Chapter 6

Finite Volume Method and Conservative Discretizations

The finite volume method was apparently introduced into the field of numerical fluid dynamics independently by McDonald (1971) and MacCormack and Paullay (1972) for the solution of two-dimensional, time-dependent Euler equations and extended by Rizzi and Inouye (1973) to three-dimensional flows. This is the name given to the technique by which the integral formulation of the conservation laws are discretized directly in the physical space. Although, according to one's point of view, it can be considered as a finite difference method applied to the differential, conservative form of the conservation laws, written in arbitrary co-ordinates, or as a variant of a weak formulation as described in the previous chapter, its importance and wide range of application justifies a separate presentation here.

The method takes full advantage of an arbitrary mesh, where a large number of options are open for the definition of the control volumes around which the conservation laws are expressed. Modifying the shape and location of the control volumes associated with a given mesh point, as well as varying the rules and accuracy for the evaluation of the fluxes through the control surfaces, gives considerable flexibility to the finite volume method. In addition, by the direct discretization of the integral form of the conservation laws we can ensure that the basic quantities mass, momentum and energy will also remain conserved at the discrete level. This is a most fundamental property for numerical schemes, and its precise meaning will be discussed prior to the introduction of the finite volume method.

6.1 THE CONSERVATIVE DISCRETIZATION

From the general presentation of Chapter 1 we know that the flow equations are the expression of a conservation law. Their general form for a scalar quantity $U$, with volume sources $Q$, is given by equation (1.1.1):

$$\frac{\partial}{\partial t} \int_{\Omega} U d\Omega + \oint_{S} \vec{F} \cdot d\vec{S} = \int_{\Omega} Q d\Omega \quad (6.1.1)$$

The essential significance of this formulation lies in the presence of the surface
integral and the fact that the time variation of $U$ inside the volume only depends on the surface values of the fluxes. Hence for an arbitrary subdivision of the volume $\Omega$ into, say, three subvolumes we can write the conservation law for each subvolume and recover the global conservation law by adding up the three subvolume conservation laws. Indeed, referring to Figure 6.1.1, the above equation for the subvolumes $\Omega_1, \Omega_2, \Omega_3$ becomes

\[
\frac{\partial}{\partial t} \int_{\Omega_1} U \, d\Omega + \oint_{\text{ABCA}} \vec{F} \cdot d\vec{S} = \int_{\Omega_1} Q \, d\Omega
\]

\[
\frac{\partial}{\partial t} \int_{\Omega_2} U \, d\Omega + \oint_{\text{DEBD}} \vec{F} \cdot d\vec{S} = \int_{\Omega_2} Q \, d\Omega
\]  \hspace{1cm} (6.1.2)

\[
\frac{\partial}{\partial t} \int_{\Omega_3} U \, d\Omega + \oint_{\text{AEDA}} \vec{F} \cdot d\vec{S} = \int_{\Omega_3} Q \, d\Omega
\]

When summing the surface integrals the contributions of the internal lines ADB and DE always appear twice but with opposite signs, and will cancel in the addition of the three subvolume conservation laws. Indeed, for column $\Omega_2$, for instance, we have a contribution of the fluxes

\[
\oint_{\text{DE}} \vec{F} \cdot d\vec{S}
\]

while for $\Omega_3$ we have a similar term:

\[
\oint_{\text{ED}} \vec{F} \cdot d\vec{S} = - \oint_{\text{DE}} \vec{F} \cdot d\vec{S}
\]

This essential property has to be satisfied by the numerical discretization of the flux contributions in order for a scheme to be conservative. When this is not the case, that is, when, after summation of the discretized equations over a certain number of adjacent mesh cells, the resulting equation still contains flux contributions from inside the total cell, the discretization is said to be non-conservative, and the internal flux contributions appear as numerical internal volume sources.

![Figure 6.1.1 Conservation laws for subvolumes of volume $\Omega$](image)
Let us illustrate this on a one-dimensional form of the conservation law, written here as follows, where $f$ is the $x$-component of the flux vector:

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = q \quad (6.1.3)$$

With a central difference applied to the mesh of Figure 6.1.2, the following discretized equation is obtained at point $i$:

$$\frac{\partial u_i}{\partial t} + \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = q_i \quad (6.1.4)$$

The same discretization applied to point $(i + 1)$ will give

$$\frac{\partial u_{i+1}}{\partial t} + \frac{f_{i+3/2} - f_{i+1/2}}{\Delta x} = q_{i+1} \quad (6.1.5)$$

and for $(i - 1)$

$$\frac{\partial u_{i-1}}{\partial t} + \frac{f_{i-1/2} - f_{i-3/2}}{\Delta x} = q_{i-1} \quad (6.1.6)$$

The sum of these three equations is a consistent discretization of the conservation law for the cell $AB = (i - 3/2, i + 3/2)$:

$$\frac{\partial}{\partial t} \left( \frac{u_i + u_{i+1} + u_{i-1}}{3} \right) + \frac{f_{i+3/2} - f_{i-3/2}}{3\Delta x} = \frac{1}{3} (q_i + q_{i+1} + q_{i-1}) \quad (6.1.7)$$

since the flux contributions at internal points have cancelled out. This is sometimes called the 'telescoping property' for the flux terms (Roache, 1972).

On the other hand, the non-conservative form (equation (1.1.7)) can be written as

$$\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = q \quad (6.1.8)$$

where the flux derivative has been expressed as

$$\frac{\partial f}{\partial x} = \left( \frac{\partial f}{\partial u} \right) \frac{\partial u}{\partial x}$$

and the function $a(u) = \partial f/\partial u$ is the derivative of the flux function with respect to the variable $u$. For instance, if $f = u^2/2$, $a(u) = u$.

![Figure 6.1.2 Subdivision of the one-dimensional space into mesh cells](image)
Both formulations (6.1.3) and (6.1.8) are mathematically equivalent for arbitrary, non-linear fluxes, but their numerical implementation is not. Applying, for instance, a second-order central difference at mesh point $i$ would give

$$\frac{\partial u_i}{\partial t} + a_i \frac{(u_{i+1/2} - u_{i-1/2})}{\Delta x} = q_i \quad (6.1.9)$$

where $a_i$ can be estimated as $a_i = (a_{i+1/2} + a_{i-1/2})/2$.

If similar equations are written for $(i+1)$ and $(i-1)$ and summed, a discretized equation for the cell AB in Figure 6.1.2 is obtained:

$$\frac{\partial}{\partial t} \left( \frac{u_i + u_{i+1} + u_{i-1}}{3} \right) + \frac{(a_{i+3/2} + a_{i-3/2})}{6} \frac{u_{i+3/2} - u_{i-3/2}}{\Delta x} - \frac{q_i + q_{i+1} + q_{i-1}}{3} \quad (6.1.10)$$

$$= - \frac{(a_{i+1/2} - a_{i-1/2})}{6} \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} + \frac{(a_{i+3/2} - a_{i-1/2})}{6} \frac{u_{i+1/2} - u_{i-3/2}}{\Delta x}$$

A direct discretization of equation (6.1.8) on the cell AB would have given the left-hand side of equation (6.1.10) with a vanishing right-hand side. It is therefore seen that the discretization of the non-conservative form of the equation gives rise to internal sources, equal in this case to the right-hand side of equation (6.1.10). These terms can be considered (by performing a Taylor expansion) as a discretization to second order of a term proportional to $\Delta x^2 [(axux)_x - (axuxx)]$ at mesh point $i$. For continuous flows, these numerical source terms are of the same order as the truncation error and hence could be neglected. However, numerical experiments and comparisons consistently show that non-conservative formulations are generally less accurate than conservative ones, particularly in the presence of strong gradients.

