"According to Abramowitz and Stegun" or How arccoth can be Couth

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1 Introduction

The OpenMath definitions of elementary functions (contained in the Content Dictionaries[1] `transc1` and `transc2` [8]) generally state that the elementary functions are "as defined in Abramowitz and Stegun" [1]. It is the contention of this paper that more needs to be done to define the elementary functions precisely, particularly (but not exclusively) on their branch cuts. There are also implications for the definition of functions on $\mathbf{R}$. This affects the logarithm (and therefore non-integer powers such as square root) and the inverse trigonometric and hyperbolic functions, as well as relations between these functions, and relations between them and the forward functions.

Throughout this paper, z and its decorations indicate complex variables, x and y real variables. The precise OpenMath proposals are listed at the end.

2 Logarithms

Let us begin with the simplest example.

$\ln z$ (and z^a for $a \notin \mathbf{Z}$). [1, p. 67] gives the branch cut $(-\infty, 0]$, and the rule (4.1.2) that

$$-\pi < \Im \ln z \leq \pi. \tag{1}$$

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[1] In the process of being reshuffled to ensure alignment with MathML version 2 [11], but this does not affect the thrust of this paper.

This then completely specifies the behaviour of ln: on the branch cut it is closed on the positive imaginary side of the cut, i.e. counter-clockwise continuous in the sense of [7].

What are the consequences of this definition[2]? From the existence of branch cuts, we get the problem of a lack of continuity:

$$\lim_{y\to 0^-} \ln(x+iy) \neq \ln(x) : \tag{2}$$

for $x<0$ the limit is $\ln(x)-2\pi i$. Related to this is the fact that

$$\ln\overline{z} \neq \overline{\ln z} \tag{3}$$

on the branch cut: instead $\ln\overline{z} = \overline{\ln z} + 2\pi i$ on the cut. Similarly,

$$\ln(\frac{1}{z}) \neq -\ln z \tag{4}$$

on the branch cut: instead $\ln(\frac{1}{z}) = -\ln z + 2\pi i$ on the cut.

Two families of solutions have been mooted to these problems.

- [7] points out that the concept of a "signed zero"[3] [6] (for clarity, we write the positive zero as 0^+ and the negative one as 0^-) can be used to solve problems such as the above, if we say that, for $x<0$, $\ln(x+0^+i) = \ln(x)+\pi i$ whereas $\ln(x+0^-i) = \ln(x)-\pi i$. Equation (2) then becomes an equality for all x, interpreting the x on the right as $x+0^-i$. Similarly, (3) and (4) become equalities throughout. Attractive though this proposal is (and OpenMath should probably not inhibit systems that do have signed zeros), it does not answer the fundamental question: what to do if the user types $\ln(-1)$.

- [3] points out that most "equalities" do not hold for the complex logarithm, e.g. $\ln(z^2) \neq 2\ln z$ (try $z=-1$), and its generalisation

$$\ln(z_1 z_2) \neq \ln(z_1) + \ln(z_2). \tag{5}$$

  The most fundamental of all non-equalities is $z = \ln\exp z$, whose most obvious violation is at $z = 2\pi i$. They therefore propose to introduce the *unwinding number* $\mathcal{K}$, defined[4] by

$$\mathcal{K}(z) = \frac{z-\ln\exp z}{2\pi i} = \left\lceil \frac{\Im z - \pi}{2\pi} \right\rceil \in \mathbf{Z} \tag{6}$$

[2]Which we do not contest: it seems that few people today would support the rule one of us (JHD) was taught, viz. that $0 \leq \Im \ln z < 2\pi$. The placement of the branch cut is "merely" a notational convention, but an important one. If we wanted a function that behaves like ln but with this cut, we could consider $\ln(-1) - \ln(-1/z)$ instead. We note that, until 1925, astronomers placed the branch cut between one day and the next at noon [4] vol. 15 p. 417.

[3]One could ask why zero should be special and have two values. The answer is that all the branch cuts we need to consider are on either the real or imaginary axes, so the side to which the branch cut adheres depends on the sign of the imaginary or real part, including the sign of zero.

[4]Note that the sign convention here is the opposite to that of [3], which defined $\mathcal{K}(z)$ as $\lfloor \frac{\pi - \Im z}{2\pi} \rfloor$: the authors of [3] recanted later to keep the number of -1s occurring in formulae to a minimum.

(note that the apparently equivalent definition $\lfloor \frac{\Im z+\pi}{2\pi} \rfloor$ differs precisely on the branch cut for ln as applied to $\exp z$). (5) can then be rescued as

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2) - 2\pi i \mathcal{K}(\ln(z_1) + \ln(z_2)). \tag{7}$$

Similarly (3) can be rescued as

$$\ln \overline{z} = \overline{\ln z} - 2\pi i \mathcal{K}(\overline{\ln z}). \tag{8}$$

Note that, as part of the algebra of $\mathcal{K}$, $\mathcal{K}(\overline{\ln z}) = \mathcal{K}(-\ln z) \neq \mathcal{K}(\ln \frac{1}{z})$.

These problems translate into difficulties with the powering operation, even in cases as apparently simple as square roots. In fact,

$$\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}(-1)^{\mathcal{K}(\ln(z_1)+\ln(z_2))}. \tag{9}$$

3 General Principles

There is no "right" answer to the specialisation of the one–many inverse trigonometric and hyperbolic functions. Here we outline various principles that could be used to justify choices of one branch over another.

1. *For a trigonometric or hyperbolic function f, it should be the case that*

   $$f(f^{-1}(z)) = z, \tag{10}$$

   at least off the branch cuts, and preferably on it.

   Since the functions f are many–one, asking for the converse $f^{-1}(f(z)) = z$ is impossible. However, we could ask that the domain of validity of

   $$f^{-1}(f(z)) = z \tag{11}$$

   be as "natural" as possible.

