Effective solution of two simultaneous Pell equations by the Elliptic Logarithm Method

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To my wife Maro

on the occasion of the silver jubilee of our common life

1 Introduction

The purpose of this paper is to present a hopefully *practical* and *uniform* general method[1] for solving explicitly any specific system of two Pell equations.

Clearly, any such system can be reduced to one of the form

$$X^2 - D_1Y^2 = A_1 \ , \ Z^2 - D_2Y^2 = A_2 \ \text{ in positive integers } X, Y, Z, \qquad (1)$$

where A_1, A_2 are non-zero integers, D_1, D_2 are non-square positive integers and $A_1D_2 - A_2D_1 \neq 0$. Note that, if the last condition is not satisfied, then $A_2/A_1 = D_2/D_1 = q^2$ for some $q \in \mathbb{Q}$ and the solutions (X, Y, Z) of the system of equations are given by (X, Y, qX), where (X, Y) is a solution of the first equation of the system.

A large number of papers is devoted to the study of simultaneous Pell equations[2]; they could be classified into three categories:

The first one includes papers, in which results are proved of the type *"there are no solutions"*, or *"there are at most k solutions"*, where k is a number between, say, 1 and 4; such are the papers [4],[5],[6],[7],[8],[20],[21],[27],[38],[39],[40]. The generality of such results is obtained at the cost of restricting at least two parameters among A_1, D_1, A_2, D_2. Of course, if a system (1) falls within the scope of such a paper *and* [3] one is so lucky that

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[1]Throughout this paper the term "algorithm", in a broad sense though, could also be used in place of the word "method".

[2]Often, solving (1) is reduced to finding common terms of two distinct second order recurrence sequences, a problem studied in a number of papers without any direct reference to (1); in our bibliography, it is natural to include such papers.

[3]This is the very optimistic case!

the number of solutions *he actually knows* is equal to the maximum number of solutions proved in the paper, the system is completely solved.

In a second category we put papers dealing with the *explicit* solution of some system(s), in which each parameter A_1, D_1, A_2, D_2 has either a specific numerical value or a certain special form. In these papers, either Baker's Theory is applied or, quite often, elegant elementary arguments; see[2],[3],[9],[13],[14],[15],[16],[18],[24],[25],[28],[37].

A third category includes papers in which effective but not explicit results are proved; see [19],[22],[23].

From the content and the results of the above papers it is clear that their objects are distinct from that of the present paper[4]. To the best of the author's knowledge, only two works in the literature deal with a similar object[5]:

First, W.S. Anglin's book [1], Section 4.6, where a method is proposed for solving (1) when all parameters are absolutely less than 1000. In principle, that method, which is based on the *Theory of Linear Forms in Complex Logarithms*, can work in general, even whithout the above restriction on the size of the parameters. Its complexity, however, depends upon the number of prime factors of A_1 and A_2, while the method proposed in the present paper, based on *Theory of Linear Forms in Elliptic Logarithms*, might be free of this defect[6]. Nice examples with a vivid discussion on the comparison of the *Complex Logarithm* versus the *Elliptic Logarithm* method are found in [34].

Second, the paper [26] by R.G.E. Pinch , where the solution of (1) is reduced to a finite number of problems of the form: *"Given two distinct second order recurrence sequences* (X_k) *and* (Y_l) *assuming integer values and an integer* g*, to show that* $(0,0)$ *is the only pair* (n,m) *satisfying* $X_n = Y_m + g \quad (*)$*"*. A first step of the method consists in obtaining an upper bound B of n, using again the *Theory of Linear Forms in Complex Logarithms*. Then, an algorithm builds up, step by step, sets P and Q of primes with certain properties. The main property of the set P is that its elements are the prime factors of an integer N such that, if $(*)$ is satisfied, then $n \equiv 0 \pmod{N}$. One expects to build P so large that $N > B$, which will force $n = 0$. It is not known *a priori* how large the primes in the sets P, Q have to be[7]; on the other hand, there is no theoretical guarantee that, in any specific case, the method will work with sets P, Q including primes of "reasonable" size.

[4] Cf. with the very begining of our Introduction.

[5] We do not include among them paper [24] in which an elementary interesting method is proposed which, however, could not be characterized "general".

[6] I owe this remark to the referee.

[7] In [35], where we had to solve the system $U^2 - 5V^2 = -4\mu$, $U^2 - 10Z^2 = \mu$, $\mu = \pm 1$, we applied Pinch's method; the sets P and Q turned out to include the 40 primes from 2 to 173 and the 145 primes from 3 to 997, respectively.

In the present paper, we adopt a quite different approach, using *Elliptic Curves* and *Linear Forms of Elliptic Logarithms*. The method that we propose depends upon the theoretically non-effective computation of a Mordell-Weil basis for a certain elliptic curve. This, in practice, cause no real problems, given the current state of the art on the actual computation of Mordell-Weil bases and the large positive experience from the extended application of the elliptic curves techniques to the explicit solution of diophantine equations. A more real problem from the application of the method can possibly arise when the rank of the related elliptic curve is high, say, larger than 7 or 8. In such a case[8], the final and easiest, from the theoretical point of view, step, which we describe after relation (21), may turn out a difficult computational task.

Finally, we wish to stress the following fact: As the reader will realize, for the application of our method we assume the knowledge of a non-trivial *rational solution* (X_0, Y_0, Z_0). What if we can neither detect such a solution nor prove its non-existence by, say, congruence considerations? Note that there are very efficient methods for deciding whether each equation in (1), considered separately, has a *rational* solution; see, for example, [12] and the references therein. Thus, if we know that both equations have rational solutions, but we are unable to decide whether the system of the two equations has a rational solution [9], we can work, at least in principle, as follows: We effectively obtain a rational solution (X_0, Y_0), say, of the first equation, so that (X_0, Y_0, Z_0), $Z_0 = \sqrt{A_2 + D_2 Y_0^2}$ is a solution belonging to at most a quadratic extension $\mathbb{K}$ of $\mathbb{Q}$ and the method of this paper remains exactly the same with $\mathbb{K}$ in place of $\mathbb{Q}$. In practice, however, this implies working with elliptic curves over $\mathbb{K}$ and the state of the art of this subject, from the computational point of view, is not satisfactory today. Thus, although *the method developed in the present paper still works in this case*, we cannot, at present, be very optimistic about the happy outcome of the computations related to the relevant elliptic curve.

