Wester's test suite in MuPAD 1.3

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In December 1994, Michael Wester made a review [5] of the mathematical capabilities of different computer algebra systems, namely Axiom, Derive, Macsyma, Maple, Mathematica and Reduce. This review, which is available by anonymous ftp from `math.unm.edu`, file `pub/cas/Paper.ps`, consists of 131 tests in different domains of mathematics (arithmetic, algebraic equations, differential equations, integration, operator computation, series expansions, limits).

We describe in this paper the problems that can be solved with MuPAD 1.3, and how to solve them. The problems marked as [New] are solved using new functionalities of the version 1.3 with respect to 1.2.2.

1 Numbers

Problem 1: compute 50!.

```
>> fact(50);
```

```
30414093201713378043612608166064768844377641568960512000000000000
```

Problem 2: compute the prime decomposition of 50!.

```
>> ifactor(fact(50));
```

```
[1, 2, 47, 3, 22, 5, 12, 7, 8, 11, 4, 13, 3, 17, 2, 19, 2, 23, 2, 29, 1,

    31, 1, 37, 1, 41, 1, 43, 1, 47, 1]
```

The function `ifactor(n)` returns a list $[\epsilon, p_1, \alpha_1, \ldots, p_k, \alpha_k]$ where $\epsilon = \pm 1$, and $n = \epsilon p_1^{\alpha_1} \ldots p_k^{\alpha_k}$.

Problem 3: compute $1/2 + \cdots + 1/10$.

```
>> _plus(1/i$i=2..10);
```

```
4861/2520
```

In MuPAD each arithmetic operator has a functional form, here `_plus` for addition, so for example `_plus(a,b)` is the same as `a+b`.

Problem 4: compute a numerical approximation of $e^{\pi\sqrt{163}}$ to 50 digits.

```
>> DIGITS:=50: float(exp(PI*sqrt(163)));
```

```
2.6253741264076874399999999999992500725971981856888 7e17
```

The floating-point numbers are output in scientific notation. This notation can be used as input too.

Problem 5: compute a numerical approximation of $J_2(1+i)$, where J is the Bessel function of the first kind.

```
>> besselJ(2,1.0+I);
```

```
                    0.4157988694e-1 + 0.2473976415 I
```

Please note that floating-point numbers are "contagious": no `float` instruction is needed here.

Problem 6: compute an infinite decimal representation of 1/7.

```
>> export(numlib):
>> decimal(1/7);
```

```
                              0, [1, 4, 2, 8, 5, 7]
```

This result means that the decimal expansion of 1/7 is $0.142857142857142\ldots$. The command `export(numlib)` exports the functions of the `numlib` package, i.e. enables one to call them with a short name (otherwise we should write `numlib::decimal`).

Problem 7: compute the first terms of the continued fraction of π.

```
>> contfrac(PI,5);
```

```
                                  1
                ---------------------- + 3
                                1
                ------------------ + 7
                              1
                ------------- + 15
                            1
                --------- + 1
                 1
                --- + 292
                ...
```

The second (optional) argument of `contfrac` specifies the number of digits to be taken into account before computing the continued fraction.

Problem 8: simplify $\sqrt{2\sqrt{3}+4}$.

```
>> radsimp(sqrt(2*sqrt(3)+4));
```

```
                                  1/2
                                 3    + 1
```

Problem 9: simplify $\sqrt{14+3\sqrt{3+2\sqrt{5-12\sqrt{3-2\sqrt{2}}}}}$.

```
>> radsimp(sqrt(14 + 3*sqrt(3 + 2*sqrt(5 - 12*sqrt(3 - 2*sqrt(2))))));
```

```
                                  1/2
                                 2    + 3
```

Problem 10: simplify $2\infty - 3$.

```
>> 2*infinity-3;
```

```
                                  infinity
```

The symbol `infinity` is implemented as a domain in MuPAD. This allows the overloading of the basic arithmetic operations, together with the comparisons. For example we can write in a MuPAD program `if a<infinity then ... end_if`.

2 Statistics

Problem 11: compute the standard deviation of the sample $[1,2,3,4,5]$.

```
>> stats::stdev([1,2,3,4,5],Sample);
```

$$\frac{2^{1/2}\ 5^{1/2}}{2}$$

Here we have shown the other way to call package functions, without exporting them.

[New] **Problem 12:** determine how good of a guess is 5 for the mean of x, where 10 experiments gave $1,-2,3,-4,5,-6,7,-8,9,10$, using the t distribution.

```
>> l:=[1, -2, 3, -4, 5, -6, 7, -8, 9, 10]:
>> stats::meanTest(l,5);
```

$$\frac{105\ \mathrm{PI}}{128} - \frac{2416017695\ 10^{1/2}}{256} + \frac{105\ \mathrm{atan}\left(\frac{7\ 10^{1/2}}{5\ 2^{1/2}\ 29^{1/2}}\right)}{128}$$

$$105\ \mathrm{PI}$$

```
>> float(%);
```

$$5.756660091e\text{-}2$$

[New] **Problem 13:** same problem, but with the normal distribution.

```
>> l:=[1, -2, 3, -4, 5, -6, 7, -8, 9, 10]:
>> stats::meanTest(l,5,stats::normal);
```

$$-\frac{\mathrm{erf}\left(\frac{21\ 10^{1/2}}{10\ 10^{1/2}\ 29^{1/2}}\right)}{2} + \frac{1}{2}$$

```
>> float(%);
```

$$4.058346175e\text{-}2$$

3 Algebra

Problem 14: compute the normal form of $(x^2-4)/(x^2+4x+4)$.

```
>> normal((x^2 - 4)/(x^2 + 4*x + 4));
```

$$\frac{x-2}{x+2}$$

Problem 15: simplify $(e^x - 1)/(e^{x/2} + 1)$.

```
>> simplify((exp(x)-1)/(exp(x/2)+1),exp);
```

$$\exp\left(\frac{x}{2}\right) - 1$$

Problem 16: expand $(x+1)^{20}$, then differentiate and factorize.

```
>> Factor(diff(expand((x + 1)^20),x));
```

$$20\,(x+1)^{19}$$

MuPAD provides two factorization functions: the toplevel function `Factor` returns an expression, and the library function `factor` returns a list $[a, p_1, \alpha_1, \ldots, p_k, \alpha_k]$ where a is a scalar, the p_i are irreducible polynomials and the α_i are the corresponding powers. In the above example, `factor` would return `[20, x + 1, 19]`.