2. *All these functions should be mathematically*[5] *defined in terms of* ln*, thus inheriting their branch cuts from the chosen branch cut for* ln *(equation 1).*

   This does not define the function uniquely: far from that, but it does ensure that the functions can be unambiguously reduced to logarithms, a bonus for simplification programs. This rule is essentially the same as the "Principal Expression" rule of [7]; the difference is that he allows $\sqrt{\ }$ as a constructor. In practice we will use it as well, but will consider it to be defined by $\sqrt{z} = \exp(\frac{1}{2}\ln(z))$.

[5] This does not imply that it is always right to compute them this way. There may be reasons of efficiency, numerical stability or plain economy (it is wasteful to compute a real arcsin in terms of complex logarithms and square roots) why a numerical, or even symbolic, implementation should be different, but the *semantics* should be those of this definition in tems of logarithms, possibly augmented by exceptional values when the logarithm formula is ill-defined.

3. *The choice made for the branch cuts over* $\mathbf{C}$ *should be consistent with the "well-known" behaviour over* $\mathbf{R}$.

   Again, this doesn't disambiguate completely, and may only be a weak guide: see the discussion of arccot below.

4. *As many as possible of the "well-known" identities should hold, at least off the branch cuts.*

   Even more than item (3) above, this is subjective, but still worth adhering to. The decision as to which identity is "more fundamental" than another can, unfortunately, be very controversial.

5. *Occam's razor.*

   The plethora of trigonometric and hyperbolic functions is due partly to the requirements of mathematical tables and their users. For example, cot is frequently tabulated because, near $\pi/2$, it is much easier to use than a table of tan, in which interpolation is difficult, even though it is logically redundant in tables (and calculators) that contain reciprocals and tangents. In general, we should not create functions which are minor variants of other formulae. The programming language Pascal [5] takes this principle to extremes: the only trigonometric functions defined are sin, cos and arctan. No hyperbolic functions are defined.

   In a similar vein, we should relate the hyperbolic and trigonometric functions. If h is any hyperbolic function, and t the corresponding trigonometric function, we have a relation

$$t(z) = ch(iz) \text{ where } c = \begin{cases} 1 & \cos, \sec \\ i & \cot, \operatorname{cosec} . \\ -i & \sin, \tan \end{cases} \tag{12}$$

   From this it follows *formally* that

$$h^{-1}(\frac{1}{c}z') = it^{-1}(z'). \tag{13}$$

   Definition 1 *A choice of branch cuts for* h^{-1} *and* t^{-1} *is said to be* couth *if equation (13) holds except possibly at finitely many points.*

4 (Inverse) Trigonometric Functions

arcsin z [1] page 79 gives the branch cuts[6] $(-\infty, -1)$ and $(1, \infty)$, but does not specify the values on the branch cut. Two obvious identities that we want arcsin to satisfy are

$$\arcsin(-z) = -\arcsin(z) \tag{14}$$

[6] In fact, [1] does not specify the closure of the branch cuts except in the case of ln: we have completed the definitions.

and

$$\overline{\arcsin(z)} = \arcsin(\overline{z}). \tag{15}$$

In fact the second cannot be satisfied on these branch cuts: $\arcsin(2) = \frac{\pi}{2} - 1.3169\ldots i$, so $\arcsin(\overline{2})$ would have to be $\frac{\pi}{2} + 1.3169\ldots i$, but $2 = \overline{2}$. As in the case of the logarithm, these problems can be resolved by *signed zeros* [7] or by unwinding numbers [3]. For signed zeros, $2 = 2 + 0^+i$, and $\overline{2 + 0^+i} = 2 + 0^-i$, and we are at liberty to define $\arcsin(2 + 0^+i) = \frac{\pi}{2} - 1.3169i$ but $\arcsin(2 + 0^-i) = \frac{\pi}{2} + 1.3169i$. The unwinding number solution to (15) is given by the following result.

Lemma 1 $\overline{\arcsin z}$ *can be "simplified":*

$$\overline{\arcsin z} = (-1)^{\mathcal{K}(-\ln(1-z^2))} \arcsin \overline{z} + \pi\mathcal{K}(-\ln(1-z^2)) \tag{16}$$

The proof is in Appendix A.

We should note that, as functions $\mathbf{R} \to \mathbf{R}$, on $[-1, 1]$ $\sin(\arcsin(x))$ is the identity, and $\arcsin(\sin(x))$ is the identity on $[-\pi/2, \pi/2]$ (and cannot be on a greater range, since sin ceases to be injective).

The suggestion in [7] is that arcsin be defined by the expression

$$\arcsin z = -i \ln\left(\sqrt{1-z^2} + iz\right). \tag{17}$$

This has the correct behaviour on $\mathbf{R}$, and satisfies (14), even on the branch cut.

$\arccos z$ This has the same branch cuts as arcsin in [1], and therefore the same problems. As functions $\mathbf{R} \to \mathbf{R}$, on $[-1, 1]$ $\cos(\arccos(x))$ is the identity, and $\arccos(\cos(x))$ is the identity on $[0, \pi]$ (and cannot be on a greater range, since cos ceases to be injective).

The suggestion in [7] is to define

$$\arccos(z) = \frac{\pi}{2} - \arcsin(z) = \frac{2}{i} \ln\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right). \tag{18}$$

This satisfies $\arccos(-z) = \pi - \arccos(z)$ everywhere, as we might expect from $\mathbf{R}$.

Lemma 2 *An alternative formulation is*

$$\arccos(z) = -i \ln\left(z + i\sqrt{1-z^2}\right). \tag{19}$$

This is trivially equal to $-i \ln\left(z + i\sqrt{(1-z)(1+z)}\right)$, which may be numerically more stable near $z = \pm 1$. This issue, and the choice between

(18) and (19), which may depend on the availability of hadware square-root capability, are beyond the scope of this paper.

