CHAPTER

TWO

PARTIAL DIFFERENTIAL EQUATIONS

2.1 INTRODUCTION

Many important physical processes in nature are governed by partial differential equations (PDEs). For this reason, it is important to understand the physical behavior of the model represented by the PDE. In addition, knowledge of the mathematical character, properties, and solution of the governing equations is required. In this chapter we will discuss the physical significance and the mathematical behavior of the most common types of PDEs encountered in fluid mechanics and heat transfer. Examples are included to illustrate important properties of the solutions of these equations. In the last sections we extend our discussion to systems of PDEs and present a number of model equations, many of which are used in Chapter 4 to demonstrate the application of various discretization methods.

2.2 PHYSICAL CLASSIFICATION

2.2.1 Equilibrium Problems

Equilibrium problems are problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions (see Fig. 2.1). Equilibrium problems are boundary value problems. Examples of such problems include steady-state temperature distributions, incompressible inviscid flows, and equilibrium stress distributions in solids.

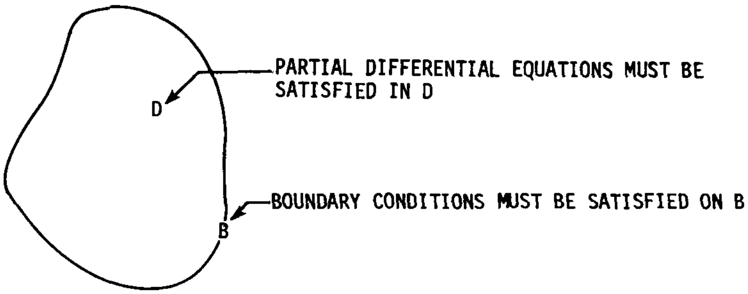

Figure 2.1 Domain for an equilibrium problem.

Sometimes equilibrium problems are referred to as jury problems. This is an apt name, since the solution of the PDE at every point in the domain depends upon the prescribed boundary condition at every point on B. In this sense the boundary conditions are certainly the jury for the solution in D. Mathematically, equilibrium problems are governed by elliptic PDEs.

Example 2.1 The steady-state temperature distribution in a conducting medium is governed by Laplace's equation. A typical problem requiring the steady-state temperature distribution in a two-dimensional (2-D) solid with the boundaries held at constant temperatures is defined by the equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 \leqslant x \leqslant 1 \qquad 0 \leqslant y \leqslant 1 \tag{2.1}$$

with boundary conditions

$$T(0, y) = 0$$
$$T(1, y) = 0$$
$$T(x, 0) = T_0$$
$$T(x, 1) = 0$$

The 2-D configuration is shown in Fig. 2.2.

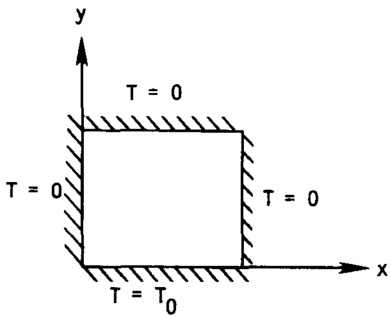

Figure 2.2 Unit square with fixed boundary temperatures.

Solution One of the standard techniques used to solve a linear PDE is separation of variables (Greenspan, 1961). This technique assumes that the unknown temperature can be written as the product of a function of x and a function of y, i.e.,

$$T(x, y) = X(x)Y(y)$$

If a solution of this form can be found that satisfies both the PDE and the boundary conditions, then it can be shown (Weinberger, 1965) that this is the one and only solution to the problem. After this form of the temperature is substituted into Laplace's equation, two ordinary differential equations (ODEs) are obtained. The resulting equations and homogeneous boundary conditions are

$$\begin{aligned} X'' + \alpha^2 X &= 0 \qquad & Y'' - \alpha^2 Y &= 0 \\ X(0) &= 0 & & \\ X(1) &= 0 & Y(1) &= 0 \end{aligned} \tag{2.2}$$

The prime denotes differentiation, and the factor α^2 arises from the separation process and must be determined as part of the solution to the problem. The solutions of the two differential equations given in Eq. (2.2) may be written

$$X(x) = A \sin(n\pi x) \qquad Y(y) = C \sinh[n\pi(y - 1)]$$

the boundary conditions enter the solution in the following way:

1.
$$T(0, y) = 0 \rightarrow X(0) = 0$$
$$T(x, 1) = 0 \rightarrow Y(1) = 0$$

These two conditions determine the kinds of functions allowed in the expression for $T(x, y)$. The boundary condition $T(0, y) = 0$ is satisfied if the solution of the separated ODE satisfies $X(0) = 0$. Since the solution in general contains sine and cosine terms, this boundary condition eliminates the cosine terms. A similar behavior is observed by satisfying $T(x, 1) = 0$ through $Y(1) = 0$ for the separated equation.

2.
$$T(1, y) = 0 \rightarrow X(1) = 0$$

This condition identifies the eigenvalues, i.e., the particular values of α that generate eigenfunctions satisfying this required boundary condition. Since the solution of the first separated equation, Eq. (2.2), was

$$X(x) = A \sin(\alpha x)$$

a nontrivial solution for $X(x)$ exists that satisfies $X(1) = 0$ only if $\alpha = n\pi$, where $n = 1, 2, \ldots$.

3.
$$T(x, 0) = T_0$$

The prescribed temperature on the x axis determines the manner in which the eigenfunctions are combined to yield the correct solution to the problem.

The solution of the present problem is written

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh[n\pi(y - 1)] \tag{2.3}$$

In this case, functions of the form $\sin(n\pi x)\sinh[n\pi(y - 1)]$ satisfy the PDE and three of the boundary conditions. In general, an infinite series composed of products of trigonometric sines and cosines and hyperbolic sines and cosines is required to satisfy the boundary conditions. For this problem, the fourth boundary condition along the lower boundary of the domain is given as

$$T(x, 0) = T_0$$

We use this to determine the coefficients A_n of Eq. (2.3). Thus we find (see Prob. 2.1)

$$A_n = \frac{2T_0}{n\pi} \frac{[(-1)^n - 1]}{\sinh(n\pi)}$$

The solution $T(x, y)$ provides the steady temperature distribution in the solid. It is clear that the solution at any point interior to the domain of interest depends upon the specified conditions at all points on the boundary. This idea is fundamental to all equilibrium problems.