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2 The method

We assume that we know a rational solution (X_0, Y_0, Z_0) with $Y_0 > 0$ and $X_0, Z_0 \geq 0$, *not both zero.*

[8]Very unlikely to happen, we believe, if the parameters in (1) are not constructed on purpose.

[9]It is worth asking whether such a case can occur in practice.

2.1 The related elliptic curve and the birational transformation

Elimination of A_1, A_2 from the two equations gives

$$A_2X^2 - A_1Z^2 + (A_1D_2 - A_2D_1)Y^2 = 0 \ . \tag{2}$$

We put

$$u = \frac{Y_0X - X_0Y}{Y_0Z - Z_0Y} \ , \tag{3}$$

so that

$$Y = \frac{Y_0(X - uZ)}{X_0 - uZ_0} \ ,$$

and substitute in (2) to obtain a quadratic equation in X/Z. Obviously, $(X/Z = X_0/Z_0)$ is one root of this quadratic equation, whence it follows that the second root is

$$\frac{X}{Z} = \frac{-A_2X_0u^2 + 2A_1Z_0u - A_1X_0}{A_2Z_0u^2 - 2A_2X_0u + A_1Z_0} \ . \tag{4}$$

Then, by the expression of Y in terms of X, Z and u, above, we obtain

$$\frac{Y}{Z} = \frac{(-A_2u^2 + A_1)Y_0}{-A_2Z_0u^2 + 2A_2X_0u - A_1Z_0} \ . \tag{5}$$

We have $1 - D_2\left(\frac{Y}{Z}\right)^2 = \frac{A_2}{Z^2}$, hence, by (5) and the relations

$$X_0^2 - D_1Y_0^2 = A_1 \ \ , \ \ Z_0^2 - D_2Y_0^2 = A_2 \ , \tag{6}$$

it follows that

$$au^4 + bu^3 + cu^2 + du + e^2 = v^2 \ , \tag{7}$$

where

$$v = \sigma\frac{-A_2Z_0u^2 + 2A_2X_0u - A_1Z_0}{Z} \ , \ \ \sigma = \mathrm{sgn}(A_1) \ . \tag{8}$$

and

$$\begin{gathered} a = A_2^2 \ , \ b = -4A_2X_0Z_0 \ , \ c = 4A_1Z_0^2 - 2A_1A_2 + 4A_2X_0^2 \ , \\ d = -4A_1X_0Z_0 \ , \ e = |A_1| \ . \end{gathered} \tag{9}$$

Later we will justify the choice of σ in (8). Using (3) in (8) and taking into account (6) we express v as a rational function of X, Y, Z:

$$v = 2\sigma Y_0^2\frac{A_2X_0X + (D_2A_1 - D_1A_2)Y_0Y - A_1Z_0Z}{(Y_0Z - Z_0Y)^2} \ . \tag{10}$$

Conversely, we need also express X, Y, Z as rational functions of u, v. By the definition of v,

$$Z = \sigma \frac{-A_2 Z_0 u^2 + 2A_2 X_0 u - A_1 Z_0}{v} \tag{11}$$

and this, combined with (5), (4) and (6) gives

$$Y = \sigma \frac{(-A_2 u^2 + A_1) Y_0}{v} \quad , \quad X = \sigma \frac{A_2 X_0 u^2 - 2A_1 Z_0 u + A_1 X_0}{v} \; . \tag{12}$$

We consider now (7). Following Section 2 of [36], we define the rational functions

$$\mathbf{x}(u, v) = \frac{2ev + du + 2e^2}{u^2} + \frac{c}{3} \tag{13}$$

$$\sigma \cdot \mathbf{y}(u, v) = \frac{8e^3 v + 8e^4 + 4e^2 du + (4e^2 c - d^2) u^2}{2eu^3} + \frac{d}{2e} \mathbf{x}(u, v) - \frac{cd}{6e} + be \tag{14}$$

$$\mathbf{u}(x, y) = \frac{12e^2 x + 8e^2 c - 3d^2}{2(6\sigma ye - 3dx + cd - 6be^2)} \tag{15}$$

$$\mathbf{v}(x, y) = \frac{3x - c}{6e} \mathbf{u}(x, y)^2 - \frac{d}{2e} \mathbf{u}(x, y) - e \tag{16}$$

where, as before, $\sigma = \operatorname{sgn}(A_1)$. The functions

$$(u, v) \longrightarrow (x, y) = (\mathbf{x}(u, v), \mathbf{y}(u, v)) \tag{17}$$

$$(x, y) \longrightarrow (u, v) = (\mathbf{u}(x, y), \mathbf{v}(x, y)) \tag{18}$$

establish a birational transformation between (7) and

$$y^2 = x^3 + Ax + B \; , \tag{19}$$

where

$$\begin{aligned} A &= -\frac{1}{3}c^2 + bd - 4ae^2 \\ B &= \frac{2}{27}c^3 - \frac{1}{3}bcd - \frac{8}{3}ace^2 + b^2 e^2 + ad^2 \; . \end{aligned}$$

So far, our discussion sums up to the following:

Proposition 2.1. *Let (X, Y, Z) be a solution of the initial system (1). Using (3) and (10) we obtain (u, v) satisfying (7) and then, by (17), we obtain (x, y) satisfying (19). This gives a birational transformation from (1) to (19), say $x = \mathcal{X}(X, Y, Z)$, $y = \mathcal{Y}(X, Y, Z)$. Conversely, if (x, y) satisfies (19), then, using (18), we obtain (u, v) satisfying (7) and then, by (11) and (12), we obtain a solution (X, Y, Z) of (1).*

2.2 Some technical results

The following result is useful for the practical application of our method.