Problem 17: factor $x^{100} - 1$.

```
>> Factor(x^100-1);
```

$$(x-1)\,(x+1)\,(x^2+1)\,(x^4+x^3+x^2+x+1)\,(-x^4+x^3-x^2+x+1)\,(x^8-x^6+x^4-x^2+1)\,(x^{20}+x^{15}+x^{10}+x^5+1)\,(-x^{20}+x^{15}-x^{10}+x^5+1)\,(x^{40}-x^{30}+x^{20}-x^{10}+1)$$

[New] **Problem 18:** factor the polynomial $x^4 - 3x^2 + 1$ over the algebraic extension $Q(\phi)$ where ϕ is a root of the polynomial $X^2 - X - 1$.

```
>> Qphi:=Dom::AlgebraicExtension(Dom::Rational,Phi^2-Phi-1):
>> Qphi::name:="Qphi":
>> Factor(poly(x^4-3*x^2+1,Qphi));

poly(x + Phi, [x], Qphi) poly(x + Phi - 1, [x], Qphi)

poly(x - Phi, [x], Qphi) poly(x - Phi + 1, [x], Qphi)
```

We have first defined the algebraic extension `Qphi` of the rationals with the polynomial $X^2 - X - 1$, with `Phi` being an alias for the roots of this polynomial. The command `Qphi::name:="Qphi"` was only used to have a simpler output for the algebraic extension name.

Problem 19: factor the polynomial $x^4 - 3x^2 + 1$ modulo 5.

```
>> factor(poly(x^4 - 3*x^2 + 1,[x],IntMod(5)));

    [1, poly(x - 2, [x], IntMod(5)), 2, poly(x + 2, [x], IntMod(5)), 2]
```

This result means that $x^4 - 3x^2 + 1$ factors into $(x-2)^2(x+2)^2$ modulo 5. MuPAD provides a special representation `poly` for polynomials, that enables the user to specify the variables and the coefficient field. It should be noted that the `factor` function of MuPAD implements the currently best-known algorithm to factorize polynomials over a finite field [4].

Problem 20: compute the partial fraction decomposition of $(x^2 + 2x + 3)/(x^3 + 4x^2 + 5x + 2)$.

```
>> partfrac((x^2+2*x+3)/(x^3+4*x^2+5*x+2));

                                2        3              - 2
                            - ----- + ----- + 2 (x + 1)
                              x + 1    x + 2
```

4 Inequalities

[New] **Problem 23:** solve the inequality $|x-1|>2$.

```
>> solve(abs(x-1)>2,x);

          {[-infinity <= x, x < -1], [3 < x, x <= infinity]}
```

[New] **Problem 24:** solve the inequality $(x-1)(x-2)(x-3)(x-4)(x-5)<0$.

```
>> solve(expand((x-1)*(x-2)*(x-3)*(x-4)*(x-5))<0,x);

        {[2 < x, x < 3], [-infinity <= x, x < 1], [4 < x, x < 5]}
```

5 Trigonometry

Problem 25: rewrite $\cos(3x)/\cos(x)$ in terms of $\cos(x)$ and $\sin(x)$.

```
>> expand(cos(3*x)/cos(x));

                                   2
                            4 cos(x)  - 3
```

Problem 26: linearize $\cos(3x)/\cos(x)$.

```
>> combine(expand(cos(3*x)/cos(x)),sincos);

                            2 cos(2 x) - 1
```

6 Determining Zero Equivalence

Problem 28: simplify $\sqrt{997}-(997^3)^{1/6}$.

```
>> sqrt(997)-(997^3)^(1/6);

                                 1/6      1/2
                          - 991026973    + 997

>> simplify(%);

                                   0
```

Problem 29: simplify $\sqrt{999983}-(999983^3)^{1/6}$.

```
>> sqrt(999983)-(999983^3)^(1/6);

                          1/2                      1/6
                    999983    - 999949000866995087

>> simplify(%);

                                   0
```

Problem 30: recognize that $(2^{1/3}+4^{1/3})^3-6(2^{1/3}+4^{1/3})-6$ is 0.

```
>> simplify(expand((2^(1/3)+4^(1/3))^3-6*(2^(1/3)+4^(1/3))-6));
```

$$0$$

[New] **Problem 31:** simplify $\log\tan(x/2+\pi/4)-\sinh^{-1}\tan x$ to 0. In fact, this simplification is not valid in general in the complex plane:

```
>> f := ln(tan(x/2+PI/4))-asinh(tan(x)):
>> float(subs(f,x=-2));
```

$$-3.046904887+3.141592653\,I$$

and whence it is not possible to achieve it in MuPAD.

Problem 32: show that the function $\log(\tan(x/2+\pi/4))-\sinh^{-1}(\tan x)$ is 0 in $x=0$ and that its derivative is 0.

```
>> f := ln(tan(1/2*x + PI/4)) - asinh(tan(x)):
>> eval(subs(f,x=0));
```

$$0$$

```
>> combine(normal(expand(diff(f,x))),sincos);
```

$$\frac{-\cos(x)\left(\frac{1}{\cos(x)^2}\right)^{1/2}+1}{\cos(x)}$$

As said previously, the function f is not zero everywhere in the complex plane. The above expression tells us that the derivative vanishes when the real part of $\cos x$ is positive, i.e. we can conclude that $f(x)$ is really zero when $-\pi/2<\mathrm{Re}(x)<\pi/2$.

