# STATISTICS OF ENERGY LEVELS AND EIGENFUNCTIONS IN DISORDERED SYSTEMS 

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#### Abstract

The article reviews recent developments in the theory of fluctuations and correlations of energy levels and eigenfunction amplitudes in diffusive mesoscopic samples. Various spatial geometries are considered, with emphasis on low-dimensional (quasi-1D and 2D) systems. Calculations are based on the supermatrix $\sigma$-model approach. The method reproduces, in so-called zero-mode approximation, the universal random matrix theory (RMT) results for the energy-level and eigenfunction fluctuations. Going beyond this approximation allows us to study system-specific deviations from universality, which are determined by the diffusive classical dynamics in the system. These deviations are especially strong in the far "tails" of the distribution function of the eigenfunction amplitudes (as well as of some related quantities, such as local density of states, relaxation time, etc.). These asymptotic "tails" are governed by anomalously localized states which are formed in rare realizations of the random potential. The deviations of the level and eigenfunction statistics from their RMT form strengthen with increasing disorder and become especially pronounced at the Anderson metal-insulator transition. In this regime, the wave functions are multifractal, while the level statistics acquires a scale-independent form with distinct critical features. Fluctuations of the conductance and of the local intensity of a classical wave radiated by a point-like source in the quasi-1D geometry are also studied within the $\sigma$-model approach. For a ballistic system with rough surface an appropriately modified ("ballistic") $\sigma$-model is used. Finally, the interplay of the fluctuations and the electron-electron interaction in small samples is discussed, with application to the Coulomb blockade spectra. © 2000 Elsevier Science B.V. All rights reserved.


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## 1. Introduction

Statistical properties of energy levels and eigenfunctions of complex quantum systems have been attracting a lot of interest of physicists since the work of Wigner [1], who formulated a statistical point of view on nuclear spectra. In order to describe excitation spectra of complex nuclei, Wigner proposed to replace a complicated and unknown Hamiltonian by a large $N \times N$ random matrix. This was a beginning of the random matrix theory (RMT) further developed by Dyson and Mehta in the early 1960s [2,3]. This theory predicts a universal form of the spectral correlation functions determined solely by some global symmetries of the system (time-reversal invariance and value of the spin).

Later it was realized that the random matrix theory is not restricted to strongly interacting many-body systems, but has a much broader range of applicability. In particular, Bohigas et al. [4] put forward a conjecture (strongly supported by accumulated numerical evidence) that the RMT describes adequately statistical properties of spectra of quantum systems whose classical analogs are chaotic.

Another class of systems to which the RMT applies and which is of special interest to us here is that of disordered systems. More specifically, we mean a quantum particle (an electron) moving in a random potential created by some kind of impurities. It was conjectured by Gor'kov and Eliashberg [5] that statistical properties of the energy levels in such a disordered granule can be described by the random matrix theory. This statement had remained in the status of conjecture until 1982, when it was proved by Efetov [6]. This became possible due to development by Efetov of a very powerful tool of treatment of the disordered systems under consideration - the supersymmetry method (see the review [6] and the recent book [7]). This method allows one to map the problem of the particle in a random potential onto a certain deterministic field-theoretical model (supermatrix $\sigma$-model), which generates the disorder-averaged correlation functions of the original problem. As Efetov showed, under certain conditions one can neglect spatial variation of the $\sigma$-model supermatrix field (so-called zero-mode approximation), which allows one to calculate the correlation functions. The corresponding results for the two-level correlation function reproduced precisely the RMT results of Dyson.

The supersymmetry method can be also applied to the problems of the RMT-type. In this connection, we refer the reader to the paper [8], where the technical aspects of the method are discussed in detail.

More recently, focus of the research interest was shifted from the proof of the applicability of RMT to the study of system-specific deviations from the universal (RMT) behavior. For the problem of level correlations in a disordered system, this question was addressed for the first time by Altshuler and Shklovskii [9] in the framework of the diffuson-cooperon diagrammatic perturbation theory. They showed that the diffusive motion of the particle leads to a high-frequency behavior of the level correlation function completely different from its RMT form. Their perturbative treatment was however restricted to frequencies much larger than the level spacing and was not able to reproduce the oscillatory contribution to the level correlation function. Inclusion of non-zero spatial modes (which means going beyond universality) within the $\sigma$-model treatment of the level correlation function was performed in Ref. [10]. The method developed in [10] was later used for calculation of deviations from the RMT of various statistical characteristics of a disordered system. For the case of level statistics, the calculation of [10] valid for not too large
frequencies (below the Thouless energy equal to the inverse time of diffusion through the system) was complemented by Andreev and Altshuler [11] whose saddle-point treatment was, in contrast, applicable for large frequencies. Level statistics in diffusive disordered samples is discussed in detail in Section 2 of the present article.

Not only the energy levels statistics but also the statistical properties of wave functions are of considerable interest. In the case of nuclear spectra, they determine fluctuations of widths and heights of the resonances [12]. In the case of disordered (or chaotic) electronic systems, eigenfunction fluctuations govern, in particular, statistics of the tunnel conductance in the Coulomb blockade regime [13]. Note also that the eigenfunction amplitude can be directly measured in microwave cavity experiments [14-16] (though in this case one considers the intensity of a classical wave rather than of a quantum particle, all the results are equally applicable; see also Section 7). Within the random matrix theory, the distribution of eigenvector amplitudes is simply Gaussian, leading to $\chi^{2}$ distribution of the "intensities" $\left|\psi_{i}\right|^{2}$ (Porter-Thomas distribution) [12].

A theoretical study of the eigenfunction statistics in a disordered system is again possible with use of the supersymmetry method. The corresponding formalism, which was developed in Refs. [17-20] (see Section 3.1), allows one to express various distribution functions characterizing the eigenfunction statistics through the $\sigma$-model correlators. As in the case of the level correlation function, the zero-mode approximation to the $\sigma$-model reproduces the RMT results, in particular the Porter-Thomas distribution of eigenfunction amplitudes. However, one can go beyond this approximation. In particular, in the case of a quasi-one-dimensional geometry, considered in Section 3.2, this $\sigma$-model has been solved exactly using the transfer-matrix method, yielding exact analytical results for the eigenfunction statistics for arbitrary length of the system, from weak to strong localization regime $[17,18,21-23]$. The case of a quasi-1D geometry is of great interest not only from the point of view of condensed matter theory (as a model of a disordered wire) but also for quantum chaos.

In Section 3.3 we consider the case of arbitrary spatial dimensionality of the system. Since for $d>1$ an exact solution of the problem cannot be found, one has to use some approximate methods. In Refs. $[24,25]$ the scheme of [10] was generalized to the case of the eigenfunction statistics. This allowed us to calculate the distribution of eigenfunction intensities and its deviation from the universal (Porter-Thomas) form. Fluctuations of the inverse participation ratio and long-range correlations of the eigenfunction amplitudes, which are determined by the diffusive dynamics in the corresponding classical system [25-27], and are considered in Section 3.3.3.

Section 4 is devoted to the asymptotic "tails" of the distribution functions of various fluctuating quantities (local amplitude of an eigenfunction, relaxation time, local density of states) characterizing a disordered system. It turns out that the asymptotics of all these distribution functions are determined by rare realizations of disorder leading to formation of anomalously localized eigenstates. These states show some kind of localization while all "normal" states are ergodic; in the quasi-one-dimensional case they have an effective localization length much shorter than the "normal" one. Existence of such states was conjectured by Altshuler et al. [28] who studied distributions of various quantities in $2+\varepsilon$ dimensions via the renormalization group approach. More recently, Muzykantskii and Khmelnitskii [29] suggested a new approach to the problem. Within this method, the asymptotic "tails" of the distribution functions are obtained by finding a non-trivial saddle-point configuration of the supersymmetric $\sigma$-model. Further development and generalization of the method allowed one to calculate the asymptotic behavior of the distribution
functions of relaxation times [29-31], eigenfunction intensities [32,33], local density of states [34], inverse participation ratio [35,36], level curvatures [37,38], etc. The saddle-point solution describes directly the spatial shape of the corresponding anomalously localized state [29,36].

Section 5 deals with statistical properties of the energy levels and wave functions at the Anderson metal-insulator transition point. As is well known, in $d>2$ dimensions a disordered system undergoes, with increasing strength of disorder, a transition from the phase of extended states to that of localized states (see, e.g. [39] for review). This transition changes drastically the statistics of energy levels and eigenfunctions. While in the delocalized phase the levels repel each other strongly and their statistics is described by RMT (up to the deviations discussed above and in Section 2), in the localized regime the level repulsion disappears (since states nearby in energy are located far from each other in real space). As a result, the levels form an ideal 1D gas (on the energy axis) obeying the Poisson statistics. In particular, the variance of the number $N$ of levels in an interval $\Delta E$ increases linearly, $\operatorname{var}(N)=\langle N\rangle$, in contrast to the slow logarithmic increase in the RMT case. What happens to the level statistics at the transition point? This question was addressed for the first time by Altshuler et al. [40], where a Poisson-like increase, $\operatorname{var}(N)=\chi\langle N\rangle$, was found numerically with a spectral compressibility $\chi \simeq 0.3$. More recently, Shklovskii et al. [41] put forward the conjecture that the nearest level spacing distribution $P(s)$ has a universal form at the critical point, combining the RMT-like level repulsion at small $s$ with the Poisson-like behavior at large $s$. However, these results were questioned by Kravtsov et al. [42] who developed an analytical approach to the problem and found, in particular, a sublinear increase of $\operatorname{var}(N)$. This controversy was resolved in [43,44] where the consideration of [42] was critically reconsidered and the level number variance was shown to have generally a linear behavior at the transition point. By now, this result has been confirmed by numerical simulations done by several groups [45-48]. Recently, a connection between this behavior and multifractal properties of eigenfunctions has been conjectured [49].

