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Abstract

A number of recent papers have been concerned with algorithms to decide the limiting behaviour of functions of a single variable. Here we make a corresponding study of a class of functions of two variables, namely the exp-log functions. As in the one-variable case, we need to make certain assumptions regarding the handling of constants.

Two of the main tools in the one-variable case are Hardy fields and nested forms. Here, we show how to compute some asymptotic estimates for two-variable exp-log functions (modulo a constant oracle). This method is then used to give an algorithm for computing the nested forms of real implicit functions.

Asymptotique automatique :
Fonctions de deux variables, fonctions implicites

Résumé

De nombreux travaux récents ont considéré des algorithmes de calcul du comportement asymptotique de fonctions d'une variable. Nous menons ici une étude correspondante pour le cas d'une classe de fonctions de deux variables, les fonctions exp-log. Comme dans le cas univarié, nous devons faire certaines hypothèses sur la gestion des constantes.

Deux des outils principaux dans le cas univarié sont les corps de Hardy et les "nested forms". Nous montrons ici comment calculer des estimations asymptotiques pour des fonctions exp-log en deux variables (modulo un oracle pour les constantes). Cette méthode est ensuite utilisée pour donner un algorithme de calcul de "nested form" de fonctions implicites réelles.

SYMBOLIC ASYMPTOTICS: FUNCTIONS OF TWO VARIABLES, IMPLICIT FUNCTIONS

BRUNO SALVY AND JOHN SHACKELL

Abstract. A number of recent papers have been concerned with algorithms to decide the limiting behaviour of functions of a single variable. Here we make a corresponding study of a class of functions of two variables, namely the exp-log functions. As in the one-variable case, we need to make certain assumptions regarding the handling of constants.

Two of the main tools in the one-variable case are Hardy fields and nested forms. Here, we show how to compute some asymptotic estimates for two-variable exp-log functions (modulo a constant oracle). This method is then used to give an algorithm for computing the nested forms of real implicit functions.

This work is part of a global effort to automate the formal aspects of asymptotic expansions. It is possible to mechanize some techniques of asymptotics and build a computer algebra toolbox of these. A lot of work in symbolic asymptotics follows this approach and most existing facilities for asymptotic expansions in computer algebra systems have been obtained in this way. An alternative approach aims at studying the asymptotics of whole classes of problems, investigating all the possible asymptotic scales that may occur. The main tools here are nested forms and expansions, zero-equivalence methods and the theory of Hardy fields. The present paper follows this path.

Nested forms and nested expansions were introduced in [22]. A formal definition is given in Section 1. An example of a nested form is

$$e^{\log^2 x\, e^{\sqrt{\log\log x}(c+\phi_1(x))}},$$

where c is a real constant and $\phi_1(x)$ tends to 0 when x tends to infinity. In some cases, one can compute the nested form of ϕ_1, introducing a new function ϕ_2 and then repeat the process, thus generating a sequence of nested forms; this sequence is called a *nested expansion*.

The field $\mathcal{H}(x)$ of *exp-log functions* of a single variable, x, is formed of expressions built from x and real constants by means of arithmetic operations and the operations:

$$f \mapsto \exp(f), \qquad f \mapsto \log|f|. \tag{1}$$

In previous works [21, 22, 24, 25, 6, 13], an algorithmic treatment in terms of nested expansions was given to the asymptotics of *i)* exp-log functions; *ii)* Liouvillian functions and *iii)* Hardy-field solutions of algebraic differential equations. All these algorithms require the use of a method for deciding zero equivalence in the class of functions concerned. This brings particular difficulties regarding constants. As is the normal practice in this area we shall use an oracle for the determination of signs of these. We discuss this matter more fully in Section 1.4.

Inverse functions have long been problematic in asymptotics [5, 8]. However in [18], the authors gave an algorithm for inverting nested forms which solves the problem of expressing the asymptotic behaviour of inverse functions. In the present paper we treat the more general problem of implicit functions. More precisely, let $\mathcal{H}(x, y)$ denote the field obtained by closing $\mathbb{R}(x, y)$ under the operations (1). Let $h \in \mathcal{H}(x, y)$. We will say that a real function $y(x)$ defined on some interval of the form (a, ∞) of the real line *satisfies* the equation $h(x, y) = 0$ if $h(x, y(x)) = 0$ for all sufficiently large x. One of our main interests in this paper is in giving asymptotic expressions which describe the growth of these solutions as $x \to \infty$.

In Section 1 of the present paper, we give the background needed for later sections. We recall some of the properties of Hardy fields, and in particular of comparability classes. We similarly recall the definition and basic properties of nested forms, and introduce partial nested forms. As their name would suggest, the latter give some of the information available from the corresponding nested form. They will be used in Section 3. It will be convenient to make use of the z-functions from [21], so we define these and recall some of their properties. We close Section 1 with a slightly more extensive discussion of zero equivalence.

In Section 2, we give the basis of our algorithm. We assume that we are presented with a two-variable exp-log function h. We compute the asymptotic behaviour of $h(x, y)$ in various cases, specified by asymptotic relations involving x and y. To do this we define certain sets of basic expressions, such that in each case, the conditions are given in terms of these expressions, and h is asymptotic to a monomial in them. The method is illustrated by some examples. Then in Section 3 we use this framework to give a set of possible nested forms of solutions of $h(x, y) = 0$. Again we illustrate our method with examples.

A forthcoming paper [19] gives another approach to the asymptotics of two-variable exp-log functions. This approach is less general and does not cover all bivariate exp-log functions, but it gives some possibly faster methods for calculating nested forms of implicit functions which work "most of the time".

Our research on this problem was mooted during a visit by John Shackell to Inria-Rocquencourt in late 1991, and begun with a visit by Bruno Salvy to Canterbury in 1992. Work on it continued through further visits by John Shackell to Inria-Rocquencourt in 1995 and 1996. These visits were funded by Inria, the University of Kent at Canterbury, EPSRC, the Long Term Research Project *Alcom-IT* (# 20244) of the European Union and the Alliance 96 programme. The authors wish to thank these organizations for their support.

1. Background

Computing limits of simple exp-log functions is easily done by hand. However, the automation of the process requires dealing with the indefinite cancellation problem, as exemplified by $\exp(1/x + e^{-x}) - \exp(1/x)$ at $x = +\infty$. The problem here is that if the exponentials are expanded, and terms collected, the powers of x^{-1} dominate the exponentials but forever cancel out. There is thus a risk of non-termination if such examples are not handled with care. It was shown in 1984 by B. Dahn and P. Göring [4] that limit computation could be reduced to the so-called constant problem; that is to say the problem of finding an algorithm to determine the signs of constant expressions. Then in [21], an actual *algorithm* was given to perform limit computation (modulo the constant problem). The underlying tool, which was not explicit at that time, was *nested forms*, and the theory required to prove that the algorithm works and terminates for the whole class of exp-log functions is the theory of Hardy fields. In this section, we give the basic definitions on Hardy fields and nested forms. We introduce a notion of partial nested form which will be used in Section 3. We also recall notation and basic properties on a class of exp-log functions called z-functions. The section finishes with a discussion of matters relating to zero equivalence.

1.1. Hardy fields. Let $\mathcal{X}$ be the ring of germs at ∞ of $\mathcal{C}^\infty$ functions. (Think of it as the set of possible asymptotic behaviours.) A *Hardy field* is a subring of $\mathcal{X}$ which is a field closed under differentiation.

The main constraint here is that non-zero elements of Hardy fields have to be invertible, and thus cannot have arbitrarily large zeros; so they are ultimately positive or ultimately negative. In addition, the difference of any two (germs of) functions in a Hardy field is also in the field. Consequently, the field may be ordered by setting $f > g$ when $f(x) - g(x)$ is positive for sufficiently large x. Much of the power of the theory comes from this order. For since the derivatives of elements belong to the Hardy field, the elements have to be ultimately monotonic. Hence they tend to limits, which are possibly infinite. This guarantee of the existence of a limit greatly simplifies the analysis. Often one only needs to know whether the limit concerned is infinite, zero or some other real number.

Many of the functions one meets in asymptotics turn out to have germs lying in some Hardy field. The following result is of particular importance for our present purpose. It can be found, in different notation, in [26], and also follows from the work of Khovanskii [9].