[New] **Problem 33:** simplify $\log\frac{2\sqrt{r}+1}{\sqrt{4r+4\sqrt{r}+1}}$.

```
>> simplify(ln((2*sqrt(r)+1)/sqrt(4*r+4*sqrt(r)+1)));
```

$$0$$

[New] **Problem 34:** simplify $-2\sqrt{r}+(2\sqrt{r}+1)^{\frac{1}{2\sqrt{r}+1}}\,(4r+4\sqrt{r}+1)^{\sqrt{r}\frac{1}{2\sqrt{r}+1}}-1$.

```
>> e:=(4*r+4*sqrt(r)+1)^(sqrt(r)/(2*sqrt(r)+1))
     *(2*sqrt(r)+1)^(1/(2*sqrt(r)+1))-2*sqrt(r)-1:
>> assume(r>0):
>> simplify(e);
```

$$0$$

Without the assumption $r>0$, MuPAD cannot simplify the expression because the transformation $(z^a)^b\leftarrow x^{ab}$ is not true in general for z complex (here we have $z=2\sqrt{r}+1$, $a=2$, and $b=sqrtr/(2\sqrt{r}+1)$.

7 The Complex Domain

Problem 35: compute the rectangular form $a+ib$ of $\log(3+4i)$.

```
>> rectform(ln(3+4*I));
```

```
                          ln(5) + I atan(4/3)
```

The real and imaginary part are obtained with `Re(..)` and `Im(..)` respectively.

Problem 36: compute the rectangular form of $\tan(x+iy)$.

```
>> rectform(tan(x+I*y),{x,y});
```

```
                sin(2 x)                I sinh(2 y)
------------------------------- + -------------------------
cos(2 x) + cosh(2 y)              cos(2 x) + cosh(2 y)
```

Here the set `{x,y}` given as second argument specifies that the variables x and y are real. By default all variables are considered as complex.

Problem 37: simplify $\sqrt{xy|z|^2}/\sqrt{x}/|z|$.

```
>> sqrt(x*y*abs(z)^2)/(sqrt(x)*abs(z));
```

```
                          -1/2      1/2
                         x     (x y)
```

Problem 39: do not simplify $\sqrt{e^z} - e^{z/2}$ to 0 without any assumption on the imaginary part of z.

```
>> sqrt(exp(z))-exp(z/2);
```

```
                      / z \          1/2
                - exp | - | + exp(z)
                      \ 2 /
```

[New] **Problem 40:** simplify $\sqrt{e^{6i}}$ into its correct value $-e^{3i}$.

```
>> sqrt(exp(6*I));
```

```
                              -exp(3 I)
```

[New] **Problem 41:** simplify $\log e^z$ into z only for $-\pi < \mathrm{Im} z \leq \pi$.

```
>> simplify(ln(exp(z)));
```

```
                              ln(exp(z))
```

```
>> assume(x,Type::RealNum): ln(exp(x));
```

```
                                   x
```

[New] **Problem 42:** compute the correct principal value of $\log e^{10i}$, namely $(10-4\pi)i$.

```
>> ln(exp(10*I));
```

```
                          (- 4 I) PI + 10 I
```

[New] **Problem 43:** simplify $(xy)^{1/n} - x^{1/n}y^{1/n}$ with respect to the domain of x, y, n.

```
>> e:=(x*y)^(1/n)-x^(1/n)*y^(1/n): simplify(e);
```

```
                              1    1  1
                              -    -  -
                              n    n  n
                          (x y)  - x  y
```

```
>> assume(x>0): simplify(e);

                                    0

>> unassume(x): assume(y>0): simplify(e);

                                    0
```

Here the command `unassume(x)` deletes all assumptions about the variable x.

[New] **Problem 44:** simplify $\tan^{-1}(\tan z)$ into z only for $-\pi/2 < z \leq \pi/2$.

```
>> atan(tan(z));

                                atan(tan(z))

>> assume(x>-PI/2): assume(x<=PI/2): atan(tan(x));

                                    x
```

[New] **Problem 45:** compute the correct value of $\tan^{-1}(\tan 4)$, namely $4 - \pi$.

```
>> atan(tan(4));

                                - PI + 4
```

[New] **Problem 46:** convert $(x = 0)/2 + 1$ to $x/2 + 1 = 1$.

```
>> simplify( (x=0)/2+1, relation );

                                    x
                                    - + 1 = 1
                                    2
```

8 Equations

Problem 47: solve the polynomial equation $3x^3 - 18x^2 + 33x - 19 = 0$.

```
>> s:=solve(3*x^3-18*x^2+33*x-19):
>> simplify(rectform(op(s,i))) $ i=1..3;

   1/2    / PI \                     1/2    / PI \
2 3    cos| -- |                   3    cos| -- |
          \ 18 /           / PI \          \ 18 /
---------------- + 2, - sin| -- | - -------------- + 2,
       3                  \ 18 /         3

                1/2    / PI \
               3    cos| -- |
     / PI \           \ 18 /
  sin| -- | - -------------- + 2
     \ 18 /         3
```

[New] **Problem 48:** solve the equation $x^4 + x^3 + x^2 + x + 1 = 0$.

```
>> p:=x^4+x^3+x^2+x+1: s:=solve(p,x);

{        /    1/2       \1/2               /    1/2       \1/2
```

```
{        |   5          |                  |   5          |
{  1/2   | - ---- - 5/2 |            1/2   | - ---- - 5/2 |
{ 5      \    2         /           5      \    2         /
{ ---- - ------------------ - 1/4, ---- + ------------------ - 1/4,
{  4             2                  4             2

            /  1/2       \1/2                /  1/2       \1/2       }
            | 5          |                   | 5          |          }
      1/2   | ---- - 5/2 |             1/2   | ---- - 5/2 |          }
     5      \  2         /            5      \  2         /          }
   - ---- - ----------------- - 1/4, - ---- + ----------------- - 1/4 }
     4             2                  4             2               }
```

[New] **Problem 49:** verify a solution of the above equation.

```
>> expand(simplify(expand(subs(p,x=s[1]))));

                                    0
```

