# Many-Body Physics 

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## Preface

Some useful textbooks:
A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics
G. Mahan, Many-Particle Physics
A. Fetter and J. Walecka, Quantum Theory of Many-Particle Systems
S. Doniach and Sondheimer, Green's Functions for Solid State Physicists
J. R. Schrieffer, Theory of Superconductivity
J. Negele and H. Orland, Quantum Many-Particle Systems
E. Fradkin, Field Theories of Condensed Matter Systems
A. M. Tsvelik, Field Theory in Condensed Matter Physics
A. Auerbach, Interacting Electrons and Quantum Magnetism

A useful review article:
R. Shankar, Rev. Mod. Phys. 66, 129 (1994).

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## Part I

## Preliminaries

## Chapter 1

## Introduction

In this course, we will be developing a formalism for quantum systems with many degrees of freedom. We will be applying this formalism to the $\sim 10^{23}$ electrons and ions in crystalline solids. It is assumed that you have already had a course in which the properties of metals, insulators, and semiconductors were described in terms of the physics of non-interacting electrons in a periodic potential. The methods described in this course will allow us to go beyond this and tackle the complex and profound phenomena which arise from the Coulomb interactions between electrons and from the coupling of the electrons to lattice distortions.

The techniques which we will use come under the rubric of many-body physics or quantum field theory. The same techniques are also used in elementary particle physics, in nuclear physics, and in classical statistical mechanics. In elementary particle physics, large numbers of real or virtual particles can be excited in scattering experiments. The principal distinguishing feature of elementary particle physics which actually simplifies matters - is relativistic invariance. Another simplifying feature is that in particle physics one often considers systems at zero-temperature with applications of particle physics to astrophysics and cosmology being the notable exception - so that there are quantum fluctuations but no thermal fluctuations. In classical statistical mechanics, on the other hand, there are only thermal fluctuations,
but no quantum fluctuations. In describing the electrons and ions in a crystalline solid - and quantum many-particle systems more generally - we will be dealing with systems with both quantum and thermal fluctuations.

The primary difference between the systems considered here and those considered in, say, a field theory course is the physical scale. We will be concerned with:

- $\omega, T \ll 1 e V$
- $\left|x_{i}-x_{j}\right|, \frac{1}{q} \gg 1 \AA$
as compared to energies in the $M e V$ for nuclear matter, and $G e V$ or even $T e V$, in particle physics.

Special experimental techniques are necessary to probe such scales.

- Thermodynamics: measure the response of macroscopic variables such as the energy and volume to variations of the temperature, volume, etc.
- Transport: set up a potential or thermal gradient, $\nabla \varphi, \nabla T$ and measure the electrical or heat current $\vec{j}, \vec{j}_{Q}$. The gradients $\nabla \varphi, \nabla T$ can be held constant or made to oscillate at finite frequency.
- Scattering: send neutrons or light into the system with prescribed energy, momentum and measure the energy, momentum of the outgoing neutrons or light.
- NMR: apply a static magnetic field, $B$, and measure the absorption and emission by the system of magnetic radiation at frequencies of the order of $\omega_{c}=g e B / m$.

As we will see, the results of these measurements can be expressed in terms of correlation functions. Fortunately, these are precisely the quantities which our fieldtheoretic techinques are designed to calculate. By developing the appropriate theoretical toolbox in this course, we can hope to learn not only a formalism, but also a language for describing the systems of interest.

Systems containing many particles exhibit properties - reflected in their correlation functions - which are special to such systems. Such properties are emergent. They are fairly insensitive to the details at length scales shorter than $1 \AA$ and energy scales higher than 1 eV - which are quite adequately described by the equations of non-relativistic quantum mechanics. For example, precisely the same microscopic equations of motion - Newton's equations - can describe two different systems of $10^{23}$ $\mathrm{H}_{2} \mathrm{O}$ molecules.

$$
\begin{equation*}
m \frac{d^{2} \vec{x}_{i}}{d t^{2}}=-\sum_{j \neq i} \vec{\nabla}_{i} V\left(\vec{x}_{i}-\vec{x}_{j}\right) \tag{1.1}
\end{equation*}
$$

Or, perhaps, the Schrödinger equation:

$$
\begin{equation*}
\left(-\frac{\hbar^{2}}{2 m} \sum_{i} \nabla_{i}^{2}+\sum_{i, j} V\left(\vec{x}_{i}-\vec{x}_{j}\right)\right) \psi\left(\vec{x}_{1}, \ldots, \vec{x}_{N}\right)=E \psi\left(\vec{x}_{1}, \ldots, \vec{x}_{N}\right) \tag{1.2}
\end{equation*}
$$

However, one of these systems might be water and the other ice, in which case the properties of the two systems are completely different, and the similarity between their microscopic dsecriptions is of no practical consequence. As this example shows, many-particle systems exhibit various phases - such as ice and water - which are not, for the most part, usefully described by the microscopic equations. Instead, new low-energy, long-wavelength physics emerges as a result of the interactions among large numbers of particles. Different phases are separated by phase transitions, at which the low-energy, long-wavelength description becomes non-analytic and exhibits singularities. In the above example, this occurs at the freezing point of water, where its entropy jumps discontinuously.

As we will see, different phases of matter are often distinguished on the basis of symmetry. The microscopic equations are often highly symmetrical - for instance, Newton's laws are translationally and rotationally invariant - but a given phase may exhibit much less symmetry. Water exhibits the full translational and rotational symmetry of Newton's laws; ice, however, is only invariant under the discrete translational and rotational group of its crystalline lattice. We say that the translational and
rotational symmetries of the microscopic equations have been spontaneously broken. As we will see, it is also possible for phases (and phase transition points) to exhibit symmetries which are not present in the microscopic equations.

The liquid - with full translational and rotational symmetry - and the solid which only preserves a discrete subgroup - are but two examples of phases. In a liquid crystalline phase, translational and rotational symmetry is broken to a combination of discrete and continuous subgroups. For instance, a nematic liquid crystal is made up of elementary units which are line segments. In the nematic phase, these line segments point, on average, in the same direction, but their positional distribution is as in a liquid. Hence, a nematic phase breaks rotational invariance to the subgroup of rotations about the preferred direction and preserves the full translational invariance. In a smectic-A phase, on the other hand, the line segments arrange themselves into evenly spaced layers, thereby partially breaking the translational symmetry so that discrete translations perpendicular to the layers and continuous translations along the layers remain unbroken. In a magnetic material, the electron spins can order, thereby breaking the spin-rotational invariance. In a ferromagnet, all of the spins line up in the same direction, thereby breaking the spin-rotational invariance to the subgroup of rotations about this direction while preserving the discrete translational symmetry of the lattice. In an antiferromagnet, neighboring spins are oppositely directed, thereby breaking spin-rotational invariance to the subgroup of rotations about the preferred direction and breaking the lattice translational symmetry to the subgroup of translations by an even number of lattice sites.

These different phases are separated by phase transitions. Often these phase transitions are first-order, meaning that there is a discontinuity in some first derivative of the free energy. Sometimes, the transition is second-order, in which case the discontinuity is in the second derivative. In such a transition, the system fluctuates at all length scales, and new techniques are necessary to determine the bahvior of the
system. A transition from a paramagnet to a ferromagnet can be second-order.
The calculational techniques which we will develop will allow us to quantitatively determine the properties of various phases. The strategy will be to write down a simple soluble model which describes a system in the phase of interest. The system in which we are actually interested will be accessed by perturbing the soluble model. We will develop calculational techniques - perturbation theory - which will allow us to use our knowledge of the soluble model to compute the physical properties of our system to, in principle, any desired degree of accuracy. These techniques break down if our system is not in the same phase as the soluble model. If this occurs, a new soluble model must be found which describes a model system in the same phase or at the same phase transition as our system. Of course, this presupposes a knowledge of which phase our system is in. In some cases, this can be determined from the microscopic equations of motion, but it must usually be inferred from experiment. Critical points require a new techniques. These techniques and the entire phase diagram can be understood in terms of the renormalization group.

After dealing with some preliminaries in the remainder of Part I, we move on, in Part II, to develop the perturbative calculational techniques which can be used to determine the properties of a system in some stable phase of matter. In Part III, we discuss spontaneous symmetry breaking, a non-perturbative concept which allows us to characterize phases of matter. In Part IV, we discuss critical points separating stable phases of matter. The discussion is centered on the Fermi liquid, which is a critical line separating various symmetry-breaking phases. We finally focus on one of these, the superconductor.

## Chapter 2

## Conventions, Notation, Reminders

### 2.1 Units, Physical Constants

We will use a system of units in which

$$
\begin{equation*}
\hbar=k_{B}=e=1 \tag{2.1}
\end{equation*}
$$

In such a system of units, we measure energies, temperatures, and frequencies in electron volts. The basic rule of thumb is that $1 \mathrm{eV} \sim 10,000 \mathrm{~K}$ or $1 \mathrm{meV} \sim 10 \mathrm{~K}$, while a frequency of 1 Hz corresponds to $\sim 6 \times 10^{-16} \mathrm{eV}$. The Fermi energy in a typical metal is $\sim 1 \mathrm{eV}$. In a conventional, 'low-temperature' superconductor, $T_{c} \sim 0.1-1 \mathrm{meV}$. This corresponds to a frequency of $10^{11}-10^{12} \mathrm{~Hz}$ or a wavelength of light of $\sim 1 \mathrm{~cm}$. We could set the speed of light to 1 and measure distances in $(e v)^{-1}$, but most of the velocities which we will be dealing with are much smaller than the speed of light, so this is not very useful. The basic unit of length is the angstrom, $1 \AA=10^{-10} \mathrm{~m}$. The lattice spacing in a typical crystal is $\sim 1-10 \AA$.

### 2.2 Mathematical Conventions

Vectors will be denoted in boldface, $\mathbf{x}, \mathbf{E}$, or with a Latin subscript $x_{i}, E_{i}, i=$ $1,2, \ldots, d$. Unless otherwise specified, we will work in $d=3$ dimensions. Occasionally,
we will use Greek subscripts, e.g. $j_{\mu}, \mu=0,1, \ldots, d$ where the 0 -component is the time-component as in $x_{\mu}=(t, x, y, z)$. Unless otherwise noted, repeated indices are summed over, e.g. $a_{i} b_{i}=a_{1} b_{1}+a_{2} b_{2}+a_{3} b_{3}=\mathbf{a} \cdot \mathbf{b}$

We will use the following Fourier transform convention:

$$
\begin{align*}
f(t) & =\int_{-\infty}^{\infty} \frac{d \omega}{(2 \pi)^{1 / 2}} \tilde{f}(\omega) e^{-i \omega t} \\
\tilde{f}(\omega) & =\int_{-\infty}^{\infty} \frac{d t}{(2 \pi)^{1 / 2}} f(t) e^{i \omega t} \tag{2.2}
\end{align*}
$$

### 2.3 Quantum Mechanics

A quantum mechanical system is defined by a Hilbert space, $\mathcal{H}$, whose vectors are states, $|\psi\rangle$. There are linear operators, $\mathcal{O}_{i}$ which act on this Hilbert space. These operators correspond to physical observables. Finally, there is an inner product, which assigns a complex number, $\langle\chi \mid \psi\rangle$, to any pair of states, $|\psi\rangle,|\chi\rangle$. A state vector, $|\psi\rangle$ gives a complete description of a system through the expectation values, $\langle\psi| \mathcal{O}_{i}|\psi\rangle$ (assuming that $|\psi\rangle$ is normalized so that $\langle\psi \mid \psi\rangle=1$ ), which would be the average values of the corresponding physical observables if we could measure them on an infinite collection of identical systems each in the state $|\psi\rangle$.

The adjoint, $\mathcal{O}^{\dagger}$, of an operator is defined according to

$$
\begin{equation*}
\langle\chi|(\mathcal{O}|\psi\rangle)=\left(\langle\chi| \mathcal{O}^{\dagger}\right)|\psi\rangle \tag{2.3}
\end{equation*}
$$

In other words, the inner product between $|\chi\rangle$ and $\mathcal{O}|\psi\rangle$ is the same as that between $\mathcal{O}^{\dagger}|\chi\rangle$ and $|\psi\rangle$. An Hermitian operator satisfies

$$
\begin{equation*}
\mathcal{O}=\mathcal{O}^{\dagger} \tag{2.4}
\end{equation*}
$$

while a unitary operator satisfies

$$
\begin{equation*}
\mathcal{O} \mathcal{O}^{\dagger}=\mathcal{O}^{\dagger} \mathcal{O}=1 \tag{2.5}
\end{equation*}
$$

If $\mathcal{O}$ is Hermitian, then

$$
\begin{equation*}
e^{i \mathcal{O}} \tag{2.6}
\end{equation*}
$$

is unitary. Given an Hermitian operator, $\mathcal{O}$, its eigenstates are orthogonal,

$$
\begin{equation*}
\left\langle\lambda^{\prime}\right| \mathcal{O}|\lambda\rangle=\lambda\left\langle\lambda^{\prime} \mid \lambda\right\rangle=\lambda^{\prime}\left\langle\lambda^{\prime} \mid \lambda\right\rangle \tag{2.7}
\end{equation*}
$$

For $\lambda \neq \lambda^{\prime}$,

$$
\begin{equation*}
\left\langle\lambda^{\prime} \mid \lambda\right\rangle=0 \tag{2.8}
\end{equation*}
$$

If there are $n$ states with the same eigenvalue, then, within the subspace spanned by these states, we can pick a set of $n$ mutually orthogonal states. Hence, we can use the eigenstates $|\lambda\rangle$ as a basis for Hilbert space. Any state $|\psi\rangle$ can be expanded in the basis given by the eigenstates of $\mathcal{O}$ :

$$
\begin{equation*}
|\psi\rangle=\sum_{\lambda} c_{\lambda}|\lambda\rangle \tag{2.9}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{\lambda}=\langle\lambda \mid \psi\rangle \tag{2.10}
\end{equation*}
$$

A particularly important operator is the Hamiltonian, or the total energy, which we will denote by $H$. Schrödinger's equation tells us that $H$ determines how a state of the system will evolve in time.

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}|\psi\rangle=H|\psi\rangle \tag{2.11}
\end{equation*}
$$

If the Hamiltonian is independent of time, then we can define energy eigenstates,

$$
\begin{equation*}
H|E\rangle=E|E\rangle \tag{2.12}
\end{equation*}
$$

which evolve in time according to:

$$
\begin{equation*}
|E(t)\rangle=e^{-i \frac{E t}{\hbar}}|E(0)\rangle \tag{2.13}
\end{equation*}
$$

An arbitrary state can be expanded in the basis of energy eigenstates:

$$
\begin{equation*}
|\psi\rangle=\sum_{i} c_{i}\left|E_{i}\right\rangle \tag{2.14}
\end{equation*}
$$

It will evolve according to:

$$
\begin{equation*}
|\psi(t)\rangle=\sum_{j} c_{j} e^{-i \frac{E_{j} t}{\hbar}}\left|E_{j}\right\rangle \tag{2.15}
\end{equation*}
$$

When we have a system with many particles, we must now specify the states of all of the particles. If we have two distinguishable particles whose Hilbert spaces are spanned by the bases

$$
\begin{equation*}
|i, 1\rangle \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
|\alpha, 2\rangle \tag{2.17}
\end{equation*}
$$

Then the two-particle Hilbert space is spanned by the set:

$$
\begin{equation*}
|i, 1 ; \alpha, 2\rangle \equiv|i, 1\rangle \otimes|\alpha, 2\rangle \tag{2.18}
\end{equation*}
$$

Suppose that the two single-particle Hilbert spaces are identical, e.g. the two particles are in the same box. Then the two-particle Hilbert space is:

$$
\begin{equation*}
|i, j\rangle \equiv|i, 1\rangle \otimes|j, 2\rangle \tag{2.19}
\end{equation*}
$$

If the particles are identical, however, we must be more careful. $|i, j\rangle$ and $|j, i\rangle$ must be physically the same state, i.e.

$$
\begin{equation*}
|i, j\rangle=e^{i \alpha}|j, i\rangle \tag{2.20}
\end{equation*}
$$

Applying this relation twice implies that

$$
\begin{equation*}
|i, j\rangle=e^{2 i \alpha}|i, j\rangle \tag{2.21}
\end{equation*}
$$

so $e^{i \alpha}= \pm 1$. The former corresponds to bosons, while the latter corresponds to fermions. The two-particle Hilbert spaces of bosons and fermions are respectively spanned by:

$$
\begin{equation*}
|i, j\rangle+|j, i\rangle \tag{2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
|i, j\rangle-|j, i\rangle \tag{2.23}
\end{equation*}
$$

The $n$-particle Hilbert spaces of bosons and fermions are respectively spanned by:

$$
\begin{equation*}
\sum_{\pi}\left|i_{\pi(1)}, \ldots, i_{\pi(n)}\right\rangle \tag{2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{\pi}(-1)^{\pi}\left|i_{\pi(1)}, \ldots, i_{\pi(n)}\right\rangle \tag{2.25}
\end{equation*}
$$

In position space, this means that a bosonic wavefunction must be completely symmetric:

$$
\begin{equation*}
\psi\left(x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n}\right)=\psi\left(x_{1}, \ldots, x_{j}, \ldots, x_{i}, \ldots, x_{n}\right) \tag{2.26}
\end{equation*}
$$

while a fermionic wavefunction must be completely antisymmetric:

$$
\begin{equation*}
\psi\left(x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n}\right)=-\psi\left(x_{1}, \ldots, x_{j}, \ldots, x_{i}, \ldots, x_{n}\right) \tag{2.27}
\end{equation*}
$$

### 2.4 Statistical Mechanics

In statistical mechanics, we deal with a situation in which even the quantum state of the system is unknown. The expectation value of an observable must be averaged over:

$$
\begin{equation*}
\langle\mathcal{O}\rangle=\sum_{i} w_{i}\langle i| \mathcal{O}|i\rangle \tag{2.28}
\end{equation*}
$$

where the states $|i\rangle$ form an orthonormal basis of $\mathcal{H}$ and $w_{i}$ is the probability of being in state $|i\rangle$. The $w_{i}$ 's must satisfy $\sum w_{i}=1$. The expectation value can be written in
a basis-independent form:

$$
\begin{equation*}
\langle\mathcal{O}\rangle=\operatorname{Tr}\{\rho \mathcal{O}\} \tag{2.29}
\end{equation*}
$$

where $\rho$ is the density matrix. In the above example, $\rho=\sum_{i} w_{i}|i\rangle\langle i|$. The condition, $\sum w_{i}=1$, i.e. that the probabilities add to 1 , is:

$$
\begin{equation*}
\operatorname{Tr}\{\rho\}=1 \tag{2.30}
\end{equation*}
$$

We usually deal with one of three ensembles: the microcanonical emsemble, the canonical ensemble, or the grand canonical ensemble. In the microcanonical ensemble, we assume that our system is isolated, so the energy is fixed to be $E$, but all states with energy $E$ are taken with equal probability:

$$
\begin{equation*}
\rho=C \delta(H-E) \tag{2.31}
\end{equation*}
$$

$C$ is a normalization constant which is determined by (2.30). The entropy is given by,

$$
\begin{equation*}
S=-\ln C \tag{2.32}
\end{equation*}
$$

Recall that the inverse temperature, $\beta$, is given by $\beta=\partial S / \partial E$. and the pressure, $P$, is given by $P=T \partial S / \partial V$, where $V$ is the volume of the system.

In the canonical ensemble, on the other hand, we assume that our system is in contact with a heat reservoir so that the temperature is constant. Then,

$$
\begin{equation*}
\rho=C e^{-\beta H} \tag{2.33}
\end{equation*}
$$

It is useful to drop the normalization constant, $C$, and work with an unnormalized density matrix so that we can define the partition function:

$$
\begin{equation*}
Z=\operatorname{Tr}\{\rho\} \tag{2.34}
\end{equation*}
$$

$Z$ is related to the free energy, $F$, through:

$$
\begin{equation*}
F=-T \ln Z \tag{2.35}
\end{equation*}
$$

which, in turn, is related to $S, P$ through

$$
\begin{align*}
S & =-\frac{\partial F}{\partial T} \\
P & =-\frac{\partial F}{\partial V} \tag{2.36}
\end{align*}
$$

The chemical potential, $\mu$ is defined by

$$
\begin{equation*}
\mu=\frac{\partial F}{\partial N} \tag{2.37}
\end{equation*}
$$

where $N$ is the particle number.
In the grand canonical ensemble, the system is in contact with a reservoir of heat and particles. Thus, the temperature and chemical potential are held fixed and

$$
\begin{equation*}
\rho=C e^{-\beta(H-\mu N)} \tag{2.38}
\end{equation*}
$$

If we again work with an unnormalized density matrix, we have:

$$
\begin{equation*}
P V=T \ln \operatorname{Tr}\{\rho\} \tag{2.39}
\end{equation*}
$$

## Part II

## Basic Formalism

## Chapter 3

## Phonons and Second Quantization

### 3.1 Classical Lattice Dynamics

Consider the lattice of ions in a solid. Suppose the equilibrium positions of the ions are the sites $\vec{R}_{i}$. Let us describe small displacements from these sites by a displacement field $\vec{u}\left(\vec{R}_{i}\right)$. We will imagine that the crystal is just a system of masses connected by springs of equilibrium length $a$.

At length scales much longer than its lattice spacing, a crystalline solid can be modelled as an elastic medium. We replace $\vec{u}\left(\vec{R}_{i}\right)$ by $\vec{u}(\vec{r})$ (i.e. we replace the lattice vectors, $\vec{R}_{i}$, by a continuous variable, $\left.\vec{r}\right)$. Such an approximation is valid at length scales much larger than the lattice spacing, $a$, or, equivalently, at wavevectors $q \ll$ $2 \pi / a$.


Figure 3.1: A crystalline solid viewed as an elastic medium.

The potential energy of the elastic medium must be translationally and rotationally invariant (at shorter distances, these symmetries are broken to discrete lattice symmetries, but let's focus on the long-wavelength physics for now). Translational invariance implies $V\left[\vec{u}+\vec{u}_{0}\right]=V[\vec{u}]$, so $V$ can only be a function of the derivatives, $\partial_{i} u_{j}$. Rotational invariance implies that it can only be a function of the symmetric combination,

$$
\begin{equation*}
u_{i j} \equiv \frac{1}{2}\left(\partial_{i} u_{j}+\partial_{j} u_{i}\right) \tag{3.1}
\end{equation*}
$$

There are only two possible such terms, $u_{i j} u_{i j}$ and $u_{k k}^{2}$ (repeated indices are summed). A third term, $u_{k k}$, is a surface term and can be ignored. Hence, the action of a crystalline solid to quadratic order, viewed as an elastic medium, is:

$$
\begin{equation*}
S_{0}=\int d t d^{3} \vec{r} \mathcal{L}=\frac{1}{2} \int d t d^{3} \vec{r}\left[\rho\left(\partial_{t} u_{i}\right)^{2}-2 \mu u_{i j} u_{i j}-\lambda u_{k k}^{2}\right] \tag{3.2}
\end{equation*}
$$

where $\rho$ is the mass density of the solid and $\mu$ and $\lambda$ are the Lamé coefficients. Under a dilatation, $\vec{u}(\vec{r})=\alpha \vec{r}$, the change in the energy density of the elastic medium is $\alpha^{2}(\lambda+2 \mu / 3) / 2$; under a shear stress, $u_{x}=\alpha y, u_{y}=u_{z}=0$, it is $\alpha^{2} \mu / 2$. In a crystal - which has only a discrete rotational symmetry - there may be more parameters than just $\mu$ and $\lambda$, depending on the symmetry of the lattice. In a crystal with cubic symmetry, for instance, there are, in general, three independent parameters. We will make life simple, however, and make the approximation of full rotational invariance.

### 3.2 The Normal Modes of a Lattice

Let us expand the displacement field in terms of its normal-modes. The equations of motion which follow from (3.2) are:

$$
\begin{equation*}
\rho \partial_{t}^{2} u_{i}=(\mu+\lambda) \partial_{i} \partial_{j} u_{j}+\mu \partial_{j} \partial_{j} u_{i} \tag{3.3}
\end{equation*}
$$

The solutions,

$$
\begin{equation*}
u_{i}(\vec{r}, t)=\epsilon_{i} e^{i(\vec{k} \cdot \vec{r}-\omega t)} \tag{3.4}
\end{equation*}
$$

where is a unit polarization vector, satisfy

$$
\begin{equation*}
-\rho \omega^{2} \epsilon_{i}=-(\mu+\lambda) k_{i}\left(k_{j} \epsilon_{j}\right)-\mu k^{2} \epsilon_{i} \tag{3.5}
\end{equation*}
$$

For longitudinally polarized waves, $k_{i}=k \epsilon_{i}$,

$$
\begin{equation*}
\omega_{k}^{l}= \pm \sqrt{\frac{2 \mu+\lambda}{\rho}} k \equiv \pm v_{l} k \tag{3.6}
\end{equation*}
$$

while transverse waves, $k_{j} \epsilon_{j}=0$ have

$$
\begin{equation*}
\omega_{k}^{t}= \pm \sqrt{\frac{\mu}{\rho}} k \equiv \pm v_{s} k \tag{3.7}
\end{equation*}
$$

Hence, the general solution of (3.3) is of the form:

$$
\begin{equation*}
u_{i}(\vec{r}, t)=\sum_{k, s} \frac{1}{\sqrt{2 \rho \omega_{k}^{s}}} \epsilon_{i}^{s}\left(a_{\vec{k}, s} e^{i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}+a_{\vec{k}, s}^{\dagger} e^{-i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}\right) \tag{3.8}
\end{equation*}
$$

$s=1,2,3$ corresponds to the longitudinal and two transverse polarizations. The normalization factor, $1 / \sqrt{2 \rho \omega_{k}^{s}}$, was chosen for later convenience.

The allowed $\vec{k}$ values are determined by the boundary conditions in a finite system. For periodic boundary conditions in a cubic system of size $V=L^{3}$, the allowed $\vec{k}$ 's are $\frac{2 \pi}{L}\left(n_{1}, n_{2}, n_{3}\right)$. Hence, the $\vec{k}$-space volume per allowed $\vec{k}$ is $(2 \pi)^{3} / V$. Hence, we can take the infinite-volume limit by making the replacement:

$$
\begin{align*}
\sum_{k} f(\vec{k}) & =\frac{1}{(\Delta \vec{k})^{3}} \sum_{k} f(\vec{k})(\Delta \vec{k})^{3} \\
& =\frac{V}{(2 \pi)^{3}} \int d^{3} \vec{k} f(\vec{k}) \tag{3.9}
\end{align*}
$$

It would be natural to use this in defining the infinite-volume limit, but we will, instead, use the following, which is consistent with our Fourier transform convention:

$$
\begin{equation*}
u_{i}(\vec{r}, t)=\int \frac{d^{3} \vec{k}}{(2 \pi)^{3 / 2}} \sum_{s} \frac{1}{\sqrt{2 \rho \omega_{k}^{s}}} \epsilon_{i}^{s}\left(a_{\vec{k}, s} e^{i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}+a_{\vec{k}, s}^{\dagger} e^{-i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}\right) \tag{3.10}
\end{equation*}
$$

### 3.3 Canonical Formalism, Poisson Brackets

The canonical conjugate to our classical field, $u_{i}$, is

$$
\begin{equation*}
\pi_{i} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{t} u_{i}\right)}=\rho \partial_{t} u_{i} \tag{3.11}
\end{equation*}
$$

The Hamiltonian is given by

$$
\begin{align*}
H & =\int d^{3} \vec{r} \pi_{i} \partial_{t} u_{i}-\mathcal{L} \\
& =\frac{1}{2} \int d^{3} \vec{r}\left[\rho\left(\partial_{t} u_{i}\right)^{2}+2 \mu u_{i j} u_{i j}+\lambda u_{k k}^{2}\right] \\
& =\frac{1}{2} \int d^{3} \vec{r}\left[\frac{1}{\rho} \pi_{i}^{2}+2 \mu u_{i j} u_{i j}+\lambda u_{k k}^{2}\right] \tag{3.12}
\end{align*}
$$

Let us define the functional derivative,

$$
\begin{equation*}
\frac{\delta}{\delta \eta} \int d^{3} \vec{r} \mathcal{F}(\vec{r})=\frac{\partial \mathcal{F}}{\partial \eta}-\frac{\partial \mathcal{F}}{\partial\left(\partial_{i} \eta\right)} \tag{3.13}
\end{equation*}
$$

Then the equation of motion for $\pi_{i}$ can be written

$$
\begin{equation*}
\partial_{t} \pi_{i}=-\frac{\delta H}{\delta u_{i}} \tag{3.14}
\end{equation*}
$$

while

$$
\begin{equation*}
\partial_{t} u_{i}=\frac{\delta H}{\delta \pi_{i}} \tag{3.15}
\end{equation*}
$$

From these equations, we see that it is natural to define the Poisson brackets:

$$
\begin{equation*}
[U, V]_{\mathrm{PB}}=\int d^{3} \vec{r}\left(\frac{\delta U}{\delta u_{i}} \frac{\delta V}{\delta \pi_{i}}-\frac{\delta U}{\delta \pi_{i}} \frac{\delta V}{\delta u_{i}}\right) \tag{3.16}
\end{equation*}
$$

With this definition,

$$
\begin{equation*}
\left[u_{j}(\vec{r}), \pi_{i}\left(\overrightarrow{r^{\prime}}\right)\right]_{\mathrm{PB}}=\delta\left(\vec{r}-\vec{r}^{\prime}\right) \tag{3.17}
\end{equation*}
$$

and

$$
\begin{align*}
\partial_{t} \pi_{i} & =\left[\pi_{i}, H\right]_{\mathrm{PB}} \\
\partial_{t} u_{i} & =\left[u_{i}, H\right]_{\mathrm{PB}} \tag{3.18}
\end{align*}
$$

As we will see shortly, the normalization chosen above for the normal mode expansion of $u_{i}(\vec{r})$ is particularly convenient since it leads to:

$$
\begin{equation*}
\left[a_{\vec{k}, s}, a_{\vec{k}^{\prime}, s^{\prime}}^{\dagger}\right]_{\mathrm{PB}}=-i \delta_{s s^{\prime}} \delta\left(\vec{k}-\vec{k}^{\prime}\right) \tag{3.19}
\end{equation*}
$$

When we quantize a classical field theory, we will promote the Poisson brackets to commutators, [, $]_{\mathrm{PB}} \rightarrow i[$,$] .$

### 3.4 Motivation for Second Quantization

The action (3.2) defines a classical field theory. It has 3 degrees of freedom per spatial point - i.e. it has infinitely many degrees of freedom. This is a consequence of the continuum limit which we took. A real finite-size sample of a solid has a finite number of degrees of freedom: if there are $N$ ions, there are $3 N$ degrees of freedom, $\vec{r}_{1}, \vec{r}_{2}, \ldots, \vec{r}_{N}$. However, it is extremely convenient to take the continuum limit and ignore the difference between $3 N$ and $\infty$. Furthermore, we will also be concerned with the electromagnetic field, $\vec{E}=-\nabla \varphi-\partial_{t} \vec{A}, \vec{B}=\nabla \times \vec{A}$, which does have infinitely many degrees of freedom ( 2 per spatial point when gauge invariance is taken into account). By going to the continuum limit, we can handle the electromagnetic field and an elastic medium in a parallel fashion which greatly facilitates calculations. We thereby make a transition from classical particle mechanics (with a discrete number of degrees of freedom) to classical field theory (with continuously many degrees of freedom):

$$
\begin{align*}
\vec{r}_{a} & \leftrightarrow \vec{u}(\vec{x}, t) \\
t & \leftrightarrow t \\
a & \leftrightarrow \vec{x} \tag{3.20}
\end{align*}
$$

At the quantum level, we will be dealing with wavefunctionals of the form $\Psi[\vec{u}(\vec{r})]$ or $\Psi[\vec{A}(\vec{r})]$ rather than $\psi\left(\vec{r}_{1}, \vec{r}_{2}, \ldots, \vec{r}_{N}\right)$. The coordinates $\vec{r}$ are no more than indices
(but continuous ones) on the fields. Hence, the operators of the theory will be $\vec{u}(\vec{r})$, $\partial_{t} \vec{u}(\vec{r})$ or $\vec{A}(\vec{r}), \partial_{t} \vec{A}(\vec{r})$ rather than $\vec{r}_{a}, \vec{p}_{a}$.

In this approach, the basic quantities will be the normal modes of the displacement field, rather than the ionic coordinates. As we will see below, the collective excitations of an elastic medium are particle-like objects - phonons - whose number is not fixed. Phonons are an example of the quasiparticle concept. In order to deal with particles whose number is not fixed (in contrast with the ions themselves, whose number is fixed), we will have to develop the formalism of second quantization. ${ }^{1}$

### 3.5 Canonical Quantization of Continuum Elastic Theory: Phonons

### 3.5.1 Review of the Simple Harmonic Oscillator

No physics course is complete without a discussion of the simple harmonic oscillator. Here, we will recall the operator formalism which will lead naturally to the Fock space construction of quantum field theory.

The harmonic oscillator is defined by the Hamiltonian,

$$
\begin{equation*}
H=\frac{1}{2} \omega\left(p^{2}+q^{2}\right) \tag{3.21}
\end{equation*}
$$

and the commutation relations,

$$
\begin{equation*}
[p, q]=-i \tag{3.22}
\end{equation*}
$$

We define raising and lowering operators:

$$
\begin{aligned}
a & =(q+i p) / \sqrt{2} \\
a^{\dagger} & =(q-i p) / \sqrt{2}
\end{aligned}
$$

[^0]The Hamiltonian and commutation relations can now be written:

$$
\begin{align*}
H & =\omega\left(a^{\dagger} a+\frac{1}{2}\right) \\
{\left[a, a^{\dagger}\right] } & =1 \tag{3.24}
\end{align*}
$$

The commutation relations,

$$
\begin{gather*}
{\left[H, a^{\dagger}\right]=\omega a^{\dagger}} \\
{[H, a]=-\omega a} \tag{3.25}
\end{gather*}
$$

imply that there is a ladder of states,

$$
\begin{array}{r}
H a^{\dagger}|E\rangle=(E+\omega) a^{\dagger}|E\rangle \\
H a|E\rangle=(E-\omega) a|E\rangle \tag{3.26}
\end{array}
$$

This ladder will continue down to negative energies (which it can't since the Hamiltonian is manifestly positive definite) unless there is an $E_{0} \geq 0$ such that

$$
\begin{equation*}
a\left|E_{0}\right\rangle=0 \tag{3.27}
\end{equation*}
$$

Such a state has $E_{0}=\omega / 2$.
We label the states by their $a^{\dagger} a$ eigenvalues. We have a complete set of $H$ eigenstates, $|n\rangle$, such that

$$
\begin{equation*}
H|n\rangle=\omega\left(n+\frac{1}{2}\right)|n\rangle \tag{3.28}
\end{equation*}
$$

and $\left(a^{\dagger}\right)^{n}|0\rangle \propto|n\rangle$. To get the normalization, we write $a^{\dagger}|n\rangle=c_{n}|n+1\rangle$. Then,

$$
\begin{align*}
\left|c_{n}\right|^{2} & =\langle n| a a^{\dagger}|n\rangle \\
& =n+1 \tag{3.29}
\end{align*}
$$

Hence,

$$
\begin{align*}
a^{\dagger}|n\rangle & =\sqrt{n+1}|n+1\rangle \\
a|n\rangle & =\sqrt{n}|n-1\rangle \tag{3.30}
\end{align*}
$$

### 3.5.2 Fock Space for Phonons

A quantum theory is made of the following ingredients:

- A Hilbert Space $\mathcal{H}$ of states $|\psi\rangle \in \mathcal{H}$.
- Operators $\mathcal{O}_{i}$ on $\mathcal{H}$, corresponding to physical observables.
- An Inner Product $\langle\chi \mid \psi\rangle$ which must be defined so that $\mathcal{O}_{i}$ is Hermitian with respect to it if the corresponding physical observable is real.

In order to construct these objects for an elastic medium - thereby quantizing our classical field theory - we employ the following procedure. We replace the classical variables, $u_{i}, \pi_{i}$ by quantum operators satisfying the canonical commutation relations:

$$
\begin{array}{r}
{\left[u_{i}(\vec{x}, t), u_{j}\left(\vec{x}^{\prime}, t\right)\right]=\left[\pi_{i}(\vec{x}, t), \pi_{j}\left(\vec{x}^{\prime}, t\right)\right]=0} \\
{\left[u_{i}(\vec{x}, t), \pi_{j}\left(\vec{x}^{\prime}, t\right)\right]=i \delta_{i j} \delta\left(\vec{x}-\vec{x}^{\prime}\right)} \tag{3.31}
\end{array}
$$

We can now define the operators $a_{\vec{k}, s}, a_{\vec{k}, s}^{\dagger}$ according to:

$$
\begin{align*}
& a_{\vec{k}, s}=\frac{1}{2} \epsilon_{i}^{s} \sqrt{2 \rho \omega_{k}^{s}} \int \frac{d^{3} \vec{x}}{(2 \pi)^{3 / 2}}\left(u_{i}(\vec{x}, 0)+\frac{i}{\omega_{k}^{s}} \partial_{t} u_{i}(\vec{x}, 0)\right) e^{i \vec{k} \cdot \vec{x}} \\
& a_{\vec{k}, s}^{\dagger}=\frac{1}{2} \epsilon_{i}^{s} \sqrt{2 \rho \omega_{k}^{s}} \int \frac{d^{3} \vec{x}}{(2 \pi)^{3 / 2}}\left(u_{i}(\vec{x}, 0)-\frac{i}{\omega_{k}^{s}} \partial_{t} u_{i}(\vec{x}, 0)\right) e^{-i \vec{k} \cdot \vec{x}} \tag{3.32}
\end{align*}
$$

These expressions can be inverted to give the normal-mode expansion, (9.72). Using $\pi_{i}=\rho \partial_{t} u_{i}$, and the above commutation relations, we see that $a_{\vec{k}, s}$ and $a_{\vec{k}, s}^{\dagger}$ satisfy the commutation relations:

$$
\begin{aligned}
{\left[a_{\vec{k}, s}, a_{\overrightarrow{k^{\prime}, s^{\prime}}}^{\dagger}\right]=} & \frac{1}{2} \rho \sqrt{\omega_{k}^{s} \omega_{k^{\prime}}^{s^{\prime}}} \epsilon_{i}^{s} \epsilon_{j}^{s^{\prime}} \int \frac{d^{3} \vec{x}}{(2 \pi)^{3 / 2}} \int \frac{d^{3} \vec{x}^{\prime}}{(2 \pi)^{3 / 2}} \\
& {\left[\left(u_{i}(\vec{x}, 0)+\frac{i}{\omega_{k}^{s}} \partial_{t} u_{i}(\vec{x}, 0)\right) e^{i \vec{k} \cdot \vec{x}},\left(u_{j}\left(\overrightarrow{x^{\prime}}, 0\right)-\frac{i}{\omega_{k^{\prime}}^{s^{\prime}}} \partial_{t} u_{j}\left(\overrightarrow{x^{\prime}}, 0\right)\right) e^{-i \vec{k}^{\prime} \cdot \overrightarrow{x^{\prime}}}\right] } \\
= & \frac{1}{2} \rho \sqrt{\omega_{k}^{s} \omega_{k^{\prime}}^{s^{\prime}}} \epsilon_{i}^{s} \epsilon_{j}^{s^{\prime}} \int \frac{d^{3} \vec{x}}{(2 \pi)^{3 / 2}} \int \frac{d^{3} \vec{x}^{\prime}}{(2 \pi)^{3 / 2}} e^{i \vec{k} \cdot \vec{x}-i \overrightarrow{k^{\prime}} \cdot \overrightarrow{x^{\prime}}}\left(\left[u_{i}(\vec{x}, 0),-\frac{i}{\omega_{k^{\prime}}^{s^{\prime}}} \partial_{t} u_{j}\left(\overrightarrow{x^{\prime}}, 0\right)\right]+\right. \\
& {\left.\left[\frac{i}{\omega_{k}^{s}} \partial_{t} u_{i}(\vec{x}, 0), u_{j}\left(\overrightarrow{x^{\prime}}, 0\right)\right]\right) }
\end{aligned}
$$

$$
\begin{align*}
& =\frac{1}{2} \sqrt{\omega_{k}^{s} \omega_{k^{\prime}}^{s^{\prime}}} \epsilon_{i}^{s} \epsilon_{j}^{s^{\prime}} \int \frac{d^{3} \vec{x}}{(2 \pi)^{3 / 2}} \int \frac{d^{3} \vec{x}^{\prime}}{(2 \pi)^{3 / 2}}\left(\frac{1}{\omega_{k}^{s}}+\frac{1}{\omega_{k^{\prime}}^{s^{\prime}}}\right) \delta_{i j} \delta\left(\vec{x}-\overrightarrow{x^{\prime}}\right) e^{i \vec{k} \cdot \vec{x}-i \overrightarrow{k^{\prime}} \cdot \overrightarrow{x^{\prime}}} \\
& =\delta_{s s^{\prime}} \delta\left(\vec{k}-\vec{k}^{\prime}\right) \tag{3.33}
\end{align*}
$$

We can similarly show that

$$
\begin{equation*}
\left[a_{\vec{k}, s}, a_{\vec{k}^{\prime}, s^{\prime}}\right]=\left[a_{\vec{k}, s}^{\dagger}, a_{\vec{k}^{\prime}, s^{\prime}}^{\dagger}\right]=0 \tag{3.34}
\end{equation*}
$$

We can re-write the Hamiltonian, $H$, in terms of $a_{\vec{k}, s}$ and $a_{\vec{k}, s}^{\dagger}$ by substituting (9.72) into (3.12).

$$
\begin{align*}
H=\frac{1}{2} \int \frac{d^{3} \vec{k}}{2 \rho \omega_{k}^{s}} & \left(a_{\vec{k}, s} a_{-\vec{k}, s} e^{-2 i \omega_{k}^{s} t}\left(-\rho\left(\omega_{k}^{s}\right)^{2}+\mu k^{2}+\delta_{s 1}(\mu+\lambda) k^{2}\right)\right. \\
& +a_{\vec{k}, s} a_{\vec{k}, s}^{\dagger}\left(\rho\left(\omega_{k}^{s}\right)^{2}+\mu k^{2}+\delta_{s 1}(\mu+\lambda) k^{2}\right) \\
& +a_{\overrightarrow{\vec{k}, s},}^{\dagger} a_{\vec{k}, s}\left(\rho\left(\omega_{k}^{s}\right)^{2}+\mu k^{2}+\delta_{s 1}(\mu+\lambda) k^{2}\right) \\
& \left.\quad+a_{\vec{k}, s}^{\dagger} a_{-\vec{k}, s}^{\dagger} e^{2 i \omega_{k}^{s} t}\left(-\rho\left(\omega_{k}^{s}\right)^{2}+\mu k^{2}+\delta_{s 1}(\mu+\lambda) k^{2}\right)\right) \\
= & \frac{1}{2} \int d^{3} \vec{k} \omega_{k}^{s}\left(a_{\vec{k}, s} a_{\vec{k}, s}^{\dagger}+a_{\vec{k}, s}^{\dagger} a_{\vec{k}, s}\right) \\
= & \int d^{3} \vec{k} \omega_{k}^{s}\left(a_{\vec{k}, s}^{\dagger} a_{\vec{k}, s}+\frac{1}{2} \delta(0)\right) \tag{3.35}
\end{align*}
$$

Hence, the elastic medium can be treated as a set of harmonic oscillators, one for each $\vec{k}$. There is a ground state, or vacuum state, $|0\rangle$ which satisfies:

$$
\begin{equation*}
a_{k, s}|0\rangle=0 \tag{3.36}
\end{equation*}
$$

for all $k, s$. The full set of energy eigenstates can built on the vacuum state:

$$
\begin{equation*}
\left(a_{k_{1}, s_{1}}^{\dagger}\right)^{n_{1}}\left(a_{k_{2}, s_{2}}^{\dagger}\right)^{n_{2}} \ldots\left(a_{k_{j}, s_{j}}^{\dagger}\right)^{n_{j}}|0\rangle \tag{3.37}
\end{equation*}
$$

The Hilbert space spanned by these states is called Fock space. We demand that the states of the form (3.37) are an orthogonal basis of Fock space, thereby defining the inner product. The state (3.37), which has energy

$$
\begin{equation*}
\sum_{i} n_{i} \omega_{k_{i}, s_{i}} \tag{3.38}
\end{equation*}
$$

can be thought of as a state with $n_{1}$ phonons of momentum $k_{1}$ and polarization $s_{1} ; n_{2}$ phonons of momentum $k_{2}$ and polarization $s_{2} ; \ldots ; n_{j}$ phonons of momentum $k_{j}$ and polarization $s_{j}$. The creation operator $a_{k_{i}, s_{i}}^{\dagger}$ creates a phonon of momentum $k_{i}$ and polarization $s_{i}$ while the annihilation operator $a_{k_{1}, s_{1}}$ annihilates such a phonon. At the quantum level, the normal-mode sound-wave oscillations have aquired a particle-like character; hence the name phonons.

You may have observed that the above Hamiltonian has an infinite constant. This constant is the zero-point energy of the system; it is infinite because we have taken the continuum limit in an infinite system. If we go back to our underlying ionic lattice, we will find that this energy, which is due to the zero-point motion of the ions, is finite. The sum over $k$ really terminates when $\omega_{k}$ is the Debye energy. For the most part, we will not be interested in this energy (see, however, the problem set), so we will drop it. This can be done by introducing the notion of a normal-ordered product, which will be useful later. The normal-ordered product of a set of $a_{k_{i}, s_{i}}^{\dagger}$ 's and $a_{k_{j}, s_{j}}$ 's is the product with all of the $a_{k_{i}, s_{i}}^{\dagger}$ 's to the left and all of the $a_{k_{i}, s_{i}}$ 's to the right. It is denoted by a pair of colons. For example,

$$
\begin{equation*}
: a_{k_{1}, s_{1}} a_{k_{1}, s_{1}}^{\dagger} a_{k_{2}, s_{2}}:=a_{k_{1}, s_{1}}^{\dagger} a_{k_{1}, s_{1}} a_{k_{2}, s_{2}} \tag{3.39}
\end{equation*}
$$

Since creation operators commute with one another and annihilation operators do as well, we do not need to speficy their orderings. Hence in the above example, the ordering of $a_{k_{1}, s_{1}}$ and $a_{k_{2}, s_{2}}$ above is unimportant. The normal ordered product can be defined for any free fields, i.e. for any fields which can be expanded in creation and annihilation operators with time-dependence of the form (9.72). Suppose $A$ is such an operator. Then we can always write $A=A^{(+)}+A^{(-)}$where $A^{(+)}$is the part of the expansion of $A$ which contains positive frequencies, $e^{i \omega t}$ and $A^{(-)}$is the part which contains the negative frequencies. Normal-ordering puts the $A^{(+)}$'s to the left and the $A^{(-)}$'s to the right. If we define the quantum Hamiltonian to be : $H:$, then
we eliminate the zero-point energy.
The divergent zero-point energy is the first of many ultra-violet divergences which we will encounter. They occur when we extend the upper limit of $k$-integrals to infinity. In fact, these integrals are always cutoff at some short length scale. In most of the problems which we will be discussing in this course, this cutoff is the inverse of the lattice scale. In the above example, this is the wavevector corresponding to the Debye energy. When we turn to electrons, the cutoff will be at a scale of electron volts. When field theory is applied to electrodynamics, it must be cutoff at the scale at which it becomes unified with the weak interactions, approximately 100 GeV .

### 3.5.3 Fock space for $\mathrm{He}^{4}$ atoms

We can use the same formalism to discuss a system of bosons, say $\mathrm{He}^{4}$ atoms. This is particularly convenient when the number of $\mathrm{He}^{4}$ atoms is not fixed, as for instance in the grand canonical ensemble, where the chemical potential, $\mu$, is fixed and the number of particles, $N$, is allowed to vary.

Suppose we have a $\mathrm{He}^{4}$ atom with Hamiltonian

$$
\begin{equation*}
H=\frac{P^{2}}{2 m} \tag{3.40}
\end{equation*}
$$

The energy eigenstates $|\vec{k}\rangle$ have energies and momenta

$$
\begin{align*}
H|\vec{k}\rangle & =\frac{k^{2}}{2 m}|\vec{k}\rangle \\
\vec{P}|\vec{k}\rangle & =\vec{k}|\vec{k}\rangle \tag{3.41}
\end{align*}
$$

They are orthogonal:

$$
\begin{equation*}
\left\langle\vec{k}^{\prime} \mid \vec{k}\right\rangle=\delta\left(\vec{k}-\vec{k}^{\prime}\right) \tag{3.42}
\end{equation*}
$$

If we have two particles, we have eigenstates $\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle$ with

$$
H\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle=\left(\frac{k_{1}^{2}}{2 m}+\frac{k_{2}^{2}}{2 m}\right)\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle
$$

$$
\begin{equation*}
\vec{P}\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle=\left(\vec{k}_{1}+\vec{k}_{2}\right)\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle \tag{3.43}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
\left\langle\vec{k}_{1}, \vec{k}_{2} \mid \vec{k}_{3}, \vec{k}_{3}\right\rangle=\delta\left(\vec{k}_{1}-\vec{k}_{3}\right) \delta\left(\vec{k}_{2}-\vec{k}_{4}\right)+\delta\left(\vec{k}_{1}-\vec{k}_{4}\right) \delta\left(\vec{k}_{2}-\vec{k}_{3}\right) \tag{3.44}
\end{equation*}
$$

We can continue in this way to 3-particle, 4-particle,..., etc. states. There is also a no-particle state: the vacuum, $|0\rangle$.

The Hilbert space spanned by all of these states is Fock space. We can define creation and annihilation operators $a_{k}^{\dagger}, a_{k}$ satisfying:

$$
\begin{array}{r}
{\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}^{\dagger}\right]=\delta\left(\vec{k}-\vec{k}^{\prime}\right)} \\
{\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}\right]=\left[a_{\vec{k}}^{\dagger}, a_{\vec{k}^{\prime}}^{\dagger}\right]=0} \tag{3.45}
\end{array}
$$

so that

$$
\begin{align*}
|\vec{k}\rangle= & a_{\vec{k}}^{\dagger}|0\rangle \\
\left|\vec{k}_{1}, \vec{k}_{2}\right\rangle= & a_{\vec{k}_{1}}^{\dagger} a_{\vec{k}_{1}}^{\dagger}|0\rangle \\
& \text { etc. } \tag{3.46}
\end{align*}
$$

Writing

$$
\begin{align*}
H & =\int d^{3} \vec{k} \omega_{k} a_{\vec{k}}^{\dagger} a_{\vec{k}} \\
P & =\int d^{3} \vec{k} \vec{k} a_{\vec{k}}^{\dagger} a_{\vec{k}} \tag{3.47}
\end{align*}
$$

where $\omega_{k}=k^{2} / 2 m$, we see that we recover the correct energies and momenta. From the commutation relations, we see that the correct orthonormality properties (3.42) (3.44) are also recovered.

We can now construct the fields, $\psi(x), \psi^{\dagger}(x)$ :

$$
\psi(x)=\int \frac{d^{3} \vec{k}}{(2 \pi)^{3 / 2}} a_{\vec{k}} e^{-i\left(\omega_{k} t-\vec{k} \cdot \vec{x}\right)}
$$

$$
\begin{equation*}
\psi^{\dagger}(x)=\int \frac{d^{3} \vec{k}}{(2 \pi)^{3 / 2}} a_{\vec{k}}^{\dagger} e^{i\left(\omega_{k} t-\vec{k} \cdot \vec{x}\right)} \tag{3.48}
\end{equation*}
$$

$\psi(x)$ satisfies the equation:

$$
\begin{equation*}
i \frac{\partial}{\partial t} \psi(x)=-\frac{1}{2 m} \nabla^{2} \psi(x) \tag{3.49}
\end{equation*}
$$

In other words, suppose we view the Schödinger equation as a classical wave equation - analogous to the wave equation of an elastic medium (3.3) - which can be derived from the action

$$
\begin{equation*}
S=\int d t d^{3} \vec{r} \psi^{\dagger}\left(i \frac{\partial}{\partial t}+\frac{1}{2 m} \nabla^{2}\right) \psi \tag{3.50}
\end{equation*}
$$

Then, we can second quantize this wave equation and arrive at the Fock space description.

## Chapter 4

## Perturbation Theory: Interacting Phonons

### 4.1 Higher-Order Terms in the Phonon Lagrangian

The second quantization procedure described in the previous chapter can be immediately applied to any classical field theory which has a Lagrangian which is quadratic in its basic fields. However, most systems have Lagrangians with higher-order terms. For instance, there are certainly terms in the phonon Lagrangian which we have neglected which are cubic, quartic, and higher-order in the displacement fields, $u_{i}$. An example of a phonon Lagrangian with such a term included is

$$
\begin{equation*}
S=S_{0}-\frac{g}{4!} \int d t d^{3} \vec{x}\left(\partial_{k} u_{k}\right)^{4} \tag{4.1}
\end{equation*}
$$

The Hamiltonian corresponding to (4.1) is:

$$
\begin{align*}
H & =\frac{1}{2} \int d^{3} \vec{r}\left[\frac{1}{\rho} \pi_{i}^{2}+2 \mu u_{i j} u_{i j}+\lambda u_{k k}^{2}\right]+\frac{g}{4!} \int d t d^{3} \vec{x}\left(\partial_{k} u_{k}\right)^{4} \\
& =H_{0}+H^{\prime} \tag{4.2}
\end{align*}
$$

We use this phonon Lagrangian as an illustrative example; it is not intended to be a realistic phonon Lagrangian.

Classically, the presence of such terms means that different solutions can no longer be superposed. Hence, there is no normal mode expansion, and we cannot follow the steps which we took in chapter 2 . When $g$ is small, we can, however, hope to use perturbation theory to solve this Hamiltonian. In this chapter, we develop a perturbation theory for $H^{\prime}$ using the solution of $H_{0}$ presented in chapter 2. As we will see, higher-order terms in the phonon Lagrangian lead to interactions between the phonons which cause them to scatter off each other.

In order to facilitate the construction of the perturbation theory, we will need several technical preliminaries: the interaction picture, the time-ordered product, and Wick's theorem.

### 4.2 Schrödinger, Heisenberg, and Interaction Pictures

In the Schrödinger picture, states evolve in time according to:

$$
\begin{equation*}
i \frac{\partial}{\partial t}|\psi(t)\rangle_{S}=H(t)|\psi(t)\rangle_{S} \tag{4.3}
\end{equation*}
$$

while operators are time-independent unless they have explicit time dependence. For example, if we have a particle in $1 D, p$ and $x$ do not depend on time, but we can switch on a time-dependent driving force in which case the Hamiltonian, $H(t)=$ $p^{2} / 2 m+x \cos \omega t$, is time-dependent. The time-evolution operator, $U\left(t, t^{\prime}\right)$ acts on states in the following way:

$$
\begin{equation*}
|\psi(t)\rangle_{S}=U\left(t, t^{\prime}\right)\left|\psi\left(t^{\prime}\right)\right\rangle_{S} \tag{4.4}
\end{equation*}
$$

It satisfies the equation

$$
\begin{equation*}
i \frac{\partial}{\partial t} U\left(t, t^{\prime}\right)=H(t) U\left(t, t^{\prime}\right) \tag{4.5}
\end{equation*}
$$

subject to the initial condition, $U(t, t)=1$. If $H$ is time-independent, then

$$
\begin{equation*}
U\left(t, t^{\prime}\right)=e^{-i\left(t-t^{\prime}\right) H} \tag{4.6}
\end{equation*}
$$

In the Heisenberg picture, on the other hand, states are time-independent,

$$
\begin{equation*}
|\psi(t)\rangle_{H}=|\psi(0)\rangle_{S}=|\psi(0)\rangle_{H} \tag{4.7}
\end{equation*}
$$

while operators contain all of the time-dependence. Suppose $O_{S}(t)$ is an operator in the Schrödinger picture (we have allowed for explicit time dependence as in $H(t)$ above). Then the corresponding Heisenberg picture operator is:

$$
\begin{equation*}
O_{H}(t)=U(0, t) O_{S}(t)(U(0, t))^{\dagger} \tag{4.8}
\end{equation*}
$$

Finally, we turn to the interaction picture, which we will use extensively. This picture can be defined when the Hamiltonian is of the form $H=H_{0}+H^{\prime}$ and $H_{0}$ has no explicit time-dependence. The interaction picture interpolates between the Heisenberg and Schrödinger pictures. Operators have time-dependence given by $H_{0}$ :

$$
\begin{equation*}
O_{I}(t)=e^{i t H_{0}} O_{S}(t) e^{-i t H_{0}} \tag{4.9}
\end{equation*}
$$

This includes the interaction Hamiltonian, $H^{\prime}$ which now has time-dependence due to $H_{0}$ :

$$
\begin{equation*}
H_{I}(t)=e^{i t H_{0}} H_{S}^{\prime}(t) e^{-i t H_{0}} \tag{4.10}
\end{equation*}
$$

(We will drop the prime and simply call it $H_{I}$.) The states lack this part of the time-dependence,

$$
\begin{equation*}
|\psi(t)\rangle_{I}=e^{i t H_{0}}|\psi(t)\rangle_{S} \tag{4.11}
\end{equation*}
$$

Hence, states satisfy the differential equation

$$
\begin{aligned}
i \frac{\partial}{\partial t}|\psi(t)\rangle_{I} & =i \frac{\partial}{\partial t}\left(e^{i t H_{0}}|\psi(t)\rangle_{S}\right) \\
& =e^{i t H_{0}}\left(-H_{0}+H_{S}\right)|\psi(t)\rangle_{S}
\end{aligned}
$$

$$
\begin{align*}
& =e^{i t H_{0}} H_{S}^{\prime}(t) e^{-i t H_{0}}|\psi(t)\rangle_{I} \\
& =H_{I}(t)|\psi(t)\rangle_{I} \tag{4.12}
\end{align*}
$$

We can define an Interaction picture time-evolution operator, $U_{I}\left(t, t^{\prime}\right)$, satisfying

$$
\begin{equation*}
i \frac{\partial}{\partial t} U_{I}\left(t, t^{\prime}\right)=H_{I}(t) U_{I}\left(t, t^{\prime}\right) \tag{4.13}
\end{equation*}
$$

which evolves states according to

$$
\begin{equation*}
|\psi(t)\rangle_{I}=U_{I}\left(t, t^{\prime}\right)\left|\psi\left(t^{\prime}\right)\right\rangle_{I} \tag{4.14}
\end{equation*}
$$

### 4.3 Dyson's Formula and the Time-Ordered Product

If we can find $U_{I}\left(t, t^{\prime}\right)$, then we will have solved to full Hamiltonian $H_{0}+H^{\prime}$, since we will know the time dependence of both operators and states. A formal solution was written down by Dyson:

$$
\begin{equation*}
U_{I}\left(t, t^{\prime}\right)=T\left\{e^{-i \int_{t^{\prime}}^{t}, t^{\prime \prime} H_{I}\left(t^{\prime \prime}\right)}\right\} \tag{4.15}
\end{equation*}
$$

where the time-ordered product, T , of a string of operators, $O_{1}\left(t_{1}\right) O_{2}\left(t_{2}\right) \ldots O_{n}\left(t_{n}\right)$, is their product arranged sequentially in the order of their time arguments, with operators with earlier times to the right of operators with later times:

$$
\begin{align*}
T\left\{O_{1}\left(t_{1}\right) O_{2}\left(t_{2}\right) \ldots O_{n}\left(t_{n}\right)\right\}= & O_{i_{1}}\left(t_{i_{1}}\right) O_{i_{2}}\left(t_{i_{2}}\right) \ldots O_{i_{n}}\left(t_{i_{n}}\right) \\
& \text { if } t_{i_{1}}>t_{i_{2}}>\ldots>t_{i_{n}} \tag{4.16}
\end{align*}
$$

There is some ambiguity if $t_{i}=t_{j}$ and $O\left(t_{i}\right)$ and $O\left(t_{j}\right)$ do not commute. In (4.15), however, all of the $O_{i}$ 's are $H_{I}$, so we do not need to worry about this.

To see that it satisfies the differential equation (4.13), observe that all operators commute under the time-ordering symbol, so we can take the derivative naively:

$$
\begin{equation*}
i \frac{\partial}{\partial t} T\left\{e^{-i \int_{t^{\prime}}^{t} d t^{\prime \prime} H_{I}\left(t^{\prime \prime}\right)}\right\}=T\left\{H_{I}(t) e^{-i \int_{t^{\prime}}^{t} d t^{\prime \prime} H_{I}\left(t^{\prime \prime}\right)}\right\} \tag{4.17}
\end{equation*}
$$

Since $t$ is the upper limit of integration, it is greater than or equal to any other $t^{\prime \prime}$ which appears under the time-ordering symbol. Hence, we can pull it out to the left:

$$
\begin{equation*}
i \frac{\partial}{\partial t} T\left\{e^{-i \int_{t^{\prime}}^{t}, t t^{\prime \prime} H_{I}\left(t^{\prime \prime}\right)}\right\}=H_{I}(t) T\left\{e^{-i \int_{t^{\prime}}^{t} d t^{\prime \prime} H_{I}\left(t^{\prime \prime}\right)}\right\} \tag{4.18}
\end{equation*}
$$

With Dyson's formula in hand, we can - at least in principle - compute transition amplitudes. For example, let us suppose that we have a system which is in its ground state, $|0\rangle$. Suppose we perform a neutron scattering experiment in which a neutron is fired into the system with momentum $\vec{P}$ at time $t^{\prime}$ and then interacts with our system according to $H_{I}$. The probability (which is the square of the amplitude) for the system to undergo a transition to an excited state $\langle 1|$ so that the neutron is detected with momentum $\vec{P}^{\prime}$ at time $t$ is:

$$
\begin{equation*}
\left.\left|\left\langle 1 ; \vec{P}^{\prime}\right| U_{I}\left(t, t^{\prime}\right)\right| 0 ; \vec{P}\right\rangle\left.\right|^{2} \tag{4.19}
\end{equation*}
$$

Of course, we can rarely evaluate $U_{I}\left(t, t^{\prime}\right)$ exactly, so we must often expand the exponential. The first-order term in the expansion of the exponential is:

$$
\begin{equation*}
-i \int_{t^{\prime}}^{t} d t_{1} H_{I}\left(t_{1}\right) \tag{4.20}
\end{equation*}
$$

Hence, if we prepare an initial state $|i\rangle$ at $t^{\prime}=-\infty$, we measure the system in a final state $\langle f|$ at $t=\infty$ with amplitude:

$$
\begin{equation*}
\langle f| U_{I}(\infty,-\infty)|i\rangle=-i\langle f| \int_{-\infty}^{\infty} d t H_{I}(t)|i\rangle \tag{4.21}
\end{equation*}
$$

Squaring this, we recover Fermi's Golden Rule. There is a slight subtlety in that the $t$ integral leads to an amplitude proportional to $\delta\left(E_{i}-E_{f}\right)$. This appears to lead to a transition probability which is proportional to the square of a $\delta$-function. We understand, however, that this is a result of taking the limits of integration to infinity carelessly: the square of the $\delta$-function is actually a single $\delta$-function multiplied by the difference between the initial and final times. Hence, this implies that the transition
rate is:

$$
\begin{equation*}
\left.\frac{d P}{d t}=\left|\langle f| \int_{-\infty}^{\infty} d t H_{I}(t)\right| i\right\rangle\left.\right|^{2} \tag{4.22}
\end{equation*}
$$

with one $\delta$-function dropped.
To get a sense of the meaning of the $T$ symbol, it is instructive to consider the second-order term in the expansion of the exponential:

$$
\begin{equation*}
\frac{(-i)^{2}}{2!} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t} d t_{2} T\left(H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right)\right)=(-i)^{2} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t_{1}} d t_{2} H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right) \tag{4.23}
\end{equation*}
$$

### 4.4 Wick's Theorem

We would like to evaluate the terms of the perturbation series obtained by expanding Dyson's formula (4.15). To do this, we need to compute time-ordered products $T\left\{H_{I} H_{I} \ldots H_{I}\right\}$. This can be done efficiently if we can reduce the time-ordered products to normal-ordered products (which enjoy the relative simplicity of annihilating the vacuum).

To do this, we define the notion of the contraction of free fields (remember that, in the interaction picture, the operators are free and the states have complicated time-dependence), which we will denote by an overbrace:

$$
\begin{equation*}
\overbrace{A\left(t_{1}\right) B\left(t_{2}\right)}=T\left(A\left(t_{1}\right) B\left(t_{2}\right)\right)-: A\left(t_{1}\right) B\left(t_{2}\right): \tag{4.24}
\end{equation*}
$$

Dividing $A$ and $B$ into their positive- and negative-frequency parts, $A^{( \pm)}, B^{( \pm)}$, we see that:

$$
\begin{equation*}
\overbrace{A\left(t_{1}\right) B\left(t_{2}\right)}=\left[A^{(-)}, B^{(+)}\right] \tag{4.25}
\end{equation*}
$$

if $t_{1}>t_{2}$ and

$$
\begin{equation*}
\overbrace{A\left(t_{1}\right) B\left(t_{2}\right)}=\left[B^{(-)}, A^{(+)}\right] \tag{4.26}
\end{equation*}
$$

if $t_{1}<t_{2}$. This is a $c$-number (i.e. it is an ordinary number which commutes with everything) since $\left[a, a^{\dagger}\right]=1$. Hence, it is equal to its vacuum expectation value:

$$
\overbrace{A\left(t_{1}\right) B\left(t_{2}\right)}=\langle 0| \overbrace{A\left(t_{1}\right) B\left(t_{2}\right)}|0\rangle
$$

$$
\begin{align*}
& =\langle 0| T\left(A\left(t_{1}\right) B\left(t_{2}\right)\right)|0\rangle-\langle 0|: A\left(t_{1}\right) B\left(t_{2}\right):|0\rangle \\
& =\langle 0| T\left(A\left(t_{1}\right) B\left(t_{2}\right)\right)|0\rangle \tag{4.27}
\end{align*}
$$

The following theorem, due to Gian-Carlo Wick, uses the contraction to reduce time-ordered products to normal-ordered products:

$$
\begin{align*}
T\left\{u_{1} u_{2} \ldots u_{n}\right\}= & : u_{1} u_{2} \ldots u_{n}: \\
+ & : \overbrace{u_{1}}^{u_{2}} \ldots u_{n}:+ \text { other terms with one contraction } \\
+ & : \overbrace{u_{1} u_{2}} \overbrace{u_{3} u_{4}} \ldots u_{n}:+ \text { other terms with two contractions } \\
& \vdots \\
& : \overbrace{u_{1} u_{2}} \ldots \overbrace{u_{n-1} u_{n}}: \\
& + \text { other such terms if } n \text { is even } \\
+ & : \overbrace{u_{1} u_{2}} \ldots \overbrace{u_{n-2} u_{n-1}} u_{n}:  \tag{4.28}\\
& + \text { other such terms if } n \text { is odd }
\end{align*}
$$

The right-hand-side is normal-ordered. It contains all possible terms with all possible contractions appear, each with coefficient 1 . The proof proceeds by induction. Let us call the right-hand-side $w\left(u_{1} u_{2} \ldots u_{n}\right)$. The equality of the left and right-hand sides is trivial for $n=1,2$. Suppose that it is true for time-ordered products of $n-1$ fields. Let us further suppose, without loss of generality, that $t_{1}$ is the latest time. Then,

$$
\begin{align*}
T\left\{u_{1} u_{2} \ldots u_{n}\right\} & =u_{1} T\left\{u_{2} \ldots u_{n}\right\} \\
& =u_{1} w\left(u_{2}, \ldots, u_{n}\right) \\
& =u_{1}^{(+)} w\left(u_{2}, \ldots, u_{n}\right)+u_{1}^{(-)} w\left(u_{2}, \ldots, u_{n}\right) \\
& =u_{1}^{(+)} w\left(u_{2}, \ldots, u_{n}\right)+w\left(u_{2}, \ldots, u_{n}\right) u_{1}^{(-)}+\left[u_{1}^{(-)}, w\right] \\
& =w\left(u_{1}, u_{2}, \ldots, u_{n}\right) \tag{4.29}
\end{align*}
$$

The equality between the last two lines follows from the fact that the final expression is normal ordered and contains all possible contractions: the first two terms contain all contractions in which $u_{1}$ is not contracted while the third term contains all contractions in which $u_{1}$ is contracted.

A concise way of writing down Wick's theorem is the following:

$$
\begin{equation*}
T\left\{u_{1} u_{2} \ldots u_{n}\right\}=: e^{\frac{1}{2} \sum_{i, j=1}^{n} \overbrace{u_{i} u_{j}} \frac{\partial}{\partial u_{i}} \frac{\partial}{\partial u_{j}}} u_{1} u_{2} \ldots u_{n}: \tag{4.30}
\end{equation*}
$$

### 4.5 The Phonon Propagator

As a result of Wick's theorem, the contraction of two phonon fields, $\overbrace{u_{i} u_{j}}$, is the basic building block of perturbation theory. Matrix elements of the time-evolution operator will be given by integrals of products of contractions. The contraction $\overbrace{u_{i} u_{j}}$ is also called the phonon propagator. In the problem set, you will compute the propagator in two different ways. First, you will calculate it directly from:

$$
\begin{equation*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle=\left[u_{i}^{(+)}, u_{j}^{(-)}\right] \tag{4.31}
\end{equation*}
$$

You will also calculate it by noting that

$$
\begin{equation*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle=\theta\left(t_{1}-t_{2}\right)\left\langle u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right\rangle+\theta\left(t_{2}-t_{1}\right)\left\langle u_{j}\left(\vec{x}_{2}, t_{2}\right) u_{i}\left(\vec{x}_{1}, t_{1}\right)\right\rangle \tag{4.32}
\end{equation*}
$$

and acting on this with (3.3) to obtain,

$$
\begin{align*}
& \left(\rho \delta_{i k} \partial_{t}^{2}-(\mu+\lambda) \partial_{i} \partial_{k}-\mu \delta_{i k} \partial_{l} \partial_{l}\right)\left\langle T\left(u_{k}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle \\
& =-i \delta\left(\vec{x}_{1}-\vec{x}_{2}\right) \delta\left(t_{1}-t_{2}\right) \delta_{i j} \tag{4.33}
\end{align*}
$$

By Fourier transforming this equation, we find:

$$
\begin{equation*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle=\frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega^{2}-\left(\omega_{p}^{s}\right)^{2}} \tag{4.34}
\end{equation*}
$$

Here, we have used $\epsilon_{i}^{s} \epsilon_{j}^{s}=\delta_{i j}$. However, the singularities at $\omega^{2}=\left(\omega_{p}^{s}\right)^{2}$ are unresolved by this expression. As you will show in the problem set, the correct expression is:

$$
\begin{equation*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle=\frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{2}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega^{2}-\left(\omega_{p}^{s}\right)^{2}+i \delta} \tag{4.35}
\end{equation*}
$$

Since $\epsilon_{i}^{1} k_{i}=k$ while $\epsilon_{i}^{2,3} k_{i}=0$,

$$
\begin{align*}
\epsilon_{i}^{1} \epsilon_{j}^{1} & =\frac{k_{i} k_{j}}{k^{2}} \\
\epsilon_{i}^{2} \epsilon_{j}^{2}+\epsilon_{i}^{3} \epsilon_{j}^{3} & =\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}} \tag{4.36}
\end{align*}
$$

Hence, using $\omega_{p}^{1}=v_{l} p, \omega_{p}^{2,3}=v_{s} p$, we can rewrite the phonon propagator as:

$$
\begin{align*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle= & \frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i k_{i} k_{j} / k^{2}}{\omega^{2}-v_{l}^{2} p^{2}+i \delta} \\
& +\frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i\left(\delta_{i j}-k_{i} k_{j} / k^{2}\right)}{\omega^{2}-v_{t}^{2} p^{2}+i \delta} \tag{4.37}
\end{align*}
$$

For some purposes, it will be more convenient to consider a slightly different phonon field,

$$
\begin{equation*}
\varphi_{i}(\vec{r}, t)=\int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \sum_{s} \frac{1}{\sqrt{\rho}} \epsilon_{i}^{s}\left(a_{\vec{k}, s} e^{i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}+a_{\vec{k}, s}^{\dagger} e^{-i\left(\vec{k} \cdot \vec{r}-\omega_{k}^{s} t\right)}\right) \tag{4.38}
\end{equation*}
$$

The difference with $u_{i}$ is the missing $1 / \sqrt{2 \omega_{k}^{S}}$. This field has propagator:

$$
\begin{align*}
\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle & =\frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{2 i \omega_{p}^{s} \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega^{2}-\left(\omega_{p}^{s}\right)^{2}+i \delta} \\
& =\frac{1}{\rho} \int d^{3} \vec{p} d \omega e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)}\left(\frac{i \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega-\omega_{p}^{s}+i \delta}+\frac{i \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega+\omega_{p}^{s}-i \delta}\right) \tag{4.39}
\end{align*}
$$

### 4.6 Perturbation Theory in the Interaction Pic-

 tureWe are now in position to start looking at perturbation theory. Since transverse phonons are unaffected by the interaction (4.1), we only need to discuss longitudinal phonons. Consider the second-order contribution in our theory of phonons with a quartic anharmonicity (4.1),

$$
U(-\infty, \infty)=\frac{(-i)^{2}}{2!} \int_{-\infty}^{\infty} d t_{1} \int_{-\infty}^{\infty} d t_{2} T\left(H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right)\right)
$$

$$
\begin{equation*}
=\frac{(-i g / 4!)^{2}}{2!} \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2} T\left(\left(\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right)\right)^{4}\left(\partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right)\right)^{4}\right) \tag{4.40}
\end{equation*}
$$

When we apply Wick's theorem, we get such terms as:

$$
\begin{equation*}
\frac{(-i g / 4!)^{2}}{2!} \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2}: \partial_{k} u_{k} \partial_{k} u_{k} \overbrace{\partial_{k} u_{k} \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}} \partial_{k} u_{k}} \partial_{k} u_{k} \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right): \tag{4.41}
\end{equation*}
$$

This term will contribute to such physical processes as the scattering between two longitudinal phonons. If we look at

$$
\begin{align*}
& \left\langle\vec{k}_{3}, l ; \vec{k}_{4}, l ; t=\infty\right| U(-\infty, \infty)\left|\vec{k}_{1}, l ; \vec{k}_{2}, l ; t=-\infty\right\rangle= \\
& \ldots+\frac{(-i g / 4!)^{2}}{2!} \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2} \overbrace{\partial_{k} u_{k} \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}} \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right)} \times \\
& \quad\left\langle\vec{k}_{3}, l ; \vec{k}_{4}, l ; t=\infty\right|: \partial_{k} u_{k} \partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k} \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right):\left|\vec{k}_{1}, l ; \vec{k}_{2}, l ; t=-\infty\right\rangle \\
& +\ldots \tag{4.42}
\end{align*}
$$

this will give a non-vanishing contribution since two of the uncontracted $u_{i}$ 's can annihilate the phonons in the initial state and the other two can create the phonons in the final state. Let's suppose that the incoming phonons are annihilated by $\partial_{k} u_{k}^{(-)} \partial_{k} u_{k}^{(-)}$ at $\left(\vec{x}_{1}, t_{1}\right)$ and the outgoing phonons are created by the $\partial_{k} u_{k}^{(+)} \partial_{k} u_{k}^{(+)}$at $\left(\vec{x}_{2}, t_{2}\right)$. Since

$$
\begin{equation*}
\partial_{k} u_{k}^{(-)} \partial_{k} u_{k}^{(-)}\left(\vec{x}_{1}, t_{1}\right)\left|\vec{k}_{1}, l ; \vec{k}_{2}, l ; t=-\infty\right\rangle=-\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left(e^{i\left(\vec{k}_{1} \cdot \vec{x}_{1}-\omega_{k_{1}}^{l} t_{1}\right)} e^{i\left(\vec{k}_{2} \cdot \vec{x}_{1}-\omega_{k_{2}}^{l} t_{1}\right)}\right)|0\rangle \tag{4.43}
\end{equation*}
$$

we obtain a contribution to (4.42) of the form:

$$
\begin{align*}
\frac{(-i g / 4!)^{2}}{2!} & \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right. \\
& \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right)} \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right)\}}^{\left(e^{\left.i\left(\vec{k}_{1}+\vec{k}_{2}\right) \cdot \vec{x}_{1}-\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}\right) t_{1}\right)}\right)\left(e^{-i\left(\left(\vec{k}_{3}+\vec{k}_{4}\right) \cdot \vec{x}_{2}-\left(\omega_{k_{3}}^{l}+\omega_{k_{4}}^{l}\right) t_{2}\right)}\right) \times} \text {, } \tag{4.44}
\end{align*}
$$

Substituting the expression for $\overbrace{u_{k}\left(\vec{x}_{1}, t_{1}\right) u_{k}\left(\vec{x}_{2}, t_{2}\right)}$, we find:

$$
\frac{(-i g / 4!)^{2}}{2!} \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2} \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d \omega_{1}}{2 \pi} \frac{d^{3} \vec{p}_{2}}{(2 \pi)^{3}} \frac{d \omega_{2}}{2 \pi}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right.
$$

$$
\begin{align*}
& \left(e^{i\left(\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}-\vec{p}_{2}\right) \cdot \vec{x}_{1}-\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{1}-\omega_{2}\right) t_{1}\right)}\right)\left(e^{-i\left(\left(\vec{k}_{3}+\vec{k}_{4}-\vec{p}_{1}-\vec{p}_{2}\right) \cdot \vec{x}_{2}-\left(\omega_{k_{3}}^{l}+\omega_{k_{4}}^{l}-\omega_{1}-\omega_{2}\right) t_{2}\right)}\right) \times \\
& \left.\left.\frac{1}{\rho}\left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-\left(\omega_{p_{1}}^{l}\right)^{2}+i \delta}\right|^{2} \frac{i}{\omega_{2}^{2}-\left(\omega_{p_{2}}^{l}\right)^{2}+i \delta}\right\} \tag{4.45}
\end{align*}
$$

The $x$ and $t$ integrals give $\delta$ functions which enforce momentum- and energy-conservation.

$$
\begin{align*}
\frac{(-i g / 4!)^{2}}{2!} & \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d \omega_{1}}{2 \pi} \frac{d^{3} \vec{p}_{2}}{(2 \pi)^{3}} \frac{d \omega_{2}}{2 \pi}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right. \\
& (2 \pi)^{3} \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}-\vec{p}_{2}\right) 2 \pi \delta\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{1}-\omega_{2}\right) \\
& (2 \pi)^{3} \delta\left(\vec{k}_{3}+\vec{k}_{4}-\vec{p}_{1}-\vec{p}_{2}\right) 2 \pi \delta\left(\omega_{k_{3}}^{l}+\omega_{k_{4}}^{l}-\omega_{1}-\omega_{2}\right) \\
& \left.\frac{1}{\rho^{2}}\left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-\left(\omega_{p_{1}}^{l}\right)^{2}+i \delta}\left|\vec{p}_{2}\right|^{2} \frac{i}{\omega_{2}^{2}-\left(\omega_{p_{2}}^{l}\right)^{2}+i \delta}\right\} \tag{4.46}
\end{align*}
$$

which, finally, gives us

$$
\begin{align*}
\frac{(-i g / 4!)^{2}}{2!} & \frac{1}{\rho^{2}} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d \omega_{1}}{2 \pi}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right. \\
& \left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-\left(\omega_{p_{1}}^{l}\right)^{2}+i \delta}\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2} \frac{i}{\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{1}\right)^{2}-\left(\omega_{k_{1}+k_{2}-p_{1}}^{l}\right)^{2}+i \delta} \\
& \left.(2 \pi)^{3} \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) 2 \pi \delta\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{k_{3}}^{l}-\omega_{k_{4}}^{l}\right)\right\} \tag{4.47}
\end{align*}
$$

There are actually several ways in which an identical contribution can be obtained. By an identical contribution, we mean one in which there are two contractions of the form $\overbrace{u_{k}\left(\vec{x}_{1}, t_{1}\right) u_{k}\left(\vec{x}_{2}, t_{2}\right)}$; the incoming phonons are annihilated at the same point (which can be either $\left(\vec{x}_{1}, t_{1}\right)$ or $\left(\vec{x}_{2}, t_{2}\right)$ since these are dummy variables which are integrated over); and the outgoing phonons are created at the same point. The incoming phonons are annihilated by the $u_{i}$ 's at $\left(\vec{x}_{1}, t_{1}\right)$ and the outgoing phonons are annihilated by the $u_{i}$ 's at $\left(\vec{x}_{2}, t_{2}\right)$, which can be done in $(4 \cdot 3)(4 \cdot 3)$ ways. There are 2 ways in which we can choose how the remaining $u_{i}$ 's at $\left(\vec{x}_{1}, t_{1}\right)$ are contracted with the remaining $u_{i}$ 's at $\left(\vec{x}_{2}, t_{2}\right)$, giving us a multiplicity of $(4 \cdot 3)(4 \cdot 3) 2$. It is now clear why we included a factor of $1 / 4$ ! in our definition of $g$ : the above multiplicity almost cancels the two factors of $1 / 4$ !. Only a factor of $1 / 2$ remains. If we permute ( $\vec{x}_{1}, t_{1}$ ) and $\left(\vec{x}_{2}, t_{2}\right)$, then the incoming phonons are annihilated by the $u_{i}^{\prime}$ 's at $\left(\vec{x}_{2}, t_{2}\right)$ and
the outgoing phonons are annihilated by the $u_{i}^{\prime}$ 's at $\left(\vec{x}_{1}, t_{1}\right)$. This gives an identical contribution, thereby cancelling the $1 / 2$ ! which we get at second-order. Hence, the sum of all such contributions is:

$$
\begin{align*}
\frac{(-i g)^{2}}{2} & \frac{1}{\rho^{2}} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d \omega_{1}}{2 \pi}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right. \\
& \left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-\left(\omega_{p_{1}}^{l}\right)^{2}+i \delta}\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2} \frac{i}{\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{1}\right)^{2}-\left(\omega_{k_{1}+k_{2}-p_{1}}^{l}\right)^{2}+i \delta} \\
& \left.(2 \pi)^{3} \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) 2 \pi \delta\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{k_{3}}^{l}-\omega_{k_{4}}^{l}\right)\right\} \tag{4.48}
\end{align*}
$$

There are, of course, other, distinct second-order contributions to the two phonon $\rightarrow$ two phonon transition amplitude which result, say, by contracting fields at the same point or by annihilating the incoming phonons at different points. Consider the latter contributions. Ther is a contirbution of the form:

$$
\begin{align*}
\frac{(-i g)^{2}}{2} & \frac{1}{\rho^{2}} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d \omega_{1}}{2 \pi}\left\{\left|\vec{k}_{1}\right|\left|\vec{k}_{2}\right|\left|\vec{k}_{3}\right|\left|\vec{k}_{4}\right| \times\right. \\
& \left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-\left(\omega_{p_{1}}^{l}\right)^{2}+i \delta}\left|\vec{k}_{1}-\vec{k}_{3}-\vec{p}_{1}\right|^{2} \frac{i}{\left(\omega_{k_{1}}^{l}-\omega_{k_{3}}^{l}-\omega_{1}\right)^{2}-\left(\omega_{k_{1}-k_{3}-p_{1}}^{l}\right)^{2}+i \delta} \\
& \left.(2 \pi)^{3} \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) 2 \pi \delta\left(\omega_{k_{1}}^{l}+\omega_{k_{2}}^{l}-\omega_{k_{3}}^{l}-\omega_{k_{4}}^{l}\right)\right\} \tag{4.49}
\end{align*}
$$

and one with $\vec{k}_{3} \rightarrow \vec{k}_{4}$.
The cancellation which we obtained by permuting the different $\left(\vec{x}_{i}, t_{i}\right)$ 's does not always occur. For instance, the following contraction at second-order makes a contribution to the amplitude for the vacuum at $t=-\infty$ to go into the vacuum at $t=\infty$ :

$$
\begin{align*}
& \langle 0 ; t=\infty| U(-\infty, \infty)|0 ; t=-\infty\rangle= \\
& \ldots+\overbrace{\partial_{k} u_{k} \partial_{k} u_{k}}^{\frac{(-i g / 4!)^{2}}{2!}} \overbrace{\partial_{k} u_{k} \partial_{k} u_{k}}^{\int}\langle 0 ; t=\infty \mid 0 ; t=-\infty\rangle \\
& +\ldots
\end{align*}
$$

We have written this term with contracted fields adjacent in order to avoid clutter. There are no distinct permutations, so there is nothing to cancel the $1 / 2$ !. In addition, there are only 4 ! ways to do the contractions, so there is an uncancelled factor of $1 / 4$ ! as well, and hence, an overall factor of $1 / 2!4!$. Consider, for a moment, how this works at $n^{\text {th }}$ order. There will be a factor of $n!$. If this is incompletely cancelled by permutations of the $\left(\vec{x}_{i}, t_{i}\right)$ 's, there will be a factor, $1 / S$ (in the above, $S=2$ ). In the next chapter, we will see that the symmetry factor, $S$, is related to the symmetries of Feynman diagrams. In addition, there will be factors arising from the incomplete cancellation of the $(1 / 4!)^{n}$. In the above, this additional factor is $1 / 4!$.

Again, there are other second-order contributions to the vacuum-to-vacuum amplitude which result from contracting fields at the same point, but they will give a different contribution which is different in form from the one above. One such is the following:

$$
\begin{align*}
& \langle 0 ; t=\infty| U(-\infty, \infty)|0 ; t=-\infty\rangle= \\
& \ldots+\frac{(-i g / 4!)^{2}}{2!} \int d^{3} \vec{x}_{1} d t_{1} d^{3} \vec{x}_{2} d t_{2} \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right)} \overbrace{\partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right) \partial_{k} u_{k}\left(\vec{x}_{2}, t_{2}\right)} \times \\
& \quad \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}}\left(\vec{x}_{2}, t_{2}\right) \overbrace{\partial_{k} u_{k}\left(\vec{x}_{1}, t_{1}\right) \partial_{k} u_{k}}\left(\vec{x}_{2}, t_{2}\right)\langle 0 ; t=\infty \mid 0 ; t=-\infty\rangle \\
& +\ldots \tag{4.51}
\end{align*}
$$

There are $4 \cdot 3 / 2$ ways of choosing the two fields at $\left(\vec{x}_{1}, t_{1}\right)$ which are contracted and $4 \cdot 3 / 2$ ways of choosing the two fields at $\left(\vec{x}_{2}, t_{2}\right)$ which are contracted. Finally, there are 2 ways of contracting the remaining fields at $\left(\vec{x}_{1}, t_{1}\right)$ with those at $\left(\vec{x}_{2}, t_{2}\right)$. This multiplicity incompletely cancels the $1 / 2!(4!)^{2}$.

As another example, consider


This contributes to the amplitude for processes in which both the initial and final
states contain one longitudinal phonon. There are 4.4 ways of choosing the $u_{i}$ 's which create the incoming phonon at $\left(\vec{x}_{1}, t_{1}\right)$ and annihilate the outgoing phonon at $\left(\vec{x}_{2}, t_{2}\right)$. There are 3 ! ways of contracting the remaining $u_{i}$ 's. Finally, $\left(\vec{x}_{1}, t_{1}\right)$ and $\left(\vec{x}_{2}, t_{2}\right)$ can be permuted. This gives an overall factor of $4 \cdot 4 \cdot 3!\cdot 2$, which incompletely cancels the $(1 / 2!) \cdot(1 / 4!)^{2}$, leaving $1 / 3$ !.

## Chapter 5

## Feynman Diagrams and Green Functions

### 5.1 Feynman Diagrams

Feynman introduced a diagrammatic notation which will help us systematically enumerate all of the perturbative contributions which we generate using Wick's theorem. This diagrammatic notation will have the added benefit of having a simple physical interpretation which will guide our intuition about physical processes.

Suppose we want to construct a matrix element at $n^{\text {th }}$ order in perturbation theory. We draw a diagram containing $n$ vertices with 4 lines emanating from each vertex. Each such vertex represents a factor of $\left(\partial_{k} u_{k}\right)^{4}$. The lines emanating from the vertices can be connected. Each such connection represents a contraction. We will call such a line an internal line. The remaining (uncontracted) lines - external lines - represent incoming and outgoing phonons. We will adopt the convention that incoming phonon lines enter at the left of the diagram while outgoing phonon lines exit at the right of the diagram.

The first contribution which we considered in chapter 3 (4.47) can be represented as:


Figure 5.1: The diagram corresponding to (4.47).
Given such a diagram - a Feynman diagram - you can immediately reconstruct the expression which it represents according to the following rules:

- Assign a directed momentum and energy to each line. For external lines, the momentum is directed into or out of the diagram for, respectively, incoming and outgoing phonons.
- For each external line with momentum $\vec{k}$, write $|\vec{k}|$.
- For each internal line with momentum and energy $\vec{p}, \omega$ write:

$$
\frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi}|\vec{p}|^{2} \frac{i}{\omega^{2}-v_{l}^{2} p^{2}+i \delta}
$$

- For each vertex with momenta, energies $\left(\vec{p}_{1}, \omega_{1}\right), \ldots,\left(\vec{p}_{4}, \omega_{4}\right)$ directed into the vertex, write:

$$
g(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right) 2 \pi \delta\left(\omega_{1}+\omega_{2}+\omega_{3}+\omega_{4}\right)
$$

- Imagine labelling the vertices $1,2, \ldots, n$. Vertex $i$ will be connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices $i, i$ is still connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. If $S$ is the number of permutations which leave the diagram invariant, we assign a factor $1 / S$ to the diagram.
- If two vertices are connected by $l$ lines, we assign a factor $1 / l$ ! to the diagram.

You can verify that by applying these rules to figure (5.1) we recover (4.47).
For the particular interaction we have chosen, we can ignore the transverse phonons - since they don't interact - and consider only the longitudinal phonons. If we were to consider a model in which both longitudinal and transverse phonons interact, our Feynman diagrams would have to have internal and external indices corresponding to the vector indices of the fields $u_{i}, u_{j}$, etc., and our Feynman rules would have to tell us how to contract or route these indices.

In figure (5.2) we display all of the connected diagrams which appear to $O\left(g^{2}\right)$ in the theory given by (4.1). In the problem set, you will write down expressions for them.

The Feynman diagram representation for transition amplitudes suggests a beautiful visualization of perturbative processes. External lines correspond to 'real phonons' or simply phonons, while internal lines correspond to 'virtual phonons'. For the diagram of figure 5.1, we say that the incoming phonons with momenta $\vec{k}_{1}, \vec{k}_{2}$ interact at $x_{1}$, propagate as virtual phonons with momenta $\vec{p}_{1}, \vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}$, and finally interact again at $x_{2}$, thereby scattering into the outgoing phonons with momenta $\vec{k}_{3}, \vec{k}_{4}$. For the first diagram of figure 5.2 b , we say that the incoming phonons with momenta $\vec{k}_{1}, \vec{k}_{2}$ exchange a pair of virtual phonons, thereby scattering into $\vec{k}_{3}, \vec{k}_{4}$. External lines correspond to initial or final states with phonons of momentum, energy $(\vec{p}, \omega)$. These satisfy $\omega^{2}=\left(\omega_{p}^{l}\right)^{2}$. Such a phonon is said to be 'on-shell'. Virtual phonons need not be 'on-shell'. Indeed, the phonon propagator diverges if a virtual phonon is on-shell, thereby signalling a resonance.
(a)

(b)


Figure 5.2: All connected Feynman diagrams for the theory (4.1) to $O\left(g^{2}\right)$. In (a), we have the diagrams of order $g$. In (b), we have the diagrams of order $g^{2}$.

### 5.2 Loop Integrals

Suppose we have a Feynman diagram with $E$ external lines, $I$ internal lines, and $V$ vertices. Suppose, further, that this diagram has $L$ loops (e.g. the first diagram in figure 5.2 has one loop, while the third, fourth, and fifth have two loops. The second has no loops.). Then, let's imagine connecting all of the external lines at a single point so that the Feynman diagram defines a polyhedron with $E+I$ edges; $V+1$ vertices - the extra vertex being the one at which the external lines are connected; and $L+E$ faces - with $E$ faces formed as a result of connecting the external lines. According to Euler's fomula,

$$
\begin{equation*}
(\# \text { faces })+(\# \text { vertices })-(\# \text { edges })=2 \tag{5.1}
\end{equation*}
$$

or,

$$
\begin{equation*}
L=I-V+1 \tag{5.2}
\end{equation*}
$$

The number of loops is given by the number of internal lines - i.e. the number of propagators, each coming with an integral - minus the number of vertices, each coming with momentum and energy-conserving $\delta$-functions, plus 1 for the overall $\delta$-functions satisfied by the external momenta, energies. In short, there are as many $(\vec{p}, \omega)$ pairs to be integrated over as there are loops. A diagram with no loops has as many $\delta$ functions as integrals, so the integrals can all be evaluated trivially, and there are no remaining integrals to be evaluated. Such a diagram is said to be a tree level diagram. The tree-level diagrams are indicated in figure 5.3 These diagrams can be evaluated without doing any integrals. Note that most of these are not connected diagrams. In order to evaluate a one-loop diagram, we need to do one $\int d \omega d^{3} \vec{p}$ integral; to evaluate a two-loop diagram we need to do two such integrals, i.e. $\iint d \omega_{1} d^{3} \vec{p}_{1} d \omega_{2} d^{3} \vec{p}_{2}$; and so on. The expansion in loops is actually an expansion in powers of Planck's constant, since, as you will show in the problem set each propagator comes with a factor of
(a)
(b)

(c) $\quad+\ldots$

Figure 5.3: The tree-level Feynman diagrams of the theory (4.1).
$\hbar$ and each vertex comes with a factor of $1 / \hbar$. An $L$-loop diagram comes with a coefficient of $\hbar^{L-1}$.