Theorem 1. *Let the real function $y = y(x)$ satisfy the equation $h(x, y) = 0$, where h is some element of $\mathcal{H}(x, y)$. Then (the germ of) $y(x)$ belongs to a Hardy field.*

If f and g are two elements of a Hardy field tending to infinity, they are said to be *comparable* when there exists a positive integer n such that

$$f < g^n \quad \text{and} \quad g < f^n,$$

where the order is that of the field. Extending this by saying that $\pm f$ and $\pm f^{-1}$ are all comparable and that two elements tending to a non-zero finite limit are comparable yields a decomposition of the non-zero elements of the Hardy field into equivalence classes called *comparability classes*; the comparability class of f is denoted $\gamma(f)$. One should think of these classes as basic functions of an asymptotic scale. Their number minus one is called the *rank* of the field. Comparability classes are ordered. If f and g are two functions tending to infinity, then $\gamma(f) < \gamma(g)$ when $f^n = o(g)$ for all fixed $n \in \mathbb{N}$. The comparability class of 1 is taken as the smallest comparability class. This relation clearly depends only on the comparability classes.

An important special type of Hardy field, namely one of finite rank which is closed under $f \to f^c$ for all real c and all $f \neq 0$, was considered by M. Rosenlicht in [16]. In [22], such fields were called *Rosenlicht fields*. The importance of Rosenlicht fields is firstly that any function which belongs to a Hardy field and satisfies an algebraic differential equation over $\mathbb{R}$ automatically belongs to a Rosenlicht field. Secondly, it was shown in [22] that *any element of a Rosenlicht field has a nested expansion*. These latter objects are defined in the next section.

More information on Hardy fields can be found in the papers of M. Boshernitzan [1, 2] and M. Rosenlicht [14, 15, 16, 17].

1.2. Nested forms, nested expansions and pnfs. If $f \to \infty$ and $g \to 1$, we know that $f/g \to \infty$, but if $g \to \infty$ we can of course make no deduction regarding the limit of f/g. In order to obtain a calculus, we need some measure of the rapidity with which a function tends to its limit. This has long been recognized; the problem was extensively studied by Hardy [7, 8], and some of the ideas used there go back to the work of du Bois-Reymond. Nested forms and expansions are based on Hardy's orders of infinity, but have a more formal recursive structure which is suitable for algorithmic work.

We use the classical notations $e_k(x)$ for the exponential iterated k times, and likewise $l_k(x)$ (or sometimes just $l_k x$) for the iterated logarithm. A *partial nested form*, or pnf, of a positive function tending to zero or infinity is a finite sequence $\{(s_i, \epsilon_i, m_i, d_i, \phi_i); i = 1, \ldots, n\}$ where s_i and m_i are non-negative integers, ϵ_i is ± 1, d_i is a positive real number, and ϕ_i is an element of a Hardy field. Such a sequence gives a representation of the function ϕ as

$$\phi(x) = e_{s_1}^{\epsilon_1}(l_{m_1}^{d_1}(x)\phi_1(x)), \tag{2}$$

and recursively

$$\phi_{i-1}(x) = e_{s_i}^{\epsilon_i}(l_{m_i}^{d_i}(x)\phi_i(x)), \quad i = 2, \ldots, n, \tag{3}$$

with the additional constraint that each ϕ_i is of a smaller order of growth than l_{m_i} (i.e. $\gamma(\phi_i) < \gamma(l_{m_i})$). The number n will be called *the length* of the pnf. The pnf for a negative function tending to zero or minus infinity is a finite sequence as above prefixed by a minus sign. Usually the ϕ_i's will only be specified indirectly (i.e. using the fact that (2) and (3) hold), whereas the ϵ_i, s_i, m_i and d_i are given explicitly. Nonetheless it is useful to be able to refer to the ϕ_i's.

The *nested form* of a function tending to zero or $\pm\infty$ is a pnf satisfying two extra requirements. Firstly, if n is the length, ϕ_n must tend to a finite non-zero constant, which will generally be explicitly specified. Thus we have $\phi_n = K + o(1)$ for some constant $K \neq 0$. Secondly, d_n must not be equal to 1 unless at least one of s_n or m_n is zero. This second condition disallows such expressions as $\exp(\log x(K + o(1)))$, which would instead be written $x^K\phi$ with $\gamma(\phi) < \gamma(x)$. One may think of a nested form being approximated by partial nested forms of shorter length.

A function tending to a finite non-zero limit can only have one pnf, which must then be the nested form. It is given as $K + o(1)$ where K is the limit.

Having thus defined a nested form, one defines a *nested expansion* by repeatedly giving a nested form for the residual '$o(1)$' part of the expression. In other words, a nested expansion is a sequence of nested forms F_k, such that F_{k+1} is the nested form of $\phi^{\langle k+1\rangle} = \phi_{n_k}^{\langle k\rangle} - \lim \phi_{n_k}^{\langle k\rangle}$, where we have set $\phi^{\langle 0\rangle} = \phi$, and used n_k to denote the length of $\phi^{\langle k\rangle}$.

When we come to compute nested forms and expansions of implicit functions, we shall, in some cases, obtain initially only a pnf of length 1. Then by substituting this into the defining equation, we get an implicit equation for ϕ_1. We can repeat the process to obtain pnfs of greater length, but the question arises as to whether this sequence of pnfs must terminate with a nested form. An affirmative answer is given by the following lemma.

Lemma 1. *Let $h(x, y) \in \mathcal{H}$ and suppose the real function $y(x)$ satisfies $h(x, y) = 0$. Then y has a nested form, whose length is bounded in terms of the structure of h.*

We already know from Theorem 1 that $y(x)$ belongs to a Hardy field. We show that it also satisfies an algebraic differential equation with constant coefficients.

We can build a tower of function fields,

$$\mathbb{R} = \mathcal{F}_0 \subset \mathbb{R}(x) = \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k,$$

where $\mathcal{F}_i = \mathcal{F}_{i-1}(g_i)$ for $i = 2, \ldots, k$. Each g_i will be an exponential or a logarithm of an element in $\mathcal{F}_{i-1}$, and h will belong to $\mathcal{F}_k$. By multiplying through by the denominator, we may take h to be a polynomial in g_k with coefficients in $\mathcal{F}_{k-1}$. By replacing h with an appropriate factor if necessary, we may further suppose that h is irreducible. We regard y as a function of x, and differentiate the identity $h = 0$ with respect to x. We can then eliminate g_k between the equations $dh/dx = 0$ and $h = 0$ to obtain an equation $h_1 = 0$, where $h_1 \in \mathcal{F}_{k-1}(y')$. Then we use the fact that $\mathcal{F}_{k-1} = \mathcal{F}_{k-2}(g_{k-1})$ and differentiate again to eliminate g_{k-1}. Continuing in this way, we see that any solution $y = y(x)$ of $h = 0$ satisfies an algebraic differential equation over $\mathbb{R}$ of order at most k.

Now since $y(x)$ also belongs to a Hardy field, then by Theorem 7 of [22], y has nested expansion $\{(\epsilon_i, s_i, m_i, d_i, \phi_i), i = 1, \ldots, I\}$ where

$$\sum_{i=1}^{I} s_i + \delta_I + m_I \leq k, \tag{4}$$

with δ_I equal to 0 or 1. This suffices to prove the lemma.

The relation (4) implies in particular that $m_I \leq k$. However the conditions $\gamma(\phi_i) < \gamma(l_i)$ ensures that $m_{i+1} > m_i$. So we if we generate a pnf of length greater than k, we may reject it as not corresponding to a possible solution. We note that k will be known from our construction of the tower $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k$. Thus repeated pnf calculations will terminate and give us a nested form for $y(x)$.

1.3. z-functions. If $f \to \pm\infty$ then $\exp(f)$ will have a different comparability class from f. However if $f \to 0$, we have the well known series for $\exp(f)$ in powers of f, and similarly if f tends to some other finite limit. For this reason exponentials whose arguments tend to finite limits are dealt with in a completely different way from those whose arguments tend to plus or minus infinity. Similar comments apply to logarithms and to negative powers. We stress the distinctions by introducing the z-function notation from [21]. Let t be a function which tends to zero. We write

$$\mathrm{zexp}(t) = \exp(t) - 1, \qquad \mathrm{zlog}(t) = \log(1+t), \qquad \mathrm{zpow}(r,t) = (1+t)^r - 1,$$

where $r \in \mathbb{R} \setminus \mathbb{N}$. We have $\mathrm{zexp}(t) \sim t$, $\mathrm{zlog}(t) \sim t$ and $\mathrm{zpow}(r,t) \sim rt$. Also for $n > 0$, we make the following definitions.

$$\begin{aligned}
\mathrm{zexp}_n(t) &= t^{-n}\left\{\mathrm{zexp}(t) - \left(t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right)\right\},\\
\mathrm{zlog}_n(t) &= t^{-n}\left\{\mathrm{zlog}(t) - \left(t - \frac{t^2}{2} + \cdots + (-1)^{n-1}\frac{t^n}{n}\right)\right\},\\
\mathrm{zpow}_n(r,t) &= t^{-n}\left\{\mathrm{zpow}(r,t) - \left(rt + \frac{r(r-1)}{2}t^2 + \cdots + \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n+1)}t^n\right)\right\},
\end{aligned}$$

$r \in \mathbb{R} \setminus \mathbb{N}$. We take $\mathrm{zexp}_0 = \mathrm{zexp}$, $\mathrm{zlog}_0 = \mathrm{zlog}$ and $\mathrm{zpow}_0 = \mathrm{zpow}$. The functions zexp_n, zlog_n and $\mathrm{zpow}_n(r,.)$, $n \geq 0$ are referred to collectively as *z-functions*. We stress that in all cases the arguments of z-functions tend to zero. Then the z-functions themselves also tend to zero, and moreover they are analytic at the origin.