[New] **Problem 50:**

```
>> solve(exp(2*x)+2*exp(x)+1=z,x);

                                  1/2
{(- 2 I) PI (0, 1, -1, ...) + ln(z    - 1),

                                    1/2
   (- 2 I) PI (0, 1, -1, ...) + ln(- z    - 1)}
```

[New] **Problem 51:** solve the equation $(x+1)(\sin^2 x + 1)^2 \cos^3 3x = 0$.

```
>> solve((x+1)*(sin(x)^2+1)^2*cos(3*x)^3=0,x);

{
{ I asinh(1) + 2 PI (0, 1, -1, ...), (- I) asinh(1) + 2 PI (0, 1, -1, ...), -1,
{

   PI   PI (0, 1, -1, ...)
   -- + ------------------, PI + (- I) asinh(1) + 2 PI (0, 1, -1, ...),
   6            3

                                          }
   PI + I asinh(1) + 2 PI (0, 1, -1, ...) }
                                          }
```

[New] **Problem 52:** solve the equation $e^z = 1$.

```
>> solve(exp(z)=1,z);

                          {(- 2 I) PI (0, 1, -1, ...)}
```

[New] **Problem 53:** solve the equation $\sin x = \cos x$.

```
>> solve(sin(x)=cos(x),x);

          { PI                          3 PI                         }
          { -- + 2 PI (0, 1, -1, ...), - ---- + 2 PI (0, 1, -1, ...) }
          { 4                            4                           }
```

[New] **Problem 54:** solve the equation $\tan x = 1$.

```
>> solve(tan(x)=1,x);

                          { PI                      }
                          { -- + PI (0, 1, -1, ...) }
                          { 4                       }
```

[New] **Problem 55:** solve the equation $\sin x = \tan x$.

```
>> solve(sin(x)=tan(x),x);

                  {PI (0, 1, -1, ...), 2 PI (0, 1, -1, ...)}
```

[New] **Problem 56:** solve the equation $\sqrt{x^2+1} = x - 2$.

```
>> solve(sqrt(x^2+1)=x-2,x);

                                    {}
```

The function `sqrt` in MuPAD represents the principal square root, and the possible solution $x = 3/4$ does not correspond to a principal square root.

[New] **Problem 57:** solve the equation $e^{2-x^2} = e^{-x}$.

```
>> solve(exp(2-x^2)=exp(-x),x);

{                              1/2                                        1/2       }
{ (8 I PI (0, 1, -1, ...) + 9)             (8 I PI (0, 1, -1, ...) + 9)           }
{ ------------------------------ + 1/2, - ------------------------------ + 1/2 }
{               2                                        2                     }

>> assume(x,Type::RealNum):
>> solve(exp(2-x^2)=exp(-x),x);

                                  {-1, 2}
```

The first solution assumes that x represents any complex number, whereas in the second call MuPAD was asked to return only real solutions, which correspond to the index 0 of the general solution.

[New] **Problem 58:** solve the equation $\sqrt{\log x} = \log \sqrt{x}$.

```
>> solve(sqrt(ln(x))=ln(sqrt(x)),x);

                                {1, exp(4)}
```

The above answer is returned without any assumption about x, therefore 1 and e^4 are the only roots in the whole complex plane.

[New] **Problem 59:** solve the equation $|x - 1| = 2$.

```
>> solve(abs(x-1)=2,x);

                                  {-1, 3}
```

Problem 60: solve the linear (dependent) system $x + y + z = 6, 2x + y + 2z = 10, x + 3y + z = 10$.

```
>> linsolve({x+y+z=6,2*x+y+2*z=10,x+3*y+z=10},{x,y,z});

                              [{y = 2, x = - z + 4}]
```

Problem 61: solve the system of nonlinear equations $x^2y + 3yz - 4 = 0, -3x^2z + 2y^2 + 1 = 0, 2yz^2 - z^2 - 1 = 0$.

```
>> s:=solve({ x^2*y + 3*y*z - 4, -3*x^2*z + 2*y^2 + 1, 2*y*z^2 - z^2 - 1});

-- --          2       4
|  |        5 z    21 z        5      6                           2
|  |  y = - ---- - ----- + 24 z  - 9 z  - 1, x = RootOf(19 z + 6 x  - 48 z
-- --        2       2


2      3       4      5       6       7                     2      4
  + 2 z  + 48 z  + 3 z  - 48 z  + 18 z , x), z = RootOf(3 z  + 5 z  + 21 z


                                                        -- --
                        6       7       8              |  |
                          - 48 z  + 18 z  + 1)         |  |
                                                        -- --
```

Here the solution is expressed with the help of the `RootOf` notation. There are 8 solutions, each one corresponding to z being one of the roots of the polynomial $18z^8 - 48z^7 + 21z^6 + 5z^4 + 3z^2 + 1$, y is a polynomial in z as shown above, and x being one root of a degree two polynomial dependent on z.

9 Matrix Algebra

Problem 62: invert the 2×2 matrix $[[a, b], [1, ab]]$.

```
>> export(Dom): export(linalg):
>> M := Matrix( ExpressionField(normal) ):
>> m := M( [[a,b],[1,a*b]] ):
>> 1/m;
```

```
+-                                    -+
|    a             1                  |
|  ------  ,  ---------                |
|   2              2                   |
|  a  - 1      - a  + 1                |
|                                      |
|    1             a                   |
| -------- , ----------- |
|      2             2                 |
| b - a  b   - b + a  b                |
+-                                    -+
```

Here we first export the necessary packages, then we define the domain `M` of matrices whose coefficients are in the domain `ExpressionField(normal)`, which means the set of all MuPAD expressions after simplification by `normal`. We could have written as well `Matrix( ExpressionField )` if we do not want the expressions to be normalized. Please note that no special command is necessary to invert the matrix: we can use the standard arithmetic operators due to the overloading mechanism of MuPAD.

