only in some particular cases, Mora's tangent cone algorithm, and A. Logar's algorithms give complete membership tests (see [MPT 92], [Lo 87]).

D.6.1 Pseudo-reduction

Let $\mathfrak{A} = \mathfrak{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates over an effective field $\mathfrak{C}$. We use the pure lexicographical order on monomials, with $x_1 < \cdots < x_n$. Let $\mathfrak{S} \supseteq \mathfrak{C}^\star$ be a multiplicative subset of $\mathfrak{C}$ with an effective membership test.

In order to compute "pseudo-bases" of ideals of $\mathfrak{A}/\mathfrak{S}$, we use a classical reduction-completion approach. The keystone of our method lies in the non-classical definitions of the head $H(P)$ and the leading-coefficient $C(P)$ of non-zero polynomials P: they are inspired both by Ritt-Wu's work and Buchberger's terminology.

Each non constant polynomial P in $\mathfrak{A}$ can be written $P = I_P x_P^{d_P} + R_P$, where x_P is the greatest indeterminate involved in P, and d_P the highest order of P with respect to x_P. I_P is usually called the **initial** of P. Now we define $H(P)$ and $C(P)$ for non-zero polynomials:

- if $P \in \mathfrak{S}$ then $H(P) = 1$ and $C(P) = P$;
- if $P \notin \mathfrak{S}$ and $I_P \in \mathfrak{S}$ then $H(P) = x_P^{d_P}$ and $C(P) = I_P$;
- if $P \notin \mathfrak{S}$ and $I_P \notin \mathfrak{S}$ then $H(P) = x_P^{d_P} H(I_P)$ and $C(P) = C(I_P)$.

Example D.11. Let $\mathfrak{S}$ be the set of polynomials that do not vanish at $x_1 = \ldots = x_n = 0$. If $P = (x_1 + 1)x_2 + x_1$, then $H(P) = x_2$ and $C(P) = x_1 + 1$. Now if $P = ((x_1 + 1)x_2 + x_1)x_3^2 + x_3 x_2$, then $H(P) = x_2 x_3^2$ and $C(P) = x_1 + 1$.

Suppose $Q \neq 0$, $Q \notin \mathfrak{S}$ and $P \neq 0$. We say that P is **reducible** with respect to Q if $H(P)$ is divisible by $H(Q)$. In this case, write $P = UH(Q) + V$, where $U, V \in \mathfrak{A}$, and no monomial appearing in V is divisible by $H(Q)$. P is then **elementary reduced** to $red(P, Q) = C(Q)P - UQ$ by Q. If $Q \in \mathfrak{S}$, then P is reducible with respect to Q and $red(P, Q) = 0$. It can be easily checked, although this is a little technical, that $H(red(P, Q)) < H(P)$ ($H(0) = -\infty$ by convention). Repeating the elementary reduction of P by Q, that is

$$P \to P_1 = red(P, Q) \to red(P_1, Q) \to \cdots,$$

we end up with a polynomial R such that $H(Q)$ does not divide $H(R)$ or $R = 0$. This process stops because the heads of the intermediate polynomials strictly decrease. This polynomial R is called the **reduction** of P by Q and is denoted by $Red(P, Q)$. More generally, we can reduce P by a set E of polynomials by reducing P by $Q \in E$ as long as we can. Although the result R of this procedure is not necessarily unique, we will abusively denote $R = \mathit{Red}(P, E)$. Note that R belongs to the ideal $(P, E)_{\mathfrak{A}/\mathfrak{S}}$ generated by P and E in $\mathfrak{A}/\mathfrak{S}$ and if $R = 0$, then $P \in (E)_{\mathfrak{A}/\mathfrak{S}}$.

Let P, Q be nonzero elements of $\mathfrak{C}[x_1, \ldots, x_n]$. Let i be the highest index such that $C(P)$ and $C(Q)$ are both in $\mathfrak{A}_i = \mathfrak{C}[x_1, \cdots, x_{i-1}]$. We write $P = C_i(P)H_i(P)+R(P)$, where $H_i(P)$ is highest monomial occurring in P, when considered as a polynomial in $x_i, \cdots, x_n$ with coefficients in $\mathfrak{A}_i$. Similarly, we write $Q = C_i(Q)H_i(Q) + R(Q)$. Then the **S-polynomial** of P and Q is defined by

$$SPol(P, Q) := \frac{C_i(Q)H_i(Q)}{\gcd(H_i(P), H_i(Q))}P - \frac{C_i(P)H_i(P)}{\gcd(H_i(P), H_i(Q))}Q.$$

This definition enables us to assert that $H(SPol(P, Q)) < \mathrm{scm}(H(P), H(Q))$. Note also that $Spol(P, Q) \in (P, Q)$ and *a fortiori* $Spol(P, Q) \in (P, Q)_{\mathfrak{A}/\mathfrak{S}}$.

D.6.2 The algorithm

We now apply Buchberger's algorithm (see [CLO 92], [Buch 65]) with our alternative definitions of heads, leading coefficients, reduction, and S-polynomials. We recall hereafter a compact but non optimized version of this algorithm.

Algorithm `Pseudo-Groebner-basis`(E)
INPUT: A finite set E of non zero polynomials in $\mathfrak{A}$.
OUTPUT: A pseudo-Groebner basis G of the ideal generated by E in $\mathfrak{A}/\mathfrak{S}$.

$G := E$
repeat
  $G' := G$
  for each $P \in G'$ **do**
    $P := \mathit{Red}(P, G - \{P\})$
    if $R \neq 0$ **then** $G := G \cup \{R\}$
  for each pair $P \neq Q$ in G' **do**
    $R := \mathit{Red}(SPol(P, Q), G')$
    if $R \neq 0$ **then** $G := G \cup \{R\}$
until $G = G'$

The ideals generated by the heads of the elements of the successive values of G form a strictly increasing sequence of ideals, whence the algorithm terminates. The subsets E and G of $\mathfrak{A}/\mathfrak{S}$ generate the same ideal $I_{\mathfrak{A}/\mathfrak{S}}$. Indeed, we only insert elements that are already in $(E)_{\mathfrak{A}/\mathfrak{S}}$ into G. G is not a Groebner basis, but has the property that if P is in $\mathfrak{A}$ and $\mathit{Red}(P, G) = 0$, then $P \in I_{\mathfrak{A}/\mathfrak{S}}$. The computation of G enables us to quickly extract much information about $I_{\mathfrak{A}/\mathfrak{S}}$, without obtaining a complete description of $I_{\mathfrak{A}/\mathfrak{S}}$. Notice that if G contains a polynomial in $\mathfrak{S}$, then $I_{\mathfrak{A}/\mathfrak{S}}$ is trivial, and every polynomial in $\mathfrak{A}$ is reduced to zero by G.

Our algorithm reduces to the usual Buchberger algorithm if $\mathfrak{S} = \mathfrak{C}^*$; that is the reason why we call G a pseudo-Groebner basis rather than a pseudo-Ritt basis.

D.7 References

[Buch 65] B. Buchberger. Ein Algorithmus zum auffinden der Basiselemente des Restklassenringes nach einem null-dimensionalen Polynomideal. *PhD. thesis, University of Innsbruck.*

[Bui 92] A. Buium. Differential algebraic groups of finite dimension. *Lecture Notes in Mathematics 1506, Springer-Verlag.*

[CC 85] G.W. Cherry, B.F. Caviness. Integration in finite terms with special functions. *A progress report, Proc. EUROSAM 84, Lect. Notes in Comp. Sc. 174, Springer-Verlag (p. 351-359).*

[Ch 83] G.W. Cherry. Algorithms for integrating elementary functions in terms of logarithmic integrals and error functions. *PhD. Thesis, Univ. of Delaware.*

[CLO 92] D. Cox, J. Little, D. O' Shea. *Ideals, varieties and algorithms.* Springer-Verlag, New York.

[DL 89] J. Denef, L. Lipshitz. Decision problems for differential equations. *Journal of Symbolic Logic, Vol 54(3) (p. 941- 950).*

[Fau 94] J.-C. Faugère. Résolution de systèmes algébriques. *PhD. thesis, Univ. Paris VI.*

[GHL 87] S. Gallot, D. Hulin, J. Lafontaine. *Riemannian geometry.* Springer-Verlag.

[GoDi 94] T. Gomez-Diaz. Quelques applications de l'évaluation dynamique. *PhD. Thesis, Univ. of Limoges, France.*

[Jan 20] M. Janet. *Systèmes d'équations aux dérivées partielles.* J. de Maths, Series 8, Vol. 3.

[Kap 76] I. Kaplanski. *An introduction to differential algebra.* Hermann (2-nd ed.).