Example 2.2 The irrotational flow of an incompressible inviscid fluid is governed by Laplace's equation. Determine the velocity distribution around the 2-D cylinder shown in Fig. 2.3 in an incompressible inviscid fluid flow. The flow is governed by

$$\nabla^2\phi = 0$$

where ϕ is defined as the velocity potential, i.e., $\nabla\phi = \mathbf{V}$ = velocity vector. The boundary condition on the surface of the cylinder is

$$\mathbf{V} \cdot \nabla F = 0 \tag{2.4}$$

where $F(r, \theta) = 0$ is the equation of the surface of the cylinder. In addition, the *velocity must* approach the free stream value as distance from the body becomes large, i.e., as $(x, y) \to \infty$,

$$\nabla\phi = \mathbf{V}_\infty \tag{2.5}$$

Solution This problem is solved by combining two elementary solutions of Laplace's equation that satisfy the boundary conditions. This superposition of two elementary solutions is an acceptable way of obtaining a third solution only because Laplace's equation is linear. For a linear PDE, any linear combination of solutions is also a solution (Churchill, 1941). In this case, the flow around a cylinder can be simulated by adding the velocity potential for a uniform flow to

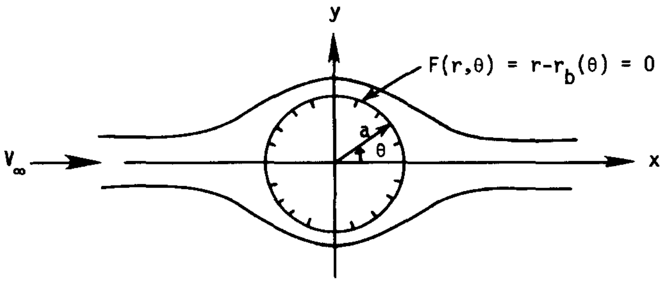

Figure 2.3 Two-dimensional flow around a cylinder.

that for a doublet (Karamcheti, 1966). The resulting solution becomes

$$\phi = V_\infty x + \frac{K \cos\theta}{\sqrt{x^2 + y^2}} = V_\infty x + \frac{Kx}{x^2 + y^2} \tag{2.6}$$

where the first term is the uniform oncoming flow, and the second term is a solution for a doublet of strength $2\pi K$.

2.2.2 Marching Problems

Marching or propagation problems are transient or transient-like problems where the solution of a PDE is required on an open domain subject to a set of initial conditions and a set of boundary conditions. Figure 2.4 illustrates the domain and marching direction for this case. Problems in this category are initial value or initial boundary value problems. The solution must be computed by marching outward from the initial data surface while satisfying the boundary conditions. Mathematically, these problems are governed by either hyperbolic or parabolic PDEs.

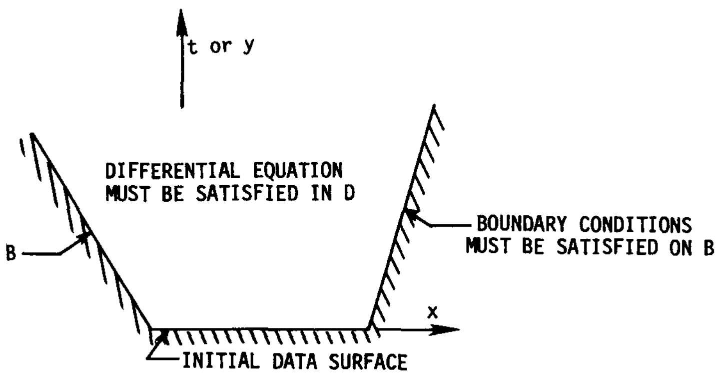

Figure 2.4 Domain for a marching problem.

Example 2.3 Determine the transient temperature distribution in a 1-D solid (Fig. 2.5) with a thermal diffusivity α if the initial temperature in the solid is 0° and if at all subsequent times, the temperature of the left side is held at 0° while the right side is held at T_0.

Solution The governing differential equation is the 1-D heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{2.7}$$

with boundary conditions

$$T(0,t) = 0 \qquad T(1,t) = T_0$$

and initial condition

$$T(x,0) = 0$$

Again, for this linear equation, separation of variables will lead to a solution. Because of the nonhomogeneous boundary conditions in this problem, it is helpful to use the principle of superposition to determine the solution as the sum of the solution to the steady problem that results as the time becomes very large and a transient solution that dies out at large times. Thus we let $T(x,t) = u(x) + v(x,t)$. Substituting this decomposition into the governing PDE, we find that because u is independent of time,

$$\frac{d^2 u}{dx^2} = 0 \tag{2.8}$$

with boundary conditions

$$u(0) = 0 \qquad u(1) = T_0$$

The solution for the steady problem is thus $u(x) = T_0 x$. We find also that the transient solution must satisfy

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} \tag{2.9}$$

with associated boundary conditions

$$v(0,t) = v(1,t) = 0$$

and initial condition

$$v(x,0) = -T_0 x$$

The initial condition for v is required in order that the sum of u and v satisfy the initial conditions of the problem. Separation of variables may be used to solve Eq. (2.9), and the solution is written in the form

$$v(x,t) = V(t)X(x)$$

If we denote the separation constant by $-\beta^2$, it is necessary to solve the ODEs

$$V' + \alpha\beta^2 V = 0 \qquad X'' + \beta^2 X = 0$$

$$X(0) = X(1) = 0$$

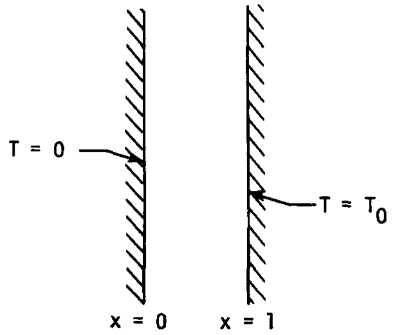

Figure 2.5 One-dimensional solid.

with the initial distribution on v as noted above. The general solution for V is readily obtained as

$$V(t) = e^{-\alpha\beta^2 t}$$

A solution for X that satisfies the boundary conditions is of the form

$$X(x) = \sin \beta x$$

where β must equal $n\pi$ ($n = 1, 2, \ldots$), so that the boundary conditions on X are met. The general solution that satisfies the PDE for v and the boundary conditions is then of the form

$$v(x, t) = e^{-\alpha n^2 \pi^2 t} \sin(n\pi x)$$

The orthogonality properties of the trigonometric functions (Weinberger, 1965) are used to meet the initial conditions as a Fourier sine series. This leads to the final solution for T, obtained by adding the solutions for u and v together:

$$T = T_0 x + \sum_{n=1}^{\infty} \frac{2T_0(-1)^n}{n\pi} e^{-n^2\pi^2\alpha t} \sin(n\pi x) \tag{2.10}$$

Example 2.4 Find the displacement $y(x, t)$ of a string of length l stretched between $x = 0$ and $x = l$ if it is displaced initially into position $y(x, 0) = \sin \pi x / l$ and released from rest. Assume no external forces act on the string.

Solution In this case the motion of the string is governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{2.11}$$

where a is a positive constant. The boundary conditions are

$$y(0, t) = y(l, t) = 0 \tag{2.12}$$

and initial conditions

$$y(x,0) = \sin\frac{\pi x}{l} \qquad \frac{\partial}{\partial t}y(x,t)|_{t=0} = 0 \tag{2.13}$$

The solution for this particular example is

$$y(x,t) = \sin\left(\pi\frac{x}{l}\right)\cos\left(a\pi\frac{t}{l}\right) \tag{2.14}$$

Solutions for problems of this type usually require an infinite series to correctly approximate the initial data. In this case, only one term of this series survives because the initial displacement requirement is exactly satisfied by one term.