Problem 63: compute the determinant of the 4×4 Vandermonde matrix over w, x, y, z.

```
>> M := Dom::SquareMatrix(4):
>> m := M([[1,1,1,1],[w,x,y,z],[w^2,x^2,y^2,z^2],[w^3,x^3,y^3,z^3]]):
>> Factor(linalg::det(m));
```

```
(w - x) (w - z) (x - z) (x - y) (w - y) (y - z)
```

Problem 64: find the eigenvalues of the matrix $[[5,-3,-7],[-2,1,2],[2,-3,-4]]$.

```
>> m := Dom::SquareMatrix(3)([[5, -3, -7],[-2,  1,  2],[ 2, -3, -4]]):
>> linalg::eigenValues(m);
```

```
                              {1, -2, 3}
```

10 Sums and products

Problem 67: find a closed-form for $\sum_{k=1}^{n} k^3$.

```
>> Factor(sum(k^3,k=1..n));
```

```
                                 2        2
                          1/4 n  (n + 1)
```

Problem 68: find a closed-form for $\sum_{k=1}^{\infty}(1/k^2 + 1/k^3)$.

```
>> sum(1/k^2+1/k^3,k=1..infinity);
```

```
                                    2
                                  PI
                        zeta(3) + ---
                                   6
```

[New] **Problem 69:** evaluate numerically the above sum.

```
>> misc::freeze(sum)(1/k^2+1/k^3,k=1..infinity);
```

```
                        / 1    1                    \
                        | -- + --, k = 1..infinity |
                    sum|  2    3                    |
                        \ k    k                    /
```

```
>> float(%);
```

```
                              2.84699097
```

The function `misc::freeze` enables one to make a MuPAD function like `sum` or `int` inert, but the attributes of the function (here the `float` attribute) are loaded. Here MuPAD uses Euler-Maclaurin formula.

[New] **Problem 70:** compute the product of k from $k = 1$ to n.

```
>> product(k,k=1..n);
```

```
                             gamma(n + 1)
```

The command `rewrite(%,fact)` enables one to rewrite the answer in terms of the factorial function.

11 Calculus

Problem 71: compute the limit of $(1 + 1/n)^n$ when n goes to infinity, and the limit of $(1 - \cos x)/x^2$ when x goes to zero.

```
>> limit((1+1/n)^n,n=infinity);
```

$$E$$

```
>> limit((1-cos(x))/x^2,x=0);
```

$$1/2$$

It should be noted that the limit functions of MuPAD implements the best known algorithm to this date. It is based on the concept of most-rapidly-varying subexpressions (MRV), and a comparison made by Dominik Gruntz on 19 exp-log expressions [3] showed that the algorithms implemented in other computer algebra systems (except Maple and MuPAD) fail on many examples, and even return wrong values !

[New] **Problem 72:** compute the second derivative of $y(x(t))$ with respect to t.

```
>> diff(y(x(t)),t$2);
```

```
                                                    2
D(D(x))(t) D(y)(x(t)) + D(x)(t)  D(D(y))(x(t))
```

12 Indefinite Integrals

Problem 73: integrate $1/(x^3+2)$, then differentiate and simplify.

```
>> int(1/(x^3+2),x);
```

```
                                                  3
sum(X2 ln(x + 6 X2), X2 = RootOf(108 X3  - 1))
```

```
>> diff(%,x);
```

$$\frac{1}{x^3+2}$$

The output from `int` means that an antiderivative of $1/(x^3+2)$ is the sum of the three functions $\alpha_i \ln(x+6\alpha_i)$, where α_i is one of the roots of the polynomial $108X^3-1$. The `sum` expression can be differentiated, and the result is a sum of three rational functions, namely $\alpha_i/(x+6\alpha_i)$, which is automatically normalized. More generally, the sum of any rational function over the roots of a polynomial simplifies to an expression involving the polynomial coefficients. The MuPAD syntax for this is `sum(f,t=RootOf(..))`; for example `sum(t^2,t=RootOf(x^3+a*x^2+b*x+c,x))` returns `a^2-2*b`.

[New] **Problem 74:** compute an antiderivative of $1/(a+b\cos x)$ where $a<b$.

```
>> assume(a^2<b^2):
>> int(1/(a+b*cos(x)),x);
```

```
      2          2         2    2 1/2                2    2 1/2
ln((I a  + (- I) b  + a (- a  + b )    + b cos(x) (- a  + b )    +

                     2    2 1/2              2    2 1/2             2
   (- I) b sin(x) (- a  + b )    )   / b) / (- a  + b )    - ln(((- I) a  +
```

```
   2          2    2 1/2                  2    2 1/2
I b  + a (- a  + b )    + b cos(x) (- a  + b )    +

                     2    2 1/2                2    2 1/2
(- I) b sin(x) (- a  + b )    )   / b) / (- a  + b )
```

[New] **Problem 75:** differentiate the above antiderivative and find again $1/(a + b\cos x)$.

```
>> normal(expand(combine(normal(diff(%,x)),sincos)));
```

$$\frac{1}{a + b\cos(x)}$$

Problem 76: compute the derivative of $|x|$.

```
>> diff(abs(x),x);
```

$$\mathrm{sign}(x)$$

The extension of the functionality of MuPAD by such rules is very easy. For example to "teach" to MuPAD that the derivative of $\sin(f)$ with respect to x is $\cos(f)f'$, we simply write:

```
>> sin := funcattr(sin,"diff",proc(f,x) begin cos(f)*diff(f,x) end_proc);
```

Problem 77: compute an antiderivative of $|x|$.

```
>> int(abs(x),x);
```

$$\frac{x\,\mathrm{abs}(x)}{2}$$

[New] **Problem 78:** compute the derivative of $|x|$ (piecewise defined).

```
>> a:=proc(x) begin if x < 0 then -x else x end_if end_proc:
>> D(a);
```

```
proc(x)
  name a;
begin
  if x < 0 then
    -1
  else
    1
  end_if
end_proc
```

[New] **Problem 79:** compute the anderivative of $|x|$ (piecewise defined).

```
>> a:=proc(x) begin if x < 0 then -x else x end_if end_proc:
>> int(a);
```

```
proc(x)
  name a;
begin
  if x < 0 then
    x^2*(-1/2)
  else
    x^2*1/2
  end_if
end_proc
```

Problem 80: compute an antiderivative of $x/(\sqrt{1+x}+\sqrt{1-x})$.

```
>> int(x/(sqrt(1 + x) + sqrt(1 - x)), x);
```

```
       / ln(x + 1) \ / x       \       / ln(- x + 1 ) \ /   x        \
   exp | --------- | | - + 1/3 | + exp | ------------ | | - - + 1/3  |
       \     2     / \ 3       /       \      2       / \   3        /
```

The result can be simplified using the `expand` command.