[Kol 73] E.R. Kolchin. *Differential algebra and algebraic groups.* Academic press, New-York.

[Krei 64] H.F. Kreimer. The foundations for an extension of differential algebra. *Transactions A.M.S. 111 (p 482-492).*

[Li 1837] J. Liouville. Mémoire sur la classification des transcendantes, et sur les racines de certaines équations en fonction finie explicite des coefficients. *J. Math. Pures et Appl. 2 (p. 56-104).*

[Li 1838] J. Liouville. Suite du mémoire sur la classification des transcendantes, et sur les racines de certaines équations en fonction finie explicite des coefficients. *J. Math. Pures et Appl. 3 (p. 523-546).*

[Lo 87] A. Logar. Constructions over localizations of rings. *Le Mathematiche, Vol. 42 (p. 131-149).*

[MPT 92] T. Mora, G. Pfister, C. Traverso. An introduction to the tangent cone algorithm. *Advances in computing research (1992), vol. 6 (p. 199-270).*

[NiWe 82] W. Nichols, B. Weisfeiler. Differential formal groups of J.F. Ritt. *Amer. Jour. Math. 104(5) (p 943-1005).*

[Pél 95] A. Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948.*

[Pél 97] A. PÉLADAN-GERMA. Tests effectifs de nullité dans des extensions d'anneaux différentiels. *PhD. Thesis, École Polytechnique, France.*

[PV 96] A. PÉLADAN-GERMA, J. VAN DER HOEVEN. Un algorithme de Buchberger local. *Submitted to C.R.A.S.*

[Riq 10] RIQUIER. *Les systèmes d'équations aux dérivés partielles.* Gauthiers Villars, Paris.

[Ris 75] R.H. RISCH. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math. 4, 101 (p. 743-759).*

[Ritt 50] J.F. RITT. *Differential algebra.* Amer. Math. Soc, New-York.

[Sh 89] J. SHACKELL. Zero-equivalence in function fields defined by algebraic differential equations. *Trans. of the AMS, Vol. 336, Number 1, pp 151-171.*

[VdH 96c] J. VAN DER HOEVEN. Differential and mixed differential difference equations from the effective viewpoint. *Research report LIX/RR/96/11, Ecole polytechnique, France.*

[Wu 87] W.T. WU. A Zero structure theorem for polynomial equation solving and its applications. *Proceedings of ISSAC'88, Roma, Springer Verlag.*

[Zeil 82] D. ZEILBERGER. Sister Celine's technique and its generalizations. *Journal of mathematical analysis and applications 82, p 114-145.*

Conclusion

Étant arrivé au terme de la thèse, on peut toujours se poser les questions suivantes :

- Quelles sont les applications ?

- Peut-on simplifier les résultats obtenus, les formuler ou expliquer autrement, ou les réorganiser d'une façon différente ?

- Comment aller plus loin ?

En ce qui concerne les applications, j'avoue pleinement que cela n'a pas encore vraiment été à l'ordre du jour. La cause en revient partiellement à ce que nous avons considéré jusqu'ici surtout des comportements fortements monotones — une restriction sévère pour des applications en physique. Néanmoins, cette thèse peut d'ores et déjà avoir des applications significatives en combinatoire et en analyse d'algorithmes, car dans ces domaines, les comportements fortement monotones sont plus fréquents. Le lecteur pourra se rapporter à [FSZ 89], [Sal 91], [Sor 90] et [FS 96] pour plus de détails.

Considérons maintenant la deuxième question. Avec un peu de recul, il m'est apparu à l'issue de cet épais travail, que si certaines parties de la thèse étaient à refaire, alors nous nous y prendrons différemment. Premièrement, nous sommes de plus en plus convaincus que les deux parties de la thèse auraient pu faire l'objet de deux publications différentes : le choix de travailler avec des transséries bien ordonnées est surtout justifié d'un point de vue théorique. En effet, ceci permet d'étudier des équations fonctionnelles bien plus générales que les équations différentielles algébriques (voir aussi plus bas). En revanche, d'un point de vue pratique, c'est surtout les applications au calcul différentiel qui intéressent les mathématiciens et informaticiens.

Quoi qu'il en soit, il faut noter que la restriction aux transséries réticulées n'apporte pas autant de simplifications que l'on pourrait penser dans les chapitres 3, 4 et 5. En effet, on ne pourra se débarrasser des récurrences transfinies que dans le calcul des monômes dominants des solutions distinguées, mais pas dans la construction de ces solutions elles-mêmes. En outre, dans ce cadre il faut vérifier que la propriété des supports réticulés se préserve lors de la résolution d'équations différentielles algébriques (voir la section 12.3).

D'autre part, les transséries réticulées interviennent essentiellement dans l'étude de singularités plus générales que celles qui interviennent dans la résolution d'équations différentielles linéaires à coefficients dans les séries. En effet, c'est l'extension la plus simple du corps des séries qui a les propriétés de clôture appropriées pour une étude asymptotique des singularités qui interviennent lors de la résolution d'équations différentielles algébriques. Donc toute restriction supplémentaire sur le type de transséries considérées réduirait notre théorie à une théorie plus « banale » et porterait essentiellement atteinte au type de résultats obtenus.

Mais venons en maintenant à la troisième question, car nous avons surtout imaginé notre thèse comme ouvrant la porte vers des théories plus générales, inaccessibles jusqu'alors. Nous diviserons la présentation en trois parties : d'abord nous avons obtenu quelques « résultats », que nous avons pas eu le temps d'écrire en détail et qui étaient originalement destinés à faire partie de la thèse. Deuxièmement, nous prévoyons quelques extensions de la théorie, sur lesquelles nous avons déjà des idées assez précises. Enfin, nous rêverons un peu à des extensions plus lointaines, mais pas pour autant farfelues.

Extensions dans un avenir proche.

1. Dans [VdH 95a], nous avons donné une première approche visant à utiliser les méthodes de cette thèse pour calculer les signes de constantes exp-logs comme

$$c = e^{e^{10^{100}}+10^{-200}} - e^{e^{10^{100}}+e^{10^{99}}},$$

et plus généralement pour obtenir des renseignements sur leur ordre de grandeur. Ceci est en fait un problème plus difficile que le développement de fonctions exp-logs, car la substitution d'une valeur très grande à x dans une fonction exp-log $f(x)$ nécessite en particulier une connaissance précise du comportement asymptotique de f. Pour résoudre ce problème, nous avons introduit dans [VdH 95a] des « développements asymptotiques avec estimation de l'erreur ». Après, nous avons su raffiner et simplifier ces résultats, mais ceci reste à être développé complètement.

2. Dans la partie B de cette thèse nous supposons fréquemment l'existence d'un oracle pour tester si un système d'équations et inégalités exp-logs sur les constantes admet une solution. En généralisant les techniques évoquées au point de 1. et en employant les techniques d'élimination exp-logs du chapitre 11, nous préconisons une réduction de ce problème au problème du test à zéro pour les « constantes élémentaires » de Richardson, qui a été réduit à la conjecture de Schanuel dans [Rich 95]. Nous avons récemment appris dans [Mar 96] que ce problème peut-être résolu par d'autres techniques (voir [Wil *], [MW *]). Quoi qu'il en soit, notre approche peut conduire à des algorithmes efficaces pour ce problème.

3. On peut regretter le résultat un peu vague de l'existence d'un algorithme théorique pour résoudre n'importe quelle équation différentielle algébrique dans les transséries dans le chapitre 11. Nous avons aussi un résultat, qui bien que moins

fort, est plus frappant : si P est un polynôme différentiel à coefficients dans $\mathbb{T}$, et $f < g$ sont telles que $P(f) < 0$ et $P(g) > 0$, alors il existe un $h \in \mathbb{T}$ avec $f < h < g$ telle que $P(h) = 0$. On pourrait dire que $\mathbb{T}$ est D-réellement clos. Ce théorème se montre naturellement à l'aide des techniques du chapitre 5.

4. Bien que ceci introduise quelques difficultés techniques supplémentaires, la théorie des chapitres 4 et 5 se généralise au cas d'équations différentielles aux différences algébriques. Ici, on exige que les opérateurs aux différences soient des compositions à droite par des transséries d'exponentialité zéro. Le résultat mentionné au point 3. se généralise également à ce cas. En fait, ces résultats ont été à l'origine de l'emploi systématique des transséries bien ordonnées dans la partie B de cette thèse, mais leur rédaction n'a malheureusement pas pu aboutir à temps.

