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Numerical Computation of INTERNAL AND EXTERNAL FLOWS

Volume 2: Computational Methods for Inviscid and Viscous Flows

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To A. F.
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Preface

This volume, divided into Parts V to VII, is a continuation of the first one which was devoted to fundamentals of numerical discretizations. It contains a presentation of computational methods for inviscid and viscous flow models as they have evolved over the last decade.

Over the last twenty to thirty years considerable progress has been achieved and the field of Computational Fluid Dynamics (CFD) is reaching a mature stage, where most of the basic methodology is, and will remain, well established. Basically, the 1970s can be considered as the development period for the foundations of the discretization methods for transonic potential models and for the foundations of the central discretization methods for the Euler and Navier–Stokes equations, following on the landmark introduction of the Lax–Wendroff scheme.

Although prepared by earlier fundamental developments in the line of Godunov's method for physically based discretizations of the Euler equations, the upwind, high resolution methods have reached their maturity and been established on solid theoretical grounds in the 1980s. They are by now as firmly established as the central methods. Hence a large variety of techniques are available and a considerable experience has already been accumulated with various discretizations of the Euler equations.

The concomitant tremendous development of computer performance over the same period has resulted in the present capacity of solving two-dimensional Euler equations in seconds of computer time, and simple three-dimensional problems in minutes of CPU times, with the best available codes on the powerful supercomputers. Hence more attention can be given to the validation, accuracy and reliability of numerical flow simulations and to their extensions to complex industrial design and analysis applications.

Another consequence is the current possibility of obtaining Navier–Stokes solutions, within the Reynolds-averaged approximation, in rather short computer times (at least for two-dimensional problems and simple three-dimensional configurations). Although the accumulated experience with Navier–Stokes solutions is not yet as large as with the inviscid models, it is rapidly building up. Due to the strong connection between Euler and Navier–Stokes equations at high Reynolds numbers, most of the inviscid methods are of application to the viscous flows. The major topic of uncertainty remains
essentially connected to the fundamental problems of turbulence and its model-
lization within the Reynolds-averaged approximation.

The content of this volume reflects in a certain way the situation just described.

Part V deals with the simplest inviscid approximation which is, in certain
flow regimes, equivalent to the full system of Euler equations, namely the full
potential model. It contains three chapters, 13 to 15, covering the mathematical
formulations (Chapter 13), the discretization of subsonic potential flows
(Chapter 14) and the treatment of transonic situations (Chapter 15).

Part VI is devoted to a detailed presentation of the Euler equations and of
the basic numerical techniques developed in order to discretize the complex
system of inviscid, compressible conservation laws. It covers Chapters 16 to 21,
dealing with the algebra of the Euler equations (Chapter 16), the central schemes
(Chapter 17 and 18), the treatment of boundary conditions (Chapter 19) and
the upwind methods (Chapters 20 and 21).

Part VII finally introduces the discretization methods for the Navier–Stokes
equations and contains two chapters, 22 and 23. Chapter 22 covers the basic
mathematical formulation of Reynolds-averaged Navier–Stokes equations with
an introduction to turbulence models and the last chapter summarizes the
approaches for compressible and incompressible viscous conservation laws.

The present text is directed at students at the graduate level as well as at
scientists and engineers already engaged, or starting to be engaged, in
Computational Fluid Dynamics. Although Computational Fluid Dynamics
requires a good theoretical base, it remains for the large part an experimental
science since many properties depend on the non-linear character of the flow
equations and cannot be fully analysed. Therefore, a fraction of the problems
added to each chapter request the writing of a program, mainly for the
one-dimensional flow equations.

Since the development of a code covers many aspects: selection of a scheme,
implementation of boundary conditions, selection of a time integration method,
definition of control mechanisms of non-linear instabilities,...., it is recom-
mended to experiment intensively with as many variants as possible, either
individually or by sharing the number of selected options and different test
cases within a group or a class of students. A single modular code with many
options is a remarkably effective and instructive 'numerical laboratory'.

Initial versions of some chapters have been written while holding the NAVAIR
Research Chair at the Naval Postgraduate School in Monterey. I am particularly
grateful to Ray Shreeve for this opportunity and for his friendship.

Some sections on Euler equations have been written during a summer stay
at ICASE, NASA Langley, and I would like to acknowledge particularly
Dr Milton Rose, former Director of ICASE, for his hospitality and the
stimulating atmosphere.

I have also had the privilege to benefit from results of computations performed,
at my request, on different test cases by several groups and I would like to
thank D. Caughey at Cornell University, T. Holst at NASA Ames, A. Jameson
at Princeton University, M. Salas at NASA Langley, and J. South and C. Gumbert also at NASA Langley, for their willingness and effort.

During the redaction of this book, I have had some stimulating discussions on the subject of the Kutta condition with T. Pulliam and A. Rizzi for which I am grateful.

I have also the pleasure to thank my coworkers C. Lacor and G. Van Dijck for their comments and support, as well as my secretary J. D'haes for her considerable help with figures and text.