Proof. Now

$$\left(\sqrt{\frac{1+z}{2}}+i\sqrt{\frac{1-z}{2}}\right)^2 = z+i\sqrt{1-z}\sqrt{1+z} = z+1\sqrt{1-z^2}$$

since the imaginary parts of $1-z$ and $1+z$ have opposite signs. Also $2\ln a = \ln(a^2)$ if $\mathcal{K}(2\ln a)=0$, so we need only show this last stipulation i.e. that

$$-\frac{\pi}{2} < \arg\left(\sqrt{\frac{1+z}{2}}+i\sqrt{\frac{1-z}{2}}\right) \le \frac{\pi}{2}.$$

This is trivially true at $z=0$. If it is false, then we have to pass through $\left|\arg\left(\sqrt{\frac{1+z}{2}}+i\sqrt{\frac{1-z}{2}}\right)\right| = \frac{\pi}{2}$, i.e. $\sqrt{\frac{1+z}{2}}+i\sqrt{\frac{1-z}{2}} = it$ for $t\in\mathbf{R}$. Squaring both sides, $z+i\sqrt{1-z^2}=-t^2$, i.e. $(z+t^2)^2=-(1-z^2)$. Hence $2zt^2+t^4=-1$, so $z=\frac{-(1+t^4)}{2t^2}\le -1$, and in particular is real. On this half-line, the argument in question is $+\pi/2$, which is acceptable. Hence the argument never leaves the desired range, and the lemma is proved.

$\arctan z$ [1] gives the following branch cuts: $(-i\infty,-i]$ and $[i,i\infty)$.

[7] suggests the following definition:

$$\frac{1}{2i}\left(\ln(1+iz)-\ln(1-iz)\right). \tag{20}$$

This satisfies

$$\arctan(-z) = -\arctan(z), \tag{21}$$

and is now the definition adopted by Common Lisp [9] p. 309, replacing

$$-i\ln\left((1+iz)\sqrt{1/(1+z^2)}\right),$$

which does not satisfy (21).

$\operatorname{arccot} z$ [1] (page 79) gives the branch cut $[-i,i]$ from the ninth printing on, with the definition

$$\operatorname{arccot} z = \arctan(1/z). \tag{22}$$

However, according to [7][7], earlier printings gave the definition

$$\operatorname{arccot} z = \frac{\pi}{2} - \arctan z. \tag{23}$$

[7] Atypically, Kahan is guilty of over-simplification here. The definition (equation 4.4.8) was always (22), with the branch cuts as listed. However the claimed range was $0 \le \operatorname{arccot} z \le \pi$ (note the $\le$ rather than $<$), with the formula $\operatorname{arccot}(-z) = \pi - \operatorname{arccot}(z)$, which is algebraically inconsistent with (22) and with (21), which was quoted in all printings. This inconsistency has been reproduced in as recent a source as [12] on pages 465–466.

It should be noted that definition (22), regarded as the definition of a purely real function, has a range of $(-\pi/2, \pi/2) \setminus \{0\}$, with a singularity at $z = 0$ (probably intended to be given the value $\pi/2$, but this has to be inferred from [1]), whereas (23) has a range of $(0, \pi)$ and no singularity. The branch cuts for (23) are clearly those for arctan.

There is no clear standard for the definition of arccot as a function $\mathbf{R} \to \mathbf{R}$: the standard shibboleth is to ask the definition of $\operatorname{arccot}(-1)$. This gives the following results.

[1]	1st printing	$\frac{3\pi}{4}$	inconsistent
[1]	9th printing	$\frac{-\pi}{4}$	
[12]	30th edition	$\frac{3\pi}{4}$	inconsistent
Maple	V release 5	$\frac{3\pi}{4}$	
Axiom	2.1	$\frac{3\pi}{4}$	
Mathematica	[10]	$\frac{-\pi}{4}$	
Reduce	3.4.1	$\frac{-\pi}{4}$	in floating point
Matlab	5.3.0	$\frac{-\pi}{4}$	in floating point

In the absence of any consensus, we appeal to Occam's razor, and, since $\cot(z) = 1/\tan(z)$, we define $\operatorname{arccot}(z)$ by (22). In fact, we need to be slightly more careful, since this does not define $\operatorname{arccot}(0)$. To solve this problem and have $\operatorname{arccot}(0) = \pi/2$, we follow [7] and define

$$\operatorname{arccot} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right). \tag{24}$$

The apparently similar equation[8] "$\operatorname{arccot} z = \frac{i}{2} \ln\left(\frac{z-i}{z+i}\right)$" in fact defines $-\operatorname{arccot}(-z)$, which takes the value $-\pi/2$ rather than $\pi/2$ at $z = 0$, but is equal everywhere else real. In fact, the two differ only on the branch cuts.

$\operatorname{arcsec} z$ [1] gives the branch cut $(-1, 1)$. The parsimony principle mentioned above would define this in terms of arccos, viz.

$$\operatorname{arcsec}(z) = \arccos(1/z) = -i \ln(1/z + i\sqrt{1 - 1/z^2}). \tag{25}$$

This satisfies the rule that $\operatorname{arcsec}(-z) = \pi - \operatorname{arcsec}(z)$, but does have a spurious singularity at $z = 0$ (where the value should be $\pi/2$).

$\operatorname{arccsc} z$ The branch cuts are as for arcsec. Again, we should define this in terms of arcsin, viz.

$$\operatorname{arccsc}(z) = \arcsin(1/z) = -i \ln(i/z + \sqrt{1 - 1/z^2}). \tag{26}$$

This satisfies the rule that $\operatorname{arccsc}(-z) = -\operatorname{arccsc}(z)$, but does have a spurious singularity at $z = 0$ (where the value should be 0).

[8] This, in its equivalent form $\frac{-1}{2i} \ln \frac{iz+1}{iz-1}$, is quoted in [2, p. 579] as the definition of arccot. The authors are grateful to Prof. Pohst for this information.

