The Painlevé Integrability Test

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The Painlevé test is a widely applied and quite successful technique to investigate the integrability [8] of nonlinear ODEs and PDEs by analyzing the singularity structure of the solutions. The test is named after the French mathematician Paul Painlevé (1863-1933) [18], who classified second order differential equations that are solvable in terms of known elementary functions or new transcendental functions [12].

The Painlevé test, allows one to verify whether or not a differential equation (perhaps after a change of variables) satisfies the necessary conditions for having the Painlevé property. If so, the equation is prime candidate for being completely integrable [1].

As originally formulated by Ablowitz *et al.* [2], the Painlevé conjecture asserts that all similarity reductions of a completely integrable PDE should have the Painlevé property (or be of Painlevé-type), i.e. their general solutions should have no movable singularities other than poles in the complex plane.

A later version of the Painlevé test due to Weiss *et al.* [23] allows testing of PDEs directly, without recourse to the reduction(s) to ODEs. A PDE is said to have the Painlevé property if its solutions in the complex plane are single-valued in the neighborhood of all its movable singularities. In other words, the equation must have a solution without any branching around the singular points whose positions depend on the initial conditions. The traditional Painlevé test does not test for essential singularities and therefore cannot determine whether or not branching occurs about these.

The algorithm

The Painlevé test can be applied to nonlinear polynomial system of ODEs or PDEs with (real) polynomial terms. For brevity, we give the three steps of the test for a single PDE, $\mathcal{F}(x, t, u(x, t)) = 0$, in two independent variables x and t.

Following [23], the Laurent expansion of the solution $u(x, t)$,

$$u(x,t) = g^{\alpha}(x,t) \sum_{k=0}^{\infty} u_k(x,t)\; g^k(x,t), \tag{1}$$

should be single-valued in the neighborhood of a non-characteristic, movable singular manifold $g(x, t)$, which can be viewed as the surface of the movable poles in the complex plane. In (1), $u_0(x, t) \neq 0$, α is a negative integer, and $u_k(x, t)$ are analytic functions in a neighborhood of $g(x, t)$.

Note that for ODEs the singular manifold is $g(x, t) = x - x_0$, where x_0 is the initial value for x. For PDEs, if $u(x, t)$ has simple zeros and $g_x(x, t) \neq 0$, one may apply the implicit function theorem near the singularity manifold and set $g(x, t) = x - h(t)$, for an arbitrary function $h(t)$ [16, 20]. This considerably simplifies the computations.

Step 1: Leading order analysis

Determine the (negative) integer α and u_0 by balancing the minimal power terms after substitution of $u = u_0 g^\alpha$ into the given PDE. There may be several branches for u_0, and for each the next two steps must be performed.

Step 2: Determination of the resonances

For a selected α and u_0, calculate the non-negative integers r, called the *resonances*, at which arbitrary functions u_r enter the series (1). To do so, substitute $u = u_0 g^\alpha + u_r g^{\alpha+r}$ into the equation, only retaining its most singular terms. Require that the coefficient u_r is arbitrary by equating its coefficient to zero. Compute the integer roots of the resulting polynomial. For (1) to represent the general solution, the number of roots (including $r = -1$) must match the order of the given equation. The root $r = -1$ corresponds to the arbitrariness of the manifold $g(x, t)$.

Step 3: Verification of the compatibility conditions

Verify that a solution of the form (1) is indeed admissible, and that it has the necessary number of free coefficients u_r. Substitute (1), truncated a the largest resonance, into the PDE. Determine u_k at non-resonance levels k. At resonance levels, u_r should be arbitrary, and since we are dealing with a nonlinear equation, a *compatibility condition* must be unconditionally satisfied.

An equation for which these three steps can be carried out consistently and unambiguously passes the Painlevé test.

In the case of systems, for every dependent variable u_i one substitutes

$$u_i = g(x,t)^{\alpha_i} \sum_{k=0}^{\infty} u_k^{(i)} g(x,t)^k, \tag{2}$$

and carefully determines all branches of dominant behavior corresponding to various choices of α_i and/or $u_0^{(i)}$. For each branch, the single-valuedness of the corresponding Laurent expansion must be tested, i.e. the resonances must be computed and the compatibility conditions must be verified. Details and an abundance of worked examples can be found in [1, 5, 6, 8, 16, 20, 22].

Simple Examples

Consider the PDE, $u_{tx} + a(t)u_x + 6uu_{xx} + 6u_x^2 + u_{xxxx} = 0$, and ask under what condition for $a(t)$ the equation passes the Painlevé test.

