CHAPTER

FIVE

GOVERNING EQUATIONS OF FLUID MECHANICS AND HEAT TRANSFER

In this chapter, the governing equations of fluid mechanics and heat transfer (i.e., fluid dynamics) are described. Since the reader is assumed to have some background in this field, a complete derivation of the governing equations is not included. The equations are presented in order of decreasing complexity. For the most part, only the classical forms of the equations are given. Other forms of the governing equations, which have been simplified primarily for computational purposes, are presented in later chapters. Also included in this chapter is an introduction to turbulence modeling.

5.1 FUNDAMENTAL EQUATIONS

The fundamental equations of fluid dynamics are based on the following universal laws of conservation:

Conservation of Mass
Conservation of Momentum
Conservation of Energy

The equation that results from applying the Conservation of Mass law to a fluid flow is called the continuity equation. The Conservation of Momentum law is nothing more than Newton's Second Law. When this law is applied to a fluid flow, it yields a vector equation known as the momentum equation. The

Conservation of Energy law is identical to the First Law of Thermodynamics, and the resulting fluid dynamic equation is named the energy equation. In addition to the equations developed from these universal laws, it is necessary to establish relationships between fluid properties in order to close the system of equations. An example of such a relationship is the equation of state, which relates the thermodynamic variables pressure p, density ρ, and temperature T.

Historically, there have been two different approaches taken to derive the equations of fluid dynamics: the phenomenological approach and the kinetic theory approach. In the phenomenological approach, certain relations between stress and rate of strain and heat flux and temperature gradient are postulated, and the fluid dynamic equations are then developed from the conservation laws. The required constants of proportionality between stress and rate of strain and heat flux and temperature gradient (which are called transport coefficients) must be determined experimentally in this approach. In the kinetic theory approach (also called the mathematical theory of nonuniform gases), the fluid dynamic equations are obtained with the transport coefficients defined in terms of certain integral relations, which involve the dynamics of colliding particles. The drawback to this approach is that the interparticle forces must be specified in order to evaluate the collision integrals. Thus a mathematical uncertainty takes the place of the experimental uncertainty of the phenomenological approach. These two approaches will yield the same fluid dynamic equations if equivalent assumptions are made during their derivations.

The derivation of the fundamental equations of fluid dynamics will not be presented here. The derivation of the equations using the phenomenological approach is thoroughly treated by Schlichting (1968), and the kinetic theory approach is described in detail by Hirschfelder et al. (1954). The fundamental equations given initially in this chapter were derived for a uniform, homogeneous fluid without mass diffusion or finite-rate chemical reactions. In order to include these later effects it is necessary to consider extra relations, called the species continuity equations, and to add terms to the energy equation to account for diffusion. These additional equations and terms are given in Section 5.1.5. Further information on reacting flows can be found in the works by Dorrance (1962) and Anderson (1989).

5.1.1 Continuity Equation

The Conservation of Mass law applied to a fluid passing through an infinitesimal, fixed control volume (see Fig. 5.1) yields the following equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{5.1}$$

where ρ is the fluid density and $\mathbf{V}$ is the fluid velocity. The first term in this equation represents the rate of increase of the density in the control volume, and the second term represents the rate of mass flux passing out of the control surface (which surrounds the control volume) per unit volume. It is convenient

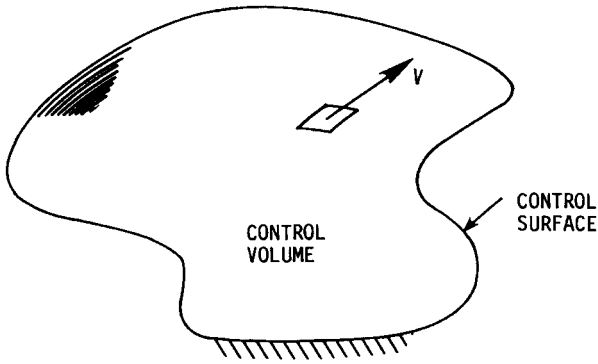


Figure 5.1 Control volume for Eulerian approach.

to use the substantial derivative

$$\frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + \mathbf{V} \cdot \nabla(\ ) \tag{5.2}$$

to change Eq. (5.1) into the form

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{V}) = 0 \tag{5.3}$$

Equation (5.1) was derived using the *Eulerian approach.* In this approach, a fixed control volume is utilized, and the changes to the fluid are recorded as the fluid passes through the control volume. In the alternative *Lagrangian approach,* the changes to the properties of a fluid element are recorded by an observer moving with the fluid element. The Eulerian viewpoint is commonly used in fluid mechanics.

For a Cartesian coordinate system, where u, v, w represent the x, y, z components of the velocity vector, Eq. (5.1) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{5.4}$$

Note that this equation is in conservation-law (divergence) form.

A flow in which the density of each fluid element remains constant is called *incompressible.* Mathematically, this implies that

$$\frac{D\rho}{Dt} = 0 \tag{5.5}$$

which reduces Eq. (5.3) to

$$\nabla \cdot \mathbf{V} = 0 \tag{5.6}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.7}$$

for the Cartesian coordinate system. For steady air flows with speed $V < 100$ m/s or $M < 0.3$ the assumption of incompressibility is a good approximation.

5.1.2 Momentum Equation

Newton's Second Law applied to a fluid passing through an infinitesimal, fixed control volume yields the following momentum equation:

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \nabla \cdot \rho \mathbf{V}\mathbf{V} = \rho \mathbf{f} + \nabla \cdot \mathbf{\Pi}_{ij} \tag{5.8}$$

The first term in this equation represents the rate of increase of momentum per unit volume in the control volume. The second term represents the rate of momentum lost by convection (per unit volume) through the control surface. Note that $\rho \mathbf{V}\mathbf{V}$ is a tensor, so that $\nabla \cdot \rho \mathbf{V}\mathbf{V}$ is not a simple divergence. This term can be expanded, however, as

$$\nabla \cdot \rho \mathbf{V}\mathbf{V} = \rho \mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{V}(\nabla \cdot \rho \mathbf{V}) \tag{5.9}$$

When this expression for $\nabla \cdot \rho \mathbf{V}\mathbf{V}$ is substituted into Eq. (5.8), and the resulting equation is simplified using the continuity equation, the momentum equation reduces to

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{\Pi}_{ij} \tag{5.10}$$

The first term on the right-hand side of Eq. (5.10) is the body force per unit volume. Body forces act at a distance and apply to the entire mass of the fluid. The most common body force is the gravitational force. In this case, the force per unit mass ($\mathbf{f}$) equals the acceleration of gravity vector $\mathbf{g}$:

$$\rho \mathbf{f} = \rho \mathbf{g} \tag{5.11}$$

The second term on the right-hand side of Eq. (5.10) represents the surface forces per unit volume. These forces are applied by the external stresses on the fluid element. The stresses consist of normal stresses and shearing stresses and are represented by the components of the stress tensor $\mathbf{\Pi}_{ij}$.

The momentum equation given above is quite general and is applicable to both continuum and noncontinuum flows. It is only when approximate expressions are inserted for the shear-stress tensor that Eq. (5.8) loses its generality. For all gases that can be treated as a continuum, and most liquids, it has been observed that the stress at a point is linearly dependent on the rates of strain (deformation) of the fluid. A fluid that behaves in this manner is called a *Newtonian fluid*. With this assumption, it is possible to derive (Schlichting, 1968) a general deformation law that relates the stress tensor to the pressure and

velocity components. In compact tensor notation, this relation becomes

$$\Pi_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\mu'\frac{\partial u_k}{\partial x_k} \qquad i,j,k = 1,2,3 \quad (5.12)$$

where δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$); u_1, u_2, u_3 represent the three components of the velocity vector $\mathbf{V}$; x_1, x_2, x_3 represent the three components of the position vector; μ is the coefficient of viscosity (dynamic viscosity), and μ' is the second coefficient of viscosity. The two coefficients of viscosity are related to the coefficient of bulk viscosity κ by the expression

$$\kappa = \tfrac{2}{3}\mu + \mu' \quad (5.13)$$

In general, it is believed that κ is negligible except in the study of the structure of shock waves and in the absorption and attenuation of acoustic waves. For this reason, we will ignore bulk viscosity for the remainder of the text. With $\kappa = 0$, the second coefficient of viscosity becomes

$$\mu' = -\tfrac{2}{3}\mu \quad (5.14)$$

and the stress tensor may be written as

$$\Pi_{ij} = -p\delta_{ij} + \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] \qquad i,j,k = 1,2,3 \quad (5.15)$$

The stress tensor is frequently separated in the following manner:

$$\Pi_{ij} = -p\delta_{ij} + \tau_{ij} \quad (5.16)$$

where τ_{ij} represents the viscous stress tensor given by

$$\tau_{ij} = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] \qquad i,j,k = 1,2,3 \quad (5.17)$$

Upon substituting Eq. (5.15) into Eq. (5.10), the famous *Navier-Stokes equation* is obtained:

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\mu\frac{\partial u_k}{\partial x_k}\right] \quad (5.18)$$

For a Cartesian coordinate system, Eq. (5.18) can be separated into the following three scalar Navier-Stokes equations:

$$\rho\frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \tag{5.19}$$

Utilizing Eq. (5.8), these equations can be rewritten in conservation-law form as

$$\begin{aligned}
&\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p - \tau_{xx}) + \frac{\partial}{\partial y}(\rho uv - \tau_{xy}) + \frac{\partial}{\partial z}(\rho uw - \tau_{xz}) = \rho f_x \\
&\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho uv - \tau_{xy}) + \frac{\partial}{\partial y}(\rho v^2 + p - \tau_{yy}) + \frac{\partial}{\partial z}(\rho vw - \tau_{yz}) = \rho f_y \\
&\frac{\partial \rho w}{\partial t} + \frac{\partial}{\partial x}(\rho uw - \tau_{xz}) + \frac{\partial}{\partial y}(\rho vw - \tau_{yz}) + \frac{\partial}{\partial z}(\rho w^2 + p - \tau_{zz}) = \rho f_z
\end{aligned} \tag{5.20}$$

where the components of the viscous stress tensor τ_{ij} are given by

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)$$

$$\tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)$$

$$\tau_{zz} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \tau_{zx}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

The Navier-Stokes equations form the basis upon which the entire science of viscous flow theory has been developed. Strictly speaking, the term Navier-Stokes equations refers to the components of the viscous momentum equation [Eq. (5.18)]. However, it is common practice to include the continuity equation and the energy equation in the set of equations referred to as the Navier-Stokes equations.

If the flow is incompressible and the coefficient of viscosity (μ) is assumed constant, Eq. (5.18) will reduce to the much simpler form

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + \mu\nabla^2\mathbf{V} \tag{5.21}$$

It should be remembered that Eq. (5.21) is derived by assuming a constant viscosity, which may be a poor approximation for the nonisothermal flow of a liquid whose viscosity is highly temperature dependent. On the other hand, the viscosity of gases is only moderately temperature dependent, and Eq. (5.21) is a good approximation for the incompressible flow of a gas.

5.1.3 Energy Equation

The First Law of Thermodynamics applied to a fluid passing through an infinitesimal, fixed control volume yields the following energy equation:

$$\frac{\partial E_t}{\partial t} + \nabla \cdot E_t\mathbf{V} = \frac{\partial Q}{\partial t} - \nabla \cdot \mathbf{q} + \rho\mathbf{f} \cdot \mathbf{V} + \nabla \cdot (\mathbf{\Pi}_{ij} \cdot \mathbf{V}) \tag{5.22}$$

where E_t is the total energy per unit volume given by

$$E_t = \rho\left(e + \frac{V^2}{2} + \text{potential energy} + \cdots\right) \tag{5.23}$$

and e is the internal energy per unit mass. The first term on the left-hand side of Eq. (5.22) represents the rate of increase of E_t in the control volume, while the second term represents the rate of total energy lost by convection (per unit volume) through the control surface. The first term on the right-hand side of Eq. (5.22) is the rate of heat produced per unit volume by external agencies, while the second term ($\nabla \cdot \mathbf{q}$) is the rate of heat lost by conduction (per unit volume) through the control surface. Fourier's law for heat transfer by conduction will be assumed, so that the heat transfer $\mathbf{q}$ can be expressed as

$$\mathbf{q} = -k \nabla T \tag{5.24}$$

where k is the coefficient of thermal conductivity and T is the temperature. The third term on the right-hand side of Eq. (5.22) represents the work done on the control volume (per unit volume) by the body forces, while the fourth term represents the work done on the control volume (per unit volume) by the surface forces. It should be obvious that Eq. (5.22) is simply the First Law of Thermodynamics applied to the control volume. That is, the increase of energy in the system is equal to heat added to the system plus the work done on the system.

For a Cartesian coordinate system, Eq. (5.22) becomes

$$\begin{aligned}\frac{\partial E_t}{\partial t} - \frac{\partial Q}{\partial t} - \rho(f_x u + f_y v + f_z w) + \frac{\partial}{\partial x}(E_t u + pu - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} + q_x) \\ + \frac{\partial}{\partial y}(E_t v + pv - u\tau_{xy} - v\tau_{yy} - w\tau_{yz} + q_y) \\ + \frac{\partial}{\partial z}(E_t w + pw - u\tau_{xz} - v\tau_{yz} - w\tau_{zz} + q_z) = 0\end{aligned} \tag{5.25}$$

which is in conservation-law form. Using the continuity equation, the left-hand side of Eq. (5.22) can be replaced by the following expression:

$$\rho \frac{D(E_t/\rho)}{Dt} = \frac{\partial E_t}{\partial t} + \nabla \cdot E_t \mathbf{V} \tag{5.26}$$

which is equivalent to

$$\rho \frac{D(E_t/\rho)}{Dt} = \rho \frac{De}{Dt} + \rho \frac{D(V^2/2)}{Dt} \tag{5.27}$$

if only internal energy and kinetic energy are considered significant in Eq. (5.23). Forming the scalar dot product of Eq. (5.10) with the velocity vector $\mathbf{V}$

allows one to obtain

$$\rho \frac{D\mathbf{V}}{Dt} \cdot \mathbf{V} = \rho \mathbf{f} \cdot \mathbf{V} - \nabla p \cdot \mathbf{V} + (\nabla \cdot \tau_{ij}) \cdot \mathbf{V} \tag{5.28}$$

Now if Eqs. (5.26), (5.27), and (5.28) are combined and substituted into Eq. (5.22), a useful variation of the original energy equation is obtained:

$$\rho \frac{De}{Dt} + p(\nabla \cdot \mathbf{V}) = \frac{\partial Q}{\partial t} - \nabla \cdot \mathbf{q} + \nabla \cdot (\tau_{ij} \cdot \mathbf{V}) - (\nabla \cdot \tau_{ij}) \cdot \mathbf{V} \tag{5.29}$$

The last two terms in this equation can be combined into a single term, since

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = \nabla \cdot (\tau_{ij} \cdot \mathbf{V}) - (\nabla \cdot \tau_{ij}) \cdot \mathbf{V} \tag{5.30}$$

This term is customarily called the *dissipation function* Φ and represents the rate at which mechanical energy is expended in the process of deformation of the fluid due to viscosity. After inserting the dissipation function, Eq. (5.29) becomes

$$\rho \frac{De}{Dt} + p(\nabla \cdot \mathbf{V}) = \frac{\partial Q}{\partial t} - \nabla \cdot \mathbf{q} + \Phi \tag{5.31}$$

Using the definition of enthalpy,

$$h = e + \frac{p}{\rho} \tag{5.32}$$

and the continuity equation, Eq. (5.31) can be rewritten as

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \frac{\partial Q}{\partial t} - \nabla \cdot \mathbf{q} + \Phi \tag{5.33}$$

For a Cartesian coordinate system, the dissipation function, which is always positive if $\mu' = -(2/3)\mu$, becomes

$$\Phi = \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \right] \tag{5.34}$$

If the flow is incompressible, and if the coefficient of thermal conductivity is assumed constant, Eq. (5.31) reduces to

$$\rho \frac{De}{Dt} = \frac{\partial Q}{\partial t} + k \nabla^2 T + \Phi \tag{5.35}$$

5.1.4 Equation of State

In order to close the system of fluid dynamic equations it is necessary to establish relations between the thermodynamic variables (p, ρ, T, e, h) as well as to relate the transport properties (μ, k) to the thermodynamic variables. For

example, consider a compressible flow without external heat addition or body forces and use Eq. (5.4) for the continuity equation, Eqs. (5.19) for the three momentum equations, and Eq. (5.25) for the energy equation. These five scalar equations contain seven unknowns ρ, p, e, T, u, v, w, provided that the transport coefficients μ, k can be related to the thermodynamic properties in the list of unknowns. It is obvious that two additional equations are required to close the system. These two additional equations can be obtained by determining relations that exist between the thermodynamic variables. Relations of this type are known as equations of state. According to the *state principle* of thermodynamics, the local thermodynamic state is fixed by any two independent thermodynamic variables, provided that the chemical composition of the fluid is not changing owing to diffusion or finite-rate chemical reactions. Thus for the present example, if we choose e and ρ as the two independent variables, then equations of state of the form

$$p = p(e, \rho) \qquad T = T(e, \rho) \tag{5.36}$$

are required.

For most problems in gas dynamics, it is possible to assume a *perfect gas*. A perfect gas is defined as a gas whose intermolecular forces are negligible. A perfect gas obeys the perfect gas equation of state,

$$p = \rho RT \tag{5.37}$$

where R is the gas constant. The intermolecular forces become important under conditions of high pressure and relatively low temperature. For these conditions, the gas no longer obeys the perfect gas equation of state, and an alternative equation of state must be used. An example is the Van der Waals equation of state,

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) = RT$$

where a and b are constants for each type of gas.

For problems involving a perfect gas at relatively low temperatures, it is possible to also assume a *calorically perfect gas*. A calorically perfect gas is defined as a perfect gas with constant specific heats. In a calorically perfect gas, the specific heat at constant volume c_v, the specific heat at constant pressure c_p, and the ratio of specific heats γ all remain constant, and the following relations exist:

$$e = c_v T \qquad h = c_p T \qquad \gamma = \frac{c_p}{c_v} \qquad c_v = \frac{R}{\gamma - 1} \qquad c_p = \frac{\gamma R}{\gamma - 1}$$

For air at standard conditions, $R = 287\ \text{m}^2/(\text{s}^2\ \text{K})$ and $\gamma = 1.4$. If we assume that the fluid in our example is a calorically perfect gas, then Eqs. (5.36) become

$$p = (\gamma - 1)\rho e \qquad T = \frac{(\gamma - 1)e}{R} \tag{5.38}$$

For fluids that cannot be considered calorically perfect, the required state relations can be found in the form of tables, charts, or curve fits.

The coefficients of viscosity and thermal conductivity can be related to the thermodynamic variables using kinetic theory. For example, Sutherland's formulas for viscosity and thermal conductivity are given by

$$\mu = C_1 \frac{T^{\frac{3}{2}}}{T + C_2} \qquad k = C_3 \frac{T^{\frac{3}{2}}}{T + C_4}$$

where C_1–C_4 are constants for a given gas. For air at moderate temperatures, $C_1 = 1.458 \times 10^{-6}$ kg/(m s $\sqrt{\text{K}}$), $C_2 = 110.4$ K, $C_3 = 2.495 \times 10^{-3}$ (kg m)/(s^{3}K$^{\frac{3}{2}}$), and $C_4 = 194$ K. The Prandtl number

$$\text{Pr} = \frac{c_p \mu}{k}$$

is often used to determine the coefficient of thermal conductivity k once μ is known. This is possible because the ratio (c_p/Pr), which appears in the expression

$$k = \frac{c_p}{\text{Pr}} \mu$$

is approximately constant for most gases. For air at standard conditions, Pr = 0.72.

5.1.5 Chemically Reacting Flows

The assumption of a calorically perfect gas is valid if the intermolecular forces are negligible and the temperature is relatively low. The equations governing a calorically perfect gas are given in the previous section. As the temperature of the gas increases to higher values, the gas can no longer be considered calorically perfect. At first, the vibrational energy of the molecules becomes excited and the specific heats c_p and c_v are no longer constant but are functions of temperature. For air, this occurs at temperatures above 800 K, where the air first becomes *thermally perfect*. By definition, a thermally perfect gas is a perfect gas whose specific heats are functions only of temperature. As the temperature of the gas is increased further, chemical reactions begin to take place, and the gas is no longer thermally perfect. For air at sea level pressure, the dissociation of molecular oxygen ($O_2 \rightarrow 2O$) starts at about 2000 K, and the molecular oxygen is totally dissociated at about 4000 K. The dissociation of molecular nitrogen ($N_2 \rightarrow 2N$) then begins, and total dissociation occurs at about 9000 K. Above 9000 K, ionization of the air takes place ($N \rightarrow N^+ + e^-$ and $O \rightarrow O^+ + e^-$), and the gas becomes a partially ionized plasma (Anderson, 1989).

For most chemically reacting gases, it is possible to assume that the intermolecular forces are negligible, and hence each individual species obeys the perfect gas equation of state. In addition, each individual species can be assumed to be thermally perfect. In this case, the gas is a chemically reacting mixture of thermally perfect gases, and this assumption will be employed for the

remainder of this section. The equation of state for a mixture of perfect gases can be written as

$$p = \rho \frac{\mathscr{R}}{\mathscr{M}} T \tag{5.39}$$

where $\mathscr{R}$ is the universal gas constant [8314.34 J/(kg mol K)] and $\mathscr{M}$ is the molecular weight of the mixture of gases. The molecular weight of the mixture can be calculated using

$$\mathscr{M} = \left(\sum_{i=1}^{n} \frac{c_i}{\mathscr{M}_i} \right)^{-1}$$

where c_i is the mass fraction of species i and $\mathscr{M}_i$ is the molecular weight of each species.

The species mass fractions in a reacting mixture of gases are determined by solving the species continuity equations, which are given by

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot [\rho_i(\mathbf{V} + \mathbf{U}_i)] = \dot{\omega}_i \qquad i = 1, 2, \ldots, n \tag{5.40}$$

where ρ_i is the species density, $\mathbf{U}_i$ is the species diffusion velocity, and $\dot{\omega}_i$ is the rate of production of species i due to chemical reactions. The species mass fraction is related to the species density by

$$c_i = \rho_i / \rho$$

If ρ_i is replaced with ρc_i and the global continuity equation, Eq. (5.1), is employed, the species continuity equation can be rewritten as

$$\rho \left(\frac{\partial c_i}{\partial t} + \mathbf{V} \cdot \nabla c_i \right) + \nabla \cdot (\rho_i \mathbf{U}_i) = \dot{\omega}_i \qquad i = 1, 2, \ldots, n \tag{5.41}$$

The mass flux of species $i(\rho_i \mathbf{U}_i)$ due to diffusion can be approximated for most applications using Fick's law:

$$\rho_i \mathbf{U}_i = -\rho \mathscr{D}_{\text{im}} \nabla c_i$$

where $\mathscr{D}_{\text{im}}$ is the multicomponent diffusion coefficient for each species. The multicomponent diffusion coefficient is often replaced with a binary diffusion coefficient $\mathscr{D}$ for mixtures of gases like air. The binary diffusion coefficient is assumed to be the same for all species in the mixture.

The rate of production of each species $\dot{\omega}_i$ is evaluated by using an appropriate chemistry model to simulate the reacting mixture. A chemistry model consists of m reactions, n species, and n_t reactants and can be symbolically represented as

$$\sum_{i=1}^{n_t} \nu'_{l,i} A_i \rightleftarrows \sum_{i=1}^{n_t} \nu''_{l,i} A_i \qquad l = 1, 2, \ldots, m$$

where $\nu'_{l,i}$ and $\nu''_{l,i}$ are the stoichiometric coefficients and A_i is the chemical symbol of the ith reactant. For example, a widely used chemistry model for air is

due to Blottner et al. (1971) and consists of molecular oxygen (O_2), atomic oxygen (O), molecular nitrogen (N_2), nitric oxide (NO), nitric oxide ion (NO^+), atomic nitrogen (N), and electrons (e^-), which are reacting according to the chemical reactions

$$\begin{aligned} O_2 + M_1 &\rightleftarrows 2O + M_1 \\ N_2 + M_2 &\rightleftarrows 2N + M_2 \\ N_2 + N &\rightleftarrows 2N + N \\ NO + M_3 &\rightleftarrows N + O + M_3 \\ NO + O &\rightleftarrows O_2 + N \\ N_2 + O &\rightleftarrows NO + N \\ N + O &\rightleftarrows NO^+ + e^- \end{aligned}$$

where M_1, M_2, and M_3 are catalytic third bodies. This chemistry model involves 7 reactions ($m = 7$), 6 species ($n = 6$) excluding electrons, and 10 reactants ($n_t = 10$), which include species, electrons, and catalytic third bodies.

Once the chemistry model is specified, the rate of production of species i can be computed using the Law of Mass Action (Vincenti and Kruger, 1965):

$$\dot{\omega}_i = \mathscr{M}_i \sum_{l=1}^{m} (\upsilon''_{l,i} - \upsilon'_{l,i}) \left[K_{f_l} \prod_{j=1}^{n_t} (\rho\gamma_j)^{\upsilon'_{l,j}} - K_{b_l} \prod_{j=1}^{n_t} (\rho\gamma_j)^{\upsilon''_{l,j}} \right] \tag{5.42}$$

where K_{f_l} and K_{b_l} are the forward and backward reaction rates for the lth reaction and γ_j is the mole-mass ratios of the reactants, defined by

$$\gamma_j = \begin{cases} c_j/\mathscr{M}_j & j = 1, 2, \ldots, n \\ \sum_{i=1}^{n} Z_{j-n,i}\gamma_i & j = n+1, n+2, \ldots, n_t \end{cases}$$

where $Z_{j-n,i}$ are the third-body efficiencies. The forward and backward reaction rates are functions of temperature and can be expressed in the modified Arrhenius form as

$$K_{f_l} = \exp\left(\ln K_1 + \frac{K_2}{T} + K_3 \ln T\right)$$

$$K_{b_l} = \exp\left(\ln K_4 + \frac{K_5}{T} + K_6 \ln T\right)$$

For the present air chemistry model, the constants for each reaction ($K_1, K_2, K_3, K_4, K_5, K_6$) and the third-body efficiencies ($Z_{j-n,i}$) are given by Blottner et al. (1971).

A chemically reacting mixture of gases can be classified as being *frozen*, in *equilibrium*, or in *nonequilibrium*, depending on the reaction rates. If the reaction rates are essentially zero, the mixture is said to be frozen, and the rate of production of species i ($\dot{\omega}_i$) is zero. If the reaction rates approach infinity, the

mixture is said to be in *chemical equilibrium.* If the reaction rates are finite, the mixture is in *chemical nonequilibrium,* and Eq. (5.42) can be used to find $\dot{\omega}_i$. At high velocities and rarefield conditions, the mixture of gases may also be in *thermal nonequilibrium.* In this case, the translational, rotational, vibrational, and electronic modes of the thermal energy are not in equilibrium. As a consequence, the modeling of the chemistry will require a multitemperature approach as opposed to the usual single-temperature formulation. For the present discussion, the mixture will be assumed to be in thermal equilibrium. See Anderson (1989), Park (1990), and Vincenti and Kruger (1965) for information on flows in thermal nonequilibrium.

The thermodynamic properties for a reacting mixture in chemical nonequilibrium are functions of both temperature and the mass fractions. For example, the enthalpy and internal energy of the mixture can be expressed as

$$h = h(T, c_1, c_2, \ldots, c_n)$$

$$e = e(T, c_1, c_2, \ldots, c_n)$$

If the mixture is in chemical equilibrium, the thermodynamic properties are a unique function of any two thermodynamic variables, such as temperature and pressure. In this case, h and e can be expressed as

$$h = h(T, p)$$

$$e = e(T, p)$$

Computer programs are available (i.e., Gordon and McBride, 1971) that can be used to compute the composition and thermodynamic properties of equilibrium mixtures of gases. Also available are computer programs that obtain properties by interpolating values from tables of equilibrium data or by using simplified curve fits of the data. Included in the latter approach are the curve fits of Srinivasan et al. (1987) for the thermodynamic properties of equilibrium air. These curve fits include correlations for $p(e, \rho)$, $a(e, \rho)$, $T(e, \rho)$, $s(e, \rho)$, $h(p, \rho)$, $T(p, \rho)$, $\rho(p, s)$, $e(p, s)$, and $a(p, s)$. The curve fits are based on the data from the NASA RGAS (Real GAS) program (Bailey, 1967) and are valid for temperatures up to 25,000 K and densities from 10^{-7} to 10^3 amagats (ρ/ρ_0). In addition, Srinivasan and Tannehill (1987) have developed simplified curve fits for the transport properties of equilibrium air. These curve fits include correlations for $\mu(e, \rho)$ $k(e, \rho)$, $\mu(T, \rho)$, and $\Pr(T, \rho)$. The curve fits are based on the data of Peng and Pindroh (1962) and are valid for temperatures up to 15,000 K and densities from 10^{-5} to 10^3 amagats.

The thermodynamic properties for a reacting mixture in chemical nonequilibrium can be determined once the mass fractions of each species are known. If each species is assumed to be thermally perfect, the species enthalpy and specific heat at constant pressure are given by

$$h_i = C_{1,i}T + h_i^0$$

$$c_{p_i} = C_{2,i}$$

where h_i^0 is the enthalpy of formation for species i. The coefficients $C_{1,i}$ and $C_{2,i}$ are functions of temperature and can be interpolated from the tabulated data of Blottner et al. (1971) or McBride et al. (1963). The mixture enthalpy and

frozen specific heat at constant pressure are then given by

$$h = \sum_{i=1}^{n} c_i h_i$$

$$c_{p_f} = \sum_{i=1}^{n} c_i c_{p_i}$$

The transport properties for a chemically reacting mixture can be determined in a similar manner. The viscosity for each species is given by Svehla (1962) in the form of curve fits. Using the viscosity for each species, the species thermal conductivity can be evaluated using Eucken's semi-empirical formula (Prabhu et al., 1987a, 1987b). The mixture viscosity and thermal conductivity can then be determined using Wilke's mixing rule (Wilke, 1950). Further details on these methods can be found in the works by Prabhu et al. (1987a, 1987b) and Buelow et al. (1991).

For chemically reacting flows, it is also necessary to modify the energy equation to include the effect of mass diffusion. This effect is accounted for by adding the following component to the heat flux vector **q**:

$$\rho \sum_{i=1}^{n} c_i h_i \mathbf{U}_i$$

where $\mathbf{U}_i$ is the diffusion velocity for each species.