For discontinuous flows, such as transonic flows with shock waves, these numerical source terms can become important across the discontinuity and give rise to large errors. This is indeed the case, and the discretization of the non-conservative form will not lead to the correct shock intensities. Therefore in order to obtain, in the numerical computation, the correct discontinuities (such as the Rankine–Hugoniot relations for the Euler equations) it has been shown by Lax (1954) that it is necessary to discretize the conservative form of the flow equations.

**Formal expression of a conservative discretization**

The conservativity requirement on equation (6.1.3) will be satisfied if the scheme can be written as

$$\frac{\partial u_i}{\partial t} + \frac{(f_{i+1/2}^* - f_{i-1/2}^*)}{\Delta x} = q_i \quad (6.1.11)$$

where $f^*$ is called the numerical flux and is a function of the values of $u$ at
(2k - 1) neighbouring points:

\[ f_{i+1/2}^* = f^*(u_{i+k}, \ldots, u_{i-k+1}) \]  

(6.1.12)

In addition, the consistency of equation (6.1.11) with the original equation requires that, when all the \( u_{i+j} \) are equal, we should have

\[ f^*(u, \ldots, u) = f(u) \]  

(6.1.13)

The generalization to multi-dimensions is straightforward, and the above conditions must hold separately for all the components of the flux vector. The importance of this formalization of the conservativity condition is expressed by the following fundamental theorem of Lax and Wendroff (1960):

**Theorem** If the solution \( u_i \) of the discretized equation (6.1.11) converges boundedly almost everywhere to some function \( u(x, t) \) when \( \Delta x, \Delta t \) tend to zero, then \( u(x, t) \) is a weak solution of equation (6.1.3).

This theorem guarantees that when the numerical solution converges it will do so to a solution of the basic equations, with the correct satisfaction of the Rankine-Hugoniot relations in the presence of discontinuities. Indeed, by comparing the derivation of the Rankine-Hugoniot relations in Section 2.7.1 with the weak formulation of the basic flow equations, with \( W = 1 \) in equation (5.3.14), it is obvious that these relations are satisfied by the weak solutions, since the starting point of the derivation is the integral form of the conservation law.

### 6.2 THE FINITE VOLUME METHOD

The integral conservation laws are written for a discrete volume,

\[ \frac{\partial}{\partial t} \int_{\Omega} U \, d\Omega + \oint_{\Gamma} \vec{F} \cdot d\vec{S} = \int_{\Omega} Q \, d\Omega \]  

(6.2.1)

and applied to a control volume \( \Omega_J \), when the discretized equation associated with \( U_J \) is to be defined. Equation (6.2.1) is replaced by the discrete form:

\[ \frac{\partial}{\partial t} (U_J \Omega_J) + \sum_{\text{sides}} (\vec{F} \cdot \vec{S}) = Q_J \Omega_J \]  

(6.2.2)

where the sum of the flux terms refers to all the external sides of the control cell \( \Omega_J \). Referring to Figure 6.2.2(a) and to cell 1(i, j), we would identify \( U_J \) with \( U_{i,j}, \Omega_J \) with the area of ABCD, and the flux terms are summed over the four sides AB, BC, CD, DA. On the mesh of Figure 6.2.2(d) \( \Omega_J \) is the dotted area of the triangles having node \( J \) in common, and the flux summation extends over the six sides 12, 23, 34, 45, 56, 61. This is the general formulation of the finite volume method, and the user has to define, for a selected \( \Omega_J \), how to estimate the volume and cell face areas of the control volume \( \Omega_J \) and how to approximate the fluxes at the faces. We will discuss some of the most current options, in two and three dimensions.
The following constraints on the choice of the $\Omega_j$ volumes for a conservative finite volume method have to be satisfied:

1. Their sum should cover the whole domain $\Omega$;
2. Adjacent $\Omega_j$ may overlap if each internal surface $\Gamma_I$ is common to two volumes;
3. Fluxes along a cell surface have to be computed by formulas independent of the cell in which they are considered.

Requirement (3) ensures that the conservative property is satisfied, since the flux contributions of internal boundaries will cancel when the contributions of the associated finite volumes are added.

Referring to Figure 6.2.1, cells 1–4 have no common sides and their sum does not cover the whole volume. In addition, the sides are not common to two volumes. Cells 5–7 overlap, but have no common surfaces. Hence the conservative property will not be satisfied.

Equation (6.2.2) shows several interesting features which distinguish the interpretation of finite volume methods from the finite difference and finite element approaches:

1. The co-ordinates of point $J$, that is, the precise location of the variable $U$ inside the control volume $\Omega_j$, do not appear explicitly. Consequently, $U_J$ is not necessarily attached to a fixed point inside the control volume and can be considered as an *average value* of the flow variable $U$ over the
Figure 6.2.2 Two-dimensional finite volume mesh systems. (a) Cell centred structured finite volume mesh; (b) cell vertex structured finite volume mesh; (c) cell centred unstructured finite volume mesh; (d) cell vertex unstructured finite volume mesh
control cell. This is the interpretation taken in Figure 6.2.2(a). The first term of equation (6.2.2) therefore represents the time rate of change of the averaged flow variable over the selected finite volume.

(2) The mesh co-ordinates appear only in the determination of the cell volume and side areas. Hence, referring to Figure 6.2.2(a), and considering, for instance, the control cell ABCD around point 1, only the co-ordinates of A, B, C, D will be needed.

(3) In the absence of source terms, the finite volume formulation expresses that the variation of the average value $U$ over a time interval $\Delta t$ is equal to the sum of the fluxes exchanged between neighbouring cells. For stationary flows the numerical solution is obtained as a result of the balance of all the fluxes entering the control volume. That is,

$$\sum_{\text{sides}} (\vec{F} \cdot \vec{S}) = 0$$  \hspace{1cm} (6.2.3)

When adjacent cells are considered, for instance cells ABCD and AEFB in Figure 6.2.2(a), the flux through face AB contributes to the two cells but with opposite signs. It is therefore convenient to program the method by sweeping through the cell faces and, when calculating the flux through side AB, to add this contribution to the flux balance of cell 1 and subtract it from the flux balance of cell 8. This automatically guarantees global conservation.

(4) The finite volume method also allows a natural introduction of boundary conditions, for instance at solid walls where certain normal components are zero. For the mass conservation equation, $\vec{F} = \rho \vec{v}$ and at a solid boundary $\vec{F} \cdot d\vec{S} = 0$. Hence the corresponding contribution to equations (6.2.2) or (6.2.3) would vanish.

**Mesh and control volume definitions**

Due to its generality, the finite volume method can handle any type of mesh and has therefore the same flexibility, in this respect, as the finite element method, restricted to elements with rectilinear sides. Two types of meshes can be considered:

(1) A 'finite difference' type mesh, where all mesh points lie on the intersection of two (or three) families of lines, considered as defining curvilinear co-ordinate lines. They are currently designated as **structured** meshes, and examples are shown in Figures 6.2.2(a) and 6.2.2(b).

(2) A 'finite element' type mesh formed by combinations of triangular and quadrilateral cells (or tetrahedra and pyramids in three dimensions), where the mesh points cannot be identified with co-ordinate lines. Therefore they cannot be represented by a set of integers, such as $i, j$ (or $i, j, k$ in three dimensions), but have to be numbered individually in a certain order. This type of mesh is designated as **unstructured**, and
examples are shown in Figures 6.2.2(c) and 6.2.2(d). Although requiring a more complicated bookkeeping, unstructured meshes can offer greater flexibility for complicated geometrical configurations.

