Symbolic Software for Lie Symmetry Analysis

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To appear as Chapter 13
CRC Handbook of Lie Group Analysis of Differential Equations
Volume 3: New Trends in Theoretical Developments and Computational Methods
Edited by N.H. Ibragimov
CRC Press, Boca Raton, Florida, 1995

November 15, 1999

MCS-94-07 August 1994

1991 Mathematics Subject Classification.

Primary 68Q40, 35-04; Secondary 22E60, 34-04

[1]Research Supported in Part by NSF under Grant CCR-9300978

Chapter 13

Abstract

A survey of techniques and symbolic programs for the determination of Lie symmetry groups of systems of differential equations is presented. The purpose, methods and algorithms of symmetry analysis are outlined. An exhaustive review of the literature, including old and modern books and papers presenting key concepts is given. Special attention is paid to methods for reducing the determining equations into standard form, and their subsequent integration. Several examples illustrate the use of the Lie symmetry software. Throughout the paper, new trends in the development of symbolic packages for Lie symmetry analysis are indicated.

13.1. INTRODUCTION AND LITERATURE

The Norwegian mathematician Marius Sophus Lie (1842-1899) and Felix Klein (1849-1925), a German geometer, pioneered the study of transformation groups that leave systems of equations invariant. Klein's work focused on the role of finite groups in the study of regular bodies and algebraic equations. Lie founded the theory of continuous transformations groups [179, 180, 181, 182, 183, 184, 185] and Lie groups. Although Lie's starting point had been geometry, the inspirational source for his group theoretic investigations was the field of differential equations. His goal was to establish a theory of integration of differential equations that would mirror Abel's theory for the solution of algebraic equations. His work brought many diverse and ad hoc integration methods for solving special classes of differential equations under a common conceptual umbrella.

Later on, the concept of *symmetry* evolved into one of the most explosive developments of mathematics and physics throughout the twentieth century. The theory of Lie groups and Lie algebras is now applied to diverse fields of mathematics including differential geometry, algebraic topology, bifurcation theory, numerical analysis, special functions, to name a few; and to nearly any area of theoretical physics, in particular classical, continuum and quantum mechanics, fluid dynamics, relativity, and particle physics.

Lie's infinitesimal transformation method provides the most widely applicable technique to find closed form solutions of ordinary differential equations (ODEs). Standard solution methods for first-order or linear ODEs can be characterized in terms of symmetries. For nonlinear ODEs, Lie's method, when it succeeds, provides a means of reducing the solution to a series of quadratures and can be implemented in symbolic programs [41]. Under certain conditions, first- and second-order ODEs can be linearized via Lie group transformations [149, 151]. Through the group classification of ODEs, Lie also succeeded in identifying all ODEs that can either be reduced to lower-order ones, or completely integrated via group theoretic techniques.

Applied to partial differential equations (PDEs), the method leads to group-invariant solutions, conservation laws, invariant center manifolds in bifurcation theory [56, 111, 112, 252, 299], etc. Exploiting the symmetries of PDEs, new solutions can be derived from old ones, and PDEs can be classified into equivalence classes. Special, physically significant solutions arising from symmetry methods allow one to investigate the asymptotic or physical behavior of general solutions. The group-invariant solutions obtained via Lie's approach provide insight into the physical models themselves. Explicit solutions also serve as benchmarks in the design, accuracy testing, and comparison of numerical algorithms.

Lie's original ideas greatly influenced the study of physically important systems of differential equations in classical mechanics, fluid dynamics, elasticity, and many other applied areas. Currently, Lie symmetry methods are applied to difference, differential-difference equations [177, 301] and integro-differential equations. For a good review of the present state of affairs see [60].

The application of Lie groups methods to concrete physical systems involves tedious, mechanical computations. Even the calculation of the continuous symmetry group of a modest system of differential equations is prone to fail, if done with pencil and paper. Programmable computer algebra systems (CAS) such as Mathematica, MACSYMA, Maple, REDUCE, AXIOM and MuPAD, are extremely useful aides in such computations. Symbolic packages, written in the language of CAS, can find the defining (or determining) equations of the Lie symmetry group. The most sophisticated packages then reduce the determining system into an equivalent but more suitable system, subsequently solve that system in closed form, and go on to calculate the infinitesimal generators that span the Lie algebra of symmetries.

A large body of literature exists on the topic of Lie symmetries. We list some older books [8, 9, 10, 26, 29, 35, 48, 51, 55] [73, 84, 130, 140, 161, 174, 196, 197, 198, 220, 222, 272], more modern books [12, 13, 18, 21, 36, 45, 78] [80, 90, 103, 105, 106, 107, 108] [110, 112, 144, 145, 147, 148, 149, 153, 162, 178] [175, 193, 214, 221, 224, 227, 229, 244, 245, 247] [248, 252, 253, 275, 281, 285, 305, 310, 316, 326], some recent conference proceedings [7, 43, 102, 155, 166, 301], published lecture notes [230], recent Ph.D. theses [54, 120, 141, 146, 164, 219, 276, 278, 311], and papers presenting key concepts [6, 11, 14, 15, 28, 109] [150, 171, 251, 264, 290, 291, 292, 303, 304, 306, 312, 313, 315]. Also of interest are the lecture notes and preprints published by The International Sophus Lie Center in Oslo (for example, [169]), and the notes of the courses organized by ERCIM on "Partial Differential Equations and Group Theory" [230, 270].

Extensive tables of symmetry group generators for well-known equations from mathematical physics have been gathered by Rogers and Ames [244] and Ibragimov [150, 153, 154].

Many conferences addressed issues related to group analysis of differential equations. We single out the latest volumes of "Group Theoretical Methods in Physics" [75, 76], wherein a list of previous colloquia may be found. As of 1994, a new journal [155] covers topics in Lie groups and their applications.

We highly recommend the special issues of *Acta Applicandae Mathematicae* on "Symmetries of Partial Differential Equations" [304], now available in book form [305], and the other two volumes in this series edited by Ibragimov [153, 154].

Biographies of Lie and his contemporaries are found in [19, 134, 150, 152, 226, 326]. Delightful historical notes about symmetry research in general, and Lie's work in particular can be found in the new, updated edition of Olver's classic [214]. Complete references to Lie's original work are also given in this book, and in [153]. Translations of some of Lie's fundamental papers are available in [154, 179, 180].

Readers interested in the differential geometrical foundation of Lie symmetry analysis, including topics like Lie-Bäcklund mappings, Cartan forms, supersymmetry, graded differential geometry, and gauge theories, may want to consult [83, 96, 111, 171, 172, 193, 214, 216] [224, 279, 307, 308, 310, 328], and [123, 124] for related REDUCE algorithms.

In this review, we will sparsely address the computation of conservation laws, which through Noether's famous theorem [200] and its extensions [36, 249], is intrinsically connected with the investigation of variational symmetries. Indeed, recall that variational symmetries admitted by a Lagrangian system—symmetries that leave the action integral invariant—can be obtained by computing the Lie-Bäcklund symmetries of the corresponding Euler-Lagrange equations. Once the Lie-Bäcklund symmetries are obtained and the variational symmetries are singled out, Boyer's formulation of Noether's theorem can be used to calculate the conservation laws.

In Section 13.2 we discuss the purpose, methods and algorithms used in the computation of Lie symmetries. Apart from a detailed description of the methods for computing and solving the determining equations, we address the reduction of systems of PDEs into standard and passive forms. This topic, in turn, ties in with the computation of the size of the symmetry group. In this section we also address some of the newest trends in the development of symbolic software for Lie symmetries.

In Section 13.3 we look beyond Lie-point symmetries, addressing contact and generalized symmetries, as well as nonclassical or conditional symmetries.

Section 13.4 is devoted to a detailed review of modern symbolic packages that aid in the investigation of Lie symmetries for systems of differential equations. The software is grouped according to the underlying CAS. The review of the available code intentionally focuses on packages written after 1985. More details about pioneering efforts and software written prior to 1985 can be found elsewhere [138]. In Section 13.5, three examples illustrate the results that can be obtained with the software packages. Finally, in Section 13.6 we draw some conclusions.

Although no survey can be considered exhaustive, our study intended to cover all the Lie symmetry software, with the exclusion of software for Lie group computations, such as LiE [300], which is outside the scope of this review.

13.2. PURPOSE, METHODS AND ALGORITHMS

13.2.1. PURPOSE

The classical "Lie symmetry group of a system of differential equations" is a local group of point transformations, meaning diffeomorphisms on the space of independent and dependent variables, which map solutions of the system to solutions. Various other types of local symmetries [14] and nonlocal symmetries [6, 31, 32, 35, 36, 305] have been studied, as well as approximate symmetries [20]. The software reviewed in this paper attempts to compute the classical Lie-point symmetries. Several packages go beyond that in computing contact (or dynamical) and generalized (or Lie-Bäcklund) symmetries, nonclassical (or conditional) symmetries. Several programs could be modified to compute Cartan's dynamical symmetries [51, 285] and hidden symmetries. Two programs that utilize point symmetries to generalize special solutions or find similarity variables recently became available [295, 296, 297, 322]. The latter programs actually try to integrate the characteristic system of first-order differential equations.

Loosely speaking, contact symmetries are generalized symmetries of order one; i.e., the coefficients in the vector field include first derivatives of the dependent variables.

The name Lie-Bäcklund symmetries, commonly used for generalized symmetries, is somewhat misleading for its connection with Bäcklund transformations [199]. Bäcklund transformations constitute a set of case-specific equations that allow one to transform one solution into another, whereas generalized symmetries are infinitesimal symmetries involving higher-order derivatives of the dependent variables. Also, note that Bäcklund transformations do not have group properties. Among the several ways of computing Lie-Bäcklund symmetries, this paper focuses on an extension of the original methods due to Lie. For a discussion of the difference with other symmetry approaches, including other computer algebra methods, we refer the reader to [97, 99, 117, 118, 327, 329].

Conditional symmetries are found by the "nonclassical method of group-invariant solutions," as introduced in 1969 by Bluman and Cole [34]. Further generalizations of the nonclassical method lead to the less practical "weak symmetries" [217, 218], and the method of differential constraints [213, 214].

Nonlocal (potential) symmetries [31, 32, 33, 36, 173, 221, 305] for a system of PDEs are computed as follows. First, one replaces (one or more of) the PDEs in the given system by equivalent conserved forms; second, one introduces auxiliary potential variables; finally, one determines the point symmetries of the resulting auxiliary system of PDEs. The form of an infinitesimal generator then determines whether or not it defines a nonlocal symmetry. The technique of potential symmetries leads to interesting linearizations involving non-invertible mappings.

"Hidden" symmetries [2, 3, 4, 128] show up when, for example, the order of an ODE is increased by one and the number of symmetries (of the now higher-order ODE) is increased at least by two. Upon subsequent reduction, a symmetry may be lost if the transformation is done in non-normal subgroup variables. Hidden symmetries also occur when an ODE is reduced in order, and when this induces an additional symmetry that was not a symmetry of the original ODE.

Somewhat related to this is the following: If a higher-order ODE is rewritten as an equivalent system of first-order equations, and one analyzes the system of first-order equations, rather than the single equation of higher order, the class of symmetries (and consequently solutions) can be substantially enlarged [220, 221, 251]. A similar situation arises with PDEs. When a PDE is reduced to a lower-dimensional PDE one can lose or gain symmetries. Clarkson [60] provides some examples.

Discrete symmetries of differential equations and disconnected Lie-point groups have been considered by Reid, Weih and Wittkopf [240].

Precise definitions and a discussion of internal symmetries (also called dynamical symmetries) and external symmetries are given in [14, 15]. It is beyond the scope of this survey.

13.2.2. METHODS FOR COMPUTING DETERMINING EQUATIONS

Although Lie's theory is geometric, one has to resort to differential-algebraic methods to compute Lie symmetries. Indeed, the criterion for a vector field to be a generator is purely geometric, namely, certain functions must vanish on a submanifold. But differential geometry gives us no tools to implement such criteria in concrete applications. The use of differential algebra imposes some restrictions on the type of problems that can be handled. For instance, the differential equations must be polynomial in their variables (so that one can work in the ring of differential polynomials); the differential equations must be solvable for some derivatives, etc.