5.1.6 Vector Form of Equations

Before applying a numerical algorithm to the governing fluid dynamic equations, it is often convenient to combine the equations into a compact vector form. For example, the compressible Navier-Stokes equations in Cartesian coordinates without body forces, mass diffusion, finite-rate chemical reactions, or external heat addition can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0 \tag{5.43}$$

where **U**, **E**, **F**, and **G** are vectors given by

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho uv - \tau_{xy} \\ \rho uw - \tau_{xz} \\ (E_t + p)u - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} + q_x \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \rho v \\ \rho uv - \tau_{xy} \\ \rho v^2 + p - \tau_{yy} \\ \rho vw - \tau_{yz} \\ (E_t + p)v - u\tau_{xy} - v\tau_{yy} - w\tau_{yz} + q_y \end{bmatrix} \tag{5.44}$$

$$\mathbf{G} = \begin{bmatrix} \rho w \\ \rho uw - \tau_{xz} \\ \rho vw - \tau_{yz} \\ \rho w^2 + p - \tau_{zz} \\ (E_t + p)w - u\tau_{xz} - v\tau_{yz} - w\tau_{zz} + q_z \end{bmatrix}$$

The first row of the vector Eq. (5.43) corresponds to the continuity equation as given by Eq. (5.4). Likewise, the second, third, and fourth rows are the momentum equations, Eqs. (5.20), while the fifth row is the energy equation, Eq. (5.25). With the Navier-Stokes equations written in this form, it is often easier to code the desired numerical algorithm. Other fluid dynamic equations that are written in conservation-law form can be placed in a similar vector form.

5.1.7 Nondimensional Form of Equations

The governing fluid dynamic equations are often put into nondimensional form. The advantage in doing this is that the characteristic parameters such as Mach number, Reynolds number, and Prandtl number can be varied independently. Also, by nondimensionalizing the equations, the flow variables are "normalized," so that their values fall between certain prescribed limits such as 0 and 1. Many different nondimensionalizing procedures are possible. An example of one such procedure is

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L} \qquad t^* = \frac{t}{L/V_\infty}$$

$$u^* = \frac{u}{V_\infty} \qquad v^* = \frac{v}{V_\infty} \qquad w^* = \frac{w}{V_\infty} \qquad \mu^* = \frac{\mu}{\mu_\infty}$$

$$\rho^* = \frac{\rho}{\rho_\infty} \qquad p^* = \frac{p}{\rho_\infty V_\infty^2} \qquad T^* = \frac{T}{T_\infty} \qquad e^* = \frac{e}{V_\infty^2}$$

where the nondimensional variables are denoted by an asterisk, free stream conditions are denoted by ∞, and L is the reference length used in the Reynolds number:

$$\mathrm{Re}_L = \frac{\rho_\infty V_\infty L}{\mu_\infty}$$

If this nondimensionalizing procedure is applied to the compressible Navier-Stokes equations given previously by Eqs. (5.43) and (5.44), the following nondimensional equations are obtained:

$$\frac{\partial \mathbf{U}^*}{\partial t^*} + \frac{\partial \mathbf{E}^*}{\partial x^*} + \frac{\partial \mathbf{F}^*}{\partial y^*} + \frac{\partial \mathbf{G}^*}{\partial z^*} = 0 \tag{5.45}$$

where $\mathbf{U}^*$, $\mathbf{E}^*$, $\mathbf{F}^*$, and $\mathbf{G}^*$ are the vectors

$$\mathbf{U}^* = \begin{bmatrix} \rho^* \\ \rho^* u^* \\ \rho^* v^* \\ \rho^* w^* \\ E_t^* \end{bmatrix}$$

$$\mathbf{E}^* = \begin{bmatrix} \rho^* u^* \\ \rho^* u^{*2} + p^* - \tau_{xx}^* \\ \rho^* u^* v^* - \tau_{xy}^* \\ \rho^* u^* w^* - \tau_{xz}^* \\ (E_t^* + p^*)u^* - u^*\tau_{xx}^* - v^*\tau_{xy}^* - w^*\tau_{xz}^* + q_x^* \end{bmatrix}$$

$$\mathbf{F}^* = \begin{bmatrix} \rho^* v^* \\ \rho^* u^* v^* - \tau_{xy}^* \\ \rho^* v^{*2} + p^* - \tau_{yy}^* \\ \rho^* v^* w^* - \tau_{yz}^* \\ (E_t^* + p^*)v^* - u^*\tau_{xy}^* - v^*\tau_{yy}^* - w^*\tau_{yz}^* + q_y^* \end{bmatrix} \tag{5.46}$$

$$\mathbf{G}^* = \begin{bmatrix} \rho^* w^* \\ \rho^* u^* w^* - \tau_{xz}^* \\ \rho^* v^* w^* - \tau_{yz}^* \\ \rho^* w^{*2} + p^* - \tau_{zz}^* \\ (E_t^* + p^*)w^* - u^*\tau_{xz}^* - v^*\tau_{yz}^* - w^*\tau_{zz}^* + q_z^* \end{bmatrix}$$

and

$$E_t^* = \rho^* \left(e^* + \frac{u^{*2} + v^{*2} + w^{*2}}{2} \right)$$

The components of the shear-stress tensor and the heat flux vector in nondimensional form are given by

$$\tau_{xx}^* = \frac{2\mu^*}{3\,\mathrm{Re}_L} \left(2\frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} - \frac{\partial w^*}{\partial z^*} \right)$$

$$\tau_{yy}^* = \frac{2\mu^*}{3\,\mathrm{Re}_L} \left(2\frac{\partial v^*}{\partial y^*} - \frac{\partial u^*}{\partial x^*} - \frac{\partial w^*}{\partial z^*} \right)$$

$$
\begin{aligned}
\tau_{zz}^* &= \frac{2\mu^*}{3\,\mathrm{Re}_L}\left(2\frac{\partial w^*}{\partial z^*} - \frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*}\right) \\
\tau_{xy}^* &= \frac{\mu^*}{\mathrm{Re}_L}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right) \\
\tau_{xz}^* &= \frac{\mu^*}{\mathrm{Re}_L}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right) \\
\tau_{yz}^* &= \frac{\mu^*}{\mathrm{Re}_L}\left(\frac{\partial v^*}{\partial z^*} + \frac{\partial w^*}{\partial y^*}\right) \\
q_x^* &= -\frac{\mu^*}{(\gamma-1)M_\infty^2\,\mathrm{Re}_L\,\mathrm{Pr}}\frac{\partial T^*}{\partial x^*} \\
q_y^* &= -\frac{\mu^*}{(\gamma-1)M_\infty^2\,\mathrm{Re}_L\,\mathrm{Pr}}\frac{\partial T^*}{\partial y^*} \\
q_z^* &= -\frac{\mu^*}{(\gamma-1)M_\infty^2\,\mathrm{Re}_L\,\mathrm{Pr}}\frac{\partial T^*}{\partial z^*}
\end{aligned}
\tag{5.47}
$$

where M_∞ is the free stream Mach number,

$$M_\infty = \frac{V_\infty}{\sqrt{\gamma R T_\infty}}$$

and the perfect gas equations of state [Eqs. (5.38)] become

$$p^* = (\gamma - 1)\rho^* e^*$$

$$T^* = \frac{\gamma M_\infty^2 p^*}{\rho^*}$$

Note that the nondimensional forms of the equations given by Eqs. (5.45) and (5.46) are identical (except for the asterisks) to the dimensional forms given by Eqs. (5.43) and (5.44). For convenience, the asterisks can be dropped from the nondimensional equations, and this is usually done.

5.1.8 Orthogonal Curvilinear Coordinates

The basic equations of fluid dynamics are valid for any coordinate system. We have previously expressed these equations in terms of a Cartesian coordinate system. For many applications it is more convenient to use a different orthogonal coordinate system. Let us define x_1, x_2, x_3 to be a set of generalized orthogonal curvilinear coordinates whose origin is at point P and let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be the corresponding unit vectors (see Fig. 5.2). The rectangular Cartesian coordinates are related to the generalized curvilinear coordinates by

$$
\begin{aligned}
x &= x(x_1, x_2, x_3) \\
y &= y(x_1, x_2, x_3) \\
z &= z(x_1, x_2, x_3)
\end{aligned}
\tag{5.48}
$$

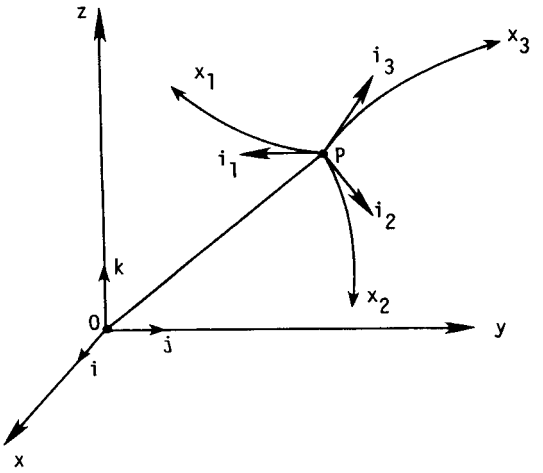


Figure 5.2 Orthogonal curvilinear coordinate system.

so that if the Jacobian

$$\frac{\partial(x, y, z)}{\partial(x_1, x_2, x_3)}$$

is nonzero, then

$$\begin{aligned} x_1 &= x_1(x, y, z) \\ x_2 &= x_2(x, y, z) \\ x_3 &= x_3(x, y, z) \end{aligned} \tag{5.49}$$

The elemental arc length ds in Cartesian coordinates is obtained from

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{5.50}$$

If Eq. (5.48) is differentiated and substituted into Eq. (5.50), the following result is obtained:

$$(ds)^2 = (h_1\, dx_1)^2 + (h_2\, dx_2)^2 + (h_3\, dx_3)^2 \tag{5.51}$$

where

$$(h_1)^2 = \left(\frac{\partial x}{\partial x_1}\right)^2 + \left(\frac{\partial y}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_1}\right)^2$$

$$(h_2)^2 = \left(\frac{\partial x}{\partial x_2}\right)^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2$$

$$(h_3)^2 = \left(\frac{\partial x}{\partial x_3}\right)^2 + \left(\frac{\partial y}{\partial x_3}\right)^2 + \left(\frac{\partial z}{\partial x_3}\right)^2$$

If ϕ is an arbitrary scalar and $\mathbf{A}$ is an arbitrary vector, the expressions for the gradient, divergence, curl, and Laplacian operator in the generalized curvilinear coordinates become

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\mathbf{i}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\mathbf{i}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\mathbf{i}_3 \tag{5.52}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3A_1) + \frac{\partial}{\partial x_2}(h_3h_1A_2) + \frac{\partial}{\partial x_3}(h_1h_2A_3)\right] \tag{5.53}$$

$$\nabla \times \mathbf{A} = \frac{1}{h_1h_2h_3}\left\{h_1\left[\frac{\partial(h_3A_3)}{\partial x_2} - \frac{\partial(h_2A_2)}{\partial x_3}\right]\mathbf{i}_1 + h_2\left[\frac{\partial(h_1A_1)}{\partial x_3} - \frac{\partial(h_3A_3)}{\partial x_1}\right]\mathbf{i}_2 + h_3\left[\frac{\partial(h_2A_2)}{\partial x_1} - \frac{\partial(h_1A_1)}{\partial x_2}\right]\mathbf{i}_3\right\} \tag{5.54}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right] \tag{5.55}$$

The expression $\mathbf{V} \cdot \nabla\mathbf{V}$, which is contained in the momentum equation term $D\mathbf{V}/Dt$, can be evaluated as

$$\mathbf{V} \cdot \nabla\mathbf{V} = \left(\frac{u_1}{h_1}\frac{\partial}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial}{\partial x_3}\right)(u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3)$$

where u_1, u_2, u_3 are the velocity components in the x_1, x_2, x_3 coordinate directions. After taking into account the fact that the unit vectors are functions of the coordinates, the final expanded form becomes

$$\begin{aligned}
\mathbf{V} \cdot \nabla\mathbf{V} = {} & \left(\frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial x_3} + \frac{u_1u_2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \frac{u_2^2}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \frac{u_3^2}{h_1h_3}\frac{\partial h_3}{\partial x_1}\right)\mathbf{i}_1 \\
& + \left(\frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_2}{\partial x_3} - \frac{u_1^2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_1u_2}{h_1h_2}\frac{\partial h_2}{\partial x_1} + \frac{u_2u_3}{h_2h_3}\frac{\partial h_2}{\partial x_3} - \frac{u_3^2}{h_2h_3}\frac{\partial h_3}{\partial x_2}\right)\mathbf{i}_2 \\
& + \left(\frac{u_1}{h_1}\frac{\partial u_3}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_3}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial x_3} - \frac{u_1^2}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \frac{u_2^2}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \frac{u_2u_3}{h_2h_3}\frac{\partial h_3}{\partial x_2}\right)\mathbf{i}_3
\end{aligned}$$

The components of the stress tensor given by Eq. (5.15) can be expressed in terms of the generalized curvilinear coordinates as

$$\begin{aligned}
\Pi_{x_1x_1} &= -p + \tfrac{2}{3}\mu(2e_{x_1x_1} - e_{x_2x_2} - e_{x_3x_3}) \\
\Pi_{x_2x_2} &= -p + \tfrac{2}{3}\mu(2e_{x_2x_2} - e_{x_1x_1} - e_{x_3x_3}) \\
\Pi_{x_3x_3} &= -p + \tfrac{2}{3}\mu(2e_{x_3x_3} - e_{x_1x_1} - e_{x_2x_2}) \\
\Pi_{x_2x_3} &= \Pi_{x_3x_2} = \mu e_{x_2x_3} \\
\Pi_{x_1x_3} &= \Pi_{x_3x_1} = \mu e_{x_1x_3} \\
\Pi_{x_1x_2} &= \Pi_{x_2x_1} = \mu e_{x_1x_2}
\end{aligned} \tag{5.56}$$

where the expressions for the strains are

$$\begin{aligned}
e_{x_1x_1} &= \frac{1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_3}{h_1h_3}\frac{\partial h_1}{\partial x_3} \\
e_{x_2x_2} &= \frac{1}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{u_3}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{u_1}{h_1h_2}\frac{\partial h_2}{\partial x_1} \\
e_{x_3x_3} &= \frac{1}{h_3}\frac{\partial u_3}{\partial x_3} + \frac{u_1}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \frac{u_2}{h_2h_3}\frac{\partial h_3}{\partial x_2} \\
e_{x_2x_3} &= \frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{u_3}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{u_2}{h_2}\right) \\
e_{x_1x_3} &= \frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{u_1}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{u_3}{h_3}\right) \\
e_{x_1x_2} &= \frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{u_2}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{u_1}{h_1}\right)
\end{aligned} \tag{5.57}$$

The components of $\nabla \cdot \Pi_{ij}$ are

$$\begin{aligned}
x_1 : \frac{1}{h_1h_2h_3}&\left[\frac{\partial}{\partial x_1}(h_2h_3\Pi_{x_1x_1}) + \frac{\partial}{\partial x_2}(h_1h_3\Pi_{x_1x_2}) + \frac{\partial}{\partial x_3}(h_1h_2\Pi_{x_1x_3})\right] \\
&+ \Pi_{x_1x_2}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \Pi_{x_1x_3}\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \Pi_{x_2x_2}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \Pi_{x_3x_3}\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1} \\
x_2 : \frac{1}{h_1h_2h_3}&\left[\frac{\partial}{\partial x_1}(h_2h_3\Pi_{x_1x_2}) + \frac{\partial}{\partial x_2}(h_1h_3\Pi_{x_2x_2}) + \frac{\partial}{\partial x_3}(h_1h_2\Pi_{x_2x_3})\right] \\
&+ \Pi_{x_2x_3}\frac{1}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \Pi_{x_1x_2}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \Pi_{x_3x_3}\frac{1}{h_2h_3}\frac{\partial h_3}{\partial x_2} - \Pi_{x_1x_1}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial x_2}
\end{aligned}$$

$$x_3: \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3\Pi_{x_1x_3}) + \frac{\partial}{\partial x_2}(h_1h_3\Pi_{x_2x_3}) + \frac{\partial}{\partial x_3}(h_1h_2\Pi_{x_3x_3})\right]$$
$$+ \Pi_{x_1x_3}\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \Pi_{x_2x_3}\frac{1}{h_2h_3}\frac{\partial h_3}{\partial x_2} - \Pi_{x_1x_1}\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \Pi_{x_2x_2}\frac{1}{h_2h_3}\frac{\partial h_2}{\partial x_3} \tag{5.58}$$

In generalized curvilinear coordinates, the dissipation function becomes

$$\Phi = \mu\left[2\left(e_{x_1x_1}^2 + e_{x_2x_2}^2 + e_{x_3x_3}^2\right) + e_{x_2x_3}^2 + e_{x_1x_3}^2 + e_{x_1x_2}^2 - \tfrac{2}{3}(e_{x_1x_1} + e_{x_2x_2} + e_{x_3x_3})^2\right] \tag{5.59}$$

The above formulas can now be used to derive the fluid dynamic equations in any orthogonal curvilinear coordinate system. Examples include

Cartesian coordinates

$x_1 = x$	$h_1 = 1$	$u_1 = u$
$x_2 = y$	$h_2 = 1$	$u_2 = v$
$x_3 = z$	$h_3 = 1$	$u_3 = w$

Cylindrical coordinates

$x_1 = r$	$h_1 = 1$	$u_1 = u_r$
$x_2 = \theta$	$h_2 = r$	$u_2 = u_\theta$
$x_3 = z$	$h_3 = 1$	$u_3 = u_z$

Spherical coordinates

$x_1 = r$	$h_1 = 1$	$u_1 = yu_r$
$x_2 = \theta$	$h_2 = r$	$u_2 = u_\theta$
$x_3 = \phi$	$h_3 = r \sin\theta$	$u_3 = u_\phi$

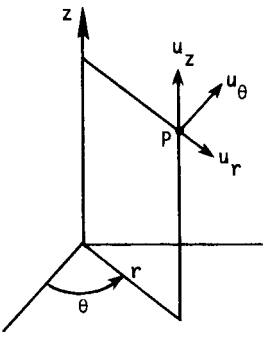
Two-dimensional (2-D) or axisymmetric body intrinsic coordinates

$x_1 = \xi$	$h_1 = 1 + K(\xi)\eta$	$u_1 = u$
$x_2 = \eta$	$h_2 = 1$	$u_2 = v$
$x_3 = \phi$	$h_3 = [r(\xi) + \eta\cos\alpha(\xi)]^m$	$u_3 = mw$

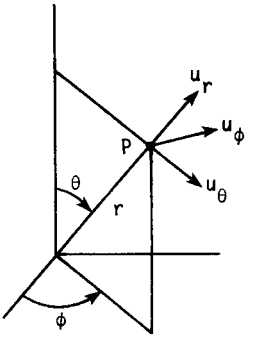
where $K(\xi)$ is the local body curvature, $r(\xi)$ is the cylindrical radius, and

$$m = \begin{cases} 0 & \text{for 2-D flow} \\ 1 & \text{for axisymmetric flow} \end{cases}$$

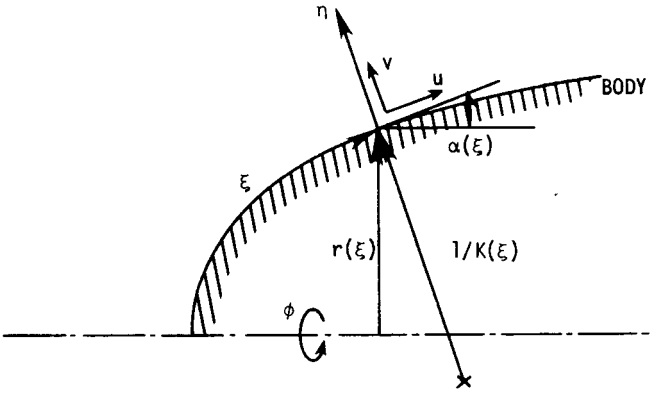
These coordinate systems are illustrated in Fig. 5.3.



(a) CYLINDRICAL COORDINATES (r, θ, z)



(b) SPHERICAL COORDINATES (r, θ, ϕ)



(c) 2-D OR AXISYMMETRIC BODY INTRINSIC COORDINATES (ξ, η, ϕ)

Figure 5.3 Curvilinear coordinate systems. (a) Cylindrical coordinates (r, θ, z); (b) spherical coordinates (r, θ, ϕ); (c) 2-D or axisymmetric body intrinsic coordinates (ξ, η, ϕ).

5.2 AVERAGED EQUATIONS FOR TURBULENT FLOWS

5.2.1 Background

For more than 60 years it has been recognized that our understanding of turbulent flows is incomplete. A quotation attributed to Sir Horace Lamb in 1932 might still be appropriate: "I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motion of fluids. And about the former I am rather optimistic."

According to Hinze (1975), "Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates so that statistically distinct average values can be discerned."

We are all familiar with some of the differences between laminar and turbulent flows. Usually, higher values of friction drag and pressure drop are associated with turbulent flows. The diffusion rate of a scalar quantity is usually greater in a turbulent flow than in a laminar flow (increased "mixing"), and turbulent flows are usually noisier. A turbulent boundary layer can normally negotiate a more extensive region of unfavorable pressure gradient prior to separation than can a laminar boundary layer. Users of dimpled golf balls are well aware of this.

The unsteady Navier-Stokes equations are generally considered to govern turbulent flows in the continuum regime. If this is the case, then we might wonder why turbulent flows cannot be solved numerically as easily as laminar flows. Perhaps the wind tunnels can be dismantled once and for all. This is indeed a possibility, but it is not likely to happen very soon. To resolve a turbulent flow by *direct numerical simulation* (DNS) requires that all relevant length scales be resolved from the smallest eddies to scales on the order of the physical dimensions of the problem domain. The computation needs to be 3-D even if the time-mean aspects of the flow are 2-D, and the time steps must be small enough that the small-scale motion can be resolved in a time-accurate manner even if the flow is steady in a time-mean sense. Such requirements place great demands on computer resources, to the extent that only relatively simple flows at low Reynolds numbers can be computed directly with present-day machines.

The computations of Kim et al. (1987) provide an example of required resources. They computed a nominally fully developed incompressible channel flow at a Reynolds number based on channel height of about 6000 using grids of 2 and 4 million points. For the finer grid, 250 hours of Cray XMP time were required. For channel flow the number of grid points needed can be estimated from the expression (Wilcox, 1993)

$$N_{\mathrm{DNS}} = (0.088\,\mathrm{Re}_h)^{9/4}$$

where Re_h is the Reynolds number based on the mean channel velocity and channel height. Turbulent wall shear flows that have been successfully simulated directly include planar and square channel flows, flow over a rearward facing step, and flow over a flat plate. Much useful information has been gained from

such simulations, since many of the statistical quantities of interest cannot be measured experimentally, but can be evaluated from the simulations. Recently, simulations have been extended to include compressible and transitional flows.

Another promising approach is known as *large-eddy simulation* (LES), in which the large-scale structure of the turbulent flow is computed directly and only the effects of the smallest (subgrid-scale) and more nearly isotropic eddies are modeled. This is accomplished by "filtering" the Navier-Stokes equations to obtain a set of equations that govern the "resolved" flow. This filtering, to be defined below, is a type of space averaging of the flow variables over regions approximately the size of the computational control volume (cell). The computational effort required for LES is less than that of DNS by approximately a factor of 10 using present-day methods. Clearly, with present-day computers, it is not possible to simulate directly or on a large-eddy basis most of the turbulent flows arising in engineering applications. However, the frontier is advancing relentlessly as computer hardware and algorithms improve. With each advance in computer capability, it becomes possible to apply DNS and LES to more and more flows of increasing complexity. Sometime during the twenty-first century it is highly likely that DNS and LES will replace the more approximate modeling methods currently used as the primary design procedure for engineering applications.

The main thrust of present-day research in computational fluid mechanics and heat transfer in turbulent flows is through the time-averaged Navier-Stokes equations. These equations are also referred to as the Reynolds equations of motion or the *Reynolds averaged Navier-Stokes* (RANS) equations. Time averaging the equations of motion gives rise to new terms, which can be interpreted as "apparent" stress gradients and heat flux quantities associated with the turbulent motion. These new quantities must be related to the mean flow variables through turbulence models. This process introduces further assumptions and approximations. Thus this attack on the turbulent flow problem through solving the Reynolds equations of motion does not follow entirely from first principles, since additional assumptions must be made to "close" the system of equations.

The Reynolds equations are derived by decomposing the dependent variables in the conservation equations into time-mean (obtained over an appropriate time interval) and fluctuating components and then time averaging the entire equation. Two types of averaging are presently used, the classical Reynolds averaging and the mass-weighted averaging suggested by Favre (1965). For flows in which density fluctuations can be neglected, the two formulations become identical.

5.2.2 Reynolds Averaged Navier-Stokes Equations

In the conventional averaging procedure, following Reynolds, we define a time-averaged quantity $\bar{f}$ as

$$\bar{f} \equiv \frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} f \, dt \tag{5.60}$$

We require that Δt be large compared to the period of the random fluctuations associated with the turbulence, but small with respect to the time constant for any slow variations in the flow field associated with ordinary unsteady flows. The Δt is sometimes indicated to approach infinity as a limit, but this should be interpreted as being relative to the characteristic fluctuation period of the turbulence. For practical measurements, Δt must be finite.

In the conventional Reynolds decomposition, the randomly changing flow variables are replaced by time averages plus fluctuations (see Fig. 5.4) about the average. For a Cartesian coordinate system, we may write

$$\begin{aligned} u &= \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w' \qquad \rho = \bar{\rho} + \rho' \\ p &= \bar{p} + p' \qquad h = \bar{h} + h' \qquad T = \bar{T} + T' \qquad H = \bar{H} + H' \end{aligned} \tag{5.61}$$

where total enthalpy H is defined by $H = h + u_i u_i/2$. Fluctuations in other fluid properties such as viscosity, thermal conductivity, and specific heat are usually small and will be neglected here.

By definition, the time average of a fluctuating quantity is zero:

$$\bar{f}' = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} f'\, dt \equiv 0 \tag{5.62}$$

It should be clear from these definitions that for symbolic flow variables f and g, the following relations hold:

$$\overline{\bar{f}g'} = 0 \qquad \overline{\bar{f}\bar{g}} = \bar{f}\bar{g} \qquad \overline{f+g} = \bar{f} + \bar{g} \tag{5.63}$$

It should also be clear that, whereas $\bar{f}' \equiv 0$, the time average of the product of two fluctuating quantities is, in general, not equal to zero, i.e., $\overline{f'f'} \neq 0$. In fact, the root mean square of the velocity fluctuations is known as the turbulence intensity.

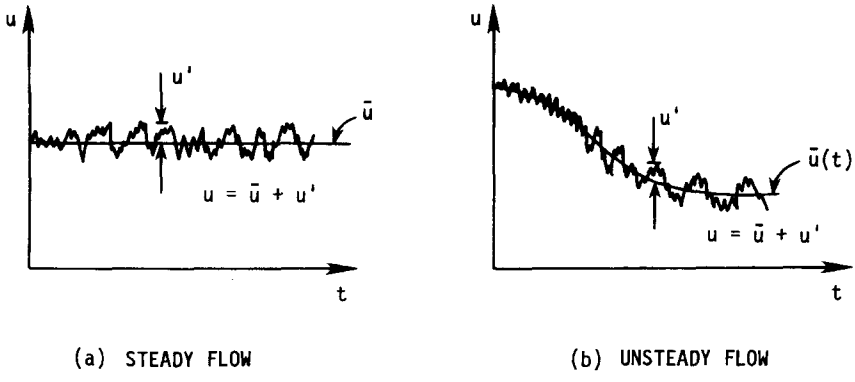


Figure 5.4 Relation between u, $\bar{u}$, and u'. (a) Steady flow. (b) Unsteady flow.

For treatment of compressible flows and mixtures of gases in particular, mass-weighted averaging is convenient. In this approach we define mass-averaged variables according to $\tilde{f} = \overline{\rho f}/\bar{\rho}$. This gives

$$\tilde{u} = \frac{\overline{\rho u}}{\bar{\rho}} \qquad \tilde{v} = \frac{\overline{\rho v}}{\bar{\rho}} \qquad \tilde{w} = \frac{\overline{\rho w}}{\bar{\rho}} \qquad \tilde{h} = \frac{\overline{\rho h}}{\bar{\rho}} \qquad \tilde{T} = \frac{\overline{\rho T}}{\bar{\rho}} \qquad \tilde{H} = \frac{\overline{\rho H}}{\bar{\rho}} \tag{5.64}$$

We note that only the velocity components and thermal variables are mass averaged. Fluid properties such as density and pressure are treated as before.

To substitute into the conservation equations, we define new fluctuating quantities by

$$u = \tilde{u} + u'' \qquad v = \tilde{v} + v'' \qquad w = \tilde{w} + w'' \qquad h = \tilde{h} + h'' \qquad T = \tilde{T} + T''$$
$$H = \tilde{H} + H'' \tag{5.65}$$

It is very important to note that the time averages of the doubly primed fluctuating quantities ($\overline{u''}, \overline{v''}$, etc.) *are not* equal to zero, in general, unless $\rho' = 0$. In fact, it can be shown that $\overline{u''} = -\overline{\rho' u'}/\bar{\rho}$, $\overline{v''} = -\overline{\rho' v'}/\bar{\rho}$, etc. Instead, the time average of the doubly primed fluctuation multiplied by the density is equal to zero:

$$\overline{\rho f''} \equiv 0 \tag{5.66}$$

The above identity can be established by expanding $\overline{\rho f} = \overline{\rho(\tilde{f} + f'')}$ and using the definition of $\tilde{f}$.

5.2.3 Reynolds Form of the Continuity Equation

Starting with the continuity equation in the Cartesian coordinate system as given by Eq. (5.4), we first decompose the variables into the conventional time-averaged variables plus fluctuating components as given by Eqs. (5.61).