The physical phenomena governed by the heat equation and the wave equation are different, but both are classified as marching problems. The behavior of the solutions to these equations and methods used to obtain these solutions are also quite different. This will become clear as the mathematical character of these equations is studied.

Typical examples of marching problems include unsteady inviscid flow, steady supersonic inviscid flow, transient heat conduction, and boundary-layer flow.

2.3 MATHEMATICAL CLASSIFICATION

The classification of PDEs is based on the mathematical concept of characteristics that are lines (in two dimensions) or surfaces (in three dimensions) along which certain properties remain constant or certain derivatives may be discontinuous. Such *characteristic* lines or surfaces are related to the directions in which "information" can be transmitted in physical problems governed by PDEs. Equations (single or system) that admit wave-like solutions are known as hyperbolic. If the equations admit solutions that correspond to damped waves, they are designated parabolic. If solutions are not wave-like, the equation or system is designated as elliptic. Although first-order equations or a system of first-order equations can be classified as indicated above, it is instructive at this point to develop classification concepts through consideration of the following general second-order PDE:

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g(x,y) \tag{2.15a}$$

where a, b, c, d, e, and f are functions of (x,y), i.e., we consider a linear equation. While this restriction is not essential, this form is convenient to use. Frequently, consideration is given to quasi-linear equations, which are defined as equations that are linear in the highest derivative. In terms of Eq. (2.15*a*), this means that a, b, and c could be functions of x, y, ϕ, ϕ_x, and ϕ_y. For our discussion, however, we assume that Eq. (2.15*a*) is linear and the coefficients depend only upon x and y.

We will indicate how equations having the general form of Eq. (2.15*a*) can be classified as hyperbolic, parabolic, or elliptic and how a standard or canonical form can be identified for each class by making use of the characteristic curves associated with the PDE. This will be discussed for equations with two independent variables, but the concepts can be extended to equations involving more independent variables, such as would be encountered in 3-D unsteady physical problems.

The classification of a second-order PDE depends only on the second-derivative terms of the equation, so we may rearrange Eq. (2.15*a*) as

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = -(d\phi_x + e\phi_y + f\phi - g) = H \tag{2.15b}$$

The characteristics, if they exist and are real curves within the solution domain, represent the locus of points along which the second derivatives may not be continuous. Along such curves, discontinuities in the solution, such as shock waves in supersonic flow, may appear. To identify such curves, we proceed as follows. For the general second-order PDE under consideration, the initial and boundary conditions are specified in terms of the function ϕ and first derivatives of ϕ. Assuming that ϕ and first derivatives of ϕ are continuous, we inquire if there may be any locations where this information would not uniquely determine the solution. In other words, may there be locations where the second derivatives are discontinuous?

Let τ be a parameter that varies along a curve C in the x-y plane. That is, on C, $x = x(\tau)$ and $y = y(\tau)$. The curve C may be on the boundary. For convenience, on C, we define

$$\begin{aligned} \phi_x &= p(\tau) & \qquad \phi_{xx} &= u(\tau) \\ \phi_y &= q(\tau) & \qquad \phi_{xy} &= v(\tau) \\ & & \qquad \phi_{yy} &= w(\tau) \end{aligned}$$

We suppose that ϕ, p, and q are given along C, as they might be given as boundary or initial conditions. With these definitions, Eq. (2.15*b*) becomes

$$au(\tau) + bv(\tau) + cw(\tau) = H \tag{2.15c}$$

Using the chain rule, we observe that

$$\frac{dp}{d\tau} = u\frac{dx}{d\tau} + v\frac{dy}{d\tau} \tag{2.15d}$$

$$\frac{dq}{d\tau} = v\frac{dx}{d\tau} + w\frac{dy}{d\tau} \tag{2.15e}$$

Equations (2.15*c*)–(2.15*e*) can be considered a system of three equations from which the second derivatives (u, v, and w) might be determined from the specified values of ϕ and the first derivatives of ϕ along C. These can be written in matrix form ($[A]\mathbf{x} = \mathbf{c}$) as

$$\begin{bmatrix} a & b & c \\ \dfrac{dx}{d\tau} & \dfrac{dy}{d\tau} & 0 \\ 0 & \dfrac{dx}{d\tau} & \dfrac{dy}{d\tau} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} H \\ \dfrac{dp}{d\tau} \\ \dfrac{dq}{d\tau} \end{bmatrix}$$

If the determinant of the coefficient matrix is zero, then there may be no unique solution for the second derivatives u, v, w along C for the given values of ϕ and its first derivatives. Thus we can write the condition for discontinuity (or nonuniqueness) in the highest order derivatives as

$$a\left(\frac{dy}{d\tau}\right)^2 - b\left(\frac{dx}{d\tau}\right)\left(\frac{dy}{d\tau}\right) + c\left(\frac{dx}{d\tau}\right)^2 = 0$$

or

$$a(dy)^2 - b\,dx\,dy + c(dx)^2 = 0 \tag{2.16}$$

Letting $h = dy/dx$, we can write Eq. (2.16) as

$$ah^2(dx)^2 - bh(dx)^2 + c(dx)^2 = 0$$

which, after division by $(dx)^2$, reduces to a quadratic equation in h:

$$ah^2 - bh + c = 0 \tag{2.17}$$

Solving for $h = dy/dx$ gives

$$h = \frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.18}$$

The curves $y(x)$ that satisfy Eq. (2.18) are called the characteristics of the PDE. Along these curves, the second derivatives are not uniquely determined by specified values of ϕ and first derivatives of ϕ, and discontinuities in the highest order derivatives may exist. Note that when the coefficients a, b, and c are constants, the solution has a particularly simple form. In passing, we note that other useful relationships, known as the *compatibility relations*, can be developed from the system Eqs. (2.15c–2.15e). These are discussed in Chapter 6. See also Hirsch (1988).

We notice that the parameter $(b^2 - 4ac)$ plays a major role in the nature of the characteristic curves. If $(b^2 - 4ac)$ is positive, two distinct families of real characteristic curves exist. If $(b^2 - 4ac)$ is zero, only a single family of characteristic curves exist. If $(b^2 - 4ac)$ is negative, the right-hand side of Eq. (2.18) is complex, and no real characteristics exist. As in the classification of general second-degree equations in analytic geometry, the PDE is classified as (1) hyperbolic if $(b^2 - 4ac)$ is positive, (2) parabolic if $(b^2 - 4ac)$ is zero, and (3) elliptic if $(b^2 - 4ac)$ is negative. Note that if a, b, c are not constants, the classification may change from point to point in the problem domain.

Equations of each class can be reduced to a representative *canonical* or *characteristic coordinate* form by a coordinate transformation that makes use of the characteristic curves. We state these forms here and illustrate the transformations needed to obtain them in examples to follow.