Ch. HIRSCH
Brussels, July 1988
Nomenclature

\begin{itemize}
\item \( a \) convection velocity or wave speed
\item \( \overrightarrow{A} \) jacobian of flux vector with respect to conservative variables, with components \( A, B, C \)
\item \( c \) speed of sound
\item \( c_p \) specific heat at constant pressure
\item \( c_v \) specific heat at constant volume
\item \( D \) artificial dissipation function
\item \( e \) internal energy per unit mass
\item \( E \) total energy per unit mass
\item \( f \) scalar flux function
\item \( f^* \) numerical flux function
\item \( f_e \) external force vector
\item \( \overrightarrow{F} \) flux vector with components \( f, g, h \)
\item \( g^{\alpha\beta}, g_{\alpha\beta} \) contravariant and covariant metric tensor
\item \( G \) amplification factor/matrix; convergence operator of iterative schemes
\item \( h \) enthalpy per unit mass
\item \( H \) stagnation enthalpy per unit mass
\item \( I \) rothalpy
\item \( J \) Jacobian of coordinate transformation
\item \( k \) coefficient of thermal conductivity
\item \( k \) wave number
\item \( K \) stiffness matrix
\item \( K = \overrightarrow{A} \cdot \overrightarrow{e} \) projection of jacobian matrix on propagation direction \( \overrightarrow{e} \)
\item \( K_T \) jacobian matrix of differential operator \( L \)
\item \( \ell^{(j)} \) left eigenvector of jacobian matrix
\item \( L \) differential operator
\item \( M \) Mach number
\item \( n \) normal distance
\item \( \overrightarrow{n} \) normal vector
\item \( N_1 \) finite element interpolation function for node \( I \)
\item \( p \) pressure
\item \( P \) convergence or conditioning operator
\item \( Pr \) Prandtl number
\item \( q \) modulus of velocity; source term
\item \( Q \) source term column-vector
\item \( r \) gas constant per unit mass
\item \( r^{(j)} \) right eigenvector of jacobian matrix
\item \( R \) residual of iterative scheme
\end{itemize}
Re
Reynolds number

s
entropy per unit mass

$S$
characteristic surface, area of nozzle cross-section

$\overline{S}$
surface vector

t
time

$T$
temperature

$u$
scalar dependent variable

$U$
column-vector of conservative variables

$\vec{u}$
velocity vector cartesian components $u, v, w$

$V$
column-vector of primitive variables

$w$
characteristic variable

$W$
column-vector of characteristic variables

$\vec{x}$
position vector

$x, y, z$
cartesian coordinates

$\alpha$
diffusivity coefficient

$\gamma$
specific heat ratio

$\Gamma$
circulation; boundary of domain $\Omega$

$\delta$
central-difference operator: $\delta u_i = u_{i+1/2} - u_{i-1/2}$

$\overline{\delta}$
central-difference operator: $\overline{\delta} u_i = (u_{i+1} - u_{i-1})/2$

$\delta^+$
forward difference operator $\delta^+ u_i = u_{i+1} - u_i$

$\delta^-$
backward difference operator $\delta^- u_i = u_i - u_{i-1}$

$\Delta$
Laplace operator

$\Delta t$
time step

$\Delta x, \Delta y$
spatial mesh size in $x$ and $y$ directions

$\epsilon$
turbulence dissipation rate

$\epsilon_D$ dissipation or diffusion error

$\epsilon_{\phi}$ dispersion error

$\vec{\zeta}$
vorticity vector

$\overline{K}$
wave-number vector; wave propagation direction

$\lambda(A)$ eigenvalue of matrix $A$

$\mu$
coefficient of dynamic viscosity

$\mu$
averaging difference operator: $\mu u_i = (u_{i+1/2} + u_{i-1/2})/2$

$\mu$
switching function for transonic potential flow

$\xi, \eta, \zeta$
curvilinear coordinates

$\rho$
density

$\rho(A)$ spectral radius of matrix $A$

$\sigma$
Courant number

$\vec{\sigma}$
internal stress tensor

$\tau$
ratio $\Delta t/\Delta x$

$\xi$
viscous shear stress tensor

$\nu$
kinematic viscosity

$\phi$
velocity potential function

$\phi$
phase angle in Von Neumann analysis

$\Phi$
phase angle of amplification factor

$\omega$
time frequency of plane wave

$\omega$
overrelaxation parameter

$\Omega$
volume

$\overline{1}_x, \overline{1}_y, \overline{1}_z$
unit vectors along the $x, y, z$ directions
Subscripts

\(e\)  
external variable

\(i,j\)  
mesh point locations in \(x, y\) directions

\(I, J\)  
nodal point index

\(J\)  
eigenvalue number

\(L, R\)  
left and right states

\(\text{min}\)  
minimum

\(\text{max}\)  
maximum

\(n\)  
normal or normal component

\(o\)  
stagnation values

\(v\)  
viscous term

\(x, y, z\)  
components in \(x, y, z\) directions

\(\partial\)  
partial differentiation with respect to \(x, y, z\)

\(\infty\)  
freestream value

\(\zeta, \eta, \zeta\)  
components in \(\zeta, \eta, \zeta\) directions

Superscripts

\(AV\)  
artificial viscosity

\(n\)  
iteration level

\(n\)  
time level

\(\text{-}\)  
exact solution of discretized equation

\(\text{\text{-\text{-}}}\)  
exact solution of differential equation

Symbols

\(x\)  
vector product of two vectors

\(\otimes\)  
tensor product of two vectors

\(\nabla\)  
gradients or divergence operator