The entire equation is then time averaged, yielding in summation notation

$$\frac{\partial \bar{\rho}}{\partial t} + \cancelto{0}{\frac{\partial \bar{\rho}'}{\partial t}} + \frac{\partial}{\partial x_j}\left(\overline{\bar{\rho}\bar{u}_j}\right) + \cancelto{0}{\frac{\partial}{\partial x_j}\left(\overline{\rho' u_j}\right)} + \cancelto{0}{\frac{\partial}{\partial x_j}\left(\overline{\bar{\rho} u_j'}\right)} + \frac{\partial}{\partial x_j}\left(\overline{\rho' u_j'}\right) = 0 \tag{5.67}$$

Three of the terms are identically zero as indicated because of the identity given by Eq. (5.62). Finally, the Reynolds form of the continuity equation in conventionally averaged variables can be written

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j + \overline{\rho' u_j'}\right) = 0 \tag{5.68}$$

Substituting the mass-weighted averaged variables plus the doubly primed fluctuations given by Eqs. (5.65) into Eq. (5.4) and time averaging the entire

equation gives

$$\frac{\partial \bar{\rho}}{\partial t} + \cancelto{0}{\frac{\partial \rho'}{\partial t}} + \frac{\partial}{\partial x_j}\overline{(\bar{\rho}\tilde{u}_j)} + \frac{\partial}{\partial x_j}\cancelto{0}{\overline{(\rho' u_j'')}} + \frac{\partial}{\partial x_j}\overline{(\bar{\rho}u_j'')} + \frac{\partial}{\partial x_j}\overline{(\rho' u_j'')} = 0 \tag{5.69}$$

Two of the terms in Eq. (5.69) are obviously identically zero as indicated. In addition, the last two terms can be combined, i.e.,

$$\frac{\partial}{\partial x_j}\overline{(\bar{\rho}u_j'')} + \frac{\partial}{\partial x_j}\overline{(\rho' u_j'')} = \frac{\partial}{\partial x_j}\overline{\rho u_j''}$$

which is equal to zero by Eq. (5.66). This permits the continuity equation in mass-weighted variables to be written as

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j) = 0 \tag{5.70}$$

We note that Eq. (5.70) is more compact in form than Eq. (5.68). For incompressible flows, $\rho' = 0$, and the differences between the conventional and mass-weighted variables vanish, so that the continuity equation can be written as

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0 \tag{5.71}$$

5.2.4 Reynolds Form of the Momentum Equations

The development of the Reynolds form of the momentum equations proceeds most easily when we start with the Navier-Stokes momentum equations in divergence or conservation-law form as in Eq. (5.20). Working first with the conventionally averaged variables, we replace the dependent variables in Eq. (5.20) with the time averages plus fluctuations according to Eq. (5.61). As an example, the resulting x component of Eq. (5.20), after neglecting body forces, becomes

$$\frac{\partial}{\partial t}[(\bar{\rho} + \rho')(\bar{u} + u')] + \frac{\partial}{\partial x}[(\bar{\rho} + \rho')(\bar{u} + u')(\bar{u} + u') + (\bar{p} + p') - \tau_{xx}]$$
$$+ \frac{\partial}{\partial y}[(\bar{\rho} + \rho')(\bar{u} + u')(\bar{v} + v') - \tau_{yx}]$$
$$+ \frac{\partial}{\partial z}[(\bar{\rho} + \rho')(\bar{u} + u')(\bar{w} + w') - \tau_{zx}] = 0$$

Next, the entire equation is time averaged. Terms that are linear in fluctuating quantities become zero when time averaged, as they did in the continuity equation. Several terms disappear in this manner, while others can be

grouped together and found to be zero through use of the continuity equation. The resulting Reynolds x-momentum equation can be written as

$$\begin{aligned}
&\frac{\partial}{\partial t}(\bar{\rho}\bar{u} + \overline{\rho' u'}) + \frac{\partial}{\partial x}(\bar{\rho}\bar{u}\bar{u} + \bar{u}\overline{\rho' u'}) \\
&\quad + \frac{\partial}{\partial y}(\bar{\rho}\bar{u}\bar{v} + \bar{u}\overline{\rho' v'}) + \frac{\partial}{\partial z}(\bar{\rho}\bar{u}\bar{w} + \bar{u}\overline{\rho' w'}) \\
&\quad = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{u}_k}{\partial x_k}\right) - \bar{u}\overline{\rho' u'} - \bar{\rho}\overline{u' u'} - \overline{\rho' u' u'}\right] \\
&\qquad + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) - \bar{v}\overline{\rho' u'} - \bar{\rho}\overline{u' v'} - \overline{\rho' u' v'}\right] \\
&\qquad + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) - \bar{w}\overline{\rho' u'} - \bar{\rho}\overline{u' w'} - \overline{\rho' u' w'}\right]
\end{aligned} \tag{5.72}$$

The complete Reynolds momentum equations (all three components) can be written

$$\begin{aligned}
&\frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i + \overline{\rho' u_i'}) + \frac{\partial}{\partial x_j}(\bar{\rho}\bar{u}_i\bar{u}_j + \bar{u}_i\overline{\rho' u_j'}) \\
&\qquad = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} - \bar{u}_j\overline{\rho' u_i'} - \bar{\rho}\overline{u_i' u_j'} - \overline{\rho' u_i' u_j'})
\end{aligned} \tag{5.73}$$

where

$$\bar{\tau}_{ij} = \mu\left[\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right] \tag{5.74}$$

To develop the Reynolds momentum equation in mass-weighted variables, we again start with Eq. (5.20) but use the decomposition indicated by Eq. (5.65) to represent the instantaneous variables. As an example, the resulting x component of Eq. (5.20) becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}[(\bar{\rho} + \rho')(\tilde{u} + u'')] + \frac{\partial}{\partial x}[(\bar{\rho} + \rho')(\tilde{u} + u'')(\tilde{u} + u'') + (\bar{p} + p') - \tau_{xx}] \\
&\quad + \frac{\partial}{\partial y}[(\bar{\rho} + \rho')(\tilde{u} + u'')(\tilde{v} + v'') - \tau_{yx}] \\
&\quad + \frac{\partial}{\partial z}[(\bar{\rho} + \rho')(\tilde{u} + u'')(\tilde{w} + w'') - \tau_{zx}] = 0
\end{aligned} \tag{5.75}$$

Next, the entire equation is time averaged, and the identity given by Eq. (5.66) is used to eliminate terms. The complete Reynolds momentum equation in mass-

weighted variables becomes

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_i\tilde{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}) \tag{5.76}$$

where, neglecting viscosity fluctuations, $\bar{\tau}_{ij}$ becomes

$$\bar{\tau}_{ij} = \mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right] + \mu\left[\left(\frac{\partial \overline{u_i''}}{\partial x_j} + \frac{\partial \overline{u_j''}}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \overline{u_k''}}{\partial x_k}\right] \tag{5.77}$$

The momentum equation, Eq. (5.76), in mass-weighted variables is simpler in form than the corresponding equation using conventional variables. We note, however, that even when viscosity fluctuations are neglected, $\bar{\tau}_{ij}$ is more complex in Eq. (5.77) than the $\bar{\tau}_{ij}$ that appeared in the conventionally averaged equation [Eq. (5.74)]. In practice, the viscous terms involving the doubly primed fluctuations are expected to be small and are likely candidates for being neglected on the basis of order of magnitude arguments.

For incompressible flows the momentum equation can be written in the simpler form

$$\frac{\partial}{\partial t}(\rho\bar{u}_i) + \frac{\partial}{\partial x_j}(\rho\bar{u}_i\bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} - \rho\overline{u_i' u_j'}) \tag{5.78}$$

where $\bar{\tau}_{ij}$ takes on the reduced form

$$\bar{\tau}_{ij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{5.79}$$

As we noted in connection with the continuity equation, there is no difference between the mass-weighted and conventional variables for incompressible flow.

5.2.5 Reynolds Form of the Energy Equation

The thermal variables H, h, and T are all related, and the energy equation takes on different forms, depending upon which one is chosen to be the transported thermal variable. To develop one common form, we start with the energy equation as given by Eq. (5.22). The generation term, $\partial Q/\partial t$, will be neglected. Assuming that the total energy is composed only of internal energy and kinetic energy, and replacing E_t by $\rho H - p$, we can write Eq. (5.22) in summation notation as

$$\frac{\partial}{\partial t}(\rho H) + \frac{\partial}{\partial x_j}(\rho u_j H + q_j - u_i\tau_{ij}) = \frac{\partial p}{\partial t} \tag{5.80}$$

To obtain the Reynolds energy equation in conventionally averaged variables, we replace the dependent variables in Eq. (5.80) with the decomposition

indicated by Eq. (5.61). After time averaging, the equation becomes

$$\frac{\partial}{\partial t}\left(\bar{\rho}\bar{H} + \overline{\rho' H'}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j\bar{H} + \bar{\rho}\overline{u_j' H'} + \overline{\rho' u_j'}\bar{H} + \overline{\rho' u_j' H'} + \bar{u}_j\overline{\rho' H'} - k\frac{\partial \bar{T}}{\partial x_j}\right)$$

$$= \frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x_j}\left[\bar{u}_i\left(-\frac{2}{3}\mu\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right) + \mu\bar{u}_i\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right)\right.$$

$$\left. - \frac{2}{3}\mu\delta_{ij}\overline{u_i'\frac{\partial u_k'}{\partial x_k}} + \mu\left(\overline{u_i'\frac{\partial u_j'}{\partial x_i}} + \overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)\right] \tag{5.81}$$

It is frequently desirable to utilize static temperature as a dependent variable in the energy equation. We will let $h = c_p T$ and write Eq. (5.33) in conservative form to provide a convenient starting point for the development of the Reynolds averaged form:

$$\frac{\partial}{\partial t}(\rho c_p T) + \frac{\partial}{\partial x_j}\left(\rho c_p u_j T - k\frac{\partial T}{\partial x_j}\right) = \frac{\partial p}{\partial t} + u_j\frac{\partial p}{\partial x_j} + \Phi \tag{5.82}$$

The dissipation function Φ [see Eq. (5.34)] can be written in terms of the velocity components using summation convention as

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu\left[-\frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)^2 + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2\right] \tag{5.83}$$

The variables in Eq. (5.83) are then replaced with the decomposition indicated by Eq. (5.61), and the resulting equation is time averaged. After eliminating terms known to be zero, the Reynolds energy equation in terms of temperature becomes

$$\frac{\partial}{\partial t}\left(c_p\bar{\rho}\bar{T} + c_p\overline{\rho' T'}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}c_p\bar{T}\bar{u}_j + c_p\bar{T}\overline{\rho' u_j'}\right)$$

$$= \frac{\partial \bar{p}}{\partial t} + \bar{u}_j\frac{\partial \bar{p}}{\partial x_j} + \overline{u_j'\frac{\partial p'}{\partial x_j}}$$

$$+ \frac{\partial}{\partial x_j}\left(k\frac{\partial \bar{T}}{\partial x_j} - \bar{\rho}c_p\overline{T' u_j'} - c_p\overline{\rho' T' u_j'} - \bar{u}_j c_p\overline{\rho' T'}\right) + \bar{\Phi} \tag{5.84}$$

where

$$\bar{\Phi} = \overline{\tau_{ij}\frac{\partial u_i}{\partial x_j}} = \bar{\tau}_{ij}\frac{\partial \bar{u}_i}{\partial x_j} + \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}} \tag{5.85}$$

The $\bar{\tau}_{ij}$ in Eq. (5.85) should be evaluated as indicated by Eq. (5.74).

To develop the Reynolds form of the energy equation in mass-weighted variables, we replace the dependent variables in Eq. (5.80) with the

decomposition indicated by Eq. (5.65) and time average the entire equation. The result can be written

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{H}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{H} + \overline{\rho u_j'' H''} - k\frac{\partial \bar{T}}{\partial x_j}\right) = \frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{u}_i\bar{\tau}_{ij} + \overline{u_i''\tau_{ij}}\right) \tag{5.86}$$

where $\bar{\tau}_{ij}$ can be evaluated as given by Eq. (5.77) in terms of mass-weighted variables.

In terms of static temperature, the Reynolds energy equation in mass-weighted variables becomes

$$\frac{\partial}{\partial t}\left(\bar{\rho}c_p\tilde{T}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}c_p\tilde{T}\tilde{u}_j\right) = \frac{\partial \bar{p}}{\partial t} + \tilde{u}_j\frac{\partial \bar{p}}{\partial x_j} + \overline{u_j''\frac{\partial p}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(k\frac{\partial \tilde{T}}{\partial x_j} + k\frac{\partial \overline{T''}}{\partial x_j} - c_p\overline{\rho T'' u_j''}\right) + \bar{\Phi} \tag{5.87}$$

where

$$\bar{\Phi} = \overline{\tau_{ij}\frac{\partial u_i}{\partial x_j}} = \bar{\tau}_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} + \overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}} \tag{5.88}$$

For incompressible flows the energy equation can be written in terms of total enthalpy as

$$\frac{\partial \rho \overline{H}}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\overline{H} + \rho\overline{u_j'H'} - k\frac{\partial \overline{T}}{\partial x_j}\right) = \frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x_j}\left[\mu\bar{u}_i\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right) + \mu\left(\overline{u_i'\frac{\partial u_j'}{\partial x_i}} + \overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)\right] \tag{5.89}$$

and in terms of static temperature as

$$\frac{\partial}{\partial t}\left(\rho c_p\overline{T}\right) + \frac{\partial}{\partial x_j}\left(\rho c_p\overline{T}\bar{u}_j\right) = \frac{\partial \bar{p}}{\partial t} + \bar{u}_j\frac{\partial \bar{p}}{\partial x_j} + \overline{u_j'\frac{\partial p'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(k\frac{\partial \overline{T}}{\partial x_j} - \rho c_p\overline{T'u_j'}\right) + \bar{\Phi} \tag{5.90}$$

where $\bar{\Phi}$ is reduced slightly in complexity owing to the vanishing of the volumetric dilatation term in $\bar{\tau}_{ij}$ for incompressible flow.

5.2.6 Comments on the Reynolds Equations

At first glance, the Reynolds equations are likely to appear quite complex, and we are tempted to question whether or not we have made any progress toward

solving practical problems in turbulent flow. Certainly, a major problem in fluid mechanics is that more equations can be written than can be solved. Fortunately, for many important flows, the Reynolds equations can be simplified. Before we turn to the task of simplifying the equations, let us examine the Reynolds equations further.

We will consider an incompressible turbulent flow first and interpret the Reynolds momentum equation in the form of Eq. (5.78). The equation governs the time-mean motion of the fluid, and we recognize some familiar momentum flux and laminar-like stress terms plus some new terms involving fluctuations that must represent apparent turbulent stresses. These apparent turbulent stresses originated in the momentum flux terms of the Navier-Stokes equations. To put this another way, the equations of mean motion relate the particle acceleration to stress gradients, and since we know how acceleration for the time-mean motion is expressed, anything new in these equations must be apparent stress gradients due to the turbulent motion. To illustrate, we will utilize the continuity equation to arrange Eq. (5.78) in a form in which the particle (substantial) derivative appears on the left-hand side,

$$\underbrace{\rho \frac{D\bar{u}_i}{Dt}}_{\substack{\text{Particle acceleration} \\ \text{of mean motion}}} = \underbrace{-\frac{\partial \bar{p}}{\partial x_i}}_{\substack{\text{Mean pressure} \\ \text{gradient}}} + \underbrace{\frac{\partial (\bar{\tau}_{ij})_{\text{lam}}}{\partial x_j}}_{\substack{\text{Laminar-like} \\ \text{stress gradients} \\ \text{for the mean motion}}} + \underbrace{\frac{\partial (\bar{\tau}_{ij})_{\text{turb}}}{\partial x_j}}_{\substack{\text{Apparent stress} \\ \text{gradients due to} \\ \text{transport of} \\ \text{momentum by} \\ \text{turbulent fluctuations}}} \tag{5.91}$$

where $(\bar{\tau}_{ij})_{\text{lam}}$ is the same as Eq. (5.79) and has the same form in terms of the time-mean velocities as the stress tensor for a laminar incompressible flow. The apparent turbulent stresses can be written as

$$(\bar{\tau}_{ij})_{\text{turb}} = -\rho \overline{u_i' u_j'} \tag{5.92}$$

These apparent stresses are commonly called the Reynolds stresses.

For compressible turbulent flow, labeling the terms according to the acceleration of the mean motion and apparent stresses becomes more of a challenge. Using conventional averaging procedures, the presence of terms like $\overline{\rho' u_i'}$ can result in the flux of momentum across mean flow streamlines, frustrating our attempts to categorize terms. The use of mass-weighted averaging eliminates the $\overline{\rho' u_i'}$ terms and provides a compact expression for the particle acceleration but complicates the separation of stresses into purely laminar-like and apparent turbulent categories. When conventionally averaged variables are used, the fluctuating components of $\bar{\tau}_{ij}$ vanish when the equations are time averaged. They do not vanish, however, when mass-weighted averaging is used. To illustrate, we will arrange Eq. (5.76) (using the continuity equation) in a form that utilizes

the substantial derivative and label the terms as follows:

$$\underbrace{\bar{\rho}\frac{D\tilde{u}_i}{Dt}}_{\substack{\text{Particle acceleration of mean motion}}} = \underbrace{-\frac{\partial \bar{p}}{\partial x_i}}_{\substack{\text{Mean pressure gradient}}} + \underbrace{\frac{\partial(\bar{\tau}_{ij})_{\text{lam}}}{\partial x_j}}_{\substack{\text{Laminar-like stress gradients for the mean motion}}} + \underbrace{\frac{\partial(\bar{\tau}_{ij})_{\text{turb}}}{\partial x_j}}_{\substack{\text{Apparent stress gradients due to transport of momentum by turbulent fluctuations and deformations attributed to fluctuations}}} \tag{5.93a}$$

The form of Eq. (5.93*a*) is identical to that of Eq. (5.91) except that $\tilde{u}_i$ replaces the $\bar{u}_i$ used in Eq. (5.91). If we insist that $(\bar{\tau}_{ij})_{\text{lam}}$ have the same *form* as for a laminar flow, then the second half of the $\bar{\tau}_{ij}$ of Eq. (5.77) should be attributed to turbulent transport, resulting in

$$(\bar{\tau}_{ij})_{\text{lam}} = \mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right] \tag{5.93b}$$

and

$$(\bar{\tau}_{ij})_{\text{turb}} = -\overline{\rho u_i'' u_j''} + \mu\left[\left(\frac{\partial \overline{u_i''}}{\partial x_j} + \frac{\partial \overline{u_j''}}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \overline{u_k''}}{\partial x_k}\right] \tag{5.93c}$$

As before, viscosity fluctuations have been neglected in obtaining Eq. (5.93*a*). The second term in the expression for $(\bar{\tau}_{ij})_{\text{turb}}$ involving the molecular viscosity is expected to be much smaller than the $-\overline{\rho u_i'' u_j''}$ component.

We can perform a similar analysis on the Reynolds form of the energy equation and identify certain terms involving temperature or enthalpy fluctuations as apparent heat flux quantities. For example, in Eq. (5.84) the molecular "laminar-like" heat flux term is

$$-(\nabla \cdot \mathbf{q})_{\text{lam}} = \frac{\partial}{\partial x_j}\left(k\frac{\partial \bar{T}}{\partial x_j}\right) \tag{5.94a}$$

and the apparent turbulent (Reynolds) heat flux component is

$$-(\nabla \cdot \mathbf{q})_{\text{turb}} = \frac{\partial}{\partial x_j}\left(-\bar{\rho}c_p\overline{T'u_j'} - c_p\overline{\rho' T' u_j'} - \bar{u}_j c_p \overline{\rho' T'}\right) \tag{5.94b}$$

Further examples illustrating the form of the Reynolds stress and heat flux terms will be given in subsequent sections that consider reduced forms of the Reynolds equations.

The Reynolds equations cannot be solved in the form given because the new apparent turbulent stresses and heat flux quantities must be viewed as new unknowns. To proceed further, we need to find additional equations involving the new unknowns or make assumptions regarding the relation between the new apparent turbulent quantities and the time-mean flow variables. This is known

as the closure problem, which is most commonly handled through *turbulence modeling*, which is discussed in Section 5.4.

5.2.7 Filtered Navier-Stokes Equations for Large-Eddy Simulation

At the present time, DNS and LES require such enormous computer resources that it is not feasible to use them for design calculations. However, during the professional careers of the present generation of students, it is very likely that the use of LES as a design tool will become as commonplace as the more refined closure schemes used today for the Reynolds averaged equations. In light of these anticipated advances, it seems appropriate to describe the LES approach briefly in this section. Because the goal is to only introduce the LES approach, the incompressible equations will be considered for the sake of brevity.

The methodology for LES parallels that employed for computing turbulent flows through closure of the Reynolds averaged equations in many respects. First, a set of equations is derived from the Navier-Stokes equations by performing a type of averaging. For LES a spatial average is employed instead of the temporal average used in deriving the Reynolds equations. The averaged equations contain stress terms that must be evaluated through modeling to achieve closure. The equations are then solved numerically. The basic difference between the RANS and LES approaches arises in the choice of quantities to be resolved.

The equations solved in LES are formally developed by "filtering" the Navier-Stokes equations to remove the small spatial scales. The resulting equations describe the evolution of the large eddies and contain the subgrid-scale stress tensor that represents the effects of the unresolved small scales.

Following Leonard (1974), flow variables are decomposed into large (filtered, resolved) and subgrid (residual) scales, as follows:

$$u_i = \bar{u}_i + u_i' \tag{5.94c}$$

Note that the bar and prime have a different meaning here than when previously used in connection with the Reynolds equations. The filtered variable is defined in the general case by the convolution integral

$$\bar{u}_i(x_1, x_2, x_3) = \iiint_D \left[\prod_{j=1}^{3} G_j(x_j, x_j') \right] u_i(x_1', x_2', x_3')\, dx_1'\, dx_2'\, dx_3' \tag{5.94d}$$

over the entire flow domain, where x_i and x_i' are position vectors and G is the general filter function. To return the correct value when u is constant, G is normalized by requiring that

$$\iiint_D \left[\prod_{j=1}^{3} G_j(x_j, x_j') \right] dx_1'\, dx_2'\, dx_3' = 1$$

Although several filter functions, G, have been employed (Aldama, 1990), the volume-averaged "box" (also known as "top-hat") filter is most frequently used with finite-difference and finite-volume methods. The box filter is given by

$$G_j(x_j - x_j') = \begin{cases} 1/\Delta_j & |x_j - x_j'| \leqslant \Delta_j/2 \\ 0 & \text{otherwise} \end{cases} \tag{5.94e}$$

This gives

$$\bar{u}_i(\mathbf{x}, t) = \frac{1}{\Delta^3} \int_{x_1 - \Delta x_1/2}^{x_1 + \Delta x_1/2} \int_{x_2 - \Delta x_2/2}^{x_2 + \Delta x_2/2} \times \int_{x_3 - \Delta x_3/2}^{x_3 + \Delta x_3/2} u(x_1 - x_1', x_2 - x_2', x_3 - x_3')\, dx_1'\, dx_2'\, dx_3' \tag{5.94f}$$

where $\Delta = (\Delta_1\, \Delta_2\, \Delta_3)^{1/3}$ and $\Delta_1, \Delta_2, \Delta_3$ are increments in x_1, x_2, x_3, respectively. Filtering the Navier-Stokes continuity and momentum equations gives

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0 \tag{5.94g}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k\, \partial x_k} \tag{5.94h}$$

However, we cannot solve the system for both $\bar{u}_i$ and $\overline{u_i u_j}$, so we represent the convective flux in terms of decomposed variables, as follows:

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j + \tau_{ij}$$

resulting in

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k\, \partial x_k} - \frac{\partial \tau_{ij}}{\partial x_j} \tag{5.94i}$$

where τ_{ij} is the subgrid-scale stress tensor,

$$\tau_{ij} = \left(\overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j\right) + \left(\overline{u_i' \bar{u}_j} + \overline{\bar{u}_i u_j'}\right) + \overline{(u_i' u_j')} \tag{5.94j}$$

The first term in parentheses on the right-hand side is known as the Leonard stress, the second term, the cross-term stress, and the third term, the Reynolds stress. Note that if time averaging were being employed instead of filtering, the first two terms would be zero, leaving only the Reynolds stress. Thus an important difference between time averaging and filtering is that for filtering, $\overline{\bar{u}_i} \neq \bar{u}_i$. That is, a second averaging yields a different result from the first averaging.

Although the Leonard term can be computed from the resolved flow, it is not easily done with most finite-difference and finite-volume schemes. Furthermore, the Leonard stresses can be shown to be (Shaanan et al., 1975) of the same order as the truncation error for second-order schemes. As a result,

most present-day finite-difference and finite-volume approaches consider that the subgrid-scale model accounts for the important effects of all three terms in Eq. (5.94*i*). Another point in defense of this approach is that the sum of the cross-term stress and the Reynolds stress is not Galilean invariant (Speziale, 1985), whereas the sum of all three terms is.

Solving the filtered Navier-Stokes equations gives the time-dependent solution for the resolved variables. In applications that operate nominally at steady state, we are usually interested in the time-mean motion of the flow and the drag, lift, etc., in a steady sense. Thus the solution to the filtered equations must be averaged in time (and/or averaged spatially in directions in which the flow is assumed to be homogeneous) to obtain the time-averaged values of the variables. If it is desired to compute turbulence statistics to compare with experimental measurements, the fluctuations in the time-averaged sense are computed from the difference between the resolved flow and the time-averaged results, as for example, $\bar{u}_i'' = \bar{u}_i - \langle \bar{u}_i \rangle$, where the double prime designates a time-basis fluctuation and the angle brackets indicate a time- or ensemble-averaged quantity. Of course, the solution to the filtered equations resolves the motion of the large eddies and is not quite the same as the true instantaneous solution to the full Navier-Stokes equations. Thus we cannot expect perfect agreement when turbulence statistics computed from LES are compared with DNS results or experimental data.

Modeling for the subgrid-scale stress is addressed in Section 5.4.8. Information on the extension of LES to compressible flows can be found in the works of Erlebacher et al. (1992) and Moin et al. (1991).

5.3 BOUNDARY-LAYER EQUATIONS

5.3.1 Background

The concept of a boundary layer originated with Ludwig Prandtl in 1904 (Prandtl, 1926). Prandtl reasoned from experimental evidence that for sufficiently large Reynolds numbers a thin region existed near a solid boundary where viscous effects were at least as important as inertia effects no matter how small the viscosity of the fluid might be. Prandtl deduced that a much reduced form of the governing equations could be used by systematically employing two constraints. These were that the viscous layer must be thin relative to the characteristic streamwise dimension of the object immersed in the flow, $\delta/L \ll 1$, and that the largest viscous term must be of the same approximate magnitude as any inertia (particle acceleration) term. Prandtl used what we now call an order of magnitude analysis to reduce the governing equations. Essentially, his conclusions were that second derivatives of the velocity components in the streamwise direction were negligible compared to corresponding derivatives transverse to the main flow direction and that the entire momentum equation for the transverse direction could be neglected.

In the years since 1904, we have found that a similar reduction can often be made in the governing equations for other flows for which a primary flow direction can be identified. These flows include jets, wakes, mixing layers, and the developing flow in pipes and other internal passages. Thus the terminology "boundary-layer flow" or "boundary-layer approximation" has taken on a more general meaning, which refers to circumstances that permit the neglect of the transverse momentum equation and the streamwise second-derivative term in the remaining momentum equation (or equations in the case of 3-D flow). It is increasingly common to refer to these reduced equations as the "thin-shear-layer" equations. This terminology seems especially appropriate in light of the applicability of the equations to free-shear flows such as jets and wakes as well as flows along a solid boundary. We will use both designations, boundary layer and thin-shear layer, interchangeably in this book.

5.3.2 Boundary-Layer Approximation for Steady Incompressible Flow

It is useful to review the methodology used to obtain the boundary-layer approximations to the Navier-Stokes and Reynolds equations for steady 2-D incompressible constant-property flow along an isothermal surface at temperature T_w. First, we define the nondimensional variables (much as was done in Section 5.1.7):

$$u^* = \frac{u}{u_\infty} \qquad v^* = \frac{v}{u_\infty} \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad p^* = \frac{p}{\rho u_\infty^2} \qquad \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{5.95}$$

and introduce them into the Navier-Stokes equations by substitution. After rearrangement, the results can be written as

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5.96}$$

x momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{5.97}$$

y momentum:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{5.98}$$

energy:

$$u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} = \frac{1}{\mathrm{Re}_L \,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}}\right) + \mathrm{Ec}\left(u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*}\right) + \frac{\mathrm{Ec}}{\mathrm{Re}_L}\left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\left(\frac{\partial v^*}{\partial y^*}\right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2\right] \tag{5.99}$$

In the above,

$$\text{Re}_L = \text{Reynolds number} = \frac{\rho u_\infty L}{\mu}$$

$$\text{Pr} = \text{Prandtl number} = \frac{c_p \mu}{k}$$

$$\text{Ec} = \text{Eckert number} = 2\frac{T_0 - T_\infty}{T_w - T_\infty}$$

and u_∞, T_∞ are the free stream velocity and temperature, respectively, and T_0 is the stagnation temperature. The product Re Pr is also known as the Peclet number Pe.

Following Prandtl, we assume that the thicknesses of the viscous and thermal boundary layers are small relative to a characteristic length in the primary flow direction. That is, $\delta/L \ll 1$ and $\delta_t/L \ll 1$ (see Fig. 5.5). For convenience, we let $\epsilon = \delta/L$ and $\epsilon_t = \delta_t/L$. Since ϵ and ϵ_t are both assumed to be small, we will take them to be of the same order of magnitude. We are assured that ϵ and ϵ_t are small over L if $\partial\delta/\partial x$ and $\partial\,\delta_t/\partial x$ are everywhere small. At a distance L from the origin of the boundary layer, we now estimate typical or expected sizes of terms in the equation.

As a general rule, we estimate sizes of derivatives by using the "mean value" provided by replacing the derivative by a finite difference over the expected range of the variables in the boundary-layer flow. For example, we estimate the size of $\partial u^*/\partial x^*$ by noting that for flow over a flat plate in a uniform stream u^* ranges between 1 and 0 as x^* ranges between 0 and 1; thus we say that we expect $\partial u^*/\partial x^*$ to be of the order of magnitude of 1. That is,

$$\left|\frac{\partial u^*}{\partial x^*}\right| \approx \left|\frac{0-1}{1-0}\right| = 1$$

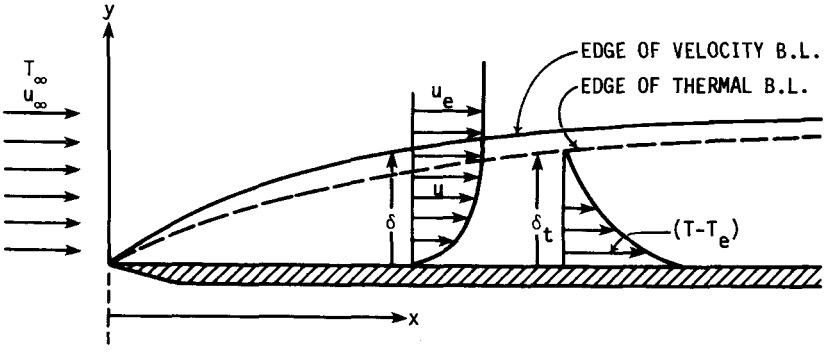


Figure 5.5 Notation and coordinate system for a boundary layer (B.L.) on a flat plate.

A factor of 2 or so does not matter in our estimates, but a factor of 10–100 does and represents an order of magnitude. It should be noted that the velocity at the outer edge of the boundary layer may deviate somewhat from u_∞ (as would be the case for flows with a pressure gradient) without changing the order of magnitude of $\partial u^*/\partial x^*$. Having established $(\partial u^*/\partial x^*) \cong 1$, we now consider the $\partial v^*/\partial y^*$ term in the continuity equation. We require that this term be of the same order of magnitude as $\partial u^*/\partial x^*$, so that mass can be conserved. Since y^* ranges between 0 and ϵ in the boundary layer, we expect from the continuity equation that v^* will also range between 0 and ϵ. Thus, $v^* \cong \epsilon$. If $\partial\delta/\partial x$ should locally become large owing to some perturbation, then the continuity equation suggests that v^* could also become large locally. The nondimensional thermal variable θ clearly ranges between 0 and 1 for incompressible constant-property flow.

We are now in a position to establish the order of magnitudes for the terms in the Navier-Stokes equations. The estimates are labeled underneath the terms in Eqs. (5.100)–(5.103).