5. Nous voulons également implanter une partie plus importante des algorithmes de cette thèse. Un problème majeur que l'on rencontre ici, est que la stratégie de la séparation automatique des cas ne puisse pas s'implanter de façon naturelle dans la plupart des langages de programmation actuels. Durant la préparation de cette thèse, nous avons consacré beaucoup de temps à la mise au point d'une extension de C++ pour remédier à ce (et d'autres) problème. Nous espérons transformer nos idées sur ce sujet dans un logiciel concret.

Extensions dans un avenir moyen.

6. Nous nous sommes récemment rendu compte que notre algorithme de résolution d'équations différentielles algébriques peut être interprété d'une façon différente, en l'intégrant dans le cadre de l'algorithme de développement de transséries multivariées du chapitre 11. En effet, considérons $f, f^{\langle 1\rangle}, \cdots, f^{\langle r\rangle}$ et x comme des transséries génériques, avec l'ordre d'élimination $f > f^{\langle 1\rangle} > \cdots > f^{\langle r\rangle} > x$. Initialement, les $f^{\langle i\rangle}$ sont les dérivées logarithmiques, itérées r fois, de f. Mais les $f^{\langle i\rangle}$ peuvent être raffinés durant l'exécution. Pour résoudre l'équation différentielle, on élimine f en utilisant l'algorithme du chapitre 11 avec les changements suivants :

Premièrement, on impose toujours les contraintes $f \ggg \cdots \ggg f^{\langle r\rangle}$. Deuxièmement, lorsque l'on raffine

$$f^{\langle i\rangle} = \text{ц}(x)(\varphi(x) + \widetilde{f^{\langle i\rangle}})\ (\widetilde{f^{\langle i\rangle}} \prec\!\!\prec 1),$$

avec $\varphi(x) \asymp 1$, on vérifie que les dérivés logarithmiques itérées de $\text{ц}(x)$ vérifient bien les contraintes imposés sur les dérivés logarithmiques de $f^{\langle i\rangle}$. Après, on « oublie » $f^{\langle i+1\rangle}, \cdots, f^{\langle r\rangle}$ et on les remplace par des $\widetilde{f^{\langle i+1\rangle}}, \cdots, \widetilde{f^{\langle r\rangle}}$ « neufs », qui correspondent aux dérivés logarithmiques itérés de $\widetilde{f^{\langle i\rangle}}$. Troisièmement, lorsque l'on impose la contraite ci-dessus, on sépare les cas où $\text{ц}(x)\widetilde{f^{\langle i\rangle}} \succeq\!\!\succeq 1$ et $\text{ц}(x)\widetilde{f^{\langle i\rangle}} \prec\!\!\prec 1$. Dans le dernier cas, on impose la contrainte

$$f^{\langle i-1\rangle} \asymp \exp(\int \text{ц}(x)\varphi(x)).$$

Quatrièmement, si l'on veut imposer une contrainte de la forme

$$(f^{\langle i\rangle})^{\alpha_i}\cdots(f^{\langle r\rangle})^{\alpha_r}\asymp\varphi(x)$$

on le transforme en une contrainte équivalente qui ne fait pas intervenir $f^{\langle i+1\rangle},\cdots,$ $f^{\langle r\rangle}$, en utilisant le fait que $f^{\langle i\rangle}\ggg\cdots\ggg f^{\langle r\rangle}$, et des propriétés différentielles. Enfin, on prend soin à faire les mouvements montants nécessaires pour rester dans le cas purement exponentiel.

7. L'avantage de l'approche décrite ci-dessus est qu'elle se généralise à des systèmes d'équations plus générales, comme :

- Des systèmes d'équations différentielles algébriques aux dérivés partielles. Notons que dans ce cas, le rôle des « constantes » sera joué par des fonctions arbitraires en moins de variables, vérifiant des équations aux dérivés partielles.
- Des systèmes d'équations différentielles ordinaires non nécessairement algébriques. Dans le cas extrême, on fera intervenir simultanément la dérivation et l'exponentation.
- Des mélanges de ces deux choses.

8. Dans le cas où l'on considère des équations différentielles non algébriques de la forme

$$\sum_{\alpha_0,\cdots,\alpha_r\in\mathbb{N}}P_{\alpha_0,\cdots,\alpha_r}f^{\alpha_0}\cdots(f^{(r)})^{\alpha_i},$$

il est important pour des raisons d'effectivité d'avoir une théorie d'élimination analogue à la théorie de Ritt pour ce genre d'équations. En supposant que les coefficients sont « sympathiques », nous pensons que l'on peut développer une telle théorie, justement à l'aide des techniques exposées dans cette thèse.

9. On peut enrichir la classe des expressions $\mathfrak{L}$-exp-logs avec la composition à gauche par des fonctions réelles analytiques sur des intervalles fermés. D'un point de vue théorique, cette extension est facile, mais d'un point de vue effectif, il faut trouver des classes de ce genre de fonctions réelles qui se comportent bien.

10. On peut donner le développement du n-ième zéro positif de $\tan x = x$, pour $x\to\infty$. On bénéficie ici du fait que les singularités de la fonction tangante se trouvent dans les points $\pi/2+\pi\mathbb{Z}$. Or considérons maintenant la fonction

$$f(x)=\sin x+\sin ex-1-x^{-1}.$$

Clairement, on peut à nouveau exprimer le n-ième zéro positif de f en fonction du n-ième zéro de $\sin x+\sin ex-1$. Plus généralement, nous pouvons considérer les fonctions

$$f_{a_1,\cdots,a_n,b_1,\cdots,b_n,\lambda}(x)=\sin(a_1x+b_1)+\cdots+\sin(a_nx+b_n)-\lambda,$$

et les suites $N_{a_1,\cdots,a_n,b_1,\cdots,b_n,\lambda}$ qui donnent le n-ième zéro de $f_{a_1,\cdots,a_n,b_1,\cdots,b_n,\lambda}$ (quand il y a une infinité de zéros). Alors se posent plusieurs questions. Quelles relations algébriques sont vérifiées par ces suites ? Quelles sont les relations avec les développements en fractions continues ? Quelles sont les relations avec le groupe linéaire sur $\mathbb{Q}^n$?

11. L'algorithme de résolution asymptotique d'équations différentielles algébriques se généralise vraisemblablement au cas des équations aux différences algébriques, si on ne recherche que les solutions réticulées. Ceci tient au faite que la théorie de Ritt s'adapte à ce cas (voir par exemple [VdH 96c]), et que l'on peut se ramener au cas où $f(g(x))$ se développe par la formule de Taylor dans le cas réticulé.

12. Il est aussi possible de donner des algorithmes pour calculer des transformées intégrales. Écalle a donné des formules pour les transformées de Borel et Laplace formelles pour les transséries dans [Éc 92], et Salvy a donné des algorithmes dans des classes plus restreintes (voir [Sal 91]), mais il prouve la validité analytique des résultats. Il reste à étendre ses travaux à des classes plus étendues de fonctions.

Discussion finale.

Grossièrement parlant, on peut résoudre les équations différentielles de trois façons différentes:

- Résolution numérique.
- Résolution asymptotique.
- Recherche des solutions sous forme close.

Nous pensons que sur long terme, ces trois méthodes de résolution se mélangerons. De plus, notre ultime espoir et conviction est qu'au moins dans le cas des équations différentielles ordinaires, il existe une théorie de résolution asymptotique complète.

Dans notre vision des choses, des algorithmes de résolution asymptotique seront utiles d'un point de vue numérique de trois façons. Premièrement, ils permettrons de déterminer dans quelles régions de l'espace des méthodes de résolution numérique classiques échouerons à cause de l'imprécision des calculs. Deuxièmement, ils permettront de savoir quand une résolution numérique est *possible* d'un point de vue pratique (penser à la détermination du signe de $\sin 10^{10^{10^{10}}}$!). Finalement, dans des régions proches de singularités violentes, on pourra appliquer la théorie de resommation numérique pour faire des calculs numériques fiables et efficaces.

Des techniques asymptotiques pourraient aussi s'avérer utiles lorsque l'on cherche des solutions sous forme close. Par exemple, étant données des transséries $f_1, \cdots, f_n$, on peut donner des algorithmes efficaces pour déterminer les relations linéaires, $\mathbb{Q}$-linéaires, voire algébriques qu'elles vérifient. En effet, les propriétés asymptotiques de $f_1, \cdots, f_n$ servent à « diriger » ce genre de calculs, tout comme la réduction modulo p peut par exemple servir pour diriger les calculs de bases de Groebner (voir

[Fau 94]). La détermination des relations $\mathbb{Q}$-linéaires est par exemple importante dans l'algorithme d'intégration formelle de Risch (voir [Ris 75]) et dans le test à zéro de Richardson (voir [Rich 95]) pour les constantes exp-logs (voir le point 1.). Des techniques asymptotiques ont aussi été appliquées avec succès à la factorisation d'opérateurs différentielles linéaires (voir [VH 96]).

