What is a closed-form number?

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1. Introduction

When I was a high-school student, I liked giving *exact* answers to numerical problems whenever possible. If the answer to a problem were $2/7$ or $\pi\sqrt{5}$ or $\arctan 3$ or $e^{1/e}$, I would always leave it in that form instead of giving a decimal approximation.

Certain problems frustrated me because there did not seem to be any way to express their solutions exactly. For example, consider the following problems.

Question 1. The equation

$$x + e^x = 0 \tag{1.1}$$

has exactly one real root; call it R. Is there a closed-form expression for R?

Question 2. The equation

$$2x^5 - 10x + 5 = 0 \tag{1.2}$$

has five distinct roots r_1, r_2, r_3, r_4, and r_5. Are there closed-form expressions for them?

Questions like this seemed to have a negative answer, but I continued hoping that the answer was yes, and that I just did not know enough mathematics yet.

In college I learned about Galois theory, and that the Galois group of equation (1.2) is S_5 [7, Section 5.8]. So the r_i are provably not expressible in terms of radicals. But although this probably should have satisfied me, it did not. Consider the equation

$$x^4 - (6\sqrt{3})x^3 + 8x^2 + (2\sqrt{3})x - 1 = 0.$$

Its roots are $\tan(\pi/15)$, $\tan(4\pi/15)$, $\tan(7\pi/15)$, and $\tan(13\pi/15)$. These seemed to me to be perfectly good closed-form expressions. Although in this particular case the roots could also be expressed in terms of radicals, it seemed to me that there might exist algebraic numbers that were not expressible using radicals but that could still be expressed in closed form—say, using trigonometric or exponential or logarithmic functions. So as far as I was concerned, Galois theory was not the end of the story.

When students ask for a closed-form expression for $\int \exp(x^2)\,dx$, we all know the standard answer: the given function is not an elementary function. Curiously, though, Question 1 (as well as Question 2, if you accept my dissatisfaction with the Galois-theoretic answer) does not seem to have a standard answer that "everybody knows." At most we might mutter vaguely that equation (1.1) is a "transcendental equation," but this is not very helpful.

This nonexistence of a standard answer to such a simple and common question seems almost scandalous to me. The main purpose of this paper is to eliminate this scandal by suggesting a precise definition of a "closed-form expression for a number." This will enable us to restate Questions 1 and 2 precisely, and will let us see how they are related to existing work in logic, computer algebra, and transcendental number theory. My hope is that this definition of a closed-form expression for a number will become standard, and that many readers will be lured into working on the many attractive open problems in this area.

2. From elementary functions to EL numbers

How can we make Questions 1 and 2 precise? Our first inclination might be to turn to the notion of an *elementary function*. Recall that a function is *elementary* if it can be constructed using only a finite combination of constant functions, field operations, and algebraic, exponential, and logarithmic functions. This class of functions has been studied a great deal in connection with the problem of symbolic integration or "integration in finite terms" [4], and it does a rather good job of capturing "high-school intuitions" about what a closed-form expression should look like. For example, in Question 1 above, it turns out that $R = -W(1)$, where W, the *Lambert W function* [6], is the (multivalued) function defined by the equation

$$W(x)e^{W(x)} = x.$$

But since W is not an elementary function [5], this is not an answer that would satisfy most high-school students. Similarly, if we allow various special functions—e.g., elliptic, hypergeometric, or theta functions—then we can explicitly express the r_i in Question 2, or indeed the roots of any polynomial equation, in terms of the coefficients [3, 10]. But this again feels unsatisfactory because these special functions are not elementary.

The concept of an elementary function is certainly on the right track, but observe that what we need for Questions 1 and 2 is a notion of a closed-form *number* rather than a closed-form *function*. The distinction is important; we cannot, for example, simply define an "elementary number" to be any number obtainable by evaluating an elementary function at a point, because all constant functions are elementary, and this definition would make all numbers elementary. Furthermore, even if a function (like W) is not elementary, it is conceivable that each particular value that it takes $(W(1), W(2), \ldots)$ could have an elementary expression, but with different-looking expressions at different points. So we will not try to define closed-form numbers *in terms of* elementary functions, but will give an *analogous* definition.

We mention one more technical point. By convention, all algebraic functions are considered to be elementary, but this is not suitable for our purposes. Intuitively, "closed-form" implies "explicit," and most algebraic functions have no simple explicit expression.

So the set of *purely transcendental elementary functions* is a better prototype for our purposes than the set of elementary functions. ("Purely transcendental" simply means that the word "algebraic" is dropped from the definition.)