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5.100}$$

$$1 \qquad 1$$

x momentum:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{5.101}$$

$$1 \quad 1 \qquad \epsilon \quad \frac{1}{\epsilon} \qquad 1 \qquad \epsilon^2 \quad 1 \qquad \frac{1}{\epsilon^2}$$

y momentum:

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{5.102}$$

$$1 \quad \epsilon \qquad \epsilon \quad 1 \qquad \epsilon \qquad \epsilon^2 \qquad \epsilon \qquad \frac{1}{\epsilon}$$

energy:

$$u^*\frac{\partial \theta}{\partial x^*} + v^*\frac{\partial \theta}{\partial y^*} = \frac{1}{\mathrm{Re}_L\,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}}\right) + \mathrm{Ec}\left(u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*}\right)$$

$$1 \quad 1 \qquad \epsilon \quad \frac{1}{\epsilon} \qquad \epsilon^2 \qquad 1 \qquad \frac{1}{\epsilon^2} \qquad 1 \quad 1 \quad 1 \qquad \epsilon \quad \epsilon$$

$$+ \frac{\mathrm{Ec}}{\mathrm{Re}_L}\left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\left(\frac{\partial v^*}{\partial y^*}\right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2\right]$$

$$\epsilon^2 \qquad 1 \qquad\qquad 1 \qquad\qquad \epsilon^2 \quad 1 \quad \frac{1}{\epsilon^2}$$

(5.103)

Some comments are in order. In Eq. (5.101), the order of magnitude of the pressure gradient was established by the observation that the Navier-Stokes equations reduce to the Euler equations (see Section 5.5) at the outer edge of the viscous region. The pressure gradient must be capable of balancing the inertia terms. Hence the pressure gradient and the inertia terms must be of the same order of magnitude. We are also requiring that the largest viscous term be of the same order of magnitude as the inertia terms. For this to be true, Re_L must be of the order of magnitude of $1/\epsilon^2$, as can be seen from Eq. (5.101).

The order of magnitude of all terms in Eq. (5.102) can be established in a straight-forward manner except for the pressure gradient. Since the pressure gradient must be balanced by other terms in the equation, its order of magnitude cannot be greater than any of the others in Eq. (5.102). Accordingly, its maximum order of magnitude must be ϵ, as recorded in Eq. (5.102).

In the energy equation, we have assumed somewhat arbitrarily that the Eckert number Ec was of the order of magnitude of 1. This should be considered a typical value. Ec can become an order of magnitude larger or smaller in certain applications. The order of magnitude of the Peclet number, Re Pr, was set at $1/\epsilon^2$. Since we have already assumed $\mathrm{Re}_L \cong (1/\epsilon^2)$ in dealing with the momentum equations, this suggests that $\mathrm{Pr} \cong 1$. This is consistent with our original hypothesis that ϵ and ϵ_t were both small, i.e., $\delta \cong \delta_t$. In other words, we are assuming that the Pe is of the same order of magnitude as Re. We expect that the present results will be applicable to flows in which Pr does not vary from 1 by more than an order of magnitude. The exact limitation of the analysis must be determined by comparisons with experimental data. Three orders are specified for the last term in parentheses in Eq. (5.103) to account for the cross-product term that results from squaring the quantity in parentheses.

Carrying out the multiplication needed to establish the order of each term in Eqs. (5.101)–(5.103), we observe that all terms in the x-momentum equation are of the order of 1 in magnitude except for the streamwise second-derivative (diffusion) term, which is of the order of ϵ^2. No term in the y-momentum equation is larger than ϵ in estimated magnitude. Several terms in the energy equation are of the order 1 in magnitude, although several in the compression work and viscous dissipation terms are smaller. Keeping terms whose order of magnitude estimates are equal to 1 gives the boundary-layer equations. These are recorded below in terms of dimensional variables.

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.104}$$

momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.105}$$

energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\beta T u}{\rho c_p}\frac{dp}{dx} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{5.106}$$

where ν is the kinematic viscosity μ/ρ and α is the thermal diffusivity $k/\rho c_p$.

The form of the energy equation has been generalized to include nonideal gas behavior through the introduction of β, the volumetric expansion coefficient:

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg)_p$$

For an ideal gas, $\beta = 1/T$, where T is absolute temperature. It should be pointed out that the last two terms in Eq. (5.106) were retained from the order of magnitude analysis on the basis that Ec $\sim$ 1. Should Ec become of the order of ϵ or smaller for a particular flow, neglecting these terms should be permissible.

To complete the mathematical formulation, initial and boundary conditions must be specified. The steady boundary-layer momentum and energy equations are parabolic with the streamwise direction being the marching direction. Initial distributions of u and T must be provided. The usual boundary conditions are

$$u(x,0) = v(x,0) = 0$$

$$T(x,0) = T_w(x) \quad \text{or} \quad -\frac{\partial T}{\partial y}\bigg)_{y=0} = \frac{q(x)}{k} \tag{5.107}$$

$$\lim_{y\to\infty} u(x,y) = u_e(x) \qquad \lim_{y\to\infty} T(x,y) = T_e(x)$$

where the subscript e refers to conditions at the edge of the boundary layer. The pressure gradient term in Eqs. (5.105) and (5.106) is to be evaluated from the given boundary information. With $u_e(x)$ specified, dp/dx can be evaluated from an application of the equations that govern the inviscid outer flow (Euler's equations), giving $dp/dx = -\rho u_e\, du_e/dx$.

It is not difficult to extend the boundary-layer equations to variable-property and/or compressible flows. The constant-property restriction was made only as a convenience as we set about the task of illustrating principles that can frequently be used to determine a reduced but approximate set of governing equations for a flow of interest. The compressible form of the boundary-layer equations, which will also account for property variations, is presented in Section 5.3.3.

Before moving on from our order of magnitude deliberations for laminar flows, we should raise the question as to which terms neglected in the boundary-layer approximation should first become important as δ/L becomes larger and larger. Terms of the order of ϵ will next become important and then, eventually, terms of the order of ϵ^2. We note that the second-derivative term neglected in the streamwise momentum equation is of the order of ϵ^2, whereas most terms in the y-momentum equation are of the order of ϵ. This means that

contributions through the transverse momentum equation are expected to become important before additional terms need to be considered in the x-momentum equation. The set of equations that results from retaining terms of both orders of 1 and ϵ in the order of magnitude analysis, while neglecting terms of the order of ϵ^2 and higher, has proven to be useful in computational fluid dynamics. Such steady flow equations, which neglect all streamwise second-derivative terms, are known as the "parabolized" Navier-Stokes equations for supersonic applications and are known as the "partially parabolized" Navier-Stokes equations for subsonic applications. These are but two examples from a category of equations frequently called parabolized Navier-Stokes equations. These equations are intermediate in complexity between the Navier-Stokes and boundary-layer equations and are discussed in Chapter 8.

We next consider extending the boundary-layer approximation to an incompressible constant-property 2-D turbulent flow. Under our incompressible assumption, $\rho' = 0$ and the Reynolds equations simplify considerably. We will nondimensionalize the incompressible Reynolds equations very much in the same manner as for the Navier-Stokes equations, letting

$$u^* = \frac{\bar{u}}{u_\infty} \qquad v^* = \frac{\bar{v}}{u_\infty} \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad p^* = \frac{\bar{p}}{\rho u_\infty^2} \qquad (u')^* = \frac{u'}{u_\infty}$$

$$(v')^* = \frac{v'}{u_\infty} \qquad \theta = \frac{\bar{T} - T_\infty}{T_w - T_\infty} \qquad H^* = \frac{\bar{H} - H_\infty}{H_w - H_\infty} \qquad \theta' = \frac{T'}{T_w - T_\infty} \tag{5.108}$$

Asterisks appended to parentheses indicate that all quantities within the parentheses are dimensionless; that is, instead of $\overline{u'^* v'^*}$, we will use the more convenient notation $(\overline{u'v'})^*$.

As before, we assume that $\delta/L \ll 1$, $\delta_t/L \ll 1$, and let $\epsilon = \delta/L \approx \delta_t/L$. We rely on experimental evidence to guide us in establishing the magnitude estimates for the Reynolds stress and heat flux terms. Experiments indicate that the Reynolds stresses can be at least as large as the laminar counterparts. This requires that $(\overline{u'v'})^* \sim \epsilon$. Measurements suggest that $(\overline{u'^2})^*, (\overline{u'v'})^*, (\overline{v'^2})^*$, while differing in magnitudes and distribution somewhat, are nevertheless of the same order of magnitude in the boundary layer. That is, we cannot stipulate that the magnitudes of any of these terms are different by a factor of 10 or more. A similar observation can be made for the energy equation, leading to the conclusion that $(\overline{\theta' v'})^*$ and $(\overline{\theta' u'})^*$ are of the order of magnitude of ϵ. Triple correlations such as $(\overline{u'u'u'})^*$ are clearly expected to be smaller than double correlations, and they will be taken to be of the order ϵ^2 (Schubauer and Tchen, 1959). It will be expedient to invoke the boundary-layer approximation to the form of the energy equation that employs the total enthalpy as the transported thermal variable, Eq. (5.89). We will, however, substitute for H' according to

$$H' = c_p T' + u_i' \bar{u}_i + \frac{u_i' u_i'}{2} - \frac{1}{2}\overline{u_i' u_i'}$$

The above expression for H' may look suspicious because a time-averaged expression appears on the right-hand side. However, this is necessary so that $\overline{H'} = 0$. The correct expression for H' can be derived by expanding H in terms of decomposed temperature and velocity variables and then subtracting $\overline{H}$. That is, let H' be defined as $H' = H - \overline{H}$, where

$$H = c_p T + \frac{u_i u_i}{2} = c_p \overline{T} + c_p T' + \frac{(\overline{u}_i + u_i')(\overline{u}_i + u_i')}{2}$$

The incompressible nondimensional Reynolds equations are given below along with the order of magnitude estimates for the individual terms:

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5.109}$$

$$1 \qquad 1$$

x momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) - \frac{\partial}{\partial y^*}(\overline{u'v'})^* - \frac{\partial}{\partial x^*}\left(\overline{u'^2}\right)^* \tag{5.110}$$

$$1 \quad 1 \qquad \epsilon \quad \frac{1}{\epsilon} \qquad 1 \qquad \epsilon^2 \quad 1 \qquad \frac{1}{\epsilon^2} \qquad \frac{\epsilon}{\epsilon} \qquad \epsilon$$

y momentum:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) - \frac{\partial(\overline{v'u'})^*}{\partial x^*} - \frac{\partial\left(\overline{v'^2}\right)^*}{\partial y^*} \tag{5.111}$$

$$1 \quad \epsilon \qquad \epsilon \quad 1 \qquad 1 \qquad \epsilon^2 \qquad \epsilon \qquad \frac{1}{\epsilon} \qquad \epsilon \qquad 1$$

energy:

$$u^* \frac{\partial H^*}{\partial x^*} + v^* \frac{\partial H^*}{\partial y^*} = \frac{T_w - T_\infty}{T_w - T_0}\left[-\frac{\partial}{\partial x^*}(\overline{u'\theta'})^* - \frac{\partial}{\partial y^*}(\overline{v'\theta'})^*\right] + \frac{1}{\mathrm{Re}_L \,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}}\right) + \mathrm{Ec}\left[-\frac{\partial}{\partial x^*}(u\overline{u'u'})^* - \frac{\partial}{\partial x^*}(v\overline{v'u'})^*\right]$$

$$1 \quad 1 \qquad \epsilon \quad \frac{1}{\epsilon} \qquad 1 \qquad \epsilon \qquad \frac{\epsilon}{\epsilon} \qquad\qquad \epsilon^2 \quad 1 \qquad \frac{1}{\epsilon^2} \qquad 1 \qquad \epsilon \qquad \epsilon^2$$

$$
\begin{array}{cccc}
-\dfrac{\partial}{\partial y^*}(u\overline{u'v'})^* & -\dfrac{\partial}{\partial y^*}(v\overline{v'v'})^* & -\dfrac{1}{2}\dfrac{\partial}{\partial x^*}(\overline{u'u'u'})^* & -\dfrac{1}{2}\dfrac{\partial}{\partial x^*}(\overline{v'v'u'})^* \\
\epsilon & \epsilon^2 & \epsilon^2 & \epsilon^2 \\
-\dfrac{1}{2}\dfrac{\partial}{\partial y^*}(\overline{u'u'v'})^* & -\dfrac{1}{2}\dfrac{\partial}{\partial y^*}(\overline{v'v'v'})^* \Big] + \dfrac{\mathrm{Ec}}{\mathrm{Re}_L}\Big[2\dfrac{\partial}{\partial x^*}\Big(u^*\dfrac{\partial u^*}{\partial x^*}\Big) & & & \\
\epsilon^2 & \epsilon^2 & \epsilon^2 & 1 \\
+\dfrac{\partial}{\partial x^*}\Big(v^*\dfrac{\partial u^*}{\partial y^*}\Big) + \dfrac{\partial}{\partial x^*}\Big(v^*\dfrac{\partial v^*}{\partial x^*}\Big) & +\dfrac{\partial}{\partial y^*}\Big(u^*\dfrac{\partial v^*}{\partial y^*}\Big) & +\dfrac{\partial}{\partial y^*}\Big(u^*\dfrac{\partial u^*}{\partial y^*}\Big) & \\
\epsilon & \epsilon^2 & \epsilon & \dfrac{1}{\epsilon^2} \\
+2\dfrac{\partial}{\partial y^*}\Big(v^*\dfrac{\partial v^*}{\partial y^*}\Big) + 2\dfrac{\partial}{\partial x^*}\Big(\overline{u'\dfrac{\partial u'}{\partial x}}\Big)^* & +\dfrac{\partial}{\partial x^*}\Big(\overline{v'\dfrac{\partial u'}{\partial y}}\Big)^* & +\dfrac{\partial}{\partial x}\Big(\overline{v'\dfrac{\partial v'}{\partial x}}\Big)^* & \\
\epsilon^2 & \epsilon & \epsilon & \epsilon \\
+\dfrac{\partial}{\partial y^*}\Big(\overline{u'\dfrac{\partial v'}{\partial x}}\Big)^* & +\dfrac{\partial}{\partial y^*}\Big(\overline{u'\dfrac{\partial u'}{\partial y}}\Big)^* & +2\dfrac{\partial}{\partial y^*}\Big(\overline{v'\dfrac{\partial v'}{\partial y}}\Big)^*\Big] & \\
\epsilon & \epsilon^2 & \epsilon^2 &
\end{array}
\tag{5.112}
$$

Again we assume that Pr and Ec are near 1 in order of magnitude. The 2-D boundary-layer equations are obtained by retaining only terms of the order of 1. They can be written in dimensional variables as follows:

continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

momentum:

$$\rho\bar{u}\frac{\partial \bar{u}}{\partial x} + \rho\bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{d\bar{p}}{dx} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \rho\frac{\partial}{\partial y}(\overline{u'v'}) \tag{5.113}$$

energy:

$$\rho\bar{u}\frac{\partial \bar{H}}{\partial x} + \rho\bar{v}\frac{\partial \bar{H}}{\partial y} = k\frac{\partial^2 \bar{T}}{\partial y^2} - \rho c_p\frac{\partial}{\partial y}(\overline{v'T'}) - \rho\frac{\partial}{\partial y}(\bar{u}\overline{v'u'}) + \mu\frac{\partial}{\partial y}\Big(\bar{u}\frac{\partial \bar{u}}{\partial y}\Big) \tag{5.114}$$

It should be noted that terms of the order of 1 do remain in the y-momentum equation for turbulent flow, namely,

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} = -\frac{\partial}{\partial y}\left(\overline{v'^2}\right)$$

These terms have not been listed above with the boundary-layer equations because they contribute no information about the mean velocities. The pressure variation across the boundary layer is of the order of ϵ (negligible in comparison with the streamwise variation). The boundary-layer energy equation can be easily written in terms of static temperature by substituting

$$c_p\bar{T} + \frac{\bar{u}^2}{2}$$

for $\bar{H}$ in Eq. (5.114). In doing this, we are neglecting $\bar{v}^2$ compared to $\bar{u}^2$ in the kinetic energy of the mean motion. Examination of the way in which $\bar{H}$ appears in Eq. (5.114) reveals that this is permissible in the boundary-layer approximation. Utilizing Eq. (5.113) to eliminate the kinetic energy terms permits the boundary-layer form of the energy equation to be written as

$$\rho\bar{u}c_p\frac{\partial \bar{T}}{\partial x} + \rho\bar{v}c_p\frac{\partial \bar{T}}{\partial y} = k\frac{\partial^2 \bar{T}}{\partial y^2} - \rho c_p\frac{\partial}{\partial y}(\overline{v'T'}) + \bar{u}\frac{d\bar{p}}{dx} + \left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{v'u'}\right)\frac{\partial \bar{u}}{\partial y} \tag{5.115}$$

The last two terms on the right-hand side of Eq. (5.115) can be neglected in some applications. However, it is not correct to categorically neglect these terms for incompressible flows. The last term on the right-hand side, for example, represents the viscous dissipation of energy, which obviously is important in incompressible lubrication applications, where the major heat transfer concern is to remove the heat generated by the viscous dissipation. In some instances, it is also possible to neglect one or both of the last two terms on the right-hand side of Eq. (5.114). Both Eq. (5.114) and (5.115) can be easily treated with finite-difference/finite-volume methods in their entirety, so the temptation to impose further reductions should generally be resisted unless it is absolutely clear that the terms neglected will indeed be negligible. The boundary conditions remain unchanged for turbulent flow.

In closing this section on the development of the thin-shear-layer approximation, it is worthwhile noting that for turbulent flow, the largest term neglected in the streamwise momentum equation, the Reynolds normal stress term, was estimated to be an order of magnitude larger, ϵ, than the largest term neglected in the laminar flow analysis. We note also that only one Reynolds stress term and one Reynolds heat flux term remain in the governing equations after the boundary-layer approximation is invoked.

For any steady internal flow application of the thin-shear-layer equations, it is possible to develop a global channel mass flow constraint. This permits the pressure gradient to be computed rather than requiring that it be given, as in the case for external flow. This point is developed further in Chapter 7.

5.3.3 Boundary-Layer Equations for Compressible Flow

The order of magnitude reduction of the Reynolds equations to boundary-layer form is a lengthier process for compressible flow. Only the results will be presented here. Details of the arguments for elimination of terms are given by Schubauer and Tchen (1959), van Driest (1951), and Cebeci and Smith (1974). As was the case for incompressible flow, guidance must be obtained from experimental observations in assessing the magnitudes of turbulence quantities. An estimate must be made for $\rho'/\bar{\rho}$ for compressible flows.

Measurements in gases for Mach numbers (M) less than about 5 indicate that temperature fluctuations are nearly isobaric for adiabatic flows. This suggests that $T'/\bar{T} \approx -\rho'/\bar{\rho}$. However, there is evidence that appreciable pressure fluctuations exist (8–10% of the mean wall static pressure) at $M = 5$ and it is speculated that $p'/\bar{p}$ increases with increasing M. In the absence of specific experimental evidence to the contrary, it is common to base the order of magnitude estimates of fluctuating terms on the assumption that the pressure fluctuations are small. This appears to be a safe assumption for $M \leqslant 5$, and good predictions based on this assumption have been noted for M as high as 7.5. We will adopt the isobaric assumption here. It is primarily the correlation terms involving the density fluctuations that may increase in magnitude with increasing Mach number above $M \approx 5$.

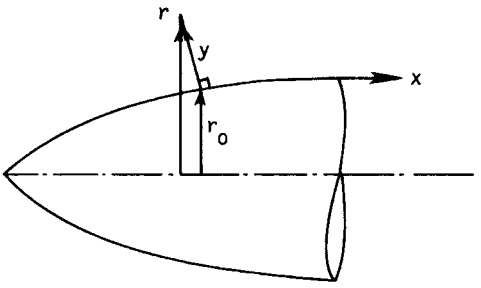
We find that the difference between $\tilde{u}$ and $\bar{u}$ vanishes under the boundary-layer approximation. This follows because $\overline{\rho' u'}$ is expected to be small compared to $\bar{\rho}\bar{u}$ and can be neglected in the momentum equation. We also find $\bar{T} = \tilde{T}$ and $\bar{H} = \tilde{H}$ to be consistent with the boundary-layer approximation. On the other hand, $\overline{\rho' v'}$ and $\bar{\rho}\bar{v}$ are both of about the same order of magnitude in a thin shear layer. Thus $\bar{v} \neq \tilde{v}$. Below, the unsteady boundary-layer equations for a compressible fluid are written in a form applicable to both 2-D and axisymmetric turbulent flow. For convenience, we will drop the use of bars over time-mean quantities and make use of $\tilde{v} = (\bar{\rho}\bar{v} + \overline{\rho' v'})/\bar{\rho}$. The equations are also valid for laminar flow when the terms involving fluctuating quantities are set equal to zero. The coordinate system is indicated in Fig. 5.6. The equations are as follows.

continuity:

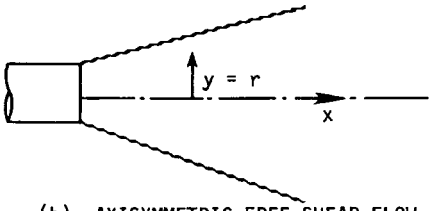
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(r^m \rho u) + \frac{\partial}{\partial y}(r^m \rho \tilde{v}) = 0 \tag{5.116}$$

momentum:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho \tilde{v} \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{r^m} \frac{\partial}{\partial y}\left[r^m \left(\mu \frac{\partial u}{\partial y} - \rho \overline{u' v'} \right) \right] \tag{5.117}$$



(a) EXTERNAL BOUNDARY LAYER



(b) AXISYMMETRIC FREE SHEAR FLOW

(c) CONFINED AXISYMMETRIC FLOW

Figure 5.6 Coordinate system for axisymmetric thin-shear-layer equations. (a) External boundary layer; (b) axisymmetric free-shear flow; (c) confined axisymmetric flow.

energy:

$$\rho\frac{\partial H}{\partial t} + \rho u\frac{\partial H}{\partial x} + \rho\tilde{v}\frac{\partial H}{\partial y}$$

$$= \frac{1}{r^m}\frac{\partial}{\partial y}\left(r^m\left\{\frac{\mu}{\mathrm{Pr}}\frac{\partial H}{\partial y} - \rho c_p\overline{v'T'} + u\left[\left(1 - \frac{1}{\mathrm{Pr}}\right)\mu\frac{\partial u}{\partial y} - \rho\overline{v'u'}\right]\right\}\right) + \frac{\partial p}{\partial t} \tag{5.118}$$

state:

$$\rho = \rho(p, T) \tag{5.119}$$

In the preceding equations, m is a flow index equal to unity for axisymmetric flow ($r^m = r$) and equal to zero for 2-D flow ($r^m = 1$). Other forms of the energy equation will be noted in subsequent sections.

We note that the boundary-layer equations for compressible flow are not significantly more complex than for incompressible flow. Only one Reynolds stress and one heat flux term appear, regardless of whether the flow is compressible or incompressible. As for purely laminar flows, the main difference is in the property variations of μ, k, and ρ for the compressible case, which nearly always requires that a solution be obtained for some form of the energy equation. When properties can be assumed constant (as for many incompressible flows), the momentum equation is independent of the energy equation, and as a result, the energy equation need not be solved for many problems of interest.

The boundary-layer approximation remains valid for a flow in which the turning of the main stream results in a 3-D flow as long as velocity derivatives with respect to only one coordinate direction are large. That is, the 3-D boundary layer is a flow that remains "thin" with respect to only one coordinate direction. The 3-D unsteady boundary-layer equations in Cartesian coordinates, applicable to a compressible turbulent flow, are given below. The y direction is normal to the wall.

continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho \tilde{v}}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{5.120}$$

x-momentum:

$$\frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho \tilde{v} \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} - \rho \overline{u'v'} \right) \tag{5.121}$$

z-momentum:

$$\frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial x} + \rho \tilde{v} \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y} - \rho \overline{w'v'} \right) \tag{5.122}$$

energy:

$$\begin{aligned} \frac{\partial H}{\partial t} + \rho u \frac{\partial H}{\partial x} + \rho \tilde{v} \frac{\partial H}{\partial y} + \rho w \frac{\partial H}{\partial z} \\ = \frac{\partial}{\partial y}\left[\frac{\mu}{\mathrm{Pr}} \frac{\partial H}{\partial y} - \rho c_p \overline{v'T'} + \mu\left(1 - \frac{1}{\mathrm{Pr}}\right)\left(u \frac{\partial u}{\partial y} + w \frac{\partial w}{\partial y}\right) \right. \\ \left. - \rho u \overline{v'u'} - \rho w \overline{v'w'} \right] \end{aligned} \tag{5.123}$$

For a 3-D flow, the boundary-layer approximation permits H to be written as

$$H = c_p T + \frac{u^2}{2} + \frac{w^2}{2}$$

The 3-D boundary-layer equations are used primarily for external flows. This usually permits the pressure gradient terms to be evaluated from the solution to the inviscid flow (Euler) equations. Three-dimensional internal flows are normally computed from slightly different equations, discussed in Chapter 8.

It is common to employ body intrinsic curvilinear coordinates to compute the 3-D boundary layers occurring on wings and other shapes of practical interest. Often, this curvilinear coordinate system is nonorthogonal. An example of this can be found in the work by Cebeci et al. (1977). The orthogonal system is somewhat more common (see, for example, Blottner and Ellis, 1973). One coordinate, x_2, is almost always taken to be orthogonal to the body surface. This convention will be followed here. Below we record the 3-D boundary-layer equations in the orthogonal curvilinear coordinate system described in Section 5.1.8. Typically, x_1 will be directed roughly in the primary flow direction and x_3 will be in the crossflow direction. The metric coefficients (h_1, h_2, h_3) are as defined in Section 5.1.8; however, h_2 will be taken as unity as a result of the boundary-layer approximation. In addition, we will make use of the geodesic curvatures of the surface coordinate lines,

$$K_1 = \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3} \qquad K_3 = \frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} \tag{5.124}$$

With this notation, the boundary-layer form of the conservation equations for a compressible turbulent flow can be written as follows:

continuity:

$$\frac{\partial}{\partial x_1}(\rho h_3 u_1) + \frac{\partial}{\partial x_2}(h_1 h_3 \rho \tilde{u}_2) + \frac{\partial}{\partial x_3}(\rho h_1 u_3) = 0 \tag{5.125}$$

x_1-momentum:

$$\frac{\rho u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \rho \tilde{u}_2 \frac{\partial u_1}{\partial x_2} + \frac{\rho u_3}{h_3}\frac{\partial u_1}{\partial x_3} + \rho u_1 u_3 K_1 - \rho u_3^2 K_3 = -\frac{1}{h_1}\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial u_1}{\partial x_2} - \rho \overline{u_1' u_2'}\right) \tag{5.126}$$

x_3-momentum:

$$\frac{\rho u_1}{h_1}\frac{\partial u_3}{\partial x_1} + \rho \tilde{u}_2 \frac{\partial u_3}{\partial x_2} + \frac{\rho u_3 \partial u_3}{h_3 \partial x_3} + \rho u_1 u_3 K_3 - \rho u_1^2 K_1 = -\frac{1}{h_3}\frac{\partial p}{\partial x_3} + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial u_3}{\partial x_2} - \rho \overline{u_3' u_2'}\right) \tag{5.127}$$

energy:

$$\frac{\rho u_1}{h_1}\frac{\partial H}{\partial x_1} + \rho\bar{u}_2\frac{\partial H}{\partial x_2} + \frac{\rho u_3}{h_3}\frac{\partial H}{\partial x_3} = \frac{\partial}{\partial x_2}\left[\frac{\mu}{\mathrm{Pr}}\frac{\partial H}{\partial x_2} - \rho c_p\overline{u_2'T'} + \mu\left(1 - \frac{1}{\mathrm{Pr}}\right)\left(u_1\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_2}\right) - \rho u_1\overline{u_2'u_1'} - \rho u_3\overline{u_2'u_3'}\right] \quad (5.128)$$

As always, an equation of state, $\rho = \rho(p, T)$, is needed to close the system of equations for a compressible flow. The above equations remain valid for a laminar flow when the fluctuating quantities are set equal to zero.

5.4 INTRODUCTION TO TURBULENCE MODELING

5.4.1 Background

The need for turbulence modeling was pointed out in Section 5.2. In order to predict turbulent flows by numerical solutions to the Reynolds equations, it becomes necessary to make closing assumptions about the apparent turbulent stress and heat flux quantities. All presently known turbulence models have limitations: the ultimate turbulence model has yet to be developed. Some argue philosophically that we have a system of equations for turbulent flows that is both accurate and general in the Navier-Stokes set, and therefore, to hope to develop an alternative system having the same accuracy and generality (but being simpler to solve) through turbulence modeling is being overly optimistic. If this premise is accepted, then our expectations in turbulence modeling are reduced from seeking the ultimate to seeking models that have reasonable accuracy over a limited range of flow conditions.

It is important to remember that turbulence models must be verified by comparing predictions with experimental measurements. Care must be taken in interpreting predictions of models outside the range of conditions over which they have been verified by comparisons with experimental data.

The purpose of this section is to introduce the methodology commonly used in turbulence modeling. The intent is not to present all models in sufficient detail that they can be used without consulting the original references, but rather to outline the rationale for the evolution of modeling strategy. Simpler models will be described in sufficient detail to enable the reader to formulate a "baseline" model applicable to simple thin shear layers.

5.4.2 Modeling Terminology

Boussinesq (1877) suggested, more than 100 year ago, that the apparent turbulent shearing stresses might be related to the rate of mean strain through an apparent scalar turbulent or "eddy" viscosity. For the general Reynolds stress

tensor, the Boussinesq assumption gives

$$-\overline{\rho u_i' u_j'} = 2\mu_T S_{ij} - \frac{2}{3}\delta_{ij}\left(\mu_T \frac{\partial u_k}{\partial x_k} + \rho \bar{k}\right) \tag{5.129a}$$

where μ_T is the turbulent viscosity, $\bar{k}$ is the kinetic energy of turbulence, $\bar{k} = \overline{u_i' u_i'}/2$, and the rate of the mean strain tensor S_{ij} is given by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5.129b}$$

Following the convention introduced in Section 5.3.2, we are omitting bars over the time-mean variables.

By analogy with kinetic theory, by which the molecular viscosity for gases can be evaluated with reasonable accuracy, we might expect that the turbulent viscosity can be modeled as

$$\mu_T = \rho v_T l \tag{5.130}$$

where v_T and l are characteristic velocity and length scales of the turbulence, respectively. The problem, of course, is to find suitable means for evaluating v_T and l.

Turbulence models to close the Reynolds equations can be divided into two categories, according to whether or not the Boussinesq assumption is used. Models using the Boussinesq assumption will be referred to as Category I, or turbulent viscosity models. These are also known as first-order models. Most models currently employed in engineering calculations are of this type. Experimental evidence indicates that the turbulent viscosity hypothesis is a valid one in many flow circumstances. There are exceptions, however, and there is no physical requirement that it hold. Models that affect closure to the Reynolds equations without this assumption will be referred to as Category II models and include those known as Reynolds stress or stress-equation models. The stress-equation models are also referred to as second-order or second-moment closures.