An interesting example is provided by the internal flow in a circular duct where a structured mesh, formed by circles and radial lines as shown in Figure 6.2.3(a), cannot avoid the 'singular' point at the centre, which requires special treatment. This difficulty is avoided with the unstructured mesh of Figure 6.2.3(b).

Once the mesh is selected, we have to decide where to define the variables:

Figure 6.2.3 Mesh configurations for flow computation in a circular duct. (a) Structured and (b) unstructured mesh
(1) When the variables are associated with a cell, as in Figures 6.2.2(a) and 6.2.2(c), a *cell-centred* finite volume method is defined. The flow variables are averaged values over the cell and can be considered as representative of some point inside the cell (for instance, the central point of the cell).

(2) When the variables are attached to the mesh points, that is, to the cell vertices, we speak of a *cell-vertex* finite volume method, as shown in Figures 6.2.2(b) and 6.2.2(d).

With the first choice the mesh cells coincide with the control volume. With the second, a larger flexibility exists for the definition of the control volumes. Referring to Figure 6.2.2(b), an obvious choice would be to consider the four cells having mesh point \((i, j)\) in common as the control volume \(GHKEFBCDG\), associated with point \((i, j)\). Many other choices are, however, possible and two of them are shown in Figure 6.2.4. Figure 6.2.4(a) is from McDonald (1971), who selected an hexagonal control volume, while Denton (1975) used a trapezoidal control surface covering two half-mesh cells (Figure 6.2.4(b)).

### 6.2.1 Two-dimensional finite volume method

Equation (6.2.1), considered for control cell ABCD of Figure 6.2.2, can be written as

\[
\frac{\partial}{\partial t} \int_{a_u} U \, d\Omega + \oint_{ABCD} (f \, dy - g \, dx) = \int_{a_u} Q \, d\Omega \quad (6.2.4)
\]

where \(f\) and \(g\) are the Cartesian components of the flux vector \(\vec{F}\). Equation (6.2.4) is the most appropriate for a direct discretization. The surface vector for a side AB can be defined as

\[
\vec{S}_{AB} = \Delta y_{AB} \vec{i}_x - \Delta x_{AB} \vec{i}_y = (y_B - y_A) \vec{i}_x - (x_B - x_A) \vec{i}_y \quad (6.2.5)
\]
Figure 6.2.5 Area of an arbitrary plane quadrilateral

and we obtain the finite volume equation for cell \( \Omega_{ij} \):

\[
\frac{\partial}{\partial t} (U\Omega)_{ij} + \sum_{ABCD} [f_{AB}(y_B - y_A) - g_{AB}(x_B - x_A)] = (Q\Omega)_{ij} \quad (6.2.6)
\]

The sum \( \sum_{ABCD} \) extends over the four sides of the quadrilateral ABCD.

For a general quadrilateral ABCD the area \( \Omega \) can be evaluated from the vector products of the diagonals. As seen from Figure 6.2.5, the parallelogram 1234 built on the diagonals is twice the area of the quadrilateral ABCD. Hence with \( \vec{x}_{AB} = \vec{x}_B - \vec{x}_A \), where \( \vec{x}_A \) is the position vector of point A,

\[
\Omega_{ABCD} = \frac{1}{2} | \vec{x}_{AC} \times \vec{x}_{BD} | = \frac{1}{2} [(x_C - x_A)(y_D - y_B) - (y_C - y_A)(x_D - x_B)] \quad (6.2.7)
\]

The right-hand side of equation (6.2.7) should be positive for a cell ABCD, where A, B, C, D are located counterclockwise.

**Evaluation of fluxes through cell faces**

The evaluation of flux components along the sides, such as \( f_{AB}, g_{AB} \), depends on the selected scheme as well as on the location of the flow variables with respect to the mesh. As will be seen more in detail in the following chapters and in Volume 2 for the systems of Euler and Navier–Stokes equations, we can distinguish essentially between central and upwind discretization schemes. Central schemes are based on local flux estimations, while upwind schemes determine the cell face fluxes according to the propagation direction of the wave components.

For **central schemes** and **cell-centred** finite volume methods the following alternatives can be considered:
(1) Average of fluxes:
\[ f_{AB} = \frac{1}{2} (f_{ij} + f_{i+1,j}) \quad (6.2.8) \]
\[ f_{ij} = f(U_{ij}) \quad (6.2.9) \]

(2) Since the flux components are generally non-linear functions of \( U \), the following choice is not identical to equation (6.2.8):
\[ f_{AB} = f\left(\frac{U_{ij} + U_{i+1,j}}{2}\right) \quad (6.2.10) \]

(3) Take \( f \) as the average of the fluxes in A and B:
\[ f_{AB} = \frac{1}{2} (f_A + f_B) \quad (6.2.11) \]
where either the variables are evaluated in A and B
\[ U_A = \frac{1}{4} (U_{ij} + U_{i+1,j} + U_{i+1,j-1} + U_{i,j-1}) \quad (6.2.12) \]
and
\[ f_A = f(U_A) \quad (6.2.13) \]
or the fluxes are averaged, as
\[ f_A = \frac{1}{4} (f_{ij} + f_{i+1,j} + f_{i+1,j-1} + f_{i,j-1}) \quad (6.2.14) \]

Observe that equations (6.2.10) and (6.2.13) will generally lead to schemes requiring a lower number of flux evaluations compared with the application of equations (6.2.8) and (6.2.14).

For central schemes and cell-vertex finite volume methods equations (6.2.10) or (6.2.11) are straightforward approximations to the flux \( f_{AB} \). The choice (6.2.11) corresponds to the application of a trapezium formula for the integral \( \int_{AB} f \, dy = (f_A + f_B)(y_B - y_A)/2 \).

By summing the contributions of these integrals over the four sides of cell ABCD of Figure 6.2.2(b) we obtain the various discretizations already derived in Example 5.3.3; for instance, the flux terms in equation (E5.3.23), which become here:
\[ \int_{ABCD} \vec{F} \cdot d\vec{S} = \frac{1}{4} [(f_A - f_C) \Delta y_{DB} + (f_B - f_D) \Delta y_{AC} - (g_A - g_C) \Delta x_{DB} - (g_B - g_D) \Delta x_{AC}] = 0 \quad (6.2.15) \]

This also shows the equivalence of the flux evaluations in the cell-vertex, finite volume approach, with the finite element Galerkin method on linear triangles or bilinear quadrilaterals.

**Example 6.2.1 Central scheme on a Cartesian mesh**

Over a Cartesian, uniform mesh the above finite volume formulation is
identical to a finite difference formula. Indeed, with
\[ \Delta y_{AB} = y_{i+1/2,j+1/2} - y_{i+1/2,j-1/2} = \Delta y \]
\[ \Delta x_{AB} = 0 \quad \Delta x_{BC} = -\Delta x \quad \Omega_{ij} = \Delta x \cdot \Delta y \quad \Delta y_{CB} = 0 \]  
(E6.2.1)

we obtain, writing \( f_{AB} = f_{i+1/2,j} \) and similarly for the other components,
\[ \frac{\partial}{\partial t} U_{ij} \Delta x \Delta y + (f_{i+1/2,j} - f_{i-1/2,j}) \Delta y + (g_{i,j+1/2} - g_{i,j-1/2}) \Delta x = Q_{ij} \Delta x \Delta y \]  
(E6.2.2)

After division by \( \Delta x \Delta y \) this reduces to the central difference form:
\[ \frac{\partial U_{ij}}{\partial t} + \frac{(f_{i+1/2,j} - f_{i-1/2,j})}{\Delta x} + \frac{(g_{i,j+1/2} - g_{i,j-1/2})}{\Delta y} = Q_{ij} \]  
(E6.2.3)

We have still to define how to calculate the flux components at the side centres \( f_{i \pm 1/2,j}, g_{i,j \pm 1/2} \). With the choice (6.2.8) applied to Figure 6.2.2(a), equation (E6.2.3) becomes
\[ \frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = Q_{ij} \]  
(E6.2.4)

while equation (6.2.11) with equation (6.2.14) leads to
\[ \frac{\partial U_{ij}}{\partial t} + \frac{1}{4} \left[ \frac{2 f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{2 f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x} + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x} \right] \]  
(E6.2.5)

\[ + \frac{1}{4} \left[ \frac{2 g_{i,j+1} - g_{i,j-1}}{2\Delta y} + \frac{2 g_{i+1,j} - g_{i-1,j}}{2\Delta y} + \frac{g_{i+1,j+1} - g_{i-1,j+1}}{2\Delta y} \right] = Q_{ij} \]

The central finite volume method therefore leads to second-order accurate space discretizations on Cartesian meshes.