With all these considerations in mind, we propose the following fundamental definition.

Definition. A subfield F of $\mathbb{C}$ is *closed under* exp *and* log if (1) $\exp(x) \in F$ for all $x \in F$ and (2) $\log(x) \in F$ for all nonzero $x \in F$, where log is the branch of the natural logarithm function such that $-\pi < \operatorname{Im}(\log x) \leq \pi$ for all x. The field $\mathbb{E}$ of *EL numbers* is the intersection of all subfields of $\mathbb{C}$ that are closed under exp and log.

Before discussing $\mathbb{E}$, let us make some remarks about terminology. It might seem more natural to call $\mathbb{E}$ the field of *elementary numbers,* but unfortunately this term is already taken. It seems to have been first used by Ritt [18, p. 60]. By analogy with elementary functions, Ritt thought of elementary numbers as the smallest *algebraically closed* subfield $\mathbb{L}$ of $\mathbb{C}$ that is closed under exp and log. It so happens that terminology has evolved since Ritt, so that $\mathbb{L}$ is now known as the field of *Liouvillian numbers,* and "elementary numbers" are now numbers that can be specified *implicitly* as well as explicitly by exponential, logarithmic, and algebraic operations [16]. But either way, calling $\mathbb{E}$ the field of elementary numbers would conflict with existing usage. The "EL" in the term "EL number" is intended to be an abbreviation for "Exponential-Logarithmic" as well as a diminutive of "ELementary," reminding us that $\mathbb{E}$ is a subfield of the elementary numbers.

I should also remark that I am certainly not the first person ever to have considered the field $\mathbb{E}$, but it has received surprisingly little attention in the literature and nobody seems to have lobbied for it as a fundamental object of interest, which in my opinion it is (as illustrated by my temerity in using "blackboard bold" for it).

Let us do a few warmup exercises to familiarize ourselves with $\mathbb{E}$. We can construct $\mathbb{E}$ as follows. Set $\mathbb{E}_0 = \{0\}$, and for each $n > 0$ let $\mathbb{E}_n$ be the set of all complex numbers obtained either by applying a field operation to any pair of (not necessarily distinct) elements of $\mathbb{E}_{n-1}$ or by applying exp or log to any element of $\mathbb{E}_{n-1}$. (Of course, division by zero and taking the logarithm of zero are forbidden.) Then it is clear that $\mathbb{E}$ is the union of all the $\mathbb{E}_n$. This shows in particular that $\mathbb{E}$ is countable, and that every element of $\mathbb{E}$ admits an explicit finite expression in terms of rational numbers, field operations, exp, and log.

Most familiar constants lie in $\mathbb{E}$, e.g.,

$$\begin{aligned} e &= \exp(\exp(0)) \\ i &= \exp\left(\frac{\log(-1)}{2}\right) \\ \pi &= -i\log(-1) \end{aligned}$$

Since $2\pi i \in \mathbb{E}$, we actually have access to all branches of the logarithm and not just the principal one, so all n of the nth roots of any $x \in \mathbb{E}$ are also in $\mathbb{E}$. It follows that all the roots of any polynomial equation with rational coefficients that is solvable in radicals lie in $\mathbb{E}$. Finally, formulas like

$$\begin{aligned} x^{2/3} &= \exp\left(\frac{2\log x}{3}\right) \\ \sin x &= \frac{\exp(ix) - \exp(-ix)}{2i} \\ \tanh x &= \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} \\ \arccos x &= -i\log\left(x + \exp\left(\frac{\log(x^2-1)}{2}\right)\right) \end{aligned}$$

show that any expression involving "high-school" functions and elements of $\mathbb{E}$ is also in $\mathbb{E}$.

We hope that this brief discussion has persuaded the reader that $\mathbb{E}$ is the "right" precise definition of "the set of all complex numbers that can be written in closed form." Accepting this, we can reformulate Questions 1 and 2 as follows.

Conjecture 1. *The real root R of $x + e^x = 0$ is not in $\mathbb{E}$.*

Conjecture 2. *The roots r_1, r_2, r_3, r_4, and r_5 of $2x^5 - 10x + 5 = 0$ are not in $\mathbb{E}$.*

As far as I know, Conjecture 1 and Conjecture 2 are—perhaps surprisingly—still open. Thus we are still frustrated, but at least our frustration has been raised to a higher plane. The next section of this paper is devoted to partial results.