Turning now to the evaluation of multi-loop diagrams, we find the following trick (due to - you guessed it - Feynman) very useful. When we integrate the momenta in closed loops, we often encounter integrals of products of propagators. These are more easily evaluated if we combine the denominators of the propagators using the following formula:

$$
\begin{equation*}
\int_{0}^{1} d x \frac{1}{[a x+b(1-x)]^{2}}=\frac{1}{a b} \tag{5.3}
\end{equation*}
$$

For the more general case of the product of several propagators, which can occur in higher orders of perturbation theory, we will use the following formula:

$$
\begin{equation*}
\frac{\Gamma(\alpha)}{A^{\alpha}}=\int_{0}^{\infty} d t t^{\alpha-1} e^{-A t} \tag{5.4}
\end{equation*}
$$

Using this formula, we can write:

$$
\begin{equation*}
\frac{1}{\prod_{j} A_{j}^{\alpha_{j}}}=\prod_{j} \frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{0}^{\infty} d t_{j} t_{j}^{\alpha_{j}-1} e^{-A_{j} t_{j}}\left[\int_{0}^{\infty} d s \delta\left(s-\sum_{j} t_{j}\right)\right] \tag{5.5}
\end{equation*}
$$

Here, the integral in brackets is equal to 1 . Changing variables from $t_{i}$ to $x_{i}$ according to $t_{j}=s x_{j}$, we have:

$$
\begin{align*}
\frac{1}{\prod_{j} A_{j}^{\alpha_{j}}} & =\int_{0}^{\infty} d s \prod_{j} \frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{0}^{\infty} d x_{j} s^{\alpha_{j}} x_{j}^{\alpha_{j}-1} e^{-s A_{j} x_{j}} \frac{1}{s} \delta\left(1-\sum_{j} x_{j}\right) \\
& =\prod_{j} \frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{0}^{1} d x_{j} x_{j}^{\alpha_{j}-1} \int_{0}^{\infty} d s s^{\sum_{j} \alpha_{j}-1} e^{-s \sum_{j} A_{j} x_{j}} \delta\left(1-\sum_{j} x_{j}\right) \\
& =\frac{\Gamma\left(\sum_{j} \alpha_{j}\right)}{\prod_{j} \Gamma\left(\alpha_{j}\right)} \int_{0}^{1} d x_{1} \ldots d x_{n} \delta\left(1-\sum_{j} x_{j}\right) \frac{\prod_{j} x^{\alpha_{j}-1}}{\left(\sum_{j} x_{j} A_{j}\right)^{\sum \alpha_{j}}} \tag{5.6}
\end{align*}
$$

To see why these formulas are useful, consider the evaluation of diagram 5.1. We have

$$
\begin{equation*}
\frac{g^{2}}{2 \rho} \int \frac{d \omega_{1}}{2 \pi} \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2}}{\left(\epsilon_{1}+\epsilon_{2}-\omega_{1}\right)^{2}-v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right)^{2}+i \delta} \frac{\left|\vec{p}_{1}\right|^{2}}{\omega_{1}^{2}-v_{l}^{2} \vec{p}_{1}^{2}+i \delta} \tag{5.7}
\end{equation*}
$$

This can be brought into a more useful form using (5.3) with

$$
\begin{align*}
a & =\left(\epsilon_{1}+\epsilon_{2}-\omega_{1}\right)^{2}-v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right)^{2}+i \delta \\
b & =\omega_{1}^{2}-v_{l}^{2} \vec{p}_{1}^{2}+i \delta \tag{5.8}
\end{align*}
$$

Using (5.3), we can write

$$
\begin{aligned}
& \frac{\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2}}{\left(\epsilon_{1}+\epsilon_{2}-\omega_{1}\right)^{2}-v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}+i \delta} \frac{\left|\vec{p}_{1}\right|^{2}}{\omega_{1}^{2}-v_{l}^{2} \vec{p}_{1}^{2}+i \delta} \\
= & \int_{0}^{1} d x \frac{\left|\vec{p}_{1}\right|^{2}\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2}}{\left[\left(\left(\epsilon_{1}+\epsilon_{2}-\omega_{1}\right)^{2}-v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right)^{2}+i \delta\right) x+\left(\omega_{1}^{2}-v_{l}^{2} \vec{p}_{1}^{2}+i \delta\right)(1-x)\right]^{2}} \\
= & \int_{0}^{1} d x \frac{\left|\vec{p}_{1}\right|^{2}\left|\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right|^{2}}{\left[\omega_{1}^{2}-v_{l}^{2} \vec{p}_{1}^{2}+x\left(\left(\epsilon_{1}+\epsilon_{2}\right)^{2}-v_{l}^{2} \vec{p}_{1}^{2}\right)-2 x \omega_{1}\left(\epsilon_{1}+\epsilon_{2}\right)+2 x v_{l}^{2} \vec{p}_{1} \cdot\left(\vec{k}_{1}+\vec{k}_{2}-\vec{p}_{1}\right)+i \delta\right]^{2}}
\end{aligned}
$$

If these integrals were from $-\infty$ to $\infty$, then we could shift the variables of integration without worrying. In condensed matter physics, these integrals are always cutoff, so we must be a little more careful; in our phonon theory, the momentum cutoff, $\Lambda$, is the inverse lattice spacing and the frequency cutoff, $\Lambda_{\omega}$, is the Debye energy. However, so long as the external momenta and energies are much smaller than the cutoffs, i.e. $k_{i} \ll \Lambda, \omega_{i} \ll \Lambda_{\omega}$, we can shift the variables of integration and neglect the effect of this shift on the range of integration. Thus, we proceed by changing the variables of integration to $\omega=\omega_{1}-x\left(\epsilon_{1}+\epsilon_{2}\right), \vec{q}=\vec{p}_{1}-x\left(\vec{k}_{1}+\vec{k}_{2}\right)$. Writing $a=x(1-x)\left(\left(\epsilon_{1}+\epsilon_{2}\right)^{2}-v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}\right)$ we can write the loop integral as:

$$
\begin{align*}
\int \frac{d \omega}{2 \pi} \frac{d^{3} q}{(2 \pi)^{3}} \frac{\vec{q}^{4}}{\left(\omega^{2}-v_{l}^{2} \vec{q}^{2}+a\right)^{2}} & +\int \frac{d \omega}{2 \pi} \frac{d^{3} q}{(2 \pi)^{3}} \frac{\vec{q}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}\left(x^{2}+(1-x)^{2}+\frac{4}{3} x(1-x)\right)}{\left(\omega^{2}-v_{l}^{2} \vec{q}^{2}+a\right)^{2}} \\
& +\int \frac{d \omega}{2 \pi} \frac{d^{3} q}{(2 \pi)^{3}} \frac{x^{2}(1-x)^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{4}}{\left(\omega^{2}-v_{l}^{2} \vec{q}^{2}+a\right)^{2}} \tag{5.9}
\end{align*}
$$

Integrals of this form are often divergent. If we forget about the momentum and frequency cutoffs, then these integrals are ultraviolet divergent. If we're careful and we remember that these integrals are cutoff in the ultraviolet, then we will get finite
(albeit cutoff-dependent) answers. On the other hand, these integrals are infrared divergent if $a=0$ - i.e. if $\epsilon_{i}, \vec{k}_{i}$ vanish. This is a real, physical effect: phonon Green functions do diverge when the momenta, energies tend to zero. We will study the power-law forms of these divergences when we turn to the renormalization group in chapter 11.

It will sometimes be important to distinguish between the frequency cutoff, $\Lambda_{\omega}$, and the momentum cutoff, $\Lambda$. Often, however, the distinction is unimportant and we can assume that there is a single cutoff, $\Lambda_{\omega}=v_{l} \Lambda$. In such a case, we can simplify (5.9) by using analytic continuation.

For either sign of $|\vec{q}|^{2}-a$, the poles in $\omega$ are in the second and fourth quadrants. Hence, the contour of integration can be harmlessly rotated in an anti-counterclockwise direction from the real $\omega$ axis to the imaginary $\omega$ axis. If we write $q_{4}=-i \omega$ and $q^{2}=q_{4}^{2}+v_{l}^{2} \vec{q}^{2}$, then (5.9) is equal to

$$
\begin{align*}
\frac{i}{v_{l}^{7}} \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{q^{4}}{\left(-q^{2}+a\right)^{2}} & +\frac{i}{v_{l}^{5}} \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\frac{3}{4} q^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}\left(x^{2}+(1-x)^{2}+\frac{4}{3} x(1-x)\right)}{\left(-q^{2}+a\right)^{2}} \\
& +\frac{i}{v_{l}^{3}} \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{x^{2}(1-x)^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{4}}{\left(-q^{2}+a\right)^{2}} \tag{5.10}
\end{align*}
$$

or

$$
\begin{array}{r}
\frac{1}{v_{l}^{7}} I_{2,2}(a)+\frac{1}{v_{l}^{5}} \frac{3}{4}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}\left(x^{2}+(1-x)^{2}+\frac{4}{3} x(1-x)\right) I_{2,1}(a) \\
+\frac{1}{v_{l}^{3}}\left(x^{2}(1-x)^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{4}\right) I_{2,0}(a) \tag{5.11}
\end{array}
$$

where the integrals which we need to study are:

$$
\begin{equation*}
I_{n, m}(a)=i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{q^{2 m}}{\left(-q^{2}+a\right)^{n}} \tag{5.12}
\end{equation*}
$$

or, setting $z=q^{2}$, and $V\left(S^{3}\right)=2 \pi^{2}$,

$$
I_{n, m}(a)=\frac{i}{16 \pi^{2}} \int_{0}^{\Lambda^{2}} \frac{z^{m+1} d z}{(-z+a)^{n}}
$$

$$
\begin{align*}
& =\frac{(-1)^{n}}{(n-1)!} \frac{d^{n-1}}{d a^{n-1}}\left(\frac{i}{16 \pi^{2}} \int_{0}^{\Lambda^{2}} \frac{z^{m+1} d z}{(-z+a)}\right) \\
& =-\frac{(-1)^{n}}{(n-1)!} \frac{d^{n-1}}{d a^{n-1}}\left(\frac{i}{16 \pi^{2}} \int_{-a}^{\Lambda^{2}-a} \frac{(u+a)^{m+1} d u}{u}\right) \\
& =-\frac{i}{16 \pi^{2}} \frac{(-1)^{n}}{(n-1)!} \frac{d^{n-1}}{d a^{n-1}}\left(\sum_{k=0}^{m+1}\binom{m+1}{k} u^{k-1} a^{m+1-k}\right) \\
& =--\frac{i}{16 \pi^{2}} \frac{(-1)^{n}}{(n-1)!} \frac{d^{n-1}}{d a^{n-1}}\left(a^{m+1} \ln u+\sum_{k=0}^{m+1} \frac{1}{k}\binom{m+1}{k} u^{k} a^{m+1-k}\right)_{-a}^{\Lambda^{2}-a} \\
& =-\frac{i}{16 \pi^{2}} \frac{(-1)^{n}}{(n-1)!} \frac{d^{n-1}}{d a^{n-1}}\left(a^{m+1} \ln \left(\frac{\Lambda^{2}-a}{-a}\right)+\right. \\
& \left.\sum_{k=0}^{m+1} \frac{1}{k}\binom{m+1}{k} a^{m+1-k}\left[\left(\Lambda^{2}-a\right)^{k}-(-a)^{k}\right]\right) \tag{5.13}
\end{align*}
$$

Hence, we finally obtain:

$$
\left.\begin{array}{r}
-\frac{i}{16 \pi^{2}} \int_{0}^{1} d x\left[\frac { 1 } { v _ { l } ^ { 7 } } ( x ^ { 2 } ( 1 - x ) ^ { 2 } ( \vec { k } _ { 1 } + \vec { k } _ { 2 } ) ^ { 4 } ) \left\{-\frac{1}{2} \Lambda^{4}-3 \Lambda^{2} x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)\right.\right. \\
\left.+3\left(x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)\right)^{2} \ln \left(\frac{\left.-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)\right)^{2}}{x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)}\right)\right\} \\
+\frac{1}{v_{l}^{5}} \frac{3}{4}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}\left(x^{2}+(1-x)^{2}+\frac{4}{3} x(1-x)\right)\left\{\Lambda^{2}+x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)\right. \\
\left.-x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right) \ln \left(\frac{\Lambda^{2}}{x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)}\right)\right\} \\
+\frac{1}{v_{l}^{3}}\left(x^{2}(1-x)^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{4}\right)\left\{\ln \left(\frac{\Lambda^{2}}{x(1-x)\left(v_{l}^{2}\left(\vec{k}_{1}+\vec{k}_{2}\right)^{2}-\left(\epsilon_{1}+\epsilon_{2}\right)^{2}\right)}\right)-1\right\}
\end{array}\right]
$$

To summarize, we evaluate a Feynman diagram by the following steps:

- Use the Feynman rules to obtain a loop integral.
- Combine the denominators using Feynman's trick.
- Shift the variables of integration to make the denominator invariant under $\omega \rightarrow$ $\omega, \vec{p} \rightarrow-\vec{p}$.


Figure 5.4: The definition of the 4 point Green function

- Analytically continue to imaginary frequencies.
- Use rotational invariance to reduce the integral to an integral over a single variable for each $\omega, \vec{p}$.


### 5.3 Green Functions

In the preceding discussion, we have implicitly assumed that external phonon lines are 'on shell', i.e. they satisfy $\omega^{2}=\left(\omega_{p}^{l}\right)^{2}$. It does, however, make sense to relax this requirement and allow even the external phonons to be "off-shell". One reason is that we may want to define a Feynman diagram - or a set of diagrams - which can be part of a larger diagram. In such a case, the lines which enter this part might not be on-shell.

Consider the diagram of figure 5.4a. The shaded circle represents all possible diagrams with 4 external legs. The first few are shown in figure 5.4 b . We will call
such an object

$$
\begin{equation*}
G\left(p_{1}, p_{2}, p_{3}, p_{4}\right) \tag{5.15}
\end{equation*}
$$

(We will use $p$ as a shorthand for $\vec{p}, \omega$.) $G\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ is defined to include the momentum conserving $\delta$ functions,

$$
(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right) 2 \pi \delta\left(\omega_{1}+\omega_{2}+\omega_{3}+\omega_{4}\right)
$$

and a propagator

$$
\left|\vec{p}_{i}\right|^{2} \frac{i}{\omega_{i}^{2}-v_{l}^{2} p_{i}^{2}+i \delta}
$$

on each external leg.
We can define similar objects - called Green functions - for any number of external legs:

$$
\begin{equation*}
G\left(p_{1}, p_{2}, \ldots, p_{n}\right) \tag{5.16}
\end{equation*}
$$

It is given by the sum of all diagrams with $n$ external legs with (possibly off-shell) momenta and energies $p_{1}, p_{2}, \ldots, p_{n}$ with a propagator assigned to each external leg. We can Fourier transform the Green function to obtain the real-space n-point Green function, $G\left(x_{1}, \ldots, x_{n}\right)$.

While the notation $G\left(x_{1}, \ldots, x_{n}\right)$ is generically used for Green functions, the phonon two-point Green function is often denoted $D\left(x_{1}, x_{2}\right)$. However, we will reserve this notation for the two-point function of the other phonon field $\varphi_{i}$, which is more natural in some contexts.

The name 'Green function' is due to the fact that when the interaction is turned off, i.e. $g=0$, the two-point Green function is a Green function of the differential operator

$$
\begin{equation*}
\rho \delta_{i k} \partial_{t}^{2}-(\mu+\lambda) \partial_{i} \partial_{k}-\mu \delta_{i k} \partial_{j} \partial_{j} \tag{5.17}
\end{equation*}
$$

This follows since the two-point Green function is just the derivative of the propagator:

$$
G\left(x_{1}, x_{2}\right)=\partial_{i} \partial_{j}\left\langle T\left(u_{i}\left(\vec{x}_{1}, t_{1}\right) u_{j}\left(\vec{x}_{2}, t_{2}\right)\right)\right\rangle
$$

$$
\begin{equation*}
=\partial_{i} \partial_{j} \frac{1}{\rho} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi} e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i \epsilon_{i}^{s} \epsilon_{j}^{s}}{\omega^{2}-\left(\omega_{p}^{s}\right)^{2}+i \delta} \tag{5.18}
\end{equation*}
$$

It therefore satisfies

$$
\begin{equation*}
\left(\rho \delta_{i k} \partial_{t}^{2}-(\mu+\lambda) \partial_{i} \partial_{k}-\mu \delta_{i k} \partial_{j} \partial_{j}\right) G\left(x_{1}, x_{2}\right)=i \delta\left(\vec{x}_{1}-\vec{x}_{2}\right) \delta\left(t_{1}-t_{2}\right) \tag{5.19}
\end{equation*}
$$

as you showed in the first problem set.

### 5.4 The Generating Functional

Let's modify our Hamiltonian by adding a 'source term',

$$
\begin{equation*}
H \rightarrow H+\int d^{3} \vec{x} j(\vec{x}, t) \partial_{k} u_{k}(\vec{x}, t) \tag{5.20}
\end{equation*}
$$

The source, $j$, is some arbitrary, prescribed function. We can think of $j(\vec{x}, t)$ as a knob which we can turn in order to set up compressional waves in the solid. By measuring the system at $\left(\vec{x}^{\prime}, t^{\prime}\right)$, we can study the propagation of sound waves.

Our interaction Hamiltonian is now $H_{I}+\int d^{3} \vec{x} j(\vec{x}) \partial_{k} u_{k}(\vec{x})$, so our Feynman rules must be expanded to include a new vertex - which we will call a 'source vertex' - with only one line emerging from it. If this line has momentum, energy $\vec{p}, \omega$, we assign $-i \tilde{j}(\vec{p}, \omega)$ to it ( $\tilde{j}$ is the Fourier transform of $j$ ). Let us now look at the vacuum-tovacuum amplitude, which we will call $Z[j]$ :

$$
\begin{equation*}
Z[j]=\langle 0| T\left\{e^{-i \int H_{I}+j \partial_{k} u_{k}}\right\}|0\rangle \tag{5.21}
\end{equation*}
$$

This is given by the sum of all diagrams with no external legs. Several of these are shown in figure (5.5). We have denoted the new vertex by a large dot with a $j$ next to it. To make life easy, let us shift the zero of energy by adding a constant to the Hamiltonian, $H \rightarrow H+E_{0}$ and choose $E_{0}$ so that:

$$
\begin{equation*}
Z[0]=1 \tag{5.22}
\end{equation*}
$$



Figure 5.5: Some vacuum-to-vacuum diagrams in the presence of an external source.

Hence, the sum of all of the diagrams with no source vertices if 1 . Consider a diagram with $n$ source vertices. It will have an amplitude proportional to $\tilde{j}\left(\vec{p}_{1}, \omega_{1}\right) \ldots \tilde{j}\left(\vec{p}_{n}, \omega_{n}\right)$. Each $\tilde{j}\left(\vec{p}_{i}, \omega_{i}\right)$ creates a phonon with momentum $\vec{p}_{i}, \omega_{i}$. These $n$ phonons enter the diagram along the external legs of the Green function $G\left(p_{1}, \ldots, p_{n}\right)$. We then have to integrate over all of the $p_{i}$ 's with a factor of $1 / n$ ! to avoid overcounting due to permutations of the $p_{i}$ 's. Hence,

$$
\begin{equation*}
Z[j]=1+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int d^{3} \vec{p}_{1} d \omega_{1} \ldots d^{3} \vec{p}_{n} d \omega_{n} j\left(p_{1}\right) \ldots j\left(p_{n}\right) G\left(p_{1}, \ldots, p_{n}\right) \tag{5.23}
\end{equation*}
$$

We can Fourier transform this expression into real space:

$$
\begin{equation*}
Z[j]=1+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int d^{3} \vec{x}_{1} d t_{1} \ldots d^{3} \vec{x}_{n} d t_{n} j\left(x_{1}\right) \ldots j\left(x_{n}\right) G\left(x_{1}, \ldots, x_{n}\right) \tag{5.24}
\end{equation*}
$$

We can understand the Green function in another way by considering the Hamiltonian with a source term,

$$
\begin{align*}
H \rightarrow & H+j \partial_{k} u_{k} \\
& =H_{0}+H^{\prime}+j \partial_{k} u_{k} \\
& =\left(H_{0}+H^{\prime}\right)+j \partial_{k} u_{k} \tag{5.25}
\end{align*}
$$

We can now treat $H_{0}+H^{\prime}$ as our 'free' Hamiltonian and $j \partial_{k} u_{k}$ as our interaction Hamiltonian. Since $H_{0}+H^{\prime}$ is not actually free, we can't use Wick's theorem, but
we we can still use Dyson's formula. The 'interaction' representation for this 'free' Hamiltonian is actually what we would call the Heisenberg representation for $j=0$, so we will put an $H$ superscript on all fields. Using Dyson's formula, we can express $Z[j]$ as:

$$
\begin{align*}
Z[j] & =\langle 0| T\left\{e^{-i \int d^{3} \vec{x} d t j(\vec{x}, t) \partial_{k} u_{k}^{H}(\vec{x}, t)}\right\}|0\rangle \\
& =1+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int d^{3} \vec{x}_{1} d t_{1} \ldots d^{3} \vec{x}_{n} d t_{n} j\left(x_{1}\right) \ldots j\left(x_{n}\right)\langle 0| T\left(\partial_{k} u_{k}^{H}\left(x_{1}\right) \ldots \partial_{k} u_{k}^{H}\left(x_{n}\right)\right)|0\rangle \tag{5.26}
\end{align*}
$$

Comparing this with our earlier expression for $Z[j]$, we see that the Green function is given by:

$$
\begin{equation*}
G\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\langle 0| T\left(\partial_{k} u_{k}^{H}\left(x_{1}\right) \ldots \partial_{k} u_{k}^{H}\left(x_{n}\right)\right)|0\rangle=\frac{\delta^{n} Z[j]}{\delta j\left(x_{1}\right) \ldots \delta j\left(x_{n}\right)} \tag{5.27}
\end{equation*}
$$

In other words, the Green functions are the vacuum expectation values of the $T$ ordered product of a string of (Heisenberg picture) fields. These vacuum expectation values are the coefficients of the Taylor expansion of the vacuum-to-vacuum transition amplitude in the presence of an external source field. While our earlier definition - as a sum of Feynman diagrams - is convenient for perturbative calculation, the present definition as a vacuum expectation value is far more general since it is nonperturbative and can be compared with experiments. These vacuum expectation values are also called time-ordered correlation functions.

### 5.5 Connected Diagrams

There is a very useful theorem which states that

$$
\begin{equation*}
Z[j]=e^{W[j]} \tag{5.28}
\end{equation*}
$$

where $Z[j]$ is the sum of vacuum-to-vacuum Feynman diagrams we defined above and $W[j]$ is the sum of connected vacuum-to-vacuum diagrams. To prove this theorem,
observe that a given diagram which contributes to $Z[j]$ will, in general, be made up of several different connected diagrams and it will factor into the contributions from each of them. We can assemble the set of all vacuum-to-vacuum diagrams by putting together $n_{1}$ connected diagrams of type $1, n_{2}$ connected diagrams of type $2, \ldots, n_{r}$ connected diagrams of type $r$, etc. The contribution of such a diagram to $Z[j]$ will be the product of the contributions, $C_{r}$, of its connected components:

$$
\begin{equation*}
Z[j]=\sum_{n_{1}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \cdots \prod_{r=1}^{\infty} \frac{C_{r}^{n_{r}}}{n_{r}!} \tag{5.29}
\end{equation*}
$$

The $n_{r}$ ! in the denominator is the symmetry factor resulting from the permutations of the $n_{r}$ identical connected components of type $r$. Commuting the sums and the product:

$$
\begin{align*}
Z[j] & =\prod_{r=1}^{\infty}\left(\sum_{n_{r}=0}^{\infty} \frac{C_{r}^{n_{r}}}{n_{r}!}\right) \\
& =\prod_{r=1}^{\infty} e^{C_{r}} \\
& =e^{\sum_{r=1}^{\infty} C_{r}} \\
& =e^{W[j]} \tag{5.30}
\end{align*}
$$

This theorem - sometimes called the linked cluster theorem - will be particularly useful when we construct a diagrammatic expansion for the partition function, $Z=$ $\operatorname{tr}\left(e^{-\beta H}\right)$. In order to compute the free energy, $F=-T \ln Z$, we need only compute the connected diagrams.

The Taylor expansion of $W[j]$ is:

$$
\begin{equation*}
W[j]=W[0]+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int d^{3} \vec{x}_{1} d t_{1} \ldots d^{3} \vec{x}_{n} d t_{n} j\left(x_{1}\right) \ldots j\left(x_{n}\right) G_{c}\left(x_{1}, \ldots, x_{n}\right) \tag{5.31}
\end{equation*}
$$

where the $G_{c}$ 's are connected Green functions. The two-point connected Green function is given by:

$$
G_{c}\left(x_{1}, x_{2}\right)=\langle 0| T\left(\partial_{k} u_{k}\left(x_{1}\right) \partial_{k} u_{k}\left(x_{2}\right)\right)|0\rangle-\langle 0| \partial_{k} u_{k}\left(x_{1}\right)|0\rangle\langle 0| \partial_{k} u_{k}\left(x_{2}\right)|0\rangle
$$

$$
\left.=\langle 0| T\left(\left(\partial_{k} u_{k}\left(x_{1}\right)-\left\langle\partial_{k} u_{k}\left(x_{1}\right)\right\rangle\right)\left(\partial_{k} u_{k}\left(x_{2}\right)\right)-\left\langle\partial_{k} u_{k}\left(x_{2}\right)\right\rangle\right)|0\rangle 5.32\right)
$$

This correlation function is often more useful since it measures fluctuations around mean values.

### 5.6 Spectral Representation of the Two-Point Green function

The spectral representation of the two-point Green function has the advantage of being intuitive yet well-suited for rigorous statements. It is obtained by inserting a complete set of states into

$$
\begin{align*}
\left\langle T\left(\partial_{i} u_{i}(\vec{x}, t) u_{j}(0,0)\right)\right\rangle= & \theta(t)\left\langle\partial_{i} u_{i}(\vec{x}, t) \partial_{j} u_{j}(0,0)\right\rangle+\theta(-t)\left\langle\partial_{j} u_{j}(0,0) \partial_{i} u_{i}(\vec{x}, t)\right\rangle \\
= & \theta(t) \sum_{i}\langle 0| \partial_{i} u_{i}(\vec{x}, t)|i\rangle\langle i| \partial_{j} u_{j}(0,0)|0\rangle \\
& +\theta(-t) \sum_{i}\langle 0| \partial_{j} u_{j}(0,0)|i\rangle\langle i| \partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right)|0\rangle \tag{5.33}
\end{align*}
$$

By translational invariance,

$$
\begin{equation*}
\langle 0| \partial_{i} u_{i}(\vec{x}, t)|i\rangle=e^{i \vec{p}_{i} \cdot \vec{x}-i \omega_{i} t}\langle 0| \partial_{i} u_{i}(0,0)|i\rangle \tag{5.34}
\end{equation*}
$$

where $\vec{p}_{i}$ and $\omega_{i}$ are the momentum and energy of the state $|i\rangle$.
Hence, we can write the Green function as

$$
\begin{align*}
\left\langle T\left(\partial_{i} u_{i}(\vec{x}, t) u_{j}(0,0)\right)\right\rangle= & \left.\sum_{i}\left|\langle i| \partial_{j} u_{j}(0,0)\right| 0\right\rangle\left.\right|^{2}\left(\theta(t) e^{i\left(\overrightarrow{p_{i}} \cdot \vec{x}-\omega_{i} t\right)}+\theta(-t) e^{-i\left(\overrightarrow{p_{i}} \cdot \vec{x}-\omega_{i} t\right)}\right) \\
= & \left.\left.\int d^{3} \vec{P} d E\left[\sum_{i}\left|\langle i| \partial_{j} u_{j}(0,0)\right| 0\right\rangle\right|^{2} \delta\left(\vec{P}-\overrightarrow{p_{i}}\right) \delta\left(\omega_{i}-E\right)\right] \\
& \times\left(\theta(t) e^{i(\vec{P} \cdot \vec{x}-E t)}+\theta(-t) e^{-i(\vec{P} \cdot \vec{x}-E t)}\right) \\
\equiv & \int d^{3} \vec{P} d E|P|^{2} B(\vec{P}, E)\left(\theta(t) e^{i(\vec{P} \cdot \vec{x}-E t}+\theta(-t) e^{-i(\vec{P} \cdot \vec{x}-E t}\right) \tag{5.35}
\end{align*}
$$

Here, we have introduced the spectral function, $B(\vec{P}, E)$, so that $|P|^{2} B(\vec{P}, E)$ is given by the quantity in brackets. In order to take the Fourier transform, we have to add $i \delta$ s to make the $t$ integral convergent, so that we obtain:

$$
\begin{align*}
G(\vec{x}, \omega) & =\int d t \int d^{3} \vec{P} d E|P|^{2} B(\vec{P}, E)\left(\theta(t) e^{i(\vec{P} \cdot \vec{x}+(\omega-E+i \delta) t)}+\theta(-t) e^{-i(\vec{P} \cdot \vec{x}-(\omega+E-i \delta) t)}\right) \\
& =\int d t \int d^{3} \vec{P} d E|P|^{2} B(\vec{P}, E)\left(\theta(t) e^{i(\omega-E) t-\delta t} e^{i \vec{P} \cdot \vec{x}}+\theta(-t) e^{i(\omega+E) t+\delta t} e^{-i \vec{P} \cdot \vec{x}}\right) \\
& =\int d^{3} \vec{P} d E i|p|^{2} B(\vec{p}, E)\left(\frac{e^{i \vec{P} \cdot \vec{x}}}{\omega-E+i \delta}-\frac{e^{-i \vec{P} \cdot \vec{x}}}{\omega+E-i \delta}\right) \tag{5.36}
\end{align*}
$$

or

$$
\begin{equation*}
G(\vec{p}, \omega)=\int d E i|p|^{2}\left(\frac{B(\vec{p}, E)}{\omega-E+i \delta}-\frac{B(-\vec{p}, E)}{\omega+E-i \delta}\right) \tag{5.37}
\end{equation*}
$$

For a parity-invariant system, $B(\vec{p}, E)=B(-\vec{p}, E)$, so

$$
\begin{equation*}
G(\vec{p}, \omega)=\int d E i|p|^{2} \frac{2 E B(\vec{p}, E)}{\omega^{2}-E^{2}+i \delta} \tag{5.38}
\end{equation*}
$$

From its definition, $B(\vec{p}, E)$ is non-negative. If the phonons are non-interacting, $B(\vec{p}, E)=\delta\left(E^{2}-\omega_{p}^{2}\right)$, and we recover the free-phonon two-point function.

We can split the sum over $i$ into the vacuum state, the one-phonon states, and all other states. Now, let us assume that

$$
\begin{equation*}
\langle 0| u_{i}\left(\vec{x}_{1}, t_{1}\right)|0\rangle=0 \tag{5.39}
\end{equation*}
$$

If it didn't, this would be the statement that there is some kind of static distortion present in the ground state. We could shift $u_{i}\left(\vec{x}_{1}, t_{1}\right)$ by $u_{i}\left(\vec{x}_{1}, t_{1}\right) \rightarrow u_{i}\left(\vec{x}_{1}, t_{1}\right)-$ $\langle 0| u_{i}\left(\vec{x}_{1}, t_{1}\right)|0\rangle$ and we would have the above for our new displacement field.

Consider a one-phonon state of momentum $\vec{p}$. Then, we will write:

$$
\begin{equation*}
\left.|p|^{2} Z=\left|\langle 0| \partial_{i} u_{i}(0,0)\right| \vec{p}\right\rangle\left.\right|^{2} \tag{5.40}
\end{equation*}
$$

Rotational and Galilean invariance imply that the left-hand-side is independent of the direction of $\vec{p}$.

Then the spectral function can be broken into a piece, carrying weight $Z$, which looks like a non-interacting phonon, and the remaining 'incoherent' weight:

$$
\begin{equation*}
B(\vec{p}, E)=Z \delta\left(E^{2}-\omega_{p}^{2}\right)+B_{\mathrm{inc}}(\vec{p}, E) \tag{5.41}
\end{equation*}
$$

The phonon propagates as a free phonon with probability $Z$ and as a multi-phonon state of energy $E$ with probability $B_{\text {inc }}(\vec{p}, E)$.

### 5.7 The Self-Energy and Irreducible Vertex

The two-point Green function $G\left(p_{1}, p_{2}\right)$ is given by the diagrams in figure (??). To zeroth order in $g$, it is simply the free phonon propagator. There is an $O(g)$ correction given by the diagram of figure (5.2) which leads to

$$
\begin{align*}
G\left(p_{1}, p_{2}\right)= & (2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}\right) 2 \pi \delta\left(\omega_{1}+\omega_{2}\right)\left(\left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-v_{l}^{2} p_{1}^{2}+i \delta}\right. \\
& \left.+\frac{g}{2}\left(\left|\vec{p}_{1}\right|^{2} \frac{i}{\omega_{1}^{2}-v_{l}^{2} p_{1}^{2}+i \delta}\right)^{2} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{d \omega}{2 \pi}|\vec{p}|^{2} \frac{i}{\omega^{2}-v_{l}^{2} p^{2}+i \delta}+O\left(g^{2}\right)\right) \tag{5.42}
\end{align*}
$$

For the two-point Green function, we can do better without doing much more work. Let us define the one-particle irreducible, or 1PI $n$-point Green function as the sum of all the Feynman graphs contributing to the $n$-point Green function which cannot be made disconnected by cutting a single internal line (this is a subset of the set of connected diagrams). For the 1PI $n$-point function, we do not include propagators on the external legs. The 1PI two-point Green function is given by $\Pi(p, \omega) / p^{2} ; \Pi(p, \omega)$ is called the self-energy because the two-point Green function can be expressed in terms of it according to the graphical relation of figure 5.6. Summing this geometrical series, we have:

$$
\begin{equation*}
G\left(p_{1}, p_{2}\right)=(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}\right) 2 \pi \delta\left(\omega_{1}+\omega_{2}\right)\left|\overrightarrow{p_{1}}\right|^{2} \frac{i}{\omega_{1}^{2}-v_{l}^{2} p_{1}^{2}-\Pi\left(\vec{p}_{1}, \omega_{1}\right)+i \delta} \tag{5.43}
\end{equation*}
$$



Figure 5.6: The relation between $\Pi$ and $G$.

From our calculation above, we see that the self-energy is given by:

$$
\begin{equation*}
\Pi(p, \omega)=\frac{g}{2}|\vec{p}|^{2} \int \frac{d^{3} \vec{q}}{(2 \pi)^{3}} \frac{d \epsilon}{2 \pi}|\vec{q}|^{2} \frac{i}{\epsilon^{2}-v_{l}^{2} q^{2}+i \delta}+O\left(g^{2}\right) \tag{5.44}
\end{equation*}
$$

In the problem set, you will show that $\operatorname{Im}\{\Pi(p, \omega)\}$ is related to the phonon lifetime.
The coherent weight in the phonon spectral function, $Z$, is given by:

$$
\begin{equation*}
Z^{-1}=1-\left(\frac{\partial}{\partial \omega^{2}} \operatorname{Re}(\Pi)\right)_{\omega=v p} \tag{5.45}
\end{equation*}
$$

We can also define a 1PI 4-point Green function, $\Gamma\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$. The full Green function $G\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ can be expressed in terms of $\Gamma\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ and the twopoint Green function $G\left(p_{1}, p_{2}\right)$ according to the graphical relation of figure 5.7.


Figure 5.7: The relation between the regular and 1PI four point Green functions.

## Chapter 6

## Imaginary-Time Formalism

### 6.1 Finite-Temperature Imaginary-Time Green Functions

In the previous chapter, we found that the mathematical trick of analytically continuing to imaginary frequencies, $\omega \rightarrow i \omega$, facilitated the calculation of the integrals arising from Feynman diagrams. As we will demonstrate in this chapter, it is extremely convenient to work with imaginary-time from the outset. Such an imaginary-time formalism will have the advantage of having a natural extension to arbitrary temperature. It can also, in many cases, serve as a preliminary step in the calculation of retarded correlation functions which - as we will discuss in the next chapter - are the quantities most closely related to physical measurements.

We make the analytic continuation it $\rightarrow \tau$ and define the following object for $0<\tau<\beta:$

$$
\begin{aligned}
\mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, \tau-\tau^{\prime}\right)= & \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H} e^{\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime} H}\right\} \\
& +\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H} e^{\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime} H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H}\right\} \\
= & \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} \partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(\vec{x}^{\prime}, \tau^{\prime}\right)\right\} \\
& +\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H} \partial_{j} u_{j}\left(\vec{x}^{\prime}, \tau^{\prime}\right) \partial_{k} u_{k}(\vec{x}, \tau)\right\}
\end{aligned}
$$

$$
\begin{equation*}
\equiv\left\langle T_{\tau}\left(\partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(\vec{x}^{\prime}, \tau^{\prime}\right)\right)\right\rangle \tag{6.1}
\end{equation*}
$$

We have passed from the Schrödinger representation in the first line to an imaginarytime Heisenberg representation in the third line. In the final line, we have defined the imaginary-time-ordering symbol, $T_{\tau}$, by analogy with the real-time symbol, $T$ : operators are arranged from right to left in order of increasing $\tau$. In a similar way, we can define the imaginary-time-ordered product of strings of fields. If $\tau_{1}>\tau_{2}>$ $\ldots>\tau_{n}$, then

$$
\begin{equation*}
\left\langle T_{\tau}\left(O_{1} \ldots O_{n}\right)\right\rangle=\operatorname{Tr}\left\{e^{-\beta H} O_{1} \ldots O_{n}\right\} \tag{6.2}
\end{equation*}
$$

Rather than take the expectation value in the gound state, we are averaging the expectation value over all energy eigenstates, $|n\rangle$, weighted by $e^{-\beta E_{n}}$. If we had used it rather than $\tau$, we would have the finite-temperature time-ordered Green function. By working with $\tau$, we will construct a Green function from which the retarded Green function can be constructed by analytic continuation. We will also exploit the formal analogy between the time-evolution operator $e^{-i t H} \rightarrow e^{-\tau H}$ and the Boltzmann weight, $e^{-\beta H}$.

In analogy wth the real-time case, we write

$$
\begin{equation*}
U\left(\tau_{2}, \tau_{1}\right)=e^{-\left(\tau_{2}-\tau_{1}\right) H} \tag{6.3}
\end{equation*}
$$

We will add a source to the Hamiltonian and view the partition function, $Z[j]=$ $\operatorname{Tr}\left\{e^{-\beta H}\right\}=\operatorname{Tr}\{U(\beta, 0)\}$, as the generating functional for imaginary-time Green functions. In the perturbative expansion of $U(\beta, 0)$, we will only encounter fields with imaginary-time arguments in the interval $[0, \beta]$.

There is a further condition which follows from the cyclic property of the trace. Since $0<\tau, \tau^{\prime}<\beta$, it follows that $-\beta<\tau-\tau^{\prime}<\beta$. Now suppose that $\tau<\tau^{\prime}$. Then,

$$
\begin{aligned}
\mathcal{G}\left(\tau-\tau^{\prime}<0\right) & =\operatorname{Tr}\left\{e^{-\beta H} e^{\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime} H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H}\right\} \\
& =\operatorname{Tr}\left\{e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H} e^{-\beta H} e^{\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime} H}\right\}
\end{aligned}
$$

$$
\begin{align*}
& =\operatorname{Tr}\left\{e^{-\beta H} e^{\beta H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H} e^{-\beta H} e^{\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime} H}\right\} \\
& =\mathcal{G}\left(\tau-\tau^{\prime}+\beta\right) \tag{6.4}
\end{align*}
$$

The first equality follows from the cyclic property of the trace. The final equality follows from the fact that $\tau-\tau^{\prime}+\beta>0$.

As a result of periodicity in imaginary-time, we can take the Fourier transform over the interval $[0, \beta]$ :

$$
\begin{equation*}
\mathcal{G}\left(i \omega_{n}\right)=\int_{0}^{\beta} d \tau e^{i \omega_{n} \tau} \mathcal{G}(\tau) \tag{6.5}
\end{equation*}
$$

where the Matsubara frequencies $\omega_{n}$, are given by:

$$
\begin{equation*}
\omega_{n}=\frac{2 n \pi}{\beta} \tag{6.6}
\end{equation*}
$$

Inverting the Fourier transform, we have:

$$
\begin{equation*}
\mathcal{G}(\tau)=\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \omega_{n}\right) e^{-i \omega_{n} \tau} \tag{6.7}
\end{equation*}
$$

In the absence of interactions, we can evaluate the imaginary-time two-point Green function directly. Using the Planck distribution,

$$
\begin{equation*}
\operatorname{Tr}\left\{e^{-\beta H_{0}} a_{k}^{\dagger} a_{k}\right\}=n_{B}\left(\omega_{k}\right)=\frac{1}{e^{\beta \omega_{k}}-1} \tag{6.8}
\end{equation*}
$$

and substituting the mode expansion of $u_{k}$, we have:

$$
\begin{align*}
\mathcal{G}(\vec{x}, \tau)= & \theta(\tau) \operatorname{Tr}\left\{e^{-\beta H_{0}} \partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}(0,0)\right\} \\
& +\theta(-\tau) \operatorname{Tr}\left\{e^{-\beta H_{0}} \partial_{j} u_{j}(0,0) \partial_{k} u_{k}(\vec{x}, \tau)\right\} \\
= & \int \frac{d^{3} \vec{k}}{(2 \pi)^{3} 2 \omega_{k}}|\vec{k}|^{2}\left[\theta(\tau)\left(\left(n_{B}\left(\omega_{k}\right)+1\right) e^{i \vec{k} \cdot \vec{x}-\omega_{k} \tau}+n_{B}\left(\omega_{k}\right) e^{-i \vec{k} \cdot \vec{x}+\omega_{k} \tau}\right)\right. \\
& \left.+\theta(-\tau)\left(n_{B}\left(\omega_{k}\right) e^{i \vec{k} \cdot \vec{x}-\omega_{k} \tau}+\left(n_{B}\left(\omega_{k}\right)+1\right) e^{-i \vec{k} \cdot \vec{x}+\omega_{k} \tau}\right)\right] \tag{6.9}
\end{align*}
$$

We can now compute the Fourier representation of the Green function:

$$
\mathcal{G}\left(\vec{p}, i \omega_{n}\right)=\int d^{3} \vec{x} e^{i \vec{p} \cdot \vec{x}} \int_{0}^{\beta} d \tau e^{i \omega_{n} \tau} \mathcal{G}(\vec{x}, \tau)
$$

$$
\begin{align*}
& =\frac{|\vec{p}|^{2}}{2 \omega_{k}}\left(\frac{\left(n_{B}\left(\omega_{p}\right)+1\right)\left(e^{-\beta \omega_{p}}-1\right)}{i \omega_{n}-\omega_{p}}+\frac{n_{B}\left(\omega_{p}\right)\left(e^{\beta \omega_{p}}-1\right)}{i \omega_{n}+\omega_{p}}\right) \\
& =\frac{|\vec{p}|^{2}}{2 \omega_{k}}\left(\frac{-1}{i \omega_{n}-\omega_{p}}+\frac{1}{i \omega_{n}+\omega_{p}}\right) \\
& =-|\vec{p}|^{2} \frac{1}{\omega_{n}^{2}+\omega_{p}^{2}} \\
& =-|\vec{p}|^{2} \frac{1}{\omega_{n}^{2}+v_{l}^{2} p^{2}} \tag{6.10}
\end{align*}
$$

In real-space, this is:

$$
\begin{equation*}
\mathcal{G}(\vec{x}, \tau)=-\frac{1}{\beta} \sum_{n} \int \frac{d^{3} \vec{k}}{(2 \pi)^{3}}|\vec{k}|^{2} \frac{e^{-i \vec{k} \cdot \vec{x}-i \omega_{n} \tau}}{\omega_{n}^{2}+v_{l}^{2} k^{2}} \tag{6.11}
\end{equation*}
$$

As we can see from the above derivation of the Green function, in the imaginary-time formalism, we have the decaying exponential $e^{-\omega \tau}$ rather than the oscillatory $e^{i \omega t}$, so we can dispense with the $i \delta$ 's which are needed to ensure convergence in the real-time formulation. Indeed, from a mathematical point of view, imaginary-time is simpler than real-time; sadly, nature forces us to live in real-time.

### 6.2 Perturbation Theory in Imaginary Time

Following our real-time development, we consider a Hamiltonian of the form $H=H_{0}+$ $H_{\text {int }}$, and go to the imaginary-time interaction representation. In this representation,

$$
\begin{equation*}
U\left(\tau_{1}, \tau_{2}\right)=T_{\tau}\left\{e^{-\int_{\tau_{2}}^{\tau_{1}} d \tau H_{\mathrm{int}}(\tau)}\right\} \tag{6.12}
\end{equation*}
$$

Hence, the imaginary-time Green function, which takes the Schrödinger picture form:

$$
\begin{align*}
\mathcal{G}\left(\vec{x}, \tau-\tau^{\prime}\right)= & \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H} e^{\tau^{\prime} H} \partial_{j} u_{j}(0) e^{-\tau^{\prime} H}\right\} \\
& +\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H} e^{\tau^{\prime} H} \partial_{j} u_{j}(0) e^{-\tau^{\prime} H} e^{\tau H} \partial_{k} u_{k}(\vec{x}) e^{-\tau H}\right\}(6 \tag{6.13}
\end{align*}
$$

can be written in the interaction picture as:

$$
\mathcal{G}\left(\vec{x}, \tau-\tau^{\prime}\right)=\theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H_{0}} U(\beta, 0) U^{-1}(\tau, 0) \partial_{k} u_{k}(\vec{x}, \tau) U(\tau, 0) \times\right.
$$

$$
\begin{gathered}
\left.U^{-1}\left(\tau^{\prime}, 0\right) \partial_{j} u_{j}\left(0, \tau^{\prime}\right) U\left(\tau^{\prime}, 0\right)\right\} \\
+\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H_{0}} U(\beta, 0) U^{-1}\left(\tau^{\prime}, 0\right) \partial_{j} u_{j}\left(0, \tau^{\prime}\right) U\left(\tau^{\prime}, 0\right) \times\right. \\
\left.U^{-1}(\tau, 0) \partial_{k} u_{k}(\vec{x}, \tau) U(\tau, 0)\right\} \\
=\quad \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H_{0}} U(\beta, \tau) \partial_{k} u_{k}(\vec{x}, \tau) U\left(\tau, \tau^{\prime}\right) \partial_{j} u_{j}\left(0, \tau^{\prime}\right) U\left(\tau^{\prime}, 0\right)\right\} \\
+\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H_{0}} U\left(\beta, \tau^{\prime}\right) \partial_{j} u_{j}(0) U\left(\tau, \tau^{\prime}\right) \partial_{k} u_{k}(\vec{x}) U(\tau, 0)\right\}(6.14)
\end{gathered}
$$

or, simply, as

$$
\begin{align*}
\mathcal{G}\left(\vec{x}, \tau-\tau^{\prime}\right) & =\operatorname{Tr}\left\{e^{-\beta H_{0}} T_{\tau}\left(U(\beta, 0) \partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(0, \tau^{\prime}\right)\right)\right\} \\
& \equiv\left\langle T_{\tau}\left(U(\beta, 0) \partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(0, \tau^{\prime}\right)\right)\right\rangle \tag{6.15}
\end{align*}
$$

As we noted earlier, only imaginary times $\tau \in[0, \beta]$ appear.
To evaluate this perturbatively, we expand $U(\beta, 0)$, as in the real-time case:

$$
\begin{align*}
\mathcal{G}\left(\vec{x}, \tau-\tau^{\prime}\right)= & \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \int_{0}^{\beta} \ldots \int_{0}^{\beta} d \tau_{1} \ldots d \tau_{n} \\
& \left\langle T_{\tau}\left(\partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(0, \tau^{\prime}\right) H_{\mathrm{int}}\left(\tau_{1}\right) \ldots H_{\mathrm{int}}\left(\tau_{n}\right)\right)\right\rangle \tag{6.16}
\end{align*}
$$

We can show that Wick's theorem holds for the imaginary-time-ordered product and use it to evaluate the above expectation values. Following our real-time development in this way, we can use Feynman diagrams to evaluate the perturbation series. The differences are that in imaginary-time,

- To each line, we associate a momentum, $\vec{p}$ and a Matsubara frequency, $\omega_{n}$.
- For each external line with momentum $\vec{p}$, write $|\vec{p}|$.
- The propagator assigned to each internal line is:

$$
-\frac{1}{\beta} \sum_{n} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}}|\vec{p}|^{2} \frac{1}{\omega_{n}^{2}+v_{l}^{2} p^{2}}
$$

- For each vertex with momenta, Matsubara frequencies $\left(\vec{p}_{1}, \omega_{n_{1}}\right), \ldots,\left(\vec{p}_{4}, \omega_{n_{4}}\right)$ directed into the vertex, we write

$$
g(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right) \beta \delta_{n_{1}+n_{2}+n_{3}+n_{4}, 0}
$$

- Imagine labelling the vertices $1,2, \ldots, n$. Vertex $i$ will be connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices $i, i$ is still connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. If $S$ is the number of permutations which leave the diagram invariant, we assign a factor $1 / S$ to the diagram.
- If two vertices are connected by $l$ lines, we assign a factor $1 / l$ ! to the diagram.

Using our result on connected Feynman diagrams from chapter 5, we see that the free energy, $F$, is given by

$$
\begin{align*}
-\beta F & =\ln [\operatorname{Tr}\{U(\beta, 0)\}] \\
& =\sum \text { All connected diagrams with no external legs } \tag{6.17}
\end{align*}
$$

### 6.3 Analytic Continuation to Real-Time Green Functions

In spite of their many charms, imaginary-time Green functions cannot be directly measured in experiments. Hence, we must contemplate real-time Green functions. In fact, it is useful to consider $\tau$ as a complex variable, and to analyze the properties of $\mathcal{G}(\tau)$ as $\tau$ varies through the complex plane. When $\tau$ lies on the real axis, we have the imaginary-time Green function:

$$
\mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, \tau-\tau^{\prime}\right) \quad=\quad \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} e^{-\tau H} \partial_{k} u_{k}(\vec{x}) e^{\tau H} e^{-\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{\tau^{\prime} H}\right\}
$$

$$
\begin{equation*}
+\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H} e^{-\tau^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{\tau^{\prime} H} e^{-\tau H} \partial_{k} u_{k}(\vec{x}) e^{\tau H}\right\} \tag{6.18}
\end{equation*}
$$

When $\tau$ is on the imaginary axis, $\tau=i t$, we have the real-time Green function:

$$
\begin{align*}
\mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right)= & \theta\left(t-t^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} e^{-i t H} \partial_{k} u_{k}(\vec{x}) e^{i t H} e^{-i t^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{i t^{\prime} H}\right\} \\
& +\theta\left(t^{\prime}-t\right) \operatorname{Tr}\left\{e^{-\beta H} e^{-i t^{\prime} H} \partial_{j} u_{j}\left(\vec{x}^{\prime}\right) e^{i t^{\prime} H} e^{-i t H} \partial_{k} u_{k}(\vec{x}) e^{i t H}\right\} \tag{6.19}
\end{align*}
$$

For arbitrary complex $\tau, \mathcal{G}(\tau)$ interpolates between these two. $\mathcal{G}(\tau)$ is not, however, an analytic function over the entire complex plane, as we can see from its spectral representation. We follow our earlier derivation of the spectral representation for the $T=0$ real-time ordered Green function. The principal difference is that we now have $e^{-\beta E_{n}}|n\rangle$ rather than $|0\rangle$. Hence, by inserting a complete set of intermediate states, $|m\rangle\langle m|$, we have, in lieu of (5.35),

$$
\begin{align*}
\mathcal{G}(\vec{x}, \tau)=\int d^{3} \vec{p} d \omega\left[\sum_{n, m}\right. & \delta\left(\vec{p}-\vec{p}_{m}+\vec{p}_{n}\right) \delta\left(\omega-\omega_{n m}\right)\left(\theta(\tau) e^{i \vec{p} \cdot \vec{x}-\omega \tau} e^{-\beta E_{n}}\right. \\
+ & \left.\left.\left.\theta(-\tau)) e^{-i \vec{p} \cdot \vec{x}+\omega \tau} e^{-\beta E_{m}}\right)\left|\langle m| \partial_{i} u_{i}(0,0)\right| n\right\rangle\left.\right|^{2}\right] \tag{6.20}
\end{align*}
$$

The Fourier transform,

$$
\begin{equation*}
\mathcal{G}\left(\vec{p}, i \omega_{j}\right)=\int d^{3} \vec{x} \int_{0}^{\beta} d \tau \mathcal{G}(\vec{x}, \tau) e^{i \omega_{j} \tau} \tag{6.21}
\end{equation*}
$$

is given by:

$$
\begin{align*}
\mathcal{G}\left(\vec{p}, i \omega_{j}\right)= & \left.\int d E\left[\sum_{n, m}\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right)\left|\langle m| \partial_{i} u_{i}(0,0)\right| n\right\rangle\right|^{2} \\
& \left.\times \delta\left(\vec{p}-\vec{p}_{m}+\vec{p}_{n}\right) \delta\left(E-E_{m}+E_{n}\right)\right] \frac{1}{E-i \omega_{j}} \tag{6.22}
\end{align*}
$$

Writing

$$
\begin{equation*}
\left.p^{2} B(\vec{p}, E)=\sum_{n, m}\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right)\left|\langle m| \partial_{i} u_{i}(0,0)\right| n\right\rangle\left.\right|^{2} \delta\left(\vec{p}-\vec{p}_{m}+\vec{p}_{n}\right) \delta\left(E-E_{m n}\right) \tag{6.23}
\end{equation*}
$$

we have the spectral representation of $\mathcal{G}$ :

$$
\begin{equation*}
\mathcal{G}\left(\vec{p}, i \omega_{n}\right)=\int_{-\infty}^{\infty} d E \frac{p^{2} B(\vec{p}, E)}{E-i \omega_{j}} \tag{6.24}
\end{equation*}
$$

As usual, the spectral function $B(\vec{p}, E)$ is real and positive.
$\mathcal{G}$ is not analytic as a result of the singularities in (13.46). Hence, it does not satisfy the Kramers-Kronig relations. However, the functions

$$
\begin{equation*}
\int_{-\infty}^{\infty} d E \frac{p^{2} B(\vec{p}, E)}{E-\omega \pm i \delta} \tag{6.25}
\end{equation*}
$$

are analytic functions of $\omega$ in the lower and upper-half planes respectively. Consequently, they do satisfy the Kramers-Kronig relations. As you will show in the problem set, these are the advanced and retarded correlation functions defined in the next section:

$$
\begin{align*}
G_{\mathrm{ret}}(\vec{p}, \omega) & =\int_{-\infty}^{\infty} d E \frac{p^{2} B(\vec{p}, E)}{E-\omega-i \delta} \\
G_{\mathrm{adv}}(\vec{p}, \omega) & =\int_{-\infty}^{\infty} d E \frac{p^{2} B(\vec{p}, E)}{E-\omega+i \delta} \tag{6.26}
\end{align*}
$$

Note that the spectral function is the difference between the retarded and advanced correlation functions.

$$
\begin{equation*}
G_{\mathrm{ret}}(\vec{p}, \omega)-G_{\mathrm{adv}}(\vec{p}, \omega)=2 \pi i p^{2} B(\vec{p}, \omega) \tag{6.27}
\end{equation*}
$$

### 6.4 Retarded and Advanced Correlation Functions

In the previous chapter, we dealt with the time-ordered two-point correlation function,

$$
\begin{align*}
G\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right) \quad & =\theta\left(t_{1}-t_{2}\right)\left\langle\partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right) \partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right)\right\rangle \\
& +\theta\left(t_{2}-t_{1}\right)\left\langle\partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right) \partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right)\right\rangle \tag{6.28}
\end{align*}
$$

In this chapter, we have introduced the imaginary-time two-point correlation function:

$$
\mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, \tau-\tau^{\prime}\right) \quad=\quad \theta\left(\tau-\tau^{\prime}\right) \operatorname{Tr}\left\{e^{-\beta H} \partial_{k} u_{k}(\vec{x}, \tau) \partial_{j} u_{j}\left(\vec{x}^{\prime}, \tau^{\prime}\right)\right\}
$$

$$
\begin{equation*}
+\theta\left(\tau^{\prime}-\tau\right) \operatorname{Tr}\left\{e^{-\beta H} \partial_{j} u_{j}\left(\vec{x}^{\prime}, \tau^{\prime}\right) \partial_{k} u_{k}(\vec{x}, \tau)\right\} \tag{6.29}
\end{equation*}
$$

To this family of Green functions, we have now added the retarded and advanced correlation function. As we will see in the next chapter, the retarded correlation function is often more useful for comparison with experiments. At zero temperature, the retarded and advanced correlation functions are given by:

$$
\begin{align*}
G_{\mathrm{ret}}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right) & =\theta\left(t_{1}-t_{2}\right)\langle 0|\left[\partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right), \partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right)\right]|0\rangle \\
G_{\mathrm{adv}}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right) & =\theta\left(t_{2}-t_{1}\right)\langle 0|\left[\partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right), \partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right)\right]|0\rangle \tag{6.30}
\end{align*}
$$

At finite temperature, these are generalized to:

$$
\begin{align*}
G_{\mathrm{ret}}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right) & =\theta\left(t_{1}-t_{2}\right) \operatorname{Tr}\left\{e^{-\beta H}\left[\partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right), \partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right)\right]\right\} \\
G_{\mathrm{adv}}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right) & =\theta\left(t_{2}-t_{1}\right) \operatorname{Tr}\left\{e^{-\beta H}\left[\partial_{j} u_{j}\left(\vec{x}_{2}, t_{2}\right), \partial_{i} u_{i}\left(\vec{x}_{1}, t_{1}\right)\right]\right\} \tag{6.31}
\end{align*}
$$

For free phonons, the zero-temperature advanced and retarded correlation functions can be obtained by choosing the correct $i \delta$ prescription for the poles:

$$
\begin{align*}
& G_{\text {ret }}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right)=\int d^{3} \vec{p} d \omega e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i p^{2}}{(\omega+i \delta)^{2}-v_{l}^{2} p^{2}}  \tag{6.32}\\
& G_{\mathrm{adv}}\left(\vec{x}_{1}, t_{1} ; \vec{x}_{2}, t_{2}\right)=\int d^{3} \vec{p} d \omega e^{i\left(\vec{p} \cdot\left(\vec{x}_{1}-\vec{x}_{1}\right)-\omega\left(t_{1}-t_{2}\right)\right)} \frac{i p^{2}}{(\omega-i \delta)^{2}-v_{l}^{2} p^{2}} \tag{6.33}
\end{align*}
$$

For interacting phonons, the situation is not so simple. From (13.46), we see that for $i \omega_{n}$ in the upper-half-plane, we can obtain $\mathcal{G}$ from $G_{\text {ret }}$ by taking $\omega \rightarrow i \omega_{n}$. From (13.46), we see that $\mathcal{G}\left(-i \omega_{n}\right)=\mathcal{G}^{*}\left(i \omega_{n}\right)$, from which we can obtain $\mathcal{G}$ for $i \omega_{n}$ in the lower-half-plane. In other words, we can always obtain $\mathcal{G}$ from $G_{\text {ret }}$. What we would like to do, however, is obtain $G_{\text {ret }}$ from $\mathcal{G}$. This analytic continuation from the Matsubara frequencies $i \omega_{n}$ to the entire upper-half-plane can often be done by simply taking $i \omega_{n} \rightarrow \omega+i \delta$. In the specific examples which we will look at, this procedure works. However, there is no general theorem stating that this can always be done.

In the next chapter, we will see why retarded correlation functions are intimately related to experimental measurements.

### 6.5 Evaluating Matsubara Sums

We can use contour integration and the fact that the poles of $n_{B}(\omega)$ are precisely the Matsubara frequencies, $\omega_{n}=2 n \pi / \beta$, to convert sums over Matsubara frequencies into integrals. As we will see, it is natural to rewrite these integrals in terms of advanced and retarded Green functions.


Figure 6.1: The contour of integration in (6.35).

Consider the sum

$$
\begin{equation*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}-i \omega_{n}, \vec{p}-\vec{q}\right) \mathcal{G}\left(i \omega_{n}, \vec{q}\right) \tag{6.34}
\end{equation*}
$$

This sum is equal to the following contour integral (see figure 6.1) since the integral avoids the singularities of the Green functions; consequently, it picks up only the poles of $n_{B}(\omega)$, thereby leading to the Matsubara sum.

$$
\begin{equation*}
\oint_{C} \frac{d \omega}{2 \pi i} n_{B}(\omega) \mathcal{G}\left(i \Omega_{m}-\omega, \vec{p}-\vec{q}\right) \mathcal{G}(\omega, \vec{q})=\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}-i \omega_{n}, \vec{p}-\vec{q}\right) \mathcal{G}\left(i \omega_{n}, \vec{q}\right) \tag{6.35}
\end{equation*}
$$

The singularities of the Green functions occur when $\omega$ or $i \Omega_{m}-\omega$ are real, as may be seen from the spectral representation (13.46). The only non-vanishing segments of the contour integral are those which run on either side of the lines $\omega=E$ (the first term on the right-hand-side below) or $\omega=i \Omega_{m}-E$ (the second term) where $E$ is real:

$$
\begin{array}{r}
\oint_{C} \frac{d \omega}{2 \pi i} n_{B}(\omega) \mathcal{G}\left(\omega+i \Omega_{m}\right) \mathcal{G}(\omega)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} d E n_{B}(E) \mathcal{G}\left(i \Omega_{m}-E\right)(\mathcal{G}(E+i \delta)-\mathcal{G}(E-i \delta)) \\
+\frac{1}{2 \pi i} \int_{\infty}^{-\infty} d E n_{B}\left(i \Omega_{m}-E\right)(\mathcal{G}(E+i \delta)-\mathcal{G}(E-i \delta)) \mathcal{G}\left(i \Omega_{m}-E\right)
\end{array}
$$

Note the reverse limits of integration in the second integral; they arise from the fact that $E$ and $\omega$ are oppositely directed.

If we assume that the analytic continuation is straightforward (as it often is), then we can use (13.49) to write this as:

$$
\begin{aligned}
\oint_{C} \frac{d \omega}{2 \pi i} n_{B}(\omega) \mathcal{G}(\omega+ & \left.i \Omega_{m}\right) \mathcal{G}(\omega)=\int_{-\infty}^{\infty} d E n_{B}(E) \mathcal{G}\left(i \Omega_{m}-E, \vec{p}-\vec{q}\right) q^{2} B(E, \vec{q}) \\
& -\int_{-\infty}^{\infty} d E n_{B}\left(i \Omega_{m}-E\right) \mathcal{G}\left(i \Omega_{m}-E, \vec{q}\right)(p-q)^{2} B(E, \vec{p}-\vec{q})
\end{aligned}
$$

Since $n_{B}\left(i \Omega_{m}-E\right)=-\left(1+n_{B}(E)\right)$ for Matsubara frequencies $i \Omega_{n}$, we finally have:

$$
\begin{align*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \omega_{n}+\right. & \left.i \Omega_{m}\right) \mathcal{G}\left(i \omega_{n}\right)=\int_{-\infty}^{\infty} d E n_{B}(E) \mathcal{G}\left(i \Omega_{m}-E, \vec{p}-\vec{q}\right) q^{2} B(E, \vec{q}) \\
+ & \int_{-\infty}^{\infty} d E\left(n_{B}(E)+1\right) \mathcal{G}\left(i \Omega_{m}-E, \vec{q}\right)(p-q)^{2} B(E, \vec{p}-\vec{q}) \tag{6.36}
\end{align*}
$$

If we also continue $i \Omega_{m} \rightarrow \Omega+i \delta$, then we have:

$$
\begin{array}{r}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}-i \omega_{n}, \vec{p}-\vec{q}\right) \mathcal{G}\left(i \omega_{n}, \vec{q}\right)=\int_{-\infty}^{\infty} d E n_{B}(E) G_{\mathrm{ret}}(\Omega-E, \vec{p}-\vec{q}) q^{2} B(E, \vec{q}) \\
+\int_{-\infty}^{\infty} d E\left(n_{B}(E)+1\right) G_{\mathrm{ret}}\left(\Omega_{m}-E, \vec{q}\right)(p-q)^{2} B(E, \vec{p}-\vec{q}) \tag{6.37}
\end{array}
$$

It is important that we did the analytic continuation of the external frequency, $\Omega$, at the end of the calculation, rather than during some intermediate step. This type of contour integral trick can be used rather generally to bring Matsubara sums to a convenient form, as you will see in the problem set.

### 6.6 The Schwinger-Keldysh Contour

The formalism which we have thus far constructed was designed to determine the transition amplitudes from some given intial state at $t=-\infty$ to some final state at $t=\infty$. In the previous chapter, we were able to relate these amplitudes to the amplitude for a system to remain in its vacuum state in the presence of an external source $j(x, t)$ and, hence, to correlation functions in the vacuum state. We may, however, wish to consider a situation in which the system is in a given intial state say the vacuum state or the state at thermal equilibrium at inverse temperature $\beta$ but we make no assumptions about the final state. We may then wish to compute the correlation functions of some observable such as $\partial_{i} u_{i}(x, t)$ at time $t$.

In order to do this, we imagine evolving the system from some initial state $|i\rangle$ at $t=-\infty$ to $t=\infty$ and then back to $t=-\infty$. The evolution operator for such a process is:

$$
\begin{equation*}
U(\infty,-\infty) U(\infty,-\infty) \tag{6.38}
\end{equation*}
$$

Clearly, this is simply equal to 1 and $^{1}$

$$
\begin{equation*}
\langle i| U(-\infty, \infty) U(\infty,-\infty)|i\rangle=1 \tag{6.39}
\end{equation*}
$$

Suppose, however, that we switch on a source field, $j(x, t)$, only during the forward propagation in time. Then

$$
\begin{equation*}
\langle i| U(-\infty, \infty) U_{j}(\infty,-\infty)|i\rangle \neq 1 \tag{6.40}
\end{equation*}
$$

[^1]and, by differentiating with respect to $j(x, t)$, we can obtain correlation functions.
If we wish to work with a system at zero-temperature which is in its ground state at $t=-\infty$, then we define the generating functional:
\[

$$
\begin{equation*}
Z[j]=\langle 0| U(-\infty, \infty) U_{j}(\infty,-\infty)|0\rangle \tag{6.41}
\end{equation*}
$$

\]

At finite temperature, we have

$$
\begin{equation*}
Z[j]=\operatorname{Tr}\left\{e^{-\beta H} U(-\infty, \infty) U_{j}(\infty,-\infty)\right\} \tag{6.42}
\end{equation*}
$$

or

$$
\begin{equation*}
Z[j]=\operatorname{Tr}\left\{U(-\infty-i \beta,-\infty) U(-\infty, \infty) U_{j}(\infty,-\infty)\right\} \tag{6.43}
\end{equation*}
$$

i.e. we evolve the system from $t=-\infty$ to $t=\infty$ then back to $t=-\infty$ and thence to $t=-\infty-i \beta$.

This generating functional allows us to calculate non-equilibrium correlation functions: $j(x, t)$ can drive the system out of equilibrium since we make no assumptions about the final state of the system. The price which must be paid is the doubling of the number of the fields in the theory; the second copy of each field propagates backwards in time.

The Keldysh contour which we just described is just one example of a possible contour along which the time-evolution is performed. There is a more general class of contours, $C$, which go from $-\infty$ to $\infty$; from $\infty$ to $\infty-i \sigma$; from $\infty-i \sigma$ to $-\infty-i \sigma$; and thence to $\infty-i \beta$. We make the choice $\sigma=\beta / 2$ for which the propagator takes a particularly simple form; however, this is a matter of taste. All choices of $\sigma$ share the advantage of being real-time formulations and thereby obviating the need for potentially ill-defined analytical continuations.

There is an important factorization property, which we won't prove here, for the contributions from each piece of the contour to the functional integral $Z$ :

$$
\begin{equation*}
Z=Z^{C_{1} \cup C_{2}} Z^{C_{3} \cup C_{4}} \tag{6.44}
\end{equation*}
$$



Figure 6.2: Real-time contour separated into four parts that factorize into separate contributions: $C_{1} \cup C_{2}$ and $C_{3} \cup C_{4}$.
where

$$
\begin{align*}
Z & =\operatorname{Tr}\left\{U(-\infty-i \beta,-\infty-i \sigma) U(-\infty-i \sigma, \infty-i \sigma) U(\infty-i \sigma, \infty) U_{j}(\infty,-\infty)\right\} \\
& =\operatorname{Tr}\left\{T_{c} e^{-i \int_{C} d t H_{\mathrm{int}}(t)}\right\} \tag{6.45}
\end{align*}
$$

Only $C_{1}$ and $C_{2}$ are important in obtaining correlation functions. Using Dyson's formula for $U\left(t_{f}, t_{i}\right)$, we can expand $Z=Z^{C_{1} \cup C_{2}}$ perturbatively as we did in the equilibrium zero-temperature and imaginary-time formalisms. We can use Wick's theorem to evaluate $T_{c}$-ordered products. To construct the resulting perturbation theory, it is useful to denote the fields on the upper $\left(C_{1}\right)$ and the lower $\left(C_{2}\right)$ pieces of the countour by

$$
\begin{equation*}
u_{i}^{1}(t)=u_{i}(t), \quad u_{i}^{2}(t)=u_{i}(t-i \sigma), \quad(t=\text { real }) \tag{6.46}
\end{equation*}
$$

The Feynman rules are similar to those of our equilibrium zero-temperature theory with the major difference being each vertex is labelled by an index $a=1,2$ and the amplitude is assigned a factor of -1 for each $a=2$ vertex. The vertex resulting from the source field $j(x, t)$ is assigned $a=1$. The propagator between two vertices with indices $a$ and $b$ is given by:

$$
\begin{equation*}
-i \Delta^{a b}\left(t-t^{\prime}, x-x^{\prime}\right)=<T_{c}\left[\partial_{i} u_{i}^{a}(t, x) \partial_{j} u_{j}^{b}\left(t^{\prime}, x^{\prime}\right)\right]> \tag{6.47}
\end{equation*}
$$

where $T_{c}$ denotes ordering of fields according their position along the contour of Fig.
6.2. For the Keldysh contour, the diagonal elements of the propagator are the realtime zero-temperature time- and anti-time ordered propagators. The off-diagonal element contains all of the information about the occupation numbers of states; in thermal equilibrium the occupation numbers are given by $n_{B}(\omega)$, but they can be more general. For our choice, $\sigma=\beta / 2$, the dynamical information contained in the zero-temperature propagator and the information about occupation numbers can be untangled by the parametrization:

$$
\begin{equation*}
i \Delta(\omega, k)=u(\omega) i \Delta_{0}(\omega, k) u^{\dagger}(\omega) \tag{6.48}
\end{equation*}
$$

where

$$
i \Delta_{0}(\omega, k)=\left(\begin{array}{cc}
i G_{0}(\omega, k) & 0  \tag{6.49}\\
0 & -i G_{0}^{*}(\omega, k)
\end{array}\right)
$$

with $i G_{0}(\omega, k)$ the usual time-ordered propagator

$$
\begin{equation*}
i G_{0}\left(t-t^{\prime}, x-x^{\prime}\right)=\left\langle T\left[\partial_{i} u_{i}(t, x) \partial_{j} u_{j}\left(t^{\prime}, x^{\prime}\right)\right]\right\rangle \tag{6.50}
\end{equation*}
$$

and $-i G_{0}^{*}(\omega, k)$, consequently, the anti-time-ordered one

$$
\begin{equation*}
-i G_{0}^{*}\left(t-t^{\prime}, x-x^{\prime}\right)=\left(\left\langle T\left[\partial_{i} u_{i}(t, x) \partial_{i} u_{i}\left(t^{\prime}, x^{\prime}\right)\right]\right\rangle\right)^{*}=\left\langle\bar{T}\left[\partial_{i} u_{i}\left(t^{\prime}, x^{\prime}\right) \partial_{i} u_{i}(t, x)\right]\right\rangle \tag{6.51}
\end{equation*}
$$

The matrix $u$ contains the information about the temperature. This matrix is given by

$$
\begin{gather*}
u(\omega)=\left(\begin{array}{cc}
\cosh \Delta \theta_{\omega} & \sinh \Delta \theta_{\omega} \\
\sinh \Delta \theta_{\omega} & \cosh \Delta \theta_{\omega}
\end{array}\right), \quad \text { where } \Delta \theta_{\omega}=\theta_{\omega}^{T}-\theta_{\omega}^{T=0}, \\
\text { and } \quad \cosh ^{2} \theta_{\omega}^{T}=\frac{1}{1-e^{-\omega / T}} . \tag{6.52}
\end{gather*}
$$

Notice that at zero temperature $u=\mathbf{1}$.
To summarize,

- To each line, we associate a momentum, $\vec{p}$ and a frequency, $\omega$.
- To each vertex we assign an index $a=1,2$. External lines are assigned $a=1$ at their free end.
- For each external line with momentum $\vec{p}$, write $|\vec{p}|$.
- The propagator assigned to an internal line carrying momentum $\vec{p}$ and frequency $\omega$ which connectes vertices labelled by indices $a$ and $b$ is:

$$
\Delta_{0}^{a b}(\vec{p}, \omega)
$$

- For each vertex carrying index $a$ with momenta, frequencies $\left(\vec{p}_{1}, \omega_{1}\right), \ldots,\left(\vec{p}_{4}, \omega_{4}\right)$ directed into the vertex, we write

$$
(3-2 a) g(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right) \delta\left(\omega_{1}+\omega_{2}+\omega_{3}+\omega_{4}\right)
$$

- Imagine labelling the vertices $1,2, \ldots, n$. Vertex $i$ will be connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices $i, i$ is still connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. If $S$ is the number of permutations which leave the diagram invariant, we assign a factor $1 / S$ to the diagram.
- If two vertices are connected by $l$ lines, we assign a factor $1 / l$ ! to the diagram.

In equilibrium, the Schwinger-Keldysh formalism gives results which are identical to those of the Matsubara formalism. Out of equilibrium, however, the two formalisms give different results; only the Schwinger-Keldysh is correct.

## Chapter 7

## Measurements and Correlation Functions

### 7.1 A Toy Model

We will now take a break from our development of techniques for calculating correlation functions to relate retarded correlation functions to experimental measurements. We will also discuss those properties of retarded correlation functions which follow from causality, symmetries, and conservation laws.

Let us take a look at a toy model to see why retarded correlation functions are useful objects which are simply related to experimentally measurable quantities. Consider a single damped harmonic oscillator, with equation of motion

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}+\gamma \frac{d x}{d t}+\omega_{0}^{2} x=F^{\mathrm{ext}}(t) \tag{7.1}
\end{equation*}
$$

We define the retarded response function, $\chi_{\mathrm{ret}}\left(t-t^{\prime}\right)$ by

$$
\begin{equation*}
x(t)=\int_{-\infty}^{\infty} d t^{\prime} \chi_{\mathrm{ret}}\left(t-t^{\prime}\right) F^{\mathrm{ext}}\left(t^{\prime}\right) \tag{7.2}
\end{equation*}
$$

By causality, $\chi_{\mathrm{ret}}\left(t-t^{\prime}\right)$ must vanish for $t-t^{\prime}<0$. Substituting the definition of $\chi_{\mathrm{ret}}\left(t-t^{\prime}\right),(7.2)$ into the equation of motion (7.1), we see that $\chi_{\mathrm{ret}}\left(t-t^{\prime}\right)$ satisfies:

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \chi_{\mathrm{ret}}\left(t-t^{\prime}\right)+\gamma \frac{d}{d t} \chi_{\mathrm{ret}}\left(t-t^{\prime}\right)+\omega_{0}^{2} \chi_{\mathrm{ret}}\left(t-t^{\prime}\right)=\delta\left(t-t^{\prime}\right) \tag{7.3}
\end{equation*}
$$

Thus, $\chi_{\text {ret }}\left(t-t^{\prime}\right)$ is the Green function of the differential operator on the left-hand-side of (7.1) subject to the boundary condition that $\chi_{\text {ret }}\left(t-t^{\prime}\right)=0$ for $t-t^{\prime}<0$.

We can define the Fourier transform of $\chi$ :

$$
\begin{equation*}
\chi(\omega)=\int_{-\infty}^{\infty} d t e^{i \omega t} \chi(t)=\int_{0}^{\infty} d t e^{i \omega t} \chi(t) \tag{7.4}
\end{equation*}
$$

Since $\chi(t)$ vanishes for $t<0$, the integral is well-defined for $\omega$ anywhere in the upper-half-plane. Therefore, $\chi(\omega)$ is an analytic function in the upper-half-plane.

Substituting the Fourier representation of $\chi(t)$ in the equation of motion, we find $\chi(\omega)$ :

$$
\begin{equation*}
\chi(\omega)=\frac{1}{\omega_{0}^{2}-\omega^{2}-i \gamma \omega} \tag{7.5}
\end{equation*}
$$

We can break $\chi(\omega)$ into its real and imaginary parts:

$$
\begin{equation*}
\chi(\omega)=\chi^{\prime}(\omega)+i \chi^{\prime \prime}(\omega) \tag{7.6}
\end{equation*}
$$

From (7.5), we have in our toy model:

$$
\begin{align*}
\chi^{\prime}(\omega) & =\frac{\omega_{0}^{2}-\omega^{2}}{\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+(\gamma \omega)^{2}} \\
\chi^{\prime \prime}(\omega) & =\frac{\gamma \omega}{\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+(\gamma \omega)^{2}} \tag{7.7}
\end{align*}
$$

From the above definition,

$$
\begin{align*}
\chi^{\prime \prime}(\omega) & =\operatorname{Im}\left\{\int_{-\infty}^{\infty} d t e^{i \omega t} \chi(t)\right\} \\
& =\int_{-\infty}^{\infty} d t \frac{1}{2 i}\left(e^{i \omega t}-e^{-i \omega t}\right) \chi(t) \\
& =\int_{-\infty}^{\infty} d t e^{i \omega t} \frac{1}{2 i}(\chi(t)-\chi(-t)) \tag{7.8}
\end{align*}
$$

Similarly,

$$
\begin{equation*}
\chi^{\prime}(\omega)=\int_{-\infty}^{\infty} d t e^{i \omega t} \frac{1}{2}(\chi(t)+\chi(-t)) \tag{7.9}
\end{equation*}
$$

Thus, $\chi^{\prime \prime}(\omega)$ is the Fourier transform of the part of $\chi(\omega)$ which is not invariant under $t \rightarrow-t$ while $\chi^{\prime}(\omega)$ is the Fourier transform of the part of $\chi(\omega)$ which is invariant
under $t \rightarrow-t$. In other words, $\chi^{\prime \prime}(\omega)$ knows about the arrow of time which is implicit in the condition $\chi\left(t-t^{\prime}\right)$ vanishes for $t-t^{\prime}<0 . \chi^{\prime}(\omega)$, on the other hand, does not.

This is reflected in the fact that $\chi^{\prime \prime}(\omega)$ determines the dissipative response. To see this, suppose that we apply a force, $F(t)$. The work done by this force is the energy which is tranferred to the system - i.e. the dissipation:

$$
\begin{align*}
\frac{d W}{d t} & =F(t) \frac{d x}{d t} \\
& =F(t) \frac{d}{d t} \int_{-\infty}^{\infty} d t^{\prime} \chi_{\mathrm{ret}}\left(t-t^{\prime}\right) F\left(t^{\prime}\right) \\
& =F(t) \int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} e^{i \omega t} i \omega \chi(\omega) F(\omega) \\
& =\int \frac{d \omega}{2 \pi} \frac{d \omega^{\prime}}{2 \pi} e^{i\left(\omega+\omega^{\prime}\right) t} i \omega \chi(\omega) F(\omega) F\left(\omega^{\prime}\right) \tag{7.10}
\end{align*}
$$

If we assume $F(t)=F_{0} \cos \Omega_{0} t$ and compute the zero-frequency part of $d W / d t$ (rather than the part which oscillates at $2 \omega_{0}$ ), we find:

$$
\begin{align*}
\frac{d W}{d t}(\omega=0) & =\frac{1}{4} F_{0}^{2} i\left(\Omega_{0} \chi\left(\Omega_{0}\right)-\Omega_{0} \chi\left(-\Omega_{0}\right)\right) \\
& =\frac{1}{2} F_{0}^{2} \Omega_{0} \chi^{\prime \prime}\left(\Omega_{0}\right) \tag{7.11}
\end{align*}
$$

The essential reason that $\chi^{\prime}$ doesn't enter the dissipation is that the time-reversal symmetry of $\chi^{\prime}$ implies that the energy gain and loss due to $\chi^{\prime}$ are the same. $\chi^{\prime}$ is often called the reactive part of the susceptibility while $\chi^{\prime \prime}$ is the dissipative or absorptive part.

For our toy model, the energy dissipation,

$$
\begin{equation*}
\frac{d W}{d t}=\frac{1}{2} F_{0}^{2} \Omega_{0} \frac{\gamma \Omega_{0}}{\left(\omega_{0}^{2}-\Omega_{0}^{2}\right)^{2}+\left(\gamma \Omega_{0}\right)^{2}} \tag{7.12}
\end{equation*}
$$

is maximum at $\Omega_{0}= \pm \omega_{0}$, i.e. on resonance. Consider the resonance $\Omega_{0} \approx \omega_{0}$. Approximating $\Omega_{0}+\omega_{0} \approx 2 \omega_{0}$, we have a Lorentzian lineshape:

$$
\begin{equation*}
\frac{d W}{d t}=\frac{1}{2} F_{0}^{2} \frac{\gamma}{4\left(\omega_{0}-\Omega_{0}\right)^{2}+\gamma^{2}} \tag{7.13}
\end{equation*}
$$



Figure 7.1: The countour of integration taken for the Kramers-Kronig relation.
with full-width $\gamma$ at half-maximum.
As a result of the analyticity of $\chi$ in the upper-half-plane - i.e. as a result of causality - we have the Kramers-Kronig relations. The analyticity of $\chi(z)$ in the upper-half-plane implies that:

$$
\begin{equation*}
\oint_{C} \frac{d z}{\pi i} \frac{\chi(z)}{z-\omega}=0 \tag{7.14}
\end{equation*}
$$

for the contour of figure 7.1 The integral of the semicircular part of the contour vanishes, so we have:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d \omega^{\prime}}{\pi i} \frac{\chi\left(\omega^{\prime}\right)}{\omega^{\prime}+i \epsilon-\omega}=0 \tag{7.15}
\end{equation*}
$$

Using the relation:

$$
\begin{equation*}
\frac{1}{\omega^{\prime}+i \epsilon-\omega}=\mathcal{P}\left(\frac{1}{\omega^{\prime}-\omega}\right)-i \pi \delta\left(\omega^{\prime}-\omega\right) \tag{7.16}
\end{equation*}
$$

we have

$$
\begin{equation*}
\chi(\omega)=-i \mathcal{P} \int_{-\infty}^{\infty} \frac{d \omega^{\prime}}{\pi} \frac{\chi\left(\omega^{\prime}\right)}{\omega^{\prime}-\omega} \tag{7.17}
\end{equation*}
$$

Comparing real and imaginary parts, we have the Kramers-Kronig relations:

$$
\begin{align*}
\chi^{\prime}(\omega) & =\mathcal{P} \int_{-\infty}^{\infty} \frac{d \omega^{\prime}}{\pi} \frac{\chi^{\prime \prime}\left(\omega^{\prime}\right)}{\omega^{\prime}-\omega} \\
\chi^{\prime \prime}(\omega) & =-\mathcal{P} \int_{-\infty}^{\infty} \frac{d \omega^{\prime}}{\pi} \frac{\chi^{\prime}\left(\omega^{\prime}\right)}{\omega^{\prime}-\omega} \tag{7.18}
\end{align*}
$$

### 7.2 General Formulation

We can cull the essential features of the toy model and apply them to a many-body system such as our phonon theory. It is fairly clear that this can be done for a free field theory which is, after all, just a set of harmonic oscillators. As we will see momentarily, this can be done - at least at a formal level - for any theory.

Suppose our external probe couples to our system through a term in the Hamiltonian of the form:

$$
\begin{equation*}
H_{\text {probe }}=\int d^{3} \vec{x} \phi(\vec{x}, t) f(\vec{x}, t) \tag{7.19}
\end{equation*}
$$

$f(\vec{x}, t)$ is our external probe, which we control, and $\phi(\vec{x}, t)$ is some quantum field describing the system. In our phonon theory, we could take $\phi(\vec{x}, t)=\partial_{k} u_{k}(\vec{x}, t)$ in which case our probe compresses the solid. Alternatively, we could take $\phi(\vec{x}, t)=$ $u_{3}(\vec{x}, t)$, causing displacements along the 3 -direction. We will work - as in the last chapter when we defined Green functions - in an interaction representation in which $H_{\text {probe }}$ is the interaction Hamiltonian and the rest of the Hamiltonian is the 'free' Hamiltonian. Let us suppose that we now measure the field $\eta(\vec{x}, t)$, which may or may not be the same as $\phi(\vec{x}, t)$. Its expectation value is given by:

$$
\begin{equation*}
\langle\eta(\vec{x}, t)\rangle=\langle 0| U_{I}^{-1}(t,-\infty) \eta(\vec{x}, t) U_{I}(t,-\infty)|0\rangle \tag{7.20}
\end{equation*}
$$

where

$$
\begin{align*}
U_{I}(t,-\infty) & =T\left\{e^{-i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)}\right\} \\
& =1-i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)+\ldots \tag{7.21}
\end{align*}
$$

If we keep only terms up to first-order in $H_{\text {probe }}$, then we have:

$$
\begin{aligned}
\langle\eta(\vec{x}, t)\rangle & =\langle 0|\left(1+i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)\right) \eta(\vec{x}, t)\left(1-i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)\right)|0\rangle \\
& =\langle 0| \eta(\vec{x}, t)|0\rangle_{0}+\langle 0| i \int_{-\infty}^{t} d t^{\prime}\left[H_{\text {probe }}\left(t^{\prime}\right), \eta(\vec{x}, t)\right]|0\rangle
\end{aligned}
$$

$$
\left.=\langle 0| \eta(\vec{x}, t)|0\rangle_{0}+i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{t} d t^{\prime} f\left(\vec{x}^{\prime}, t^{\prime}\right)\langle 0|\left[\phi\left(\vec{x}^{\prime}, t^{\prime}\right), \eta(\vec{x}, t)\right] \mid 0 \nmid 7.22\right)
$$

We have added a subscript 0 to emphasize that these are interaction picture states - i.e. these are expectation values in the absence of the probe. Let us assume, as is usually the case, that

$$
\begin{equation*}
\langle 0| \eta(\vec{x}, t)|0\rangle_{0}=0 \tag{7.23}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\langle\eta(\vec{x}, t)\rangle=i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{t} d t^{\prime} f\left(\vec{x}^{\prime}, t^{\prime}\right)\langle 0|\left[\phi\left(\vec{x}^{\prime}, t^{\prime}\right), \eta(\vec{x}, t)\right]|0\rangle \tag{7.24}
\end{equation*}
$$

The commutator on the right-hand-side, $\langle 0|\left[\phi\left(\vec{x}^{\prime}, t^{\prime}\right), \eta(\vec{x}, t)\right]|0\rangle$, is an example of a response function.

Let us specialize to the case $\eta(\vec{x}, t)=\phi\left(\vec{x}^{\prime}, t^{\prime}\right)$. Then, we write:

$$
\begin{equation*}
\langle\eta(\vec{x}, t)\rangle=-i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{t} d t^{\prime} f\left(\vec{x}^{\prime}, t^{\prime}\right) \chi\left(\vec{x}, \vec{x} ; t, t^{\prime}\right) \tag{7.25}
\end{equation*}
$$

If the Hamiltonian is space- and time-translationally in the absence of $H_{\text {probe }}$, then we can write:

$$
\begin{equation*}
\chi\left(\vec{x}, \vec{x}^{\prime} ; t, t^{\prime}\right)=\chi\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right) \tag{7.26}
\end{equation*}
$$

We can also extend the $d t^{\prime}$ integral to $\infty$

$$
\begin{equation*}
\langle\eta(\vec{x}, t)\rangle=-i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{\infty} d t^{\prime} f\left(\vec{x}^{\prime}, t^{\prime}\right) \chi\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right) \tag{7.27}
\end{equation*}
$$

if we define

$$
\begin{equation*}
\chi\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right) \equiv i \theta\left(t-t^{\prime}\right)\langle 0|\left[\phi(\vec{x}, t), \phi\left(\vec{x}^{\prime}, t^{\prime}\right)\right]|0\rangle \tag{7.28}
\end{equation*}
$$

As in our toy model, we can define the Fourier transform with respect to time, $\chi\left(\vec{x}-\vec{x}^{\prime}, \omega\right)$ and its real and imaginary parts,

$$
\begin{equation*}
\chi\left(\vec{x}-\vec{x}^{\prime}, \omega\right)=\chi^{\prime}\left(\vec{x}-\vec{x}^{\prime}, \omega\right)+i \chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, \omega\right) \tag{7.29}
\end{equation*}
$$

Following the steps of (7.8), we see that $\chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, \omega\right)$ is the Fourier transform of the commutator without a $\theta$-function:

$$
\begin{equation*}
\chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right)=\int \frac{d \omega}{2 \pi} e^{-i \omega\left(t-t^{\prime}\right)} \chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, \omega\right) \tag{7.30}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right)=\langle 0|\left[\phi(\vec{x}, t), \phi\left(\vec{x}^{\prime}, t^{\prime}\right)\right]|0\rangle \tag{7.31}
\end{equation*}
$$

As in our toy model, $\chi^{\prime \prime}$, satisfies the antisymmetry properties:

$$
\begin{align*}
\chi^{\prime \prime}\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right) & =-\chi^{\prime \prime}\left(\vec{x}^{\prime}-\vec{x}, t^{\prime}-t\right) \\
\chi^{\prime \prime}\left(\vec{x}^{\prime}-\vec{x}, \omega\right) & =-\chi^{\prime \prime}\left(\vec{x}^{\prime}-\vec{x},-\omega\right) \tag{7.32}
\end{align*}
$$

These properties follow from the fact that $\chi^{\prime \prime}$ is a commutator.
Following the steps which we took in our toy model, the dissipation rate under the influence of a periodic probe, $f(\vec{x}, t)=f_{\omega}(\vec{x}) \cos \omega t$ is given by

$$
\begin{equation*}
\frac{d w}{d t}=\int d^{3} \vec{x} d^{3} \vec{x}^{\prime} \omega \chi^{\prime \prime}\left(\vec{x}^{\prime}-\vec{x}, \omega\right) f_{\omega}(\vec{x}) f_{\omega}\left(\vec{x}^{\prime}\right) \tag{7.33}
\end{equation*}
$$

We can also follow the derivation for our toy model to show that $\chi\left(\vec{x}^{\prime}-\vec{x}, \omega\right)$ satisfies the Kramers-Kronig relations (7.18). The Kramers-Kronig relations can be approached from a different angle by constructing a spectral representation for the response function (7.28). Following our earlier derivations of spectral representations, we have:

$$
\begin{equation*}
\chi(\vec{q}, \omega)=\int_{-\infty}^{\infty} d E \frac{\rho(\vec{q}, E)}{E-\omega-i \delta} \tag{7.34}
\end{equation*}
$$

where

$$
\begin{equation*}
\left.\rho(\vec{q}, E)=\sum_{m}|\langle m| \phi(0,0)| 0\right\rangle\left.\right|^{2} \delta\left(\vec{q}-\vec{p}_{m}\right) \delta\left(E-\omega_{m}\right) \tag{7.35}
\end{equation*}
$$

If we construct a spectral representation for $\chi^{\prime \prime}(\vec{q}, \omega)$ - which can be done trivially since there are no $\theta$-functions - or simply take the imaginary part of (7.34), we see that:

$$
\begin{equation*}
\chi^{\prime \prime}(\vec{q}, \omega)=\pi \rho(\vec{q}, \omega) \tag{7.36}
\end{equation*}
$$

In other words, $\chi^{\prime \prime}(\vec{q}, \omega)$ is the spectral function for $\chi(\vec{q}, \omega)$ :

$$
\begin{equation*}
\chi(\vec{q}, \omega)=\int_{-\infty}^{\infty} \frac{d E}{\pi} \frac{\chi^{\prime \prime}(\vec{q}, E)}{E-\omega-i \delta} \tag{7.37}
\end{equation*}
$$

which is the Kramers-Kronig relation. According to $(7.37) \chi(\vec{q}, \omega)$ is singular whenever $\chi^{\prime \prime}(\vec{q}, \omega)$ is non-vanishing. These are the regions of phase space where there are states of the system with which the external probe can resonate, thereby causing dissipation.

### 7.3 The Fluctuation-Dissipation Theorem

Consider the correlation function:

$$
\begin{equation*}
S_{\eta \phi}(\vec{x}, t)=\operatorname{Tr}\left(e^{-\beta H} \eta(\vec{x}, t) \phi(0,0)\right) \tag{7.38}
\end{equation*}
$$

The response function can be expressed in terms of the correlation function:

$$
\begin{equation*}
\chi_{\eta \phi}(\vec{x}, t)=\theta(t)\left(S_{\eta \phi}(\vec{x}, t)-S_{\phi \eta}(-\vec{x},-t)\right) \tag{7.39}
\end{equation*}
$$

and its dissipative part is simply

$$
\begin{equation*}
\chi_{\eta \phi}^{\prime \prime}(\vec{x}, t)=\left(S_{\eta \phi}(\vec{x}, t)-S_{\phi \eta}(-\vec{x},-t)\right) \tag{7.40}
\end{equation*}
$$

or

$$
\begin{equation*}
\chi_{\eta \phi}^{\prime \prime}(\vec{x}, \omega)=\left(S_{\eta \phi}(\vec{x}, \omega)-S_{\phi \eta}(-\vec{x},-\omega)\right) \tag{7.41}
\end{equation*}
$$

By the cyclic property of the trace,

$$
\begin{aligned}
S_{\eta \phi}(\vec{x}, t) & =\operatorname{Tr}\left(e^{-\beta H} \eta(\vec{x}, t) \phi(0,0)\right) \\
& =\operatorname{Tr}\left(\phi(0,0) e^{-\beta H} \eta(\vec{x}, t)\right) \\
& =\operatorname{Tr}\left(e^{-\beta H} e^{\beta H} \phi(0,0) e^{-\beta H} \eta(\vec{x}, t)\right) \\
& =\operatorname{Tr}\left(e^{-\beta H} \phi(0,-i \beta) \eta(\vec{x}, t)\right)
\end{aligned}
$$

$$
\begin{equation*}
=S_{\phi \eta}(-\vec{x},-t-i \beta) \tag{7.42}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
S_{\eta \phi}(\vec{x}, \omega)=e^{\beta \omega} S_{\phi \eta}(-\vec{x},-\omega) \tag{7.43}
\end{equation*}
$$

Thus, we finally have:

$$
\begin{equation*}
\chi_{\eta \phi}{ }^{\prime \prime}(\vec{x}, \omega)=\left(1-e^{-\beta \omega}\right) S_{\eta \phi}(\vec{x}, \omega) \tag{7.44}
\end{equation*}
$$

Since the right-hand-side is a measure of the dissipation and the left-hand-side is a measure of the fluctuation, (7.44) is called the fluctuation-dissipation theorem. As we will see shortly, neutron scattering experiments measure $S_{\rho \rho}(q, \omega)$, and the fluctuation-dissipation theorem relates this to a quantity which we can attempt to calculate using the imaginary-time formalism: the imaginary part of a retarded correlation function, $\chi_{\rho \rho}{ }^{\prime \prime}(q, \omega)$.

### 7.4 Perturbative Example

Let us consider the case in which $\phi=\eta=\partial_{t} u_{1} . \partial_{t} u_{1}$ is the current in the $x_{1}$-direction carried by the ions in the solid, so we are driving the solid in the $x_{1}$-direction and measuring the subsequent flow of the positive ions in this direction. Then $\chi_{\phi \phi}$ is given by the retarded correlation function of $\partial_{t} u_{1}$ with itself. Let us make the simplifying assumption that $v_{l}=v_{t}$. According to the $i \delta$ prescription of (6.32), for free phonons this is:

$$
\begin{equation*}
\chi_{\phi \phi}(\omega, \vec{q})=\frac{\omega^{2}}{(\omega+i \delta)^{2}-v_{l}^{2} q^{2}} \tag{7.45}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\chi_{\phi \phi}^{\prime \prime}=\omega^{2} \operatorname{sgn}(\omega) \delta\left(\omega^{2}-v_{l}^{2} q^{2}\right) \tag{7.46}
\end{equation*}
$$

In other words, there will only be dissipation if the compressional force has a component with $\omega^{2}=v_{l}^{2} q^{2}$. Thus a measurement of this response function is a direct measurment of $v_{l}$, i.e. of the phonon spectrum.

If the phonons are interacting,

$$
\begin{equation*}
\chi_{\phi \phi}(\omega, \vec{q})=\frac{\omega^{2}}{(\omega+i \delta)^{2}-v_{l}^{2} q^{2}+\Pi_{r e t}(\omega, q)} \tag{7.47}
\end{equation*}
$$

and the $\delta$-function is broadened:

$$
\begin{equation*}
\chi_{\phi \phi}^{\prime \prime}=\frac{\omega^{2} \operatorname{Im}\left\{\Pi_{r e t}(\omega, q)\right\}}{\left(\omega^{2}-v_{l}^{2} q^{2}+\operatorname{Re}\left\{\Pi_{r e t}(\omega, q)\right\}\right)^{2}+\left(\operatorname{Im}\left\{\Pi_{r e t}(\omega, q)\right\}\right)^{2}} \tag{7.48}
\end{equation*}
$$

Consider a perturbative computation of $\Pi_{r e t}(\omega, q)$. At $O(g)$, there is diagram (a) of figure ??. This diagram gives a purely real contribution

$$
\begin{equation*}
\Pi\left(i \omega_{n}, q\right)=-\frac{g}{2} q^{2} \frac{1}{\beta} \sum_{n} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}}|\vec{p}|^{2} \frac{1}{\omega_{n}^{2}+v_{l}^{2} p^{2}}+O\left(g^{2}\right) \tag{7.49}
\end{equation*}
$$

In the problem set, you will compute the Matsubara sum. At zero temperature, we find:

$$
\begin{align*}
\Pi\left(i \omega_{n}, q\right) & =\frac{g}{2} q^{2} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{p}{v}+O\left(g^{2}\right) \\
& =\text { (const.) } g q^{2} \Lambda^{4} / v+O\left(g^{2}\right) \\
& =\left(\delta v_{l}^{2}\right) q^{2} \tag{7.50}
\end{align*}
$$

where $\left(\delta v_{l}^{2}\right)=$ (const.) $g \Lambda^{4} / v$. The analytic continuation to the retarded Green function is trivial.