Proposition 2.2. *The right-hand side of (19) has three distinct rational roots, namely*

$$\begin{aligned}\rho_1 &= \frac{4}{3}(D_2X_0^2 - 2D_1D_2Y_0^2 + D_1Z_0^2)Y_0^2 = \frac{4}{3}(A_1D_2 + A_2D_1)Y_0^2\\ \rho_2 &= \frac{4}{3}(-2D_2X_0^2 + D_1D_2Y_0^2 + D_1Z_0^2)Y_0^2 = \frac{4}{3}(A_2D_1 - 2A_1D_2)Y_0^2\\ \rho_3 &= \frac{4}{3}(D_2X_0^2 + D_1D_2Y_0^2 - 2D_1Z_0^2)Y_0^2 = \frac{4}{3}(A_1D_2 - 2A_2D_1)Y_0^2\ ,\end{aligned}$$

hence, the curve $\mathcal{E}$ defined by (19) is an elliptic curve. Denote by e_1, e_2, e_3, respectively, the largest, the middle and the least among the three roots above. Then, the following table gives the values of e_1, e_2, e_3:

A_1	A_2	$A_2D_1 - A_1D_2$	e_3	e_2	e_1
>0	>0	>0	ρ_3	ρ_2	ρ_1
>0	>0	<0	ρ_2	ρ_3	ρ_1
>0	<0	any sign	ρ_2	ρ_1	ρ_3
<0	>0	any sign	ρ_3	ρ_1	ρ_2
<0	<0	>0	ρ_1	ρ_3	ρ_2
<0	<0	<0	ρ_1	ρ_2	ρ_3

Moreover, we define

$$\begin{aligned}x_0 &= \lim_{Y\to+\infty} \mathcal{X}(\sqrt{A_1 + D_1Y^2},\, Y, -\sqrt{A_2 + D_2Y^2}\,)\\ y_0 &= \lim_{Y\to+\infty} \mathcal{Y}(\sqrt{A_1 + D_1Y^2},\, Y, -\sqrt{A_2 + D_2Y^2}\,)\ .\end{aligned}$$

Then $x_0, y_0 \in \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$, $x_0 > e_1, y_0 > 0$ and (x_0, y_0) is a point on $\mathcal{E}_0$, the zero-component of the elliptic curve $\mathcal{E}$.

Finally, for any positive solution (X, Y, Z) of (1), with Y sufficiently large, the point $(x, y) = (\mathcal{X}(X, Y, -Z), \mathcal{Y}(X, Y, -Z))$ belongs to $\mathcal{E}_0$ and $y > 0$; how large Y must be, can be made explicit in any particular case (1).

Proof. With MAPLE V we easily calculated the three roots. These are distinct, in view of our initial restrictions on the parameters of (1). Every claim in the table can be checked without much difficulty, but this is probably a rather tedious task without the aid of a Computer Algebra package.

The same is true for the computation of x_0 and y_0, the values of which are

$$\begin{aligned} x_0 &= \frac{4Y_0^2}{3}(D_2X_0^2 + D_1D_2Y_0^2 + D_1Z_0^2 + 3\sqrt{D_1}D_2X_0Y_0 \\ &\qquad +3D_1\sqrt{D_2}Y_0Z_0 + 3\sqrt{D_1D_2}X_0Z_0) \\ y_0 &= \frac{8\sigma|A_1|Y_0^3\sqrt{D_1}(Z_0\sqrt{D_1} + X_0\sqrt{D_2})(Z_0 + Y_0\sqrt{D_2})}{X_0 - Y_0\sqrt{D_1}} . \end{aligned}$$

It is clear now that x_0 is positive and larger than all three roots ρ_1, ρ_2, ρ_3. On the other hand, $\mathrm{sgn}(X_0 - Y_0\sqrt{D_1}) = \mathrm{sgn}(A_1) = \sigma$, hence $y_0 > 0$. Finally, if (X, Y, Z) is a positive solution of (1), then $X = \sqrt{A_1 + D_1Y^2}$, $Z = \sqrt{A_2 + D_2Y^2}$, hence, for sufficiently large Y, the number $\mathcal{X}(X, Y, -Z)$ is, sufficiently close to x_0, which, in turn is larger than e_1; this, obviously, proves that $(x, y) \in \mathcal{E}_0$ and $y > 0$ in view of the fact that, for sufficiently large Y, y is near y_0. □

We also need the following

Lemma 2.3. *The substitution* $(\sqrt{A_1 + D_1Y^2}, -\sqrt{A_2 + D_2Y^2}) \leftarrow (X, Z)$ *in (10) results in a function* $v = v(Y)$*, which, for sufficiently large* Y*, assumes positive values.*

Proof. In (10), the factor that determines the sign of v is (after the substitution mentioned in the Lemma)

$$\sigma(A_1Z_0\sqrt{A_2 + D_2Y^2} + (D_2A_1 - D_1A_2)Y_0Y + A_2X_0\sqrt{A_1 + D_1Y^2}) .$$

Dividing this by Y and letting $Y \to +\infty$, we find the limit

$$\sigma(X_0 - Y_0\sqrt{D_1})(Z_0\sqrt{D_1} + X_0\sqrt{D_2})(Z_0 + Y_0\sqrt{D_2}) ,$$

which is positive, as we saw in the proof of Proposition 2.2. This, certainly, implies that, for sufficiently large Y, the corresponding v is positive. □

The following result is implicitly needed for our method.