We shall use the following lemma from [21, Lemma 4]. The proof is elementary.

Lemma 2. *Let t_0 and t_1 be elements of a Hardy field which tend to zero, and suppose that $\gamma(t_1) > \gamma(t_0)$. Then*

(1) $\mathrm{zexp}(t_0 + t_1) = \mathrm{zexp}(t_0)\,\mathrm{zexp}(t_1) + \mathrm{zexp}(t_0) + \mathrm{zexp}(t_1)$*;*
(2) $\mathrm{zlog}(t_0 + t_1) = \mathrm{zlog}(t_0) + \mathrm{zlog}(t_1(1 - \mathrm{zpow}(-1, t_0)))$*;*
(3) $\mathrm{zpow}(r, t_0 + t_1) = \mathrm{zpow}(r, t_0) + \mathrm{zpow}\{r, t_1(1 + \mathrm{zpow}(-1, t_0))\}\{1 + \mathrm{zpow}(r, t_0)\}$.

There is an analogue of Lemma 2 for the functions zexp_n, zlog_n and zpow_n with $n \geq 1$. This was essentially given in [21, Lemma 4]; we reproduce the result below for completeness.

Lemma 3. *Let t_0 and t_1 be as in the previous lemma, Then we have the following formulae:*

$$\frac{\mathrm{zexp}_n(t_0 + t_1)}{\mathrm{zpow}(-n, t_1/t_0) + 1} = \mathrm{zexp}_n(t_0)(1 + \mathrm{zexp}(t_1)) + \frac{t_1}{t_0^n}\{t_1^{n-1}\,\mathrm{zexp}_n(t_1) P_n(t_0) - Q_n(t_0, t_1)\},$$

where $P_n(t_0) = 1 + t_0 + \cdots + t_0^n/n!$ *and* $Q_n(t_0, t_1) = t_1^{-1}\{P_n(t_0 + t_1) - P_n(t_0)P_n(t_1)\}$.

$$\frac{\mathrm{zlog}_n(t_0 + t_1)}{\mathrm{zpow}(-n, t_1/t_0) + 1} = \mathrm{zlog}_n(t_0) + \frac{t_1}{t_0^n}\{(1 + \mathrm{zpow}(-1, t_0)) \times \\ \times (1 + \mathrm{zlog}_1(t_1(1 + \mathrm{zpow}(-1, t_0)))) + S_n(t_0, t_1)\},$$

where $S_n(t_0, t_1) = (R_n(t_0 + t_1) - R_n(t_0))/t_1$ *with* $R_n(t_0) = -t_0 + t_0^2/2 + \cdots + (-t_0)^n/n$.

$$\frac{\mathrm{zpow}_n(r, t_0 + t_1)}{\mathrm{zpow}(-n, t_1/t_0) + 1} = \mathrm{zpow}_n(r, t_0) \\ + \left\{\mathrm{zpow}\left(r, \frac{t_1}{1 + t_0}\right)(1 + \mathrm{zpow}(r, t_0)) - \{T_n(t_0 + t_1) - T_n(t_0)\}\right\} t_0^{-n},$$

where

$$T_n(t_0) = rt_0 + \frac{r(r-1)}{2}t_0^2 + \cdots + \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n+1)}t_0^n.$$

As regards the proof, the first two identities are given in [21], and the third is similar and equally elementary.

Apart from their notational use mentioned above, the z-functions give us explicit expressions for the tails of series. It will be important that we have these since we shall sometimes need to know whether expressions involving them are functionally equivalent to zero.

1.4. Zero-equivalence problem. In fact, there will be a number of different occasions when we shall need to be able to decide whether a given expression represents the zero function or not. Firstly if our given expression, $h(x, y)$, is actually equivalent to the zero function of the two variables x and y, then of course every function $y(x)$ satisfies the equation $h(x, y) = 0$, and nothing useful can be said about the asymptotics of solutions! Assuming that $h(x, y) \not\equiv 0$, we shall need to know whether the function $y = 0$ satisfies our equation; this can be done by checking $h(x, 0)$ for zero equivalence. Similar problems appear in more subtle guise when we develop expansions of solutions. Coefficients will be given by exp-log expressions, and we shall need to know if these are equivalent to zero or not.

Zero equivalence is of fundamental importance in the entire area of exact computation with transcendental functions, so it is hardly surprising that it plays a role in determining the asymptotics of implicit functions. However it brings a special difficulty concerning constants. Alas, there is no known algorithm for deciding whether a constant exp-log expression represents zero, although there do exist algorithms based on the Schanuel conjecture [3, 11, 12]. The normal practice in this area is to postulate the existence of an oracle which decides such questions. Modulo that assumption, there are a number of approaches to the zero-equivalence problem for functions. In particular the methods of [20, 23] can be easily adapted for use with functions of several variables, and this capability is already present in [10]. For the rest of the paper, we shall treat the problem of zero equivalence as one that can be solved, and not repeat the caveat that this requires the above-mentioned assumptions concerning constants.

2. Bivariate exp-log functions

Introduction. We now consider bivariate exp-log functions $h(x, y)$. We assume throughout this section that x is tending to infinity. Other cases may be obtained by change of variable. Our method is based on manipulation of bivariate asymptotic estimates. These estimates contain hypotheses on the relative growth of x and y, where y is assumed to be a function of x belonging to a Hardy field. For instance $\exp(xy) - x$ has asymptotic estimate $\exp(xy)$ if y tends to infinity or to a positive real value, or if $\lim \log y / \log x > -1$. The asymptotic estimate is $-x$ in all the other cases, except when $\lim \log y / \log x = -1$, where the estimate is left as a question mark ('?'). The question-mark case is of particular interest with regard to implicit functions, since it is the only one which can be associated with a solution of $h(x, y) = 0$. In this section we show how to compute such an estimate and how to handle the splitting into different cases automatically. In Section 3 we use the conditions associated with the question-mark estimates to compute the nested forms of solutions of bivariate exp-log equations. We also show there how one can refine a question-mark estimate.

The algorithm in this section can be viewed as a generalization of the univariate algorithm from [13] (itself a descendant of [21]).

2.1. Sketch of the method. We assume that a function $h(x, y) \in \mathcal{H}(x, y)$ is given by an expression tree E in which the leaves are either constants or one of the variables, x, y, and the nodes are operations which are either arithmetic or an application of the logarithm or exponential function. We begin by checking that h is not the zero function of x and y.

If $t_1, \dots, t_k$ are elements of $\mathcal{H}(x, y)$, we write $\mathcal{Z}(t_1, \dots, t_k)$ for the set of functions that can be built from the constants and real powers of the $t_1, \dots, t_k$ using arithmetic operations and the application of z-functions (to arguments tending to zero of course). The z-functions will be used to provide expansions in $t_1, \dots, t_k$, while keeping an explicit form for the remainders.

As in the univariate case, we build up asymptotic estimates for subexpressions of E.

Definition 1. A bivariate asymptotic estimate is a set of pairs $\{(\text{condition,expression})\}$ such that the conditions are mutually exclusive and together cover all the possibilities.

In the cases when the limit of y is zero or infinite, the expressions are either question marks or monomials in certain elements $t_1, \dots, t_k$ of $\mathcal{H}(x, y)$ obtained by giving the expression in question as an element of $\mathcal{Z}(t_1, \dots, t_k)$. The associated conditions will be conjunctions of basic conditions of one of the three forms

$$\begin{aligned}
&\gamma(t_i) > \gamma(t_j),\\
&\gamma(t_i) = \gamma(t_j) \quad \& \quad \operatorname{Pred}(\lim \log t_i / \log t_j, K_0),\\
&\gamma(t_i) = \gamma(t_j) = \gamma(t_k) \quad \& \quad \operatorname{Pred}(\lim \log t_j / \log t_i + r \lim \log t_k / \log t_i, K_0).
\end{aligned}$$

Here Pred is one of '$\neq$', '$<$', '$>$', '$=$', and K_0 and r are given non-zero constants. A question-mark estimate is always associated with a condition of one of the last two types, with Pred an '$=$' sign.

