NUMERICAL COMPUTATION OF INTERNAL AND EXTERNAL FLOWS

Volume 1
Fundamentals of Numerical Discretization

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Numerical Computation of INTERNAL AND EXTERNAL FLOWS
Volume 1: Fundamentals of Numerical Discretization

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A Wiley–Interscience Publication
To the memory of

Leon Hirsch
and
Czipa Zugman,

my parents,
struck by destiny
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Preface

This book, which is published in two volumes, aims at introducing the reader to the essential steps involved in the numerical simulation of fluid flows by providing a guide from the initial step of the selection of a mathematical model to practical methods for their numerical discretization and resolution.

The first volume, divided into four parts, is devoted to the fundamentals of numerical discretization techniques and attempts a systematic presentation of the successive steps involved in the definition and development of a numerical simulation. The second, on the other hand, presents the applications of numerical methods and algorithms to selected flow models, from the full potential flow model to the systems of Euler and Navier–Stokes equations.

Part I, covering Chapters 1 to 3, introduces the mathematical models corresponding to various levels of approximation of a flow system. We hope hereby to draw, if necessary, the reader’s attention to the range of validity and limitations of the different available flow models so that the user will be in a position to make a choice in full awareness of its implications. Part II is devoted to a presentation of the essentials of the most frequently applied discretization methods for differential equations, the finite difference (Chapter 4), finite element (Chapter 5) and finite volume methods (Chapter 6). Part III introduces the next step in the development of an algorithm, namely the methods for the analysis of the stability, convergence and accuracy properties of a selected discretization. This is covered in Chapters 7 and 10, dealing, respectively, with basic definitions, the Von Neumann method, the method of the equivalent differential equation and the matrix method. Finally, Part IV covers the resolution methods for discretized equations. More particularly, integration methods which can be applied to systems of ordinary differential equations (in time) are discussed in Chapter 11 and iterative methods for the resolution of algebraic systems are discussed in Chapter 12.

No attempt has been made towards an exhaustive presentation of the material covered and several important topics are not treated in the text for objective as well as subjective reasons. To explain a few of them, spectral discretization methods applied to flow problems are an important technique, which is treated in existing textbooks, but also we have no practical experience with the method. Stability analysis methods, such as the energy method, require a mathematical background which is not often found in the engineering community, and it was not felt appropriate to introduce this subject in a
text which is addressed mainly to engineers and physicists with an interest in flow problems. The computational techniques for boundary layers are largely covered in recent textbooks, and we thought that there was not much to add to the existing, well-documented material.

This text is directed at students at the graduate level as well as at scientists and engineers already engaged, or starting to be engaged, in computational fluid dynamics. With regard to the material for a graduate course, we have aimed at allowing a double selection. For an introductory course, one can consider an ‘horizontal’ reading, by selecting subsections of different chapters in order to cover a wider range of topics. An alternative ‘vertical’ reading would select fewer chapters, with a more complete treatment of the selected topics.

Parts of this book have been written while holding the NAVAIR Research Chair at the Naval Postgraduate School in Monterey, during the academic year 1983–4, for which I am particularly indebted to Ray Shreeve, Professor at the Areonautical Department and Director of the Turbopropulsion Laboratory. The pleasant and encouraging atmosphere during this period and during subsequent summer stays at NPS, where some additional writing could partly be done, is, for a large part, the basis of having brought this task to an end.

Some sections on Euler equations were written during a summer stay at ICASE, NASA Langley and I would like to acknowledge particularly Dr Milton Rose, former Director of ICASE, for his hospitality and the stimulating atmosphere. I have also had the privilege of benefiting from results of computations performed, at my request, on different test cases by several groups, and I would like to thank D. Caughey at Cornell University, T. Holst at NASA Ames, A. Jameson at Princeton University, M. Salas at NASA Langley, and J. South and C. Gumbert also at NASA Langley, for their willingness and effort.

Finally, I would like to thank my colleagues S. Wajc and G. Warzee as well as present and former coworkers H. Deconinck, C. Lacor and J. Peuteman for various suggestions, comments and contributions. I have also the pleasure to thank my secretaries L. Vandenbosche and J. D’haes for the patience and the effort of typing a lengthy manuscript.