There are three major methods to compute Lie symmetries. The first one uses prolonged vector fields, the second utilizes differential forms (wedge products) due to Cartan [51]. The third one uses the notion of "formal symmetry" [37, 38, 195]. Although restricted to evolution systems with two independent variables, the latter method provides a very quick way to compute canonical generalized symmetries. Due to its limited scope we will not elaborate on that technique.

13.2.2.1. PROLONGED VECTOR FIELDS

The first method is used in the algorithm or our program SYMMGRP.MAX [52] and in most of the other Lie symmetry packages. Instead of looking for the Lie group G, one looks for its Lie algebra $\mathcal{L}$, realized by the vector field. From the Lie algebra of symmetry generators, one can obtain the Lie group of point transformations upon integration of a system of first-order characteristic equations (see Section 13.5.1 for a simple example).

For notational simplicity, let us consider the case of Lie-point symmetries [35, 36, 214]. We follow the method, notations and terminology used in [214].

We start with a system of m differential equations,

$$\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m, \tag{13.1}$$

of order k, with p independent and q dependent (real) variables, denoted by $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, $u = (u^1, u^2, ..., u^q) \in \mathbb{R}^q$. We stress that m, k, p and q are *arbitrary* positive integers. The partial derivatives of u^l are represented using a multi-index notation, for $J = (j_1, j_2, ..., j_p) \in \mathbb{N}^p$, we denote

$$u^l_J \equiv \frac{\partial^{|J|} u^l}{\partial {x_1}^{j_1} \partial {x_2}^{j_2} ... \partial {x_p}^{j_p}}\ , \tag{13.2}$$

where $|J| = j_1 + j_2 + ... + j_p$. Finally, let $u^{(k)}$ denote a vector whose components are all the partial derivatives of order 0 up to k of all the u^l.

The group transformations have the form $\tilde{x} = \Lambda_g(x,u)$, $\tilde{u} = \Omega_g(x,u)$, where the functions Λ_g and Ω_g are to be determined. Note that the subscript g refers to the group parameters. Instead of looking for a Lie group G, we look for its Lie algebra $\mathcal{L}$, realized by vector fields of the form

$$\alpha = \sum_{i=1}^{p} \eta^i(x,u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x,u) \frac{\partial}{\partial u^l}. \tag{13.3}$$

The problem is now reduced to finding the coefficients $\eta^i(x,u)$ and $\varphi_l(x,u)$. In essence, the computer constructs the k^{th} prolongation $\text{pr}^{(k)}\alpha$ of the vector field α, applies it to the system (13.1), and requests that the resulting expression vanishes on the solution set of (13.1).

This sounds straightforward, but the method involves tedious calculations because the length and complexity of the expressions increase rapidly as p, q, m, and especially k, increase. The number of determining equations then also rises dramatically as is shown in [244] on page 346. Here are the details and the steps to be performed:

1. Construct the k^{th} prolongation of the vector field α in (13.3) by means of the formula

$$\text{pr}^{(k)}\alpha = \alpha + \sum_{l=1}^{q} \sum_{J} \psi^J_l(x, u^{(k)}) \frac{\partial}{\partial u^l_J}, \quad 1 \le |J| \le k, \tag{13.4}$$

where the coefficients ψ^J_l are defined as follows. The coefficients of the first prolongation are

$$\psi^{J_i}_l = D_i \varphi_l(x,u) - \sum_{j=1}^{p} u^l_{J_j} D_i \eta^j(x,u), \tag{13.5}$$

where J_i is a $p-$tuple with 1 on the i^{th} position and zeros elsewhere, and D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} u^l_{J+J_i} \frac{\partial}{\partial u^l_J}, \quad 0 \le |J| \le k. \tag{13.6}$$

The higher-order prolongations are defined recursively as

$$\psi^{J+J_i}_l = D_i \psi^J_l - \sum_{j=1}^{p} u^l_{J+J_j} D_i \eta^j(x,u), \quad |J| \ge 1. \tag{13.7}$$

2. Apply the prolonged operator $\text{pr}^{(k)}\alpha$ to each equation $\Delta^i(x, u^{(k)})$ and require that

$$\text{pr}^{(k)}\alpha\ \Delta^i \ |_{\Delta^j = 0}\ = 0 \quad i,j = 1, ..., m. \tag{13.8}$$

The meaning of condition (13.8) is that $\text{pr}^{(k)}\alpha$ vanishes on the solution set of the originally given system (13.1). Precisely, this condition assures that α is an infinitesimal symmetry generator of the group transformation; $\tilde{x} = \Lambda_g(x,u)$, $\tilde{u} = \Omega_g(x,u)$, i.e., that $u(x)$ is a solution of (13.1) whenever $\tilde{u}(\tilde{x})$ is one.

3. Choose, if possible, m components of the vector $u^{(k)}$, say $v^1, ..., v^m$, such that:

   (a) Each v^i is equal to a derivative of a u^l $(l = 1, ..., q)$ with respect to at least one variable x_i $(i = 1, ..., p)$.

   (b) None of the v^i is the derivative of another one in the set.

   (c) The system (13.1) can be solved algebraically for the v^i in terms of the remaining components of $u^{(k)}$, which we denote by w. Hence, $v^i = S^i(x, w)$, $i = 1, ..., m$.

   (d) The derivatives of v^i, $v^i_J = D_J S^i(x, w)$, where $D_J \equiv D_1^{j_1} D_2^{j_2} ... D_p^{j_p}$, can all be expressed in terms of the components of w and their derivatives, without ever reintroducing the v^i or their derivatives.

   The requirements in step 3 put some restrictions on the system (13.1), but for many systems the choice of the appropriate v^i is quite obvious. For example, for a system of evolution equations

$$\frac{\partial u^i}{\partial t}(x_1, ..., x_{p-1}, t) = F^i(x_1, ..., x_{p-1}, t, u^{(k)}), \quad i = 1, ..., m, \tag{13.9}$$

   where $u^{(k)}$ involves derivatives with respect to the variables x_i but not t, an appropriate choice is $v^i = \dfrac{\partial u^i}{\partial t}$.

4. Use $v^i = S^i(x, w)$ to eliminate all v^i and their derivatives from the expression (13.8), so that all the remaining variables are now independent of each other. It is tacitly assumed that the resulting expression is now a polynomial in the u^l_J.

5. Obtain the determining equations for $\eta^i(x, u)$ and $\varphi_l(x, u)$ by equating to zero the coefficients of all functionally independent expressions (monomials) in the remaining derivatives u^l_J.

In the above algorithm the variables x_i, u^l, and u^l_J are treated as independent; the dependent ones are η^i and φ_l.

In summary, the result of implementing (13.4) is a system of linear homogeneous PDEs for η^i and φ_l, in which x and u are independent variables. These are the so-called determining or defining equations for the symmetries of the system. Solving these by hand, interactively or automatically with a symbolic package, will give the explicit forms of the $\eta^i(x, u)$ and $\varphi_l(x, u)$.

The procedure, which is thoroughly discussed in [214], consists of two major steps: *deriving* the determining equations, and *solving* them explicitly.

We note that in [30], Bluman proves theorems about the forms of admitted infinitesimal transformations, which cover a large class of scalar ODEs and PDEs. In essence, the theorems in [36] imply that the coefficients of the vector field are either free or depend linearly on the dependent variable. Use of these theorems can significantly simplify the tedious work involved in setting up and solving the determining equations.

13.2.2.2. DIFFERENTIAL FORMS

A differential geometric approach to invariance groups and solutions of PDEs was presented by Harrison and Estabrook [131]. They showed how to derive infinitesimal symmetries using Cartan's exterior differential calculus. The isovector fields, which are the generators of geometric transformations with suitable algebraic invariance properties, are then used to obtain invariant solutions of several PDEs, including the heat equation, the Korteweg-de Vries equation, and the vacuum Maxwell equations.

Briefly, this method to find infinitesimal symmetries proceeds as follows. The system of differential equations is rewritten as a Pfaffian system, that is a system of one-forms. The condition for an isovector is then that the contraction of every form in the closed ideal generated by the exterior system with the isovector remains in the ideal. That is completely equivalent to the condition in the jet bundle approach. From a computational point of view there is a disadvantage: the exterior system usually consists of more forms than the PDE system of equations. To reduce the number of forms, one can opt to formulate the system of differential equations as a closed exterior differential system of two-forms. The underlying manifold is prolonged by new coordinates, the so-called prolonged variables. The exterior differential system is then prolonged by special one-forms in these prolonged variables. The condition on these added one-forms is that the prolonged exterior differential systems remain closed. This condition leads to the determining equations for the coefficients of the manifold.

For the mathematical formulation and several worked examples, the reader is referred to [49, 86, 125, 131, 165, 279]. Olver's forthcoming book [216] addresses the application of various approaches, particularly those of Lie and Cartan, to equivalence and symmetry problems. That book also discusses the determination of invariants, the classification of differential equations, variational problems, and so on.

It should also be noted that routines [77, 254, 255, 309] for Cartan's exterior calculus [132] are included as standard packages of CAS. For instance, CARTAN in the MACSYMA Share Library, EXCALC in REDUCE, and DifferentialForms [330, 331] in Mathematica. For a list of other symbolic packages that manipulate objects of differential geometry we refer the reader to [255].

13.2.3. METHODS FOR REDUCING DETERMINING EQUATIONS

To design a reliable and powerful integration algorithm for a system of determining equations the system needs to be brought into a canonical form. Various related concepts appear in the literature, were authors refer to the normal, orthonomic, involutive and passive forms [45, 174]. All of these have slightly different definitions, for which the reader should consult the later cited literature. Involution is a geometric concept, independent of coordinate systems. Passive systems are defined with respect to a given coordinate system and a given ordering of terms. The same holds for differential Gröbner bases (DGBs).

The original theory of involutive systems goes back to Cartan [51], Janet [159, 160], Kähler [161], Kolchin [167, 168], Riquier [242], Ritt [243], Thomas [286], and Tresse [288, 289]. In the introduction to his lecture notes [230], Pommaret discusses the basic ideas of the theory of differential elimination. His description, which contains interesting historical remarks (see also [228]), relates the work of the French to the formal cohomology approach advocated by D.C. Spencer and collaborators in the USA. A concise survey of the theory of involutive systems with complete proofs is also given by Finikov [95].

Roughly speaking, the methods that reduce systems of linear homogeneous PDEs into an equivalent, but much simpler standard form, can be viewed as generalizations of triangulation by Gauss-Jordan elimination, but are applied here to systems of linear PDEs. First, the original system is appended by all its differential consequences. Second, highest derivatives are eliminated, and, if they occur, integrability conditions are added to the system. The procedure is then repeated until the new system is in involution.

Although Schwarz did valuable pioneer work on solving systems of linear homogeneous PDEs [259], he is by far not the only one working on innovative ways of classifying, subsequently reducing, and finally solving overdetermined systems of linear homogeneous PDEs. Early efforts to implement Cartan's exterior-forms approach to involutive systems, and the method of Riquier and Janet, are due to Russian teams [16, 113, 114, 287].

Kornyak, Fedorova and Fushchich [92, 170], Bocharov [40], Bocharov and Bronstein [42], Ganzha and Yanenko [114], Gerdt and Zharkov [119], Pankrat'ev [223] and Topunov [287], amongst others, partially implemented algorithms to reduce systems of PDEs.

The GRBASES algorithm of Pankrat'ev and the implementation of the Riquier-Janet method by Topunov are both in REFAL. Both programs can only reduce linear systems of PDEs, this in contrast with the program DIFFGROB2 of Mansfield discussed below.

A good description of the relation between the Riquier-Janet theory and the modern implementations (Gröbner bases) is given by Topunov [287] (see also [266]). For an introduction of Gröbner bases, including Buchberger's algorithm [46, 47], we recommend the new books by Cox, Little and O'Shea [74], and Becker and Weispfenning [25]. Most of this work directly relates to Lie symmetry computations, with the exception of Pankrat'ev, who offers general algorithms for the computation of Gröbner bases in differential and difference modules.

In the West, Reid and collaborators [234, 235, 237, 238, 241], Schwarz [259, 266, 267], and Wolf and Brand [320, 321, 324, 323, 325], partially implemented algorithms to reduce systems of PDEs. Their work led to sophisticated symbolic code in MACSYMA, Maple, and REDUCE for that purpose.