When the limit of y is non-zero and finite, the expressions are again either question marks or monomials, but the field of constants is now $\mathcal{H}(K)$ (the exp-log functions of K) rather than $\mathbb{R}$, where K is the (as yet undetermined) limit of y. The conditions are as above with the addition of predicates involving elements of $\mathcal{H}(K)$. In this case a question-mark estimate is always associated with an equality for a non-zero element of $\mathcal{H}(K)$.

Note that the above asymptotic estimates are not canonical, but depend on the choice of the t_i's. However, they do give some asymptotic information on the function and make it possible to solve the asymptotic implicit function problem in Section 3.

Our aim is to prove the following.

Theorem 2. *Let $h(x, y)$ be a bivariate exp-log function, then a bivariate asymptotic estimate for h can be computed.*

To compute the different possible cases, we build a tower of differential fields, $\mathbb{R}(x, y) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k$, with our given expression, E, an element of the top field and each $\mathcal{F}_i$ being a simple extension of $\mathcal{F}_{i-1}$ by an exponential or a logarithm. By working up the tower, we obtain a set $\mathcal{D}_j = \{(\mathcal{T}_i^j, \mathcal{C}_i^j), i = 1, \dots, \beta_j\}$, for each $j = 0, \dots, k$. Here each $\mathcal{T}_i^j$ is a set of elements of

$\mathcal{F}_j$ such as $t_1, \ldots, t_k$ above, with the property that $\mathcal{F}_j \subset \mathcal{Z}(\mathcal{T}_i^j)$. Similarly C_i^j is a conjunction of conditions of the type indicated above, concerning the comparability classes of the elements of $\mathcal{T}_i^j$. The conditions C_i^j for different i cover all cases and are mutually exclusive; that is to say, they give a breakdown into cases.

Once we have $\mathcal{D}_j$, then in the case C_i^j, we attempt to obtain a $\mathbb{K}$-power $\mathcal{T}_i^j$-monomial asymptotic to each subexpression f of h with $f \in \mathcal{F}_j$, where $\mathbb{K}$ is either $\mathbb{R}$ (in the cases when y tends to zero or $\pm\infty$) or $\mathcal{H}(K)$ (when y tends to a non-zero finite limit). These monomials will thus be of the form $At_1^{r_1}t_2^{r_2}\cdots t_k^{r_k}$ where $t_1, \ldots, t_k \in \mathcal{T}_i^j$ and $A, r_1, \ldots, r_k \in \mathbb{K}\setminus\{0\}$. If all the elements of $\mathcal{T}_i^j$ are of different comparability classes, then the asymptotic estimate of f can be inferred from the above monomial, which solves the problem in the case C_i^j. When two elements, t_a and t_b, of $\mathcal{T}_i^j$ have the same comparability class, we may write $t_b = t_a^\lambda$ with $\lim \lambda \in \mathbb{K}\setminus\{0\}$. Then just one value, say Λ, of $\lim\lambda$ will allow the possibility of cancellation between the powers of t_a and t_b in the monomial asymptotic to f, and this is the only value which can lead to a question-mark estimate for f. Similar considerations apply when three elements of $\mathcal{T}_i^j$ share the same comparability class. Because of the way our construction works, there cannot be more than three such elements. At the final stage, $\mathcal{D}_k$ gives us the information from which to obtain the asymptotic estimate of h.

To complete our description of the method, we must do the following:

(1) Show how to obtain each $\mathcal{D}_j$.
(2) In each case, show how to obtain a $\mathcal{T}_i^j$-monomial asymptotic to the given element of $\mathcal{F}_j$, or else obtain a question-mark estimate as described above.
(3) Specify precisely when in a pair $(\mathcal{T}_i^k, C_i^k)$ several elements of $\mathcal{T}_i^k$ share the same comparability class, and show how to handle this.
(4) Describe how new conditions are added to a C_i^k.

2.2. The cases when y tends to 0 or $\pm\infty$. By making a change of variable of the form $y \to \pm y^{\pm 1}$, we may confine our attention to the case $y \to +\infty$.

2.2.1. *Obtaining the $\mathcal{D}_j$.* Recall that we have a tower of fields $\mathbb{R}(x,y) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k$, with our given expression, E, defining an element, h, of $\mathcal{F}_k$. We use induction on the index j of the fields $\mathcal{F}_j$.

We also require some extra conditions on the $\mathcal{T}_i^j$. Suppose that $(\mathcal{T}_i^j, C_i^j) \in \mathcal{D}_j$ with $1 \le j \le k$. We insist that each element of $\mathcal{T}_i^j$ be of one of the forms $l_n(x)$, $l_m(y)$, $\exp(w)$, where $n, m \ge 0$ and w is an element of $\mathcal{F}_{j-1}$ which tends to infinity and is not asymptotic to a constant multiple of any $l_p(x)$ or $l_p(y)$ with $p > 0$. We also assume that if $l_n(x) \in \mathcal{T}_i^j$ with $n > 0$ then also $l_{n-1}(x) \in \mathcal{T}_i^j$, and similarly for $l_m(y)$ with $m > 0$. Initially, $\mathcal{T}_i^0 = \{x, y\}$ for $i = 1, 2, 3$ and $C_1^0 = \{\gamma(x) < \gamma(y)\}$, $C_2^0 = \{\gamma(x) = \gamma(y)\}$, $C_3^0 = \{\gamma(x) > \gamma(y)\}$.

Now suppose that we have defined $\mathcal{D}_{j-1}$ satisfying the conditions above, and that $\mathcal{F}_j = \mathcal{F}_{j-1}(g)$, where g is a subexpression of h which is either an exponential or a logarithm of an element, f, of $\mathcal{F}_{j-1}$.

Extension by a logarithm. We consider first the case when $g = \log(f)$. By induction we may suppose that $f \in \mathcal{Z}(\mathcal{T}_i^{j-1})$ and hence that we can compute a real-power $\mathcal{T}_i^{j-1}$-monomial asymptotic to f (In practice we usually compute several according to the various cases, see §2.2.2); thus

$$f = At_{a_1}^{r_1}t_{a_2}^{r_2}\cdots t_{a_p}^{r_p}(1 + f_0), \tag{5}$$

with $t_{a_1}, t_{a_2}, \ldots, t_{a_p} \in \mathcal{T}_i^{j-1}$, $r_1, r_2, \ldots, r_p \in \mathbb{R}\setminus\{0\}$, $f_0 \in \mathcal{F}_{j-1}$ and $f_0 \to 0$. Now $\log A$ and $\log(1+f_0)$ belong to $\mathcal{Z}(\mathcal{T}_i^{j-1})$, as does the logarithm of any t_a which is an exponential. The other $\log t_a$ are of the forms $l_{n+1}(x)$ or $l_{m+1}(y)$. Many of these will already belong to $\mathcal{T}_i^{j-1}$, but others will have to be adjoined. In view of our assumptions regarding $\mathcal{T}_i$, we have only to add one or both of a single $l_{n+1}(x)$ and a single $l_{m+1}(y)$, where n and m are the largest values for which $l_n(x)$ and $l_m(y)$ respectively belong to $\mathcal{T}_i^{j-1}$. It then remains to consider the possible order relations involving the comparability classes of the new elements.

We look in more detail at this for the case of $l_{n+1}(x)$, the case of $l_{m+1}(y)$ being similar. We split the condition C_i^{j-1} into the various cases according to the possible position of $\gamma(l_{n+1}(x))$ in the ordering of the existing comparability classes. First, $l_n x$ is necessarily already present in $\mathcal{T}_i^{j-1}$. We therefore just have to consider the comparability classes less than $\gamma(l_n(x))$. Also, existing relations involving $\gamma(l_n(x))$ may have implications for $\gamma(l_{n+1}(x))$. For example if $\gamma(l_n(x)) < \gamma(l_\beta(y))$, then $\gamma(l_{n+1}(x)) \leq \gamma(l_{\beta+1}(y))$. After these checks, we create a new condition for each possible position of $\gamma(l_{n+1}(x))$ among the elements of $\mathcal{T}_i^{j-1}$. Note that we do not claim that all the C_i^j's necessarily correspond to a possible solution. The only property we need is that there are a finite number of them, which are distinct and together cover all cases.