Two characteristic coordinate forms exist for a hyperbolic PDE:

$$\phi_{\xi\xi} - \phi_{\eta\eta} = h_1(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.19}$$

$$\phi_{\xi\eta} = h_2(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.20}$$

The canonical form for a parabolic PDE can be written as either

$$\phi_{\xi\xi} = h_3(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.21}$$

or

$$\phi_{\eta\eta} = h_4(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.22}$$

For elliptic PDEs the canonical form is

$$\phi_{\xi\xi} + \phi_{\eta\eta} = h_5(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.23}$$

In the preceding equations, the coordinates ξ and η are functions of x and y. In a coordinate transformation of the form $(x, y) \to (\xi, \eta)$, a one-to-one relationship must exist between points specified by (x, y) and (ξ, η). We are assured of a nonsingular mapping, provided that the Jacobian of the transformation

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x \eta_y - \xi_y \eta_x \tag{2.24}$$

is nonzero (Taylor, 1955). In order to apply this transformation to Eq. (2.15*a*), each derivative is replaced by repeated application of the chain rule. For example,

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \xi_x \frac{\partial \phi}{\partial \xi} + \eta_x \frac{\partial \phi}{\partial \eta} \\ \frac{\partial^2 \phi}{\partial x^2} &= \xi_x^2 \frac{\partial^2 \phi}{\partial \xi^2} + 2\xi_x \eta_x \frac{\partial^2 \phi}{\partial \xi\, \partial \eta} + \eta_x^2 \frac{\partial^2 \phi}{\partial \eta^2} + \xi_{xx} \frac{\partial \phi}{\partial \xi} + \eta_{xx} \frac{\partial \phi}{\partial \eta}\end{aligned} \tag{2.25}$$

Substitution into Eq. (2.15*a*) yields

$$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + \cdots = g(\xi, \eta)$$

where $A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$
$B = 2a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + 2c\xi_y\eta_y$
$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$

An important result of applying this transformation is immediately clear. The discriminant of the transformed equation becomes

$$B^2 - 4AC = (b^2 - 4ac)(\xi_x\eta_y - \xi_y\eta_x)^2 \tag{2.26}$$

where

$$\xi_x \eta_y - \xi_y \eta_x = J = \frac{\partial(\xi, \eta)}{\partial(x, y)}$$

Therefore, any real nonsingular transformation does not change the type of PDE.

2.3.1 Hyperbolic PDEs

From Eq. (2.18), we observe that two distinct families of characteristics exist for a hyperbolic equation. These can be found by first writing Eq. (2.18) as

$$\frac{dy}{dx} = \lambda_1 \qquad \frac{dy}{dx} = \lambda_2 \tag{2.27}$$

where the λ represent the right-hand side of Eq. (2.18) and a, b, and c are assumed constant. Upon solving the ODEs for the characteristic curves, we obtain

$$y - \lambda_1 x = k_1 \qquad y - \lambda_2 x = k_2 \tag{2.28}$$

A hyperbolic PDE in (x, y) can be written in canonical form,

$$\phi_{\xi\eta} = f(\xi, \eta, \phi, \phi_\xi, \phi_\eta) \tag{2.29}$$

by using the characteristic curves as the transformed coordinates $\xi(x, y)$ and $\eta(x, y)$. That is, we let

$$\xi = y - \lambda_1 x \qquad \eta = y - \lambda_2 x \tag{2.30}$$

In order to obtain the alternative canonical form for a hyperbolic equation,

$$\phi_{\bar{\xi}\bar{\xi}} - \phi_{\bar{\eta}\bar{\eta}} = f\left(\bar{\xi}, \bar{\eta}, \phi, \phi_{\bar{\xi}}, \phi_{\bar{\eta}}\right) \tag{2.31}$$

we can introduce linear combinations of ξ and η:

$$\bar{\xi} = \frac{\xi + \eta}{2} \qquad \bar{\eta} = \frac{\xi - \eta}{2}$$

An example utilizing the second-order wave equation is instructive.

Example 2.5 Solve the second-order wave equation

$$u_{tt} = c^2 u_{xx} \tag{2.32}$$

on the interval

$$-\infty < x < +\infty$$

with initial data

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

Solution The transformation to characteristic coordinates permits simple integration of the wave equation

$$u_{\xi\eta} = 0$$

where $\xi = x + ct$, $\eta = x - ct$.

We integrate to obtain the solution

$$u(x,t) = F_1(x + ct) + F_2(x - ct) \tag{2.33}$$

This is called the D'Alembert (Wylie, 1951) solution of the wave equation. The particular forms for F_1 and F_2 are determined from the initial data:

$$u(x,0) = f(x) = F_1(x) + F_2(x)$$

$$u_t(x,0) = g(x) = cF_1'(x) - cF_2'(x)$$

This results in a solution of the form

$$u(x,t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau \tag{2.34}$$

A distinctive property of hyperbolic PDEs can be deduced from the solution of Eq. (2.32) and the geometry of the physical domain of interest. Figure 2.6 shows the characteristics that pass through the point (x_0, t_0). The right running characteristic has a slope $+(1/c)$, while the left running one has slope $-(1/c)$. The solution $u(x,t)$ at (x_0, t_0) depends only upon the initial data contained in the interval

$$x_0 - ct_0 \leqslant x \leqslant x_0 + ct_0$$

The first term of the solution given by Eq. (2.34) represents propagation of the initial data along the characteristics, while the second term represents the effect of the data within the closed interval at $t = 0$.

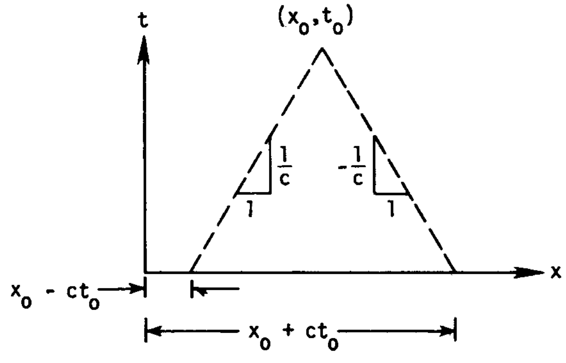

Figure 2.6 Characteristics for the wave equation.

A fundamental property of hyperbolic PDEs is the limited domain of dependence exhibited in Example 2.5. This domain of dependence is bounded by the characteristics that pass through the point (x_0, t_0). Clearly, the solution $u(x_0, t_0)$ depends only upon information in the interval bounded by these characteristics. This means that any disturbance that occurs outside of this interval can never influence the solution at (x_0, t_0). This behavior is common to all hyperbolic equations and is nicely demonstrated through the solution of the second-order wave equation. The basis for the term "initial value" or "marching problem" is clear. Initial conditions are specified, and the solution is marched outward in time or in a time-like direction.

The term "pure initial value problem" is frequently encountered in the study of hyperbolic PDEs. Example 2.5 is a pure initial value problem, i.e., there are no boundary conditions that must be applied at $x = \text{const}$. The solution at (x_0, t_0) depends only upon initial data.