Problem 81: compute an antiderivative of $(\sqrt{1+x}-\sqrt{1-x})/2$, which is the same as above.

```
>> int((sqrt(1 + x) - sqrt(1 - x))/2, x);
```

```
       / ln(x + 1) \ / x       \       / ln(- x + 1 ) \ /   x        \
   exp | --------- | | - + 1/3 | + exp | ------------ | | - - + 1/3  |
       \     2     / \ 3       /       \      2       / \   3        /
```

13 Definite Integrals

[New] **Problem 82:** compute the Cauchy principal value of the integral $\int_{a-1}^{a+1} 1/(x-a)dx$.

```
>> int(1/(x-a),x=a-1..a+1), int(1/(x-a),x=a-1..a+1,PrincipalValue);
```

```
                          undefined, 0
```

[New] **Problem 83:** check that the integral $\int_{a-1}^{a+1} 1/(x-a)^2 dx$ is divergent.

```
>> int(1/(x-a)^2,x=a-1..a+1);
```

```
                            infinity
```

[New] **Problem 84:** compute the integral of $\sqrt{x+1/x-2}$ from 0 to 1.

```
>> int(sqrt(x+1/x-2),x=0..1);
```

```
                              4/3
```

[New] **Problem 85:** compute the integral of $\sqrt{x+1/x-2}$ from 1 to 2.

```
>> int(sqrt(x+1/x-2),x=1..2);
```

```
                     (1/2)
                  2 2
                - -------- + 4/3
                     3
```

[New] **Problem 86:** compute the integral of $\sqrt{x+1/x-2}$ from 0 to 2.

```
>> int(sqrt(x+1/x-2),x=0..2);
```

```
                     (1/2)
                  2 2
                - -------- + 8/3
                     3
```

[New] **Problem 87:** compute the integral $\int_{-\infty}^{\infty} \cos x/(x^2+a^2)dx$ for $a > 0$.

```
>> sign(a):=1: int(cos(x)/(x^2+a^2),x=-infinity..infinity);
```

$$\frac{\pi\, e^{-a}}{a}$$

[New] **Problem 88:** compute the integral $\int_0^\infty t^{a-1}/(1+t)dt$ for $0 \le \mathrm{Re}(a) \le 1$.

```
>> assume(Re(a)>0): assume(Re(a)<1):
>> int(t^(a-1)/(1+t),t=0..infinity);
```

$$\Gamma(a)\,\Gamma(1-a)$$

This result equals $\pi/\sin(\pi a)$ due to the reflection formula $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$.

Problem 90: compute the integral $\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} 1\,dz\,dy\,dz$.

```
>> int(int(int(1,z=0..c*(1-x/a-y/b)),y=0..b*(1-x/a)),x=0..a);
```

$$\frac{a\,b\,c}{6}$$

14 Series

Problem 91: compute the first terms of the Taylor expansion of $1/\sqrt{1-v^2/c^2}$ at $v = 0$.

```
>> series(1/sqrt(1-(v/c)^2),v=0);
```

$$1 + \frac{c^{-2}\,v^2}{2} + \frac{3\,c^{-4}\,v^4}{8} + O(v^6)$$

Problem 92: compute the inverse of the square of the above expansion.

```
>> 1/%^2;
```

$$1 - c^{-2}\,v^2 + O(v^6)$$

This is possible in MuPAD because of the domain facility, and the fact that standard arithmetic operations can be overloaded for domain elements. This allows to get $O(x)$ as the result of $O(x) - O(x)$, whereas for example Maple gives 0.

Problem 93: Compute the Taylor expansion of $\tan(x)$ at $x = 0$ by dividing the expansion of $\sin(x)$ by that of $\cos(x)$.

```
>> series(sin(x),x)/series(cos(x),x)=series(tan(x),x);
```

$$x + \frac{x^3}{3} + \frac{2\,x^5}{15} + O(x^6) = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + O(x^6)$$

```
>> bool(%);
```

TRUE

Here again, the results of a `series` command can be combined with any arithmetic operation.

Problem 95: compute the expansion of $\log\sinh(z) + \log\cosh(z+w)$ at $z = 0$.

```
>> series(ln(sinh(z))+ln(cosh(z+w)),z,3);
```

```
                              z sinh(w)     /  2 \
ln(z) + ln(cosh(w)) + --------- + O \ z  /
                       cosh(w)
```

Problem 96: subtract from the above expansion that of $\log(\sinh(z)\cosh(z+w))$.

```
>> %-series(ln(sinh(z)*cosh(z+w)),z,3);
```

```
                                 /  2 \
                               O \ z  /
```

Problem 97: compute the expansion of $\log(\sin x/x)$ at $x = 0$.

```
>> series(ln(sin(x)/x),x);
```

```
                                 2    4
                                x    x        5
                              - -- - --- + O(x )
                                6    180
```

[New] **Problem 99:** get a closed-form for the nth coefficient in the Taylor expansion of $e^{-x}\sin x$ at $x = 0$.

```
>> series(exp(-x)*sin(x),x,infinity);
```

```
                  /  n   n 1/2    / 3 n PI \                  \
                  | x  (2 )    sin| ------ |                  |
                sum|               \   4    /                  |
                  | ------------------------, n = 0..infinity |
                  \         fact(n)                            /
```

Problem 100: revert the Taylor expansion of $\sin y + \cos y$ at $y = 0$.

```
>> s:=series(sin(y)+cos(y),y=0);
```

```
                                2    3    4    5
                               y    y    y    y        6
                     1 + y - -- - -- + -- + --- + O(y )
                               2    6    24   120
```

```
>> t:=revert(s);
```

```
                    2            3                          5
             (y - 1)    2 (y - 1)           4   17 (y - 1)             6
  (y - 1) + -------- + ---------- + (y - 1)  + ----------- + O((y - 1) )
               2           3                        10
```

We can check the result by asking for `s @ t` that computes the composition of `s` and `t`.