Observe that \( f_{ij}, g_{ij} \) do not appear in equation (E6.2.4), and if \( (i+j) \) is even, this equation contains only nodes with \( (i+j) \) odd. Hence even- and odd-numbered nodes are separated, and this could lead to oscillations in the solution. This separation is not present with equation (E6.2.5). For applications to cell-vertex meshes, the reader is referred to Problems 6.1–6.4.

For upwind schemes and cell-centred finite volume methods a convective flux is evaluated as a function of the propagation direction of the associated convection speed. The latter is determined by the flux Jacobian
\[ \overline{A}(U) = \frac{\partial F}{\partial U} = a \overline{1}_x + b \overline{1}_y \]  
(6.2.16)

with \( a(U) = \partial f/\partial U \) and \( b(U) = \partial g/\partial U \).
The simplest upwind scheme takes the cell side flux equal to the flux generated in the upstream cell. This expresses that the cell side flux is fully determined by contributions transported in the direction of the convection velocity.

Considering Figure 6.2.2(a), we could define

\[ (\bar{F} \cdot \bar{S})_{AB} = (\bar{F} \cdot \bar{S})_{ij} \quad \text{if} \quad (\bar{A} \cdot \bar{S})_{AB} > 0 \]  
\[ (\bar{F} \cdot \bar{S})_{AB} = (\bar{F} \cdot \bar{S})_{i+1,j} \quad \text{if} \quad (\bar{A} \cdot \bar{S})_{AB} < 0 \]  

(6.2.17)

For \textit{upwind schemes} and \textit{cell-vertex} finite volume methods (Figure 6.2.2(b)) we could define

\[ (\bar{F} \cdot \bar{S})_{AB} = (\bar{F} \cdot \bar{S})_{CD} \quad \text{if} \quad (\bar{A} \cdot \bar{S})_{AB} > 0 \]  
\[ (\bar{F} \cdot \bar{S})_{AB} = (\bar{F} \cdot \bar{S})_{EF} \quad \text{if} \quad (\bar{A} \cdot \bar{S})_{AB} < 0 \]  

(6.2.18)

When applied to the control volume GHKEFBCD of Figure 6.2.2(b), we obtain contributions from points such as \((i-2, j)\) and \((i, j-2)\) for positive convection speeds. This leads to schemes with an unnecessary large support for the same accuracy. Therefore this option is not applied in practice (see Problem 6.14).

\textbf{Example 6.2.2 Upwind scheme on a Cartesian mesh}

We consider the discretization of the two-dimensional linear convection equation

\[ \frac{\partial U}{\partial t} + a \frac{\partial U}{\partial x} + b \frac{\partial U}{\partial y} = 0 \quad \text{with} \quad a > 0 \quad \text{and} \quad b > 0 \]  
\[ \text{(E6.2.6)} \]

by a finite volume formulation on the cell ABCD of Figure 6.2.2(a), defined as a Cartesian cell following Example 6.2.1.

The fluxes are defined by \( f = aU \) and \( g = bU \) and with the choice of equation (6.2.17) we have, for AB and CD taken as vertical sides,

\[ (\bar{F} \cdot \bar{S})_{AB} = f_{ij} \Delta y = aU_{ij} \Delta y \]  
\[ (\bar{F} \cdot \bar{S})_{CD} = - f_{i-1,j} \Delta y = - aU_{i-1,j} \Delta y \]  

(E6.2.7)

and similarly for the two horizontal sides BC and DA:

\[ (\bar{F} \cdot \bar{S})_{BC} = g_{ij} \Delta x = bU_{ij} \Delta x \]  
\[ (\bar{F} \cdot \bar{S})_{DA} = - g_{i,j-1} \Delta x = - bU_{i-1,j} \Delta x \]  

(E6.2.8)

The resulting scheme, obtained after division by the cell area \( \Delta x \Delta y \), is only first-order accurate, and is a straightforward generalization of the first-order upwind scheme, to be introduced in Chapter 7 (equation (7.2.7)):

\[ \frac{\partial U_{ij}}{\partial t} + \frac{1}{\Delta x} (f_{ij} - f_{i-1,j}) + \frac{1}{\Delta y} (g_{ij} - g_{i,j-1}) = 0 \]  
\[ \text{(E6.2.9)} \]
Non-uniform mesh Although the finite volume formulation applies to arbitrary grids, the above equations for the determination of the fluxes nevertheless imply some regularity of the mesh. Referring, for instance, to equations (6.2.8) or (6.2.10) as applied to cell-centred finite volume methods, and interpreting the cell-averaged values $U_{ij}$ in Figure 6.2.2(a) as mid-cell values, it is seen that these equations perform an arithmetic average of the fluxes (or the variables) on both sides of the cell face AB. This leads to a second-order approximation on a Cartesian mesh (see Example 6.2.1) if AB is at mid-distance from the cell centres 1 and 8. However, this will seldom be the case on non-uniform meshes, as shown in Figure 6.2.6(a), and a loss of accuracy will result from the application of these equations. Similar considerations apply to equations (6.2.12) and (6.2.14), based on the assumption that point A is in the centre of cell 1678. An analysis of the truncation errors for certain finite volume discretizations on non-uniform meshes can be found in Arts (1984), and more general analysis can be found in Turkel (1985), Turkel et al. (1985) and Roe (1987).

A straightforward generalization of equations (6.2.8) and (6.2.10) can be defined through a linear interpolation of $f_{AB}(or U_{AB})$ between the cell values $f_{ij}$ and $f_{i+1,j}$ (or $U_{ij}$ and $U_{i+1,j}$). This is uniquely defined on a one-dimensional basis, which is of application for an orthogonal mesh such as shown in Figure 6.2.6(b). Hence we can define

\[
\frac{\partial U_{ij}}{\partial t} + \frac{a}{\Delta x} (U_{ij} - U_{i-1,j}) + \frac{b}{\Delta y} (U_{ij} - U_{i,j-1}) = 0
\]  

(E6.2.10)

\[
\frac{\partial U_{ij}}{\partial t} + \frac{a}{\Delta x} (U_{ij} - U_{i-1,j}) + \frac{b}{\Delta y} (U_{ij} - U_{i,j-1}) = 0
\]  

For more general meshes, the distances $a$ and $b$ could be defined as shown in Figure 6.2.6(a), where M is the mid-point of AB, but in any case the second-order accuracy can only be maintained for sufficiently smooth varying mesh sizes (see Turkel, 1985, and Turkel et al., 1985, for a more detailed discussion).