In the classification of PDEs, many well-known names are associated with the specific problem types. The most well-known problem in the hyperbolic class is the Cauchy problem. This problem requires that one obtain a solution u to a hyperbolic PDE with initial data specified along a curve C. A very important theorem in mathematics assures us that a solution to the Cauchy problem exists. This is the Cauchy-Kowalewsky theorem. This theorem asserts that if the initial data are analytic in the neighborhood of (x_0, y_0) and the function u_{xx} (applied to our second-order wave equation of Example 2.5) is analytic there, a unique analytic solution for u exists in the neighborhood of (x_0, y_0).

Some discussion is warranted regarding the type of problem specification that is allowed for hyperbolic equations. For our second-order wave equation, initial conditions are required on the unknown function and its first derivatives along some curve C. It is important to observe that the curve C must not coincide with a characteristic of the differential equation. If an attempt is made to solve an initial value problem with characteristic initial data, a unique solution cannot be obtained (see Example 2.6). As is discussed further in Section 2.4, the problem is said to be "ill-posed."

Example 2.6 Solve the second-order wave equation in characteristic coordinates,

$$u_{\xi\eta} = 0$$

subject to initial data

$$u(0, \eta) = \phi(\eta) \qquad u_\xi(0, \eta) = \psi(\eta)$$

Solution The characteristics of the governing PDE are defined by $\xi = \text{const}$ and $\eta = \text{const}$. In this case the initial data are prescribed along a characteristic.

Suppose we attempt to write a Taylor-series expansion in ξ to obtain a solution for u in the neighborhood of the initial data surface $\xi = 0$. Our solution must be in the form

$$u(\xi, \eta) = u(0, \eta) + \xi u_\xi(0, \eta) + \frac{\xi^2}{2} u_{\xi\xi}(0, \eta) + \cdots$$

From the given initial data, $u(0, \eta)$ and $u_\xi(0, \eta)$ are known. It remains to determine $u_{\xi\xi}(0, \eta)$.

The governing differential equation requires

$$u_{\xi\eta}(0, \eta) = 0$$

However, we already have the condition that

$$u_{\xi\eta}(0, \eta) = \psi'(\eta) = 0$$

Therefore

$$\psi(\eta) = \text{const} = c_1$$

We may also write

$$\frac{\partial u_{\xi\eta}}{\partial \xi} = \frac{\partial u_{\xi\xi}}{\partial \eta} = 0$$

Integration of this equation yields

$$u_{\xi\xi} = f(\xi)$$

In view of the given initial data, we conclude that

$$u_{\xi\xi}(0, \eta) = \text{const} = c_2$$

and

$$u(\xi, \eta) = \phi(\eta) + \xi c_1 + \frac{\xi^2}{2} c_2$$

or

$$u(\xi, \eta) = \phi(\eta) + g(\xi)$$

We are unable to uniquely determine the function $g(\xi)$ when the initial data are given along the characteristic $\xi = 0$.

Proper specification of initial data or boundary conditions is very important in solving a PDE. Hadamard (1952) provided insight in noting that a well-posed problem is one in which the solution depends continuously upon the initial data. The concept of the well-posed problem is equally appropriate for elliptic and parabolic PDEs. An example for an elliptic problem is presented in Section 2.4.

2.3.2 Parabolic PDEs

A study of the solution of a simple hyperbolic PDE provided insight on the behavior of the solution of that type of equation. In a similar manner, we will now study the solution to parabolic equations. Referring to Eq. (2.15*a*), the parabolic case occurs when

$$b^2 - 4ac = 0$$

For this case the characteristic differential equation is given by

$$\frac{dy}{dx} = \frac{b}{2a} \tag{2.35}$$

The canonical form for the parabolic case is

$$\phi_{\xi\xi} = g(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.36}$$

If a and b are constant, this form may be obtained by identifying ξ and η as

$$\eta = y - \lambda_1 x \qquad \xi = y - \lambda_2 x$$

where λ_1 is given by the right-hand side of Eq. (2.35). In view of Eq. (2.35), we obtain only one characteristic. We must choose λ_2 to ensure linear independence of ξ and η. This requires that the Jacobian be nonzero:

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = f(\lambda_1, \lambda_2) \neq 0 \tag{2.37}$$

When λ_2 is selected, satisfying this requirement, and the transformation to (ξ, η) coordinates is completed, the canonical form given by Eq. (2.36) is obtained.

Parabolic PDEs are associated with diffusion processes. The solutions of parabolic equations clearly show this behavior. While the PDEs controlling diffusion are marching problems, i.e., we solve them starting at some initial data plane and march forward in time or in a time-like direction, they do not exhibit the limited zones of influence that hyperbolic equations have. In contrast, the solution of a parabolic equation at time t_1 depends upon the entire physical domain ($t \leqslant t_1$), including any side boundary conditions. To illustrate further, Example 2.3 required that we solve the heat equation for transient conduction in a 1-D solid. The initial temperature distribution was specified, as were the temperatures at the boundaries. Figure 2.7 illustrates the domain of dependence for this parabolic problem at t_1.

This shows that the solution at $t = t_1$ depends upon everything that occurred in the physical domain at all earlier times. The solution given by Eq. (2.10) also exhibits this behavior. Another example illustrating the behavior of a solution of a parabolic equation is of value.

Example 2.7 The unsteady motion due to the impulsive acceleration of an infinite flat plate in a viscous incompressible fluid is known as the Rayleigh problem and may be solved exactly. If the flow is 2-D, only the velocity component parallel to the plate will be nonzero. Let y be the coordinate normal to the plate and x be the coordinate along the plate. The equation that governs the velocity distribution is

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} \tag{2.38}$$

where v is the kinematic viscosity. The time derivative term is the local acceleration of the fluid, while the right-hand side is the resisting force provided by the shear stress in the fluid ($\tau = v\rho\, \partial u/\partial y$). This equation is subject to the boundary conditions

$$u(0, y) = 0$$
$$u(t, 0) = U \qquad t > 0$$
$$u(t, \infty) = 0$$

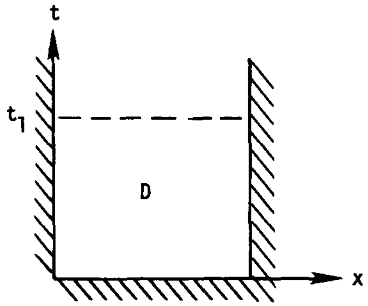

Figure 2.7 Domain of dependence for a simple parabolic problem.