Ch. Hirsch
Brussels, January 1987
# Nomenclature

\( a \) convection velocity of wave speed
\( A \) Jacobian of flux function
\( c \) speed of sound
\( c_p \) specific heat at constant pressure
\( c_v \) specific heat at constant volume
\( C \) discretization operator
\( D \) first derivative operator
\( e \) internal energy per unit mass
\( e \) vector (column matrix) of solution errors
\( E \) total energy per unit volume
\( E \) finite difference displacement (shift) operator
\( f \) flux function
\( f_e \) external force vector
\( \hat{F}(f, g, h) \) flux vector with components \( f, g, h \)
\( g \) gravity acceleration
\( G \) amplification factor/matrix
\( h \) enthalpy per unit mass
\( H \) total enthalpy
\( I \) rothalpy
\( J \) Jacobian
\( k \) coefficient of thermal conductivity
\( k_\text{rad} \) wavenumber
\( L \) differential operator
\( M \) Mach number
\( M_{x}, M_{y}, M_{z} \) Mach number of cartesian velocity components
\( n \) normal distance
\( \hat{n} \) normal vector
\( N \) finite element interpolation function
\( p \) pressure
\( P \) convergence or conditioning operator
\( Pr \) Prandtl number
\( q \) non-homogeneous term
\( q_\text{H} \) heat source
\( Q \) source term; matrix of non-homogeneous terms
\( r \) gas constant per unit mass
\( R \) residual of iterative scheme
\( Re \) mesh Reynolds (Peclet) number
\( Re \) Reynolds number
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>s</td>
<td>entropy per unit mass</td>
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<tr>
<td>S</td>
<td>characteristic surface</td>
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<td>S</td>
<td>space-discretization operator</td>
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<tr>
<td>$\bar{S}$</td>
<td>surface vector</td>
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<tr>
<td>t</td>
<td>time</td>
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<tr>
<td>T</td>
<td>temperature</td>
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<tr>
<td>u</td>
<td>dependent variable</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>entrainment velocity</td>
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<tr>
<td>$U$</td>
<td>vector (column matrix) of dependent variables</td>
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<tr>
<td>$\vec{v}(u, v, w)$</td>
<td>velocity vector with cartesian components $u, v, w$</td>
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<tr>
<td>$U$</td>
<td>vector of conservative variables</td>
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<tr>
<td>$\vec{w}$</td>
<td>relative velocity</td>
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<tr>
<td>W</td>
<td>weight function</td>
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<tr>
<td>x, y, z</td>
<td>Cartesian co-ordinates</td>
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<tr>
<td>z</td>
<td>amplification factor of time-integration scheme</td>
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<tr>
<td>$\alpha$</td>
<td>diffusivity coefficient</td>
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<tr>
<td>$\beta$</td>
<td>dimensionless diffusion coefficient $\beta = \alpha \Delta t / \Delta x$</td>
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<tr>
<td>$\gamma$</td>
<td>ratio of specific heats</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>circulation; boundary of domain $\Omega$</td>
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<tr>
<td>$\delta$</td>
<td>central-difference operator</td>
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<tr>
<td>$\delta^+$</td>
<td>forward difference operator</td>
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<tr>
<td>$\delta^-$</td>
<td>backward difference operator</td>
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<tr>
<td>$\Delta$</td>
<td>Laplace operator</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time step</td>
</tr>
<tr>
<td>$\Delta U$</td>
<td>variation of solution $U$ between levels $n + 1$ and $n$</td>
</tr>
<tr>
<td>$\Delta x, \Delta y$</td>
<td>spatial mesh size in x and y directions</td>
</tr>
<tr>
<td>$\eta$</td>
<td>non-dimensional difference variable in local co-ordinates</td>
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<tr>
<td>$\epsilon$</td>
<td>error of numerical solution</td>
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<tr>
<td>$\epsilon_v$</td>
<td>turbulence dissipation rate</td>
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<tr>
<td>$\epsilon_d$</td>
<td>dissipation or diffusion error</td>
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<tr>
<td>$\epsilon_\delta$</td>
<td>dispersion error</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>vorticity vector</td>
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<tr>
<td>$\xi$</td>
<td>magnitude of vorticity vector</td>
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<tr>
<td>$\theta$</td>
<td>parameter controlling type of difference scheme</td>
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<tr>
<td>$\kappa$</td>
<td>wavenumber vector; wave-propagation direction</td>
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<tr>
<td>$\lambda$</td>
<td>eigenvalue of amplification matrix</td>
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<tr>
<td>$\mu$</td>
<td>coefficient of dynamic viscosity</td>
</tr>
<tr>
<td>$\mu$</td>
<td>averaging difference operator</td>
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<tr>
<td>$\xi$</td>
<td>non-dimensional distance variable in local co-ordinates</td>
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<tr>
<td>$\rho$</td>
<td>density</td>
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<td>$\rho$</td>
<td>spectral radius</td>
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<tr>
<td>$\sigma$</td>
<td>Courant number</td>
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<td>$\sigma$</td>
<td>shear stress tensor</td>
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<tr>
<td>$\tau$</td>
<td>relaxation parameter</td>
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<tr>
<td>$\tau$</td>
<td>stress tensor</td>
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<tr>
<td>$\nu$</td>
<td>kinematic viscosity</td>
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<tr>
<td>$\phi$</td>
<td>velocity potential</td>
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</tbody>
</table>
\( \phi \) phase angle in Von Neumann analysis

\( \Phi \) phase angle of amplification factor

\( \psi \) rotational function

\( \omega \) time frequency of plane wave

\( \omega \) overrelaxation parameters

\( \Omega \) eigenvalue of space discretization matrix

\( \Omega \) volume

\( \hat{I}_x, \hat{I}_y, \hat{I}_z \) unit vectors along the \( x, y, z \) directions

\( \hat{I}_n \) unit vector along the normal direction

Subscripts

\( e \) external variable

\( i, j \) mesh point locations in \( x, y \) directions

\( I, J \) nodal point index

\( j \) eigenvalue number

min minimum

max maximum

n normal or normal component

o stagnation values

v viscous term

\( x, y, z \) components in \( x, y, z \) directions

\( \partial_x, \partial_y, \partial_z \) partial differentiation with respect to \( x, y, z \)

\( \infty \) freestream value

Superscripts

\( n \) iteration level

\( n \) time-level

\( - \) exact solution of discretized equation

\( \ast \) exact solution of differential equation

Symbols

\( \times \) vector product of two vectors

\( \otimes \) tensor product of two vectors