15 Transforms

[New] **Problem 101:** compute the Padé approximation to order 2 of e^{-x} at $x = 0$.

```
>> Dom::Pade(exp(-x),x,3);
```

$$\frac{-x+2}{x+2}$$

The above fraction has the same Taylor expansion as e^{-x} up to order 2. The order given to `Pade` is the order of the error term, like in the command `series`, therefore one more than the wanted order.

[New] **Problem 102:** compute the Laplace transform of $\cos((\omega-1)t)$ with respect to t.

```
>> laplace(cos((w-1)*t),t,s);
```

$$\frac{s}{s^2+(w-1)^2}$$

[New] **Problem 103:** compute the inverse Laplace transform of the above result.

```
>> ilaplace(%,s,t);
```

$$\cos(t\ (w-1))$$

16 Recurrences and Differential Equations

[New] **Problem 104:** solve the recurrence $r_{n+2} - 2r_{n+1} + r_n = 2$ with initial conditions $r_0 = -1$ and $r_1 = m$.

```
>> solve(rec(r(n+2)-2*r(n+1)+r(n)=2,r(n),{r(0)=-1,r(1)=m}));
```

$$\{m\ n + n^2 - 1\}$$

Like for differential equations (see below), the `rec` command only defines the recurrence, then `solve` actually solves it.

Problem 105: solve the differential equation $f''(t)+4f(t) = \sin(2t)$ with the initial conditions $f(0) = f'(0) = 0$.

```
>> ode({diff(f(t),t,t)+4*f(t)=sin(2*t),f(0)=0,D(f)(0)=0},f(t));

 ode({D(f)(0) = 0, f(0) = 0, 4 f(t) + diff(f(t), t, t) = sin(2 t)}, f(t))

>> solve(%);
```

$$\left\{\frac{\sin(2\ t)}{8} - \frac{t\ \cos(2\ t)}{4}\right\}$$

Here we first define the equation with the `ode` command, and simply call `solve` on this equation.

[New] **Problem 106:** solve the above equation using Laplace transforms.

```
>> laplace( diff(f(t),t,t)+4*f(t)=sin(2*t), t, s);

                                                                    2
4 laplace(f(t), t, s) - D(f)(0) + s (s laplace(f(t), t, s) - f(0)) = ------
                                                                     2
                                                                    s  + 4

>> solve( subs(%,f(0)=0,D(f)(0)=0), laplace(f(t), t, s));

                              --     2     --
                              |  ---------  |
                              |    2     2  |
                              -- (s  + 4)  --

>> ilaplace(op(%),s,t);

                          / sin(2 t)   t cos(2 t) \
                        2 | -------- - ---------- |
                          \    16          8      /
```

Problem 107: solve the equation $x^2y'(x) + 3xy(x) = \sin x/x$.

```
>> solve(ode(x^2*diff(y(x),x)+3*x*y(x)=sin(x)/x,y(x)));

                          --     - 3    - 3        --
                          |  C3 x    - x    cos(x)  |
                          --                       --
```

Problem 108: solve the equation $y'' + y{y'}^3 = 0$ for $y(x)$.

```
>> solve(ode(diff(y(x),x,x)+y(x)*diff(y(x),x)^3=0,y(x)));

                --            /                          3    \ --
                |  C5, RootOf \ 6 x + 6 C6 + 6 C4 y - y , y /  |
                --                                             --
```

This result says that the solution is either a constant function $y(x) = C_5$, or an algebraic function satisfying the equation $y(x)^3 = 6C_4y(x) + 6x + 6C_6$ where C_4 and C_6 are arbitrary constants.

Problem 109: solve the equation $\frac{d}{dx}y(x, a) = ay(x, a)$.

```
>> solve(ode(diff(y(x,a),x)=a*y(x,a),y(x,a)));

                              [C1 exp(a x)]
```

MuPAD recognizes equations with only one differential variable as ordinary differential equations.

[New] **Problem 110:** solve the equation $y''(x) + k^2y(x) = 0$ with the initial conditions $y(0) = 0$ and $y'(1) = 0$.

```
>> solve(ode({diff(y(x),x,x)+k^2*y(x),y(0)=0,D(y)(1)=0},y(x)));

                                    {0}

>> solve(ode({diff(y(x),x,x)+(PI/2)^2*y(x),y(0)=0,D(y)(1)=0},y(x)));

            {- C19 exp(1/2 I x PI) + C19 exp((- 1/2 I) x PI)}
```

The general equation has no solution valid for all values of k. For k of the form $\pi/2 + n\pi$ where n is an integer, it has solutions that MuPAD finds.

Problem 111: solve the system $x'(t) = x(t) - y(t), y'(t) = x(t) + y(t)$.

```
>> sys := {diff(x(t),t)-x(t)+y(t)=0,diff(y(t),t)-x(t)-y(t)=0}:
>> solve(ode(sys, {x(t),y(t)}));

 [x(t) = (I) C5 exp((1 + I) t) + (- I) C6 exp((1 - I) t), y(t) = C5 exp((

                     1 + I) t) + C6 exp((1 - I) t)]
```

Problem 112: verify the above is a solution.

```
>> eval(subs([op(sys)],op(%)));

                              [0 = 0, 0 = 0]
```

[New] **Problem 113:** solve the triangular system $x'(t) = x(t)(1 + \frac{\cos t}{2+\sin t}), y'(t) = x(t) - y(t)$.

```
>> solve(ode({diff(x(t),t)=x(t)*(1+cos(t)/(2+sin(t))),
&>      diff(y(t),t)=x(t)-y(t)},{x(t),y(t)}));

{ {                                C1 cos(t) exp(t)   2 C1 sin(t) exp(t)
{ { y(t) = C1 exp(t) + C2 exp(-t) - ---------------- + ------------------,
{ {                                       5                  5

                                } }
x(t) = C1 exp(t) (sin(t) + 2) } }
                                } }
```