In [266, 267], Schwarz describes the algorithm *InvolutionSystem*, based on the theory of differential equations due to Riquier [242] and Janet [159, 160], to transform a linear system of PDEs into involutive form. In modern language: the involutive form is a DGB with respect to the selected term ordering. If all consistent orderings for the terms in the system of PDEs are known, the algorithm *InvolutionSystem* may be applied repeatedly to determine a universal Gröbner base [267]. Schwarz designed his algorithm *InvolutionSystem* for a specific purpose, namely to determine the size of the Lie symmetry group of a given system of PDEs without having to integrate the determining equations. We devote Section 13.2.5 to this important topic, where details about Schwarz's program SYMSIZE can be found.

As defined by Reid [235], a standard form of a system is obtained by repeating the following steps: (i) write each equation in solved form with respect to its highest order derivative, (ii) replace these highest order derivatives throughout the rest of the system, (iii) add any new equations arising from integrability conditions.

As said, standard-form algorithms have their origins in the work of Riquier and Janet for passive forms. Note that the Riquier-Janet form is not a standard form, but can be fairly simply transformed into a standard form. Also, in method, design of algorithm, and in actual implementation, modern standard form algorithms are quite different from the original inefficient methods and algorithms proposed by Riquier and Janet.

A first, but brief account of Reid's algorithm *standard_form* [235], which also has it roots in the classical Riquier-Janet theory, appeared in [233] and [234]. The algorithm was first implemented in MACSYMA, and later on a MAPLE version became available [241].

The algorithm *standard_form* reduces systems of PDEs to a simplified standard form. Again, the procedure can be viewed as a generalization to linear differential equations of the Gaussian reduction method for matrices or linear systems. The algorithm now takes as input the system of PDEs and a matrix which specifies a complete ordering on the derivatives appearing in the system.

It then reduces the system of PDEs to an equivalent simplified ordered triangular system with all integrability conditions included and all redundancies (differential and algebraic) eliminated. Reid's algorithm implements an equivalence class approach to the problem of bringing a system of PDEs into a standard form. For that purpose, Reid developed a new completion method based on a free direction index (rather than the monomials of the Riquier-Janet theory).

Within *standard_form*, Reid uses an "update strategy" based on updating lists of equations: one term equations, easy equations, hard (or yet-to-be classified) equations; with special user-defined tuning knobs (parameters), so that the user can control the flow of equations between the various lists. Thus *standard_form* works on easier parts of the system first, a strategy that becomes crucial when dealing with large systems.

Further details about Reid's algorithm and examples of its use can be found in [237], where it is shown how directed graphs representing the dependencies amongst the system's variables can be used to simplify or numerically integrate the system. Once the system is in standard form, one can continue with the automatic determination of a Taylor series solution of the system to a specified finite degree.

Reid and Wittkopf's package [241] facilitates automated interfacing with major symmetry packages such as DIMSYM [277], LIESYMM [49], and SYMMGRP.MAX [52], and also with the differential Gröbner basis package DIFFGROB2 [189]. A TEX interface between *standard_form* and Hickman's program [142, 143] that uses physical variable notation has been provided by Lisle. Full details and many illustrative examples of the package, which, besides the function *standard_form*, includes other powerful algorithms for symmetry analysis of PDEs, are given in [241].

Reid and McKinnon developed a recursive algorithm called *Rsolve_Pdesys* [239] that builds on Reid's *standard_form* [235] and on algorithms of Abramov and Kvashenko [5] and Bronstein [44]. Reid and McKinnon's algorithm *RSolve_Pdesys* now finds particular solutions of linear systems of PDEs using only ODE solution techniques. Applied to symmetry problems, their algorithm will find all polynomial/rational solutions of the determining equations provided the symmetry group is finite-dimensional.

Several other approaches, and consequently implementations, are possible to complete a given system of PDEs to an involutive system. Schü, Seiler and Calmet [256] present an algorithm in AXIOM to perform this task. Their method is highly geometrical and their implementation is based in part on the Cartan-Kuranishi theorem [45], which assures that the integrability conditions for the determining equations can be found in a finite number of steps. A detailed description of their programs, called JET, is given in Seiler's thesis [274]. JET can be viewed as an environment for computations within the geometric theory of PDEs based on the jet bundle formalism. Some standard tasks are put into AXIOM packages. One such package is called CartanKuranishi, which completes a given system to an involutive one. Another package, in development at the time of writing, contains a procedure to set up determining equations for classical and non-classical symmetries.

Hartley and Tucker [132] implemented an algorithm (in REDUCE) to analyse involutive systems of exterior forms, based on the Cartan-Kähler theory. Later they extended their program [133], originally in REDUCE, now partly rewritten in Maple, to non-involutive systems. For such systems their completion procedure constructs the needed integrability conditions. Their work corresponds that of Schü, Seiler and Calmet on involutive systems, however, Hartley and Tucker use exterior systems.

According to my sources, a student of Fackerell (Sydney) has implemented the Vessiot approach to involution. Vessiot's method can be viewed as a dual version to the exterior system approach.

In the full computer implementation of "triangulation" algorithms, one takes advantage of a "differential" generalization of Buchberger's algorithm for Gröbner bases. Buchberger's algorithm, which is included as a standard package with modern CAS [47, 136, 194] (also see Appendix C in [74]), provides a technique for canonically simplifying polynomial nonlinear systems of algebraic equations. The "differential" generalization of that algorithm allows one to reduce systems of nonlinear (and, consequently, also linear) PDEs into standard form. In the "differential" analogue of Buchberger's algorithm, one has to replace cross-multiplication by cross-differentiation, algebraic reduction by differential reduction, etc.

Carrá-Ferro [50] was the first to define DGBs for PDE systems. She gave a method based on differential reduction for the attempted construction of such DGBs, but showed that they may be infinite (unlike the case for polynomial algebraic equations and linear systems of PDEs). Subsequently, Ollivier [212] gave a method which could, in a finite number of steps, construct a DGB up to a given order of derivation (even when the DGB was infinite). But criteria were not given for determining when the DGB had been constructed up to the given order. Recently Mansfield [188] has given an algorithm (and a computer-algebra implementation) that uses pseudo-reduction instead of reduction to attempt to construct DGBs. The advantage of her technique is that it always terminates; however, her algorithm may not always terminate with a DGB. In short, the differential ideal membership problem remains unsolved; and there are even disagreements on the definition of DGBs.

The Maple program DIFFGROB [188, 192] and its second version DIFFGROB2 by Mansfield [189], are designed to compute the DGB of a finitely generated ideal of PDEs with polynomial terms. With respect to this basis, every member of the ideal pseudo-reduces to zero. In pseudo-reduction one is allowed to multiply expressions by differential coefficients of the highest derivative terms that occur in the system to be reduced. This is needed for nonlinear systems where otherwise standard reduction algorithms would not necessarily terminate. DIFFGROB2 allows one to calculate in a systematic way: the elimination ideals, integrability conditions, and compatibility conditions of a system of nonlinear PDEs of polynomial type, up to certain technical constraints fully explained in [189, 192].

The fundamental tools in Mansfield's package are the Kolchin-Ritt algorithm [94], which is a differential analogue of Buchberger's algorithm, with pseudo-reduction instead of reduction, and the *diffgbasis* algorithm, which takes into account algebraic as well as differential consequences of nonlinear systems. These two algorithms allow one to compute the DGB for a wide range of systems of PDEs. A detailed discussion of these algorithms is beyond the scope of this survey paper. For more information about DIFFGROB2 and illustrations of its use, the reader should consult [68, 71, 192, 190, 191], and in particular the manual [189].

For fairly simple examples, such as the Boussinesq equation, DIFFGROB2 is able to automatically reduce the nonlinear determining equations (corresponding to nonclassical symmetries) in standard form. For more complicated cases, DIFFGROB2 may need to be used interactively. Nevertheless, the package has proven to be an effective tool [60, 68, 69, 71, 190, 191] in solving overdetermined systems of linear and nonlinear PDEs arising in the study of classical and nonclassical symmetries. Needless to say, DIFFGROB2 can be used in applications other than symmetry analysis. Such applications include finding the compatibility conditions for inhomogeneous systems, testing the consistency of systems of PDEs, and finding the least amount of necessary data for a formal power series solution of a linear system (the "initial data" problem). Obviously, DIFFGROB2 can also be used to bring the input equations into involutive form. For some examples this is indeed necessary to be able to compute nonclassical symmetries. In passing, within DIFF-

GROB2 a package called DIRMETH is available to compute the determining equations related to the symmetry reduction of PDEs via the direct method of Clarkson and Kruskal [65].

The program CRACK by Wolf and Brand [323, 324, 325] also carries out a Gröbner Basis analysis but in slightly modified form. First, the algorithm is enriched by the integration of PDEs whenever possible, but only in such a way, that the new integrated PDEs are still polynomial in the Gröbner basis. In other words, the 'critical pair completion steps' of the Gröbner basis algorithm and the integrations used within CRACK are consistent. Selective integration can reduce the complexity and aid in solving the determining equations, in particular for systems for which pure Gröbner basis methods would be unfeasible. Second, for efficiency reasons, only a restricted completion algorithm is used, although it is the authors intention to extend it to a complete Gröbner basis algorithm in the future.

According to a recent paper [240], Wittkopf is also developing an algorithm, called *diff_reduce*, which attempts to reduce polynomially nonlinear systems of PDEs to the form of a DGB. In essence, the algorithm is a differential analogue of Buchberger's elimination algorithm for polynomial equations. Wittkopf's algorithm uses reduction rather than pseudo-reduction, and incorporates strategies for efficient memory management.

Finally, Oaku [211] is designing and implementing software in the computer algebra system Risa/asir to automatically find the structure of the solution space of systems of linear PDEs. Oaku's method is based on the notion of Gröbner basis and the Buchberger algorithm applied to rings of differential operators with polynomial coefficients (Weyl algebra).

13.2.4. METHODS FOR SOLVING DETERMINING EQUATIONS

The most challenging part of Lie symmetry analysis by computer, involves the design of an "integrator" for the overdetermined systems of linear homogeneous PDEs. This topic is also of importance in the study of so-called adjoint symmetries of differential equations [250], and in many other areas where determining equations of the same type occurs. Ideally, good integration code should be applicable to generic systems of linear differential equations, which do not necessarily come from symmetry analysis.

In the context of Lie symmetry analysis, one can aim at the design of faster and more powerful algorithms that work for *large* systems of determining equations, typically a few hundred, and that automatically reduce systems to where they can be handled interactively with the computer, or by hand.

Since the early developments [135, 257, 260, 283, 284, 285] of semi-heuristic methods to solve determining equations, substantial progress has been made in understanding the mathematics of this problem and a new breed of algorithms is now available. These algorithms attempt to close the gap between solution techniques for ODEs and PDEs (consult [280] for an impressive review and large bibliography).

Two other important topics tie in with the integration of the determining equations: (i) the transformation of the determining equations into standard and passive forms; and (ii) the computation of the size of the symmetry group discussed in Sections 13.2.3 and 13.2.5, respectively.

The design of algorithms and programs to bring the determining equations into standard form were a major step forward. Once systems are reduced into standard involutive form or decoupled, subsequent integration is more tractable and reliable. One could use separation of variables, standard techniques for linear differential equations, and specific heuristic rules given below. The only determining equations left for manual handling should be the "constraint" equations or any other equations whose general solutions cannot be written explicitly in closed form.

In order to be able to make the determination of certain types of Lie generators into a decision procedure, one needs an algorithm for solving linear homogeneous ODEs. Such equations are always obtained as the lowest equation of the reduced determining system, with reduction based on lexicographical term ordering. An important step towards this goal is the factorization as it is applied in SPDE [264, 265]. An in-depth review of issues related to the implementation of this and other algorithms is given in [269].

After searching the relevant literature [42, 233, 234, 235, 239, 259, 266, 267, 287, 320, 321] [323, 324, 325], it is clear that many mathematical questions remain open. Despite the innovative efforts of Reid, Schwarz, Wolf and Brand, and many others, there is no general algorithm available to integrate an arbitrary (overdetermined) system of determining equations that consists of linear homogeneous PDEs for the η's and the ϕ's.

Most integration algorithms are based on a set of heuristic rules [135, 165, 257, 260, 277, 284]. In the computer programs reviewed in Section 13.4, the following rules are used.

1. Integrate single term equations of the form
$$\frac{\partial^{|I|} f(x_1, x_2, ..., x_n)}{\partial {x_1}^{i_1} \partial {x_2}^{i_2} ... \partial {x_n}^{i_n}} = 0, \tag{13.10}$$
where $|I| = i_1 + i_2 + ... + i_n$, to obtain the solution
$$f(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} \sum_{j=0}^{i_k - 1} h_{kj}(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n)(x_k)^j, \tag{13.11}$$
thus, introducing functions h_{kj} with fewer variables.