5 (Inverse) Hyperbolic Functions

arcsinh z [1] gives the branch cuts: $(-i\infty, -i)$ and $(i, i\infty)$. [7] gives the principal expression

$$\operatorname{arcsinh}(z) = \ln\left(z + \sqrt{1+z^2}\right), \tag{27}$$

and this satisfies the rule $\arcsin(z) = \operatorname{arcsinh}(iz)/i$, which is the converse of $\sin(z) = \sinh(iz)/i$, as well as the symmetry rule $\operatorname{arcsinh}(z) = -\operatorname{arcsinh}(-z)$.

arccosh z [1] gives the branch cut $(-\infty, 1)$: in addition there is a branch point at $z = -1$. [7] gives the primary expression

$$\operatorname{arccosh}(z) = 2\ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right). \tag{28}$$

arctanh z [1] gives the branch cuts: $(-\infty, -1]$ and $[1, \infty)$. [7] gives the principal expression

$$\operatorname{arctanh}(z) = \frac{1}{2}\left(\ln(1+z) - \ln(1-z)\right). \tag{29}$$

This satisfies $\operatorname{arctanh}(z) = -\operatorname{arctanh}(-z)$, and well as the relationship with arctan: $\arctan(z) = \frac{1}{i}\operatorname{arctanh}(iz)$.

arccoth z [1] gives the branch cut $[-1, 1]$. The principal expression derived from that of arctanh would be

$$\operatorname{arccoth}(z) = \frac{1}{2}\left(\ln(1+\frac{1}{z}) - \ln(1-\frac{1}{z})\right). \tag{30}$$

However, it is better to choose

$$\operatorname{arccoth}(z) = \frac{1}{2}\left(\ln(-1-z) - \ln(1-z)\right). \tag{31}$$

Superficially, this has a branch cut on $(-\infty, 1]$, but in fact the branch cuts of the two logarithms cancel on $(-\infty, -1)$, so that the branch cut is in fact $[-1, 1]$. This satisfies the couthness identity $\operatorname{arccot}(z) = i\operatorname{arccoth}(iz)$ (see Lemma 3 and the argument before it), and $\operatorname{arccoth}(-z) = -\operatorname{arccoth}(z)$ except at $z = 0$.

arcsech z [1] gives the branch cuts: $(-\infty, 0]$ and $(1, \infty)$. Definition in terms of the formula for arccosh would give

$$\operatorname{arcsech}(z) = 2\ln\left(\sqrt{\frac{z+1}{2z}} + \sqrt{\frac{1-z}{2z}}\right). \tag{32}$$

arccsch z [1] gives the branch cut $(-1, 1)$. Definition in terms of the formula for arcsinh would give

$$\operatorname{arccsch}(z) = \ln\left(\frac{1}{z} + \sqrt{1 + \left(\frac{1}{z}\right)^2}\right), \tag{33}$$

6 How Couth are We?

In this section, we examine how the various choices we have outlined in the previous two section fit in with the definition of couthness given above.

arcsin / arcsinh In this case, equation(13) would translate into

$$\operatorname{arcsinh}(iz) \stackrel{?}{=} i \arcsin(z). \tag{34}$$

Substituting in equations (17) and (27) gives

$$\ln\left(iz + \sqrt{1 + (iz)^2}\right) \stackrel{?}{=} i\left(-i \ln\left(\sqrt{1 - z^2} + iz\right)\right),$$

which is patently true. So these two definitions are couth.

arccos / arccosh In this case, equation(13) would translate into

$$\operatorname{arccosh}(z) \stackrel{?}{=} i \arccos(z). \tag{35}$$

Substituting in equations (18) and (28) gives

$$2 \ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right) \stackrel{?}{=} i\left(\frac{2}{i} \ln\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right)\right).$$

The difference between the two is that the left-hand side has $\sqrt{\frac{z-1}{2}}$, whereas the right-hand side has $i\sqrt{\frac{1-z}{2}}$. Now, a consequence of the branch cut for ln (1) is that every square root has its argument in the range $-\frac{\pi}{2} < \arg \leq \frac{\pi}{2}$, so these two cannot be equal everywhere. In fact, they are equal when $\arg\left(\sqrt{\frac{1-z}{2}}\right) \leq 0$, which simplifies to $\arg(1 - z) \leq 0$, i.e. when $\Im z > 0$ or $\Im z = 0$ and $\Re(1 - z) \geq 0$, i.e. $\Re z \leq 1$. This region is the upper half-plane, including the real axis for $\Re z \leq 1$.

arctan / arctanh In this case, equation(13) would translate into

$$\operatorname{arctanh}(iz) \stackrel{?}{=} i \arctan(z). \tag{36}$$

Substituting in equations (20) and (29) gives

$$\frac{1}{2}(\ln(1 + iz) - \ln(1 - iz)) \stackrel{?}{=} i\left(\frac{1}{2i}(\ln(1 + iz) - \ln(1 - iz))\right).$$

These are clearly equal everywhere.

arccot / arccoth In this case, equation(13) would translate into

$$\operatorname{arccoth}(-iz) \stackrel{?}{=} i \operatorname{arccot}(z). \tag{37}$$

Substituting in equations (24) and (31) gives

$$\frac{1}{2}\left(\ln(-1-(-iz))-\ln(1-(-iz))\right)\stackrel{?}{=}i\left(\frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)\right).$$

This reduces to

$$\left(\ln(iz-1)-\ln(iz+1)\right)\stackrel{?}{=}\ln\left(\frac{z+i}{z-i}\right).$$

Lemma 3

$$\left(\ln(iz-1)-\ln(iz+1)\right)=\ln\left(\frac{iz-1}{iz+1}\right)=\ln\left(\frac{z+i}{z-i}\right).$$

Proof. The seond is clearly equal to the third, so we need only prove that the first is equal to the second.

$$\ln\left(\frac{iz-1}{iz+1}\right)=\ln(iz-1)-\ln(iz+1)-2\pi i\mathcal{K}\left(\ln(iz-1)-\ln(iz+1)\right). \quad (38)$$