Solution The solution of this problem provides the velocity distribution on a 2-D flat plate impulsively accelerated to a velocity U from rest. An interesting method frequently used in solving parabolic equations is to seek a similarity solution (Hansen, 1964). In finding a similarity solution, we introduce a change in variables, which results in reducing the number of independent variables in the original PDE (Churchill, 1974). In this case we attempt to reduce the PDE in (y, t) to an ODE in a new independent variable η. For this problem, let

$$f(\eta) = \frac{u}{U}$$

and

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

The governing differential equation becomes

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$$

with boundary conditions

$$f(0) = 1$$
$$f(\infty) = 0$$

This ODE may be solved directly to yield the solution

$$u = U\left(1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2}\, d\eta\right) \tag{2.39}$$

Using the definition of the error function

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2}\, d\eta \tag{2.40}$$

the solution becomes

$$u = U[1 - \operatorname{erf}(\eta)]$$

This shows that the layer of fluid that is influenced by the moving plate increases in thickness with time. In fact, the layer of fluid has thickness proportional to $\sqrt{\nu t}$. This indicates that the growth of this layer is controlled by the kinematic viscosity ν and the velocity change in the layer is induced by diffusion of the plate velocity into the initially undisturbed fluid. We see that this is a diffusion process, as is 1-D transient heat conduction.

2.3.3 Elliptic PDEs

The third type of PDE is elliptic. As we previously noted, jury problems are governed by elliptic PDEs. If Eq. (2.15*a*) is elliptic, the discriminant is negative, i.e.,

$$b^2 - 4ac < 0 \tag{2.41}$$

and the characteristic differential equation has no real solution. For this case, the solutions to Eq. (2.18) take the form (assuming a, b, and c are constant)

$$\begin{aligned} y - c_1 x + ic_2 x &= k_1 \\ y - c_1 x - ic_2 x &= k_2 \end{aligned} \tag{2.42}$$

The transformation to the canonical form

$$\phi_{\xi\xi} + \phi_{\eta\eta} = h_5(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \tag{2.43}$$

can be achieved by selecting ξ and η to be the real and imaginary parts of the complex conjugate functions in Eqs. (2.42). This gives

$$\xi = y - c_1 x \qquad \eta = c_2 x \tag{2.44}$$

The dependence of the solution upon the boundary conditions for elliptic PDEs has been previously discussed and demonstrated in Example 2.1. However, another example is presented here to reinforce this basic idea.

Example 2.8 Given Laplace's equation on the unit disk

$$\nabla^2 u = 0 \qquad 0 \leqslant r < 1 \qquad -\pi \leqslant \theta \leqslant \pi$$

subject to boundary conditions

$$\frac{\partial u}{\partial r}(1, \theta) = f(\theta) \qquad -\pi \leqslant \theta \leqslant \pi$$

what is the solution $u(r, \theta)$?

Solution This problem can be solved by assuming a solution of the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

The correct expressions for a_n and b_n can be developed using standard techniques (Garabedian, 1964). For this example, the expressions for a_n and b_n depend upon the boundary conditions at all points on the unit disk. This dependence on the boundary conditions should be expected for all elliptic problems. The important point of this example is that a solution of this problem exists only if

$$\int f(\theta)\, dl = 0$$

over the boundary of the unit disk (Zachmanoglou and Thoe, 1976). This may be demonstrated by applying Green's theorem to the unit disk. In this problem the boundary conditions are not arbitrarily chosen but must satisfy the integral constraint shown above.

2.4 THE WELL-POSED PROBLEM

The previous section discussed the mathematical character of the different PDEs. The examples illustrated the dependence of the solution of a particular problem upon the initial data and boundary conditions. In our discussion of hyperbolic PDEs, it was noted that a unique solution to a hyperbolic PDE cannot be obtained if the initial data are given on a characteristic. Similar examples showing improper use of boundary conditions can be constructed for elliptic and parabolic equations.

The difficulty encountered in solving our hyperbolic equation subject to characteristic initial data had to do with the question of whether or not the problem was "well-posed." In order for a problem involving a PDE to be well-posed, the solution to the problem must exist, must be unique, and must depend continuously upon the initial or boundary data. Example 2.6 led to a uniqueness question. Hadamard (1952) has constructed a simple example that demonstrates the problem of continuous dependence on boundary data.

Example 2.9 A solution of Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty \qquad y \geqslant 0$$

is desired subject to the boundary conditions ($y = 0$)

$$u(x,0) = 0$$

$$u_y(x,0) = \frac{1}{n}\sin(nx) \qquad n > 0$$

Solution Using separation of variables, we obtain

$$u = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

If our problem is well-posed, we expect the solution to depend continuously upon the boundary conditions. For the data given, we must have

$$u_y(x,0) = \frac{1}{n} \sin(nx)$$

We see that u_y becomes small for large values of n. The solution behaves in a different fashion for large n. As n becomes large, u approaches e^{ny}/n^2 and grows without bound even for small y. However, $u(x,0) = 0$, so that continuity with the initial data is lost. Thus we have an ill-posed problem. This is evident from our earlier discussions. Since Laplace's equation is elliptic, the solution depends upon conditions on the entire boundary of the closed domain. The problem given in this example requires the solution of an elliptic differential equation on an open domain. Boundary conditions were given only on the $y = 0$ line.

Problems requiring the solution of Laplace's equation subject to different types of boundary conditions are identified with specific names. The first of these is the *Dirichlet problem* (Fig. 2.8). In this problem, a solution of Laplace's equation is required on a closed domain subject to boundary conditions that require the solution to take on prescribed values on the boundary. The *Neumann problem* also requires the solution of Laplace's equation in D. However, the

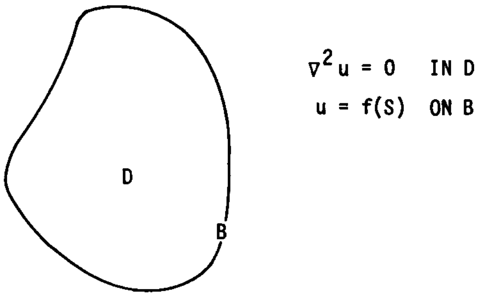

Figure 2.8 Dirichlet problem.

2.5 SYSTEMS OF EQUATIONS

In applying numerical methods to physical problems, systems of equations are frequently encountered. It is the exceptional case when a physical process is governed by a single equation. In those cases where the process is governed by a higher-order PDE, the PDE can usually be converted to a system of first-order equations. This can be most easily demonstrated by two simple examples.

The wave equation [Eq. (2.32)] can be written as a system of two first-order equations. Let

$$v = \frac{\partial u}{\partial t} \qquad w = c\frac{\partial u}{\partial x}$$

Then we may write

$$\begin{aligned} \frac{\partial v}{\partial t} &= c\frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial t} &= c\frac{\partial v}{\partial x} \end{aligned} \tag{2.45}$$

If we introduce u as one of the variables in place of either w or v, then u can be seen to satisfy the second-order wave equation.

Many physical processes are governed by Laplace's equation [Eq. (2.1)]. As in the previous example, Laplace's equation can be replaced by a system of first-order equations. In this case, let u and v represent the unknown dependent variables. We require that

$$\begin{aligned} \frac{\partial u}{\partial x} &= +\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{2.46}$$

These are the famous Cauchy-Riemann equations (Churchill, 1960). These equations are extensively used in conformally mapping one region onto another.*

* It should be noted that some differences exist in solving Laplace's equation and the Cauchy-Riemann equations. A solution of the Cauchy-Riemann equations is a solution of Laplace's equation, but the converse is not necessarily true.