2. Replace equations of type
$$\sum_{j=0}^{n} f_j(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n)(x_k)^j = 0, \tag{13.12}$$
by $f_j = 0 \ \ (j = 0, 1, ..., n)$. More generally, this method of splitting equations (via polynomial decomposition) into a set of smaller equations is also allowed when f_j are differential equations themselves, if the variable x_k is missing.

3. Integrate linear differential equations of first and second order with constant coefficients. Integrate first-order equations with variable coefficients via the integrating factor technique, provided the resulting integrals can be computed in closed form.

4. Integrate higher-order equations of type
$$\frac{\partial^n f(x_1, x_2, ..., x_n)}{\partial {x_k}^n} = g(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n), \tag{13.13}$$
n successive times to obtain
$$\begin{aligned} f(x_1, x_2, ..., x_n) &= \frac{(x_k)^n}{n!}\, g(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n) \\ &+ \frac{x_k^{n-1}}{(n-1)!}\, h(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n) \\ &+ \ ... \ + r(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n), \end{aligned} \tag{13.14}$$
where $h, ..., r$ are arbitrary functions.

5. Solve any simple equation (without derivatives) for a function (or a derivative of a function) provided that it both (i) occurs linearly and only once, and (ii) depends on all the variables that occur as arguments in the remaining terms.

6. Explicitly integrate exact equations.

7. Substitute the solutions obtained above in all the equations.

8. Add differences, sums or other linear combinations of equations (with similar terms) to the system, provided these combinations are shorter than the original equations.

With these simple rules, and perhaps a few more, the determining system can often be drastically simplified. Amazingly, in many cases nothing beyond the above heuristic rules is needed to solve the determining equations completely. If that is not possible, after simplification, the programs return the remaining unsolved differential equations for further inspection. In most programs, the user can then interactively simplify and fully solve the determining equations on the computer, thereby minimizing human errors.

At least for Lie-point symmetries, solving the determining equations can usually be done by hand, using elementary results from the theory of linear PDEs. Solving them on a computer may be time consuming, since the simplest approach varies greatly from case to case. Furthermore, a computer program may accidentally not catch the most general result and therefore may return an incomplete symmetry group. The author is aware of this problem, which occurred when testing some of the reviewed symmetry programs. Even worse, the computer algorithm may not be able to determine the solution of the determining equations in a finite number of steps.

13.2.5. COMPUTATION OF THE SIZE OF THE SYMMETRY GROUP

Schwarz [266, 267] and Reid [234, 235, 236] independently developed algorithms for determining the size of the Lie symmetry groups of differential equations without integrating the determining equations explicitly.

Schwarz's algorithm SYMSIZE [266, 267] is available with the computer algebra system REDUCE, as part of the package SPDE (see Section 13.4.2). Schwarz also translated SYMSIZE into the language of Scratchpad II, the predecessor of AXIOM. Use of SYMSIZE circumvents some of the shortcomings mentioned at the end of the previous section. Indeed, if a differential equation has no other than obvious symmetries or if the symmetry group is small (because all generators are algebraic and of low degree), SYMSIZE will greatly help in completely solving the symmetry problem.

In contrast to the *heuristic* algorithms for the explicit computation of the symmetry generators, the size of the symmetry group can always be determined with SYMSIZE in a finite number of steps. SYMSIZE accepts a system of PDEs as input, and allows one to compute *a priori* the number of free parameters if the group is finite and the number of unspecified functions of the group is infinite. In turn, SYMSIZE allows one to test *a posterior* if the solution of the determining equations is complete. In cases where some, perhaps all, symmetries are known by inspection or from the physics of the problem, the knowledge of the size of the symmetry group can evade an expensive search for more, perhaps nonexistent, symmetries.

At the heart of SYMSIZE is the procedure *InvolutionSystem*, which transforms the determining system into an involutive system by means of a critical pair/completion algorithm. Similar algorithms are applied in computing Gröbner bases in polynomial ideal theory (see Section 13.2.3).

Concurrently, yet independent of Schwarz, Reid [234, 235, 236] realized that triangularization algorithms may allow bypassing the explicit solution of the determining equations and compute the size of the symmetry group and the commutators immediately. Reid developed the program SYMCAL [235], written originally in MACSYMA, but now converted by Reid and Wittkopf into Maple [241].

In [233, 236], a non-heuristic algorithm *structure constant* is presented, based on *Taylor* and *standard_form*, which always determines (in a finite number of steps) the dimension and the structure constants of the finite part of the Lie symmetry algebra. An extension of the algorithm [233] also allows one to classify differential equations (with variable coefficients) according to the structure of their symmetry groups. Furthermore, the approach advocated by Reid applies to the determination of symmetries of Lie-contact and Lie-Bäcklund types, as well as potential symmetries.

The algorithm described in [236], provides information about the dimension and commutators of the Lie symmetry algebra. It is based on explicit Taylor expansions of the symmetry generators, and therefore is computationally expensive and restricted to finite dimensional Lie algebras. The newest Maple algorithm [238] allows one to compute the dimension and the commutation relations without Taylor expansions; hence, it is applicable to infinite-dimensional Lie algebras.

Readers interested in the problem of determining the "size" of the solution space for arbitrary involutive systems should consult a recent paper by Seiler [273]. Seiler's results can be applied to linear and nonlinear determining systems.

Finally, we should mention that skillful use of the tools for reducing systems of linear homogeneous PDEs, available within the package CRACK [323, 324, 325], can also greatly assist in the investigation of the size of the symmetry group.

13.3. BEYOND LIE-POINT SYMMETRIES

The discussion of symmetries other than point symmetries is limited here to those for which symmetry software is already available. For a general review of various types of symmetries we refer to the new edition of Olver's book [214] and to Clarkson [60].

13.3.1. CONTACT AND GENERALIZED SYMMETRIES

For the computation of generalized symmetries or Lie-Bäcklund symmetries [13, 214] the use of symbolic programs is even more appropriate. The procedure to determine symmetries is essentially the same as that for point symmetries, although the calculations are lengthier and more time consuming. In a generalized vector field, which still takes the form of (13.3), the functions η^i and ϕ_l may now depend on a finite number of derivatives of u, i.e.,

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u^{(k)}) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u^{(k)}) \frac{\partial}{\partial u^l}. \tag{13.15}$$

If $k = 1$ the generalized symmetry determines a classical contact symmetry and vice versa, at least in the case of one dependent variable. The even simpler case $k = 0$, with $u^{(0)} = u$, leads to point symmetries. Olver [214] discusses various possibilities to simplify the calculations, for example by putting the symmetries in evolutionary form, or by fixing the order of derivation on which the η's and ϕ's may depend.

13.3.2. NONCLASSICAL OR CONDITIONAL SYMMETRIES

Recently it was shown that the "nonclassical method of group-invariant solutions," originally introduced in [34], can determine new solutions of various physically significant nonlinear PDEs.

Examples include the nonlinear Schrödinger (NLS) equation [58], and its cylindrical version [62]; the Boussinesq equation [176]; the Kadomtsev-Petviashvili equation [72] and other members of its hierarchy [314]; the Burgers equation [88, 232]; the telegraph equation [207]; the Fitzhugh-Nagumo equation [87, 88, 210], and other reaction-diffusion equations [66]; the Helmholtz [209] and shallow water wave equations [69, 70]; and a class of nonlinear heat equations [67, 68]. Levi and Winternitz recently showed how conditional symmetries can be determined for the 2D-Toda lattice, a differential-difference equation [177]. For a well-documented perspective on the computation of nonclassical symmetries we recommend [71] and [59, 60].

An example is the Boussinesq equation where the new reductions that follow from application of the nonclassical method were discovered earlier with a direct method [58, 57, 65]. The direct method was also applied to the Zabolotskaya-Khokhlov equation [61] and the Davey Stewartson system [63, 64]. Recently, Olver [215] proved that both methods are equivalent in the case of fiber preserving transformations, which means that the new independent variables depend only on the old independent variables, not on the original dependent variables. Arrigo, Broadbridge and Hill [17] explicitly derive criteria for which the direct method and the nonclassical symmetry method lead to the same results; they use the Burgers and Boussinesq equations as illustrative examples. For shallow water wave equations, Clarkson and Mansfield [69] have shown that the nonclassical symmetry method can lead to particular solutions which can not be obtained via the singular manifold method. Estévez [87, 88] and Pucci [232] pointed out some interesting connections between the direct method of Clarkson and Kruskal [65]. Estévez [87, 88] and Estévez and Gordoa [89] also compare the nonclassical symmetry method with the singular manifold approach from Painlevé analysis. Nucci [206] shows how Bäcklund transformations can be obtained via the investigation of nonclassical symmetries.

In contrast to Lie-point symmetries, for example, the transformations corresponding to nonclassical (or conditional) symmetries neither leave the differential equation invariant, nor transform all the solutions into other solutions. They merely transform a subset of solutions into other solutions.

Accounting for "nonclassical symmetries," the program should automatically add the q invariant surface conditions [34, 213],

$$Q^l(x, u^{(1)}) = \sum_{i=1}^{p} \eta^i(x,u) \frac{\partial u^l}{\partial x_i} - \varphi_l(x,u) = 0, \quad l = 1, ..., q, \tag{13.16}$$

and their differential consequences, to the system (13.1). However, the inclusion of nonclassical symmetries, and perhaps other types of symmetries as discussed in [14, 15, 213], requires solving systems of determining equations which are *no longer linear*. Consequently, new integration algorithms must be designed.

It should be noted that various other types of conditional symmetries could be considered. For instance, one could ask under what extra conditions a class of PDEs would admit a symmetry chosen beforehand. Extensive work on this problem, which we will not address here, was done by Fushchich [100].

13.4. REVIEW OF SYMBOLIC SOFTWARE

In this section we review the most modern Lie symmetry programs, classified according to the underlying computer algebra system. Focusing on new trends, packages written before 1985 are only briefly mentioned. Researchers interested in more details about some of the pioneering work could consult [138].

13.4.1. COMPUTER IMPLEMENTATION

Ideally, a fully automated software package for Lie symmetries should consist of effective, powerful algorithms and fast procedures for the following tasks:

1. derivation of the determining equations for large or complicated systems of equations;
2. reduction of determining equations into so-called standard form;
3. finding the size of the symmetry group;
4. determining any obvious symmetry generators;
5. simplifying and integrating the determining equations to compute the generators, if not all the generators have been found yet.

Then the program should be able to execute the following steps in the order relevant to the specific application:

(a) calculation of commutator tables, based on the results of 1, 2 and 3;

(b) calculation of group invariant solutions.

The program should be able to handle: calculation of nonlocal (potential) symmetries [31, 32, 36, 221, 305]; calculation of nonclassical reductions (conditional symmetries) and resulting solutions; calculation of generalized symmetries; and calculation of equivalent conservation laws.

Furthermore, it should be able to accept systems with free unknown (classification) functions [186, 236, 277] as input. If so, questions such as "for which values of these parameters or parametric functions does a given ODE or PDE have prescribed symmetries or special solutions" could be answered.

Other ideas could be incorporated in the design of faster and more efficient symbolic software for Lie symmetry analysis. Let me give a couple of examples. Lengthy calculations should be broken up into smaller pieces by consistently taking advantage of the "linear algebra" structure of the Lie symmetry problem. For instance, prolongations should be applied to vector fields of single equations or subsets of equations, and not to the whole system at once. Furthermore, full expansions of the prolonged vector fields should be halted until the explicit forms are actually needed. Avoiding lengthy, redundant expansions will make the generation of the determining equations much more efficient, particularly for nonclassical and Lie-Bäcklund symmetries.

Currently, no software handles this entire ambitious program. Many Lie symmetry programs carry out parts of the listed tasks. Making matters worse, the available Lie symmetry programs, with the exception of Lie by Head [135], work with specific CAS, which has to be bought separately.

In Table 1 we list the most modern software packages, along with information about developers and distributors. In Table 2 we summarize the scope of these packages.

With the exception of DELiA and DIMSYM, all the programs listed in Table 1 are public domain software. Potential users can obtain the software from the developers or through the referenced sources. The main cost in using these packages is related to the cost of the underlying CAS. As a rule of thumb, individual copies of Mathematica, MACSYMA, Maple, REDUCE, and the like, cost about 10% of the price of the platform you buy them for.