Here, $\alpha = -2$ and $u_0 = -2g_x^2$. Apart from $r = -1$, the roots are $r = 4, 5,$ and 6. The latter three are resonances. Furthermore, u_1, u_2 and u_3 can uniquely be determined in terms of derivatives of $g(x, t)$.

The compatibility conditions at resonances $r = 4$ and $r = 5$ are satisfied. Hence, u_4 and u_5 are arbitrary. The compatibility condition at resonance $r = 6$ is $a_t + 2a^2 = 0$. Hence, $a = \frac{1}{2t}$ and the PDE becomes the cylindrical KdV equation which is indeed completely integrable [1].

As a second example, consider the famous Lorenz system from meteorology,

$$u_1' = a(u_2 - u_1), \quad u_2' = -u_1u_3 + bu_1 - u_2, \quad u_3' = u_1u_2 - cu_3, \tag{3}$$

where a, b, and c are positive constants.

For each dependent variable, one substitutes a Laurent series (2) and determines the leading orders: $\alpha_1 = -1, \alpha_2 = \alpha_3 = -2$. The first coefficients are $u_0^{(1)} = \pm 2i, u_0^{(2)} = \mp 2i/a, u_0^{(3)} = -2/a$. The roots are $r = -1, 2, 4$. The expressions for $u_1^{(1)}, u_1^{(2)}$ and $u_1^{(3)}$ are readily computed.

The compatibility conditions at resonances $r = 2$ and $r = 4$ are not satisfied. At resonance $r = 2$ one encounters $a(c - 2a)(c + 3a - 1) = 0$. Investigating all cases, it turns out that for $c = 2a$ the compatibility condition at $r = 4$ is not satisfied. For $c = 1 - 3a$, the compatibility condition at $r = 4$ is satisfied if $a = \frac{1}{3}$. The Lorenz system (3) thus passes the Painlevé test when $a = \frac{1}{3}$ and $c = 0$ [10].

In the last example, we consider a coupled system of KdV equations,

$$u_{1,t} - 6au_1u_{1,x} + 6u_2u_{2,x} - au_{1,xxx} = 0, \qquad u_{2,t} + 3u_1u_{2,x} + u_{2,xxx} = 0, \tag{4}$$

where a is a nonzero parameter. System (4) is known to be completely integrable if $a = \frac{1}{2}$. This is confirmed by the Painlevé test. Indeed, with a Laurent series for u_1 and u_2 one obtains $\alpha_1 = \alpha_2 = -2$ and $r = -2, -1, 3, 4, 6$ and 8. Furthermore, $u_0^{(1)} = -4$ and $u_0^{(2)} = \pm 2\sqrt{2a}$ determine the coefficients $u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)}$ unambiguously. At resonances 3 and 4 there is one free function and no condition for a. The coefficients $u_5^{(1)}$ and $u_5^{(2)}$ are unique determined. At resonance 6, the compatibility condition is only satisfied if $a = \frac{1}{2}$. For this value, the compatibility condition at $r = 8$ is also satisfied.

Symbolic Programs

The Painlevé test, although algorithmic, is cumbersome when done by hand. Several computer implementations of the Painlevé test exist [5, 10, 11]. A brief review is given in [21]. These symbolic codes are particularly useful for the verification of the self-consistency (compatibility) conditions, and in exploring all possibilities of balancing singular terms. Applied to equations with parameters, the software can determine the conditions on the parameters so that the equations pass the Painlevé test (see [10, 11]).

Further Reading

There is a vast amount of literature about the test and its applications to specific differential equations. Several well-documented surveys [3, 5, 8, 9, 14, 15, 17, 19] and books [4, 6, 22] discuss the basics, as well as subtleties and pathological cases of the test. The survey papers also deal with the many interesting connections with other properties of PDEs and by-products of the Painlevé test. They show, for example, how truncated Laurent series expansions allow one to construct Lax pairs, Bäcklund and Darboux transformations, and closed-form particular solutions of PDEs.

Some shortcomings of the traditional Painlevé test have been identified by Kruskal and others [13, 14, 15]. Improved versions of the Painlevé test have been proposed, such as the poly-Painlevé test [14]. Besides, other variants of the test exist [5, 6, 13, 15], e.g the weak Painlevé test [20], and a perturbative Painlevé approach [7] which allows for a deeper analysis of equations with negative resonances.

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