Multifractality is a formal way to characterize strong fluctuations of the wave function amplitude at the mobility edge. It follows from the renormalization group calculation of Wegner [50] (though the term "multifractality" was not used there). Later the multifractality of the critical wave functions was discussed in [51] and confirmed by numerical simulations of the disordered tight-binding model [52-56]. It implies, in very rough terms, that the eigenfunction is effectively located in a vanishingly small portion of the system volume. A natural question then arises: why do such extremely sparse eigenfunctions show the same strong level repulsion as the ergodic states in the RMT? This problem is addressed in Section 5.1. It is shown there that the wavefunctions of nearby-in-energy states exhibit very strong correlations (they have essentially the same multifractal structure), which preserves the level repulsion despite the sparsity of the wave functions.

In Section 5.2 we consider a "power-law random banded matrix ensemble" (PRBM) which describes a kind of one-dimensional system with a long-range hopping whose amplitude decreases as $r^{-\alpha}$ with distance [57]. Such a random matrix ensemble arises in various contexts in the theory of quantum chaos $[58,59]$ and disordered systems [60-62]. The problem can again be mapped onto a supersymmetric $\sigma$-model. It is further shown that at $\alpha=1$ the system is at a critical point of the localization-delocalization transition. More precisely, there exists a whole family of such critical points labeled by the coupling constant of the $\sigma$-model (which can be in turn related to the parameters of the microscopic PRBM ensemble). Statistics of levels and eigenfunctions in this model are studied. At the critical point they show the critical features discussed above (such as the multifractality of eigenfunctions and a finite spectral compressibility $0<\chi<1$ ).

The energy level and eigenfunction statistics characterize the spectrum of an isolated sample. For an open system (coupled to external conducting leads), different quantities become physically relevant. In particular, we have already mentioned the distributions of the local density of states and of the relaxation times discussed in Section 4 in connection with anomalously localized states. In Section 6 we consider one of the most famous issues in the physics of mesoscopic systems, namely that of conductance fluctuations. We focus on the case of the quasi-one-dimensional geometry. The underlying microscopic model describing a disordered wire coupled to freely propagating modes in the leads was proposed by Iida et al. [63]. It can be mapped onto a 1D $\sigma$-model with boundary terms representing coupling to the leads. The conductance is given in this approach by the multichannel Landauer-Büttiker formula. The average conductance $\langle g\rangle$ of this system for arbitrary value of the ratio of its length $L$ to the localization length $\xi$ was calculated by Zirnbauer [64], who developed for this purpose the Fourier analysis on supersymmetric manifolds. The variance of the conductance was calculated in [65] (in the case of a system with strong spin-orbit interaction there was a subtle error in the papers [64,65] corrected by Brouwer and Frahm [66]). The analytical results which describe the whole range of $L / \xi$ from the weak localization $(L \ll \xi)$ to the strong localization $(L \gtrdot \xi)$ regime were confirmed by numerical simulations $[67,68]$.

As has been already mentioned, the $\sigma$-model formalism is not restricted to quantum-mechanical particles, but is equally applicable to classical waves. Section 7 deals with a problem of intensity distribution in the optics of disordered media. In an optical experiment, a source and a detector of the radiation can be placed in the bulk of disordered media. The distribution of the detected intensity is then described in the leading approximation by the Rayleigh law [69] which follows from the assumption of a random superposition of independent traveling waves. This result can be also reproduced within the diagrammatic technique [70]. Deviations from the Rayleigh distribution governed by the diffusive dynamics were studied in [71] for the quasi-1D geometry. When the source and the detector are moved toward the opposite edges of the sample, the intensity distribution transforms into the distribution of transmission coefficients [72-74].

Recently, it has been suggested by Muzykantskii and Khmelnitskii [75] that the supersymmetric $\sigma$-model approach developed previously for the diffusive systems is also applicable in the case of ballistic systems. Muzykantskii and Khmelnitskii derived the "ballistic $\sigma$-model" where the diffusion operator was replaced by the Liouville operator governing the ballistic dynamics of the corresponding classical system. This idea was further developed by Andreev et al. [76,77] who derived the same action via the energy averaging for a chaotic ballistic system with no disorder. (There are some indications that one has to include in consideration certain amount of disorder to justify the derivation of [76,77].) Andreev et al. replaced, in this case, the Liouville operator by its regularization known as Perron-Frobenius operator. However, this approach has failed to provide explicit analytical results for any particular chaotic billiard so far. This is because the eigenvalues of the Perron-Frobenius operator are usually not known, while its eigenfunctions are highly singular. To overcome these difficulties and to make a further analytical progress possible, a ballistic model with surface disorder was considered in [78,79]. The corresponding results are reviewed in Section 8. It is assumed that roughness of the sample surface leads to the diffusive surface scattering, modelling a ballistic system with strongly chaotic classical dynamics. Considering the simplest (circular) shape of the system allows one to find the spectrum of the corresponding Liouville operator and to study statistical properties of energy levels and eigenfunctions. The
results for the level statistics show important differences as compared to the case of a diffusive system and are in agreement with arguments of Berry [80,81] concerning the spectral statistics in a generic chaotic billiard.

In Section 9 we discuss a combined effect of the level and eigenfunction fluctuations and the electron-electron interaction on thermodynamic properties of quantum dots. Section 9.1 is devoted to statistics of the so-called addition spectrum of a quantum dot in the Coulomb blockade regime. The addition spectrum, which is determined by the positions of the Coulomb blockade conductance peaks with varying gate voltage, corresponds to a successive addition of electrons to the dot coupled very weakly to the outside world [82]. The two important energy scales characterizing such a dot are the charging energy $e^{2} / C$ and the electron level spacing $\Delta$ (the former being much larger than the latter for a dot with large number of electrons). Statistical properties of the addition spectrum were experimentally studied for the first time by Sivan et al. [83]. It was conjectured in Ref. [83] that fluctuations in the addition spectrum are of the order of $e^{2} / C$ and are thus of classical origin. However, it was found in Refs. [84,85] that this is not the case and that the magnitude of fluctuations is set by the level spacing $\Delta$, as in the non-interacting case. The interaction modifies, however, the shape of the distribution function. In particular, it is responsible for breaking the spin degeneracy of the quantum dot spectrum. These results have been confirmed recently by thorough experimental studies [86,87].
The research activity in the field of disordered mesoscopic systems, random matrix theory, and quantum chaos has been growing enormously during the recent years, so that a review article clearly cannot give an account of the progress in the whole field. Many of the topics which are not covered here have been extensively discussed in the recent reviews by Beenakker [88] and by Guhr et al. [89].

## 2. Energy level statistics: random matrix theory and beyond

### 2.1. Supersymmetric $\sigma$-model formalism

The problem of energy level correlations has been attracting a lot of research interest since the work of Wigner [1]. The random matrix theory (RMT) developed by Wigner et al. [2,3] was found to describe well the level statistics of various classes of complex systems. In particular, in 1965 Gor'kov and Eliashberg [5] put forward a conjecture that the RMT is applicable to the problem of energy level correlations of a quantum particle moving in a random potential. To prove this hypothesis, Efetov developed the supersymmetry approach to the problem [6,7]. The quantity of primary interest is the two-level correlation function ${ }^{2}$

$$
\begin{equation*}
R(s)=\frac{1}{\langle v\rangle^{2}}\langle v(E-\omega / 2) v(E+\omega / 2)\rangle \tag{2.1}
\end{equation*}
$$

[^1]where $v(E)=V^{-1} \operatorname{Tr} \delta(E-\hat{H})$ is the density of states at the energy $E, V$ is the system volume, $\hat{H}$ is the Hamiltonian, $\Delta=1 /\langle v\rangle V$ is the mean level spacing, $s=\omega / \Delta$, and $\langle\ldots\rangle$ denote averaging over realizations of the random potential. As was shown by Efetov [6], the correlator (2.1) can be expressed in terms of a Green function of certain supermatrix $\sigma$-model. Depending on whether the time reversal and spin rotation symmetries are broken or not, one of three different $\sigma$-models is relevant, with unitary, orthogonal or symplectic symmetry group. We will consider first the technically simplest case of the unitary symmetry (corresponding to the broken time reversal invariance); the results for two other cases will be presented at the end.
We give only a brief sketch of the derivation of the expression for $R(s)$ in terms of the $\sigma$-model. One begins with representing the density of states in terms of the Green's functions,
\[

$$
\begin{equation*}
v(E)=\frac{1}{2 \pi \mathrm{i} V} \int \mathrm{~d}^{d} \boldsymbol{r}\left[G_{\mathrm{A}}^{E}(\boldsymbol{r}, \boldsymbol{r})-G_{\mathrm{R}}^{E}(\boldsymbol{r}, \boldsymbol{r})\right], \tag{2.2}
\end{equation*}
$$

\]

where

$$
\begin{equation*}
G_{\mathrm{R}, \mathrm{~A}}^{E}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\left\langle\boldsymbol{r}_{1}\right|(E-\hat{H} \pm \mathrm{i} \eta)^{-1}\left|\boldsymbol{r}_{2}\right\rangle, \quad \eta \rightarrow+0 . \tag{2.3}
\end{equation*}
$$

The Hamiltonian $\hat{H}$ consists of the free part $\hat{H}_{0}$ and the disorder potential $U(\boldsymbol{r})$ :

$$
\begin{equation*}
\hat{H}=\hat{H}_{0}+U(\boldsymbol{r}), \quad \hat{H}_{0}=\frac{1}{2 m} \hat{\boldsymbol{p}}^{2}, \tag{2.4}
\end{equation*}
$$

the latter being defined by the correlator

$$
\begin{equation*}
\left\langle U(\boldsymbol{r}) U\left(\boldsymbol{r}^{\prime}\right)\right\rangle=\frac{1}{2 \pi v \tau} \delta\left(\boldsymbol{r}-\boldsymbol{r}^{\prime}\right) . \tag{2.5}
\end{equation*}
$$