Extension by an exponential. Now consider the case when $g = \exp(f)$. Since $f \in \mathcal{Z}(\mathcal{T}_i^{j-1})$, we can compute a $\mathcal{T}_i^{j-1}$-monomial asymptotic to f as in (5), and hence determine the limit of f in the various cases. If $\lim f \in \mathbb{R}$, we can express $\exp(f)$ as an element of $\mathcal{Z}(\mathcal{T}_i^{j-1})$ using the z-function zexp, and in this case no addition to $\mathcal{T}_i^{j-1}$ is required. Otherwise, we may assume that $f \to \infty$. Then for each $t_a \in \mathcal{T}_i^{j-1}$, we consider the limit of $f/\log t_a$. If $\log(t_a) \in \mathcal{F}_{j-1}$ then this limit may be calculated in $\mathcal{Z}(\mathcal{T}_i^{j-1})$, and the order between the comparability classes $\gamma(\exp(f))$ and $\gamma(t_a)$ obtained. If we find that $f \sim K_0 \log t_a$ with $K_0 \in \mathbb{R} \setminus \{0\}$ and $\log t_a \in \mathcal{Z}(\mathcal{T}_i^{j-1})$, we rewrite $\exp(f)$ as $t_a^{K_0} \exp(f_1)$ where $f_1 = f - K_0 \log t_a$, and consider $\exp(f_1)$. This step can only be repeated a finite number of times since $f_1 \in \mathcal{Z}(\mathcal{T}_i^{j-1})$ and $\gamma(\exp(f_1)) < \gamma(t_a)$. In general, decreasing comparability class is not sufficient to ensure termination since a set of comparability classes does not have to be well ordered, but here this set is finite. If f is not asymptotic to a constant multiple of any $\log t_a$, we take $\mathcal{T}_{i'}^j = \mathcal{T}_i^{j-1} \cup \{\exp(f)\}$.

If $\log(t_a) \notin \mathcal{F}_{j-1}$, then $\log t_a$ is of one of forms $l_{n+1}(x)$, $l_{m+1}(y)$ not already in $\mathcal{T}_i^{j-1}$. Then we split C_i^j by adding each of the three conditions

$$\gamma(\exp(f)) < \gamma(t_a), \qquad \gamma(\exp(f)) = \gamma(t_a), \qquad \gamma(\exp(f)) > \gamma(t_a),$$

and in each case we take $\mathcal{T}_{i'}^j = \mathcal{T}_i^{j-1} \cup \{\exp(f)\}$.

Thus to obtain the ordering $C_{i'}^j$ from C_i^j, we consider the order between the comparability class of $\exp(f)$ and that of the elements $t_s \in \mathcal{T}_i^{j-1}$. This is governed by the limit of $f/\log(t_s)$, which will already have been calculated.

We note that the requirements on the new $\mathcal{T}_i^j$ will be met by our constructions; in particular, we obtain $\mathcal{F}_j \subset \mathcal{Z}(\mathcal{T}_i^j)$ and when two t_i's have the same comparability class, only one of them can be an exponential. A consequence of this last remark is that there can be at most three t_i's in the same comparability class.

2.2.2. *Expansions of Elements of $\mathcal{Z}(\mathcal{T}_i^j)$.* We have just shown how to obtain the $\mathcal{D}_j$ on the inductive assumption that it is possible to compute $\mathcal{T}_i^{j-1}$-monomials asymptotic to subexpressions f of h belonging to $\mathcal{F}_{j-1}$. To maintain the induction, we now have to show how to calculate similar $\mathcal{T}_i^j$-monomials for elements of $\mathcal{F}_j$. Essentially we just use the known expansions of the exponential and logarithmic functions and the function $t \to (1+t)^r$, $r \in \mathbb{R}$, extending the ideas of [21, 13] to the bivariate case. A new difficulty is that several elements of $\mathcal{T}_i^j$ may share the same comparability class. The first step is to check whether f is functionally equivalent to zero, and we assume in the sequel that the answer to this is negative.

Suppose that under the condition C_i^j, we have $\mathcal{T}_i^j = \mathcal{T}_{i,1} \cup \cdots \cup \mathcal{T}_{i,\eta}$, where the elements of each $\mathcal{T}_{i,s}$ have the same comparability class, $\gamma(\mathcal{T}_{i,s})$, and $\gamma(\mathcal{T}_{i,1}) > \gamma(\mathcal{T}_{i,2}) > \cdots > \gamma(\mathcal{T}_{i,\eta})$. We may also suppose that f contains some variables in $\mathcal{T}_{i,1}$, and by renumbering if necessary we may take these to be $t_1, \ldots, t_q$, with $q \in \{1, 2, 3\}$.

If $q = 1$, then we proceed as in the one-variable algorithm [13]: using Lemmas 2 and 3 where necessary, we can expand f as a series in t_1 with real exponents and coefficients belonging to $\mathcal{Z}(t_2, \ldots, t_n)$. The reason why Lemmas 2 and 3 may be needed is that, for example, the coefficient of t_1 in $\operatorname{zexp}(t_0 + t_1)$ will involve all the terms of the series expansion of zexp. Lemma 2 allows us to avoid this problem by supplying an explicit expression for the coefficient. Once we have the relevant coefficients, they are tested for zero-equivalence. We stop the expansion process as soon as we reach a non-zero coefficient. By induction, it is then possible to compute a $\mathcal{Z}(t_2, \ldots, t_n)$-monomial asymptotic to this coefficient, which we multiply by the appropriate power of t_1 to get a monomial asymptotic to f.

If $q \in \{2, 3\}$, then we write $t_i = t_1^{\lambda_i}$ for $i = 2, \ldots, q$, and put $\Lambda_i = \lim \lambda_i$; then $\Lambda_i \in \mathbb{R} \setminus \{0\}$. The Λ_i are initially undetermined. We then proceed as before to compute a series expansion in t_1, except that now the exponents are $\mathbb{Q}$-linear combinations of 1 and the λ_i's, $i = 2, \ldots, q$. Every time we have to compare two such exponents, we compute their difference $d(\lambda_2, \ldots, \lambda_q)$. Assuming d involves the λ_i's, we split condition C_j^i into three subcases depending on the sign of $d(\Lambda_2, \ldots, \Lambda_q)$ being positive, negative or zero. In the latter case, we return a question-mark estimate for f. As regards termination, we note that only finitely many splittings can take place during such an expansion, their number being bounded in terms of the number of nodes in the expression tree for f.

Note that as an optimization, it is sometimes possible to use some information on the Λ_i's. In particular, if $t_1 = l_m x$ and $t_2 = l_n y$ and it is known that $\gamma(l_{m-1} x) = \gamma(l_{n-1} y)$, then $\Lambda_2 = 1$.

If it is desired, further terms can be calculated by applying the same process to the difference between f and the terms already obtained.

2.3. Finite non-zero limits. In order to compute a bivariate asymptotic estimate of $h(x, y)$, it is also necessary to consider the case when y tends to a finite non-zero limit K, and discuss the possible asymptotic behaviours of $h(x, y)$ according to the value of K. Our algorithm in this case consists in treating y as if it were a finite non-zero constant (and not merely asymptotic to it). We apply the one-variable variant of the algorithm described above, except that the basic operations now take place in the field $\mathcal{H}(y)$ of exp-log functions in y. Every time a comparison is necessary, we shall produce a splitting into several cases depending on the sign of some exp-log function of K. Special values of K, which might give a singularity for example, cause no fundamental problem provided there are only finitely-many of them and they can be calculated. For they can always be associated with question-mark estimates.

So we proceed as above to build a tower of differential fields $\mathcal{Z}(x) = \mathcal{Z}(t_1) \subset \cdots \subset \mathcal{Z}(t_1, \ldots, t_n)$ such that the last one contains h. Now the functions t_i's will be functions of x only, so that we can insure that the t_i's have different comparability classes (except possibly for a finite number of explicitly determined special values of K attached to a question-mark estimate). The extensions of the tower of fields work almost as before.

When we consider a new logarithm $\log(f)$, we first expand its argument in the previous field, to get an estimate

$$f \sim A t_1^{r_1} \cdots t_k^{r_k}. \tag{6}$$

Here the r_i's and A are exp-log functions of y. Since A is an exp-log function, it is continuous except at a finite number of points. So from $\lim A(y) = 0$ or $\lim A(y) = \pm\infty$ as y tends to K, we can get an exp-log equation satisfied by K. We first output these equations with a question-mark estimate. Otherwise, if one of the $\log t_i$'s is not already in the field, which can only happen if $t_i = l_m x$ for some m, then we add $l_{m+1}x$ to the field and the expansion of $\log f$ is obtained by expanding the product (6).