The first contribution to the imaginary part of the self-energy comes from diagram (b) of ??. It is given by

$$
\begin{array}{r}
\operatorname{Im}\left\{\Pi\left(i \omega_{n}, q\right)\right\}=\operatorname{Im}\left\{q^{2} \frac{g^{2}}{6} \frac{1}{\beta^{2}} \sum_{n_{1}, n_{2}} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d^{3} \vec{p}_{2}}{(2 \pi)^{3}} \mathcal{G}\left(i \omega_{n}-i \omega_{n_{1}}-i \omega_{n_{2}}, q-p_{1}-p_{2}\right)\right. \\
\left.\mathcal{G}\left(i \omega_{n_{1}}, p_{1}\right) \mathcal{G}\left(i \omega_{n_{2}}, p_{2}\right)\right\}_{i \omega_{n} \rightarrow \omega+i \delta}+O\left(g^{3}\right) \tag{7.51}
\end{array}
$$

Evaluating this integral is tedious, but we can make some simple observations. When we convert the two Matsubara sums to integrals, we will convert the Green functions to spectral functions; taking the imaginary part will convert the final one to a spectral function:

$$
\operatorname{Im}\left\{\Pi\left(i \omega_{n}, q\right)\right\}=\operatorname{Im}\left\{q^{2} \frac{g^{2}}{6} \int d \omega_{1} \int d \omega_{2} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d^{3} \vec{p}_{2}}{(2 \pi)^{3}} \mathcal{G} B B\right\}_{i \omega_{n} \rightarrow \omega+i \delta}+O\left(g^{3}\right)
$$

$$
\begin{align*}
&= \operatorname{Im}\left\{q^{2} \frac{g^{2}}{6} \int d \omega_{1} \int d \omega_{2} \int \frac{d^{3} \vec{p}_{1}}{(2 \pi)^{3}} \frac{d^{3} \vec{p}_{2}}{(2 \pi)^{3}} B B B\right\}_{i \omega_{n} \rightarrow \omega+i \delta} \\
&+O\left(g^{3}\right) \tag{7.52}
\end{align*}
$$

By taking the imaginary part, we have put the three internal phonon lines on-shell. There is no phase space for the $\delta$-functions to be satisfied if $\omega=0$ (since $\vec{p}_{1}$ and $\vec{p}_{2}$ will have to be collinear), so the integral is propotional to $\omega$. Hence, we have:

$$
\begin{equation*}
\operatorname{Im}\left\{\Pi\left(i \omega_{n}, q\right)\right\}=D q^{2} \omega+O\left(q^{4} \omega\right)+O\left(g^{3}\right) \tag{7.53}
\end{equation*}
$$

where $D=$ (const.) $g^{2} \Lambda^{7}$. Keeping only the first term, we can now write the spectral function as:

$$
\begin{equation*}
\chi_{\phi \phi}^{\prime \prime}=\frac{\omega^{2}\left(D q^{2} \omega\right)}{\left(\omega^{2}-\tilde{v}_{l}^{2} q^{2}\right)^{2}+\left(D q^{2} \omega\right)^{2}} \tag{7.54}
\end{equation*}
$$

where $\tilde{v}_{l}^{2}=v_{l}^{2}+\delta v_{l}^{2}$.
This is the form of the response function which we expect when $g$ is small and $\Pi_{r e t}(\omega, q)$ can be calculated perturbatively. In such a case, the corrections due to $\Pi_{r e t}(\omega, q)$ are small and lead to a small damping of a propagating mode. In general, however, the calculation of the response function, $\chi$, is a difficult problem. Nevertheless, we can often say something about $\chi$ since some of its general features follow from conservation laws and symmetries. The resulting equations satisfied by physical quantities (including response functions) are hydrodynamic equations.

### 7.5 Hydrodynamic Examples

Let us consider as an example some particles dissolved in a fluid. The density, $\rho$, and current, $\vec{J}$ of these particles will satisfy a conservation law:

$$
\begin{equation*}
\frac{\partial}{\partial t} \rho+\vec{\nabla} \cdot \vec{J}=0 \tag{7.55}
\end{equation*}
$$

Unlike in the case of a propagating mode, $\rho$ and $\vec{J}^{M}$ will satisfy a constitutive relation:

$$
\begin{equation*}
\vec{J}=-D \vec{\nabla} \rho+\vec{f}_{\mathrm{ext}} \tag{7.56}
\end{equation*}
$$

where $D$ is the diffusion constant and $B_{\text {ext }}$ is an external force acting on the particles (such as gravity). Ideally, we would like to compute $D$ (perturbatively or by going beyond perturbation theory), but in many cases we must leave it as a phenomenological parameter. As a result of the constitutive relation, $\rho$ satisfies the diffusion equation:

$$
\begin{equation*}
\frac{\partial}{\partial t} \rho-D \nabla^{2} \rho=\vec{\nabla} \cdot \vec{f}_{\mathrm{ext}} \tag{7.57}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\chi_{\rho \rho}(\omega, q)=\frac{i q}{-i \omega+D q^{2}} \tag{7.58}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{\rho \rho}^{\prime \prime}(\omega, q)=\frac{D q^{3}}{\omega^{2}+\left(D q^{2}\right)^{2}} \tag{7.59}
\end{equation*}
$$

Thus, $\chi_{\rho \rho}^{\prime \prime} / q$ is a Lorentzian centered at $\omega=0$ with width $D q^{2}$. Similarly,

$$
\begin{equation*}
\chi_{J J}(\omega, q)=\frac{i \omega}{-i \omega+D q^{2}} \tag{7.60}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{J J}^{\prime \prime}(\omega, q)=\frac{D q^{2} \omega}{\omega^{2}+\left(D q^{2}\right)^{2}} \tag{7.61}
\end{equation*}
$$

Note that this is precisely the same as (7.54) for $\tilde{v}=0$.
In this example, we have seen how the low $q, \omega$ behavior of response functions of conserved quantities and their associated currents can be determined by a knowledge of the hydrodynamic modes of the system. In general, there will be one hydrodynamic mode for each conservation law satisfied by the system. The conservation law, together with a constitutive relation, leads to hydrodynamic equations satisfied by the conserved quantity and its current. These equations, in turn, determine the correlation functions. Note that such constraints usually only hold for conserved quantities; correlation functions of arbitrary fields are typically unconstrained.

Observe that (7.61) is of precisely the same form as (7.54) above, but with $\tilde{v}_{l}=0$. In this section and the last, we have seen how response functions which are calculated perturbatively (as we imagined doing in the first example) are often of the
same form as those which are deduced from the hydrodynamic - or long-wavelength, low-frequency - equations which they satisfy. Hydrodynamic laws hold in the longwavelength, low-frequency limit in which local equilibrium is maintained so that constitutive relations such as (7.56) hold. Linear response theory holds in the limit of small $f_{\text {ext }}$. When both of these conditions are satisfied, we can sometimes perturbatively calculate response functions which satisfy the constraints imposed by hydrodynamic relations. In chapter 7 , we will see examples of this in the context of spin systems.

In a solid, we might be interested in the response functions for the energy density and the mass density. It turns out that these quantities are coupled so that the "normal modes" of the solid are a combination of the energy and mass. One of thse normal modes diffuses while the other is a (damped) propagating mode. Consequently, $\chi_{\rho \rho}$, $\chi_{\rho E}$, and $\chi_{E E}$ are given by linear combinations of functions of the form of (7.54) and (7.59).

### 7.6 Kubo Formulae

Transport measurements fit naturally into the paradigm of linear response theory: a weak external probe - such as a potential or temperature gradient - is applied and the resulting currents are measured. Transport coefficients relate the resulting currents to the applied gradients. These coefficients - or the corresponding response functions - may be derived by following the steps of section 2 .

We have already encountered one example of a transport coefficient which can be obtained from a response function, namely the diffusion constant, $D$, which can be obtained from (7.54) or (7.61) by:

$$
\begin{equation*}
D=\lim _{\omega \rightarrow 0} \lim _{q \rightarrow 0}\left(\frac{\omega}{q^{2}} \chi_{J J}^{\prime \prime}(q, \omega)\right) \tag{7.62}
\end{equation*}
$$

To see why transport properties should, in general, be related to such limits of response functions, let us derive the corresponding relation, or Kubo formula, for the electrical conductivity of a system. Let $\vec{j}$ denote the current in our condensed matter system when the external vector and scalar potentials, $\vec{A}$ and $\varphi$, are zero. Let $\rho$ be the charge density. Then, when we turn on the electromagnetic field, the current is given by $\vec{J}=\vec{j}-\frac{n e^{2}}{m} \vec{A}$. Meanwhile, $H_{\text {probe }}$ is given by:

$$
\begin{equation*}
H_{\text {probe }}=\int d^{3} \vec{x}\left(-\rho(\vec{x}, t) \varphi(\vec{x}, t)+\vec{j}(\vec{x}, t) \cdot \vec{A}(\vec{x}, t)+\frac{e}{m} \rho(\vec{x}, t) A^{2}(\vec{x}, t)\right) \tag{7.63}
\end{equation*}
$$

Following our derivation of the response function in section 2, we have

$$
\begin{align*}
\langle\vec{J}(\vec{x}, t)\rangle & =\langle 0|\left(1+i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)\right) \vec{J}(\vec{x}, t)\left(1-i \int_{-\infty}^{t} d t^{\prime} H_{\text {probe }}\left(t^{\prime}\right)\right)|0\rangle \\
& =\langle 0| \vec{J}(\vec{x}, t)|0\rangle_{0}+\langle 0| i \int_{-\infty}^{t} d t^{\prime}\left[H_{\text {probe }}\left(t^{\prime}\right), \vec{J}(\vec{x}, t)\right]|0\rangle \tag{7.64}
\end{align*}
$$

Using

$$
\begin{align*}
\langle 0| \vec{J}|0\rangle_{0} & =\langle 0| \vec{j}|0\rangle_{0}-\frac{n e^{2}}{m} \vec{A} \\
& =-\frac{n e^{2}}{m} \vec{A} \tag{7.65}
\end{align*}
$$

and the expression (7.63) for $H_{\text {probe }}$ and keeping only terms linear in $\vec{A}$, we have:

$$
\begin{align*}
\left\langle J_{i}(\vec{x}, t)\right\rangle= & i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{t} d t^{\prime}\left[j_{j}\left(\vec{x}^{\prime}, t^{\prime}\right), j_{i}(\vec{x}, t)\right] A_{i}\left(\vec{x}^{\prime}, t^{\prime}\right) \\
& -i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{t} d t^{\prime}\left[j_{i}\left(\vec{x}^{\prime}, t^{\prime}\right), \rho(\vec{x}, t)\right] \varphi\left(\vec{x}^{\prime}, t^{\prime}\right) \\
& -\frac{n e^{2}}{m} A_{i}(\vec{x}, t) \tag{7.66}
\end{align*}
$$

We are free to choose any gauge we want, so let's take $\varphi=0$ gauge. Then,

$$
\begin{equation*}
\left\langle J_{i}(\vec{x}, t)\right\rangle=i \int d^{3} \vec{x}^{\prime} \int_{-\infty}^{\infty} d t^{\prime} \theta\left(t-t^{\prime}\right)\left[j_{j}\left(\vec{x}^{\prime}, t^{\prime}\right), j_{i}(\vec{x}, t)\right] A_{i}\left(\vec{x}^{\prime}, t^{\prime}\right)-\frac{n e^{2}}{m} A_{i}(\vec{x}, t) \tag{7.67}
\end{equation*}
$$

In this gauge, $\vec{E}=d \vec{A} / d t$, so we naively have:

$$
\begin{equation*}
\left\langle J_{i}(\vec{x}, t)\right\rangle=\int d^{3} \vec{x}^{\prime} \int_{-\infty}^{\infty} d t^{\prime} \sigma_{i j}\left(\vec{x}-\vec{x}^{\prime}, t-t^{\prime}\right) \vec{E}\left(\vec{x}^{\prime}, t^{\prime}\right) \tag{7.68}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle J_{i}(\vec{q}, \omega)\right\rangle=\sigma_{i j}(\vec{q}, \omega) E_{j}(\vec{q}, \omega) \tag{7.69}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{i j}(\vec{q}, \omega)=\frac{1}{\omega} \int_{0}^{\infty} d t e^{-i \omega t}\left[j_{j}(-\vec{q}, 0), j_{i}(\vec{q}, t)\right]-\frac{1}{i \omega} \frac{n e^{2}}{m} \tag{7.70}
\end{equation*}
$$

In terms of the response function,

$$
\begin{equation*}
\sigma_{i j}(\vec{q}, \omega)=\frac{1}{i \omega} \chi_{j j}(\vec{q}, \omega)-\frac{1}{i \omega} \frac{n e^{2}}{m} \tag{7.71}
\end{equation*}
$$

The first term on the right-hand-side leads to the real part of the conductivity:

$$
\begin{equation*}
\operatorname{Re} \sigma_{i j}(\vec{q}, \omega)=\frac{1}{\omega} \chi_{j j}^{\prime \prime}(\vec{q}, \omega) \tag{7.72}
\end{equation*}
$$

while the second term - if it's not cancelled by the imaginary part of the first term leads to superconductivity.

The DC conductivity is obtained by taking the $q \rightarrow 0$ limit first, to set up a spatially uniform current, and then taking the DC limit, $\omega \rightarrow 0$.

$$
\begin{equation*}
\sigma_{i j}^{D C}=\lim _{\omega \rightarrow 0} \lim _{q \rightarrow 0}\left(\frac{1}{\omega} \chi_{j j}^{\prime \prime}(\vec{q}, \omega)\right) \tag{7.73}
\end{equation*}
$$

If we take $\omega \rightarrow 0$, then we'll get a static, inhomogenous charge distribution and the $q \rightarrow 0$ limit won't tell us anything about the conductivity.

The above formulas are almost right. The problem with them is that they give the response to the applied electric field. In fact, we want the response to the total electric field. Using Maxwell's equations and our linear response result for $\vec{J}$, we can compute the total field and thereby find the correction to (7.73). This issue is most relevant in the context of interacting electrons, so we will defer a thorough discussion of it to that chapter.

### 7.7 Inelastic Scattering Experiments

Another way of experimentally probing a condensed matter system involves scattering a neutron off the system and studying the energy and angular dependence of the resulting cross-section. This is typically (but not exclusively) done with neutrons rather than photons - for which the requisite energy resolution has not yet been achieved - or electrons - which have the complication of a form factor arising from the long-range Coulomb interactions.

Let us assume that our system is in thermal equilibrium at inverse temperature $\beta$ and that neutrons interact with our system via the Hamiltonian $H^{\prime}$. Suppose that neutrons of momentum $\vec{k}_{i}$, and energy $\omega_{i}$ are scattered by our system. The differential cross-section for the neutrons to be scattered into a solid angle $d \Omega$ centered about $\vec{k}_{f}$ and into the energy range between $\omega_{f} \pm d \omega$ is:

$$
\begin{equation*}
\left.\frac{d^{2} \sigma}{d \Omega d \omega}=\sum_{m, n} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2}\left|\left\langle\vec{k}_{f} ; m\right| H^{\prime}\right| \vec{k}_{i} ; n\right\rangle\left.\right|^{2} e^{-\beta E_{n}} \delta\left(\omega+E_{n}-E_{m}\right) \tag{7.74}
\end{equation*}
$$

where $\omega=\omega_{i}-\omega_{f}$ and $n$ and $m$ label the initial and final states of our system.
For simplicity, let us assume that there are simple $\delta$-function interactions between the neutrons and the particles in our system:

$$
\begin{equation*}
H^{\prime}=V \sum_{j} \delta\left(\mathbf{x}-\mathbf{X}_{j}\right)=V \rho(x) \tag{7.75}
\end{equation*}
$$

Then

$$
\begin{align*}
\left\langle\vec{k}_{f} ; m\right| H^{\prime}\left|\vec{k}_{i} ; n\right\rangle & =V \int d^{3} \vec{x} e^{i \vec{q} \cdot \vec{x}}\langle m| \rho(x)|n\rangle \\
& =V\langle m| \rho(\vec{q})|n\rangle \tag{7.76}
\end{align*}
$$

If we use the Fourier representation of the $\delta$-function,

$$
\begin{equation*}
\delta\left(\omega+E_{n}-E_{m}\right)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} d t e^{i\left(\omega+E_{n}-E_{m}\right) t} \tag{7.77}
\end{equation*}
$$

and pass from the Schrödinger to the Heisenberg representation,

$$
\begin{align*}
e^{i\left(E_{n}-E_{m}\right) t}\langle m| \rho(\vec{q})|n\rangle & =\langle n| e^{i H t} \rho(\vec{q}) e^{-i H t}|m\rangle \\
& =\langle n| \rho(\vec{q}, t)|m\rangle \tag{7.78}
\end{align*}
$$

then we can rewrite (7.74) as

$$
\begin{align*}
\frac{d^{2} \sigma}{d \Omega d \omega} & \left.=\sum_{m} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2}\left|\left\langle\vec{k}_{f} ; m\right| H^{\prime}\right| \vec{k}_{i} ; n\right\rangle\left.\right|^{2} e^{-\beta E_{n}} \delta\left(\omega+E_{n}-E_{m}\right) \\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty} d t e^{i \omega t} \sum_{m} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2} e^{-\beta E_{n}}\langle n| \rho(\vec{q}, t)|m\rangle\langle m| \rho(-\vec{q}, 0)|n\rangle(7 . \tag{7.79}
\end{align*}
$$

We can now use $|m\rangle\langle m|=1$ and write this as

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \Omega d \omega}=\frac{1}{2 \pi} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2} \int_{-\infty}^{\infty} d t e^{i \omega t} \operatorname{Tr}\left\{e^{-\beta H} \rho(\vec{q}, t) \rho(-\vec{q}, 0)\right\} \tag{7.80}
\end{equation*}
$$

If we define the dynamic structure factor, $S(\vec{q}, \omega) \equiv S_{\rho \rho}(\vec{q}, \omega)$ :

$$
\begin{equation*}
S(\vec{q}, \omega) \delta\left(\omega+\omega^{\prime}\right)=\operatorname{Tr}\left\{e^{-\beta H} \rho(\vec{q}, \omega) \rho\left(-\vec{q}, \omega^{\prime}\right)\right\} \tag{7.81}
\end{equation*}
$$

then we can write the inelastic scattering cross-section as:

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \Omega d \omega}=\frac{1}{2 \pi} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2} S(\vec{q}, \omega) \tag{7.82}
\end{equation*}
$$

According to the fluctuation-dissipation theorem, this can be written

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \Omega d \omega}=\frac{1}{2 \pi} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2}\left(\frac{1}{1-e^{-\beta \omega}}\right) \chi^{\prime \prime}(\vec{q}, \omega) \tag{7.83}
\end{equation*}
$$

In our elastic theory of a solid,

$$
\begin{gather*}
\rho(\vec{x})=\sum_{i} \delta\left(\vec{x}-\vec{R}_{i}-\vec{u}\left(\vec{R}_{i}\right)\right)  \tag{7.84}\\
\rho(\vec{q})=\sum_{i} e^{i \vec{q} \cdot\left(\vec{R}_{i}-\vec{u}\left(\vec{R}_{i}\right)\right)} \tag{7.85}
\end{gather*}
$$

Let us assume that the displacements of the ions are small and expand the exponential,

$$
\rho(\vec{q})=\sum_{i} e^{i \vec{q} \cdot \vec{R}_{i}}\left(1-i \vec{q} \cdot \vec{u}\left(\vec{R}_{i}\right)\right)
$$

$$
\begin{equation*}
\approx \sum_{Q} \delta(\vec{q}-\vec{Q})-i \vec{q} \cdot \vec{u}(\vec{q}) \tag{7.86}
\end{equation*}
$$

where $\vec{Q}$ is the set of reciprocal lattice vectors. By dropping the higher-order terms in the expansion of the exponential, we are neglecting multi-phonon emission processes. Hence, the scattering cross-section is given by the sum of the contributions of the Bragg peaks together with the contributions of one-phonon emission processes:

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \Omega d \omega}=\frac{1}{2 \pi} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2}\left(\left[\sum_{Q} \delta(\vec{q}-\vec{Q}) \delta(\omega)\right]+\left[\operatorname{Tr}\left\{e^{-\beta H} \vec{q} \cdot \vec{u}(\vec{q}, \omega) \vec{q} \cdot \vec{u}(-\vec{q},-\omega)\right\}\right]\right) \tag{7.87}
\end{equation*}
$$

Recognizing our longitudinal phonon Green function on the right-hand-side and using the fluctuation-dissipation theorem, we can write this as:

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \Omega d \omega}=\frac{1}{2 \pi} \frac{k_{f}}{k_{i}}\left(\frac{M}{2 \pi}\right)^{2}\left(\left[\sum_{Q} \delta(\vec{q}-\vec{Q}) \delta(\omega)\right]+\left[\left(\frac{1}{1-e^{-\beta \omega}}\right) \operatorname{Im}\left\{G_{\mathrm{ret}}(q, \omega)\right\}\right]\right) \tag{7.88}
\end{equation*}
$$

Hence, the quantity which our imaginary-time perturbation theory is designed to compute $-G_{\text {ret }}(q, \omega)$ is precisely the quantity which is measured in inelastic scattering experiments. If we assume a self-energy as we did in (7.54), then there will be Lorentzian peaks at $\omega= \pm \tilde{v}_{l} q$ of width $D q^{2}$.

### 7.8 NMR Relaxation Rate

In nuclear magnetic resonance, or NMR, experiments, a material is placed in a constant magnetic field. As a result of this magnetic field, there is an energy splitting $\omega_{0}$ between the up-spin excited state and the down-spin ground state of the nuclei (let's assume spin- $1 / 2$ nuclei). If an up-spin state were an energy eigenstate, then electromagnetic radiation at frequency $\omega_{0}$ would be perfectly resonant with the nuclear spins; the absorption cross section would have a $\delta$-function at $\omega_{0}$. As a result of the interaction between nuclear spins and the other excitations in the system (electrons, phonons, magnons), the up-spin state has a finite lifetime, $T_{1}$. The width of
the resonance is, therefore, $1 / T_{1}$. A measurement of $T_{1}$ is an important probe of the spin-carrying excitations of a system.

The interaction Hamiltonian for the coupling between a nuclear spin and the other excitations is:

$$
\begin{equation*}
H_{\mathrm{int}}=\int \frac{d^{2} q}{(2 \pi)^{2}} A(q)\left[I_{+} S_{-}(q)+I_{-} S_{+}(q)\right] \tag{7.89}
\end{equation*}
$$

$A(q)$ is the hyperfine coupling between the the nuclear spin $\vec{I}$ and the spin density $\vec{S}(q)$ due to the excitations of the system. The lifetime of the up-spin state is given by:

$$
\begin{equation*}
\left.\frac{1}{T_{1}}=\sum_{m, n}\left|\langle\downarrow ; m| H^{\prime}\right| \uparrow ; n\right\rangle\left.\right|^{2} e^{-\beta E_{n}} \delta\left(\omega_{0}+E_{n}-E_{m}\right) \tag{7.90}
\end{equation*}
$$

Following the steps which we used in the derivation of the scattering cross-section, we rewrite this as:

$$
\begin{align*}
\frac{1}{T_{1}} & =\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \frac{1}{2 \pi} \int_{-\infty}^{\infty} d t e^{i\left(\omega_{0}+E_{n}-E_{m}\right) t} \sum_{m, n}\langle n| S_{-}(q)|m\rangle\langle m| S_{+}(q)|n\rangle e^{-\beta E_{n}} \\
& =\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \frac{1}{2 \pi} \int_{-\infty}^{\infty} d t e^{i \omega_{0} t} \sum_{m, n}\langle n| S_{-}(-q, 0)|m\rangle\langle m| S_{+}(q, t)|n\rangle e^{-\beta E_{n}} \\
& =\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \sum_{m, n}\langle n| S_{-}\left(-q,-\omega_{0}\right)|m\rangle\langle m| S_{+}\left(q, \omega_{0}\right)|n\rangle e^{-\beta E_{n}} \\
& =\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \sum_{n}\langle n| S_{-}\left(-q,-\omega_{0}\right) S_{+}\left(q, \omega_{0}\right)|n\rangle e^{-\beta E_{n}} \\
& =\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \operatorname{Tr}\left\{e^{-\beta H} S_{-}\left(-\vec{q},-\omega_{0}\right) S_{+}\left(\vec{q}, \omega_{0}\right)\right\} \tag{7.91}
\end{align*}
$$

or, using the fluctuation-dissipation theorem,

$$
\begin{equation*}
\frac{1}{T_{1}}=\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \frac{\chi_{+-}^{\prime \prime}\left(q, \omega_{0}\right)}{1-e^{-\beta \omega_{0}}} \tag{7.92}
\end{equation*}
$$

$\omega_{0}$ is usually a very small frequency, compared to the natural frequency scales of electrons, spin waves, etc., so we can take $\omega_{0} \rightarrow 0$ :

$$
\begin{equation*}
\frac{1}{T_{1} T}=\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \lim _{\omega \rightarrow 0} \frac{1}{\omega} \chi_{+-}^{\prime \prime}(q, \omega) \tag{7.93}
\end{equation*}
$$

# Chapter 8 

## Functional Integrals

### 8.1 Gaussian Integrals

We will now shift gears and develop a formalism which will give us a fresh perspective on many-body theory and its associated approximation methods. This formalism functional integration - will also reveal the underlying similarity and relationship between quantum and classical statistical mechanics.

Consider the Gaussian integral,

$$
\begin{equation*}
\int_{-\infty}^{\infty} d x e^{-\frac{1}{2} a x^{2}}=\left(\frac{2 \pi}{a}\right)^{1 / 2} \tag{8.1}
\end{equation*}
$$

This integral is well-defined for any complex $a$ so long as $\operatorname{Re}\{a\}>0$. We can generalize this to integration over $n$ variables,

$$
\begin{equation*}
\int d^{n} \vec{x} e^{-\frac{1}{2} x_{i} A_{i j} x_{j}}=(2 \pi)^{n / 2}(\operatorname{det} A)^{-1 / 2} \tag{8.2}
\end{equation*}
$$

and even to integration over complex variables $z_{i}$ with $d^{n} \vec{z} d^{n} \vec{z}^{*} \equiv d^{n}(\operatorname{Re} \vec{z}) d^{n}(\operatorname{Im} \vec{z})$,

$$
\begin{equation*}
\int d^{n} \vec{z} d^{n} \vec{z}^{*} e^{-z_{i}^{*} A_{i j} z_{j}}=(4 \pi)^{n}(\operatorname{det} A)^{-1} \tag{8.3}
\end{equation*}
$$

so long as $A_{i j}$ is a symmetric matrix with $\operatorname{Re}\left\{A_{i j}\right\}>0$.
By completing the square,

$$
\begin{equation*}
\frac{1}{2} x_{i} A_{i j} x_{j}+b_{i} x_{i}=\frac{1}{2}\left(x_{i}-x_{i}^{0}\right) A_{i j}\left(x_{j}-x_{j}^{0}\right)-\frac{1}{2} b_{i}\left(A^{-1}\right)_{i j} b_{j} \tag{8.4}
\end{equation*}
$$

where $x_{i}^{0}=\left(A^{-1}\right)_{i j} b_{j}$, we can do the integral of the exponential of a quadratic form:

$$
\begin{equation*}
\int d^{n} \vec{x} e^{-\frac{1}{2} x_{i} A_{i j} x_{j}+b_{i} x_{i}}=(2 \pi)^{n / 2}(\operatorname{det} A)^{-1 / 2} e^{\frac{1}{2} b_{i}\left(A^{-1}\right)_{i j} b_{j}} \tag{8.5}
\end{equation*}
$$

By differentiating with respect to $b_{i}$, we can also do the integrals of polynomials multiplying Gaussians:

$$
\begin{equation*}
\int d^{n} \vec{x} P\left(x_{1}, \ldots, x_{n}\right) e^{-\frac{1}{2} x_{i} A_{i j} x_{j}+b_{i} x_{i}}=P\left(\frac{\partial}{\partial b_{1}}, \ldots, \frac{\partial}{\partial b_{n}}\right)\left((2 \pi)^{n / 2}(\operatorname{det} A)^{-1 / 2} e^{\frac{1}{2} b_{i}\left(A^{-1}\right)_{i j} b_{j}}\right) \tag{8.6}
\end{equation*}
$$

A non-Gaussian integral can often be approximated by a Gaussian integral using the saddle-point approximation:

$$
\begin{align*}
\int d^{n} \vec{x} e^{-\lambda f\left(x_{i}\right)} & \approx \int d^{n} \vec{x} e^{-\lambda f\left(x_{i}^{0}\right)-\lambda\left(\partial_{i} \partial_{j} f\right)_{x_{i}=x_{i}^{0}}\left(x_{i}-x_{i}^{0}\right)\left(x_{j}-x_{j}^{0}\right)} \\
& =(2 \pi)^{n / 2} e^{-\lambda f\left(x_{i}^{0}\right)}\left(\operatorname{det}\left(\lambda \partial_{i} \partial_{j} f\right)_{x_{i}=x_{i}^{0}}\right)^{-\frac{1}{2}} \tag{8.7}
\end{align*}
$$

Where $x_{i}^{0}$ is a stationary point of $f\left(x_{i}\right)$, i.e. $\partial_{j} f\left(x_{i}^{0}\right)=0$ for all $j$. This approximation is good in the $\lambda \rightarrow \infty$ limit where the minimum of $f\left(x_{i}\right)$ dominates the integral.

Nothing that we have done so far depended on having $n$ finite. If we blithely allow $n$ to be infinite (ignoring the protests of our mathematician friends), we have the Gaussian functional integral. In the next section, we will do this by making the replacement $i \rightarrow t, x_{i} \rightarrow x(t)$, and

$$
\begin{equation*}
\int d^{n} \vec{x} e^{-\frac{1}{2} x_{i} A_{i j} x_{j}} \rightarrow \int D x(t) e^{-\int_{t_{i}}^{t_{f}} d t \frac{1}{2} x(t)\left(-\frac{d^{2}}{d t^{2}}\right) x(t)} \tag{8.8}
\end{equation*}
$$

As we will see, the generating functional, $Z$, can be expressed in this way. Such an expression for the generating functional will facilitate many formal manipulations such as changes of variables and symmetry transformations. It will also guide our intuition about quantum mechanical processes and emphasize the connections with classical statistical mechanics.

### 8.2 The Feynman Path Integral

In this section, we will - following Feynman - give an argument relating the matrix elements of the evolution operator, $U$, of a free particle to a Gaussian functional integral. This derivation can be made more or less rigorous, but shouldn't be taken overly seriously. We could just as well write down the functional integral without any further ado and justify it by the fact that it gives the same result as canonical quantization - ultimately, this is its real justification. Here, our reason for discussing Feynman's derivation lies its heuristic value and its intuitive appeal.

Suppose that we have a particle in one dimension moving in a potential $V(x)$. Then

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}+V(x) \tag{8.9}
\end{equation*}
$$

Then the imaginary-time evolution operator is given by

$$
\begin{equation*}
U\left(t_{f}, t_{i}\right)=e^{-\left(\tau_{f}-\tau_{i}\right) H} \tag{8.10}
\end{equation*}
$$

We would like to compute

$$
\begin{equation*}
\left\langle x_{f}\right| U\left(\tau_{f}, \tau_{i}\right)\left|x_{i}\right\rangle \tag{8.11}
\end{equation*}
$$

In order to do this, we will write (the Trotter product formula)

$$
\begin{equation*}
e^{-\left(\tau_{f}-\tau_{i}\right) H} \rightarrow\left(e^{-\delta \tau H}\right)^{N_{\tau}} \tag{8.12}
\end{equation*}
$$

in the limit $\delta \tau=\left(\tau_{f}-\tau_{i}\right) / N_{\tau} \rightarrow 0$. We will now take the desired matrix element (8.11),

$$
\begin{equation*}
\left\langle x_{f}\right| e^{-\delta \tau H} \ldots e^{-\delta \tau H}\left|x_{i}\right\rangle \tag{8.13}
\end{equation*}
$$

and insert a resolution of the identity,

$$
\begin{equation*}
\int d x|x\rangle\langle x|=1 \tag{8.14}
\end{equation*}
$$

between each factor of $e^{-\delta \tau H}$ :

$$
\begin{equation*}
e^{-\delta \tau H}=\int \ldots \int d x_{1} \ldots d x_{N_{\tau}-1}\left\langle x_{f}\right| e^{-\delta \tau H}\left|x_{N_{\tau}-1}\right\rangle \ldots\left\langle x_{1}\right| e^{-\delta \tau H}\left|x_{i}\right\rangle \tag{8.15}
\end{equation*}
$$

When we repeat this derivation for a quantum mechanical spin in the next chapter, we will insert a resolution of the identity in terms of an overcomplete set of states. As a result, the integration measure will be non- trivial.

We now make an approximation which is accurate to $O\left(\delta \tau^{2}\right)$. In the $\delta \tau \rightarrow 0$ limit, we will have an exact (though formal for arbitrary $V(x)$ ) expression for the desired matrix element. Observe that

$$
\begin{equation*}
e^{-\delta \tau\left(\frac{p^{2}}{2 m}+V(x)\right)}=e^{-\delta \tau\left(\frac{p^{2}}{2 m}\right)} e^{-\delta \tau(V(x))}+O\left(\delta \tau^{2}\right) \tag{8.16}
\end{equation*}
$$

Hence, we can write

$$
\begin{equation*}
\left\langle x_{n}\right| e^{-\delta \tau H}\left|x_{n-1}\right\rangle=\left\langle x_{n}\right| e^{-\delta \tau\left(\frac{p^{2}}{2 m}\right)}\left|x_{n-1}\right\rangle e^{-\delta \tau\left(V\left(x_{n-1}\right)\right)}+O\left(\delta \tau^{2}\right) \tag{8.17}
\end{equation*}
$$

We now insert a complete set of momentum eigenstates into the right-hand-side of this expression:

$$
\begin{align*}
\left\langle x_{n}\right| e^{-\delta \tau \frac{p^{2}}{2 m}}\left|x_{n-1}\right\rangle e^{-\delta \tau V\left(x_{n-1}\right)} & =\int d p_{n}\left\langle x_{n} \mid p_{n}\right\rangle\left\langle p_{n}\right| e^{-\delta \tau \frac{p^{2}}{2 m}}\left|x_{n-1}\right\rangle e^{-\delta \tau V\left(x_{n-1}\right)} \\
& =\int d p_{n} e^{i p_{n}\left(x_{n}-x_{n-1}\right)} e^{-\delta \tau \frac{p_{n}^{2}}{2 m}} e^{-\delta \tau V\left(x_{n-1}\right)} \tag{8.18}
\end{align*}
$$

Note that the second line is an ordinary integral of $c$-numbers. Doing the Gaussian $p_{n}$ integral, we have

$$
\begin{equation*}
e^{-\delta \tau\left(\frac{m}{2}\left(\frac{x_{n}-x_{n-1}}{\delta \tau}\right)^{2}+V\left(x_{n-1}\right)\right)} \tag{8.19}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left\langle x_{f}\right| U\left(\tau_{f}, \tau_{i}\right)\left|x_{i}\right\rangle=\int \ldots \int d x_{1} \ldots d x_{N_{\tau}-1} e^{-\sum_{n} \delta \tau\left(\frac{m}{2}\left(\frac{x_{n}-x_{n-1}}{\delta \tau}\right)^{2}+V\left(x_{n-1}\right)\right)} \tag{8.20}
\end{equation*}
$$

In the $\delta \tau \rightarrow 0$ limit, we write

$$
n \delta \tau \quad \rightarrow \quad \tau
$$

$$
\begin{align*}
x_{n} & \rightarrow x(\tau) \\
\sum_{n} \delta \tau & \rightarrow \int d \tau \tag{8.21}
\end{align*}
$$

(but we make no assumption that $x(\tau)$ is differentiable or even continuous) and

$$
\begin{equation*}
\left\langle x_{f}\right| U\left(\tau_{f}, \tau_{i}\right)\left|x_{i}\right\rangle=\int \mathcal{D} x(\tau) e^{-S_{E}} \tag{8.22}
\end{equation*}
$$

where $S_{E}$ is the imaginary-time - or 'Euclidean' - action:

$$
\begin{equation*}
S_{E}[x(\tau)]=\int_{t_{i}}^{t_{f}} d \tau\left(\frac{1}{2} m\left(\frac{d x}{d \tau}\right)^{2}+V(x)\right) \tag{8.23}
\end{equation*}
$$

The path integral representation suggests a beautiful interpretation of the quantummechanical transition amplitude: the particle takes all possible trajectories with each trajectory $x(\tau)$ contributing $e^{-S_{E}[x(\tau)]}$ to the amplitude.

If the particle is free or is in a harmonic oscillator potential, $V(x)=m \omega^{2} x^{2} / 2$, this is a Gaussian functional integral:

$$
\begin{equation*}
\int \mathcal{D} x(\tau) e^{-\int_{t_{i}}^{t_{f}} d \tau \frac{1}{2} m\left(\frac{d x}{d \tau}\right)^{2}+\frac{1}{2} m \omega^{2} x^{2}}=N \operatorname{det}\left(-\frac{d^{2}}{d t^{2}}+\omega^{2}\right) \tag{8.24}
\end{equation*}
$$

where the determinant is taken over the space of functions satisfying $x\left(t_{i}\right)=x_{i}$, $x\left(t_{f}\right)=x_{f}$ and $N$ is a 'normalization constant' into which we have absorbed factors of $m, \pi$, etc.

For a more general potential, the path integral can be defined perturbatively using

$$
\begin{align*}
& \int \mathcal{D} x(\tau) e^{-\int_{t_{i}}^{t_{f}} d \tau \frac{1}{2} m\left(\frac{d x}{d \tau}\right)^{2}+V(x)}=  \tag{8.6}\\
& \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}\left(\int_{t_{i}}^{t_{f}} d t^{\prime} V\left(\frac{\partial}{\partial b\left(t^{\prime}\right)}\right)\right)^{n} \int \mathcal{D} x(\tau) e^{-\int_{t_{i}}^{t_{f}} d \tau\left(\frac{1}{2} m\left(\frac{d x}{d \tau}\right)^{2}+b(t) x(t)\right)} \tag{8.25}
\end{align*}
$$

Time-ordered expectation values can be simply handled by the path-integral formalism:

$$
\begin{equation*}
\left\langle x_{f}\right| T\left(x\left(\tau_{1}\right) \ldots x\left(\tau_{n}\right) U\left(\tau_{f}, \tau_{i}\right)\right)\left|x_{i}\right\rangle=\int \mathcal{D} x(\tau) x\left(\tau_{1}\right) \ldots x\left(\tau_{n}\right) e^{-\int_{t_{i}}^{t_{f}} d \tau\left(\frac{1}{2} m\left(\frac{d x}{d \tau}\right)^{2}+V(x)\right)} \tag{8.26}
\end{equation*}
$$

As you will show in the problem set, this follows because the $T$-symbol puts the operators in precisely the right order so that they can act on the appropriate resolutions of the identity and become $c$-numbers.

### 8.3 The Functional Integral in Many-Body Theory

Instead of following the steps of the previous section to derive the functional integral for a field theory, we will simply demonstrate that the generating functional, $Z[j]$, is given ${ }^{1}{ }^{1}$ :

$$
\begin{equation*}
Z[j]=N \int \mathcal{D} u e^{-S_{E}[j]} \tag{8.27}
\end{equation*}
$$

where $S_{E}[j]$ is the imaginary-time - or Euclidean - action in the presence of an external source field $j(\vec{x}, \tau)$ and $N$ is a normalization factor.

To show that this is true, we will first show that it is true for a free field, i.e. $g=0$ in our phonon Lagrangian. The generating functional, $Z[j]$ is given by the exponential of the generating functional for connected Green functions, $W[j]$, which, in turn, is given by a single diagram:

$$
\begin{equation*}
Z_{0}[j]=e^{W_{0}[j]}=e^{\frac{1}{2} \int j(x) G_{0}(x-y) j(y)} Z[0] \tag{8.28}
\end{equation*}
$$

The functional integral, on the other hand, is given by

$$
\begin{equation*}
N \int \mathcal{D} u e^{-S_{E}[j]}=N\left[\operatorname{det}\left(-\delta_{i j} \partial_{\tau}^{2}-(\mu+\lambda) \partial_{i} \partial_{j}-\mu \delta_{i j} \partial_{k} \partial_{k}\right)\right]^{-\frac{1}{2}} e^{\frac{1}{2} \int j(x) G_{0}(x-y) j(y)} \tag{8.29}
\end{equation*}
$$

which follows because

$$
\begin{equation*}
G_{0}=\partial_{l} \partial_{l}\left(-\delta_{i j} \partial_{\tau}^{2}-(\mu+\lambda) \partial_{i} \partial_{j}-\mu \delta_{i j} \partial_{k} \partial_{k}\right)^{-1} \tag{8.30}
\end{equation*}
$$

If we choose $N$ to cancel the determinant, we have the desired result.

[^2]Now consider the interacting case, $S=S_{0}+\int \mathcal{L}_{i n t}\left(\partial_{k} u_{k}\right)$. Then, following (8.6), we can write the functional integral as:

$$
\begin{equation*}
N \int \mathcal{D} u e^{-S_{0}[j]-\int \mathcal{L}_{\text {int }}\left(\partial_{k} u_{k}\right)}=\frac{N}{N_{0}} e^{-\int \mathcal{L}_{i n t}\left(\frac{\delta}{\delta j}\right)} e^{W_{0}[j]} \tag{8.31}
\end{equation*}
$$

According to Dyson's formula, we have precisely the same thing for the generating functional. Hence (8.27) is true even for an interacting theory. By straightforward extension, we can show that the same relation holds at finite-temperature, where imaginary-time integrals run from 0 to $\beta$.

An important result which follows from this discussion is that the propagator is simply the inverse of the differential operator in the quadratic term in the action. The inverse is almost always most easily taken in momentum space.

With the functional integral representation of $Z[j]$ in hand, we can give simple proofs of Wick's theorem and the Feynman rules. To do this, it's helpful to use the following identity (which we state for finite-dimensional vector spaces)

$$
\begin{equation*}
F\left(-i \frac{\partial}{\partial \mathbf{x}}\right) G(\mathbf{x})=\left(G\left(-i \frac{\partial}{\partial \mathbf{y}}\right) F(\mathbf{y}) e^{i \mathbf{x} \cdot \mathbf{y}}\right)_{\mathbf{y}=0} \tag{8.32}
\end{equation*}
$$

This identity can be proven by expanding $F$ and $G$ in plane waves.

$$
\begin{align*}
& F(\mathbf{x})=e^{i \mathbf{a} \cdot \mathbf{x}} \\
& G(\mathbf{y})=e^{i \mathbf{b} \cdot \mathbf{y}} \tag{8.33}
\end{align*}
$$

The left-hand-side of (8.32) is

$$
\begin{equation*}
e^{i \mathbf{a} \cdot \partial / \partial \mathbf{x}} e^{i \mathbf{b} \cdot \mathbf{x}}=e^{i \mathbf{b} \cdot(\mathbf{x}+\mathbf{a})} \tag{8.34}
\end{equation*}
$$

while the right-hand-side is:

$$
\begin{equation*}
e^{i \mathbf{b} \cdot \partial / \partial \mathbf{y}} e^{i(\mathbf{x}+\mathbf{a}) \cdot \mathbf{y}}=e^{i(\mathbf{x}+\mathbf{a}) \cdot(\mathbf{y}+\mathbf{b})} \tag{8.35}
\end{equation*}
$$

from which (8.32) follows.

Using this, we can compute the time-ordered product of a string of fields in a free field theory:

$$
\begin{align*}
\langle 0| T\left(\partial_{k} u_{k}\left(x_{1}\right) \ldots \partial_{k} u_{k}\left(x_{n}\right)\right)|0\rangle & =\frac{\delta^{n} Z[j]}{\delta j\left(x_{1}\right) \ldots \delta j\left(x_{n}\right)} \\
& =\left[\frac{\delta}{\delta j\left(x_{1}\right)} \cdots \frac{\delta}{\delta j\left(x_{1}\right)} e^{\frac{1}{2} \int j(x) G_{0}(x-y) j(y)}\right]_{j=0} \\
& =\left[e^{\frac{1}{2} \int G_{0}(x-y) \frac{\delta}{\delta\left(\partial_{i} u_{i}(x)\right)} \frac{\delta}{\delta\left(\partial_{j} u_{j}(y)\right)}} \partial_{i} u_{i}\left(x_{1}\right) \ldots \partial_{i} u_{i}\left(x_{n}\right)\right]_{u_{i}=0} \tag{8.36}
\end{align*}
$$

This is Wick's theorem in the form in which we rewrote it at the end of section 4.4.
In the same way, we can derive the Feynman rules for an interacting theory. Using our expression (8.31),

$$
\begin{align*}
Z[j] & =N e^{-\int \mathcal{L}_{i n t}\left(\frac{\delta}{\delta j}\right)} e^{W_{0}[j]} \\
& =N\left[e^{-\int \mathcal{L}_{i n t}\left(\frac{\delta}{\delta j}\right)} e^{\frac{1}{2} \int j(x) G_{0}(x-y) j(y)}\right] \tag{8.37}
\end{align*}
$$

Using our identity, (8.32) we can rewrite this as:

$$
\begin{equation*}
Z[j]=N e^{\frac{1}{2} \int G_{0}(x-y) \frac{\delta}{\delta \partial_{k} u_{k}(x)} \frac{\delta}{\delta \partial_{j} u_{j}(y)}} e^{-\int \mathcal{L}_{i n t}\left(\partial_{k} u_{k}\right)+j \partial_{k} u_{k}} \tag{8.38}
\end{equation*}
$$

which is a compact expression of the Feynman rules for the generating functional.

### 8.4 Saddle Point Approximation, Loop Expansion

As we pointed out in chapter 5 , the loop expansion is an expansion in powers of $\hbar$. Using the functional integral, we can obtain another perspective on the expansion in powers of $\hbar$. Restoring the $\hbar$, the functional integral is

$$
\begin{equation*}
Z[j]=N \int \mathcal{D} u e^{-\frac{1}{\hbar}\left(S_{E}[u]+\int j \partial_{k} u_{k}\right)} \tag{8.39}
\end{equation*}
$$

At saddle-point level, this is given by:

$$
\begin{equation*}
Z[j]=N e^{-\frac{1}{\hbar}\left(S_{E}\left[u^{c}\right]+\int j \partial_{k} u_{k}^{c}\right)} \tag{8.40}
\end{equation*}
$$

where $u^{c}$ is a classical solution of the equation of motion:

$$
\begin{equation*}
K_{i j}(x-y) u_{j}(y)-\frac{g}{3!} \partial_{i}\left(\partial_{k} u_{k}\right)^{3}=-\partial_{i} j(x) \tag{8.41}
\end{equation*}
$$

and

$$
\begin{equation*}
K_{i j}\left(x-x^{\prime}\right)=\delta\left(x-x^{\prime}\right)\left(\delta_{i j} \partial_{\tau}^{2}+(\mu+\lambda) \partial_{i} \partial_{j}+\mu \delta_{i j} \partial_{k} \partial_{k}\right) \tag{8.42}
\end{equation*}
$$

If we use the classical solution which is obtained by starting with the $g=0$ solution:

$$
\begin{equation*}
u_{i}(x)=-K_{i j}^{-1}(x-y) \partial_{j} j(y) \tag{8.43}
\end{equation*}
$$

and solve this iteratively:

$$
\begin{equation*}
u_{i}(x)=-K_{i j}^{-1}(x-y) \partial_{j} j(y)+\frac{g}{3!} K_{i l}^{-1}\left(x-x^{\prime}\right) \partial_{l}\left(\partial_{k} K_{k j}^{-1}\left(x^{\prime}-x^{\prime \prime}\right) \partial_{j} j\left(x^{\prime \prime}\right)\right)^{3}+\ldots \tag{8.44}
\end{equation*}
$$

we obtain:

$$
\begin{align*}
W[j]=\ln Z[j]= & \frac{1}{\hbar}\left(\frac{1}{2} \partial_{i} j(x) K_{i j}^{-1}(x-y) \partial_{j} j(y)\right. \\
& \left.-\frac{1}{2} \partial_{i} j(x) \frac{g}{3!} K_{i l}^{-1}\left(x-x^{\prime}\right) \partial_{l}\left(\partial_{k} K_{k j}^{-1}\left(x^{\prime}-x^{\prime \prime}\right) \partial_{j} j\left(x^{\prime \prime}\right)\right)^{3}+\ldots\right) \tag{8.45}
\end{align*}
$$

This is the contribution to the generating functional for connected diagrams coming from tree-level diagrams. In the terms not shown, for each additional vertex carrying a $g$ we have one extra internal line - i.e. a propagator $K_{i j}^{-1}$ which is not attached to a $\partial_{i} j$.

The Gaussian fluctuations about the saddle point contribute to the functional integral a factor of

$$
\begin{equation*}
\operatorname{det}\left(-K_{i j}+\frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)=e^{T r \ln \left(-K_{i j}+\frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)} \tag{8.46}
\end{equation*}
$$

This gives the following contribution to the generating functional of connected diagrams:
$\operatorname{Tr} \ln \left(-K_{i j}+\frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)=\operatorname{Tr} \ln \left(-K_{i j}\right)+\operatorname{Tr} \ln \left(1-K_{j i}^{-1} \frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)$

$$
\begin{align*}
= & \operatorname{Tr} \ln \left(-K_{i j}\right)-\operatorname{Tr}\left(K_{j i}^{-1} \frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)+ \\
& \frac{1}{2} \operatorname{Tr}\left(K_{n m}^{-1} \frac{g}{2} \partial_{m} \partial_{l}\left(\partial_{k} u_{k}^{c}\right)^{2} K_{l i}^{-1} \frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\right)^{2}\right)+\ldots \tag{8.47}
\end{align*}
$$

or, writing out the traces,

$$
\begin{align*}
W_{1-\text { loop }}[J]= & \int d^{3} x d t \ln \left(-K_{i i}\right)(0)- \\
& \int d^{3} x d t\left(K_{j i}^{-1}(0) \frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}(x, t)\right)^{2}\right)+ \\
& \frac{1}{2} \int d^{3} x d t d^{3} x^{\prime} d t^{\prime}\left(K_{j m}^{-1}\left(x^{\prime}-x, t^{\prime}-t\right) \frac{g}{2} \partial_{m} \partial_{l}\left(\partial_{k} u_{k}^{c}(x, t)\right)^{2} \times\right. \\
& \left.K_{l i}^{-1}\left(x-x^{\prime}, t-t^{\prime}\right) \frac{g}{2} \partial_{i} \partial_{j}\left(\partial_{k} u_{k}^{c}\left(x^{\prime}, t^{\prime}\right)\right)^{2}\right)+\ldots \tag{8.48}
\end{align*}
$$

The first term is independent of $j(x, t)$ and can be absorbed in the normalization. The rest of the series gives the connected one-loops contributions. The second term is the loop obtained by connecting a point to itself with a propagator; the $u_{k}^{c}$ 's attach all possible tree diagrams to this loop. The third term is the loop obtained by connecting two points by two propagators and again attaching all possible tree diagrams. The next term (not written) is the loop obtained by connecting three points with three propagators, and so on.

Hence, the tree-level diagrams give the $O(1 / \hbar)$ contribution to $W[J]$ while the one-loop diagrams give the $O(1)$ contribution. To see that the $L$-loop diagrams give the $O\left(\hbar^{L-1}\right)$ contribution to $W[J]$, observe that each propagator comes with a factor of $\hbar$ (since it is the inverse of the quadratic part of the action) while each vertex comes with a factor of $1 / \hbar$ (from the perturbative expansion of $e^{S / \hbar}$ ). Hence a diagram with $I$ internal lines and $V$ vertices is $O\left(\hbar^{I-V}\right)$. According to the graphical argument we gave in chapter $5, I-V=L-1$, which proves the claim.

Note that we have chosen a particular saddle-point, $u_{i}^{c}$, namely the one which can be obtained by solving the classical equations perturbatively about $g=0$. In
principle, we must, of course, sum the contributions from all saddle-points of the functional integral.

### 8.5 The Functional Integral in Statistical Mechanics

### 8.5.1 The Ising Model and $\varphi^{4}$ Theory

The functional integral representation of the generating functional of a quantum mechanical many-body system bears a strong resemblance to the partition function of a classical statistical mechanical system. Indeed the formal similarity between the two allows us to use the same language and calculational techniques to analyze both. To see the correspondance, let's consider the Ising model,

$$
\begin{equation*}
H=-\frac{1}{2} \sum_{i, j} J_{i j} \sigma_{i} \sigma_{j} \tag{8.49}
\end{equation*}
$$

where the spins $\sigma_{i}= \pm 1$ lie on a lattice. The classical partition function is:

$$
\begin{equation*}
\mathcal{Z}=\sum_{\sigma_{i}} e^{\frac{1}{2 T} \sum_{i, j} J_{i j} \sigma_{i} \sigma_{j}} \tag{8.50}
\end{equation*}
$$

We can introduce auxiliary variables, $\varphi_{i}$, to rewrite this as:

$$
\begin{equation*}
\mathcal{Z}=N \sum_{\sigma_{i}} \int \mathcal{D} \varphi_{i} e^{\sum_{i} \varphi_{i} \sigma_{i}} e^{-\frac{T}{2} \sum_{i, j}\left(J^{-1}\right)_{i j} \varphi_{i} \varphi_{j}} \tag{8.51}
\end{equation*}
$$

The sum over the $\sigma_{i}$ 's can be done, giving:

$$
\begin{equation*}
\mathcal{Z}=N \sum_{\sigma_{i}} \int \mathcal{D} \varphi_{i} e^{-\frac{T}{2} \sum_{i, j}\left(J^{-1}\right)_{i j} \varphi_{i} \varphi_{j}+\sum_{i} \ln \cosh \varphi_{i}} \tag{8.52}
\end{equation*}
$$

If $J_{i j}=J(i-j)$, then the first term in the exponential can be brought to a more convenient form by Fourier transforming:

$$
\begin{equation*}
\sum_{i, j}\left(J^{-1}\right)_{i j} \varphi_{i} \varphi_{j}=\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{J(q)} \varphi(q) \varphi(-q) \tag{8.53}
\end{equation*}
$$

The momenta are cutoff at large $q$ by the inverse lattice spacing. If we are interested in small $q$, this cutoff is unimportant, and we can write

$$
\begin{equation*}
J(q)=J_{0}-\frac{1}{2} J_{2} q^{2}+O\left(q^{4}\right) \tag{8.54}
\end{equation*}
$$

so long as $J(i-j)$ falls off sufficiently rapidly. Hence, we can write

$$
\begin{align*}
\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{J(q)} \varphi(q) \varphi(-q) & =\int \frac{d^{d} q}{(2 \pi)^{d}}\left(\frac{1}{J_{0}} \varphi(q) \varphi(-q)+\frac{J_{2}}{2 J_{0}^{2}} q^{2} \varphi(q) \varphi(-q)+O\left(q^{4}\right)\right) \\
& =\int d^{d} x\left(\frac{J_{2}}{2 J_{0}^{2}}(\nabla \varphi)^{2}+\frac{1}{J_{0}} \varphi^{2}+O\left((\nabla \varphi)^{4}\right)\right) \tag{8.55}
\end{align*}
$$

In the second line, we have gone back to real space and taken the continuum limit, $\varphi_{i} \rightarrow \varphi(x)$. Expanding the other term,

$$
\begin{equation*}
\ln \cosh \varphi_{i}=\ln 2+\frac{1}{2} \varphi^{2}-\frac{1}{12} \varphi^{4}+O\left(\varphi^{6}\right) \tag{8.56}
\end{equation*}
$$

If we neglect the $O\left((\nabla \varphi)^{4}\right)$ and higher gradient terms - which seems reasonable in the small $q$ limit - as well as powers of $\varphi$ higher than the quartic term - which is not obviously the right thing to do, but later will be shown to be reasonable - then we can write the partition function as the following functional integral.

$$
\begin{equation*}
\mathcal{Z}=N \int \mathcal{D} \varphi e^{-\int d^{d} x\left(\frac{1}{2} K(\nabla \varphi)^{2}+\frac{1}{2} r \varphi^{2}+\frac{1}{4} u \varphi^{4}\right)} \tag{8.57}
\end{equation*}
$$

where $K=T J_{2} / 2 J_{0}^{2}, r=T / J_{0}-1, u=2$. Hence, the imaginary-time functional integral which we have introduced for the generating functional of quantum-mechanical correlation functions is analogous to the classical partition function. The weighted sum over all possible classical histories is in direct analogy with the sum over all classical configurations. The similarity between the functional integral (8.57) and the functional integral for our theory of interacting phonons allows us to immediately deduce its Feynman rules:

- Assign a directed momentum to each line. For external lines, the momentum is directed into the diagram.
- For each internal line with momentum $\vec{q}$ write:

$$
-\int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{K p^{2}+r}
$$

- For each vertex with momenta $\vec{p}_{1}, \ldots, \vec{p}_{4}$ directed into the vertex, write:

$$
u(2 \pi)^{d} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right)
$$

- Imagine labelling the vertices $1,2, \ldots, n$. Vertex $i$ will be connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices $i, i$ is still connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$. If $S$ is the number of permutations which leave the diagram invariant, we assign a factor $1 / S$ to the diagram.
- If two vertices are connected by $l$ lines, we assign a factor $1 / l$ ! to the diagram.

In classical equilibrium statistical mechanics, time plays no role, so there are momenta but no frequencies (unlike in quantum statistical mechanics where, as we have seen, statics and dynamics are intertwined). So long as the system is rotationally invariant, the theory will have Euclidean invariance in $d$ spatial dimensions, and therefore be formally the same as the imaginary-time description of a quantum system in $d-1$-spatial dimensions (and one time dimension). ${ }^{2}$ The classical analog of a quantum system at finite-temperature is a classical system which has a finite extent, $\beta$, in one direction. Much of what we have to say about quantum-mechanical many-body systems can be applied to classical systems with little modification. Of course, classical statistical mechanics should be contained within quantum statistical mechanics in the small $\beta$ limit.

[^3]
### 8.5.2 Mean-Field Theory and the Saddle-Point Approximation

We can apply the saddle-point - or 0-loop - approximation to our functional integral (8.57). The classical equation of motion is:

$$
\begin{equation*}
-K \nabla^{2} \varphi(x)+r \varphi(x)+\frac{u}{6} \varphi^{3}(x)=0 \tag{8.58}
\end{equation*}
$$

Let's look for spatially uniform solutions $\varphi(x)=\varphi$. If $r>0$, there is only

$$
\begin{equation*}
\varphi=0 \tag{8.59}
\end{equation*}
$$

However, for $r<0$ there are also the solutions

$$
\begin{equation*}
\varphi= \pm \sqrt{\frac{-6 r}{u}} \tag{8.60}
\end{equation*}
$$

These latter solutions have larger saddle-point contribution: $\exp \left(3 r^{2} / 2 u\right)$ compared to 1 and are therefore more important.
$r=0$ occurs at $T=T_{c}=J_{0}$. For $T>T_{c}$, there is only one saddle-point solution, $\varphi=0$. According to the saddle-point approximation, at $T=T_{c}$, a phase transition occurs, and for $T<T_{c}$ there is spontaneous magnetization in the system, $\varphi \sim \pm \sqrt{T_{c}-T}$. Of course, we shouldn't stop with the saddle-point approximation but should include higher-loop processes. In the problem set, you will find the Ginzburg criterion which determines whether higher-loop processes invalidate the saddle-point analysis.

The saddle-point analysis reproduces and is completely equivalent to the standard mean-field-theory. To make this more obvious, let's use the full potential (8.56) rather than the one truncated at quartic order:

$$
\begin{equation*}
\mathcal{Z}=N \int \mathcal{D} \varphi e^{-\int d^{d} x\left(\frac{1}{2} K(\nabla \varphi)^{2}+\frac{T}{2 J_{0}} \varphi^{2}-\ln \cosh \varphi\right)} \tag{8.61}
\end{equation*}
$$

The saddle-point equation is:

$$
\begin{equation*}
\frac{T}{J_{0}} \varphi=\tanh \varphi \tag{8.62}
\end{equation*}
$$

This is the usual self-consistency condition of mean-field theory which predicts $T_{c}=$ $J_{0}$.

For purposes of comparison, let's recapitulate mean-field theory. We replace the effective field which each spins sees as a result of its interaction with its neighbors,

$$
\begin{equation*}
H=-\frac{1}{2} \sum_{i, j} J_{i j} \sigma_{i} \sigma_{j} \tag{8.63}
\end{equation*}
$$

by a mean-field, $h$ :

$$
\begin{equation*}
H=-\sum_{i} h \sigma_{i} \tag{8.64}
\end{equation*}
$$

with $h$ given by

$$
\begin{equation*}
h=\sum_{i} J_{i j}\left\langle\sigma_{i}\right\rangle=J_{0}\left\langle\sigma_{i}\right\rangle \tag{8.65}
\end{equation*}
$$

In this field, the partition function is just $2 \cosh \frac{h}{T}$ and

$$
\begin{equation*}
\langle\sigma\rangle=\tanh \frac{h}{T} \tag{8.66}
\end{equation*}
$$

Using the self-consistency condition, this is:

$$
\begin{equation*}
\langle\sigma\rangle=\tanh \frac{J_{0}\langle\sigma\rangle}{T} \tag{8.67}
\end{equation*}
$$

which is the saddle-point condition above.

## Part III

## Goldstone Modes and Spontaneous Symmetry Breaking

## Chapter 9

## Spin Systems and Magnons

### 9.1 Coherent-State Path Integral for a Single Spin

Let us follow Feynman's derivation of the functional integral to formulate a functional integral for a quantum-mechanical spin. A quantum mechanical spin of magnitude $s$ has a $2 s+1$-dimensional Hilbert space of states. One basis is:

$$
\begin{equation*}
S^{z}\left|s^{z}\right\rangle=s^{z}\left|s^{z}\right\rangle, \quad s^{z}=-s,-s+1, \ldots, s-1, s \tag{9.1}
\end{equation*}
$$

For the functional integral representation, it is more convenient to use the overcomplete coherent state basis $|\vec{\Omega}\rangle$ :

$$
\begin{equation*}
\vec{S} \cdot \vec{\Omega}|\vec{\Omega}\rangle=s|\vec{\Omega}\rangle \tag{9.2}
\end{equation*}
$$

where $\vec{\Omega}$ is a unit vector. For $s=1 / 2$, we can write this basis in terms of the spinor $z^{\alpha}, \alpha= \pm 1 / 2:$

$$
\begin{equation*}
|\vec{\Omega}\rangle=z^{\alpha}|\alpha\rangle \tag{9.3}
\end{equation*}
$$

where:

$$
\begin{equation*}
\vec{\Omega}=z^{* \alpha} \vec{\sigma}_{\alpha \beta} z^{\beta} \tag{9.4}
\end{equation*}
$$

in terms of the spherical angles $\theta, \phi$ of $\vec{\Omega}$

$$
\begin{equation*}
z=\binom{e^{-i \phi / 2} \cos \frac{\theta}{2}}{e^{i \phi / 2} \sin \frac{\theta}{2}} \tag{9.5}
\end{equation*}
$$

and $z^{* \alpha} z^{\alpha}=1$. Of course, there is arbitrariness in our choice of the overall phase of $z^{\alpha}$, but so long as we choose a phase and stick with it, there is no problem. Therefore, the states $|\vec{\Omega}\rangle$ and $\left|\overrightarrow{\Omega^{\prime}}\right\rangle$ have overlap:

$$
\begin{equation*}
\langle\vec{\Omega} \mid \vec{\Omega}\rangle=z^{\prime * \alpha} z^{\alpha} \tag{9.6}
\end{equation*}
$$

To obtain larger $s$, we can simply symmetrize $2 s$ spin-1/2's:

$$
\begin{equation*}
|\vec{\Omega}\rangle=z^{\alpha_{1}} \ldots z^{\alpha_{2 s}}\left|\alpha_{1}, \ldots, \alpha_{2 s}\right\rangle \tag{9.7}
\end{equation*}
$$

with

$$
\begin{equation*}
\left\langle\overrightarrow{\Omega^{\prime}} \mid \vec{\Omega}\right\rangle=\left(z^{\prime * \alpha} z^{\alpha}\right)^{2 s} \tag{9.8}
\end{equation*}
$$

In terms of the spherical angles,

$$
\begin{align*}
\langle\hat{\mathbf{z}} \mid \vec{\Omega}\rangle & =\left(\frac{1}{2}(1+\cos \theta) e^{-i \phi}\right)^{s} \\
& =\left(\frac{1}{2}(1+\hat{\mathbf{z}} \cdot \vec{\Omega}) e^{-i \phi}\right)^{s} \tag{9.9}
\end{align*}
$$

Hence the general relation is:

$$
\begin{equation*}
\left\langle\overrightarrow{\Omega^{\prime}} \mid \vec{\Omega}\right\rangle=\left(\frac{1}{2}\left(1+\vec{\Omega}^{\prime} \cdot \vec{\Omega}\right) e^{-i \phi}\right)^{s} \tag{9.10}
\end{equation*}
$$

where $\phi$ is the phase of $z^{\prime * \alpha} z^{\alpha}$. In this basis, the resolution of the identity is given by:

$$
\begin{equation*}
I=\frac{2 s+1}{4 \pi} \int d^{2} \vec{\Omega}|\vec{\Omega}\rangle\langle\vec{\Omega}| \tag{9.11}
\end{equation*}
$$

as may be seen by taking its matrix elements between states $\langle s|$ and $|s-n\rangle$. The usefulness of this basis lies in the following property:

$$
\begin{equation*}
\langle\vec{\Omega}| f(\vec{S})|\vec{\Omega}\rangle=f(s \vec{\Omega}) \tag{9.12}
\end{equation*}
$$

To see this, use (9.7) to write this as:

$$
\langle\vec{\Omega}| f(\vec{S})|\vec{\Omega}\rangle=z^{* \beta_{1}} \ldots z^{* \beta_{2 s}}\left\langle\beta_{1}, \ldots, \beta_{2 s}\right| f\left(\frac{\vec{\sigma}_{1}}{2}+\ldots+\frac{\vec{\sigma}_{2 s}}{2}\right) z^{\alpha_{1}} \ldots z^{\alpha_{2 s}}\left|\alpha_{1}, \ldots, \alpha_{2 s}\right\rangle
$$

$$
\begin{equation*}
=f(s \vec{\Omega}) \tag{9.13}
\end{equation*}
$$

where we have used $\vec{\Omega}=z^{* \alpha} \vec{\sigma}_{\alpha \beta} z^{\beta}$ in the second line.
Let us construct the functional integral representation for the partition function of a single spin. Following our derivation of the path integral in chapter 8 , we write the imaginary-time evolution operator as

$$
\begin{equation*}
e^{-\beta H(\vec{S})}=\left(e^{-\Delta \tau H(\vec{S})}\right)^{N} \tag{9.14}
\end{equation*}
$$

where $N \Delta \tau=\beta$. Then we can write the partition function as:

$$
\begin{equation*}
\operatorname{Tr}\left\{\left(e^{-\Delta \tau H(\vec{S})}\right)^{N}\right\}=\int \prod_{i=1}^{N} \frac{2 s+1}{4 \pi} d^{2} \vec{\Omega}_{i}\left\langle\vec{\Omega}_{i+1}\right| e^{-\Delta \tau H(\vec{S})}\left|\vec{\Omega}_{i}\right\rangle \tag{9.15}
\end{equation*}
$$

Taking $\Delta \tau \rightarrow 0$, we have

$$
\begin{equation*}
\left\langle\vec{\Omega}_{i+1}\right| e^{-\Delta \tau H(\vec{S})}\left|\vec{\Omega}_{i}\right\rangle \approx e^{-\Delta \tau H\left(s \vec{\Omega}_{i}\right)-s \frac{1}{4}(\Delta \tau)^{2}\left(\frac{d \Omega}{d \tau}\right)^{2}+s\left(z_{i}^{* \alpha} z_{i+1}^{\alpha}-z_{i+1}^{* \alpha} z_{i}^{\alpha}\right)} \tag{9.16}
\end{equation*}
$$

The second term in the exponent was obtained by making the approximation

$$
\begin{equation*}
\left(1+\vec{\Omega}^{\prime} \cdot \vec{\Omega}\right)^{s}=\left(2-\frac{1}{2}\left(\vec{\Omega}-\vec{\Omega}^{\prime}\right)^{2}\right)^{s} \approx 2^{s}\left(1-\frac{1}{4}(\Delta \tau)^{2}\left(\frac{d \Omega}{d \tau}\right)^{2}\right)^{s} \approx 2^{s} e^{-\frac{s}{4}(\Delta \tau)^{2}\left(\frac{d \Omega}{d \tau}\right)^{2}} \tag{9.17}
\end{equation*}
$$

while the third term follows from

$$
\begin{align*}
e^{s \phi} & \approx e^{s \sin \phi} \\
& =e^{s\left(z_{i}^{* \alpha} z_{i+1}^{\alpha}-z_{i+1}^{* \alpha} z_{i}^{\alpha}\right)} \tag{9.18}
\end{align*}
$$

Hence, we have

$$
\begin{equation*}
\operatorname{Tr}\left\{\left(e^{-\Delta \tau H(\vec{S})}\right)^{N}\right\}=N \lim _{\Delta \tau \rightarrow 0} \int \prod_{i=1}^{N} \frac{2 s+1}{4 \pi} d^{2} \vec{\Omega}_{i} e^{\sum_{i}\left(-\Delta \tau H\left(s \vec{\Omega}_{i}\right)-\frac{s}{4}(\Delta \tau)^{2}\left(\frac{d \Omega}{d \tau}\right)^{2}+2 s \Delta \tau z^{* \alpha} \frac{d z^{\alpha}}{d \tau}\right)} \tag{9.19}
\end{equation*}
$$

The $(\Delta \tau)^{2}$ term can be dropped and we can write the partition function as

$$
\begin{equation*}
\operatorname{Tr}\left\{e^{-\beta H(\vec{S})}\right\}=N \int D \vec{\Omega}(\tau) e^{-S_{\text {spin }}[\vec{\Omega}]} \tag{9.20}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{\mathrm{spin}}[\vec{\Omega}]=\int d \tau\left(H(s \vec{\Omega})-2 s z^{* \alpha} \frac{d z^{\alpha}}{d \tau}\right) \tag{9.21}
\end{equation*}
$$

In terms of the spherical angles, the second term can be written

$$
\begin{align*}
2 s \int z^{* \alpha} \frac{d z^{\alpha}}{d \tau} & =i s \int d \tau \frac{d \phi}{d \tau} \cos \theta(\tau) \\
& =i s \int d \phi \cos \theta(\phi) \\
& =i s \int d \phi\left(1-\int_{\cos \theta(\phi)}^{1} d(\cos \theta)\right) \\
& =i s(2 \pi-A) \tag{9.22}
\end{align*}
$$

where $A$ is the area of the region of the sphere enclosed by the curve traced out by $\vec{\Omega}(\tau)$. Actually, the curve traced out by $\vec{\Omega}(\tau)$ divides the surface of the sphere into two pieces, so there is some ambiguity in the definition of $A$. However, these two areas add up to $4 \pi$ and $e^{4 \pi i s}=1$, so we can take either choice of $A$.

We can check that we get the correct equation of motion from this Lagrangian. As you will show in the problem set,

$$
\begin{equation*}
\delta\left(z^{* \alpha} \frac{d z^{\alpha}}{d \tau}\right)=\delta z^{* \alpha} \frac{d z^{\alpha}}{d \tau}-\delta z^{\alpha} \frac{d z^{* \alpha}}{d \tau} \tag{9.23}
\end{equation*}
$$

while

$$
\begin{equation*}
\delta \vec{\Omega}=\delta z^{* \alpha} \vec{\sigma}_{\alpha \beta} z^{\beta}+z^{* \alpha} \vec{\sigma}_{\alpha \beta} \delta z^{\beta} \tag{9.24}
\end{equation*}
$$

so that

$$
\begin{equation*}
\delta\left(z^{* \alpha} \frac{d z^{\alpha}}{d \tau}\right)=\delta \vec{\Omega} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau} \tag{9.25}
\end{equation*}
$$

Hence, the equation of motion following from our Lagrangian is:

$$
\begin{equation*}
i s \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}=\frac{\partial H}{\partial \vec{\Omega}} \tag{9.26}
\end{equation*}
$$

Since $\vec{\Omega}$ is a unit vector, $\vec{\Omega}$ and $d \vec{\Omega} / d \tau$ are perpendicular and the equation of motion can be written

$$
\begin{equation*}
i s \frac{d \vec{\Omega}}{d \tau}=\vec{\Omega} \times \frac{\partial H}{\partial \vec{\Omega}} \tag{9.27}
\end{equation*}
$$

which is what we expect from $d \vec{\Omega} / d \tau=[\vec{\Omega}, H]$. If the spin is in a magnetic field, but is otherwise free, $H=\mu \vec{B} \cdot \vec{S}$, then the spin precesses about the magnetic field:

$$
\begin{equation*}
i \frac{d \vec{\Omega}}{d \tau}=\mu \vec{\Omega} \times \vec{B} \tag{9.28}
\end{equation*}
$$

The time-derivative term,

$$
\begin{equation*}
S=s \int d \tau\left(-i \cos \theta \frac{d \phi}{d \tau}\right) \tag{9.29}
\end{equation*}
$$

can be written in the alternative form:

$$
\begin{equation*}
S=s \int d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \frac{d \vec{\Omega}}{d \tau}\right) \tag{9.30}
\end{equation*}
$$

where $\vec{A}(\vec{\Omega})$ satisfies

$$
\begin{equation*}
\vec{\nabla}_{\Omega} \times \vec{A}(\vec{\Omega})=s \vec{\Omega} \tag{9.31}
\end{equation*}
$$

This clearly leads to the same equation of motion and leads to the following interpretation of a spin: the spin can be modelled by a charged particle moving on the surface of a sphere with a magnetic monopole of magnetic charge $s$ located at the origin. By Stokes' theorem, the action is given by the magnetic flux through the area enclosed by the orbit.

It can also be written in the form

$$
\begin{equation*}
S=s \int_{0}^{\beta} d \tau \int_{0}^{1} d r\left(-i \frac{d \vec{\Omega}}{d r} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}\right) \tag{9.32}
\end{equation*}
$$

where $\vec{\Omega}(r, \tau)$ interpolates between the north pole, $\hat{\mathbf{z}}$ and $\vec{\Omega}(\tau): \vec{\Omega}(0, \tau)=\hat{\mathbf{z}}, \vec{\Omega}(1, \tau)=$ $\vec{\Omega}(\tau) . \vec{\Omega}(r, \tau)$ thereby covers the region of the sphere enclosed by $\vec{\Omega}(\tau)$. To see that the action (9.32) gives the area of this region, observe that the integrand is equal to the Jacobian of the map from the $(r, \tau)$ plane to the surface of the sphere. Hence, the action depends only on $\vec{\Omega}(1, \tau)=\vec{\Omega}(\tau)$ and is independent of the particular
interpolating function $\vec{\Omega}(r, \tau)$. To see that the same equation of motion results from this form of the action, imagine varying the upper limit of integration:

$$
\begin{equation*}
S=s \int_{0}^{\beta} d \tau \int_{0}^{r^{\prime}} d r\left(-i \frac{d \vec{\Omega}}{d r} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}\right) \tag{9.33}
\end{equation*}
$$

so that $\delta \vec{\Omega}(\tau)=\vec{\Omega}\left(r^{\prime}, \tau\right)-\vec{\Omega}(1, \tau)$. Then, writing $\delta r=r^{\prime}-1$, we have $\delta \vec{\Omega}(\tau)=$ $\delta r(d \Omega / d r)$ and:

$$
\begin{align*}
\delta S & =s \int_{0}^{\beta} d \tau \delta r\left(-i \frac{d \vec{\Omega}}{d r} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}\right) \\
& =s \int_{0}^{\beta} d \tau\left(-i \delta \vec{\Omega} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}\right) \tag{9.34}
\end{align*}
$$

from which the correct equation of motion follows. Terms of this form are often called Wess-Zumino terms.

### 9.2 Ferromagnets

### 9.2.1 Spin Waves

Suppose that we have the ferromagnetic Heisenberg Hamiltonian:

$$
\begin{equation*}
H=-J \sum_{i, j} \vec{S}_{i} \cdot \vec{S}_{i} \tag{9.35}
\end{equation*}
$$

with $J>0$ and the sum restricted to nearest neighbors. The equation of motion is:

$$
\begin{equation*}
i s \frac{d \vec{\Omega}_{i}}{d \tau}=-J s^{2} \sum_{j} \vec{\Omega}_{i} \times \vec{\Omega}_{j} \tag{9.36}
\end{equation*}
$$

The ground state of the Heisenberg ferromagnet is one in which the $S U(2)$ spinrotational symmetry is spontaneously broken. The system chooses a direction along which to order. Let us call this direction $\vec{\Omega}_{0}$. Linearizing about the uniform ground state, $\vec{\Omega}_{i}=\vec{\Omega}_{0}+\delta \vec{\Omega}_{i}$, we have:

$$
\begin{equation*}
i s \frac{d \delta \vec{\Omega}_{i}}{d \tau}=-J s^{2} \sum_{j} \vec{\Omega}_{0} \times\left(\delta \vec{\Omega}_{j}-\delta \vec{\Omega}_{i}\right) \tag{9.37}
\end{equation*}
$$

Substituting a plane-wave solution, $\delta \vec{\Omega}=\vec{\epsilon} e^{i \vec{q} \cdot \vec{R}_{i}}$, we have the dispersion relation of spin-wave theory:

$$
\begin{equation*}
E(\vec{q})=J s\left(z-\sum_{i=1}^{z} e^{i \vec{q} \cdot \vec{\delta}_{i}}\right) \tag{9.38}
\end{equation*}
$$

where the sum is over the $z$ nearest neighbors. For a $d$-dimensional hypercubic lattice, the cordination number, $z$, is $2 d$. For a square lattice of spacing $a$ in $d$-dimensions, this gives:

$$
\begin{equation*}
E(\vec{q})=2 J s\left(d-\sum_{i=1}^{d} \cos q_{i} a\right) \tag{9.39}
\end{equation*}
$$

In the small $\vec{q}$ limit, the spin-waves have quadratic dispersion:

$$
\begin{equation*}
E(\vec{q})=2 J s a^{2} q^{2} \tag{9.40}
\end{equation*}
$$

### 9.2.2 Ferromagnetic Magnons

The small $\vec{q}$ behavior can be obtained directly from the continuum limit of the Lagrangian. Since $S_{i}^{2}=s(s+1)$,

$$
\begin{equation*}
\vec{S}_{i} \cdot \vec{S}_{i}=\frac{1}{2}\left(\vec{S}_{i}-\vec{S}_{j}\right)^{2}+\text { const. } \tag{9.41}
\end{equation*}
$$

the action takes the following form in the continuum limit:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \frac{d \vec{\Omega}}{d \tau}+\frac{1}{2} D(\nabla \vec{\Omega})^{2}\right) \tag{9.42}
\end{equation*}
$$

where $D=2 J s a^{2}$.
As in the previous section, we linearize the Lagrangian about an ordered state which, without loss of generality, we take to be $\vec{\Omega}=\hat{\mathbf{z}}$. We write

$$
\begin{equation*}
\vec{\Omega}=\left(m_{x}, m_{y}, \sqrt{1-m_{x}^{2}-m_{y}^{2}}\right) \tag{9.43}
\end{equation*}
$$

and assume that $m_{x}, m_{y}$ are small so that we can neglect all terms in the action higher than quadratic. We can now write

$$
\begin{equation*}
A\left(m_{x}, m_{y}\right)=\frac{s}{2}\left(-m_{y}, m_{x}, 0\right) \tag{9.44}
\end{equation*}
$$

since

$$
\begin{equation*}
\nabla_{m} \times A\left(m_{x}, m_{y}\right)=s(0,0,1)=s \Omega \tag{9.45}
\end{equation*}
$$

Hence, we can write the action as:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(\frac{1}{2} i \epsilon_{j i} m_{i} \frac{\partial m_{j}}{\partial \tau}+\frac{1}{2} D\left(\nabla m_{i}\right)^{2}\right) \tag{9.46}
\end{equation*}
$$

Introducing the fields $m_{ \pm}=m_{x} \pm i m_{y}$, we can write:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(\frac{1}{2} m_{+} \frac{\partial m_{-}}{\partial \tau}+\frac{1}{2} D \vec{\nabla} m_{+} \cdot \vec{\nabla} m_{-}\right) \tag{9.47}
\end{equation*}
$$

In chapter 8, we learned that the propagator is simply the inverse of the differential operator in the quadratic part of the action. The inverse can be taken trivially in momentum space:

$$
\begin{equation*}
\left\langle T_{\tau}\left(m_{+}(\vec{x}, \tau) m_{-}(0,0)\right)\right\rangle=\frac{2}{s} \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{e^{i\left(\vec{p} \cdot \vec{x}-\omega_{n} \tau\right)}}{i \omega_{n}-D p^{2}} \tag{9.48}
\end{equation*}
$$

Alternatively, we can expand $m_{ \pm}$in normal modes:

$$
\begin{align*}
m_{-}(x, \tau) & =\int \frac{d^{d} \vec{k}}{(2 \pi)^{d / 2}} a_{\vec{k}}^{\dagger} e^{-D k^{2} \tau+i \vec{k} \cdot \vec{x}} \\
m_{+}(x, \tau) & =\int \frac{d^{d} \vec{k}}{(2 \pi)^{d / 2}} a_{\vec{k}} e^{D k^{2} \tau-i \vec{k} \cdot \vec{x}} \tag{9.49}
\end{align*}
$$

and compute the propagator directly as we did for phonons. $a_{\vec{k}}^{\dagger}$ and $a_{\vec{k}}$ are called magnon creation and annihilation operators. Magnons are the quantum particles which correspond to spin waves in analogy with the correspondence between phonons and sound waves or photons and electromagnetic waves. In the ground state, all of the spins point up. To create a magnon, we flip one spin down with $a^{\dagger}$; to annihilate it, we flip the spin back up.

Using the propagator (9.48), we can compute the magnetization as a function of temperature. To lowest order in $m_{ \pm}$,

$$
\begin{equation*}
\Omega_{z}=\sqrt{1-m_{+} m_{-}} \approx 1-\frac{1}{2} m_{+} m_{-} \tag{9.50}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\left\langle\Omega_{z}(x, \tau)\right\rangle & =1-\frac{1}{2}\left\langle m_{+}(x, \tau) m_{-}(x, \tau)\right\rangle \\
& =1-\frac{1}{s} \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{i \omega_{n}-D p^{2}} \\
& =1-\frac{1}{s} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \int \frac{d \omega}{2 \pi i} n_{B}(\omega)\left(\frac{1}{\omega+i \delta-D p^{2}}-\frac{1}{\omega-i \delta-D p^{2}}\right) \\
& =1-\frac{1}{s} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \int d \omega n_{B}(\omega) \delta\left(\omega-D p^{2}\right) \\
& =1-\frac{1}{s} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} n_{B}\left(D p^{2}\right) \\
& =1-\frac{1}{2 s} \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma(d / 2)}\left(\frac{T}{D}\right)^{d / 2} \int_{0}^{\infty} \frac{x^{\frac{d}{2}-1} d x}{e^{x}-1} \tag{9.51}
\end{align*}
$$

In the third line, we have converted the sum over Matsubara frequencies to an integral, as usual, obtaining a contribution only from the real axis.

Hence in $d=3$, the magnetization decreases as $M_{z}(0)-M_{z}(T) \sim T^{\frac{3}{2}}$. In $d \leq 2$, however, the integral is divergent so we cannot trust the approximation (9.50). In fact, this divergence is a sign that the magnetization vanishes for any finite temperature in $d \leq 2$. Note that at $T=0$, the exact ground state of the ferromagnetic Heisenberg Hamiltonian is fully polarized for arbitrary $d$. For $d>2$, the magnetization decreases continuously from its full value as the temperature is increased. For $d \leq 2, M_{z}$ discontinuously jumps to zero as the temperature is raised above zero.

Thus far, we have neglected anharmonic terms in the magnon Lagrangian. By including these terms, we would have interactions between the magnons. Magnonmagnon interactions affect, for instance, the magnetization as a function of temperature. We will not discuss these interactions here, but we will discuss the analogous interactions in the next section in the context of antiferromagnetism.

### 9.2.3 A Ferromagnet in a Magnetic Field

Suppose we place our ferromagnet in a magnetic field, $\vec{B}$. At zero temperature, the magnetization will line up along the direction of the field. Let us suppose that the field is in the $\hat{\mathbf{z}}$ direction. Then the action is:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \frac{d \vec{\Omega}}{d \tau}+\frac{1}{2} D(\nabla \vec{\Omega})^{2}+\mu s B \Omega_{z}\right) \tag{9.52}
\end{equation*}
$$

where $\mu$ is the gyromagnetic ratio. If we expand about the ordered state, $\vec{\Omega}=\Omega_{z} \hat{\mathbf{z}}$, then we have the quadratic action:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(\frac{1}{2} m_{+} \frac{\partial m_{-}}{\partial \tau}+D \vec{\nabla} m_{+} \cdot \vec{\nabla} m_{-}-\frac{1}{2} \mu s B m_{+} m_{-}\right) \tag{9.53}
\end{equation*}
$$

The propagator is now

$$
\begin{equation*}
\left\langle T_{\tau}\left(m_{+}(\vec{x}, \tau) m_{-}(0,0)\right)\right\rangle=\frac{2}{s} \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{e^{i\left(\vec{p} \cdot \vec{x}-\omega_{n} \tau\right)}}{i \omega_{n}-D p^{2}-\mu s B / 2} \tag{9.54}
\end{equation*}
$$

As a result of the magnetic field, there is a minumum energy cost $\mu B$ to flip a spin. If we repeat the calculation of the magnetization as a function of temperature, we find:

$$
\begin{align*}
\left\langle\Omega_{z}(x, \tau)\right\rangle & =1-\frac{1}{2}\left\langle m_{+}(x, \tau) m_{-}(x, \tau)\right\rangle \\
& =1-\frac{1}{s} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} n_{B}\left(D p^{2}+\mu s B / 2\right) \tag{9.55}
\end{align*}
$$

Unlike in the $B=0$ case, this integral is not infrared divergent: the magnetic field stabilizes the ferromagnetic state against thermal fluctuations.

### 9.3 Antiferromagnets

### 9.3.1 The Non-Linear $\sigma$-Model

Let us now consider the antiferromagnetic case,

$$
\begin{equation*}
H=J \sum_{i, j} \vec{S}_{i} \cdot \vec{S}_{i} \tag{9.56}
\end{equation*}
$$

with $J>0$ and the sum restricted to nearest neighbors. We expand about the state in which the spins are staggered - the Néel state:

$$
\begin{equation*}
\vec{\Omega}_{i}=(-1)^{i} \vec{n}\left(x_{i}\right)+\frac{1}{s} \vec{l}\left(x_{i}\right) \tag{9.57}
\end{equation*}
$$

$\vec{l}$ is the $q=0$ part of the spin field while $\vec{n}$ is the $\vec{q}=(\pi / a, \ldots, \pi / a)$ part. We only keep the Fourier modes near these wavevectors. $\vec{n}$ and $\vec{l}$ are assumed to be slowly varying and $\vec{n} \cdot \vec{l}=0$. Then $\Omega_{i}^{2}=1$ is satisfied to $O\left(1 / s^{2}\right)$ if $n^{2}=1$. With this decomposition, we can write

$$
\begin{align*}
H & =J s^{2} \sum_{i, j}\left(-\vec{n}\left(x_{i}\right) \cdot \vec{n}\left(x_{j}\right)+\frac{1}{s^{2}} \vec{l}\left(x_{i}\right) \cdot \vec{l}\left(x_{j}\right)\right) \\
& =J s^{2} \sum_{i, j}\left(\left(\vec{n}\left(x_{i}\right)-\vec{n}\left(x_{j}\right)\right)^{2}+\frac{1}{s^{2}}\left(\vec{l}\left(x_{i}\right)+\vec{l}\left(x_{j}\right)\right)^{2}\right)+\text { const. } \tag{9.58}
\end{align*}
$$

going to the continuum limit, we have:

$$
\begin{equation*}
H=v a^{-d} \int d^{d} \vec{x}\left(\frac{1}{s a} \vec{l}^{2}+\frac{s a}{4}(\nabla \vec{n})^{2}\right) \tag{9.59}
\end{equation*}
$$

where $v=2 J$ sa.
The corresponding action is:

$$
\begin{gather*}
S=a^{-d} \int d^{d} \vec{x} d \tau\left(-\frac{i}{a} \vec{A}\left((-1)^{i} \vec{n}+\frac{1}{s} \vec{l}\right) \cdot\left((-1)^{i} \frac{d \vec{n}}{d \tau}+\frac{1}{s} \frac{d \vec{l}}{d \tau}\right)\right. \\
\left.+\frac{1}{s a} v \vec{l}^{2}+\frac{s a}{4} v(\nabla \vec{n})^{2}\right) \tag{9.60}
\end{gather*}
$$

Using (9.57) and

$$
\begin{align*}
\vec{\nabla}_{\Omega} \times\left((-1)^{i} \vec{n} \times \vec{l}\right) & =(-1)^{i+1} \vec{n}+\frac{1}{s} \vec{l} \\
& =(-1)^{i} \vec{n}+\frac{1}{s} \vec{l}+(-1)^{i+1} \partial_{x} \vec{n} \\
& \approx \vec{\Omega} \tag{9.61}
\end{align*}
$$

we can express $\vec{A}$ in terms of $\vec{n}$ and $\vec{l}$ if we drop the gradient term in the penultimate line (this cannot be done in $d=1$, where it is absolutely crucial, but can in higher
dimensions). Neglecting oscillatory terms in the action, we have:

$$
\begin{equation*}
S=a^{-d} \int d^{d} \vec{x} d \tau\left(\frac{i}{a} \vec{n} \times \vec{l} \cdot \frac{d \vec{n}}{d \tau}+\frac{1}{s a} v \vec{l}^{2}+\frac{s a}{4} v(\nabla \vec{n})^{2}\right) \tag{9.62}
\end{equation*}
$$

The functional integral

$$
\begin{equation*}
\int \mathcal{D} \vec{l} \mathcal{D} \vec{n} e^{-S[\vec{l}, \vec{n}]} \tag{9.63}
\end{equation*}
$$

is a Gaussian integral in $\vec{l}$, so we can perform the $\vec{l}$ integral. Integrating out $\vec{l}$, we have:

$$
\begin{equation*}
S=a^{-d} \int d^{d} \vec{x} d \tau\left(\frac{s a}{4 v}\left(\frac{d \vec{n}}{d \tau}\right)^{2}+\frac{s a}{4} v(\nabla \vec{n})^{2}\right) \tag{9.64}
\end{equation*}
$$

Or, writing $g=a^{d-2} / J s^{2}$,

$$
\begin{equation*}
S=\frac{1}{g} \int d^{d} \vec{x} d \tau\left(\frac{1}{2 v^{2}}\left(\frac{d \vec{n}}{d \tau}\right)^{2}+\frac{1}{2}(\nabla \vec{n})^{2}\right) \tag{9.65}
\end{equation*}
$$

This action is called the $O(3)$ Non-Linear $\sigma$ Model, or $O(3) \mathrm{NL} \sigma \mathrm{M}$ for short. The $O(3)$ refers to the fact that the field $\vec{n}$ is a three-component field which transforms as a vector under the rotation group, $O(3)$. The model is non-linear because there is a non-linear constraint, $n^{2}=1$.

### 9.3.2 Antiferromagnetic Magnons

Let us, for simplicity work in a system of units in which $v=1$. We can always rescale our time coordinate at the end of any calculation so as to restore $v$. Let us also employ the notation $\mu=0,1, \ldots, d$. with 0 referring to the time direction so that $\partial_{0}=\partial_{t}$. Then we can write the action of the $O(3) \mathrm{NL} \sigma \mathrm{M}$ as:

$$
\begin{equation*}
S=\frac{1}{g} \int d^{d} \vec{x} d \tau\left(\partial_{\mu} \vec{n}\right)^{2} \tag{9.66}
\end{equation*}
$$

If, as in the ferromagnetic case, we expand about an ordered state,

$$
\begin{equation*}
\vec{n}=\left(n_{x}, n_{y}, \sqrt{1-n_{x}^{2}-n_{y}^{2}}\right) \tag{9.67}
\end{equation*}
$$

then we can write the action as:

$$
\begin{equation*}
S=\frac{1}{g} \int d^{d} \vec{x} d \tau\left(\left(\partial_{\mu} n_{i}\right)^{2}+\frac{n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j}}{1-n_{i} n_{i}}\right) \tag{9.68}
\end{equation*}
$$

where $i=1,2$ and $n_{1}=n_{x}, n_{2}=n_{y}$. Let us rescale the fields so that $n_{i} \rightarrow \sqrt{g} n_{i}$. Then the action becomes:

$$
\begin{equation*}
S=\int d^{d} \vec{x} d \tau\left(\left(\partial_{\mu} n_{i}\right)^{2}+g \frac{n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j}}{1-g n_{i} n_{i}}\right) \tag{9.69}
\end{equation*}
$$

In order to do perturbation theory in $g$, which we can hope to do when it is small, we divide the action into two parts,

$$
\begin{equation*}
S_{\mathrm{free}}=\int d^{d} \vec{x} d \tau\left(\partial_{\mu} n_{i}\right)^{2} \tag{9.70}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{\mathrm{int}}=\int d^{d} \vec{x} d \tau g \frac{n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j}}{1-g n_{i} n_{i}} \tag{9.71}
\end{equation*}
$$

Note that $S_{\text {int }}$ contains all powers of $g$ when the denominator is expanded in a geometric series.
$S_{\text {free }}$ is very similar to the phonon action; as in that case, we can expand $n_{i}$ in normal modes:

$$
\begin{equation*}
n_{i}(\vec{r}, t)=\int \frac{d^{d} \vec{k}}{(2 \pi)^{d / 2}} \frac{1}{\sqrt{2 k}}\left(a_{\vec{k}, i} e^{i \vec{k} \cdot \vec{r}+\omega_{k}^{s} \tau}+a_{\vec{k}, i}^{\dagger} e^{-i \vec{k} \cdot \vec{r}-v k \tau}\right) \tag{9.72}
\end{equation*}
$$

$a_{ \pm}^{\dagger}=a_{x}^{\dagger} \pm a_{y}^{\dagger}$ create, respectively, up- and down-spin antiferromagnetic magnons. Note the difference with the ferromagnetic case, in which there was only one type of magnon. Since there is no net uniform magnetization in the ground state, we can either flip a spin up - thereby creating an up-spin magnon - or flip a spin down thereby creating a down spin magnon.