A non-trivial part of the calculation is the averaging of the $G_{\mathrm{R}} G_{\mathrm{A}}$ terms entering the correlation function $\langle v(E+\omega / 2) v(E-\omega / 2)\rangle$. The following steps are:
(i) to write the product of the Green's functions in terms of the integral over a supervector field $\Phi=\left(S_{1}, \chi_{1}, S_{2}, \chi_{2}\right)$ :

$$
\begin{align*}
& G_{\mathrm{R}}^{E+\omega / 2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{1}\right) G_{\mathrm{A}}^{E-\omega / 2}\left(\boldsymbol{r}_{2}, \boldsymbol{r}_{2}\right)=\int D \Phi D \Phi^{\dagger} S_{1}\left(\boldsymbol{r}_{1}\right) S_{\mathrm{i}}^{*}\left(\boldsymbol{r}_{1}\right) S_{2}\left(\boldsymbol{r}_{2}\right) S_{2}^{*}\left(\boldsymbol{r}_{\mathbf{2}}\right) \\
& \quad \times \exp \left\{\mathrm{i} \int \mathrm{~d} \boldsymbol{r}^{\prime} \Phi^{\dagger}\left(\boldsymbol{r}^{\prime}\right) \Lambda^{1 / 2}[E+(\omega / 2+\mathrm{i} \eta) \Lambda-\hat{H}] \Lambda^{1 / 2} \Phi\left(\boldsymbol{r}^{\prime}\right)\right\}, \tag{2.6}
\end{align*}
$$

where $\Lambda=\operatorname{diag}\{1,1,-1,-1\}$,
(ii) to average over the disorder;
(iii) to introduce a $4 \times 4$ supermatrix variable $\mathscr{R}_{\mu \nu}(\boldsymbol{r})$ conjugate to the tensor product $\Phi_{\mu}(\boldsymbol{r}) \Phi_{v}^{\dagger}(\boldsymbol{r})$;
(iv) to integrate out the $\Phi$ fields;
(v) to use the saddle-point approximation which leads to the following equation for $\mathscr{R}$ :

$$
\begin{align*}
& \mathscr{R}(\boldsymbol{r})=\frac{1}{2 \pi v \tau} g(\boldsymbol{r}, \boldsymbol{r})  \tag{2.7}\\
& g\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\left\langle\boldsymbol{r}_{\mathbf{1}}\right|\left(E-\hat{H}_{0}-\mathscr{R}\right)^{-1}\left|\boldsymbol{r}_{2}\right\rangle \tag{2.8}
\end{align*}
$$

The relevant set of the solutions (the saddle-point manifold) has the form:

$$
\begin{equation*}
\mathscr{R}=\sigma \cdot I-\frac{\mathrm{i}}{2 \tau} Q \tag{2.9}
\end{equation*}
$$

where $I$ is the unity matrix, $\sigma$ is certain constant, and the $4 \times 4$ supermatrix $Q=T^{-1} \Lambda T$ satisfies the condition $Q^{2}=1$, with $T$ belonging to the coset space $U(1,1 \mid 2) / U(1 \mid 1) \times U(1 \mid 1)$. The expression for the two-level correlation function $R(s)$ then reads

$$
\begin{equation*}
R(s)=\left(\frac{1}{4 V}\right)^{2} \operatorname{Re} \int D Q(\boldsymbol{r})\left[\int \mathrm{d}^{d} \boldsymbol{r} \operatorname{Str} Q \Lambda k\right]^{2} \exp \left\{-\frac{\pi v}{4} \int \mathrm{~d}^{d} \boldsymbol{r} \operatorname{Str}\left[-D(\nabla Q)^{2}-2 \mathrm{i} \omega \Lambda Q\right]\right\} \tag{2.10}
\end{equation*}
$$

Here $k=\operatorname{diag}\{1,-1,1,-1\}$, Str denotes the supertrace, and $D$ is the classical diffusion constant. We do not give here a detailed description of the model and mathematical entities involved, which can be found, e.g. in Refs. [6-8,90], and restrict ourselves to a qualitative discussion of the structure of the matrix $Q$. The size 4 of the matrix is due to (i) two types of the Green functions (advanced and retarded) entering the correlation function (2.1), and (ii) necessity to introduce bosonic and fermionic degrees of freedom to represent these Green's function in terms of a functional integral. The matrix $Q$ consists thus of four $2 \times 2$ blocks according to its advanced-retarded structure, each of them being a supermatrix in the boson-fermion space.

To proceed further, Efetov [6] neglected spatial variation of the supermatrix field $Q(r)$ and approximated the functional integral in Eq. (2.10) by an integral over a single supermatrix $Q$ (so-called zero-mode approximation). The resulting integral can be calculated yielding precisely the Wigner-Dyson distribution: ${ }^{3}$

$$
\begin{equation*}
R_{\mathrm{WD}}^{\mathrm{U}}(s)=1-\frac{\sin ^{2}(\pi s)}{(\pi s)^{2}} \tag{2.11}
\end{equation*}
$$

the superscript $U$ standing for the unitary ensemble. The corresponding results for the orthogonal $(\mathrm{O})$ and the symplectic $(\mathrm{Sp})$ ensemble are

$$
\begin{equation*}
R_{\mathrm{WD}}^{\mathrm{O}}(s)=1-\frac{\sin ^{2}(\pi s)}{(\pi s)^{2}}-\left[\frac{\pi}{2} \operatorname{sgn}(s)-\operatorname{Si}(\pi s)\right]\left[\frac{\cos \pi s}{\pi s}-\frac{\sin \pi s}{(\pi s)^{2}}\right] \tag{2.12}
\end{equation*}
$$

[^2]\[

$$
\begin{aligned}
& R_{\mathrm{WD}}^{\mathrm{Sp}}(s)=1-\frac{\sin ^{2}(2 \pi s)}{(2 \pi s)^{2}}+\operatorname{Si}(2 \pi s)\left[\frac{\cos 2 \pi s}{2 \pi s}-\frac{\sin 2 \pi s}{(2 \pi s)^{2}}\right] \\
& \mathrm{Si}(x)=\int_{0}^{x} \frac{\sin y}{y} \mathrm{~d} y .
\end{aligned}
$$
\]

The aim of Section 2.2 will be to study the deviations of the level correlation function from the universal RMT results (2.11)-(2.13).

### 2.2. Deviations from universality

The procedure we are using in order to calculate deviations from the universality is as follows [10]. We first decompose $Q$ into the constant part $Q_{0}$ and the contribution $\widetilde{Q}$ of higher modes with non-zero momenta. Then we use the renormalization group ideas and integrate out all fast modes. This can be done perturbatively provided the dimensionless (measured in units of $e^{2} / h$ ) conductance $g=2 \pi E_{c} / \Delta=2 \pi \nu D L^{d-2} \gg 1$ (here $E_{c}=D / L^{2}$ is the Thouless energy). As a result, we get an integral over the matrix $Q_{0}$ only, which has to be calculated non-perturbatively.

We begin with presenting the correlator $R(s)$ in the form

$$
\begin{align*}
& R(s)=\left.\frac{1}{(2 \pi i)^{2}} \frac{\partial^{2}}{\partial u^{2}} \int D Q \exp \{-S[Q]\}\right|_{u=0}, \\
& S[Q]=-\frac{1}{t} \int \operatorname{Str}(\nabla Q)^{2}+\tilde{s} \int \operatorname{Str} \Lambda Q+\tilde{u} \int \operatorname{Str} Q \Lambda k \tag{2.14}
\end{align*}
$$

where $1 / t=\pi v D / 4, \tilde{s}=\pi s / 2 \mathrm{i} V, \tilde{u}=\pi u / 2 \mathrm{i} V$. Now we decompose $Q$ in the following way:

$$
\begin{equation*}
Q(r)=T_{0}^{-1} \widetilde{Q}(r) T_{0} \tag{2.15}
\end{equation*}
$$

where $T_{0}$ is a spatially uniform matrix and $\widetilde{Q}$ describes all modes with non-zero momenta. When $\omega \ll E_{c}$, the matrix $\widetilde{Q}$ fluctuates only weakly near the origin $\Lambda$ of the coset space. In the leading order, $\tilde{Q}=\Lambda$, thus reducing (2.14) to a zero-dimensional $\sigma$-model, which leads to the WignerDyson distribution (2.13). To find the corrections, we should expand the matrix $\widetilde{Q}$ around the origin $\Lambda$ :

$$
\begin{equation*}
\widetilde{Q}=\Lambda(1+W / 2)(1-W / 2)^{-1}=\Lambda\left(1+W+\frac{W^{2}}{2}+\frac{W^{3}}{4}+\cdots\right) \tag{2.16}
\end{equation*}
$$

where $W$ is a supermatrix with the following block structure:

$$
W=\left(\begin{array}{cc}
0 & t_{12}  \tag{2.17}\\
t_{21} & 0
\end{array}\right)
$$

Substituting this expansion into Eq. (2.14), we get

$$
S=S_{0}+S_{1}+\mathrm{O}\left(W^{3}\right)
$$

$$
\begin{align*}
& S_{0}=\int \operatorname{Str}\left[\frac{1}{t}(\nabla W)^{2}+\tilde{s} Q_{0} \Lambda+\tilde{u} Q_{0} \Lambda k\right], \\
& S_{1}=\frac{1}{2} \int \operatorname{Str}\left[\tilde{s} U_{0} \Lambda W^{2}+\tilde{u} U_{0 k} \Lambda W^{2}\right], \tag{2.18}
\end{align*}
$$

where $Q_{0}=T_{0}^{-1} \Lambda T_{0}, U_{0}=T_{0} \Lambda T_{0}^{-1}, U_{0 k}=T_{0} \Lambda k T_{0}^{-1}$. Let us define $S_{\text {eff }}\left[Q_{0}\right]$ as a result of elimination of the fast modes:

$$
\begin{equation*}
\mathrm{e}^{-S_{\text {eff }}\left[Q_{0}\right]}=\mathrm{e}^{-S_{0}\left[Q_{0}\right]}\left\langle\mathrm{e}^{-S_{1}\left[\varrho_{0}, W\right]+\ln J[W]}\right\rangle_{W}, \tag{2.19}
\end{equation*}
$$

where $\langle\ldots\rangle_{W}$ denote the integration over $W$ and $J[W]$ is the Jacobian of the transformation (2.15), (2.16) from the variable $Q$ to $\left\{Q_{0}, W\right\}$ (the Jacobian does not contribute to the leading order correction calculated here, but is important for higher-order calculations [25,91]). Expanding up to the order $W^{4}$, we get

$$
\begin{equation*}
S_{\text {eff }}=S_{0}+\left\langle S_{1}\right\rangle-\frac{1}{2}\left\langle S_{1}^{2}\right\rangle+\frac{1}{2}\left\langle S_{1}\right\rangle^{2}+\cdots \tag{2.20}
\end{equation*}
$$