Lemma 2.4. *There exists a positive constant* Ψ_0*, explicitly computable in every particular instance of the system (1), such that, the bijection*

$$\begin{aligned} [\Psi_0\,, +\infty) \ni Y \quad &\longrightarrow \quad \mathcal{X}(\sqrt{A_1 + D_1Y^2},\, Y,\, -\sqrt{A_2 + D_2Y^2}\,) \\ &\qquad \in \left[\mathcal{X}(\sqrt{A_1 + D_1\Psi_0^2}\,,\, \Psi_0,\, -\sqrt{A_2 + D_2\Psi_0^2}\,),\, x_0\right) \end{aligned}$$

is strictly decreasing.

Proof. The proof is elementary, but quite tedious; once again we turned to the aid of MAPLE V. The function $\mathcal{X}(\sqrt{A_1+D_1Y^2},\, Y,\, -\sqrt{A_2+D_2Y^2}\,)$ is the composition $h_2 \circ h_1$ of the functions

$$\begin{array}{lll} h_1 & : & Y \longrightarrow u = \dfrac{X_0Y - Y_0\sqrt{A_1+D_1Y^2}}{Y_0\sqrt{A_2+D_2Y^2}+Z_0Y} \qquad \text{(cf. (3))} \\ h_2 & : & u \longrightarrow \mathbf{x}(u, \sqrt{au^4+bu^3+cu^2+du+e^2}) \quad \text{(cf. (13) and (10))}\ , \end{array}$$

with a,b,c,d,e as defined in (9). A small explanation on the definition of h_2: The arguments of the function $\mathbf{x}$ in (13), are $u = h_1(Y)$ and the v obtained from (10) after the substitution $(X,Z) \leftarrow (\sqrt{A_1+D_1Y^2},\, -\sqrt{A_2+D_2Y^2})$. By Lemma 2.3, this value of v is positive, for sufficiently large Y, hence, by (7), it is expressed as $\sqrt{au^4+bu^3+cu^2+du+e^2}$ [10]. A computation shows that (primed letters denote derivatives)

$$\lim_{Y\to+\infty} h_1'(Y) = \frac{Y_0(X_0 - Y_0\sqrt{D_1})(Z_0\sqrt{D_1}+X_0\sqrt{D_2})}{\sqrt{D_1D_2}(Z_0+Y_0\sqrt{D_2})}\ .$$

Since $\mathrm{sgn}(X_0 - Y_0\sqrt{D_1}) = \mathrm{sgn}(A_1) = \sigma$, this shows that, for sufficiently large Y, the function h_1 is strictly increasing if $\sigma = +1$ and strictly decreasing if $\sigma = -1$. We note also that

$$\lim_{Y\to+\infty} h_1(Y) = \frac{X_0 - Y_0\sqrt{D_1}}{Z_0+Y_0\sqrt{D_2}} \stackrel{\text{def}}{=} u_0\ .$$

Next, we consider the function $h_2(u)$ for u sufficiently near to u_0. We write $h_2(u) = 2e\sqrt{h_3(u)} + h_4(u)$, where

$$h_3(u) = a + \frac{b}{u} + \frac{c}{u^2} + \frac{d}{u^3} + \frac{e^2}{u^4}\quad ,\quad h_4(u) = \frac{d}{u} + \frac{2e^2}{u^2} + \frac{c}{3}\ .$$

We computed

$$\begin{aligned} h_3'(u_0) &= -\frac{4Y_0}{A_1}(X_0+Y_0\sqrt{D_1D_2}+Z_0\sqrt{D_1})(X_0+Y_0\sqrt{D_1})^2(Z_0+Y_0\sqrt{D_2})^2 \\ h_4'(u_0) &= -\frac{8Y_0^3}{A_1^3}(Z_0\sqrt{D_1}+X_0\sqrt{D_2})(X_0+Y_0\sqrt{D_1})^3(Z_0+Y_0\sqrt{D_2})^3\sqrt{D_1D_2}\ . \end{aligned}$$

Thus, $\mathrm{sgn}(h_3'(u_0)) = \mathrm{sgn}(h_4'(u_0)) = -\sigma$, which shows that, near u_0 $h_2(u)$ is strictly decreasing if $\sigma = +1$ and strictly increasing if $\sigma = -1$. If we combine with the analogous conclusion for $h_1(u)$, we see that, for sufficiently large Y, the function $h_2 \circ h_1(Y)$ is strictly decreasing, as claimed. □

[10] We always use the symbol $\surd$ to denote the *positive* square root.

2.3 Elliptic Integrals and Linear Forms in Elliptic Logarithms

Differentiating (3) and taking into account that $X\mathrm{d}X = D_1\mathrm{d}Y$ and $Z\mathrm{d}X = D_2\mathrm{d}Y$ we get

$$\mathrm{d}u = -\frac{Y_0(-A_2D_1Y_0Y + A_2X_0X + A_1D_2Y_0Y - A_1Z_0Z)}{XZ(Y_0Z - Z_0Y)^2}\mathrm{d}Y\ .$$

Then, taking into account (10) and the relation

$$\frac{\mathrm{d}x}{y} = -\sigma\frac{\mathrm{d}u}{v}$$

(see e.g. relation (2) of [33][11]), we conclude that

$$\frac{\mathrm{d}x}{y} = \frac{1}{2Y_0}\cdot\frac{\mathrm{d}Y}{XZ}\ . \tag{20}$$

From now on we will assume that (X, Y, Z) is a positive integer solution of (1) with Y sufficiently large (cf. Proposition 2.2 and Lemmas 2.3 and 2.4) and we will consider $P = (x(P), y(P)) = (\mathcal{X}(X, Y, -Z), \mathcal{Y}(X, Y, -Z))$, which, by Proposition 2.2, belongs to $\mathcal{E}_0(\mathbb{Q})$ and has $y(P) > 0$. If necessary, we may also write $X(P), Y(P), Z(P)$ instead of $X, Y, -Z$. Conversely, by Proposition 2.1, any point $P \in \mathcal{E}(\mathbb{Q})$[12], corresponds to a rational solution $(X(P), Y(P), Z(P))$ of (1).