3. Schanuel's conjecture

Conjecture 1 is essentially due to Ritt, except that he asked the question with $\mathbb{L}$ instead of $\mathbb{E}$, since he was motivated by different considerations from ours. The best partial result I am aware of is due to Ferng-Ching Lin [12]. To state Lin's theorem, we must first recall *Schanuel's conjecture.*

Schanuel's Conjecture. *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are complex numbers linearly independent over $\mathbb{Q}$, then the transcendence degree of the field $\mathbb{Q}(\alpha_1, e^{\alpha_1}, \alpha_2, e^{\alpha_2}, \ldots, \alpha_n, e^{\alpha_n})$ over $\mathbb{Q}$ is at least n.*

To orient the reader who has not seen Schanuel's conjecture before, we mention that it implies many famous theorems and conjectures about transcendental numbers. For example, Schanuel's conjecture implies the Lindemann-Weierstrass theorem [2, Theorem 1.4] that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$. Schanuel's conjecture also implies the Gelfond-Schneider theorem that if α_1 and α_2 are algebraic numbers for which there exist $\mathbb{Q}$-linearly independent numbers β_1 and β_2 such that $\alpha_1 = e^{\beta_1}$ and $\alpha_2 = e^{\beta_2}$, then β_1

and β_2 are linearly independent over the algebraic numbers. Baker's generalization [2, Theorem 2.1] of Gelfond-Schneider to an arbitrarily large finite number of α_i also follows from Schanuel's conjecture. It is an easy exercise (using $e^{\pi i} = -1$) to show that Schanuel's conjecture implies that e and π are algebraically independent, which is currently not known. (It is not even known that $e + \pi$ is transcendental.) A proof of Schanuel's conjecture would be big news, although at present it seems to be out of reach.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. Then Lin's result is the following.

Theorem 1. *If Schanuel's conjecture is true and $f(x, y) \in \overline{\mathbb{Q}}[x, y]$ is an irreducible polynomial involving both x and y and $f\big(\alpha, \exp(\alpha)\big) = 0$ for some nonzero $\alpha \in \mathbb{C}$, then $\alpha \notin \mathbb{L}$.*

By taking $f(x) = x + y$ and noting that $\mathbb{E} \subseteq \mathbb{L}$, we see at once that Schanuel's conjecture implies Conjecture 1.

Conjecture 2 seems to be new. The literature does contain some negative results about the insolubility of *general* polynomial equations in terms of the exponential and logarithmic functions, although it is difficult to find a satisfactory reference (the best I know are [9, paragraph 513] and [1, p. 114]). The inexpressibility of an algebraic function in terms of exp and log does not, however, imply that *particular values* of an algebraic function cannot be expressed in terms of exp and log, just as some quintic equations with rational coefficients are solvable in radicals even though the general quintic is not.

The remainder of this section is devoted to proving the following result.

Theorem 2. *Schanuel's conjecture implies Conjecture 1 and Conjecture 2.*

As we just remarked, Lin has already shown that Schanuel's conjecture implies Conjecture 1, but we shall exploit the fact that Conjecture 1 is weaker than the conclusion of Theorem 1 to give a shorter proof. The proof of Theorem 2 below is joint work with Daniel Richardson. Inspection of the proof shows that it is readily generalized to other transcendental equations that are similar to (1.1) and to any algebraic number whose minimum polynomial has a Galois group that is not solvable. In particular, if Schanuel's conjecture is true, then our notion of a "closed-form algebraic number" coincides with the usual one, i.e., solvability in radicals.

Although Conjecture 1 and Conjecture 2 involve quite different kinds of equations, it turns out that there is a single concept (that of a *reduced tower,* defined below) that is the key to both. The reader who is not interested in the details of the proof may skip directly to the next section now without loss of continuity.

We need some preliminaries. If $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a finite sequence of complex numbers then for brevity we write A_i for the field $\mathbb{Q}(\alpha_1, e^{\alpha_1}, \alpha_2, e^{\alpha_2}, \ldots, \alpha_i, e^{\alpha_i})$. In particular, $A_0 = \mathbb{Q}$.

Definition. A *tower* is a finite sequence $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonzero complex numbers such that for all $i \in \{1, 2, \ldots, n\}$, there exists some integer $m_i > 0$ such that $\alpha_i^{m_i} \in A_{i-1}$ or

$e^{\alpha_i m_i} \in A_{i-1}$ (or both). A tower is *reduced* if the set $\{\alpha_i\}$ is linearly independent over $\mathbb{Q}$. If $\beta \in \mathbb{C}$, then a *tower for* β is a tower $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\beta \in A_n$.