The integral over the fast modes can be calculated now using the Wick theorem and the contraction rules [6,28]:

$$
\begin{align*}
& \left\langle\operatorname{Str} W(\boldsymbol{r}) P W\left(\boldsymbol{r}^{\prime}\right) R\right\rangle=\Pi\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)(\operatorname{Str} P \operatorname{Str} R-\operatorname{Str} P \Lambda \operatorname{Str} R \Lambda),  \tag{2.21}\\
& \left\langle\operatorname{Str}[W(\boldsymbol{r}) P] \operatorname{Str}\left[W\left(\boldsymbol{r}^{\prime}\right) R\right]\right\rangle=\Pi\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right) \operatorname{Str}(P R-P \Lambda R \Lambda),
\end{align*}
$$

where $P$ and $R$ are arbitrary supermatrices. The diffusion propagator $\Pi$ is the solution of the diffusion equation

$$
\begin{equation*}
-D \nabla^{2} \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=(\pi v)^{-1}\left[\delta\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right)-V^{-1}\right] \tag{2.22}
\end{equation*}
$$

with the Neumann boundary condition (normal derivative equal to zero at the sample boundary) and can be presented in the form

$$
\begin{equation*}
\Pi\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=\frac{1}{\pi v_{\mu ; \varepsilon_{\mu} \neq 0}} \sum_{0} \frac{1}{\varepsilon_{\mu}} \phi_{\mu}(\boldsymbol{r}) \phi_{\mu}\left(\boldsymbol{r}^{\prime}\right) \tag{2.23}
\end{equation*}
$$

where $\phi_{\mu}(r)$ are the eigenfunctions of the diffusion operator $-D \nabla^{2}$ corresponding to the eigenvalues $\varepsilon_{\mu}$ (equal to $D \boldsymbol{q}^{2}$ for a rectangular geometry). As a result, we find

$$
\begin{align*}
& \left\langle S_{1}\right\rangle=0, \\
& \left\langle S_{1}^{2}\right\rangle=\frac{1}{2} \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime} \Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)\left(\tilde{s} \operatorname{Str} Q_{0} \Lambda+\tilde{u} \operatorname{Str} Q_{0} \Lambda k\right)^{2} . \tag{2.24}
\end{align*}
$$

Substitution of Eq. (2.24) into Eq. (2.20) yields

$$
\begin{align*}
& S_{\text {eff }}\left[Q_{0}\right]=\frac{\pi}{2 \mathrm{i}} s \operatorname{Str} Q_{0} \Lambda+\frac{\pi}{2 \mathrm{i}} u \operatorname{Str} Q_{0} \Lambda k+\frac{\pi^{2} a_{d}}{4 g^{2}}\left(s \operatorname{Str} Q_{0} \Lambda+u \operatorname{Str} Q_{0} \Lambda k\right)^{2}, \\
& a_{d}=\frac{g^{2}}{4 V^{2}} \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime} \Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=\frac{1}{\pi^{4}} \sum_{\substack{n_{1}, \ldots, n_{d}=0 \\
n_{1}^{2}+\ldots+n_{d}^{2}>0}}^{\infty} \frac{1}{\left(n_{1}^{2}+\cdots+n_{d}^{2}\right)^{2}} . \tag{2.25}
\end{align*}
$$

The value of the coefficient $a_{d}$ depends on spatial dimensionality $d$ and on the sample geometry; in the last line of Eq. (2.25) we assumed a cubic sample with hard-wall boundary conditions. Then for $d=1,2,3$ we have $a_{1}=1 / 90 \simeq 0.0111, a_{2} \simeq 0.0266$, and $a_{3} \simeq 0.0527$ respectively. In the case of a cubic sample with periodic boundary conditions we get instead

$$
\begin{equation*}
a_{d}=\frac{1}{(2 \pi)^{4}} \sum_{\substack{n_{1}, \ldots, n_{d}=-\infty \\ n_{1}^{2}+\ldots+n_{d}^{2}>0}}^{\infty} \frac{1}{\left(n_{1}^{2}+\cdots+n_{d}^{2}\right)^{2}}, \tag{2.26}
\end{equation*}
$$

so that $a_{1}=1 / 720 \simeq 0.00139, a_{2} \simeq 0.00387$, and $a_{3} \simeq 0.0106$. Note that for $d<4$ the sum in Eqs. (2.25) and (2.26) converges, so that no ultraviolet cut-off is needed.

Using now Eq. (2.14) and calculating the remaining integral over the supermatrix $Q_{0}$, we finally get the following expression for the correlator to the $1 / g^{2}$ order:

$$
\begin{equation*}
R(s)=1-\frac{\sin ^{2}(\pi s)}{(\pi s)^{2}}+\frac{4 a_{d}}{g^{2}} \sin ^{2}(\pi s) . \tag{2.27}
\end{equation*}
$$

The last term in Eq. (2.27) just represents the correction of order $1 / g^{2}$ to the Wigner-Dyson distribution. The formula (2.27) is valid for $s<g$. Let us note that the smooth (non-oscillating) part of this correction in the region $1 \ll s<g$ can be found by using purely perturbative approach of Altshuler and Shklovskii [9,40]. For $s \gg 1$ the leading perturbative contribution to $R(s)$ is given by a two-diffuson diagram:

$$
\begin{equation*}
R_{\mathrm{AS}}(s)-1=\frac{\Delta^{2}}{2 \pi^{2}} \operatorname{Re} \sum_{\substack{q_{i}=\pi n_{i} / L \\ n_{i}=0,1,2, \ldots}} \frac{1}{\left(D q^{2}-\mathrm{i} \omega\right)^{2}}=\frac{1}{2 \pi^{2}} \operatorname{Re} \sum_{n_{i} \geq 0} \frac{1}{\left[-\mathrm{i} s+(\pi / 2) g \boldsymbol{n}^{2}\right]^{2}} . \tag{2.28}
\end{equation*}
$$

At $s<g$ this expression is dominated by the $\boldsymbol{q}=0$ term, with other terms giving a correction of order $1 / g^{2}$ :

$$
\begin{equation*}
R_{\mathrm{As}}(s)=1-\frac{1}{2 \pi^{2} s^{2}}+\frac{2 a_{d}}{g^{2}}, \tag{2.29}
\end{equation*}
$$

where $a_{d}$ was defined in Eq. (2.25). This formula is obtained in the region $1 \ll s<g$ and is perturbative in both $1 / s$ and $1 / g$. It does not contain oscillations (which cannot be found perturbatively) and gives no information about actual small-s behavior of $R(s)$. The result (2.27) is much stronger: it represents the exact (non-perturbative in $1 / s$ ) form of the correction in the whole region $s \ll g$.

The important feature of Eq. (2.27) is that it relates corrections to the smooth and oscillatory parts of the level correlation function (represented by the contributions to the last term proportional to unity and to $\cos 2 \pi s$ respectively). While appearing naturally in the framework of the supersymmetric $\sigma$-model, this fact is highly non-trivial from the point of view of semiclassical theory [80,81], which represents the level structure factor $K(\tau)$ (Fourier transform of $R(s)$ ) in terms of a sum over periodic orbits. The smooth part of $R(s)$ corresponds then to the small- $\tau$ behavior of $K(\tau)$, which is related to the properties of short periodic orbits. On the other hand, the oscillatory part of $R(s)$ is related to the behavior of $K(\tau)$ in the vicinity of the Heisenberg time $\tau=2 \pi(t=2 \pi / \Delta$ in dimensionful unit), and thus to the properties of long periodic orbits.

The calculation presented above can be straightforwardly generalized to the other symmetry classes. The result can be presented in a form valid for all the three cases: ${ }^{4}$

$$
\begin{equation*}
R^{(\beta)}(s)=\left(1+\frac{2 a_{d}}{\beta g^{2}} \mathrm{~d}^{2} \mathrm{ds}^{2} s^{2}\right) R_{\mathrm{WD}}^{(\beta)}(s) \tag{2.30}
\end{equation*}
$$

where $\beta=1(2,4)$ for the orthogonal (unitary, symplectic) symmetry; $R_{W \mathrm{~W}}^{(\beta)}$ denotes the corresponding Wigner-Dyson distribution (2.11)-(2.13).

For $s \rightarrow 0$ the Wigner-Dyson distribution has the following behavior:

$$
\begin{align*}
& R_{\mathrm{WD}}^{(\beta)} \simeq c_{\beta} s^{\beta}, \quad s \rightarrow 0 \\
& c_{1}=\frac{\pi^{2}}{6}, \quad c_{2}=\frac{\pi^{2}}{3}, \quad c_{4}=\frac{(2 \pi)^{4}}{135} . \tag{2.31}
\end{align*}
$$

As is clear from Eq. (2.30), the found correction does not change the exponent $\beta$, but renormalizes the prefactor $c_{\beta}$ :

$$
\begin{equation*}
R^{(\beta)}(s)=\left(1+\frac{2(\beta+2)(\beta+1)}{\beta} \frac{a_{d}}{g^{2}}\right) c_{\beta} s^{\beta}, \quad s \rightarrow 0 . \tag{2.32}
\end{equation*}
$$

The correction to $c_{\beta}$ is positive, which means that the level repulsion becomes weaker. This is related to a tendency of eigenfunctions to localization with decreasing $g$.