We look at the curves where $\mathcal{K}$ can change value, i.e. the boundaries of the "clear-cut region" $\mathcal{K}=0$:

$$\ln(iz-1)-\ln(iz+1)=t\pm i\pi \qquad t\in\mathbf{R}.$$

This is $\frac{iz-1}{iz+1}=-\exp(t)$. This gives

$$z=i\frac{e^t-1}{e^t+1},$$

a parameterisation of the line segment $\Re(z)=0$, $-1<\Im(z)<1$. Taking a point z^* off this line, say $z^*=1$, $\mathcal{K}\left(\ln(iz^*-1)-\ln(iz^*+1)\right)=0$, so $\mathcal{K}=0$ everywhere except on this segment. On the critical segment, $\mathcal{K}=0$, so the $\mathcal{K}$ term in (38) vanishes and the lemma is proved.

arcsec / arcsech In this case, equation(13) would translate into

$$\operatorname{arcsech}(z)\stackrel{?}{=}i\operatorname{arcsec}(z). \quad (39)$$

Substituting in equations (25) and (32) gives

$$2\ln\left(\sqrt{\frac{z+1}{2z}}+\sqrt{\frac{1-z}{2z}}\right)\stackrel{?}{=}\ln\left(1/z+i\sqrt{1-1/z^2}\right). \quad (40)$$

At $z=\frac{1}{2}$, this becomes

$$2\ln\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)\stackrel{?}{=}\ln\left(2+i\sqrt{-3}\right):$$

a palpable falsehood. In fact, one side is the negation of the other. Now, lemma 2 shows that

$$\ln\left(z+i\sqrt{1-z^2}\right) = 2\ln\left(\sqrt{\frac{z+1}{2}}+i\sqrt{frac1-z2}\right).$$

Replacing z by $1/z$ gives

$$\ln\left(\frac{1}{z}+i\sqrt{1-\frac{1}{z^2}}\right) = 2\ln\left(\sqrt{\frac{z+1}{2z}}+i\sqrt{\frac{z-1}{2z}}\right).$$

Substituting this into equation (40) gives

$$2\ln\left(\sqrt{\frac{z+1}{2z}}+\sqrt{\frac{1-z}{2z}}\right) \stackrel{?}{=} 2\ln\left(\sqrt{\frac{z+1}{2z}}+i\sqrt{\frac{z-1}{2z}}\right).$$

Since $\ln a = \ln b$ if and only if $a = b$, this reduces to $\sqrt{\frac{1-z}{2z}} \stackrel{?}{=} i\sqrt{\frac{z-1}{2z}}$. These are equal when $\arg\sqrt{\frac{z-1}{2z}} \leq 0$, equivalent to $\arg(1-1/z) \leq 0$ This is true precisely when $\Im z < 0$ or $\Im z = 0$ and $\Re z \geq 1$. Hence the pair arcsec / arcsech is not couth.

arccsc / arccsch In this case, equation(13) would translate into

$$\operatorname{arccsch}(-iz) \stackrel{?}{=} i\operatorname{arccsc}(z). \tag{41}$$

Substituting in equations (26) and (33) gives

$$\ln\left(\frac{1}{-iz}+\sqrt{1+\left(\frac{1}{-iz}\right)^2}\right) \stackrel{?}{=} \ln\left(i/z+\sqrt{1-1/z^2}\right).$$

These are patently equal.

7 Implications for Phrase-book Writers

An OpenMath phrase-book is actually a piece of software that translates between the semantics of OpenMath (as defined in the Content Dictionaries) and the semantics of a particular application, as well as simply translating names. Of course, life for the phrase-book writer is simplest if the semantics of the application are the same as those of OpenMath, but this will not always be the case.

A classic example of this would be the translation of arccot between OpenMath, whose semantics are defined by (22), and a system, such as Maple or Axiom, where the semantics of arccot are defined by (23).

Application→OpenMath	$\operatorname{arccot} z \mapsto \frac{\pi}{2} - \arctan z$
OpenMath→Application	$\operatorname{arccot} z \mapsto \arctan\left(\frac{1}{z}\right)$

8 Proposals

This section lists the concrete suggestions that the authors have for OpenMath.

1. OpenMath should *define* a^b (in the case of non-integer b) via

$$a^b = \exp(b \ln a), \tag{42}$$

   rather than the current weasel words: "When the second argument is not an integer care should be taken as to the meaning of this function; however it is here to represent general powering" (`arith1.ocd` [8]).

2. OpenMath should base its definitions of all the multi-valued elementary functions on the logarithm function. This means that it would be reasonable to define all the forward functions in terms of the exponential function.

3. OpenMath should be parsimonious in terms of the number of (subtly) different concepts it introduces: in particular cot, sec and cosec, and their inverses and hyperbolic analogues should be defined[9] in terms of the other functions and reciprocals.

4. OpenMath should use the "unwinding number" formalism for stating any mathematical properties of these functions (and therefore OpenMath should define the unwinding number).

5. OpenMath should not forbid the use of signed zeros, and therefore range restrictions such as (1) have to be relaxed to non-strict inequalities in this case.

6. OpenMath should consider whether it wants (using different symbols, e.g. `Log`) to represent the multi-valued inverse functions as well as the single-valued ones.

Revisions to the content dictionaries are attached as appendices.

References

[1] Abramowitz,M. & Stegun,I., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. US Government Printing Office, 1964. 10th Printing December 1972. Note that [8] does not define which printing to use.

[9] One could argue that these functions need not be defined at all. Indeed tan could be replaced by sin / cos etc. This would create problems for renderers, but these are not insuperable, and getting good-looking mathematics out of OpenMath already requires a certain amount of intelligence on the part of the renderers. However, not having these functions would, we argue, actually *increase* the risk of the sort of subtle mis-understanding that this paper is meant to point out (e.g. defining arccot in terms of arctan leaves arccot(0) undefined, and the definition is non-trivial: see the discussion after (24)), and makes it harder to write numerically-accurate expressions.

[2] Bronstein, Semendjajew & Musiol, *Taschenbuch der Mathematik*. Mühlig in Harri Deutsch Verlag, Frankfurt 1995.

[3] Corless,R.M. & Jeffrey,D.J., The Unwinding Number. SIGSAM Bulletin **30** (1996) 2, pp. 28–35.