The other common classification of models is according to the number of supplementary partial differential equations that must be solved in order to supply the modeling parameters. This number ranges from zero for the simplest algebraic models to 12 for the most complex of the Reynolds stress models (Donaldson and Rosenbaum, 1968).

Category III models will be defined as those that are not based entirely on the Reynolds equations. Large-eddy simulations fall into this category, since it is a filtered set of conservation equations that is solved instead of the Reynolds equations.

As we turn to examples of specific turbulence models, it will be helpful to keep in mind an example set of conservation equations for which turbulence modeling is needed. The thin-shear-layer equations, Eqs. (5.116)–(5.119), will serve this purpose reasonably well. In the incompressible 2-D or axisymmetric

thin-shear-layer equations, the modeling task reduces to finding expressions for $-\rho\overline{v'u'}$ and $\rho c_p\overline{v'T'}$.

5.4.3 Simple Algebraic or Zero-Equation Models

Algebraic turbulence models invariably utilize the Boussinesq assumption. One of the most successful of this type of model was suggested by Prandtl in the 1920s:

$$\mu_T = \rho l^2 \left| \frac{\partial u}{\partial y} \right| \tag{5.131a}$$

where l, a "mixing length," can be thought of as a transverse distance over which particles maintain their original momentum, somewhat on the order of a mean free path for the collision or mixing of globules of fluid. The product $l|\partial u/\partial y|$ can be interpreted as the characteristic velocity of turbulence, v_T. In Eq. (5.131*a*), u is the component of velocity in the primary flow direction, and y is the coordinate transverse to the primary flow direction.

For 3-D thin shear layers, Prandtl's formula is usually interpreted as

$$\mu_T = \rho l^2 \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]^{1/2} \tag{5.131b}$$

This formula treats the turbulent viscosity as a scalar and gives qualitatively correct trends, especially near the wall. There is increasing experimental evidence, however, that in the outer layer, the turbulent viscosity should be treated as a tensor (i.e., dependent upon the direction of strain) in order to provide the best agreement with measurements. For flows in corners or in other geometries where a single "transverse" direction is not clearly defined, Prandtl's formula must be modified further (see, for example, Patankar et al., 1979).

The evaluation of l in the mixing-length model varies with the type of flow being considered, wall boundary layer, jet, wake, etc. For flow along a solid surface (internal or external flow), good results are observed by evaluating l according to

$$l_i = \kappa y(1 - e^{-y^+/A^+}) \tag{5.132}$$

in the inner region closest to the solid boundaries and switching to

$$l_0 = C_1\delta \tag{5.133}$$

when l_i predicted by Eq. (5.132) first exceeds l_0. The constant C_1 in Eq. (5.133) is usually assigned a value close to 0.089, and δ is the velocity boundary-layer thickness.

In Eq. (5.132), κ is the von Kármán constant, usually taken as 0.41, and A^+ is the damping constant, most commonly evaluated as 26. The quantity in parentheses is the van Driest damping function (van Driest, 1956) and is the most common expression used to bridge the gap between the fully turbulent

region where $l = \kappa y$ and the viscous sublayer where $l \to 0$. The parameter y^+ is defined as

$$y^+ = \frac{y(|\tau_w|/\rho_w)^{1/2}}{\nu_w}$$

Numerous variations on the exponential function have been utilized in order to account for effects of property variations, pressure gradients, blowing, and surface roughness. A discussion of modifications to account for several of these effects can be found in the work by Cebeci and Smith (1974). It appears reasonably clear from comparisons in the literature, however, that the inner layer model as stated [Eq. (5.132)] requires no modification to accurately predict the variable-property flow of gases with moderate pressure gradients on smooth surfaces.

The expression for l_i, Eq. (5.132), is responsible for producing the inner, "law-of-the-wall" region of the turbulent flow, and l_0 [Eq. (5.133)] produces the outer "wake-like" region. These two zones are indicated in Fig. 5.7, which depicts a typical velocity distribution for an incompressible turbulent boundary

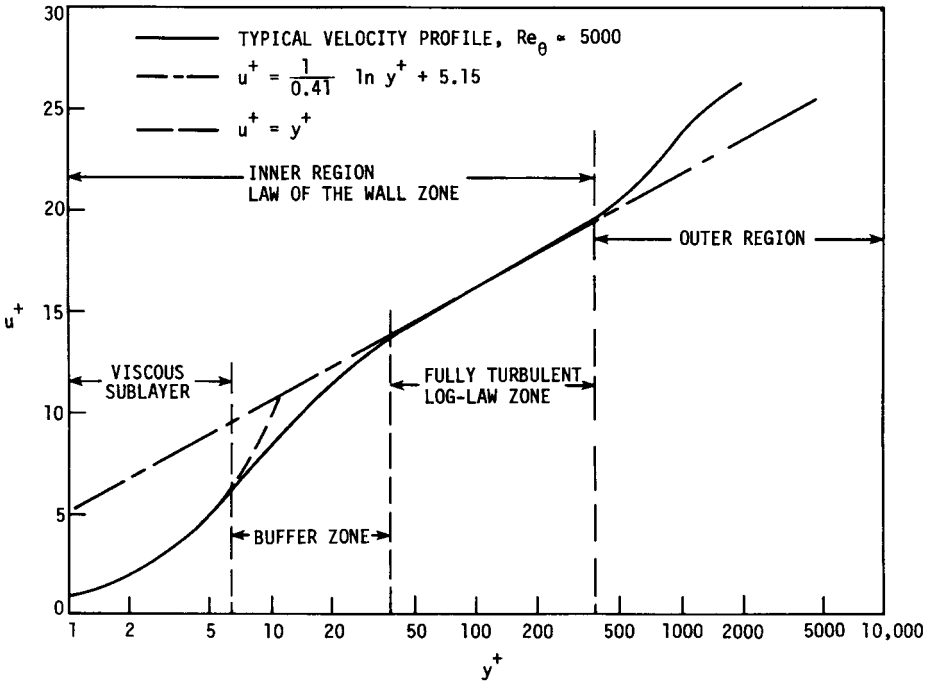


Figure 5.7 Zones in the turbulent boundary layer for a typical incompressible flow over a smooth flat plate.

layer on a smooth impermeable plate using law-of-the-wall coordinates. In Fig. 5.7, Re_θ is the Reynolds number based on momentum thickness, $\rho_e u_e \theta/\mu_e$, where for 2-D flow, the momentum thickness is defined as

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

The nondimensional velocity u^+ is defined as $u^+ = u/(|\tau_w|/\rho_w)^{1/2}$. The inner and outer regions are indicated on the figure. Under normal conditions, the inner law-of-the-wall zone only includes about 20% of the boundary layer. The log-linear zone is the characteristic "signature" of a turbulent wall boundary layer, although the law-of-the-wall plot changes somewhat in general appearance as Re and M are varied.

It is worth noting that for low Re_θ, i.e., relatively near the origin of the turbulent boundary layer, both inner and outer regions are tending toward zero, and problems might be expected with the two-region turbulence model employing Eqs. (5.132) and (5.133). The difficulty occurs because the smaller δ occurring near the origin of the turbulent boundary layer are causing the switch to the outer model to occur before the wall damping effect has permitted the fully turbulent law-of-the-wall zone to develop. This causes the numerical scheme using such a model to underpredict the wall shear stress. The discrepancy is nearly negligible for incompressible flow, but the effect is more serious for compressible flows, persisting at higher and higher Re as M increases owing to the relative thickening of the viscous sublayer from thermal effects (Pletcher, 1976). Naturally, details of the effect are influenced by the level of wall cooling present in the compressible flow.

Predictions can be brought into good agreement with measurements at low Re by simply delaying the switch from the inner model. Eq. (5.132), to the outer, Eq. (5.133), until $y^+ \geqslant 50$. If, at $y^+ = 50$ in the flow, $l/\delta \leqslant 0.089$, then no adjustment is necessary. On the other hand, if Eq. (5.132) predicts $l/\delta > 0.089$, then the mixing length becomes constant in the outer region at the value computed at $y^+ = 50$ by Eq. (5.132). This simple adjustment ensures the existence of the log-linear region in the flow, which is in agreement with the preponderance of measurements.

Other modeling procedures have been used successfully for the inner and outer regions. Some workers advocate the use of wall functions based on a Couette flow assumption (Patankar and Spalding, 1970) in the near-wall region. This approach probably has not been quite as well refined as the van Driest function to account for variable properties, transpiration, and other near-wall effects.

An alternative treatment to Eq. (5.133) is often used to evaluate the turbulent viscosity in the outer region (Cebeci and Smith, 1974). This follows the Clauser formulation,

$$\mu_{T(\text{outer})} = \alpha \rho u_e |\delta_k^*| \tag{5.134}$$

where α accounts for low-Re effects. Cebeci and Smith (1974) recommend

$$\alpha = 0.0168 \frac{(1.55)}{1 + \pi} \tag{5.135}$$

where $\pi = 0.55[1 - \exp(-0.243z^{1/2} - 0.298z)]$ and $z = (\mathrm{Re}_\theta/425) - 1$. For $\mathrm{Re}_\theta > 5000$, $\alpha \cong 0.0168$. The parameter δ_k^* is the kinematic displacement thickness, defined as

$$\delta_k^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy \tag{5.136}$$

Closure for the Reynolds heat flux term, $\rho c_p \overline{v'T'}$, is usually handled in algebraic models by a form of the Reynolds analogy, which is based on the similarity between the transport of heat and momentum. The Reynolds analogy is applied to the apparent turbulent conductivity in the assumed Boussinesq form:

$$\rho c_p \overline{v'T'} = -k_T \frac{\partial T}{\partial y}$$

In turbulent flow this additional transport of heat is caused by the turbulent motion. Experiments confirm that the ratio of the diffusivities for the turbulent transport of heat and momentum, called the turbulent Prandtl number, $\mathrm{Pr}_T = \mu_T c_p / k_T$, is a well-behaved function across the flow. Most algebraic turbulence models do well by letting the Pr_T be a constant near 1; most commonly, $\mathrm{Pr}_T = 0.9$. Experiments indicate that for wall shear flows, Pr_T varies somewhat from about 0.6–0.7 at the outer edge of the boundary layer to about 1.5 near the wall, although the evidence is not conclusive. Several semi-empirical distributions for Pr_T have been proposed; a sampling is found in the works by Cebeci and Smith (1974), Kays (1972), and Reynolds (1975). Using Pr_T, the apparent turbulent heat flux is related to the turbulent viscosity and mean flow variables as

$$-\rho c_p \overline{v'T'} = \frac{c_p \mu_T}{\mathrm{Pr}_T} \frac{\partial T}{\partial y} \tag{5.137a}$$

and closure has been completed.

For other than thin shear flows, it may be necessary to model other Reynolds heat flux terms. To do so, the turbulent conductivity, $k_T = c_p \mu_T / \mathrm{Pr}_T$, is normally considered as a scalar, and the Boussinesq-type approximation is extended to other components of the temperature gradient. As an example, we would evaluate $-\rho c_p \overline{u'T'}$ as

$$-\rho c_p \overline{u'T'} = \frac{c_p \mu_T}{\mathrm{Pr}_T} \frac{\partial T}{\partial x}$$

To summarize, a recommended baseline algebraic model for wall boundary layers consists of evaluating the turbulent viscosity by Prandtl's mixing-length formula, Eq. (5.131*a*), where l is given by Eq. (5.132) for the inner region, and is

given by Eq. (5.133) for the outer region. Alternatively, the Clauser formulation, Eq. (5.134), could be used in the outer region. The apparent turbulent heat flux can be evaluated through Eq. (5.137*a*) using $\mathrm{Pr}_T = 0.9$. This simplest form of modeling has employed four empirical, adjustable constants as given in Table 5.1.

Note that outer-region models used with the boundary-layer equations typically require the determination of the boundary-layer thickness or displacement thickness. Such measures are satisfactory for the boundary-layer equations because the streamwise velocity from the boundary-layer solution smoothly reaches the specified outer-edge velocity at the outer edge of the viscous region. As a consequence, the edge of the viscous region is reasonably well defined when solving the boundary-layer equations. This is not the case when solving more complete forms of the conservation equations, as for example, the full Reynolds averaged Navier-Stokes (RANS) equations, because the solution domain extends well outside the viscous region and the solution to the remaining Euler terms often results in a streamwise velocity that varies with distance from the solid boundary. This makes it difficult to identify a viscous layer thickness. A widely used algebraic model introduced by Baldwin and Lomax (1978) avoids the use of boundary-layer parameters such as displacement thickness or boundary-layer thickness. Basically, the Baldwin-Lomax model is a two-region algebraic model similar to the Cebeci-Smith model but formulated with the requirements of RANS solution schemes in mind.

The inner region is resolved as for many other algebraic models but with the use of the full measure of vorticity:

Inner:

$$\mu_{T_i} = \rho l^2 |\omega| \tag{5.137b}$$

$$|\omega| = \sqrt{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2} \tag{5.137c}$$

where $l = \kappa y(1 - e^{-(y^+/A^+)})$.

Table 5.1 Empirical constants employed in algebraic turbulence models for wall boundary layers

Symbol	Description
κ	von Kármán constant, used for inner layer ≈ 0.41
A^+	van Driest constant for damping function ≈ 26, but frequently modified to account for complicating effects
C_1 or α	constant for outer-region model $C_1 \approx 0.089$, $\alpha \approx 0.0168$, but usually includes $f(\mathrm{Re}_\theta)$ in α
Pr_T	turbulent Prandtl number, $= 0.9$ *is most common*

Outer:

$$\mu_{T_0} = 0.0168 \rho V_{T_0} L_0 \tag{5.137d}$$

where

$$V_{T_0} = \min\left(F_{\max}, 0.25 q^2_{\text{Diff}}/F_{\max}\right) \tag{5.137e}$$

$$q^2_{\text{Diff}} = (u^2 + v^2 + w^2)_{\max} - (u^2 + v^2 + w^2)_{\min} \tag{5.137f}$$

$$F_{\max} = \max\left[y|\omega|(1 - e^{-(y^+/A^+)})\right] \tag{5.137g}$$

$$L_0 = 1.6 y_{\max} I^k \tag{5.137h}$$

$$I^k = \left[1 + 5.5\left(\frac{0.3y}{y_{\max}}\right)^6\right]^{-1} \tag{5.137i}$$

and $y_{\max}$ = value of y for which $F_{\max}$ occurs.

Algebraic models have accumulated an impressive record of good performance for relatively simple viscous flows but need to be modified in order to accurately predict flows with "complicating" features. It should be noted that compressible flows do not represent a "complication" in general. The turbulence structure of the flow appears to remain essentially unchanged for Mach numbers up through at least 5. Naturally, the variation of density and other properties must be accounted for in the form of the conservation equations used with the turbulence model. Table 5.2 lists several flow conditions requiring alterations or extensions to the simplest form of algebraic models cited above. Some key references are also tabulated where such model modifications are discussed.

The above discussion of algebraic models for wall boundary layers is by no means complete. Over the years, dozens of slightly different algebraic models have been suggested. Eleven algebraic models were compared in a study by

Table 5.2 Effects requiring alterations or additions to simplest form of algebraic turbulence models

Effect	References
Low Reynolds number	Cebeci and Smith (1974), Pletcher (1976), Bushnell et al. (1975), Herring and Mellor (1968), McDonald (1970)
Roughness	Cebeci and Smith (1974), Bushnell et al. (1976), McDonald and Fish (1973), Healzer et al. (1974), Adams and Hodge (1977)
Transpiration	Cebeci and Smith (1974), Bushnell et al. (1976), Pletcher (1974), Baker and Launder (1974), Kays and Moffat (1975)
Strong pressure gradients	Cebeci and Smith (1974), Bushnell et al. (1976), Adams and Hodge (1977), Pletcher (1974), Baker and Launder (1974), Kays and Moffat (1975), Jones and Launder (1972), Kreskovsky et al. (1974), Horstman (1977)
Merging shear layers	Bradshaw et al. (1973), Stephenson (1976), Emery and Gessner (1976), Cebeci and Chang (1978), Malik and Pletcher (1978)

McEligot et al. (1970) for turbulent pipe flow with heat transfer. None were found superior to the van Driest–damped mixing-length model presented here.

Somewhat less information is available in the literature on algebraic turbulence models for free-shear flows. This category of flows has historically been more difficult to model than wall boundary layers, especially if model generality is included as a measure of merit. Some discussion of the status of the simple models for round jets can be found in the works of Madni and Pletcher (1975b, 1977a). The initial mixing region of round jets can be predicted fairly well using Prandtl's mixing-length formulation, Eq. (5.131*a*), with

$$l = 0.0762\delta_m \tag{5.138}$$

where δ_m is the width of the mixing zone. This model does not perform well after the shear layers have merged, and a switch at that point to models of the form (Hwang and Pletcher, 1978)

$$\nu_T = \mu_T/\rho = \gamma F y_{1/2}(u_{\max} - u_{\min}) \tag{5.139}$$

or (Madni and Pletcher, 1975b)

$$\nu_T = \frac{2F}{a}\int_y^\infty |u_e - u|\, y\, dy \tag{5.140}$$

has provided good agreement with measurements for round co-flowing jets. Equation (5.139) is a modification of the model suggested for jets by Prandtl (1926). In the above equations, a is the jet discharge radius and γ is an intermittency function,

$$\gamma = 1 \qquad 0 \leqslant \frac{y}{y_{1/2}} \leqslant 0.8$$

$$\gamma = (0.5)^z \qquad \frac{y}{y_{1/2}} > 0.8 \qquad \text{where} \qquad z = \left(\frac{y}{y_{1/2}} - 0.8\right)^{2.5} \tag{5.141}$$

F is a function of the ratio (R) of the stream velocity to the jet discharge velocity given by $F = 0.015(1 + 2.13R^2)$. The distance y is measured from the jet centerline, and $y_{1/2}$ is the "velocity half width," the distance from the centerline to the point at which the velocity has decreased to the average of the centerline and external stream velocities.

Philosophically, the strongest motivation for turning to more complex models is the observation that the algebraic model evaluates the turbulent viscosity only in terms of *local* flow parameters, yet we feel that a turbulence model ought to provide a mechanism by which effects upstream can influence the turbulence structure (and viscosity) downstream. Further, with the simplest models, ad hoc additions and corrections are frequently required to handle specific effects, and constants need to be changed to handle different classes of shear flows. To many investigators, it is appealing to develop a model general enough that specific modifications to the constants are not required to treat different classes of flows.

If we accept the general form for the turbulent viscosity, $\mu_T = \rho v_T l$, then a logical way to extend the generality of turbulent viscosity models is to permit v_T and perhaps l to be more complex (and thus more general) functions of the flow capable of being influenced by upstream (historic) effects. This rationale serves to motivate several of the more complex turbulence models.

5.4.4 One-Half-Equation Models

A *one-half-equation model* will be defined as one in which the value of one model parameter (v_T, l, or μ_T itself) is permitted to vary with the primary flow direction in a manner determined by the solution to an *ordinary* differential equation (ODE). The ODE usually results from either neglecting or assuming the variation of the model parameter with one coordinate direction. Extended mixing-length models and relaxation models fall into this category. A *one-equation model* is one in which an additional *partial* differential equation (PDE) is solved for a model parameter. The main features of several one-half-equation models are tabulated in Table 5.3.

The first three models in Table 5.3 differ in detail, although all three utilize an integral form of a transport equation for turbulence kinetic energy as a basis for letting flow history influence the turbulent viscosity. Models of this type have been refined to allow prediction of transition, roughness effects, transpiration, pressure gradients, and qualitative features of relaminarization. Most of the test cases reported for the models have involved external rather than channel flows.

Although models 5D, 5E, and 5F appear to be purely empirical relaxation or lag models, Birch (1976) shows that models of this type are actually equivalent to 1-D versions of transport PDEs for the quantities concerned except that these transport equations are not generally derivable from the Navier-Stokes equations. This is no serious drawback, since transport equations cannot be

Table 5.3 Some one-half-equation models

Model	Transport equation used as basis for ODE	Model parameter determined by ODE solution	References
5A	Turbulence kinetic energy	l_∞	McDonald and Camerata (1968), Kreskovsky et al. (1974), McDonald and Kreskovsky (1974)
5B	Turbulence kinetic energy	l_∞	Chan (1972)
5C	Turbulence kinetic energy	l_∞	Adams and Hodge (1977)
5D	Empirical ODE for $\mu_{T(\text{outer})}$	$\mu_{T(\text{outer})}$	Shang and Hankey (1975)
5E	Empirical ODE for $\mu_{T(\text{outer})}$	$\mu_{T(\text{outer})}$	Reyhner (1968)
5F	Empirical ODE for l_∞	l_∞	Malik and Pletcher (1978). Pletcher (1978)
5G	Empirical ODE for $\tau_{\max}$	$\tau_{\max}$	Johnson and King (1985)

solved without considerable empirical simplification and modeling of terms. In the end, these transport equations tend to have similar forms characterized by generation, dissipation, diffusion, and convection terms, regardless of the origin of the equation.

To illustrate, for wall boundary layers, model 5F utilizes the expression for mixing length given by Eq. (5.132) for the inner region. In the outer part of the flow, the mixing length is calculated according to

$$l_0 = 0.12L \tag{5.142}$$

where L is determined by the solution to an ODE. For a shear layer of constant width, the natural value of L is presumed to be the thickness δ of the shear layer. When δ is changing with the streamwise direction, x, L will lag δ in a manner controlled by the relaxation time for the large-eddy structure, which is assumed to be equal to $\delta/\bar{u}_\tau$, where $\bar{u}_\tau$ is a characteristic turbulence velocity. If it is further assumed that the fluid in the outer part of the shear layer travels at velocity u_e, then the streamwise distance traversed by the flow during the relaxation time is $L^* = C_2 u_e \delta/\bar{u}_\tau$. A rate equation can be developed by assuming that L will tend toward δ, according to

$$\frac{dL}{dx} = \frac{\delta - L}{L^*} \tag{5.143}$$

This model has been extended to free-shear flows (Minaie and Pletcher, 1982) by interpreting δ as the distance between the location of the maximum shear stress and the outer edge of the shear flow and replacing u_e by the average streamwise velocity over the shear layer. The optimum evaluation for $\bar{u}_\tau$ appears unsettled. The expression $\bar{u}_\tau = (L/\delta)(|\tau_w|/\rho_w)^{1/2}$ has been employed successfully for flows along solid surfaces, whereas $\bar{u}_\tau = (\tau_{\max}/\rho_w)^{1/2}$ has proven satisfactory for free-shear flows predicted to date. It might be speculated that this latter evaluation would work reasonably well for wall boundary layers also. The final form of the transport ODE for L used for separating wall boundary layers (Pletcher, 1978) and merging shear layers in annular passages (Malik and Pletcher, 1978) can be written as

$$u_e \frac{dL}{dx} = 1.25 \left| \frac{\tau_w}{\rho_w} \right|^{1/2} \left[\frac{L}{\delta} - \left(\frac{L}{\delta} \right)^2 \right] \tag{5.144}$$

Although the simple one-half-equation model described above provides remedies for shortcomings in zero-equation models for several flows, it performs badly in predicting shock-separated flows. The one-half-equation model proposed by Johnson and King (1985), and modified by Johnson and Coakley (1990), 5G in Table 5.3, was developed to excel especially for the nonequilibrium conditions present in transonic separated flows. Basically, it provides an extension of algebraic models to nonequilibrium flows and effectively reduces to the Cebeci-Smith model under equilibrium conditions.

5.4.5 One-Equation Models

One obvious shortcoming of algebraic viscosity models that normally evaluate v_T in the expression $\mu_T = \rho v_T l$ by $v_T = l|\partial u/\partial y|$ is that $\mu_T = k_T = 0$ whenever $\partial u/\partial y = 0$. This would suggest that μ_T and k_T would be zero at the centerline of a pipe, in regions near the mixing of a wall jet with a main stream and in flow through an annulus or between parallel plates where one wall is heated and the other cooled. Measurements (and common sense) indicate that μ_T and k_T are *not* zero under all conditions whenever $\partial u/\partial y = 0$. The mixing-length models can be "fixed up" to overcome this deficiency, but this conceptual shortcoming provides motivation for considering other interpretations for μ_T and k_T. In fairness to the algebraic models, we should mention that this defect is not always crucial because Reynolds stresses and heat fluxes are frequently small when $\partial u/\partial y = 0$. For some examples illustrating this point, see Malik and Pletcher (1981).

It was the suggestion of Prandtl and Kolmogorov in the 1940s to let v_T in $\mu_T = \rho v_T l$ be proportional to the square root of the kinetic energy of turbulence, $\bar{k} = \frac{1}{2}\overline{u_i' u_i'}$. Thus the turbulent viscosity can be evaluated as

$$\mu_T = C_k \rho l (\bar{k})^{1/2} \tag{5.145}$$

and μ_T no longer becomes equal to zero when $\partial u/\partial y = 0$. The kinetic energy of turbulence is a measurable quantity and is easily interpreted physically. We naturally inquire how we might predict $\bar{k}$.

A transport PDE can be developed (Prob. 5.22) for $\bar{k}$ from the Navier-Stokes equations. For incompressible flows, the equation takes the form

$$\rho\frac{\partial \bar{k}}{\partial t} + \rho u_j \frac{\partial \bar{k}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{k}}{\partial x_j} - \frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right) - \rho\overline{u_i' u_j'}\frac{\partial u_i}{\partial x_j} - \mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{5.146}$$

The term $-\frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}$ is typically modeled as a gradient diffusion process,

$$-\frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'} = \frac{\mu_T}{\mathrm{Pr}_k}\frac{\partial \bar{k}}{\partial x_j}$$

where Pr_k is a turbulent Prandtl number for turbulent kinetic energy and, as such, is purely a closure constant. Using the Boussinesq eddy viscosity assumption, the second term on the right-hand side readily becomes

$$-\rho\overline{u_i' u_j'}\frac{\partial u_i}{\partial x_j} = \left(2\mu_T S_{ij} - \frac{2}{3}\rho\bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j}$$

The last term on the right-hand side of Eq. (5.146) is the dissipation rate of turbulent kinetic energy per unit volume, $\rho\varepsilon$. Based on dimensional arguments, the dissipation rate of turbulent kinetic energy is given by $\varepsilon = C_D \bar{k}^{3/2}/l$. Thus

the modeled form of the turbulent kinetic energy equation becomes

$$\underbrace{\rho \frac{D\bar{k}}{Dt}}_{\substack{\text{Particle rate} \\ \text{of increase} \\ \text{of } \bar{k}}} = \underbrace{\frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_k)\frac{\partial \bar{k}}{\partial x_j}\right]}_{\substack{\text{Diffusion rate} \\ \text{for } \bar{k}}} + \underbrace{\left(2\mu_T S_{ij} - \frac{2}{3}\rho\bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j}}_{\substack{\text{Generation} \\ \text{rate for } \bar{k}}} - \underbrace{C_D \rho \bar{k}^{3/2}/l}_{\substack{\text{Dissipation} \\ \text{rate for } \bar{k}}} \tag{5.147}$$

The physical interpretation of the various terms is indicated for Eq. (5.147). This modeled transport equation is then added to the system of PDEs to be solved for the problem at hand. Note that a length parameter, l, needs to be specified algebraically. In the above, $\mathrm{Pr}_k \approx 1.0$ and $C_D \simeq 0.164$ if l is taken as the ordinary mixing length.

The above modeling for the $\bar{k}$ transport equation is only valid in the fully turbulent regime, i.e., away from any wall damping effects. For typical wall flows, this means for y^+ greater than about 30. Inner boundary conditions for the $\bar{k}$ equations are often supplied through the use of wall functions (Launder and Spalding, 1974).

Wall functions are based on acceptance of the law of the wall as the link between the near-wall velocity and the wall shear stress. The logarithmic portion of the law of the wall follows exactly from Prandtl's mixing-length hypothesis and is confirmed by experiments under a fairly wide range of conditions. In the law of the wall region, experiments also indicate that convection and diffusion of $\bar{k}$ are negligible. Thus generation and dissipation of $\bar{k}$ are in balance, and it can be shown (Prob. 5.23) that the turbulence kinetic energy model for the turbulent viscosity reduces to Prandtl's mixing-length formulation, Eq. (5.131*a*), under these conditions if we take $C_k = (C_D)^{1/3}$ in Eq. (5.145). At the location where the diffusion and convection are first neglected, we can establish (Prob. 5.26) an inner boundary condition for $\bar{k}$ in a 2-D flow as

$$\bar{k}(x, y_c) = \frac{\tau(y_c)}{\rho C_D^{2/3}} \tag{5.148a}$$

where y_c is a point within the region where the logarithmic law of the wall is expected to be valid. For $y < y_c$ the Prandtl-type algebraic inner-region model [Eqs. (5.131*a*) and (5.132)] can be used.

Other one-equation models have been suggested that deviate somewhat from the Prandtl-Kolmogorov pattern. One of the most successful of these is by Bradshaw et al. (1967). The turbulence energy equation is used in the Bradshaw model, but the modeling is different in both the momentum equation, where the

turbulent shearing stress is assumed proportional to $\bar{k}$, and in the turbulence energy equation. The details will not be given here, but an interesting feature of the Bradshaw method is that, as a consequence of the form of modeling used for the turbulent transport terms, the system of equations becomes hyperbolic and can be solved by a procedure similar to the method of characteristics. The Bradshaw method has enjoyed good success in the prediction of wall boundary layers. Even so, the predictions have not been notably superior to those of the algebraic models, one-half-equation models, or other one-equation models.

Not all one-equation models have been based on the turbulent kinetic energy equation. Nee and Kovasznay (1968) and, more recently, Baldwin and Barth (1990) and Spalart and Allmaras (1992) have devised model equations for the transport of the turbulent viscosity or a parameter proportional to the turbulent viscosity. To illustrate the approach, the model of Spalart and Allmaras will be outlined below.

The Spalart and Allmaras turbulent kinematic viscosity is given by

$$\nu_T = \tilde{\nu} f_{v1} \tag{5.148b}$$

The parameter $\tilde{\nu}$ is obtained from the solution of the transport equation,

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = \frac{1}{\sigma} \frac{\partial}{\partial x_k} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_k} \right] + c_{b1}(1 - f_{v2}) \tilde{S} \tilde{\nu} - c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2 + \frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_k} \frac{\partial \tilde{\nu}}{\partial x_k} \tag{5.148c}$$

where the closure coefficients and functions are given by

$$c_{b1} = 0.1355 \qquad c_{b2} = 0.622 \qquad c_{v1} = 7.1 \qquad \sigma = 2/3$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{(1 + c_{b2})}{\sigma} \qquad c_{w2} = 0.3 \qquad c_{w3} = 2 \qquad \kappa = 0.41$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \qquad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \qquad f_w = g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right) \qquad \chi = \frac{\tilde{\nu}}{\nu}$$

$$g = r + c_{w2}(r^6 - r) \qquad r = \frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2} \qquad \tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

$$S = \sqrt{2 \Omega_{ij} \Omega_{ij}} \qquad \Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

and d is the distance from the closest surface. Corrections for transition can be found in the work by Spalart and Allmaras (1992). A comparison of the performance of the Baldwin-Barth and Spalart-Allmaras models has been reported by Mani et al. (1995).