What is the behavior of the level correlation function in its high-frequency tail $s \gg g$ ? The non-oscillatory part of $R(s)$ in this region follows from the Altshuler-Shklovskii perturbative formula (2.28). For $s \gg g$ summation can be replaced by integration, yielding

$$
\begin{equation*}
R_{\mathrm{As}}(s)-1 \propto g^{-d / 2} s^{d / 2-2} \tag{2.33}
\end{equation*}
$$

(note that in 2D the coefficient of the term (2.33) vanishes, and the result for $R_{\text {As }}$ is smaller by an additional factor $1 / g$, see [44]). What is the fate of the oscillations in $R(s)$ in this regime? The answer to this question was given by Andreev and Altshuler [11] who calculated $R(s)$ using the stationarypoint method for the $\sigma$-model integral (2.10). Their crucial observation was that on top of the trivial stationary point $Q=\Lambda$ (expansion around which is just the usual perturbation theory), there exists another one, $Q=k \Lambda$, whose vicinity generates the oscillatory part of $R(s)$. (In the case of symplectic symmetry there exists an additional family of stationary points, see [11]). The saddle-point approximation of Andreev and Altshuler is valid for $s \gg 1$; at $1 \ll s \ll g$ it reproduces the above results of Ref. [10] (we remind that the method of [10] works for all $s<g$ ). The result of [11] has the following form:

$$
\begin{equation*}
R_{\mathrm{osc}}^{\mathrm{U}}(s)=\frac{\cos 2 \pi s}{2 \pi^{2}} D(s), \tag{2.34}
\end{equation*}
$$

[^3]\[

$$
\begin{align*}
& R_{\mathrm{osc}}^{\mathrm{O}}(s)=\frac{\cos 2 \pi s}{2 \pi^{4}} D^{2}(s)  \tag{2.35}\\
& R_{\mathrm{osc}}^{\mathrm{Sp}}(s)=\frac{\cos 2 \pi s}{4} D^{1 / 2}(s)+\frac{\cos 4 \pi s}{32 \pi^{4}} D^{2}(s) \tag{2.36}
\end{align*}
$$
\]

where $D(s)$ is the spectral determinant

$$
\begin{equation*}
D(s)=\frac{1}{s^{2}} \prod_{\mu}\left(1+\frac{s^{2} \Delta^{2}}{\varepsilon_{\mu}^{2}}\right)^{-1} \tag{2.37}
\end{equation*}
$$

The product in Eq. (2.37) goes over the non-zero eigenvalues $\varepsilon_{\mu}$ of the diffusion operator (which are equal to $D \boldsymbol{q}^{2}$ for the cubic geometry). This demonstrates again the relation between $R_{\text {osc }}(s)$ and the perturbative part (2.28), which can be also expressed through $D(s)$,

$$
\begin{equation*}
R_{\mathrm{AS}}^{(\beta)}(s)-1=-\frac{1}{2 \beta \pi^{2}} \frac{\partial^{2} \ln D(s)}{\partial s^{2}} \tag{2.38}
\end{equation*}
$$

In the high-frequency region $s \gg g$ the spectral determinant is found to have the following behavior:

$$
\begin{equation*}
D(s) \sim \exp \left\{-\frac{\pi}{\Gamma(d / 2) d \sin (\pi d / 4)}\left(\frac{2 s}{g}\right)^{d / 2}\right\} \tag{2.39}
\end{equation*}
$$

so that the amplitude of the oscillations vanishes exponentially with $s$ in this region.
Taken together, the results of $[10,11]$ provide complete description of the deviations of the level correlation function from universality in the metallic regime $g \gg 1$. They show that in the whole region of frequencies these deviations are controlled by the classical (diffusion) operator governing the dynamics in the corresponding classical system.

## 3. Statistics of eigenfunctions

### 3.1. Eigenfunction statistics in terms of the supersymmetric $\sigma$-model

Within the RMT, the distribution of eigenfunction amplitudes is simply Gaussian, leading to the $\chi^{2}$ distribution of the "intensities" $y_{i}=N\left|\psi_{i}^{2}\right|$ (we normalized $y_{i}$ in such a way that $\langle y\rangle=1$ ) [12]

$$
\begin{align*}
\mathscr{P}^{\mathrm{U}}(y) & =\mathrm{e}^{-y},  \tag{3.1}\\
\mathscr{P}^{\mathrm{O}}(y) & =\frac{\mathrm{e}^{-y / 2}}{\sqrt{2 \pi y}} \tag{3.2}
\end{align*}
$$

Eq. (3.2) is known as the Porter-Thomas distribution; it was originally introduced to describe fluctuations of widths and heights of resonances in nuclear spectra [12].

Recently, interest in properties of eigenfunctions in disordered and chaotic systems has started to grow. On the experimental side, it was motivated by the possibility of fabrication of small systems (quantum dots) with well resolved electron energy levels [92,93,82]. Fluctuations in the tunneling
conductance of such a dot measured in recent experiments [94,95] are related to statistical properties of wave function amplitudes [13,96-98]. When the electron-electron Coulomb interaction is taken into account, the eigenfunction fluctuations determine the statistics of matrix elements of the interaction, which is in turn important for understanding the properties of excitation and addition spectra of the dot $[99,100,84]$. Furthermore, the microwave cavity technique allows one to observe experimentally spatial fluctuations of the wave amplitude in chaotic and disordered cavities [13-15].

Theoretical study of the eigenfunction statistics in a $d$-dimensional disordered system is again possible with use of the supersymmetry method [17-20]. The distribution function of the eigenfunction intensity $u=\left|\psi^{2}\left(\boldsymbol{r}_{0}\right)\right|$ in a point $\boldsymbol{r}_{0}$ is defined as

$$
\begin{equation*}
\mathscr{P}(u)=\frac{1}{\nu V}\left\langle\sum_{\alpha} \delta\left(\left|\psi_{\alpha}\left(\boldsymbol{r}_{0}\right)\right|^{2}-u\right) \delta\left(E-E_{\alpha}\right)\right\rangle . \tag{3.3}
\end{equation*}
$$

The moments of $\mathscr{P}(u)$ can be written through the Green's functions in the following way:

$$
\begin{equation*}
\left.\left.\langle | \psi\left(\boldsymbol{r}_{0}\right)\right|^{2 q}\right\rangle=\frac{\mathrm{i}^{q-2}}{2 \pi \nu V} \lim _{\eta \rightarrow 0}(2 \eta)^{q-1}\left\langle G_{\mathrm{R}}^{q-1}\left(\boldsymbol{r}_{0}, \boldsymbol{r}_{0}\right) G_{\mathrm{A}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}_{0}\right)\right\rangle . \tag{3.4}
\end{equation*}
$$

The product of the Green's functions can be expressed in terms of the integral over a supervector field $\Phi=\left(S_{1}, \chi_{1}, S_{2}, \chi_{2}\right)$,

$$
\begin{align*}
G_{\mathrm{R}}^{q-1}\left(\boldsymbol{r}_{0}, \boldsymbol{r}_{0}\right) G_{\mathrm{A}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}_{0}\right)= & \frac{\mathrm{i}^{2-q}}{(q-1)!} \int D \Phi D \Phi^{\dagger}\left(S_{1}\left(\boldsymbol{r}_{0}\right) S_{1}^{*}\left(\boldsymbol{r}_{0}\right)\right)^{q-1} S_{2}\left(\boldsymbol{r}_{0}\right) S_{2}^{*}\left(\boldsymbol{r}_{0}\right) \\
& \times \exp \left\{\mathrm{i} \int \mathrm{~d} \boldsymbol{r}^{\prime} \Phi^{\dagger}\left(\boldsymbol{r}^{\prime}\right) \Lambda^{1 / 2}(E+\mathrm{i} \eta \Lambda-\hat{H}) \Lambda^{1 / 2} \Phi\left(\boldsymbol{r}^{\prime}\right)\right\} . \tag{3.5}
\end{align*}
$$

Proceeding now in the same way as in the case of the level correlation function (Section 2.1), we represent the r.h.s. of Eq. (3.5) in terms of a $\sigma$-model correlation function. As a result, we find ${ }^{5}$

$$
\begin{equation*}
\left.\left.\langle | \psi\left(\boldsymbol{r}_{0}\right)\right|^{2 q}\right\rangle=-\frac{q}{2 V} \lim _{n \rightarrow 0}(2 \pi v \eta)^{q-1} \int D Q Q_{11, b b}^{q-1} Q_{22, b b} \mathrm{e}^{-S[\varrho]}, \tag{3.6}
\end{equation*}
$$

where $S[Q]$ is the $\sigma$-model action,

$$
\begin{equation*}
S[Q]=-\frac{\beta}{2} \int \mathrm{~d}^{d} r \operatorname{Str}\left[\frac{\pi v D}{4}(\nabla Q)^{2}-\pi v \eta \Lambda Q\right] \tag{3.7}
\end{equation*}
$$

( $\beta=2$ for the considered case of the unitary symmetry). Let us define now the function $Y\left(Q_{0}\right)$ as

$$
\begin{equation*}
Y\left(Q_{0}\right)=\int_{Q\left(r_{0}\right)=Q_{0}} D Q(r) \exp \{-S[Q]\} . \tag{3.8}
\end{equation*}
$$

[^4]Here $\boldsymbol{r}_{0}$ is the spatial point, in which the statistics of eigenfunction amplitudes is studied. For the invariance reasons, the function $Y\left(Q_{0}\right)$ turns out to be dependent in the unitary symmetry case on the two scalar variables $1 \leq \lambda_{1}<\infty$ and $-1 \leq \lambda_{2} \leq 1$ only, which are the eigenvalues of the "retarded-retarded" block of the matrix $Q_{0}$. Moreover, in the limit $\eta \rightarrow 0$ (at a fixed value of the system volume) only the dependence on $\lambda_{1}$ persists:

$$
\begin{equation*}
Y\left(Q_{0}\right) \equiv Y\left(\lambda_{1}, \lambda_{2}\right) \rightarrow Y_{a}\left(2 \pi v \eta \lambda_{1}\right) . \tag{3.9}
\end{equation*}
$$

With this definition, Eq. (3.6) takes the form of an integral over the single matrix $Q_{0}$,

$$
\begin{equation*}
\left.\left.\langle | \psi\left(\boldsymbol{r}_{0}\right)\right|^{2 q}\right\rangle=-\frac{q}{2 V} \lim _{\eta \rightarrow 0}(2 \pi v \eta)^{q-1} \int D Q_{0} Q_{0 ; 11, b b}^{q-1} Q_{0 ; 22, b b} Y\left(Q_{0}\right) . \tag{3.10}
\end{equation*}
$$

Evaluating this integral, we find

$$
\begin{equation*}
\left.\left.\langle | \psi\left(\boldsymbol{r}_{0}\right)\right|^{2 q}\right\rangle=\frac{1}{V} q(q-1) \int \mathrm{d} u u^{q-2} Y_{a}(u) . \tag{3.11}
\end{equation*}
$$