We can obtain the antiferromagnetic magnon propagator using this mode expansion or by inverting $S_{\text {free }}$ :

$$
\begin{equation*}
\left\langle T_{\tau}\left(n_{i}(\vec{x}, \tau) n_{j}(0,0)\right)\right\rangle=\delta_{i j} \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{e^{i\left(\vec{p} \cdot \vec{x}-\omega_{n} \tau\right)}}{\omega_{n}^{2}+p^{2}} \tag{9.73}
\end{equation*}
$$

or, restoring $v$,

$$
\begin{equation*}
\left\langle T_{\tau}\left(n_{i}(\vec{x}, \tau) n_{j}(0,0)\right)\right\rangle=\delta_{i j} \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{e^{i\left(\vec{p} \cdot \vec{x}-\omega_{n} \tau\right)}}{\omega_{n}^{2}+v^{2} p^{2}} \tag{9.74}
\end{equation*}
$$

The corresponding spectral function is:

$$
\begin{equation*}
B(\omega, \vec{p})=\frac{1}{2 v p} \delta(\omega-v p)-\frac{1}{2 v p} \delta(\omega+v p) \tag{9.75}
\end{equation*}
$$

With this propagator in hand, we can compute the staggered magnetization in the $g \rightarrow 0$ limit. It is given by

$$
\begin{equation*}
n_{z}=\sqrt{1-g n_{i} n_{i}} \approx 1-\frac{1}{2} g n_{i} n_{i} \tag{9.76}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\left\langle n_{z}(x, \tau)\right\rangle & \approx 1-g \frac{1}{2}\left\langle n_{i}(x, \tau) n_{i}(x, \tau)\right\rangle \\
& =1-g \frac{1}{2} \cdot 2 \cdot \frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{\omega_{n}^{2}+v^{2} p^{2}} \\
& =1-g \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \int \frac{d \omega}{2 \pi i} n_{B}(\omega) 2 \pi i B(\omega, \vec{p}) \\
& =1-g \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{2 v p}\left(n_{B}(v p)-n_{B}(-v p)\right) \\
& =1-g \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{2 v p} \operatorname{coth}\left(\frac{\beta v p}{2}\right) \tag{9.77}
\end{align*}
$$

Unlike in the ferromagnetic case, the second term does not vanish in the $T \rightarrow 0$ limit. For $T=0$, we have:

$$
\begin{equation*}
\left\langle n_{z}(x, \tau)\right\rangle \approx 1-g \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{2 v p} \tag{9.78}
\end{equation*}
$$

If we approximate the Brillouin zone by a sphere, $|p|<\pi / a$, then we find

$$
\begin{equation*}
\left\langle n_{z}(x, \tau)\right\rangle \approx 1-g \frac{1}{d-1}\left(\frac{\pi}{a}\right)^{d-1} \frac{(2 \pi)^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma(d / 2)} \tag{9.79}
\end{equation*}
$$

Hence, the staggered magnetization of an antiferromagnet is less than 1 even at $T=0$, unlike the uniform magnetization in the ferromagnetic case. The Néel state,
with neighboring spins oppositely oriented, is not the exact ground state. In $d=1$, the integral is actually logarithmically divergent. This divergence hints at the impossibility of antiferromagnetic order in $d=1$, which is a consequence of a theorem which we will prove in the next chapter. For $T$ finite, the integral (9.77) is logarithmically divergent at small $p$ in $d=2$, just as in the ferromagnetic case. Again, it is a sign of the impossibility of antiferromagnetic order at finite temperatures in $d=2$.

### 9.3.3 Magnon-Magnon-Interactions

Thus far, we have ignored the higher-order terms in the NL $\sigma \mathrm{M}$. These terms lead to interactions between magnons. To get an idea of the nature of these terms, let's expand $S_{\text {int }}$ to $O\left(g^{2}\right)$ :

$$
\begin{equation*}
S_{g^{2}}=\int d^{d} \vec{x} d \tau\left(\left(\partial_{\mu} n_{i}\right)^{2}+g n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j}+g^{2} n_{k} n_{k} n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j} 1-g n_{i} n_{i}+\ldots\right) \tag{9.80}
\end{equation*}
$$

We need only these terms in order to do computations to order $O\left(g^{2}\right)$. The Feynman rules for this action are:

- Assign a directed momentum and Matsubara frequency to each line. Assign an index $i=1,2$ to each line. For external lines, the momentum and frequency are directed into the diagram.
- For each internal line with momentum, frequency $\vec{q}, i \omega_{n}$ write:

$$
-\frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{1}{\omega_{n}^{2}+v^{2} p^{2}}
$$

- For each 4-leg vertex with momenta, Matsubara frequencies $\left(\vec{p}_{1}, \omega_{n_{1}}\right), \ldots,\left(\vec{p}_{4}, \omega_{n_{4}}\right)$ directed into the vertex and indices $i_{1}, \ldots, i_{4}$ associated to these incoming lines, write:

$$
g \delta_{i_{1} i_{2}} \delta_{i_{3} i_{4}}\left(\omega_{i_{1}} \omega_{i_{3}}+\vec{p}_{1} \cdot \vec{p}_{3}\right)(2 \pi)^{d} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}\right) \beta \delta_{n_{1}+n_{2}+n_{3}+n_{4}, 0}
$$

- For each 6-leg vertex with momenta momenta, Matsubara frequencies $\left(\vec{p}_{1}, \omega_{n_{1}}\right), \ldots,\left(\vec{p}_{6}, \omega_{n_{6}}\right)$ directed into the vertex and indices $i_{1}, \ldots, i_{6}$ associated to these incoming lines, write:
$g^{2} \delta_{i_{1} i_{2}} \delta_{i_{3} i_{4}} \delta_{i_{5} i_{6}}\left(\omega_{i_{1}} \omega_{i_{3}}+\vec{p}_{1} \cdot \vec{p}_{3}\right)(2 \pi)^{d} \delta\left(\vec{p}_{1}+\vec{p}_{2}+\vec{p}_{3}+\vec{p}_{4}+\vec{p}_{5}+\vec{p}_{6}\right) \beta \delta_{n_{1}+n_{2}+n_{3}+n_{4}+n_{5}+n_{6}, 0}$
- We assign a factor $1 / S$ to the diagram if there are $S$ permutations of the vertices and external momenta which leave the diagram invariant.


### 9.4 Spin Systems at Finite Temperatures

In the previous two sections, we saw that ferromagnetic and antiferromagnetic order are suppressed by thermal fluctuations. Let us examine this more closely. Let us, for the sake of concreteness, consider the case of antiferromagnetism. Let us re-write the action in the following dimensionless form:

$$
\begin{equation*}
S=\frac{a^{d-1}}{g v} \int d^{d} \vec{y} \int_{0}^{\beta v / a} d \tau\left(\frac{1}{2}\left(\frac{d \vec{n}}{d u}\right)^{2}+\frac{1}{2}(\nabla \vec{n})^{2}\right) \tag{9.81}
\end{equation*}
$$

where $u=v \tau / a$ and $y=x / a$. If we go to momentum space,

$$
\begin{equation*}
S=\frac{a^{d-1}}{g v} \int_{0}^{\pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\beta} \sum_{n}\left(\frac{1}{2}\left|\omega_{n} \vec{n}\left(\omega_{n}, q\right)\right|^{2}+\frac{1}{2}\left|q \vec{n}\left(\omega_{n}, q\right)\right|^{2}\right) \tag{9.82}
\end{equation*}
$$

then the cutoff is just $\pi$ (for a spherical Brillouin zone; more generally, it's some number of order 1). The Matsubara frequencies are $\omega_{n}=2 \pi n a / \beta v$. When $a / \beta v \gg 1$ - i.e. at temperatures which are large compared to $v / a$ (the energy of a magnon at the cutoff wavevector) - the configurations with $\omega_{n} \neq 0$ are strongly suppressed and give very little contribution to the functional integral. Therefore, the functional integral is dominated by configurations which are independent of $\tau$ and we can replace $\int_{0}^{\beta} d \tau \rightarrow \beta$. Hence, we may make the approximation:

$$
Z=\int \mathcal{D} \vec{n} e^{-\frac{a^{d-1}}{g v} \int_{0}^{\pi} \frac{d^{d} \overrightarrow{\vec{a}}}{(2 \pi)^{d}} \int_{0}^{\beta} d \tau\left(\frac{1}{2}\left(\partial_{\tau} \vec{n}\right)^{2}+\frac{1}{2}|q \vec{n}(q)|^{2}\right)}
$$

$$
\begin{equation*}
\approx \int \mathcal{D} \vec{n} e^{-\frac{\left\langle a^{d-1}\right.}{g v} \int_{0}^{\pi} d^{d} \vec{q}\left(\frac{1}{2}\left|q \vec{n}\left(\omega_{n}=0, q\right)\right|^{2}\right)} \tag{9.83}
\end{equation*}
$$

We can similarly write the ferromagnetic functional integral with momentum cutoff of order 1 and Matsubara frequencies $\omega_{n}=2 \pi n a^{2} / \beta D$ :

$$
\begin{align*}
Z & =\int \mathcal{D} \vec{\Omega} e^{-s a^{d} \int \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \int_{0}^{\beta} d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \partial_{\tau} \vec{\Omega}+\frac{1}{2}(q \vec{\Omega})^{2}\right)} \\
& =\int \mathcal{D} \vec{\Omega} e^{-s a^{d} \beta \int \frac{d^{d} \vec{q}}{(2 \pi)^{d}}\left(\frac{1}{2}\left(q \vec{\Omega}\left(\omega_{n}=0, q\right)\right)^{2}\right)} \tag{9.84}
\end{align*}
$$

Hence, the functional integrals for the ferromagnet and antiferromagnet are identical at temperatures large compared to $v / a$ or $D / a^{2}$. Both systems are described by the $d$-dimensional $\mathrm{NL} \sigma \mathrm{M}$. The differences between ferromagnets and antiferromagnets, which have to do with their dynamics, are unimportant in the limit of classical statistical mechanics, which is the limit which we have just taken. Thus we can kill two birds with one stone by studying the functional integral

$$
\begin{equation*}
Z=\int \mathcal{D} \vec{n} e^{-\beta \int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}\right)} \tag{9.85}
\end{equation*}
$$

This functional integral would be a trivial Gaussian integral if it were not for the constraint $n^{2}=1$. To impose this constraint, let's intoduce a Lagrange multiplier:

$$
\begin{equation*}
Z=\int \mathcal{D} \vec{n} \mathcal{D} \lambda e^{-\beta \int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}+\lambda(\vec{x})\left(n^{2}-1\right)\right)} \tag{9.86}
\end{equation*}
$$

Now the functional integral is, indeed, a Gaussian integral in $\vec{n}$ which we can do:

$$
\begin{align*}
Z & =\int \mathcal{D} \lambda \operatorname{det}\left(\nabla^{2}+\lambda(\vec{x})\right)^{-1 / 2} e^{-\beta \int d^{d} \vec{x} \lambda(\vec{x})} \\
& =\int \mathcal{D} \lambda e^{-\frac{1}{2} T r \ln \left(-\nabla^{2}+\lambda(\vec{x})\right)-\beta \int d^{d} \vec{x} \lambda(\vec{x})} \tag{9.87}
\end{align*}
$$

Unfortunately, we can't do the resulting integral exactly, but we can try to use the saddle-point approximation. The saddle-point of the argument of the exponential is given by:

$$
\begin{equation*}
\frac{\delta}{\delta \lambda}\left(\operatorname{Tr} \ln \left(-\nabla^{2}+\lambda(\vec{x})\right)-\beta \int d^{d} \vec{x} \lambda(\vec{x})\right)=0 \tag{9.88}
\end{equation*}
$$

If we look for a saddle-point solution $\lambda_{0}$ for which $\lambda(x)$ is independent of position, then this is simply

$$
\begin{equation*}
\frac{d}{d \lambda}\left(\int \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \ln \left(q^{2}+\lambda\right)-\beta \lambda\right)=0 \tag{9.89}
\end{equation*}
$$

or,

$$
\begin{equation*}
\int \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{q^{2}+\lambda}=\beta \tag{9.90}
\end{equation*}
$$

If we aproximate the integral by using $\lambda_{0}$ as an infrared cutoff, then we have:

$$
\begin{equation*}
\frac{1}{a^{d-2}}-\lambda_{0}^{\frac{d-2}{2}} \sim \beta \tag{9.91}
\end{equation*}
$$

For $T \rightarrow 0$, there is no spatially homogeneous saddle-point solution, $\lambda_{0}$, if $d>2$. For high-temperature, however, there is a solution of the form:

$$
\begin{equation*}
\lambda_{0}^{\frac{d-2}{2}} \sim \frac{1}{a^{d-2}}-\beta \tag{9.92}
\end{equation*}
$$

In $d=2$, there is always a solution:

$$
\begin{equation*}
\lambda_{0} \sim \frac{1}{a^{2}} e^{-(\text {const. }) \beta} \tag{9.93}
\end{equation*}
$$

When the functional integral is dominated by a non-zero saddle-point value $\lambda_{0}$, we can approximate it by:

$$
\begin{equation*}
Z=\int \mathcal{D} \vec{n} e^{-\beta \int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}+\lambda_{0} \vec{n}^{2}\right)} \tag{9.94}
\end{equation*}
$$

which is a Gaussian integral. This Gaussian theory is called a linear $\sigma$-model
This describes the high-temperature phase in which thermal fluctuations have disordered the magnet. From (9.94), we can see that

$$
\begin{equation*}
\langle\vec{n}(x)\rangle=0 \tag{9.95}
\end{equation*}
$$

Furthermore, using the real-space Green function which you have calculated in the problem set, we see that correlation functions of the magnetization decay exponentially with distance

$$
\begin{equation*}
\langle\vec{n}(\vec{x}) \vec{n}(0)\rangle \sim \frac{1}{|x|^{(d-1) / 2}} e^{-|x| / \xi} \tag{9.96}
\end{equation*}
$$

where we have defined the correlation length $\xi \sim \sqrt{\lambda_{0}}$. As the temperature is lowered and $\lambda_{0} \rightarrow 0$, the correlation length grows. Finally, a transition takes place and the magnet orders. In the saddle-point - or mean-field - approximation, this occurs at $\lambda_{0}=0$. The saddle-point approximation would tell us that

$$
\begin{equation*}
\langle\vec{n}(\vec{x}) \vec{n}(0)\rangle \sim \frac{1}{|x|^{(d-1) / 2}} \tag{9.97}
\end{equation*}
$$

at the critical point, $T=T_{c}$. However, as we will discuss in chapter 11 - and as you have investigated in the problem set - the saddle-point approximation is often incorrect. For temperatures below $T_{c}$ the magnet is ordered, and we can expand about the ordered state, as we did in the previous two sections.

To summarize, there are 4 regimes for a ferro- or antiferromagnet:

- High temperature, $T>T_{c}$, where the system is described by a linear $\sigma$-model,

$$
Z=\int \mathcal{D} \vec{n} e^{-\beta \int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}+\lambda_{0} \vec{n}^{2}\right)}
$$

with $\lambda_{0}>0$. Correlation functions fall off exponentially with correlation length $\xi \sim \sqrt{\lambda_{0}}$.

- The critical point, $T=T_{c}$, at which correlation functions have power-law falloff.
- The ordered phase, $0<T<T_{c}$, where the magnetization (or staggered magnetization) has a non-zero expectation value. This regime is described by the $d$-dimensional $\mathrm{NL} \sigma \mathrm{M}$ :

$$
Z=\int \mathcal{D} \vec{n} e^{-\beta \int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}\right)}
$$

which can be expanded perturbatively about the ordered state.

- The ordered state at $T=0$ which is described by the $d+1$-dimensional NL $\sigma \mathrm{M}$ in the antiferromagnetic case,

$$
Z=\int \mathcal{D} \vec{n} e^{-\frac{1}{g} \int d^{d} \vec{x} d \tau\left(\frac{1}{2 v^{2}}\left(\frac{d \vec{r}}{d \tau}\right)^{2}+\frac{1}{2}(\nabla \vec{n})^{2}\right)}
$$

and by the following functional integral in the ferromagnetic case.

$$
\begin{equation*}
Z=Z=\int \mathcal{D} \vec{\Omega} e^{-s \int d^{d} \vec{x} d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \frac{d \vec{\Omega}}{d \tau}+\frac{1}{2} D(\nabla \vec{\Omega})^{2}\right)} \tag{9.98}
\end{equation*}
$$

### 9.5 Hydrodynamic Description of Magnetic Systems

In the limit in which magnon-magnon interactions are strong, it is hopeless to try to expand perturbatively about a quadratic action in either the ferromagnetic or antiferromagnetic cases. However, some properties of correlation functions can be deduced from hydrodynamic equations.

A ferromagnet satisfies hydrodynamic equations very similar to those of a conserved particle density which we discussed in chapter 7 . The magnetization, $\vec{\Omega}$ is a conserved quantity, so the deviation from an ordered state, $\delta \Omega=\Omega-\Omega_{0}$ satisfies a conservation law:

$$
\begin{equation*}
\frac{\partial}{\partial t} \delta \Omega_{i}+\vec{\nabla} \cdot \vec{J}^{i}=0 \tag{9.99}
\end{equation*}
$$

and a constitutive relation:

$$
\begin{equation*}
\overrightarrow{J^{\imath}}=-\chi_{0}^{-1}(T) \vec{\nabla} \delta \Omega_{i}+\vec{\nabla} B_{i} \tag{9.100}
\end{equation*}
$$

( $\chi_{0}$ is the static magnetic susceptibility) from which it follows that the magnetization has diffusive correlation functions:

$$
\begin{equation*}
\chi_{\delta \Omega_{i} \delta \Omega_{i}}(\omega, q)=\frac{q^{2}}{-i \omega+\chi_{0}^{-1}(T) q^{2}} \tag{9.101}
\end{equation*}
$$

In the case of an antiferromagnet, however, the staggered magnetization, $\vec{n}$, is not conserved, so it does not diffuse. If the system is ordered with $\vec{n}=\hat{\mathbf{z}}$, then the correct hydrodynamic equations for $i=1,2$ are:

$$
\frac{\partial l_{i}}{\partial t}=\rho_{s}(T) \vec{\nabla} \cdot\left(\epsilon_{i j k} n_{j} \vec{\nabla} n_{k}\right)
$$

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\epsilon_{i j k} n_{j} \vec{\nabla} n_{k}\right)=\chi_{\perp}^{-1}(T) \vec{\nabla} l_{i} \tag{9.102}
\end{equation*}
$$

These equations are the equations of motion for $\vec{n}, \vec{l}$ which follow from the $\mathrm{NL} \sigma \mathrm{M}$ with $\rho_{s}=1 / g$ and $\chi_{\perp}=1 / g v^{2}$. Rotational invariance dictates that the hydrodynamic equations must hold generally albeit with different values of $\rho_{s}$ and $\chi_{\perp}$. These equations can be combined to give:

$$
\begin{equation*}
\frac{\partial^{2}}{\partial t^{2}}\left(\epsilon_{i j k} n_{j} \vec{\nabla} n_{k}\right)=\rho_{s}(T) \chi_{\perp}^{-1}(T) \nabla^{2}\left(\epsilon_{i j k} n_{j} \vec{\nabla} n_{k}\right) \tag{9.103}
\end{equation*}
$$

from which it follows that there is a propagating mode of velocity $\sqrt{\rho_{s}(T) \chi_{\perp}^{-1}(T)}$. A more refined analysis, which includes higher-order terms which have been neglected above leads to a small damping of this mode which goes as $q^{2}$.

## Chapter 10

## Symmetries in Many-Body Theory

### 10.1 Discrete Symmetries

Symmetries constrain the behavior of physical systems and are, therefore, a useful tool in the analysis of a many-body system. In general, a more symmetrical system is more highly constrained and, consequently, more easily solved. The most useful symmetries are continuous symmetries - ie. symmetries which belong to continuous families - which we discuss in the remainder of this chapter. The simplest symmetries are discrete, and we focus on them in this section. We will focus on the archetypal discrete symmetries, parity, $P$, and time-reversal, $T$.

A discrete symmetry is a transformation of the fields in a theory, $\varphi(\vec{x}, \tau) \rightarrow \varphi^{\prime}(\vec{x}, \tau)$ which leaves the action unchanged. Since the classical equations of motion are just the stationarity conditions on the action, a discrete symmetry takes one solution of the equations of motion and transforms it into another. At the quantum level, we would like such a transformation to be effected by a unitary operator, $U$ :

$$
\begin{equation*}
U^{\dagger} \varphi(\vec{x}, \tau) U=\varphi^{\prime}(\vec{x}, \tau) \tag{10.1}
\end{equation*}
$$

Parity is an example of such a symmetry. We will call any transformation of the form

$$
\begin{equation*}
\varphi^{a}(\vec{x}, \tau) \rightarrow M_{a b} \varphi^{b}(-\vec{x}, \tau) \tag{10.2}
\end{equation*}
$$

a parity transformation, but $M_{a b}$ is typically $\pm \delta_{a b}$. Let us consider, as an example, our action for interacting phonons.

$$
\begin{equation*}
S=\int d \tau d^{3} \vec{r}\left[\frac{1}{2} \rho\left(\partial_{t} u_{i}\right)^{2}+\mu u_{i j} u_{i j}+\frac{1}{2} \lambda u_{k k}^{2}+\frac{g}{4!}\left(\partial_{k} u_{k}\right)^{4}\right] \tag{10.3}
\end{equation*}
$$

The parity transformation,

$$
\begin{equation*}
u_{i}(\vec{x}, \tau) \rightarrow-u_{i}(-\vec{x}, \tau) \tag{10.4}
\end{equation*}
$$

leaves (10.3) invariant. If we define the unitary operator $U_{P}$ by

$$
\begin{equation*}
U_{P}^{\dagger} u_{i}(\vec{x}, \tau) U_{P}=-u_{i}(-\vec{x}, \tau) \tag{10.5}
\end{equation*}
$$

then $U$ has the following effect on the creation and annihilation operators:

$$
\begin{align*}
U_{P}^{\dagger} a_{\vec{k}, s} U_{P} & =-a_{-\vec{k}, s} \\
U_{P}^{\dagger} a_{\vec{k}, s}^{\dagger} U_{P} & =-a_{-\vec{k}, s}^{\dagger} \tag{10.6}
\end{align*}
$$

Hence the vacuum state of a free phonon system, $g=0$, which is defined by:

$$
\begin{equation*}
a_{\vec{k}, s}|0\rangle=0 \tag{10.7}
\end{equation*}
$$

is invariant under parity:

$$
\begin{equation*}
U|0\rangle=|0\rangle \tag{10.8}
\end{equation*}
$$

If we assume that the ground state evolves continuously as $g$ is increased from 0 so that the $g \neq 0$ ground state is also invariant under parity, then parity constrains the correlation functions of the interacting theory:

$$
\begin{equation*}
\langle 0| T_{\tau}\left(u_{i_{1}}\left(x_{1}, \tau_{1}\right) \ldots u_{i_{n}}\left(x_{n}, \tau_{n}\right)\right)|0\rangle=(-1)^{n}\langle 0| T_{\tau}\left(u_{i_{1}}\left(-x_{1}, \tau_{1}\right) \ldots u_{i_{n}}\left(-x_{n}, \tau_{n}\right)\right)|0\rangle \tag{10.9}
\end{equation*}
$$

Note that

$$
\begin{equation*}
u_{i}(\vec{x}, \tau) \rightarrow-u_{i}(\vec{x}, \tau) \tag{10.10}
\end{equation*}
$$

is also a symmetry of (10.3), so we can take any combination of this and parity, such as

$$
\begin{equation*}
u_{i}(\vec{x}, \tau) \rightarrow u_{i}(-\vec{x}, \tau) \tag{10.11}
\end{equation*}
$$

It doesn't really matter what we call these various symmetries so long as we realize that there are two independent symmetries. Realistic phonon Lagrangians have cubic terms which are not invariant under (10.10), so usually the parity transformation (10.4) is the only symmetry. The symmetry (10.10), when it is present, leads to the relation

$$
\begin{equation*}
\langle 0| T_{\tau}\left(u_{i_{1}}\left(x_{1}, \tau_{1}\right) \ldots u_{i_{n}}\left(x_{n}, \tau_{n}\right)\right)|0\rangle=(-1)^{n}\langle 0| T_{\tau}\left(u_{i_{1}}\left(x_{1}, \tau_{1}\right) \ldots u_{i_{n}}\left(x_{n}, \tau_{n}\right)\right)|0\rangle \tag{10.12}
\end{equation*}
$$

which implies that correlation functions of odd numbers of phonon fields must vanish.
A spin, on the other hand, transforms as:

$$
\begin{equation*}
U_{P}^{\dagger} \vec{\Omega}(\vec{x}, t) U_{P}=\vec{\Omega}(-\vec{x}, t) \tag{10.13}
\end{equation*}
$$

The time-derivative term in the Lagrangian is not invariant under $\Omega \rightarrow-\Omega$, so there is no arbitrariness in our choice of parity transformation.

Time-reversal is a symmetry which does not quite fit into this paradigm. Reversing the direction of time, $t \rightarrow-t$ takes one solution of the equations of motion into another, but it does not necessarily leave the action $S=\int_{\tau_{i}}^{\tau_{f}} d \tau L(\tau)$ invariant. Nevertheless, we might expect that there is a unitary operator, $U_{T}$, which transforms the phonon field $u_{i}(t) \rightarrow u_{i}(-t)$

$$
\begin{equation*}
U_{T}^{-1} u_{i}(t) U_{T}=u_{i}(-t) \tag{10.14}
\end{equation*}
$$

In fact, this operator cannot be unitary. To see this, differentiate both sides of (10.14):

$$
\begin{equation*}
U_{T}^{-1} \partial_{t} u_{i}(t) U_{T}=-\partial_{t} u_{i}(-t) \tag{10.15}
\end{equation*}
$$

Act on

$$
\begin{equation*}
\left[u_{i}(x, t), \rho \partial_{t} u_{j}\left(x^{\prime}, t\right)\right]=i \delta_{i j} \delta\left(x-x^{\prime}\right) \tag{10.16}
\end{equation*}
$$

with $U_{T}^{-1}$ and $U_{T}$ :

$$
\begin{align*}
U_{T}^{-1}\left[u_{i}(x, t), \rho \partial_{t} u_{j}\left(x^{\prime}, t\right)\right] U_{T} & =U_{T}^{-1} i \delta_{i j} \delta\left(x-x^{\prime}\right) U_{T} \\
{\left[u_{i}(x,-t),-\rho \partial_{t} u_{j}\left(x^{\prime},-t\right)\right] } & =i \delta_{i j} \delta\left(x-x^{\prime}\right) U_{T}^{-1} U_{T} \\
-\left[u_{i}(x, t), \rho \partial_{t} u_{j}\left(x^{\prime}, t\right)\right] & =i \delta_{i j} \delta\left(x-x^{\prime}\right) \\
-i \delta_{i j} \delta\left(x-x^{\prime}\right) & =i \delta_{i j} \delta\left(x-x^{\prime}\right) \tag{10.17}
\end{align*}
$$

which is a contradiction.
The time-reversal operator must actually be an antiunitary operator. An antiunitary operator is a type of antilinear operator. While a linear operator $\mathcal{O}$ satisfies:

$$
\begin{equation*}
\mathcal{O}(\alpha|\psi\rangle+\beta|\chi\rangle)=\alpha \mathcal{O}|\psi\rangle+\beta \mathcal{O}|\chi\rangle \tag{10.18}
\end{equation*}
$$

an antilinear operator satisfies

$$
\begin{equation*}
\mathcal{O}(\alpha|\psi\rangle+\beta|\chi\rangle)=\alpha^{*} \mathcal{O}|\psi\rangle+\beta^{*} \mathcal{O}|\chi\rangle \tag{10.19}
\end{equation*}
$$

An antiunitary operator is an antilinear operator $\tilde{\mathcal{O}}$ which satisfies

$$
\begin{equation*}
(\tilde{\mathcal{O}} \chi, \tilde{\mathcal{O}} \psi)=(\chi, \psi) \tag{10.20}
\end{equation*}
$$

where we have used the notation $(\chi, \psi)$ to denote the inner product between the states $|\psi\rangle$ and $|\chi\rangle$.

The time-reversal operator, $U_{T}$, is an anti-unitary operator, which explains how the paradox (10.17) is avoided. While the phonon field has the time-reversal property (10.14), a spin must transform under time-reversal so as to leave invariant the timederivative term in the action:

$$
\begin{equation*}
s \int_{0}^{\beta} d \tau \int_{0}^{1} d r\left(-i \frac{d \vec{\Omega}}{d r} \cdot \vec{\Omega} \times \frac{d \vec{\Omega}}{d \tau}\right) \tag{10.21}
\end{equation*}
$$

Evidently, the correct transformation property is:

$$
\begin{equation*}
U_{T}^{-1} \vec{\Omega}(t) U_{T}=-\vec{\Omega}(-t) \tag{10.22}
\end{equation*}
$$

Hence, the ferromagnetic, $\langle\vec{\Omega}(x, t)\rangle=\vec{\Omega}_{0}$, and antiferromagnetic, $\langle\vec{\Omega}(x, t)\rangle=(-1)^{i} \vec{n}_{0}$, ground states are not time-reversal invariant, i.e. they spontaneously break timereversal invariance, unlike the phonon ground state which does not. The antiferromagnetic state also breaks the discrete symmetry of translation by one lattice spacing. However, the product of $T$ and a translation by one lattice spacing is unbroken.

### 10.2 Noether's Theorem: Continuous Symmetries and Conservation Laws

Before looking at continuous symmetries in quantum systems, let us review one of the basic results of classical field theory: Noether's theorem. This theorem relates symmetries of the action to the existence of conserved currents.

Suppose we have a classical field theory defined by an action and Lagrangian density:

$$
\begin{equation*}
S=\int d t d^{3} \vec{r} \mathcal{L}(\phi, \partial \phi, \vec{r}) \tag{10.23}
\end{equation*}
$$

where $\phi$ is the classical field. Consider a transformation

$$
\begin{equation*}
\phi(\vec{r}, t) \rightarrow \phi(\vec{r}, t, \lambda), \phi(\vec{r}, t, 0)=\phi(\vec{r}, t) \tag{10.24}
\end{equation*}
$$

and define the infinitesimal transformation

$$
\begin{equation*}
D K=\left(\frac{\partial K}{\partial \lambda}\right)_{\lambda=0} \tag{10.25}
\end{equation*}
$$

Then, this transformation is a symmetry of the action if and only if

$$
\begin{equation*}
D \mathcal{L}=\partial_{\mu} F_{\mu} \tag{10.26}
\end{equation*}
$$

for any $\phi$ (i.e. not only for $\phi$ satisfying the equations of motion). (Greek indices take the values $0,1,2,3$ where $0=i \tau$; relativistic invariance is not implied.)

Now, a general expression for $D \mathcal{L}$ can be obtained from the chain rule:

$$
\begin{align*}
D \mathcal{L} & =\frac{\partial \mathcal{L}}{\partial \phi} D \phi+\pi_{\mu} D\left(\partial_{\mu} \phi\right) \\
& =\partial_{\mu} \pi_{\mu} D \phi+\pi_{\mu} D\left(\partial_{\mu} \phi\right) \\
& =\partial_{\mu}\left(\pi_{\mu} D \phi\right) \tag{10.27}
\end{align*}
$$

We used the equations of motion to go from the first line to the second and the equality of mixed partials, $D \partial \phi=\partial D \phi$, to go from the second to the third.

Setting these two expressions equal to each other, we have
Noether's theorem: for every transformation which is a symmetry of the action - i.e. $D \mathcal{L}=\partial_{\mu} F_{\mu}$ - there is a current $j_{\mu}=(\rho, \vec{j})$,

$$
\begin{equation*}
j_{\mu}=\pi_{\mu} D \phi-F_{\mu} \tag{10.28}
\end{equation*}
$$

which is conserved,

$$
\begin{equation*}
\partial_{\mu} j_{\mu}=\partial_{t} \rho+\nabla \cdot \vec{j}=0 \tag{10.29}
\end{equation*}
$$

The extension to theories with multiple fields is straightforward and can be accomodated by decorating the preceding formulas with extra indices.

As an example, let's consider space and time translations:

$$
\begin{equation*}
\phi\left(x_{\mu}\right) \rightarrow \phi\left(x_{\mu}+\lambda e_{\mu}\right) \tag{10.30}
\end{equation*}
$$

where $x_{\mu}=(t, \vec{r})$ and $e_{\mu}$ is an arbitrary 4-vector. Then,

$$
\begin{align*}
D \phi & =e_{\alpha} \partial_{\alpha} \phi \\
D \mathcal{L} & =\partial_{\alpha}\left(e_{\alpha} \mathcal{L}\right) \tag{10.31}
\end{align*}
$$

Hence, the conserved current is

$$
\begin{equation*}
j_{\beta}=e_{\alpha} T_{\alpha \beta} \tag{10.32}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{\alpha \beta}=\pi_{\alpha} \partial_{\beta} \phi-\delta_{\alpha \beta} \mathcal{L} \tag{10.33}
\end{equation*}
$$

This is the stress-energy tensor. $T_{0 \mu}$ is the 4 -current corresponding to time-translation invariance: $T_{00}$ is the energy density and $T_{0 i}$ is the energy 3 -current. $T_{i \mu}$ are the 4 currents corresponding to spatial translational invariance: $T_{i 0}$ are the momentum densities and $T_{i j}$ are the momentum currents.

In our theory of an elastic medium, $T_{\alpha \beta}$ is given by:

$$
\begin{align*}
T_{00} & =\mathcal{H} \\
T_{0 i} & =\rho \partial_{t} u_{j} \partial_{i} u_{j} \\
T_{i j} & =-2 \mu \partial_{i} u_{k} \partial_{j} u_{k}-\partial_{k} u_{k} \partial_{j} u_{i}-\delta_{i j} \mathcal{L} \tag{10.34}
\end{align*}
$$

$T_{i j}$ is the stress tensor of the elastic medium.
Our action for a spin - as well as our actions for ferro- and anti-ferromagnets - is invariant under spin rotations, $\Omega_{a} \rightarrow \mathcal{R}_{a b} \Omega_{b}$. In the case of an ferromagnet, this leads to 3 conserved quantities corresponding to spin rotations about the three different axes:

$$
\begin{equation*}
\left(J_{0}^{i}, \vec{J}^{i}\right)=\left(\Omega^{i}, D \epsilon_{i j k} \Omega^{j} \vec{\nabla} \Omega^{k}\right) \tag{10.35}
\end{equation*}
$$

In the antiferromagnetic case, they are:

$$
\begin{equation*}
J_{\mu}^{i}=\epsilon_{i j k} n_{j} \partial_{\mu} n_{k} \tag{10.36}
\end{equation*}
$$

In general, symmetries are related to unobservable quantities. In the above, the conservation of momentum follows from the unobservability of absolute position; the conservation of energy, from the unobservability of absolute temporal position. Angular momentum is a consequence of the unobservability of absolute direction.

### 10.3 Ward Identities

In the previous section, we discussed the consequences of continuous symmetries and conservations laws for classical systems. We now turn to the quantum theory, where the existence of continuous symmetries and their associated conservation laws leads to important constraints on correlation functions. These constraints are called Ward identities. The Ward identity relates the divergence of a time-ordered correlation function of a conserved current, $j_{\mu}$, with some other fields, $\varphi_{i}$ to the variations of those field under the symmetry generated by $j_{0}$. The variation of $\varphi(x, t)$ under such a symmetry operation is:

$$
\begin{equation*}
D \varphi(x, t)=\int d^{d} x^{\prime}\left[J^{0}\left(x^{\prime}, t\right), \varphi(x, t)\right] \tag{10.37}
\end{equation*}
$$

To derive the Ward identities, we consider a correlation function of $j_{\mu}$ with the $\varphi_{i}$ 's:

$$
\begin{align*}
& \left\langle T_{\tau}\left(j_{\mu}(x, \tau) \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle= \\
& \quad \theta\left(\tau-\tau_{1}\right) \theta\left(\tau_{1}-\tau_{2}\right) \ldots \theta\left(\tau_{n-1}-\tau_{n}\right)\left\langle j_{\mu}(x, \tau) \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right\rangle+ \\
& \quad \theta\left(\tau_{1}-\tau\right) \theta\left(\tau_{1}-\tau_{2}\right) \ldots \theta\left(\tau_{n-1}-\tau_{n}\right)\left\langle\varphi_{1}\left(x_{1}, \tau_{1}\right) j_{\mu}(x, \tau) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right\rangle \\
& \quad+\ldots \tag{10.38}
\end{align*}
$$

If we differerentiate this with respect to $x_{\mu}$, the derivative operator can act on a $\theta$-function which has $\tau$ in its argument or it can act on $j_{\mu}(x, \tau)$. If the symmetry is conserved in the classical field theory, then we can ordinarily conclude that $\partial_{\mu} j_{\mu}(x, \tau)=0$. However, it is possible for this equation to be violated in the quantum theory when there is a cutoff (the conservation law can be violated when the conserved quantity flows to wavevectors beyond the cutoff) and this violation can remain finite even as the cutoff is taken to infinity. Such a symmetry is called anomalous. If the symmetry is not anomalous, then the right-hand-side contains only terms
resulting from the derivative acting on the $\theta$-function to give a $\delta$-function:

$$
\begin{align*}
& \partial_{\mu}\left\langle T_{\tau}\left(j_{\mu}(x, \tau) \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle \\
&= \delta\left(\tau-\tau_{1}\right) \theta\left(\tau_{1}-\tau_{2}\right) \ldots \theta\left(\tau_{n-1}-\tau_{n}\right)\left\langle j_{\mu}(x, \tau) \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right\rangle \\
& \quad-\delta\left(\tau_{1}-\tau\right) \theta\left(\tau_{1}-\tau_{2}\right) \ldots \theta\left(\tau_{n-1}-\tau_{n}\right)\left\langle\varphi_{1}\left(x_{1}, \tau_{1}\right) j_{\mu}(x, \tau) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right\rangle+\ldots \\
&= \delta\left(\tau-\tau_{1}\right) \theta\left(\tau_{1}-\tau_{2}\right) \ldots \theta\left(\tau_{n-1}-\tau_{n}\right)\left\langle\left[j_{\mu}(x, \tau), \varphi_{1}\left(x_{1}, \tau_{1}\right)\right] \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right\rangle+\ldots \\
&= \delta\left(x-x_{1}\right) \delta\left(\tau-\tau_{1}\right)\left\langle T_{\tau}\left(D \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle \\
& \quad+\delta\left(x-x_{1}\right) \delta\left(\tau-\tau_{1}\right)\left\langle T_{\tau}\left(\varphi_{1}\left(x_{1}, \tau_{1}\right) D \varphi_{2}\left(x_{2}, \tau_{2}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle+\ldots \text { (10.39) } \tag{10.39}
\end{align*}
$$

The final equality is the Ward identity:

$$
\begin{align*}
\partial_{\mu}\left\langle T_{\tau}( \right. & \left.\left.j_{\mu}(x, \tau) \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle \\
= & \delta\left(x-x_{1}\right) \delta\left(\tau-\tau_{1}\right)\left\langle T_{\tau}\left(D \varphi_{1}\left(x_{1}, \tau_{1}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle \\
& +\delta\left(x-x_{2}\right) \delta\left(\tau-\tau_{2}\right)\left\langle T_{\tau}\left(\varphi_{1}\left(x_{1}, \tau_{1}\right) D \varphi_{2}\left(x_{2}, \tau_{2}\right) \ldots \varphi_{n}\left(x_{n}, \tau_{n}\right)\right)\right\rangle \\
& +\ldots \tag{10.40}
\end{align*}
$$

As an example of the Ward identity, consider an antiferromagnet, for which the spin currents are:

$$
\begin{equation*}
J_{\mu}^{i}=\epsilon_{i j k} n_{j} \partial_{\mu} n_{k} \tag{10.41}
\end{equation*}
$$

Then the Ward identity tells us that:

$$
\begin{align*}
& \partial_{\mu}\left\langle T_{\tau}\left(\epsilon_{i j s} n_{j}(x, \tau) \partial_{\mu} n_{s}(x, \tau) n_{k}\left(x_{1}, \tau_{1}\right) n_{l}\left(x_{2}, \tau_{2}\right)\right)\right\rangle \\
&= \delta\left(x-x_{1}\right) \delta\left(\tau-\tau_{1}\right) \epsilon_{i k m}\left\langle T_{\tau}\left(n_{m}\left(x_{1}, \tau_{1}\right) n_{l}\left(x_{2}, \tau_{2}\right)\right)\right\rangle \\
&+\delta\left(x-x_{2}\right) \delta\left(\tau-\tau_{2}\right) \epsilon_{i l r}\left\langle T_{\tau}\left(n_{l}\left(x_{1}, \tau_{1}\right) n_{r}\left(x_{2}, \tau_{2}\right)\right)\right\rangle \tag{10.42}
\end{align*}
$$

This is a non-trivial constraint when imposed order-by-order in perturbation theory, since the correlation function on the left-hand-side is given by diagrams such as those in figure 10.1a while the right-hand-side is given by diagrams such as those of 10.1 b .

As another example, consider a ferromagnet which is ordered along the $\hat{\mathbf{z}}$ axis, $\left\langle\Omega_{z}\right\rangle=1$ :. $\Omega_{x}$ generates rotations about the $x$-axis, so the following correlation


Figure 10.1: Diagrams contributing to the (a) left-hand-side and (b) right-hand side of the Ward identity (10.42).
function is of the form for which the Ward identity is applicable:

$$
\begin{equation*}
\left\langle\Omega_{x}\left(i \omega_{n}, 0\right) \Omega_{y}\left(i \omega_{n}, 0\right)\right\rangle \tag{10.43}
\end{equation*}
$$

with $J_{0}=\Omega_{x}$ and $\partial_{\mu} \rightarrow p_{\mu}=\left(i \omega_{n}, 0\right)$. Hence, the Ward identity tells us that:

$$
\begin{equation*}
i \omega_{n}\left\langle\Omega_{x}\left(i \omega_{n}, 0\right) \Omega_{y}\left(i \omega_{n}, 0\right)\right\rangle=\left\langle\Omega_{z}\right\rangle=1 \tag{10.44}
\end{equation*}
$$

or,

$$
\begin{equation*}
\left\langle\Omega_{x}\left(i \omega_{n}, 0\right) \Omega_{y}\left(i \omega_{n}, 0\right)\right\rangle=\frac{1}{i \omega_{n}} \tag{10.45}
\end{equation*}
$$

We found the same result earlier for a linearized theory in which magnon-magnon interactions. The Ward identity shows that this result is exact, i.e. the tree-level result is unchanged by the inclusion of magnon-magnon interactions. The divergence of this correlation function at low frequency is an example of Goldstone's theorem in action, as we will see in the next section.

### 10.4 Spontaneous Symmetry-Breaking and Goldstone's Theorem

Often, the ground state is invariant under the symmetries of the Lagrangian. Our phonon Lagrangian, for instance, is invariant under parity and time-reversal, and the ground state is as well. However, this is not the only possibility, as we have already seen.It is possible that there is not an invariant ground state, but rather a multiplet of degenerate symmetry-related ground states, in which case we say that the symmetry is spontaneously broken. In terms of correlation functions, the statement of spontaneous symmetry-breaking is

$$
\begin{equation*}
\langle\phi(x, \tau)\rangle \neq 0 \tag{10.46}
\end{equation*}
$$

where $\phi(x, \tau)$ is a field which is not invariant under the symmetry, $\phi(x, \tau) \neq U^{\dagger} \phi(x, \tau) U$. Such a field is called an order parameter.

For instance, our field theory for the Ising model,

$$
\begin{equation*}
\mathcal{Z}=N \int \mathcal{D} \varphi e^{-\int d^{d} x\left(\frac{1}{2} K(\nabla \varphi)^{2}+\frac{1}{2} r \varphi^{2}+\frac{1}{4} u \varphi^{4}\right)} \tag{10.47}
\end{equation*}
$$

is invariant under the $Z_{2}$ symmetry $\varphi \rightarrow-\varphi$ which is broken for $T<T_{c}$ (i.e. $r<$ $0),\langle\varphi\rangle= \pm \sqrt{6 r / u}$ and unbroken for $T>T_{c},\langle\varphi\rangle=0$. As the temperature is lowered through $T_{c}$, the system spontaneously chooses one of the two symmetryrelated configurations $\langle\varphi\rangle= \pm \sqrt{6 r / u}$, either as a result of a random fluctuation or some weak external perturbation. The ferromagnetic and antiferromagnetic ground states are two more examples of spontaneous symmetry breaking: the Heisenberg model and the field theories derived from it,

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(-i \vec{A}(\vec{\Omega}) \cdot \frac{d \vec{\Omega}}{d \tau}+\frac{1}{2} D(\nabla \vec{\Omega})^{2}\right) \tag{10.48}
\end{equation*}
$$

and

$$
\begin{equation*}
S=\frac{1}{g} \int d^{d} \vec{x} d \tau\left(\frac{1}{2 v^{2}}\left(\frac{d \vec{n}}{d \tau}\right)^{2}+\frac{1}{2}(\nabla \vec{n})^{2}\right) \tag{10.49}
\end{equation*}
$$

are invariant under $S U(2)$ spin-rotational symmetry, $\Omega_{i} \rightarrow \mathcal{R}_{i j} \Omega_{j}, n_{i} \rightarrow \mathcal{R}_{i j} n_{j}$, but the ground states are not invariant since the magnetization or staggered magnetization chooses a particular direction. The signal of spontaneous symmetry breakdown is the non-invariant expectation value $\langle\vec{\Omega}\rangle \neq 0$ or $\langle\vec{n}\rangle \neq 0$. At high-temperature, $T>T_{c}$, the symmetry is restored and $\langle\vec{\Omega}\rangle=0$ or $\langle\vec{n}\rangle=0$. These expectation values also break the discrete $T$ symetry. A ferromagnetic in a magnetic field does not have an $S U(2)$ or $T$ invariant Lagrangian, so its ferromagnetic ground state is an example of an explicitly broken symmetry rather than a spontaneously broken one. The $\mu \vec{B} \cdot \vec{\Omega}$ term is called a symmetry-breaking term. The phonon Lagrangian is actually another example: the translational symmetry of the continuum is broken to the discrete translational symmetry of the lattice. At high temperature (when our continuum elastic theory is no longer valid), the lattice melts and the resulting fluid state has the full translational symmetry.

In the first example, the Ising model, the broken symmetry is discrete, and there are no gapless excitations in the symmetry-broken phase. In the other two examples, magnets and ionic lattices, the broken symmetry, since it is continuous, leads to gapless excitations - magnons and phonons. This is a general feature of field theories with broken symmetries: broken continuous symmetries lead to gapless excitations - called Goldstone modes or Goldstone bosons - but broken discrete symmetries do not.

Physically, the reason for the existence of Goldstone bosons is that by applying the generator of the broken symmetry, we obtain another state with the same energy as the ground state. For instance, if we take a magnet aligned along the $\hat{\mathbf{z}}$ axis and rotate all of the spins away from the $\hat{\mathbf{z}}$ axis then we obtain another state of the same energy. If the spins instead vary slowly in space with wavevector $\vec{q}$, then the energy of the resulting state vanishes as $\vec{q} \rightarrow 0$. These states are the Goldstone modes.

The number of Goldstone modes is at most $\operatorname{dim} G-\operatorname{dim} H$ if $G$ is the symmetry
group of the theory and $H$ is subgroup of $G$ which is left unbroken. If $H=G$, i.e. the symmetry is completely unbroken, then there no Goldstone bosons. In the case of an antiferromagnet, $G=S U(2)$ and $H=U(1)$ - the group of rotations about staggered magnetization axis - so there are $\operatorname{dim} G-\operatorname{dim} H=2$ gapless modes. A ferromagnet in zero field has only one Goldstone mode while $\operatorname{dim} G-\operatorname{dim} H=2$. A ferromagnet in a finite field has no Goldstone modes; $G=H=U(1)$, the group of rotations about the direction of the field, so $\operatorname{dim} G-\operatorname{dim} H=0$. A crytal has only three Goldstone modes, the $u_{i}$ 's, while $G$ is the group of translations and rotations, $\operatorname{dim} G=6$, and $H$ is a discrete subgroup, $\operatorname{dim} H=0$.

We will now give a precise statement and proof of Goldstone's theorem. Suppose we have a conserved quantity, $J^{0}$, and its associated current, $J^{i}$, so that $\partial_{\mu} J^{\mu}=0$. Let $\varphi(x, t)$ be some field in the theory. $\varphi(x, t)$ transforms as

$$
\begin{equation*}
D \varphi(x, t)=\int d^{d} x^{\prime}\left[J^{0}\left(x^{\prime}, t\right), \varphi(x, t)\right] \tag{10.50}
\end{equation*}
$$

under an infinitesimal symmetry operation corresponding to the conserved quantity $J^{0}$. Then, the following theorem holds.

Goldstone's Theorem: If there is an energy gap, $\Delta$, then

$$
\begin{equation*}
\langle 0| D \varphi(k=0, t)|0\rangle=0 \tag{10.51}
\end{equation*}
$$

Conversely, if $\langle 0| D \varphi(k=0, t)|0\rangle \neq 0$, then there must be gapless excitations. These gapless excitations are the Goldstone modes.

The proof proceeds by constructing a spectral representation for $\langle 0| J^{0}\left(x^{\prime}, t^{\prime}\right) \varphi(x, t)|0\rangle$ :

$$
\begin{equation*}
\langle 0| J^{0}\left(x^{\prime}, t^{\prime}\right) \varphi(x, t)|0\rangle=\int d^{d} k d \omega e^{-i\left(k \cdot\left(x-x^{\prime}\right)-\omega\left(t-t^{\prime}\right)\right)} \rho(\omega, k) \tag{10.52}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho(\omega, k)=\sum_{n}\langle 0| J^{0}(0,0)|n\rangle\langle n| \varphi(0,0)|0\rangle \delta\left(\omega-E_{n}\right) \delta\left(\vec{k}-\vec{P}_{n}\right) \tag{10.53}
\end{equation*}
$$

By unitarity, $\rho(\omega, k) \geq 0$. The existence of an energy gap, $\Delta$, implies that $\rho(\omega, k)=0$ for $\omega<\Delta$. Applying the conservation law to the correlation function of (10.52), we have:

$$
\begin{equation*}
\langle 0| \partial_{\mu} J^{\mu}\left(x^{\prime}, t^{\prime}\right) \varphi(x, t)|0\rangle=0 \tag{10.54}
\end{equation*}
$$

Fourier transforming and taking the $\vec{k} \rightarrow 0$ limit, we have:

$$
\begin{equation*}
\omega\langle 0| J^{0}(k=0,-\omega) \varphi(k=0, \omega)|0\rangle=0 \tag{10.55}
\end{equation*}
$$

The left-hand-side can be rewritten using our spectral representation (10.52):

$$
\begin{equation*}
\omega \rho(k=0, \omega)=0 \tag{10.56}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\rho(k=0, \omega)=0 \tag{10.57}
\end{equation*}
$$

or

$$
\begin{equation*}
\omega=0 \tag{10.58}
\end{equation*}
$$

Hence, $\rho(k=0, \omega)=0$ for all $\omega>0$. However, the existence of an energy gap, $\Delta$ implies that $\rho(k=0, \omega)=0$ for $\omega<\Delta$ and, in particular, for $\omega=0$. Hence, $\rho(k=0, \omega)=0$ for all $\omega$. Therefore,

$$
\begin{equation*}
\langle 0| J^{0}\left(x^{\prime}, t^{\prime}\right) \varphi(k=0, t)|0\rangle=0 \tag{10.59}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\langle 0| \varphi(k=0, t) J^{0}\left(x^{\prime}, t^{\prime}\right)|0\rangle=0 \tag{10.60}
\end{equation*}
$$

and, consequently,

$$
\begin{equation*}
\langle 0| D \varphi(k=0, t)|0\rangle=0 \tag{10.61}
\end{equation*}
$$

When the symmetry is broken, $\rho(\omega, k=0)$ does not vanish when $\omega=0$; instead, there is a contribution to $\rho(\omega, k=0)$ coming from the Goldstone modes of the form
$\rho(\omega, k=0)=\sigma \delta(\omega)$. Then $\langle 0| D \varphi(k=0, t)|0\rangle=\sigma$. Note that this proof depended on unitarity and translational invariance.

A field $\varphi$ which satisfies

$$
\begin{equation*}
\langle 0| D \varphi(k=0, t)|0\rangle \neq 0 \tag{10.62}
\end{equation*}
$$

is an order parameter: it signals the development of an ordered state. The order parameter of a ferromagnet is $\vec{\Omega}$ while the order parameter of an antiferromagnet is $\vec{n}$. It is not necessary for the order parameter of a theory to be the fundamental field of the theory. The order parameters of broken translational invariance in a crystal are:

$$
\begin{equation*}
\rho_{G}=e^{i \vec{G} \cdot \vec{u}(x, \tau)} \tag{10.63}
\end{equation*}
$$

where $G$ is a reciprocal lattice vector of the crystal and $\vec{u}$ is the phonon field.
If the order parameter, $\varphi$, is itself a conserved quantity, $J_{0}=\varphi$, which generates a spontaneously broken symmetry, then $D \varphi$ vanishes identically and the associated Goldstone boson doesn't exist. This is the reason why a ferromagnet has only 1 Goldstone mode. If the ferromagnet is ordered along the $\hat{\mathbf{z}}$ axis, then the symmetries generated by $\Omega_{x}$ and $\Omega_{y}$ are broken. If we look at the spectral functions for $\left\langle\Omega_{x} \Omega_{y}\right\rangle$ and $\left\langle\Omega_{y} \Omega_{y}\right\rangle$, only the former has a $\delta(\omega)$ contribution; the latter vanishes. In the case of an antiferromagnet, the spectral functions for both $\left\langle L_{x} n_{y}\right\rangle$ and $\left\langle L_{y} n_{y}\right\rangle$, have $\delta(\omega)$ contributions.

### 10.5 The Mermin-Wagner-Coleman Theorem

In chapter 9, we encountered hints that neither ferromagnets nor antiferromagnets could order at finite $T$ in $d \leq 2$ and that antiferromagnets could not even order at zero temperature in $d=1$. Let us now discuss the difficulties involved in breaking a symmetry in a low dimensional system. Consider the simplest example, namely the

Ising model. Suppose the system is ordered with all of the spins pointing up. What is the energy cost to create a size $L^{d}$ region of down spins in $d$-dimensions? It is simply

$$
\begin{equation*}
E_{\text {fluct }} \sim L^{d-1} \tag{10.64}
\end{equation*}
$$

i.e. the energy cost of a domain of reversed spins is proportional to the surface area of the domain wall since that is the only place where unlike spins are neighbors. For $d>1$, this grows with $L$, so large regions of reversed spins are energetically disfavored. At low temperature, this energy cost implies that such fluctuations occur with very low probability

$$
\begin{equation*}
P_{\text {fluc }} \sim e^{-(\text {const. }) \beta L^{d-1}} \tag{10.65}
\end{equation*}
$$

and hence the orderd phase is stable. In $d=1$, however, the energy cost of a fluctuation is independent of the size of the fluctuation. It is simply $4 J$. Hence, a fluctuation in which a large fraction of the system consists of reversed spins can occur with probability $\sim \exp (-4 \beta J)$. As a result of these fluctuations, it is impossible for the system to order at any finite temperature. This is clearly true for any discrete symmetry.

Let us now consider a continuous symmetry at finite temperature. For the sake of concreteness, let us consider a $d$-dimensional magnet in an ordered phase in which the magnetization (or staggered magnetization) is aligned along the $\hat{\mathbf{z}}$ axis. (Recall that ferro- and antiferromagnets have the same description at finite temperature.) The energy cost for a size $L^{d}$ region of reversed magnetization is less than in the case of a discrete symmetry since the magnetization at the domain wall need not jump from one degenerate ground state to another. Rather, the spins can interpolate continuously between one ground state and another. The energy cost will be the gradient energy,

$$
\begin{equation*}
\int d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}\right) \tag{10.66}
\end{equation*}
$$

For a fluctuation of linear size $L,(\nabla \vec{n})^{2} \sim 1 / L^{2}$, so

$$
\begin{equation*}
\int_{L^{d}} d^{d} \vec{x}\left(\frac{1}{2}(\nabla \vec{n})^{2}\right) \sim L^{d-2} \tag{10.67}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
P_{\text {fluc }} \sim e^{-(\text {const. }) \beta L^{d-2}} \tag{10.68}
\end{equation*}
$$

and we conclude that a continuous symmetry can be broken for $T>0$ in $d>2$, but not for $d \leq 2$. This confirms our suspicion that magnets can't order for $T>0$ in $d=2$. The dimension below which symmetry-breaking is impossible is called the lower critical dimension. For $T>0$, the lower critical dimension is 2 for continuous symmetries and 1 for discrete dymmetries.

These qualitative considerations can be made rigorous. Let us consider our finitetemperature order parameter, $\varphi(x, \omega=0)$, in $d=2$. We will show that

$$
\begin{equation*}
\langle D \varphi\rangle=0 \tag{10.69}
\end{equation*}
$$

i.e. the symmetry is necessarily unbroken for $d=2$. Define:

$$
\begin{align*}
F(k) & =\int d^{2} x e^{i k \cdot x}\langle\varphi(x) \varphi(0)\rangle \\
F_{i}(k) & =\int d^{2} x e^{i k \cdot x}\left\langle j_{i}(x) \varphi(0)\right\rangle \\
F_{i l}(k) & =\int d^{2} x e^{i k \cdot x}\left\langle j_{i}(x) j_{l}(0)\right\rangle \tag{10.70}
\end{align*}
$$

where $i=0,1$. The conservation law $k_{i} F_{i}=0$ implies that

$$
\begin{equation*}
F_{i}(k)=\sigma k_{i} \delta\left(k^{2}\right)+\epsilon_{i j} k_{j} \rho(k) \tag{10.71}
\end{equation*}
$$

This decomposition is clearly special to two dimensions. Substituting this decomposition into the definition of $D \varphi$, we have:

$$
\begin{equation*}
\langle D \varphi\rangle=\sigma \tag{10.72}
\end{equation*}
$$

By unitarity,

$$
\begin{equation*}
\int d^{2} x h(x)\left(a j_{0}(x)+b \varphi(x)\right)|0\rangle \tag{10.73}
\end{equation*}
$$

has positive norm. From the special cases $a=0$ and $b=0$, we see that

$$
\begin{align*}
\int d^{2} k|h(k)|^{2} F(k) & \geq 0  \tag{10.74}\\
\int d^{2} k|h(k)|^{2} F_{00}(k) & \geq 0
\end{align*}
$$

positivity of the norm also implies that

$$
\begin{equation*}
\left(\int d^{2} k|h(k)|^{2} F(k)\right)\left(\int d^{2} k|h(k)|^{2} F_{00}(k)\right) \geq\left(\int d^{2} k|h(k)|^{2} F_{0}(k)\right)^{2} \tag{10.75}
\end{equation*}
$$

If we take an $h(k)$ which is even in $x_{1}$ - and, therefore, even in $k_{1}$ - then the right-hand-side will be:

$$
\begin{equation*}
\sigma \int d^{2} k|h(k)|^{2} k_{0} \delta\left(k^{2}\right)=\sigma \int d k_{0}\left|h\left(k_{0},\left|k_{0}\right|\right)\right|^{2} \tag{10.76}
\end{equation*}
$$

We can make the left-hand-side vanish by making $|h(k)|^{2}$ sharply peaked at very high $k$ where $F(k)$ and $F_{00}(k)$ must vanish. Consequently, $\sigma=0$ and the symmetry is unbroken. The proof works by essentially taking the spatial points very far apart in the correlation functions on the left-hand-side. In the presence of long-range forces, the left-hand-side need not vanish, and spontaneous symmetry-breaking is possible.

Thus far, our discussion has focussed on thermal fluctuations. Can quantum fluctuations prevent order at $T=0$ ? In the case of an antiferromagnet, the answer is clearly yes. The quantum theory of an $d$-dimensional antiferromagnet at $T=0$ is the same as the classical statistical theory of a magnet in $d+1$-dimensions. Hence, we conclude that a quantum antiferromagnet can order at $T=0$ in $d+1>2$, i.e. in $d>1$, but not in $d=1$.

A ferromagnet, on the other hand, can order in any number of dimensions. The exact ground state of a Heisenberg ferromagnet is a state in which all of the spins are aligned. The reason that the above arguments about fluctuations do not apply to the ferromagnet is that it has a fluctuationless ground state. This can be said somewhat differently as follows. The order parameter for a ferromagnet, $\Omega\left(q=0, \omega_{n}=0\right)$ is
a conserved quantity: the components of $\Omega\left(q=0, \omega_{n}=0\right)$ are the components of the total spin along the different axes. Thus, while it is true that there is very little energy cost for a state with reversed spins, such a state will never be reached at $T=0$ since the dynamics conserves the the total spin. In the case of an antiferromagnet, on the other hand, $\vec{n}$ is not conserved; hence, the dynamics of the system can lead to fluctuations which destroy the order. At finite temperature, we must average over all of the states in the canonical ensemble, so the fluctuations can destroy the ordered state of the ferromagnet. To summarize, if the order parameter is a conserved quantity, then there can always be order at $T=0$ in any $d$. If it is not, then quantum fluctuations can destroy the order at $T=0$. In the case of antiferomagnets - or phonons - this occurs in $d=1$. More generally, it occurs when $d+z=2$ for a continuous symmetry or $d+z=1$ for a discrete symmetry.

## Chapter 11

## XY Magnets and Superfluid ${ }^{4} \mathrm{He}$

### 11.1 XY Magnets

Broken $U(1)$ symmetries are canonical examples of Goldstone's theorem because they are simpler than the $O(3)$ magnets which we have already studied.

Suppose we modify the Heisenberg Hamiltonian by adding a term:

$$
\begin{equation*}
H=-J \sum_{i, j} \vec{S}_{i} \cdot \vec{S}_{i}+a \sum_{i}\left(S_{i}^{z}\right)^{2} \tag{11.1}
\end{equation*}
$$

where $J>0$ for a ferromagnet and $J<0$ for an antiferromagnet. If $a<0$, we call this Ising anisotropy; for $a>0$, we call this $X Y$ anisotropy. We will focus on the case $a>0$.

Consider first an antiferromagnet. Keeping only quadratic terms and neglect higher-order terms such as $\left(\vec{n} \times \partial_{\tau} \vec{n}\right)^{2}$, the anisotropy results in the following modification of the action:

$$
\begin{equation*}
S=\int d^{d} x d \tau\left(\left(\partial_{\mu} \vec{n}\right)^{2}+a n_{z}^{2}\right) \tag{11.2}
\end{equation*}
$$

As a result of this term, the field $n_{z}$ aquires a gap. Introducing a Lagrange multiplier to enforce $\vec{n}^{2}=1$,

$$
\begin{equation*}
S=\int d^{d} x d \tau\left(\left(\partial_{\mu} \vec{n}\right)^{2}+\lambda\left(\vec{n}^{2}-1\right)+a n_{z}^{2}\right) \tag{11.3}
\end{equation*}
$$

we integrate out $n_{z}$. Ignoring the functional determinant which results when $n_{z}$ is integrated out, we have the following action for $n_{x}, n_{y}$.

$$
\begin{equation*}
S=\int d^{d} x d \tau\left(\left(\partial_{\mu} n_{x}\right)^{2}+\left(\partial_{\mu} n_{x}\right)^{2}+\lambda\left(n_{x}^{2}+n_{y}^{2}-1\right)\right) \tag{11.4}
\end{equation*}
$$

Since $n_{x}+i n_{y}=e^{i \theta}$ solves the constraint, $n_{x}^{2}+n_{y}^{2}=1$, imposed by the Lagrange multiplier, we can re-write the action in terms of the field $\theta$.

$$
\begin{equation*}
S=\int d^{d} \vec{x} d \tau\left(\partial_{\mu} \theta\right)^{2} \tag{11.5}
\end{equation*}
$$

We now consider a ferromagnet with $X Y$ anisotropy. The action is now given by.

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(-i \cos \theta \partial_{\tau} \varphi+\frac{1}{2} D(\nabla \vec{\Omega})^{2}+a \Omega_{z}^{2}\right) \tag{11.6}
\end{equation*}
$$

or, since $\Omega_{z}=\cos \theta$,

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(-i \Omega_{z} \partial_{\tau} \varphi+\frac{1}{2} D(\nabla \vec{\Omega})^{2}+a \Omega_{z}^{2}\right) \tag{11.7}
\end{equation*}
$$

$\Omega_{z}$ aquires a gap as a result of the anisotropy. In the long-wavelength limit, we can neglect the $D\left(\nabla \Omega_{z}\right)^{2}$ term compared to the $a \Omega_{z}^{2}$ term. The action is:

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(\frac{1}{a}\left(\partial_{\tau} \varphi\right)^{2}+\frac{1}{2} D\left(\left(\nabla \Omega_{x}\right)^{2}+\left(\nabla \Omega_{y}\right)^{2}\right)\right) \tag{11.8}
\end{equation*}
$$

or, writing $\Omega_{x}+i \Omega_{y}=e^{i \varphi}$,

$$
\begin{equation*}
S=s \int d^{d} \vec{x} d \tau\left(\partial_{\mu} \varphi\right)^{2} \tag{11.9}
\end{equation*}
$$

where we have rescaled the time units according to $\tau \rightarrow \tau / \sqrt{a}$ so that the velocity is 1.

This action is clearly the same as the action for an $X Y$ antiferromagnet. Hence, we can compute the propagator

$$
\begin{equation*}
\left\langle T_{\tau}(\varphi(x, \tau) \varphi(0,0))\right\rangle=\frac{1}{\beta} \sum_{n} \int \frac{d^{d} \vec{p}}{(2 \pi)^{d}} \frac{e^{i\left(\vec{p} \cdot \vec{x}-\omega_{n} \tau\right)}}{\omega_{n}^{2}+p^{2}} \tag{11.10}
\end{equation*}
$$

for an $X Y$ ferromagnet or antiferromagnet. $\varphi$ is the Goldstone boson for the spontaneously broken $U(1)$ symmetry of rotations about the $\hat{\mathbf{z}}$ axis. Whereas $O(3)$ ferromagnets differ from $O(3)$ antiferromagnets in that the former has an order parameter, $\vec{\Omega}$, which commutes with the Hamiltonian and is therefore conserved while the latter has an order parameter, $\vec{n}$ which does not, $X Y$ ferromagnets and antiferromagnets have order parameters which do not commute with the Hamiltonian. The $X Y$ ferromagnet has an order parameter, $\Omega_{x}+i \Omega_{y}=e^{i \varphi}$, which does not commute with the $a \Omega_{z}^{2}$ anisotropy term in the Hamiltonian.

### 11.2 Superfluid ${ }^{4} \mathrm{He}$

We will now look at ${ }^{4} \mathrm{He}$ atoms with repulsive interactions

$$
\begin{equation*}
H_{\mathrm{int}}=V \sum_{i, j} \delta\left(x_{i}-x_{j}\right) \tag{11.11}
\end{equation*}
$$

We work at finite chemical potential, so it is necessary to make the replacement:

$$
\begin{equation*}
H \rightarrow H-\mu N \tag{11.12}
\end{equation*}
$$

where $\mu$ is the chemical potential and $N$ is the total number operator,

$$
\begin{equation*}
N=\int d^{d} x \psi^{\dagger}(x, \tau) \psi(x, \tau) \tag{11.13}
\end{equation*}
$$

The functional integral representation for the grand canonical partition function,

$$
\begin{equation*}
\mathcal{Z}=\operatorname{Tr}\left\{e^{-\beta(H-\mu N)}\right\} \tag{11.14}
\end{equation*}
$$

is given by

$$
\begin{equation*}
\mathcal{Z}=\int \mathcal{D} \psi \mathcal{D} \psi^{\dagger} e^{-S} \tag{11.15}
\end{equation*}
$$

where

$$
S=\int d \tau d^{d} x\left(\psi^{*} \partial_{\tau} \psi+|\nabla \psi|^{2}-\mu|\psi|^{2}+V|\psi|^{4}\right)
$$

$$
\begin{equation*}
=\int d \tau d^{d} x\left(\psi^{*} \partial_{\tau} \psi+|\nabla \psi|^{2}+V\left(|\psi|^{2}-\frac{\mu}{2 V}\right)^{2}\right) \tag{11.16}
\end{equation*}
$$

For $\mu>0$, the ground state has $\psi=0$. For $\mu<0$, however, the ground state has $|\psi|^{2}=\frac{\mu}{2 V}$. The system spontaneously chooses one of these ground states, say $\psi=\sqrt{\mu / 2 V}$, thereby breaking the $U(1)$ symmetry:

$$
\begin{aligned}
\psi & \rightarrow e^{i \alpha} \psi \\
\psi^{\dagger} & \rightarrow e^{-i \alpha} \psi^{\dagger}
\end{aligned}
$$

By varying the chemical potential, $\mu$, the system can be tuned through a quantum phase transition. At the saddle-point level, $\mu=\mu_{c}=0$ is the quantum critical point. The $\mu<0$ phase corresponds to a system with zero density because all states are above the chemical potential. The $\mu>0$ phase has non-vanishing density and is, therefore, a superfluid at $T=0$ in $d \geq 2$.

In the $\mu>0$ phase, we can study the symmetry-breaking by writing

$$
\begin{equation*}
\psi=\sqrt{\left(\frac{\mu}{2 V}+\rho\right)} e^{i \theta} \tag{11.17}
\end{equation*}
$$

We can rewrite the action as:

$$
\begin{align*}
S= & \int d \tau d^{d} x\left(\frac{1}{2} \partial_{\tau} \rho+\frac{\mu}{2 V} i \partial_{\tau} \theta+\rho i \partial_{\tau} \theta\right. \\
& \left.+\frac{1}{2(\rho+(\mu / 2 V))}(\nabla \rho)^{2}+\frac{\mu}{2 V}(\nabla \theta)^{2}+\rho(\nabla \theta)^{2}+V \rho^{2}\right) \tag{11.18}
\end{align*}
$$

The first two terms can (naively) be neglected since they are total derivatives, so the free part of this action is

$$
\begin{equation*}
S=\int d \tau d^{d} x\left(\rho i \partial_{\tau} \theta+\frac{V}{\mu}(\nabla \rho)^{2}+\frac{\mu}{2 V}(\nabla \theta)^{2}+V \rho^{2}\right) \tag{11.19}
\end{equation*}
$$

From the first term, we see that $\rho$ and $\theta$ are canonical conjugates and the Hamiltonian:

$$
\begin{equation*}
H=\int d^{d} k\left(\left(\frac{V}{\mu} k^{2}+V\right)\left|\rho_{k}\right|^{2}+\frac{\mu}{2 V} k^{2}\left|\theta_{k}\right|^{2}\right) \tag{11.20}
\end{equation*}
$$

is of the harmonic osciallator form

$$
\begin{equation*}
H=\int d^{d} k\left(\frac{1}{2 m}\left|P_{k}\right|^{2}+\frac{1}{2} m \omega_{k}^{2}\left|X_{k}\right|^{2}\right) \tag{11.21}
\end{equation*}
$$

with

$$
\begin{equation*}
\omega_{k}=\sqrt{4\left(\frac{\mu}{2 V} k^{2}\right)\left(V+\frac{V}{\mu} k^{2}\right)} \tag{11.22}
\end{equation*}
$$

In the long-wavelength limit, $k \rightarrow 0$,

$$
\begin{equation*}
\omega_{k}=\sqrt{2 \mu} k \tag{11.23}
\end{equation*}
$$

i.e. there is a gapless mode. Alternatively, we can integrate out the gapped field $\rho$, thereby arriving at the same action as we derived for an $X Y$ magnet in the previous section:

$$
\begin{equation*}
S=\frac{2 V}{\mu} \int d^{d} \vec{x} d \tau\left(\partial_{\mu} \varphi\right)^{2} \tag{11.24}
\end{equation*}
$$

where we have rescaled the time units according to $\tau \rightarrow \tau / \sqrt{2 \mu}$ so that the velocity is 1. While this analysis has neglected interaction terms, Godstone's theorem guarantees us a gapless mode which survives even when interactions are taken into account.

If we wish to go beyond the saddle-point approximation by computing one-loop corrections, the quantum critical point will occur at some $\mu_{c} \neq 0$.

## Part IV

## Critical Fluctuations and Phase Transitions

## Chapter 12

## The Renormalization Group

### 12.1 Low-Energy Effective Field Theories

In our earlier discussions, we focussed on the low (compared to some cutoff $\Lambda$ ) $T$, low $\omega, q$ properties of the systems at which we looked. Why? The principal reason is that these properties are universal - i.e. independent of many of the details of the systems. Sometimes universal properties are the most striking and interesting aspect of a physical system, but not always (certainly not for many practical, e.g. engineering, applications). We like universal properties because we can understand them using effective field theories.

Suppose we have a system defined by the following functional integral:

$$
\begin{equation*}
Z=\int \mathcal{D} \phi e^{-S[\phi]} \tag{12.1}
\end{equation*}
$$

with correlation functions:

$$
\begin{equation*}
\left\langle\phi\left(p_{1}\right) \ldots \phi\left(p_{n}\right)\right\rangle=\int \mathcal{D} \phi \phi\left(p_{1}\right) \ldots \phi\left(p_{n}\right) e^{-s[\phi]} \tag{12.2}
\end{equation*}
$$

The long-wavelength, universal properties of the system are determined by these correlation functions in the $p_{i} \rightarrow 0$ limit, i.e. in the limit that the $p_{i}$ 's are smaller than any other scales in the problem.
$Z$ contains a great deal of 'extraneous' information about correlation functions for large $p_{i}$. We would like an effective action, $S_{\text {eff }}$, so that

$$
\begin{equation*}
Z_{\mathrm{eff}}=\int \mathcal{D} \phi e^{-S_{\mathrm{eff}}[\phi]} \tag{12.3}
\end{equation*}
$$

only contains the information necessary to compute the long-wavelength correlation functions. The reason that this a worthwhile program to pursue is that $Z_{\text {eff }}$ is often simple or, at least, simpler than $Z$. On the other hand, this is not a comletely straightforward program because no one tells us how to derive $Z_{\text {eff }}$. We have to use any and all tricks available to us (sometimes we can find a small parameter which enables us to get $Z_{\text {eff }}$ approximately and often we simply have to guess.

At a formal level, we can make the division:

$$
\begin{align*}
\phi_{L}(p) & =\phi(p) \theta\left(\Lambda^{\prime}-|p|\right) \\
\phi_{H}(p) & =\phi(p) \theta\left(|p|-\Lambda^{\prime}\right) \tag{12.4}
\end{align*}
$$

so that

$$
\begin{equation*}
\phi(p)=\phi_{L}(p)+\phi_{H}(p) \tag{12.5}
\end{equation*}
$$

where $\Lambda^{\prime}$ is some scale such that we're interested in $|p|<\Lambda^{\prime}$. Then

$$
\begin{equation*}
Z=\int \mathcal{D} \phi_{L} \mathcal{D} \phi_{H} e^{-S\left[\phi_{L}, \phi_{H}\right]} \tag{12.6}
\end{equation*}
$$

The effective field theory for long-wavelength correlation functions, i.e. correlation functions of $\phi_{L}$, is

$$
\begin{equation*}
Z_{\mathrm{eff}}=\int \mathcal{D} \phi_{L} e^{-S_{\mathrm{eff}}\left[\phi_{L}\right]} \tag{12.7}
\end{equation*}
$$

where

$$
\begin{equation*}
e^{-S_{\mathrm{eff}}\left[\phi_{L}\right]}=\int \mathcal{D} \phi_{H} e^{-S\left[\phi_{L}, \phi_{H}\right]} \tag{12.8}
\end{equation*}
$$

$S_{\text {eff }}\left[\phi_{L}\right]$ has cutoff $\Lambda^{\prime}$
Occasionally, we will be in the fortunate situation in which

$$
\begin{equation*}
S\left[\phi_{L}, \phi_{H}\right]=S_{L}\left[\phi_{L}\right]+S_{H}\left[\phi_{H}\right]+\lambda S_{\mathrm{int}}\left[\phi_{L}, \phi_{H}\right] \tag{12.9}
\end{equation*}
$$

with $\lambda$ small so that we can compute $S_{\text {eff }}$ perturbatively in $\lambda$ :

$$
\begin{equation*}
S_{\mathrm{eff}}=S_{L}\left[\phi_{L}\right]+\lambda S_{1}\left[\phi_{L}\right]+\lambda^{2} S_{2}\left[\phi_{L}\right]+\ldots \tag{12.10}
\end{equation*}
$$

In general, we have no such luck, and we have to work much harder to derive $S_{\text {eff }}$. However, even without deriving $S_{\text {eff }}$, we can make some statements about it on general grounds.

### 12.2 Renormalization Group Flows

Let's suppose that we have somehow derived $S_{\text {eff }}$ with cutoff $\Lambda$. Let's call it $S_{\Lambda}[\phi]$. $S[\phi]$ itself may have had all kinds of structure, but this doesn't interest us now; we're only interested in $S_{\Lambda}[\phi]$.

We expand $S_{\Lambda}[\phi]$ as

$$
\begin{equation*}
S_{\Lambda}[\phi]=\sum_{i} g_{i} \mathcal{O}_{i} \tag{12.11}
\end{equation*}
$$

where the $g_{i}$ 's are 'coupling constants' and the $\mathcal{O}_{i}$ are local operators. For instance, our phonon Lagrangian can be written as:

$$
\begin{equation*}
S=\rho \mathcal{O}_{1}+\mu \mathcal{O}_{2}+\lambda \mathcal{O}_{3}+g \mathcal{O}_{4} \tag{12.12}
\end{equation*}
$$

with

$$
\begin{align*}
\mathcal{O}_{1} & =\frac{1}{2} \int d \tau d^{3} \vec{x}\left(\partial_{t} u_{i}\right)^{2} \\
\mathcal{O}_{2} & =\int d \tau d^{3} \vec{x} u_{i j} u_{i j} \\
\mathcal{O}_{3} & =\frac{1}{2} \int d \tau d^{3} \vec{x} u_{k k}^{2} \\
\mathcal{O}_{3} & =\frac{1}{4!} \int d \tau d^{3} \vec{x}\left(\partial_{k} u_{k}\right)^{4} \tag{12.13}
\end{align*}
$$

while the $\mathrm{NL} \sigma \mathrm{M}$ for an antiferromagnet can be written as:

$$
\begin{equation*}
S=\mathcal{O}_{0}+g \mathcal{O}_{1}+g^{2} \mathcal{O}_{2}+\ldots \tag{12.14}
\end{equation*}
$$

with

$$
\begin{align*}
\mathcal{O}_{0} & =\int d^{d} \vec{x} d \tau\left(\partial_{\mu} n_{i}\right)^{2} \\
\mathcal{O}_{1} & =\int d^{d} \vec{x} d \tau n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j} \\
\mathcal{O}_{2} & =\int d^{d} \vec{x} d \tau n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j} n_{i} n_{i} \\
& \vdots \tag{12.15}
\end{align*}
$$

We now pick one term in the action - call it $\mathcal{O}_{\text {free }}$ even though it need not be quadratic - and use this term to assign dimensions to the various fields in the theory by requiring $\mathcal{O}_{\text {free }}$ to be dimensionless. For instance, if we choose $\mathcal{O}_{\text {free }}=\mathcal{O}_{1}$ in our phonon theory, then $\left[u_{i}\right]=1 / 2$. If we choose $\mathcal{O}_{\text {free }}=\mathcal{O}_{3}$ then $\left[u_{i}\right]=0$. Typically, we choose the term which we believe to be most 'important'. If we choose the 'wrong' one (i.e. an inconvenient one) then we will find out in the next step. Let's call $\delta_{\phi}$ the dimension of the field $\phi$ and $\delta_{i}$ the dimension of the operator $\mathcal{O}_{i} . \delta_{\text {free }}=0$ by construction.

We now rescale all momenta and fields by the cutoff $\Lambda$,

$$
\begin{align*}
q & \rightarrow q \Lambda \\
\phi & \rightarrow \phi \Lambda^{\delta_{\phi}} \tag{12.16}
\end{align*}
$$

so that the momenta and fields are now dimensionless. Then

$$
\begin{equation*}
S_{\Lambda}[\phi]=\sum_{i} g_{i} \Lambda^{\delta_{i}} \mathcal{O}_{i}=\sum_{i} \lambda_{i} \mathcal{O}_{i} \tag{12.17}
\end{equation*}
$$

Ordinarily, the dimensionless couplings $\lambda_{i}$ will be $O(1)$. On dimensional grounds, at energy scale $\omega, \mathcal{O}_{i} \sim(\omega / \Lambda)^{\delta_{i}}$, and the action

$$
\begin{equation*}
S_{\Lambda}[\phi]=\sum_{i} \lambda_{i}\left(\frac{\omega}{\Lambda}\right)^{\delta_{i}} \tag{12.18}
\end{equation*}
$$

If $\delta_{i}>0$, this term becomes less important at low energies. Such a term is called irrelevant. If $\delta_{i}=0$, the term is called marginal; it remains constant as $\omega \rightarrow 0$. If $\delta_{i}<0$, the term is relevant; it grows in importance at low energies. If $S_{\text {eff }}$ is simple,
it is only because there might be a finite number of relevant operators. At lower and lower energies, $\omega \ll \Lambda$, it becomes a better and better approximation to simply drop the irrelevant operators.

Let's formalize this by putting together the notion of a low-energy effective field theory with the above considerations about scaling:

- We have an effective action, $S_{\Lambda}[\phi]$ and a choice of $\mathcal{O}_{\text {free }}$.
- We divide

$$
\phi(p)=\phi_{L}(p) \theta\left(b \Lambda^{\prime}-|p|\right)+\phi_{H}(p) \theta\left(|p|-b \Lambda^{\prime}\right)
$$

where $b<1$.

- The next step is to obtain (by hook or by crook) $S_{b \Lambda}$

$$
e^{-S_{b \Lambda}\left[\phi_{L}\right]}=\int \mathcal{D} \phi_{H} e^{-S_{\Lambda}\left[\phi_{L}, \phi_{H}\right]}
$$

- We now rescale

$$
\begin{aligned}
q & \rightarrow q b \\
\omega & \rightarrow \omega b^{z} \\
\phi & \rightarrow \phi b^{\zeta}
\end{aligned}
$$

where $\zeta$ and $z$ are chosen to preserve $\mathcal{O}_{\text {free }}$. In general, $\zeta$ and $z$ will depend on the couplings $g_{i}, \zeta=\zeta\left(g_{i}\right), z=z\left(g_{i}\right)$. In equilibrium classical statistical mechanics, there are no frequencies, so we do not need to worry about $g$; in the theories which have examined thus far $\omega$ and $q$ are on the same footing, so $z$ is fixed to $z=1$. In general, however, one must allow for arbitrary $z$. The rescaling yields $S_{\Lambda}^{b}[\phi]$ which also has cutoff $\Lambda$.

- The physics of the system can be obtained from $S_{\Lambda}^{b}[\phi]$ by a rescaling. For instance, the correlation length is given by

$$
(\xi)_{S_{\Lambda}}=\frac{1}{b}(\xi)_{S_{\Lambda}^{b}}
$$

- If

$$
S_{\Lambda}[\phi]=\sum_{i} g_{i}^{0} \mathcal{O}_{i}
$$

then

$$
S_{\Lambda}^{b}[\phi]=\sum_{i} g_{i}(b) \mathcal{O}_{i}
$$

where $g_{i}(1)=g_{i}^{0}$. Let $b=e^{-\ell}$. Then we can define flow equations:

$$
\frac{d g}{d \ell}=-\delta_{i} g_{i}+\ldots
$$

which describe the evolution of $S_{\Lambda}$ under an infinitesimal transformation. These equations are called Renormalization Group (RG) equations or flow equations.

If we can neglect the $\ldots$, then for $\delta_{i}<0, g_{i}$ grows as $\ell$ increases, i.e. $\mathcal{O}_{i}$ is more important at low energies $\left(\ell \rightarrow \infty\right.$. For $\delta_{i}>0, g_{i}$ decreases, i.e. $\mathcal{O}_{i}$ is less important at low energies. Of course, the ... need not be small. In fact, it can dominate the first term. In the case of a marginal operator, $\delta_{i}=0$, the $\ldots$ is the whole story. For example, if

$$
\begin{equation*}
\frac{d g}{d \ell}=g^{2} \tag{12.19}
\end{equation*}
$$

Then

$$
\begin{equation*}
g(\ell)=\frac{g_{0}}{1-g_{0}\left(\ell-\ell_{0}\right)} \tag{12.20}
\end{equation*}
$$

so $g(\ell)$ grows as $\ell$ grows.

### 12.3 Fixed Points

If, for some values of the couplings, $g_{i}=g_{i}^{*}$,

$$
\begin{equation*}
\left(\frac{d g_{i}}{d \ell}\right)_{g_{k}=g_{k}^{*}}=0 \tag{12.21}
\end{equation*}
$$

then we call $g_{k}=g_{k}^{*}$ a fixed point. At a fixed point, $S_{\Lambda}=S_{\Lambda}^{b^{\prime}}$, so the physics is the same at all scales. Hence,

$$
\begin{equation*}
\xi=\frac{1}{b} \xi \tag{12.22}
\end{equation*}
$$

i.e. $\xi=\infty$ - the low-energy physics is gapless - or $\xi=0$ - there is no low-energy physics.

The notion of universality is encapsulated by the observation that different physical systems with very different 'microscopic' actions $S_{\Lambda}, S_{\Lambda}^{\prime}$, $S_{\Lambda}^{\prime \prime}$ can all flow into the same fixed point action, $S_{\Lambda}^{*}$. If this happens, these systems have the same asymptotic long-wavelength physics, i.e. the same universal behavior.

At a fixed point, we can linearize the $R G$ equations:

$$
\begin{equation*}
\frac{d}{d \ell}\left(g_{i}-g_{i}^{*}\right)=A_{i j}\left(g_{j}-g_{j}^{*}\right) \tag{12.23}
\end{equation*}
$$

This can be diagonalized to give:

$$
\begin{equation*}
\frac{d u_{i}}{d \ell}=y_{i} u_{i} \tag{12.24}
\end{equation*}
$$

where $u_{i}=O_{i j}\left(g_{j}-g_{j}^{*}\right)$. The corresponding operators, $\tilde{\mathcal{O}}_{i}=O_{i j} \mathcal{O}_{i} j$,

$$
\begin{equation*}
S_{\Lambda}[\phi]=\sum_{i} u_{i} \tilde{\mathcal{O}}_{i} \tag{12.25}
\end{equation*}
$$

are called eigenoperators. If $y_{i}>0$, we say that $u_{i}$ and $\tilde{\mathcal{O}}_{i}$ are relevant at this fixed point. If $y_{i}=0$, we say that $u_{i}$ is marginal. If $y_{i}<0, u_{i}$ is irrelevant at this fixed point.

Earlier, we characterized $\mathcal{O}_{i}$ as relevant, marginal, or irrelevant according to whether $\delta_{i}<0, \delta_{i}=0$, or $\delta_{i}>0$. What this really means is that $\mathcal{O}_{i}$ has this property at the fixed point $S^{*}=\mathcal{O}_{\text {free }}$. It is possible for a coupling constant, $g$, to be relevant at one fixed point, $S_{1}^{*}$, but irrelevant at another fixed point, $S_{2}^{*}$, as shown in figure 12.1.


Figure 12.1: The coupling $g_{1}$ is relevant at the fixed point on the left but irrelevant at the fixed point on the right.

### 12.4 Phases of Matter and Critical Phenomena

If $y_{i}<0$ for all $i$ at a given fixed point, then we call this fixed point an attractive or stable fixed point. Theories will generically flow into such a fixed point. Stable fixed points represent phases of matter. In this course, we have already looked at a number of stable fixed points. Our phonon theory,

$$
\begin{equation*}
S_{0}=\int d t d^{3} \vec{x} \mathcal{L}=\frac{1}{2} \int d t d^{3} \vec{r}\left[\rho\left(\partial_{t} u_{i}\right)^{2}-2 \mu u_{i j} u_{i j}-\lambda u_{k k}^{2}\right] \tag{12.26}
\end{equation*}
$$

is a stable fixed point (you can check that $g$ is an irrelevant coupling) at $T=0$ for $d>1$. This stable fixed point corresponds to a stable phase of matter: a crystal. For $T>0$, it is stable for $d>2$. Our theories of non-interacting magnons

$$
\begin{gather*}
S=s \int d^{d} \vec{x} d \tau\left(\frac{1}{2} m_{+} \frac{\partial m_{-}}{\partial \tau}+\frac{1}{2} D \vec{\nabla} m_{+} \cdot \vec{\nabla} m_{-}\right)  \tag{12.27}\\
S=\int d^{d} \vec{x} d \tau\left(\partial_{\mu} n_{i}\right)^{2} \tag{12.28}
\end{gather*}
$$

are also stable fixed points corresponding, repectively, to ferromagnetic and antiferromagnetic phases. The ferromagnet is always stable at $T=0$, while the antiferromagnet is stable at $T=0$ for $d>1$. For $T>0$, both are stable for $d>2$. Similarly,
$X Y$ magnets and superfluid ${ }^{4} \mathrm{He}$

$$
\begin{equation*}
S=\int d^{d} \vec{x} d \tau\left(\partial_{\mu} \theta\right)^{2} \tag{12.29}
\end{equation*}
$$

are phases of matter at $T=0$ for $d>1$. For $T>0$, they are stable for $d \geq 2(d=2$ is a special case).

The stable phases described above are all characterized by gapless modes - i.e. $\xi=\infty$ which are a consequence of spontaneous symmetry breaking. There are also stable phases without gapless modes - i.e. with $\xi=0$. The ${ }^{4} \mathrm{He}$ action with $\mu<\mu_{c}$ (in the saddle-point approximation, $\mu_{c}=0$ ) describes an empty system with a gap $\mu_{c}-\mu$ to all excitations.

$$
\begin{equation*}
S=\int d \tau d^{d} x\left(\psi^{*} \partial_{\tau} \psi+|\nabla \psi|^{2}+V\left(|\psi|^{2}-\frac{\mu}{2 V}\right)^{2}\right) \tag{12.30}
\end{equation*}
$$

Similarly, $\varphi^{4}$ theory

$$
\begin{equation*}
\int d^{d} x\left(\frac{1}{2} K(\nabla \varphi)^{2}+\frac{1}{2} r \varphi^{2}+\frac{1}{4!} u \varphi^{4}\right) \tag{12.31}
\end{equation*}
$$

has two stable phases - ordered and disordered phases - corresponding to the fixed points $r \rightarrow \pm \infty$. At both of these fixed points, $\xi=0$. Similarly, the high-temperature disordered states of magnets are stable phases with gaps.

It makes sense to do perturbation theory in an irrelevant coupling because this perturbation theory gets better at low $q, \omega$. Essentially, the expansion parameter for perturbation theory is the dimensionless combination $g \omega^{-y_{g}}$ for some correlation function at characteristic frequency $\omega$. Hence, our perturbative calculations of correlation functions in the phonon and magnon theories were sensible calculations. Similarly, perturbation theory in the coupling $u$ in $\varphi^{4}$ is sensible for $d>4$ (the Ginzburg criterion) as you showed in the problem set. However, it does not make sense to perturb in a relevant coupling such as $u$ in $\varphi^{4}$ for $d<4$. In such a case, the effective expansion parameter grows at low $q, \omega$. The low $q, \omega$ physics is, in fact controlled by some other fixed point.


Figure 12.2: The flow diagram of a critical point $C$ and two stable fixed points $S_{1}, S_{2}$ which it separates.

If some of the $y_{i}>0$ then the fixed point is repulsive or unstable. The relevant couplings must be tuned to zero in order for the theory to flow into an unstable fixed point. Unstable fixed points represent (multi-)critical points separating these phases. The unstable directions correspond to the parameters which must be tuned to reach the critical point. Superfluid ${ }^{4} \mathrm{He}$

$$
\begin{equation*}
S=\int d \tau d^{d} x\left(\psi^{*} \partial_{\tau} \psi+|\nabla \psi|^{2}+V\left(|\psi|^{2}-\frac{\mu}{2 V}\right)^{2}\right) \tag{12.32}
\end{equation*}
$$

has a critical point at $\mu=\mu_{c}$. The corresponding fixed point is at $\mu=\mu^{*}, V=V^{*}$. This critical point separates two stable phases: the superfluid and the empty system. There is one relevant direction at this fixed point. By tuning this relevant direction, we can pass from one phase through the critical point to the other phase. Similarly, the Ising model has a fixed point with one relevant direction which we discuss in a later section. By tuning the relevant coupling, we can pass from the ordered state through the critical point and into the disordered state. Figure 12.2 depicts the flow diagram for a critical point and two stable fixed points which it separates.

### 12.5 Scaling Equations

Let us consider the the implications of this framework for physical quantities. Suppose $C\left(p_{i}, g_{i}\right)$ is some physical quantity such as a correlation function of $n$ fields $\varphi$. It will, in general, depend on some momenta $p_{i}$ and on the coupling constants of the system. Suppose that the couplings are all close to their fixed point values, $g_{i} \approx g_{i}^{*}$, so we will
write $C$ as $C\left(p_{i}, g_{i}-g_{i}^{*}\right)$ and suppose that the linearized flow equations read:

$$
\begin{equation*}
\frac{d}{d \ell}\left(g_{i}-g_{i}^{*}\right)=\lambda_{i}\left(g_{i}-g_{i}^{*}\right) \tag{12.33}
\end{equation*}
$$

Then we can perform an RG transformation, according to which:

$$
\begin{equation*}
C\left(p_{i}, g_{i}-g_{i}^{*}\right)=b^{-n \zeta^{*}} C\left(\frac{p_{i}}{b},\left(g_{i}-g_{i}^{*}\right) b^{-\lambda_{i}}\right) \tag{12.34}
\end{equation*}
$$

Suppose that we are in the vicinity of a stable fixed point, so that all of the $\lambda_{i}<0$. Then, in the $b \rightarrow 0$ limit

$$
\begin{equation*}
C\left(p_{i}, g_{i}-g_{i}^{*}\right)=b^{-n \zeta^{*}} C\left(\frac{p_{i}}{b}, 0\right) \tag{12.35}
\end{equation*}
$$

If, for instance, we are interested in the two-point correlation function at low $p$, we can take $b=p$ and:

$$
\begin{equation*}
C\left(p, g_{i}-g_{i}^{*}\right) \rightarrow p^{-2 \zeta^{*}} C(1,0) \tag{12.36}
\end{equation*}
$$

A similar result follows if we are in the vicinity of an unstable fixed point, but we have set all of the relevant couplings equal to zero. This scaling relation may seem to contradict simple dimensional analysis, which would predict $C\left(p, g_{i}-g_{i}^{*}\right) \sim p^{-2 \delta_{\phi}}$. In fact, there is no contradiction. The missing powers of $p$ are made up by the dependence on the cutoff:

$$
\begin{equation*}
C\left(p, g_{i}-g_{i}^{*}\right) \sim p^{-2 \zeta^{*}} \Lambda^{-2 \delta_{\phi}+2 \zeta^{*}} \tag{12.37}
\end{equation*}
$$

These observations can be reformulated as follows. Consider a correlation function $C_{n}\left(p_{i}, g_{i}\right)$ of $n \varphi$ fields. Up to a rescaling, this correlation function is equal to its value after an RG transformation:

$$
\begin{equation*}
C_{n}\left(p_{i}, g_{i}\right)=\left(e^{\ell}\right)^{n \zeta\left(g_{i}\right)} C_{n}\left(p_{i} e^{\ell}, g_{i}(\ell)\right) \tag{12.38}
\end{equation*}
$$

The left-hand-side is independent of $\ell$, so differentiating both sides with respect to $\ell$ yields the $R G$ equation:

$$
\begin{equation*}
\left(\frac{\partial}{\partial \ell}+n\left(\delta_{\varphi}-\frac{1}{2} \eta\left(g_{j}\right)\right)+\beta_{i}\left(g_{j}\right) \frac{\partial}{\partial g_{i}}\right) C_{n}\left(p_{i} e^{\ell}, g_{i}(\ell)\right)=0 \tag{12.39}
\end{equation*}
$$

where

$$
\begin{equation*}
2 \delta_{\varphi}-\eta\left(g_{j}\right)=\frac{d}{d \ell}\left(e^{\ell}\right)^{\zeta\left(g_{i}\right)} \tag{12.40}
\end{equation*}
$$

$\delta_{\varphi}$ is the naive scaling dimension of $\varphi . \eta$ is called the anomalous dimension of $\varphi$. The $\beta$ functions are the right-hand-sides of the flow-equations for the couplings:

$$
\begin{equation*}
\beta_{i}\left(g_{j}\right)=\frac{d g_{i}}{d \ell} \tag{12.41}
\end{equation*}
$$

At a fixed point, the $\beta$-functions vanish, $\beta_{i}=0$ and $\eta$ is a constant,

$$
\begin{equation*}
2 \delta_{\varphi}-\eta=2 \zeta\left(g_{i}^{*}\right) \tag{12.42}
\end{equation*}
$$

so the RG equation reads:

$$
\begin{equation*}
\left(\frac{\partial}{\partial \ell}-n\left(\delta_{\varphi}-\frac{1}{2} \eta\right)\right) C_{n}\left(p_{i} e^{-\ell}\right)=0 \tag{12.43}
\end{equation*}
$$

In other words, the correlation function is a power-law in $p_{i}$ with exponent $n \delta_{\varphi}-n \eta / 2$.
Suppose, instead, that we are near a fixed point with one relevant direction. Call this coupling $u$ and the other irrelevant couplings $g_{i}$. Then,

$$
\begin{equation*}
C\left(p_{i}, u-u^{*}, g_{i}-g_{i}^{*}\right)=b^{-n \zeta^{*}} C\left(p_{i} / b,\left(u-u^{*}\right) b^{-\lambda_{u}},\left(g_{i}-g_{i}^{*}\right) b^{-\lambda_{i}}\right) \tag{12.44}
\end{equation*}
$$

If $u-u^{*}$ is small, then we can take $b=\left(u-u^{*}\right)^{1 / \lambda_{u}}$ and be in the $b \rightarrow 0$ limit:

$$
\begin{equation*}
C\left(p_{i}, u-u^{*}, g_{i}-g_{i}^{*}\right) \rightarrow \frac{1}{\left(u-u^{*}\right)^{n \zeta^{*} / \lambda_{u}}} C\left(p_{i}\left(u-u^{*}\right)^{-1 / \lambda_{u}}, 1,0\right) \tag{12.45}
\end{equation*}
$$

or

$$
\begin{equation*}
C\left(p_{i}, u-u^{*}, g_{i}-g_{i}^{*}\right) \rightarrow \frac{1}{\left(u-u^{*}\right)^{n \zeta^{*} / \lambda_{u}}} F\left(p_{i}\left(u-u^{*}\right)^{-1 / \lambda_{u}}\right) \tag{12.46}
\end{equation*}
$$

where $F(x)$ is called a scaling function. If the stable phase to which the system flows is trivial - i.e. has a gap - then $C\left(p_{i}\left(u-u^{*}\right)^{-1 / \lambda_{u}}, 1,0\right)$ can be calculated and does not have any interesting structure. The non-trivial physics is entirely in the prefactor.