Assume now that we know a basis $P_1, \ldots, P_r$ for the torsion-free part of the Mordell-Weil group $\mathcal{E}(Q)$[13]. Observe that, for any P_i, if $P_i \notin \mathcal{E}_0(\mathbb{Q})$, then $P_i + (e_2, 0) \in \mathcal{E}_0(\mathbb{Q})$. Therefore,

> *We can always choose the points of our basis so that $P_1, \ldots, P_r$ belong to $\mathcal{E}_0(\mathbb{Q})$.*

We put

$$P = m_1P_1 + \cdots + m_rP_r + T\ , \tag{21}$$

where T denotes a torsion point which, necessarily, belongs to $\mathcal{E}_0(\mathbb{Q})$ and the m_i's are unknown integers. If we manage to bound

$$M = \max\{|m_1|, \cdots, |m_r|\}\ ,$$

then we can *effectively* solve, at least in principle, our system (1). Indeed, for any $(m_1, \cdots, m_r)$ satisfying this bound and any torsion point $T \in \mathcal{E}_0(\mathbb{Q})$, we

[11] In the notation of that paper, $f(u, v) = au^4 + bu^3 + cu^2 + du + e^2 - v^2$ and a (symbolic) computation shows that $G(u, v) = -\sigma$.

[12] Actually, we deal only with points $P \in \mathcal{E}_0(\mathbb{Q})$.

[13] This is, of course, the non-effective part of the method of this paper.

compute the point $P=(x(P),y(P))$ and then, by the procedure described in Proposition 2.1, we obtain a rational solution $(X(P),Y(P),Z(P))$, which we accept if and only if all three $X(P),Y(P),Z(P)$ are integers.

A basic tool for our method is the usual isomorphism

$$\phi:\mathcal{E}_0(\mathbb{R})\longrightarrow[0,1)=\mathbb{R}/\mathbb{Z}$$

(see e.g. Section 2 of [31]). Let also $\omega=2\int_{e_1}^{+\infty}\frac{\mathrm{d}x}{\sqrt{x^3+Ax+B}}$, the fundamental real period. From (21) we see that

$$\phi(P)=m_1\phi(P_1)+\cdots+m_r\phi(P_r)+m_0+\frac{s}{t}, \tag{22}$$

where $m_0\in\mathbb{Z}$ and $s/t=\phi(T)$ with $0<s<t$ relatively prime integers, effectively bounded by a small number (see e.g. section 4 of [31]). Moreover, since the ϕ-values are positive and <1, it follows that

$$|m_0|\leq rM\ . \tag{23}$$

We exhibit now the close relation of $\phi(P)$ with an elliptic integral $\int\frac{\mathrm{d}x}{y}$, where $y=\sqrt{x^3+Ax+B}$. We will need the point $Q_0=(x_0,y_0)\in\mathcal{E}_0(\sqrt{D_1},\sqrt{D_2})$, with x_0,y_0 as defined in Proposition 2.2. We have

$$\begin{aligned}\int_{x(P)}^{x_0}\frac{\mathrm{d}x}{y} &= \int_{x(P)}^{+\infty}\frac{\mathrm{d}x}{y}-\int_{x_0}^{+\infty}\frac{\mathrm{d}x}{y}\\ &= \omega\phi(P)-\omega\phi(Q_0)\ ,\end{aligned}$$

by the definition of ϕ. This, combined with (22) shows that the integral in the left-hand side of the above relation is equal to a linear form in elliptic logarithms:

$$\int_{x(P)}^{x_0}\frac{\mathrm{d}x}{y}=\mathcal{L}(P)\stackrel{\mathrm{def}}{=}-\omega\phi(Q_0)+(m_0+\frac{s}{t})\omega+m_1\omega\phi(P_1)+\cdots+m_r\omega\phi(P_r).$$

A symbolic computation using MAPLE V shows that

$$2\cdot Q_0=\left(\frac{4}{3}(D_2X_0^2+D_1D_2Y_0^2+D_1Z_0^2)Y_0^2\ ,\ 8D_1D_2X_0Y_0^4Z_0\right)\ .$$

Therefore, $2\cdot Q_0\in\mathcal{E}(\mathbb{Q})$, hence $2\cdot Q_0=k_1P_1+\cdots k_rP_r+T'$, where $k_1,\ldots,k_r$ are explicitly known rational integers and $T'\in\mathcal{E}_0(\mathbb{Q})$ is a torsion point. Then,

$$\phi(Q_0)=\frac{k_1}{2}\phi(P_1)+\cdots+\frac{k_r}{2}\phi(P_r)+\frac{k_0}{2}+\frac{s'}{2t'}\ ,$$

where k_0 is an explicitly known integer and s', t' have analogous properties with those of s, t (see immediately after (22). This transforms the expression of the linear form $\mathcal{L}(P)$ into

$$\int_{x(P)}^{x_0} \frac{\mathrm{d}x}{y} = \mathcal{L}(P) = \frac{p_0}{q_0}\omega + \frac{p_1}{2}\omega\phi(P_1) + \cdots + \frac{p_r}{2}\omega\phi(P_r) \, , \tag{24}$$

where $p_0, p_1, \ldots, p_r$ are rational (unknown) integers and q_0 is an explicitly known "small" integer[14]. Moreover, it is clear that[15],

$$N \stackrel{\text{def}}{=} \max_{0 \le i \le r} |p_i| \le \alpha M + \beta \, , \tag{25}$$

where the "small" positive constants α, β can be explicitly calculated in every specific case.

2.4 Estimating the Linear Form in Elliptic Logarithms

The following steps lead now to an upper bound for $|\mathcal{L}(P)|$ in terms of M.