Consequently, the distribution function of the eigenfunction intensity is given by [17]

$$
\begin{equation*}
\mathscr{P}(u)=\frac{1}{V} \frac{\mathrm{~d}^{2}}{\mathrm{~d} u^{2}} Y_{a}(u) \quad(\mathrm{U}), \tag{3.12}
\end{equation*}
$$

where $V$ is the sample volume.
In the case of the orthogonal symmetry, $Y\left(Q_{0}\right) \equiv Y\left(\lambda_{1}, \lambda_{2}, \lambda\right)$, where $1 \leq \lambda_{1}, \lambda_{2}<\infty$ and $-1 \leq \lambda \leq 1$. In the limit $\eta \rightarrow 0$, the relevant region of values is $\lambda_{1} \gg \lambda_{2}$, $\lambda$, where

$$
\begin{equation*}
Y\left(Q_{0}\right) \rightarrow Y_{a}\left(\pi v \eta \lambda_{1}\right) . \tag{3.13}
\end{equation*}
$$

The distribution of eigenfunction intensities is expressed in this case through the function $Y_{a}$ as follows [17]:

$$
\begin{align*}
\mathscr{P}(u) & =\frac{1}{\pi V u^{1 / 2}} \int_{u / 2}^{\infty} \mathrm{d} z(2 z-u)^{-1 / 2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} z^{2}} Y_{a}(z) \\
& =\frac{2 \sqrt{2}}{\pi V u^{1 / 2}} \frac{\mathrm{~d}^{2}}{\mathrm{~d} u^{2}} \int_{0}^{\infty} \frac{\mathrm{d} z}{z^{1 / 2}} Y_{a}(z+u / 2) \quad(\mathrm{O}) . \tag{3.14}
\end{align*}
$$

In the diffusive sample, typical configurations of the $Q$-field are nearly constant in space, so that one can approximate the functional integral (3.8) by an integral over a single supermatrix $Q$. This procedure, which makes the problem effectively zero dimensional and is known as zero-mode approximation (see Section 2.1), gives

$$
\begin{equation*}
Y_{a}(z) \approx \mathrm{e}^{-V z} \quad(\mathrm{O}, \mathrm{U}), \tag{3.15}
\end{equation*}
$$

and consequently,

$$
\begin{align*}
\mathscr{P}(u) & \approx V \mathrm{e}^{-u V} \quad(\mathrm{U}),  \tag{3.16}\\
\mathscr{P}(u) & \approx \sqrt{\frac{V}{2 \pi u}} \mathrm{e}^{-u V / 2} \quad(\mathrm{O}), \tag{3.17}
\end{align*}
$$

which are just the RMT results for the Gaussian Unitary Ensemble (GUE) and Gaussian Orthogonal Ensemble (GOE) respectively, Eqs. (3.1) and (3.2).

Therefore, like in the case of the level correlations, the zero mode approximation yields the RMT results for the distribution of the eigenfunction amplitudes. To calculate deviations from RMT, one has to go beyond the zero-mode approximation and to evaluate the function $Y_{a}(z)$ determined by Eqs. (3.8) and (3.9) for a $d$-dimensional diffusive system. In the case of a quasi-1D geometry this can be done exactly via the transfer-matrix method, see Section 3.2. For higher $d$, the exact solution is not possible, and one should rely on approximate methods. Corrections to the "main body" of the distribution can be found by treating the non-zero modes perturbatively, while the asymptotic "tail" can be found via a saddle-point method (see Sections 3.3 and 4).

Let us note that the formula (3.8), (3.9) can be written in a slightly different, but completely equivalent form [32,33]. Making in (3.8) the transformation

$$
Q(\boldsymbol{r}) \rightarrow \widetilde{Q}(\boldsymbol{r})=V^{-1}\left(\boldsymbol{r}_{0}\right) Q(\boldsymbol{r}) V\left(\boldsymbol{r}_{0}\right),
$$

where the matrix $V\left(\boldsymbol{r}_{0}\right)$ is defined from $Q\left(\boldsymbol{r}_{0}\right)=V\left(\boldsymbol{r}_{0}\right) \Lambda V^{-1}\left(\boldsymbol{r}_{0}\right)$, one gets in the unitary case

$$
\begin{equation*}
Y_{a}(u)=\int_{Q\left(r_{0}\right)=\Lambda} D Q \exp \left\{\int \mathrm{~d}^{d} r \operatorname{Str}\left[\frac{\pi v D}{4}(\nabla Q)^{2}-\frac{u}{2} \Lambda Q P_{b b}\right]\right\}, \tag{3.18}
\end{equation*}
$$

where $P_{b b}$ denotes the projector onto the boson-boson sector, and a similar formula in the orthogonal case.

The above derivation can be extended to a more general correlation function representing a product of eigenfunction amplitudes in different points

$$
\begin{equation*}
\Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)=\frac{1}{\nu V}\left\langle\sum_{\alpha}\right| \psi_{\alpha}^{2 q_{1}}\left(\boldsymbol{r}_{1}\right)| | \psi_{\alpha}^{2 q_{2}}\left(\boldsymbol{r}_{2}\right)|\ldots| \psi_{\alpha}^{2 q_{k}}\left(\boldsymbol{r}_{k}\right)\left|\delta\left(E-E_{\alpha}\right)\right\rangle . \tag{3.19}
\end{equation*}
$$

If all the points $\boldsymbol{r}_{i}$ are separated by sufficiently large distances (much larger than the mean free path $l$ ), one finds for the unitary ensemble [18]

$$
\begin{align*}
\Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)= & -\frac{1}{2 V} \frac{q_{1}!q_{2}!\ldots q_{k}!}{\left(q_{1}+q_{2}+\cdots+q_{k}-1\right)!} \lim _{\eta \rightarrow 0}(2 \pi v \eta)^{q_{1}+\cdots+q_{k}-1} \\
& \times \int D Q Q_{11, b b}^{q_{1}-1}\left(\boldsymbol{r}_{1}\right) Q_{22, b b}\left(\boldsymbol{r}_{1}\right) Q_{11, b b}^{q_{2}}\left(\boldsymbol{r}_{2}\right) \ldots Q_{11, b b}^{q_{k}}\left(\boldsymbol{r}_{k}\right) \mathrm{e}^{-S[Q]} . \tag{3.20}
\end{align*}
$$

In the case of the quasi-1D system one can again evaluate Eq. (3.20) via the transfer matrix method, while in higher $d$ one has to use approximate schemes. The correlation functions of the type (3.19) appear, in particular, when one calculates the distribution of the inverse participation ratio (IPR) $P_{2}=\int \mathrm{d}^{d} r\left|\psi_{\alpha}^{4}(r)\right|$, the moments of which are given by Eq. (3.19) with $q_{1}=q_{2}=\cdots=q_{k}=2$. We will discuss the IPR distribution function below in Sections 3.2.4 and 3.3.3. The case of $k=2$ in Eq. (3.19) corresponds to the correlations of the amplitudes of an eigenfunction in two different points; we will discuss such correlations in Sections 3.3.3 and 4.1.1 (where they will describe the shape of an anomalously localized state).

### 3.2. Quasi-one-dimensional geometry

### 3.2.1. Exact solution of the $\sigma$-model

In the case of quasi-1D geometry an exact solution of the $\sigma$-model is possible due to the transfer-matrix method. The idea of the method, quite general for the one-dimensional problems, is in reducing the functional integral (3.8) or (3.20) to solution of a differential equation. This is completely analogous to constructing the Schrödinger equation from the quantum-mechanical Feynman path integral. In the present case, the role of the time is played by the coordinate along the wire, while the role of the particle coordinate is played by the supermatrix $Q$. In general, at finite value of the frequency $\eta$ in Eq. (3.7) (more precisely, $\eta$ plays a role of imaginary frequency), the corresponding differential equation is too complicated and cannot be solved analytically [6]. However, a simplification appearing in the limit $\eta \rightarrow 0$, when only the non-compact variable $\lambda_{1}$ survives, allows to find an analytical solution [18] of the 1D $\sigma$-model. ${ }^{6}$
There are several different microscopic models which can be mapped onto the 1D supermatrix $\sigma$-model. First of all, this is a model of a particle in a random potential (discussed above) in the case of a quasi-1D sample geometry. Then one can neglect transverse variation of the $Q$-field in the $\sigma$-model action, thus reducing it to the 1D form [101,6]. Secondly, this is the random banded matrix (RBM) model [102,17,18] which is relevant to various problems in the field of quantum chaos [103,104]. In particular, the evolution operator of a kicked rotor (paradigmatic model of a periodically driven quantum system) has a structure of a quasi-random banded matrix, which makes this system to belong to the "quasi-1D universality class" [18,105]. Finally, the Iida-Weidenmüller-Zuk random matrix model [63] of the transport in a disordered wire (see Section 6 for more detail) can be also mapped onto the 1D $\sigma$-model.