13.4.2. REDUCE PROGRAMS

In the early '80s, Schwarz developed his well-documented program SPDE [257, 258, 260, 261, 262, 263, 264], The program automatically derives and often successfully solves the determining equations for Lie-point symmetries with minimal intervention by the user. Since 1986 SPDE is distributed together with REDUCE for various types of computers, ranging from PCs to CRAYs. In 1994 version 1.0 of SPDE became available. Although Schwarz decided to keep the old name, the new program is drastically different. According to the documentation [268], SPDE 1.0 guarantees that all infinitesimal symmetry generators with algebraic coefficients will be obtained if the equations are nonlinear and of order higher than one. Concerning the input, the equations must be algebraic in their arguments. There is no restriction on the number of independent and dependent variables, and the equations can have any number of constant parameters (no arbitrary functions). The program computes the determining equations, then generates a Gröbner basis for the determining system in a term ordering specified by the user (total degree, lexicographic, etc.). The integration of the reduced system is carried out automatically, the symmetry generators and their commutator table are displayed in LaTeX (if so desired).

Based on Cartan's exterior calculus [51, 95, 140], Edelen [79, 80, 81, 82], Gragert [120], and Gragert, Kersten and Martini [125] used computer algebra systems to calculate the classical Lie symmetries of differential equations. More recently, Gragert [121, 122] added a package for more general Lie algebra computations, including code for higher-order and super symmetries and super prolongations. Kersten [164, 165] further perfected the software package for the calculation of the Lie algebra of infinitesimal symmetries (including Lie-Bäcklund symmetries) of exterior differential systems.

Eliseev, Fedorova and Kornyak [85], wrote code in REDUCE-2 to generate (but not solve) the system of determining equations for point and contact symmetries. Their paper discusses the algorithm and shows three worked examples. Fedorova and Kornyak [91, 93] generalized the algorithm to include the case of Lie-Bäcklund symmetries.

The interactive REDUCE program NUSY by Nucci [202, 203, 204, 205], included with this book [208], generates determining equations for Lie-point, non-classical, Lie-Bäcklund and approximate symmetries and provides interactive tools to solve them. The manual [202, 204, 208] gives a clear description of the various routines with their scope and limitations, and has several worked examples.

The package CRACK by Wolf and Brand [323, 324, 325] solves overdetermined systems of differential equations with polynomial terms. To do this, it uses code for decoupling, separating and simplifying PDEs. Integration of exact PDEs and differential factorization are also possible. CRACK has many applications that are facilitated via special tools. For instance, the function LIEORD can aid in the investigation of Lie symmetries of ODEs. With CRACK one can also construct the Lagrangian for a given second-order ODE, and find first integrals via the integrating factor method. In attempting to solve standard ODEs, the program makes use of the REDUCE package ODESOLVE written by MacCallum [187]. The functions and tools available within CRACK allow simplification and integration of linear homogeneous PDEs, beyond those derivable via symmetry analysis.

Upon completion of CRACK, Wolf went on to develop three new REDUCE programs, called LIEPDE, QUASILINPDE and APPLYSYM, which all make use of the tools of CRACK.

LIEPDE [321] finds Lie-point and contact symmetries of PDEs by deriving and solving a few simple determining equations, before continuing with the computation of the more complicated determining equations. This idea, which makes the program highly efficient, was used in Wolf's FORMAC program [318, 319, 320], and is also implemented in the design of the feedback mechanism of SYMMGRP.MAX [52]. For solving the determining equations, LIEPDE makes use of modules of the package CRACK discussed above. The difference is that within LIEPDE the steps are carried out automatically, without intervention by the user. This approach is particularly useful when applied to large systems of PDEs, or in the computation of higher-order symmetries, where space and memory limitations come into play.

The aim of QUASILINPDE [322] is to find the solutions of quasi-linear PDEs. These solutions are then used by APPLYSYM [322], which applies the symmetry to lower the order of ODEs, to calculate similarity variables for PDEs, to effectively reduce the number of independent variables of a system of PDEs, and to generalize special known solutions of ODEs and PDEs. To our knowledge, APPLYSYM is one of the first symbolic programs that truly applies point symmetries that can be calculated with the program LIEPDE. The program APPLYSYM is automatic but can also be used interactively. Thus far, APPLYSYM is only applicable to point symmetries for which the generators are at worst rational. The actual problem solving is done in all these programs through a call to the package CRACK for solving overdetermined systems of PDEs.

In [115, 116], Gerdt introduced the program HSYM for the explicit computation of higher-order symmetries for PDEs. If the given system of equations has arbitrary parameters, the necessary conditions for the existence of higher order symmetries will lead to a system of algebraic equations in the parameters. Via the program ASYS, that algebraic system is reduced into standard form via a Gröbner basis algorithm. The focus in Gerdt's work is on the investigation of the integrability of polynomial type nonlinear evolution equations, by verifying the existence of higher order symmetries and their associated conservation laws.

Sarlet and Vanden Bonne [250] offer specific procedures to assist in the computation of adjoint symmetries of second-order ODEs. This assistance, however, is limited to the construction of determining equations for certain classes of adjoint symmetries, which are of the same nature as determining equations for (generalized) symmetries, and relies on other packages such as DIMSYM below for solving these determining equations. In addition, procedures have been written for testing whether a given adjoint symmetry can give rise to a Lagrangian or a first integral for the original second-order equations.

The program DIMSYM by Sherring [277], in collaboration with Prince, was inspired by Head's symmetry program LIE [135], in turn influenced by SPDE [260, 264], but is much larger and grew independently of it during development. It is capable of finding various types of symmetries, currently, point symmetries, Lie-Bäcklund, and conditional symmetries. DIMSYM can isolate special cases, bring the determining equations in standard form for example, and aid in the solution of group classification problems. It attempts to determine the generators and allows one to check whether or not the generators are correct. It allows the user to specify the dependence of the symmetry vector field coefficients, which is particularly practical if one wants to compute Lie-Bäcklund symmetries. DIMSYM provides the user with a lot of flexibility: ansätze can be made, simplification routines can be called separately, manual intervention is possible, etc. Quite often such interventions indeed allow the user to complete the desired computations, whereas the DIMSYM in auto-pilot mode may not.

DIMSYM has routines that convert the input equations into standard form. Another attractive feature of the package is that the integrator for the determining equations also works for systems of linear homogeneous differential equations not necessarily obtained from symmetry analysis. The overall strategy of the solver is to put the system of determining equations into standard form based on Reid's algorithm (see Section 13.2.3), while solving explicitly any equations in the system that the algorithm is capable of solving.

Finally, we mention the programs by Ito [156, 157, 158] for the determination of symmetries and conservation laws of systems of evolution equations. Ito's program does not use any of the algorithms discussed in Section 13.2.2, but uses infinitesimal symmetries to determine the form of conservation laws.

13.4.3. MACSYMA PROGRAMS

One of the first programs was written by Schwarzmeier and Rosenau [246, 271]. Their program calculates the determining equations, simplifies them a bit, but does not solve them automatically.

The MACSYMA version of the program SYMCON [294], which was originally written in muMATH, tries to compute Lie-point and Bessel-Haagen generalized symmetries (of any order) and their conservation laws. Vafeades later produced PDELIE [295, 296, 297, 298], which is a drastically improved version of SYMCON [294]. The package PDELIE attempts to produce similarity solutions of ODEs, analyze PDEs with a multiplicative or additive scalar parameter, and compute the commutator table and the structure constants of the Lie algebra. PDELIE also allows one to compute the Noether conservation laws of variational systems.

PDELIE consists of several subroutines. Let's discuss the main ones. The function PL_SYM, produces the determining equations and the generators of the Lie group. It uses a standard form algorithm by Reid and a set of heuristic rules to facilitate the integration. The function PL_SOLVE tries to find the invariants of the symmetry group. Using these invariants, it then dimensionally reduces the given differential equation. In cases where the reduced equation is an ODE, it tries to integrate explicitly, thus arriving at special similarity solutions of the original equation. The functions PL_COMTAB performs computations with elements in the commutator table of structure constants of the Lie algebra. The function PL_CON computes the densities of the Noether conservation laws of systems of variational and divergence type.

Just as PDELIE, the program SYM_DE by Steinberg [282, 283, 284] was recently added to the out-of-core library of MACSYMA. Steinberg's program computes infinitesimal symmetry operators and the explicit form of the infinitesimal transformations for simple systems. In cases where the program cannot automatically finish the computation, the user can intervene and, for instance, ask for infinitesimal symmetries of polynomial form. The program solves some (or all) of the determining equations automatically and, if needed, the user can (interactively) add extra information. Steinberg intends to extend his program so that it would include the calculation of generalized symmetries.

The program SYMMGRP.MAX written by Champagne, Hereman and Winternitz [52, 137], is a modification of an earlier package [53] that has been extensively used over the last decade at the University of Montréal and in many places elsewhere. It has been tested on hundreds of systems of equations and has thus been solidly debugged.

The flexibility within SYMMGRP.MAX and the possibility of using it interactively, allows the user to find the symmetry group of arbitrarily large and complicated systems of equations on relatively small computers. For example, whenever the prolongation can be applied successfully to the complete system, or a subset thereof, it produces a list of determining equations. This list is free of trivial factors, duplication and differential redundancies.

To make SYMMGRP.MAX work for *large* systems of differential equations, the designers followed the path that would be taken in manual calculations. That is, obtain in as simple a manner as possible the simplest determining equations, solve them and feed the information back to the computer. Partial information can be extracted very rapidly. For instance, one can derive a subset of the determining equations, such as those that occur as coefficients in the highest derivatives in the independent variables. These are usually single-term equations, which express that the coefficients of the vectorfield are independent of some variables or depend linearly on some of the other variables.

A feedback mechanism facilitates the solution of the determining equations step by step on the computer; hence, avoiding human error in the algebraic simplifications. Typically, users will provide information about the η's and φ's, as it becomes available from solving the determining equations step by step. The amount of interaction by the user will depend on the complexity of the system of differential equations and on the capacity of the computer used. A worked example showing the use of the feedback mechanism is given in [52]. Needless to say, with the feedback mechanism, the program SYMMGRP.MAX can also be used to verify previously calculated solutions of the determining equations and, hence, detect errors in the literature on the subject.

Although not designed for that purpose, the program SYMMGRP.MAX can be easily adapted to compute the determining equations corresponding to nonclassical symmetries [60, 67, 68, 69, 71]. In [71], Clarkson and Mansfield give a detailed explanation of such an adaptation. Their proof of correctness of the proposed adaptation is based on the theory of Gröbner bases.

13.4.4. MAPLE PROGRAMS

In [49], Carminati, Devitt and Fee present LIESYMM for creating the determining equations via the Harrison-Estabrook procedure. Within LIESYMM various interactive tools are available for integrating the determining equations, and for working with Cartan's differential forms. Their program is independent of Donsig's differential forms package *difforms*, also available in Maple.

Khai T. Vu (Department of Mathematics, Monash University, Clayton, Victoria, Australia) has translated Head's muMATH program LIE [135], discussed in Section 13.4.7, into Maple syntax. In 1994, the Maple version of LIE, which computes Lie-point symmetries, was still being tested and, therefore, was not yet released. The β-version of the program computes the determining equations, solves them, and gives the explicit forms of the (vectorfield) coefficients together with the generators.

Hickman [142, 143] offers a collection of Maple routines that aid in the computation of Lie-point symmetries, non-local symmetries, and Wahlquist-Estabrook-type prolongations. The tools for symmetry analysis include user-friendly procedures to enter names of variables, to create total derivatives, to generate and prolong vector fields, and to derive and partially solve determining equations. Program and documentation are available via anonymous FTP from math.canterbury.ac.nz.

Mansfield has developed the package DIRMETH for the computation of symmetries via the direct method proposed by Clarkson and Kruskal [65]. The program DIRMETH is part of DIFFGROB2, discussed in detail in Section 13.2.3. Other efforts in the design of packages for the direct method are given in the doctoral thesis of Williams [311].

13.4.5. MATHEMATICA PROGRAMS

Herod [141] developed MathSym for deriving the determining equations corresponding to Lie-point symmetries, including nonclassical (or conditional) symmetries. Upon derivation of the determining equations, the program reduces these equations via an algorithm based on the method of Riquier and Janet. Herod's doctoral thesis contains the well-documented code of MathSym and applications to various equations from fluid dynamics.