The one-equation model has been extended to compressible flow (Rubesin, 1976) and appears to provide a definite improvement over algebraic models. The

recent one-equation models of (Baldwin-Barth and Spalart-Allmaras) have provided better agreement with experimental data for some separated flows than has generally been possible with algebraic models. On the whole, however, the performance of most one-equation models (for both incompressible and compressible flows) has been disappointing, in that relatively few cases have been observed in which these models offer an improvement over the predictions of the algebraic models.

5.4.6 One-and-One-Half- and Two-Equation Models

One conceptual advance made by moving from a purely algebraic mixing-length model to a one-equation model was that the latter permitted one model parameter to vary throughout the flow, being governed by a PDE of its own. In most one-equation models, a length parameter still appears that is generally evaluated by an algebraic expression dependent upon only *local* flow parameters. Researchers in turbulent flow have long felt that the length scale in turbulence models should also depend upon the upstream "history" of the flow and not just on local flow conditions. An obvious way to provide more complex dependence of l on the flow is to derive a transport equation for the variation of l. If the equation for l added to the system is an ODE, such as given by Eq. (5.144) for model 5F, the resulting model might well be termed a *one-and-one-half-equation model.* Such a model has been employed to predict separating external turbulent boundary layers (Pletcher, 1978), flow in annular passages with heat transfer (Malik and Pletcher, 1981), and plane and round jets (Minaie and Pletcher, 1982).

Frequently, the equation from which the length scale is obtained is a PDE, and the model is then referred to as a *two-equation turbulence model.* Although a transport PDE can be developed for a length scale, the terms in this equation are not easily modeled, and some workers have experienced better success by solving a transport equation for a length-scale-related parameter rather than the length scale itself.

Several combinations of variables have been used as the transported quantity, including the dissipation rate ε, $\bar{k}l$, ω, τ, and $\bar{k}\tau$, where $\omega = \varepsilon/\bar{k}$ is known as the specific dissipation rate and τ is a dissipation time, $\tau = 1/\omega$. Thus the length scale needed in the expression for turbulent viscosity, $\mu_T = C_D^{1/3}\rho l\bar{k}^{1/2}$, can be obtained from the solution for any of the combinations listed above. For example,

$$l = C_D\bar{k}^{3/2}/\varepsilon \qquad l = C_D\bar{k}^{1/2}/\omega \qquad l = C_D\bar{k}^{1/2}\tau$$

The most commonly used variable for a second transport equation is the dissipation rate ε. The versions of the $\bar{k}$-ε model used today can largely be traced back to the early work of Harlow and Nakayama (1968) and Jones and Launder (1972). The description here follows the work of Jones and Launder (1972) and Launder and Spalding (1974).

In terms of $\bar{k}$ and ε, the turbulent viscosity can be evaluated as $\mu_T = C_\mu \rho \bar{k}^2/\varepsilon$, where in terms of C_D introduced earlier, $C_\mu = C_D^{4/3}$. Although an exact equation can be derived for the transport of ε (see, for example, Wilcox, 1993), it provides relatively little guidance for modeling except to support the idea that the modeled equation should allow for the production, dissipation, diffusion, and convective transport of ε. The turbulent kinetic energy equation used in the $\bar{k}$-ε equation is given by Eq. (5.147) except that ε is maintained as an unknown. The transport equations used in the "standard" $\bar{k}$-ε model are as follows:

turbulent kinetic energy:

$$\rho \frac{D\bar{k}}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_k)\frac{\partial \bar{k}}{\partial x_j}\right] + \left(2\mu_T S_{ij} - \frac{2}{3}\rho \bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - \rho\varepsilon \quad (5.149)$$

dissipation rate:

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_\varepsilon)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\varepsilon}{\bar{k}}\left(2\mu_T S_{ij} - \frac{2}{3}\rho \bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - C_{\varepsilon 2}\rho\frac{\varepsilon^2}{\bar{k}} \quad (5.150a)$$

The terms on the right-hand side of Eq. (5.150*a*) from left to right can be interpreted as the diffusion, generation, and dissipation rates of ϵ. Typical values of the model constants are tabulated in Table 5.4.

The most common $\bar{k}$-ε closure for the Reynolds heat flux terms utilizes the same turbulent Prandtl number formulation as used with algebraic models, Eq. (5.137*a*). A number of modifications or adjustments to the $\bar{k}$-ε model have been suggested to account for effects not accounted for in the standard model such as buoyancy and streamline curvature.

Numerous other two-equation models have been proposed and evaluated. Several of these are described and discussed by Wilcox (1993). Of these other two-equation models, the $\bar{k}$-ω model in the form prescribed by Wilcox (1988) has probably been developed and tested the most extensively.

The standard $\bar{k}$-ε model given above is not appropriate for use in the viscous sublayer because the damping effect associated with solid boundaries has not been included in the model. Closure can be achieved with the use of the standard model by assuming that the law of the wall holds in the inner region and either using wall functions of a form described by Launder and Spalding (1974) or using a traditional damped mixing-length algebraic model and matching

Table 5.4 Model constants for $\bar{k}$-ϵ two-equation model

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	Pr_k	Pr_ϵ	Pr_T
0.09	1.44	1.92	1.0	1.3	0.9

with the two-equation model by neglecting the convection and diffusion of $\bar{k}$ and ε. This provides a boundary condition for $\bar{k}$ at a point y_c within the law of the wall region given by Eq. (5.148*a*). Applying the strategy to ε gives

$$\varepsilon = \frac{C_D \bar{k}^{3/2}}{l} = \frac{C_D \left[\bar{k}(y_c)\right]^{3/2}}{\kappa y} \tag{5.150b}$$

A frequently used alternative is to employ a model based on transport equations for $\bar{k}$ and ε that have been modified by the addition of damping terms to extend applicability to the near-wall region. The models proposed by Jones and Launder (1972), Launder and Sharma (1974), Lam and Bremhorst (1981), and Chien (1982) are among the most commonly used of these *low Reynolds number* $\bar{k}$-ε *models*. The Re that is "low" in these models is the Re of turbulence, $\mathrm{Re}_T = \bar{k}^2/\varepsilon\nu$, where $\bar{k}^{3/2}/\varepsilon$ is used as a length scale for dissipation of kinetic energy. As an example, the low Reynolds number $\bar{k}$-ε model of Chien (1982) is given below with the modified or added terms identified by labels underneath.

turbulent kinetic energy:

$$\rho \frac{D\bar{k}}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_k)\frac{\partial \bar{k}}{\partial x_j}\right] + \left(2\mu_T S_{ij} - \frac{2}{3}\rho\bar{k}\,\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - \rho\varepsilon - \underbrace{2\mu\frac{\bar{k}}{y^2}}_{\text{added term}} \tag{5.150c}$$

dissipation rate:

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_\varepsilon)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\varepsilon}{\bar{k}}\left(2\mu_T S_{ij} - \frac{2}{3}\rho\bar{k}\,\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - \underbrace{C_{\varepsilon 2} f_1 \rho\frac{\varepsilon^2}{\bar{k}}}_{\text{modified term}} - \underbrace{f_2}_{\text{added term}} \tag{5.150d}$$

where the turbulent viscosity is evaluated as

$$\mu_T = C_\mu f_\mu \rho \bar{k}^2/\varepsilon \tag{5.150e}$$

and the additional functions are given by

$$f_\mu = 1 - e^{-0.0115y^+} \qquad f_1 = 1 - 0.22e^{-(\mathrm{Re}_T/6)^2} \qquad f_2 = -2\mu\frac{\varepsilon}{y^2}e^{-y^+/2}$$

The constants C_μ, Pr_k, Pr_ε have the same values as in the standard $\bar{k}$-ε model, but $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are assigned slightly different values of 1.35 and 1.80, respectively.

Only a few of the large number of two-equation models proposed in the literature have been discussed in this section. A number of those omitted, particularly those based on renormalization group theory (Yakhot and Orszag, 1986), show much promise.

The literature on the application of two-equation turbulence models to compressible flows is relatively sparse (see Viegas et al., 1985; Viegas and Horstman, 1979; Coakley, 1983a; Wilcox, 1993). Details of modeling considerations for the compressible RANS equations will not be given here, but the compressible form of the $\bar{k}$-ε model equations (Coakley, 1983a) will be listed below in Favre variables for completeness.

turbulent kinetic energy:

$$\frac{\partial(\bar{\rho}\tilde{k})}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j\tilde{k}) = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_k)\frac{\partial \tilde{k}}{\partial x_j}\right] + \left[2\mu_T\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right) - \frac{2}{3}\bar{\rho}\tilde{k}\,\delta_{ij}\right]\frac{\partial \tilde{u}_i}{\partial x_j} - \bar{\rho}\tilde{\varepsilon} \tag{5.150f}$$

dissipation rate:

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{\varepsilon}) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j\tilde{\epsilon}) = \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\mathrm{Pr}_\varepsilon)\frac{\partial \tilde{\varepsilon}}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\tilde{\varepsilon}}{\tilde{k}}\left[2\mu_T\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right) - \frac{2}{3}\bar{\rho}\tilde{k}\,\delta_{ij}\right]\frac{\partial \tilde{u}_i}{\partial x_j} - C_{\varepsilon 2}\bar{\rho}\frac{\tilde{\varepsilon}^2}{\tilde{k}} \tag{5.150g}$$

Despite the enthusiasm that is noted from time to time over two-equation models, it is perhaps appropriate to point out again the two major restrictions on this type of model. First, two-equation models of the type discussed herein are merely turbulent *viscosity* models, which assume that the Boussinesq approximation [Eq. (5.129*a*)] holds. In algebraic models, μ_T is a local function, whereas in two-equation models, μ_T is a more general and complex function governed by two additional PDEs. If the Boussinesq approximation fails, then even two-equation models fail. Obviously, in many flows the Boussinesq approximation models reality closely enough for engineering purposes.

The second shortcoming of two-equation models is the need to make assumptions in evaluating the various terms in the model transport equations, especially in evaluating the third-order turbulent correlations. This same

shortcoming, however, plagues all higher-order closure attempts. These model equations contain no magic; they only reflect the best understanding and intuition of the originators. We can be optimistic, however, that the models can be improved by improved modeling of these terms.

5.4.7 Reynolds Stress Models

By Reynolds stress models (sometimes called stress-equation models), we are referring to those Category II (second-order) closure models that *do not* assume that the turbulent shearing stress is proportional to the rate of mean strain. That is, for a 2-D incompressible flow,

$$-\rho\overline{u'v'} \neq \mu_T\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Such models are more general than those based on the Boussinesq assumption and can be expected to provide better predictions for flows with sudden changes in the mean strain rate or with effects such as streamline curvature or gradients in the Reynolds normal stresses. For example, without specific ad hoc adjustments, two-equation viscosity models are not able to predict the existence of the secondary flow patterns observed experimentally in turbulent flow through channels having a noncircular cross section.

Exact transport equations can be derived (Prob. 5.21) for the Reynolds stresses. This is accomplished for 2-D incompressible flow by multiplying the Navier-Stokes form of the ith momentum equation by the fluctuating velocity component u_j' and adding to it the product of u_i' and the jth momentum equation and then time averaging the result. That is, if we write the Navier-Stokes momentum equations in a form equal to zero, we form

$$\overline{u_i' N_j + u_j' N_i} = 0$$

where the N_i and N_j denote the ith and jth components of the Navier-Stokes momentum equation, respectively. Before the time averaging is carried out, the variables in the Navier-Stokes equations are replaced by the usual decomposition, $u_i = \bar{u}_i + u_i'$, etc. Intermediate steps are outlined by Wilcox (1993). The result can be written in the form

$$\rho\frac{D\overline{(-u_i'u_j')}}{Dt} = -\rho\overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} - \rho\overline{u_j'u_k'}\frac{\partial u_i}{\partial x_k} + \varepsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left(\mu\frac{\partial\overline{(-u_i'u_j')}}{\partial x_k} + C_{ijk}\right) \quad (5.150h)$$

where

$$\Pi_{ij} = \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \qquad \varepsilon_{ij} = 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

$$C_{ijk} = \rho\overline{u_i'u_j'u_k'} + \overline{p'u_i'}\,\delta_{jk} + \overline{p'u_j'}\,\delta_{ik}$$

Thus, in principle, transport equations can be solved for the six components of the Reynolds stress tensor. However, the equations contain 22 new unknowns that must be modeled first. Thus, we see again that in turbulence modeling, transport equations can be written for nearly anything of interest but none of them can be solved exactly. Models that employ transport PDEs for the Reynolds stresses are often referred to as second-order or second-moment closures.

Modeling for stress transport equations has followed the pioneering work of Rotta (1951). A commonly used example of the stress equation approach is the Reynolds stress model proposed by Launder et al. (1975), although numerous variations have since been suggested. Because of their computational complexity, the Reynolds stress models have not been widely used for engineering applications. Because they are not restricted by the Boussinesq approximation and because the closure contains the greatest number of model PDEs and constants of all the models considered, it would seem that the Reynolds stress models would have the best chance of emerging as "ultimate" turbulence models. Such ultimate turbulence models may eventually appear, but after more than 20 years of serious numerical research with these models, the results have been somewhat disappointing, considering the computational effort needed to implement the models.

Another Category II approach that has shown considerable promise recently is known as the *algebraic Reynolds stress model* (ASM). The idea here is to allow a nonlinear constitutive relationship between the Reynolds stresses and the rate of mean strain while avoiding the need to solve full PDEs for each of the six stresses. To do this, of course, requires modeling assumptions, just as modeling assumptions are needed to close the transport equations that arise in the development of a full Reynolds stress model. Many of the ASM models employ $\overline{k}$ and ε as parameters, and the working forms of the models often have the appearance of a two-equation model but with the constant in the expression for the turbulent viscosity being replaced by a function. Thus the computational effort required for such models is only slightly greater than that for a traditional two-equation model.

Two approaches have been followed in developing ASM models. Several researchers have simply proposed nonlinear relationships between the Reynolds stresses and the rate of mean strain, usually in the form of a series expansion having the Boussinesq approximation as the leading term (see for example, Lumley, 1970; Speziale, 1987, 1991). Such constitutive relationships are usually required to satisfy several physical and mathematical principles, such as Galilean invariance and "realizability." Realizability (Schumann, 1977; Lumley, 1978) requires that the turbulent normal stresses be positive and that Schwarz's inequality hold for fluctuating quantities, as, for example, $|\overline{ab}/\sqrt{\overline{a^2b^2}}| \leqslant 1$, where a and b are fluctuating quantities. Without the realizability constraint, models (even the standard $\overline{k}$-ϵ model) may lead to nonphysical results, such as negative values for the turbulent kinetic energy. As an example, the ASM model of Shih et al. (1994) for incompressible flow will be summarized below.

The Reynolds stress in the Shih et al. (1994) ASM model is evaluated as

$$-\overline{u_i u_j} = 2C_\mu \frac{\bar{k}^2}{\varepsilon} S_{ij}^* - \frac{2}{3}\bar{k}\,\delta_{ij} - 2C_2 \frac{\bar{k}^3}{\varepsilon^2}(-S_{ik}^*\Omega_{kj} + \Omega_{ik}S_{kj}^*) \quad (5.150i)$$

where

$$S_{ij}^* = S_{ij} - \frac{1}{3}S_{kk}\delta_{ij} \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

The constants in Eq. (5.150*i*) are given by

$$C_\mu = \frac{1}{6.5 + A_s^* U^* \bar{k}/\varepsilon} \qquad C_2 = \frac{\sqrt{1 - 9C_\mu^2 (S^*\bar{k}/\varepsilon)^2}}{1.0 + 6S^*\Omega^*\bar{k}^2/\varepsilon^2}$$

where

$$S^* = \sqrt{S_{ij}^* S_{ij}^*} \qquad \Omega^* = \sqrt{\Omega_{ij}\Omega_{ij}} \qquad U^* = \sqrt{S_{ij}^* S_{ij}^* + \Omega_{ij}\Omega_{ij}}$$

and

$$A_s^* = \sqrt{6}\cos\phi \qquad \phi = \cos^{-1}(\sqrt{6}W^*) \qquad W^* = \frac{S_{ij}^* S_{jk}^* S_{ki}^*}{(S^*)^3}$$

The values of $\bar{k}$ and ε are determined according to the standard $\bar{k}$-ε model given by Eqs. (5.149) and (5.150*a*).

The second approach, and the one followed by Rodi (1976), is to deduce a nonlinear algebraic equation for the Reynolds stresses by simplifying the full Reynolds stress PDE, Eq. (5.150*h*). Rodi argued that the convection minus the diffusion of Reynolds stress was proportional to the convection minus the diffusion of turbulent kinetic energy, with the proportionality factor being the ratio of the stress to the turbulent kinetic energy. That is,

$$\begin{aligned}
&\rho\frac{D\left(-\overline{u_i' u_j'}\right)}{Dt} - \frac{\partial}{\partial x_k}\left(\mu\frac{\partial\left(-\overline{u_i' u_j'}\right)}{\partial x_k} + C_{ijk}\right) \\
&\quad = \frac{-\rho\overline{u_i' u_j'}}{\rho\bar{k}}\left[\rho\frac{D\bar{k}}{Dt} - \frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{k}}{\partial x_j} - \frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right)\right] \\
&\quad = \frac{-\rho\overline{u_i' u_j'}}{\rho\bar{k}}\left(-\rho\overline{u_i' u_j'}\frac{\partial u_i}{\partial x_j} - \mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}\right) \qquad (5.150j)
\end{aligned}$$

Simply put, this relates the difference between production and dissipation of Reynolds stresses to the difference between production and dissipation of turbulent kinetic energy. Thus, by defining

$$P = -\rho\overline{u_i' u_j'}\frac{\partial u_i}{\partial x_j} \qquad P_{ij} = -\rho\overline{u_i' u_k'}\frac{\partial u_j}{\partial x_k} - \rho\overline{u_j' u_k'}\frac{\partial u_i}{\partial x_k}$$

and recalling the definitions of ε, ε_{ij}, and Π_{ij}, we can directly write the following nonlinear algebraic relation:

$$\frac{-\rho\overline{u_i'u_j'}}{\rho\bar{k}}(P - \rho\varepsilon) = P_{ij} + \varepsilon_{ij} - \Pi_{ij}$$

With closure approximations for ε_{ij} and Π_{ij}, Rodi (1976) developed a nonlinear constitutive equation relating Reynolds stresses and rates of mean strain that contained $\bar{k}$ and ε as parameters. For thin shear layers the constitutive relation reduces to

$$-\overline{u'v'} = \frac{2}{3}\frac{(1 - C_2)}{C_1}\frac{C_1 - 1 + C_2(P/\varepsilon)}{[C_1 - 1 + (P/\varepsilon)]^2}\frac{\bar{k}^2}{\varepsilon}\frac{\partial u}{\partial y} \qquad (5.150k)$$

where $C_1 \cong 1.5$ and $C_2 \cong 0.6$. The above can be viewed as a variation of the $\bar{k}$-ε eddy viscosity model in which the "constant" (C_μ) is replaced by a function of P/ε. Notice that when $P = \varepsilon$ (production and dissipation of turbulent kinetic energy are in balance), the function takes on the standard value of C_μ in the $\bar{k}$-ε model, 0.09.

Because the Boussinesq assumption is not invoked directly in the ASM models, they are considered to belong in Category II. However, they are not generally considered to be second-order or second-moment models because they are not based on solutions to modeled forms of the full-transport PDEs for Reynolds stresses.

The algebraic Reynolds stress models have provided a means of accounting for a number of effects, including streamline curvature, rotation, and buoyancy that cannot be predicted without ad hoc adjustments to standard two-equation models. They can predict the Reynolds-stress-driven secondary flow patterns in noncircular ducts, for example. However, they have generally not performed as well as full Reynolds stress models for flows with sudden changes in mean strain rate.

5.4.8 Subgrid-Scale Models for Large-Eddy Simulation

In LES the effect of the subgrid-scale (SGS) stresses must be modeled. Because the small-scale motion tends to be fairly isotropic and universal, there is hope that a relatively simple model will suffice. The earliest and simplest model was proposed by Smagorinsky (1963). It takes the form of a mixing-length or gradient diffusion model with the length, $l_s = C_s\Delta$, being proportional to the filter width. Thus the SGS stress tensor is represented by

$$\tau_{ij} = 2\mu_T S_{ij} \qquad (5.150l)$$

where S_{ij} is the rate of strain tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \qquad (5.150m)$$

as before, and

$$\mu_T = \rho(C_s\Delta)^2\sqrt{2S_{ij}S_{ij}} \tag{5.150n}$$

The Smagorinsky constant C_s, unfortunately, is not universal. Values ranging from 0.1 to 0.24 have been reported. For wall shear flows it is common to multiply the mixing length by a Van Driest–type exponential damping function (Moin and Kim, 1982; Piomelli, 1988) to force the length scale to approach zero at the wall.

A number of more complex models have been proposed for LES. One of the most promising model variations is the dynamic SGS model proposed by Germano et al. (1991). In its most basic form, the dynamic model follows the Smagorinsky mixing-length format but provides a basis for the value of the modeling constant C_s to be computed as part of the solution by using filters of two sizes. This approach has been extended to the transport of a scalar and compressible flow (Moin et al., 1991). When applied to flows with heat transfer, the dynamic model permits the turbulent Prandtl number to be computed as part of the solution rather than being specified a priori, and the van Driest damping function is not required to provide the correct limiting behavior near solid boundaries. Very good results have been reported with the use of the model to date (see, for example, Yang and Ferziger, 1993; Akselvoll and Moin, 1993; Wang and Pletcher 1995a, 1995b).

5.5 EULER EQUATIONS

Prandtl discovered in 1904 (see Section 5.3.1) that for sufficiently large Re the important viscous effects are confined to a thin boundary layer near the surface of a solid boundary. As a consequence of this discovery, the inviscid (non-viscous, nonconducting) portion of the flow field can be solved independently of the boundary layer. Of course, this is only true if the boundary layer is very thin compared to the characteristic length of the flow field, so that the interaction between the boundary layer and the inviscid portion of the flow field is negligible. For flows in which the interaction is not negligible, it is still possible to use separate sets of equations for the two regions, but the equations must be solved in an iterative fashion. This iterative procedure can be computationally inefficient, and as a result, it is sometimes desirable to use a single set of equations that remain valid throughout the flow field. Equations of this latter type are discussed in Chapter 8.

In the present section, a reduced set of equations will be discussed that are valid only in the inviscid portion of the flow field. These equations are obtained by dropping both the viscous terms and the heat-transfer terms from the complete Navier-Stokes equations. The resulting equations can be numerically solved (see Chapter 6) using less computer time than is required for the complete Navier-Stokes equations. We will refer to these simplified equations as the *Euler equations*, although strictly speaking, Euler's name should be attached

only to the inviscid momentum equation. In addition to the assumption of inviscid flow, it will also be assumed that there is no external heat transfer, so that the heat-generation term $\partial Q/\partial t$ in the energy equation can be dropped.

5.5.1 Continuity Equation

The continuity equation does not contain viscous terms or heat transfer terms, so that the various forms of the continuity equation given in Section 5.1.1 cannot be simplified for an inviscid flow. However, if the steady form of the continuity equation reduces to two terms for a given coordinate system, it becomes possible to discard the continuity equation by introducing the so-called *stream function* ψ. This holds true whether the flow is viscous or nonviscous. For example, the continuity equation for a 2-D steady compressible flow in Cartesian coordinates is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{5.151}$$

If the stream function ψ is defined such that

$$\begin{aligned} \rho u &= \frac{\partial \psi}{\partial y} \\ \rho v &= -\frac{\partial \psi}{\partial x} \end{aligned} \tag{5.152}$$

it can be seen by substitution that Eq. (5.151) is satisfied. Hence the continuity equation does not need to be solved, and the number of dependent variables is reduced by 1. The disadvantage is that the velocity derivatives in the remaining equations are replaced using Eqs. (5.152), so that these remaining equations will now contain derivatives that are one order higher. The physical significance of the stream function is obvious when we examine

$$\begin{aligned} d\psi &= \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = -\rho v\,dx + \rho u\,dy \\ &= \rho \mathbf{V} \cdot d\mathbf{A} = d\dot{m} \end{aligned} \tag{5.153}$$

We see that lines of constant $\psi (d\psi = 0)$ are lines across which there is no mass flow ($d\dot{m} = 0$). A *streamline* is defined as a line in the flow field whose tangent at any point is in the same direction as the flow at that point. Hence lines of constant ψ are streamlines, and the difference between the values of ψ for any two streamlines represents the mass flow rate per unit width between those streamlines.

For an incompressible 2-D flow the continuity equation in Cartesian coordinates is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.154}$$

and the stream function is defined by

$$u = \frac{\partial \psi}{\partial y}$$
$$v = -\frac{\partial \psi}{\partial x} \tag{5.155}$$

For a steady, axially symmetric compressible flow in cylindrical coordinates (see Section 5.1.8), the continuity equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{5.156}$$

and the stream function is defined by

$$\rho u_r = \frac{1}{r}\frac{\partial \psi}{\partial z}$$
$$\rho u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{5.157}$$

For the case of 3-D flows, it is possible to use stream functions to replace the continuity equation. However, the complexity of this approach usually makes it less attractive than using the continuity equation in its original form.

5.5.2 Inviscid Momentum Equations

When the viscous terms are dropped from the Navier-Stokes equations [Eq. (5.18)], the following equation results:

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p \tag{5.158}$$

This equation was first derived by Euler in 1755 and has been named Euler's equation. If we neglect body forces and assume steady flow, Euler's equation reduces to

$$\mathbf{V} \cdot \nabla\mathbf{V} = -\frac{1}{\rho}\nabla p \tag{5.159}$$

Integrating this equation along a line in the flow field gives

$$\int(\mathbf{V} \cdot \nabla\mathbf{V}) \cdot d\mathbf{r} = -\int\frac{1}{\rho}\nabla p \cdot d\mathbf{r} \tag{5.160}$$

where $d\mathbf{r}$ is the differential length of the line. For a Cartesian coordinate system, $d\mathbf{r}$ is defined by

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} \tag{5.161}$$

Let us assume that the line is a streamline. Hence **V** has the same direction as $d\mathbf{r}$, and we can simplify the integrand on the left side of Eq. (5.160) in the following manner:

$$(\mathbf{V} \cdot \nabla \mathbf{V}) \cdot d\mathbf{r} = V \frac{\partial \mathbf{V}}{\partial r} \cdot d\mathbf{r} = V \frac{\partial V}{\partial r} dr = V dV = d\left(\frac{V^2}{2}\right)$$

Likewise, the integrand on the right-hand side becomes

$$\frac{1}{\rho} \nabla p \cdot d\mathbf{r} = \frac{dp}{\rho}$$

and Eq. (5.160) reduces to

$$\frac{V^2}{2} + \int \frac{dp}{\rho} = \text{const} \tag{5.162}$$

The integral in this equation can be evaluated if the flow is assumed *barotropic*. A barotropic fluid is one in which ρ is a function only of p (or a constant), i.e., $\rho = \rho(p)$. Examples of barotropic flows are as follows.

1. steady incompressible flow:

$$\rho = \text{const} \tag{5.163}$$

2. isentropic (constant entropy) flow (see Section 5.5.4):

$$\rho = (\text{const}) p^{1/\gamma} \tag{5.164}$$

Thus for an incompressible flow, the integrated Euler's equation [Eq. (5.162)]

irrotational flow,

$$\boldsymbol{\zeta} = \nabla \times \mathbf{V} = 0 \tag{5.168}$$

and as a result, we can express $\mathbf{V}$ as the gradient of a single-valued point function ϕ, since

$$\nabla \times \mathbf{V} = \nabla \times (\nabla\phi) = 0 \tag{5.169}$$

The scalar ϕ is called the *velocity potential.* Also, from kinematics, the acceleration of a fluid particle, $D\mathbf{V}/Dt$, is given by

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times \boldsymbol{\zeta} \tag{5.170}$$

which is called *Lagrange's acceleration formula.* For an irrotational flow, this equation reduces to

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right)$$

which can be substituted into Euler's equation to give

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) = \mathbf{f} - \frac{1}{\rho}\nabla p \tag{5.171}$$

If we again neglect body forces and assume steady flow, Eq. (5.171) can be rewritten as

$$\nabla\left(\frac{V^2}{2} + \int \frac{dp}{\rho}\right) = 0 \tag{5.172}$$

since

$$\nabla \int \frac{dp}{\rho} \cdot d\mathbf{r} = \frac{\nabla p}{\rho} \cdot d\mathbf{r}$$

Integrating Eq. (5.172) along any arbitrary line in the flow field yields

$$\frac{V^2}{2} + \int \frac{dp}{\rho} = \text{const} \tag{5.173}$$

The constant in this equation now has the same value everywhere in the flow field, since Eq. (5.173) was integrated along any arbitrary line. The incompressible Bernoulli equation [Eq. (5.165)] and the compressible Bernoulli equation [Eq. (5.166)] follow directly from Eq. (5.173) in the same manner as before. The only difference is that the resulting equations are now valid everywhere in the inviscid flow field because of our additional assumption of irrotationality.

For the special case of an inviscid incompressible irrotational flow, the continuity equation

$$\nabla \cdot \mathbf{V} = 0 \tag{5.174}$$

can be combined with

$$\mathbf{V} = \nabla\phi \tag{5.175}$$

to give Laplace's equation

$$\nabla^2\phi = 0 \tag{5.176}$$

5.5.3 Inviscid Energy Equations

The inviscid form of the energy equation given by Eq. (5.22) becomes

$$\frac{\partial E_t}{\partial t} + \nabla \cdot E_t\mathbf{V} = \rho\mathbf{f} \cdot \mathbf{V} - \nabla \cdot (p\mathbf{V}) \tag{5.177}$$

which is equivalent to

$$\frac{\partial}{\partial t}(\rho H) + \nabla \cdot (\rho H\mathbf{V}) = \rho\mathbf{f} \cdot \mathbf{V} + \frac{\partial p}{\partial t} \tag{5.178}$$

Additional forms of the inviscid energy equation can be obtained from Eq. (5.29),

$$\rho\frac{De}{Dt} + p(\nabla \cdot \mathbf{V}) = 0 \tag{5.179}$$

and from Eq. (5.33),

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} \tag{5.180}$$

If we use the continuity equation and ignore the body force term, Eq. (5.178) can be written as

$$\frac{DH}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{5.181}$$

which for a steady flow becomes

$$\mathbf{V} \cdot \nabla H = 0 \tag{5.182}$$

This equation can be integrated along a streamline to give

$$H = h + \frac{V^2}{2} = \text{const} \tag{5.183}$$

The constant will remain the same throughout the inviscid flow field for the special case of an isoenergetic (homenergic) flow.

For an incompressible flow, Eq. (5.179) reduces to

$$\frac{De}{Dt} = 0 \tag{5.184}$$

which, for a steady flow, implies that the internal energy is constant along a streamline.