- By (20) and Lemma 2.4 we conclude[16]

$$\int_{x(P)}^{x_0} \frac{\mathrm{d}x}{y} = -\frac{1}{2Y_0} \int_{Y(P)}^{+\infty} \frac{\mathrm{d}Y}{\sqrt{(A_1 + D_1 Y^2)(A_2 + D_2 Y^2)}} \, , \tag{26}$$

  therefore, for sufficiently large Y [17], (24) and (26) imply that

$$|\mathcal{L}(P)| \le c_1 Y^{-1} \, , \tag{27}$$

  where c_1 is an explicit positive constant.

- $x(P)$ is an explicit rational function $\mathcal{X}(X, Y, Z)$, hence, for the height $\mathrm{h}(x(P))$, we can find an explicit upper bound in terms of X, Y, Z [18]. Then, since $X = \sqrt{A_1 + D_1 Y^2}$, $Z = -\sqrt{A_2 + D_2 Y^2}$, it follows that

$$\mathrm{h}(x(P)) \le c_2 + 2\log Y \, , \tag{28}$$

  where c_2 is an explicit positive constant.

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[14] Actually, $\frac{p_0}{q_0} = m_0 - \frac{k_0}{2} + \frac{s}{t} - \frac{s'}{2t'}$ and $p_i = m_i - \frac{k_i}{2}$ for $i = 1, \ldots, r$.
[15] We use also (23) at this point.
[16] We always have in mind that $X = \sqrt{A_1 + D_1 Y^2}$ and $Z = -\sqrt{A_2 + D_2 Y^2}$.
[17] This is easily made explicit.
[18] The fact that X, Y, Z are integers is a crucial "detail" here.

- We have an inequality of the form

$$\hat{\mathrm{h}}(P) - \frac{1}{2}\mathrm{h}(x(P)) \leq c_3 \; , \tag{29}$$

where $\hat{\mathrm{h}}(\cdot)$ denotes the canonical height, as defined, for example, in Silverman's book [29], and c_3 is an explicit positive constant; see e.g. [30].

- We refer to (21). It is a well-know fact that $\hat{\mathrm{h}}(P)$ is a *positive-definite quadratic form* $q(m_1, \ldots, m_r)$, hence.

$$\hat{\mathrm{h}}(P) \geq \lambda M^2 \; , \tag{30}$$

where λ is the least eigenvalue of the matrix of the quadratic form q, hence an explicit positive constant.

- An obvious combination of (27),(28),(29) and (30) gives

$$|\mathcal{L}(P)| \leq c_1 \exp(0.5c_2 + c_3 - \lambda M^2) \; . \tag{31}$$

Next, an explicit lower bound for $|\mathcal{L}(P)|$ in terms of N is obtained by a deep theorem due to S. David [11]. A version of this theorem adapted to our needs is found in the Appendix of [36]. According to Theorem 5 of that Appendix, either N is "very small", or

$$|\mathcal{L}(P)| \geq \exp(-c_4(\log N + c_5)(\log\log N + c_6)^{r+2}) \; ,$$

where c_4, c_5, c_6 are explicit positive constants. Taking into account (25), we write the last relation

$$|\mathcal{L}(P)| \geq \exp(-c_4(\log(\alpha M + \beta) + c_5)(\log\log(\alpha M + \beta) + c_6)^{r+2}) \; . \tag{32}$$

Since the lower bound in (32) is larger than the upper bound in (31) if M is sufficiently large, it follows that M must be bounded by some explicit upper bound. The upper bound for M, which is obtained in this way, is huge. The rather large experimental experience accumulated so far shows that this bound is of the size of $10^{(5r^2+15r+28)/2}$; see Section 3 of [32]. This bound is dramatically reduced with the use of the LLL Basis Reduction Algorithm. The reduction process has been extensively discussed in many relevant already published papers, that there is no point to discuss it here once again; we refer the reader to Section 5 of [31], Section 5 of [36]and Section 6 of [17][19].

[19]These are just three indicative references.

3 Examples

In the examples of this section, all computations related to elliptic curves have been done by APECS of Ian Connell, a package based on MAPLE V. In all cases, the Mordell-Weil bases were computed unconditionally (no conjectures were assumed) and care was taken to transform the points of the basis to ones belonging to the zero component of the curve, by adding the point $(e_2, 0)$, if necessary (cf. just before (21)).

Also, the results of Subsection 2.2 require that Y be "sufficiently large". In every example, computations, very simple in practice, show that $Y \geq 10$ is more than enough.

3.1 The system with $D_1 = 5,\ A_1 = 19,\ D_2 = 3,\ A_2 = -11.$

A rational solution: $(X_0, Y_0, Z_0) = (8, 3, 4)$. Corresponding Weierstrass model is

$$y^2 = x^3 - 4064688x + 97538688\ . \tag{33}$$

Roots of the right-hand side,

$$e_3 = -2028,\ e_2 = 24,\ e_1 = 2004\ .$$

Relations (3) and (10) now become

$$u = \frac{3X - 8Y}{3Z - 4Y}\ ,\ v = -72\frac{22X - 84Y + 19Z}{(3Z - 4Y)^2}\ ,$$

hence, by (13) and (17),

$$\begin{aligned} x = \frac{-12}{(3X - 8Y)^2}&(7477Y^2 - 19152Y - 4008XY - 3420YZ \\ &+1824ZX + 5016X + 4332Z + 11571) \end{aligned}$$

from which we find

$$x_0 = 4884 + 2160\sqrt{3} + 2592\sqrt{5} + 1152\sqrt{5}\sqrt{3} \approx 18882.7947568591741\ ,$$

and by (14) and (17),

$$\begin{aligned} y = \frac{16416}{(3X - 8Y)^3}&(855Y^3 + 7477Y^2 - 456XY^2 - 2140ZY^2 - 4008XY - 9519Y \\ &+1002XZY + 456ZY + 190XZ + 3344X - 1900Z + 17347). \end{aligned}$$