D'un point de vue logique, il y a aussi quelques remarques intéressantes à faire. Pour un numéricien acharné, un algorithme est correct quand il marche dans tous les cas auquels il veut l'appliquer. Pour un mathématicien puriste, un algorithme doit être accompagné d'une preuve de correction et de terminaison à partir des axiomes de Zermelo-Fraenkel. Or, les deux approches ont des avantages et des inconvénients.

Le numéricien a raison de se fier à son expérience : si un algorithme lui rend systématiquement service pour résoudre ses problèmes, une preuve de correction est superflue. Mais le numéricien doit disposer d'un grand savoir faire pour *juger* s'il a effectivement résolu son problème initial. Et est-ce qu'il sera toujours capable de déterminer « ce qui cloche » quand son algorithme cesse de donner les bonnes réponses ?

Le mathématicien a raison de faire confiance à des algorithmes dont il a su démontrer la validité : à moins que les axiomes de Zermelo-Fraenkel soient contradictoires, son algorithme produira toujours le bon résultat. Mais le mathématicien puriste se trouve souvent « disconnecté » de la réalité : est-ce qu'il a déjà *appliqué* son algorithme pour résoudre un problème concret ? L'expérience montre que certains problèmes décidables, comme la détermination du signe de $\sin 10^{10^{10^{10}}}$ s'avèrent indécidables dans la pratique, tandis que des problèmes indécidables en analyse peuvent souvent se « résoudre » d'un point de vue pratique !

On peut alors envisager de réconcilier les deux approches et de rechercher une « déontologie » commune pour le numéricien et le mathématicien. Cette déontologie doit être régie par des axiomes précis, qui permettrons au mathématicien de valider ses algorithmes. Ces règles doivent aussi être orientées vers la pratique, pour que les algorithmes développés soient utiles pour le numéricien.

Dans le cadre plus restreint de cette thèse, une telle déontologie consistera par exemple à supposer la conjecture de Schanuel. Dans ce cas, le mathématicien sera content de pouvoir affirmer que les algorithmes dans la partie B de cette thèse terminent. De plus, on n'a toujours pas trouvé de contre-exemples à cette conjecture, tout comme on n'est jamais arrivé à une contractiction à partir des axiomes de Zermelo-Fraenkel. Et il n'est *a priori* pas moins raisonnable de faire confiance à la conjecture de Schanuel qu'à ces derniers axiomes.

Cependant, le numéricien n'est pas tout à fait satisfait encore, car le test à zéro pour les constantes exp-logs de Richardson coûte souvent très cher; en particulier, il n'y a pas de bornes de complexité. On est alors amené à la question de savoir si la conjecture de Schanuel peut être remplacée par une autre conjecture, qui servira alors de « règle de déontologie ». Proposons en une, dans le cadre restreint des constantes exp-logs. Pour tout entier $N \geqslant 3$, soit E_N la classe d'expressions construites à partir de $1, +, -, \cdot, /, \exp$ et $\log|\cdot|$, telle que la valeur absolue de toute sous-expression stricte s'évalue vers une valeur entre N^{-1} et N. Nous notons par $\chi(f)$ la taille d'une telle expression f. Nous proposons alors:

Conjecture. *Pour tout $N \geqslant 3$, il existe une constante C_N, telle que pour toute expression f dans E_N, il suffit d'évaluer f à une précision de $C_N^{\chi(f)}$ décimales, pour savoir si elle s'évalue à zéro.*

Si on remplace la fonction $C_N^{\chi(f)}$ par une fonction calculable quelconque, la conjecture est impliquée par la conjecture de Schanuel. D'une autre côté, on pourra peut-être remplacer $C_N^{\chi(f)}$ par $\chi(f)^{C_N}$. Dans la nouvelle version de l'algorithme mentionné dans le point 1., il sera possible de remplacer la conjecture de Schanuel par la conjecture ci-dessus.

Il est clair que pour des applications plus générales, il sera intéressant de faire une conjecture plausible, dans le style de notre conjecture, mais pour des classes plus vastes de constantes.

Bibliography

[AMR 89] M.E. ALONSO, T. MORA, M. RAIMONDO. Computing with algebraic series. *Proc. of ISSAC '89, A.C.M. (p. 101-111).*

[AS 64] M. ABRAMOWITZ, I.A. STEGUN. *Handbook of mathematical functions.* Nat. Bur. Stand., Washington D.C..

[AtMac 69] M.F. ATIYAH, I.G. MAC DONALD. *Introduction to commutative algebra.* Addison-Wesley.

[Bal 94] W. BALSER. *From divergent power series to analytic functions.* Lect. Notes in Math. 1582, Springer-Verlag.

[BB 1856] C.A. BRIOT, J.C. BOUQUET. Propriétés des fonctions définies par des équations différentielles. *Journal de l'École Polytechnique 36 (p. 133-198).*

[Ber 1697] J. BERNOULLI. Principia calculi exponentialium seu percurrentium. *Acta eruditorum (p. 125-133).*

[Bj 79] J.E. BJÖRK. *Rings of differential operators.* North Holland Publ. Comp.

[BK 75] R.P. BRENT, H.T. KUNG. $O((n \log n)^{3/2})$ algorithms for composition and reversion of power series. *In Analytic Computational Complexity, J.F. Traub Ed. Proc. of a symposium on analytic computational complexity held by Carnegie-Mellon University.*

[BK 78] R.P. BRENT, H.T. KUNG. Fast algorithms for manipulating formal power series. *Journal of the ACM 25 (p. 581-595).*

[BLOP 95] F. BOULIER, D. LAZARD, F. OLLIVIER, M. PETITOT. Représentation du radical d'un idéal différentiel de type fini. *Proc. ISSAC 1995, Montreal.*

[Bor 28] É. BOREL. *Leçons sur les séries divergentes.* Gauthiers Villars, Paris, 2-nd Ed., Reprinted by Jacques Gabay.

[Bos 96] H.J.M. BOS. Johann Bernoulli on exponential curves ca. 1695. Innovation and habituation in the transition from explicit constructions to implicit functions. *Nieuw archief voor wiskunde serie 4, 14(1) (p. 1-19).*

[Boul 94] F. BOULIER. Étude et implantation de quelques algorithmes en algèbre différentielle. *PhD. Thesis, University of Lille I, France.*

[Bour 61] N. BOURBAKI. *Éléments de mathématiques chap.V: Fonctions d'une variable réelle.* Hermann, 2-nd ed. (Appendix p. 36-55).

[Br 75] L.E.J. BROUWER. *Collected Works.* North-Holland, Amsterdam.

[Br 76] R.P. BRENT. Fast multiple-precision evaluation of elementary functions. *Journal of the Assoc. for Comp. Machinery 23(2) (p. 242-251).*

[Br 92] B.L.J. Braaksma. Multisummability of formal power series solutions of non linear meromorphic differential equations. *Ann. Inst. Fourier, Grenoble 42 (p. 517-540).*

[Br 58] N.G. de Bruijn. *Asymptotic methods in analysis.* North-Holland Publ. Co. Amsterdam.

[BrP 93] M. Bronstein, M. Petrovšek. On Ore rings, linear operators and factorization. *Technical Report 200, ETH Zürich.*

[Buch 65] B. Buchberger. Ein Algorithmus zum auffinden der Basiselemente des Restklassenringes nach einem null-dimensionalen Polynomideal. *PhD. thesis, University of Innsbruck.*

[BW 87] W.H. Burge, S.M. Watt. Infinite structures in Scratchpad II. *Proc. of the Europ. Conf. on Comp. Alg. EUROCAL'87 (J.H. Davenport), Lect. Notes in Comp. Sc. 378, Springer-Verlag (p. 138-148).*

[Can 1899] G. Cantor. Sur les fondéments de la théorie des ensembles infinis. *Mémoires de la Société des Sciences physiques et naturelles de Bordeaux. Reprinted by Jacques Gabay.*

[Cano 93] J. Cano. An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms. *Ann. Institut Fourier de Grenoble 43(1) (p. 125-142).*

[CC 85] G.W. Cherry, B.F. Caviness. Integration in finite terms with special functions. *A progress report, Proc. EUROSAM 84, Lect. Notes in Comp. Sc. 174, Springer-Verlag (p. 351-359).*

[Ch 83] G.W. Cherry. Algorithms for integrating elementary functions in terms of logarithmic integrals and error functions. *PhD. Thesis, Univ. of Delaware.*

[Ch 86] A.L. Chistov. Polynomial complexity of the NEwton-Puiseux algorithm. *Proc. Symp. Math. FOund. Comput. Sci. Bratislava. Springer Lect. Notes in Comp. Sc. 233 (p. 247-255).*

[CP 78] B.F. Caviness, M.J. Prelle. A note on algebraic independence of logarithmic and exponential constants. *SIGSAM Bulletin 12 (p. 18-20).*

[Chyz 95] F. Chyzak. Formal manipulations of linear operators and holonomic calculations. *D.E.A. report.*

[CLO 92] D. Cox, J. Little, D. O' Shea. *Ideals, varieties and algorithms.* Springer-Verlag, New York.