Recently, the packages *Lie.m* and *Baecklund.m* have been added by Baumann [22, 23, 24] to MathSource, the Mathematica Program Library. Baumann's program *Lie.m* [22] follows the structure of our MACSYMA program SYMMGRP.MAX [52] very closely. Users familiar with SYMMGRP.MAX will have a short learning curve with *Lie.m*. In contrast to SYMMGRP.MAX, the program *Lie.m* can handle transcendental functions in the input equations. The newest version of *Lie.m* can be used to compute point symmetries, contact symmetries and nonclassical symmetries. *Lie.m* brings the determining equations in canonical form via the procedure of Janet and Riquier, and goes on to solve the determining equations automatically. A finite set of integration rules, similar to the ones described in Section 13.2.4, are implemented.

Once the solution of the determining equations is obtained, the program can continue with the computation of the vector basis, ideals, and commutator table of the Lie algebra, its structure constants, Casimir operators, and its metric tensor.

Baumann's package *Baecklund.m* [23, 24] contains functions that attempt to compute generalized symmetries for PDEs and ODEs and invariants of ODEs only. When applied to second-order ODEs, the program attempts to verify if the computed symmetries are of variational type. If so, the program calculates the corresponding invariants (integrals of motion). For the explicit calculations to be successful, quite often one has to specify that the coefficients in the vectorfield are polynomials in the coordinates and momenta. With this "ansatz" one may not be able to obtain all the generalized symmetries, but one may successfully obtain explicit forms of invariants.

Bérubé and de Montigny [27] produced Lie-symmetry code in Mathematica. Their program *symmgroup.c* computes the determining equations for Lie-point symmetries. In its syntax and format *symmgroup.c* closely follows the structure of SYMMGRP.MAX. The data for the program may consist of DEs with arbitrary functions. Transcendental functions in both dependent and independent variables are also permitted. In [27], three well-chosen examples are given to illustrate the capabilities of the program.

Finally, Coult (while at Carleton College, Northfield, Minnesota) developed a Mathematica program, temporarily called *symmgroup.m*, for the computation of the determining equations corresponding to Lie-point symmetries of a large class of differential equations (with polynomial terms).

13.4.6. SCRATCHPAD AND AXIOM PROGRAMS

Schwarz [263] rewrote SPDE [262, 264] for use with version 1 of Scratchpad II, a symbolic manipulation program developed by IBM. Scratchpad II is now superseded by AXIOM.

Seiler and co-workers [256, 274] are designing a package that will compute determining equations for classical and non-classical symmetries. See Section 13.2.3 for a description of their program JET for geometry computations based on the jet bundle formalism.

13.4.7. MUMATH PROGRAMS

The program LIE by Head [135] is based on version 4.12 of muMATH, but is self contained and runs on IBM compatible PCs. As a matter of fact, the program comes bundled with a limited version of muMATH. Head's program calculates and solves the determining equations (for Lie-point symmetries) automatically for single equations and systems of differential equations. LIE also computes the Lie vectors and their commutators. Interventions by the user are possible but are rarely needed. The source code of the program is available, including the heuristic routines that attempt to solve the determining equations. Due to the limitations of muMATH, the program LIE is bounded by the 256 KB of memory for program and workspace. For a program of limited size, LIE is remarkable in its achievements.

Version 4.2 of LIE is freely available by FTP from various public domain software archives such as SIMTEL and associated archives. By this printing, version 4.3 of LIE also will be available. In addition to Lie-point symmetries, the new version will be able to compute contact and generalized (Lie Bäcklund) symmetries.

The SYMCON package written by Vafeades [294] also uses muMATH to calculate the determining equations (without solving them). The program is restricted to point symmetries. Furthermore, the program verifies whether the symmetry group is of variational or divergence type and computes the conservation laws associated with the symmetries. Unfortunately, these programs are confined to the 256 KB memory accessible by muMATH and, cannot presently handle very large systems of equations. This limitation motivated Vafeades to rewrite his SYMCON program in MACSYMA syntax [295, 296, 297, 298]. The MACSYMA version of SYMCON can handle generalized symmetries and their conservation laws.

For completeness, Mikhailov developed software in muMATH to verify the integrability of systems of PDEs by testing for the existence of higher symmetries. The program computes special symmetries, canonical conservation laws, and carries out conformal transformations to bring PDEs into canonical form. With their PC program, Mikhailov, Shabat and Sokolov [195] produced an exhaustive list of integrable nonlinear Schrödinger-type equations. In this context, integrable means that the equations have infinitely many conserved quantities and infinitely many local symmetries.

13.4.8. PROGRAMS FOR OTHER SYSTEMS

Kornyak and Fushchich [101, 104, 170] developed programs in Turbo C and AMP for the computation of Lie-Bäcklund symmetries. Their programs also classify equations with arbitrary parameters and functions with respect to such symmetries. It is important to note that their programs reduce the determining equations into passive form (see Section 13.2.3). All integrability conditions are then explicit and, therefore, the resulting system is in involution.

We should mention their two FORMAC programs. The first program, called *LB*, was written in the PL/1 language by Fedorova and Kornyak [91, 92]. The successor, called *LBF*, was developed by Fushchich and Kornyak [104]. Both programs create the system of determining equations for Lie-Bäcklund symmetries and attempt to solve these equations. The program *LBF*, with its 1362 lines of PL/1-FORMAC code, is completely automatic and consists of 37 subroutines, one of which brings the determining equations in passive (Riquier-Janet) form. The program *LB* [92] is available from the Computer Physics Communications Program Library in Belfast. The above programs were designed for low-memory requirements so that they could run on PCs.

The PL/1-based FORMAC package CRACKSTAR developed by Wolf [318, 319, 320] allows one to investigate Lie symmetries of systems of PDEs, besides dealing with dynamical symmetries of ODEs [225], and the like. A good overview of the capabilities of CRACKSTAR is given in [319]; a description of the routines and worked examples are in [225]. For efficiency, CRACKSTAR generates and solves first-order determining equations early on, and then continues with the higher-order determining equations. The successor of CRACKSTAR is the REDUCE package CRACK discussed already in Section 13.4.2.

Gerdt [115, 116], Gerdt and Zharkov [119] and Gerdt, Shvachka and Zharkov [117, 118] used REDUCE and PL/1-FORMAC to investigate the integrability of nonlinear evolution equations. Their program FORMINT contains algorithms to calculate Lie-Bäcklund symmetries and conserved densities, but does not use the jet bundle formalism.

The calculation of the Lie group by computer was also proposed by Popov [231], who used the program SOPHUS for the calculation of conservation laws of evolution equations.

In [42], Bocharov and Bronstein present SCoLAr, a package written in standard PASCAL that finds infinitesimal symmetries and conservation laws of arbitrary systems of differential equations. An application of SCoLAr to the Kadomtsev-Pogutse equations is given in [129].

The PC package DELiA, standing for "Differential Equations with Lie Approach," is an outgrowth of the SCoLar project [42]. DELiA, written in Turbo PASCAL by Bocharov and his collaborators [37, 38, 39, 40], is a stand-alone computer algebra system for investigating differential equations. It performs various tasks based on Lie's approach, such as the computation of Lie-point and Lie-Bäcklund symmetries, canonical conserved densities and generalized conservation laws, simplification and partial integration of overdetermined systems of differential equations, etc. The methods used in DELiA and many examples are well described in the user guides [37, 40].

In order to be able to handle large problems, DELiA first generates and solves first-order determining equations, and then continues to generate and solve the higher-order determining equations. The analyzer/integrator, which is available as a separate tool at the user level, includes a general algorithm for passivization [42], together with a set of integration rules for linear and quasi-linear systems of PDEs. Currently, a MS Windows version of DEliA, called MS Win DELiA, is under development.

Using the algorithmic language REFAL, Topunov [287] developed a software package for symmetry analysis that contains subroutines to reduce determining systems in passive form.

13.5. EXAMPLES

In this section we give three examples that illustrate the computation of Lie-point symmetries with symbolic software. The first and simplest example involves a single scalar nonlinear equation. The second example illustrates how symmetries of a nonlinear complex equation are computed by splitting the equation into a system of nonlinearly coupled equations for the real and imaginary parts of the original dependent variable. The last and most complicated example involves a system of vector equations that needs to be split into equations for its scalar components in order to compute its Lie symmetries.

13.5.1. THE HARRY DYM EQUATION

Consider the Harry Dym equation [1],

$$u_t - u^3 u_{xxx} = 0. \tag{13.17}$$

Clearly, this is one equation with two independent variables and one dependent variable. The assignments of the variables are as follows:

$$x \longmapsto x[1] \quad , \quad t \longmapsto x[2] \quad , \quad u \longmapsto u[1]. \tag{13.18}$$

This permits us to rewrite the equation (13.17) in a form accepted by the program SYMMGRP.MAX; i.e.,

```
e1 : u[1,[0,1]]-u[1]^3*u[1,[3,0]].
```

For PDELIE and SYM_DE the input form would be

```
'DIFF(U,T)-U^3*'DIFF(U,X,3).
```

For SPDE and LIE the program accepts

```
U(1,2)-U(1)^3*U(1,1,1,1).
```

Next, one selects the variable u_t for elimination, e.g.

```
v1 : u[1,[0,1]].
```

Then, the programs automatically compute the determining equations for the coefficients eta[1] $= \eta^x$, eta[2] $= \eta^t$, and phi[1] $= \varphi^u$ of the vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}. \tag{13.19}$$

There are only eight determining equations,

$$\begin{aligned}
\frac{\partial \text{eta2}}{\partial u[1]} &= 0, \\
\frac{\partial \text{eta2}}{\partial x[1]} &= 0, \\
\frac{\partial \text{eta1}}{\partial u[1]} &= 0, \\
\frac{\partial^2 \text{phi1}}{\partial u[1]^2} &= 0,
\end{aligned} \tag{13.20}$$

$$\begin{aligned}
\frac{\partial^2 \mathrm{phi1}}{\partial u[1]\partial x[1]} - \frac{\partial^2 \mathrm{eta1}}{\partial x[1]^2} &= 0, \\
\frac{\partial \mathrm{phi1}}{\partial x[2]} - u[1]^3 \frac{\partial^3 \mathrm{phi1}}{\partial x[1]^3} &= 0, \\
3u[1]^3 \frac{\partial^3 \mathrm{phi1}}{\partial u[1]\partial x[1]^2} + \frac{\partial \mathrm{eta1}}{\partial x[2]} - u[1]^3 \frac{\partial^3 \mathrm{eta1}}{\partial x[1]^3} &= 0, \\
u[1] \frac{\partial \mathrm{eta2}}{\partial x[2]} - 3u[1] \frac{\partial \mathrm{eta1}}{\partial x[1]} + 3\ \mathrm{phi1} &= 0.
\end{aligned} \tag{13.21}$$

These determining equations are easily solved explicitly, either automatically with SPDE, LIE and PDELIE, or with the feedback mechanism within SYM_DE and SYMMGRP.MAX. The general solution, rewritten in the original variables, is

$$\eta^x = k_1 + k_3\, x + k_5\, x^2\,, \tag{13.22}$$

$$\eta^t = k_2 - 3k_4\, t\,, \tag{13.23}$$

$$\varphi^u = (k_3 + k_4 + 2k_5\, x)\, u\,, \tag{13.24}$$

where $k_1, ..., k_5$ are arbitrary constants. The five infinitesimal generators then are

$$G_1 = \partial_x\,, \tag{13.25}$$

$$G_2 = \partial_t\,, \tag{13.26}$$

$$G_3 = x\partial_x + u\partial_u\,, \tag{13.27}$$

$$G_4 = -3t\partial_t + u\partial_u\,, \tag{13.28}$$

$$G_5 = x^2\partial_x + 2xu\partial_u\,. \tag{13.29}$$

Clearly, (13.17) is invariant under translations (G_1 and G_2) and scaling (G_3 and G_4). The flow corresponding to each of the infinitesimal generators can be obtained via simple integration. As an example, let us compute the flow corresponding to G_5. This requires integration of the first-order system

$$\begin{aligned}
\frac{d\tilde{x}}{d\epsilon} &= \tilde{x}^2\,, & \tilde{x}(0) &= x\,, \\
\frac{d\tilde{t}}{d\epsilon} &= 0\,, & \tilde{t}(0) &= t\,, \\
\frac{d\tilde{u}}{d\epsilon} &= 2\tilde{x}\tilde{u}\,, & \tilde{u}(0) &= u\,,
\end{aligned} \tag{13.30}$$

where ϵ is the parameter of the transformation group. One readily obtains

$$\tilde{x}(\epsilon) = \frac{x}{(1-\epsilon x)}\,, \tag{13.31}$$

$$\tilde{t}(\epsilon) = t\,, \tag{13.32}$$

$$\tilde{u}(\epsilon) = \frac{u}{(1-\epsilon x)^2}\,. \tag{13.33}$$

Therefore, we conclude that for any solution $u = f(x,t)$ of equation (13.17), the transformed solution

$$\tilde{u}(\tilde{x},\tilde{t}) = (1+\epsilon\tilde{x})^2 f(\frac{\tilde{x}}{1+\epsilon\tilde{x}},\tilde{t}) \tag{13.34}$$

will solve $\tilde{u}_{\tilde{t}} - \tilde{u}^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$.