Also, relations (11) and (12) give

$$X = -8\frac{11u^2 + 19u - 19}{v}\ ,\ Y = 3\frac{11u^2 + 19}{v}\ ,\ Z = \frac{44u^2 - 176u - 76}{v}\ ,$$

from which, by means of (15),(16) and (18), we obtain

$$X = 8\frac{x^3 + 828x^2 + 90yx + 15841008x - 20126660544 + 182520y}{x^3 - 14652x^2 + 4064688x + 19656858816 - 622080y}$$

$$Y = 3\frac{x^3 + 4932x^2 + 128yx - 12427344x + 20041387200 - 3072y}{x^3 - 14652x^2 + 4064688x + 19656858816 - 622080y}$$

$$Z = -4\frac{x^3 + 8892x^2 + 216yx - 23737104x - 19181430720 - 432864y}{x^3 - 14652x^2 + 4064688x + 19656858816 - 622080y} .$$

A basis for the torsion-free part of $\mathcal{E}(\mathbb{Q})$, where $\mathcal{E}$ is the elliptic curve defined by (33), is

$$P_1 = (7548, 632016),\ P_2 = (112884, 37920960)$$

($P_1, P_2 \in \mathcal{E}_0(\mathbb{Q})$) and $T = (2004, 0)$ is the only torsion point on $\mathcal{E}_0(\mathbb{Q})$. With the aid of the function `Expr` of Apecs we find that

$$2 \cdot Q_0 = (4884, 311040) = P_1 + P_2 .$$

We easily check that $\phi(2 \cdot Q_0) = \phi(P_1) + \phi(P_2)$ and $\phi(T) = 1/2$. Therefore, if P is a point on the elliptic curve, which corresponds to an integer solution of our system of Pell equations and we put

$$P = m_1P_1 + m_2P_2 + T\ ,\quad M = \max\{|m_1|, |m_2|\}\ ,$$

then the corresponding linear form in elliptic logarithms is

$$\mathcal{L}(P) = \frac{p_0}{2}\omega + \frac{p_1}{2}\omega\phi(P_1) + \frac{p_2}{2}\omega\phi(P_2)\ ,$$

where $p_i = 2m_i - 1$, $(i = 1, 2)$ and $p_0 = 2m_0 + s$ for some integers m_0 with $|m_0| \leq 2M$ and $s \in \{0, 1\}$ (cf. (23)). Therefore, in the notation of (25), $\alpha = 4, \beta = 1$. Also, $A_1 + D_1Y^2 = 19 + 5Y^2 > 5Y^2$ and $A_2 + D_2Y^2 = -11 + 3Y^2 > 2.9725Y^2$, provided that $Y > 20$. Therefore, in (27) we can take $c_1 = 0.04324$. In the expression of x as a rational function of X, Y, Z we make the substitution $X = \sqrt{19 + 5Y^2}, Z = -\sqrt{-11 + 3Y^2}$; then, it is straightforward to check that the maximum of the absolute values of the numerator and the denominator is $\leq 376400Y^2$, provided that $Y > 20$; therefore in (28) we can take $c_2 = \log 376400$. In (29) we use Silverman's result [30], which gives $c_3 = 5.84226214$. An easy program based on Maple V and Apecs calculates the least eigenvalue of the positive-definite matrix related to $\hat{\mathrm{h}}$, giving thus $\lambda = 0.5179388$. Following the detailed steps described in the Appendix of [36] we calculate

$$c_4 = 4.046622 \times 10^{74}\ ,\ c_5 = 2.570121\ ,\ c_6 = 30.018387$$

and the comparison of the upper bound (31) with the lower bound (32) forces $M \le 3.41 \times 10^{41}$. Following the reduction technique that uses the LLL basis reduction algorithm (as implemented in Pari), we can immediately reduce this huge upper bound down to 20 and then, by a second reduction step we reduce the bound to 7. Finally, by a simple program that uses Maple and Apecs we compute all combinations

$$(x,y) = m_1 P_1 + m_2 P_2 + \varepsilon T\ , \quad -7 \le m_i \le 7\,(i=1,2)\,,\ \varepsilon = 0,1$$

and for each one we insert the values of x and y in the previously obtained formulas, which express X, Y, Z as rational functions of x, y. The only positive integer triads (X, Y, Z) obtained in this way are $(8, 3, 4)$ and $(12, 5, 8)$. Since the computation of certain c_i's has been done under the assumption $Y > 20$, we also check which integer values in the range $1 \le Y \le 20$ are such that $\sqrt{19 + 5Y^2}$ and $\sqrt{-11 + 3Y^2}$ are both integers. No values of Y besides 3 and 5 satisfy this condition, from which we conclude that *the only positive integer solutions* (X, Y, Z) *to the system (1) in this specific case are those two mentioned above.*

3.2 The system with $D_1 = 7,\ A_1 = 2,\ D_2 = 32,\ A_2 = -23$.

This example is partially solved in [10], where it is proved that, if (X, Y, Z) is a solution, then $X + Y\sqrt{7} < (3+\sqrt{7})(8+3\sqrt{7})^{17^{192}}$. Here we give the complete solution. A rational solution: $(X_0, Y_0, Z_0) = (3, 1, 3)$. Corresponding Weierstrass model is

$$y^2 = x^3 - \frac{645136}{3}x - \frac{692527232}{27} \tag{34}$$

with right-hand side having roots

$$e_3 = -\frac{1156}{3},\ e_2 = -\frac{388}{3},\ e_1 = \frac{1544}{3}\ .$$

Relations (3), (10) and (17) combined with (13) and (14), respectively, give

$$\begin{aligned} x &= \frac{8(-235 - 207X + 675Y - 18Z + 579XY + 63YZ - 27ZX - 1448Y^2)}{3(X-3Y)^2} \\ y &= \frac{-96}{(X-3Y)^3}(-224Y^3 + 96XY^2 - 1448Y^2 + 504ZY^2 + 579XY - 54ZY \\ &\qquad +386Y - 193XZY - 138X + 63Z - 253 + 14XZ)\,, \end{aligned}$$

from the first of which we find

$$x_0 = \frac{2300}{3} + 384\sqrt{7} + 336\sqrt{2} + 144\sqrt{7}\sqrt{2} \approx 2796.609591\ .$$

