Exercises on the Lambert W Function

1. Solve the following transcendental equations for the indicated variables, in terms of W.

   (a) $y = x \exp y$ for y
   (b) $\log x = vx^{\alpha}$ for x
   (c) $2A(1 - \sqrt{w})/\log w = 1$ for w
   (d) $dy/dt = y^2(1-y)$, $y(0) = \varepsilon > 0$, for $y(t)$.
   (e) $ds/dt = -(\sigma+1)/(\sigma s + 1)$, $s(0) = 1$, for $s(t)$
   (f) $\gamma = 1 - \exp(-\alpha\gamma)$ for γ
   (g) $xb^x = a$ for x
   (h) $x^{x^a} = b$ for x
   (i) $pe^p + w(e^p - 1) = 0$ for p
   (j) $y + \ln y = z$ for y.

2. Show that Euler's series for $x^{\alpha} - x^{\beta} = (\alpha - \beta)vx^{\alpha+\beta}$,

$$
\begin{aligned}
x &= 1 + v + (1+\alpha+\beta)\frac{v^2}{2!} + \cdots \\
&\quad + (1+(n-1)\alpha+\beta)(1+(n-2)\alpha+2\beta)\cdots(1+\alpha+(n-1)\beta)\frac{v^n}{n!} \\
&\quad + \cdots
\end{aligned}
$$

   converges for all v small enough that

$$
|v| < \frac{1}{eM_E(\alpha,\beta))}\,, \tag{1}
$$

   where $M_E(a,b) = \sqrt{(a^2+b^2)/2}$ is the Euclidean mean of a and b.

3. Iterated exponentiation.

   (a) If

$$
x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2\,, \tag{2}
$$

   what is x? (High School problem)

   (b) If

$$
x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4\,, \tag{3}
$$

   what is x? Explain, comparing to the previous problem. (2nd year analysis question)

   (c) Find a closed-form expression for the tower of exponentials in the previous questions, in terms of W.

4. Show that

$$
\frac{d^n W(x)}{dx^n} = \frac{p_n(W(x))e^{-nW(x)}}{(1+W(x))^{2n-1}} \tag{4}
$$

   where the polynomials $p_n(w)$ satisfy $p_1(w) = 1$ and

$$
p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p_n'(w)\,. \tag{5}
$$