[New] **Problem 114:** solve the above system, one equation at a time.

```
>> solve(ode(diff(x(t),t)=x(t)*(1+cos(t)/(2+sin(t))),x(t)));

                        {C1 exp(t) (sin(t) + 2)}

>> solve(ode(diff(y(t),t)=op(%)-y(t),y(t)));

    {                          C1 cos(t) exp(t)   2 C1 sin(t) exp(t) }
    { C1 exp(t) + C2 exp(-t) - ---------------- + ------------------ }
    {                                5                  5          }
```

17 Operators

Problem 115: define the operator $L = (D-1)(D+2)$ where D is the differentiation operator.

```
>> L := (D-id) @ (D+2*id);

                          (- id + D )@(2 id + D)
```

In MuPAD, `id` stand for the identity, and `@` is the composition operator.

Problem 116: compute $L(f)$ where L is the above defined operator.

```
>> L(f);

                          - 2 f + D(f) + D(D(f))
```

Problem 117: compute $L(g)(y)$ where L is the above defined operator.

```
>> L(g)(y);

                      - 2 g(y) + D(g)(y) + D(D(g))(y)
```

Problem 118: apply the operator L to the function $z \to A \sin z^2$.

```
>> h := func(A*sin(z^2),z):
>> L(h)(z);
```

```
                2             2               2         2      2
  2 A cos(z ) - 2 A sin(z ) + 2 A z cos(z ) - 4 A z  sin(z )
```

Problem 119: define the operator T such that $T(f)(x,a) = f(a) + f'(x)(x-a) + f''(a)(x-a)^2/2$.

```
>> T:=proc(f) begin
&>    eval(subsop(hold(func(f,x,a)),
&>      1=_plus(f(a),_fconcat(D$k)(f)(a)/fact(k)*(x-a)^k$k=1..2)))
&> end_proc:
```

Problem 120: evaluate T for an unknown function f.

```
>> T(f);
```

```
     func(f(a) + (-a + x)*D(f)(a) + (-a + x)^2*D(D(f))(a)*1/2, x, a)
```

Problem 121: evaluate T for an unknown function g and a generic point (y,b).

```
>> T(g)(y,b);
```

```
                                                          2
                                              D(D(g))(b) (- b + y )
          g(b) + D(g)(b) (- b + y ) + ----------------------
                                                  2
```

Problem 122: evaluate T for the function sin and a point (z,c).

```
>> T(sin)(z,c);
```

```
                                                        2
                                            sin(c) (- c + z )
          sin(c) + cos(c) (- c + z ) - ------------------
                                              2
```

18 Programming

Problem 123: compute the Legendre polynomials directly.

```
>> p:= proc(n, x) begin
    if n = 0 then 1
    else normal(1/(2^n*fact(n)) * diff((x^2 - 1)^n, x$n))
    end_if
    end_proc:
>> p(i,x) $ i=0..4;
```

```
                2                   3         2       4
             3 x           3 x   5 x      15 x    35 x
       1, x, ---- - 1/2, - --- + ---- , - ----- + ----- + 3/8
              2             2     2         4       8
```

Problem 124: compute the Legendre polynomials recursively, using their recurrence of order 2.

```
>> pp:= proc(n, x) option remember; begin
     if n=0 then 1
     elif n=1 then x
     else normal(((2*n - 1)*x*pp(n - 1, x) - (n - 1)*pp(n - 2, x))/n)
     end_if
     end_proc:
>> pp(i,x) $ i=0..4;
```

$$1,\ x,\ \frac{3x^2}{2} - 1/2,\ -\frac{3x^3}{2} + \frac{5x}{2},\ -\frac{15x^2}{4} + \frac{35x^4}{8} + 3/8$$

Problem 125: evaluate the fourth Legendre polynomial at 1.

```
>> pp(4,1);
```

$$1$$

The Legendre polynomials are already defined in the `orthpoly` package, whence one can write `orthpoly::legendre(4,1)` too.

19 Translation

Problem 126: define the polynomial $p = \sum_{i=1}^{5} a_i x^i$.

```
>> p:= poly(_plus(a[i] * x^i $ i=1..5), [x]);
```

$$\mathtt{poly}(x\ a_1 + x^2\ a_2 + x^3\ a_3 + x^4\ a_4 + x^5\ a_5, [x])$$

Problem 127: apply Horner's rule to the above polynomial.

```
>> p(x);
```

$$x\ (a_1 + x\ (a_2 + x\ (a_3 + x\ (a_4 + x\ a_5))))$$

[New] **Problem 128:** convert the above result to Fortran syntax.

```
>> generate::fortran(hold(p)=%);
      p = x*(a(1)+x*(a(2)+x*(a(3)+x*(a(4)+x*a(5)))))
```

20 Boolean Logic

Problem 129: evaluate `TRUE and FALSE`.

```
>> TRUE and FALSE;
```

$$\mathtt{FALSE}$$

Problem 130: simplify `x or (not x)`.

```
>> x or (not x);
```

$$\mathtt{TRUE}$$

[New] **Problem 131:** simplify the boolean expression `x or y or (x and y)`.

```
>> simplify(x or y or (x and y), logic);

                                x or y
```

Number of solved problems (either directly or indirectly): 122 over 131 (52 new problems with respect to MuPAD 1.2.2). As a comparison, Reduce can solve 65 problems, Axiom 75 problems, Derive 88 problems, Mathematica 102 problems, Maple 110 problems and PC Macsyma 2.1 122 problems, but none of these systems is public domain ! You can get MuPAD by anonymous ftp from `ftp.uni-paderborn.de` (main site) or in France from `ftp.inria.fr`.

Acknowledgement. Many thanks to Michael Wester for his very detailed comments on a first version of this paper.

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