The equations most frequently encountered in CFD may be written as first-order systems. We must be able to classify systems of first-order equations in order to correctly treat them. Consider the linear system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + [A]\frac{\partial \mathbf{u}}{\partial x} + [B]\frac{\partial \mathbf{u}}{\partial y} + \mathbf{r} = 0 \tag{2.47}$$

We assume for simplicity that the coefficient matrices $[A]$ and $[B]$ are functions of t, x, y, and we restrict our attention to two space dimensions. The dependent variable $\mathbf{u}$ is a column vector of unknowns, and $\mathbf{r}$ depends upon $\mathbf{u}$, x, y.

According to Zachmanoglou and Thoe (1976), there are two cases that can be definitely identified for first-order systems. The system given in Eq. (2.47) is said to be hyperbolic at a point in (x, t) if the eigenvalues of $[A]$ are all real and distinct. Richtmyer and Morton (1967) define a system to be hyperbolic if the eigenvalues are all real and $[A]$ can be written as $[T][\lambda][T]^{-1}$, where $[\lambda]$ is a diagonal matrix of eigenvalues of $[A]$ and $[T]^{-1}$ is the matrix of left eigenvectors. The same can be said of the behavior of the system in (y, t) with respect to the eigenvalues of the B matrix.

This point can be illustrated by writing the system of equations given in Eq. (2.45) as

$$\frac{\partial \mathbf{u}}{\partial t} + [A]\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{2.48a}$$

where

$$\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$$

$$[A] = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}$$

The eigenvalues λ of the $[A]$ matrix are found from

$$\det \|[A] - \lambda[I]\| = 0$$

Thus

$$\begin{vmatrix} -\lambda & -c \\ -c & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - c^2 = 0$$

The roots of this characteristic equation are

$$\lambda_1 = +c$$

$$\lambda_2 = -c$$

These are the characteristic differential equations for the wave equation, i.e.,

$$\left(\frac{dx}{dt}\right)_1 = +c$$

$$\left(\frac{dx}{dt}\right)_2 = -c$$

The system of equations in this example is hyperbolic, and we see that the eigenvalues of the $[A]$ matrix represent the characteristic differential equations of the wave equation.

The second case that can be identified for the system given in Eq. (2.47) is elliptic. Equation (2.47) is said to be elliptic at a point in (x, t) if the eigenvalues of $[A]$ are all complex. An example illustrating this behavior is given by the Cauchy-Riemann equations.

Example 2.10 Classify the system given in Eq. (2.46), which may be written as

$$\frac{\partial \mathbf{w}}{\partial x} + [A]\frac{\partial \mathbf{w}}{\partial y} = 0$$

where

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution The eigenvalues of $[A]$ are

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Since both eigenvalues of $[A]$ are complex, we identify the system as elliptic. Again, this is consistent with the behavior we are familiar with in Laplace's equation.

The first-order system represented by Eq. (2.47) can exhibit hyperbolic behavior in (x, t) space and elliptic behavior in (y, t) space, depending upon the eigenvalue structure of the A and B matrices. This is a result of evaluating the behavior of the PDE by examining the eigenvalues in (x, t) or (y, t) independently.

Note that a single first-order equation can be considered as a special case of the above development. That is, we can let $[A]$ and $[B]$ in Eq. (2.47) be real scalars a and b and the vector $\mathbf{u}$ be a scalar variable u. The conclusion is that such a single first-order equation would be classified as hyperbolic because there is only one root and it is real.

Since second-order PDEs can be represented as a system of first-order equations, one might wonder if such systems can also be identified as hyperbolic, parabolic, or elliptic by using a procedure that inquires about the continuity of the highest order derivatives. This seems reasonable, since discontinuities in second derivatives would show up as discontinuities in first derivatives in any first-order system that was developed from a second-order equation.

Consider a system of two first-order equations in two independent variables of the form

$$\begin{aligned}
a_1\frac{\partial u}{\partial x} + b_1\frac{\partial v}{\partial x} + c_1\frac{\partial u}{\partial y} + d_1\frac{\partial v}{\partial y} &= f_1 \\
a_2\frac{\partial u}{\partial x} + b_2\frac{\partial v}{\partial x} + c_2\frac{\partial u}{\partial y} + d_2\frac{\partial v}{\partial y} &= f_2
\end{aligned} \tag{2.48b}$$

This system may be written as a matrix system of the form

$$[A]\frac{\partial \mathbf{w}}{\partial x} + [C]\frac{\partial \mathbf{w}}{\partial y} = \mathbf{F} \tag{2.49a}$$

where

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

and

$$[A] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad [C] = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$

As before, we consider curves C on which all but the highest order derivatives are specified (in this case, we consider u and v specified) and inquire about conditions that will indicate that the highest derivatives are not uniquely determined. Again, we let a parameter τ vary along curves C and use the chain rule to write

$$\begin{aligned}
\frac{du}{d\tau} &= \frac{\partial u}{\partial x}\frac{dx}{d\tau} + \frac{\partial u}{\partial y}\frac{dy}{d\tau} \\
\frac{dv}{d\tau} &= \frac{\partial v}{\partial x}\frac{dx}{d\tau} + \frac{\partial v}{\partial y}\frac{dy}{d\tau}
\end{aligned} \tag{2.49b}$$

Writing the four equations (2.48b) and (2.49b) in matrix form with the prescribed data on the right-hand side gives

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \dfrac{dx}{d\tau} & 0 & \dfrac{dy}{d\tau} & 0 \\ 0 & \dfrac{dx}{d\tau} & 0 & \dfrac{dy}{d\tau} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dfrac{du}{d\tau} \\ \dfrac{dv}{d\tau} \end{bmatrix}$$

A unique solution for the first derivatives of u and v with respect to x and y does not exist if the determinant of the coefficient matrix is zero. We can write this determinant in different ways. However, a Laplace development of the determinant on the elements of the last row followed by another development on the last rows of the third-order determinants allows the determinant to be written as

$$-\left(\frac{dy}{d\tau}\right)^2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \frac{dx}{d\tau}\frac{dy}{d\tau}\left(\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}\right) - \left(\frac{dx}{d\tau}\right)^2\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

Letting

$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$|C| = \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

and setting the determinant equal to zero gives the conditions under which first partial derivatives are not uniquely determined on C:

$$|A|\left(\frac{dy}{dx}\right)^2 - |B|\frac{dy}{dx} + |C| = 0$$

Notice that this expression has the same form as Eq. (2.17) except that a, b, and c have now become determinants. The classification of the first-order system is also similar to that of the second-order PDE. Letting

$$D = |B|^2 - 4|A|\,|C|$$

we find that the system is hyperbolic if $D > 0$, parabolic if $D = 0$, and elliptic if $D < 0$.