The result for the function $Y_{a}(u)$ determining the distribution of the eigenfunction intensity $u=\left|\psi^{2}\left(\boldsymbol{r}_{0}\right)\right|$ reads (for the unitary symmetry)

$$
\begin{equation*}
Y_{a}(u)=\frac{1}{V} W^{(1)}\left(u A \xi, \tau_{+}\right) W^{(1)}\left(u A \xi, \tau_{-}\right) . \tag{3.21}
\end{equation*}
$$

Here $A$ is the wire cross-section, $\xi=2 \pi v D A$ the localization length, $\tau_{+}=L_{+} / \xi, \tau_{-}=L_{-} / \xi$, with $L_{+}, L_{-}$being the distances from the observation point $\boldsymbol{r}_{0}$ to the right and left edges of the sample. For the orthogonal symmetry, $\xi$ is replaced by $\xi / 2$. The function $W^{(1)}(z, \tau)$ satisfies the equation

$$
\begin{equation*}
\frac{\partial W^{(1)}(z, \tau)}{\partial \tau}=\left(z^{2} \frac{\partial^{2}}{\partial z^{2}}-z\right) W^{(1)}(z, \tau) \tag{3.22}
\end{equation*}
$$

and the boundary condition

$$
\begin{equation*}
W^{(1)}(z, 0)=1 . \tag{3.23}
\end{equation*}
$$

The solution to Eqs. (3.22) and (3.23) can be found in terms of the expansion in eigenfunctions of the operator $z^{2} \partial^{2} / \partial z^{2}-z$. The functions $2 z^{1 / 2} K_{i \mu}\left(2 z^{1 / 2}\right)$, with $K_{v}(x)$ being the modified Bessel function

[^5](Macdonald function), form the proper basis for such an expansion [106], which is known as the Lebedev-Kontorovich expansion; the corresponding eigenvalues are $-\left(1+\mu^{2}\right) / 4$. The result is
\[

$$
\begin{equation*}
W^{(1)}(z, \tau)=2 z^{1 / 2}\left\{K_{1}\left(2 z^{1 / 2}\right)+\frac{2}{\pi} \int_{0}^{\infty} \mathrm{d} \mu \frac{\mu}{1+\mu^{2}} \sinh \frac{\pi \mu}{2} K_{i \mu}\left(2 z^{1 / 2}\right) \mathrm{e}^{-\left(\left(1+\mu^{2}\right) 4\right) \tau}\right\} . \tag{3.24}
\end{equation*}
$$

\]

The formulas (3.12), (3.14), (3.21) and (3.24) give therefore the exact solution for the eigenfunction statistics for arbitrary value of the parameter $X=L / \xi$ (ratio of the total system length $L=L_{+}+L_{-}$to the localization length). The form of the distribution function $\mathscr{P}(u)$ is essentially different in the metallic regime $X \ll 1$ (in this case $X=1 / g$ ) and in the insulating one $X \gg 1$. We will discuss these two limiting cases below, in Sections 3.2.3 and 3.2.4 respectively.

### 3.2.2. Global statistics of eigenfunctions

The multipoint correlation functions (3.19) determining the global statistics of eigenfunctions can be also computed in a similar way. Let us first assume that the points $\boldsymbol{r}_{i}$ lie sufficiently far from each other, $\left|\boldsymbol{r}_{i}-\boldsymbol{r}_{j}\right| \gg l$. We order the points according to their coordinates $x_{i}$ along the wire, $0<x_{1}<\cdots<x_{k}<L$, and define $t_{i}=x_{i} / \xi, \tau_{i}=t_{i}-t_{i-1}, \tau_{1}=t_{1}$. Then we find from Eq. (3.20) for the unitary symmetry

$$
\begin{align*}
V \Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)= & \frac{q_{1}!\ldots q_{k}!}{\left(q_{1}+\cdots+q_{k}-2\right)!(\xi A)^{q_{1}+\cdots+q_{k}-1}} \\
& \times \int_{0}^{\infty} \mathrm{d} z z^{q_{k}-2} W^{(1)}\left(z ; X-\sum_{s=1}^{k} \tau_{s}\right) W^{(k)}\left(z ; \tau_{1}, \ldots, \tau_{k}\right), \tag{3.25}
\end{align*}
$$

where the functions $W^{(s)}\left(z ; \tau_{1}, \ldots, \tau_{s}\right)$ are defined by the equation (identical to Eq. (3.22))

$$
\begin{equation*}
\frac{\partial W^{(s)}\left(z ; \tau_{1}, \ldots, \tau_{s}\right)}{\partial \tau_{s}}=\left(z^{2} \frac{\partial^{2}}{\partial z^{2}}-z\right) W^{(s)}\left(z ; \tau_{1}, \ldots, \tau_{s}\right) \tag{3.26}
\end{equation*}
$$

and the boundary conditions

$$
\begin{equation*}
W^{(s)}\left(z ; \tau_{1}, \tau_{2}, \ldots, \tau_{s-1}, 0\right)=z^{q_{s-1}} W^{(s-1)}\left(z ; \tau_{1}, \tau_{2}, \ldots, \tau_{s-1}\right) . \tag{3.27}
\end{equation*}
$$

Solving these equations consecutively via the Lebedev-Kontorovich transformation, one can find all the correlation functions $V \Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)$ (in the form of multiple integrals over $\mu_{s}$ ). This will be in particular used in Section 4.2.1, where we will study the joint distribution function of the wave function intensities in two points ( $k=2$ ).

We show now that the correlation functions (3.25) allow to represent the statistics of eigenfunction envelops in a very compact form. Making the substitution of the variable $z=\mathrm{e}^{\theta}$ and defining the functions $\tilde{W}^{(s)}\left(\theta ; \tau_{1}, \ldots, \tau_{s}\right)=z^{-1 / 2} W^{(s)}\left(z ; \tau_{1}, \ldots, \tau_{s}\right)$, we can rewrite (3.26) in the form of the imaginary time Schrödinger equation,

$$
\begin{equation*}
-\frac{\partial \tilde{W}^{(s)}}{\partial \tau_{s}}=\hat{H} \tilde{W}^{(s)}, \quad \hat{H}=-\partial_{\theta}^{2}+\mathrm{e}^{\theta}+\frac{1}{4} \tag{3.28}
\end{equation*}
$$

with the boundary conditions

$$
\begin{align*}
& \tilde{W}^{(s)}\left(\theta ; \tau_{1}, \ldots, \tau_{s-1}, 0\right)=\mathrm{e}^{q_{s-1} \theta} \tilde{W}^{(s-1)}\left(\theta ; \tau_{1}, \ldots, \tau_{s-1}\right), \\
& \tilde{W}^{(1)}(\theta ; 0)=\mathrm{e}^{-\theta / 2} \tag{3.29}
\end{align*}
$$

This allows to rewrite Eq. (3.25) as a matrix element,

$$
\begin{align*}
V \Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)= & \frac{q_{1}!\ldots q_{k}!}{\left(q_{1}+\cdots+q_{k}-2\right)!(\xi A)^{q_{1}+\ldots+q_{k}-1}} \\
& \times\left\langle\mathrm{e}^{-\theta / 2}\right| \mathrm{e}^{-\left(X-\sum_{s=1}^{k} \tau_{s}\right) \hat{\mathrm{A}}} \mathrm{e}^{q_{s} \theta} \mathrm{e}^{-\tau_{s} \hat{\mathrm{H}}^{q_{s-1}}{ }^{q_{s} \theta}} \ldots \mathrm{e}^{q_{1} \theta} \mathrm{e}^{-\tau_{1} \hat{H}}\left|\mathrm{e}^{-\theta / 2}\right\rangle \tag{3.30}
\end{align*}
$$

(these transformations are completely analogous to those performed by Kolokolov in [107], where the eigenfunction statistics in the strictly 1D case was studied). Furthermore, the matrix element can be represented as a Feynman path integral,

$$
\begin{align*}
V \Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)= & \frac{q_{1}!\ldots q_{k}!}{\left(q_{1}+\cdots+q_{k}-2\right)!(\xi A)^{q_{1}+\cdots+q_{k}-1}} \int \mathrm{~d} \theta_{l} \mathrm{~d} \theta_{r} \mathrm{e}^{-\left(\left(\theta_{l}+\theta_{r}\right) / 2\right)} \\
& \times \mathscr{N} \int_{\theta(0)=\theta_{l}, \theta(X)=\theta_{r}} D \theta(t) \exp \left\{-\int \mathrm{d} t\left(\frac{1}{4} \dot{\theta}^{2}+\mathrm{e}^{\theta}+\frac{1}{4}\right)\right\} \\
& \times \mathrm{e}^{q_{1} \theta\left(t_{1}\right)+q_{2} \theta\left(t_{2}\right)+\cdots+q_{k} \theta\left(t_{k}\right)}, \tag{3.31}
\end{align*}
$$

with $\mathcal{N}$ being the normalization constant, $\mathcal{N}^{-1}=\int D \theta \exp \left\{-\frac{1}{4} \int \mathrm{~d} t \dot{\theta}^{2}\right\}$. The quantum mechanics defined by the Hamiltonian (3.28) (or, equivalently, by the path integral (3.31)) is known as Liouville quantum mechanics [108,109]; the corresponding spectral expansion is obviously equivalent to the Lebedev-Kontorovich expansion.

Inserting here the decomposition of unity, $1=\int \mathrm{d} w \delta\left(X^{-1} \int \mathrm{~d} t \mathrm{e}^{\theta}-w\right)$ and making a shift $\theta \rightarrow \theta+\ln w$, we get

$$
\begin{align*}
\Gamma^{\{q\}}\left(\boldsymbol{r}_{1}, \ldots, \boldsymbol{r}_{k}\right)= & \frac{q_{1}!\ldots q_{k}!}{V^{q_{1}+\ldots+q_{k}}} \mathrm{e}^{-X / 4} \cdot \mathcal{N} \int D \theta(t) \mathrm{e}^{-(\theta(0)+\theta(X)) / 2} \\
& \times \exp \left\{-\frac{1}{4} \int \mathrm{~d} t \dot{\theta}^{2}\right\} \mathrm{e}^{q_{1} \theta\left(t_{1}\right)+\cdots+q_{k} \theta\left(t_{k}\right)} \delta\left(X^{-1} \int \mathrm{~d} t \mathrm{e}^{\theta}-1\right) . \tag{3.32}
\end{align*}
$$

According to (3.32), the eigenfunction intensity can be written as a product

$$
\begin{equation*}
\psi(r)=\Phi(r) \Psi(t), \tag{3.33}
\end{equation*}
$$

where $\Phi(r)$ is a quickly fluctuating (in space) function, which has the Gaussian Ensemble statistics, $\langle | \Phi^{2 q}| \rangle=q!/ V^{q}$, and fluctuates independently in the points separated by a distance larger than the mean free path. The function $\Psi(t)$ determines, in contrast, a smooth envelope of the wave function. Its fluctuations are long-range correlated and are described by the probability density

$$
\begin{equation*}
\mathscr{P}\left\{\theta(t)=\ln \Psi^{2}(t)\right\}=\mathscr{N} \mathrm{e}^{-X / 4} \mathrm{e}^{-(\theta(0)+\theta(X)) / 2} \exp \left\{-\frac{1}{4} \int \mathrm{~d} t \dot{\theta}^{2}\right\} . \tag{3.34}
\end{equation*}
$$

The above calculation can be repeated for the case, when some of the points $\boldsymbol{r}_{i}$ lie closer than $l$ to each other. The result (3.33), (3.34) is reproduced also in this case, with the function $\Phi(r)$ having the ideal metal statistics given by the zero-dimensional $\sigma$-model. This statistics [110-112] is Gaussian and is determined by the (short-range) correlation function

$$
\begin{equation*}
V\left\langle\Phi^{*}(\boldsymbol{r}) \Phi\left(\boldsymbol{r}^{\prime}\right)\right\rangle=k_{\mathrm{d}}^{1 / 2}\left(\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|\right), \tag{3.35}
\end{equation*}
$$

see Eq. (3.71) below.
The physics of these results is as follows. The short-range fluctuations of the wave function (described by the function $\Phi(r)$ ) have the same origin as in a strongly chaotic system, where superposition of plane waves with random amplitudes and phases leads to the Gaussian fluctuations of eigenfunctions with the correlation function (3.35) and, in particular, to the RMT statistics of the local amplitude, $\langle | \Phi^{2 q}| \rangle=q!/ V^{q}$. The second factor $\Psi(t)$ in the decomposition (3.33) describes the smooth envelope of the eigenfunction (changing on a scale $>l$ ), whose statistics is given by (3.34) and is determined by diffusion and localization effects.