If we are interested in a correlation function at $p_{i}=0$ such as the magnetization of a ferromagnet, then we can write:

$$
\begin{equation*}
C_{0}\left(u-u^{*}, g_{i}-g_{i}^{*}\right) \rightarrow \frac{1}{\left(u-u^{*}\right)^{n \zeta^{*} / \lambda_{u}}} F\left(\left(g_{i}-g_{i}^{*}\right)\left(u-u^{*}\right)^{-\lambda_{i} / \lambda_{u}}\right) \tag{12.47}
\end{equation*}
$$

Now imagine that there is a second relevant coupling, $g$, or, even that $g$ is the leading irrelevant coupling (i.e. the least irrelevant of the irrelevant couplings) so that we are interested in the $g$ dependence of the correlation function. Then, setting the other couplings to their fixed point values in the small $u-u^{*}$ limit:

$$
\begin{equation*}
C_{0}\left(u-u^{*}, g_{i}-g_{i}^{*}\right) \rightarrow \frac{1}{\left(u-u^{*}\right)^{n \zeta^{*} / \lambda_{u}}} F\left(\frac{g_{i}-g_{i}^{*}}{\left(u-u^{*}\right)^{-\lambda_{i} / \lambda_{u}}}\right) \tag{12.48}
\end{equation*}
$$

### 12.6 Finite-Size Scaling

Temperature plays a very different role in classical and quantum statistical mechanics. In the classical theory, temperature is one of the couplings in the theory. The temperature dependence of physical quantities can be determined from the scaling behavior of the temperature. Classical statistical mechanics can be used to calculate a correlation function at wavevector the temperature is larger than the important excitation energies since the $n \neq 0$ Matsubara frequencies can then be ignored. In the quantum theory, temperature is the size of the system in the imaginary time direction, and the temperature dependence of physical quantities can be determined from finite-size scaling which we discuss below. Finite-size scaling can be used in the limit in which $\beta$ is large. An alternative, related way of dealing with finite-temperature is dicussed in the context of the $\mathrm{NL} \sigma \mathrm{M}$ in the last section of this chapter.

Finite-size scaling is also useful for dealing with systems which are finite in one or more spatial directions. Since numerical calculations must be done in such systems, finite-size scaling allows us to compare numerics to analytical calculations which are more easily done in the infinite-size limit.

Since renormalization group equations describe the evolution of effective Lagrangians as one integrates out short-distance physics, it is clear that these equations are insensitive to finite-size effects so long as the finite-size is much larger than the inverse of the cutoff. While the equations themselves are unchanged, the solutions are modified because they depend on an additional dimensionful parameter, namely the size of the system (in our case, $\beta$ ). For simplicity, let us consider a theory with a single relevant coupling (say, $\phi^{4}$ theory), which satisfies a renormalization group equation with a low-energy fixed point:

$$
\begin{equation*}
\left(\frac{\partial}{\partial \ell}+\beta(g) \frac{\partial}{\partial g}+n\left(\delta_{\varphi}-\frac{1}{2} \eta(g)\right)\right) G^{(n)}\left(p_{i} e^{\ell}, g(\ell), L e^{-\ell}\right)=0 \tag{12.49}
\end{equation*}
$$

$G^{(n)}$ is an n-point Green function, $L$ is the finite size of the system. We may take $e^{\ell}=L$, and we find

$$
\begin{equation*}
G^{(n)}\left(p_{i}, g, L\right)=L^{n\left(\delta_{\varphi}-\frac{1}{2} \eta\left(g_{j}\right)\right)} G^{(n)}\left(p_{i} L, g(\ln L), 1\right) \tag{12.50}
\end{equation*}
$$

Then in the large-size limit, $L \rightarrow \infty$, we have $g(\ln L) \rightarrow g^{*}$. As a result, we have the scaling form:

$$
\begin{equation*}
G^{(n)}\left(p_{i}, g, L\right)=L^{n\left(\delta_{\varphi}-\frac{1}{2} \eta\left(g_{j}\right)\right)} G^{(n)}\left(p_{i} L, g^{*}, 1\right) \tag{12.51}
\end{equation*}
$$

We will be primarily concerned with the case in which the finite size, $L$, will be the inverse temperature, $\beta$, so (12.51) will give the temperature dependence of Green functions in the low-temperature limit.

$$
\begin{equation*}
G^{(n)}\left(p_{i}, g, \beta\right) \sim \beta^{n\left(\delta_{\varphi}-\frac{1}{2} \eta\right)} G^{(n)}\left(p_{i} \beta, g^{*}, 1\right) \tag{12.52}
\end{equation*}
$$

### 12.7 Non-Perturbative RG for the 1D Ising Model

In the next two sections, we will look at two examples of RG transformations, one non-perturbative and one perturbative. First, we look at the $1 D$ Ising model,

$$
\begin{equation*}
H=J \sum_{i} \sigma_{i} \sigma_{i+1} \tag{12.53}
\end{equation*}
$$



Figure 12.3: The flow diagram of the $1 D$ Ising model.

Our RG transformation will be done in 'real space' by integrating out the spins on the even sites. This procedure is called decimation. Whereas the original model has wavevectors $-\pi / a<k<\pi / a$, the resulting theory has $-\pi / 2 a<k<\pi / 2 a$. We can then rescale momenta by 2 to obtain an RG tranformation.

$$
\begin{align*}
\mathcal{Z} & =\sum_{\sigma_{i}= \pm 1} \prod_{i} e^{\frac{J}{T} \sigma_{i} \sigma_{i+1}} \\
& =\sum_{\sigma_{2+1}= \pm 1} \prod_{i}\left(2 \cosh \frac{J}{T}\left(\sigma_{2 i+1}+\sigma_{2(i+1)+1}\right)\right) \\
& =\sum_{\sigma_{2+1}= \pm 1} \prod_{i} K e^{\left(\frac{J}{T}\right)^{\prime} \sigma_{2 i+1} \sigma_{2(i+1)+1}} \tag{12.54}
\end{align*}
$$

where $K=2 e^{\left(\frac{J}{T}\right)^{\prime}}$ and

$$
\begin{equation*}
\left(\frac{J}{T}\right)^{\prime}=\frac{1}{2} \ln \cosh 2 \frac{J}{T} \tag{12.55}
\end{equation*}
$$

This RG transformation has only 2 fixed points, $\frac{J}{T}=0$ and $\frac{J}{T}=\infty \cdot \frac{J}{T}$ is relevant at the $\frac{J}{T}=\infty$ fixed point but irrelevant at the $\frac{J}{T}=0$ fixed point. The flow diagram is shown in 12.3. This flow diagram shows that for any $T>0$, the system is controlled by the disordered $\frac{J}{T}=0$ fixed point. Only at $T=0$ can the system be ordered.

### 12.8 Perturbative RG for $\varphi^{4}$ Theory in $4-\epsilon$ Dimensions

Our second example is $\varphi^{4}$ theory.

$$
S=\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} q^{2}|\varphi(q)|^{2}+\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} r|\varphi(q)|^{2}
$$

$$
\begin{equation*}
+\frac{u}{4!} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \varphi\left(q_{1}\right) \varphi\left(q_{2}\right) \varphi\left(q_{3}\right) \varphi\left(-q_{1}-q_{2}-q_{3}\right) \tag{12.56}
\end{equation*}
$$

We take the first term as $\mathcal{O}_{\text {free }}$. Under a rescaling $q \rightarrow q b$, we must take

$$
\begin{equation*}
\varphi \rightarrow \varphi b^{-\frac{d+2}{2}} \tag{12.57}
\end{equation*}
$$

Using this rescaling, we immediately see that the leading terms in the RG equations are:

$$
\begin{align*}
\frac{d r}{d \ell} & =2 r+\ldots \\
\frac{d u}{d \ell} & =(4-d) u+\ldots \tag{12.58}
\end{align*}
$$

We immediately find one fixed point, the Gaussian fixed point: $r=u=0$. At this fixed point, $r$ is always relevant while $u$ is irrelevant for $d>4$ and relevant for $d<4$. You will recognize that this is the same as the Ginzburg criterion which determines when the saddle-point approximation is valid for this theory: the saddlepoint approximation is valid when the quartic interaction is irrelevant. When the quartic interaction is irrelevant, the correct theory of the critical point is simply.

$$
\begin{equation*}
S=\int \frac{d^{d} q}{(2 \pi)^{d}} q^{2}|\varphi(q)|^{2} \tag{12.59}
\end{equation*}
$$

which has critical correlation functions

$$
\begin{equation*}
\langle\varphi(\vec{x}) \varphi(0)\rangle \sim \frac{1}{|x|^{d-2}} \tag{12.60}
\end{equation*}
$$

The one relevant direction at the Gaussian fixed point in $d>4$ is the temperature, $r$. At the Gaussian fixed point, $r$ has scaling dimension 2. Hence, $\xi \sim 1 / \sqrt{r}$. As we discussed in the context of the Ising model, $r \sim T-T_{c}$. Hence,

$$
\begin{equation*}
\xi \sim \frac{1}{\left|T-T_{c}\right|^{1 / 2}} \tag{12.61}
\end{equation*}
$$

Of course, we should also allow $\varphi^{6}, \varphi^{8}$, etc. terms. If we don't include them initially in our action, they will be generated by the RG transformation. However,
the $\varphi^{6}$ operator is only relevant below 3 dimensions, the $\varphi^{8}$ operator is only relevant below $8 / 3$ dimensions, etc. Hence, for $d>3$, we can ignore the higher order terms in the asymptotic $q \rightarrow 0$ limit because they are irrelevant. (Actually, we have only shown that they are irrelevant at the Gaussian fixed point; in fact, they are also irrelevant at the fixed point which we find below.)

For $d<4$, the Gaussian fixed point has two unstable directions, $r$ and $u$. We can compute the RG equations to the one-loop level to find other fixed points. We make the division of $\varphi$ into $\varphi_{L}$ and $\varphi_{H}$ and integrate out $\varphi_{H}$ at the one-loop level. At a schematic level, this works as follows:

$$
\begin{equation*}
e^{-S_{\Lambda}^{b}\left[\varphi_{L}\right]}=e^{-S_{L}\left[\varphi_{L}\right]} \int \mathcal{D} \varphi_{H} e^{-S_{H}^{0}\left[\varphi_{H}\right]} e^{-S_{\mathrm{int}}\left[\varphi_{L}, \varphi_{H}\right]} \tag{12.62}
\end{equation*}
$$

where

$$
\begin{align*}
S_{L}\left[\varphi_{L}\right]= & \int_{0}^{b \Lambda} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} q^{2}\left|\varphi_{L}(q)\right|^{2}+\int_{0}^{b \Lambda} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} r\left|\varphi_{L}(q)\right|^{2} \\
& +\frac{u}{4!} \int_{0}^{b \Lambda} \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \varphi_{L}\left(q_{1}\right) \varphi_{L}\left(q_{2}\right) \varphi_{L}\left(q_{3}\right) \varphi_{L}\left(q_{4}\right)  \tag{12.63}\\
S_{H}^{0}\left[\varphi_{H}\right]= & \int_{b \Lambda}^{\Lambda} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} q^{2}\left|\varphi_{H}(q)\right|^{2}+\int_{b \Lambda}^{\Lambda} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{2} r\left|\varphi_{H}(q)\right|^{2}  \tag{12.64}\\
S_{\mathrm{int}}\left[\varphi_{L}, \varphi_{H}\right]= & \frac{u}{4!} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \varphi_{H}\left(q_{1}\right) \varphi_{H}\left(q_{2}\right) \varphi_{H}\left(q_{3}\right) \varphi_{H}\left(q_{4}\right) \\
& +\frac{u}{4} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \varphi_{H}\left(q_{1}\right) \varphi_{H}\left(q_{2}\right) \varphi_{L}\left(q_{3}\right) \varphi_{L}\left(q_{4} \chi_{1} 1\right.
\end{align*}
$$

$S_{\text {int }}$ also contains $\varphi_{H} \varphi_{H} \varphi_{H} \varphi_{L}$ and $\varphi_{L} \varphi_{L} \varphi_{L} \varphi_{H}$ terms, but the phase space for these terms is very small since it is difficult for three large momenta to add up to a small momentum or the reverse. Hence, we can safely ignore these terms. Expanding perturbatively, there is a contribution of the form:
$e^{-S_{\Lambda}^{b}\left[\varphi_{L}\right]}=e^{-S_{L}\left[\varphi_{L}\right]} \int \mathcal{D} \varphi_{H} e^{-S_{H}^{0}\left[\varphi_{H}\right]}(1-$

$$
\begin{align*}
& \left.\quad \frac{u}{4} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \varphi_{H}\left(q_{1}\right) \varphi_{H}\left(q_{2}\right) \varphi_{L}\left(q_{3}\right) \varphi_{L}\left(q_{4}\right)+\ldots\right) \\
& =e^{-S_{L}\left[\varphi_{L}\right]}\left(1-\frac{u}{4} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}}\left\langle\varphi_{H}\left(q_{1}\right) \varphi_{H}\left(q_{2}\right)\right\rangle \varphi_{L}\left(q_{3}\right) \varphi_{L}\left(q_{4}\right)+\ldots\right) \\
& =e^{-\left(S_{L}\left[\varphi_{L}\right]+\frac{u}{4} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}}\left\langle\varphi_{H}\left(q_{1}\right) \varphi_{H}\left(q_{2}\right)\right\rangle \varphi_{L}\left(q_{3}\right) \varphi_{L}\left(q_{4}\right)\right)}+O\left(u^{2}\right) \tag{12.66}
\end{align*}
$$

We can do this diagrammatically by computing one-loop diagrams with internal momenta restricted to the range $b \Lambda<|q|<\Lambda$. The external legs must be $\varphi_{L}$ fields, i.e. must have momenta $q<b \Lambda$. (Note that the contribution to $S_{b \Lambda}$ is the negative of the value of the diagram since we are absorbing it into $e^{-S_{\Lambda}^{b}\left[\varphi_{L}\right]}$.) The contribution described above results from the first-order diagram with two external legs. Such diagrams give a contribution to $S_{b \Lambda}$ of the form:

$$
\begin{equation*}
\int \frac{d^{d} q}{(2 \pi)^{d}} c(q)\left|\varphi_{L}(q)\right|^{2} \tag{12.67}
\end{equation*}
$$

where $c(q)=c_{0}+c_{2} q^{2}+\ldots$. Diagrams with four external legs give a contribution to $S_{b \Lambda}$ of the form:

$$
\begin{equation*}
\frac{1}{4!} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} v\left(q_{1}, q_{2}, q_{3}\right) \varphi\left(q_{1}\right) \varphi\left(q_{2}\right) \varphi\left(q_{3}\right) \varphi\left(-q_{1}-q_{2}-q_{3}\right) \tag{12.68}
\end{equation*}
$$

The one-loop contribution to the RG equations is given by the diagrams of figure 12.4. The diagram of $12.4(\mathrm{a})$ gives the contribution of (12.66), namely

$$
\begin{equation*}
\delta S_{b \Lambda}=-\frac{1}{2} u \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}+r} \int \frac{d^{d} p}{(2 \pi)^{d}} \frac{1}{2}|\varphi(p)|^{2} \tag{12.69}
\end{equation*}
$$

Dropping higher-order terms in $r$ (because we are interested in the vicinity of the Gaussian fixed point, $r=u=0$ ), we can rewrite the integral as

$$
\begin{aligned}
\int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}+r} & =\int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}}-r \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}\left(q^{2}+r\right)} \\
& =\int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}}-r \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{4}}+O\left(r^{2}\right) \\
& =\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-2}\left(1-b^{d-2}\right)
\end{aligned}
$$

(a)

(b)


Figure 12.4: The one-loop diagrams which determine the RG equations for (a) $r$ and (b) $u$.

$$
\begin{equation*}
-\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} r \frac{1}{d-4}\left(1-b^{d-4}\right) \tag{12.70}
\end{equation*}
$$

Meanwhile, the three diagrams of figure 12.4(b) each give a contribution
$-\frac{1}{2} u^{2} \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left(q^{2}+r\right)^{2}} \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \frac{1}{4!} \varphi\left(q_{1}\right) \varphi\left(q_{2}\right) \varphi\left(q_{3}\right) \varphi\left(-q_{1}-q_{2}-q_{3}\right)$
or, adding them together and evaluating the integral in the $r=0$ limit,
$-\frac{3}{2} u^{2} \frac{1}{(2 \pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-4}\left(1-b^{d-4}\right) \int \frac{d^{d} q_{1}}{(2 \pi)^{d}} \frac{d^{d} q_{2}}{(2 \pi)^{d}} \frac{d^{d} q_{3}}{(2 \pi)^{d}} \frac{1}{4!} \varphi\left(q_{1}\right) \varphi\left(q_{2}\right) \varphi\left(q_{3}\right) \varphi\left(q_{4}\right)$

Observe that there is no one-loop contribution to the

$$
\begin{equation*}
\int \frac{d^{d} q}{(2 \pi)^{d}} q^{2}|\varphi(q)|^{2} \tag{12.73}
\end{equation*}
$$

term. Hence the correct rescaling is still

$$
\begin{equation*}
\varphi \rightarrow \varphi b^{-\frac{d+2}{2}} \tag{12.74}
\end{equation*}
$$

As a result, we find the one-loop RG equations:

$$
r+d r=b^{-2}\left(r+\frac{1}{2} u \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-2}\left(1-b^{d-2}\right)\right.
$$



Figure 12.5: The flow diagram of a $\varphi^{4}$ theory in $4-\epsilon$ dimensions.

$$
\begin{gather*}
\left.-\frac{1}{2} u r \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-4}\left(1-b^{d-4}\right)\right) \\
u+d u=b^{d-4}\left(u-\frac{3}{2} u^{2} \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-4}\left(1-b^{d-4}\right)\right) \tag{12.75}
\end{gather*}
$$

Writing $b=e^{-d \ell}$, and taking the limit of small $\epsilon=4-d$ we have:

$$
\begin{align*}
& \frac{d r}{d \ell}=2 r+\frac{1}{16 \pi^{2}} u-\frac{1}{16 \pi^{2}} u r+\ldots \\
& \frac{d u}{d \ell}=\epsilon u-\frac{3}{16 \pi^{2}} u^{2}+\ldots \tag{12.76}
\end{align*}
$$

The corresponding flow diagram is shown in figure 12.5. These RG equations have a fixed point at $O(\epsilon)$ :

$$
\begin{align*}
r^{*} & =-\frac{1}{6} \epsilon \\
u^{*} & =\frac{16 \pi^{2}}{3} \epsilon \tag{12.77}
\end{align*}
$$

At this fixed point, the eigenoperators are:

$$
\begin{align*}
\frac{d}{d \ell}\left(r-r^{*}\right) & =\left(2-\frac{1}{3} \epsilon\right)\left(r-r^{*}\right) \\
\frac{d}{d \ell}\left(u-u^{*}\right) & =-\epsilon\left(u-u^{*}\right) \tag{12.78}
\end{align*}
$$

There is only one relevant direction (corresponding to the temperature) with scaling dimension $1 / \nu=2-\frac{1}{3} \epsilon$. The correlation length scales as:

$$
\begin{equation*}
\xi \sim \frac{1}{\left|T-T_{c}\right|^{\nu}} \tag{12.79}
\end{equation*}
$$

At the critical point, the correlation function has the power law decay

$$
\begin{equation*}
\langle\varphi(\vec{x}) \varphi(0)\rangle \sim \frac{1}{x^{d-2+\eta}} \tag{12.80}
\end{equation*}
$$

At order $\epsilon, \eta=0$, as we have seen. However, this is an artifact of the $O(\epsilon)$ calculation. At the next order, we find a non-vanishing contribution: $\eta=\epsilon^{2} / 54$.

For $\epsilon$ small, our neglect of higher-loop contributions is justified. To compute in $d=3$, however, we must go to higher loops or, equivalently, higher-order in $\epsilon$.

Several remarks are in order:

- As we showed in chapter 8 , the Ising model can be mapped onto a $\varphi^{4}$ theory if higher powers of $\varphi$ are neglected. We can now justify their neglect: these terms are irrelevant.
- There are many different ways of implementing the cutoff, i.e. regularizing a theory: by putting the theory on a lattice (as in the $1 D$ Ising model above), by introducing a hard cutoff (as in $\varphi^{4}$ theory above), or by introducing a soft cutoff - e.g. by multiplying all momentum integrals by $e^{-q^{2} / \Lambda^{2}}-$ just to name a few.
- Corresponding to these different regularization schemes, there are different renormalization group transformations, such as real-space decimation on the lattice or momentum shell-integration for a momentum-space cutoff. Since all of these different cutoff theories will flow, ultimately, into the same fixed point under the different RG transformations, it is a matter of convenience which scheme we choose.
- Both RG equations and the fixed point values of the couplings are scheme dependent. The universal properties, such as exponents, are scheme independent. e.g. in the RG equation

$$
\begin{equation*}
\frac{d g}{d \ell}=\lambda_{g}\left(g-g^{*}\right)+\ldots \tag{12.81}
\end{equation*}
$$

$\lambda_{g}$ is scheme independent and independent of all microscopic details, but $g^{*}$ is scheme-dependent, and can depend on microscopic details such as the cutoff.

- Integrals which are logarithmically divergent at large $q$ are proportional to $\ln b$ and are independent of the cutoff $\Lambda$. Consequently they are scheme independent. Integrals which are more strongly ultra-violet divergent are scheme-dependent.
- The term in the RG equation for $r$ which is independent of $r$ determines only the fixed point value of $r$, i.e. $r^{*}$. It does not affect the scaling exponents. In fact, it is renormalization scheme dependent; in some schemes, it vanishes, so it can be dropped and we can work with:

$$
\begin{align*}
\frac{d r}{d \ell} & =2 r-\frac{1}{16 \pi^{2}} u r+\ldots \\
\frac{d u}{d \ell} & =\epsilon u-\frac{3}{16 \pi^{2}} u^{2}+\ldots \tag{12.82}
\end{align*}
$$

### 12.9 The $O(3) \mathrm{NL} \sigma \mathrm{M}$

In order to study the phase diagram of a quantum system we would like to consider both $T=0$ and finite $T$ in the same phase diagram. In particular, we would like to consider even the high temperatures at which classical phase transitions can occur. Finite-size scaling - which is useful for asymptotically low temperatures - is not appropriate for such an analysis. Instead, we carry out the renormalization group
transformation directly on the finite-temperature quantum-mechanical functional integral. We will show how this is done for an antiferromagnet:

$$
\begin{equation*}
\int \mathcal{D} \vec{n} e^{-\frac{1}{g} \int d^{d} \vec{x} \int_{0}^{\beta} d \tau\left(\partial_{\mu} \vec{n}\right)^{2}} \tag{12.83}
\end{equation*}
$$

The requirement of $O(3)$ symmetry together with the constraint $\vec{n}^{2}=1$ implies that we can add to this action irrelevant terms such as

$$
\begin{equation*}
\int d^{d} \vec{x} d \tau \int_{0}^{\beta} d \tau\left(\partial_{\mu} \vec{n} \cdot \partial_{\mu} \vec{n}\right)^{2} \tag{12.84}
\end{equation*}
$$

but no relevant terms for $d \geq 1$.
As usual, we rewrite the action as:

$$
\begin{equation*}
S=\int d^{d} \vec{x} \int_{0}^{\beta} d \tau\left(\left(\partial_{\mu} n_{i}\right)^{2}+g \frac{n_{i} \partial_{\mu} n_{i} n_{j} \partial_{\mu} n_{j}}{1-g n_{i} n_{i}}\right) \tag{12.85}
\end{equation*}
$$

We now define an RG transformation in which we integrate out $n_{i}\left(\vec{q}, \omega_{n}\right)$ with wavevectors $e^{-\ell} \Lambda<|q|<\Lambda$ but arbitrary Matsubara frequency $\omega_{n}$. This is different from the RG which we defined earlier, but it is still perfectly well-defined. In the evaluation of diagrams, the internal momenta are restricted to the shell $e^{-\ell} \Lambda q<\Lambda$, but the Matsubara frequencies can run from $n=-\infty$ to $n=\infty$. However, in the rescaling step, we rescale both momenta and frequencies:

$$
\begin{align*}
\vec{q} & \rightarrow \vec{q} e^{-\ell} \\
\omega_{n} & \rightarrow \omega_{n} e^{-\ell} \tag{12.86}
\end{align*}
$$

The second equation means that the temperature is rescaled:

$$
\begin{equation*}
\beta \rightarrow \beta e^{\ell} \tag{12.87}
\end{equation*}
$$

$n_{i}$ must also be rescaled:

$$
\begin{equation*}
n_{i} \rightarrow n_{i} e^{-\ell \zeta} \tag{12.88}
\end{equation*}
$$

In the problem set, you will compute the one-loop RG equation for $g$. It is:


Figure 12.6: The flow diagram of an antiferromagnet in $d>2$.

$$
\begin{equation*}
\frac{d g}{d \ell}=(1-d) g+\frac{1}{2}\left(\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)}\right) g^{2} \operatorname{coth} \frac{\beta}{2} \tag{12.89}
\end{equation*}
$$

$\beta$ changes trivially since it is only affected by the rescaling.

$$
\begin{equation*}
\frac{d \beta}{d \ell}=-\beta \tag{12.90}
\end{equation*}
$$

Hence, if we define the parameter $t=g / \beta$, we can write a scaling equation for $t$ :

$$
\begin{align*}
\frac{d}{d \ell}\left(\frac{g}{\beta}\right) & =\frac{1}{\beta} \frac{d g}{d \ell}+g \frac{d}{d \ell}\left(\frac{1}{\beta}\right) \\
& =\frac{1}{\beta}\left((1-d) g+\frac{1}{2} \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} g^{2} \operatorname{coth} \frac{\beta}{2}\right)+\frac{g}{\beta} \\
& =(2-d) \frac{g}{\beta}+\frac{1}{2} \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} g \frac{g}{\beta} \operatorname{coth} \frac{\beta}{2} \tag{12.91}
\end{align*}
$$

In other words, we can rewrite the RG equation for $g$ and the trivial rescaling for $\beta$ as the two equations:

$$
\begin{align*}
\frac{d g}{d \ell} & =(1-d) g+\frac{1}{2}\left(\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)}\right) g^{2} \operatorname{coth} \frac{g}{2 t} \\
\frac{d t}{d \ell} & =(2-d) t+\frac{1}{2}\left(\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)}\right) g t \operatorname{coth} \frac{g}{2 t} \tag{12.92}
\end{align*}
$$



Figure 12.7: The flow diagram of an antiferromagnet in $d=2$.

At zero temperature, $t=0$, the first equation shows that there is a stable fixed point at $g^{*}=0$ for $d>1$. This is the antiferromagnetically ordered phase. For $t, g$ small, the system flows into the $g^{*}=t^{*}=0$ fixed point. We will discuss the basin of attraction of this fixed point below.

There is an unstable fixed point at

$$
\begin{equation*}
g_{c}=\frac{(d-1)(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \tag{12.93}
\end{equation*}
$$

For $g>g_{c}, g$ flows to $g=\infty$. At this fixed point, the antiferromagnet is disordered by quantum fluctuations. Such fixed points are called quantum critical points. $g$ can be varied by introducing a next-neighbor coupling $J^{\prime}$ which frustrates the nearestneighbor coupling $J$. Increasing $J^{\prime}$ increases $g$.

At finite temperature, $t>0$, there is a fixed point for $d>2$ at,

$$
\begin{align*}
& g^{*}=0 \\
& t^{*}=t_{c}=\frac{(d-2)(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)}{2 \pi^{\frac{d}{2}}} \tag{12.94}
\end{align*}
$$

For $d \leq 2$, there is no fixed point at finite temperature; all flows go to $t=\infty$. The flow diagrams are shown in figures 12.6 and 12.7.

The region underneath the dark line in figure 12.6 is the antiferromagnetically ordered phase controlled by the $g^{*}=0, t^{*}=0$ fixed point. For given $g$-i.e. for a given system - there is a range of $t$ for which the system is antiferromagnetically ordered. This range of $t$ translates into a range of temperatures, $0<T<T_{c}$. For $g \rightarrow 0, T_{c} \rightarrow \infty$.

At both the zero and finite-temperature fixed points, the correlation functions exhibit power-law decay. As these fixed points are approached, the correlation length diverges. In the zero-temperature case, the divergence is:

$$
\begin{equation*}
\xi \sim\left|g-g_{c}\right|^{-\nu_{d+1}} \tag{12.95}
\end{equation*}
$$

while, at finite temperture, it is:

$$
\begin{equation*}
\xi \sim\left|t-t_{c}\right|^{-\nu_{d}} \tag{12.96}
\end{equation*}
$$

In the problem set, you will calculate $\nu_{d}$.
At the finite-temperature critical point, the correlation functions have power-law decay:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}) n_{j}(0)\right\rangle \sim \frac{1}{x^{d-2+\eta_{d}}} \delta_{i j} \tag{12.97}
\end{equation*}
$$

while at the zero-temperature critical point, they decay as:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}, \tau) n_{j}(0)\right\rangle \sim \frac{1}{\left(x^{2}+\tau^{2}\right)^{\left(d-1+\eta_{d+1}\right) / 2}} \delta_{i j} \tag{12.98}
\end{equation*}
$$

In the problem set, you will calculate $\eta_{d}$.
To summarize, in $d>2$, an antiferromagnet described by the $O(3) \mathrm{NL} \sigma \mathrm{M}$ exhibits the following physics:

- An antiferromagnetic phase controlled by the $g^{*}=t^{*}=0$ fixed point. This phase is characterized by an $O(3)$ symmetry which is spontaneously broken to $U(1)$,

$$
\begin{equation*}
\langle\vec{n}\rangle \neq 0 \tag{12.99}
\end{equation*}
$$

yielding two Goldstone modes.

- A zero-temperature quantum critical point $g^{*}=g_{c}, t^{*}=0$ characterized by power-law correlation functions:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}, \tau) n_{j}(0)\right\rangle \sim \frac{1}{\left(x^{2}+\tau^{2}\right)^{\left(d-1+\eta_{d+1}\right) / 2}} \delta_{i j} \tag{12.100}
\end{equation*}
$$

- A zero-temperature paramagnetic phase controlled by a fixed point at $g^{*}=\infty$, $t^{*}=0$ and characterized by exponentially-decaying correlation functions:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}, \tau) n_{j}(0)\right\rangle \sim \frac{e^{-|x| / \xi}}{\left(x^{2}+\tau^{2}\right)^{(d-1) / 2}} \delta_{i j} \tag{12.101}
\end{equation*}
$$

As $g_{c}$ is approached at $t=0$, the correlation length diverges as:

$$
\begin{equation*}
\xi \sim\left|g-g_{c}\right|^{-\nu_{d+1}} \tag{12.102}
\end{equation*}
$$

- A finite-temperature critical point at $t^{*}=t_{c}, g^{*}=0$ characterized by power-law correlation functions:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}) n_{j}(0)\right\rangle \sim \frac{1}{x^{d-2+\eta_{d}}} \delta_{i j} \tag{12.103}
\end{equation*}
$$

Near 4 dimensions, this critical point can can be studied with an $O(3)\left(\varphi_{a} \varphi_{a}\right)^{2}$ theory using a an $\epsilon=4-d$ expansion.

- A finite-temperature paramagnetic phase controlled by a fixed point at $t^{*}=\infty$, $g^{*}=0$ and characterized by exponentially-decaying correlation functions:

$$
\begin{equation*}
\left\langle n_{i}(\vec{x}) n_{j}(0)\right\rangle \sim \frac{e^{-|x| / \xi}}{x^{d-2}} \delta_{i j} \tag{12.104}
\end{equation*}
$$

As $t \rightarrow t_{c}$, the correlation length diverges as;

$$
\begin{equation*}
\xi \sim\left|t-t_{c}\right|^{-\nu_{d}} \tag{12.105}
\end{equation*}
$$

In $d=2$, we have an antiferromagnetic phase only at $T=0$ for $0<g<g_{c}$. The system is paramagnetic in the rest of the phase diagram.
(a)


(b)


Figure 12.8: (a) $O(1)$ diagrams and (b) $O(1 / N)$ diagrams with two external legs.

The calculations of this section and the problem set are all to lowest order in $d-1$ at zero-temperature and $d-2$ at finite temperature. In the next section, we will generalize the $O(3) \mathrm{NL} \sigma \mathrm{M}$ to the $O(N) \mathrm{NL} \sigma \mathrm{M}$ and derive the RG equations to lowest order in $1 / N$.

### 12.10 Large $N$

Suppose we generalize $\varphi^{4}$ theory to

$$
\begin{equation*}
S=\int d^{d} x\left(\frac{1}{2} \nabla \varphi_{a} \nabla \varphi_{a}+\frac{1}{2} r \varphi_{a} \varphi_{a}+\frac{1}{N} \frac{u}{8}\left(\varphi_{a} \varphi_{a}\right)^{2}\right) \tag{12.106}
\end{equation*}
$$

where $a=1,2, \ldots, N$. For $N=1$, this theory has the $Z_{2}$ symmetry of the Ising model. For $N=2$, the theory has the $O(2)=U(1)$ symmetry of ${ }^{4} \mathrm{He}$ or an $X Y$ magnet. For arbitrary $N$, the action has $O(N)$ symmetry. The RG equations simplify for $N \rightarrow \infty$ as we now show.

Let's classify diagrams according to powers of $N$. Each vertex gets a factor of $1 / N$. Every time we sum over an index $a$, we get a factor of $N$. First, let's consider the diagrams with two external legs. These diagrams renormalize $r$ (and, possibly, $\zeta)$. Figure 12.8a contains some $O(1)$ two-leg diagrams.
(a)

(b)


Figure 12.9: (a) Some $O(1 / N)$ diagrams and (b) an $O\left(1 / N^{2}\right)$ diagram with four external legs.

Let's now turn to the diagrams with 4 external legs. Figure 12.9a contains some $O(1 / N)$ diagrams with 4 external legs. Other diagrams, such as that of figure 12.9 b are down by powers of $1 / N$.

To organize the diagrams in powers of $1 / N$, it is useful to perform a HubbardStratonovich transformation. We introduce a field $\sigma$ and modify the action by adding a term:

$$
\begin{equation*}
S \rightarrow S-\frac{N}{2 u}\left(\sigma-\frac{u}{2 N} \varphi_{a} \varphi_{a}\right)^{2} \tag{12.107}
\end{equation*}
$$

Since the action is quadratic in $\sigma$, we could integrate out $\sigma$ without affecting the functional integral for $\varphi_{a}$. However, it is also possible to expand the square, which leads to the action:

$$
\begin{equation*}
S=\int d^{d} x\left(\frac{1}{2} \nabla \varphi_{a} \nabla \varphi_{a}+\frac{1}{2} r \varphi_{a} \varphi_{a}-\frac{N}{2 u} \sigma^{2}+\frac{1}{2} \sigma \varphi_{a} \varphi_{a}\right) \tag{12.108}
\end{equation*}
$$

Notice that integrating out $\sigma$ restores the $\left(\varphi_{a} \varphi_{a}\right)^{2}$ term.
This is now a quadratic action for $\varphi_{a}$. Hence, we can integrate out $\varphi_{a}$ :

$$
\begin{equation*}
S_{\mathrm{eff}}[\sigma]=\int \frac{d^{d} x}{(2 \pi)^{d}}\left(-\frac{N}{2 u} \sigma^{2}+N \operatorname{Tr} \ln \left(\nabla^{2}+r+\sigma\right)\right) \tag{12.109}
\end{equation*}
$$

Dropping a constant, the logarithm can be expanded to give:

$$
\begin{align*}
S_{\mathrm{eff}}[\sigma]= & -\int \frac{d^{d} p}{2 \pi} \frac{N}{2 u} \sigma(p) \sigma(-p) \\
& +N \sum_{n} \frac{(-1)^{n}}{n} \int\left(\prod_{i=1}^{n} \frac{d^{d} p_{i}}{2 \pi} \frac{d^{d} k_{i}}{2 \pi} \frac{1}{k_{i}^{2}+r}\right) \sigma\left(p_{1}\right) \ldots \sigma\left(p_{n}\right) \prod \delta\left(p_{i}+k_{i}-k_{i+1}\right) \tag{12.110}
\end{align*}
$$

Since there is a factor of $N$ in front of $S_{\text {eff }}[\sigma]$, each $\sigma$ propagator carries a $1 / N$, while each vertex carries an $N$. Hence, a diagram goes as $N^{V-I-E}=N^{-E+1-L}$. The lowest order in $1 / N$ for the $E$-point $\sigma$ correlation function is a tree-level in $S_{\text {eff }}[\sigma]$ diagram. To compute the $\varphi_{a} k$-point correlation function, we need to compute a diagram with $k / 2$ external $\sigma$ legs. Hence, the lowest order in $1 / N$ contribution to the $\varphi_{a}$ two-point correlation function is obtained from the $\sigma$ one-point correlation function (which is determined by the diagram obtained by joining the one-point function to one end of the two-point function). It is $O(1)$ and it is given by the graph of figure ??a. The lowest order in $1 / N$ contribution to the $\varphi_{a}$ four-point correlation function is obtained from the $\sigma$ two-point correlation function. It is $O(1 / N)$ and it is given by the graphs of figure ??b.

Since the $\sigma$ one-point function is:

$$
\begin{equation*}
N \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}+r} \tag{12.111}
\end{equation*}
$$

while the $\sigma$ two-point function at zero momentum is

$$
\begin{equation*}
\left[\frac{N}{2 u}-N \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left(q^{2}+r\right)^{2}}\right]^{-1} \tag{12.112}
\end{equation*}
$$

we have:

$$
\begin{align*}
r+d r & =b^{-2}\left(r+\frac{N \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}+r}}{\frac{N}{2 u}-N \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left(q^{2}+r\right)^{2}}}\right) \\
u+d u & =b^{d-4}\left(\frac{N}{2 u}-3 N \int_{b}^{1} \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left(q^{2}+r\right)^{2}}\right)^{-1} \tag{12.113}
\end{align*}
$$

Differentiating these equations, we obtain the RG equations:

$$
\begin{align*}
\frac{d r}{d \ell} & =2 r-\frac{1}{16 \pi^{2}} u r+\ldots \\
\frac{d u}{d \ell} & =\epsilon u-\frac{3}{16 \pi^{2}} u^{2}+\ldots \tag{12.114}
\end{align*}
$$

In other words, the one-loop RG equations contain the same information as the geometric series of $O(1)$ and $O(1 / N)$ diagrams! In the $N \rightarrow \infty$ limit, the one-loop RG equations are valid even when $\epsilon$ is not small.

We can also consider the $O(N)$ generalization of the $\mathrm{NL} \sigma \mathrm{M}$ :

$$
\begin{equation*}
\int \mathcal{D} \vec{n} e^{-\frac{1}{g} \int d^{d} \vec{x} \int_{0}^{\beta} d \tau\left(\partial_{\mu} \vec{n}\right)^{2}} \tag{12.115}
\end{equation*}
$$

where $\vec{n}$ is an $N$ component vector. Imposing the constraint $\vec{n}^{2}=1$ with a Lagrange multiplier, we have:

$$
\begin{equation*}
S=\frac{1}{2 g} \int d^{d} \vec{x} \int_{0}^{\beta} d \tau\left((\nabla \vec{n})^{2}+\lambda\left(n^{2}-1\right)\right) \tag{12.116}
\end{equation*}
$$

Integrating out $\vec{n}$, we have:

$$
\begin{equation*}
S=\frac{1}{2 g} \int d^{d} \vec{x} \int_{0}^{\beta} d \tau\left(\frac{1}{2 g} \lambda\left(n^{2}-1\right)+\frac{1}{2} N \operatorname{Tr} \ln \left(-\nabla^{2}+\lambda(\vec{x})\right)\right) \tag{12.117}
\end{equation*}
$$

In the $N \rightarrow \infty$ limit, the saddle-point approximation becomes exact, so:

$$
\begin{equation*}
N \sum_{n} \int \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega_{n}^{2}+q^{2}+\lambda}=\frac{1}{g} \tag{12.118}
\end{equation*}
$$

Let's specialize to the case $T=0$ :

$$
\begin{equation*}
N \int^{\Lambda} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda}=\frac{1}{g} \tag{12.119}
\end{equation*}
$$

This integral equation can be solved using the RG transformation. First, we integrate out momenta $b \Lambda<|q|<\Lambda$, assuming that $\lambda \ll \Lambda^{2}$ :

$$
\begin{equation*}
N \int^{b} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda}+N \int_{b}^{1} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda}=\frac{1}{g} \tag{12.120}
\end{equation*}
$$

or,

$$
\begin{equation*}
N \int^{b} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda}+N \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1}\left(1-b^{d-1}\right)=\frac{1}{g} \tag{12.121}
\end{equation*}
$$

If we bring the second term on the left-hand-side to the right-hand-side, we have:

$$
\begin{equation*}
N \int^{b} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda}=\frac{1}{g}-N \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1}\left(1-b^{d-1}\right) \tag{12.122}
\end{equation*}
$$

Rescaling the momenta in the integral, $q \rightarrow q b, \omega \rightarrow \omega b$ we have:

$$
\begin{equation*}
N \int^{1} \frac{d \omega}{2 \pi} \frac{d^{d} \vec{q}}{(2 \pi)^{d}} \frac{1}{\omega^{2}+q^{2}+\lambda b^{2}}=b^{d-1}\left(\frac{1}{g}-N \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1}\left(1-b^{d-1}\right)\right) \tag{12.123}
\end{equation*}
$$

In other words,

$$
\begin{equation*}
\frac{1}{g}+d\left(\frac{1}{g}\right)=b^{d-1}\left(\frac{1}{g}-N \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1}\left(1-b^{d-1}\right)\right) \tag{12.124}
\end{equation*}
$$

writing $b=e^{-d \ell}$, this gives:

$$
\begin{equation*}
\frac{d g}{d \ell}=(d-1) g-N \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} g^{2} \tag{12.125}
\end{equation*}
$$

Again, the large- $N$ RG equation is essentially a one-loop RG equation.
As we will see again in the context of interacting fermions, the large- $N$ limit is one in which RG equations can be calculated with minimum fuss.

### 12.11 The Kosterlitz-Thouless Transition

We turn now to the RG analysis of an XY magnet or, equivalently, ${ }^{4} \mathrm{He}$ at zerotemperature in 1D

$$
\begin{equation*}
S=\frac{1}{2} \rho_{s} \int d \tau d x\left(\partial_{\mu} \theta\right)^{2} \tag{12.126}
\end{equation*}
$$

or at finite temperature in 2 D ,

$$
\begin{equation*}
S=\frac{\rho_{s}}{2 T} \int d^{2} x(\nabla \theta)^{2} \tag{12.127}
\end{equation*}
$$

We will use the notation

$$
\begin{equation*}
S=\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta\right)^{2} \tag{12.128}
\end{equation*}
$$

to encompass both cases. This is an $O(2)$ non-linear sigma model with $K=1 / g$. Clearly, $d K / d \ell=0$ to all orders in $K$.

The two-point correlation function of the order parameter may be calculated using:

$$
\begin{equation*}
\left\langle e^{i \theta(x)} e^{-i \theta(0)}\right\rangle=\int \mathcal{D} \theta e^{\int d^{2} x\left(\frac{1}{2} K\left(\partial_{\mu} \theta\right)^{2}+J(x) \theta(x)\right)} \tag{12.129}
\end{equation*}
$$

where

$$
\begin{equation*}
J(y)=i \delta(y-x)-i \delta(y) \tag{12.130}
\end{equation*}
$$

Hence it is given by,

$$
\begin{align*}
\left\langle e^{i \theta(x)} e^{-i \theta(0)}\right\rangle & =e^{\frac{1}{2} \int d^{2} x d^{2} x^{\prime} J(x) G\left(x-x^{\prime}\right) J\left(x^{\prime}\right)} \\
& =e^{G(x)-G(0)} \tag{12.131}
\end{align*}
$$

Now,

$$
\begin{align*}
G(x)-G(0) & =\langle\theta(x) \theta(0)\rangle-\langle\theta(0) \theta(0)\rangle \\
& =\int \frac{d^{2} q}{(2 \pi)^{2}}\left(e^{i q \cdot x}-1\right) \frac{1}{K} \frac{1}{q^{2}} \\
& =-\frac{1}{K} \int_{\frac{1}{|x|}}^{\Lambda} \frac{d^{2} q}{(2 \pi)^{2}} \frac{1}{q^{2}} \\
& =-\frac{1}{2 \pi K} \ln |x / a| \tag{12.132}
\end{align*}
$$

where $a=1 / \Lambda$ is a short-distance cutoff. Hence,

$$
\begin{equation*}
\left\langle e^{i \theta(x)} e^{-i \theta(0)}\right\rangle=\frac{1}{|x|^{\frac{1}{2 \pi K}}} \tag{12.133}
\end{equation*}
$$

Similarly if $\sum_{i} n_{i}=0$,

$$
\begin{equation*}
\left\langle e^{i \sum_{i} n_{i} \theta\left(x_{i}\right)}\right\rangle=e^{-\frac{1}{2 \pi K} \sum_{i, j} n_{i} n_{j} \ln \left|x_{i}-x_{j}\right|} \tag{12.134}
\end{equation*}
$$

In other words, the correlation function has the form of the Boltzmann weight for a Coulomb gas.

Thus far, we have neglected the periodicity of $\theta$, i.e. the fact that $0<\theta<2 \pi$. However, for $|x|$ large,

$$
\begin{equation*}
e^{-\frac{1}{2}\left\langle(\theta(x)-\theta(0))^{2}\right\rangle}=\frac{1}{|x|^{\frac{1}{2 \pi K}}} \tag{12.135}
\end{equation*}
$$

tells us that $(\theta(x)-\theta(0))^{2}$ becomes large for $|x|$ large. This means that $\theta$ must wind around $2 \pi$, i.e. that there are vortices.

A vortex is a singular configuration of the field $\theta(x)$ such that the vector field $\partial_{\mu} \theta(x)$ twists around an integer number, $n$, times as the vortex is encircled. In the context of ${ }^{4} \mathrm{He}$, a vortex is a swirl of current. In an $X Y$ magnet, it is a point about which the spins rotate. In other words,

$$
\begin{equation*}
\oint_{P} \partial_{\mu} \theta d x_{\mu}=2 \pi n \tag{12.136}
\end{equation*}
$$

for any path, $P$, which encloses the vortex. $n$ is the winding number of the vortex.
We can understand this qualitatively by calculating the contribution of a vortex configuration to the functional integral. If there is a vortex at the origin with winding number $n$, then (12.136) implies that

$$
\begin{equation*}
\partial_{\mu} \theta \sim \frac{2 \pi n}{r} \tag{12.137}
\end{equation*}
$$

So the action of a vortex at the origin is:

$$
\begin{align*}
A_{v} & =\frac{K}{2} \int\left(\frac{2 \pi n}{r}\right)^{2} d^{2} r \\
& =\pi K n^{2} \ln \frac{R}{a}+E_{c} \tag{12.138}
\end{align*}
$$

where $R$ is the size of the system, $a$ is the size of the core of the vortex and $E_{c}$ is the core energy. Meanwhile a vortex-anti-vortex pair separated by a distance $r$ has energy

$$
\begin{equation*}
A_{\mathrm{pair}}=\pi K \ln \frac{r}{a}+2 E_{c} \tag{12.139}
\end{equation*}
$$

To calculate the contribution of a vortex to the functional integral, we must take into account the fact that the vortex can be placed anywhere in the system. Hence, the contribution to the functional integral is proportional to the area of the system:

$$
\begin{align*}
Z_{v} & \sim\left(\frac{R}{a}\right)^{2} e^{-A_{v}} \\
& \sim e^{(2-\pi K) \ln \frac{R}{a}} \tag{12.140}
\end{align*}
$$

For $K<2 / \pi$, this is a large contribution, so vortices can proliferate. The proliferation of vortices destroys the power-law correlation functions.

Let us now study this transition more systematically. We break $\theta$ into a smooth piece, $\theta_{s}$, and a piece that contains the vortices $\theta_{V}$,

$$
\begin{equation*}
\theta_{V}(x)=\sum_{i} n_{i} \arctan \left(\frac{\left(x-x_{i}\right)_{2}}{\left(x-x_{i}\right)_{1}}\right) \tag{12.141}
\end{equation*}
$$

where the $i^{\text {th }}$ vortex has winding number $n_{i}$ and position $x_{i}$. Using

$$
\begin{align*}
\partial_{\mu} \theta_{V}(x) & =\sum_{i} n_{i} \epsilon_{\mu \nu} \frac{\left(x-x_{i}\right)_{\mu}}{\left(x-x_{i}\right)^{2}} \\
& =\sum_{i} n_{i} \epsilon_{\mu \nu} \partial_{\mu} \ln \left|x-x_{i}\right| \tag{12.142}
\end{align*}
$$

we can rewrite the action as (the cross term between $\theta_{s}$ and $\theta_{V}$ vanishes upon integration by parts)

$$
\begin{aligned}
S & =\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta\right)^{2} \\
& =\frac{1}{2} K \int d^{2} x\left(\partial_{\mu}\left(\theta_{s}+\theta_{V}\right)\right)^{2} \\
& =\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{s}\right)^{2}+\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{V}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
& =\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{s}\right)^{2}+\frac{1}{2} K \int d^{2} x \sum_{i, j} n_{i} n_{j} \partial_{\mu} \ln \left|x-x_{i}\right| \partial_{\mu} \ln \left|x-x_{j}\right| \\
& =\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{s}\right)^{2}-2 \pi K \sum_{i, j} n_{i} n_{j} \ln \left|x_{i}-x_{j}\right|+n_{v} E_{c} \tag{12.143}
\end{align*}
$$

In the last line, we have restored the core energies of the $n_{v}$ vortices.
Hence, the partition function is:

$$
\begin{align*}
\mathcal{Z}= & \int \mathcal{D} \theta_{s} e^{-\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{s}\right)^{2}} \times \\
& \sum_{n_{v}=0}^{\infty} e^{-n_{v} E_{c}} \sum_{i=1}^{n_{v}} \sum_{n_{i}=0, \pm 1, \pm 2, \ldots} \int \prod_{i} \frac{1}{N_{1}!N_{-1}!N_{2}!N_{-2}!\ldots} d^{2} x_{i} e^{-\pi K \sum_{i, j} n_{i} n_{j} \ln \left|x_{i}-x_{j}\right|} \tag{12.144}
\end{align*}
$$

where $N_{k}$ is the number of vortices of strength $k$ in a given configuration.
Observe that (12.134) implies that this can be rewritten as:

$$
\begin{equation*}
\mathcal{Z}[\phi]=\int \mathcal{D} \theta_{s} \mathcal{D} \phi e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+\sum_{m} y_{m} \cos m \phi\right)} e^{-\frac{1}{2} K \int d^{2} x\left(\partial_{\mu} \theta_{s}\right)^{2}} \tag{12.145}
\end{equation*}
$$

where $y_{m}=e^{-n_{v} E_{c}}$ is the vortex fugacity. The perturbative expansion of $\mathcal{Z}[\phi]$ function is the sum over all vortex configurations of $\mathcal{Z}[\theta]$. Expanding perturbatively in the $y_{i}$ 's and using (12.134), we have:
$\mathcal{Z}[\phi]=\sum_{n_{v}=0}^{\infty} e^{-n_{v} E_{c}} \sum_{i=1}^{n_{v}} \sum_{n_{i}=0, \pm 1, \pm 2, \ldots} \int \prod_{i} \frac{y_{1}^{N_{1}+N_{-1}} y_{2}^{N_{2}+N_{-2}} \cdots}{N_{1}!N_{-1}!N_{2}!N_{-2}!\ldots} d^{2} x_{i} e^{-\pi K \sum_{i, j} n_{i} n_{j} \ln \left|x_{i}-x_{j}\right|}$

Integrating out $\theta_{s}$, we are left with:

$$
\begin{equation*}
\mathcal{Z}=\int \mathcal{D} \phi e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+\sum_{m} y_{m} \cos m \phi\right)} \tag{12.147}
\end{equation*}
$$

Notice that we have transformed the partition function for the vortices in the field $\theta$ into the partition function for another scalar field, $\phi$. This is an example of a duality transformation. The action for $\phi$ is a sine-Gordon model. Let us consider the $\cos m \phi$ term in the action:

$$
\begin{equation*}
S=\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+\sum_{m} y_{m} \cos m \phi\right) \tag{12.148}
\end{equation*}
$$

Is this term relevant or irrelavant? At $y_{m}=0$, we can determine the first term in the RG equation for $y_{m}$ from its scaling dimension. This can be determined from the correlation function:

$$
\begin{equation*}
\langle\cos \theta(x) \cos \theta(0)\rangle \sim \frac{1}{|x|^{2 \pi m K}} \tag{12.149}
\end{equation*}
$$

which tells us that $\cos m \phi$ has dimension $\pi m K$. Hence, the RG equation for $y$ is:

$$
\begin{equation*}
\frac{d y_{m}}{d \ell}=(2-\pi m K) y_{m}+\ldots \tag{12.150}
\end{equation*}
$$

Consequently, $y \equiv y_{1}$ is the most relevant operator. As $K$ is decreased, $y_{1}$ becomes relevant first - i.e. at $K=2 / \pi$. Let us, therefore, focus on the action with only the $m=1$ term:

$$
\begin{equation*}
S=\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+y \cos \phi\right) \tag{12.151}
\end{equation*}
$$

In order to study the flow of $K$ resulting from the presence of $y$, let us expand the functional integral perturbatively in $y$.

$$
\begin{aligned}
\mathcal{Z}= & \int \mathcal{D} \phi e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+y \cos \phi\right)} \\
= & \ldots+\int \mathcal{D} \phi\left(\frac{y^{2}}{2} \int d^{2} x \cos \phi(x) \int d^{2} y \cos \phi(y)\right) e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}\right)}+\ldots \\
= & \ldots+\int \mathcal{D} \phi\left[\left(\frac{y^{2}}{2} \int_{|x-y|>1 / b \Lambda} d^{2} x \cos \phi(x) \int d^{2} y \cos \phi(y)\right)+\right. \\
& \left.\left(y^{2} \int_{1 / \Lambda<|x-y|<1 / b \Lambda} d^{2} x \int d^{2} y \frac{1}{2} e^{i \phi(x)} e^{-i \phi(y)}\right)\right] e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}\right)}+\ldots \\
= & \ldots+\int \mathcal{D} \phi\left(y^{2} \int_{1 / \Lambda<|x-y|<1 / b \Lambda} d^{2} x \int d^{2} y \frac{1}{2} \frac{1}{(x-y)^{2 \pi K}} e^{i \phi(x)-i \phi(y)}\right) \\
& \times e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}(\partial \mu \phi)^{2}+y \cos \phi\right)}+\ldots \\
= & \ldots+\int \mathcal{D} \phi\left(y^{2} \int_{1 / \Lambda<|x-y|<1 / b \Lambda} d^{2} x \int d^{2} y \frac{1}{2} \frac{1}{(x-y)^{2 \pi K}}\right. \\
& \left.\times\left(1+i(x-y) \partial \phi(y)-\frac{1}{2}(x-y)^{2}(\partial \phi(y))^{2}\right)\right) e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+y \cos \phi\right)}+\ldots \\
& \ldots+\int \mathcal{D} \phi\left(-\frac{1}{4} y^{2}\left(\int_{1 / \Lambda<|x-y|<1 / b \Lambda} d^{2} x \frac{1}{\left.(x-y)^{2 \pi K-2}\right)}\right.\right. \\
& \left.\times\left(\int d^{2} y(\partial \phi(y))^{2}\right)\right) e^{-\int d^{2} x\left(\frac{1}{8 \pi^{2} K}\left(\partial_{\mu} \phi\right)^{2}+y \cos \phi\right)}+\ldots
\end{aligned}
$$

$$
\begin{equation*}
=\ldots+\int \mathcal{D} \phi e^{-\int d^{2} x\left[\left(\frac{1}{8 \pi^{2} K}+\frac{\pi}{2} y^{2} \ln b\right)\left(\partial_{\mu} \phi\right)^{2}+y \cos \phi\right]} \tag{12.152}
\end{equation*}
$$

In the last line, we have done the integral at $\pi K=2$ (since we are interested in the vicinity of the transition) where it is logarithmic, and re-exponentiated the result. Hence, $K^{-1}$ flows as a result of $y$ :

$$
\begin{equation*}
\frac{d}{d \ell} K^{-1}=4 \pi^{3} y^{2}+O\left(y^{4}\right) \tag{12.153}
\end{equation*}
$$

Together with the flow equation for $y$,

$$
\begin{equation*}
\frac{d y}{d \ell}=(2-\pi K) y+O\left(y^{3}\right) \tag{12.154}
\end{equation*}
$$

these RG equations determine the physics of an XY model in 2 dimensions at finite temperature or in 1 dimension at zero temperature. These equations may be analyzed by defining $u=\pi K-2$ and $v=4 \pi y$, in terms of which the RG equations are:

$$
\begin{align*}
& \frac{d u}{d \ell}=-v^{2}+O\left(u v^{2}\right) \\
& \frac{d v}{d \ell}=-u v+O\left(v^{3}\right) \tag{12.155}
\end{align*}
$$

Observe that $u^{2}-v^{2}$ is an RG invariant to this order:

$$
\begin{equation*}
\frac{d}{d \ell}\left(u^{2}-v^{2}\right)=0 \tag{12.156}
\end{equation*}
$$

Hence, the RG trajectories in the vicinity of $K=K_{c}=2 / \pi$ are hyperbolae which asymptote the lines $u= \pm v$. The resulting Kosterlitz-Thouless flow diagram is shown in figure 12.10

These RG flows feature a line of fixed points - or a fixed line $-y^{*}=0, K>K_{c}$. Any point below the the asymptote $u=v$ - or, equivalently, $\pi K-2=4 \pi y$ - flows into one of these fixed points. Correlation functions exhibit power-law falloff at these fixed points:

$$
\begin{equation*}
\left\langle e^{i \theta(x)} e^{-i \theta(0)}\right\rangle=\frac{1}{|x|^{\frac{1}{2 \pi K}}} \tag{12.157}
\end{equation*}
$$



Figure 12.10: The Kosterlitz-Thouless flow diagram.

The line $\pi K-2=4 \pi y$ which separates these power-law phases from the exponentially decaying phase is called the Kosterlitz-Thouless separatrix. At the critical point,

$$
\begin{equation*}
\left\langle e^{i \theta(x)} e^{-i \theta(0)}\right\rangle=\frac{1}{|x|^{\frac{1}{4}}} \tag{12.158}
\end{equation*}
$$

When the system is above the line $\pi K-2=4 \pi y$, it flows away to large $y$ : the system is disordered by the proliferation of vortices and has exponentially decaying correlation functions. Since the $\cos \phi$ term is relevant, it bounds the fluctuations of $\phi$, just as an $r \phi^{2}$ term would. In the problem set, you will show that as $K_{c}$ is approached from below, the correlation length diverges as:

$$
\begin{equation*}
\xi \sim e^{\frac{1}{\left(K_{c}-K\right)^{1 / 2}}} \tag{12.159}
\end{equation*}
$$

Hence, at finite temperature in $2 D$ or at zero-temperature in $1 D,{ }^{4} \mathrm{He}$ and $X Y$ magnets have a phase transition between a disordered phase and a power-law ordered 'phase'.

## Chapter 13

## Fermions

### 13.1 Canonical Anticommutation Relations

In the remainder of this course, we will be applying the field-theoretic techniques which we have developed to systems of interacting electrons. In order to do this, we will have to make a detour into formalism so that we can handle systems of fermions.

Let us first consider a system of non-interacting spinless fermions at chemical potential $\mu$. As in the case of ${ }^{4} \mathrm{He}$, we must modify the Hamiltonian by $H \rightarrow H-\mu N$. The action is the same as for a system of free bosons:

$$
\begin{equation*}
S=\int d \tau d^{3} \vec{x} \psi^{\dagger}\left(\frac{\partial}{\partial \tau}+\frac{1}{2 m} \nabla^{2}-\mu\right) \psi \tag{13.1}
\end{equation*}
$$

The difference is that we want the associated Fock space to be fermionic, i.e. we would like the Pauli exclusion principle to hold. This can be accomplished by imposing the canonical anticommutation relations.

$$
\begin{gather*}
\left\{\psi(\vec{x}, t), \psi^{\dagger}\left(x^{\prime}, t\right)\right\}=\delta\left(\vec{x}-\vec{x}^{\prime}\right)  \tag{13.2}\\
\left\{\psi(\vec{x}, t), \psi\left(\vec{x}^{\prime}, t\right)\right\}=\left\{\psi^{\dagger}(\vec{x}, t), \psi^{\dagger}\left(\vec{x}^{\prime}, t\right)\right\}=0 \tag{13.3}
\end{gather*}
$$

Performing a mode expansion

$$
\psi(x)=\int \frac{d^{3} \vec{k}}{(2 \pi)^{3 / 2}} c_{\vec{k}} e^{\xi_{k} \tau+i \vec{k} \cdot \vec{x}}
$$

$$
\begin{equation*}
\psi^{\dagger}(x)=\int \frac{d^{3} \vec{k}}{(2 \pi)^{3 / 2}} c_{\vec{k}}^{\dagger} e^{-\xi_{k} \tau-i \vec{k} \cdot \vec{x}} \tag{13.4}
\end{equation*}
$$

where $\xi_{k}=\epsilon_{k}-\mu=k^{2} / 2 m-\mu$, we see that the creation and annihilation operators satisfy:

$$
\begin{array}{r}
\left\{c_{\vec{k}}, c_{\vec{k}^{\prime}}^{\dagger}\right\}=\delta\left(\vec{k}-\vec{k}^{\prime}\right) \\
\left\{c_{\vec{k}}, c_{\vec{k}^{\prime}}\right\}=\left\{c_{\vec{k}}^{\dagger}, c_{\vec{k}^{\prime}}^{\dagger}\right\}=0 \tag{13.5}
\end{array}
$$

Hence, $\left(c_{k}^{\dagger}\right)^{2}=c_{k}^{2}=0$, i.e. no state can be doubly occupied.
The Green function is:

$$
\begin{align*}
\mathcal{G}(\vec{x}, \tau)= & \theta(\tau) \operatorname{Tr}\left\{e^{-\beta\left(H_{0}-\mu N\right)} \psi^{\dagger}(\vec{x}, \tau) \psi(0,0)\right\} \\
& -\theta(-\tau) \operatorname{Tr}\left\{e^{-\beta\left(H_{0}-\mu N\right)} \psi(0,0) \psi^{\dagger}(\vec{x}, \tau)\right\} \tag{13.6}
\end{align*}
$$

Note the -sign in the definition of the Green function. It is necessary because the fermions satisfy canonical anticommutation relations. You may verify that $\mathcal{G}$ as defined above satisfies:

$$
\begin{equation*}
\left(\frac{\partial}{\partial \tau}+\frac{1}{2 m} \nabla^{2}-\mu\right) \mathcal{G}(\vec{x}, \tau)=\delta(\tau) \delta(\vec{x}) \tag{13.7}
\end{equation*}
$$

As in the bosonic case, we find a further condition which follows from the cyclic property of the trace. Since $0<\tau, \tau^{\prime}<\beta$, it follows that $-\beta<\tau-\tau^{\prime}<\beta$. Now suppose that $\tau<\tau^{\prime}$. Then,

$$
\begin{align*}
\mathcal{G}\left(\tau-\tau^{\prime}<0\right) & =-\operatorname{Tr}\left\{e^{-\beta(H-\mu N)} e^{\tau^{\prime}(H-\mu N)} \psi\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime}(H-\mu N)} e^{\tau(H-\mu N)} \psi^{\dagger}(\vec{x}) e^{-\tau(H-\mu N)}\right\} \\
& =-\operatorname{Tr}\left\{e^{\tau(H-\mu N)} \psi^{\dagger}(\vec{x}) e^{-\tau(H-\mu N)} e^{-\beta(H-\mu N)} e^{\tau^{\prime}(H-\mu N)} \psi\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime}(H-\mu N)}\right\} \\
& =-\operatorname{Tr}\left\{e^{-\beta(H-\mu N)} e^{\beta(H-\mu N)} e^{\tau(H-\mu N)} \psi^{\dagger}(\vec{x}) e^{-\tau(H-\mu N)} e^{-\beta(H-\mu N)}\right. \\
& \left.\quad e^{\tau^{\prime}(H-\mu N)} \psi\left(\vec{x}^{\prime}\right) e^{-\tau^{\prime}(H-\mu N)}\right\} \\
& =-\mathcal{G}\left(\tau-\tau^{\prime}+\beta\right) \tag{13.8}
\end{align*}
$$

The first equality follows from the cyclic property of the trace. The final equality follows from the fact that $\tau-\tau^{\prime}+\beta>0$. Hence, a fermion Green function is antiperiodic in imaginary time.

As a result of antiperiodicity in imaginary-time, we can take the Fourier transform over the interval $[0, \beta]$ :

$$
\begin{equation*}
\mathcal{G}\left(i \epsilon_{n}\right)=\int_{0}^{\beta} d \tau e^{-i \epsilon_{n} \tau} \mathcal{G}(\tau) \tag{13.9}
\end{equation*}
$$

where the Matsubara frequencies $\epsilon_{n}$, are given by:

$$
\begin{equation*}
\epsilon_{n}=\frac{(2 n+1) \pi}{\beta} \tag{13.10}
\end{equation*}
$$

Inverting the Fourier transform, we have:

$$
\begin{equation*}
\mathcal{G}(\tau)=\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \epsilon_{n}\right) e^{i \epsilon_{n} \tau} \tag{13.11}
\end{equation*}
$$

Using the mode expansion and the Fermi-Dirac distribution,

$$
\begin{equation*}
\operatorname{Tr}\left\{e^{-\beta\left(H_{0}-\mu N\right)} c_{k}^{\dagger} c_{k}\right\}=n_{F}\left(\xi_{k}\right)=\frac{1}{e^{\beta \xi_{k}}+1} \tag{13.12}
\end{equation*}
$$

we can compute the propagator:

$$
\begin{align*}
\mathcal{G}(\vec{x}, \tau)= & \theta(\tau) \operatorname{Tr}\left\{e^{-\beta\left(H_{0}-\mu N\right)} \psi^{\dagger}(\vec{x}, \tau) \psi(0,0)\right\} \\
& -\theta(-\tau) \operatorname{Tr}\left\{e^{-\beta\left(H_{0}-\mu N\right)} \psi(0,0) \psi^{\dagger}(\vec{x}, \tau)\right\} \\
= & \int \frac{d^{3} \vec{k}}{(2 \pi)^{3} 2 \omega_{k}} e^{-i \vec{k} \cdot \vec{x} \cdot \xi_{k} \tau}\left(\theta(\tau) n_{F}\left(\xi_{k}\right)-\theta(-\tau)\left(1-n_{F}\left(\xi_{k}\right)\right)\right) \tag{13.13}
\end{align*}
$$

We can now compute the Fourier representation of the Green function:

$$
\begin{align*}
\mathcal{G}\left(\vec{p}, i \epsilon_{n}\right) & =\int d^{3} \vec{x} e^{i \vec{p} \cdot \vec{x}} \int_{0}^{\beta} d \tau e^{-i \epsilon_{n} \tau} \mathcal{G}(\vec{x}, \tau) \\
& =-\frac{n_{F}\left(\xi_{k}\right)\left(e^{\beta\left(-i \epsilon_{n}+\xi_{k}\right)}-1\right)}{-i \epsilon_{n}+\xi_{k}} \\
& =\frac{1}{i \epsilon_{n}-\xi_{k}} \tag{13.14}
\end{align*}
$$

### 13.2 Grassman Integrals

Fermionic systems can also be described by functional integrals. In order to do this, we will need the concept of a Grassman number. Grassman numbers are objects $\psi_{i}$
which can be multiplied together and anticommute under multiplication:

$$
\begin{equation*}
\psi_{i} \psi_{j}=-\psi_{i} \psi_{j} \tag{13.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{i}^{2}=0 \tag{13.16}
\end{equation*}
$$

Grassman numbers can be multiplied by complex numbers; multiplication by a complex number is distributive:

$$
\begin{equation*}
a\left(\psi_{1}+\psi_{2}\right)=a \psi_{1}+b \psi_{2} \tag{13.17}
\end{equation*}
$$

$\psi=\psi_{1}+i \psi_{2}$ and $\bar{\psi}=\psi_{1}-i \psi_{2}$ can be treated as independent Grassman variables,

$$
\begin{equation*}
\psi \bar{\psi}=-\bar{\psi} \psi \tag{13.18}
\end{equation*}
$$

Since the square of a Grassman number vanishes, the Taylor expansion of a function of Grassman variables has only two terms. For instance,

$$
\begin{equation*}
e^{\psi}=1+\psi \tag{13.19}
\end{equation*}
$$

Integration is defined for Grassman numbers as follows:

$$
\begin{align*}
\int d \psi & =0  \tag{13.20}\\
\int d \psi \psi & =1
\end{align*}
$$

Similarly,

$$
\begin{align*}
\int d \psi d \bar{\psi} & =0 \\
\int d \psi d \bar{\psi} \psi & =0 \\
\int d \psi d \bar{\psi} \bar{\psi} & =0  \tag{13.21}\\
\int d \psi d \bar{\psi} \bar{\psi} \psi & =1
\end{align*}
$$

As a result of the anticommutation, the order is important in the last line:

$$
\begin{equation*}
\int d \psi d \bar{\psi} \psi \bar{\psi}=-1 \tag{13.22}
\end{equation*}
$$

Since the square of a Grassman number vanishes, these rules are sufficient to define integration.

With these definitions, we can do Grassman integrals of Gaussians. Suppose $\theta_{i}$ and $\bar{\theta}_{i}$ are independent Grassman variables. Then

$$
\begin{align*}
\int d \theta_{1} d \bar{\theta}_{1} \ldots d \theta_{n} d \bar{\theta}_{n} e^{\sum_{i, j} \bar{\theta}_{i} A_{i j} \theta_{j}} & =\int d \theta_{1} d \bar{\theta}_{1} \ldots d \theta_{n} d \bar{\theta}_{n} \prod_{i, j} e^{\bar{\theta}_{i} A_{i j} \theta_{j}} \\
& =\int d \theta_{1} d \bar{\theta}_{1} \ldots d \theta_{n} d \bar{\theta}_{n} \prod_{i, j}\left(1+\bar{\theta}_{i} A_{i j} \theta_{j}\right) \\
& =\int d \theta_{1} d \bar{\theta}_{1} \ldots d \theta_{n} d \bar{\theta}_{n} \sum_{\sigma}\left(A_{1 \sigma(1)} A_{2 \sigma(2)} \ldots A_{n \sigma(n)}\right) \\
& \times \bar{\theta}_{1} \theta_{\sigma(1)} \ldots \bar{\theta}_{n} \theta_{\sigma(n)} \\
& =\sum_{\sigma}(-1)^{\sigma}\left(A_{1 \sigma(1)} A_{2 \sigma(2)} \ldots A_{n \sigma(n)}\right) \\
& =\operatorname{det}(A) \tag{13.23}
\end{align*}
$$

We can prove Wick's theorem for Grassman integrals:

$$
\begin{equation*}
Z\left(\eta_{i}, \bar{\eta}_{i}\right)=\int \prod_{i} d \theta_{i} d \bar{\theta}_{i} e^{\sum_{i, j} \bar{\theta}_{i} A_{i j} \theta_{j}+\sum_{i}\left(\overline{\bar{i}}_{i} \theta_{i}+\bar{\theta}_{i} \eta_{i}\right)} \tag{13.24}
\end{equation*}
$$

By making the change of variables,

$$
\begin{align*}
\theta_{i} & =\theta_{i}{ }^{\prime}+\left(A^{-1}\right)_{i j} \eta_{j} \\
\bar{\theta}_{i} & =\bar{\theta}_{i}{ }^{\prime}+\bar{\eta}_{j}\left(A^{-1}\right)_{j i} \tag{13.25}
\end{align*}
$$

we get

$$
\begin{align*}
Z\left(\eta_{i}, \bar{\eta}_{i}\right) & =\int \prod_{i} d \theta_{i}^{\prime} d \bar{\theta}_{i}^{\prime} e^{\sum_{i, j}\left(\bar{\theta}_{i}^{\prime} A_{i j} \theta_{j}^{\prime}+\bar{\eta}_{i}\left(A^{-1}\right)_{i j} \eta_{j}\right)} \\
& =\operatorname{det}(A) e^{\sum_{i, j} \bar{\eta}_{i}\left(A^{-1}\right)_{i j} \eta_{j}} \tag{13.26}
\end{align*}
$$

Hence,

$$
\left\langle\bar{\theta}_{i_{1}} \theta_{j_{1}} \ldots \bar{\theta}_{i_{k}} \theta_{j_{k}}\right\rangle=\frac{\int \prod_{i} d \theta_{i} d \bar{\theta}_{i} \bar{\theta}_{i_{1}} \theta_{j_{1}} \ldots \bar{\theta}_{i_{k}} \theta_{j_{k}} e^{\sum_{i, j} \bar{\theta}_{i} A_{i j} \theta_{j}}}{\int \prod_{i} d \theta_{i} d \bar{\theta}_{i} e^{\sum_{i, j} \bar{\theta}_{i} A_{i j} \theta_{j}}}
$$

$$
\begin{align*}
& =\frac{1}{\operatorname{det}(A)}\left(\frac{\partial}{\partial \eta_{i_{1}}} \frac{\partial}{\partial \bar{\eta}_{j_{1}}} \ldots \frac{\partial}{\partial \eta_{i_{k}}} \frac{\partial}{\partial \bar{\eta}_{j_{k}}} Z\left(\eta_{i}, \bar{\eta}_{i}\right)\right)_{\eta_{i}=\bar{\eta}_{i}=0} \\
& =\sum_{\sigma}(-1)^{\sigma}\left(A_{j_{\sigma(1) i_{1}}} A_{j_{\sigma(2)} i_{2}} \ldots A_{j_{\sigma}(k) i_{k}}\right) \\
& =\sum_{\sigma}(-1)^{\sigma}\left\{\left\langle\bar{\theta}_{i_{1}} \theta_{j_{\sigma(1)}}\right\rangle \ldots\left\langle\bar{\theta}_{i_{k}} \theta_{j_{\sigma(k)}}\right\rangle\right\} \tag{13.27}
\end{align*}
$$

In other words, we sum over all possible Wick contractions, multiplying by -1 every time the contraction necessitates a reordering of the fields by an odd permutation.

Thus far, we have considered finite-dimensional Grassman integrals. However, the generalization to functional Grassman integrals is straightforward.

### 13.3 Feynman Rules for Interacting Fermions

Let us now turn to a system of fermions with a $\delta$-function interaction. The grand canonical partition function is given by:

$$
\begin{equation*}
\mathcal{Z}=\mathcal{N} \int \mathcal{D} \psi \mathcal{D} \psi^{\dagger} e^{-S} \tag{13.28}
\end{equation*}
$$

where the functional integral is over all Grassman-valued functions which are antiperiodic in the interval $[0, \beta]$ (so that $\epsilon_{n}=(2 n+1) \pi / \beta$ ).

$$
\begin{equation*}
S=\int_{0}^{\beta} d \tau \int d^{d} x\left(\psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi+V \psi^{\dagger} \psi \psi^{\dagger} \psi\right) \tag{13.29}
\end{equation*}
$$

This action has the $U(1)$ symmetry $\psi \rightarrow e^{i \theta} \psi, \psi^{\dagger} \rightarrow e^{-i \theta} \psi^{\dagger}$. According to Noether's theorem, there is a conserved density,

$$
\begin{equation*}
\rho=\psi^{\dagger} \psi \tag{13.30}
\end{equation*}
$$

and current

$$
\begin{equation*}
\vec{j}=\psi^{\dagger} \frac{\nabla}{m} \psi-\left(\frac{\nabla}{m} \psi^{\dagger}\right) \psi \tag{13.31}
\end{equation*}
$$

satisfying the conservation law

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\vec{\nabla} \cdot \vec{j}=0 \tag{13.32}
\end{equation*}
$$

For $V=0$, this is the free fermion functional integral:

$$
\begin{equation*}
\int \mathcal{D} \psi \mathcal{D} \psi^{\dagger} e^{-\int_{0}^{\beta} d \tau \int d^{d} x \psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi}=\operatorname{det}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \tag{13.33}
\end{equation*}
$$

The Green function is:

$$
\begin{align*}
\mathcal{G}(\vec{x}, \tau) & =\mathcal{N} \int \mathcal{D} \psi \mathcal{D} \psi^{\dagger} \psi^{\dagger}(\vec{x}, \tau) \psi(0,0) e^{-\int_{0}^{\beta} d \tau \int d^{d} x \psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi} \\
& =\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right)^{-1} \tag{13.34}
\end{align*}
$$

The difference between this Green function and the Green function of a bosonic system with the same Hamiltonian is that this is the inverse of this operator on the space of functions with antiperiodic boundary conditions on $[0, \beta]$. The Fourier transform of the Green function is:

$$
\begin{equation*}
\mathcal{G}\left(\vec{k}, \epsilon_{n}\right)=\frac{1}{i \epsilon_{n}-\left(\frac{k^{2}}{2 m}-\mu\right)} \tag{13.35}
\end{equation*}
$$

In the presence of source fields $\eta, \bar{\eta}$,

$$
\begin{align*}
Z_{0}[\eta, \bar{\eta}] & =\mathcal{N} \int \mathcal{D} \psi \mathcal{D} \psi^{\dagger} e^{-\int_{0}^{\beta} d \tau \int d^{d} x \psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi+\eta \psi^{\dagger}+\psi \bar{\eta}} \\
& =e^{-\int_{0}^{\beta} d \tau d \tau^{\prime} \int d^{d} x d^{d} x^{\prime} \bar{\eta}(\vec{x}, \tau) \mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, \tau-\tau^{\prime}\right) \eta\left(\vec{x}^{\prime}, \tau^{\prime}\right)} \tag{13.36}
\end{align*}
$$

For interacting fermions, it is straightforward to generalize (8.31) to Grassman integrals, so that

$$
\begin{equation*}
Z[\eta, \bar{\eta}]=\frac{N}{N_{0}} e^{-\int \mathcal{L}_{i n t}\left(\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}\right)} Z_{0}[\eta, \bar{\eta}] \tag{13.37}
\end{equation*}
$$

In applying this formula, we must remember that $\eta$ and $\bar{\eta}$ are Grassman numbers so a - sign results every time they are anticommuted. As in the bosonic case, we can use (8.32) to rewrite this as:

$$
\begin{equation*}
Z[\eta, \bar{\eta}]=e^{-\int_{0}^{\beta} d \tau d \tau^{\prime} \int d^{d} x d^{d} x^{\prime} \frac{\delta}{\delta \psi^{\dagger}} \mathcal{G}\left(\vec{x}-\vec{x}^{\prime}, \tau-\tau^{\prime}\right) \frac{\delta}{\delta \psi}} e^{-\int \mathcal{L}_{i n t}\left(\psi^{\dagger}, \psi\right)+\eta \psi^{\dagger}+\psi \bar{\eta}} \tag{13.38}
\end{equation*}
$$

By expanding the

$$
\begin{equation*}
e^{-\int \mathcal{L}_{i n t}\left(\psi^{\dagger}, \psi\right)+\eta \psi^{\dagger}+\psi \bar{\eta}} \tag{13.39}
\end{equation*}
$$



Figure 13.1: The graphical representation of the fermion propagator and vertex.
we derive the following Feynman rules for fermions with $\delta$-function interactions. The lines of these Feynman diagrams have a direction which we denote by an arrow. Each vertex has two lines directed into it and two lines directed out of it. Momenta and Matrsubara frequencies are directed in the direction of the arrows. The propagator and vertex are shown in figure 13.1

- To each line, we associate a momentum, $\vec{p}$ and a Matsubara frequency, $\epsilon_{n}$.
- The propagator assigned to each internal line is:

$$
-\frac{1}{\beta} \sum_{n} \int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} \frac{1}{i \epsilon_{n}-\left(\frac{k^{2}}{2 m}-\mu\right)}
$$

- For each vertex with momenta, Matsubara frequencies $\left(\vec{p}_{1}, \epsilon_{n_{1}}\right),\left(\vec{p}_{2}, \epsilon_{n_{2}}\right)$ directed into the vertex and $\left(\vec{p}_{3}, \epsilon_{n_{3}}\right),\left(\vec{p}_{4}, \epsilon_{n_{4}}\right)$ directed out of the vertex, we write

$$
V(2 \pi)^{3} \delta\left(\vec{p}_{1}+\vec{p}_{2}-\vec{p}_{3}-\vec{p}_{4}\right) \beta \delta_{n_{1}+n_{2}, n_{3}+n_{4}}
$$

- Imagine labelling the vertices $1,2, \ldots, n$. Vertex $i$ will be connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$ by directed lines. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices $i, i$ is still connected to vertices $j_{1}, \ldots, j_{m}(m \leq 4)$ and to external momenta $p_{1}, \ldots, p_{4-m}$ by lines in the same direction. If $S$ is the
number of permutations which leave the diagram invariant, we assign a factor $1 / S$ to the diagram.
- If two vertices are connected by $l$ lines in the same direction, we assign a factor $1 / l$ ! to the diagram.
- To each closed loop, we assign a factor of -1 .

The final rule follows from the necessity of performing an odd number of anticommutations in order to contract fermion fields around a closed loop.

For $\mu<0$, (13.29) describes an insulating state. There is a gap to all excited states. For $V=0$, the gap is simply $-\mu$. In the problem set, you will compute the gap for $V \neq 0$ perturbatively.

For $\mu>0$, the ground state has a Fermi surface. For $V=0$, this Fermi surface is at $k_{F}=\sqrt{2 m \mu}$. In the problem set, you will compute the Fermi momentum for $V \neq 0$ perturbatively. Since this phase has gapless excitations, we must worry whether the interaction term is relevant. If the interactions are irrelevant, then we can perturbatively compute corrections to free fermion physics. If interactions are relevant, however, the system flows away from the free fermion fixed point, and we must look for other fixed points. Such an analysis is taken up in the next chapter, where we see the importance of the new feature that the low-energy excitations are not at $k=0$, but, rather, at $k=k_{F}$. We construct the renormalization group which is appropriate to such a situation, thereby arriving at Fermi liquid theory. First, however, we will investigate the two-point function of the interacting Fermi gas perturbatively.

Fermion lines with arrows that point to the right represent fermions above the Fermi surface. Those which point to the left represent holes below the Fermi surface. This is analogous to electrons and positrons in QED. However, unlike in QED, where a positron can have any momentum, fermions must have $k>k_{F}$ and holes must have
a.

b.


Figure 13.2: A one-loop diagram with an intermediate (a) particle-hole pair and (b) particle-particle pair.
$k<k_{F}$ at $T=0$ (at finite-temperature, this is smeared out by the Fermi function). Hence, the diagram of figure 13.2a corresponds to the expression

$$
\begin{equation*}
\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right) \tag{13.40}
\end{equation*}
$$

When $q<k_{f}$, the second Green function represents the propagation of a hole at $\vec{q}$ while the first Green function represents the propagation of a fermion at $\vec{p}+\vec{q}$. If $\vec{p}+\vec{q}$ isn't above the Fermi surface (smeared by the Fermi function), then this expression vanishes, as we will see shortly. Similarly, when $\vec{q}$ is above the Fermi surface, $\vec{p}+\vec{q}$ must be a hole below the Fermi surface. Meanwhile, the diagram of figure 13.2 b corresponds to the expression

$$
\begin{equation*}
\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}-i \epsilon_{n}, \vec{p}-\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right) \tag{13.41}
\end{equation*}
$$

where $\vec{q}$ and $\vec{p}-\vec{q}$ are now both fermions above the Fermi surface.

### 13.4 Fermion Spectral Function

Following our earlier derivation of the phonon spectral representation, we construct a spectral representation for the fermion two-point Green function. By inserting a complete set of intermediate states, $|m\rangle\langle m|$, we have,

$$
\begin{array}{r}
\mathcal{G}(\vec{x}, \tau)=\int d^{3} \vec{p} d \epsilon\left[\sum _ { n , m } \delta ( \vec { p } - \vec { p } _ { m } + \vec { p } _ { n } ) \delta ( \epsilon - \epsilon _ { n m } ) \left(\theta(\tau) e^{-i \vec{p} \cdot \vec{x}+\epsilon \tau} e^{-\beta E_{n}}\right.\right. \\
\left.\left.\left.-\theta(-\tau)) e^{i \vec{p} \cdot \vec{x}-\epsilon \tau} e^{-\beta E_{m}}\right)\left|\langle m| \psi^{\dagger}(0,0)\right| n\right\rangle\left.\right|^{2}\right] \tag{13.42}
\end{array}
$$

The Fourier transform,

$$
\begin{equation*}
\mathcal{G}\left(\vec{p}, i \epsilon_{j}\right)=\int d^{3} \vec{x} \int_{0}^{\beta} d \tau \mathcal{G}(\vec{x}, \tau) e^{-i \epsilon_{j} \tau} \tag{13.43}
\end{equation*}
$$

is given by:

$$
\begin{align*}
\mathcal{G}\left(\vec{p}, i \epsilon_{j}\right)= & {\left.\left[\sum_{n, m}\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right)\left|\langle m| \psi^{\dagger}(0,0)\right| n\right\rangle\right|^{2} } \\
& \left.\times \delta\left(\vec{p}-\vec{p}_{m}+\vec{p}_{n}\right) \delta\left(E-E_{m}+E_{n}\right)\right] \frac{1}{E-i \epsilon_{j}} \tag{13.44}
\end{align*}
$$

Writing

$$
\begin{equation*}
\left.A(\vec{p}, E)=\sum_{n, m}\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right)\left|\langle m| \psi^{\dagger}(0,0)\right| n\right\rangle\left.\right|^{2} \delta\left(\vec{p}-\vec{p}_{m}+\vec{p}_{n}\right) \delta\left(E-E_{m n}\right) \tag{13.45}
\end{equation*}
$$

we have the spectral representation of $\mathcal{G}$ :

$$
\begin{equation*}
\mathcal{G}\left(\vec{p}, i \epsilon_{n}\right)=\int_{-\infty}^{\infty} d E \frac{A(\vec{p}, E)}{E-i \epsilon_{j}} \tag{13.46}
\end{equation*}
$$

As usual, the spectral function $A(\vec{p}, E)$ is real and positive. It also satisfies the sum rule:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d E}{2 \pi} A(\vec{p}, E)=1 \tag{13.47}
\end{equation*}
$$

$\mathcal{G}$ is not analytic since it does not satisfy the Kramers-Kronig relations. However, the advanced and retarded correlation functions,

$$
G_{\mathrm{ret}}(\vec{p}, \epsilon)=\int_{-\infty}^{\infty} d E \frac{A(\vec{p}, E)}{E-\epsilon-i \delta}
$$



Figure 13.3: The spectral function in free and interacting Fermi systems.

$$
\begin{equation*}
G_{\mathrm{adv}}(\vec{p}, \epsilon)=\int_{-\infty}^{\infty} d E \frac{A(\vec{p}, E)}{E-\epsilon+i \delta} \tag{13.48}
\end{equation*}
$$

are analytic functions of $\epsilon$ in the upper- and lower-half-planes, respectively.
As usual, the spectral function is the difference between the retarded and advanced correlation functions.

$$
\begin{equation*}
G_{\mathrm{ret}}(\vec{p}, \epsilon)-G_{\mathrm{adv}}(\vec{p}, \epsilon)=2 \pi i A(\vec{p}, \epsilon) \tag{13.49}
\end{equation*}
$$

The spectral function of a free Fermi gas is a $\delta$-function:

$$
\begin{equation*}
A(\vec{p}, \epsilon)=\delta\left(\epsilon-\left(\frac{p^{2}}{2 m}-\mu\right)\right) \tag{13.50}
\end{equation*}
$$

In an interacting Fermi gas, the spectral weight is not concentrated in a $\delta$ function but spread out over a range of frequencies as in figure 13.3.

### 13.5 Frequency Sums and Integrals for Fermions

To compute fermion Green functions perturbatively, we will need to do summations over Matsubara frequencies. Sums over fermion Matsubara frequencies can be done
using contour integrals, as in the bosonic case. Consider the Matsubara sum:

$$
\begin{equation*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right)=\oint_{C} \frac{d \epsilon}{2 \pi i} n_{F}(\epsilon) \mathcal{G}\left(i \Omega_{m}+\epsilon, \vec{p}+\vec{q}\right) \mathcal{G}(\epsilon, \vec{q}) \tag{13.51}
\end{equation*}
$$

where the contour avoids the singularties of the Green functions, as in chapter 6 . $\epsilon_{n}$ and $\Omega_{m}+\epsilon_{n}$ are fermionic Matsubara frequencies, so $\Omega_{m}$ is a bosonic one. The contour integration is given by two contributions: $\epsilon$ real and $i \Omega_{m}+\epsilon$ real. Hence,

$$
\begin{array}{r}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} d E n_{F}(E) \mathcal{G}\left(E+i \Omega_{m}\right)(\mathcal{G}(E+i \delta)-\mathcal{G}(E-i \delta)) \\
+\frac{1}{2 \pi i} \int_{-\infty}^{\infty} d E n_{F}\left(E-i \Omega_{m}\right)(\mathcal{G}(E+i \delta)-\mathcal{G}(E-i \delta)) \mathcal{G}\left(E-i \Omega_{m}\right)
\end{array}
$$

Analytically continuing the imaginary-time Green functions, we have:

$$
\begin{align*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right) & =\int_{-\infty}^{\infty} d E n_{F}(E) \mathcal{G}\left(E+i \Omega_{m}, \vec{p}+\vec{q}\right) A(E, \vec{q}) \\
& +\int_{-\infty}^{\infty} d E n_{F}(E) \mathcal{G}\left(E-i \Omega_{m}, \vec{q}\right) A(E, \vec{p}+\vec{q}) \tag{13.52}
\end{align*}
$$

In the case of free fermions, the spectral function is a $\delta$-function, so the $d E$ integrals can be done:

$$
\begin{equation*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right)=\frac{n_{F}\left(\xi_{q}\right)-n_{F}\left(\xi_{p+q}\right)}{i \Omega_{m}+\xi_{q}-\xi_{p+q}} \tag{13.53}
\end{equation*}
$$

At zero-temperature, the discrete frequency sum becomes a frequency integral,

$$
\begin{equation*}
\frac{1}{\beta} \sum_{n} \rightarrow \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \tag{13.54}
\end{equation*}
$$

so

$$
\begin{equation*}
\frac{1}{\beta} \sum_{n} \mathcal{G}\left(i \Omega_{m}+i \epsilon_{n}, \vec{p}+\vec{q}\right) \mathcal{G}\left(i \epsilon_{n}, \vec{q}\right) \rightarrow \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \mathcal{G}(i \Omega+i \epsilon, \vec{p}+\vec{q}) \mathcal{G}(i \epsilon, \vec{q}) \tag{13.55}
\end{equation*}
$$

Using the spectral representation of $\mathcal{G}$ we can rewrite this as:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \mathcal{G}(i \Omega+i \epsilon, \vec{p}+\vec{q}) \mathcal{G}(i \epsilon, \vec{q})=\int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \int_{-\infty}^{\infty} d E_{1} \int_{-\infty}^{\infty} d E_{2} \frac{A\left(\vec{p}+\vec{q}, E_{1}\right)}{E_{1}-i \Omega-i \epsilon} \frac{A\left(\vec{q}, E_{2}\right)}{E_{2}-i \epsilon} . \tag{3.56}
\end{equation*}
$$

The $d \epsilon$ integral can be done by closing the contour in the upper-half-plane. The pole at $\epsilon=-i E_{1}-\Omega$ is enclosed by the contour when $E_{1}<0$; the pole at $\epsilon=-i E_{2}$ is enclosed when $E_{2}<0$. Hence,

$$
\int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \mathcal{G}(i \Omega+i \epsilon, \vec{p}+\vec{q}) \mathcal{G}(i \epsilon, \vec{q})=\int_{-\infty}^{\infty} d E_{1} \int_{-\infty}^{\infty} d E_{2} \frac{\theta\left(-E_{2}\right)-\theta\left(-E_{1}\right)}{E_{2}-E_{1}+i \omega} A\left(\vec{p}+\vec{q}, E_{1}\right) A(\vec{q}, \text { els.57) }
$$

In the case of free fermions, the $d E_{i}$ integrals may be done:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \mathcal{G}(i \Omega+i \epsilon, \vec{p}+\vec{q}) \mathcal{G}(i \epsilon, \vec{q})=\frac{\theta\left(-\xi_{q}\right)-\theta\left(-\xi_{p+q}\right)}{i \Omega+\xi_{q}-\xi_{p+q}} \tag{13.58}
\end{equation*}
$$

which is the zero-temperature limit of (13.53).