[4] *Encyclopedia Britannica*, 15th. edition. Encyclopedia Britannica Inc., Chicago etc., 15th ed., 1995 printing.

[5] IEEE Standard Pascal Computer Programming Language. IEEE Inc., 1983.

[6] IEEE Standard 754 for Binary Floating-Point Arithmetic. IEEE Inc., 1985.

[7] Kahan,W., Branch Cuts for Complex Elementary Functions. *The State of Art in Numerical Analysis* (ed. A. Iserles & M.J.D. Powell), Clarendon Press, Oxford, 1987, pp. 165–211.

[8] The OpenMath Consortium. Draft Content Dictionaries. `http://www.nag.co.uk/projects/OpenMath/corecd/`.

[9] Steele,G.L.,Jr., *Common LISP: The Language*, 2nd. edition. Digital Press, 1990.

[10] Wolfram,S., *The Mathematica Book*. Wolfram Media/C.U.P., 1999.

[11] World-Wide Web Consortium, Mathematical Markup Language (MathML [tm]) 2.0 Draft Specification. W3C Draft, revision of 11 February 2000. `http://www.w3.org/TR/2000/WD-MathML2-20000211`.

[12] Zwillinger,D. (ed.), *CRC Standard Mathematical Tables and Formulae*. 30th. ed., CRC Press, Boca Raton, 1996.

A Proof of Lemma 1

Our starting point is the set of equations:

$$\begin{aligned} \overline{\ln z} &= \ln \overline{z} + 2\pi i \mathcal{K}\left(\overline{\ln z}\right) && (43)\\ &= \ln \overline{z} + 2\pi i \mathcal{K}\left(-\ln z\right) && (44)\\ \arcsin(z) &= -i \ln\left(iz + \sqrt{1-z^2}\right) && (45) \end{aligned}$$

So now we want to obtain a formula for $\overline{\arcsin z}$, preferably in terms of $\arcsin \overline{z}$. We attempt to obtain it using formal rules for the unwinding number.

$$\begin{aligned} \overline{\arcsin z} &= i\,\overline{\ln(iz + \sqrt{1-z^2})} && (46)\\ &= i\left[\ln \overline{iz + \sqrt{1-z^2}} + 2\pi i \mathcal{K}\left(-\ln(iz + \sqrt{1-z^2}\,)\right)\right] && (47) \end{aligned}$$

This unwinding number can be dismissed quickly by observing that it can only be nonzero when the argument of log is real and negative. Therefore any nonzero

imaginary part of the log argument makes $\mathcal{K}$ zero. The only way the first term can be real is if $z = iy$, but then the log argument is always positive. So we continue:

$$\overline{\arcsin z} \quad = \quad i \ln\left(-i\overline{z} + \overline{\sqrt{1-z^2}}\right) \tag{48}$$

Now $\overline{z^{1/2}} = \overline{z}^{1/2} e^{i\pi\mathcal{K}(-\ln z)}$. Therefore

$$\overline{\arcsin z} \quad = \quad i \ln\left(-i\overline{z} + \sqrt{1-\overline{z}^2}\, e^{i\pi\mathcal{K}(-\ln(1-\overline{z}^2))}\right) \tag{49}$$

Notice that the expression $\mathcal{K}(-\ln(1-z^2))$ is the most concise characteristic function for the branch cuts of arcsine available in this notation. So for $|z| < 1$ or $z \notin R$, the square root is positive and the unwinding number is zero and the right side is an arcsine again (see below). However, for $z > 1$, the unwinding number is 1 and we have

$$\overline{\arcsin z} \quad = \quad i \ln\left(-i\overline{z} - \sqrt{1-\overline{z}^2}\right) \tag{50}$$

Although (49) is a complete answer to the simplification, the answer does not look like arcsine, so I suppose we must battle on here and obtain an expression in terms of arcsine. Using $\ln z = -\ln z^{-1} - 2\pi i\mathcal{K}(-\ln z)$, we continue. To save typing I shall write z for $\overline{z}$ and $E = e^{i\pi\mathcal{K}(-\ln(1-z^2))}$

$$\begin{aligned} i \ln\left(-iz + E\sqrt{1-z^2}\right) \quad &= \quad i\left[-\ln\left(\frac{1}{-iz + E\sqrt{1-z^2}}\right)\right. \\ &\qquad \left. -2\pi i\mathcal{K}\left(-\ln(-iz + E\sqrt{1-z^2})\right)\right] \end{aligned}$$

As above, the last term is zero, and rationalizing the first term:

$$\frac{1}{-iz + E\sqrt{1-z^2}} = \frac{E\sqrt{1-z^2} + iz}{E^2(1-z^2) + z^2} = E\sqrt{1-z^2} + iz\ ,$$

because $E^2 = 1$. Therefore

$$\overline{\arcsin z} \quad = \quad -i \ln\left(i\overline{z} + \sqrt{1-\overline{z}^2}\, e^{i\pi\mathcal{K}(-\ln(1-z^2))}\right) \tag{51}$$

If z is not on the branch cut, then $\mathcal{K}$ is zero and we have immediately $\overline{\arcsin z} = \arcsin\overline{z}$. If z is on the branch cut, then we can check the correctness with a numerical example: for $z = 2$, both sides of the equation correctly reduce to $\pi/2 - i\ln(2-\sqrt{3})$. Therefore (51) is correct and could be the final result, as could (49). If we insist on having arcsine in our final expression, then we can write

$$\begin{aligned} \overline{\arcsin z} \quad &= \quad e^{i\pi\mathcal{K}(-\ln(1-z^2))} \arcsin\overline{z} + \pi\mathcal{K}(-\ln(1-z^2)) && (52) \\ &= \quad (-1)^{\mathcal{K}(-\ln(1-z^2))} \arcsin\overline{z} + \pi\mathcal{K}(-\ln(1-z^2)) && (53) \end{aligned}$$

B Differences in `arith1.ocd` (with W.A. Naylor, Bath)

We have taken some liberties with the line-breaking in the textual parts, but the differences are as shown apart from this change.