[Cohn 65] P.M. Cohn. *Universal algebra.* M.I.A. 6, Reidel, Dordrecht.

[Col 75] G.E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Proc. 2-nd conf. on automata theory and formal languages, Springer Lect. Notes in Comp. Sc. 33 (p. 134-183).*

[Con 76] J.H. Conway. *On numbers and games.* Academic Press.

[Dahn 84] B.I. Dahn. The limiting behaviour of exponential terms. *Fund. Math. 124 (p. 169-186).*

[Dav 77] M. Davis. *Applied nonstandard analysis.* J. Wiley, New York.

[DBR 1875] P. Du Bois-Raymond. Über asymptotische Werthe, infinitäre Approximationen und infinitäre Auflösung von Gleichungen. *Math. Ann. 8 (p. 363-414).*

[DBR 1877] P. Du Bois-Raymond. Über die Paradoxen des Infinitärscalcüls. *Math. Ann. 11 (p. 149-167).*

[DDD 85] J. Della Dora, C. Dicrescenzo, D. Duval. A new method for computing in algebraic number fields. *Eurocal '85 (2), Springer Lect. Notes in Comp. Sc. 358, ed. G. Goos, J. Hartmanis (p. 321-326).*

[DDT 82] J. Della Dora, C. Dicrescenzo, E. Tournier. An algorithm to obtain formal solutions of a linear homogeneous differential equation at an irregular singular point. *Eurocal '82, Lect. Notes in Comp. Sc. 174, Springer-Verlag (p. 273-280).*

[DG 86] B.I. Dahn, P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae 127 (p. 45-50).*

[DL 89] J. Denef, L. Lipshitz. Decision problems for differential equations. *The Journ. of Symb. Logic 55(3) (p. 941-950).*

[DST 86] J. Davenport, Y. Siret, E. Tournier. *Calcul formel.* Masson, Paris.

[DuR 94] D. Duval, J-.C. Reynaud. Sketches and computation I: basic definitions and static evaluation, II: dynamic evaluation and applications. *Math. Struct. in Comp. Sc. 4 (p. 185-238, 239-271).*

[Ec 81a] J. Écalle. *Les fonction résurgentes. Tome I: les algèbres de fonctions résurgentes.* Publ. Math. d'Orsay.

[Ec 81b] J. Écalle. *Les fonction résurgentes. Tome II: les fonctions résurgentes appliquées à l'itération.* Publ. Math. d'Orsay.

[Ec 85] J. Écalle. *Les fonction résurgentes. Tome III: l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay.

[Ec 92] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Hermann, collection: Actualités Mathématiques.

[Rib] P. Ribenboim. *Generalized power series rings.* Preprint of Queen's Univ. Kingston, Ontario, Canada.

[Erd 49] P. Erdős. Problem 4358. *Amer. Math. Monthly 56 (p. 480).*

[Fab 1885] M.E. Fabry. Sur les intégrales des équations différentielles linéaires à coefficients rationnels. *Thèse à la faculté des sciences de Paris, Gauthier-Villars.*

[Fau 94] J.-C. Faugère. Résolution de systèmes algébriques. *PhD. thesis, Univ. Paris VI.*

[Fi 1889] H.B. Fine. On the functions defined by differential equations, with an extension of the Puiseux polygon construction to these equations. *Amer. Journ. of Math. 12 (p. 295-322).*

[Fit 73] J.P. Fitch. On algebraic simplification. *The Computer Journal 16 (1) (p. 23-27).*

[FS 96] P. Flajolet, R. Sedgewick. *An introduction to the analysis of algorithms.* Addison Wesley Publ. Comp., Reading.

[FSZ 89] P. Flajolet, B. Salvy, P. Zimmermann. Automatic average-case analysis of algorithms. *Theoretical Computer Science 79 (p. 37-109).*

[Gal 87] H. Gallaire. Boosting logic programming. *In Proc. of the fourth Intern. Conf. on Logic Programming, Melbourne, Australia (ed. J.-L. Lassez), MIT. Press, Cambridge (p. 928-962).*

[GeGo 88] K.O. Geddes, G.H. Gonnet. A new algorithm for computing symbolic limits using hierarchical series. *Proc. ISSAC 1988, Lecture Notes in Computer Science 358, (p. 490-495).*

[GHL 87] S. Gallot, D. Hulin, J. Lafontaine. *Riemannian geometry.* Springer-Verlag.

[GoDi 94] T. Gomez-Diaz. Quelques applications de l'évaluation dynamique. *PhD. Thesis, Univ. of Limoges, France.*

[GoGr 92] G.H. Gonnet, D. Gruntz. Limit computation in computer algebra. *Tech. Report 187 of the ETH, Zürich.*

[Gokh 93a] D. Gokhman. Regular growth of solutions to the Riccati equation $W' + W^2 = e^{2z}$ in the complex plane. *Tech. Rep. of the Univ. of Texas at San Antonio.*

[Gokh 93b] D. Gokhman. Limits in differential fields of holomorphic germs. *Tech. Rep. of the Univ. of Texas at San Antonio.*

[Gr 94] D. Gruntz. Infinite structures in Maple. *Maple Tech 1 (2) (p. 19-30).*

[Gr 96] D. Gruntz. On computing limits in a symbolic manipulation system. *PhD. Thesis, E.T.H. Zürich, Switzerland.*

[Gri 91] D. Y. Grigoriev. Complexity of factoring and calculating the gcd of linear ordinary differential operators. *J. Symb. Comp. 10 (p. 7-37).*

[Gri 94] D. Y. Grigoriev. Deviation theorems for solutions to differential equations and applications to lower bounds on parallel complexity of sigmoids. *Theoretical Comp. Sc. 133(1) (p. 23-33).*

[Gri 95] D. Y. Grigoriev. Deviation theorems for solutions of linear ordinary differential equations and applications to parallel complexity of sigmoids. *St.Petersburg Math. Jorn. 6(1) (p. 89-106).*

[GS 91] D. Grigoriev, M. Singer. Solving ordinary differential equations in terms of series with real exponents. *Trans. of the A.M.S. 327(1) (p 329-351).*

[GSZ 95] Cl. Gomez, B. Salvy, P. Zimmermann. Calcul formel: mode d'emploi, exemples en MAPLE. *Masson.*

[Hahn 07] H. Hahn. Über die nichtarchimedischen Grössensysteme. *Sitz. Akad. Wiss. Wien 116 (p. 601-655).*

[Har 10] G.H. Hardy. *Orders of infinity.* Cambridge Univ. Press.

[Har 11] G.H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London Mathematical Society 10,2 (p. 54-90).*

[Har 49] G.H. Hardy. *Divergent series.* Oxford Univ. Press. Reprinted by Jacques Gabay.

[Har 77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New-York.

[Haus 08] F. Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Math. Ann. 65 (p. 435-505).*

[Hig 52] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. 2 (p. 326-336).*

[Hir 64] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Annals of Mathematics 79 (p. 109-326).*

[HW 38] G.H. Hardy, E.M. Wright. *An introduction to the theory of numbers (chapter XXIII).* Oxford Science Publications.

[Il 84] Yu. S. Il'yashenko. Limit cycles of polynomial vector fields with non degenerate singular points on the real plane. *Funk. anal. i ego prim., 18(3) (p. 32-34). Func. Anal. and Appl. 18(3), 1985 (p. 199-209).*

[Il 91] Yu. S. Il'yashenko. Finiteness theorems for limit cycles. *Transl. Math. Monographs 94, Amer. Math. Soc.*

[Jan 20] M. Janet. *Systèmes d'équations aux dérivées partielles.* J. de Maths, Series 8, Vol. 3.

[Kap 76] I. Kaplanski. *An introduction to differential algebra.* Hermann (2-nd ed.).