### 13.6 Fermion Self-Energy

We can begin to understand the role played by the Fermi surface when we start computing perturbative corrections to the behavior of free fermions. Let us look first at the fermion two-point Green function. As in the bosonic case, we can define the self-energy, $\Sigma(\epsilon, k)$, as the 1PI two-point function and sum the geometric series to obtain:

$$
\begin{equation*}
\mathcal{G}\left(\vec{p}, i \epsilon_{n}\right)=\frac{1}{i \epsilon_{n}-\left(\frac{p^{2}}{2 m}-\mu\right)-\Sigma\left(\epsilon_{n}, p\right)} \tag{13.59}
\end{equation*}
$$

The retarded Green function is defined by analytic continuation:

$$
\begin{equation*}
G_{\mathrm{ret}}(\vec{p}, \epsilon)=\frac{1}{\epsilon-\left(\frac{p^{2}}{2 m}-\mu\right)-\Sigma_{\mathrm{ret}}(\epsilon, p)} \tag{13.60}
\end{equation*}
$$

The spectral function can be written as:

$$
\begin{equation*}
A(\vec{p}, \epsilon)=\frac{1}{\pi} \frac{-2 \operatorname{Im} \Sigma_{\mathrm{ret}}(\epsilon, p)}{\left(\epsilon-\left(\frac{p^{2}}{2 m}-\mu\right)-\operatorname{Re} \Sigma_{\mathrm{ret}}(\epsilon, p)\right)^{2}+\left(\operatorname{Im} \Sigma_{\mathrm{ret}}(\epsilon, p)\right)^{2}} \tag{13.61}
\end{equation*}
$$

When $\operatorname{Im} \Sigma_{\text {ret }}(\epsilon, p)=0$, the spectral function can be rewritten as:

$$
\begin{equation*}
A(\vec{p}, \epsilon)=Z(p) \delta\left(\epsilon-\xi_{p}\right) \tag{13.62}
\end{equation*}
$$

where $\xi_{p}$ is the location of the pole, defined by the implicit equation

$$
\begin{equation*}
\xi_{p}=\frac{p^{2}}{2 m}-\mu-\operatorname{Re} \Sigma_{\mathrm{ret}}\left(\xi_{p}, p\right) \tag{13.63}
\end{equation*}
$$

and $Z(p)$ is its residue

$$
\begin{equation*}
Z(p)=\left|1-\left(\frac{\partial}{\partial \epsilon} \operatorname{Re} \Sigma_{\mathrm{ret}}(\epsilon, p)\right)_{\epsilon=\xi_{p}}\right|^{-1} \tag{13.64}
\end{equation*}
$$

$\xi_{p}$ can be expanded about the Fermi surface:

$$
\begin{equation*}
\xi_{p}=v_{F}^{*}\left(p-p_{F}\right)+O\left(\left(p-p_{F}\right)^{2}\right) \tag{13.65}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{F}^{*}=\frac{p_{F}}{m^{*}} \tag{13.66}
\end{equation*}
$$

and $m^{*}$ is the effective mass. $\xi_{p}$ and $v_{F}^{*}$ define the one-particle density of states, $N\left(\epsilon_{F}\right)$, of the interacting problem.

$$
\begin{align*}
\int \frac{d^{d} k}{(2 \pi)^{d}} & =\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \int k^{d-1} d k \\
& \approx \frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} k_{F}^{d-1} \int d k \\
& =\frac{2 \pi^{\frac{d}{2}}}{(2 \pi)^{d} \Gamma\left(\frac{d}{2}\right)} \frac{k_{F}^{d-1}}{v_{F}^{*}} \int d \xi_{k} \\
& \equiv N\left(\epsilon_{F}\right) \int d \xi_{k} \tag{13.67}
\end{align*}
$$

The lowest-order contribution to $\operatorname{Im} \Sigma_{\text {ret }}(\epsilon, p)$ comes from the diagram of figure ??. We can do the zero-temperature calculation by contour integration:

$$
\begin{equation*}
\Sigma(i \epsilon, k)=V^{2} \int \frac{d^{d} q}{(2 \pi)^{d}} \int \frac{d^{d} p}{(2 \pi)^{d}} \int \frac{d \omega}{2 \pi} \int \frac{d \zeta}{2 \pi} \mathcal{G}(i \zeta, p) \mathcal{G}(i \zeta+i \omega, p+q) \mathcal{G}(i \epsilon-i \omega, k-q) \tag{13.68}
\end{equation*}
$$

The $d \zeta$ integral may be done by contour integration, as in (13.58):

$$
\begin{equation*}
\Sigma(i \epsilon, k)=V^{2} \int \frac{d^{d} q}{(2 \pi)^{d}} \int \frac{d^{d} p}{(2 \pi)^{d}} \int \frac{d \omega}{2 \pi} \mathcal{G}(i \epsilon-i \omega, k-q) \frac{\theta\left(-\xi_{p}\right)-\theta\left(-\xi_{p+q}\right)}{i \omega+\xi_{p}-\xi_{p+q}} \tag{13.69}
\end{equation*}
$$

The $d \omega$ integral may be done the same way:

$$
\begin{equation*}
\Sigma(i \epsilon, k)=V^{2} \int \frac{d^{d} q}{(2 \pi)^{d}} \int \frac{d^{d} p}{(2 \pi)^{d}} \frac{\left(\theta\left(-\xi_{p}\right)-\theta\left(-\xi_{p+q}\right)\right)\left(\theta\left(\xi_{k-q}\right)-\theta\left(\xi_{p}-\xi_{p+q}\right)\right)}{i \omega+\xi_{p}-\xi_{p+q}-\xi_{k-q}} \tag{13.70}
\end{equation*}
$$

Hence, the imaginary part of the self-energy at zero-temperature is:

$$
\begin{align*}
\operatorname{Im} \Sigma_{\text {ret }}(\epsilon, k) & =V^{2} \frac{k_{F}^{2(d-1)}}{v_{F}^{2}} \int_{-\infty}^{0} d \xi_{p} \int_{0}^{\infty} d \xi_{k-q} \frac{d \Omega_{p}}{(2 \pi)^{d}} \frac{d \Omega_{k-q}}{(2 \pi)^{d}} \delta\left(\xi_{p}-\xi_{p+q}-\xi_{k-q}+\epsilon\right) \\
& \leq V^{2} \frac{k_{F}^{2(d-1)}}{v_{F}^{2}} S_{d-1}^{2} \int_{-\epsilon}^{0} d \xi_{p} \int_{0}^{\epsilon} d \xi_{k-q} \\
& =V^{2} \frac{k_{F}^{2(d-1)}}{v_{F}^{2}} S_{d-1}^{2} \epsilon^{2} \tag{13.71}
\end{align*}
$$

Hence, we have seen that the phase space restrictions (imposed by the $\delta$ function above) due to the existence of a Fermi surface severely restricts $\operatorname{Im} \Sigma_{\text {ret }}(\epsilon, k)$. For $\epsilon \rightarrow 0, \operatorname{Im} \Sigma_{\text {ret }}(\epsilon, k) \sim \epsilon^{2}$. In other words, for $\epsilon$ small, the decay rate is much smaller than the energy: near the Fermi surface, single-fermion states are long-lived.

### 13.7 Luttinger's Theorem

Up until now, the Fermi surface has essentially been a tree-level, or free fermion, concept. However, the notion of a Fermi surface is not tied to perturbation theory. In fact, the existence and location of a Fermi surface is constrained by a non-perturbative theorem due to Luttinger, which we now discuss. Luttinger's theorem defines the Fermi surface as the surface in $\vec{k}$-space at which $G(0, \vec{k})$ changes sign. Inside the Fermi surface, $G(0, \vec{k})$ is positive; outside the Fermi surface, $G(0, \vec{k})$ is negative. In a free fermion system, $G(0, \vec{k})$ diverges at the Fermi surface. (In a superconductor, $G(0, \vec{k})$ vanishes at the Fermi surface, as we will see later.) According to Luttinger's theorem, the volume enclosed by the Fermi surface is equal to the electron density, $N / V$, so long as $\operatorname{Im} \Sigma(0, k)=0$.

To prove this, we begin with

$$
\begin{align*}
\frac{N}{V} & =\int \frac{d^{d} k}{(2 \pi)^{d}}\left\langle\psi^{\dagger}(k, t) \psi(k, t)\right\rangle \\
& =-i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon) \tag{13.72}
\end{align*}
$$

In the second line, we have the time-ordered Green function; the advanced and retarded Green functions vanish at equal times. If we write

$$
\begin{equation*}
G(k, \epsilon)=\frac{1}{\epsilon-\left(\frac{k^{2}}{2 m}-\mu\right)-\Sigma(\epsilon, k)} \tag{13.73}
\end{equation*}
$$

then

$$
\begin{align*}
\frac{N}{V} & =-i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon) \\
& =-i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon)\left[\frac{\partial}{\partial \epsilon}\left(\epsilon-\left(\frac{k^{2}}{2 m}-\mu\right)\right)\right] \\
& =-i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon)\left[\frac{\partial}{\partial \epsilon}\left(G^{-1}(k, \epsilon)+\Sigma(\epsilon, k)\right)\right] \\
& =i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi}\left[\frac{\partial}{\partial \epsilon} \ln G(k, \epsilon)-G(k, \epsilon) \frac{\partial}{\partial \epsilon} \Sigma(\epsilon, k)\right] \tag{13.74}
\end{align*}
$$

We will now use the following 'lemma' which we will prove later:

$$
\begin{equation*}
\int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon) \frac{\partial}{\partial \epsilon} \Sigma(\epsilon, k)=0 \tag{13.75}
\end{equation*}
$$

Then

$$
\begin{align*}
\frac{N}{V} & =i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} \frac{\partial}{\partial \epsilon} \ln G(k, \epsilon) \\
& =i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} \frac{\partial}{\partial \epsilon} \ln G_{\mathrm{ret}}(k, \epsilon)+i \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} \frac{\partial}{\partial \epsilon} \ln \frac{G(k, \epsilon)}{G_{\mathrm{ret}}(k, \epsilon)} \tag{13.76}
\end{align*}
$$

Since $G_{\text {ret }}$ is analytic in the upper-half-plane, the first integral vanishes. Also note that $G=G_{\mathrm{ret}}$ for $\epsilon>0$, while $G^{*}=G_{\mathrm{ret}}$ for $\epsilon<0$. Hence,

$$
\frac{N}{V}=i \int \frac{d^{d} k}{(2 \pi)^{d}} \int_{-\infty}^{0} \frac{d \epsilon}{2 \pi} \frac{\partial}{\partial \epsilon} \ln \frac{G(k, \epsilon)}{G_{\mathrm{ret}}(k, \epsilon)}
$$

$$
\begin{align*}
& =i \int \frac{d^{d} k}{(2 \pi)^{d}}\left[\ln \frac{G(k, \epsilon)}{G_{\mathrm{ret}}(k, \epsilon)}\right]_{-\infty}^{0} \\
& =-\int \frac{d^{d} k}{(2 \pi)^{d}}[\varphi(0, k)-\varphi(-\infty, k)] \tag{13.77}
\end{align*}
$$

From the spectral representation,

$$
\begin{equation*}
G(\vec{p}, \epsilon)=\int_{-\infty}^{\infty} d E \frac{A(\vec{p}, E)}{\epsilon-E+i \delta \operatorname{sgn}(\epsilon)} \tag{13.78}
\end{equation*}
$$

and the normalization property of the spectral function, we see that $\varphi(-\infty, k)=\pi$. Hence,

$$
\begin{equation*}
\frac{N}{V}=-\int \frac{d^{d} k}{(2 \pi)^{d}}[\varphi(0, k)-\pi] \tag{13.79}
\end{equation*}
$$

Since $\operatorname{Im} \Sigma(0, k)=0$ by assumption, $\varphi(0, k)$ is equal to 0 or $\pi$. The integral only receives contributions from the former case:

$$
\begin{equation*}
\frac{N}{V}=\int_{\mathcal{R}} \frac{d^{d} k}{(2 \pi)^{d}} \tag{13.80}
\end{equation*}
$$

where $\mathcal{R}=\{\vec{k} \mid G(0, \vec{k})>0\}$. In other words, the volume enclosed by the Fermi surface is equal to the electron density.

To complete the proof of the theorem, we must prove that

$$
\begin{equation*}
\int \frac{d^{d} k}{(2 \pi)^{d}} \frac{d \epsilon}{2 \pi} G(k, \epsilon) \frac{\partial}{\partial \epsilon} \Sigma(\epsilon, k)=0 \tag{13.81}
\end{equation*}
$$

To do this, we prove that there exists a functional $X[G]$ defined by:

$$
\begin{equation*}
\delta X=\int \frac{d \omega}{2 \pi} \int \frac{d^{3} k}{(2 \pi)^{3}} \Sigma(\omega, k) \delta G(\omega, k) \tag{13.82}
\end{equation*}
$$

According to this definition,

$$
\begin{align*}
\delta X & =\int \frac{d \omega}{2 \pi} \int \frac{d^{3} k}{(2 \pi)^{3}} \Sigma(\omega, k) \delta G(\omega, k) \\
& =\int \frac{d \omega}{2 \pi} \int \frac{d^{3} k}{(2 \pi)^{3}} \Sigma(\omega+\epsilon, k) \delta G(\omega+\epsilon, k) \tag{13.83}
\end{align*}
$$

Hence,

$$
\begin{align*}
\frac{\delta X}{\delta \epsilon} & =\int \frac{d \omega}{2 \pi} \int \frac{d^{3} k}{(2 \pi)^{3}} \Sigma(\omega+\epsilon, k) \frac{\partial}{\partial \epsilon} \delta G(\omega+\epsilon, k) \\
& =\int \frac{d \omega}{2 \pi} \int \frac{d^{3} k}{(2 \pi)^{3}} \Sigma(\omega, k) \frac{\partial}{\partial \omega} \delta G(\omega, k) \tag{13.84}
\end{align*}
$$

However, $X$ is independent of $\epsilon$, so $\delta X / \delta \epsilon=0$ which proves (13.81).
To see that $X$ actually exists, observe that

$$
\begin{equation*}
\frac{\delta X}{\delta G(p)}=\Sigma(p) \tag{13.85}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\frac{\delta^{2} X}{\delta G(p) \delta G(q)}=\frac{\delta \Sigma(p)}{\delta G(q)} \tag{13.86}
\end{equation*}
$$

$X$ exists if and only if the derivatives can be commuted:

$$
\begin{equation*}
\frac{\delta^{2} X}{\delta G(p) \delta G(q)}=\frac{\delta^{2} X}{\delta G(q) \delta G(p)} \tag{13.87}
\end{equation*}
$$

Since

$$
\begin{equation*}
\frac{\delta \Sigma(p)}{\delta G(q)}=\Gamma(p, q) \tag{13.88}
\end{equation*}
$$

where $\Gamma(p, q)$ is the irreducible 4-point function with external momenta $p, p, q, q$ and $\Gamma(p, q)=\Gamma(q, p)$, the existence of $X$ follows.

In the case of a free Fermi gas, $n_{k}=\left\langle c_{k}^{\dagger} c_{k}\right\rangle$ is a step function, $n_{k}=\theta\left(k_{F}-k\right)$ with $\int n_{k}=N / V$. One might imagine that, in an interacting Fermi gas, it would be possible to have $n_{k}=\lambda \theta\left(k_{F}-k \lambda\right)$ with $\lambda<1$ which would preserve $\int n_{k}=N / V$ while moving the location of the singularity in $n_{k}$ to $k_{F} / \lambda$. Luttinger's theorem tells us that this cannot happen. The singularity in $n_{k}=\int d \epsilon G(\epsilon, k)$ is fixed by the density to be at $k_{F}$.

## Chapter 14

## Interacting Neutral Fermions: Fermi Liquid Theory

### 14.1 Scaling to the Fermi Surface

We now consider a rotationally invariant system of interacting spinless fermions with $\mu>0$ in $D \geq 2$. The RG analysis of such systems was pioneered by Shankar, Polchinski, .... The Fermi sea is filled up to some $k_{F}$. First, let us examine the free part of the action,

$$
\begin{equation*}
\int d \tau d^{d} x \psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi \tag{14.1}
\end{equation*}
$$

or, in momentum space,

$$
\begin{equation*}
\int \frac{d \epsilon}{2 \pi} \frac{d^{d} k}{(2 \pi)^{d}} \psi^{\dagger}\left(i \epsilon-\left(\frac{k^{2}}{2 m}-\mu\right)\right) \psi \tag{14.2}
\end{equation*}
$$

If the kinetic energy is much larger than the potential energy, then it makes sense to focus first on it and use it to determine the scaling of $\psi$ from the kinetic energy.

Since we will be interested in low-energy, i.e. the vicinity of the Fermi surface, we make the approximation

$$
\frac{k^{2}}{2 m}-\mu \approx v_{F}\left(k-k_{F}\right)
$$

$$
\begin{equation*}
\equiv v_{F} l \tag{14.3}
\end{equation*}
$$

where $l=k-k_{F}$. We can also make the approximation

$$
\begin{align*}
d^{d} k & =k_{F}^{d-1} d k d^{d} \Omega \\
& =k_{F}^{d-1} d l d^{d} \Omega \tag{14.4}
\end{align*}
$$

Hence, the action can be written as:

$$
\begin{equation*}
\frac{k_{F}^{d-1}}{(2 \pi)^{d}} \int d l d^{d} \Omega \frac{d \epsilon}{2 \pi} \psi^{\dagger}\left(i \epsilon-v_{F} l\right) \psi \tag{14.5}
\end{equation*}
$$

Restoring the cutoffs,

$$
\begin{equation*}
\frac{k_{F}^{d-1}}{(2 \pi)^{d}} \int_{-\Lambda}^{\Lambda} d l d^{d} \Omega \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \psi^{\dagger}\left(i \epsilon-v_{F} l\right) \psi \tag{14.6}
\end{equation*}
$$

The momentum integral is restricted to a shell of thickness $2 \Lambda$ about the Fermi surface. We leave the frequency integral unrestricted (as we did in the case of the $O(3) \mathrm{NL} \sigma \mathrm{M})$. The angular integral has no cutoff, of course.

Our RG transformation now takes the following form:

- Integrate out $\psi(l, \vec{\Omega}, \epsilon), \psi^{\dagger}(l, \vec{\Omega}, \epsilon)$ for $b \Lambda<|l|<\Lambda$ and $\epsilon, \vec{\Omega}$ arbitrary.
- Rescale:

$$
\begin{array}{rlll}
\omega & \rightarrow & b \omega \\
l & \rightarrow & b l \\
\vec{\Omega} & \rightarrow \vec{\Omega} & \\
\psi & \rightarrow & b^{-\frac{3}{2}} \psi \tag{14.7}
\end{array}
$$

The principal difference between the renormalization group applied to a system with a Fermi surface and its application to more familiar contexts is that the lowenergy degrees of freedom are located in the neighborhood of a surface in momentum
space, rather than in the vicinity of a point. Hence, we do not scale to the origin of momentum space, $\vec{k}=0$, but to the Fermi surface, $\vec{l}=0$.

The free fermion action (14.5) is evidently a fixed point of this RG transformation. Thus, as we would expect, a free fermion system looks the same at any energy scale: it is always just a free theory.

To this action, we can add the following perturbation:

$$
\begin{equation*}
\frac{k_{F}^{d-1}}{(2 \pi)^{d}} \int_{-\Lambda}^{\Lambda} d l d^{d} \Omega \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \delta \mu \psi^{\dagger} \psi \tag{14.8}
\end{equation*}
$$

Under the scaling (14.7), $\delta \mu$ scales as:

$$
\begin{equation*}
\delta \mu \rightarrow b^{-1} \delta \mu \tag{14.9}
\end{equation*}
$$

Since $\delta \mu$ is a relevant operator, we cannot study it perturbatively. Relevant operators typically bring about a fundamental change of the ground state, and $\delta \mu$ is no different. Changing the chemical potential shifts the Fermi surface. If we change coordinates to $\overrightarrow{l^{\prime}}=\vec{l}+\frac{\delta \mu}{v_{F}}$, then we recover (14.5).

In a system which is not rotationally invariant, $\delta \mu$ can depend on the angle around the Fermi surface, $\delta \mu(\vec{\Omega}) \cdot \delta \mu(\vec{\Omega})$ is an example of a 'coupling function', which is a generalization of a coupling constant. Such a perturbation can change the shape of the Fermi surface.

A second perturbation is

$$
\begin{equation*}
\frac{k_{F}^{d-1}}{(2 \pi)^{d}} \int_{-\Lambda}^{\Lambda} d l d^{d} \Omega \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} \delta v_{F} \vec{l} \psi^{\dagger} \psi \tag{14.10}
\end{equation*}
$$

$\delta v_{F}$ shifts the Fermi velocity. It is clearly a marginal perturbation.

### 14.2 Marginal Perturbations: Landau Parameters

Let us now consider four-fermion interactions. Consider the term

$$
S_{4}=\quad \int d \omega_{1} d \omega_{2} d \omega_{3} d^{d} k_{1} d^{d} k_{2} d^{d} k_{3} u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}, \vec{k}_{4}\right) \times
$$

$$
\begin{equation*}
\psi^{\dagger}\left(k_{4}, \omega_{4}\right) \psi^{\dagger}\left(k_{3}, \omega_{3}\right) \psi\left(k_{2}, \omega_{2}\right) \psi\left(k_{1}, \omega_{1}\right) \tag{14.11}
\end{equation*}
$$

Rather than a single coupling constant, $u$, we have a coupling function, $u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}, \vec{k}_{4}\right)$. The RG equations for a coupling function are called functional $R G$ equations. We will assume that $u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}, \vec{k}_{4}\right)$ is non-singular or, in other words, that the fermions have short-ranged interactions, as in ${ }^{3} \mathrm{He}$. We will deal with the complications resulting from Coulomb interactions in the next section.

If $u\left(k_{1}, k_{2}, k_{2}, k_{1}\right)$ is a function only of the angular variables, $u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{3}, \vec{\Omega}_{4}\right)$, then it is marginal, i.e. it does not scale under (14.7). If it depends on the $\left|k_{i}\right|-k_{F} \mid$ 's, then it is irrelevant, so we ignore this possibility. Similarly, six-fermion, eight-fermion, etc. interactions are neglected because they are highly irrelevant. Furthermore, momentum conservation implies that

$$
\begin{equation*}
u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}, \vec{k}_{4}\right)=u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) \tag{14.12}
\end{equation*}
$$

In the last section, we considered the case of $\delta$ function interactions, for which $u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}\right)=V$. Here, we are considering the more general case of arbitrary (nonsingular) $u\left(\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}\right)$.

The crucial observation underlying Fermi liquid theory, which is depicted in Figure 14.1, is the following. Consider, for simplicity, the case of $D=2$. For $\Lambda \ll k_{F}$, $u\left(k_{1}, k_{2}, k_{3}\right)=0$ for generic $k_{1}, k_{2}, k_{3}$ because $k_{4}$ typically does not lie within the cutoff. The constraint of momentum conservation, $\vec{k}_{1}+\vec{k}_{2}=\vec{k}_{3}+\vec{k}_{4}$ together with the restriction that $\vec{k}_{1}, \vec{k}_{2}, \vec{k}_{3}, \vec{k}_{4}$ lie within $\Lambda$ of the Fermi surface severely limits the phase space for scattering. As we scale to the $\Lambda \rightarrow 0$ limit, only forward scattering, $u\left(k_{1}, k_{2}, k_{1}, k_{2}\right)$ and exchange scattering, $u\left(k_{1}, k_{2}, k_{2}, k_{1}\right)=-u\left(k_{1}, k_{2}, k_{1}, k_{2}\right)$, can satisfy momntum conservation. At small but non-zero $\Lambda$, a small subset of the $u$ 's are non-zero. As $\Lambda$ is decreased, some of these are set discontinuously to zero; the rest do not scale. As $\Lambda$ becomes smaller, fewer non-zero $u$ 's remain until, finally, at $\Lambda=0$, only the three mentioned above remain. It is this drastic simplification which makes


Figure 14.1: (a) If all of the momenta are constrained to lie on the Fermi surface, incoming momenta $k_{1}, k_{2}$ can only scatter into $k_{3}=k_{1}, k_{4}=k_{2}$ or $k_{3}=k_{2}, k_{4}=k_{1}$, unless (b) $k_{1}=-k_{2}$.

Fermi liquid theory soluble.
In three dimensions, the angle between $\vec{k}_{3}$ and $\vec{k}_{4}$ is the same as the angle between $\vec{k}_{1}$ and $\vec{k}_{2}$

$$
\begin{equation*}
\theta\left(\vec{k}_{1}, \vec{k}_{2}\right)= \pm \theta\left(\vec{k}_{3}, \vec{k}_{4}\right) \tag{14.13}
\end{equation*}
$$

but the plane of $\vec{k}_{3}$ and $\vec{k}_{4}$ can be rotated relative to the plane of $\vec{k}_{1}$ and $\vec{k}_{2}$ by and angle $\phi$ as in figure 14.2.

These phase space restrictions imply that in two dimensions, we should focus on

$$
\begin{equation*}
F\left(\theta_{1}-\theta_{2}\right) \equiv u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{2}, \vec{\Omega}_{1}\right) \tag{14.14}
\end{equation*}
$$

Fermi statistics dictates that exchange scattering is related to forward scattering by:

$$
\begin{align*}
u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{2}, \vec{\Omega}_{1}\right) & =-u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{1}, \vec{\Omega}_{2}\right) \\
& =-F\left(\theta_{1}-\theta_{2}\right) \tag{14.15}
\end{align*}
$$



Figure 14.2: In three dimensions, the outgoing momenta can be rotated relative to the incoming momenta.

In three dimensions, we should focus on:

$$
\begin{equation*}
F\left(\vec{\Omega}_{1} \cdot \vec{\Omega}_{2}, \phi\right) \equiv u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{3}, \vec{\Omega}_{4}\right) \tag{14.16}
\end{equation*}
$$

in the $\Lambda \rightarrow 0$ limit. The Fourier components of $F$ are called Landau Parameters.
There is one loophole in the preceeding analysis, depicted in figure 14.1b. If $\vec{k}_{1}=-\vec{k}_{2}$, then $\vec{k}_{3}=-\vec{k}_{4}$ is arbitrary. This is the Cooper pairing channel. We write:

$$
\begin{equation*}
V\left(\vec{\Omega}_{1} \cdot \vec{\Omega}_{3}\right) \equiv u\left(k_{1},-k_{1}, k_{3},-k_{3}\right) \tag{14.17}
\end{equation*}
$$

In $D=2$, this can be written as:

$$
\begin{equation*}
V\left(\theta_{1}-\theta_{3}\right) \tag{14.18}
\end{equation*}
$$

Then, at tree-level, in the $\Lambda \rightarrow 0$ limit, we have the following action:

$$
S=\frac{k_{F}^{d-1}}{(2 \pi)^{d}} \int d l d^{d} \Omega \frac{d \epsilon}{2 \pi} \psi^{\dagger}\left(i \epsilon-v_{F} l\right) \psi
$$



Figure 14.3: The phase space available to $k_{3}$ and $k_{4}$ when $\Lambda$ is small but non-zero is the region bounded by the arcs which has area $\sim \Lambda^{2}$.

$$
\begin{array}{r}
+\int d \epsilon_{1} d \epsilon_{2} d \epsilon_{3} d k_{1} d k_{2} d k_{3} d^{d} \Omega_{1} d^{d} \Omega_{2} F\left(\vec{\Omega}_{1} \cdot \vec{\Omega}_{2}, \phi\right) \times \\
\psi^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi\left(k_{2}, \epsilon_{2}\right) \psi\left(k_{1}, \epsilon_{1}\right) \\
+\int d \epsilon_{1} d \epsilon_{2} d \epsilon_{3} d k_{1} d k_{2} d k_{3} d^{d} \Omega_{1} d^{d} \Omega_{3} V\left(\vec{\Omega}_{1} \cdot \vec{\Omega}_{3}\right) \times \\
\psi^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi\left(k_{2}, \epsilon_{2}\right) \psi\left(k_{1}, \epsilon_{1}\right) \tag{14.19}
\end{array}
$$

For $\Lambda$ finite, we have to keep the full coupling function $u\left(\vec{\Omega}_{1}, \vec{\Omega}_{2}, \vec{\Omega}_{3}, \vec{\Omega}_{4}\right)$ with

$$
\begin{equation*}
\left|\vec{\Omega}_{1}-\vec{\Omega}_{3}\right|<\frac{\Lambda}{k_{F}} \tag{14.20}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|\vec{\Omega}_{2}-\vec{\Omega}_{3}\right|<\frac{\Lambda}{k_{F}} \tag{14.21}
\end{equation*}
$$

as in figure 14.3


Figure 14.4: The one-loop diagrams which can contribute to the renormalization of $F$.

### 14.3 One-Loop

At tree-level, $F\left(\vec{\Omega}_{1}-\vec{\Omega}_{2}\right)$ and $V\left(\vec{\Omega}_{1}-\vec{\Omega}_{3}\right)$ are marginal. We would now like to compute the one-loop RG equations for $F\left(\vec{\Omega}_{1}-\vec{\Omega}_{2}\right)$ and $V\left(\vec{\Omega}_{1}-\vec{\Omega}_{3}\right)$.

First, consider the renormalization of $F$. The one-loop diagrams are in figure 14.4. Since $F$ is independent of the frequencies and the $l_{i}$, we can set the external frequencies to zero and put the external momenta on the Fermi surface. The first diagram gives a contribution

$$
\begin{equation*}
d F\left(\vec{\Omega}_{1}-\vec{\Omega}_{2}\right)=\int d l d \Omega d \epsilon F(\vec{\Omega}) F\left(\vec{\Omega}+\vec{\Omega}_{1}-\vec{\Omega}_{2}\right) \frac{1}{i \epsilon-v_{F} l} \frac{1}{i \epsilon-v_{F} l} \tag{14.22}
\end{equation*}
$$

which vanishes since both poles in $\epsilon$ are on the same side of the axis.
The internal momenta in these diagrams must lie in thin shells at the cutoff, $\Lambda-d \Lambda<|p|-k_{F}<\Lambda$. In the second diagram, $\vec{p}$ and $\vec{p}+\vec{k}_{1}-\vec{k}_{2}$ must both satisfy this condition. The condition on $|p|$ restricts its magnitude; the condition on
$\left|\vec{p}+\vec{k}_{1}-\vec{k}_{2}\right|$ restricts the direction of $\vec{p}$. The kinematic restriction is essentially the same as that depicted in figure 14.3. As a result, the $d l$ and $d \Omega$ integrals each give a contribution proportional to $d \Lambda$, and therefore

$$
\begin{equation*}
d F \sim(d \Lambda)^{2} \tag{14.23}
\end{equation*}
$$

in the $d \Lambda \rightarrow 0$ limit, this gives a vanishing contribution to $d F / d \ell$. The third diagram gives a vanishing contribution for the same reason. Hence, at one-loop,

$$
\begin{equation*}
\frac{d}{d \ell} F\left(\vec{\Omega}_{1}-\vec{\Omega}_{2}\right)=0 \tag{14.24}
\end{equation*}
$$

The Landau parameters are strictly marginal; they remain constant as we scale to lower energies.

We now turn to the one-loop RG equations for $V$. The relevant diagrams are analogous to those of 14.4. The first two diagrams are proportional to $(d \Lambda)^{2}$ and, therefore, do not contribute to the RG equation. However, the third diagram gives the contribution

$$
\begin{align*}
d V\left(\vec{\Omega}_{1}-\vec{\Omega}_{3}\right) & =-\int \frac{d \epsilon}{2 \pi} \frac{d l}{2 \pi} \frac{d^{d} \Omega}{(2 \pi)^{d-1}} V\left(\vec{\Omega}_{1}-\vec{\Omega}\right) V\left(\vec{\Omega}-\vec{\Omega}_{3}\right) \frac{1}{i \epsilon-v_{F} l} \frac{1}{-i \epsilon-v_{F} l} \\
& =\int \frac{d l}{2 \pi} \frac{d^{d} \Omega}{(2 \pi)^{d-1}} V\left(\vec{\Omega}_{1}-\vec{\Omega}\right) V\left(\vec{\Omega}-\vec{\Omega}_{3}\right) \frac{1}{2 v_{F} p} \\
& =\frac{1}{2 \pi v_{F}} d \ell \int \frac{d^{d} \Omega}{(2 \pi)^{d-1}} V\left(\vec{\Omega}_{1}-\vec{\Omega}\right) V\left(\vec{\Omega}-\vec{\Omega}_{3}\right) \tag{14.25}
\end{align*}
$$

In two dimensions, we write

$$
\begin{equation*}
V_{m}=\int_{0}^{2 \pi} \frac{d \theta}{2 \pi} e^{i m \theta} V(\theta) \tag{14.26}
\end{equation*}
$$

The renormalization group flow equation for $V_{l}$ is:

$$
\begin{gather*}
\frac{d V_{l}}{d \ln \Lambda}=-\frac{1}{2 \pi v_{F}} V_{l}^{2}  \tag{14.27}\\
V_{l}(\Lambda)=\frac{V_{l}\left(\Lambda_{0}\right)}{1+\frac{1}{2 \pi v_{F}} V_{l}\left(\Lambda_{0}\right) \ln \left(\Lambda_{0} / \Lambda\right)} \tag{14.28}
\end{gather*}
$$

Therefore, repulsive BCS interactions are marginally irrelevant, while attractive BCS interactions are marginally relevant. From (14.28), we see that an attractive BCS interaction will grow as we go to lower scales, until it reaches the scale:

$$
\begin{equation*}
\Lambda \sim \Lambda_{0} e^{-2 \pi v_{F} /\left|V_{l}\left(\Lambda_{0}\right)\right|} \tag{14.29}
\end{equation*}
$$

As we will discuss later, at this scale, pairing takes place. In the BCS theory of a phonon-mediated superconductor, this leads to a critical temperature or zerotemperature gap given by

$$
\begin{equation*}
T_{c} \sim \Delta(T=0) \sim \omega_{D} e^{-2 \pi v_{F} /\left|V_{0}\right|} \tag{14.30}
\end{equation*}
$$

Bardeen, Cooper, and Schrieffer found the paired ground state using a variational ansatz. A more general formalism, which can be used when the interactions are retarded was pioneered by Nambu, Gor'kov, and Eliashberg. Both of these approaches owe their success to the kinematic constraints of the problem. There are no relevant interactions other than $F\left(\Omega_{1}, \Omega_{2}\right)$ and $V\left(\Omega_{1}, \Omega_{3}\right)$. $F\left(\Omega_{1}, \Omega_{2}\right)$ does not contribute to the running of $V\left(\Omega_{1}, \Omega_{3}\right)$, so the diagram of Figure ?? is essentially the only diagram which must be taken into account. BCS theory and its refinement by Nambu, Gor'kov, and Eliashberg are mean-field theories which evaluate this diagram self-consistently. These theories will be discussed in chapter 16 .

## $14.41 / N$ and All Loops

The one-loop structure of a system of interacting fermions is actually stable to all orders in perturbation theory. The essential reason for this (which was first recognized in this language by Shankar) is that $\Lambda / k_{F}$ is a small parameter like $1 / N$. To see this, consider the case $D=2$. Break the angular integration into pieces, $\Delta \theta=\Lambda / k_{F}$ with
$\theta_{j}=2 \pi j\left(\Lambda / k_{F}\right)$ and $j=0,1, \ldots, k_{F} / \Lambda$.

$$
\begin{equation*}
\int_{0}^{2 \pi} d \theta \rightarrow \sum_{i} \int_{\theta_{i}}^{\theta_{i+1}} d \theta \tag{14.31}
\end{equation*}
$$

Then, we can write

$$
\begin{aligned}
S= & \sum_{i} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{i}^{\dagger}\left(i \epsilon-v_{F} l\right) \psi_{i} \\
& +\sum_{i . j} \int \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{d^{2} k_{1}}{2 \pi} \frac{d^{2} k_{2}}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& +\sum_{i, j} \int \frac{d \epsilon_{1}}{2 \pi} \frac{F_{i j} \psi_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{\psi_{j}\left(k_{3}, \epsilon_{3}\right) \psi_{j}\left(k_{2}, \epsilon_{2}\right) \psi_{i}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right)}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& V_{i j} \psi_{j}^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi_{j+\frac{k_{F}}{2 \Lambda}}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{2}, \epsilon_{2}\right) \psi_{i+\frac{k_{F}}{2 \Lambda}}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\left(\vec{A}_{4} A\right) .32\right)
\end{aligned}
$$

We have broken the angular integral into a summation over Fermi surface 'patches' and an integral over each patch. Hence, $F\left(\theta_{i}-\theta_{j}\right)$ has been replaced by $F_{i j}$. By restricting to $F_{i j}$ rather than allowing $u_{i j k l}$, we have automatically restricted to nearly forward scattering - i.e. to scattering from one point to another within the same patch. Furthermore, the $\delta$-function,

$$
\begin{equation*}
\delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) \tag{14.33}
\end{equation*}
$$

does not contain any momenta of $O\left(k_{F}\right)$; the $\vec{k}_{i}$ 's live within patches and, therefore, are all less than the cutoff.

In this expression,

$$
\begin{align*}
\int d^{2} k & =\int_{-\Lambda}^{\Lambda} d l \int_{0}^{2 \pi \Lambda / k_{f}} d \theta \\
& =\int_{-\Lambda}^{\Lambda} d k_{\perp} \int_{-\pi \Lambda}^{\pi \Lambda} d k_{\|} \tag{14.34}
\end{align*}
$$

so both momenta have cutoff $\sim \Lambda$.
Hence, if we rescale all momenta by $\Lambda$ and the field, $\psi$, as well:

$$
\omega \rightarrow \frac{\omega}{\Lambda}
$$

$$
\begin{align*}
k_{\perp} & \rightarrow \frac{k_{\perp}}{\Lambda} \\
k_{\perp} & \rightarrow \frac{k_{\|}}{\Lambda} \\
\psi & \rightarrow \Lambda^{2} \psi \tag{14.35}
\end{align*}
$$

we can rewrite the action as:

$$
\begin{aligned}
S= & \sum_{i=0}^{k_{F} / \Lambda} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{i}^{\dagger}\left(i \epsilon-v_{F} l\right) \psi_{i} \\
& +\sum_{i, j=0}^{k_{F} / \Lambda} \int \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{d^{2} k_{1}}{2 \pi} \frac{d^{2} k_{2}}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& \frac{\Lambda}{k_{F}} F_{i j} \psi_{j}^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi_{j}\left(k_{2}, \epsilon_{2}\right) \psi_{i}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) \\
& +\sum_{i, j=0}^{k_{F} / \Lambda} \int \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{d^{2} k_{1}}{2 \pi} \frac{d^{2} k_{2}}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& \frac{\Lambda}{k_{F}} V_{i j} \psi_{j}^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi_{j+\frac{k_{F}}{2 \Lambda}}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{2}, \epsilon_{2}\right) \psi_{i+\frac{k_{F}}{2 \Lambda}}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}+\left(1 \vec{k}_{4}, \beta 6\right)\right.
\end{aligned}
$$

In other words, if we write $N=k_{F} / \Lambda$,

$$
\begin{aligned}
S= & \sum_{i=0}^{N} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{i}^{\dagger}\left(i \epsilon-v_{F} l\right) \psi_{i} \\
& +\sum_{i, j=0}^{N} \int \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{d^{2} k_{1}}{2 \pi} \frac{d^{2} k_{2}}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& \quad \frac{1}{N} F_{i j} \psi_{j}^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi_{j}\left(k_{2}, \epsilon_{2}\right) \psi_{i}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}-\vec{k}_{4}\right) \\
& \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \frac{d^{2} k_{1}}{2 \pi} \frac{d^{2} k_{2}}{2 \pi} \frac{d^{2} k_{3}}{2 \pi} \frac{d^{2} k_{4}}{2 \pi} \times \\
& \frac{1}{N} V_{i j} \psi_{j}^{\dagger}\left(k_{3}, \epsilon_{3}\right) \psi_{j+\frac{k_{F}}{2 \Lambda}}^{\dagger}\left(k_{4}, \epsilon_{4}\right) \psi_{i}\left(k_{2}, \epsilon_{2}\right) \psi_{i+\frac{k_{F}}{2 \Lambda}}\left(k_{1}, \epsilon_{1}\right) \delta\left(\vec{k}_{1}+\vec{k}_{2}-\vec{k}_{3}+1 \vec{k}_{4}, \$ 7\right)
\end{aligned}
$$

then we see that we have a model in the large $N$ limit.
Recall the analysis of the $O(N)$ model in the large $N$ limit. The only $O(1)$ corrections to the two-point function are diagrams of the form of figure ??a. These shift the chemical potential. The non-trivial diagrams such as ??b are $O(1 / N)$. Consider now the correction to the four-point function. Only diagrams such as those of ?? a, b are $O(1 / N)$. In the case of forward scattering, ?? a vanishes because both
poles are on the same side of the axis. In the case of Cooper scattering, ??b, gives a non-trivial contribution. The other corrections, such as those of ??c are $O\left(1 / N^{2}\right)$.

As we learned earlier in the context of $O(N)$ models, the large- $N$ limit introduces the following simplifications. The only diagrams which need to be considered are the bubble diagrams. The one-loop RG is the full story. Consequently, the action (14.37) is a stable fixed point if $V_{i} j>0$. A system of fermions which is controlled by this fixed point is called a Fermi liquid.

### 14.5 Quartic Interactions for $\Lambda$ Finite

The scaling of generic four-fermi interactions is quite awkward for calculations at a finite frequency or temperature scale because the $u$ 's don't scale continuously. Thus, the scaling of a physical quantity which depends on the $u$ 's is determined not by the scaling of the $u$ 's, which is marginal, but on the number of non-zero $u$ 's, which is scale dependent (except in the important case where the quantity is determined by forward, exchange, or Cooper scattering - which do scale continuously). For such calculations, a different scaling transformation is useful. Suppose $\Lambda$ is small but finite. Then, in two dimensions, we can consider nearly forward scatering, from $\vec{k}_{1}, \vec{k}_{2}$ to $\vec{k}_{3}, \vec{k}_{4}$ with $\left|\vec{k}_{1}-\vec{k}_{3}\right|<\Lambda,\left|\vec{k}_{2}-\vec{k}_{4}\right|<\Lambda$. Since all of the action is taking place in the neighborhods of $\vec{\Omega}_{1}, \vec{\Omega}_{2}$, we focus on these points. We construct cartesian coordinates, $k_{x}$ (tangent to the Fermi surface) and $k_{y}$ (perpendicular to the Fermi surface), at these two points on the Fermi surface. In the vicinity of these points,

$$
\begin{equation*}
\epsilon\left(k_{x}, k_{y}\right)=v_{F}\left(k_{y}+\frac{k_{x}^{2}}{2 k_{F}}\right) \tag{14.38}
\end{equation*}
$$

We now scale to $\vec{\Omega}_{1}, \vec{\Omega}_{2}$, using the scaling $k_{y} \rightarrow s k_{y}, k_{x} \rightarrow s^{1 / 2} k_{x}, \omega \rightarrow s \omega$. (We have assumed $d=2$; in $d>2$, there are $d-1$ momenta which scale as $k_{x}$.) The same answers are obtained with either scaling transformation; it's just that


Figure 14.5: A one-loop self-energy correction.
some calculations are easier with this one. On the other hand, it's a less natural renormalization group transformation because it involves selecting preferred points on the Fermi surface and scaling differently at different points on the Fermi surface. Let's briefly see how this works. The quadratic part of the Lagrangian is of the form:

$$
\begin{equation*}
S_{0}=\int d \omega d k_{y} d k_{x}\left\{\psi^{\dagger}\left(i \omega-v_{F}\left(k_{y}+\frac{k_{x}^{2}}{2 k_{F}}\right)\right) \psi\right\} \tag{14.39}
\end{equation*}
$$

Hence, the field now scales as $\psi \rightarrow s^{-7 / 4} \psi$, so four-fermi interactions,

$$
\begin{equation*}
S_{4}=\int d \omega_{1} d \omega_{2} d \omega_{3} d^{2} k_{1} d^{2} k_{2} d^{2} k_{3} u\left(k_{1}, k_{2}, k_{3}\right) \psi^{\dagger}\left(k_{4}, \omega_{4}\right) \psi^{\dagger}\left(k_{3}, \omega_{3}\right) \psi\left(k_{2}, \omega_{2}\right) \psi\left(k_{1}, \omega_{1}\right) \tag{14.40}
\end{equation*}
$$

scale as $s^{1 / 2}$.The scaling is perfectly continuous. If $k_{1}, k_{2}, k_{3}, k_{4}=k_{1}+k_{2}-k_{3}$ lie within the cutoff $\Lambda$, then they continue to do so under this renormalization group transformation. If we insert a $\delta\left(k_{1 x}-k_{3 x}\right)$ or $\delta\left(k_{1 x}-k_{4 x}\right)$ into the integrand, then we get a marginal interaction, namely forward scattering, as before. ${ }^{1}$ To see why this is a useful scaling, consider the diagram in figure 14.5. It has a real part, proportional to $F_{f}^{2} \omega$ which comes from the marginal forward scattering interaction, and an imaginary part, proportional to $F_{n f}^{2} \omega^{2}$ coming from irrelevant non-forward processes, in agreement with the explicit calculation which we did in chapter 13. The above scaling immediately yields the suppression of $F_{n f}^{2}$ with respect to $F_{f}^{2}$ by one power of $\omega$, a result which is more cumbersome to derive with the other RG transformation.

[^4]
### 14.6 Zero Sound, Compressibility, Effective Mass

As a result of the preceeding analysis, the density-density correlation function can be computed by summing bubble diagrams. Other diagrams are down by powers of $1 / N$. The approximation which consists of neglecting these other diagrams is called the RPA, or random-phase approximation (for historical reasons). The bubble diagrams form a geometric series which may be summed. Let us consider the simplest case, in which $F$ is a constant, $F\left(\Omega_{1} \cdot \Omega_{2}\right)=F_{0}$ :

$$
\begin{align*}
\langle\rho(q, i \omega) \rho(-q,-i \omega)\rangle & =I(q, i \omega)+(I(q, i \omega))^{2} F_{0}+\ldots+(I(q, i \omega))^{n+1} F_{0}^{n}+\ldots \\
& =\frac{I(q, i \omega)}{1-I(q, i \omega) F_{0}} \tag{14.41}
\end{align*}
$$

where $I$ is the value of a single particle-hole bubble. In the limit of $q \ll k_{F}$, this is:

$$
\begin{align*}
I(q, i \omega) & =\int \frac{d \epsilon}{2 \pi} \frac{d^{3} k}{(2 \pi)^{3}} \mathcal{G}(i \epsilon, k) \mathcal{G}(i \epsilon+i \omega, k+q) \\
& =k_{F}^{2} \int \frac{d \epsilon}{2 \pi} \frac{d l d \varphi}{(2 \pi)^{3}} d(\cos \theta) \frac{1}{i \epsilon-v_{F} l} \frac{1}{i \epsilon+i \omega-v_{F} l-v_{F} q \cos \theta} \\
& =k_{F}^{2} \int \frac{d l}{(2 \pi)^{2}} d(\cos \theta) \frac{\theta(l)-\theta(l+q \cos \theta)}{i \omega-v_{F} q \cos \theta} \\
& =\frac{k_{F}^{2}}{(2 \pi)^{2}} \int d(\cos \theta) \frac{q \cos \theta}{i \omega-v_{F} q \cos \theta} \\
& =\frac{k_{F}^{2}}{4 \pi^{2} v_{F}} \int_{-1}^{1} d x \frac{x}{\frac{i \omega}{v_{F} q}-x} \\
& =\frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left[\frac{1}{2} \frac{i \omega}{v_{F} q} \ln \left(\frac{i \omega+v_{F} q}{i \omega-v_{F} q}\right)-1\right] \tag{14.42}
\end{align*}
$$

The retarded density-density correlation function is a response function, $-\chi_{\rho \rho}$, of the type which we discussed in chapter 7 . If we imagine changing the chemical potential by a frequency- and wavevector-dependent amount, $\delta \mu(\omega, q)$, then the action changes by

$$
\begin{equation*}
S \rightarrow S-\int \frac{d \omega}{2 \pi} \frac{d^{d} q}{(2 \pi)^{d}} \delta \mu(\omega, q) \rho(\omega, q) \tag{14.43}
\end{equation*}
$$

Hence, following the steps of chapter 7 , we have

$$
\begin{equation*}
\langle\delta \rho(\omega, q)\rangle=\chi_{\rho \rho}(\omega, q) \delta \mu(\omega, q) \tag{14.44}
\end{equation*}
$$



Figure 14.6: The geometric series of bubble diagrams which determine $\langle\rho(q, i \omega) \rho(-q,-i \omega)\rangle$ to $O(1)$.

Since it reflects the density change resulting from a variation of the chemical potential, it is called the compressibility. As usual, a pole of $\chi_{\rho \rho}(q, \omega)$ on the real axis is a propagating mode. According to (14.41), there is such a pole when:

$$
\begin{equation*}
\frac{1}{F_{0}}=\frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left[\frac{1}{2} \frac{\omega}{v_{F} q} \ln \left(\frac{\omega+v_{F} q}{\omega-v_{F} q}\right)-1\right] \tag{14.45}
\end{equation*}
$$

or,

$$
\begin{equation*}
\frac{1}{F_{0}}=\frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left[\frac{1}{2} s \ln \left(\frac{s+1}{s-1}\right)-1\right] \tag{14.46}
\end{equation*}
$$

where $s=\omega / v_{F} q$ The solution of this equation occurs for $s>1$, i.e. there is a mode with $\omega=s v_{F} q$. In other words, this mode, called zero sound, has a velocity of propagation, $s v_{F}$, which is greater than the Fermi velocity. Figure 14.7 shows the allowed phase space for a particle-hole pair. The continuum of states composed of a particle-hole pair lies beneath the line $\omega=v_{F} q$ for $q$ small. Since the zero-sound mode lies outside this continuum for $q$ small, it cannot decay; this explains why it is a propagating mode. Energy and momentum conservation do allow it to decay into multiple particle-hole pairs, but the interactions which would allow such decay are six-fermion and higher interactions which are highly irrelevant.

q

Figure 14.7: The allowed $\omega, q$ values of the the zero sound mode and the continuum of particle-hole excitations.

According to (14.41),

$$
\begin{equation*}
\chi_{\rho \rho}(q, \omega)=\frac{\chi_{\rho \rho}^{0}(q, \omega)}{1+F_{0} \chi_{\rho \rho}^{0}(q, \omega)} \tag{14.47}
\end{equation*}
$$

where $\chi_{\rho \rho}^{0}$ is the compressibility in the absence of interactions. Let us consider the static compressibility, $\chi_{\rho \rho}(q \rightarrow 0,0)$, for both the interacting and non-interacting systems. From (14.42),

$$
\begin{align*}
\chi_{\rho \rho}^{0}(q \rightarrow 0,0) & =\frac{k_{F}^{2}}{2 \pi^{2} v_{F}} \\
& =\frac{m^{*} k_{F}}{2 \pi^{2}} \tag{14.48}
\end{align*}
$$

where $m^{*} \equiv k_{F} / v_{F}$. Hence,

$$
\begin{equation*}
\frac{\chi_{\rho \rho}(q \rightarrow 0,0)}{\chi_{\rho \rho}^{0}(q \rightarrow 0,0)}=\frac{1}{1+\frac{k_{F}^{2}}{2 \pi^{2} v_{F}} F_{0}} \tag{14.49}
\end{equation*}
$$

The compressibility is decreased by interactions if we assume that $m^{*}$ is the same in both the interacting and non-interacting systems. However, this is usually not the case.

Consider the behavior of a Fermi liquid under a Galilean boost by $\delta \vec{v}=\delta \vec{p} / m$. The kinetic term in the action transforms, but the potential energy is invariant.

$$
\begin{equation*}
S=\int d \tau d^{d} x\left(\psi^{\dagger}\left(\partial_{\tau}-\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi+\psi^{\dagger}(\vec{x}) \psi(\vec{x}) V\left(\vec{x}-\vec{x}^{\prime}\right) \psi^{\dagger}\left(\vec{x}^{\prime}\right) \psi\left(\vec{x}^{\prime}\right)\right) \tag{14.50}
\end{equation*}
$$

Similarly for the Hamiltonian,

$$
\begin{equation*}
H=\int d^{d} x\left(\psi^{\dagger}\left(\left(\frac{\nabla^{2}}{2 m}-\mu\right)\right) \psi+\psi^{\dagger}(\vec{x}) \psi(\vec{x}) V\left(\vec{x}-\vec{x}^{\prime}\right) \psi^{\dagger}\left(\vec{x}^{\prime}\right) \psi\left(\vec{x}^{\prime}\right)\right) \tag{14.51}
\end{equation*}
$$

so the energy transforms as:

$$
\begin{equation*}
\delta E=\vec{P} \cdot \delta \vec{p} / m \tag{14.52}
\end{equation*}
$$

where $\vec{P}$ is the total momentum. If we consider a state with a filled Fermi sea which has momentum zero - and a quasiparticle at the Fermi energy, and we boost the system in the direction of the quasiparticle's momentum, then

$$
\begin{equation*}
\delta E=k_{F} \delta p / m \tag{14.53}
\end{equation*}
$$

On the other hand, we can compute the energy change using Fermi liquid theory. The boost shifts the quasiparticle momentum by $\delta p$ and also moves the Fermi sea by this amount. This doesn't affect its momentum to lowest order in $\delta p$, but it does change $\psi_{i}^{\dagger} \psi_{i}$ by $\delta p \cos \theta_{i}$. Hence, the energy shift of this state is also

$$
\begin{align*}
\delta E & =v_{F} \delta p+\delta p \int \frac{d^{3} \Omega}{(2 \pi)^{3}} F(\theta, \phi) \cos \theta \\
& =v_{F} \delta p+\delta p \frac{1}{3} F_{1} \tag{14.54}
\end{align*}
$$

Hence, comparing these two expressions and using $m^{*}=k_{F} / v_{F}$, we have

$$
\begin{equation*}
\frac{m^{*}}{m}=1+\frac{1}{3}\left(\frac{F_{1}}{2 \pi^{2} v_{F}}\right) \tag{14.55}
\end{equation*}
$$

Consequently, the ratio of the interacting and free compressibilities is:

$$
\begin{equation*}
\frac{\chi_{\rho \rho}(q \rightarrow 0,0)}{\chi_{\rho \rho}^{\text {free }}(q \rightarrow 0,0)}=\frac{1+\frac{1}{3}\left(\frac{F_{1}}{2 \pi^{2} v_{F}}\right)}{1+\frac{k_{F}^{2}}{2 \pi^{2} v_{F}} F_{0}} \tag{14.56}
\end{equation*}
$$

a ratio which depends on the relative strengths of $F_{0}$ and $F_{1}$.

## Chapter 15

## Electrons and Coulomb Interactions

### 15.1 Ground State

Thus far, we have assumed that there are only short-range interactions between the fermions in our system. This assumption is appropriate for ${ }^{3} \mathrm{He}$, but not for electrons in metals which interact through the Coulomb interaction, $V(r)=\kappa e^{2} / r$. When the Coulomb interaction energy is large compared to the kinetic energy, we expect the electrons to form a Wigner crystal. If, on the other hand, the Coulomb energy is small compared to the kinetic energy, we expect the electrons to form some kind of liquid state; later in this chapter, we will show that this liquid is a Fermi liquid.

A naive comparison of these energies estimates the kinetic energy by the kinetic energy of a free Fermi gas and the interaction energy from the average Coulomb energy of a system of electrons at that density:

$$
\begin{equation*}
\frac{E_{\text {Coulomb }}}{E_{\text {Kinetic }}}=(\text { const. }) r_{s} \tag{15.1}
\end{equation*}
$$

where $r_{s}$ is ratio of the interparticle spacing to the effective Bohr radius in the metal

$$
\begin{equation*}
r_{s}=\left(\frac{3}{4 \pi n}\right)^{\frac{1}{3}} a_{0}^{-1} \tag{15.2}
\end{equation*}
$$



Figure 15.1: The graphical representation of Coulomb interactions.
and $a_{0}=1 / m \kappa e^{2}$ is the effective Bohr radius in the metal. Stated differently, charge $e$ is enclosed within a sphere of radius $r_{s} a_{0} . r_{s}$ is the controlling parameter for many of the approximations which we make in this chapter. $r_{s}$ is small in the high-density limit where we expect a Fermi liquid and large in the low-density limit where we expect Wigner crystallization.

Better estimates of the Wigner crystal and electron liquid energies can be obtained in the low- and high-density limits for a model of electrons in a fixed uniform background of positive charge (the jellium model). In the Wigner crystal state, this can be estimated to be:

$$
\begin{equation*}
\frac{E_{g}^{W C}}{n} \approx \frac{2.2099}{r_{s}^{2}}-\frac{1.7}{r_{s}} \tag{15.3}
\end{equation*}
$$

in units of the Rydberg, 13.6 eV .
In the liquid state, this can be computed perturbatively from the Lagrangian:

$$
\begin{align*}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{1}{2 m} k^{2} \psi^{\dagger}(k) \psi(k)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} d^{3} q\left(\psi^{\dagger}(k+q) \psi(k)-n \delta(q)\right) \times \\
& \frac{4 \pi \kappa e^{2}}{q^{2}}\left(\psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right)-n \delta(q)\right) \tag{15.4}
\end{align*}
$$

The Coulomb interaction is represented by a dotted line, as in figure 15.1
a)

b)


Figure 15.2: The (a) Hartree and (b) Fock contributions to the ground state energy of the electron gas.

If we expand the ground state energy perturbatively, the zeroth-order term is the kinetic energy. The first-order terms come from diagrams of (a) and (b) of figure 15.2. The first - or Hartree - term vanishes as a result of the neutralizing background (i.e. the $n \delta(q)$ ). The second - or Fock - term is non-vanishing. In the Hartree-Fock approximation, the ground state energy is given by:

$$
\begin{equation*}
E_{g}^{L}=\frac{2.2099}{r_{s}^{2}}-\frac{0.9163}{r_{s}} \tag{15.5}
\end{equation*}
$$

where the first term is the kinetic energy and the second term is the exchange energy.
The next terms in the expansion in $r_{s}$ come from summing the diagrams of figure 15.3. The first term in this series is infrared divergent, but the sum is convergent:

$$
\begin{equation*}
E_{g}^{L}=\frac{2.2099}{r_{s}^{2}}-\frac{0.9163}{r_{s}}-0.094+0.0622 \ln r_{s} \tag{15.6}
\end{equation*}
$$

This is the sum over bubble diagrams - the Random Phase Approximation - which we encountered in the small $\Lambda / k_{F}$ approximation for a Fermi liquid. In this context, it is justified for a calculation of the ground state energy in the small $r_{s}$ limit since the neglected diagrams give contributions of $O\left(r_{s}\right)$.

For $r_{s}$ large, the ground state of the electron gas is the Wigner crystal. For $r_{s}$ small, it is the liquid state, the nature of which we discuss in this chapter.


Figure 15.3: The RPA contributions to the ground state energy of the electron gas.

### 15.2 Screening

In the presence of Coulomb interactions, naive perturbation theory is infrared divergent because the interaction $V(q)=4 \pi \kappa e^{2} / q^{2}$ (unless otherwise specified, we work in $d=3$ in this chapter) is singular in the $q \rightarrow 0$ limit. In the language of the last chapter, we cannot divide the Fermi surface into $N$ patches and justify Fermi liquid theory in the large $N$ limit because the interaction $V(q)=4 \pi \kappa e^{2} / q^{2}$ is singular within a single patch when $q \rightarrow 0$.

However, the 'bare' Coulomb interaction, $V(q)=4 \pi \kappa e^{2} / q^{2}$, is not the actual interaction between two electrons. In fact, the interaction between any two electrons will be far weaker because all of the other electrons will act to screen the Coulomb interaction. The correct strategy for dealing with electrons with Coulomb interactions is to do perturbation theory in the screened Coulomb interaction. This can be done systematically, as we show in the next two sections.

First, however, we recall the Thomas-Fermi model, a simple model for screening which illustrates the basic physics in the low $q, \omega$ limit. To understand the physics at $q \rightarrow 2 k_{F}$, we'll have to use more sophisticated approximations such as the RPA. Let us imagine that we have test charges described by $\rho_{\mathrm{ext}}$. In a metal, they will induce
a charge distribution $\rho_{\text {ind }}$. According to the Laplace equation

$$
\begin{equation*}
\frac{1}{\kappa} \nabla^{2} \phi=4 \pi \rho_{\mathrm{ext}}+4 \pi \rho_{\mathrm{ind}} \tag{15.7}
\end{equation*}
$$

$\kappa$ is the dielectric constant due to the ions and the core electrons. In the ThomasFermi aproximation, we assume that $\phi$ is very slowly-varying so that we can make the approximation that the local density is given by $n(\mu+e \phi(r))$ :

$$
\begin{align*}
\rho_{\mathrm{ind}}(r) & \approx-e(n(\mu+e \phi(r))-n(\mu)) \\
& \approx-e^{2} \frac{\partial n}{\partial \mu} \phi(r) \tag{15.8}
\end{align*}
$$

Then,

$$
\begin{equation*}
\left(\frac{1}{\kappa} q^{2}+4 \pi e^{2} \frac{\partial n}{\partial \mu}\right) \phi=4 \pi \rho_{\mathrm{ext}} \tag{15.9}
\end{equation*}
$$

In other words, the bare Coulomb interaction has been replaced by a screened Coulomb interaction:

$$
\begin{equation*}
\frac{4 \pi \kappa e^{2}}{q^{2}} \rightarrow \frac{4 \pi \kappa e^{2}}{q^{2}+k_{0}^{2}} \tag{15.10}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{1}{r} \rightarrow \frac{e^{-k_{0} r}}{r} \tag{15.11}
\end{equation*}
$$

where $k_{0}$ is the inverse of the Thomas-Fermi screening length,

$$
\begin{equation*}
k_{0}=\left(4 \pi \kappa e^{2} \frac{\partial n}{\partial \mu}\right)^{\frac{1}{2}} \tag{15.12}
\end{equation*}
$$

For a free Fermi gas, $\mu=(3 \pi n)^{2 / 3} / 2 m$, so the screening length is

$$
\begin{equation*}
k_{0}^{-1}=\left(\frac{\pi}{4}\left(\frac{4}{9}\right)^{1 / 3}\right)^{\frac{1}{2}} a_{0} r_{s}^{\frac{1}{2}} \tag{15.13}
\end{equation*}
$$

When $r_{s}$ is small, i.e. when the density is large, the screening length is short and the Coulomb interaction is effectively screened. When this is true, we expect the potential
to be slowly-varying and the Thomas-Fermi approximaton to be reasonable. When $r_{s}$ is large, however, the screening length is large and we do not expect the Thomas-Fermi approximation to be valid.

A more refined result may be obtained by by replacing the bare Coulomb interaction of figure 15.1 by the sum of the diagrams of figure ??. Restricting attention to the sum of bubble diagrams is, again, the RPA approximation. The effective interaction, $V_{\mathrm{eff}}^{\mathrm{RPA}}(q, \omega)$, is:

$$
\begin{align*}
V_{\mathrm{eff}}^{\mathrm{RPA}}(q, \omega) & =V(q)+V(q) I(q, \omega) V(q)+V(q) I(q, \omega) V(q) I(q, \omega) V(q)+\ldots \\
& =\frac{V(q)}{1-I(q, \omega) V(q)} \tag{15.14}
\end{align*}
$$

where $I(q, \omega)$ is the particle-hole bubble which we evaluated in the last chapter. For $q$ small,

$$
\begin{align*}
V_{\mathrm{eff}}^{\mathrm{RPA}}(q, 0) & =\frac{V(q)}{1+\frac{m^{*} k_{F}}{2 \pi^{2}} V(q)} \\
& =\frac{4 \pi \kappa e^{2}}{q^{2}}+4 \pi \kappa e^{2} \frac{m^{*} k_{F}}{2 \pi^{2}} \\
& =\frac{4 \pi \kappa e^{2}}{q^{2}+k_{0}^{2}} \tag{15.15}
\end{align*}
$$

which is the same asthe Thomas-Fermi result. However, for $\omega \neq 0$, the RPA result contains additional information about the dynamics of the electrons. Also, for $q \rightarrow$ $2 k_{F}$, the RPA result contains information about the Fermi surface. Of course, it is not clear why we can restrict attention to the sum of bubble diagrams. As we will see below, this sum gives the leading contribution in $r_{s}$ in the limit of small $\omega, q$. For $\omega \neq 0$ and $q \rightarrow 2 k_{F}$, the RPA approximation can be called into question.

### 15.3 The Plasmon

Although Coulomb interactions are ultimately screened and therefore allow a Fermi liquid treatment, there are non-trivial differences with the case of short-range interactions. The zero-sound mode no longer has linear dispersion, $\omega=v_{s} q$. This may be seen at a classical level from Maxwell's equations together with the continuity equation.

$$
\begin{align*}
\frac{1}{\kappa} \nabla^{2} \phi & =4 \pi \rho \\
m \frac{d \vec{j}}{d t} & =n e^{2} \vec{\nabla} \phi \\
\frac{d \rho}{d t}+\vec{\nabla} \cdot \vec{j} & =0 \tag{15.16}
\end{align*}
$$

Combining these equations for a longitudinal disturbance, $\vec{j}=|j| \vec{q} /|q|$, we have

$$
\begin{equation*}
\left(\omega^{2}+\frac{4 \pi \kappa n e^{2}}{m}\right) \rho(q, \omega)=0 \tag{15.17}
\end{equation*}
$$

Hence, the frequency of a longitudinal density modulation is the plasma frequency, $\omega_{p}$

$$
\begin{equation*}
\omega_{p}=\left(\frac{4 \pi \kappa n e^{2}}{m}\right)^{\frac{1}{2}} \tag{15.18}
\end{equation*}
$$

rather than the gapless dispersion of zero sound in a neutral Fermi liquid.
The same result may be seen by, again, considering the RPA sum of bubble diagrams which determines the density-density correlation fucntion. The Landau parameter, $F_{0}$, is replaced in (14.41) by the Coulomb interaction, $V(q)$. Consequently, the pole in this correlation function now occurs at:

$$
\begin{equation*}
\frac{1}{V(q)}=\frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left[\frac{1}{2} \frac{\omega}{v_{F} q} \ln \left(\frac{\omega+v_{F} q}{\omega-v_{F} q}\right)-1\right] \tag{15.19}
\end{equation*}
$$

On the right-hand-side, take the $q \rightarrow 0$ limit with $q \ll \omega$.

$$
\frac{q^{2}}{4 \pi \kappa e^{2}}=\frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left[\frac{1}{2} \frac{\omega}{v_{F} q}\left(2 \frac{v_{F} q}{\omega}+\frac{2}{3}\left(\frac{v_{F} q}{\omega}\right)^{3}+\ldots\right)-1\right]
$$

$$
\begin{equation*}
=\frac{1}{3} \frac{k_{F}^{2}}{2 \pi^{2} v_{F}}\left(\frac{v_{F} q}{\omega}\right)^{2} \tag{15.20}
\end{equation*}
$$

or,

$$
\begin{equation*}
\omega^{2}=4 \pi \kappa e^{2} \frac{v_{F} k_{F}^{2}}{6 \pi^{2}} \tag{15.21}
\end{equation*}
$$

which is the same as (15.18).
Since $V(q) \rightarrow \infty$ as $q \rightarrow 0,\langle\rho(q, \omega) \rho(-q,-\omega)\rangle \rightarrow 0$ in this limit. One might be tempted to conclude that the compressibility of the electron gas vanishes. However, the density-density correlation function gives the compressibility in response to the applied field:

$$
\begin{equation*}
\delta\langle\rho(q, \omega)\rangle=\langle\rho(q, \omega) \rho(-q,-\omega)\rangle \delta \phi_{\mathrm{ext}}(q, \omega) \tag{15.22}
\end{equation*}
$$

In linear response, $\phi_{\text {ind }}(q, \omega)$ is given by,

$$
\begin{equation*}
\phi_{\text {ind }}(q, \omega)=-\frac{4 \pi \kappa e^{2}}{q^{2}} \chi_{\rho \rho}^{0}(q, \omega) \phi(q, \omega) \tag{15.23}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\delta\langle\rho(q, \omega)\rangle=\langle\rho(q, \omega) \rho(-q,-\omega)\rangle\left(1+\frac{4 \pi \kappa e^{2}}{q^{2}} \chi_{\rho \rho}^{0}(q, \omega)\right) \delta \phi(q, \omega) \tag{15.24}
\end{equation*}
$$

so the compressibility is finite as $q \rightarrow 0$.
In this section, we will show, following Bohm and Pines, how to separate the plasma oscillation from the rest of the degrees of freedom of an electronic system. When this is done, the remaining electronic degrees of freedom interact through a short-ranged, screened Coulomb interaction. Essentially, gauge invariance tells us that longitudinal photons - whose exchange gives rise to the Coulomb interaction - and density fluctuations are not distinct objects. When long-wavelength density fluctuations aquire a mass gap as a result of their self-interaction, they (and the longitudinal photons to which they are equivalent) can no longer propagate over long-distances. Consequently, the Coulomb interaction becomes short-ranged.

To exhibit this clearly, we make the following manipulations:

- Electrons with Coulomb Interactions. We begin with the action of a system of electrons with Coulomb interactions:

$$
\begin{aligned}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{1}{2 m} k^{2} \psi^{\dagger}(k) \psi(k)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right)(15.25)
\end{aligned}
$$

- Electrons interacting with Longitudinal Photons. The long-range $1 / q^{2}$ interaction results from integrating out the longitudinal part of the electromagnetic field. We could equivalently write this as

$$
\begin{align*}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+e \psi^{\dagger} A_{0} \psi+\frac{1}{2 m} \psi^{\dagger}(k)(\vec{k}+e \vec{A})^{2} \psi(k)\right) \\
& +\int d \tau \int d^{3} k \frac{1}{8 \pi \kappa} E(k) E(-k) \tag{15.26}
\end{align*}
$$

The magnetic part of the electromagnetic action has been dropped since we assume that all velocities are much smaller than the speed of light; we keep only the longitudinal modes of the electromagnetic field. Equation (15.25) is obtained from (15.26) by integrating out the electromagnetic field. To do this, we will choose Coulomb gauge, $A_{0}=0$. In doing so we must, however, impose the Gauss' law constraint which is the $A_{0}$ equation of motion.

$$
\begin{align*}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{1}{2 m} \psi^{\dagger}(k)(\vec{k}+e \vec{A})^{2} \psi(k)\right) \\
& +\int d \tau \int d^{3} k \frac{1}{8 \pi \kappa}\left(\partial_{\tau} A(k)\right)^{2}+\frac{1}{4 \pi \kappa} A_{0}\left(\vec{\nabla} \cdot \vec{E}-4 \pi \kappa e \psi^{\dagger} \psi\right)( \tag{15.27}
\end{align*}
$$

Note that $A(k)$ is a scalar field because it is only the longitudinal part of the electromagnetic field - which not independent of the density fluctuations of the electrons. The real dynamics of the electromagnetic field is in its transverse components, which do not enter here. If we were to integrate out $A(k)$, then, since $A(k)$ is gapless at tree-level, we would get Coulomb interactions between electrons. However, a tree-level analysis misses the fact that $A(k)$ is, in fact, not gapless.

- Integrate out Short-Wavelength Photons. Instead of integrating out $A(k)$ fully, let us instead only integrate out those modes of $A(k)$ with $k>Q$ for some Q. Bohm and Pines did this at the Hamiltonian level, by applying a canonical transformation of the form:

$$
\begin{equation*}
U=e^{-i \int_{Q}^{\Lambda} d^{3} q a(q) \frac{\sqrt{4 \pi \kappa \kappa^{2}}}{q} \int d^{3} k \psi^{\dagger}(k+q) \psi(k)} \tag{15.28}
\end{equation*}
$$

to the Hamiltonian corresponding to (15.27). $\Lambda$ is the upper cutoff and $Q$ is a wavevector to be determined.

Then, we obtain an action of the form:

$$
\begin{aligned}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{1}{2 m} \psi^{\dagger}(k)(\vec{k}+e \vec{A})^{2} \psi(k)\right) \\
& +\int d \tau \int_{0}^{Q} d^{3} q\left(\frac{1}{8 \pi \kappa}\left(\partial_{\tau} A(q)\right)^{2}+\frac{1}{4 \pi \kappa} A_{0}\left(\vec{\nabla} \cdot \vec{E}(q)-4 \pi \kappa e \psi^{\dagger} \psi\right)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} \int_{Q}^{\Lambda} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right)(15.29
\end{aligned}
$$

Notice that the four-fermion interaction is now short-ranged since it is restricted to $|q|>Q$.

- Isolate the term which gives a gap to long-wavelength Photons. We now expand $(\vec{k}+e \vec{A})^{2}$ :

$$
\begin{align*}
S= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{k^{2}}{2 m} \psi^{\dagger}(k) \psi(k)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} \int_{Q}^{\Lambda} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right) \\
& +\int d \tau \int d^{3} k \int_{0}^{Q} d^{3} q \frac{e}{m}\left(\vec{k}+\frac{\vec{q}}{2}\right) \cdot \vec{A}(-q) \psi^{\dagger}(k+q) \psi(k) \\
& +\int d \tau \int d^{3} k \int_{0}^{Q} d^{3} q \int_{0}^{Q} d^{3} q^{\prime} \frac{e^{2}}{2 m} A(q) A\left(q^{\prime}\right) \psi^{\dagger}\left(k-q-q^{\prime}\right) \psi(k) \\
& +\int d \tau \int_{0}^{Q} d^{3} q\left(\frac{1}{8 \pi \kappa}\left|\partial_{\tau} A(q)\right|^{2}+\frac{1}{4 \pi \kappa} A_{0}\left(\vec{\nabla} \cdot \vec{E}(q)-4 \pi \kappa e \psi^{\dagger} \psi() \| \overline{\$} .\right.\right.
\end{align*}
$$

We split the third line into a part which comes from the average density, $n$, and a part resulting from fluctuations in the density:

$$
\int d \tau \int d^{3} k \int_{0}^{Q} d^{3} q \int_{0}^{Q} d^{3} q^{\prime} \frac{e^{2}}{2 m} A(q) A\left(q^{\prime}\right) \psi^{\dagger}\left(k-q-q^{\prime}\right) \psi(k)=
$$

$$
\begin{align*}
& \int d \tau \int_{0}^{Q} d^{3} q \frac{n e^{2}}{2 m} A(q) A(-q) \\
& +\int d \tau \int_{0}^{Q} d^{3} q \int_{0}^{Q} d^{3} q^{\prime} \frac{e^{2}}{2 m} A(q) A\left(q^{\prime}\right)\left(\int d^{3} k \psi^{\dagger}\left(k-q-q^{\prime}\right) \psi(k)-n \delta\left(q+q^{\prime}\right)\right) \tag{15.31}
\end{align*}
$$

the first term on the right-hand-side can be combined with the $\left|\partial_{\tau} A(q)\right|^{2}$ term to give $S_{\mathrm{P}}$ in the action:

$$
\begin{equation*}
S=S_{\mathrm{FL}}+S_{\mathrm{P}}+S_{\mathrm{Int}}+S_{\mathrm{C}} \tag{15.32}
\end{equation*}
$$

with

$$
\begin{align*}
S_{\mathrm{FL}}= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{k^{2}}{2 m} \psi^{\dagger}(k) \psi(k)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} \int_{Q}^{\Lambda} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right) \\
S_{\mathrm{P}}= & \frac{1}{8 \pi \kappa} \int d \tau \int_{0}^{Q} d^{3} q\left(\left|\partial_{\tau} A(q)\right|^{2}+\omega_{p}^{2}|A(q)|^{2}\right) \\
S_{\mathrm{Int}}= & \int d \tau \int d^{3} k \int_{0}^{Q} d^{3} q \frac{e}{m}\left(\vec{k}+\frac{\vec{q}}{2}\right) \cdot \vec{A}(-q) \psi^{\dagger}(k+q) \psi(k) \\
& +\int d \tau \int_{0}^{Q} d^{3} q \int_{0}^{Q} d^{3} q^{\prime} \frac{e^{2}}{2 m} A(q) A\left(q^{\prime}\right)\left(\int d^{3} k \psi^{\dagger}\left(k-q-q^{\prime}\right) \psi(k)-n \delta\left(q+q^{\prime}\right)\right) \\
S_{\mathrm{C}}= & \int d \tau \int_{0}^{Q} d^{3} q\left(\frac{1}{4 \pi \kappa} A_{0}\left(\vec{\nabla} \cdot \vec{E}(q)-4 \pi \kappa e \psi^{\dagger} \psi\right)\right) \tag{15.33}
\end{align*}
$$

$S_{\mathrm{FL}}$ is the action of electrons with short-range interactions. $S_{\mathrm{P}}$ is the action of plasmon modes $A(q)$ with $|q|<Q$; these modes have frequency $\omega_{p}$. If we were to integrate them out now, they would mediate a short-range interaction, not the long-range Coulomb interaction. $S_{\text {Int }}$ describes the interaction between electrons and plasmons. $S_{\mathrm{C}}$ imposes the constraints which eliminate the additional degrees of freedom introduced with the plasmons; these degrees of freedom are not gauge invariant and are, therefore, unphysical.

By separating the plasmon from the other electronic degrees of freedom, we have obtained a theory of electrons with short-range interactions. The basic
physics is already clear from (15.32). However, we are not yet in a position to make quantitative predictions. The interaction depends on a free parameter, $Q$, and is not the Thomas-Fermi interaction in the $\omega, q \rightarrow 0$ limit. To understand the electron gas at a quantitative level, we must consider $S_{\mathrm{Int}}$ and $S_{\mathrm{C}}$.

### 15.4 RPA

The conclusions which we drew at the end of the previous section were based on a neglect of $S_{\text {Int }}$ and $S_{\mathrm{C}}$. In this section, we consider $S_{\text {Int }}$ and the RPA approximation which simplifies it. We have used the term RPA in several contexts. The definition of the RPA is the following. We neglect the coupling between $\rho(\vec{q})$ and $\rho(\vec{q})$ if $\vec{q} \neq \vec{q}$. In the computation of a correlation function at $\vec{q}$, we only consider diagrams in which the dotted Coulomb interaction line carries momentum $\vec{q}$. In other words, $V(\vec{q})$ does not appear in these diagrams unless $\vec{q}=\vec{q}$. The RPA is justified in the limit of small $r_{s}$ and the limit $q \rightarrow 0$. For the density-density response function or the ground state energy, this amounts to keeping only the bubble diagrams and neglecting other diagrams.

The first step is to choose a $Q$ which optimizes $S_{\mathrm{FL}}+S_{\mathrm{P}}$, thereby making the effect of $S_{\text {Int }}$ as small as possible. Without proof, we state that we can minimize the energy of the ground state of $S_{\mathrm{FL}}+S_{\mathrm{P}}$ (computed to lowest order in the screened Coulomb interaction) by taking

$$
\begin{equation*}
Q \approx k_{F}\left(\frac{r_{s}}{4}\right)^{\frac{1}{4}} \tag{15.34}
\end{equation*}
$$

Physicaly, $Q$ must be finite since, for $q$ large, the plasmon mixes with the particle-hole continuum and is no longer a well-defined mode.

We now make the Random Phase Approximation, or RPA and completely neglect
the term:

$$
\begin{equation*}
\int d \tau \int_{0}^{Q} d^{3} q \int_{0}^{Q} d^{3} q^{\prime} \frac{e^{2}}{2 m} A(q) A\left(q^{\prime}\right)\left(\int d^{3} k \psi^{\dagger}\left(k-q-q^{\prime}\right) \psi(k)-n \delta\left(q+q^{\prime}\right)\right) \tag{15.35}
\end{equation*}
$$

in $S_{\text {Int }}$. To justify the RPA, consider the effect of this term on the ground state energy. It shifts this energy by

$$
\begin{align*}
\Delta E & \sim \frac{Q^{3}}{2 k_{F}^{3}} Q^{3} \omega_{p} \\
& \sim\left(\frac{r_{s}}{4}\right)^{\frac{3}{4}} Q^{3} \omega_{p} \tag{15.36}
\end{align*}
$$

Hence, the random phase approximation is valid in the small $r_{s}$ limit since the energy shift is small compared to the plasmon zero-point energy.

We are now left with the action

$$
\begin{align*}
S_{\mathrm{RPA}}= & \int d \tau \int d^{3} k\left(\psi^{\dagger} \partial_{\tau} \psi+\frac{k^{2}}{2 m} \psi^{\dagger}(k) \psi(k)\right) \\
& +\int d \tau \int d^{3} k d^{3} k^{\prime} \int_{Q}^{\Lambda} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right) \\
& +\frac{1}{8 \pi \kappa} \int d \tau \int_{0}^{Q} d^{3} q\left(\left|\partial_{\tau} A(q)\right|^{2}+\omega_{p}^{2}|A(q)|^{2}\right) \\
& +\int d \tau \int d^{3} k \int_{0}^{Q} d^{3} q \frac{e}{m}\left(\vec{k}+\frac{\vec{q}}{2}\right) \cdot \vec{A}(-q) \psi^{\dagger}(k+q) \psi(k) \\
& +\int d \tau \int_{0}^{Q} d^{3} q\left(\frac{1}{4 \pi \kappa} A_{0}\left(\vec{\nabla} \cdot \vec{E}(q)-4 \pi \kappa e \psi^{\dagger} \psi\right)\right) \tag{15.37}
\end{align*}
$$

If we could ignore the last line, we would have a theory of electrons with shortrange interactions together with gapped plasmons. At frequencies or temperatures much less than $\omega_{p}$, we can ignore the plasmons, so we would have a Fermi liquid. However, the constraint cannot be ignored. Treating the electrons and plasmons as fully independent would be a double-counting of the degrees of freedom of the system. What we can do, instead, is decouple the plasmon from the electrons. When this is done, the constraint will only involve the particles. If we ignore the constraint - which is now a constraint on the electrons alone - then we can apply Fermi liquid theory to the electronic action. Fermi liquid theory (as we saw in the last chapter) instructs us to compute only bubble diagrams to obtain the screened Coulomb interaction.

### 15.5 Fermi Liquid Theory for the Electron Gas

Following Bohm and Pines, we now perform a canonical transformation,

$$
\begin{align*}
\mathcal{O} & \rightarrow e^{-i \mathcal{S}} \mathcal{O} e^{i \mathcal{S}} \\
|\chi\rangle & \rightarrow e^{-i \mathcal{S}}|\chi\rangle \tag{15.38}
\end{align*}
$$

generated by $\mathcal{S}$ :

$$
\mathcal{S}=\frac{e}{m} \int_{0}^{Q} d^{3} q \int d^{3} k \frac{1}{\omega\left(q, \psi, \psi^{\dagger}\right)-\vec{q} \cdot \vec{k}+q^{2} / 2 m}\left\{\left(\vec{k}+\frac{\vec{q}}{2}\right) \cdot \vec{A}(-q) \psi^{\dagger}(k+q) \psi(k f)\right] .5 .39
$$

where

$$
\begin{equation*}
\omega\left(q, \psi, \psi^{\dagger}\right)=\omega_{p}\left(1+\frac{q^{2}}{2 n m^{2} \omega_{p}^{2}}\left(\int d^{3} k k^{2} \psi^{\dagger}(k) \psi(k)\right)+\frac{q^{4}}{8 m^{2} \omega_{p}^{2}}\right) \tag{15.40}
\end{equation*}
$$

The principal results of this canonical transformation in the limit $r_{s} \rightarrow 0$ are:

- The elimination of the $A \psi^{\dagger} \psi$ interaction between plasmons and electrons
- The modification of the plasmon action to:

$$
\begin{equation*}
S_{P}=\frac{1}{8 \pi \kappa} \int d \tau \int_{0}^{Q} d^{3} q\left(\left|\partial_{\tau} A(q)\right|^{2}+\omega\left(q, \psi, \psi^{\dagger}\right)|A(q)|^{2}\right) \tag{15.41}
\end{equation*}
$$

- The replacement of the cutoff Coulomb interaction by the RPA screened Coulomb interaction,

$$
\begin{equation*}
\int_{Q}^{\Lambda} d^{3} q \psi^{\dagger}(k+q) \psi(k) \frac{4 \pi \kappa e^{2}}{q^{2}} \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right) \rightarrow \int d^{3} q \psi^{\dagger}(k+q) \psi(k) V_{\mathrm{RPA}}(q) \psi^{\dagger}\left(k^{\prime}-q\right) \psi\left(k^{\prime}\right) \tag{15.42}
\end{equation*}
$$

- The elimination of the plasmons from the contraints. The constraints now read:

$$
\begin{equation*}
\int d^{3} k \frac{\omega\left(q, \psi, \psi^{\dagger}\right)}{\left(\omega\left(q, \psi, \psi^{\dagger}\right)\right)^{2}-\left(\vec{q} \cdot \vec{k}-q^{2} / 2 m\right)^{2}} \psi^{\dagger}(k+q) \psi(k)=0 \tag{15.43}
\end{equation*}
$$

for $|q|<Q$.

Hence, we now have a theory of weakly-coupled electrons and plasmons. The electrons interact through a short-ranged interaction (which can be obtained by summing bubble diagrams). The contraints reduce the number of degrees of freedom of the electrons. For $Q$ small, this is assumed to have a small effect on the electronic degrees of freedom.

## Chapter 16

## Electron-Phonon Interaction

### 16.1 Electron-Phonon Hamiltonian

### 16.2 Feynman Rules

### 16.3 Phonon Green Function

### 16.4 Electron Green Function

Let us consider the electron-phonon interaction,

$$
\begin{equation*}
S_{\mathrm{el}-\mathrm{ph}}=g \int d \tau d^{3} \vec{x} \psi^{\dagger} \psi \partial_{i} u_{i} \tag{16.1}
\end{equation*}
$$

which couples electrons to transverse phonons. What effect does this have on the electron Green function?

The one-loop electron self-energy is given by the diagrams of figure 16.1. The first diagram just shifts the chemical potential. At zero-temperature, the second diagram gives a contribution:

$$
\Sigma(i \epsilon, k)=g^{2} \int \frac{d \omega}{2 \pi} \frac{d^{3} q}{(2 \pi)^{3}} \mathcal{G}(i \epsilon-i \omega, k-q) \mathcal{D}(i \omega, q)
$$



Figure 16.1: The one-loop diagrams contributing to the electron-self-energy.

$$
\begin{equation*}
=g^{2} \int \frac{d \omega}{2 \pi} \frac{d^{3} q}{(2 \pi)^{3}} \frac{1}{i \epsilon-i \omega-\xi_{k-q}} \frac{q^{2}}{-\omega^{2}-v^{2} q^{2}} \tag{16.2}
\end{equation*}
$$

Closing the contour in the upper-half-plane, we pick up the pole at $i \omega=i \epsilon-\xi_{k-q}$ if $\xi_{k-q}>0$ and the pole at $i \omega=-v q$ :

$$
\begin{equation*}
\Sigma(i \epsilon, k)=g^{2} \int \frac{d^{3} q}{(2 \pi)^{3}}\left[\frac{q^{2}}{-2 q v} \frac{1}{i \epsilon+v q-\xi_{k-q}}+\frac{q^{2} \theta\left(\xi_{k-q}\right)}{\left(i \epsilon-\xi_{k-q}\right)^{2}-v^{2} q^{2}}\right] \tag{16.3}
\end{equation*}
$$

Analytically continuing $i \omega \rightarrow \omega+i \delta$ to obtain the retarded self-energy,

$$
\begin{equation*}
\operatorname{Re} \Sigma_{\text {ret }}(\epsilon, k)=g^{2} \int \frac{d^{3} q}{(2 \pi)^{3}}\left[\frac{q}{-2 v} \frac{1}{\epsilon+v q-\xi_{k-q}}+\frac{q^{2} \theta\left(\xi_{k-q}\right)}{\left(\epsilon-\xi_{k-q}\right)^{2}-v^{2} q^{2}}\right] \tag{16.4}
\end{equation*}
$$

At small $\epsilon$, dropping a constant shift in the chemical potential, we have:

$$
\begin{equation*}
\operatorname{Re} \Sigma_{\mathrm{ret}}(\epsilon, k)=g^{2} \epsilon \int \frac{d^{3} q}{(2 \pi)^{3}}\left[\frac{q}{2 v} \frac{1}{\left(v q-\xi_{k-q}\right)^{2}}-\frac{2 \xi_{k-q} q^{2} \theta\left(\xi_{k-q}\right)}{\left(\xi_{k-q}{ }^{2}-v^{2} q^{2}\right)^{2}}\right] \tag{16.5}
\end{equation*}
$$

We can take $k \rightarrow k_{F}$ in this integral to obtain the leading behavior:

$$
\begin{equation*}
\operatorname{Re} \Sigma_{\text {ret }}(\epsilon, k) \sim g^{2} \epsilon \tag{16.6}
\end{equation*}
$$

Meanwhile,

$$
\begin{aligned}
\operatorname{Im} \Sigma_{\mathrm{ret}}(\epsilon, k) & =g^{2} \int \frac{d^{3} q}{(2 \pi)^{3}} \frac{q}{2 v}\left[-\left(1-\theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon+v q-\xi_{k-q}\right)-\theta\left(\xi_{k-q}\right) \delta\left(\epsilon-v q-\xi_{k-q}\right)\right] \\
& =\frac{g^{2}}{2(2 \pi)^{2}} \int q^{3} d q d(\cos \theta)\left[-\left(1-\theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon+v q-\xi_{k-q}\right)\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.-\theta\left(\xi_{k-q}\right) \delta\left(\epsilon-v q-\xi_{k-q}\right)\right] \tag{16.7}
\end{equation*}
$$

For $\epsilon$ small, $q \sim \epsilon$, and $q^{2}$ terms in the $\delta$ function can be dropped:

$$
\begin{align*}
\operatorname{Im} \Sigma_{\mathrm{ret}}(\epsilon, k) & =\frac{g^{2}}{2(2 \pi)^{2}} \int q^{3} d q d(\cos \theta)\left[-\left(1-\theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon+\left(v+v_{F} \cos \theta\right) q\right)\right. \\
& \sim g^{2} \epsilon^{3}
\end{align*}
$$

### 16.5 Polarons

## Chapter 17

## Superconductivity

### 17.1 Instabilities of the Fermi Liquid

When a fixed point has a relevant perturbation, this perturbation generally leads to a fundamental reorganization of the ground state. We saw a trivial example of this with a shift of the chemical potential of a Fermi liquid. When the instability is due to interaction terms, the general strategy is to use the RG to go to low energies so that the irrelevant variables have all become small and the relevant variable is dominant. The problem with a single relevant interaction must then be solved by finding a new saddle-point (i.e. mean field theory), the variational method, or some other non-perturbative method. This approach has proven very successful in the study of ordering in condensed matter physics. (Sometimes, there are competing instabilities in which case it is very difficult to find a new saddle-point or an appropriate variational ansatz. This occurs in the case of a $1 D$ system of fermions.) In the case of electrons in a solid, the Fermi surface need not be rotationally symmetric, and spin- and chargedensity wave instabilities are possible when the Fermi surface satisfies certain special conditions ('nesting'). If the Fermi surface is rotationally symmetric, there is only one instability, as we found earlier: the Cooper pairing instability.

Consider the action of electrons in $D=2$ with $F=0$ but non-zero $V$,

$$
\begin{aligned}
S= & \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{\sigma}^{\dagger}(\epsilon, k)\left(i \epsilon-v_{F} k\right) \psi_{\sigma}(\epsilon, k) \\
& \left.-\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} \frac{d \epsilon_{3}}{2 \pi} \psi_{\uparrow}^{\dagger}\left(\epsilon_{4}, k^{\prime}\right) \psi_{\downarrow}^{\dagger}\left(\epsilon_{3},-k^{\prime}\right) V\left(k, k^{\prime}\right) \psi_{\uparrow}\left(\epsilon_{2},-k\right) \psi_{\downarrow}\left(\epsilon_{1}, 7\right)\right),
\end{aligned}
$$

where $V\left(k, k^{\prime}\right) \equiv V\left(\theta_{1}-\theta_{2}\right)$ is a function of the angles only. Unlike in previous chapters, where we dealt with spinless fermions, we now consider spin $1 / 2$ electrons.

In chapter 14 , we showed that the Fourier modes of $V\left(\theta_{1}-\theta_{2}\right)$ satisfy the RG equation:

$$
\begin{equation*}
\frac{d V_{m}}{d \ell}=-\frac{1}{2 \pi v_{F}} V_{m}^{2} \tag{17.2}
\end{equation*}
$$

When negative, these are relevant. In the next section, we will find the new saddle point which is appropriate for the case in which $V$ is relevant. We will also mention briefly the equivalent variational ansatz (which was the historical method of solution).

### 17.2 Saddle-Point Approximation

We introduce a Hubbard-Stratonovich field $\Psi(k, \omega)$ to decouple the BCS interaction:

$$
\begin{align*}
S= & \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{\sigma}^{\dagger}(\epsilon, k)\left(i \epsilon-v_{F} k\right) \psi_{\sigma}(\epsilon, k) \\
& -\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi} V\left(k, k^{\prime}\right)\left[\psi_{\uparrow}^{\dagger}\left(\epsilon_{1}, k^{\prime}\right) \psi_{\downarrow}^{\dagger}\left(\epsilon_{2},-k^{\prime}\right) \Psi\left(\epsilon_{1}+\epsilon_{2}, k\right)\right. \\
& \left.+\psi_{\uparrow}\left(\epsilon_{1}, k^{\prime}\right) \psi_{\downarrow}\left(\epsilon_{2},-k^{\prime}\right) \Psi^{\dagger}\left(\epsilon_{1}, \epsilon_{2}, k\right)+\Psi^{\dagger}\left(\epsilon_{1}+\epsilon_{2}, k\right) \Psi\left(\epsilon_{1}+\epsilon_{2}, k^{\prime}\right)\right] \tag{17.3}
\end{align*}
$$

We now make the change of variables:

$$
\begin{equation*}
\Delta\left(\epsilon_{1}+\epsilon_{2}, k\right)=\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} V\left(k, k^{\prime}\right) \Psi^{\dagger}\left(\epsilon_{1}+\epsilon_{2}, k^{\prime}\right) \tag{17.4}
\end{equation*}
$$

Then, the action can be rewritten:

$$
S=\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \psi_{\sigma}^{\dagger}(\epsilon, k)\left(i \epsilon-v_{F} k\right) \psi_{\sigma}(\epsilon, k)
$$

$$
\begin{aligned}
- & \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{d \epsilon_{1}}{2 \pi} \frac{d \epsilon_{2}}{2 \pi}\left[\psi_{\uparrow}^{\dagger}\left(\epsilon_{1}, k^{\prime}\right) \psi_{\downarrow}^{\dagger}\left(\epsilon_{2},-k^{\prime}\right) \Delta\left(\epsilon_{1}+\epsilon_{2}, k\right)\right. \\
& +\psi_{\uparrow}\left(\epsilon_{1}, k^{\prime}\right) \psi_{\downarrow}\left(\epsilon_{2},-k^{\prime}\right) \Delta^{\dagger}\left(\epsilon_{1}+\epsilon_{2}, k\right)+\Delta^{\dagger}\left(\epsilon_{1}+\epsilon_{2}, k\right) V^{-1}\left(k, k^{\prime}\right) \Delta\left(\epsilon_{1}+\epsilon_{\ell} \downarrow \pi_{k}^{\prime} . \$\right]
\end{aligned}
$$

where $V^{-1}\left(k, k^{\prime}\right)$ is the inverse of $V\left(k, k^{\prime}\right)$ :

$$
\begin{equation*}
\int \frac{d^{2} k}{(2 \pi)^{2}} V^{-1}\left(k, k^{\prime}\right) V\left(k^{\prime}, k^{\prime \prime}\right)=\delta\left(k-k^{\prime \prime}\right) \tag{17.6}
\end{equation*}
$$

Since the action is quadratic in the fermion fields $\psi_{\sigma}$, we can integrate out the fermions to get an effective action $S[\Delta]$ :

$$
\begin{align*}
S[\Delta]= & -\operatorname{Tr} \ln \left((i \epsilon)^{2}-\left(v_{F} k\right)^{2}-|\Delta(k)|^{2}\right)+ \\
& \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \Delta^{\dagger}(\epsilon, k) V^{-1}\left(k, k^{\prime}\right) \Delta\left(\epsilon, k^{\prime}\right) \\
= & -\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \ln \left((i \epsilon)^{2}-\left(v_{F} k\right)^{2}-|\Delta(k)|^{2}\right)+ \\
& \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{d \epsilon_{1}}{2 \pi} \Delta^{\dagger}(\epsilon, k) V^{-1}\left(k, k^{\prime}\right) \Delta\left(\epsilon, k^{\prime}\right) \tag{17.7}
\end{align*}
$$

We look for a frequency-independent solution, $\Delta(\epsilon, k)=\Delta(k)$ of the saddle point equations,

$$
\begin{equation*}
\frac{\delta S}{\delta \Delta}=0 \tag{17.8}
\end{equation*}
$$

From (17.7), we have the saddle-point equations:

$$
\begin{equation*}
\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \frac{1}{(i \epsilon)^{2}-\left(v_{F} k\right)^{2}-|\Delta(k)|^{2}}=\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} V^{-1}\left(k, k^{\prime}\right) \Delta\left(k^{\prime}\right) \tag{17.9}
\end{equation*}
$$

At zero-temperature, the $\epsilon$ integral in the first term can be done (at finite-temperature, we must do a Matsubara sum instead), giving:

$$
\begin{equation*}
\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{1}{\sqrt{\left(v_{F} k\right)^{2}+|\Delta(k)|^{2}}}=\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} V^{-1}\left(k, k^{\prime}\right) \Delta\left(k^{\prime}\right) \tag{17.10}
\end{equation*}
$$

or

$$
\begin{equation*}
\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{V\left(k, k^{\prime}\right) \Delta\left(k^{\prime}\right)}{\sqrt{\left(v_{F} k^{\prime}\right)^{2}+\left|\Delta\left(k^{\prime}\right)\right|^{2}}}=\Delta(k) \tag{17.11}
\end{equation*}
$$

The is the BCS gap equation. It determines the order parameter $\Delta$ which breaks the $U(1)$ symmetry $\Delta \rightarrow e^{i \theta} \Delta$ of the action (17.7).