B.1 Description of `power`

The text (following "for exponent not")

```
an integer care should be taken as to the meaning of this
function; however it is here to represent general powering.
```

is replaced by

```
an integer, powering is defined in terms of exponentials and
logarithms for the complex and real numbers.
This operator can represent general powering.
```

The following is added to power:

```
<CMP>
x\in C implies x^a = exp(a ln x)
</CMP>

<FMP>
<OMOBJ>
  <OMA>
    <OMS cd="logic1" name="implies"/>
    <OMA>
      <OMS cd="set1" name="in"/>
      <OMV name="x"/>
      <OMS cd="setname1" name="C"/>
    </OMA>
    <OMA>
      <OMS cd="relation1" name="eq"/>
      <OMA>
        <OMS name="power" cd="arith1"/>
        <OMV name="x"/>
        <OMV name="a"/>
      </OMA>
      <OMA>
        <OMS name="exp" cd="transc1"/>
        <OMA>
          <OMS name="times" cd="arith1"/>
          <OMV name="a"/>
          <OMA>
```

```
            <OMS name="ln" cd="transc1"/>
            <OMV name="x"/>
          </OMA>
        </OMA>
      </OMA>
    </OMA>
  </OMA>
</OMOBJ>
</FMP>
```

B.2 Description of `root`

The text

`the general notion of taking n'th roots.`

is changed to (the CMP referred to is `for all a,n, power(root(a,n),n) = a`)

`the general notion of taking n'th roots. Note in particular that the converse of the CMP is not valid in general.`

The following text is added,

```
<CMP> x\in C implies root(x,n) = exp(ln(x)/n) </CMP>
<FMP>
  <OMOBJ>
    <OMA>
      <OMS cd="logic1" name="implies"/>
      <OMA>
        <OMS cd="set1" name="in"/>
        <OMV name="x"/>
        <OMS cd="setname1" name="C"/>
      </OMA>
      <OMA>
        <OMS cd="relation1" name="eq"/>
        <OMA>
          <OMS cd="arith1" name="root"/>
          <OMV name="x"/>
          <OMV name="n"/>
        </OMA>
        <OMA>
          <OMS name="exp" cd="transc1"/>
          <OMA>
            <OMS name="divide" cd="arith1"/>
            <OMA>
              <OMS name="ln" cd="transc1"/>
              <OMV name="x"/>
```

```
            </OMA>
            <OMV name="n"/>
          </OMA>
        </OMA>
      </OMA>
    </OMA>
  </OMA>
</OMOBJ>
```

C Differences in `transc1.ocd`

There is a tightening of the definition in the header. The following

```
functions. They are defined as in Abromowitz and Stegun.
```

is replaced by

```
functions. They are defined as in Abramowitz and Stegun (ninth
printing on), with precise reductions to logs in the case of
inverse functions.

Note that, if signed zeros are supported, some strict
inequalities have to become weak
```

In addition a few bugs are fixed:

```
        <OMS cd="arith1" name="power"/>
```

becomes

```
        <OMS cd="transc1" name="exp"/>
```

C.1 Differences in the ln function

The old text

```
This symbol represents the ln function (natural logarithm) as
described in
Abramowitz and Stegun, section 4.1. It takes one argument.
```

is replaced by

```
This symbol represents the ln function (natural logarithm) as
described in Abramowitz and Stegun, section 4.1. It takes one
argument.  Note the description in the CMP/FMP of the branch
cut.  If signed zeros are in use, the inequality needs to be
non-strict.
```

The branch cut is formally defined.

```
<CMP>
-pi &lt; Im ln x &le;= pi
</CMP>
<FMP>
  <OMOBJ>
    <OMA>
      <OMS name="and" cd="logic1"/>
      <OMA>
        <OMS name="lt" cd="relation1"/>
        <OMA>
          <OMS name="unary_minus" cd="arith1"/>
          <OMS name="pi" cd="nums1"/>
        </OMA>
        <OMA>
          <OMS name="imaginary" cd="nums1"/>
          <OMA>
            <OMS name="ln" cd="transc1"/>
            <OMV name="x"/>
          </OMA>
        </OMA>
      </OMA>
      <OMA>
        <OMS name="leq" cd="relation1"/>
        <OMA>
          <OMS name="imaginary" cd="nums1"/>
          <OMA>
            <OMS name="ln" cd="transc1"/>
            <OMV name="x"/>
          </OMA>
        </OMA>
        <OMS name="pi" cd="nums1"/>
      </OMA>
    </OMA>
  </OMOBJ>
</FMP>
```

C.2 Changes to sin

The relationship with exp is documented.

```
<CMP>
sin(x) = (exp(ix)-exp(-ix))/2i
</CMP>
<FMP>
  <OMOBJ>
    <OMA>
```

```
      <OMS cd="relation1" name="eq"/>
      <OMA>
        <OMS name="sin" cd="transc1"/>
        <OMV name="x"/>
      </OMA>
      <OMA>
        <OMS name="divide" cd="arith1"/>
        <OMA>
          <OMS name="minus" cd="arith1"/>
          <OMA>
            <OMS name="exp" cd="transc1"/>
            <OMA>
              <OMS name="times" cd="arith1"/>
              <OMS name="i" cd="nums1"/>
              <OMV name="x"/>
            </OMA>
          </OMA>
          <OMA>
            <OMS name="exp" cd="transc1"/>
            <OMA>
              <OMS name="times" cd="arith1"/>
              <OMA>
                <OMS name="unary_minus" cd="arith1"/>
                <OMS name="i" cd="nums1"/>
              </OMA>
              <OMV name="x"/>
            </OMA>
          </OMA>
        </OMA>
        <OMA>
          <OMS name="times" cd="arith1"/>
          <OMI>2</OMI>
          <OMS name="i" cd="nums1"/>
        </OMA>
      </OMA>
    </OMA>
  </OMOBJ>
</FMP>
```