When we consider a new exponential $\exp(f)$, we first get an estimate (6), then again we output question-mark estimates with equations satisfied by K which make A tend to 0 or $\pm\infty$. Then we compute $f/\log t_a$ for the t_a's whose logarithm is already in the field. Every time we get an estimate like (6) for $f/\log t_a$, we output the corresponding equations for K with a question-mark estimate. If for some t_a we obtain $f/\log t_a \sim A$, we then rewrite $\exp(f) = t_a^A \exp(f - A\log t_a)$ and proceed to deal with the new exponential. If for all the t_a's we discover that $f/\log t_a$ does not tend to a constant, then we add $\exp(t_1^{r_1}\cdots t_k^{r_k})$ as a new comparability class.

Again, all this depends on the ability to compute a monomial like (6) when given a rational expression in $\mathcal{Z}(t_1,\dots,t_k)$ with exponents and coefficients which are exp-log functions in y. We use the same algorithm as before (in its univariate version), except that every time it is necessary to compute the sign of an exp-log function $f(y)$, we split the computation into three cases depending on $f(K) < 0$, $f(K) > 0$ and $f(K) = 0$. In the latter case, we stop the computation and return this condition together with a question-mark estimate. The algorithm still terminates with a finite number of cases because the number of comparisons needed to get an asymptotic estimate is bounded in terms of the number of nodes in the expression tree for h.

2.4. Example. We start with an example where we consider only the case of the finite non-zero limit.

$$F(x,\phi) = \exp(\log(x)\exp(\phi)) - \exp(\phi\log(x)) - x^2\exp(-\phi\log(x)). \tag{7}$$

Let Φ be the finite non-zero limit of ϕ. The appropriate field is readily found to be $\mathcal{Z}(\log(x), x)$. We then expand F in this field, with exponents and coefficients in $\mathcal{H}(\phi)$. What we get is

$$F(x,\phi) = x^{e^\phi} - x^\phi - x^{2-\phi}.$$

In order to decide which is the leading term in this expression, we split into several cases depending on the differences of the exponents. Possible question-mark cases are given by $e^\Phi = \Phi$, which has no real solutions, $\Phi = 2 - \Phi$, which gives $\Phi = 1 < e^\Phi$ (so $F(x,\phi) \sim x^e$) and $e^\Phi = 2 - \Phi$, which has a single root, α, lying between 0 and 1. Note that these decisions on implicit constants may require a powerful oracle, such as developed in [12]. To summarize, we obtain

$$F(x,\phi) \sim \begin{cases} x^{e^\phi} & \text{if } \Phi > \alpha, \\ ? & \text{if } \Phi = \alpha, \\ x^{2-\phi} & \text{if } \Phi < \alpha, \end{cases} \qquad \text{with} \qquad e^\alpha + \alpha - 2 = 0.$$

2.5. Example. We consider the function

$$H = \frac{e^{x^2+2x\log^2 x + y}}{e^{x^2+x\log^2 x+y} - 1}, \qquad x \to +\infty.$$

We start with $\mathcal{F}_0 = \mathbb{R}(x,y)$ with conditions $C_1^0 = \{\gamma(x) < \gamma(y)\}$, $C_2^0 = \{\gamma(x) = \gamma(y)\}$, $C_3^0 = \{\gamma(y) < \gamma(x)\}$.

The first step is to build $\mathcal{D}_1$ corresponding to $\mathcal{F}_0(\log x)$. Necessarily $\gamma(\log x) < \gamma(x)$, so that we obtain five cases. In all cases $\mathcal{T}_i^1 = \{x, y, \log x\}$, $i = 1, \ldots, 5$; the possible orderings are

$$\begin{aligned}
C_1^1 &= \{\gamma(\log x) < \gamma(x) < \gamma(y)\},\\
C_2^1 &= \{\gamma(\log x) < \gamma(x) = \gamma(y)\},\\
C_3^1 &= \{\gamma(\log x) < \gamma(y) < \gamma(x)\},\\
C_4^1 &= \{\gamma(\log x) = \gamma(y) < \gamma(x)\},\\
C_5^1 &= \{\gamma(y) < \gamma(\log x) < \gamma(x)\}.
\end{aligned}$$

Next we build $\mathcal{D}_2$ corresponding to $\mathcal{F}_2 = \mathcal{F}_1(f)$, with $f = \exp(x^2 + x\log^2 x + y)$. Since this is an extension by an exponential, we first compute a $\mathcal{T}_i^1$-monomial asymptotic to $\log f$. The first part $x^2 + x\log^2 x$ is dealt with as in the univariate case, leading to $x^2(1 + \log^2 x/x)$. We then have to determine the limit of x^2/y, in each of the cases C_i^1, $i = 1, \ldots, 5$. In cases C_3^1, C_4^1 and C_5^1, the limit is easily seen to be infinite. In case C_1^1, the limit depends on the limit of y which can be 0 or $\pm\infty$ leading to the corresponding limit for $\log f$. The last case is C_2^1, where we set $y = x^\lambda$ with $\Lambda = \lim \lambda \in \mathbb{R} \setminus \{0\}$. Then the limit depends on the sign of $\Lambda - 2$. To summarize we have the following identities

$$\log(f) = \begin{cases} y(1 + x^2/y + x\log^2 x/y) & \text{if } (C_1^1 \text{ and } y \to \pm\infty)\\ & \text{or } (C_2^1 \text{ and } \lim \log|y|/\log x > 2),\\ ? & \text{if } C_2^1 \text{ and } \lim \log|y|/\log x = 2,\\ x^2(1 + \log^2 x/x + y/x^2) & \text{if } (C_1^1 \text{ and } y \to 0) \text{ or } C_3^1 \text{ or } C_4^1 \text{ or } C_5^1\\ & \text{or } (C_2^1 \text{ and } \lim \log|y|/\log x < 2). \end{cases} \tag{8}$$

This leads to a splitting of C_1^1 into $\bar{C}_1^2$ and $\bar{C}_2^2$ according to whether $\lim y$ is $\pm\infty$ or 0. The condition C_2^1 is split into $\bar{C}_3^2$, $\bar{C}_4^2$ and $\bar{C}_5^2$ according to whether $\Lambda = \lim \log|y|/\log x$ is greater than, equal or less than 2. The conditions C_3^1, C_4^1 and C_5^1 are unaffected and become $\bar{C}_6^2$, $\bar{C}_7^2$ and $\bar{C}_8^2$. In all cases except C_4^2, $\log f$ tends to $\pm\infty$ and is not asymptotic to the logarithm of any existing t_a, we therefore insert f in the corresponding $\mathcal{T}_i^2$'s. Next, we have to compute the position of $\gamma(f)$ in their respective $\bar{C}_i^2$'s. In the cases $\bar{C}_1^2$ and $\bar{C}_3^2$, $\log f \sim y$ implies $\gamma(f) > \gamma(y)$ so that $\gamma(f)$ is the largest comparability class so far. Similarly, in all the other cases except $\bar{C}_2^2$ and $\bar{C}_4^2$, $\log f \sim x^2$ implies $\gamma(f) > \gamma(x)$ so that again $\gamma(f)$ is the largest comparability class so far. In the case $\bar{C}_2^2$, since $\log y$ does not belong to $\mathcal{T}_2^2$, it is not possible to determine the relative position of $\gamma(f)$ and $\gamma(y)$ so that a new splitting is necessary. The last case is $\bar{C}_4^2$ which leads to a question-mark estimate.

After further relabelling, we obtain

$$\begin{aligned}
C_1^2 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) < \gamma(f), y \to \pm\infty\},\\
C_2^2 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) < \gamma(f), y \to 0\},\\
C_3^2 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) = \gamma(f), y \to 0\},\\
C_4^2 &= \{\gamma(\log x) < \gamma(x) < \gamma(f) < \gamma(y), y \to 0\},\\
C_5^2 &= \{\gamma(\log x) < \gamma(x) = \gamma(y) < \gamma(f), \lim \log|y|/\log x > 2\},\\
C_6^2 &= \{\gamma(\log x) < \gamma(x) = \gamma(y), \lim \log|y|/\log x = 2\},\\
C_7^2 &= \{\gamma(\log x) < \gamma(x) = \gamma(y) < \gamma(f), \lim \log|y|/\log x < 2\},\\
C_8^2 &= \{\gamma(\log x) < \gamma(y) < \gamma(x) < \gamma(f)\},\\
C_9^2 &= \{\gamma(\log x) = \gamma(y) < \gamma(x) < \gamma(f)\},\\
C_{10}^2 &= \{\gamma(y) < \gamma(\log x) < \gamma(x) < \gamma(f)\}.
\end{aligned}$$

Although some of these cases indicate the same ordering between comparability classes, they correspond to different rewritings of $\log f$ in terms of earlier functions.