For any $\gamma \in \mathbb{E}$, there exists a tower for γ. This is best explained by example. Suppose $\gamma = 4 + \log\bigl(1 + e^{(\log 2)/3}\bigr)$. Then we may take

$$A = (\alpha_1, \alpha_2, \alpha_3) = \bigl(\log 2, (\log 2)/3, \log(1 + e^{(\log 2)/3})\bigr).$$

We can then take $m_i = 1$ for all i, because $e^{\alpha_1} = 2 \in A_0$, $\alpha_2 \in A_1$, and $e^{\alpha_3} \in A_2$. In general, we build up the expression for γ step by step, and if at step i we need to take the exponential of some number $\beta \in A_{i-1}$ we simply set $\alpha_i = \beta$; if we need to take the logarithm of some $\beta \in A_{i-1}$ then we set $\alpha_i = \log \beta$. With this construction, we never need to take $m_i > 1$, but the tower we obtain may not be reduced (as is the case in this example: $\alpha_1 - 3\alpha_2 = 0$). In order to be able to use Schanuel's conjecture, however, we need reduced towers, so our first goal is to show how to reduce a given tower.

Division Lemma. *Suppose $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a tower and $q_1, q_2, \ldots, q_n$ are nonzero integers. Then the sequence $B = (\beta_1, \beta_2, \ldots, \beta_n)$ defined by $\beta_i = \alpha_i/q_i$ is also a tower, and $A_i \subseteq B_i$ for all i.*

Proof. Given any $i \in \{1, 2, \ldots, n\}$, note that every $\gamma \in A_i$ is a rational function (with rational coefficients) of the numbers $\alpha_1, e^{\alpha_1}, \ldots, \alpha_i, e^{\alpha_i}$. Now

$$\alpha_j = (\alpha_j/q_j)q_j = \beta_j q_j \qquad \text{and} \qquad e^{\alpha_j} = e^{(\alpha_j/q_j)q_j} = (e^{\beta_j})^{q_j} \qquad \text{for all } j,$$

so γ is also a rational function with rational coefficients of the numbers $\beta_1, e^{\beta_1}, \ldots, \beta_i, e^{\beta_i}$, and hence $\gamma \in B_i$. So $A_i \subseteq B_i$ for all i.

Given any $i \in \{1, 2, \ldots, n\}$, there is some integer $m_i > 0$ such that $\alpha_i^{m_i} \in A_{i-1}$ or $e^{\alpha_i m_i} \in A_{i-1}$. Consider first the case in which $\alpha_i^{m_i} \in A_{i-1}$. Then

$$\beta_i^{m_i} = \left(\frac{\alpha_i}{q_i}\right)^{m_i} \in A_{i-1} \subseteq B_{i-1}.$$

If on the other hand $e^{\alpha_i m_i} \in A_{i-1}$, then

$$e^{\beta_i(q_i m_i)} = e^{\alpha_i m_i} \in A_{i-1} \subseteq B_{i-1}.$$

Hence there is a positive integer m_i' (for example, $m_i' = q_i m_i$) such that $\beta_i^{m_i'} \in B_{i-1}$ or $e^{\beta_i m_i'} \in B_{i-1}$. So B is a tower. $\square$

Reduction Lemma. *For any $\gamma \in \mathbb{E}$, there exists a reduced tower for γ.*

Proof. If $\gamma \in \mathbb{Q}$ then we may take A to be the empty sequence. Otherwise, suppose that every tower for γ is not reduced; we shall derive a contradiction. Choose such an A with n minimal; since $\gamma \notin \mathbb{Q}$, $n \geq 1$. Let i be the smallest integer such that $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ is linearly dependent. Then

$$\alpha_i = \sum_{j=1}^{i-1} \frac{p_j \alpha_j}{q_j} \tag{3.1}$$

for some integers $p_1, q_1, p_2, q_2, \ldots, p_{i-1}, q_{i-1}$. We claim that the sequence

$$A' = \left(\frac{\alpha_1}{q_1}, \frac{\alpha_2}{q_2}, \ldots, \frac{\alpha_{i-1}}{q_{i-1}}, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n\right)$$

is a tower for γ. Since A' is shorter than A, this contradicts the minimality of n and proves the theorem.