For $V$ attractive, i.e. $V>0$, this equation always has a solution. Consider the simplest case, of an $s$-wave attraction, $V\left(k, k^{\prime}\right)=V$. Then the gap equation reads:

$$
\begin{equation*}
\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{V \Delta}{\sqrt{\left(v_{F} k^{\prime}\right)^{2}+|\Delta|^{2}}}=\Delta \tag{17.12}
\end{equation*}
$$

or,

$$
\begin{equation*}
\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{1}{\sqrt{\left(v_{F} k^{\prime}\right)^{2}+|\Delta|^{2}}}=\frac{1}{V} \tag{17.13}
\end{equation*}
$$

Since the left-hand-side is logarithmically divergent at the Fermi surface if $\Delta=0$, there is always a non-trivial saddle-point solution when $V>0$.

$$
\begin{equation*}
\frac{m^{*}}{2 \pi} \int d \xi \frac{1}{\sqrt{\xi^{2}+\Delta^{2}}}=\frac{1}{V} \tag{17.14}
\end{equation*}
$$

or

$$
\begin{equation*}
\Delta=\frac{\Lambda}{\sinh \frac{2 \pi}{m^{*} V}} \tag{17.15}
\end{equation*}
$$

If the attraction is weak, $m^{*} V / 2 \pi \ll 1$, then

$$
\begin{equation*}
\Delta=2 \Lambda e^{-\frac{2 \pi}{m^{*} V}} \tag{17.16}
\end{equation*}
$$

Note that the gap is not analytic in $V$; it could never be discovered in perturbation theory.

As you will show in the problem set, the finite-temperature gap equation is:

$$
\begin{equation*}
\int \frac{d^{2} k^{\prime}}{(2 \pi)^{2}} \frac{V\left(k, k^{\prime}\right) \Delta\left(k^{\prime}\right)}{\sqrt{\left(v_{F} k^{\prime}\right)^{2}+\left|\Delta\left(k^{\prime}\right)\right|^{2}}} \tanh \frac{\beta E_{k^{\prime}}}{2}=\Delta(k) \tag{17.17}
\end{equation*}
$$

with

$$
\begin{equation*}
E_{k}=\sqrt{\left(v_{F} k\right)^{2}+|\Delta(k)|^{2}} \tag{17.18}
\end{equation*}
$$

For an $s$-wave attraction, this gap equation has solution $\Delta=$ when:

$$
\begin{equation*}
\frac{m^{*}}{2 \pi} \int d \xi \frac{1}{\sqrt{\xi^{2}}} \tanh \frac{\beta \xi}{2}=\frac{1}{V} \tag{17.19}
\end{equation*}
$$

So the critical temperature for the onset of superconductivity is:

$$
\begin{equation*}
T_{c}=1.14 \Lambda e^{-\frac{2 \pi}{m^{*} V}} \tag{17.20}
\end{equation*}
$$

### 17.3 BCS Variational Wavefunction

For purposes of comparison, consider the route taken by Bardeen, Cooper, and Schrieffer. They wrote down the wavefunction

$$
\begin{equation*}
\left|\Psi_{0}\right\rangle=\prod_{k}\left(u_{k}+v_{k} \psi_{k \uparrow}^{\dagger} \psi_{-k \downarrow}^{\dagger}\right)|0\rangle \tag{17.21}
\end{equation*}
$$

with where the $u_{k}$ 's and $v_{k}$ 's are variational parameters with respect to which $\left\langle\Psi_{0}\right| H\left|\Psi_{0}\right\rangle$ is minimized. The wavefunction is normalized by taking

$$
\begin{equation*}
u_{k}^{2}+v_{k}^{2}=1 \tag{17.22}
\end{equation*}
$$

For notational simplicity, we assume that the $k$ 's are discrete (as they are in a finitesize system). The Hamiltonian which follows from (17.1) is:

$$
\begin{equation*}
H=\sum_{k} \xi_{k} \psi_{k \sigma}^{\dagger} \psi_{k \sigma}-\sum_{k, k^{\prime}} V\left(k, k^{\prime}\right) \psi_{k \uparrow}^{\dagger} \psi_{-k \downarrow}^{\dagger} \psi_{k^{\prime} \uparrow} \psi_{-k^{\prime} \downarrow} \tag{17.23}
\end{equation*}
$$

This Hamiltonian is called the BCS reduced Hamiltonian. It is the Hamiltonian which only contains the relevant interaction. The irrelevant and marginal interactions have been dropped. The expectation value of the Hamiltonian is:

$$
\begin{equation*}
\left\langle\Psi_{0}\right| H\left|\Psi_{0}\right\rangle=\sum_{k} 2 v_{k}^{2} \xi_{k}-\sum_{k, k^{\prime}} V\left(k, k^{\prime}\right) u_{k} v_{k} u_{k^{\prime}} v_{k^{\prime}} \tag{17.24}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\frac{\partial}{\partial v_{k}}\left\langle\Psi_{0}\right| H\left|\Psi_{0}\right\rangle & =4 v_{k} \xi_{k}-\sum_{k^{\prime}} V\left(k, k^{\prime}\right)\left[u_{k} u_{k^{\prime}} v_{k^{\prime}}+2 \frac{\partial u_{k}}{\partial v_{k}} v_{k} u_{k^{\prime}} v_{k^{\prime}}\right] \\
& =4 v_{k} \xi_{k}-2 \sum_{k^{\prime}} V\left(k, k^{\prime}\right)\left(\frac{u_{k}^{2}-v_{k}^{2}}{u_{k}}\right) u_{k^{\prime}} v_{k^{\prime}} \tag{17.25}
\end{align*}
$$

The minimum of $\left\langle\Psi_{0}\right| H\left|\Psi_{0}\right\rangle$ occurs when

$$
\begin{equation*}
2 \xi_{k} u_{k} v_{k}=\sum_{k^{\prime}} V\left(k, k^{\prime}\right)\left(u_{k}^{2}-v_{k}^{2}\right) u_{k^{\prime}} v_{k^{\prime}} \tag{17.26}
\end{equation*}
$$

If we define $\Delta(k)$ by

$$
\begin{equation*}
u_{k} v_{k}=\frac{\Delta(k)}{2 \sqrt{\xi_{k}^{2}+|\Delta(k)|^{2}}} \tag{17.27}
\end{equation*}
$$

or, equivalently,

$$
\begin{align*}
& u_{k}=\frac{1}{\sqrt{2}}\left(1+\frac{\xi_{k}}{E_{k}}\right)^{\frac{1}{2}} \\
& v_{k}=\frac{1}{\sqrt{2}}\left(1-\frac{\xi_{k}}{E_{k}}\right)^{\frac{1}{2}} \tag{17.28}
\end{align*}
$$

with

$$
\begin{equation*}
E_{k}=\sqrt{\xi_{k}^{2}+|\Delta(k)|^{2}} \tag{17.29}
\end{equation*}
$$

Then we can rewrite the minimization condition as the BCS gap equation:

$$
\begin{equation*}
\sum_{k} V\left(k, k^{\prime}\right) \frac{\Delta\left(k^{\prime}\right)}{\sqrt{\xi_{k}^{2}+\left|\Delta\left(k^{\prime}\right)\right|^{2}}}=\Delta(k) \tag{17.30}
\end{equation*}
$$

### 17.4 Single-Particle Properties of a Superconductor

### 17.4.1 Green Functions

When $\Delta$ takes a non-zero, frequency-independent value, the action for the fermions is:

$$
\begin{align*}
S=\int & \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi}\left[\psi_{\sigma}^{\dagger}(i \epsilon, k)\left(i \epsilon-v_{F} k\right) \psi_{\sigma}(i \epsilon, k)\right. \\
& \left.-\psi_{\uparrow}^{\dagger}\left(i \epsilon, k^{\prime}\right) \psi_{\downarrow}^{\dagger}\left(-i \epsilon,-k^{\prime}\right) \Delta(k)-\psi_{\uparrow}\left(i \epsilon, k^{\prime}\right) \psi_{\downarrow}\left(-i \epsilon,-k^{\prime}\right) \Delta^{\dagger}(k)\right](17
\end{align*}
$$

$\qquad$
(b) $\longrightarrow$
(c) $\longrightarrow$

Figure 17.1: The graphical representation of (a) $\mathcal{G}$ (b) $\mathcal{F}$ and (c) $\mathcal{F}^{\dagger}$.

As usual, the propagator is obtained by inverting the quadratic part of the action. This is now a matrix, with an inverse which gives

$$
\begin{align*}
\mathcal{G}_{\sigma \sigma^{\prime}}(i \epsilon, k)=\left\langle\psi_{\sigma}^{\dagger}(i \epsilon, k) \psi_{\sigma^{\prime}}(i \epsilon, k)\right\rangle & =\delta_{\sigma \sigma^{\prime}} \frac{i \epsilon+\xi_{k}}{(i \epsilon)^{2}-\xi_{k}^{2}-|\Delta(k)|^{2}} \\
\mathcal{F}_{\sigma \sigma^{\prime}}(i \epsilon, k)=\left\langle\psi_{\sigma}(i \epsilon, k) \psi_{\sigma^{\prime}}(-i \epsilon,-k)\right\rangle & =\epsilon_{\sigma \sigma^{\prime}} \frac{\Delta(k)}{(i \epsilon)^{2}-\xi_{k}^{2}-|\Delta(k)|^{2}} \tag{17.32}
\end{align*}
$$

We denote $\mathcal{G}(i \epsilon, k)$ by a line with two arrows pointing in the same direction. We denote $\mathcal{F}(i \epsilon, k)$ by a line with two arrows pointing away from each other and $\mathcal{F}^{\dagger}(i \epsilon, k)=$ $\left\langle\psi_{\sigma}^{\dagger}(i \epsilon, k) \psi_{\sigma^{\prime}}^{\dagger}(-i \epsilon,-k)\right\rangle$ by a line with two arrows pointing towards each other. The electron spectral function is given by

$$
\begin{align*}
A(k, \epsilon) & =\operatorname{Im}\left(\frac{\epsilon+\xi_{k}}{(\epsilon+i \delta)^{2}-\xi_{k}^{2}-|\Delta(k)|^{2}}\right) \\
& =u_{k}^{2} \delta\left(\epsilon-E_{k}\right)+v_{k}^{2} \delta\left(\epsilon+E_{k}\right) \tag{17.33}
\end{align*}
$$

which shows that the electron has spectral weight $u_{k}^{2}$ at $E_{k}$ and spectral weight $v_{k}^{2}$ at $-E_{k}$.

Another way of understanding the single-particle properties of a superconductor is to diagonalize the action. The action is diagonalized by the $\gamma(k)$ 's

$$
\begin{gather*}
\gamma_{\uparrow}(k, \epsilon)=u_{k} \psi_{\uparrow}(k, \epsilon)-v_{k} \psi_{\downarrow}^{\dagger}(-k, \epsilon) \\
\gamma_{\downarrow}(k, \epsilon)=u_{k} \psi_{\downarrow}(k, \epsilon)+v_{k} \psi_{\uparrow}^{\dagger}(-k, \epsilon)  \tag{17.34}\\
S=\int \frac{d^{2} k}{(2 \pi)^{2}} \frac{d \epsilon}{2 \pi} \gamma_{\sigma}^{\dagger}(k, \epsilon)\left(i \epsilon-E_{k}\right) \gamma_{\sigma}(k, \epsilon) \tag{17.35}
\end{gather*}
$$

The $\gamma(k)$ 's have propagator:

$$
\begin{equation*}
\left\langle\gamma_{\sigma}^{\dagger}(i \epsilon, k) \gamma_{\sigma^{\prime}}(i \epsilon, k)\right\rangle=\frac{\delta_{\sigma \sigma^{\prime}}}{i \epsilon-E_{k}} \tag{17.36}
\end{equation*}
$$

The $\gamma(k)$ 's are the basic single-particle excitations - 'Bogoliubov-DeGennes quasiparticles' - of a superconductor; they are superpositions of fermions and holes. In the case of electrons, the basic excitations have indefinite charge, since they are a superposition of an electron and a hole. Although they are not charge eigenstates, they are spin eigenstates.

Note that $E_{k}>0$. When $\xi_{k} \gg \Delta, u_{k} \rightarrow 1, v_{k} \rightarrow 0$, so $\gamma_{\sigma}^{\dagger}(k)$ creates a fermion above the Fermi surface, costing positive energy. When $\xi_{k} \ll-\Delta, u_{k} \rightarrow 0, v_{k} \rightarrow 1$, so $\gamma_{\sigma}^{\dagger}(k)$ creates a hole below the Fermi surface, also costing positive energy.

For some purposes - such as the Hebel-Slichter peak in NMR - we can ignore the fact that they are a superposition of an electron and a hole and treat the superconductor as a semiconductor with energy bands $\pm E_{k}$. Since the density of single quasiparticle states,

$$
\begin{equation*}
\frac{d k}{d E}=\frac{m^{*}}{2 \pi} \frac{|E|}{\sqrt{E^{2}-\Delta^{2}}} \theta(|E|-\Delta) \tag{17.37}
\end{equation*}
$$

is divergent for $|E| \rightarrow \Delta$ and vanishing for $|E|<\Delta$, the semiconductor model predicts sharp increases in these quantities for $T \sim \Delta$ and exponential decay for $T \ll \Delta$. However, for other properties - such as the acoustic attenuation - the mixing between between electron and hole state ('coherence factors') is important. The coherence factors can cancel the density of states divergence at $|E| \rightarrow \Delta$, and there is no enhancement for $T \sim \Delta$.

### 17.4.2 NMR Relaxation Rate

According to (7.93), the NMR relaxation rate is given by:

$$
\begin{equation*}
\frac{1}{T_{1} T}=\int \frac{d^{2} q}{(2 \pi)^{2}} A(q) \lim _{\omega \rightarrow 0} \frac{1}{\omega} \chi_{+-}^{\prime \prime}(q, \omega) \tag{17.38}
\end{equation*}
$$



Figure 17.2: The two diagrams which contribute to the spin-spin correlation function of a superconductor.

The spin-spin correlation function $\left\langle S_{+}\left(q, i \omega_{m}\right) S_{-}\left(-q,-i \omega_{m}\right)\right\rangle$ is given by the sum of the two diagrams of figure 17.2. Assuming that $\Delta(k)=\Delta$, this is:

$$
\begin{align*}
\left\langle S_{+}\left(q, i \omega_{m}\right) S_{-}\left(-q,-i \omega_{m}\right)\right\rangle= & \frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \mathcal{G}_{\downarrow \downarrow}\left(i \epsilon_{n}, k\right) \mathcal{G}_{\uparrow \uparrow}\left(i \epsilon_{n}+i \omega_{m}, k+q\right) \\
& +\frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \mathcal{F}_{\uparrow \downarrow}\left(i \epsilon_{n}, k\right) \mathcal{F}_{\downarrow \uparrow}^{\dagger}\left(i \epsilon_{n}+i \omega_{m}, k+q\right) \tag{17.39}
\end{align*}
$$

or,

$$
\begin{align*}
\left\langle S_{+}\left(q, i \omega_{m}\right) S_{-}\left(-q,-i \omega_{m}\right)\right\rangle & =\frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{i \epsilon_{n}+\xi_{k}}{\left(i \epsilon_{n}\right)^{2}-\xi_{k}^{2}-|\Delta|^{2}} \frac{i \epsilon_{n}+i \omega_{m}+\xi_{k+q}}{\left(i \epsilon_{n}+i \omega_{m}\right)^{2}-\xi_{k+q}^{2}-|\Delta|^{2}} \\
& +\frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{\Delta}{\left(i \epsilon_{n}\right)^{2}-\xi_{k}^{2}-|\Delta|^{2}} \frac{\Delta}{\left(i \epsilon_{n}+i \omega_{m}\right)^{2}-\xi_{k+q}^{2}-|\Delta|^{2}} \tag{17.40}
\end{align*}
$$

If we replace the sums over Matsubara frequencies by contour integrals which avoid $z=(2 n+1) \pi i / \beta$,

$$
\begin{align*}
& \oint_{\mathcal{C}} \frac{d z}{2 \pi i} n_{F}(z) \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{z+\xi_{k}}{(z)^{2}-\xi_{k}^{2}-|\Delta|^{2}} \frac{z+i \omega_{m}+\xi_{k+q}}{\left(z+i \omega_{m}\right)^{2}-\xi_{k+q}^{2}-|\Delta|^{2}} \\
+ & \oint_{\mathcal{C}} \frac{d z}{2 \pi i} n_{F}(z) \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{\Delta}{(z)^{2}-\xi_{k}^{2}-|\Delta|^{2}} \frac{\Delta}{\left(z+i \omega_{m}\right)^{2}-\xi_{k+q}^{2}-|\Delta|^{2}} \tag{17.41}
\end{align*}
$$

these integrals receive contributions only from the poles at

$$
z= \pm \sqrt{\xi_{k}^{2}+|\Delta|^{2}}
$$

$$
\begin{equation*}
z=-i \omega_{n} \pm \sqrt{\xi_{k+q}^{2}+|\Delta|^{2}} \tag{17.42}
\end{equation*}
$$

Hence,

$$
\begin{align*}
&\left\langle S_{+}\left(q, i \omega_{m}\right) S_{-}\left(-q,-i \omega_{m}\right)\right\rangle=\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k}\right) \frac{\xi_{k}+E_{k}}{2 E_{k}} \frac{E_{k}+i \omega_{m}+\xi_{k+q}}{\left(E_{k}+i \omega_{m}\right)^{2}-E_{k+q}^{2}} \\
&+ \int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k}\right) \frac{\xi_{k}-E_{k}}{-2 E_{k}} \frac{-E_{k}+i \omega_{m}+\xi_{k+q}}{\left(-E_{k}+i \omega_{m}\right)^{2}-E_{k+q}^{2}} \\
&+ \int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k+q}\right) \frac{\xi_{k+q}+E_{k+q}}{2 E_{k+q}} \frac{E_{k+q}-i \omega_{m}+\xi_{k}}{\left(E_{k+q}-i \omega_{m}\right)^{2}-E_{k}^{2}} \\
&+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k+q}\right) \frac{\xi_{k+q}-E_{k+q}}{-2 E_{k+q}} \frac{-E_{k+q}-i \omega_{m}+\xi_{k}}{\left(E_{k+q}+i \omega_{m}\right)^{2}-E_{k}^{2}} \\
&+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k}\right) \frac{\Delta}{2 E_{k}} \frac{\Delta}{\left(E_{k}+i \omega_{m}\right)^{2}-E_{k+q}^{2}} \\
&+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k}\right) \frac{\Delta}{-2 E_{k}} \frac{\Delta}{\left(-E_{k}+i \omega_{m}\right)^{2}-E_{k+q}^{2}} \\
&+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k+q}\right) \frac{\Delta}{2 E_{k+q}} \frac{\Delta}{\left(E_{k+q}-i \omega_{m}\right)^{2}-E_{k}^{2}} \\
&+ \int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k+q}\right) \frac{\Delta}{-2 E_{k+q}} \frac{\Delta}{\left(E_{k+q}+i \omega_{m}\right)^{2}-E_{k}^{2}}(1 \tag{17.43}
\end{align*}
$$

If we now take $i \omega_{m} \rightarrow \omega+i \delta$, and $\omega<2 \Delta$ (and, thereby, dropping terms such as $\delta\left(\omega-E_{k}-E_{k+q}\right)$ which vanish for $\left.\omega<2 \Delta\right)$, we obtain:

$$
\begin{array}{r}
\chi_{+-}^{\prime \prime}(q, \omega)=\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k}\right) \frac{\left(\xi_{k}+E_{k}\right)\left(E_{k}+\xi_{k+q}\right)}{2 E_{k} E_{k+q}} \delta\left(\omega+E_{k}-E_{k+q}\right) \\
-\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k}\right) \frac{\left(\xi_{k}-E_{k}\right)\left(-E_{k}+\xi_{k+q}\right)}{2 E_{k} E_{k+q}} \delta\left(\omega+E_{k}-E_{k+q}\right) \\
+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k+q}\right) \frac{\left(\xi_{k+q}+E_{k+q}\right)\left(E_{k+q}+\xi_{k}\right)}{2 E_{k+q} E_{k}} \delta\left(\omega+E_{k}-E_{k+q}\right) \\
-\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k+q}\right) \frac{\left(\xi_{k+q}-E_{k+q}\right)\left(-E_{k+q}+\xi_{k}\right)}{2 E_{k+q} E_{k}} \delta\left(\omega+E_{k+q}-E_{k}\right) \\
+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k}\right) \frac{\Delta^{2}}{2 E_{k} E_{k+q}} \delta\left(\omega+E_{k+q}-E_{k}\right) \\
-\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k}\right) \frac{\Delta^{2}}{2 E_{k} E_{k+q}} \delta\left(\omega+E_{k}-E_{k+q}\right) \\
+\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(E_{k+q}\right) \frac{\Delta^{2}}{2 E_{k+q} E_{k}} \delta\left(\omega+E_{k}-E_{k+q}\right) \\
-\int \frac{d^{3} k}{(2 \pi)^{3}} n_{F}\left(-E_{k+q}\right) \frac{\Delta^{2}}{2 E_{k+q} E_{k}} \delta\left(\omega+E_{k+q}-E_{k}\right) \tag{17.44}
\end{array}
$$

dropping terms which are odd in $\xi_{k}$ or $\xi_{k+q}$, and using $n_{F}\left(-E_{k}\right)=1-n_{F}\left(E_{k}\right)$, we have:

$$
\chi_{+-}^{\prime \prime}(q, \omega)=\int \frac{d^{3} k}{(2 \pi)^{3}}\left(n_{F}\left(E_{k}\right)-n_{F}\left(E_{k+q}\right)\right)\left(1+\frac{\xi_{k} \xi_{k+q}+\Delta^{2}}{2 E_{k} E_{k+q}}\right) \delta\left(\omega+E_{k}-E_{k}\left(() \mathcal{l}_{1}\right) .\right.
$$

Let us assume that $A(q)=A$. Then, dropping the term linear in $\xi_{k}$ and $\xi_{k^{\prime}}$

$$
\begin{equation*}
\frac{1}{T_{1} T}=\frac{A}{\omega} \int \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \frac{d^{3} k}{(2 \pi)^{3}}\left(n_{F}\left(E_{k}\right)-n_{F}\left(E_{k^{\prime}}\right)\right)\left(1+\frac{\Delta^{2}}{2 E_{k} E_{k^{\prime}}}\right) \delta\left(\omega+E_{k}-E_{k^{\prime}}\right) 17 \tag{7.46}
\end{equation*}
$$

or, using the single-particle density of states to re-write the momentum integrals as energy integrals,

$$
\begin{align*}
\frac{1}{T_{1} T}= & \frac{A}{\omega}\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \int_{\Delta}^{\Lambda} d E^{\prime} \frac{E}{\sqrt{E^{2}-\Delta^{2}}} \frac{E^{\prime}}{\sqrt{E^{\prime 2}-\Delta^{2}}}\left(1+\frac{\Delta^{2}}{2 E E^{\prime}}\right) \\
& \times\left(n_{F}(E)-n_{F}\left(E^{\prime}\right)\right) \delta(\omega+E-E(17.47) \\
\frac{1}{T_{1} T}= & \frac{A}{\omega}\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{E}{\sqrt{E^{2}-\Delta^{2}}} \\
& \frac{E+\omega}{\sqrt{(E+\omega)^{2}-\Delta^{2}}}\left(1+\frac{\Delta^{2}}{2 E(E+\omega)}\right)  \tag{17.48}\\
& \times\left(n_{F}(E)-n_{F}(E+\omega)\right)
\end{align*}
$$

or

For $\omega \rightarrow 0$, we can write this as:

$$
\begin{equation*}
\frac{1}{T_{1} T}=A\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{E}{\sqrt{E^{2}-\Delta^{2}}} \frac{E+\omega}{\sqrt{(E+\omega)^{2}-\Delta^{2}}}\left(1+\frac{\Delta^{2}}{2 E(E+\omega)}\right) \frac{\partial}{\partial E} n_{F}(E) \tag{17.49}
\end{equation*}
$$

For $T \rightarrow 0$, the right-hand-side is exponentially suppressed as a result of the $\partial n_{F}(E) / \partial E$, and

$$
\begin{equation*}
\frac{1}{T_{1} T} \sim e^{\frac{-\Delta}{T}} \tag{17.50}
\end{equation*}
$$

For $T \sim \Delta$, the exponential suppression is not very strong so the density of states divergence is important. In fact, for $\omega=0$

$$
\begin{equation*}
\lim _{\omega \rightarrow 0} \frac{1}{T_{1} T}=A\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{E^{2}}{E^{2}-\Delta^{2}}\left(1+\frac{\Delta^{2}}{2 E^{2}}\right) \frac{\partial}{\partial E} n_{F}(E) \tag{17.51}
\end{equation*}
$$

which is a divergent integral at $E=\Delta$. For realistic values of $\omega$, there is a moderate, but clearly observable increase of $1 / T_{1}$ for $T<T_{c}$ with a maximum which is called the Hebel-Slichter peak.

### 17.4.3 Acoustic Attenuation Rate

Suppose we compute the acoustic attenuation rate, which is essentially the phonon lifetime. Phonons are coupled to the electron density, so the phonon lifetime is determined by a density-density correlation function. This, too, is given by the diagrams of figure 17.2. However, since there are density operators rather than spin operators at the vertices of these diagrams, there is a crucial minus sign arising from the ordering of the electron operators:

$$
\begin{align*}
\left\langle\rho\left(q, i \omega_{m}\right) \rho\left(-q,-i \omega_{m}\right)\right\rangle= & \frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \mathcal{G}_{\downarrow \downarrow}\left(i \epsilon_{n}, k\right) \mathcal{G}_{\uparrow \uparrow}\left(i \epsilon_{n}+i \omega_{m}, k+q\right) \\
& -\frac{1}{\beta} \sum_{n} \int \frac{d^{3} k}{(2 \pi)^{3}} \mathcal{F}_{\uparrow \downarrow}\left(i \epsilon_{n}, k\right) \mathcal{F}_{\downarrow \uparrow}^{\dagger}\left(i \epsilon_{n}+i \omega_{m}, k+q\right) \tag{17.52}
\end{align*}
$$

The acoustic attenuation rate, $\alpha$, of a phonon of frequency $\omega$ is essentially given by

$$
\begin{equation*}
\alpha=\int \frac{d^{3} q}{(2 \pi)^{3}} g(q) \chi_{\rho \rho}^{\prime \prime}(q, \omega) \tag{17.53}
\end{equation*}
$$

where $g(q)$ is the electron-phonon coupling. From our calculation of $1 / T_{1}$, we see that this is (assuming constant $g$ ):

$$
\begin{equation*}
\alpha=g\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{E}{\sqrt{E^{2}-\Delta^{2}}} \frac{E+\omega}{\sqrt{(E+\omega)^{2}-\Delta^{2}}}\left(1-\frac{\Delta^{2}}{E(E+\omega)}\right) \frac{\partial}{\partial E} n_{F}(E) \tag{17.54}
\end{equation*}
$$

As a result of the -sign, we can take the $\omega \rightarrow 0$ limit:

$$
\alpha=A\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{E^{2}}{E^{2}-\Delta^{2}}\left(1-\frac{\Delta^{2}}{E^{2}}\right) \frac{\partial}{\partial E} n_{F}(E)
$$

$$
\begin{equation*}
=A\left(\frac{m^{*}}{2 \pi}\right)^{2} \int_{\Delta}^{\Lambda} d E \frac{\partial}{\partial E} n_{F}(E) \tag{17.55}
\end{equation*}
$$

As in the case of $1 / T_{1}$, this is exponentially decreasing at low, $T$,

$$
\begin{equation*}
\alpha \sim e^{\frac{-\Delta}{T}} \tag{17.56}
\end{equation*}
$$

However, the density of states divergence has been cancelled by the quantum interference between particles and holes, so there is no enhancement for $T \sim \Delta$. Since the underlying quasiparticles are a superposition of electrons and holes such that their charge vanishes as the Fermi surface is approached, their contribution to the densitydensity correlation function is suppressed. This suppression cancels the divergence in the density of states. On the other hand, the quasiparticles carry spin $1 / 2$ (since they are a mixture of an up-spin electron and a down-spin hole) so their contribution to the spin-spin correlation function is unsuppressed; hence the density of states divergence has dramatic consequences leading to the Hebel-Slichter peak.

### 17.4.4 Tunneling

Tunneling is a classic probe of the single-particle properties of an electron system. Let us suppose we connect a superconductor on the left with another system - which may or may not be a superconductor - on the right. An approximate description of the coupling between the superconductor and the other system is given by the tunneling Hamiltonian:

$$
\begin{align*}
H_{T} & =\int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}}\left[t\left(k, k^{\prime}\right) \psi_{\sigma}^{\dagger}(k) \chi_{\sigma}(k)+t^{*}\left(k, k^{\prime}\right) \chi_{\sigma}(k) \psi_{\sigma}^{\dagger}(k)\right] \\
& \equiv B+B^{\dagger} \tag{17.57}
\end{align*}
$$

where $\psi_{\sigma}^{\dagger}(k)$ is the creation operator for an electron in the superconductor and $\chi_{\sigma}^{\dagger}(k)$ is the creation operator for an electron in the other system. $t\left(k, k^{\prime}\right)$ is the tunneling matrix element for an electron of momentum $k$ in the superconductor to tunnelin into
a momentum $k^{\prime}$ state in the other system. Tunneling occurs when there is a voltage difference, $V$, between the superconductor and the other system,

$$
\begin{align*}
H_{V} & =V \int \frac{d^{3} k}{(2 \pi)^{3}} \psi_{\sigma}^{\dagger}(k) \psi_{\sigma}(k) \\
& =V N_{L} \tag{17.58}
\end{align*}
$$

The current flowing between the superconductor and the other system is

$$
\begin{align*}
I & =i \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}}\left[t\left(k, k^{\prime}\right) \psi_{\sigma}^{\dagger}(k) \chi_{k \sigma}-t^{*}\left(k, k^{\prime}\right) \chi_{\sigma}(k) \psi_{\sigma}^{\dagger}(k)\right] \\
& \equiv i\left(B-B^{\dagger}\right) \tag{17.59}
\end{align*}
$$

Following the steps by which we derived the conductivity and other response functions in chapter 7 , we see that the current, $I(t)$ computed to linear order in $H_{T}$ is given by:

$$
\begin{align*}
\langle I(t)\rangle & =\left\langle T\left\{e^{i V t N_{L}+i \int_{-\infty}^{t} H_{T}}\right\} I(t) T\left\{e^{-i V t N_{L}-i \int_{-\infty}^{t} H_{T}}\right\}\right\rangle \\
& =i\left\langle\left[I(t), \int_{-\infty}^{t} H_{T} e^{i V t N_{L}}\right]\right\rangle \tag{17.60}
\end{align*}
$$

Substituting the above expressions for $I$ and $H_{T}$, we have:

$$
\begin{align*}
\langle I(t)\rangle= & \int_{-\infty}^{\infty} d t^{\prime} \theta\left(t-t^{\prime}\right)\left\{e^{i e V\left(t^{\prime}-t\right)} i\left\langle\left[B(t), B^{\dagger}\left(t^{\prime}\right)\right]\right\rangle-e^{i e V\left(t-t^{\prime}\right)} i\left\langle\left[B^{\dagger}(t), B\left(t^{\prime}\right)\right]\right\rangle\right\} \\
& +\int_{-\infty}^{\infty} d t^{\prime} \theta\left(t-t^{\prime}\right)\left\{e^{-i e V\left(t+t^{\prime}\right)} i\left\langle\left[B(t), B\left(t^{\prime}\right)\right]\right\rangle-e^{i e V\left(t+t^{\prime}\right)} i\left\langle\left[B^{\dagger}(t), B^{\dagger}\left(t^{\prime}\right)\right]\right\rangle\right\} \tag{17.61}
\end{align*}
$$

Suppose that $t\left(k, k^{\prime}\right)=t$. Then the real part of the current is

$$
\begin{align*}
I= & t^{2} \operatorname{Im}\left\{\int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \sum_{n} \mathcal{G}_{L}\left(k, i \epsilon_{n}\right) \mathcal{G}_{R}\left(k, i \epsilon_{n}-i \omega\right)\right\}_{i \omega \rightarrow e V+i \delta} \\
& +t^{2} \operatorname{Im}\left\{e^{2 i e V t} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \sum_{n} \mathcal{F}_{L}\left(k, i \epsilon_{n}\right) \mathcal{F}_{R}^{\dagger}\left(k, i \epsilon_{n}-i \omega\right)\right\}_{i \omega \rightarrow i \delta} \tag{17.62}
\end{align*}
$$

Converting the Matsubara sum in the first term to an integral, analytically continuing, and taking the imaginary part (as we have done so often before), we have:

$$
I=\quad t^{2} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \int_{-\infty}^{\infty} \frac{d \epsilon}{2 \pi} A_{L}(k, \epsilon+e V) A_{R}\left(k^{\prime}, \epsilon\right)\left[n_{F}(\epsilon)-n_{F}(\epsilon+e V)\right]
$$

$$
\begin{equation*}
+t^{2} \operatorname{Im}\left\{e^{2 i e V t} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \sum_{n} \mathcal{F}_{L}\left(k, i \epsilon_{n}\right) \mathcal{F}_{R}^{\dagger}\left(k, i \epsilon_{n}-i \omega\right)\right\}_{i \omega \rightarrow i \delta} \tag{17.63}
\end{equation*}
$$

Let us first focus on the first term. We will call this current $I_{E}$ since it results from the tunneling of electrons. It can be rewritten as:

$$
\begin{array}{r}
I_{E}=t^{2} \frac{m_{L}^{*} k_{F}}{2 \pi^{2}} \frac{m_{R}^{*} k_{F}}{2 \pi^{2}} \int d \xi_{k} \int d \xi_{k^{\prime}} \int
\end{array} \frac{d \epsilon}{2 \pi}\left[u_{k}^{2} \delta\left(\epsilon+e V-E_{k}\right)+v_{k}^{2} \delta\left(\epsilon+e V+E_{k}\right)\right]
$$

Suppose the system on the left is a Fermi liquid, with

$$
\begin{equation*}
A_{R}\left(k^{\prime}, \epsilon\right)=\delta\left(\epsilon-\xi_{k^{\prime}}\right) \tag{17.65}
\end{equation*}
$$

Then,

$$
\begin{equation*}
I_{E}=t^{2} \frac{m_{L}^{*} k_{F}}{2 \pi^{2}} \frac{m_{R}^{*} k_{F}}{2 \pi^{2}} \int_{\Delta}^{\infty} d E \frac{E}{\sqrt{E^{2}-\Delta^{2}}}\left[n_{F}(E)-n_{F}(E-e V)\right] \tag{17.66}
\end{equation*}
$$

For $T=0$, this vanishes for $e V<\Delta$ and asymptotes $I \propto V$ for $V$ large. For $T$ finite, $I$ is exponentially small for $V<\Delta$. If the system on the right is also a superocnductor, we find:
$I_{E}=t^{2} \frac{m_{L}^{*} k_{F}}{2 \pi^{2}} \frac{m_{R}^{*} k_{F}}{2 \pi^{2}} \int d E_{\Delta}^{\infty} \frac{E}{\sqrt{E^{2}-\Delta_{L}^{2}}} \frac{V-E}{\sqrt{(V-E)^{2}-\Delta_{R}^{2}}}\left[n_{F}(E)-n_{F}(E-e(1)\rangle .67\right)$
This is exponentially small (vanishing at $T=0$ ) for $e V<\Delta_{L}+\Delta_{R}$.
The current $I_{E}$ resulting from the tunneling of electrons can be understood in terms of the semiconductor. However, the current described by the second term in (17.63) cannot. It vanishes unless the system on the right is a superconductor. We call this current $I_{J}$, since it was first discovered by Josephson.

$$
\begin{equation*}
I_{J}=t^{2} \operatorname{Im}\left\{e^{2 i e V t} \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{d^{3} k^{\prime}}{(2 \pi)^{3}} \sum_{n} \mathcal{F}_{L}\left(k, i \epsilon_{n}\right) \mathcal{F}_{R}^{\dagger}\left(k, i \epsilon_{n}-i \omega\right)\right\}_{i \omega \rightarrow i \delta} \tag{17.68}
\end{equation*}
$$

This is one of the few cases in which it is advantageous to do the momentum integrals first. Let us assume that $\left|\Delta_{L}\right|=\left|\Delta_{R}\right|=\Delta, \Delta_{L}=\Delta_{R} e^{i \phi}$ and $m_{R}^{*}=m_{L}^{*}$.

$$
\begin{align*}
I_{J} & =t^{2}\left(\frac{m^{*} k_{F}}{2 \pi^{2}}\right)^{2} \operatorname{Im}\left\{e^{2 i e V t} \sum_{n} \int d \xi_{k} \int d \xi_{k^{\prime}} \frac{\Delta_{L}}{\left(i \epsilon_{n}\right)^{2}-\xi_{k}^{2}-|\Delta|^{2}} \frac{\Delta_{R}^{*}}{\left(i \epsilon_{n}-i \omega\right)^{2}-\xi_{k}^{2}-|\Delta|^{2}}\right\}_{i \omega \rightarrow i \delta} \\
& =t^{2}\left(\frac{m^{*} k_{F}}{2 \pi^{2}}\right)^{2} \operatorname{Im}\left\{e^{2 i e V t} \sum_{n} \frac{\pi \Delta_{L}}{\sqrt{\epsilon_{n}^{2}+\Delta^{2}}} \frac{\pi \Delta_{R}^{*}}{\sqrt{\left(\epsilon_{n}-\omega\right)^{2}+\Delta^{2}}}\right\}_{i \omega \rightarrow i \delta} \\
& =t^{2}\left(\frac{m^{*} k_{F}}{2 \pi^{2}}\right)^{2} \operatorname{Im}\left\{e^{2 i e V t+\phi} \sum_{n} \frac{(\pi \Delta)^{2}}{\epsilon_{n}^{2}+\Delta^{2}}\right\} \\
& =t^{2}\left(\frac{m^{*} k_{F}}{2 \pi^{2}}\right)^{2} \operatorname{Im}\left\{e^{2 i e V t+\phi}(\pi \Delta)^{2} \frac{1}{|\Delta|} \tanh \frac{\beta \Delta}{2}\right\} \\
& =t^{2}\left(\frac{m^{*} k_{F}}{2 \pi}\right)^{2}\left\{|\Delta| \tanh \frac{\beta \Delta}{2}\right\} \sin (2 e V t+\phi) \tag{17.69}
\end{align*}
$$

The Josephson current results from the tunneling of pairs between two superconductors. A $D C$ voltage $V$ leads to an $A C$ Josephson current at frequency $2 e V$. Even if the voltage difference is zero, there will be a $D C$ Josephson current if the superconducting order parameters onthe left and right have different phases. (The flow of this $D C$ current will feed back into the electrostatics of the problem and, eventually, turn off this current.)

The Josephson current cannot be understood with the semiconductor model since it is due to the tunneling of pairs. It can be understood as an oscillation of the phase difference between two superconductors. As such, it is an example of a collective mode in a superocnductor.

### 17.5 Collective Modes of a Superconductor

If we expand the effective action (17.7) in powers of $\Psi$ and its gradients, and include the action of the electromagnetic field, then we have:

$$
\left.\left.S=\int d t d^{3} x\left(\Psi^{\dagger}\left(i \partial_{t}-A_{0}\right) \Psi+\frac{1}{2 m^{*}}\left|\left(i \partial_{i}-A_{i}\right) \Psi\right|^{2}+V(|\Psi|)+\frac{1}{8 \pi}\left(E^{2}-B^{2}\right)\right\rangle\right) 7.70\right)
$$

$V(|\Psi|)$ is actually a complicated function, but let us, for the sake of simplicity, approximate it by:

$$
\begin{equation*}
V(|\Psi|)=a\left(|\Psi|^{2}-\rho_{s}\right)^{2} \tag{17.71}
\end{equation*}
$$

for some constants $a$ and $\rho_{s}$. This action is very similar to our effective action for ${ }^{4} \mathrm{He}$ : the $U(1)$ symmetry $\Psi \rightarrow e^{i \theta} \Psi$ is broken when $\Psi$ has an expectation value. The principal difference is the electromagnetic field.

Following our analysis of ${ }^{4} \mathrm{He}$, we write:

$$
\begin{equation*}
\psi=\sqrt{\left(\rho_{s}+\delta \rho\right)} e^{i \theta} \tag{17.72}
\end{equation*}
$$

We can rewrite the action as:

$$
\begin{align*}
S= & \int d \tau d^{d} x\left(\frac{1}{2} \partial_{\tau} \delta \rho+\rho_{s}\left(\partial_{t} \theta+A_{0}\right)+\delta \rho\left(\partial_{t} \theta+A_{0}\right)\right. \\
& +\frac{1}{2\left(\delta \rho+\rho_{s}\right)}(\nabla \delta \rho)^{2}+\rho_{s}(\vec{\nabla} \theta-\vec{A})^{2}+\delta \rho(\vec{\nabla} \theta-\vec{A})^{2}+a \delta \rho^{2} \\
& \left.+\frac{1}{8 \pi}\left(E^{2}-B^{2}\right)\right) \tag{17.73}
\end{align*}
$$

The first two terms can (naively) be neglected since they are total derivatives, so the free part of this action is

$$
\begin{align*}
S= & \int d \tau d^{d} x\left(\delta \rho\left(\partial_{t} \theta+A_{0}\right)+\frac{1}{2 \rho_{s}}(\nabla \delta \rho)^{2}+\rho_{s}(\vec{\nabla} \theta-\vec{A})^{2}+a \delta \rho^{2}\right. \\
& \left.+\frac{1}{8 \pi}\left(E^{2}-B^{2}\right)\right) \tag{17.74}
\end{align*}
$$

Let us take the gauge $\theta=0$. Then we have:

$$
\begin{align*}
S= & \int d \tau d^{d} x\left(+\frac{1}{2 \rho_{s}}(\nabla \delta \rho)^{2}+a \delta \rho^{2}\right. \\
& +(\vec{\nabla} \cdot \vec{E}-\delta \rho) A_{0} \\
& \left.+\rho_{s} \vec{A}^{2}-\frac{1}{8 \pi}(\vec{\nabla} \times \vec{A})^{2}\right) \tag{17.75}
\end{align*}
$$

From the third line, we see that the transverse electromagnetic field now aquires a gap. Its equation of motion is:

$$
\begin{equation*}
\nabla^{2} \vec{A}=\rho_{s} \vec{A} \tag{17.76}
\end{equation*}
$$

which has solutions:

$$
\begin{equation*}
\vec{A}(x)=\vec{A}(0) e^{-\lambda x} \tag{17.77}
\end{equation*}
$$

where $\lambda^{2}=1 / \rho_{s}$. This is the Meissner effect: the magnetic field vanishes in the interior of a superconductor. The action (17.75) also implies the London equation:

$$
\begin{equation*}
\vec{j} \equiv \frac{\delta S}{\delta \vec{A}}=\rho_{s} \vec{A} \tag{17.78}
\end{equation*}
$$

from which the infinite conductivity of a superconductor follows.
Although the $U(1)$ symmetry has been broken, there is no Goldstone boson. The would-be Goldstone boson, $\theta$, has been gauged away. To put this more physically, the Goldstone mode would be an oscillation of the density. However, as we saw in chapter 15 , the Coulomb interaction pushes the density oscllation up to a high frequency, the plasma frequency. Hence, the would-be Goldstone boson is at the plasma frequency.

From the first term in (17.74), we see that $\delta \rho$ and $\theta$ are canonical conjugates and the Hamiltonian is:

$$
H=\int d^{d} k\left(\left(\frac{1}{2 \rho_{s}} k^{2}+a\right)\left|\delta \rho_{k}\right|^{2}+\left(A_{0}\right)_{-k} \delta \rho_{k}+\rho_{s} k^{2}\left|\theta_{k}\right|^{2}+\frac{1}{8 \pi}\left(E^{2}+B^{2}\right)\right)(17
$$

From the constraint (the $A_{0}$ equation of motion),

$$
\begin{equation*}
k^{2}\left(A_{0}\right)_{-k}=\delta \rho_{k} \tag{17.80}
\end{equation*}
$$

we have:

$$
\begin{equation*}
\left(A_{0}\right)_{-k}=\frac{1}{k^{2}} \delta \rho_{k} \tag{17.81}
\end{equation*}
$$

Neglecting the magnetic field, since all velocities are much smaller than the speed of light, we have:

$$
\begin{equation*}
H=\int d^{d} k\left(\left(\frac{1}{2 \rho_{s}} k^{2}+a+\frac{1}{k^{2}}\right)\left|\delta \rho_{k}\right|^{2}+\rho_{s} k^{2}\left|\theta_{k}\right|^{2}\right) \tag{17.82}
\end{equation*}
$$

Since $\delta \rho$ and $\theta$ are canonical conjugates, this is of the harmonic osciallator form

$$
\begin{equation*}
H=\int d^{d} k\left(\frac{1}{2 m}\left|P_{k}\right|^{2}+\frac{1}{2} m \omega_{k}^{2}\left|X_{k}\right|^{2}\right) \tag{17.83}
\end{equation*}
$$

with

$$
\begin{equation*}
\omega_{k}=\sqrt{4\left(\rho_{s} k^{2}\right)\left(\frac{1}{k^{2}}+a+\frac{1}{2 \rho_{s}} k^{2}\right)} \tag{17.84}
\end{equation*}
$$

In the long-wavelength limit, $k \rightarrow 0$,

$$
\begin{equation*}
\omega_{k}=\sqrt{\rho_{s}} \tag{17.85}
\end{equation*}
$$

i.e. the mode is gapped.

### 17.6 Repulsive Interactions

In any real metal, there is a large repulsive force due to Coulomb intractions. This repulsion is much stronger than the weak attraction due to the exchange of phonons, so one might wonder how superconductivity can occur at all. The answer is that the repulsive interaction occurs at short time scales and high-energies. At the low energies at which superconductivity occurs, the repulsion is much weaker. Since a repulsive interaction in the BCS channel is marginally irrelevant, as we saw earlier, it will be logarithmically suppressed.

Consider the following illustrative model:

$$
V\left(k, k^{\prime}\right)= \begin{cases}V & \text { if }\left|\xi_{k}\right|>\omega_{D} \text { or }\left|\xi_{k^{\prime}}\right|>\omega_{D}- \\ \left(V-V_{a}\right) & \text { if }\left|\xi_{k}\right|,\left|\xi_{k^{\prime}}\right|<\omega_{D}\end{cases}
$$

with $V>0$ and $V-V_{a}>0$ so that the interaction is repulsive everywhere, but less repulsive near the Fermi surface - i.e. $-V_{a}$ is the weak attraction on top of the repulsion $V$. Let

$$
\Delta(k)= \begin{cases}\Delta_{1} & \text { if } \omega_{D}<\left|\xi_{k}\right|<\Lambda \text { or }\left|\xi_{k^{\prime}}\right|>\omega_{D} \\ \Delta_{2} & \text { if }\left|\xi_{k}\right|<\omega_{D}\end{cases}
$$

The gap equation is:

$$
\begin{align*}
& \Delta_{1}=-V \Delta_{1} \frac{m^{*}}{2 \pi} \int_{\omega_{D}}^{\Lambda} d \xi \frac{1}{\sqrt{\xi^{2}+\Delta_{1}^{2}}}-V \Delta_{2} \frac{m^{*}}{2 \pi} \int_{0}^{\omega_{D}} d \xi \frac{1}{\sqrt{\xi^{2}+\Delta_{2}^{2}}} \\
& \Delta_{1}=-V \Delta_{1} \frac{m^{*}}{2 \pi} \int_{\omega_{D}}^{\Lambda} d \xi \frac{1}{\sqrt{\xi^{2}+\Delta_{1}^{2}}}-\left(V-V_{a}\right) \Delta_{2} \frac{m^{*}}{2 \pi} \int_{0}^{\omega_{D}} d \xi \frac{1}{\sqrt{\xi^{2}+\Delta_{2}^{2}}}(1 \tag{17.86}
\end{align*}
$$

If we assume that $\Lambda \gg \omega_{D}$ and $\omega_{D} \gg \Delta_{2}$ then we have:

$$
\begin{align*}
\Delta_{1} & =-V \Delta_{1} \frac{m^{*}}{2 \pi} \ln \left(\frac{\Lambda}{\omega_{D}}\right)-V \Delta_{2} \frac{m^{*}}{2 \pi} \ln \left(\frac{\omega_{D}}{\Delta_{2}}\right) \\
\Delta_{1} & =-V \Delta_{1} \frac{m^{*}}{2 \pi} \ln \left(\frac{\Lambda}{\omega_{D}}\right)-\left(V-V_{a}\right) \Delta_{2} \frac{m^{*}}{2 \pi} \ln \left(\frac{\omega_{D}}{\Delta_{2}}\right) \tag{17.87}
\end{align*}
$$

From the first equation, we have:

$$
\begin{equation*}
\Delta_{1}=-\frac{V}{1+\frac{m^{*}}{2 \pi} V \ln \left(\frac{\Lambda}{\omega_{D}}\right)} \Delta_{2} \frac{m^{*}}{2 \pi} \ln \left(\frac{\omega_{D}}{\Delta_{2}}\right) \tag{17.88}
\end{equation*}
$$

Hence, $\Delta_{1}$ and $\Delta_{2}$ must have opposite signs. Substituting into the second equation, we find:

$$
\begin{equation*}
\left[V_{a}-\frac{V}{1+\frac{m^{*}}{2 \pi} V \ln \left(\frac{\Lambda}{\omega_{D}}\right)}\right] \frac{m^{*}}{2 \pi} \ln \left(\frac{\omega_{D}}{\Delta_{2}}\right)=1 \tag{17.89}
\end{equation*}
$$

This equation will have a solution if

$$
\begin{equation*}
V_{a}-\frac{V}{1+\frac{m^{*}}{2 \pi} V \ln \left(\frac{\Lambda}{\omega_{D}}\right)}>0 \tag{17.90}
\end{equation*}
$$

even if $V_{a}-V<0$. In other words, the bare interaction may be repulsive, but the effective pairing interaction can be attractive because the repulsive part will be logarithmically suppressed.

## Part V

## Gauge Fields and Fractionalization

## Chapter 18

## Topology, Braiding Statistics, and Gauge Fields

### 18.1 The Aharonov-Bohm effect

As we have discussed, systems of many particles tend to form energy gaps as a way of lowering their energy. One might be tempted to conclude that their low-energy properties are, as a result, trivial, and that interesting physics occurs only when they are gapless, either because they are tuned to a critical point or because their ground state spontaneously breaks a symmetry. However, non-trivial low-energy physics can occur even when a system is fully gapped. A fully gapped system can have non-trivial topological properties, which do not require low-energy local degrees of freedom. As we will see, such properties can be described by gauge fields. These topological properties are concomitant with the phenomenon of fractionalization, whereby the quantum numbers of the low-energy excitations of a system can be fractions of the quantum numbers of its basic microscopic constituents, presumably electrons. Phases which are characterized by fractionalization are stable against small perturbations: if the electron breaks into $n$ pieces, a small perturbation cannot change this continuously; an electron, unlike the average American family, cannot have 2.4 children. It is the
fact that fractionalization is necessarily characterized by integers which guarantees that, it is stable if it occurs. ${ }^{1}$

The basic idea can be understood by considering the Aharonov-Bohm effect. Suppose an infinitely-long, thin solenoid at the origin which is threaded by flux $\Phi$ (in units in which $\hbar=e=c=1$, one flux quantum is $\Phi=2 \pi$ ) is surrounded by an infinitely-high potential barrier. As result, electrons are prevented from entering the solenoid, and they move in a region in which the magnetic field is zero. As Aharonov and Bohm showed, the cross-section for an electron of momentum $p$ to scatter off the flux tube is:

$$
\begin{equation*}
\frac{d \sigma}{d \theta}=\frac{1}{2 \pi p \sin ^{2}(\theta / 2)} \sin ^{2} \frac{\Phi}{2} \tag{18.1}
\end{equation*}
$$

In other words, the scattering cross-section is non-trivial and depends on $\Phi$ even though the electron never enters the region in which $\mathbf{B} \neq 0$ (the interior of the solenoid).

Any description of the physics of this system in terms of the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ alone cannot be local. It must involve some kind of action-at-a-distance so that the magnetic field inside the solenoid can affect the electron outside the solenoid. However, a local description can be given in terms of the vector potential,

$$
\begin{equation*}
\mathbf{A}(\mathbf{x})=\frac{\Phi}{2 \pi} \frac{\hat{\mathbf{z}} \times \mathbf{x}}{|\mathbf{x}|^{2}} \tag{18.2}
\end{equation*}
$$

by simply including this vector potential in the Hamiltonian,

$$
\begin{equation*}
H \psi=\frac{1}{2 m}(\mathbf{p}-\mathbf{A})^{2} \psi \tag{18.3}
\end{equation*}
$$

The electromagnetic potential is an example of a gauge field. By this, we mean that the vector potential, $A_{\mu}$, is not itself a measurable quantity because all physically

[^5]measurable quantities are invariant under the gauge transformation:
\[

$$
\begin{align*}
A_{\mu}(x) & \rightarrow A_{\mu}(x)-\partial_{\mu} \chi(x) \\
\psi(x) & \rightarrow e^{i \chi(x)} \psi(x) \tag{18.4}
\end{align*}
$$
\]

The gauge field $A_{\mu}(x)$ is a redundant way of parametrizing $\mathbf{B}, \mathbf{E}$ which satisfy $\nabla \cdot \mathbf{B}=$ $0, \nabla \cdot \mathbf{E}=4 \pi \rho$. This redundancy is the price which must be paid in order to retain a local description.

In particular, when $A_{\mu}(x)=\partial_{\mu} f$ for some $f$, the electromagnetic potential is equivalent under a gauge transformation to zero. However, $\nabla \times \mathbf{A}=0$ does not always mean that an $f(x)$ exists such that $\mathbf{A}=\nabla f$. The potential (18.2) is an example of such a topologically non-trivial vector field. It is locally equivalent to zero, but not globally, as a result of the singularity at the origin.

If we were to try to gauge away the vector potential (18.2) by taking the singular function

$$
\begin{equation*}
f=\frac{\Phi}{2 \pi} \tan ^{-1} \frac{y}{x}=\frac{\Phi}{2 \pi} \theta \tag{18.5}
\end{equation*}
$$

the wavefunction would no longer be single-valued:

$$
\begin{equation*}
\psi(r, \theta) \rightarrow e^{i \Phi \theta / 2 \pi} \psi(r, \theta) \tag{18.6}
\end{equation*}
$$

This is because, as the electron encircles the origin, it aquires a gauge-invariant 'Aharonov-Bohm phase'

$$
\begin{equation*}
e^{i \oint e \mathbf{A} \cdot d \mathbf{l}}=e^{i e \Phi} \tag{18.7}
\end{equation*}
$$

which distinguishes the vector potential from a trivial one. However, as the above example shows, we can work with a vanishing vector potential at the cost of having a multi-valued wavefunction.

The phase aquired by the electron is independent of how close the electron comes to the solenoid or how fast it moves and depends only on the topology of the electron's
path, namely how many times it winds about the origin. Hence, the gauge field (18.2) gives rise to a 'topological interaction', which is felt by the electron even if it is infinitely far away from the solenoid. As we discuss below, it is customary to in certain circumstances to separate such topological interactions from ordinary ones which do depend on distance and lump them into particle 'statistics'.

As we will see, the low-energy excitations of a strongly-interacting electron system can aquire similar phases - i.e. have non-tivial braiding properties - when they encircle each other. A local description of the physics of these excitations must incorporate gauge fields for the reason which we saw above. Unlike the electromagnetic field, these gauge fields will be a dynamically generated feature of the low-energy properties of the system. Such a system can be fully gapped, in which case the nontrivial braiding properties of the excitations come into play at the finite energies at which these excitations are created. However, even at low-energies, these braiding properties are manifested in the ground state on manifolds of non-trivial topology. The ground state is degenerate, a reflection of the braiding properties of the quasiparticles. The effective field theories of these ground states and of the ground states with a fixed number of quasiparticles are called topological quantum field theories.

### 18.2 Exotic Braiding Statistics

Let us consider the braiding properties of particle trajectories in $2+1$-dimensions (2 spatial and 1 time dimension). According to Feynman, the quantum-mechanical amplitude for hard-core particles which are at $x_{1}, x_{2}, \ldots, x_{n}$ at an initial time $t_{0}$ to return to these coordinates at a later time $t$ is given by a sum over all trajectories. Each trajectory is summed with weight $e^{i S}$. This particular assignment of weights gives consistency with the classical limit. However, a peculiarity of two spatial dimensions is that the space of trajectories of hard-core particles is disconnected, as


Figure 18.1: Different trajectories of hard-core particles in $2+1$ dimensions which are not adiabatically deformable into each other.
may be seen in figure (18.1).
Consequently, at the quantum mechanical level, we have the freedom, as Leinaas and Myrrheim, Wilczek, ...observed, to weight each of these different components by a different phase factor. Since these components are not continuously deformable into each other, the stationary phase condition associated with the classical limit does not constrain these phases.

These phase factors realize an Abelian representation of the braid group, whose elements are the different components of trajectory space with a composition operation obtained by simply following one trajectory by another. Let us consider the case of two identical particles. The braid group is simply the group of integers, with integer $n$ corresponding to the number of times that one particle winds counter-clockwise about the other (negative integers are clockwise windings). If the particles are identical, then we must allow exchanges as well, which we can label by half-integer windings. The different representations of the braid group of two identical particles are labelled by a phase $\alpha$, so that a trajectory in which one particle is exchanged counter-clockwise with the other $n$ times receives the phase $e^{i n \alpha}$.

If $\alpha=0$, the particles are bosons; if $\alpha=\pi$, the particles are fermions. For intermediate values of $\alpha$, the particles are called anyons. The braid group of $N$


Figure 18.2: An assignment of phases to different disonnected components of the space of trajectories of two particles.
particles has more complicated representations which can be non-abelian, but a class of its representations is just an extension of the two-particle case: whenever any of $N$ identical particles is exchanged counter-clockwise $n$ times with another, the phase associated with this is $e^{i n \alpha}$.

In a slight abuse of terminology, we use the term 'statistics' to describe these representations of the braid group. In reality, it is more like a topological interaction since it is not limited to identical particles. Different particle species can have 'mutual statistics' when they wind about each other (since they are not identical, they cannot be exchanged). This is quite different from the case in higher dimensions, where there is no braid group, and we only have the permutation group - which acts only on identical particles - whose only abelian representations are bosonic and fermionic. To emphasize the distinction between this notion of statistics and the usual one, we will use the term 'braiding statistics'.

As we will see in the next chapter, this expanded notion of statistics is more than a mathematical curiosity; it is realized in all of its glory in the quantum Hall effect. First, however, we will discuss its field-theoretical implementation.

### 18.3 Chern-Simons Theory

Non-trivial braiding statistics can be implemented by taking wavefunctions which are multi-valued so that a phase is aquired whenever on particle is exchanged with another. However, as we saw at the beginning of this chapter, we can make these wavefunctions single-valued by introducing a gauge field a (distinct from the electromagnetic field $\mathbf{A}$ ) which gives rise to a vanishing magnetic field but is not gaugeequivalent to zero, in the spirit of (18.2).

$$
\begin{equation*}
\mathbf{a}(\mathbf{x})=\frac{\Phi}{2 \pi} \sum_{i} \frac{\hat{\mathbf{z}} \times\left(\mathbf{x}-\mathbf{x}_{i}\right)}{\left|\mathbf{x}-\mathbf{x}_{i}\right|^{2}} \tag{18.8}
\end{equation*}
$$

where $\mathbf{x}_{i}$ is the position of the $i^{\text {th }}$ particle. When one particle winds around another, it aquires a phase. An exchange is half of a wind, so half of this phase is aquired during an exchange.

Such a gauge field is produced automatically if we add a Chern-Simons term to the action. Consider the addition of such a term to the action for a system of free fermions:

$$
\begin{equation*}
S=\int\left[\psi^{\dagger}\left(i \partial_{t}-a_{0}\right) \psi+\frac{1}{2 m} \psi^{\dagger}(i \nabla-\mathbf{a})^{2} \psi\right]+\frac{1}{1 \Phi} \int d t d^{2} x \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{18.9}
\end{equation*}
$$

The action (18.9) is invariant under the gauge transformation

$$
\begin{align*}
a_{\mu}(x) & \rightarrow a_{\mu}(x)-\partial_{\mu} \chi(x) \\
\psi(x) & \rightarrow e^{i \chi(x)} \psi(x) \tag{18.10}
\end{align*}
$$

up to the boundary term

$$
\begin{equation*}
\delta S=\frac{1}{2 \Phi} \int_{\partial R} d^{2} x \chi \epsilon_{i j} \partial_{i} a_{j} \tag{18.11}
\end{equation*}
$$

In an infinite system or on a compact manifold, we can ignore this boundary term. When we consider a bounded region $R$ of the plane, this term will be important, as we will discuss in the context of the quantum Hall effect.

Since no time derivative of $a_{0}$ appears in the Lagrange multiplier, it is a Lagrange multiplier. If we vary it, we obtain the constraint:

$$
\begin{equation*}
\nabla \times \mathbf{a}=\Phi \psi^{\dagger} \psi \tag{18.12}
\end{equation*}
$$

from which the configuration (18.8). This constraint completely fixes $a_{\mu}$, up to gauge transformations. Hence, the gauge field $a_{\mu}$ has no independent dynamics of its own; it is completely determined by $\psi(x)$.

Let us consider the Chern-Simons action in the gauge $a_{0}=0$. The action is

$$
\begin{equation*}
S=\int\left[\psi^{\dagger}\left(i \partial_{t}-a_{0}\right) \psi+\frac{1}{2 m} \psi^{\dagger}(i \nabla-\mathbf{a})^{2} \psi\right]+\frac{1}{2 \Phi} \int\left(a_{1} \partial_{0} a_{2}-a_{2} \partial_{0} a_{1}\right) \tag{18.13}
\end{equation*}
$$

Thus, the Hamiltonian of the Chern-Simons gauge field vanishes. Note, however, that the Hamiltonian must be supplemented by the constraint (18.12).

Hence, the Chern-Simons term does what we want - i.e. implement anyonic braiding statistics - and it does nothing else.

### 18.4 Ground States on Higher-Genus Manifolds

Let us now imagine that the particles are all gapped, so that we can integrate them out. Let us further assume that the Chern-Simons coefficient is an integer $m$ divided by $4 \pi$. We will return to this assumption below. Then, the effective action at low energies is simply

$$
\begin{equation*}
S=\frac{m}{4 \pi} \int \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{18.14}
\end{equation*}
$$

This theory would appear to be completely trivial. The gauge field is fixed by the constraint

$$
\begin{equation*}
\nabla \times \mathbf{a}=0 \tag{18.15}
\end{equation*}
$$



Figure 18.3: The basic operators $A_{1}$ and $A_{2}$ are constructed from the line integrals of $\mathbf{a}$ around $\gamma_{1}$ and $\gamma_{2}$.
and the Hamiltonian vanishes. Thus, the effective action only describes the ground state.

On the infinite plane or the sphere, the ground state is a unique, non-degenerate state. The pure Chern-Simons theory (i.e. without any other fields to which it is coupled) has no other states. However, suppose that the theory is defined on the torus. Then $\nabla \times \mathbf{a}=0$ can still give rise to non-trivial

$$
\begin{equation*}
e^{i A_{\gamma}}=e^{\oint_{\gamma} \mathbf{a} \cdot d \mathbf{l}} \tag{18.16}
\end{equation*}
$$

if $\gamma$ winds around one of the non-trivial cycles of the torus. According to the constraint, $A_{\gamma}$ does not depend on the precise curve $\gamma$ but only on how many times it winds around the generators of the torus. Furthermore, it is clear that $A_{\gamma}$ is additive in the sense that its value for a curve $\gamma$ which winds twice around one of the generators of the torus is twice its value for a curve $\gamma$ which winds once. Hence, we have only two independent variables, $A_{1}, A_{2}$ associated with the two generators of the torus. If we take coordinates $\theta_{1}, \theta_{2} \in[0,2 \pi]$ on the torus, then

$$
\begin{equation*}
A_{i}=\int_{0}^{2 \pi} a_{i} d \theta_{i} \tag{18.17}
\end{equation*}
$$

From (18.13), we have the following equal-time commutation relations:

$$
\begin{equation*}
\left[a_{1}(x), a_{2}\left(x^{\prime}\right)\right]=i \frac{2 \pi}{m} \delta^{(2)}\left(x-x^{\prime}\right) \tag{18.18}
\end{equation*}
$$

from which it follows that

$$
\begin{equation*}
\left[A_{1}, A_{2}\right]=\frac{2 \pi i}{m} \tag{18.19}
\end{equation*}
$$

Since $A_{1}, A_{2}$ are not themselves gauge-invariant, we cannot simply use the analogy between their commutation relations and those of $p, x$ for a single particle. We must work with the gauge invariant quantities $e^{i A_{i}}$. They have more complicated comutation relations. Since

$$
\begin{equation*}
e^{i A_{1}} e^{i A_{2}}=e^{\left[A_{1}, A_{2}\right] / 2} e^{i A_{1}+i A_{2}} \tag{18.20}
\end{equation*}
$$

we have the commutation relation

$$
\begin{equation*}
e^{i A_{1}} e^{i A_{2}}=e^{2 \pi i / m} e^{i A_{2}} e^{i A_{1}} \tag{18.21}
\end{equation*}
$$

This algebra can be implemented on a space of minimum dimension $m$ :

$$
\begin{align*}
e^{i A_{1}}|n\rangle & =e^{2 \pi n i / m}|n\rangle \\
e^{i A_{2}}|n\rangle & =|n+1\rangle \tag{18.22}
\end{align*}
$$

i.e. the ground state is $m$-fold degenerate. On a genus $g$ manifold, this generalizes to $m^{g}$.

This has an interpretation in terms of the quasiparticle spectrum of the theory - about which we thought that we had lost all information by going to low energies below the quasiparticle energy gap. Imagine creating a quasihole-quasiparticle pair, taking them around one of the two non-trivial loops on the torus and annihilating them. Call the corresponding operators $T_{1}, T_{2}$. If the quasiparticles have statistics $\pi / m$, then

$$
\begin{equation*}
T_{1} T_{2}=e^{2 \pi i / m} T_{2} T_{1} \tag{18.23}
\end{equation*}
$$

because the particles wind around each other during such a process, as depicted on the right of figure 18.4. This is precisely the same algebra (18.21) which we found above, with a minimal representation of dimension $m$.


Figure 18.4: Creating a quasiparticle-quasihole pair, taking them around either of the generators of the torus and annihilating them leads yields two non-commuting operations which encode quasiparticle statistics in the ground state degeneracy.

Hence, if we know that the ground state degeneracy of a system on a genus- $g$ manifold is $m^{g}$, then one explanation of this degeneracy is that it has non-trivial quasiparticles of statistics $0, \pi / m, \ldots,(m-1) \pi / m$.

Why did we take the Chern-Simons coefficient to be an integer? This is required when we define Chern-Simons theory on compact manifolds or, equivalently, when we require invariance under large gauge transformations. On a compact manifold, the Chern-Simons action transforms under a gauge transformation defined by a function $\chi(x)$ as:

$$
\begin{equation*}
\frac{m}{4 \pi} \int \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \rightarrow \frac{m}{4 \pi} \int \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho}+2 \pi m N \tag{18.24}
\end{equation*}
$$

where $N$ is the winding number of the map from $x$ to $e^{i \chi(x)} \in U(1)$. Hence, invariance of the functional integral mandates that we take $m$ to be an integer.

## Chapter 19

## Introduction to the Quantum Hall Effect

### 19.1 Introduction

In 1879, E.H. Hall performed an experiment designed to determine the sign of the current-carrying particles in metals. If we suppose that these particles have charge $e$ (with a sign to be determined) and mass $m$, the classical equations of motion of charged particles in an electric field, $\mathbf{E}=E_{x} \hat{\mathbf{x}}+E_{y} \hat{\mathbf{y}}$, and a magnetic field, $\mathbf{B}=B \hat{\mathbf{z}}$ are:

$$
\begin{align*}
\frac{d p_{x}}{d t} & =e E_{x}-\omega_{c} p_{y}-p_{x} / \tau \\
\frac{d p_{y}}{d t} & =e E_{y}+\omega_{c} p_{x}-p_{y} / \tau \tag{19.1}
\end{align*}
$$

where $\omega_{c}=e B / m$ and $\tau$ is a relaxation rate determined by collisions with impurities, other electrons, etc. Let us, following Hall, place a wire along the $\hat{\mathbf{x}}$ direction in the above magnetic fields and run a current, $j_{x}$, through it. In the steady state, $d p_{x} / d t=d p_{y} / d t=j_{y}=0$, we must have $E_{x}=\frac{m}{n e^{2} \tau} j_{x}$ and

$$
\begin{equation*}
E_{y}=-\frac{B}{n e} j_{x}=\frac{-e}{|e|} \frac{h}{e^{2}} \frac{\Phi / \Phi_{0}}{N} j_{x} \tag{19.2}
\end{equation*}
$$



Figure 19.1: $\rho_{x x}$ and $\rho_{x y}$ vs. magnetic field, $B$, in the quantum Hall regime. A number of integer and fractional plateaus can be clearly seen. This data was taken at Princeton on a GaAs-AlGaAs heterostructure.
where $n$ and $N$ are the density and number of electrons in the wire, $\Phi$ is the magnetic flux penetrating the wire, and $\Phi_{0}=$ is the flux quantum. Hence, the sign of the charge carriers can be determined from a measurement of the transverse voltage in a magnetic field. Furthermore, according to (19.2), the density of charge carriers i.e. electrons - can be determined from the slope of the $\rho_{x y}=E_{y} / j_{x}$ vs $B$. At high temperatures, this is roughly what is observed.

In the quantum Hall regime, namely at low-temperatures and high magnetic fields, very different behavior is found in two-dimensional electron systems. $\rho_{x y}$ passes through a series of plateaus, $\rho_{x y}=\frac{1}{\nu} \frac{h}{e^{2}}$, where $\nu$ is a rational number, at which $\rho_{x x}$ vanishes [?, ?], as may be seen in Figure 19.1 (taken from [?]). The quantization
is accurate to a few parts in $10^{8}$, making this one of the most precise measurements of the fine structure constant, $\alpha=\frac{e^{2}}{\hbar c}$, and, in fact, one of the highest precision experiments of any kind.

Some insight into this phenomenon can be gained by considering the quantum mechanics of a single electron in a magnetic field. Let us suppose that the electron's motion is planar and that the magnetic field is perpendicular to the plane. For now, we will assume that the electron is spin-polarized by the magnetic field and ignore the spin degree of freedom. The Hamiltonian,

$$
\begin{equation*}
H=\frac{1}{2 m}\left(-i \hbar \nabla+\frac{e}{c} \mathbf{A}\right)^{2} \tag{19.3}
\end{equation*}
$$

takes the form of a harmonic oscillator Hamiltonian in the gauge $A_{x}=-B y, A_{y}=0$. (Here, and in what follows, I will take $e=|e|$; the charge of the electron is $-e$.) If we write the wavefunction $\psi(x, y)=e^{i k x} \phi(y)$, then:

$$
\begin{equation*}
H \psi=\left[\frac{1}{2 m}\left(\left(\frac{e B}{c}\right) y+\hbar k\right)^{2}+\frac{1}{2 m}\left(-i \hbar \partial_{y}\right)^{2}\right] \phi(y) e^{i k x} \tag{19.4}
\end{equation*}
$$

The energy levels $E_{n}=\left(n+\frac{1}{2}\right) \hbar \omega_{c}$, called Landau levels, are highly degenerate because the energy is independent of $k$. To analyze this degeneracy (and in most of what follows) it will be more convenient to work in symmetric gauge, $\mathbf{A}=\frac{1}{2} \mathbf{B} \times \mathbf{r}$ Writing $z=x+i y$, we have:

$$
\begin{equation*}
H=\frac{\hbar^{2}}{m}\left[-2\left(\partial-\frac{\bar{z}}{4 \ell_{0}^{2}}\right)\left(\bar{\partial}+\frac{z}{4 \ell_{0}^{2}}\right)+\frac{1}{2 \ell_{0}^{2}}\right] \tag{19.5}
\end{equation*}
$$

with (unnormalized) energy eigenfunctions:

$$
\begin{equation*}
\psi_{n, m}(z, \bar{z})=z^{m} L_{n}^{m}(z, \bar{z}) e^{-\frac{|z|^{2}}{4 \epsilon_{0}^{2}}} \tag{19.6}
\end{equation*}
$$

at energies $E_{n}=\left(n+\frac{1}{2}\right) \hbar \omega_{c}$, where $L_{n}^{m}(z, \bar{z})$ are the Laguerre polynomials and $\ell_{0}=$ $\sqrt{\hbar /(e B)}$ is the magnetic length.

Let's concentrate on the lowest Landau level, $n=0$. The wavefunctions in the lowest Landau level,

$$
\begin{equation*}
\psi_{n=0, m}(z, \bar{z})=z^{m} e^{-\frac{|z|^{2}}{4 \ell_{0}^{2}}} \tag{19.7}
\end{equation*}
$$

are analytic functions of $z$ multiplied by a Gaussian factor. The general lowest Landau level wavefunction can be written:

$$
\begin{equation*}
\psi_{n=0, m}(z, \bar{z})=f(z) e^{-\frac{|z|^{2}}{4 \ell_{0}^{2}}} \tag{19.8}
\end{equation*}
$$

The state $\psi_{n=0, m}$ is concentrated on a narrow ring about the origin at radius $r_{m}=$ $\ell_{0} \sqrt{2(m+1)}$. Suppose the electron is confined to a disc in the plane of area $A$. Then the highest $m$ for which $\psi_{n=0, m}$ lies within the disc is given by $A=\pi r_{m_{\max }}$, or, simply, $m_{\max }+1=\Phi / \Phi_{0}$, where $\Phi=B A$ is the total flux. Hence, we see that in the thermodynamic limit, there are $\Phi / \Phi_{0}$ degenerate single-electron states in the lowest Landau level of a two-dimensional electron system penetrated by a uniform magnetic flux $\Phi$. The higher Landau levels have the same degeneracy. Higher Landau levels can, at a qualitative level, be thought of as copies of the lowest Landau level. The detailed structure of states in higher Landau levels is different, however.

Let us now imagine that we have not one, but many, electrons and let us ignore the interactions between these electrons. To completely fill $p$ Landau levels, we need $N_{e}=$ $p\left(\Phi / \Phi_{0}\right)$ electrons. Inverting the semi-classical resistivity matrix, and substituting this electron number, we find:

$$
\begin{equation*}
\sigma_{x y}=\frac{e^{2}}{h} \frac{N_{e}}{N_{\Phi}}=\frac{e^{2}}{h} p \tag{19.9}
\end{equation*}
$$

for $p$ filled Landau levels, where $N_{\Phi}=\Phi / \Phi_{0}$.
Suppose that we fix the chemical potential, $\mu$. As the magnetic field is varied, the energies of the Landau levels will shift relative to the chemical potential. However, so long as the chemical potential lies between two Landau levels (see figure 19.2), an integer number of Landau levels will be filled, and we expect to find the quantized Hall conductance, (19.9).

These simple considerations neglected two factors which are crucial to the observation of the quantum Hall effect, namely the effects of impurities and inter-electron interactions. ${ }^{1}$ The integer quantum Hall effect occurs in the regime in which impurities dominate; in the fractional quantum Hall effect, interactions dominate. ${ }^{2}$

### 19.2 The Integer Quantum Hall Effect

Let us model the effects of impurities by a random potential in which non-interacting electrons move. Clearly, such a potential will break the degeneracy of the different states in a Landau level. More worrisome, still, is the possibility that some of the states might be localized by the random potential and therefore unable to carry any current at all. The possible effects of impurities are summarized in the hypothetical density of states depicted in Figure 19.2.

Hence, we would be led to naively expect that the Hall conductance is less than $\frac{e^{2}}{h} p$ when $p$ Landau levels are filled. In fact, this conclusion, though intuitive, is completely wrong. In a very instructive calculation (at least from a pedagogical standpoint), Prange [?] analyzed the exactly solvable model of electrons in the lowest Landau level interacting with a single $\delta$-function impurity. In this case, a single localized state, which carries no current, is formed. The current carried by each of the extended states is increased so as to exactly compensate for the localized state, and the conductance remains at the quantized value, $\sigma_{x y}=\frac{e^{2}}{h}$. This calculation gives an important hint of the robustness of the quantization, but cannot be easily generalized to the physically

[^6]

Figure 19.2: (a) The density of states in a pure system. So long as the chemical potential lies between Landau levels, a quantized conductance is observed. (b) Hypothetical density of states in a system with impurities. The Landau levels are broadened into bands and some of the states are localized. The shaded regions denote extended states. (c) As we mention later, numerical studies indicate that the extended state(s) occur only at the center of the band.


Figure 19.3: (a) The Corbino annular geometry. (b) Hypothetical distribution of energy levels as a function of radial distance.
relevant situation in which there is a random distribution of impurities. To understand the quantization of the Hall conductance in this more general setting, we will turn to the beautiful arguments of Laughlin (and their refinement by Halperin [?]), which relate it to gauge invariance.

Let us consider a two-dimensional electron gas confined to an annulus such that all of the impurities are confined to a smaller annulus, as shown in Figure 19.3. Since, as an experimental fact, the quantum Hall effect is independent of the shape of the sample, we can choose any geometry that we like. This one, the Corbino geometry, is particularly convenient. States at radius $r$ will have energies similar to to those depicted in Figure 19.3.

Outside the impurity region, there will simply be a Landau level, with energies that are pushed up at the edges of the sample by the walls (or a smooth confining potential). In the impurity region, the Landau level will broaden into a band. Let us
suppose that the chemical potential, $\mu$, is above the lowest Landau level, $\mu>\hbar \omega_{c} / 2$. Then the only states at the chemical potential are at the inner and outer edges of the annulus and, possibly, in the impurity region. Let us further assume that the states at the chemical potential in the impurity region - if there are any - are all localized.

Now, let us slowly thread a time-dependent flux $\Phi(t)$ through the center of the annulus. Locally, the associated vector potential is pure gauge. Hence, localized states, which do not wind around the annulus, are completely unaffected by the flux. Only extended states can be affected by the flux.

When an integer number of flux quanta thread the annulus, $\Phi(t)=p \Phi_{0}$, the flux can be gauged away everywhere in the annulus. As a result, the Hamiltonian in the annulus is gauge equivalent to the zero-flux Hamiltonian. Then, according to the adiabatic theorem, the system will be in some eigenstate of the $\Phi(t)=0$ Hamiltonian. In other words, the single-electron states will be unchanged. The only possible difference will be in the occupancies of the extended states near the chemical potential. Localized states are unaffected by the flux; states far from the chemical potential will be unable to make transitions to unoccupied states because the excitation energies associated with a slowly-varying flux will be too small. Hence, the only states that will be affected are the gapless states at the inner and outer edges. Since, by construction, these states are unaffected by impurities, we know how they are affected by the flux: each flux quantum removes an electron from the inner edge and adds an electron to the outer edge. Then, $\int I d t=e$ and $\int V d t=\int \frac{d \Phi}{d t}=h / e$, so:

$$
\begin{equation*}
I=\frac{e^{2}}{h} V \tag{19.10}
\end{equation*}
$$

Clearly, the key assumption is that there are no extended states at the chemical potential in the impurity region. If there were - and there probably are in samples that are too dirty to exhibit the quantum Hall effect - then the above arguments break down. Numerical studies [?] indicate that, so long as the strength of the impurity
potential is small compared to $\hbar \omega_{c}$, extended states exist only at the center of the Landau band (see Figure 19.2). Hence, if the chemical potential is above the center of the band, the conditions of our discussion are satisfied.

The other crucial assumption, emphasized by Halperin [?], is that there are gapless states at the edges of the system. In the special setup which we assumed, this was guaranteed because there were no impurities at the edges. In the integer quantum Hall effect, these gapless states are a one-dimensional chiral Fermi liquid. Impurities are not expected to affect this because there can be no backscattering in a totally chiral system. More general arguments, which we will mention in the context of the fractional quantum Hall effect, relate the existence of gapless edge excitations to gauge invariance.

One might, at first, be left with the uneasy feeling that these gauge invariance arguments are somehow too 'slick.' To allay these worries, consider the annulus with a wedge cut out, which is topologically equivalent to a rectangle (see the article by D.J. Thouless in the first reference in [?]). In such a case, some of the Hall current will be carried by the edge states at the two cuts (i.e. the edges which run radially at fixed azimuthal angle). However, probes which measure the Hall voltage between the two cuts will effectively couple these two edges leading, once again, to annular topology.

Laughlin's argument for exact quantization will apply to the fractional quantum Hall effect if we can show that the clean system has a gap. Then, we can argue that for an annular setup similar to the above there are no extended states at the chemical potential except at the edge. Then, if threading $q$ flux quanta removes $p$ electrons from the inner edge and adds $p$ to the outer edge, as we would expect at $\nu=p / q$, we would have $\sigma_{x y}=\frac{p}{q} \frac{e^{2}}{h}$.

### 19.3 The Fractional Quantum Hall Effect: The Laughlin States

A partially filled Landau level of non-interacting electrons has a highly degenerate ground state in the absence of impurities. This degeneracy is broken by fairly generic interactions, including Coulomb repulsion. As we will see below, at special filling fractions, there is a non-zero gap between the ground state and the lowest excited state. In very clean samples, the impurity potential will be a weak perturbation which pins the quasiparticles but does not drastically affect the physics of the ground state. If the sample is too dirty, however, the fractional quantum Hall effect will be destroyed. In what follows, we will try to understand the physics of a partially filled Landau level of interacting electrons in a clean system. We will further assume that we can ignore all higher Landau levels. This assumption will be valid in the limit that the cyclotron energy is much larger than the Coulomb interaction energy, $\hbar \omega_{c} \gg \frac{e^{2}}{\ell_{0}}$. In a sample with density $1.25 \times 10^{-11} \mathrm{~cm}^{-2}$, the $\nu=\frac{1}{3}$ state is seen at $15 T$, while $\hbar \omega_{c}=\frac{e^{2}}{\ell_{0}}$ at $6 T$. Hence, higher Landau levels are probably unimportant qualitatively, but could lead to some quantitative corrections.

Let us, following Haldane [?], consider the special interactions for which the Laughlin states are the exact ground states. To do this, let us first look at the two-electron problem in the lowest Landau level. We separate the center-of-mass and relative motions, $\psi=P(Z, z)$ (we will be sloppy and drop the Gaussian factors because they are unimportant for this analysis), where $Z=z_{1}+z_{2}, z=z_{1}-z_{2}$, and $z_{1}$ and $z_{2}$ are the coordinates of the two electrons. The Hamiltonian has no kinetic part in the lowest Landau level. Dropping the constant $\hbar \omega_{c}$, it is given simply by the interaction, $V$, which depends only on the relative motion

$$
\begin{equation*}
H=V\left(z, \frac{d}{d z}\right) \tag{19.11}
\end{equation*}
$$

Let us now switch to a basis of relative (canonical) angular momentum eigenstates, $L_{z}|m\rangle=m|m\rangle$, which are given in position space by $\langle z \mid m\rangle=z^{m}$. Then, we can write:

$$
\begin{equation*}
H=\sum_{m \text { odd }} V_{m} P_{m} \tag{19.12}
\end{equation*}
$$

The restriction to odd $m$ is due to Fermi statistics. $V_{m}=\langle m| V|m\rangle ;\langle m| V\left|m^{\prime}\right\rangle$ vanishes for $m \neq m^{\prime}$ if $V$ is rotationally invariant. $P_{m}$ is the projection operator onto states of relative angular momentum $m$. Suppose we take $V_{m}>0$ for $m<k$ and $V_{m}=0$ for $m \geq k$. Then the states $\psi(z)=z^{m}$ are pushed up to energies $E_{m}=V_{m}$ for $m<k$ but the states $\psi(z)=z^{m}, m \geq k$ remain degenerate at $E=0$.

The Hamiltonian for the $N$-electron problem is just:

$$
\begin{equation*}
H=\sum_{i>j} \sum_{m \text { odd }} V_{m} P_{m}^{i j} \tag{19.13}
\end{equation*}
$$

where $P_{m}^{i j}$ projects the $i-j$ pair onto a state of relative angular momentum $m$. Let us consider the simple, but unrealistic interaction $V_{1} \neq 0, V_{m}=0$ for $m>1$. Any wavefunction in the lowest Landau level, $\psi=P\left(z_{1}, z_{2}, \ldots, z_{N}\right)$ can be written:

$$
\begin{equation*}
\psi=\sum_{m \text { odd }}\left(z_{i}-z_{j}\right)^{m} F_{m}\left(z_{i}+z_{j} ; z_{k}, k \neq i, j\right) \tag{19.14}
\end{equation*}
$$

If we take $F_{m}=0$ for $m=1$, then $H \psi=0$. In this case, $\left(z_{i}-z_{j}\right)^{3}$ is a factor of $\psi$ for all $i \neq j$. Hence, the following wavefunctions all have zero energy

$$
\begin{equation*}
\psi=\prod_{i>j}\left(z_{i}-z_{j}\right)^{3} S\left(z_{1}, z_{2}, \ldots, z_{N}\right) \tag{19.15}
\end{equation*}
$$

where $S\left(z_{1}, z_{2}, \ldots, z_{N}\right)$ is a symmetric polynomial. These states describe droplets of electrons. Of these wavefunctions, the Laughlin wavefunction [?],

$$
\begin{equation*}
\psi_{3}=\prod_{i>j}\left(z_{i}-z_{j}\right)^{3} \tag{19.16}
\end{equation*}
$$

is the most spatially compact droplet. In a confining potential, this will be the ground state. The other symmetric polynomials correspond to quasiholes and edge
excitations. Had we chosen $V_{m}>0$ for $m<2 k+1$ and $V_{m}=0$ for $m \geq 2 k+1$, we would have found the wavefunction (19.16) with the power 3 replaced by $2 k+1$ :

$$
\begin{equation*}
\psi_{2 k+1}=\prod_{i>j}\left(z_{i}-z_{j}\right)^{2 k+1} \tag{19.17}
\end{equation*}
$$

The maximum power of any $z_{i}$ in the Laughlin state is $3(N-1)$. Since the singleelectron state with canonical angular momentum $m$ encloses area $2 \pi \ell_{0}^{2}(m+1)$, the Laughlin state of $N$ electrons occupies area $A=2 \pi \ell_{0}^{2}(3(N-1)+1)=2 \pi \hbar /(e B)(3(N-$ $1)+1)$. The total flux piercing this area is $\Phi=B A=\Phi_{0}(3(N-1)+1)$. Hence, the filling fraction, $\nu$ is

$$
\begin{equation*}
\nu=\frac{N}{\Phi / \Phi_{0}}=\frac{N}{3(N-1)+1} \rightarrow \frac{1}{3} \tag{19.18}
\end{equation*}
$$

in the thermodynamic limit.
To compress this state, that is, to get $\nu<1 / 3$, at least one pair of particles will have relative angular momentum $m=1$, which costs a finite amount of energy. A more precise and general way of stating this result involves calculating the compressibility, $\kappa$

$$
\begin{equation*}
\kappa=\frac{A}{N^{2}}\left(\frac{\partial N}{\partial \mu}\right)_{L} \tag{19.19}
\end{equation*}
$$

at fixed angular momentum $L$ ( $A$ is the area of the system). For our choice of interaction, $E_{0}(N)=E_{0}(N-1)=0$ but $E_{0}(N+1)>0$ for fixed total angular momentum $3 N(N-1)$. Hence, $\mu_{N}^{-}=E_{0}(N)-E_{0}(N-1)=0$ while $\mu_{N}^{+}=E_{0}(N+$ 1) $-E_{0}(N) \neq 0$. The discontinuity in the chemical potential implies incompressibility according to (19.19). For more realistic potentials, it may no longer be true that $\mu_{N}^{-}=0$, but the discontinuity will persist.

The Laughlin wavefunction (19.17) was initially proposed as a trial variational wavefunction for electrons interacting with Coulomb interactions. For small numbers of electrons, it has remarkably large overlap with the exact ground state (see, for instance, the article by F.D.M. Haldane in the first reference in [?]). At filling fraction $\nu=1 /(2 k+1)$, the wavefunction must be a homogeneous polynomial of degree
$(2 k+1) N(N-1) / 2$. In other words, if we fix the coordinates $z_{1}, z_{2}, \ldots, z_{N-1}$ of $N-1$ of the electrons, then the wavefunction, considered as a function of the remaining electron, $z_{N}$, will have $(2 k+1)$ zeroes for each of the $N-1$ electrons. Since the electrons are fermions, there must be at least one zero at the positions of the other electrons. A state at $\nu=1 /(2 k+1)$ is specified by the positions of the other zeroes. In the Laughlin state, there is a $2 k+1$-fold zero at the positions of the other electrons; all of the zeroes are at the electron locations. In the exact ground state of electrons with some other kind of interaction, say the Coulomb interaction, there are still $(2 k+1)$ zeroes bound to each electron, but they are slightly displaced from the electron. The quantum Hall effect breaks down precisely when the zeroes dissociate from the electrons.

A particularly useful technique for obtaining many properties of the Laughlin states is the plasma analogy (see, for instance, the article by R.B. Laughlin in the first reference in [?]). Since $|\psi|^{2}$ is of the form of the Boltzmann weight for a classical finite-temperature plasma of charge $2 k+1$ particles in a neutralizing background,

$$
\begin{equation*}
|\psi|^{2}=e^{\frac{1}{2 k+1}\left(2(2 k+1)^{2} \sum \ln \left|z_{i}-z_{j}\right|-(2 k+1) \sum\left|z_{i}\right|^{2} / 4 \ell_{0}^{2}\right)}=e^{-\beta H_{\text {plasma }}} \tag{19.20}
\end{equation*}
$$

the expectation value of many operators in the ground state is just given by the corresponding expectation values in the plasma. For instance, since the temperature $T=2 k+1$ is above the melting temperature for the plasma, we can conclude that the correlation functions of the density do not exhibit long-range positional order. ${ }^{3}$ Combining this result with our earlier discussion of the compressibility, we can say that the Laughlin states describe incompressible quantum liquids.

To establish the quantum Hall effect in these states, we need to understand the excitation spectrum. In particular, we must show that there is a finite energy gap

[^7]separating the ground state from excited states. If we imagine adiabatically inserting a flux tube at the origin in a Laughlin state at $\nu=1 /(2 k+1)$, then, by arguments very similar to those used in the annulus geometry, we expect charge $e /(2 k+1)$ to be transported from the insertion point to the outer edge of the system. The flux tube can be gauged away, leaving an eigenstate of the original Hamiltonian with a deficit of $1 /(2 k+1)$ of an electron at the origin [?]. ${ }^{4}$ Such an excitation is called a 'quasihole.' If the inserted flux were oppositely directed, an excitation with an excess charge of $-e /(2 k+1)$ at the origin would be created, or a 'quasiparticle.'

Laughlin suggested the following quasihole state,

$$
\begin{equation*}
\psi_{2 k+1}^{\mathrm{qh}}=\prod_{i}\left(z_{i}-\eta\right) \psi_{2 k+1} \tag{19.21}
\end{equation*}
$$

which is an exact zero-energy eigenstate of the Hamiltonian (19.13) and has a large overlap with the exact quasihole state of a system with a small number of electrons interacting through the Coulomb interactions. In this state, the angular momentum of each electron is increased by one and the net flux penetrating the electron droplet is increased by one flux quantum. The state:

$$
\begin{equation*}
\psi_{2 k+1}^{\mathrm{qh}}=\prod_{i}\left(z_{i}-\eta\right)^{2 k+1} \psi_{2 k+1} \tag{19.22}
\end{equation*}
$$

looks like the ground state of $N+1$ electrons, but with a deficit of one electron at the position $\eta$. Hence, the state (19.21) has charge $e /(2 k+1)$ at $\eta$.

A quasiparticle wavefunction which is an exact eigenstate of the Hamiltonian (19.13) has not been found. The trial wavefunction:

$$
\begin{equation*}
\psi_{2 k+1}^{\mathrm{qh}}=\prod_{i}\left(\frac{\partial}{\partial z_{i}}-\eta\right) \psi_{2 k+1} \tag{19.23}
\end{equation*}
$$

has reasonably good overlap with the exact quasihole state in systems with a small number of electrons. The quasiparticle has fractional charge $-e /(2 k+1)$. As a general

[^8]rule, exact quasiparticle eigenstates are more difficult to come by than quasihole states, so we will primarily discuss quasiholes. Most of the properties of quasiparticles can be inferred from those of quasiholes.

At $\nu=1 /(2 k+1)$, the gap between the ground state and a state with a widelyseparated quasihole-quasiparticle pair is just $\left(\mu^{+}-\mu^{-}\right) /(2 k+1)$. This follows from the definition of $\mu^{ \pm}$and the fact that a widely separated pair will have no interaction energy. $\Delta=\left(\mu^{+}-\mu^{-}\right) /(2 k+1)$ is the gap which is measured in transport experiments - for instance from $\rho_{x x} \sim e^{\Delta / T}$ - since a widely separated pair must be created to carry a longitudinal current. However, this is not the smallest gap in the system. A quasihole-quasiparticle pair at finite separation will have lower energy as a result of the Coulomb interaction between them. Suppose the distance between the quasihole and quasiparticle is parametrized by $k$ so that the distance is $k \ell_{0}^{2} \gg \ell_{0}$. Then, we can think of the quasihole and quasiparticle - which have core sizes on the order of a few magnetic lengths - as point charges; the energy of the pair will be $E(k)=\Delta-\frac{e^{2}}{k \ell_{0}^{2}}$ [?]. The pair will move in a straight line with velocity $v_{k}=\frac{\partial E(k)}{\partial k}$ perpendicular to the line connecting them since the Coulomb force between them will exactly balance the Lorentz force due to the magnetic field. At low $k$, the quasihole-quasiparticle pair evolves continuously into a collective mode, the magneto-roton [?]. The name magneto-roton stems from the fact that this collective excitation is obtained in the single-mode approximation just as the roton was in Feynman's analysis of superfluid ${ }^{4} \mathrm{He}$. As we will see later, the analogy between the quantum Hall effect and superfluidity can be further exploited.

To summarize, the Laughlin state has the following properties:

- It is a wavefunction describing electrons in a strong magnetic field. The electrons are assumed to be in the lowest Landau level.
- It is the non-degenerate ground state of a model repulsive Hamiltonian (19.13).
- It is an excellent approximation to the ground state of electrons in a magnetic field interacting through the Coulomb potential.
- The state is incompressible.
- The state does not break translational symmetry, i.e. it is a liquid.