Relations (11) and (12) combined with (15),(16) and (18), respectively, give

$$X = 3\frac{27x^3 - 108x^2 + 756yx + 6552144x + 291312y - 3080761664}{27x^3 - 62100x^2 + 5806224x - 870912y + 5836492864}$$

$$Y = \frac{27x^3 + 13716x^2 - 11161008x + 972yx + 125712y + 2373881536}{27x^3 - 62100x^2 + 5806224x - 870912y + 5836492864}$$

$$Z = -3\frac{27x^3 + 48492x^2 + 3456yx - 24422256x - 7899721664 - 1778688y}{27x^3 - 62100x^2 + 5806224x - 870912y + 5836492864}\,.$$

A basis for the torsion-free part of $\mathcal{E}(\mathbb{Q})$, where $\mathcal{E}$ is the elliptic curve defined by (34) consists of the single point $P_1 = (8444/3, 147200)$ and $T = (1544/3, 0)$ is the only torsion point on $\mathcal{E}_0(\mathbb{Q})$. With the aid of the function `Expr` of APECS we find that $2 \cdot Q_0 = (2300/3, 16128) = -P_1 + T$ and we easily check that $\phi(Q_0) = \frac{1}{4} - \frac{1}{2}\phi(P_1)$ and $\phi(T) = 1/2$. If P is a point on the elliptic curve, which corresponds to an integer solution of our system of Pell equations and we put

$$P = m_1 P_1\ , \quad M = |m_1|\ ,$$

then the corresponding linear form in elliptic logarithms is

$$\mathcal{L}(P) = \frac{p_0}{2}\omega + \frac{p_1}{2}\omega\phi(P_1)\ ,$$

where $p_1 = 2m_1 + 1$ and $p_0 = 4m_0 + 2s - 1$ for some integers m_0 with $|m_0| \leq M$ and $s \in \{0, 1\}$ (cf. (23)). Therefore, in the notation of (25), $\alpha = 4, \beta = 1$.

Calculations completely analogous to those in Example 3.1 give $c_1 = 0.03344$, $c_2 = 9.67$, $c_3 = 5.1074513$, $\lambda = 0.69249$, $c_4 = 6.251 \times 10^{44}$, $c_5 = 2.752$, $c_6 = 34.56585$ and the comparison of the upper bound (31) and the lower bound (32) forces $M \leq 5.8 \times 10^{25}$. The reduction process gives the bound $M \leq 20$. We compute all combinations

$$(x, y) = m_1 P_1 + \varepsilon T\ , \quad -20 \leq m_1 \leq 20\,,\ \varepsilon = 0, 1$$

and for each one we insert the values of x and y in the above expressions of X, Y, Z as rational functions of x, y. The only positive integer triads (X, Y, Z) obtained in this way are $(3, 1, 3)$ and $(717, 271, 1533)$. Since the computation of certain c_i's have been under the assumption $Y > 20$, we also check whether there are solutions (X, Y, Z) with $1 < Y \leq 20$, but this reveals no further solution. Therefore, *the only positive integer solutions (X, Y, Z) to the system (1) in this specific case are those two mentioned above.*

3.3 Three examples taken from the literature

In this section we discuss, in brief, the application of our method to the solution of three examples taken from the literature.

The first is the well known example of A. Baker and H. Davenport [3]. Solutions are also given in [18] and [28]. In the notation of the present paper, $D_1 = 3$, $A_1 = -2$, $D_2 = 8$, $A_2 = -7$ and $(X_0, Y_0, Z_0) = (1, 1, 1)$. The parameters in the Weierstrass model (19) are $A = -5776/3$, $B = 677248/27$. The rank of the curve is 1, $P_1 = (524/3, 2240)$ being a basic point. Then, the resolution of the system is completely analogous to that of Section 3.2 and proves that the only solutions in positive integers (X, Y, Z) are $(1, 1, 1)$ and $(19, 11, 31)$.

A second example comes from the system $z^2 - 3y^2 = -2$, $z^2 - 6x^2 = -5$, studied in [37]. The substitution $(3y, z, 6x) = (X, Y, Z)$ gives a system (1) with $D_1 = 3$, $A_1 = 6$, $D_2 = 6$, $A_2 = 30$ and $(X_0, Y_0, Z_0) = (3, 1, 6)$. The parameters in the related Weierstrass elliptic curve (19) are $A = -32832$, $B = 774144$. The rank of the curve is 1, $P_1 = (888, 25920)$ being a basic point. As in the previous example, the resolution of the system is completely analogous to that of Section 3.2 and proves that the only solutions in positive integers (X, Y, Z) are $(3, 1, 6)$ and $(123, 71, 174)$, hence the only positive integer solutions (x, y, z) are $(1, 1, 1)$ and $(29, 41, 71)$.

The third example is found in [25] and corresponds to $D_1 = 5$, $A_1 = -20$, $D_2 = 2$, $A_2 = 1$ and $(X_0, Y_0, Z_0) = (0, 2, 3)$. The parameters in the related elliptic curve (19) are $A = -467200/3$, $B = -609280000/27$ and the rank of the curve is zero. The only rational point on the zero component of the curve is the torsion point $T = (1360/3, 0)$ and the correspondence $(x, y) \to (X, Y, Z)$ gives no further solution.

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