To prove the claim, note first that by the division lemma, the sequence

$$\left(\frac{\alpha_1}{q_1}, \frac{\alpha_2}{q_2}, \ldots, \frac{\alpha_{i-1}}{q_{i-1}}\right)$$

is a tower. Next, note that equation (3.1) implies that $\alpha_i \in A'_{i-1}$ and also, by exponentiating, that e^{α_i} is a polynomial (in fact a monomial) in the numbers $e^{\alpha_1/q_1}, \ldots, e^{\alpha_{i-1}/q_{i-1}}$, so that $e^{\alpha_i} \in A'_{i-1}$. By the division lemma, $A_{i-1} \subseteq A'_{i-1}$, so

$$A'_{i-1} \supseteq A_{i-1}(\alpha_i, e^{\alpha_i}) = A_i.$$

This ensures that the tower condition for A' is satisfied at the boundary between α_{i-1}/q_{i-1} and α_{i+1}, and also that $A'_{n-1} \supseteq A_n \ni \gamma$, proving the claim. $\square$

Proof of Theorem 2. We first make a general remark. If $B = (\beta_1, \beta_2, \ldots, \beta_n)$ is a reduced tower, then Schanuel's conjecture implies that for all i, *exactly one* of β_i and e^{β_i} is algebraic over B_{i-1}. For by the definition of a tower, *at least one* of the two is algebraic over B_{i-1}; this implies that the transcendence degree of B_i over $\mathbb{Q}$ is at most i for all i. Then because B is reduced, Schanuel's conjecture may be applied, telling us that α_i and e^{α_i} cannot *both* be algebraic over B_{i-1}, and that the transcendence degree of B_i over $\mathbb{Q}$ is exactly i.

Now assume Schanuel's conjecture. We first prove Conjecture 1. Assume $R \in \mathbb{E}$; we derive a contradiction. By the reduction lemma, there is a reduced tower $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for R. Since e is transcendental, $R \notin \mathbb{Q}$ (for $R = p/q$ implies $e^p = (-p/q)^q$), so $n \geq 1$. By truncating the tower if necessary, we may assume that $R \notin A_i$ if $i < n$.

Let $A' = (\alpha_1, \alpha_2, \ldots, \alpha_n, R)$. Then $R \in A'_n$, and the relation $R + e^R = 0$ shows that $e^R \in A'_n$ as well. By our "general remark," A' cannot be reduced. But A is reduced, so

$$R = \sum_{i=1}^{n} \frac{p_i \alpha_i}{q_i}$$

for some integers $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$. Moreover, $p_n \neq 0$ because $R \notin A_i$ for $i < n$. The relation $R + e^R = 0$ becomes

$$\sum_{i=1}^{n} \frac{p_i \alpha_i}{q_i} + \prod_{i=1}^{n} \left(e^{\alpha_i/q_i}\right)^{p_i} = 0. \tag{3.2}$$

Let $A'' = (\alpha_1/q_1, \alpha_2/q_2, \ldots, \alpha_n/q_n)$. By the division lemma, A'' is a tower, and since A is reduced, A'' is reduced. But since $p_n \neq 0$, equation (3.2) shows that if α_n/q_n is algebraic over A''_{n-1} then so is e^{α_n/q_n}, and vice versa. By our "general remark," A'' cannot be reduced, and this gives our desired contradiction.

Now for Conjecture 2. We shall assume that the reader is familiar with the rudiments of Galois theory. Assume that $r_1 \in \mathbb{E}$; we derive a contradiction. By the reduction lemma there is a reduced tower $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for r_1 (of course, unrelated to the "A" in the first part of this proof). For all i, let

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i \text{ is transcendental over } A_{i-1}; \\ e^{\alpha_i}, & \text{if } e^{\alpha_i} \text{ is transcendental over } A_{i-1}. \end{cases}$$

Then the β_i are algebraically independent and form a transcendence basis for A_n over $\mathbb{Q}$. Let $F = \mathbb{Q}(\beta_1, \beta_2, \ldots, \beta_n)$. Clearly A_n is an extension by radicals of F. Let L be the Galois closure of A_n over F; then $\mathrm{Gal}(L/F)$ is solvable. If F' is the splitting field over $\mathbb{Q}$ of the polynomial (1.2), then $\mathrm{Gal}(F'/\mathbb{Q}) = S_5$ and $F' \cap F = \mathbb{Q}$ because $F'/\mathbb{Q}$ is an algebraic extension while $F/\mathbb{Q}$ is a purely transcendental extension. Therefore the compositum FF' is Galois over F with Galois group S_5 [11, Chapter 8, Theorem 1.12]. But $FF' \subseteq L$, so S_5 must be a homomorphic image of $\mathrm{Gal}(L/F)$. This is our desired contradiction, because every homomorphic image of a solvable group is solvable. $\square$