We now turn to the last extension, $\mathcal{F}_3 = \mathcal{F}_2(g)$, with $g = \exp(x^2+2x\log^2 x+y)$. Once again, this is an extension by an exponential, therefore we first compute a $\mathcal{T}_2$-monomial asymptotic to $\log g$ in each of the cases C_i^2, $i = 1,\dots,10$. Since $x^2 + 2x\log^2 x$ and $x^2 + x\log^2 x$ are asymptotically equivalent, we obtain almost the same estimates as in (8) without any new splitting. The difference with the above is that now $\log g \sim \log f$ in the cases when $\log g \to \pm\infty$, except perhaps in case C_6^2. We consequently first rewrite g as $f\exp(\log g - \log f)$ and turn to $h = \exp(\log g - \log f) = \exp(x\log^2 x)$, with $\gamma(h) < \gamma(f)$. The estimate of $\log h$ is readily obtained as $\log h = x\log^2 x$, from which follows that $\gamma(h) > \gamma(x)$. The ordering of $\gamma(h)$ and $\gamma(y)$ cannot always be determined from these two inequalities on $\gamma(h)$, and since $\log y \notin \mathcal{T}_2$ this leads to new splittings. This step thus produces

$$\begin{aligned}
C_1^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) < \gamma(h) < \gamma(f), y \to \pm\infty\},\\
C_2^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) = \gamma(h) < \gamma(f), y \to \pm\infty\},\\
C_3^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(h) < \gamma(y) < \gamma(f), y \to \pm\infty\},\\
C_4^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) < \gamma(h) < \gamma(f), y \to 0\},\\
C_5^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(y) = \gamma(h) < \gamma(f), y \to 0\},\\
C_6^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(h) < \gamma(y) < \gamma(f), y \to 0\},\\
C_7^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(h) < \gamma(y) = \gamma(f), y \to 0\},\\
C_8^3 &= \{\gamma(\log x) < \gamma(x) < \gamma(h) < \gamma(f) < \gamma(y), y \to 0\},\\
C_9^3 &= \{\gamma(\log x) < \gamma(x) = \gamma(y) < \gamma(h) < \gamma(f), \lim \log|y|/\log x > 2\}\\
C_{10}^3 &= \{\gamma(\log x) < \gamma(x) = \gamma(y), \lim \log|y|/\log x = 2\}\\
C_{11}^3 &= \{\gamma(\log x) < \gamma(x) = \gamma(y) < \gamma(h) < \gamma(f), \lim \log|y|/\log x < 2\}\\
C_{12}^3 &= \{\gamma(\log x) < \gamma(y) < \gamma(x) < \gamma(h) < \gamma(f)\},\\
C_{13}^3 &= \{\gamma(\log x) = \gamma(y) < \gamma(x) < \gamma(h) < \gamma(f)\},\\
C_{14}^3 &= \{\gamma(y) < \gamma(\log x) < \gamma(x) < \gamma(h) < \gamma(f)\}.
\end{aligned}$$

We are now finally ready to compute the possible behaviours of $H = g/(f-1)$. Depending on the cases, we have the following rewritings

$$H = \begin{cases} -fh[1+\mathrm{zpow}(-1,-f)] & \text{if } y \to -\infty \ \&\ (C_1^3 \text{ or } C_2^3 \text{ or } C_3^3), \\ ? & \text{if } C_{10}^3, \\ h[1+\mathrm{zpow}(-1,-1/f)] & \text{otherwise.} \end{cases}$$

When rewritten in terms of the original variables, what we get is that the first case and the last case correspond respectively to

$$H \sim -e^{x^2+2x\log^2 x+y}, \qquad H \sim e^{x\log^2 x}.$$

The case leading to a question-mark estimate is studied in more detail at the end of Section 3. Note that in this example, the case when y tends to a finite non-zero limit was treated as a special case of C_{14}^3.

3. Implicit functions

We now apply the algorithm of the previous section to find the asymptotic behaviour of implicit functions. Our algorithm is as follows:

Input $h(x,y)=0$, where h is a bivariate exp-log function.
Step 1 Compute a bivariate asymptotic estimate of h.
Step 2 Select the conditions that lead to a question-mark estimate.
Step 3 Use these conditions to either compute a partial nested form of the solution or reduce the problem to a simpler one and iterate.
Step 4 In order to calculate the remaining terms in the sequence which is the nested form, make the first of these a new dependent variable. Then substitute for y and iterate.

The reason for Step 2 is that question-mark estimates are the only ones consistent with the function h being equal to zero. We have seen that question-mark estimates can only occur when the corresponding condition contains a basic relation of the form

$$\sum_{j=2}^{q} r_j \lim(\log t_j/\log t_1) = \Lambda, \tag{9}$$

with $q \in \{2,3\}$ and Λ a specified constant. Our idea is to use these basic relations to get a new equation satisfied by y for which the number of exponentials occurring in the corresponding $\mathcal{T}$ is smaller than in the original problem. At the end, we shall be left with an equation of the type $l_m y = l_n^\lambda x$, with $\lim\lambda = \Lambda$ a specified constant. From this equation we deduce the partial nested form $y = e_m(l_n^\Lambda x\phi(x))$, $\gamma(\phi) < \gamma(l_n x)$. Then substituting this in the original equation and making $\phi(x)$ the new unknown function will give one more term of the nested form. Iterating this we eventually get the possible nested forms of y and as many terms as we want of the corresponding nested expansion.

Another use of this algorithm is to *refine* a bivariate asymptotic estimate. This refinement is simply obtained by stopping the algorithm after several steps. The conditions which initially lead to question-mark estimates are then replaced by several cases with corresponding asymptotic behaviours.

We shall prove the following theorem.

Theorem 3. *Let $h(x,y)$ be a bivariate exp-log function. Then the above algorithm finds nested expansions of all the real solutions $y(x)$ of $h(x,y(x))=0$.*

Note that the algorithm may also produce nested expansions which do not correspond to any solution. However, in most cases, increasing the order of the computation should eventually lead to a contradiction from which these spurious solutions can be rejected.

Note also that in many cases it is not necessary to remove all the exponentials (i.e. reduce to a relation of the form $l_m y = l_n^\lambda x$), since an estimate may become apparent earlier. This type of optimization will be described in a forthcoming paper, [19].

3.1. Obtaining the first pnf. The proof of Theorem 3 is based on an induction on the number of exponentials occurring in the $\mathcal{T}$ in which we are working.

Starting from the bivariate asymptotic estimate of h, we consider in turn each of the conditions (9) which lead to a question-mark estimate. Here, $\Lambda \in \mathbb{R} \setminus \{0\}$ is known. All the t_j's have to belong to the same $\mathcal{T}_\nu^i$. We recall that each of them must be of one of the forms, $l_n(x)$, $l_m(y)$, $\exp w$ and that at most one of them is of the form $\exp w$. Our aim is to reduce the problem to the case when none of the t_j's is an exponential, which means that $q = 2$ and we can get a partial nested form for y.

First in the case when $q = 3$, then only one of the t_j's is an exponential, which means that we can take $t_1 = l_m y$ and $t_2 = l_n x$ and $\gamma(t_1) = \gamma(t_2)$. But then, we can take $e_{m+1}(l_{n+1} x \phi(x))$ as a pnf for y, where $\gamma(\phi) < \gamma(l_{n+1})$.

The other case is $q = 2$. Suppose that $t_1 = l_m y$ and $t_2 = \exp(w)$ where $w \in \mathcal{Z}(\mathcal{T}_\nu^{i-1})$, other cases are similar. In this case, we have a relation

$$r_2 l_{m+1}(y) \sim \Lambda w, \tag{10}$$

or equivalently $r_2 l_{m+1}(y) = \Lambda w(1+\epsilon)$, with $\lim \epsilon = 0$. Then we apply the general algorithm to the corresponding equation $r_2 l_{m+1} y = \Lambda w$. Since our method is based on determining when two leading terms have the same comparability class, the question-mark estimates will also cover solutions of (10). It may be necessary to add $l_{m+1} y$ to the field, but as seen in §2.2.1, this does not increase the number of exponentials in the current $\mathcal{T}$. Then by induction, we get a partial nested form for y solution of the equation corresponding to (10), which has to be also a partial nested form for the solution of $h(x, y(x)) = 0$.

3.2. Obtaining a nested form by pnf calculations. Having obtained a pnf, say $y(x) = e_s(l_n(x)^d \phi(x))$, we substitute for y in E, making ϕ a new variable. Then we calculate a pnf for ϕ. This will yield a nested form after a finite number of repetitions, provided that the process terminates and we do not obtain an 'improper' nested form, that is to say an expression $e_s^{\pm 1}(l_n(x)(K+z))$ with $s, n > 0$, K a positive constant and $z \to 0$. In the latter eventuality, we substitute $e_{s-1}(l_{n-1}^K(x) e_1(l_n(x) z))$ for y in the expression E and take $\phi = e_1(l_n(x) z)$ as the new variable. If $K \neq 1$, then a pnf for ϕ gives us the next segment of the nested form. If $K = 1$, we are in the same situation as before, but with s and n having been reduced by one; so the situation may only recur a finite number of times.