5.5.4 Additional Equations

The conservation equations for an inviscid flow have been presented in this section. It is possible to derive additional relations that prove to be quite useful in particular applications. In some cases, these auxiliary equations can be used to replace one or more of the conservation equations. Several of the auxiliary equations are based on the First and Second Laws of Thermodynamics, which provide the relation

$$T\,ds = de + pd\left(\frac{1}{\rho}\right) \tag{5.185}$$

where s is the entropy. Using the definition of enthalpy,

$$h = e + \frac{p}{\rho}$$

it is possible to rewrite Eq. (5.185) as

$$T\,ds = dh - \frac{dp}{\rho} \tag{5.186}$$

This latter equation can also be written as

$$T\,\nabla s = \nabla h - \frac{\nabla p}{\rho}$$

since at any given instant, a fluid particle can change its state to that of a neighboring particle. Upon combining this equation with Eqs. (5.170) and (5.158) and ignoring body forces, we obtain

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times \boldsymbol{\zeta} = T\,\nabla s - \nabla h - \nabla\left(\frac{V^2}{2}\right)$$

or

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times \boldsymbol{\zeta} = T\,\nabla s - \nabla H \tag{5.187}$$

which is called *Crocco's equation.* This equation provides a relation between vorticity and entropy. For a steady flow it becomes

$$\mathbf{V} \times \boldsymbol{\zeta} = \nabla H - T\,\nabla s \tag{5.188}$$

We have shown earlier that for a steady inviscid adiabatic flow,

$$\mathbf{V} \cdot \nabla H = 0$$

which if combined with Eq. (5.188), gives

$$\mathbf{V} \cdot \nabla s = 0$$

since $\mathbf{V} \times \boldsymbol{\zeta}$ is normal to $\mathbf{V}$. Thus we have proved that entropy remains constant along a streamline for a steady, nonviscous, nonconducting, adiabatic flow. This is called an *isentropic* flow. If we also assume that the flow is irrotational and isoenergetic, then Crocco's equation tells us that the entropy remains constant everywhere (i.e., homentropic flow).

The thermodynamic relation given by Eq. (5.185) involves only changes in properties, since it does not contain path-dependent functions. For the isentropic flow of a perfect gas it can be written as

$$T ds = 0 = c_p \, dT - RT \frac{dp}{p}$$

or

$$\frac{dp}{p} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$$

The latter equation can be integrated to yield

$$\frac{p}{T^{\gamma/(\gamma-1)}} = \text{const}$$

which becomes

$$\frac{p}{\rho^{\gamma}} = \text{const} \tag{5.189}$$

after substituting the perfect gas equation of state. The latter isentropic relation was used earlier to derive the compressible Bernoulli equation [Eq. (5.166)]. It is interesting to note that the integrated energy equation, given by Eq. (5.183), can be made identical to Eq. (5.166) if the flow is assumed to be isentropic.

The speed of sound is given by

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{5.190}$$

where the subscript s indicates a constant entropy process. At a point in the flow of a perfect gas, Eqs. (5.189) and (5.190) can be combined to give

$$a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} \tag{5.191}$$

5.5.5 Vector Form of Euler Equations

The compressible Euler equations in Cartesian coordinates without body forces or external heat addition can be written in vector form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0 \tag{5.192}$$

where **U**, **E**, **F**, and **G** are vectors given by

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E_t + p)u \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (E_t + p)v \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E_t + p)w \end{bmatrix}$$

For a steady isoenergetic flow of a perfect gas, it becomes possible to remove the energy equation from the vector set and use, instead, the algebraic form of the equation given by Eq. (5.166). This reduces the overall computation time, since one less PDE needs to be solved.

5.5.6 Simplified Forms of Euler Equations

The Euler equations can be simplified by making additional assumptions. If the flow is steady, irrotational, and isentropic, the Euler equations can be combined into a single equation called the *velocity potential equation.* The velocity potential equation is derived in the following manner. In a Cartesian coordinate system, the continuity equation may be written as

$$\frac{\partial}{\partial x}(\rho\phi_x) + \frac{\partial}{\partial y}(\rho\phi_y) + \frac{\partial}{\partial z}(\rho\phi_z) = 0 \tag{5.193}$$

where the velocity components have been replaced by

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \qquad w = \frac{\partial\phi}{\partial z} \tag{5.194}$$

The momentum (and energy) equations reduce to Eq. (5.162) with the assumptions of steady, irrotational, and isentropic flow. In differential form this equation becomes

$$dp = -\rho d\left(\frac{V^2}{2}\right) = -\rho d\left(\frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{2}\right) \tag{5.195}$$

Combining Eqs. (5.190) and (5.195) yields the equation

$$d\rho = -\frac{\rho}{a^2} d\left(\frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{2}\right) \tag{5.196}$$

which may be used to find the derivatives of ρ in each direction. After substituting these expressions for ρ_x, ρ_y, and ρ_z into Eq. (5.193) and simplifying, the velocity potential equation is obtained:

$$\left(1 - \frac{\phi_x^2}{a^2}\right)\phi_{xx} + \left(1 - \frac{\phi_y^2}{a^2}\right)\phi_{yy} + \left(1 - \frac{\phi_z^2}{a^2}\right)\phi_{zz} - \frac{2\phi_x\phi_y}{a^2}\phi_{xy} - \frac{2\phi_x\phi_z}{a^2}\phi_{xz} - \frac{2\phi_y\phi_z}{a^2}\phi_{yz} = 0 \tag{5.197}$$

Note that for an incompressible flow ($a \to \infty$), the velocity potential equation reduces to Laplace's equation.

The Euler equations can be further simplified if we consider the flow over a slender body where the free stream is only slightly disturbed (perturbed). An example is the flow over a thin airfoil. An analysis of this type is an example of small-perturbation theory. In order to demonstrate how the velocity potential equation can be simplified for flows of this type, we assume that a slender body is placed in a 2-D flow. The body causes a disturbance of the uniform flow, and the velocity components are written as

$$\begin{aligned} u &= U_\infty + u' \\ v &= v' \end{aligned} \tag{5.198}$$

where the prime denotes perturbation velocity. If we let ϕ' be the perturbation velocity potential, then

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = U_\infty + \frac{\partial\phi'}{\partial x} \\ v &= \frac{\partial\phi}{\partial y} = \frac{\partial\phi'}{\partial y} \end{aligned} \tag{5.199}$$

Substituting these expressions along with Eq. (5.191) into Eq. (5.166) gives

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}\left[2u'U_\infty + (u')^2 + (v')^2\right] \tag{5.200}$$

which can then be combined with the velocity potential equation to yield

$$\begin{aligned} (1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= M_\infty^2\left[(\gamma + 1)\frac{u'}{U_\infty} + \left(\frac{\gamma + 1}{2}\right)\frac{(u')^2}{U_\infty^2} + \left(\frac{\gamma - 1}{2}\right)\frac{(v')^2}{U_\infty^2}\right]\frac{\partial u'}{\partial x} \\ &\quad + M_\infty^2\left[(\gamma - 1)\frac{u'}{U_\infty} + \left(\frac{\gamma + 1}{2}\right)\frac{(v')^2}{U_\infty^2} + \left(\frac{\gamma - 1}{2}\right)\frac{(u')^2}{U_\infty^2}\right]\frac{\partial v'}{\partial y} \\ &\quad + M_\infty^2\frac{v'}{U_\infty}\left(1 + \frac{u'}{U_\infty}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) \end{aligned} \tag{5.201}$$

Since the flow is only slightly disturbed from the free stream, we assume

$$\frac{u'}{U_\infty}, \frac{v'}{U_\infty} \ll 1$$

As a result, Eq. (5.200) simplifies to

$$a^2 = a_\infty^2 - (\gamma - 1)u'U_\infty \tag{5.202}$$

and Eq. (5.201) becomes

$$\left[\frac{1 - M_\infty^2}{M_\infty^2} - (\gamma + 1)\frac{u'}{U_\infty}\right] M_\infty^2 \phi'_{xx} + \phi'_{yy} = 0 \tag{5.203}$$

The latter equation is called the *transonic small-disturbance equation.* This nonlinear equation is either elliptic or hyperbolic, depending on whether the flow is subsonic or supersonic.

For flows in the subsonic or supersonic regimes, the magnitude of the term $M_\infty^2(\gamma + 1)(u'/U_\infty)\phi'_{xx}$ is small in comparison with $(1 - M_\infty^2)\phi'_{xx}$, and Eq. (5.203) reduces to the linear *Prandtl-Glauert equation*:

$$(1 - M_\infty^2)\phi'_{xx} + \phi'_{yy} = 0 \tag{5.204}$$

Once the perturbation velocity potential is known, the pressure coefficient can be determined from

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = -\frac{2u'}{U_\infty} \tag{5.205}$$

which is derived using Eqs. (5.166), (5.189), (5.198), and the binomial expansion theorem.

5.5.7 Shock Equations

A shock wave is a very thin region in a supersonic flow, across which there is a large variation in the flow properties. Because these variations occur in such a short distance, viscosity and heat conductivity play dominant roles in the structure of the shock wave. However, unless one is interested in studying the structure of the shock wave, it is usually possible to consider the shock wave to be infinitesimally thin (i.e., a mathematical discontinuity) and use the Euler equations to determine the changes in flow properties across the shock wave. For example, let us consider the case of a stationary straight shock wave oriented perpendicular to the flow direction (i.e., a normal shock). The flow is in the positive x direction, and the conditions upstream of the shock wave are designated with a subscript 1, while the conditions downstream are designated with a subscript 2. Since a shock wave is a weak solution to the hyperbolic Euler equations, we can apply the theory of weak solutions, described in Section 4.4, to Eq. (5.192). For the present discontinuity, this gives

$$[\mathbf{E}] = 0$$

or

$$\mathbf{E}_1 = \mathbf{E}_2$$

Thus

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2 \\ \rho_1 u_1 v_1 &= \rho_2 u_2 v_2 \\ (E_{t_1} + p_1) u_1 &= (E_{t_2} + p_2) u_2 \end{aligned}$$

Upon simplifying the above shock relations, we find that

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2 \\ v_1 &= v_2 \\ h_1 + \frac{u_1^2}{2} &= h_2 + \frac{u_2^2}{2} \end{aligned} \tag{5.206}$$

Solving these equations for the pressure ratio across the shock, we obtain

$$\frac{p_2}{p_1} = \frac{(\gamma + 1)\rho_2 - (\gamma - 1)\rho_1}{(\gamma + 1)\rho_1 - (\gamma - 1)\rho_2} \tag{5.207}$$

Equation (5.207) relates thermodynamic properties across the shock wave and is called the *Rankine-Hugoniot equation.* The label "Rankine-Hugoniot relations" is frequently applied to all equations that relate changes across shock waves.

For shock waves inclined to the free stream (i.e., oblique shocks) the shock relations becomes

$$\begin{aligned} \rho_1 V_{n_1} &= \rho_2 V_{n_2} \\ p_1 + \rho_1 V_{n_1}^2 &= p_2 + \rho_2 V_{n_2}^2 \\ V_{t_1} &= V_{t_2} \\ h_1 + \frac{V_1^2}{2} &= h_2 + \frac{V_2^2}{2} \end{aligned} \tag{5.208}$$

where V_n and V_t are the normal and tangential components of the velocity vector, respectively. These equations also apply to moving shock waves if the velocity components are measured with respect to the moving shock wave. In this case, the normal component of the flow velocity ahead of the shock (measured with respect to the shock) can be related to the pressure behind the shock by manipulating the above equations to form

$$V_{n_1}^2 = \frac{\gamma + 1}{2} \frac{p_1}{\rho_1} \left(\frac{p_2}{p_1} + \frac{\gamma - 1}{\gamma + 1} \right) \tag{5.209}$$

This latter equation is useful when attempting to numerically treat moving shock waves as discontinuities, as seen in Chapter 6. A comprehensive listing of shock relations is available in NACA Report 1135 (Ames Research Staff, 1953).

5.6 TRANSFORMATION OF GOVERNING EQUATIONS

The classical governing equations of fluid dynamics have been presented in this chapter. These equations have been written in either vector or tensor form. In Section 5.1.8, it was shown how these equations can be expressed in terms of any generalized orthogonal curvilinear coordinate system. For many applications, however, a nonorthogonal coordinate system is desirable. In this section, we will show how the governing equations can be transformed from a Cartesian coordinate system to any general nonorthogonal (or orthogonal) coordinate system. In the process, we will demonstrate how simple transformations can be used to cluster grid points in regions of large gradients such as boundary layers and how to transform a nonrectangular computational region in the physical plane into a rectangular uniformly-spaced grid in the computational plane. These latter transformations are simple examples from a very important topic of computational fluid dynamics called grid generation. A complete discussion of grid generation is presented in Chapter 10.

5.6.1 Simple Transformations

In this section, simple independent variable transformations are used to illustrate how the governing fluid dynamic equations are transformed. As a first example, we will consider the problem of clustering grid points near a wall. Refinement of the mesh near a wall is mandatory, in most cases, if the details of the boundary layer are to be properly resolved. Figure 5.8(a) shows a mesh above a flat plate in which grid points are clustered near the plate in the normal direction (y), while the spacing in the x direction is uniform. Because the spacing is not uniform in the y direction, it is convenient to apply a transformation to the y coordinate, so that the governing equations can be solved on a uniformly spaced grid in the computational plan ($\bar{x}, \bar{y}$) as seen in Fig. 5.8(b). A suitable

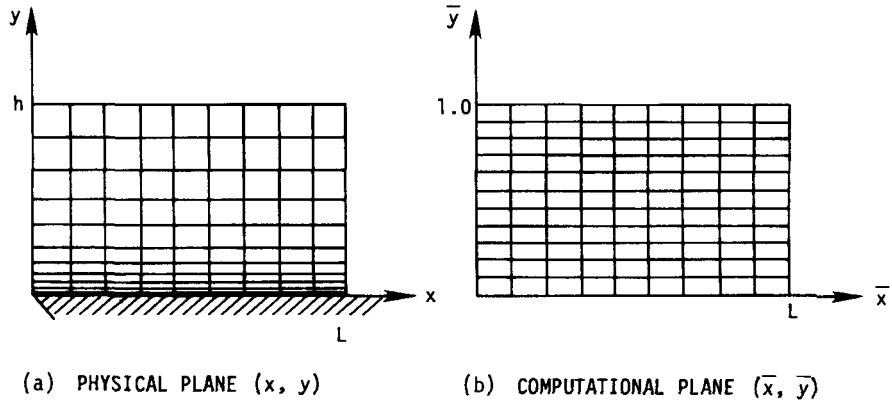


Figure 5.8 Grid clustering near a wall. (a) Physical plane (x, y). (b) Computational plane ($\bar{x}, \bar{y}$).

transformation for a 2-D boundary-layer type of problem is given by the following.

Transformation 1:

$$\bar{x} = x$$

$$\bar{y} = 1 - \frac{\ln\{[\beta + 1 - (y/h)]/[\beta - 1 + (y/h)]\}}{\ln[(\beta + 1)/(\beta - 1)]} \qquad 1 < \beta < \infty \tag{5.210}$$

This stretching transformation clusters more points near $y = 0$ as the stretching parameter β approaches 1.

In order to apply this transformation to the governing fluid dynamic equations, the following partial derivatives are formed:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x}\frac{\partial}{\partial \bar{y}} \\ \frac{\partial}{\partial y} &= \frac{\partial \bar{x}}{\partial y}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial y}\frac{\partial}{\partial \bar{y}} \end{aligned} \tag{5.211}$$

where

$$\frac{\partial \bar{x}}{\partial x} = 1 \qquad \frac{\partial \bar{y}}{\partial x} = 0$$

$$\frac{\partial \bar{x}}{\partial y} = 0 \qquad \frac{\partial \bar{y}}{\partial y} = \frac{2\beta}{h\left\{\beta^2 - [1 - (y/h)]^2\right\}\ln[(\beta + 1)/(\beta - 1)]}$$

As a result, the partial derivatives simplify to

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \bar{x}} \\ \frac{\partial}{\partial y} &= \left(\frac{\partial \bar{y}}{\partial y}\right)\frac{\partial}{\partial \bar{y}} \end{aligned} \tag{5.212}$$

If we now apply this transformation to the steady 2-D incompressible continuity equation written in Cartesian coordinates,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.213}$$

the following transformed equation is obtained:

$$\frac{\partial u}{\partial \bar{x}} + \left(\frac{\partial \bar{y}}{\partial y}\right)\frac{\partial v}{\partial \bar{y}} = 0 \tag{5.214}$$

This transformed equation can now be differenced on the uniformly spaced grid in the computational plane. The grid spacing can be computed from

$$\begin{aligned} \Delta \bar{x} &= \frac{L}{NI - 1} \\ \Delta \bar{y} &= \frac{1}{NJ - 1} \end{aligned} \tag{5.215}$$

where NI and NJ are the number of grid points in the x and y directions, respectively. We note that the expression for the metric $\partial\bar{y}/\partial y$ contains y, so that we must be able to express y as a function of $\bar{y}$. This is referred to as the inverse of the transformation. For the present transformation, given by Eqs. (5.210), the inverse can be readily found as

$$x = \bar{x}$$
$$y = h\frac{(\beta+1)-(\beta-1)\{[(\beta+1)/(\beta-1)]^{1-\bar{y}}\}}{[(\beta+1)/(\beta-1)]^{1-\bar{y}}+1} \tag{5.216}$$

The stretching transformation discussed here is from the family of general stretching transformations proposed by Roberts (1971). Another transformation from this family refines the mesh near walls of a duct, as seen in Fig. 5.9. This transformation is given by the following.

Transformation 2:

$$\bar{x} = x$$
$$\bar{y} = \alpha + (1-\alpha) \times \frac{\ln(\{\beta+[y(2\alpha+1)/h]-2\alpha\}/\{\beta-[y(2\alpha+1)/h]+2\alpha\})}{\ln[(\beta+1)/(\beta-1)]} \tag{5.217}$$

For this transformation, if $\alpha = 0$, the mesh will be refined near $y = h$ only, whereas if $\alpha = \frac{1}{2}$, the mesh will be refined equally near $y = 0$ and $y = h$. Roberts has shown that the stretching parameter β is related (approximately) to the nondimensional boundary-layer thickness (δ/h) by

$$\beta = \left(1-\frac{\delta}{h}\right)^{-1/2} \qquad 0 < \frac{\delta}{h} < 1 \tag{5.218}$$

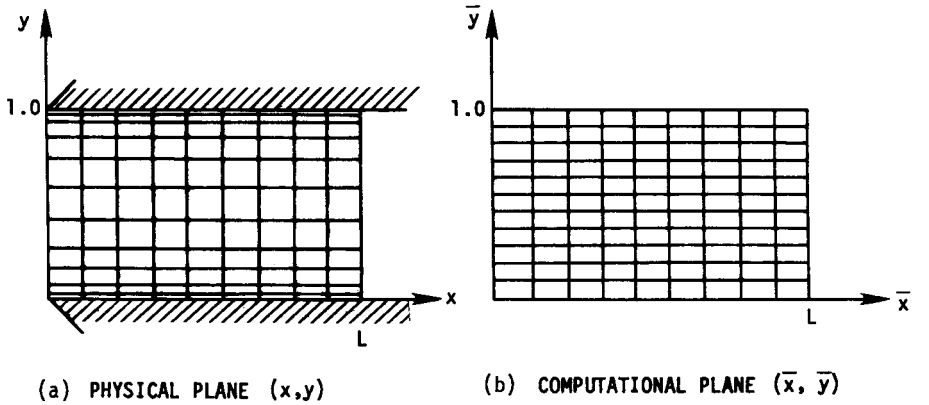


Figure 5.9 Grid clustering in a duct. (a) Physical plane (x, y). (b) Computational plane $(\bar{x}, \bar{y})$.

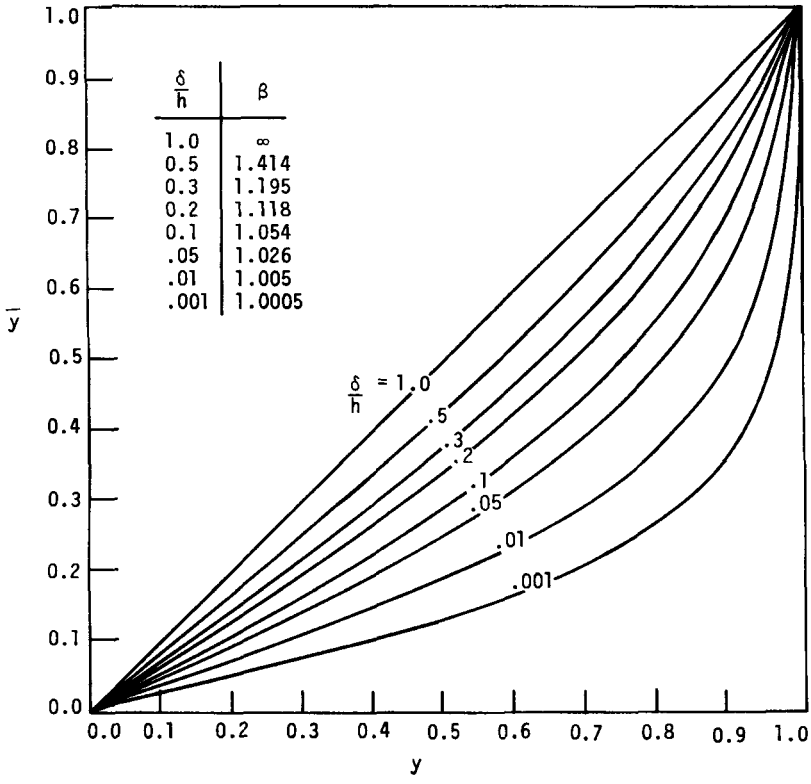


Figure 5.10 Roberts' stretching transformation ($\alpha = 0$).

where h is the height of the mesh. The amount of stretching for various values of δ/h is illustrated in Fig. 5.10 for the case where $\alpha = 0$. For the transformation given by Eqs. (5.217), the metric $\partial\bar{y}/\partial y$ is

$$\frac{\partial \bar{y}}{\partial y} = \frac{2\beta(1-\alpha)(2\alpha+1)}{h\left\{\beta^2 - [y(2\alpha+1)/h - 2\alpha]^2\right\} \ln[(\beta+1)/(\beta-1)]} \tag{5.219}$$

and the inverse transformation becomes

$$\begin{aligned} x &= \bar{x} \\ y &= h\frac{(\beta+2\alpha)[(\beta+1)/(\beta-1)]^{(\bar{y}-\alpha)/(1-\alpha)} - \beta + 2\alpha}{(2\alpha+1)\left\{1 + [(\beta+1)/(\beta-1)]^{(\bar{y}-\alpha)/(1-\alpha)}\right\}} \end{aligned} \tag{5.220}$$

A useful transformation for refining the mesh about some interior point y_c (see Fig. 5.11) is given by the following:

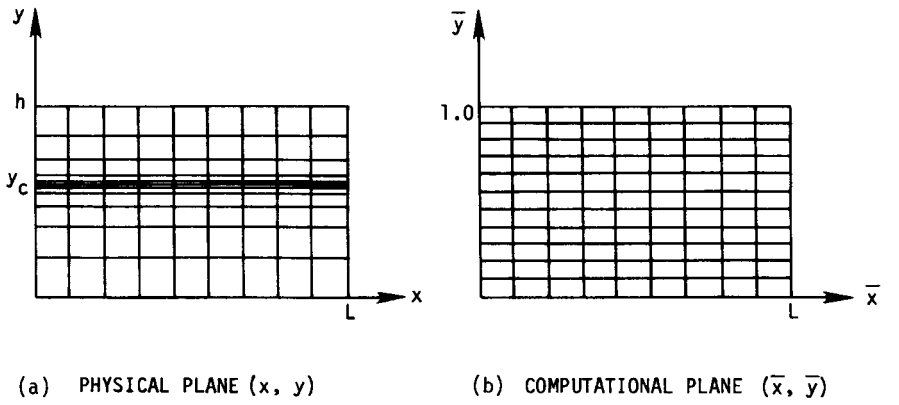


Figure 5.11 Grid clustering near an interior point. (a) Physical plane (x, y). (b) Computational plane $(\bar{x}, \bar{y})$.

Transformation 3:

$$\bar{x} = x$$
$$\bar{y} = B + \frac{1}{\tau} \sinh^{-1}\left[\left(\frac{y}{y_c} - 1\right) \sinh(\tau B)\right] \tag{5.221}$$

where

$$B = \frac{1}{2\tau} \ln\left[\frac{1 + (e^{\tau} - 1)(y_c/h)}{1 + (e^{-\tau} - 1)(y_c/h)}\right] \qquad 0 < \tau < \infty$$

In this transformation, τ is the stretching parameter, which varies from zero (no stretching) to large values that produce the most refinement near $y = y_c$. The metric $\partial\bar{y}/\partial y$ and y become

$$\frac{\partial \bar{y}}{\partial y} = \frac{\sinh(\tau B)}{\tau y_c \sqrt{1 + [(y/y_c) - 1]^2 \sinh^2(\tau B)}} \tag{5.222}$$

$$y = y_c\left\{1 + \frac{\sinh[\tau(\bar{y} - B)]}{\sinh(\tau B)}\right\} \tag{5.223}$$

For our final transformation, we will examine a simple transformation that can be used to transform a nonrectangular region in the physical plane into a rectangular region in the computational plane, as seen in Fig. 5.12. The required transformation is as follows:

Transformation 4:

$$\bar{x} = x$$
$$\bar{y} = \frac{y}{h(x)} \tag{5.224}$$

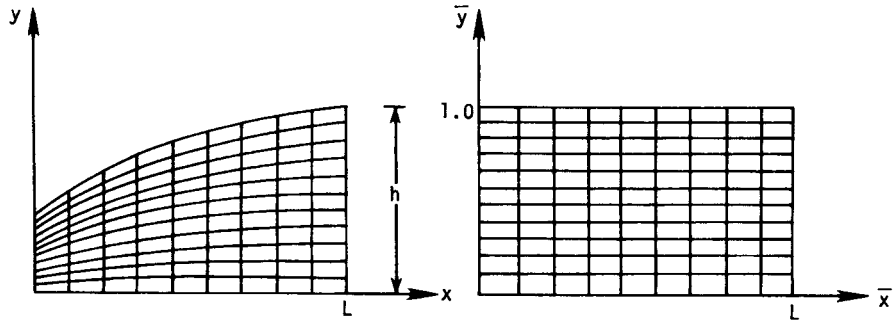


Figure 5.12 Rectangularization of computational grid. (a) Physical plane (x, y); (b) Computational plane $(\bar{x}, \bar{y})$.

The known distance between the lower boundary and the upper boundary (measured along a x = constant line) is designated by $h(x)$. The required partial derivatives are

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \bar{x}} - \bar{y}\frac{h'(x)}{h(x)}\frac{\partial}{\partial \bar{y}} \\ \frac{\partial}{\partial y} &= \frac{1}{h(x)}\frac{\partial}{\partial \bar{y}} \end{aligned} \tag{5.225}$$

where $h'(x) = dh(x)/dx$. Hence the steady 2-D incompressible continuity equation in Cartesian coordinates is transformed to

$$\frac{\partial u}{\partial \bar{x}} - \bar{y}\frac{h'(\bar{x})}{h(\bar{x})}\frac{\partial u}{\partial \bar{y}} + \frac{1}{h(\bar{x})}\frac{\partial v}{\partial \bar{y}} = 0 \tag{5.226}$$

5.6.2 Generalized Transformation

In the preceding section, we examined simple independent variable transformations that make it possible to solve the governing equations on a uniformly spaced computational grid. Let us now consider a completely general transformation of the form

$$\begin{aligned} \xi &= \xi(x, y, z) \\ \eta &= \eta(x, y, z) \\ \zeta &= \zeta(x, y, z) \end{aligned} \tag{5.227}$$

which can be used to transform the governing equations from the physical domain (x, y, z) to the computational domain (ξ, η, ζ). Using the chain rule of

partial differentiation, the partial derivatives become

$$\begin{aligned}\frac{\partial}{\partial x} &= \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} + \zeta_x \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial y} &= \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} + \zeta_y \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial z} &= \xi_z \frac{\partial}{\partial \xi} + \eta_z \frac{\partial}{\partial \eta} + \zeta_z \frac{\partial}{\partial \zeta}\end{aligned} \tag{5.228}$$

The metrics $(\xi_x, \eta_x, \zeta_x, \xi_y, \eta_y, \zeta_y, \xi_z, \eta_z, \zeta_z)$ appearing in these equations can be determined in the following manner. We first write the differential expressions

$$\begin{aligned}d\xi &= \xi_x\, dx + \xi_y\, dy + \xi_z\, dz \\ d\eta &= \eta_x\, dx + \eta_y\, dy + \eta_z\, dz \\ d\zeta &= \zeta_x\, dx + \zeta_y\, dy + \zeta_z\, dz\end{aligned} \tag{5.229}$$

which in matrix form become

$$\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{5.230}$$

In a like manner, we can write

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{5.231}$$

Therefore

$$\begin{aligned}\begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} &= \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix}^{-1} \\ &= J \begin{bmatrix} y_\eta z_\zeta - y_\zeta z_\eta & -(x_\eta z_\zeta - x_\zeta z_\eta) & x_\eta y_\zeta - x_\zeta y_\eta \\ -(y_\xi z_\zeta - y_\zeta z_\xi) & x_\xi z_\zeta - x_\zeta z_\xi & -(x_\xi y_\zeta - x_\zeta y_\xi) \\ y_\xi z_\eta - y_\eta z_\xi & -(x_\xi z_\eta - x_\eta z_\xi) & x_\xi y_\eta - x_\eta y_\xi \end{bmatrix}\end{aligned} \tag{5.232}$$

Thus the metrics are

$$\begin{aligned}\xi_x &= J(y_\eta z_\zeta - y_\zeta z_\eta) \\ \xi_y &= -J(x_\eta z_\zeta - x_\zeta z_\eta) \\ \xi_z &= J(x_\eta y_\zeta - x_\zeta y_\eta)\end{aligned}$$

$$\begin{aligned}
\eta_x &= -J(y_\xi z_\zeta - y_\zeta z_\xi) \\
\eta_y &= J(x_\xi z_\zeta - x_\zeta z_\xi) \\
\eta_z &= -J(x_\xi y_\zeta - x_\zeta y_\xi) \\
\zeta_x &= J(y_\xi z_\eta - y_\eta z_\xi) \\
\zeta_y &= -J(x_\xi z_\eta - x_\eta z_\xi) \\
\zeta_z &= J(x_\xi y_\eta - x_\eta y_\xi)
\end{aligned} \tag{5.233}$$

where J is the Jacobian of the transformation,

$$J = \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} \tag{5.234}$$

which can be evaluated in the following manner:

$$J = 1/J^{-1} = 1 \Big/ \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = 1 \Big/ \begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix}$$

$$= 1/\left[x_\xi (y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta (y_\xi z_\zeta - y_\zeta z_\xi) + x_\zeta (y_\xi z_\eta - y_\eta z_\xi) \right] \tag{5.235}$$

The metrics can be readily determined if analytical expressions are available for the inverse of the transformation:

$$\begin{aligned}
x &= x(\xi, \eta, \zeta) \\
y &= y(\xi, \eta, \zeta) \\
z &= z(\xi, \eta, \zeta)
\end{aligned} \tag{5.236}$$

For cases where the transformation is the direct result of a grid generation scheme, the metrics can be computed numerically using central differences in the computational plane. A brief discussion on the proper way to compute metrics is presented in Chapter 10.