Generalizations of equations (6.2.12) and (6.2.14) to non-uniform meshes can be considered via an area weighted average instead of an arithmetic one. For instance, we could define

\[
U_A = \sum_I \frac{\Omega_I U_I}{\Omega_T}
\]  

(6.2.21)

where the summation ranges over the four points 1, 6, 7, 8, with $\Omega_I$ being the area of cell I and $\Omega_T$ is the total area of the four cells 1, 6, 7, 8. However, this formula, although natural, has the drawback of giving the lowest weight to the
smaller cell. Applying equation (6.2.21) to Figure 6.2.6(a) will give to point A a stronger dependence on point $7(i+1, j-1)$ than on the much closer point $1(i, j)$. This should be avoided, for instance through the application of finite element interpolations. We could consider the quadrilateral 1678 as a bilinear element and define, instead of equation (6.2.14),

$$U_A = \sum_i U_i N_i(x_A, y_A)$$  \hspace{1cm} (6.2.22)

where the shape functions $N_i$ are the bilinear polynomials of the $Q_1$ elements (see Table 5.1) and where the summation ranges over the four points 1, 6, 7, 8.

![Diagram](image)

Figure 6.2.6 Non-uniform finite volume meshes. (a) Non-uniform finite volume mesh; (b) orthogonal non-uniform finite volume mesh
By definition of the finite element interpolation functions, if point A is close to one of the corner points of the cell, say point 1, than \( V_A \approx V_I \), since \( N_I(x_I, y_I) = \delta_{IJ} \) if points I and J are close enough. Alternatively, we could define a similar relation for the fluxes as a generalization of equation (6.2.12). For cell-vertex methods equation (6.2.11) can remain unchanged for arbitrary cell configurations.

In all cases, even with the above equations, a loss of accuracy will be unavoidable on strongly distorted meshes. However, cell-vertex schemes will generally maintain their accuracy for broader families of non-uniform meshes. As a general guideline one should avoid, if possible, discontinuous mesh size variations, for instance by generating grids analytically. With regard to the flux integrals, the application of finite element interpolations and integration rules can be considered as a valid guideline for the obtention of equations on strong non-uniform meshes.

### 6.2.2 General integration formulas for finite volumes

It is often necessary in finite volume discretizations to define numerically cell averages of derivatives of mesh variables. Particularly with the Navier–Stokes equations, the viscous flux components are functions of the velocity gradients, and we have to estimate appropriate values of these gradients on the cell faces. A general procedure, valid for an arbitrary control volume in two and three dimensions, can be derived by application of the divergence theorem.

This theorem can be considered as defining the average of the gradient of a scalar \( U \) as a function of its values at the boundaries of the volume under consideration. Since for an arbitrary volume \( \Omega \).

\[
\int_{\Omega} \nabla U \, d\Omega = \oint_s U \, d\vec{S} \quad (6.2.23)
\]

where \( S \) is the closed boundary surface, we can define the averaged gradients as

\[
\left( \frac{\partial U}{\partial x} \right)_{\Omega} = \frac{1}{\Omega} \int_\Omega \frac{\partial U}{\partial x} \, d\Omega = \frac{1}{\Omega} \oint_S U \, \vec{I}_x \cdot d\vec{S} \quad (6.2.24a)
\]

and

\[
\left( \frac{\partial U}{\partial y} \right)_{\Omega} = \frac{1}{\Omega} \int_\Omega \frac{\partial U}{\partial y} \, d\Omega = \frac{1}{\Omega} \oint_S U \, \vec{I}_y \cdot d\vec{S} \quad (6.2.24b)
\]

For two-dimensional control cells \( \Omega \) we obtain

\[
\left( \frac{\partial U}{\partial x} \right)_{\Omega} = \frac{1}{\Omega} \oint_S U \, dy = -\frac{1}{\Omega} \oint_S y \, dU \quad (6.2.25a)
\]

after partial integration. Similarly, the averaged \( y \)-derivatives are obtained from

\[
\left( \frac{\partial U}{\partial y} \right)_{\Omega} = -\frac{1}{\Omega} \oint_S U \, dx = \frac{1}{\Omega} \oint_S x \, dU \quad (6.2.25b)
\]
Considering the control cell of Figure 6.2.2(d) and applying trapezoidal integration formulas along each side, the following equations are obtained, in agreement with the relations of the previous section:

\[
\left( \frac{\partial U}{\partial x} \right)_\Omega = \frac{1}{\Omega} \int_\Omega \frac{\partial U}{\partial x} \, d\Omega = \frac{1}{2\Omega} \sum_I (U_I + U_{I+1})(y_{I+1} - y_I)
\]

\[
= -\frac{1}{2\Omega} \sum_I (y_I + y_{I+1})(U_{I+1} - U_I)
\]

\[
= \frac{1}{2\Omega} \sum_I U_I(y_{I+1} - y_{I-1})
\]

\[
(6.2.26a)
\]

\[
= -\frac{1}{2\Omega} \sum_I y_I(U_{I+1} - U_{I-1})
\]

where the summation extends over all the vertices, from 1 to 6 with \( U_0 = U_6 \) and \( U_7 = U_1 \). The two last relations are obtained by rearranging the sums, as has been shown in Example 5.3.3.

The corresponding relations for the \( y \)-derivatives are derived after replacing \( x \) by \( y \) and changing the signs of the various expressions:

\[
\left( \frac{\partial U}{\partial y} \right)_\Omega = \frac{1}{\Omega} \int_\Omega \frac{\partial U}{\partial y} \, d\Omega = -\frac{1}{2\Omega} \sum_I (U_I + U_{I+1})(x_{I+1} - x_I)
\]

\[
= \frac{1}{2\Omega} \sum_I (x_I + x_{I+1})(U_{I+1} - U_I)
\]

\[
= -\frac{1}{2\Omega} \sum_I U_I(x_{I+1} - x_{I-1})
\]

\[
= \frac{1}{2\Omega} \sum_I x_I(U_{I+1} - U_{I-1})
\]

\[
(6.2.26b)
\]

The area of the cells can be obtained by equations similar to the above by noting that for \( U = x \) the left-hand side of equation (6.2.26b) is equal to 1. Hence the following expressions can be used for the estimation of the area of an arbitrary cell:

\[
\Omega = \frac{1}{2} \sum_I (x_I + x_{I+1})(y_{I+1} - y_I)
\]

\[
= -\frac{1}{2} \sum_I (y_I + y_{I+1})(x_{I+1} - x_I)
\]

\[
= \frac{1}{2} \sum_I x_I (y_{I+1} - y_{I-1})
\]

\[
= -\frac{1}{2} \sum_I y_I(x_{I+1} - x_{I-1})
\]

\[
(6.2.27)
\]
For an arbitrary quadrilateral ABCD, as shown in Figure 6.2.5, an interesting formula is obtained by applying the third of the above relations, noting that the differences \( \Delta y \) can be grouped for opposite nodes, leading to

\[
\oint_{ABCD} U \, dy = \frac{1}{2} \left[ (U_A - U_C)(y_B - y_D) - (U_B - U_D)(y_A - y_C) \right] \tag{6.2.28}
\]

and

\[
\left( \frac{\partial U}{\partial x} \right)_{ABCD} = \frac{(U_A - U_C)(y_B - y_D) - (U_B - U_D)(y_A - y_C)}{(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C)} \tag{6.2.29a}
\]

with a similar relation for the \( y \)-derivative:

\[
\left( \frac{\partial U}{\partial y} \right)_{ABCD} = \frac{(x_A - x_C)(U_B - U_D) - (x_B - x_D)(U_A - U_C)}{(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C)} \tag{6.2.29b}
\]