13.5.2. THE NONLINEAR SCHRÖDINGER EQUATION

In order to compute the Lie-point symmetries of the celebrated nonlinear Schrödinger equation [1],

$$iu_t + u_{xx} + u|u|^2 = 0, \tag{13.35}$$

one needs to replace the single complex equation by a coupled system. One way of doing that is by introducing the real and imaginary parts v, w of the complex variable u via $u(x,t) = v(x,t) + iw(x,t)$. This yields

$$\begin{aligned} v_t + w_{xx} + w(v^2+w^2) &= 0, \\ w_t - v_{xx} - v(v^2+w^2) &= 0. \end{aligned} \tag{13.36}$$

One alternative is to replace (13.35) by a system consisting of the equation itself and its complex conjugate [58] and to interpret the variables u and $v = u^*$ as real,

$$\begin{aligned} k * u_t + u_{xx} + v * u^2 &= 0, \\ -k * v_t + v_{xx} + u * v^2 &= 0. \end{aligned} \tag{13.37}$$

In order to work with real quantities throughout, the imaginary unit i was temporarily replaced by the constant k during the computations. Once the determining equations are obtained, $k = i$ should be reintroduced.

Another alternative is to write $u(x,t) = R(x,t)\exp(i\Omega(x,t))$, thus replacing (13.35) by a coupled system

$$\begin{aligned} R_t + 2R_x\Omega_x + R\Omega_{xx} &= 0, \\ R_{xx} - R\Omega_t - R\Omega_x^2 + R^3 &= 0, \end{aligned} \tag{13.38}$$

for the real modulus $R(x,t)$ and real phase $\Omega(x,t)$.

Adhering to (13.36), SYMMGRP.MAX (or for that matter any other symmetry program) quickly generates the twenty determining equations for the coefficients of the vector field

$$\alpha = \eta^x\frac{\partial}{\partial x} + \eta^t\frac{\partial}{\partial t} + \varphi^v\frac{\partial}{\partial v} + \varphi^w\frac{\partial}{\partial w}. \tag{13.39}$$

The first eleven single-term determining equations are similar to (13.20), and provide information about the dependencies of the $\eta' s$ and the $\phi' s$ on x, t, v and w, and their linearity in the latter two independent variables. The remaining nine determining equations are a bit more complicated, but the entire system is readily solved.

In the original variables, the solution reads

$$\eta^x = k_1 + 2k_4\, t + k_5\, x\,, \tag{13.40}$$

$$\eta^t = k_2 + 2k_5\, t\,, \tag{13.41}$$

$$\varphi^v = k_3\, w - k_4\, xw - k_5\, v\,, \tag{13.42}$$

$$\varphi^w = -k_3\, v + k_4\, xv - k_5\, w\,, \tag{13.43}$$

where $k_1, ..., k_5$ are arbitrary constants. As in the previous examples, the complete symmetry algebra is spanned by five vector fields (generators):

$$\begin{aligned} G_1 &= \partial_x \ , & (13.44) \\ G_2 &= \partial_t \ , & (13.45) \\ G_3 &= w\partial_v - v\partial_w \ , & (13.46) \\ G_4 &= 2t\partial_x - x(w\partial_v - v\partial_w) \ , & (13.47) \\ G_5 &= x\partial_x + 2t\partial_t - v\partial_v - w\partial_w \ . & (13.48) \end{aligned}$$

If we had carried out the computations with (13.38), where $u(x,t) = R(x,t)\exp(i\Omega(x,t))$, we would have found:

$$\begin{aligned} G_1 &= \partial_x \ , & (13.49) \\ G_2 &= \partial_t \ , & (13.50) \\ G_3 &= \partial_\Omega, & (13.51) \\ G_4 &= 2t\partial_x - x\partial_\Omega \ , & (13.52) \\ G_5 &= x\partial_x + 2t\partial_t - R\partial_R \ . & (13.53) \end{aligned}$$

Either way, (13.35) is invariant under translations in space and time (G_1 and G_2). Generator G_3 corresponds to adding an arbitrary constant to the phase of u. The Galilean boost is generated by G_4. Finally, G_5 indicates invariance of the equation under scaling (or dilation). Similarity reductions can then be obtained by solving the characteristic equations,

$$\frac{dx}{\eta^x} = \frac{dt}{\eta^t} = \frac{dv}{\varphi^v} = \frac{dw}{\varphi^w}, \tag{13.54}$$

or equivalently, the invariant surface conditions

$$\begin{aligned} \eta^x v_x + \eta^t v_t - \varphi^v &= 0, \\ \eta^x w_x + \eta^t w_t - \varphi^w &= 0. \end{aligned} \tag{13.55}$$

The actual reductions can be found in [58], where a quite general nonlinear Schrödinger equation is treated. It is well known [176] that all the reductions of the NLS can be obtained from G_1 through G_5; in other words, nonclassical symmetries would not lead to new symmetry reductions. To compute nonclassical symmetries of (13.36), it suffices to replace v_t and w_t from (13.55). If $\eta^t \neq 0$, we set $\eta^t = 1$ for simplicity. Thus,

$$\begin{aligned} v_t &= -\eta^x v_x + \varphi^v, & (13.56) \\ w_t &= -\eta^x w_x + \varphi^w. & (13.57) \end{aligned}$$

The case $\eta^t = 0$ has to be considered separately. Since SYMMGRP.MAX allows the user to give information about the coefficients in the vector field, the computation can now proceed as in the classical case. For worked examples, we refer the reader to [60, 71].

13.5.3. THE MAGNETO-HYDRO-DYNAMICS EQUATIONS

As an example of a large system of differential equations, we take the equations for Magneto-Hydro-Dynamics (MHD) [98] and carry out the search for Lie-point symmetries with SYMMGRP.MAX.

The MHD equations, with or without dissipative terms, have become a benchmark for developers of Lie symmetry packages. Nucci [201] computed the classical symmetries in 1984 and commented that the calculations by hand took her a year. Michel Grundland undertook the same formidable task and also finished the job in about a year. He conveyed to the author that the long winter in Newfoundland (where he was at the time) helped.

If we neglect dissipative effects, and thus restrict the analysis to the *ideal* case, the MHD equations can be reduced to

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\rho + \rho\nabla\cdot\vec{v} &= 0\,, && (13.58)\\ \rho\Big(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\Big) + \nabla(p + \frac{1}{2}\vec{H}^2) - (\vec{H}\cdot\nabla)\vec{H} &= \vec{0}\,, && (13.59)\\ \frac{\partial \vec{H}}{\partial t} + (\vec{v}\cdot\nabla)\vec{H} + \vec{H}\nabla\cdot\vec{v} - (\vec{H}\cdot\nabla)\vec{v} &= \vec{0}\,, && (13.60)\\ \nabla\cdot\vec{H} &= 0\,, && (13.61)\\ \frac{\partial}{\partial t}(\frac{p}{\rho^{\kappa}}) + (\vec{v}\cdot\nabla)(\frac{p}{\rho^{\kappa}}) &= 0\,, && (13.62) \end{aligned}$$

with pressure p, mass density ρ, coefficient of viscosity κ, fluid velocity $\vec{v}$ and magnetic field $\vec{H}$. Using the first equation, we eliminate ρ from the last equation, hence replacing it by

$$\frac{\partial p}{\partial t} + \kappa p(\nabla\cdot\vec{v}) + (\vec{v}\cdot\nabla)p = 0. \qquad (13.63)$$

If we split the vector equations in scalar equations for the vector components, we have a system of nine equations, with four independent variables and eight dependent variables. For convenience, we denote the components of the vector $\vec{v}$ by v_x, v_y and v_z, not to be confused with partial derivatives of v.

The variables to be eliminated are selected as follows: for the first seven variables and the ninth variable we pick the partial derivatives with respect to t of $\rho, v_x, v_y, v_z, H_x, H_y$, H_z and p. From the eighth equation, we select $\partial H_x/\partial x$ for elimination.

We will only consider the case where $\kappa \neq 0$. We ran this case on a Digital VAX 4500 with 64 MB of RAM, and on an IBM Risc 6000 workstation with 32 MB of RAM. On the VAX it took 50 minutes of CPU time, on the IBM workstation 1 hour and 50 minutes, for SYMMGRP.MAX to create the 222 determining equations for the coefficients of the vector field

$$\begin{aligned} \alpha &= \eta^x\frac{\partial}{\partial x} + \eta^y\frac{\partial}{\partial y} + \eta^z\frac{\partial}{\partial z} + \eta^t\frac{\partial}{\partial t} + \varphi^{\rho}\frac{\partial}{\partial \rho} + \varphi^{p}\frac{\partial}{\partial p}\\ &+ \varphi^{v_x}\frac{\partial}{\partial v_x} + \varphi^{v_y}\frac{\partial}{\partial v_y} + \varphi^{v_z}\frac{\partial}{\partial v_z} + \varphi^{H_x}\frac{\partial}{\partial H_x} + \varphi^{H_y}\frac{\partial}{\partial H_y} + \varphi^{H_z}\frac{\partial}{\partial H_z}. \end{aligned}$$

Using SYMMGRP.MAX interactively, we then integrated the determining system and obtained the solution expressed in the original variables,

$$\eta^x = k_2 + k_5\,t - k_8\,y - k_9\,z + k_{11}\,x\,, \qquad (13.64)$$

$$\begin{aligned}
\eta^y &= k_3 + k_6\, t + k_8\, x - k_{10}\, z + k_{11}\, y\,, & (13.65)\\
\eta^z &= k_4 + k_7\, t + k_9\, x + k_{10}\, y + k_{11}\, z\,, & (13.66)\\
\eta^t &= k_1 + k_{12}\, t\,, & (13.67)\\
\varphi^\rho &= -2\,(k_{11} - k_{12} - k_{13})\,\rho\,, & (13.68)\\
\varphi^p &= 2\,k_{13}\,p\,, & (13.69)\\
\varphi^{v_x} &= k_5 - k_8\, v_y - k_9\, v_z + (k_{11} - k_{12})v_x\,, & (13.70)\\
\varphi^{v_y} &= k_6 + k_8\, v_x - k_{10}\, v_z + (k_{11} - k_{12})v_y\,, & (13.71)\\
\varphi^{v_z} &= k_7 + k_9\, v_x + k_{10}\, v_y + (k_{11} - k_{12})v_z\,, & (13.72)\\
\varphi^{H_x} &= k_{13}\, H_x - k_8 H_y - k_9 H_z\,, & (13.73)\\
\varphi^{H_y} &= k_{13}\, H_y + k_8 H_x - k_{10} H_z\,, & (13.74)\\
\varphi^{H_z} &= k_{13}\, H_z + k_9 H_x + k_{10} H_y\,. & (13.75)
\end{aligned}$$

It is clear that there is a thirteen-dimensional Lie algebra spanned by the generators:

$$\begin{aligned}
G_1 &= \partial_t\,, & (13.76)\\
G_2 &= \partial_x\,, & (13.77)\\
G_3 &= \partial_y\,, & (13.78)\\
G_4 &= \partial_z\,, & (13.79)\\
G_5 &= t\partial_x + \partial_{v_x}\,, & (13.80)\\
G_6 &= t\partial_y + \partial_{v_y}\,, & (13.81)\\
G_7 &= t\partial_z + \partial_{v_z}\,, & (13.82)\\
G_8 &= x\partial_y - y\partial x + v_x\partial_{v_y} - v_y\partial_{v_x} + H_x\partial_{H_y} - H_y\partial_{H_x}\,, & (13.83)\\
G_9 &= y\partial_z - z\partial y + v_y\partial_{v_z} - v_z\partial_{v_y} + H_y\partial_{H_z} - H_z\partial_{H_y}\,, & (13.84)\\
G_{10} &= z\partial_x - x\partial z + v_z\partial_{v_z} - v_x\partial_{v_z} + H_z\partial_{H_x} - H_x\partial_{H_z}\,, & (13.85)\\
G_{11} &= x\partial_x + y\partial_y + z\partial_z - 2\rho\partial_\rho + v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z}\,, & (13.86)\\
G_{12} &= t\partial_t + 2\rho\partial_\rho - (v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z})\,, & (13.87)\\
G_{13} &= 2\rho\partial_\rho + 2p\partial_p + H_x\partial_{H_x} + H_y\partial_{H_y} + H_z\partial_{H_z}\,. & (13.88)
\end{aligned}$$

Thus, the MHD equations (13.58)-(13.62) are invariant under translations G_2 through G_4, Galilean boosts G_5 through G_7, rotations G_8 through G_{10}, and dilations G_{11} through G_{13}. In contrast to the results obtained for the 1+1 and the 2+1 dimensional versions of the MHD problem, the dimension of the Lie algebra for (13.58)-(13.62) in the full 3+1 dimensions (x, y, z and time t) is independent of the value of the coefficient of viscosity κ. Our results confirm those in [98], and of those of Grundland and Lalague [126, 127], who computed the classical and some nonclassical symmetries of the MHD, and also classified all the subalgebras in conjugacy classes. The MHD system and our results have been used by other investigators [49, 241, 277] to test their symmetry programs.