[Kar 84] N. Karmakar. A new polynomial-time algoritm for linear programming. *Combinatorica 4(4) (p. 373-395).*

[Kat 79] L.G. Katchian. A polynomial algoritm for linear programming. *Doklady Akad. Nauk USSR 244(5) (p. 1093-1096). Soviet Math. Doklady 20 (p. 191-194).*

[Kho 83] A.G. Khovanskii. Fewnomials and Pfaff manifolds. *Proc. of the Intern. Congress of Math. August 16-24, Warszawa.*

[Kn 81] D.E. Knuth. *The art of computer programming.* Vol 2: Seminumerical algorithms. Addison-Wesley (2-nd ed.).

[KN 74] L. Kuipers, H. Niederreiter. *Uniform distribution sequences.* Wiley (New York).

[Kok 34] J.F. Koksma. Diophantische Approximationen. *Ergebnisse der Mathematik, Band IV, Heft 4, Springer, Berlin.*

[Kol 73] E.R. Kolchin. *Differential algebra and algebraic groups.* Academic Press, New-York.

[Kön 50] D. König. *Theorie der endlichen und unendlichen Graphen.* Chelsea Publ. Comp. New York.

[Kr 1884] Kronecker. Werke iii(i). *Berliner Sitzungsberichte (p. 47-110).*

[Krei 64] H.F. Kreimer. The foundations for an extension of differential algebra. *Transactions A.M.S. 111 (p 482-492).*

[Krus 60] J.B. Kruskal. Well-quasi-ordering, the tree theorem and Vázsonyi's conjecture. *Trans. Amer. Math. Soc. 95 (p. 210-225).*

[Lang 71] S. Lang. Transcendental numbers and Diophantine approximation. *Bulletin of the A.M.S. 77(5) (p. 635-677).*

[Lang 84] S. Lang. *Algebra.* Addison-Wesley.

[LC 1893] T. Levi-Civita. Sugli infiniti ed infinitesimi attuali quali elementi analitici. *Atti ist. Ven. 7 (p. 1765-1815).*

[Li 1837] J. Liouville. Mémoire sur la classification des transcendantes, et sur les racines de certaines équations en fonction finie explicite des coefficients. *J. Math. Pures et Appl. 2 (p. 56-104).*

[Li 1838] J. Liouville. Suite du mémoire sur la classification des transcendantes, et sur les racines de certaines équations en fonction finie explicite des coefficients. *J. Math. Pures et Appl. 3 (p. 523-546).*

[Lip 89] L. Lipshitz. D-finite power series. *Journal of algebra 122 (p 353-373).*

[LLL 82] A.K. Lenstra, H.W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann. 261 (p 515-534).*

[Lo 87] A. Logar. Constructions over localizations of rings. *Le Mathematiche, Vol. 42 (p. 131-149).*

[LoRo 89] D.E. Loeb, G.-C. Rota. Formal power series of logarithmic type. *Advances in Mathematics 75(1), Academic Press.*

[Mac 91] A. Macintyre. Schanuel's conjecture and free exponential rings. *Ann. Pure Appl. Logic 51 (p. 241-246).*

[Mah 40] K. Mahler. On a special functional equation. *Journal of the London Mathematical Society 15 (p. 115-123).*

[Mal 48] A.I. Malcev. On the embedding of group algebras in division algebras. *Daklady Akad. Nauk. S.S.S.R. 60 (p. 1409-1501).*

[Mal 79] B. Malgrange. Sur la réduction formelle des équations différentielles à singularités irrégulières. *Manuscript.*

[Mar 96] D. Marker. Model theory and exponentiation. *Notices of the A.M.S. 43(7) (p. 753-759).*

[Mat 70] H. Matsumara. *Commutative algebra.* W.A. Benjamin Co., New York.

[Matij 70] Yu. Matijasevic. Enumerable sets are Diophantine. *Dokl. Akad. Nauk SSSR 191 (p. 279-282). Soviet Math. Dokl. 11 (p. 354-357).*

[Men 96] F. Menous. Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle. *PhD. thesis, Univ. d'Orsay, France.*

[Mil 85] E.C. Milner. Basic wqo- and bqo-theory. *In Rival (ed.) Graphs and orders. D. Reidel Publ. Comp. (p. 487-502).*

[ML 39] S. Mac Lane. The universality of formal power series fields. *Bull. Amer. Math. Soc. 45 (p. 888-890).*

[ML 71] S. Mac Lane. *Categories for the working mathematician.* Springer-Verlag, New-York.

[MMV 94] A. Macintyre, D. Marker, L. van den Dries. The elementary theory of restricted analyic fields with exponentiation. *Annals of Math. 140 (p. 183-205).*

[MMV *] A. Macintyre, D. Marker, L. van den Dries. The elementary theory of restricted analytic fields with exponentiation. *To appear.*

[Mora 82] T. Mora. An algorithm to compute the equations of tangent cones. *Proc. of EUROCAL '83, Lect. Notes in Comp. Science 144 (p. 158-165).*

[MoRe 91] I. Moerdijk, G.E. Reyes. *Models for smooth infinitesimal analysis.* Springer-Verlag, New-York.

[MPT 92] T. Mora, G. Pfister, C. Traverso. An introduction to the tangent cone algorithm. *Advances in Comp. Research, Vol. 6 (p. 199-270).*

[MW *] A. Macintyre, A. Wilkie. On the decidability of the real exponential field. *To appear.*

[Nae 95] F. NAEGELE. Autour de quelques équations fonctionnelles analytiques. *PhD. Thesis, l'I.N.P.G., Grenoble, France.*

[NaWi 63] C. ST. J. A. NASH-WILLIAMS. On well-quasi-ordering finite trees. *Proc. Cambridge Philos. Soc 59 (p. 833-835).*

[New 1671] I. NEWTON. *De methodis serierum et Fluxionum.* Manuscript. French translation of 1740 reprinted by Librarie scientifique et technique Albert Blanchard.

[Pél 95] A. PÉLADAN-GERMA. Testing identities of series defined by algebraic partial differential equations. *Lect. Notes in Comp. Science 948 (p. 393-407).*

[Pél 96] A. PÉLADAN-GERMA. Tests effectifs de nullité dans des extensions d'anneaux différentiels. *PhD. Thesis, École Polytechnique, France.*

[Poin 1886] H. POINCARÉ. ??. *Acta Math. 8.*

[Pouz 85] M. POUZET. Applications of well quasi-ordering and better quasi-ordering. *In Rival (ed.) Graphs and orders. D. Reidel Publ. Comp. (p. 503-519).*

[Pui 1850] M.V. PUISEUX. Recherches sur les fonctions algébriques. *J. Math. Pures et Appliquées 15 (p. 365-480).*

[PV 96] A. PÉLADAN-GERMA, J. VAN DER HOEVEN. Un algorithme de Buchberger local. *Submitted to C.R.A.S.*

[Ram 78] J.P. RAMIS. Dévissage Gevrey. *Astérisque 59/60 (p. 173-204).*

[Ram 93] J.P. RAMIS. Séries divergentes et théories asymptotiques. *Suppl. au bulletin de la Soc. Math. de France 121.*

[Rib] P. RIBENBOIM. *Generalized power series rings.* Preprint of Queen's Univ. Kingston, Ontario, Canada.

[Res 93] J.P. RESSAYRE. Integer parts of real closed exponential fields. *Arithmetic, Proof Theory and Computational Complexity, P. Clote and J. Krajicek eds., Oxford Univ. Press.*

[Rich 68] D. RICHARDSON. Some undecidable problems involving elementary functions of a real variable. *The Journal of Symb. Logic 33 (4) (p. 414-520).*

[Rich 69] D. RICHARDSON. Solution of the identity problem for integral exponential functions. *Z. Math. Logik. Grundlagen 15 (p. 333-340).*

[Rich 92] D. RICHARDSON. The elementary constant problem. *Proc. ISSAC '92 (p. 108-116).*

[Rich 95] D. RICHARDSON. A simplified method for recognizing zero among elementary constants. *Proc. ISSAC '95 (p. 104-109).*

[Ris 75] R.H. RISCH. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math. 4, 101 (p. 743-759).*

[Ritt 50] J.F. RITT. *Differential algebra.* Amer. Math. Soc, New-York.