The termination of the sequence of pnfs is guaranteed by Lemma 1. We merely discard any pnf whose length exceeds k.

3.3. Example. We now look at the equation $H + 1 = 0$, where H is the function of §2.5. We see at once that the only case leading to a question-mark estimate has $\lim \log |y| / \log x = 2$. Accordingly, we put $y = x^2 \phi$ with $\gamma(\phi) < \gamma(x)$ and substitute, to get the numerator of $H + 1$ equal to

$$N = \exp\{x^2(1+\phi) + 2x \log^2 x\} + \exp\{x^2(1+\phi) + x \log^2 x\} - 1.$$

A division into cases, very similar to the existing one, now gives that $\phi \sim -1$ for a question-mark estimate, and a further substitution for $\phi + 1 = u$ with $u \to 0$ yields

$$N = \exp\{x^2 u + 2x \log^2 x\} + \exp\{x^2 u + x \log^2 x\} - 1.$$

Now, the splitting into cases is again similar to §2.5. Setting again $f = \exp(x^2 u + 2x \log^2 x)$, we get that either $f \to \pm\infty$, or we obtain a question-mark estimate, in the case $\lim \log |u| / \log x = -1$. When $f \to \infty$, then $g = \exp(x^2 u + x \log^2 x)$ also tends to infinity but at a slower rate, which implies that N cannot tend to 0. Thus we are led to set $u = x^{-1}\psi$ with $\gamma(\psi) < \gamma(x)$. We now study

$$N = \exp\{x\psi + 2x \log^2 x\} + \exp\{x\psi + x \log^2 x\} - 1.$$

Similar considerations lead to setting $\psi = \log^2(x)\theta$, with $\gamma(\theta) < \gamma(\log x)$. Repeating the same process several times eventually yields that there is only one possible solution, with the following behaviour:

$$y \approx -x^2 - 2x \log^2 x + \frac{e^{-x \log^2 x}}{x^2} - \frac{e^{-2x \log^2 x}}{2x^2} + \cdots.$$

3.4. Example. We consider the equation $H(x, y) = 0$, with

$$H(x, y) = \exp\{\log(x) \cdot \exp(\log(y)/\log(x))\} - y - x^2/y.$$

So as to avoid tedious repetition, we combine certain cases where these clearly give the same result, and shortcut a few of the steps that would be taken by an actual implementation. We write

$$G = \exp\{\log x \cdot \exp(\log y / \log x)\}.$$

In the case $\gamma(x) < \gamma(y)$, it is first found by the algorithm that

$$\gamma(\log(y)) < \gamma(\exp[\log(y)/\log(x)]) < \gamma(y).$$

Then we have

$$\frac{\log G}{\log y} = \frac{\log(x)}{\log(y)} \exp(\log(y)/\log(x)),$$

whose limit therefore depends on the limit of $\exp(\log(y)/\log(x))$. When $y \to \infty$, this limit is ∞, and it is 0 when $y \to 0$. This yields the position of the comparability class of G and thus in this case

$$H(x, y) \sim \begin{cases} G & \text{if } y \to \infty, \\ -x^2/y & \text{if } y \to 0. \end{cases}$$

The case $\gamma(x) > \gamma(y)$ is similar. We first find $\gamma(\log(x)) > \gamma(\log(y))$ from which follows that $\log y / \log x \to 0$. Then we consider the comparability class of G. We have

$$\frac{\log G}{\log x} = \frac{\log(x) \cdot \exp(\log(y)/\log(x))}{\log(x)} \sim 1.$$

So $G = xg$, where $g = \exp\{\log(x) \cdot \mathrm{zexp}(\log(y)/\log(x))\}$, and we know that $\gamma(g) < \gamma(x)$. The algorithm would then proceed to determine the comparability class of g, but we can stop here since it is easy to see that in this case

$$H(x, y) \sim -x^2/y.$$

The last case is $\gamma(x) = \gamma(y)$. Here we get a question-mark estimate at the first step. From $\gamma(x) = \gamma(y)$ we then deduce the pnf $y = \exp(\ln(x)\phi)$ with $\gamma(\phi) < \gamma(\ln(x))$. It is not difficult to see that if y is a solution of $H(x, y) = 0$, then ϕ tends to a finite limit and satisfies the equation $F(x, \phi) = 0$, with F defined by (7). From the result found in §2.4, we deduce that $\lim \phi = \alpha$, with $e^\alpha = 2 - \alpha$. We then write $\phi = \alpha + \psi$, and for convenience, we denote x^ψ by Y; so $\gamma(Y) < \gamma(x)$. Then

$$H(x, y) = h_1(x, Y) = \exp\{\log(x) \exp(\alpha + \log(Y)/\log(x))\} - x^\alpha Y - x^{2-\alpha}/Y. \tag{11}$$

If $Y \to \pm\infty$ or $Y \to 0$, the comparability classes are partially ordered by $\gamma(\log(Y)) < \gamma(Y) < \gamma(x)$ and $\gamma(\log x) < \gamma(x)$. Then the logarithm of $G = \exp\{\log(x) \exp(\alpha + \log(Y)/\log(x))\}$ is compared to $\log x$ to find that their ratio tends to a finite limit $e^\alpha = 2-\alpha$. Then G is rewritten $x^{2-\alpha} g$ with $g = \exp[(2-\alpha)\log(x)\,\mathrm{zexp}(\log Y/\log x)]$. From comparing $\log g$ and $\log Y$, it is found that $g = Y^{2-\alpha}\Psi$ with $\Psi \in \mathcal{Z}(\log Y, \log x)$ and $\lim \Psi = 1$. Thus we expand h_1 in powers of x (whose comparability class is the largest one) and get

$$h_1(x, Y) = x^{2-\alpha}[Y^{2-\alpha}\Psi - Y^{-1}] + \cdots \tag{12}$$

Then the precise form of the leading term depends on whether Y tends to 0 or ∞.

The only remaining case is when Y tends to a non-zero finite limit. Then by expanding (11) in $\mathcal{Z}(x, \log x)$ we get (12) with Ψ replaced by 1. From this it follows that the only value of $\lim Y$ leading to a question-mark estimate is 1. Then writing $Y = 1 + z$ gives

$$H(x, y) = x^{2-\alpha}\{(3 - \alpha + o(1))z - x^{2\alpha-2}(1 + o(1))\}, \tag{13}$$

and for a question-mark estimate we must have $z \sim x^{2\alpha-2}/(3-\alpha)$. Thus we obtain just one possible asymptotic form for a solution of $H(x, y(x)) = 0$, namely

$$y(x) = x^\alpha + \frac{x^{3\alpha-2}}{3-\alpha} + \cdots,$$

where α is the real root of $e^t = 2 - t$. From (13), we see that H changes sign as z passes from $x^{2\alpha-2}/(3 - \alpha + \varepsilon)$ to $x^{2\alpha-2}/(3 - \alpha - \varepsilon)$ (with ε a small positive real number), and it follows that H does indeed have a root of the above form.

Conclusion

We have given an algorithm to determine automatically the possible asymptotic behaviours of implicit exp-log functions. However, our paper leaves several issues outstanding.

As previously mentioned, the algorithm in this paper does nothing to show that solutions exist (although it may demonstrate that no real solutions exist). We expect that in many cases, after a finite number of steps, we encounter an equation like (13) from which it is possible to deduce that a solution with the specified behaviour does exist. To be able to do this in all cases, we would need to be able to handle the different comparability classes present in solutions.

A sufficient grip on the comparability classes might also bring within reach the solution of another problem, that of calculating the asymptotics of expressions containing inverse or implicit functions. For this, it is not generally sufficient to be able to generate expansions of implicit functions; one must be able to handle cancellation problems as well. Certain special cases can be reduced to soluble problems; for example if f and h are exp-log functions of one and two variables respectively, and $y(x)$ satisfies $h(x, y) = 0$, then $y(x)+f(x)$ satisfies $h(x, y-f(x)) = 0$. However what one would really like is to be able to extend asymptotic fields by implicit functions; see [25] for a definition.

Other outstanding issues include the generalization of two-variable exp-log functions, for example to include integration in the signature, handling more than two variables, and calculating the topology of solution curves.

Another matter which we have not really considered is that of efficiency. In several places of the algorithm, there are possible shortcuts under certain circumstances that we have not described here in the sake of clarity. Also, some classes of exp-log equations can be treated faster by *ad hoc* algorithms. We plan to come back to these issues in another paper [19].

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