13.5.4. OTHER INTERESTING EXAMPLES

Champagne and Winternitz [53] used SYMMGRP.MAX to compute the Lie-point symmetries of the Korteweg de Vries equation with variable coefficients,

$$u_t + f(x,t)uu_x + g(x,t)u_{xxx} = 0, \qquad (13.89)$$

illustrating that SYMMGRP.MAX can easily handle equations involving arbitrary functions.

Also in [53], the point symmetries of a modified Kadomtsev-Petviashvili equation

$$(u_{xxx} - 2u_x^3 - 4u_t)_x - 6u_{xx}u_y + 3u_{yy} = 0, \tag{13.90}$$

in 2+1 dimensions are computed with SYMMGRP.MAX. This example was chosen because it leads to an infinite-dimensional Lie algebra involving four arbitrary functions of t.

A completely worked example of the calculation of Lie-point symmetries of a system of PDEs is given in [52]. This example shows the use of the feedback mechanism within SYMMGRP.MAX to completely solve the determining equations. It involves the Karpman equations [163], for which our symmetries were independently verified with REDUCE programs by Kersten and Gragert (private communication), Sherring and Prince [278], and Wolf [321].

Finally, SYMMGRP.MAX was recently used to compute Lie-point symmetries of two large systems of equations representing classical field theories [139]. Currently, the author is adapting SYMMGRP.MAX for the calculation of Lie-point symmetries of difference-differential equations.

13.6. CONCLUSION

The various programs that we reviewed need very little data and are straightforward to use provided the user has access to and knows the basics of the underlying CAS, such as MACSYMA, Maple, Mathematica and REDUCE.

Apart from the theoretical study of the underlying mathematics, there is a need for further development and implementation of effective algorithms for generating, reducing, simplifying and fully solving the determining equations for (classical and nonclassical) Lie-point symmetries and generalized or Lie-Bäcklund symmetries.

The availability of sophisticated symbolic programs certainly will accelerate the study of symmetries of physically important systems of differential equations in classical mechanics, fluid dynamics, elasticity, and other applied areas.

ACKNOWLEDGEMENTS

While writing this survey I was able to consult with many experts in this field. I am grateful for their suggestions, comments, corrections and criticisms which greatly helped improve the manuscript.

Sincere thanks goes to M. Ablowitz, B. Abraham-Shrauner, G. Baumann, D. Bérubé, G. Bluman, A. Bocharov, F. Cantrijn, B. Champagne, P. Clarkson, R. Conte, N. Coult, B. Fuchssteiner, V. Gerdt, K. Govinder, P. Gragert, M. Grundland, F. Gungor, A. Head, B. Herbst, S. Herod, M. Hickman, D. Holm, M. Kruskal, D. Levi, M. MacCallum, F. Mahomed, E. Mansfield, L. Marchildon, A. Mikhailov, M.C. Nucci, P. Olver, J. Ondich, G. Prince, D. Rand, G. Reid, M. Roelofs, W. Sarlet, W. Seiler, F. Schwarz, J. Sherring, W.-H. Steeb, S. Steinberg, Z. Thomova, M. Torrisi, P. Vafeades, K. Vu, T. Wilcox, P. Winternitz, T. Wolf, and C. Wulfman.

Special thanks goes to S. Collart for inviting me to write a preliminary version of this survey paper for Euromath Bulletin. It is my pleasure to thank N.H. Ibragimov for giving me the opportunity to completely rework that review paper and to publish it as a chapter in this book.

I am grateful to T. Hearn and H. Melenk for providing me with a free copy of REDUCE. Research for this survey paper was supported in part by Grant # CCR-9300978 of the National Science Foundation of the United States of America.

Table 1 List of current symmetry programs

Name & System	Distributor	Developer's Address	Refs.	Email & Anonymous FTP
CRACK LIEPDE & APPLYSYM (REDUCE)	REDUCE Network Library	T. Wolf & A. Brand T. Wolf School Math. Sci. Queen Mary & Westfield College London E1 4NS, UK	[321] [323]	 T.Wolf@maths.qmw.ac.uk galois.maths.qmw.ac.uk /ftp/pub/crack
DELiA (Pascal)	Beaver Soft 715 Ocean View Ave Brooklyn NY 11235, USA Cost: $ 300	A. Bocharov et al. A. Bocharov Wolfram Research 100 Trade Center Dr. Urbana-Champaign IL 61820-7237, USA	[37]	 alexei@wri.com
DIFFGROB2 (Maple)		E. Mansfield Dept. of Maths. Univ. of Exeter Exeter EX4 4QE United Kingdom	[189]	liz@maths.exeter.ac.uk euclid.exeter.ac.uk pub/liz
DIMSYM (REDUCE)	LaTrobe University School of Maths. Cost: $ 225	J. Sherring G. Prince School of Maths. Latrobe University Bundoora, VI 3083 Australia	[277]	matjs@lure.latrobe.edu.au G.Prince@latrobe.edu.au ftp.latrobe.edu.au /ftp/pub/dimsym
LIE (REDUCE)	CPC Program Library Belfast N. Ireland Cat. No. AABS	V. Eliseev et al. V. Eliseev Lab. Comp. Tech. Aut. JINR, Dubna Moscow Region 141980 Russia	[85]	

Table 1 cont. List of current symmetry programs

Name & System	Distributor	Developer's Address	Refs.	Email & Anonymous FTP
LIE (muMath) (independent)	SIMTEL	A. Head CSIRO Div. Mat. Sci. & Tech. Clayton, Victoria 3168 Australia	[135]	head@rivett.mst.csiro.au wuarchive.wustl.edu /edu/math/msdos/.. ../adv.diff.equations/lie42
Lie & LieBaecklund (Mathematica)	Wolfram Research MathSource 0202-622 0204-680	G. Baumann Abt. Math. Phys. Universität Ulm D-7900 Ulm Germany	[22] [23] [24]	bau@theophys.physik.uni-ulm.de mathsource.wri.com /pub/PureMath/Calculus
LIEDF/INFSYM & others (REDUCE)		P. Gragert & P. Kersten P. Kersten Dept. Appl. Math. University of Twente 7500 AE Enschede The Netherlands	[120] [123]	gragert@math.utwente.nl kersten@math.utwente.nl
Liesymm (Maple)	Waterloo Maple Software (Packages)	J. Carminati et al. G. Fee Dept. Comp. Sci. University of Waterloo Waterloo, Canada	[49]	wmsi@daisy.uwaterloo.ca wmsi@daisy.waterloo.edu
MathSym (Mathematica)		S. Herod Program Appl. Math. University of Colorado Boulder, CO 80309, USA	[141]	sherod@newton.colorado.edu newton.colorado.edu pub/mathsym
NUSY (REDUCE)		M.C. Nucci Dept. di Mathematica Università di Perugia 06100 Perugia, Italy	[202]	nucci@gauss.dipmat.unipg.it

Table 1 cont. List of current symmetry programs

Name & System	Distributor	Developer's Address	Refs.	Email & Anonymous FTP
PDELIE (MACSYMA)	MACSYMA Out-of-Core Library	P. Vafeades Dept. of Eng. Sci. Trinity University San Antonio, TX 78212, USA	[295] [296] [297]	peter@engr.trinity.edu gumbo.engr.trinity.edu
SPDE & SYMSIZE (REDUCE)	REDUCE Program Lib. Rand Corp.	F. Schwarz GMD, Inst. SCAI D-53731 Sankt Augustin Germany	[264] [266]	fritz.schwarz@gmd.de reduce-netlib@rand.org
SYMCAL (Maple & MACSYMA)		G. Reid & A. Wittkopf G. Reid Math. Dept. Univ. Brit. Columbia Vancouver, BC Canada V6T IZ2	[241]	 reid@math.ubc.ca math.ubc.ca pub/reid
SYM_DE (MACSYMA)	MACSYMA Out-of-Core Library	S. Steinberg Dept. Math. & Stat. Univ. New Mexico Albuquerque, NM 87131, USA	[284]	stanly@math.unm.edu
symmgroup.c (Mathematica)		D. Bérubé & M. de Montigny M. de Montigny D. Bérubé Centre Traitement Inform. Univ. Laval, St.-Froy Canada G1K 7P4	[27]	 montigny@physics.mcgill.ca berube@genesis.ulaval.ca genesis.ulaval.ca /pub/Mathematica/symgroup
SYMMGRP.MAX (MACSYMA)	CPC Program Lib. Belfast N. Ireland Cat. No. ACBI	B. Champagne et al. W. Hereman Dept. Math. Comp. Sci. Colorado Sch. of Mines Golden, CO 80401, USA	[52]	whereman@lie.mines.colorado.edu ftp.mines.colorado.edu pub/papers/math_cs_dept/symmetry or contact: cpc@v1.am.qub.ac.uk

Table 2 Scope of symmetry programs

Name	System	Developer(s)	Point	General.	Noncl.	Solves Det. Eqs.
CRACK	REDUCE	Wolf & Brand	-	-	-	Yes
DELiA	Pascal	Bocharov et al.	Yes	Yes	No	Yes
DIFFGROB2	Maple	Mansfield	-	-	-	Reduction
DIMSYM	REDUCE	Sherring	Yes	Yes	No	Yes
LIE	REDUCE	Eliseev et al.	Yes	Yes	No	No
LIE	muMath	Head	Yes	Yes	No	Yes
Lie	Mathematica	Baumann	Yes	No	Yes	Yes
LieBaecklund	Mathematica	Baumann	No	Yes	No	Interactive
LIEDF/INFSYM	REDUCE	Gragert & Kersten	Yes	Yes	No	Interactive
LIEPDE	REDUCE	Wolf & Brand	Yes	Yes	No	Yes

Table 2 cont. Scope of symmetry programs

Name	System	Developer(s)	Point	General.	Noncl.	Solves Det. Eqs.
Liesymm	Maple	Carminati et al.	Yes	No	No	Interactive
MathSym	Mathematica	Herod	Yes	No	Yes	Reduction
NUSY	REDUCE	Nucci	Yes	Yes	Yes	Interactive
PDELIE	MACSYMA	Vafeades	Yes	Yes	No	Yes
SPDE	REDUCE	Schwarz	Yes	No	No	Yes
SYMCAL	Maple/MACSYMA	Reid & Wittkopf	-	-	-	Reduction
SYM_DE	MACSYMA	Steinberg	Yes	No	No	Partially
symgroup.c	Mathematica	Bérubé & de Montigny	Yes	No	No	No
SYMMGRP.MAX	MACSYMA	Champagne et al.	Yes	No	Yes	Interactive
SYMSIZE	REDUCE	Schwarz	-	-	-	Reduction

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