C.3 Changes to cos

Again, the relationship to exp is documented

```
<CMP>
cos(x) = (exp(ix)+exp(-ix))/2
</CMP>
```

```
<FMP>
  <OMOBJ>
    <OMA>
      <OMS cd="relation1" name="eq"/>
      <OMA>
        <OMS name="cos" cd="transc1"/>
        <OMV name="x"/>
      </OMA>
      <OMA>
        <OMS name="divide" cd="arith1"/>
        <OMA>
          <OMS name="plus" cd="arith1"/>
          <OMA>
            <OMS name="exp" cd="transc1"/>
            <OMA>
              <OMS name="times" cd="arith1"/>
              <OMS name="i" cd="nums1"/>
              <OMV name="x"/>
            </OMA>
          </OMA>
          <OMA>
            <OMS name="exp" cd="transc1"/>
            <OMA>
              <OMS name="times" cd="arith1"/>
              <OMA>
                <OMS name="unary_minus" cd="arith1"/>
                <OMS name="i" cd="nums1"/>
              </OMA>
              <OMV name="x"/>
            </OMA>
          </OMA>
        </OMA>
        <OMI>2</OMI>
      </OMA>
    </OMA>
  </OMOBJ>
</FMP>
```

C.4 Changes to arcsin

The relationship to ln is documented

```
<CMP>
arcsin(z) = -i ln (sqrt(1-z^2)-iz)
</CMP>
<FMP>
```

```
  <OMOBJ>
    <OMA>
      <OMS name="eq" cd="relation1"/>
      <OMA>
        <OMS name="arcsin" cd="transc1"/>
        <OMV name="z"/>
      </OMA>
      <OMA>
        <OMS name="times" cd="arith1"/>
        <OMA>
          <OMS name="unary_minus" cd="arith1"/>
          <OMS name="i" cd="nums1"/>
        </OMA>
        <OMA>
          <OMS name="ln" cd="transc1"/>
          <OMA>
            <OMS name="minus" cd="arith1"/>
            <OMA>
              <OMS name="root" cd="arith1"/>
              <OMA>
                <OMS name="minus" cd="arith1"/>
                <OMS name="one" cd="alg1"/>
                <OMA>
                  <OMS name="power" cd="arith1"/>
                  <OMV name="z"/>
                  <OMI> 2 </OMI>
                </OMA>
              </OMA>
              <OMI> 2 </OMI>
            </OMA>
            <OMA>
              <OMS name="times" cd="arith1"/>
              <OMS name="i" cd="nums1"/>
              <OMV name="z"/>
            </OMA>
          </OMA>
        </OMA>
      </OMA>
    </OMA>
  </OMOBJ>
</FMP>
```

C.5 Changes to arccos

The relationship to ln is documented.

```
<CMP>
arccos(z) =(2/i) log (sqrt((1+z)/2)+i\sqrt((1-z)/2))
</CMP>
<FMP>
  <OMOBJ>
    <OMA>
      <OMS name="eq" cd="relation1"/>
      <OMA>
        <OMS name="arccos" cd="transc1"/>
        <OMV name="z"/>
      </OMA>
      <OMA>
        <OMS name="times" cd="arith1"/>
        <OMA>
          <OMS name="divide" cd="arith1"/>
          <OMI> 2 </OMI>
          <OMS name="i" cd="nums1"/>
        </OMA>
        <OMA>
          <OMS name="ln" cd="transc1"/>
          <OMA>
            <OMS name="plus" cd="arith1"/>
            <OMA>
              <OMS name="root" cd="arith1"/>
              <OMA>
                <OMS name="plus" cd="arith1"/>
                <OMS name="one" cd="alg1"/>
                <OMV name="z"/>
              </OMA>
              <OMI> 2 </OMI>
            </OMA>
            <OMA>
              <OMS name="root" cd="arith1"/>
              <OMA>
                <OMS name="minus" cd="arith1"/>
                <OMS name="one" cd="alg1"/>
                <OMV name="z"/>
              </OMA>
              <OMI> 2 </OMI>
            </OMA>
          </OMA>
        </OMA>
      </OMA>
    </OMA>
  </OMOBJ>
</FMP>
```

C.6 Changes to arctan

The relationship to ln is documented, and a minor bug in the specifications corrected. The old text was the following line.

```
<CMP> x in [-(pi/2),(pi/2)] implies arctan(tan x) = x </CMP>
```

The new text is as follows.

```
<CMP>
arctan(z) = (log(1+iz)-log(1-iz))/2i
</CMP>
<FMP>
  <OMOBJ>
      <OMS name="eq" cd="relation1"/>
      <OMA>
        <OMS name="arctan" cd="transc1"/>
        <OMV name="z"/>
      </OMA>
      <OMA>
        <OMS name="divide" cd="arith1"/>
        <OMA>
          <OMS name="minus" cd="arith1"/>
          <OMA>
            <OMS name="ln" cd="transc1"/>
            <OMA>
              <OMS name="plus" cd="arith1"/>
              <OMS name="one" cd="alg1"/>
              <OMA>
                <OMS name="times" cd="arith1"/>
                <OMS name="i" cd="nums1"/>
                <OMV name="z"/>
              </OMA>
            </OMA>
          </OMA>
          <OMA>
            <OMS name="ln" cd="transc1"/>
            <OMA>
              <OMS name="minus" cd="arith1"/>
              <OMS name="one" cd="alg1"/>
              <OMA>
                <OMS name="times" cd="arith1"/>
                <OMS name="i" cd="nums1"/>
                <OMV name="z"/>
              </OMA>
            </OMA>
          </OMA>
```

```
        </OMA>
        <OMA>
          <OMS name="divide" cd="arith1"/>
          <OMI> 2 </OMI>
          <OMS name="i" cd="nums1"/>
        </OMA>
      </OMA>
  </OMOBJ>
</FMP>

<CMP> x in (-(pi/2),(pi/2)) implies arctan(tan x) = x </CMP>
```

`interval_cc` was changed to `interval_oo`.