In order to observe a fractional quantum Hall plateau with $\sigma_{x y}=\frac{1}{2 k+1} \frac{e^{2}}{h}, \sigma_{x x}=0$, we also need a small amount of impurities as well, in order to pin any quasiparticles which are produced by small changes of the magnetic field or electron density. However, we don't want too much disorder since this might simply pin the electrons and prevent them from forming a correlated state (19.16).

### 19.4 Fractional Charge and Statistics of Quasiparticles

Let us return to a discussion of the quantum numbers of the quasiholes and quasiparticles. We found earlier that these excitations carry fractional electric charge. This is remarkable, but has a precedent in polyacetylene; the statistics, to which we now turn, is perhaps even more exotic. If we suppose that the phase acquired by the wavefunction when one quasihole moves around another is $e^{i \phi}$, then the phase for taking one electron around another is $e^{i(2 k+1)^{2} \phi}$, and the phase associated with taking an electron around a quasihole is $e^{i(2 k+1) \phi}$, since $m$ quasiholes is equal to a deficit of one electron. From the wavefunction (19.21), we see that $e^{i(2 k+1) \phi}=e^{2 \pi i}$ and $e^{i(2 k+1)^{2} \phi}=e^{2 \pi(2 k+1) i}$. This would lead us to conclude that $e^{i \phi}=e^{2 \pi i /(2 k+1)}$. Similar arguments would lead us to conclude that quasiparticles have the same statistics. ${ }^{5}$

These heuristic arguments for the charge and statistics of the quasiholes and quasiparticles are inadequate even though they give the correct answers. Fortunately,

[^9]these quantum numbers can be determined directly. Following Arovas, Schrieffer, and Wilczek [?], we will calculate the Berry's phase [?] acquired when quasiholes are moved around loops. Recall that the adiabatic theorem deals with a family of Hamiltonians, $\mathbf{H}\left(\left\{\lambda_{i}\right\}\right)$, parameterized by $\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right\}$, with non-degenerate eigenstates $\left|n\left(\left\{\lambda_{i}\right\}\right)\right\rangle$ :
\[

$$
\begin{equation*}
\mathbf{H}\left(\left\{\lambda_{i}\right\}\right)\left|n\left(\left\{\lambda_{i}\right\}\right)\right\rangle=E_{n}\left(\left\{\lambda_{i}\right\}\right)\left|n\left(\left\{\lambda_{i}\right\}\right)\right\rangle \tag{19.24}
\end{equation*}
$$

\]

Suppose we vary the $\lambda_{i}$ 's slowly with time ${ }^{6}, \lambda_{i}=\lambda_{i}(t)$, such that $\mathbf{H}\left(\lambda_{i}(0)\right)=$ $\mathbf{H}\left(\lambda_{i}(T)\right)$; then $\left|n\left(\left\{\lambda_{i}(0)\right\}\right)\right\rangle=M\left|n\left(\left\{\lambda_{i}(T)\right\}\right)\right\rangle$, where $M$ is a phase. Often, we require $M=1$, but this is unnecessary. A state $|\psi(t)\rangle$ satisfying $|\psi(0)\rangle=\left|n\left(\left\{\lambda_{i}(0)\right\}\right)\right\rangle$ will evolve subject to Schrödinger's equation,

$$
\begin{equation*}
\mathbf{H}\left(\left\{\lambda_{i}(t)\right\}\right)|\psi(t)\rangle=\frac{d}{d t}|\psi(t)\rangle \tag{19.25}
\end{equation*}
$$

so that

$$
\begin{equation*}
|\psi(T)\rangle=M e^{i \gamma_{n}} e^{\frac{i}{\hbar} \int_{0}^{T} E(t) d t}|\psi(0)\rangle \tag{19.26}
\end{equation*}
$$

Berry's phase, $\gamma_{n}$, is given, according to Schrödinger's equation (19.25), by

$$
\begin{equation*}
\gamma_{n}=i \int\left\langle n\left(\left\{\lambda_{i}(t)\right\}\right)\right| \frac{d}{d t}\left|n\left(\left\{\lambda_{i}(t)\right\}\right)\right\rangle \tag{19.27}
\end{equation*}
$$

The integral (19.27) is reparameterization invariant, so Berry's phase depends only on the path in parameter space; in particular, $\gamma_{n}$ remains finite in the adiabatic limit, unlike the dynamical phase, $\int_{0}^{T} E(t) d t$. One other point worth mentioning is that Berry's phase (19.27) only depends on the Hamiltonian implicitly. In what follows, we will be interested in the Berry's phase acquired by quasihole wavefunctions as the quasiholes are moved around. We will implicitly assume that there is some Hamiltonian with a pinning potential, say, for which the state with a quasihole at $\eta$ is a non-degenerate eigenstate. As the location of the pinning potential is moved, this

[^10]eigenstate evolves, and a Berry's phase will accumulate, but we need not be concerned with the details of the Hamiltonian to do this calculation.

We consider, then, the Laughlin quasihole

$$
\begin{equation*}
|\psi(t)\rangle=\prod_{i}\left(\eta(t)-z_{i}\right) \psi_{2 k+1} \tag{19.28}
\end{equation*}
$$

and take $\eta(t)$ to move slowly around some loop as a function of $t$. Since

$$
\begin{equation*}
\frac{d}{d t}|\psi(t)\rangle=\frac{d \eta}{d t} \sum_{i} \frac{1}{\eta(t)-z_{i}}|\psi(t)\rangle \tag{19.29}
\end{equation*}
$$

we can rewrite

$$
\begin{align*}
\langle\psi(t)| \frac{d}{d t}|\psi(t)\rangle & =\int d^{2} z \frac{1}{\eta(t)-z}\langle\psi(t)| \sum_{i} \delta\left(z-z_{i}\right)|\psi(t)\rangle \\
& =\int \frac{d^{2} z}{\eta(t)-z}\langle\psi(t)| \rho(z)|\psi(t)\rangle \tag{19.30}
\end{align*}
$$

where $\rho(z)$ is the density. Then, the Berry's phase acquired in a circuit, $C$, bounding a region $R$ of area $A_{R}$ is:

$$
\begin{equation*}
\gamma_{n}=i \oint d \eta \int d^{2} z \frac{\langle\rho(z)\rangle}{\eta-z} \tag{19.31}
\end{equation*}
$$

$\rho(z)=\rho_{0}$ except in the core of the quasihole. Since $\rho_{0} A_{R}=N=\frac{1}{2 k+1} \Phi_{R} / \Phi_{0}$, where $\Phi_{R}$ is the flux in the region $R$,

$$
\begin{align*}
\gamma_{n} & =i \int d^{2} z \oint d \eta \frac{\langle\rho(z)\rangle}{\eta-z} \\
& =-2 \pi \int d^{2} z\langle\rho(z)\rangle \\
& =-\frac{2 \pi}{2 k+1}\left(\Phi_{R} / \Phi_{0}\right) \tag{19.32}
\end{align*}
$$

up to corrections of order $r_{c}^{2} / A_{R}$, where $r_{c}$ is the size of the quasihole. This is just the phase that we would expect for a particle of charge $\nu e^{6}=e /(2 k+1)$ in a magnetic field.

Suppose, now, that we had considered a multi-quasihole wavefunction. If the loop $C$ had enclosed another quasihole, $\rho(z)$ would no longer be given by $\rho_{0}=\frac{1}{A_{R}} \frac{1}{2 k+1} \Phi / \Phi_{0}$.

There would be a charge deficit at the position of the second quasihole. Then, we would find:

$$
\begin{align*}
\gamma_{n} & =i \int d^{2} z \oint d \eta \frac{\langle\rho(z)\rangle}{\eta-z} \\
& =-2 \pi \int d^{2} z\langle\rho(z)\rangle \\
& =-\frac{2 \pi}{2 k+1}\left(\Phi_{R} / \Phi_{0}\right)+\frac{2 \pi}{2 k+1} \tag{19.33}
\end{align*}
$$

Hence, there is an additional phase $2 \pi /(2 k+1)$ acquired when one quasihole winds around another. In other words, quasiholes in the Laughlin state at $\nu=1 /(2 k+1)$ have fractional statistics given by the statistics parameter $\alpha=1 /(2 k+1)$, where bosons have $\alpha=0$ and fermions, $\alpha=1$. The fractional charge and statistics of the quasiholes are the characteristic features of fractional quantum Hall states.

In chapter 4, we will be interested in non-Abelian statistics, which can occur when there is a set of degenerate states, $\left|a ;\left\{\lambda_{i}\right\}\right\rangle, a=1,2, \ldots, g$. In such a case, a state $|\psi(t)\rangle$ satisfying $|\psi(0)\rangle=\left|a ;\left\{\lambda_{i}(0)\right\}\right\rangle$ evolves into:

$$
\begin{equation*}
|\psi(T)\rangle=e^{i \gamma_{a b}} e^{\frac{i}{\hbar} \int_{0}^{T} E(t) d t} M\left|b ;\left\{\lambda_{i}(0)\right\}\right\rangle \tag{19.34}
\end{equation*}
$$

The degenerate subspaces must be equivalent at $t=0$ and $t=T$ since the Hamiltonians coincide, but the states $\left|a ;\left\{\lambda_{i}(t)\right\}\right\rangle$ at $t=0$ and $t=T$ can differ by an overall rotation; $M$ is the matrix which implements this rotation. The Berry phase matrix, $\gamma_{a b}$, is given by:

$$
\begin{equation*}
\gamma_{a b}=i \int\left\langle a ;\left\{\lambda_{i}(t)\right\}\right| \frac{d}{d t}\left|b ;\left\{\lambda_{i}(t)\right\}\right\rangle \tag{19.35}
\end{equation*}
$$

### 19.5 Fractional Quantum Hall States on the Torus

As we discussed in the last chapter the existence of anyonic quasiparticles in a system is reflected in its ground state degeneracy on higher-genus surfaces. By the arguments given there, we expect the Laughlin state for $\nu=1 / m$ to be $m$-fold degenerate on
a torus. In this section, we will construct the $m$ wavefunctions on a torus which are annihilated by the Hamiltonian with $V_{1}, \ldots, V_{m-1} \neq 0, V_{m}=V_{m+1}=\ldots=0$.

In order to do so, we will make use of the Cauchy $\vartheta$-functions, which are functions defined on the torus. Let us assume that $z$ is a complex coordinate on the torus and that the torus is defined by $z \equiv z+1, z \equiv z+\tau$, where $\tau$ is some complex number which is called the modular parameter of the torus. Then the $\vartheta$-functions are defined by:

$$
\begin{equation*}
\vartheta_{1}(z \mid \tau)=\sum_{n=-\infty}^{\infty} e^{\pi i\left(n+\frac{1}{2}\right)^{2} \tau} e^{2 \pi i\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} \tag{19.36}
\end{equation*}
$$

and

$$
\begin{align*}
& \vartheta_{2}(z \mid \tau)=\vartheta_{1}\left(\left.z+\frac{1}{2} \right\rvert\, \tau\right) \\
& \vartheta_{3}(z \mid \tau)=e^{i \pi \tau / 4} e^{i \pi z} \vartheta_{1}\left(\left.z+\frac{1}{2}(1+\tau) \right\rvert\, \tau\right) \\
& \vartheta_{4}(z \mid \tau)=\vartheta_{1}\left(\left.z+\frac{1}{2} \tau \right\rvert\, \tau\right) \tag{19.37}
\end{align*}
$$

The following properties of $\vartheta_{1}$ will be useful:

$$
\begin{align*}
\vartheta_{1}(z+1 \mid \tau) & =-\vartheta_{1}(z \mid \tau) \\
\vartheta_{1}(z+\tau \mid \tau) & =-e^{-i \pi \tau} e^{-2 \pi i z} \vartheta_{1}(z \mid \tau) \\
\vartheta_{1}(-z \mid \tau) & =-\vartheta_{1}(z \mid \tau) \tag{19.38}
\end{align*}
$$

Armed with these functions, we can generalize the Laughlin wavefunction to:

$$
\begin{equation*}
\psi=\prod_{i>j}\left[\vartheta_{1}\left(z_{i}-z_{j} \mid \tau\right)\right]^{m} e^{i K Z} \prod_{a=1}^{m} \vartheta_{1}\left(Z-Z_{a} \mid \tau\right) \tag{19.39}
\end{equation*}
$$

At short distances, $\vartheta_{1}\left(z_{i}-z_{j} \mid \tau\right) \rightarrow z_{i}-z_{j}$, so this wavefunction is annihilated by the Hamiltonian which annihilates the Laughlin wavefunction on the plane. The only remaining requireent is that it be periodic under $z \rightarrow z+1, z \rightarrow z+\tau$. These will
be satisfied if

$$
\begin{align*}
(-1)^{m N} e^{i K} & =(-1)^{N_{\phi}} e^{i K}
\end{aligned}=121 口 \begin{aligned}
& \left(-e^{i \pi \tau}\right)^{m N} e^{i K \tau} e^{2 \pi \sum_{a} Z_{a}}
\end{align*}=(-1)^{N_{\phi}} e^{i K}=1
$$

There are $m$ different choices of $K, Z_{a}$. To see this, observe that the ratio between any two wavefunctions associated with two choices of $K, Z_{a}$ is a meromorphic function of $Z$ on the torus with $m$ simple poles. By a special case of the Riemann-Roch theorem, there are $m$ linearly independent such functions. (Haldane, 1984)

### 19.6 The Hierarchy of Fractional Quantum Hall States

Thus far, we have only explained the existence of the quantized Hall plateaus at $\nu=$ $1 /(2 k+1)$. From Figure (19.1), however, we can see that there are plateaus at several other odd-denominator fractions. These other states can be thought of as descending from the Laughlin states [?, ?]. Following Halperin, let us consider a 'primary' state at $\nu=1 /(2 k+1)$ with a finite density of quasiholes or quasiparticles. Since they are charged particles in a magnetic field, we might expect that the quasiholes or quasiparticles themselves would be in a primary state (e.g. a Laughlin state) at certain preferred quasihole densities. At what densities would we expect this? Electrons form Laughlin states only at $\nu=1 /(2 k+1)$ because these are the only filling fractions at which $\Pi\left(z_{i}-z_{i}\right)^{1 / \nu}$ is an acceptable fermionic wavefunction. A Laughlin state of bosonic particles would form at $\nu=1 /(2 k)$. Following this reasoning, a Laughlin state of quasiparticles of statistics $-1 /(2 k+1)$ would be of the form $\Pi\left(z_{i}-z_{i}\right)^{2 p-\frac{1}{2 k+1}}$, while a quasihole state would be of the form $\Pi\left(\bar{z}_{i}-\bar{z}_{i}\right)^{2 p+\frac{1}{2 k+1}}$ since quasiholes have the opposite charge. Hence, the preferred filling fractions for quasiparticles and quasiholes are $1 /\left(2 p-\frac{1}{2 k+1}\right)$ and $1 /\left(2 p+\frac{1}{2 k+1}\right)$, respectively. However, we should remember
that these particles are fractionally charged as well, so their Landau levels will have $(\Phi /(2 k+1)) / \Phi_{0}$ states rather than $\Phi / \Phi_{0}$. Hence, a 'descendent' of the $\nu=1 /(2 k+1)$ primary state which has quasiholes or quasiparticles in, respectively, filling fraction $1 /\left(2 p \pm \frac{1}{2 k+1}\right)$ states has electron filling fraction:

$$
\begin{align*}
\nu & =\frac{1}{2 k+1} \mp \frac{1}{2 k+1}\left(\frac{\frac{1}{2 k+1}}{2 p \pm \frac{1}{2 k+1}}\right) \\
& =\frac{1}{(2 k+1) \pm \frac{1}{2 p}} \tag{19.41}
\end{align*}
$$

If we now imagine the quasiholes or quasiparticles of this state forming a Laughlin state, and so on, we will get the continued fractions:

$$
\begin{equation*}
\nu=\frac{1}{2 k+1 \pm \frac{1}{2 p_{1} \pm \frac{1}{2 p_{2} \pm \ddots}}} \tag{19.42}
\end{equation*}
$$

Every odd-denominator fraction less than 1 can be obtained in this way. Of course, fractional quantum Hall states are not observed at all of these fractions. As we descend through this hierarchy of states, the energy gaps become smaller and hence more easily destroyed by impurities. Furthermore, even in a pure system, the quasiholes or quasiparticles could form Wigner crystal states at some filling fractions rather than quantum Hall states.

### 19.7 Flux Exchange and 'Composite Fermions'

Another perspective on the hierarchy of fractional quantum Hall states involves mapping a fractional quantum Hall state to an integer quantum Hall state. This can be accomplished by introducing an auxiliary Chern-Simons gauge field which attaches an even number of flux tubes to each electron. The attachment of an even number of
flux tubes has no physical effect since it will change the phase acquired under braiding or exchange by a multiple of $2 \pi$. However, approximations that would have seemed unnatural without the auxiliary gauge field appear quite sensible in its presence.

Let us consider the more general problem of anyons with statistics parameter $\theta$ in a magnetic field:

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-e(\mathbf{a}+\mathbf{A}))^{2}+H_{\mathrm{int}} \tag{19.43}
\end{equation*}
$$

where

$$
\begin{equation*}
e \nabla \times \mathbf{a}=2(\theta-\pi) \sum_{i} \delta\left(\mathbf{r}-\mathbf{r}_{i}\right) \tag{19.44}
\end{equation*}
$$

Here, we have represented the anyons as fermions interacting with a Chern-Simons gauge field. If we now replace this field by its spatial average, $e \nabla \times\langle\mathbf{a}\rangle=2(\theta-\pi) \rho$, then this mean field theory is just the problem of fermions in an effective magnetic field

$$
\begin{equation*}
e B_{\mathrm{eff}}=e \nabla \times(\mathbf{a}+\mathbf{A})=e B+2(\theta-\pi) \rho \tag{19.45}
\end{equation*}
$$

If there is a state of fermions in $B_{\text {eff }}$ with a gap, then the fluctuations about mean-field theory can probably be ignored.

Suppose our anyons are actually fermions. Then, we can take $\theta=\pi$ and $e B_{\text {eff }}=$ $e B$. However, we could, instead, take $\theta=( \pm 2 k+1) \pi$, since this will give fermionic statistics as well. In such a case, $e B_{\text {eff }}=e B \pm 2 \pi(2 k) \rho$, or $1 / \nu_{\mathrm{eff}}=1 / \nu \pm 2 k$. Let us choose $B_{\text {eff }}$ so that an integral number of Landau levels, $n$, are filled; this state will have a gap. Since $\nu_{\text {eff }}=n$,

$$
\begin{equation*}
\nu=\frac{n}{2 k n \pm 1} \tag{19.46}
\end{equation*}
$$

For $n=1$, this is just the Laughlin sequence. By exchanging real magnetic flux for the fictitious statistical flux of an auxiliary Chern-Simons gauge field, we have related the Laughlin states to a single filled Landau level.

If we fix $k=1$ and consider $\nu_{\text {eff }}=1,2,3, \ldots, n$, we have $\nu=\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots, \frac{n}{2 n+1}$.

These are the filling fractions of the hierarchical sequence descending from $\nu=\frac{1}{3}$,

$$
\begin{equation*}
\nu=\frac{1}{3}, \quad \frac{2}{5}=\frac{1}{3-\frac{1}{2}}, \quad \frac{3}{7}=\frac{1}{3-\frac{1}{2-\frac{1}{2}}}, \ldots \tag{19.47}
\end{equation*}
$$

Successive levels of the hierarchy are thereby related to states with additional filled Landau levels. In somewhat misleading, but ubiquitous, jargon, the fractional quantum Hall states of electrons are integer quantum Hall states of 'composite fermions' [?]. The term 'composite fermion' refers to a composite object formed by an electron and an even number of flux quanta. This object fills an integer number of Landau levels of the remaining, uncanceled magnetic field.

At this point, we have only shown that there are quantum Hall states obtained by the 'composite fermion' construction at the same filling fractions at which there are hierarchical states. It is not clear that the two different constructions yield states in the same universality class. That they do can be shown by demonstrating that both constructions lead to states with quasiparticles of the same charge and statistics and, hence, the same ground state degeneracy on a torus. We will show this in the next chapter using the field-theoretic descriptions of these states.

Here we have considered only the simplest 'composite fermion' states. More complicated states can be constructed by introducing Chern-Simons gauge fields which only interact with electrons in particular Landau levels. Similar constructions are also available for spin-unpolarized and multi-layer systems.

Jain [?] used the 'composite fermion' construction to motivate the following trial states for the filling fractions $\nu=\frac{n}{2 k n+1}$ :

$$
\begin{equation*}
\Psi_{\frac{n}{2 k n+1}}\left(z_{k}\right)=\mathcal{P}_{L L L}\left(\prod_{i>j}\left(z_{i}-z_{j}\right)^{2 k} \Psi_{n}\left(z_{k}\right)\right) \tag{19.48}
\end{equation*}
$$

$\mathcal{P}_{L L L}$ indicates projection into the lowest Landau level. The wavefunction $\Psi_{n}\left(z_{k}\right)$
is the wavefunction of $n$ filled Landau levels, so it has vanishing projection into the lowest Landau level, and will contain powers of $\bar{z}_{i}$. However, the factor $\prod_{i>j}\left(z_{i}-z_{j}\right)^{2 k}$ will multiply this by many more powers of $z_{i}$. It may be shown that the resulting expression has large projection into the lowest Landau level. At an operational level, the lowest Landau level projection is accomplished by moving all of the factors of $\bar{z}_{i}$ to the left and making the replacement $\bar{z}_{i} \rightarrow \frac{\partial}{\partial z_{i}}$. These wavefunctions have large overlaps with the exact ground states of systems with small numbers of particles.

As we have seen, the mapping of an electron system at one filling fraction to a (presumably, weakly interacting) fermion system at a different filling fraction has shed considerable light on the fractional quantum Hall effect. This mapping has even proven to be useful starting point for a quantitative analysis. This mapping is a special case of the flux exchange process [?]: if we change the braiding statistics of the particles in a system and, at the same time, change the magnetic field, in such a way that

$$
\begin{equation*}
2 \rho \Delta \theta=e \Delta B \tag{19.49}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\Delta\left(\frac{\theta}{\pi}\right)=\Delta\left(\frac{1}{\nu}\right) \tag{19.50}
\end{equation*}
$$

then the properties of the system will not change, at the mean field level. If we assume that the fluctuations about mean-field theory are small, then the phase diagram of Figure 19.4 holds, with properties qualitatively unchanged along the diagonals [?].

In this way, we can map electron systems to other fermion systems, to Bose systems, or even to systems whose basic constituents are anyons. In the next chapter, we will see that the mapping from a fractional quantum Hall state to a Bose superfluid is the starting point for effective field theories of the quantum Hall effect.


Figure 19.4: Systems at different points along the diagonals $\Delta\left(\frac{\theta}{\pi}\right)=\Delta\left(\frac{1}{\nu}\right)$ in the magnetic field-statistics plane have the same properties at the mean-field level.

### 19.8 Edge Excitations

In our discussion of the integer quantum Hall effect, we saw that there were necessarily gapless excitations at the edge of the system. The same is true in the fractional quantum Hall effect. To see this, let us consider again our simple Hamiltonian which annihilates the Laughlin state. All of the states

$$
\begin{equation*}
\psi=S\left(z_{1}, z_{2}, \ldots, z_{N}\right) \prod_{i>j}\left(z_{i}-z_{j}\right)^{m} \tag{19.51}
\end{equation*}
$$

are also annihilated by $H$. In a more realistic model, there will be a confining potential $V(r)$ which favors states of lower total angular momentum. The Laughlin state itself, with $S=1$ is then the ground state, and the other states are edge excitations. They are spanned by:

$$
\begin{equation*}
\psi=\prod_{n}\left(s_{n}\right)^{p_{n}} \prod_{i>j}\left(z_{i}-z_{j}\right)^{m} \tag{19.52}
\end{equation*}
$$

where

$$
\begin{equation*}
s_{n}=\sum_{i} z_{i}^{n} \tag{19.53}
\end{equation*}
$$

Suppose that $H_{\text {disk }}=H+V_{\text {conf }}$. Then, to lowest order in the angular momentum, $M$, relative to the ground state:

$$
\begin{aligned}
H_{\text {disk }}\left|p_{1}, p_{2}, \ldots\right\rangle & =f(M)\left|p_{1}, p_{2}, \ldots\right\rangle \\
& \approx(\text { const. }) M\left|p_{1}, p_{2}, \ldots\right\rangle \\
& =\lambda \sum_{n} p_{n} n\left|p_{1}, p_{2}, \ldots\right\rangle \\
& =v\left(\sum_{n} p_{n} \frac{2 \pi n}{L}\right)\left|p_{1}, p_{2}, \ldots\right\rangle
\end{aligned}
$$

where $v=2 \pi \lambda / L$. This is the spectrum of a free bosonic field, but a chiral one, since only $p_{i}>0$ are allowed.

These bosonic excitations are simply the edge waves of an incompressible liquid. They will exist in any incompressible chiral fluid. To see how the edge excitations of
a given quantum Hall state depend on the particular state, consider a quantum Hall state on an annulus, rather than a disk, so that there are inner and outer edges. The Laughlin state on an annulus can be described by:

$$
\begin{equation*}
\psi=\prod_{i} z_{i}^{t m} \prod_{i>j}\left(z_{i}-z_{j}\right)^{m} \tag{19.54}
\end{equation*}
$$

where $t$ is a large integer so that the inner radius of the annulus is $\ell_{0} \sqrt{2(t m+1)}$ while the outer radius is $\ell_{0} \sqrt{2 m(N+t)}$. Essentially, we have carved out the inner hole of the annulus by removing $t$ electrons from the center of the disk. If we take a quasiparticle in the bulk and move it along a trajectory encircling the origin, it will not aquire a phase $e^{2 \pi i t}=1$.

We can now create edge excitations generated by

$$
\begin{equation*}
s_{n}=\sum_{i} z_{i}^{n} \tag{19.55}
\end{equation*}
$$

for both positive and negative $n$, so long as $n<t$. We will take $t \sim N$ so that both the inner and outer radii of the annulus are macroscopic; then for reasonable values of $n$, we will have excitations of both chiralities. Hence, the combined theory of both edges is a non-chiral boson.

This theory has free bosonic excitations which are divided into $m$ sectors, corresponding to states which are built by acting with the $s_{n}$ 's on

$$
\begin{equation*}
\psi=\prod_{i} z_{i}^{t m+r} \prod_{i>j}\left(z_{i}-z_{j}\right)^{m} \tag{19.56}
\end{equation*}
$$

where $r=0,1, \ldots, m-1$. Note that $r \equiv r+m$ by shifting $t$ by one. These different sectors correspond to transferring a quasiparticle from the inner edge to the outer edge; sectors which differ by the transference of an ineteger number of electrons from on edge to the other are equivalent. The different sectors may be distinguished by the phases which are aquired when quasiparticles encircle the origin. $r=0$ corresponds to periodic boundary conditions for quasiparticles. $r \neq 0$ corresponds to 'twisted'
boundary conditions for quasiparticles; they aquire a phase $e^{2 \pi i r / m}$ upon encircling the origin. The sectors of the edge theory correspond to the $m$-fold degenerate ground states of the theory on a torus, as may be seen by gluing the inner and outer edges of the annulus to form a torus..

## Chapter 20

# Effective Field Theories of the Quantum Hall Effect 

### 20.1 Chern-Simons Theories of the Quantum Hall Effect

The preceding discussion has been heavily dependent on Laughlin's wavefunctions. However, these wavefunctions are not the exact ground states of any real experimental system. Their usefulness lies in the fact that they are representatives of a universality class of states, all of which exhibit the fractional quantum Hall effect. What has been missing to this point is a precise sense of which properties of these wavefunctions define the universality class, and which ones are irrelevant perturbations. We alluded earlier to the binding of zeroes to electrons. We will formalize this notion and use it to find low-energy, long-wavelength effective field theories for the fractional quantum Hall effect. One formulation of these effective field theories is in the form of a LandauGinzburg theory which is strongly reminiscent of superfluidity or superconductivity. One important difference, however, is that the order parameter is not a local function of the electron variables. This is not a trivial distinction, and it is, ultimately, related to the conclusion that a novel type of ordering is present in the quantum Hall states,
namely 'topological ordering.'
Recall that in a superfluid or a superconductor the off-diagonal entries of the density matrix:

$$
\begin{equation*}
\rho\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=\left\langle\psi^{\dagger}(\mathbf{r}) \psi\left(\mathbf{r}^{\prime}\right)\right\rangle=\int d \mathbf{r}_{2} \ldots d \mathbf{r}_{N} \psi^{*}\left(\mathbf{r}, \mathbf{r}_{2}, \ldots, \mathbf{r}_{N}\right) \psi\left(\mathbf{r}^{\prime}, \mathbf{r}_{2}, \ldots, \mathbf{r}_{N}\right) \tag{20.1}
\end{equation*}
$$

exhibit off-diagonal long-range order,

$$
\begin{equation*}
\rho\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \rightarrow \phi^{*}(\mathbf{r}) \phi\left(\mathbf{r}^{\prime}\right) \tag{20.2}
\end{equation*}
$$

for some non-zero function $\phi(\mathbf{r})$. Feynman argued that the ground state wavefunction of a Bose fluid would have no zeroes, so it can be chosen everywhere real and positive. In the absence of phase variations, (20.2) will hold. As a result of (20.2), we can choose states of indefinite particle number such that $\langle\psi(\mathbf{r})\rangle=\phi(\mathbf{r}) . \quad \phi(\mathbf{r})$ can be treated as a classical field and used to analyze interference phenomena such as the Josephson effect. More importantly, off-diagonal long-range order is the hallmark of superfluidity.

What happens if we calculate (20.2) in a Laughlin state state at $\nu=1 /(2 k+1)$ ?

$$
\begin{align*}
\rho\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =\left\langle\psi^{\dagger}(\mathbf{r}) \psi\left(\mathbf{r}^{\prime}\right)\right\rangle \\
& =\int d^{2} z_{2} \ldots d^{2} z_{2} \prod_{i}\left(z-z_{i}\right)^{2 k+1}\left(\bar{z}^{\prime}-\bar{z}_{i}\right)^{2 k+1} \prod_{k>l}\left|z_{k}-z_{l}\right|^{2(2 k+1)} e^{-\sum\left|z_{j}\right|}( \tag{}
\end{align*}
$$

This correlation function does not show any signs of long-range order. The fluctuating phases of the first two terms in the integral lead to exponential falloff. On the other hand, if we consider correlation functions of:

$$
\begin{equation*}
\phi^{\dagger}(z)=e^{-i(2 k+1) \int d^{2} z^{\prime} \operatorname{Im} \ln \left(z-z^{\prime}\right)} \psi^{\dagger}(z) \tag{20.4}
\end{equation*}
$$

this phase is removed and we find algebraic falloff of correlation functions, or quasi-long-range order,

$$
\begin{equation*}
\left\langle\phi^{\dagger}\left(z^{\prime}\right) \phi(z)\right\rangle \sim \frac{1}{\left|z-z^{\prime}\right|^{(2 k+1) / 2}} \tag{20.5}
\end{equation*}
$$

as may be shown using the plasma analogy [?]. The drawback of this order parameter is that it is not an analytic function of the $z$ 's, and, hence, is not a lowest Landau level operator. We could, instead, take:

$$
\begin{equation*}
\phi^{\dagger}(z)=e^{-i m \int d^{2} z^{\prime} \ln \left(z-z^{\prime}\right)} \psi^{\dagger}(z) \tag{20.6}
\end{equation*}
$$

which not only has true long-range order, but also remains strictly within the lowest Landau level [?]. However, the field theory of this operator is more complicated, so we will use (20.4).

A Landau-Ginzburg theory may be derived for the order parameter (20.4) [?] in the following way. Begin with the Lagrangian for interacting electrons in a magnetic field such that $\nu=1 /(2 k+1)$ :

$$
\begin{align*}
\mathcal{L}_{\text {eff }}= & \psi^{*}\left(i \partial_{0}-A_{0}\right) \psi+\frac{\hbar^{2}}{2 m^{*}} \psi^{*}(i \nabla-\mathbf{A})^{2} \psi-\mu \psi^{\dagger} \psi \\
& +V\left(x-x^{\prime}\right) \psi^{\dagger}(x) \psi(x) \psi^{\dagger}\left(x^{\prime}\right) \psi\left(x^{\prime}\right) \tag{20.7}
\end{align*}
$$

We now rewrite this in terms of a bosonic field, $\phi(x)$, interacting with a gauge field. The gauge field is given a Chern-Simons action of the type discussed two chapters ago, so that its only role is to transform the bosons $\phi$ into fermionic electrons.

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}}= & \phi^{*}\left(i \partial_{0}-\left(a_{0}+A_{0}\right)\right) \phi+\frac{\hbar^{2}}{2 m^{*}} \phi^{*}(i \nabla-(\mathbf{a}+\mathbf{A}))^{2} \phi \\
-\mu \phi^{\dagger} \phi+ & V\left(x-x^{\prime}\right) \phi^{\dagger}(x) \phi(x) \phi^{\dagger}\left(x^{\prime}\right) \phi\left(x^{\prime}\right) \\
& +\frac{1}{2 k+1} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{20.8}
\end{align*}
$$

Note that the coefficient of the Chern-Simons term is $2 k+1$. We could have chosen any odd integer in order to obtain the correct statistics; the coefficient $2 k+1$ is chosen for reasons which will become clear momentarily. To see that the correct statistics are obtained, note that the $a_{0}$ equation of motion is:

$$
\begin{equation*}
\nabla \times \mathbf{a}(\mathbf{r})=2 \pi(2 k+1) \rho(\mathbf{r}) \tag{20.9}
\end{equation*}
$$

The Chern-Simons gauge field equation attaches $2 k+1$ flux tubes to each $\phi$ boson. As one boson is exchanged with another, it acquires an Aharonov-Bohm phase of $(-1)^{2 k+1}=-1$ as a result of these flux tubes.

As in the Landau-Ginzburg theory of a superconductor, long-range order in the bosonic field $\phi$ - i.e. $|\phi|^{2}=\rho$ - breaks a $U(1)$ symmetry. The Meissner effect results, i.e. $\mathbf{a}+\mathbf{A}=0$, since a non-zero constant effective magnetic field $\nabla \times(\mathbf{a}+\mathbf{A})$ would lead to badly divergent energy in (20.8) if $|\phi|^{2}=\rho$. This implies that $B=-\nabla \times \mathbf{a}=$ $2 \pi(2 k+1) \rho$, or $\nu=1 /(2 k+1)$.

If $B$ is increased or decreased from this value, vortices are created, as in a type II superconductor:

$$
\begin{aligned}
\phi(r, \theta) & =|\phi(r)| e^{i \theta} \\
b+B & =f(r)
\end{aligned}
$$

with $|\phi(0)|=0,|\phi(\infty)|=\sqrt{\rho_{0}}, f(\infty)=0$. These vortices are the Laughlin quasiholes and quasiparticles. They have one flux quantum of the a gauge field and very little real magnetic flux. As a result of the flux quantum of a which they carry, they have charge $1 /(2 k+1)$, according to (20.9).

Essentially, the electrons have become bound to $2 k+1$ flux tubes - or $2 k+1$ zeroes as we put it earlier - thereby transmuting them into bosons in zero field. These bosons undergo Bose condensation; the fractional quantum Hall liquids are these Bose condensed states. Said slightly differently, the Chern-Simons gauge field, which satisfies $\nabla \times \mathbf{a}(\mathbf{r})=(2 k+1) \rho(\mathbf{r})$, has been replaced by its spatial average, $\nabla \times\langle\mathbf{a}(\mathbf{r})\rangle=(2 k+1)\langle\rho\rangle$. The average field cancels the magnetic field so that the bosons can condense. The fluctuations of $a$ around its average value could, in principle, destabilize the Bose condensed state, but they do not because there is an energy gap.

Note that the Chern-Simons term results in an important difference between the

Landau-Ginzburg theory of the quantum Hall effect and the Landau-Ginzburg theory of a superconductor. The Chern-Simons term attaches flux to charges, so both particles (electrons) and vortices (quasiparticles) carry charge and flux. As a result, they are very much on the same footing; this can be made even more explicit, as we will see later. In a superconductor, on the other hand, particles (Cooper pairs) carry charge while vortices carry flux; they are thereby differentiated.

### 20.2 Duality in $2+1$ Dimensions

The Landau-Ginzburg theory which we have just discussed has a dual formulation which will prove useful in much of the following discussion. We will first consider duality more generally for a $U(1)$ theory in $2+1$ dimensions and then consider the particular case of the quantum Hall effect.

Consider a Landau-Ginzburg theory for a $U(1)$ symmetry:

$$
\begin{equation*}
\mathcal{L}=\psi^{*} i \partial_{0} \psi+\frac{\hbar^{2}}{2 m^{*}} \psi^{*} \nabla^{2} \psi+V\left(|\psi|^{2}\right) \tag{20.10}
\end{equation*}
$$

In chapter 11, we showed that such a theory could, in its broken symmetry phase, be simplified by writing $\psi=\sqrt{\rho_{s}} e^{i \theta}$ and integrating out the gapped fluctuations of $\rho_{s}$ :

$$
\begin{equation*}
\mathcal{L}=\frac{\rho_{s}}{2 m}\left(\partial_{\mu} \theta\right)^{2} \tag{20.11}
\end{equation*}
$$

This Lagrangian has a conserved current, $\partial_{\mu} j_{\mu}=0$ given by

$$
\begin{equation*}
j_{\mu}=\rho_{s} \partial_{\mu} \theta \tag{20.12}
\end{equation*}
$$

We have assumed that there are no fluctuations in the amplitude. However, we can allow one type of amplitude fluctuations, namely vortices, if we allow $\theta$ to have singularities. Then the vortex current takes the form:

$$
\begin{equation*}
j_{\mu}^{v}=\epsilon_{\mu \nu \lambda} \partial_{\nu} \partial_{\lambda} \theta \tag{20.13}
\end{equation*}
$$

The conservation law (20.12) can be automatically satisfied if we take

$$
\begin{equation*}
j_{\mu}=\epsilon_{\mu \nu \lambda} \partial_{\nu} a_{\lambda} \tag{20.14}
\end{equation*}
$$

Note that $a_{\lambda}$ is not uniquely defined, but is subject to the gauge transformation $a_{\lambda} \rightarrow a_{\lambda}+\partial_{\mu} \chi$. Equation (20.14) can be used to solve for $\theta$ and substituted into equation (20.13):

$$
\begin{equation*}
\partial_{\nu} f_{\mu \nu}=\rho_{s} j_{\mu}^{v} \tag{20.15}
\end{equation*}
$$

where $f_{\mu \nu}$ is the field strength associated with the gauge field $a_{\lambda}$ :

$$
\begin{equation*}
f_{\mu \nu}=\partial_{\nu} a_{\mu}-\partial_{\mu} a_{\nu} \tag{20.16}
\end{equation*}
$$

If we introduce a vortex annihilation operator, $\Phi_{v}$, then (20.15) is the equation of motion of the dual Lagrangian:

$$
\begin{equation*}
\mathcal{L}_{\text {Dual }}=\frac{\kappa}{2}\left|\left(\partial_{\mu}-i a_{\mu}\right) \Phi_{v}\right|^{2}+V_{\Phi}\left(\left|\Phi_{v}\right|\right)+\frac{1}{2 \rho_{s}} f_{\mu \nu} f_{\mu \nu} \tag{20.17}
\end{equation*}
$$

where $\kappa$ is a vortex stiffness and $V_{\Phi}\left(\left|\Phi_{v}\right|\right)$ is the vortex-vortex interaction. The final term is a Maxwell term for the gauge field,

$$
\begin{array}{r}
f_{\mu \nu} f_{\mu \nu}=e^{2}-b^{2} \\
e_{i}=\partial_{0} a_{i}-\partial_{i} a_{0} \\
b=\epsilon_{i j} \partial_{i} a_{j} \tag{20.18}
\end{array}
$$

which is of the same form as the action for the electromagnetic gauge field, $F_{\mu \nu} F_{\mu \nu}=$ $E^{2}-B^{2}$

Notice that the conservation law (20.12) which followed from the equations of motion in the original representation is a trivial, topological identity in the dual representation, following from (20.14). The definition (20.13) of the vortex current in the original representation is the equation of motion (20.15) in the dual representation.

The broken symmetry phase of our original theory (20.11) is the phase in which $\left\langle\Phi_{v}\right\rangle=0$. Vortices are gapped; the low-energy effective action in the dual language is simply

$$
\begin{equation*}
\mathcal{L}_{\text {Dual }}=\frac{1}{2 \rho_{s}} f_{\mu \nu} f_{\mu \nu} \tag{20.19}
\end{equation*}
$$

The gauge field $a_{\mu}$ is the dual formulation of the Goldstone boson. However, when the symmetry is restored by the proliferation and condensation of vortices, $\left\langle\Phi_{v}\right\rangle=\Phi_{0} \neq 0$, the dual action is in its Higgs phase:

$$
\begin{equation*}
\mathcal{L}_{\text {Dual }}=\frac{\kappa}{2}\left|\Phi_{0}\right|^{2} a_{\mu} a_{\mu}+\frac{1}{2 \rho_{s}} f_{\mu \nu} f_{\mu \nu} \tag{20.20}
\end{equation*}
$$

and the gauge field $a_{\mu}$ becomes massive. Hence, it is possible, by a duality transformation, to pass between an $X Y$ theory and a $U(1)$ Higgs theory.

Topological defects in the vortex order parameter, $\Phi$, carry one quantum of $a_{\mu}$ flux. Hence, they are simply charges - e.g. Cooper pairs if the Landau-Ginzburg theory describes a superconductor. Hence, the duality operation exchanges particles and vortices. In the original representation(20.11), the Cooper pairs are the fundamental quanta while vortices are topological defects. In the dual representations, the fundamental quanta are vortices while the topological defects are Cooper pairs.

Let us now extend this to transformation to the Chern-Simons theory of the quantum Hall effect. Suppose we consider this theory

$$
\begin{align*}
\mathcal{L}_{\text {eff }}= & \phi^{*}\left(i \partial_{0}-\left(a_{0}+A_{0}\right)\right) \phi+\frac{\hbar^{2}}{2 m^{*}} \phi^{*}(i \nabla-(\mathbf{a}+\mathbf{A}))^{2} \phi \\
& +V\left(|\phi|^{2}\right)+\frac{1}{2 k+1} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{20.21}
\end{align*}
$$

in its fractional quantized Hall phase. We write $\phi=\sqrt{\rho} e^{i \theta}$ and integrate out the gapped fluctuations of $\rho$ :

$$
\begin{equation*}
\mathcal{L}_{\mathrm{eff}}=\frac{1}{2} \rho\left(\partial_{\mu} \theta-a_{\mu}-A_{\mu}\right)^{2}+\frac{1}{2 k+1} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{20.22}
\end{equation*}
$$

This theory has a conserved current which is simply the electrical current:

$$
\begin{equation*}
j_{\mu}=\rho\left(\partial_{\mu} \theta-a_{\mu}-A_{\mu}\right) \tag{20.23}
\end{equation*}
$$

We construct the dual representation of this current with a gauge field $\alpha$ :

$$
\begin{equation*}
j_{\mu}=\epsilon_{\mu \nu \lambda} \partial_{\nu} \alpha_{\lambda} \tag{20.24}
\end{equation*}
$$

As in the derivation above, we consider vortices in the order parameter. As we saw at the beginning of this chapter, they are simply Laughlin quasiparticles and quasiholes. Their current is given by:

$$
\begin{equation*}
j_{\mu}^{\mathrm{qp}}=\epsilon_{\mu \nu \lambda} \partial_{\nu} \partial_{\lambda} \theta \tag{20.25}
\end{equation*}
$$

Using the dual expression for the current to eliminate $\partial_{\mu} \theta$ from the right-hand-side of this equation, we have:

$$
\begin{equation*}
\frac{1}{\rho_{s}} \partial_{\nu} f_{\mu \nu}=j_{\mu}^{v}-\epsilon_{\mu \nu \lambda} \partial_{\nu}\left(a_{\mu}+A_{\mu}\right) \tag{20.26}
\end{equation*}
$$

This is the equation of motion of the dual Lagrangian:

$$
\begin{align*}
\mathcal{L}_{\text {Dual }}= & \frac{\kappa}{2}\left|\left(\partial_{\mu}-i \alpha_{\mu}\right) \Phi^{\mathrm{qp}}\right|^{2}+V_{\Phi}\left(\left|\Phi^{\mathrm{qp}}\right|\right)+\frac{1}{2 \rho} f_{\mu \nu} f_{\mu \nu} \\
& +\alpha_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu}\left(a_{\lambda}+A_{\lambda}\right)+\frac{1}{2 k+1} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho} \tag{20.27}
\end{align*}
$$

where $\Phi$ is the vortex annihilation operator. Integrating out $a_{\mu}$, which appears quadratically, we find:

$$
\begin{align*}
\mathcal{L}_{\text {Dual }}= & \frac{1}{2} \kappa\left|\left(\partial_{\mu}-i \alpha_{\mu}\right) \Phi^{\mathrm{qp}}\right|^{2}+V_{\Phi}\left(\left|\Phi^{\mathrm{qp}}\right|\right)+ \\
& \frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu} \partial_{\nu} \alpha_{\rho}+\frac{1}{2 \rho} f_{\mu \nu} f_{\mu \nu}+\alpha_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} A_{\mu} \tag{20.28}
\end{align*}
$$

Since the Maxwell term for $\alpha_{\mu}$ has one extra derivative compared to the ChernSimons term, it is irrelevant in the long-wavelength limit. Let us drop the Maxwell term and consider the effect of the Chern-Simons term. Since the quasiparticle annihilation operator is coupled to the Chern-Simons gauge field, $\alpha_{\mu}$, each quasiparticle
has flux $1 /(2 k+1)$ attached to it. Hence, quasiparticles have statistics $\pi /(2 k+1)$. According to the last term in (20.28), the external electromagnetic potential $A_{0}$ is coupled to the fictitious flux $\epsilon_{i j} \partial_{i} \alpha_{j}$. Since each quasiparticle has flux $1 /(2 k+1)$ attached to it, it has charge $e /(2 k+1)$.

In the phase in which $\left\langle\Phi^{\mathrm{qp}}\right\rangle=0$, we can integrate out the quasiparticles, thereby renormalizing the Maxwell term. We cn then integrate out $\alpha_{\mu}$,

$$
\begin{align*}
\mathcal{L}_{\text {eff }} & =\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu} \partial_{\nu} \alpha_{\rho}+\frac{1}{2 \rho} f_{\mu \nu} f_{\mu \nu}+\alpha_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} A_{\mu} \\
& \equiv \frac{4 \pi}{2 k+1} A_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} A_{\mu} \tag{20.29}
\end{align*}
$$

which leaves us with an effective action for the electromagnetic field which incorporates the Hall conductance $1 /(2 k+1)$.

Hence, this duality transformation has transformed an action (20.21) in which the basic field $\phi$ represents a charge $e$ fermionic electron and the basic soliton is a charge $e /(2 k+1)$, statistics $\pi /(2 k+1)$ quasiparticle into an action (20.28) in which the basic field $\Phi^{\mathrm{qp}}$ represents a charge $e /(2 k+1)$, statistics $\pi /(2 k+1)$ quasiparticle. To complete the correspondence, we must show that the basic soliton in (20.28) is a charge $e$ fermionic electron. To do this, we must consider the state in which $\Phi^{\mathrm{qp}}$ condenses.

When $\left\langle\Phi^{\mathrm{qP}}\right\rangle=\Phi_{0}^{\mathrm{qP}} \neq 0$, there are solitons in this state

$$
\begin{aligned}
\Phi^{\mathrm{qp}}(r, \theta) & =\left|\Phi^{\mathrm{qp}}(r)\right| e^{i \theta} \\
\beta+B & =f(r)
\end{aligned}
$$

with $\left|\Phi^{\mathrm{qP}}(0)\right|=0,\left|\Phi^{\mathrm{qP}}(\infty)\right|=\sqrt{\rho_{0}}, f(\infty)=0$. They carry one flux quantum of the gauge field, so they are fermionic, charge $e$ particles - i.e. electrons are the solitonic excitations of the state in which $\Phi^{\mathrm{qp}}$ condense.

When $\left\langle\Phi^{\mathrm{qp}}\right\rangle=\Phi_{0}^{\mathrm{qP}} \neq 0$, we have the effective action:

$$
\begin{equation*}
\mathcal{L}_{\text {Dual }}=\frac{\kappa}{2}\left|\Phi_{0}^{\mathrm{qp}}\right|^{2} \alpha_{\mu} \alpha_{\mu}+\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu} \partial_{\nu} \alpha_{\rho}+\frac{1}{2 \rho} f_{\mu \nu} f_{\mu \nu}+\alpha_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} A_{\mu} \tag{20.30}
\end{equation*}
$$

The first term gives a Higgs mass to the gauge field $\alpha_{\mu}$, which can now be integrated out. In doing so, we can neglect the Chern-Simons term, which is irrelevant compared to the Higgs mass. The resulting effective action for the electromagnetic gauge field is then

$$
\begin{equation*}
\mathcal{L}_{\mathrm{eff}}=\frac{1}{2} \kappa_{0} F_{\mu \nu} F_{\mu \nu} \tag{20.31}
\end{equation*}
$$

In other words, the system is an insulator. The quantum Hall state with $\sigma_{x y}=\nu e^{2} / h$ is dual to the insulating state with $\sigma_{x y}=0$ which is formed when quasiparticles condense.

Note that this insulating state is not the only state into which quasiparticles can condense. As we saw earlier in our construction of the hierarchy, quasiparticles can also condense into fractional quantum Hall states, thereby leading to $\sigma_{x y}=$ $(2 k+1 \pm 1 / p)^{-1} e^{2} / h$. Hence, the hierarchy construction is simply a variant of duality in $2+1$ dimensions, as we discuss in the next section. In this section, we have considered the 'usual' case, $p=0$.

### 20.3 The Hierarchy and the Jain Sequence

Rather than treat the quasiparticles and quasiholes as vortices, we can require that $\phi$ be topologically trivial and introduce a new bosonic field $\phi_{q}$ to keep track of the positions of the vortices [?]. By introducing such a field, we are essentially introducing the dual field $\Phi$ of the previous section. The internal structure of the vortices is lost in such an approximation; it is only valid for wavelengths much longer than the quasiparticle size. In this limit, we obtain the following effective Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{eff}} & =\phi^{*}\left(i \partial_{0}-\left(a_{0}+A_{0}\right)\right) \phi+\frac{\hbar^{2}}{2 m^{*}} \phi^{*}(i \nabla-(\mathbf{a}+\mathbf{A}))^{2} \phi+u|\phi|^{4} \\
& +\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} a_{\rho}+\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} c_{\rho}
\end{aligned}
$$

$$
\begin{equation*}
+\phi_{q}^{*}\left(i \partial_{0}-c_{0}\right) \phi_{q}+\frac{\hbar^{2}}{2 m^{*}} \phi_{q}^{*}(i \nabla-\mathbf{c})^{2} \phi_{q} \tag{20.32}
\end{equation*}
$$

where $c$ is another auxiliary gauge field which gives the $\phi_{q}$ field the fractional statistics appropriate for a quasihole or quasiparticle. The Meissner effect requires $B=-\nabla \times \mathbf{a}$, while the $a_{0}$ and $c_{0}$ equations of motion lead to:

$$
\begin{align*}
\nabla \times \mathbf{c} & =2 \pi|\phi|^{2} \\
(2 k+1) \nabla \times \mathbf{c}+\nabla \times \mathbf{a} & =2 \pi\left|\phi_{q}\right|^{2} \tag{20.33}
\end{align*}
$$

If $\left|\phi_{q}\right|^{2}=0, \nu=1 /(2 k+1)$ as before. Each quantum of $\phi_{q}$ decreases the electron number by $1 /(2 k+1)$.

If we introduce a third gauge field, $f$ - which attaches an even number of flux tubes to each quasiparticle, thereby leaving their statistics unchanged $-\phi_{q}$ can condense as well [?],

$$
\begin{align*}
\mathcal{L}_{\text {eff }} & =\phi^{*}\left(i \partial_{0}-\left(a_{0}+A_{0}\right)\right) \phi+\frac{\hbar^{2}}{2 m^{*}} \phi^{*}(i \nabla-(\mathbf{a}+\mathbf{A}))^{2} \phi \\
& +\phi_{q}^{*}\left(i \partial_{0}-\left(c_{0}+f_{0}\right)\right) \phi_{q}+\frac{\hbar^{2}}{2 m^{*}} \phi_{q}^{*}(i \nabla-(\mathbf{c}+\mathbf{f}))^{2} \phi_{q} \\
& +\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} c_{\rho}+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} a_{\rho}-\frac{1}{4 \pi} \frac{1}{2 p} \epsilon^{\mu \nu \rho} f_{\mu} \partial_{\nu} f_{\rho} \tag{20.34}
\end{align*}
$$

There are now two Meissner effects, one for $\phi$ and one for $\phi_{q}$ :

$$
\begin{array}{r}
B+\nabla \times \mathbf{a}=0 \\
\nabla \times \mathbf{c}+\nabla \times \mathbf{f}=0 \tag{20.35}
\end{array}
$$

The $a_{0}, c_{0}$, and $f_{0}$ equations of motion are:

$$
\begin{align*}
\nabla \times \mathbf{c} & =2 \pi|\phi|^{2} \\
(2 k+1) \nabla \times \mathbf{c}+\nabla \times \mathbf{a} & =2 \pi\left|\phi_{q}\right|^{2} \\
\nabla \times \mathbf{f} & =-2 \pi(2 p)\left|\phi_{q}\right|^{2} \tag{20.36}
\end{align*}
$$

Combining these equations, we find $\nu=1 /\left(2 k+1-\frac{1}{2 p}\right)$. Continuing in this way, we can find the Landau-Ginzburg theories for all of the hierarchy states.

Let's now consider the 'composite fermion' construction for the 'Jain sequence' $\nu=n /(2 p n \pm 1)$. We represent the electrons as fermions interacting with a ChernSimons gauge field $c_{\mu}$ which attaches $2 p$ flux tubes to each fermion. Lagrangian can then be written:

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}}= & \psi^{*}\left(i \partial_{0}-c_{0}-A_{0}\right) \psi+\frac{\hbar^{2}}{2 m^{*}} \psi^{*}(i \nabla-\mathbf{c}-\mathbf{A})^{2} \psi \\
& -\mu \psi^{\dagger} \psi+V\left(x-x^{\prime}\right) \psi^{\dagger}(x) \psi(x) \psi^{\dagger}\left(x^{\prime}\right) \psi\left(x^{\prime}\right) \\
& -\frac{1}{2 p} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} c_{\rho} \tag{20.37}
\end{align*}
$$

The flux of $c_{\mu}$ is anti-aligned with the magnetic field so the effective magnetic field seen by the fermions is $\nabla \times(\mathbf{c}+\mathbf{A})$, which is such that the $\psi$ 's - the composite fermions - fill $n$ Landau levels in the effective magnetic field.

To derive the effective theory for this state, we must now construct the effective theory for $n$ Landau levels. At $\nu=1 / m$, we introduced a Chern-Simons gauge field so that we could represent each electron as a boson attached to $m$ flux quanta. At $\nu=n$, it is not useful to introduce a single Chern-Simons gauge field, which would allow us to represent each electron as an anyon attached to $1 / n$ flux quanta. Instead, it is more useful introduce $n$ gauge fields, each of which is coupled to the electrons in one of the Landau levels. We can then represent each electron as a boson attached to one flux quantum. The problem with such an approach is that we can only introduce $n$ gauge fields if there are $n$ different conserved quantities, namely the charge in each Landau level. These charges are not, in general, conserved: only the total charge is conserved. We will come back to this point later, and assume for now that this will not make a difference.

Then, the Lagrangian takes the form:

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}}= & \phi_{I}^{*}\left(i \partial_{0}-a_{0}^{I}-c_{0}-A_{0}\right) \phi_{I}+\frac{\hbar^{2}}{2 m^{*}} \phi_{I}^{*}\left(i \nabla-\mathbf{a}^{I}-\mathbf{c}-\mathbf{A}\right)^{2} \phi_{I} \\
& -\mu \phi_{I}^{*} \phi_{I}+V\left(x-x^{\prime}\right) \phi_{I}^{*}(x) \phi_{I}(x) \phi_{J}^{*}\left(x^{\prime}\right) \phi_{J}\left(x^{\prime}\right) \\
& -\frac{1}{2 p} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} c_{\rho}+\frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu}^{I} \partial_{\nu} a_{\rho}^{I} \tag{20.38}
\end{align*}
$$

where $\phi_{I}$ annihilates a boson corresponding to an electron in the $I^{\text {th }}$ Landau level.

### 20.4 K-matrices

A compact summary of the information in the Landau-Ginzburg theory is given by the dual theory (see, for instance, [?] and references therein). Consider the LandauGinzburg theory for $\nu=1 /(2 k+1)$ :
$\mathcal{L}_{\text {eff }}=\phi^{*}\left(i \partial_{0}-\left(a_{0}+A_{0}\right)\right) \phi+\frac{\hbar^{2}}{2 m^{*}} \phi^{*}(i \nabla-(\mathbf{a}+\mathbf{A}))^{2} \phi+u|\phi|^{4}+\frac{1}{2 k+1} \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho}$

Let's apply $2+1$-dimensional duality to this Lagrangian, following (20.21)-(20.28). We find the dual theory:

$$
\begin{align*}
\mathcal{L}_{\text {Dual }}= & \frac{1}{2} \kappa\left|\left(\partial_{\mu}-i \alpha_{\mu}\right) \Phi^{\mathrm{qp}}\right|^{2}+V_{\Phi}\left(\left|\Phi^{\mathrm{qp}}\right|\right)+ \\
& \frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu} \partial_{\nu} \alpha_{\rho}+\frac{1}{2 \rho} f_{\mu \nu} f_{\mu \nu}+\alpha_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} A_{\mu} \tag{20.40}
\end{align*}
$$

or, keeping only the most relevant terms, simply

$$
\begin{equation*}
\mathcal{L}_{\text {dual }}=\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu} \partial_{\nu} \alpha_{\rho}+A_{\mu} \epsilon^{\mu \nu \rho} \partial_{\nu} \alpha_{\rho}+\alpha_{\mu} j_{\text {vortex }}^{\mu}+\mathcal{L}_{\text {vortex }} \tag{20.41}
\end{equation*}
$$

This generalizes to an arbitrary abelian Chern-Simons theory:

$$
\begin{equation*}
\mathcal{L}_{\text {dual }}=\frac{1}{4 \pi} K_{I J} \epsilon^{\mu \nu \rho} \alpha_{\mu}^{I} \partial_{\nu} \alpha_{\rho}^{J}+t_{I} A_{\mu} \epsilon^{\mu \nu \rho} \partial_{\nu} \alpha_{\rho}^{I}+\alpha_{\mu}^{I} j_{\text {vortex } I}^{\mu} \tag{20.42}
\end{equation*}
$$

The Hall conductance of such a state can be obtained by inergating out the ChernSimons gauge fields, which appear quadratically:

$$
\begin{equation*}
\sigma_{H}=\sum_{I, J} t^{I} t^{J}\left(K^{-1}\right)_{I J} \tag{20.43}
\end{equation*}
$$

The charge of a vortex (i.e. a quasiparticle) of type $i$ is:

$$
\begin{equation*}
q_{I}=\sum_{J} t^{J}\left(K^{-1}\right)_{I J} \tag{20.44}
\end{equation*}
$$

and the braiding statistics between vortices of types $i$ and $j$ is:

$$
\begin{equation*}
\theta_{I J}=\left(K^{-1}\right)_{I J} \tag{20.45}
\end{equation*}
$$

Implicit in the normalizations is the assumption that the charges associated with the $j_{\mu \text { vortex }}^{I}$ are quantized in integers. Distinct quantum Hall states are therefore represented by equivalence classes of ( $K, t$ ) pairs under $S L(\kappa, Z)$ basis changes where $\kappa$ is the rank of the $K$-matrix.

Let's now construct the $K$-matrices associated with the hierarchy and the Jain sequence. First, consider the Landau-Ginzburg theory (20.34) of a hierarchy state. We assume that $\phi$ is condensed, and impose its Meissner effect, $a_{\mu}=A_{\mu}$. We ignore the gapped fluctuations of $\phi$. Then

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}} & =\phi_{q}^{*}\left(i \partial_{0}-\left(c_{0}^{1}+f_{0}\right)\right) \phi_{q}+\frac{\hbar^{2}}{2 m^{*}} \phi_{q}^{*}\left(i \nabla-\left(\mathbf{c}^{1}+\mathbf{f}\right)\right)^{2} \phi_{q} \\
& +\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} c_{\rho}^{1}+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} A_{\rho}-\frac{1}{4 \pi} \frac{1}{2 p} \epsilon^{\mu \nu \rho} f_{\mu} \partial_{\nu} f_{\rho} \tag{20.46}
\end{align*}
$$

where $c_{\mu}^{1} \equiv c_{\mu}$. We write $\phi_{\mathrm{qp}}=\left|\phi_{\mathrm{qp}}\right| e^{\varphi_{\mathrm{qp}}}$, integrate out the gapped fluctuations of $\left|\phi_{\mathrm{qp}}\right|$, and apply steps (20.21)-(20.28) to (20.34) by introducing a gauge field, $c_{\mu}^{2}$ :

$$
\begin{equation*}
\epsilon^{\mu \nu \lambda} \partial_{\nu} c_{\lambda}^{2}=J_{\mathrm{qp}}^{\mu} \equiv\left|\phi_{\mathrm{qp}}\right|^{2}\left(\partial^{\mu} \varphi_{\mathrm{qp}}-c^{\mu}-f^{\mu}\right) \tag{20.47}
\end{equation*}
$$

We use the $2+1$-dimensional duality transformation to substitute this into (20.34).

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}} & =\frac{1}{2 \pi} \epsilon^{\mu \nu \rho}\left(c_{\mu}^{1}+f_{\mu}\right) \partial_{\nu} c_{\rho}^{2}-\frac{1}{4 \pi} \frac{1}{2 p} \epsilon^{\mu \nu \rho} f_{\mu} \partial_{\nu} f_{\rho} \\
& +\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} c_{\rho}^{1}+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} A_{\rho} \tag{20.48}
\end{align*}
$$

Finally, we integrate out $f_{\mu}$ :

$$
\begin{align*}
\mathcal{L}_{\text {eff }} & =\frac{2 k+1}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} c_{\rho}^{1}+\frac{2 p}{4 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{2} \partial_{\nu} c_{\rho}^{2} \\
& +\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} c_{\rho}^{2}+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} c_{\mu}^{1} \partial_{\nu} A_{\rho} \tag{20.49}
\end{align*}
$$

Hence, a state at the first level of the hierarchy has

$$
K=\left(\begin{array}{cc}
2 k+1 & 1  \tag{20.50}\\
1 & p_{1}
\end{array}\right)
$$

and $t^{I}=(1,0)$.
Continuing in this fashion, we find the $K$-matrix of an arbitrary hierarchy state (19.42):

$$
K^{h}=\left(\begin{array}{ccccc}
2 k+1 & 1 & 0 & 0 & \ldots  \tag{20.51}\\
1 & p_{1} & 1 & 0 & \\
0 & 1 & p_{2} & 1 & \\
0 & 0 & 1 & p_{3} & \\
\vdots & & & & \ddots
\end{array}\right)
$$

and $t^{I}=\delta_{1}^{I}$.
Let's now consider the flux-exchange construction of the Jain sequence. Starting with (20.38), we write $\phi_{I}=\left|\phi_{I}\right| e^{\varphi_{I}}$, integrate out the gapped fluctuations of $\left|\phi_{I}\right|$, and apply steps (20.21)-(20.28) to (20.38) by introducing gauge fields, $\alpha_{\mu}^{I}$ :

$$
\begin{equation*}
\epsilon^{\mu \nu \lambda} \partial_{\nu} \alpha_{\lambda}^{I}=J_{I}^{\mu} \equiv\left|\phi_{I}\right|^{2}\left(\partial^{\mu} \varphi_{I}-a^{I \mu}-c^{\mu}-A^{\mu}\right) \tag{20.52}
\end{equation*}
$$

Using $2+1$-dimensional duality, we re-write this as

$$
\begin{align*}
\mathcal{L}_{\mathrm{eff}}= & \frac{1}{2 \rho}\left(f_{\mu \nu}^{I}\right)^{2}+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho}\left(\sum_{I} \alpha_{\mu}^{I}\right) \partial_{\nu}\left(c_{\rho}+A_{\rho}\right)+\frac{1}{2 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu}^{I} \partial_{\nu} a_{\rho}^{I} \\
& +\frac{1}{4 \pi} \epsilon^{\mu \nu \rho} a_{\mu}^{I} \partial_{\nu} a_{\rho}^{I}-\frac{1}{4 \pi} \frac{1}{2 p} \epsilon^{\mu \nu \rho} c_{\mu} \partial_{\nu} c_{\rho} \tag{20.53}
\end{align*}
$$

Integrating out $a_{\mu}^{I}$ and $c_{\mu}$, and dropping the subleading Maxwell terms we find

$$
\begin{align*}
\mathcal{L}_{\text {eff }}= & \frac{1}{4 \pi} \epsilon^{\mu \nu \rho} \alpha_{\mu}^{I} \partial_{\nu} \alpha_{\rho}^{I}+\frac{2 p}{2 \pi} \epsilon^{\mu \nu \rho}\left(\sum_{I} \alpha_{\mu}^{I}\right) \partial_{\nu}\left(\sum_{J} \alpha_{\mu}^{J}\right) \\
& +\frac{1}{2 \pi} \epsilon^{\mu \nu \rho}\left(\sum_{I} \alpha_{\mu}^{I}\right) \partial_{\nu} A_{\rho} \tag{20.54}
\end{align*}
$$

In other words, the flux exchange construction of the Jain sequence is summarized by the $K$-matrix:

$$
K^{\text {Jain }}=\left(\begin{array}{ccccc}
2 p+1 & 2 p & 2 p & 2 p & \cdots  \tag{20.55}\\
2 p & 2 p+1 & 2 p & 2 p & \\
2 p & 2 p & 2 p+1 & 2 p & \\
2 p & 2 p & 2 p & 2 p+1 & \\
\vdots & & & & \ddots
\end{array}\right)
$$

and $t^{I}=(1,1, \ldots, 1)$.
If we make the change of basis $K^{h}=W^{T} K^{\mathrm{Jain}} W$ and $t^{h}=W^{-1} t^{\mathrm{Jain}}$, with $W=$ $\delta_{I J}-\delta_{I+1, J}$, then (??) is transformed into (20.51) with $2 k+1=2 p+1$ and $p_{1}=$ $p_{2}=\ldots=2$. Meanwhile, $t^{\text {Jain }}$ is transformed into $t^{h}$. Hence, the two constructions are identical for the corresponding filling fractions.

The $K$-matrix formalism also applies to some quantum Hall states which we have not yet discussed. These include double-layer quantum Hall states - in which there are two parallel layers of electrons - and spin-unpolarized systems. Although, we have thus far assumed that the electrons are spin-polarized by the magnetic field, band mass and $g$ factor corrections make the ratio of Zeeman to cyclotron energies $\sim 7 / 400$, so that it may be necessary to include both spins when describing electrons in moderately strong magnetic fields, even when the filling fraction is less than unity [?]. An example of a wavefunction which can describe spin-polarized electrons in a double-layer system or spin-unpolarized electrons in a single-layer system is the ( $m, m, n$ ) wavefunction:

$$
\begin{equation*}
\Psi_{(m, m, n)}\left(w_{i}, z_{j}\right)=\prod_{\mathrm{i}<\mathrm{j}}\left(w_{i}-w_{j}\right)^{m} \prod_{\mathrm{i}<\mathrm{j}}\left(z_{i}-z_{j}\right)^{m} \prod_{\mathrm{i}, \mathrm{j}}\left(w_{i}-z_{j}\right)^{n} . \tag{20.56}
\end{equation*}
$$

The $w_{i}$ 's and $z_{j}$ 's are, respectively, the coordinates of the up and down spin electrons, or upper- and lower-layer electrons and the filling fraction is $\nu=\frac{2}{m+n}$. The notation of (20.56) is sloppy; (20.56) should be multiplied by the spin part of the wavefunction
and antisymmetrized with respect to exchanges of up- and down-spin electrons. The $K$-matrix for this state is

$$
K=\left(\begin{array}{ll}
m & n  \tag{20.57}\\
n & m
\end{array}\right)
$$

and $t^{1}=t^{2}=1$. By considering hierarchies built on the ( $m, m, n$ ) states or states of unpolarized electrons in multi-layer systems, we can imagine a cornucopia of fractional quantum Hall states specified by $K$ matrices.

What exactly do we mean when we say that a Chern-Simons theory such as (20.42) is the low-energy effective field theory of a quantum Hall state? Let us first imagine that our quantum Hall liquid is on a compact surface such as a sphere or a torus, rather than in some bounded region of the plane as it would be in a real experiment. The Hamiltonian of (20.42) vanishes, so every state in the theory has vanishing energy. In other words, the Chern-Simons theory is a theory of the ground state(s). This includes states with - essentially non-dynamical - quasiholes and quasiparticles at fixed positions, since they are the lowest energy states at a given filling fraction. This theory is only valid at energies much smaller than the gap since it ignores all of the physics above the gap. The leading irrelevant corrections to (20.42) are Maxwell terms of the form $\left(\partial^{\nu} \alpha^{\lambda}-\partial^{\lambda} \alpha^{\nu}\right)^{2}$ which, by dimensional analysis, must have a coefficient suppressed by the inverse of the gap. The quasiparticle charges (20.44) and statistics (20.45) are the essential physics of the ground state which is encapsulated in this theory. This is not all, however. On a surface of genus $g$, even the state with no quasiparticles is degenerate. Two chapters ago, we saw that a Chern-Simons with coefficient $2 k+1$ has a $2 k+1$-fold degenerate ground state on the torus. This is precisely the ground state degeneracy which we obtained in the previous chapter by adapting the Laughlin wavefunctions to the torus. This can be generalized to an arbitrary quantum Hall state by diagonalizing its $K$-matrix and multipying the degeneracies of the resulting decoupled Chern-Simons terms or; in other words, the degeneracy is simply $\operatorname{det} K$. On a genus- $g$ surface, this becomes
$(\operatorname{det} K)^{g}[?]$. Since numerical studies can be - and usually are - done on the sphere or torus, the degeneracy is an important means of distinguishing distinct quantum Hall states with different $K$-matrices at the same filling fraction.

### 20.5 Field Theories of Edge Excitations in the Quantum Hall Effect

If, instead, we look at the Chern-Simons theory (20.42) on a bounded region of the plane [?], then the variation of the action $S=\int \mathcal{L}$ is:

$$
\begin{align*}
\delta S= & \frac{1}{2 \pi} \int d^{3} x K_{I J} \delta \alpha_{\mu}^{I} \epsilon^{\mu \nu \rho} \partial_{\nu} \alpha_{\rho}^{J} \\
& +\frac{1}{2 \pi} \int_{\text {boundary }} d t d x n_{\nu}\left(K_{I J} \epsilon^{\mu \nu \rho} \alpha_{\mu}^{I} \delta \alpha_{\rho}^{J}+t_{J} \epsilon^{\mu \nu \rho} A_{\mu} \delta \alpha_{\rho}^{J}\right) \tag{20.58}
\end{align*}
$$

if we set $A_{\mu}=j_{\mu}=0$. The action is extremized if we take $\epsilon^{\mu \nu \rho} \partial_{\nu} \alpha_{\rho}^{J}=0$ subject to boundary conditions such that $K_{I J} \alpha_{\mu}^{I} \mu^{\mu \nu \rho} \delta \alpha_{\rho}^{J}=0$ at the boundary. Let us suppose that $x$ and $y$ are the coordinates along and perpendicular to the boundary. Then, the most general such boundary condition is $K_{I J} \alpha_{0}^{I}+V_{I J} \alpha_{x}^{I}=0 . V_{I J}$ is a symmetric matrix which will depend on the details of the boundary such as the steepness of the confining potential. Clearly, $K_{I J} \alpha_{0}^{I}+V_{I J} \alpha_{x}^{I}=0$ would be a sensible gauge choice since it is compatible with the boundary condition. In this gauge, the equation of motion following from the variation of $K_{I J} \alpha_{0}^{I}+V_{I J} \alpha_{x}^{I}$ in (20.42) is a constraint, which can be satisfied if $a_{i}^{I}=\partial_{i} \phi^{I}$ for some scalar field $\phi$. Substituting this into the Lagrangian and integrating by parts, we find that all of the action is at the edge:

$$
\begin{equation*}
S=\frac{1}{2 \pi} \int d t d x K_{I J} \partial_{t} \phi^{I} \partial_{x} \phi^{J}-V_{I J} \partial_{x} \phi^{I} \partial_{x} \phi^{J}+A_{\mu} \epsilon_{\mu \nu} \partial_{\nu} \phi^{I} t^{I} \tag{20.59}
\end{equation*}
$$

The Chern-Simons theory of the bulk has been reduced to a theory of (chiral) bosons at the edge of the system. These are precisely the excitations which we derived by multiplying the Laughlin state by symmetric polynomials in the previous chapter.


Figure 20.1: The Chern-Simons theory which describes the braiding of quasiparticles in the bulk is associated with a Conformal Field Theory which describes the gapless excitations at the edge.

Let's consider the simplest case, $\nu=1 / m$

$$
\begin{equation*}
S=\frac{m}{2 \pi} \int d t d x \partial_{t} \phi \partial_{x} \phi-v \partial_{x} \phi \partial_{x} \phi+\frac{1}{2 \pi} \int d t d x A_{\mu} \epsilon_{\mu \nu} \partial_{\nu} \phi^{I} t^{I} \tag{20.60}
\end{equation*}
$$

This is the action for a free chiral boson. The equations of motion (for $A_{\mu}=0$ for simplicity)

$$
\begin{equation*}
\left(\partial_{t}-v \partial_{x}\right) \partial_{x} \phi=0 \tag{20.61}
\end{equation*}
$$

are satisfied if the field is chiral, $\phi(x, t)=\phi(x+v t)$.
The equal-time commutations relations which follow from this action are:

$$
\begin{equation*}
\left[\partial_{x} \phi(x), \phi\left(x^{\prime}\right)\right]=i \frac{2 \pi}{m} \delta\left(x-x^{\prime}\right) \tag{20.62}
\end{equation*}
$$

By varying the electromagnetic field, we derive the charge and current operators:

$$
\begin{align*}
\rho & =\frac{1}{2 \pi} \partial_{x} \phi \\
j & =\frac{1}{2 \pi} \partial_{t} \phi \tag{20.63}
\end{align*}
$$

Hence, the operatosr $e^{i \phi}$ and $e^{i m \phi}$ create excitations of charge $1 / m$ :

$$
\begin{align*}
{\left[\rho(x), e^{i \phi\left(x^{\prime}\right)}\right] } & =\frac{1}{m} e^{i \phi\left(x^{\prime}\right)} \delta\left(x-x^{\prime}\right) \\
{\left[\rho(x), e^{i m \phi\left(x^{\prime}\right)}\right] } & =e^{i \phi\left(x^{\prime}\right)} \delta\left(x-x^{\prime}\right) \tag{20.64}
\end{align*}
$$

These operators create quasiparticles and electrons respectively.
To compute their correlation functions, we must first compute the $\phi-\phi$ correlation function. This is most simply obtained from the imaginary-time functional integral by inverting the quadratic part of the action. In real-space, this gives:

$$
\begin{align*}
\langle\phi(x, t) \phi(0,0)\rangle-\langle\phi(0,0) \phi(0,0)\rangle & =\int \frac{d k}{2 \pi} \frac{d \omega}{2 \pi} \frac{2 \pi}{m} \frac{1}{k(i \omega-v k)}\left(e^{i \omega \tau-i k x}-1\right) \\
& =\frac{2 \pi}{m} \int \frac{d k}{2 \pi} \frac{1}{k}\left(e^{-i k(x+i v \tau)}-1\right) \\
& =-\frac{1}{m} \int_{\frac{1}{\Lambda}}^{\Lambda} \frac{d k}{2 \pi} \frac{1}{k} \\
& =-\frac{1}{m} \ln [(x+i v \tau) / a] \tag{20.65}
\end{align*}
$$

where $a=1 / \Lambda$ is a short-distance cutoff. Hence, the quasiparticle correlation function is given by:

$$
\begin{align*}
\left\langle e^{i \phi(x, \tau)} e^{i \phi(0,0)}\right\rangle & =e^{\langle\phi(x, t) \phi(0,0)\rangle-\langle\phi(0,0) \phi(0,0)\rangle} \\
& =\frac{1}{(x+i v \tau)^{1 / m}} \tag{20.66}
\end{align*}
$$

while the electron correlation function is:

$$
\begin{align*}
\left\langle e^{i m \phi(x, \tau)} e^{i m \phi(0,0)}\right\rangle & =e^{m^{2}\langle\phi(x, t) \phi(0,0)\rangle-m^{2}\langle\phi(0,0) \phi(0,0)\rangle} \\
& =\frac{1}{(x+i v \tau)^{m}} \tag{20.67}
\end{align*}
$$

Hence, the quasiparticle creation operator has dimension $2 / m$ while the electron creation operator has dimension $m / 2$.

Let us suppose that a tunnel junction is created between a quantum Hall fluid and a Fermi liquid. The tunneling of electrons from the edge of the quantum Hall
fluid to the Fermi liquid can be described by adding the following term to the action:

$$
\begin{equation*}
S_{\mathrm{tun}}=t \int d \tau e^{i m \phi(0, \tau)} \psi(0, \tau)+\text { c.c. } \tag{20.68}
\end{equation*}
$$

Here, $x=0$ is the point at which the junction is located and $\psi(x, \tau)$ is the electron annihilation operator in the Fermi liquid. As usual, it is a dimension $1 / 2$ operator. This term is irrelevant for $m>1$ :

$$
\begin{equation*}
\frac{d t}{d \ell}=\frac{1}{2}(1-m) t \tag{20.69}
\end{equation*}
$$

Hence, it can be handled perturbatively at low-temperature. The finite-temperature tunneling conductance varies with temperature as:

$$
\begin{equation*}
G_{t} \sim t^{2} T^{m-1} \tag{20.70}
\end{equation*}
$$

while the current at zero-temperature varies as:

$$
\begin{equation*}
I_{t} \sim t^{2} V^{m} \tag{20.71}
\end{equation*}
$$

A tunnel junction between two identical quantum Hall fluids has tunneling action:

$$
\begin{equation*}
S_{\mathrm{tun}}=t \int d \tau e^{i m \phi_{1}(0, \tau)} e^{-i m \phi_{2}(0, \tau)}+\text { c.c. } \tag{20.72}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\frac{d t}{d \ell}=(1-m) t \tag{20.73}
\end{equation*}
$$

and the tunneling conductance varies with temperature as:

$$
\begin{equation*}
G_{t} \sim t^{2} T^{2 m-2} \tag{20.74}
\end{equation*}
$$

while the current at zero-temperature varies as:

$$
\begin{equation*}
I_{t} \sim t^{2} V^{2 m-1} \tag{20.75}
\end{equation*}
$$

Suppose we put a constriction in a Hall bar so that tunneling is possible from the top edge of the bar to the bottom edge. Then quasiparticles can tunnel across interior of the Hall fluid. The tunneling Hamiltonian is:

$$
\begin{equation*}
S_{\mathrm{tun}}=v \int d \tau e^{i \phi_{1}(0, \tau)} e^{-i \phi_{2}(0, \tau)}+\text { c.c. } \tag{20.76}
\end{equation*}
$$

The tunneling of quasiparticles is relevant

$$
\begin{equation*}
\frac{d t}{d \ell}=\left(1-\frac{1}{m}\right) v \tag{20.77}
\end{equation*}
$$

where $\phi_{1}$ and $\phi_{2}$ are the edge operators of the two edges. Hence, it can be treated perturbatively only at high-temperatures or large voltages. At low voltage and high temperature, the tunneling conductance varies with temperature as:

$$
\begin{equation*}
G_{t} \sim v^{2} T^{\frac{2}{m}-2} \tag{20.78}
\end{equation*}
$$

while the current at zero-temperature varies as:

$$
\begin{equation*}
I_{t} \sim v^{2} V^{\frac{2}{m}-1} \tag{20.79}
\end{equation*}
$$

so long as $V$ is not too small. If we measure the Hall conductance of the bar by running current from the left to the right, then it will be reduced by the tunneling current:

$$
\begin{equation*}
G=\frac{1}{m} \frac{e^{2}}{h}-(\text { const. }) v^{2} T^{\frac{2}{m}-2} \tag{20.80}
\end{equation*}
$$

When $T$ becomes low enough, the bar is effectively split in two so that all that remains is the tunneling of electrons from the left side to the right side:

$$
\begin{equation*}
G \sim t^{2} T^{2 m-2} \tag{20.81}
\end{equation*}
$$

In other words, the conductance is given by a scaling function of the form:

$$
\begin{equation*}
G=\frac{1}{m} \frac{e^{2}}{h} Y\left(v^{2} T^{\frac{2}{m}-2}\right) \tag{20.82}
\end{equation*}
$$

with $Y(x)-1 \sim-x$ for $x \rightarrow 0$ and $Y(x) \sim x^{-m}$ for $x \rightarrow \infty$.
For a general $K$-matrix, the edge structure is more complicated since there will be several bosonic fields, but they can still be analyzed by the basic methods of free field theory. Details can be found in .

### 20.6 Duality in $1+1$ Dimensions

At the end of the previous section, we saw that a problem which could, in the weakcoupling limit, be described by the tunneling of quasiparticles was, in the strong coupling limit, described by the tunneling of electrons. This is an example of a situation in which there are two dual descriptions of the same problem. In the quantum Hall effect, there is one description in which electrons are the fundamental objects and quasiparticles appear as vortices in the electron fluid and another description in which quasiparticles are the fundamental objects and electrons appear as aggregates of three quasiparticles. We have already discussed this duality in the $2+1$-dimensional Chern-Simons Landau-Ginzburg theory which describes the bulk. In this section, we will examine more carefully the implementation of this duality in the edge field theories. As we will see, it essentially the same as the duality which we used in our analysis of the Kosterlitz-Thouless transition. In the next section, we will look at the analogous structure in the bulk field theory.

Let us consider a free non-chiral boson $\varphi$. It can be expressed in terms of chrial fields $\phi_{L}$ and $\phi_{R}$ :

$$
\begin{align*}
\varphi & =\phi_{L}+\phi_{R} \\
\tilde{\varphi} & =\phi_{L}-\phi_{R} \tag{20.83}
\end{align*}
$$

Here, we have defined the dual field $\tilde{\varphi}$. Observe that:

$$
\begin{equation*}
\partial_{\mu} \tilde{\varphi}=\epsilon_{\mu \nu} \partial_{\nu} \varphi \tag{20.84}
\end{equation*}
$$

The free action takes the form:

$$
\begin{equation*}
S_{0}=\frac{g}{8 \pi} \int d x d \tau\left[\left(\partial_{\tau} \varphi\right)^{2}+v^{2}\left(\partial_{x} \varphi\right)^{2}\right] \tag{20.85}
\end{equation*}
$$

where $\varphi$ is an agular variable: $\varphi \equiv \varphi+2 \pi$. Let us rescale $\varphi \rightarrow \varphi / \sqrt{g}$ so that the action is of the form:

$$
\begin{equation*}
S_{0}=\frac{1}{8 \pi} \int d x d \tau\left[\left(\partial_{\tau} \varphi\right)^{2}+v^{2}\left(\partial_{x} \varphi\right)^{2}\right] \tag{20.86}
\end{equation*}
$$

As a result of the rescaling of $\varphi$ which we performed in going from (20.85) to (20.86), $\varphi$ now satisfies the identification $\varphi \equiv \varphi+2 \pi \sqrt{g}$.

Note that this theory has a conserved current, $\partial_{\mu} j_{\mu}=0$

$$
\begin{equation*}
j_{\mu}=\partial_{\mu} \varphi \tag{20.87}
\end{equation*}
$$

which is conserved by the equation of motion. It also has a current

$$
\begin{equation*}
j_{\mu}^{D}=\partial_{\mu} \tilde{\varphi} \tag{20.88}
\end{equation*}
$$

which is trivially conserved.
Let us consider the Fourier decomposition of $\phi_{R, L}$ :

$$
\begin{align*}
\phi_{R}(x, \tau) & =\frac{1}{2} x_{0}^{R}+p_{R}(i \tau-x)+i \sum_{n} \frac{1}{n} \alpha_{n} e^{-n(\tau+i x)} \\
\phi_{L}(x, \tau) & =\frac{1}{2} x_{0}^{L}+p_{L}(i \tau+x)+i \sum_{n} \frac{1}{n} \tilde{\alpha}_{n} e^{-n(\tau-i x)} \tag{20.89}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \varphi(x, \tau)=\varphi_{0}+i\left(p_{L}+p_{R}\right) \tau+\left(p_{L}-p_{R}\right) x+i \sum_{n} \frac{1}{n}\left[\alpha_{n} e^{-n(\tau+i x)}+\tilde{\alpha}_{n} e^{-n(\tau-i x)}\right] \\
& \tilde{\varphi}(x, \tau)=\tilde{\varphi}_{0}+i\left(p_{L}-p_{R}\right) \tau+\left(p_{L}+p_{R}\right) x-i \sum_{n} \frac{1}{n}\left[\alpha_{n} e^{-n(\tau+i x)}-\tilde{\alpha}_{n} e^{-n(\tau-i x)}\right] \tag{20.90}
\end{align*}
$$

where $\varphi_{0}=x_{0}^{L}+x_{0}^{R}$ and $\tilde{\varphi}_{0}=x_{0}^{L}-x_{0}^{R}$. From the identification $\varphi \equiv \varphi+2 \pi \sqrt{g}$, it follows that $\varphi_{0} \equiv \varphi_{0}+2 \pi \sqrt{g}$. From the canonical commutation relations for $\varphi$, it
follows that $\varphi_{0}$ and $\left(p_{L}+p_{R}\right) / 2$ are canonical conjugates. The periodicity condition satisfied by $\varphi_{0}$ imposes the following quantization condition on $\left(p_{L}+p_{R}\right)$ :

$$
\begin{equation*}
p_{L}+p_{R}=\frac{M}{\sqrt{g}}, \quad M \in Z \tag{20.91}
\end{equation*}
$$

Furthermore, physical operators of the theory must respect the periodicity condition of $\varphi$. The allowed exponential operators involving $\varphi$ are of the form:

$$
\begin{equation*}
\{M, 0\} \equiv e^{i \frac{M}{\sqrt{g}} \varphi(x, \tau)} \tag{20.92}
\end{equation*}
$$

and they have dimension $M^{2} / g$.
Let us assume that our edges are closed loops of finite extent, and rescale the length so that $x \in[0, \pi]$ with $\varphi(\tau, x) \equiv \varphi(\tau, x+\pi)+2 \pi N \sqrt{m}$ for some integer $N$. Then, from (20.90), we see that we must have

$$
\begin{equation*}
\left(p_{L}-p_{R}\right)=2 N \sqrt{g}, \quad N \in Z \tag{20.93}
\end{equation*}
$$

These degrees of freedom are called 'winding modes'. Hence, we have:

$$
\begin{align*}
& p_{L}=\frac{M}{2 \sqrt{g}}+N \sqrt{g} \\
& p_{R}=\frac{M}{2 \sqrt{g}}-N \sqrt{g} \tag{20.94}
\end{align*}
$$

Note that this is reversed when we consider $\tilde{\varphi}$. Momentum modes are replaced by winding modes and vice-versa. Following our earlier steps, but taking this reversal into account, the allowed exponentials of $\tilde{\varphi}$ are of the form:

$$
\begin{equation*}
\{0, N\} \equiv e^{i 2 N \sqrt{g} \tilde{\varphi}(x, \tau)} \tag{20.95}
\end{equation*}
$$

Hence, the most general exponential operator is of the form:

$$
\begin{equation*}
\{M, N\} \equiv e^{i\left(\frac{M}{2 \sqrt{g}} \varphi+N \sqrt{m} \tilde{\varphi}\right)}=e^{i\left[\left(\frac{M}{2 \sqrt{g}}+N \sqrt{m}\right) \phi_{L}+\left(\frac{M}{2 \sqrt{g}}-N \sqrt{g}\right) \phi_{R}\right]} \tag{20.96}
\end{equation*}
$$



Figure 20.2: An infinite chiral edge mode with a tunnel junction at one point can be folded into a semi-infinite nonchiral mode with a tunnel junction at its endpoint.
with scaling dimension:

$$
\begin{align*}
\operatorname{dim}(M, N) & =\frac{1}{2}\left(\frac{M}{2 \sqrt{g}}+N \sqrt{g}\right)^{2}+\frac{1}{2}\left(\frac{M}{2 \sqrt{g}}-N \sqrt{g}\right)^{2} \\
& =\frac{M^{2}}{4 g}+N^{2} g \\
& =\frac{(M / 2)^{2}}{g}+N^{2} g \tag{20.97}
\end{align*}
$$

These dimensions are invariant under the transformation $g \leftrightarrow 1 / 4 g, M \leftrightarrow N$. In fact the entire theory is invariant under this transformation. It is simply the transformation which exchanges $\varphi$ and $\tilde{\varphi}$.

When we couple two identical non-chiral bosons, $\varphi_{1}$ and $\varphi_{2}$, we form $\varphi_{ \pm}=\left(\varphi_{1} \pm\right.$ $\left.\varphi_{1}\right) / \sqrt{2}$. The factor of $\sqrt{2}$ is included so that both $\varphi_{ \pm}$have the same coefficent, $1 / 8 \pi$, in front of their actions. However, this now means that $\varphi_{ \pm} \equiv \varphi_{ \pm}+2 \pi \sqrt{g / 2}$. When we couple two bosons though exponential tunneling operators, only $\varphi_{-}$is affected. Hence, the appropriate duality is that for $\varphi_{-}:(g / 2) \leftrightarrow 1 /[4(g / 2)]$ or, simply $g \leftrightarrow 1 / g$. This duality exchanges $\cos \varphi_{-} / \sqrt{g / 2}$ and $\cos \tilde{\varphi}_{-} \sqrt{g / 2}$, which transfer, repectively, a pair of solitons (i.e. electrons) and a particle-hole pair from system 1 to system 2.

Let us now apply these considerations to quantum Hall edges. In order to apply the above duality - which applies to non-chiral bosons - to a quantum Hall edge, which is chiral, we must 'fold' the edge in order to define a non-chiral field, as depicted in figure 20.2.

If we fold the edge at $x=0$, we can define $\varphi=(\phi(x)+\phi(-x)) / \sqrt{2}$ and $\tilde{\varphi}=$
$(\phi(x)-\phi(-x)) / \sqrt{2}$. The latter vanishes at the origin; only the former is important for edge tunneling. The allowed operators are:

$$
\begin{equation*}
\left[e^{i N \varphi / \sqrt{m}}\right]=\frac{N^{2}}{2 m} \tag{20.98}
\end{equation*}
$$

The factor of $1 / 2$ on the right-hand-side comes from the $\sqrt{2}$ in the definition of $\varphi$. If we couple two edges, we can now define $\varphi_{-}$, which has allowed operators

$$
\begin{equation*}
\left[e^{i N \varphi_{-} / \sqrt{m / 2}}\right]=\frac{N^{2}}{m} \tag{20.99}
\end{equation*}
$$

and dual operators

$$
\begin{equation*}
\left[e^{2 i M \tilde{\varphi}-\sqrt{m / 2}}\right]=M^{2} m \tag{20.100}
\end{equation*}
$$

which are dual under $M \leftrightarrow N, m \leftrightarrow 1 / m$. The description in terms of $\varphi$ is equivalent to the description in terms of $\tilde{\varphi}$. However, as we saw in the previous section, the tunneling of quasiparticles between the two edges of a quantum Hall droplet is most easily discussed in terms of $\varphi$ when the tunneling is weak, i.e. in the ultraviolet, when the tunneling operator can be written $e^{i \varphi / \sqrt{m}}$. However, when the tunneling becomes strong, in the infrared, the dual description in terms of $\tilde{\varphi}$ is preferable, since the corresonding tunneling operator is $e^{i \tilde{\varphi} \sqrt{m}}$.

Chapter 21

## $P, T$-violating Superconductors

## Electron Fractionalization without $P, T$-violation

## Part VI

## Localized and Extended

Excitations in Dirty Systems

# Impurities in Solids 

23.1 Impurity States
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## Chapter 24

## Field-Theoretic Techniques for Disordered Systems

24.1 Disorder-Averaged Perturbation Theory
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## Chapter 25

## The Non-Linear $\sigma$-Model for Anderson Localization

25.1 Derivation of the $\sigma$-model
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25.4 The Metal-Insulator Transition

## Electron-Electron Interactions in Disordered Systems

26.1 Perturbation Theory
26.2 The Finkelstein $\sigma$-Model


[^0]:    ${ }^{1}$ In a relativistic theory, one must always deal with particles whose number is not fixed since there is always the possibility of pair creation, at least as a virtual process.

[^1]:    ${ }^{1}$ Contrast this with the usual zero-temperature formalism, where we compute $\langle f| U(\infty,-\infty)|i\rangle$

[^2]:    ${ }^{1}$ In the next chapter, when we turn to spin systems, however, we will need to revisit the derivation of the previous section.

[^3]:    ${ }^{2}$ Actually, one should be slightly more general and draw a parallel between a $d$-dimensional classical system and a $d$-z-dimensional quantum system. $z=1$ is somehwat special to our phonon theory since frequency and momentum enter with same power, e.g. $\omega^{2}-v_{l}^{2} p^{2}$. Ordinarily, in a system of non-relativistic bosons, the equations of motion have one time derivative and two spatial derivatives, as in the Schrödinger equation. In such a case, $\omega$ is on equal footing with $p^{2}$, so $z=2$.

[^4]:    ${ }^{1}$ In $d>2$, a four-fermi interaction generically scales as $s^{(d-1) / 2}$. A $(d-1)$-dimensional delta function which restricts to forward scattering gives a marginal operator. $\mathrm{A}(d-2)$-dimensional delta function which restricts to a plane gives an operator which scales as $s^{1 / 2}$, as in $d=2$.

[^5]:    ${ }^{1}$ Broken symmetry is similar in the sense that a symmetry is either broken or not and is therefore characterized by a binary number.

[^6]:    ${ }^{1} \mathrm{We}$ also ignored the effects of the ions on the electrons. The periodic potential due to the lattice has very little effect at the low densities relevant for the quantum Hall effect, except to replace the bare electron mass by the band mass. This can be quantitatively important. For instance, $m_{b} \simeq 0.07 m_{e}$ in GaAs.
    ${ }^{2}$ The conventional measure of the purity of a quantum Hall device is the zero-field mobility, $\mu$, which is defined by $\mu=\sigma / n e$, where $\sigma$ is the zero-field conductivity. The integer quantum Hall effect was first observed [?] in Si mosfets with mobility $\approx 10^{4} \mathrm{~cm}^{2} / \mathrm{Vs}$ while the fractional quantum Hall effect was first observed [?] in GaAs-AlGaAs heterostructures with mobility $\approx 10^{5} \mathrm{~cm}^{2} / \mathrm{Vs}$. Today, the highest quality GaAs-AlGaAs samples have mobilities of $\approx 10^{7} \mathrm{~cm}^{2} / \mathrm{Vs}$.

[^7]:    ${ }^{3}$ In the very low density limit, $2 k+1>140$, the plasma crystallizes. This is not an important limit, however, since a Wigner crystal state is already energetically favorable compared to the Laughlin state at densities $2 k+1>7$.

[^8]:    ${ }^{4}$ It should be emphasized, however, that the Laughlin quasihole does not carry one flux quantum of magnetic flux. The magnetic field generated by a quantum Hall state is, in fact, negligible. A quasihole does carry one flux quantum of 'fictitious' flux, as will be explained later.

[^9]:    ${ }^{5}$ For a discussion of the foundations of fractional statistics, see [?, ?].

[^10]:    ${ }^{6}$ Slowly, that is, compared to the inverse of the energy spacings $E_{n}-E_{m}$