If we apply the generalized transformation to the compressible Navier-Stokes equations written in vector form [Eqs. (5.43)], the following transformed equation is obtained:

$$\mathbf{U}_t + \xi_x \mathbf{E}_\xi + \eta_x \mathbf{E}_\eta + \zeta_x \mathbf{E}_\zeta + \xi_y \mathbf{F}_\xi + \eta_y \mathbf{F}_\eta + \zeta_y \mathbf{F}_\zeta + \xi_z \mathbf{G}_\xi + \eta_z \mathbf{G}_\eta + \zeta_z \mathbf{G}_\zeta = 0 \tag{5.237}$$

Viviand (1974) and Vinokur (1974) have shown that the gas dynamic equations can be put back into strong conservation-law form after a transformation has been applied. In order to do this, the transformed equation is first divided by the Jacobian and is then rearranged into conservation-law form by adding and subtracting like terms. When this procedure is applied to Eq. (5.237), the

following equation results:

$$\left(\frac{\mathbf{U}}{J}\right)_t + \left(\frac{\mathbf{E}\xi_x + \mathbf{F}\xi_y + \mathbf{G}\xi_z}{J}\right)_\xi + \left(\frac{\mathbf{E}\eta_x + \mathbf{F}\eta_y + \mathbf{G}\eta_z}{J}\right)_\eta + \left(\frac{\mathbf{E}\zeta_x + \mathbf{F}\zeta_y + \mathbf{G}\zeta_z}{J}\right)_\zeta - \mathbf{E}\left[\left(\frac{\xi_x}{J}\right)_\xi + \left(\frac{\eta_x}{J}\right)_\eta + \left(\frac{\zeta_x}{J}\right)_\zeta\right] - \mathbf{F}\left[\left(\frac{\xi_y}{J}\right)_\xi + \left(\frac{\eta_y}{J}\right)_\eta + \left(\frac{\zeta_y}{J}\right)_\zeta\right] - \mathbf{G}\left[\left(\frac{\xi_z}{J}\right)_\xi + \left(\frac{\eta_z}{J}\right)_\eta + \left(\frac{\zeta_z}{J}\right)_\zeta\right] = 0 \tag{5.238}$$

The last three terms in brackets are all equal to zero and can be dropped. This can be verified by substituting the metrics given by Eqs. (5.233) into these terms. If we now define the quantities

$$\begin{aligned} \mathbf{U}_1 &= \frac{\mathbf{U}}{J} \\ \mathbf{E}_1 &= \frac{1}{J}(\mathbf{E}\xi_x + \mathbf{F}\xi_y + \mathbf{G}\xi_z) \\ \mathbf{F}_1 &= \frac{1}{J}(\mathbf{E}\eta_x + \mathbf{F}\eta_y + \mathbf{G}\eta_z) \\ \mathbf{G}_1 &= \frac{1}{J}(\mathbf{E}\zeta_x + \mathbf{F}\zeta_y + \mathbf{G}\zeta_z) \end{aligned} \tag{5.239}$$

and substitute them into Eq. (5.238), the final equation is in strong conservation-law form:

$$\frac{\partial \mathbf{U}_1}{\partial t} + \frac{\partial \mathbf{E}_1}{\partial \xi} + \frac{\partial \mathbf{F}_1}{\partial \eta} + \frac{\partial \mathbf{G}_1}{\partial \zeta} = 0 \tag{5.240}$$

It should be kept in mind that the vectors $\mathbf{E}_1$, $\mathbf{F}_1$, and $\mathbf{G}_1$ contain partial derivatives in the viscous and heat-transfer terms. These partial derivative terms are to be transformed using Eqs. (5.228). For example, the shearing stress term, τ_{xy}, would be transformed to

$$\tau_{xy} = \mu\left(\xi_y\frac{\partial u}{\partial \xi} + \eta_y\frac{\partial u}{\partial \eta} + \zeta_y\frac{\partial u}{\partial \zeta} + \xi_x\frac{\partial v}{\partial \xi} + \eta_x\frac{\partial v}{\partial \eta} + \zeta_x\frac{\partial v}{\partial \zeta}\right) \tag{5.241}$$

The strong conservation-law form of the governing equations is a convenient form for applying finite-difference schemes. However, when using this form of the equations, caution must be exercised if the grid is changing. In this case, a constraint on the way the metrics are differenced, called the *geometric conservation law* (Thomas and Lombard, 1978), must be satisfied in order to prevent additional errors from being introduced into the solution.

5.7 FINITE-VOLUME FORMULATION

The governing equations of fluid dynamics have been mathematically expressed in differential form in this chapter. When a numerical scheme is applied to these differential equations, the computational domain is subdivided into grid points, and the finite-difference equations are solved at each point. An alternative approach is to solve the integral form of the governing equations. In this approach, the physical domain is subdivided into small volumes (or areas for a 2-D case), and the dependent variables are evaluated either at the centers of the volumes (cells) or at the corners of the volumes.

The integral approach includes both the finite-volume and finite-element methods, but only the finite-volume method will be discussed here. The finite-volume method has an obvious advantage over a finite-difference method if the physical domain is highly irregular and complicated, since arbitrary volumes can be utilized to subdivide the physical domain. Also since the integral equations are solved directly in the physical domain, no coordinate transformation is required. Another advantage of the finite-volume method is that mass, momentum, and energy are automatically conserved, since the integral forms of the governing equations are solved.

5.7.1 Two-Dimensional Finite-Volume Method

In order to explain the finite-volume method, consider the following 2-D model equation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{5.242}$$

Integrating this equation over the finite volume abcd (with unit depth) shown in Fig. 5.13 gives

$$\iiint_{\text{abcd}} \left(\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \right) d\mathscr{V} = 0 \tag{5.243}$$

where the differential volume $d\mathscr{V}$ is $dx\,dy$ (1). After applying Green's theorem, this equation becomes

$$\frac{\partial}{\partial t} \iiint_{\text{abcd}} U(1)\,dx\,dy + \oiint_{\text{abcd}} \mathbf{H} \cdot \mathbf{n}\,dS = 0 \tag{5.244}$$

where $\mathbf{n}$ is the unit normal to the surface S of the finite volume and $\mathbf{H}$ can be expressed in Cartesian coordinates as

$$\mathbf{H} = E\mathbf{i} + F\mathbf{j}$$

For the present 2-D geometry,

$$\mathbf{H} \cdot \mathbf{n}\,dS = (E\,dy - F\,dx)(1) \tag{5.245}$$

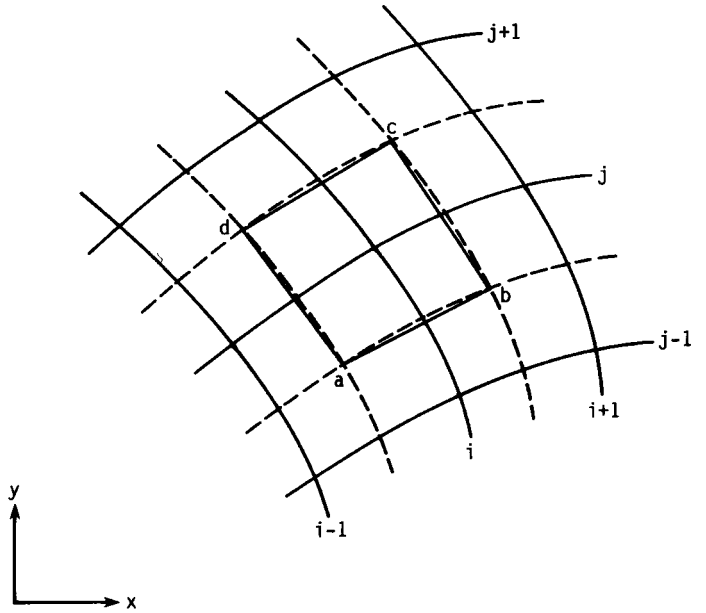


Figure 5.13 Two-dimensional finite volume.

which can be substituted into Eq. (5.244) to yield

$$\frac{\partial}{\partial t}\iint_{abcd} U\,dx\,dy + \oint_{abcd} (E\,dy - F\,dx) = 0 \tag{5.246}$$

This expression can then be approximated as

$$\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t}\right) S_{abcd} + (E_{i,j-\frac{1}{2}}\Delta y_{ab} + E_{i+\frac{1}{2},j}\Delta y_{bc} + E_{i,j+\frac{1}{2}}\Delta y_{cd} + E_{i-\frac{1}{2},j}\Delta y_{da}) - (F_{i,j-\frac{1}{2}}\Delta x_{ab} + F_{i+\frac{1}{2},j}\Delta x_{bc} + F_{i,j+\frac{1}{2}}\Delta x_{cd} + F_{i-\frac{1}{2},j}\Delta x_{da}) = 0 \tag{5.247}$$

where S_{abcd} is the area (which is assumed constant) of the quadrilateral abcd and $U_{i,j}$ is the average value of U in the quadrilateral or cell. This formulation is referred to as a *cell-centered* finite-volume scheme. An alternate approach would be to evaluate the dependent variables at the vertices of the cell, and this is called a *nodal-point* finite-volume scheme.

The increments in x and y are given by

$$\begin{aligned} \Delta x_{ab} &= x_b - x_a & \Delta x_{bc} &= x_c - x_b & \Delta x_{cd} &= x_d - x_c & \Delta x_{da} &= x_a - x_d \\ \Delta y_{ab} &= y_b - y_a & \Delta y_{bc} &= y_c - y_b & \Delta y_{cd} &= y_d - y_c & \Delta y_{da} &= y_a - y_d \end{aligned} \tag{5.248}$$

The fluxes E and F can be evaluated at time level n or $n + 1$ to provide either an explicit or implicit scheme. In addition, the spatial values of the fluxes can be determined in a variety of ways, which will lead to the various algorithms discussed in Chapter 4. As an example, let us evaluate the fluxes using average values given by

$$\begin{aligned} E_{i,j-\frac{1}{2}} &= 0.5(E_{i,j-1} + E_{i,j}) & F_{i,j-\frac{1}{2}} &= 0.5(F_{i,j-1} + F_{i,j}) \\ E_{i+\frac{1}{2},j} &= 0.5(E_{i+1,j} + E_{i,j}) & F_{i+\frac{1}{2},j} &= 0.5(F_{i+1,j} + F_{i,j}) \\ E_{i,j+\frac{1}{2}} &= 0.5(E_{i,j+1} + E_{i,j}) & F_{i,j+\frac{1}{2}} &= 0.5(F_{i,j+1} + F_{i,j}) \\ E_{i-\frac{1}{2},j} &= 0.5(E_{i-1,j} + E_{i,j}) & F_{i-\frac{1}{2},j} &= 0.5(F_{i-1,j} + F_{i,j}) \end{aligned} \tag{5.249}$$

Substituting these expressions into Eq. (5.247) yields

$$\begin{aligned} &\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t}\right) S_{\text{abcd}} + 0.5(E_{i,j-1} + E_{i,j})\,\Delta y_{\text{ab}} - 0.5(F_{i,j-1} + F_{i,j})\,\Delta x_{\text{ab}} \\ &\quad + 0.5(E_{i+1,j} + E_{i,j})\,\Delta y_{\text{bc}} - 0.5(F_{i+1,j} + F_{i,j})\,\Delta x_{\text{bc}} \\ &\quad + 0.5(E_{i,j+1} + E_{i,j})\,\Delta y_{\text{cd}} - 0.5(F_{i,j+1} + F_{i,j})\,\Delta x_{\text{cd}} \\ &\quad + 0.5(E_{i-1,j} + E_{i,j})\,\Delta y_{\text{da}} - 0.5(F_{i-1,j} + F_{i,j})\,\Delta x_{\text{da}} = 0 \end{aligned} \tag{5.250}$$

If the quadrilateral abcd is rectangular in shape and if the sides coincide with lines of constant x and y, Eq. (5.250) reduces to

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} + \frac{E_{i+1,j} - E_{i-1,j}}{2\,\Delta x} + \frac{F_{i,j+1} - F_{i,j-1}}{2\,\Delta y} = 0 \tag{5.251}$$

which we recognize as the FTCS scheme applied to our model equation. Other schemes, such as upwind algorithms, can be obtained by using appropriate expressions for the fluxes at the cell faces.

The finite-volume method described thus far in this section has been applied to a model PDE containing only first derivatives. In order to show how the finite-volume method can be applied to equations containing second derivatives, let us consider the 2-D heat equation,

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{5.252}$$

where α is assumed constant. Integrating this equation over the finite volume abcd (with unit depth) shown in Fig. 5.14 gives

$$\iiint_{\text{abcd}} \frac{\partial T}{\partial t}(1)\,dx\,dy = \alpha \iiint_{\text{abcd}} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)(1)\,dx\,dy \tag{5.253}$$

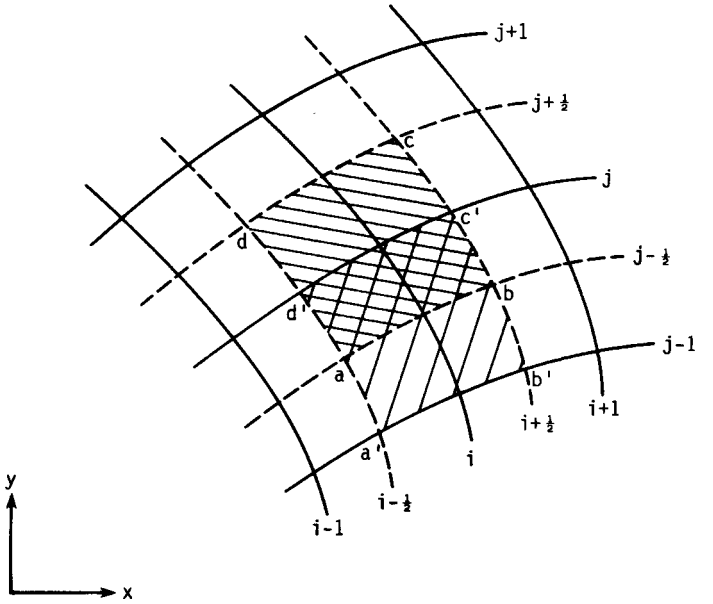


Figure 5.14 Overlapping 2-D finite volumes.

After applying Green's theorem, this equation becomes

$$\frac{\partial}{\partial t}\iiint_{\text{abcd}} T(1)\,dx\,dy = \alpha \oiint_{\text{abcd}} \mathbf{H}\cdot\mathbf{n}\,dS \tag{5.254}$$

where $\mathbf{H}$ can be expressed in Cartesian coordinates as

$$\mathbf{H} = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}$$

For the present 2-D geometry,

$$\mathbf{H}\cdot\mathbf{n}\,dS = \left(\frac{\partial T}{\partial x}\,dy - \frac{\partial T}{\partial y}\,dx\right)(1)$$

which can be substituted into Eq. (5.254) to yield

$$\frac{\partial}{\partial t}\iint_{\text{abcd}} T\,dx\,dy = \alpha \oint_{\text{abcd}} \left(\frac{\partial T}{\partial x}\,dy - \frac{\partial T}{\partial y}\,dx\right) \tag{5.255}$$

Equation (5.255) can then be approximated, as before, to obtain

$$\left(\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t}\right) S_{\text{abcd}}$$
$$= \alpha\left[\left(\frac{\partial T}{\partial x}\right)_{i,j-\frac{1}{2}} \Delta y_{\text{ab}} + \left(\frac{\partial T}{\partial x}\right)_{i+\frac{1}{2},j} \Delta y_{\text{bc}} + \left(\frac{\partial T}{\partial x}\right)_{i,j+\frac{1}{2}} \Delta y_{\text{cd}}\right.$$
$$+ \left(\frac{\partial T}{\partial x}\right)_{i-\frac{1}{2},j} \Delta y_{\text{da}} - \left(\frac{\partial T}{\partial y}\right)_{i,j-\frac{1}{2}} \Delta x_{\text{ab}} - \left(\frac{\partial T}{\partial y}\right)_{i+\frac{1}{2},j} \Delta x_{\text{bc}}$$
$$\left. - \left(\frac{\partial T}{\partial y}\right)_{i,j+\frac{1}{2}} \Delta x_{\text{cd}} - \left(\frac{\partial T}{\partial y}\right)_{i-\frac{1}{2},j} \Delta x_{\text{da}}\right] \tag{5.256}$$

where the increments in x and y are given in Eq. (5.248). Different techniques (see Peyret and Taylor, 1983) can be used to evaluate the derivatives in Eq. (5.256). A common approach is to evaluate the derivatives as a mean value over the appropriate area. For example, the derivatives $(\partial T/\partial x)_{i,j-\frac{1}{2}}$ and $(\partial T/\partial y)_{i,j-\frac{1}{2}}$ can be evaluated as their mean values over the finite volume a′b′c′d′ in Fig. 5.14. Thus

$$\left(\frac{\partial T}{\partial x}\right)_{i,j-\frac{1}{2}} = \frac{1}{S_{\text{a'b'c'd'}}} \int\int \left(\frac{\partial T}{\partial x}\right) dx\, dy = \frac{1}{S_{\text{a'b'c'd'}}} \oint T\, dy \tag{5.257}$$

where the line integral can be approximated by

$$\oint T\, dy \cong T_{i,j-1}\, \Delta y_{\text{a'b'}} + T_b\, \Delta y_{\text{b'c'}} + T_{i,j}\, \Delta y_{\text{c'd'}} + T_a\, \Delta y_{\text{d'a'}} \tag{5.258}$$

The temperatures T_a and T_b are evaluated as the average of the four surrounding temperatures:

$$\begin{aligned} T_a &= \tfrac{1}{4}(T_{i,j} + T_{i-1,j} + T_{i-1,j-1} + T_{i,j-1}) \\ T_b &= \tfrac{1}{4}(T_{i,j} + T_{i+1,j} + T_{i+1,j-1} + T_{i,j-1}) \end{aligned} \tag{5.259}$$

In a like manner,

$$\left(\frac{\partial T}{\partial y}\right)_{i,j-\frac{1}{2}} = \frac{1}{S_{\text{a'b'c'd'}}} \int\int \left(\frac{\partial T}{\partial y}\right) dx\, dy = -\frac{1}{S_{\text{a'b'c'd'}}} \oint T\, dx \tag{5.260}$$

and the line integral can be approximated as

$$\oint T\, dx = T_{i,j-1}\, \Delta x_{\text{a'b'}} + T_b\, \Delta x_{\text{b'c'}} + T_{i,j}\, \Delta x_{\text{c'd'}} + T_a\, \Delta x_{\text{d'a'}} \tag{5.261}$$

The other derivatives appearing in Eq. (5.256) can be determined in a similar manner.

5.7.2 Three-Dimensional Finite-Volume Method

The finite-volume formulation can readily be extended to three dimensions, although it does become more complicated. Consider the 3-D Navier-Stokes (or Euler) equations [Eq. (5.43)]:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0 \tag{5.262}$$

These equations can be expressed in integral form as

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \mathbf{U}\, d\mathscr{V} + \oiint_S (\bar{\mathbf{H}} \cdot \mathbf{n})\, dS = 0 \tag{5.263}$$

where the finite volume $\mathscr{V}$ is bounded by the surface S and the tensor $\bar{\mathbf{H}}$ is given in Cartesian coordinates as

$$\bar{\mathbf{H}} = \mathbf{E}\mathbf{i} + \mathbf{F}\mathbf{j} + \mathbf{G}\mathbf{k} \tag{5.264}$$

If we utilize the cell-face surface-area vector $d\mathbf{S}$ (defined as $dS\,\mathbf{n}$) and assume that the volume $\mathscr{V}$ is constant, then Eq. (5.263) can be written in discrete form for a finite volume l as

$$\mathscr{V}_l \frac{\partial}{\partial t}(\mathbf{U}_l) + \sum_{\text{sides}} \bar{\mathbf{H}} \cdot \mathbf{S} = 0 \tag{5.265}$$

where $\mathbf{U}_l$ is the value of $\mathbf{U}$ associated with the finite volume l and the summation is applied to all exterior sides of the finite volume.

In three dimensions the computational region is usually subdivided using six-sided hexahedrons such as the one shown in Fig. 5.15. The edges of the hexahedron (cell) are taken to be straight-line segments, so that a cell face can

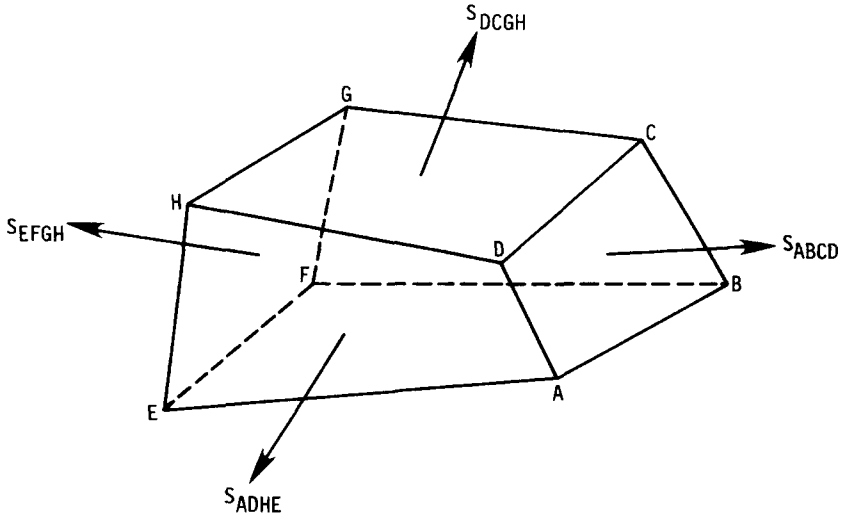


Figure 5.15 Three-dimensional hexahedral finite volume.

be considered to consist of two planar triangles. For example, the face ABCD in Fig. 5.15 can be subdivided into triangles ABC and ADC or, alternately, triangles ABD and BCD. The cell-face surface-area vector **S** is then obtained by summing the triangular area vectors. This vector is not dependent on which diagonal is used to separate the face into two triangles. Expressions for determining **S** in an efficient manner are given by Vinokur (1986). For face ABCD the surface-area vector can be determined from

$$\mathbf{S}_{ABCD} = \tfrac{1}{2}(\mathbf{r}_{AC} \times \mathbf{r}_{BD}) \tag{5.266}$$

where $\mathbf{r}_{AC}$ and $\mathbf{r}_{BD}$ are given by

$$\mathbf{r}_{AC} = \mathbf{r}_C - \mathbf{r}_A$$
$$\mathbf{r}_{BD} = \mathbf{r}_D - \mathbf{r}_B$$

and $\mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_C, \mathbf{r}_D$ are the position vectors of the points A, B, C, D, respectively. Similar expressions can be written for the other faces. For example,

$$\begin{aligned} \mathbf{S}_{ADHE} &= \tfrac{1}{2}(\mathbf{r}_{ED} \times \mathbf{r}_{AH}) \\ \mathbf{S}_{DCGH} &= \tfrac{1}{2}(\mathbf{r}_{DG} \times \mathbf{r}_{CH}) \\ \mathbf{S}_{BFGC} &= \tfrac{1}{2}(\mathbf{r}_{CF} \times \mathbf{r}_{BG}) \end{aligned} \tag{5.267}$$

The volume of a hexahedral cell can be determined in several different ways. The usual approach is to subdivide the hexahedron into tetrahedrons or pyramids. Vinokur (1986) has devised a relatively simple expression for the volume of a hexahedron, which can be expressed in terms of the notation of Fig. 5.15 as

$$\mathscr{V} = \tfrac{1}{3}(\mathbf{S}_{ABCD} + \mathbf{S}_{DCGH} + \mathbf{S}_{BFGC}) \cdot (\mathbf{r}_C - \mathbf{r}_E) \tag{5.268}$$

This formula is derived by breaking the hexahedron into three pyramids that share the main diagonal as a common edge.

PROBLEMS

5.1 Verify Eq. (5.9).
5.2 Show that for an incompressible constant-property flow, Eq. (5.18) reduces to Eq. (5.21).
5.3 Verify Eq. (5.30).
5.4 Using the nondimensionalization procedure described in Section 5.1.7, derive Eqs. (5.47).
5.5 Write the energy equation [Eq. (5.33)] in terms of axisymmetric body intrinsic coordinates.
5.6 Write the incompressible Navier-Stokes equation [Eq. (5.21)] in a spherical coordinate system.
5.7 Show that $\overline{\rho' u''} = \overline{\rho' u'}$.
5.8 Show that $\tilde{u} - \bar{u} = \overline{\rho' u'}/\bar{\rho}$.
5.9 Verify that $\overline{u''} = -\overline{\rho' u''}/\bar{\rho}$.
5.10 Starting with Eq. (5.80), show the steps in the development of Eq. (5.81).
5.11 Develop Eq. (5.84) by substitution (i.e., using $c_p \overline{T} = \overline{H} - \bar{u}_i \bar{u}_i/2 - \overline{u_i' u_i'}/2$) starting with Eq. (5.81).
5.12 Show the steps in the derivation of Eq. (5.76) starting with the Navier-Stokes equations.
5.13 Using the decomposition indicated in Section 5.2.7 for large-eddy simulation, verify the expression for τ_{ij} given in Eq. (5.94*j*).

5.14 Apply an order of magnitude analysis to the incompressible 2-D Navier-Stokes equations for the case of a planar 2-D laminar jet. Indicate which terms in the Navier-Stokes equations can be neglected in this flow.

5.15 Verify that $H' = c_p T' + u_i' \bar{u}_i + u_i' u_i'/2 - \overline{u_i' u_i'}/2$.

5.16 Explain why the boundary-layer equations may be applicable to the developing flow in a tube.

5.17 Determine the proper boundary conditions to apply to the thin-shear-layer equations for the 2-D shear layer formed by the merging of two infinite streams at uniform velocities U_a and U_b.

5.18 In the boundary-layer equations for a compressible turbulent flow, explain why $\overline{\rho u} + \overline{\rho' u'}$ has been replaced by $\overline{\rho u}$ but $\overline{\rho v} + \overline{\rho' v'}$ has been left intact.

5.19 In a flow governed by the incompressible boundary-layer equations, it is often said that the Reynolds number is of the order of $1/\epsilon^2$. What is the basis for this statement?

5.20 The boundary-layer equations, Eqs. (5.104)-(5.106), were developed for Prandtl numbers of the order of magnitude of 1. For a laminar flow over a heated flat plate, indicate what alterations should be made in these equations to properly treat flows in which the Prandtl number becomes of the order of magnitude of (*a*) ϵ, (*b*) ϵ^2, (*c*) $1/\epsilon$, (*d*) $1/\epsilon^2$.

5.21 Using the Navier-Stokes equations, develop an exact Reynolds stress transport equation applicable to an incompressible turbulent boundary layer, i.e., obtain an expression for $\rho D\overline{u_i' u_j'}/Dt$. Show the steps in your development.

5.22 Using the expression for the transport of Reynolds stresses from Prob. 5.21, let $i = j$ to obtain an expression for the transport of turbulence kinetic energy.

5.23 Using the modeled form of the turbulence kinetic energy equation, Eq. (5.147), show that when convection and diffusion of turbulence kinetic energy are negligible, the kinetic energy turbulence model reduces to the Prandtl mixing-length formula.

5.24 Assuming that convection and diffusion of turbulence kinetic energy are negligible within the log-law region of a turbulent wall boundary layer, find an expression for the turbulence kinetic energy at the outer edge of the log-law region in terms of the wall shear stress. Compare this estimate with experimental measurements of $\bar{k}$ such as those of Klebanoff (see Hinze, 1975).

5.25 Assuming the validity of the Prandtl mixing-length formula for a turbulent wall boundary layer, obtain an expression for the ratio of the apparent turbulent viscosity to the molecular viscosity in the log-law region.

5.26 Verify the inner boundary condition for $\bar{k}$ stated in Eq. (5.148*a*).

5.27 Verify that when $P = \varepsilon$ in Eq. (5.150*k*), the representation for $\overline{u'v'}$ becomes equivalent to that used in the standard $\bar{k}$-ε model.

5.28 In a 2-D body intrinsic coordinate system, define the stream function for a steady compressible flow.

5.29 Obtain Eq. (5.220).

5.30 Verify Eqs. (5.222) and (5.223).

5.31 Transform Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

into the (ξ, η) computational space using the transformation

$$\xi = x$$
$$\eta = y/h(x)$$

Note that x and y (as well as the partial derivatives with respect to these variables) should not appear in the final transformed equation.

5.32 Transform the steady 2-D incompressible continuity equation

$$u_x + v_y = 0$$

to (ξ, η) computational space using the transformation

$$\xi = x \qquad \eta = \frac{y}{x^2}$$

and display the results in strong conservation-law form using the technique of Viviand.

5.33 The 2-D physical space (x, y) is transformed to the computational space (ξ, η) by the following transformation:

$$\xi = x$$

$$\eta = \frac{y}{(x+1) - x^2}$$

(*a*) Find the Jacobian of this transformation.

(*b*) Using this transformation, transform the 2-D steady incompressible continuity equation in Cartesian coordinates. The transformed equation should contain ξ, η as the only independent variables.

5.34 Transform the 2-D incompressible Navier-Stokes equation [Eq. (5.21)] using the transformation defined by Eqs. (5.217).

5.35 Show that the transformation defined by

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

will transform the 3-D compressible continuity equation expressed in cylindrical coordinates into the compressible continuity equation in Cartesian coordinates.

5.36 Apply in a successive manner the transformations given by Eqs. (5.224) and Eqs. (5.210) to the inviscid energy equation [Eq. (5.179)] written for a 2-D steady flow.

5.37 Transform the 2-D continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

to the (τ, ξ, η) computation domain using the transformation

$$\tau = t$$

$$\xi = \xi(t, x, y)$$

$$\eta = \eta(t, x, y)$$

Use the technique of Viviand to write the transformed equation in conservation-law form. Show all intermediate steps.

5.38 Transform the steady form of Euler's equations [Eqs. (5.192)] to the (ξ, η, ζ) computational domain using the transformation

$$\xi = x$$

$$\eta = \eta(x, y, z)$$

$$\zeta = \zeta(x, y, z)$$

Using the technique of Viviand, write the transformed equations in conservation-law form.

5.39 Consider the generalized transformation

$$\tau = t$$

$$\xi = \xi(t, x, y, z)$$

$$\eta = \eta(t, x, y, z)$$

$$\zeta = \zeta(t, x, y, z)$$

(*a*) Determine suitable expressions for the Jacobian of the transformation as well as the metrics.

(*b*) Apply this transformation to the compressible Navier-Stokes equations written in vector form [Eqs. (5.43)].