**Example 6.2.3 Two-dimensional diffusion equation**

We consider the two-dimensional diffusion equation

\[
\frac{\partial U}{\partial t} + k \frac{\partial U}{\partial x} \left( \frac{\partial U}{\partial x} \right) + k \frac{\partial U}{\partial y} \left( \frac{\partial U}{\partial y} \right) = 0 \tag{E6.2.11}
\]

with diffusive flux components \( f = k \frac{\partial U}{\partial x} \) and \( g = k \frac{\partial U}{\partial y} \), where \( k \) is a constant. We would like to construct a finite volume discretization on the mesh of Figure 6.2.2(a), considered as Cartesian, by expressing the balance of fluxes around the cell ABCD with the choice

\[
f_{AB} = \frac{1}{2} (f_A + f_B) \tag{E6.2.12}
\]

and an evaluation of the derivatives \( \frac{\partial U}{\partial x} \) and \( \frac{\partial U}{\partial y} \) in the cell corners A, B. Equation (6.2.6) for cell \((i, j)\) is written here as

\[
\left( \frac{\partial U}{\partial t} \right)_{ij} \Delta x \Delta y + (f_{AB} - f_{CD}) \Delta y + (g_{BC} - g_{DA}) \Delta x = 0 \tag{E6.2.13}
\]

For point A the derivatives of \( U \) are taken as the average value over the cell 1678 and with equation (6.2.29):

\[
f_A = k \left( \frac{\partial U}{\partial x} \right)_A = \frac{k}{2 \Delta x} (U_{i+1,j} + U_{i+1,j+1} - U_{i,j} - U_{i,j+1}) \tag{E6.2.14}
\]

A similar relation is obtained for point B:

\[
f_B = k \left( \frac{\partial U}{\partial x} \right)_B = \frac{k}{2 \Delta x} (U_{i+1,j} + U_{i+1,j+1} - U_{i,j} - U_{i,j+1}) \tag{E6.2.15}
\]

and the flux contribution through the side AB is given by the sum of the two equations (E6.2.14) and (E6.2.15) multiplied by \( \Delta y \).

The contributions of the other sides are obtained in a similar way. For instance, the flux through BC is given by the sum

\[
g_{BC} \Delta x = \frac{1}{2} (g_B + g_C) \Delta x \tag{E6.2.16}
\]
with

$$g_B = k \left( \frac{\partial U}{\partial y} \right)_B = \frac{k}{2\Delta y} (U_{i+1,j+1} + U_{i,j+1} - U_{i,j} - U_{i+1,j}) \quad (E6.2.17)$$

A similar relation is obtained for point C:

$$g_C = k \left( \frac{\partial U}{\partial y} \right)_C = \frac{k}{2\Delta y} (U_{i,j+1} + U_{i-1,j+1} - U_{i,j} - U_{i-1,j}) \quad (E6.2.18)$$

Finally, equation (E6.2.13) becomes, with $\Delta x = \Delta y$,

$$\frac{\partial U_{ij}}{\partial t} + k \frac{U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} - 4U_{ij}}{4\Delta x^2} = 0 \quad (E6.2.19)$$

This scheme corresponds to the discretization of Figure 4.4.3 for the Laplace operator.

Note that the alternative, simpler choice,

$$f_{AB} = k \left( \frac{\partial U}{\partial x} \right)_{AB} = \frac{k}{\Delta x} (U_{i+1,j} - U_{i,j}) \quad (E6.2.20)$$

leads to the standard finite difference discretization of the diffusion equation, corresponding to Figure 4.4.2:

$$\frac{\partial U_{ij}}{\partial t} + \frac{U_{i+1,j} + U_{i,j-1} + U_{i-1,j} + U_{i-1,j+1} - 4U_{ij}}{\Delta x^2} = 0 \quad (E6.2.21)$$

The vector version of the divergence relation, written for an arbitrary vector $\vec{a}$, is also of interest:

$$\int_\Omega \vec{\nabla} \cdot \vec{a} \, d\Omega = \oint_\partial \vec{a} \cdot d\vec{S} \quad (6.2.30)$$

since it can be applied, particularly for the derivation of equations for cell face areas and volumes. For a two-dimensional cell, taking $\vec{a} = \vec{x}$ with $\vec{\nabla} \cdot \vec{x} = 2$, leads to

$$2\Omega = \oint_\partial \vec{x} \cdot d\vec{S} = \oint_\partial (x \, dy - y \, dx) \quad (6.2.31)$$

which reproduces the above relations when a trapezium formula is applied. Applications to three-dimensional volumes are discussed in the next section.

6.2.3 Three-dimensional finite volume method

In three dimensions the geometrical space is mostly divided into six-sided hexahedral control volumes (Figure 6.2.7), where the four points forming a cell face are not necessarily coplanar, or into tetrahedral volumes. Equation (6.2.2) remains unchanged, but some care has to be exercised in the evaluation
of volumes and cell surface areas in order to ensure that the sum of the computed volumes of adjacent cells is indeed equal to the total volume of the combined cells.

**Evaluation of cell face areas**

An important property of the area vector $\vec{S}$ attached to a cell face is derived from the divergence theorem. Equation (6.2.23), with $U = 1$, becomes

$$ \oint_S d\vec{S} = 0 \quad (6.2.32) $$

showing that the outward surface vector of a given face contained in the closed surface $S$:

$$ \vec{S}_{\text{face}} = \int_{\text{face}} d\vec{S} \quad (6.2.33) $$

is only dependent on the boundaries of the face. Hence for face ABCD of Figure 6.2.7 we could apply equation (6.2.7) or other alternatives:

$$ \vec{S}_{ABCD} = \frac{1}{2} (\vec{x}_{AC} \times \vec{x}_{BD}) \quad (6.2.34) $$

or

$$ \vec{S}_{ABCD} = \frac{1}{2} \left[ (\vec{x}_{AB} \times \vec{x}_{BC}) + (\vec{x}_{CD} \times \vec{x}_{DA}) \right] \quad (6.2.35) $$

The last equation expresses the surface vector $\vec{S}_{ABCD}$ as the average of the surface vectors of the two parallelograms constructed on the adjacent sides (AB, BC) and (CD, DA). In the general case the two normals will not be in the same direction, since (ABC) and (CDA) are not in the same plane. Hence equation (6.2.35) takes the vector $\vec{S}_{ABCD}$ as the average vector of these two normals, while equation (6.2.18) expresses $\vec{S}_{ABCD}$ as the vector product of the two diagonals. Note, however, that the two equations lead to identical results, even for non-coplanar cell faces (see Problem 6.8).
Similarly to equation (6.2.35), we have

\[ \mathbf{S}_{ABCD} = \frac{1}{2} \left[ (\mathbf{x}_{BC} \times \mathbf{x}_{CD}) + (\mathbf{x}_{DA} \times \mathbf{x}_{AB}) \right] \]  

(6.2.36)

and also, combining opposite instead of adjacent sides, by averaging equations (6.2.35) and (6.2.36):

\[ \mathbf{S}_{ABCD} = \frac{1}{4} \left[ (\mathbf{x}_{AB} + \mathbf{x}_{DC}) \times (\mathbf{x}_{BC} + \mathbf{x}_{AD}) \right] \]  

(6.2.37)

All these equations are applied in practical computations, the first one (equation (6.2.34)) being less expensive in number of arithmetic operations.