Several questions now appear regarding behavior of systems of equations with coefficient matrices where the roots of the characteristic equations contain both real and complex parts. In those cases, the system is mixed and may exhibit hyperbolic, parabolic, and elliptic behavior. The physical system under study usually provides information that is very useful in understanding the physical behavior represented by the governing PDE. Experience gained in solving mixed problems provides the best guidance in their correct treatment.

The classification of systems of second-order PDEs is very complex. It is difficult to determine the mathematical behavior of these systems except for simple cases. For example, the system of equations given by

$$\mathbf{u}_t = [A]\mathbf{u}_{xx}$$

is parabolic if all the eigenvalues of $[A]$ are real. The same uncertainties present in classifying mixed systems of first-order equations are also encountered in the classification of second-order systems.

2.6 OTHER DIFFERENTIAL EQUATIONS OF INTEREST

Our discussion in this chapter has centered on the second-order equations given by the wave equation, the heat equation, and Laplace's equation. In addition, systems of first-order equations were examined. A number of other very important equations should be mentioned, since they govern common physical phenomena or they are used as simple models for more complex problems. In many cases, exact analytical solutions for these equations exist.

1. The first-order, linear wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{2.50}$$

governs propagation of a wave moving to the right at a constant speed c. This is called the advection equation in meteorology.

2. The inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2.51}$$

is also called the nonlinear first-order wave equation. This equation governs propagation of nonlinear waves for the simple 1-D case.

3. Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{2.52}$$

is the nonlinear wave equation [Eq. (2.51)] with diffusion added. This particular form is very similar to the equations governing fluid flow and can be used as a simple nonlinear model for numerical experiments.

4. The Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.53}$$

governs problems of the mixed type such as inviscid transonic flows. The properties of the Tricomi equation include a change from elliptic to hyperbolic character, depending upon the sign of y.

5. Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{2.54}$$

governs the temperature distribution in a solid with heat sources described by the function $f(x, y)$. Poisson's equation also determines the electric field in a region containing a charge density $f(x, y)$.

6. The advection-diffusion equation

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} = \alpha\frac{\partial^2 \xi}{\partial x^2} \tag{2.55}$$

represents the advection of a quantity ξ in a region with velocity u. The quantity α is a diffusion or viscosity coefficient.

7. The Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{2.56}$$

governs the motion of nonlinear dispersive waves.

8. The Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \tag{2.57}$$

governs the motion of time-dependent harmonic waves, where k is a frequency parameter. Applications include the propagation of acoustic waves.

9. The biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0 \tag{2.58}$$

determines the stream function for a very low Reynolds number viscous (Stokes) flow and is also a governing relation in the theory of elasticity.

10. The telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu = c^2\frac{\partial^2 u}{\partial x^2} \tag{2.59}$$

governs the transmission of electrical impulses in a long wire with distributed capacitance, inductance, and resistance. If $b = 0$, the equation is called the *damped wave equation*. Applications include the motion of a string with a damping force proportional to the velocity and heat conduction with a finite thermal propagation speed.

Many of the equations cited here will be used to demonstrate the application of discretization methods in subsequent chapters. While the list of equations is not exhaustive, examples of the various types of PDEs are included.

PROBLEMS

2.1 The solution of Laplace's equation for Example 2.1 is given in Eq. (2.3). Show that the expression for the Fourier coefficients A_n is correct as given in the example. Hint: Multiply Eq. (2.3) by $\sin(m\pi x)$ and integrate over the interval $0 \leqslant x \leqslant 1$ to obtain your answer after using the boundary condition $T(x, 0) = T_0$.

2.2 Show that the velocity field represented by the potential function in Eq. (2.6) satisfies the surface boundary condition given in Eq. (2.4).
2.3 Demonstrate that Eq. (2.14) is the solution of the wave equation as required in Example 2.4. Use the separation of variables technique.
2.4 Show that the type of PDE is unchanged when any nonsingular, real transformation is used.
2.5 Derive the canonical form for hyperbolic equations [Eq. (2.29)] by applying the transformations given in Eq. (2.30) to Eq. (2.15*a*).
2.6 Show that the canonical form for parabolic equations given in Eq. (2.36) is correct.
2.7 Show that a solution to Example 2.8 exists only if

$$\int f(\theta)\, dl = 0$$

on the unit circle.
2.8 Consider the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0$$

(*a*) Discuss the mathematical character of this equation for *all* real values of x and y.
(*b*) Obtain the new coordinates ξ and η that will transform the given equation in the *first* quadrant to its canonical form.
2.9 (*a*) Classify the equation

$$2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$$

(*b*) Obtain the transformation variables required to transform the equation to its canonical form.
(*c*) Convert the equation into an equivalent system of first-order equations and write them as a matrix system.
(*d*) Apply the method for classification of a system of equations to the system determined in Prob. 2.9(*c*).
2.10 Classify the following system of equations:

$$\frac{\partial u}{\partial t} + 8\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + 2\frac{\partial v}{\partial x} = 0$$

2.11 The following system of equations is elliptic. Determine the possible range of values for a.

$$\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} + a\frac{\partial u}{\partial x} = 0$$

2.12 Determine the mathematical character of the equations given by

$$\beta^2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

2.13 Classify the following PDEs:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = -e^{-kt}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} = 4$$

2.14 Classify the behavior of the following system of PDEs in (t, x) and (t, y) space:

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2.15 (*a*) Write the Fourier cosine series for the function

$$f(x) = \sin(x) \qquad 0 < x < \pi$$

(*b*) Write the Fourier cosine series for the function

$$f(x) = \cos(x) \qquad 0 < x < \pi$$

2.16 Find the characteristics of each of the following PDEs:

(*a*)
$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

(*b*)
$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.17 Transform the PDEs given in Prob. 2.16 into canonical form.

2.18 Obtain the canonical form for the following elliptic PDEs:

(*a*)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

(*b*)
$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

2.19 Transform the following parabolic PDEs to canonical form:

(*a*)
$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x\,\partial y} + 9\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - e^{xy} = 1$$

(*b*)
$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} + 7\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} = 0$$

2.20 Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad y \geqslant 0$$

with initial data

$$u(x, 0) = 1$$
$$u_y(x, 0) = 0$$

2.21 Solve Laplace's equation,

$$\nabla^2 u = 0 \qquad 0 \leqslant x \leqslant \pi \qquad 0 \leqslant y \leqslant \pi$$

subject to boundary conditions

$$u(x, 0) = \sin x + 2\sin 2x$$
$$u(\pi, y) = 0$$
$$u(x, \pi) = 0$$
$$u(0, y) = 0$$

2.22 Repeat Prob. 2.21 with

$$u(x,0) = -\pi^2 x^2 + 2\pi x^3 - x^4$$

2.23 Determine the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \leqslant x \leqslant 1$$

with boundary conditions

$$u(t,0) = 0$$
$$u(t,1) = 0$$

and an initial distribution

$$u(0,x) = \sin(2\pi x)$$

2.24 Repeat Prob. 2.23 if the initial distribution is given by

$$u(0,x) = 1 - \cos(4\pi x)$$