[Rob 72] A. ROBINSON. On the real closure of a Hardy field. *In "Theory of sets and topology", G. Asser et al. ed., Berlin, Deut. Verlag. Wissenschaften.*

[Ros 83a] M. ROSENLICHT. Hardy fields. *Journal of Math. Analysis and Appl. 93 (p. 297-311).*

[Ros 83b] M. ROSENLICHT. The rank of a Hardy field. *Trans. Amer. Math. Soc. 280 (2) (p. 659-671).*

[Ros 87] M. Rosenlicht. Growth properties of functions in Hardy fields. *Trans. Amer. Math. Soc. 299 (1) (p. 261-272).*

[RSSV 96] D. Richardson, B. Salvy, J. Shackell, J. van der Hoeven. Asymptotic expansions of exp-log functions. *Submitted to ISSAC '96.*

[RV 94] M.F. Roy, N. Vorobjov. Finding irreducible components of some real transcendental varieties. *Comput. Complexity 4 (p. 107-132).*

[Sal 91] B. Salvy. Asymptotique automatique et fonctions génératrices. *PhD. thesis, École Polytechnique, France.*

[Seid 56] A. Seidenberg. An elimination theorem for differential algebra. *Univ. California Publ. Math. (N.S.) (p 31-38).*

[Sh 89] J. Shackell. A differential-equations approach to functional equivalence. *Proc. ISSAC '89, Portland, Oregon, A.C.M., New York, (p. 7-10).*

[Sh 90] J. Shackell. Growth estimates for exp-log functions. *Journal of Symbolic Computation 10 (p. 611-632).*

[Sh 91] J. Shackell. Limits of Liouvillian functions. *Technical report, Univ. of Kent, Canterbury. Appeared in 1996 in the Proc. of the London Math. Soc. 72 (p. 124-156).*

[Sh 92] J. Shackell. Growth orders occuring in expansions of Hardy-field solutions to algebraic differential equations. *Technical report, Univ. of Kent, Canterbury.*

[Sh 93a] J. Shackell. Rosenlicht fields. *Trans. Amer. Math. Soc. 335/2 (p. 579-595).*

[Sh 93b] J. Shackell. Zero equivalence in function fields defined by algebraic differential equations. *Trans. of the Amer. Math. Soc. 336 (1) (p. 151-172).*

[Sh 93c] J. Shackell. Inverses of Hardy L-functions. *Bull. London Math. Soc. 25 (p. 150-156).*

[Sh 95a] J. Shackell. Extensions of asymptotic fields via meromorphic functions. *Journal of the London Math. Soc. 52 (p. 356-374).*

[Sm 1875] S. Smith. On the higher singularities of plane curves. *Proc. London Math. Soc. 6 (p. 153-182).*

[Sor 90] M. Soria. Méthodes d'analyse pour les constructions combinatoires et les algorithmes. *Doctorat d'état, Univ. de Paris-sud, centre d'Orsay, France.*

[SS 71] A. Schönhage, V. Strassen. Schnelle Multiplikation grosser Zahlen. *Computing 7 (p. 281-292).*

[SS 92] B. Salvy, J. Shackell. Asymptotic expansions of functional inverses. *Proc. ISSAC '92 (p. 130-137).*

[SS 96] B. Salvy, J. Shackell. Symbolic asymptotics: functions of two variables, implicit functions. *Tech. Rep. 2883, INRIA, France.*

[St 1886] T.J. Stieltjes. Recherches sur quelques séries semi-convergentes. *Ann. Sc. École Norm. Sup. de Paris 3 (p. 201-258).*

[Stey 84] J.M. Steyaert. Structure et complexité des algorithmes. *Doctorat d'État, Univ. Paris VII.*

[Str 77] W. Strodt. A differential algebraic study of the intrusion of logarithms into asymptotic expansions. *in "Contributions to algebra", H. Bass, P.J. Cassidy, J. Kovacic eds., Academic press (p. 355-375).*

[SW 71] W. Strodt, R.K. Wright. Asymptotic behaviour of solutions and adjunction fields for nonlinear first order differential equations. *Mem. Amer. Math. Soc. 109.*

[SZ 92] B. Salvy, P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *A.C.M. transactions on mathematical software.*

[Tar 31] A. Tarski. *Sur les ensembles définissables de nombres réels.* Fund. Math. 17 (p. 210-239).

[Th 95] J. Thomann. Procédés formels et numériques de sommation de séries d'équations différentielles. *Expositiones Math. 13 (p. 223-246).*

[Tour 87] É. Tournier. Solutions formelles d'équations différentielles. *Doctorat d'État, Univ. scientifique, technologique et médicale de Grenoble.*

[VdD 84] L. van den Dries. Remarks on Tarski's problem concerning $(\mathbb{R},+,\cdot,\exp)$. *Logic Colloquium '82, G. Longi, G. Longo and A. Marcja eds., North-Holland.*

[VdH 94a] J. van der Hoeven. Outils effectifs en asymptotique et applications. *Research report LIX/RR/94/09, École polytechnique, France.*

[VdH 94b] J. van der Hoeven. General algorithms in asymptotics I, Gonnet and Gruntz's algorithm. *Research report LIX/RR/94/10, École polytechnique, France.*

[VdH 94c] J. van der Hoeven. General algorithms in asymptotics II, Common operations. *Research report LIX/RR/94/10, École polytechnique, France.*

[VdH 95a] J. van der Hoeven. Automatic numerical expansions. *Local proc. of the conference "Real numbers and computers", St-Etienne, April 4-6, France (p. 261-274).*

[VdH 95b] J. van der Hoeven. Sur certains produits infinis. *Research report LIX/RR/95/06, École polytechnique, France.*

[VdH 96a] J. van der Hoeven. Generic asymptotic expansions. *Local Proc. of the Rhine workshop on computer algebra, Saint-Louis, April 1-3, France. Submitted to AAECC.*

[VdH 96b] J. van der Hoeven. Fast evaluation of holonomic functions. *Research report LIX/RR/96/05, École polytechnique, France.*

[VdH 96c] J. van der Hoeven. Differential and mixed differential difference equations from the effective viewpoint. *Research report LIX/RR/96/11, École polytechnique, France.*

[VdH *] J. van der Hoeven. On the automatic computation of limsups. *Available at* `http://www.win.tue.nl/win/mega96/papers`*. To appear in a special issue of the Journal of Pure and Applied Algebra (eds. A.M. Cohen, M.F. Roy), Elsevier.*

[VdW 30] B.L. van der Waerden. *Moderne Algebra I.* Springer Verlag, Berlin.

[VH 96] M. van Hoeij. Factorization of linear differential operators. *PhD. Thesis, Univ. of Nijmegen, The Netherlands.*

[Wa 65] W. Wasow. *Asymptotic expansions of ordinary differential equations.* John Wiley. Reprinted by Dover in 1987.

[Wil *] A.J. WILKIE. Some model completeness results for expansions of the ordered field of reals by Pfaffian functions and exponentiation. *To appear.*

[Wu 87] W.T. WU. A Zero structure theorem for polynomial equation solving and its applications. *Proceedings of ISSAC'88, Roma, Springer Verlag.*

[WZ 92] H. WILF, D. ZEILBERGER. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Mathematicae, Vol. 108 (p. 575-633).*

[Zeil 82] D. ZEILBERGER. Sister Celine's technique and its generalizations. *Journal of Mathematical Analysis and Applications 82 (p. 114-145).*

[Zeil 90] D. ZEILBERGER. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics 32 (p. 321-368).*

Note. All my preprints are available at `http://lix.polytechnique.fr:80/~vdhoeven/`

Glossary

Conventions

$f_{i,j} = (f_i)_j$	index convention
$\leqslant_E, +_E, \cdots$	the implicit ordering, sum, etc. on a set E
$E + F$	sum of two sets: $E + F = \{x + y \mid x \in E, y \in F\}$. A similar notation is often used for other operations
$(x_i)_{i \in I}$	sequence or family notation
Id_E	the identity mapping $E \to E$
$E \amalg F$	the disjoint union or direct sum of A and B
$E \backslash F$	the set of elements in E which are not in F
$E \bigtriangleup F$	the set $E \backslash F \cup F \backslash E$
$\lvert x \rvert, \lvert E \rvert$	absolute value of x or cardinality of E
$\mathbb{N}$	the natural numbers including zero
$\mathbb{N}_k$	the set $\{1, \cdots, k\}$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	the integers, rationals, reals and complex numbers
R^*	the set of invertible (resp. non zero) elements of a ring (resp. a field)
R^+	positive elements of an ordered ring
R^+_*	positive invertible elements of an ordered ring

Part A

$f \preccurlyeq g$	f is dominated by g (Hardy's notation), 38, 43, 147
$f = O(g)$	idem (Landau's notation), 38, 43, 147
$f \prec g$	f is negligible w.r.t. g (Hardy's notation), 38, 43, 147
$f = o(g)$	idem (Landau's notation), 38, 43, 147
$f \asymp g$	f is asymptotic to g, 38, 43, 147
$f \sim g$	f is equivalent to g, 38, 43, 147
$f \prec\!\!\prec g$	f has a smaller comparability class than g, 39, 43, 147
$f \asymp\!\!\asymp g$	f and g have the same comparability classes, 39, 43, 147
$C[\![X]\!]$	ring of grid-based series over C in X, 40
$\operatorname{supp} f$	support of f, 40
$C[\![X]\!]^o$	set of infinitesimal elements of $C[\![X]\!]$, 41
$C[\![X]\!]^O$	set of bounded elements of $C[\![X]\!]$, 41
$C[\![z_1, \cdots, z_n]\!]$	ring of grid-based series in $z_1, \cdots, z_n$ over C, 41
$C[\![z_1; \cdots; z_n]\!]$	ring of lexicographical grid-based series in $z_1, \cdots, z_n$ over C, 41
M_f	dominant monomial of f, 42