**Evaluation of control cell volumes**

Different equations can be applied to obtain the volume of the hexahedral cell, the most current approach consisting of a subdivision into tetrahedra or pyramids (Figure 6.2.8). The volume of the tetrahedron \( \Omega_{PABC} \) is obtained by applying equation (6.2.30) for a vector \( \mathbf{z} \) equal to the position vector \( \mathbf{x} \). We obtain, since \( \nabla \cdot \mathbf{x} = 3 \),

\[
\Omega_{PABC} = \frac{1}{3} \oint_{PABC} \mathbf{x} \cdot d\mathbf{s} = \frac{1}{3} \sum_{\text{faces}} \mathbf{x} \cdot \mathbf{S}_{\text{faces}}
\]  

(6.2.38)

![Figure 6.2.8](image)
or

\[ \Omega_{PABC} = \frac{1}{3} \bar{x}_{(P)} \cdot \bar{s}_{ABC} \quad (6.2.39) \]

if \( \bar{x}_{(P)} \) represents a vector originating in \( P \). This relation results from the fact that when \( \bar{x}_{(P)} \) lies in the faces containing \( P \), it is orthogonal to the associated \( \bar{s} \) vector. The only remaining contribution comes from the face \( ABC \) opposite \( P \). Hence with \( \bar{x}_{(P)} = \bar{x}_{PA} \)

\[ \Omega_{PABC} = \frac{1}{6} \bar{x}_{PA} \cdot (\bar{x}_{AB} \times \bar{x}_{BC}) = \frac{1}{6} \bar{x}_{PA} \cdot (\bar{x}_{BC} \times \bar{x}_{CA}) \quad (6.2.40) \]

Equation (6.2.40) can also be expressed as a determinant:

\[ \Omega_{PABC} = \frac{1}{6} \begin{vmatrix} x_{P} & y_{P} & z_{P} & 1 \\ x_{A} & y_{A} & z_{A} & 1 \\ x_{B} & y_{B} & z_{B} & 1 \\ x_{C} & y_{C} & z_{C} & 1 \end{vmatrix} \quad (6.2.41) \]

In a similar way, for a pyramid \( PABCD \), we have

\[ \Omega_{PABCD} = \frac{1}{3} \oint_{PABCD} \bar{x} \cdot d\bar{s} \quad (6.2.42) \]

\[ = \frac{1}{3} \bar{x}_{(P)} \cdot \bar{s}_{ABCD} \]

Since \( ABCD \) is not necessarily coplanar, \( \bar{x}_{(P)} \) has to be estimated by an appropriate approximation. For instance,

\[ \bar{x}_{(P)} = \frac{1}{4} (\bar{x}_{PA} + \bar{x}_{PB} + \bar{x}_{PC} + \bar{x}_{PD}) \quad (6.2.43) \]

and with expression (6.2.34) for \( \bar{s}_{ABCD} \) we obtain

\[ \Omega_{PABCD} = \frac{1}{4} (\bar{x}_{PA} + \bar{x}_{PB} + \bar{x}_{PC} + \bar{x}_{PD}) \cdot (\bar{x}_{AC} \times \bar{x}_{BD}) \]

\[ = \frac{1}{12} (\bar{x}_{PA} + \bar{x}_{PB}) \cdot (\bar{x}_{AC} \times \bar{x}_{BD}) \quad (6.2.44) \]

If the face \( ABCD \) is coplanar, then equation (6.2.44) reduces to

\[ \Omega_{PABCD} = \frac{1}{6} \bar{x}_{PA} \cdot (\bar{x}_{AC} \times \bar{x}_{BD}) \quad (6.2.45) \]

The volume equations for the pyramids are actually expressed as the sum of the two tetrahedra.

Referring to Figure 6.2.8, the hexahedron can be divided into three pyramids, for instance with point \( D \) as summit:

\[ \Omega_{HEX} = \Omega_{DABFE} + \Omega_{DBCGF} + \Omega_{DEFGH} \quad (6.2.46) \]

Dividing each pyramid into two tetrahedra leads to a decomposition of the hexahedron into six tetrahedra, originating, for instance, in \( D \), as

\[ \Omega_{HEX} = \Omega_{DABE} + \Omega_{DBFE} + \Omega_{DBCG} + \Omega_{DBGF} + \Omega_{DEFG} + \Omega_{DEGH} \quad (6.2.47) \]
Extreme care has to be exercised in the evaluation of the tetrahedra volumes, since the sign of the volumes $\Omega_{PABC}$ in equations (6.2.39)-(6.2.41) depends on the orientation of the triangular decompositions. In addition, when the cell surfaces are not coplanar the same diagonal has to be used in the evaluations of the tetrahedra in the two cells which share this surface, otherwise gaps or overlaps would occur in the summation of volumes. A useful guideline, in order to avoid sign errors, consists of applying a right-hand rotation (screwdriver) rule from the base towards the summit of each tetrahedron.

Another alternative is to decompose the volume of the hexahedron into five tetrahedra originating in D, for instance, referring to Figure 6.2.8(b) as

$$\Omega_{\text{HEX}} = \Omega_{\text{DABE}} + \Omega_{\text{DBCG}} + \Omega_{\text{DEGH}} + \Omega_{\text{DBG}} + \Omega_{\text{FBE}} \quad (6.2.48)$$

In this decomposition four tetrahedra have D as a summit and one tetrahedron originates in point F, opposite to D. Considering the same two points D and F as references, there is a unique second decomposition into five tetrahedra, shown in Figure 6.2.8(c):

$$\Omega_{\text{HEX}} = \Omega_{\text{FACB}} + \Omega_{\text{FAEH}} + \Omega_{\text{FCHG}} + \Omega_{\text{FAHC}} + \Omega_{\text{DACH}} \quad (6.2.49)$$

For a general hexahedral volume, where points of a same cell face are not coplanar the two equations (6.2.48) and (6.2.49) will not give identical volume values. It is therefore recommended to take an average of both.

In this context it is interesting to observe that volumes of hexahedral cells can also be evaluated from a finite element isoparametric trilinear transformation, applying a $2 \times 2 \times 2$ Gauss point integration rule, as described in Section 5.4. Although very tedious to prove analytically, numerical experiments consistently show that this finite element procedure leads to volume values equal to the average of the equations (6.2.48) and (6.2.49).

An investigation of more elaborate decompositions of hexahedral volumes in pyramids can be found in Davies and Salmond (1985), while some of the above-mentioned decompositions are also discussed in Rizzi and Ericksson (1981) and Kordulla and Vinokur (1983).

References


McDonald, P. W. (1971). 'The computation of transonic flow through two-
dimensional gas turbine cascades.' ASME Paper 71-GT-89.
Roache P. J. (1972). Computational Fluid Dynamics, Albuquerque, New Mexico:
Hermosa.
Rizzi A. W., and Inouye M. (1973). 'Time split finite volume method for three-
dimensional blunt-body flows.' AIAA Journal, 11, 1478–85.
Euler equation algorithm for transonic flow around wing–body configurations.'
Proc. AIAA Fifth Computational Fluid Dynamics Conference, AIAA Paper
81-0999, pp. 43–68.
equations.' ICASE Report No. 87-6, NASA Langley Research Center.
Turkel, E. (1985). 'Accuracy of schemes with non-uniform meshes for compressible
equations with non-uniform meshes,' ICASE Report No. 85-59, NASA Langley
Research Center; also in SIAM Journal Scientific and Statistical Computing.

PROBLEMS

Problem 6.1
Apply the finite volume formula (6.2.2) to the contour ACDEGH of Figure 6.2.4(a)
and derive the discretization for node \((i, j)\). Compare with equation (6.2.5) when the
variables are defined at the nodes of the control volume, with the side fluxes defined by
the average of the corner points value; that is,
\[
\frac{1}{2} (f_A + f_C)
\]

Problem 6.2
Apply the finite volume method to the contour of Figure 6.2.4(b) and compare the
different assumptions for the evaluation of fluxes at the mid-side. Derive four schemes
by combining the two options for the vertical sides with the two options for the
horizontal sides:
\[
\frac{1}{2} (f_{i+1,j} + f_{i,j}) \quad \text{or} \quad \frac{1}{2} (f_{i+1,j+1} + f_{i+1,j-1} + f_{i,j+1} + f_{i,j-1})
\]

with
\[
f_{DA} = f_{i,j-1}
\]

or
\[
\frac{1}{4} (2f_{i,j-1} + f_{i+1,j-1} + f_{i-1,j-1})
\]

with similar expressions for \( g \) and for the two other sides.