4. Related work and open problems

Much of the work that has been done on fields like $\mathbb{E}$, $\mathbb{L}$, or the field of elementary numbers has been motivated by problems in logic and computer algebra. A typical problem is this: given a complicated expression for a number in $\mathbb{E}$, how can you tell if it exactly equals zero? Clearly this is an important problem for designers of symbolic computation software. It is harder than it might seem at first glance, and is still not fully solved, although Richardson [17] has explicitly described a procedure that takes a given elementary number and, if the procedure terminates, correctly says whether or not the number equals zero. He has also proved that if Schanuel's conjecture is true, then the procedure does in fact always terminate. This more or less solves the zero-recognition problem for elementary numbers (and *a fortiori* for $\mathbb{E}$ and $\mathbb{L}$) in practice.

The zero-recognition problem is closely related to a famous long-standing question of Tarski. Tarski proved that the first-order theory of the real numbers is decidable, which implies in particular that there is an algorithm for determining whether or not any given finite system of polynomial equations and inequalities has a solution in the reals [8, p. 340]. The proof proceeds by *quantifier elimination,* which we can think of roughly as follows: the statement that "there exists a solution" involves existential quantifiers, and quantifier elimination is a procedure for transforming such statements into ones that are quantifier-free. These are then easy to check because all that is involved is a zero-recognition problem for integers. After proving his theorem, Tarski asked if it could be extended to the first-order theory of the real numbers with exponentiation.

This problem is very hard, because it turns out that quantifier elimination is not possible in this theory. Moreover, checking quantifier-free statements involves the zero-recognition problem for expressions with exponentials, which is difficult. Great progress has recently been made, however. Macintyre [13] showed that if Schanuel's conjecture is true, then there is a decision procedure for the quantifier-free statements. Then Wilkie [19] proved in 1991 that the first-order theory of reals with exponentiation is model complete (which roughly means that quantifiers can "almost" be entirely eliminated). Building on this work, Macintyre and Wilkie [14] showed that if Schanuel's conjecture is true, then the first-order theory of the real numbers with exponentiation is decidable. In particular, from these methods one can extract a zero-recognition procedure for elementary numbers (again, contingent on Schanuel). See [15] for a splendid account of these and related results.

Zero-recognition in $\mathbb{E}$ should be easier than zero-recognition in the elementary numbers. Can one recognize zero in $\mathbb{E}$ without assuming Schanuel, or at least by assuming something weaker? The ideas of Macintyre [13] are a good starting-point here.

Another interesting open problem, posed by Thomas Colthurst (in a `sci.math` article posted on June 21, 1993), is to produce an explicit example of a number that is not in $\mathbb{E}$. Since $\mathbb{E}$ is countable, Cantor's diagonal argument gives us an algorithm for producing the decimal expansion of a non-EL number, but this is not very satisfying. Colthurst suggests that one might expect an expression of the form

$$F = \sum_{m=1}^{\infty} f(m)$$

to work, where $f(m)$ is a nonnegative function of m that approaches zero rapidly. For example, the sets $\mathbb{E}_n$ (defined above in section 2) are finite, so there exists $\epsilon_n > 0$ such that any two distinct numbers in $\mathbb{E}_n$ differ by at least ϵ_n. If one could find f with the property that, for all n,

$$\sum_{m=1}^{n} f(m) \in \mathbb{E}_n \qquad \text{and} \qquad \sum_{m=n+1}^{\infty} f(m) < \epsilon_n,$$

then F could not be in $\mathbb{E}_n$ for any n. This is probably too naïve, but it seems that something along these lines should be feasible. Can f be chosen to be elementary?

On the grounds that many high-school students are unfamiliar with complex numbers, one can ask for a "real analogue" of $\mathbb{E}$. What is the right definition? Such a real analogue would lack many of the nice properties of $\mathbb{E}$ (e.g., recall that if an irreducible cubic with rational coefficients has three distinct real roots, then they cannot be expressed using radicals alone if complex numbers are forbidden), but it might still be interesting.

Finally, we mention that Richardson (personal communication) has shown that if Schanuel's conjecture is false, then there is a counterexample involving only elementary numbers. Can this be strengthened to show that any counterexample must lie in $\mathbb{E}$?

We hope the reader will be tempted to attack these relatively untouched questions.

5. Acknowledgments

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