τ_f	dominant term of f, 42
c_f	dominant coefficient of f, 42
$\lim f$	limit of f, 43
$f = f^{\uparrow} + f^{c} + f^{\downarrow}$	canonical decomposition of a series f, 43
$C[\![X]\!]^{\uparrow}$	set of purely unbounded series with 0, 43
μ_f	dominant exponent of f, 47
$\log_k$	k-th iterated logarithm, 49, 87
$\exp_k$	k-th iterated exponential, 49, 87
$\log_{\boldsymbol{p}} x$	logarithmic monomial $x^{p_0} \cdots \log_k^{p_k} x$, 49
$\log_{C^*} x$	group of logarithmic monomials, 49
$C^{alog}[\![[x]]\!]$	field of alogarithmic transseries, 50
$C[\![[x]]\!]$	field of grid-based transseries over C in x, 50
$\mathbb{T}$	field of transseries, 51
$C_r[\![[x]]\!]$	set of transseries with exponential depth bounded by r, 51
$\mathbb{T}_{\infty}^{+}$	set of positive infinitely large transseries, 55
$f\uparrow$	upward shift of f, 56
$f\downarrow$	downward shift of f, 56
$\operatorname{expo} f$	exponentiality of f, 56
g^{inv}	functional inverse of g, 56, 78
$C[[X]]$	ring of series with Noetherian support over C in X, 62
ω	smallest infinite ordinal, 63
$C_{\alpha}^{\omega}[[[x]]]$	field of well-ordered transseries of exponential depth $\leqslant \alpha$, 64
$C_{<\alpha}^{\omega}[[[x]]]$	field of well-ordered transseries of exponential depth $< \alpha$, 64
$f \trianglelefteq g$	f is a truncation of g, 67
$\operatorname{stat\,lim}_{i \in I} f_i$	stationary limit of $(f_i)_{i \in I}$, 67
$\overline{\mathbb{T}}$	compactification of $\mathbb{T}$, 81
$\mathbb{T}^{exp}$	field of purely exponential transseries, 100
$\prec\!\!\prec_h$, $\preccurlyeq\!\!\prec_h$, $\asymp_h$	asymptotic relations $\prec\!\!\prec$, $\preccurlyeq\!\!\prec$, $\asymp$ modulo $\prec\!\!\prec\!\!\prec$ h perturbations, 101
o_h, O_h	Landau's notation for $\prec\!\!\prec_h$, $\preccurlyeq\!\!\prec_h$, 101
L	linear differential operator, 101
$\mathfrak{d}(L)$	dominant monomial of L, 101
$L_{\times h}$	multiplicative conjugate of L, 102
$L\uparrow$	upward shifting of L, 102, 120
L^{-1}	distinguished right inverse of L, 107
P	differential polynomial with transseries coefficients, 116
P_{+h}	additive conjugate of P, 118
$P_{\times h}$	multiplicative conjugate of P, 118
L	quasi-linear operator, 127
$L^{-1} g$	dominant term of distinguished solution to $Lf = g$, 129
L^{-1}	distinguished right inverse of L, 134
Σ_c^*	grid-based summation symbol, 147
GTV_C	category of grid-based transvarieties over C, 147

NTV_C	category of Noetherian transvarieties over C of finite logarithmic and exponential depths, 156

Part B

$\mathfrak{C}$	effective field of constants, 166, 187, 216
$\mathfrak{T}^{expr}$	set of exp-log expressions, 167
S_B	(effective) asymptotic scale generated by B, 167
$B = \{\text{б}_1, \cdots, \text{б}_n\}$	(effective) asymptotic (normal) basis, 167, 203, 212, 238
$\mathfrak{G}_B^{expr}$	set of exp-log Z-expressions w.r.t. B, 167
$[z_i^\alpha]u$	coefficient of z_i^α in u, 172
$Z = \{z_1, \cdots, z_k\}$	finite set of infinitesimal elements in S_B, 172, 204
$\mathfrak{L}_Z^{expr}$	set of exp-log Laurent series in Z, 172
$\overline{u}$	(trans)series represented by u, 172, 204
S_Z	set of monomials in Z, 173
$\leqslant_Z$	natural product ordering on S_Z, 173, 204
$\preccurlyeq_B$	quasi-ordering on S_Z induced by B, 174
$C[[z_1, z_2, \cdots]]$	direct limit of $C, C[[z_1]], C[[z_1, z_2]], \cdots$, 201
$C[\![z_1, z_2, \cdots]\!]$	direct limit of $C, C[\![z_1]\!], C[\![z_1, z_2]\!], \cdots$, 201
$\mathfrak{R}, \mathfrak{L}$	effective Cartesian algebra or local community, 201, 216, 226, 241
$\mathfrak{X}$	effective (quasi-ordered) monomial group, 204, 266
$\preccurlyeq_{\mathfrak{X}}$	quasi-ordering on S_Z induced by $\mathfrak{X}$, 204
$\underline{f}$	"natural" Cartesian representation of f, 208, 212, 243, 244
$\langle \text{ч} \rangle f$	pseudo-coefficient of f, 211
$B = B_1 \amalg \cdots \amalg B_q$	lexicographical decomposition of normal basis, 238
ц^{expo}	exponential part of ц, 253
ц^{free}	"free part" of ц, 253
$\operatorname{supp}_* L$	operator support of a linear operator, 271
$\mathfrak{T}$	effective differential field of transseries over $\mathfrak{C}$, 275
$S_d, S_{\boldsymbol{d}}$	shift operators, 285
$f \prec\!\!\prec^s g$	f is steadily dominated by g, 287

Appendices

(A)	final segment generated by A, 305
$E \amalg F$	disjoint union of E and F, 306
E^*	set of words over E, 307
$E^\diamond$	set of commutative words over E, 307
E^+	set of non empty words over E, 307
$E^\dagger$	set of non empty commutative words over E, 307
$x_1 \cdots x_n$	product notation for words, 307
$[x_1, \cdots, x_n]$	n-tuple notation for words, 307
$E^\top$	set of finite E-labeled trees, 308

$x[T_1, \cdots, T_n]$	tree with root labeled by x and children $T_1, \cdots, T_n$, 308
$\mathrm{root}(T)$	the root of a tree T, 308
$\mathrm{leaf}(T)$	set of leafs of a tree T, 308
$\preceq_T$	ancestor relation on a tree T, 308
Σ	signature, 315
Σ_f, Σ_R	function resp. relation symbols of Σ, 315
N_f, N_R	arity of a function f resp. a relation R, 315
$\mathrm{dom}\, f$	domain of f, 315
Σ-Alg	category of Σ-algebras, 316
Σ-PAlg	category of partial Σ-algebras, 316
I_X	free full Σ-algebra on X, 316
$t\#$	t is defined, 316
C_Σ	Horn clauses w.r.t. Σ, 316
C_E	Horn clauses valid in E, 316
(Σ, C)-PAlg	category of partial Σ-algebras modeling C, 316
$\overline{E}$	Σ-closure of E, 319
$A\langle X\rangle$	free extension of A by X, 322
$\mathcal{F}(V)$	function space of V, 323
P	category of point types, 323
Var_P	category of varieties relative to P, 324
$\mathbb{A}^X_\mathsf{P}$	affine X-space relative to P, 326
A	a D-ring, 349
D	the Lie algebra of derivations on A, 349
$[\Sigma]$	D-ideal generated by a set Σ, 351
$Q(A)$	quotient field or total ring of fractions of A, 351
Ω	free linear D-operator algebra $\Omega = A[D]$, 351
Θ	basis for $A[D]$, 352
ε	evaluation mapping, 353
$\mathfrak{A}$	effective D-ring of the form $\mathfrak{C}[f_1, \cdots, f_k]/\mathfrak{i}$, 357
$G_\mathfrak{A}$	Groebner basis for $\mathfrak{i}$, 357
$\mathfrak{D}$	effective Lie-algebra of derivations on $\mathfrak{A}$, 357
$\mathfrak{m}$	maximal ideal of $\mathfrak{A}$, which determines ε, 357

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