Problem 6.3
Determine the different formulas obtained in Problem 6.2 when the mesh is Cartesian
and compare with the results of Problem 6.1.
Problem 6.4

Develop a finite volume discretization for mesh point \( A(i, j) \), with the control volume BCDGHKEF of Figure 6.2.2(b). Compare the results from the evaluation of the side fluxes by the following three options, written, for instance, for side K(E)F:

\[
f_{KF} \Delta y_{KF} = \frac{1}{2} (f_{i+1, j+1} + f_{i+1, j-1})(y_{i+1, j+1} - y_{i+1, j-1}) \quad (a)
\]

or

\[
f_{KF} \Delta y_{KF} = f_{KE} \Delta y_{KE} + f_{EF} \Delta y_{EF}
= \frac{1}{2} (f_{i+1, j} + f_{i+1, j-1})(y_{i+1, j} - y_{i+1, j-1})
+ \frac{1}{2} (f_{i+1, j+1} + f_{i+1, j})(y_{i+1, j+1} - y_{i+1, j}) \quad (b)
\]

or

\[
f_{KF} \Delta y_{KF} = f_{E} \Delta y_{KF} = f_{i+1, j}(y_{i+1, j+1} - y_{i+1, j-1}) \quad (c)
\]

Compare the three results for a Cartesian mesh and refer also to Example 6.2.1.

Problem 6.5

Apply the results (6.2.26) in order to derive average values of the first derivatives \( \partial f / \partial x \) and \( \partial f / \partial y \) over the triangle \( J_12 \) of Figure 6.2.2(d). Compare with the expressions obtained in equations (E5.3.21) and (E5.3.22) and with the results of Problem 5.11. Note that the results are identical and comment on the reason behind the validity of the derivation by the finite element method with linear triangles.

Problem 6.6

Consider the two-dimensional diffusion equation treated in Example 6.2.3 with diffusive flux components \( f = k \partial u / \partial x \) and \( g = k \partial u / \partial y \), where \( k \) is a function of the co-ordinates. Construct the discrete equation by the finite volume approach on the mesh of Figure 6.2.2(a), considered as Cartesian, by generalizing the development of Example 6.2.3. Consider the quadrilateral control surface ABCD for the mesh point \( 1(i, j) \) and consider the values of \( k \) defined at the corners of the cell, that is in A, B, C, D. If necessary, define

\[k_{AB} = \frac{1}{2} (k_A + k_B)\]

Problem 6.7

Consider the diffusion equation of the previous problem and apply it to the cell BCDGHKEF of Figure 6.2.2(b), considered as Cartesian, with constant \( k \). Define the derivatives on the cell sides by one-sided formulas from inside the control cell. Apply successively the three options of Problem 6.4 and compare with the results of Example 6.2.3. Show in particular that options a, b and c reproduce the schemes derived in this example.

Hint: For a point F, define the derivatives as

\[
\left( \frac{\partial u}{\partial x} \right)_F = (u_{i+1, j+1} - u_{i, j+1}) / \Delta x
\]

\[
\left( \frac{\partial u}{\partial y} \right)_F = (u_{i+1, j+1} - u_{i+1, j}) / \Delta y
\]

and similar relations for the other points.
Problem 6.8
Show that equations (6.2.35)–(6.2.37) lead to results identical to the simplest expression (6.2.34).

*Hint:* Apply vector relations such as \( \vec{x}_{AC} = \vec{x}_{AB} + \vec{x}_{BC} \) and the properties of the vector products.

Problem 6.9
Show that the relation (6.2.37) for the area of a quadrilateral element ABCD can be obtained by applying a \( 2 \times 2 \) Gauss-point integration rule with bilinear interpolation functions to the area integral (6.2.33).

*Hint:* Apply the integration techniques of Section 5.4 together with relation (5.4.8). In this relation \( d\Omega \) is the magnitude of the vector \( d\vec{S} \) of equation (6.2.33). Calculate the Jacobian matrix with the isoparametric transformation.

Problem 6.10
Apply equation (6.2.31) to the quadrilateral ABCD of Figure 6.2.5 and take point A as the origin of the position vector \( \vec{x} \). Show that the contour integral reduces to the contributions along BC and CD with \( \vec{x} = \vec{x}_{AC} \) and that

\[
2\Omega = \vec{x}_{AC} \cdot \int_{BCD} d\vec{S}
\]

By working out the integral, obtain relation (6.2.7):

\[
\Omega_{ABCD} = \frac{1}{2} (\Delta x_{AC} \Delta y_{BD} - \Delta x_{BD} \Delta y_{AC})
\]

*Hint:* Observe that, with A as origin, the position vector is aligned with sides AB and AD and hence normal to the vector \( d\vec{S} \). Therefore there is no contributions from these two sides.

Problem 6.11
Repeat Problem 6.10 for triangle ABC of Figure 6.2.5 and show that we can write

\[
2\Omega_{ABC} = \vec{x}_{AB} \cdot \int_{BC} d\vec{S}
\]

obtaining

\[
\Omega_{ABC} = \frac{1}{2} (\Delta x_{AB} \Delta y_{BC} - \Delta x_{BC} \Delta y_{AB})
\]

Problem 6.12
Consider the quadrilateral BDHE in Figure 6.2.6(b) and apply relations (6.2.29) in order to define the average value of the x-derivative of a function \( U \). Consider \( y_E = y_D \) and obtain

\[
\left( \frac{\partial U}{\partial x} \right)_{BDHE} = \frac{(U_E - U_D)}{(x_E - x_D)}
\]

Comment on the accuracy of this formula when applied to point A.

Problem 6.13
Repeat Problem 6.12 for the contour BCDGHKEF of Figure 6.2.6(b) by applying
equations (6.2.26). Obtain the following approximation:

\[
\frac{\partial U}{\partial x} = \frac{(U_E - U_D) + \Delta y_{BA} (U_F - U_C) + \Delta y_{AH} (U_K - U_G)}{(x_E - x_D) + 2\Delta y_{BH} (x_F - x_C) + 2\Delta y_{BH} (x_K - x_G)}
\]

Derive also the corresponding expression for a Cartesian mesh.

**Problem 6.14**

Apply the upwind flux evaluation (equation (6.2.18)) to derive a finite volume scheme for the cell GHKEFBCD of Figure 6.2.2(b), considered as Cartesian. Compare the obtained discretization with the results of Example 6.2.2.

**Problem 6.15**

Repeat the calculations of Example 6.2.3 for the arbitrary finite volume mesh of Figure 6.2.2(a).

**Problem 6.16**

Show that the integral conservation law over the one-dimensional domain \( a \leq x \leq b \), applied to

\[
\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0
\]

with the condition \( f(b) = f(a) \), reduces to the condition:

\[
\int_a^b u \, dx
\]

is constant in time.

Apply this condition to a discretized \( x \)-space, with an arbitrary mesh point distribution, and show that this condition reduces to

\[
\frac{1}{2} \sum_i \Delta u_i (x_{i+1} - x_{i-1}) = 0
\]

where

\[
\Delta u_i = u_{i+1}^n - u_i^n = \left( \frac{\partial u_i}{\partial t} \right) \Delta t
\]

**Hint:** Apply a trapezoidal rule to evaluate the integral

\[
\frac{\partial}{\partial t} \int_a^b u \, dx = 0
\]

and rearrange the sum to isolate the \( u_i \)-terms.