Let us note that in the metallic regime, $X \ll 1$, the measure (3.34) can be approximated as

$$
\begin{equation*}
\mathscr{P}\{\theta(x)\}=\exp \left\{-\frac{\pi v A D}{2} \int \mathrm{~d} x\left(\frac{\mathrm{~d} \theta}{\mathrm{~d} x}\right)^{2}\right\} . \tag{3.36}
\end{equation*}
$$

We will see in Section 4, while studying the statistics of anomalously localized states in $d \geq 1$ dimensions, that the probability of appearance in a metallic sample of such a rare state with an envelope $\mathrm{e}^{\theta}(r)$ is given (within the exponential accuracy) by the $d$-dimensional generalization of (3.36) (see, in particular, Eqs. (4.10) and (4.77)).

Finally, we compare the eigenfunction statistics in the quasi-1D case with that in a strictly 1D disordered system. In the latter case, the eigenfunction can be written as

$$
\begin{equation*}
\psi_{1 D}(x)=\sqrt{\frac{2}{L}} \cos (k x+\delta) \Psi(x), \tag{3.37}
\end{equation*}
$$

where $\Psi(x)$ is a smooth envelope function. The local statistics of $\psi_{1 D}(x)$ (i.e. the moments (3.4)) was studied in [113], while the global statistics (the correlation functions of the type (3.19)) in [107]. Comparing the results for the quasi-1D and 1D systems, we find that the statistics of the smooth envelopes $\Psi$ is exactly the same in the two cases, for a given value of the ratio of the system length $L$ to the localization length (equal to $\beta \pi v A D$ in quasi-1D and to the mean free path $l$ in 1 D ). In particular, the moments $\Gamma^{(q)}(\boldsymbol{r})=\langle | \psi^{2 q}(\boldsymbol{r})| \rangle$ are found to be related as

$$
\begin{equation*}
A^{q} \Gamma_{Q 1 D}^{(q)}=\frac{q!^{2}}{(2 q-1)!!} \Gamma_{1 D}^{(q)}, \tag{3.38}
\end{equation*}
$$

where the factor $q!^{2} /(2 q-1)!$ ! represents precisely the ratio of the GUE moments, $\langle | \Phi^{2 q}| \rangle=q!/ V^{q}$, to the plane wave moments, $\left\langle(2 / V)^{q} \cos ^{2 q}(k x+\delta)\right\rangle=(2 q-1)!!/ q!V^{q}$. For the case of the orthogonal symmetry of the quasi-1D system, this factor is replaced by $q$ !. Equivalence of the statistics of the eigenfunction envelopes implies, in particular, that the distribution of the inverse participation ratio (IPR),

$$
\begin{equation*}
P_{2}=\int \mathrm{d} \boldsymbol{r}|\psi(\boldsymbol{r})|^{4}, \tag{3.39}
\end{equation*}
$$

is identical in the 1D [114,115] and quasi-1D [18,22] cases (the form of this distribution in the localized limit $L / \xi \gg 1$ is explicitly given in Section 3.2.4 below; for arbitrary $L / \xi$ the result is very cumbersome [115]).

### 3.2.3. Short wire

In the case of a short wire, $X=1 / g \ll 1$, Eqs. (3.12), (3.14), (3.21) and (3.24) yield [17,18,36]

$$
\begin{align*}
& \mathscr{P}^{(\mathrm{U})}(y)=\mathrm{e}^{-y}\left[1+\frac{\alpha X}{6}\left(2-4 y+y^{2}\right)+\cdots\right], \quad y \lesssim X^{-1 / 2},  \tag{3.40}\\
& \mathscr{P}^{(\mathrm{O})}(y)=\frac{\mathrm{e}^{-y / 2}}{\sqrt{2 \pi y}}\left[1+\frac{\alpha X}{6}\left(\frac{3}{2}-3 y+\frac{y^{2}}{2}\right)+\cdots\right], \quad y \lesssim X^{-1 / 2},  \tag{3.41}\\
& \mathscr{P}^{(\mathrm{U})}(y)=\exp \left\{-y+\frac{\alpha}{6} y^{2} X+\cdots\right\}, \quad X^{-1 / 2} \lesssim y \leqq X^{-1},  \tag{3.42}\\
& \mathscr{P}^{(\mathrm{O})}(y)=\frac{1}{\sqrt{2 \pi y}} \exp \left\{\frac{1}{2}\left[-y+\frac{\alpha}{6} y^{2} X+\cdots\right]\right\}, \quad X^{-1 / 2} \lesssim y \lesssim X^{-1},  \tag{3.43}\\
& \mathscr{P}(y) \sim \exp [-2 \beta \sqrt{y / X}], \quad y \gtrsim X^{-1} \tag{3.44}
\end{align*}
$$

(a more accurate formula for the far "tail" (3.44) can be found in Section 4.2.1, Eq. (4.74)). Here the coefficient $\alpha$ is equal to $\alpha=2\left[1-3 L_{-} L_{+} / L^{2}\right]$. We see that there exist three different regimes of the behavior of the distribution function. For not too large amplitudes $y$, Eqs. (3.40) and (3.41) are just the RMT results with relatively small corrections. In the intermediate range (3.42), (3.43) the correction in the exponent is small compared to the leading term but much larger than unity, so that $\mathscr{P}(y)>\mathscr{P}_{\text {Rмт }}(y)$ though $\ln \mathscr{P}(y) \simeq \ln \mathscr{P}_{\text {Rмт }}(y)$. Finally, in the large amplitude region, (3.44), the distribution function $\mathscr{P}(y)$ differs completely from the RMT prediction. Note that Eq. (3.44) is not valid when the observation point is located close to the sample boundary, in which case the exponent of (3.44) becomes smaller by a factor of 2 , see Section 4.2.3

### 3.2.4. Long wire

In the limit of a long sample, $X=L / \xi \gg 1$, Eqs. (3.12), (3.14), (3.21) and (3.24) reduce to

$$
\begin{align*}
& \mathscr{P}^{(\mathrm{U})}(u) \simeq \frac{8 \xi^{2} A}{L}\left[K_{1}^{2}(2 \sqrt{u A \xi})+K_{0}^{2}(2 \sqrt{u A \xi})\right],  \tag{3.45}\\
& \mathscr{P}^{(\mathrm{O})}(u) \simeq \frac{2 \xi^{2} A K_{1}(2 \sqrt{u A \xi})}{L} \frac{\sqrt{u A \xi}}{}, \tag{3.46}
\end{align*}
$$

with $\xi=2 \pi \nu A D$ as before. Note that in this case the natural variable is not $y=u V$, but rather $u A \xi$, since typical intensity of a localized wave function is $u \sim 1 / A \xi$ in contrast to $u \sim 1 / V$ for a delocalized one. The asymptotic behavior of Eqs. (3.45) and (3.46) at $u \gg 1 / A \xi$ has precisely the same form,

$$
\begin{equation*}
\mathscr{P}(u) \sim \exp (-2 \beta \sqrt{u A \xi}), \tag{3.47}
\end{equation*}
$$

as in the region of very large amplitude in the metallic sample, Eq. (3.44). On this basis, it was conjectured in [18] that the asymptotic behavior (3.44) is controlled by the probability to have a quasi-localized eigenstate with an effective spatial extent much less than $\xi$ ("anomalously localized state"). This conjecture was proven rigorously in [36] where the shape of the anomalously localized state (ALS) responsible for the large- $u$ asymptotics was calculated via the transfer-matrix method. We will discuss this in Section 4 devoted to ALS and to asymptotics of different distribution functions.

Distribution of the inverse participation ratio (IPR) is also found to have a simple form in the limit $L \gg \xi[22,25]$ :

$$
\begin{equation*}
\mathscr{P}(z)=2 \pi^{2} \sum_{k=1}^{\infty}\left(2 \pi^{2} z k^{4}-3 k^{2}\right) \mathrm{e}^{-\pi^{2} k^{2} z} \equiv \frac{4}{\sqrt{\pi}} \frac{\partial}{\partial z}\left\{z^{-3 / 2} \sum_{k=1}^{\infty} k^{2} \mathrm{e}^{-k^{2} / z}\right\} \tag{3.48}
\end{equation*}
$$

where $z=\pi v D A^{2} P_{2}$ in the unitary case and $z=\left(\pi v D A^{2} / 3\right) P_{2}$ in the orthogonal case. (The second line in (3.48) can be obtained from the first one by using the Poisson summation formula.) Therefore, the spatial extent of a localized eigenfunction measured by IPR fluctuates strongly (of order of $100 \%$ ) from one eigenfunction to another. More precisely, the ratio of the r.m.s. deviation of IPR to its mean value is equal to $1 / \sqrt{5}$ according to Eq. (3.48). The first form of Eq. (3.48) is more suitable for extracting the asymptotic behavior of $\mathscr{P}(z)$ at $z \gg 1$, whereas the second line gives us the leading behavior of $\mathscr{P}(z)$ at small $z \ll 1$ :

$$
\mathscr{P}(z)=\left\{\begin{array}{cc}
4 \pi^{4} z \mathrm{e}^{-\pi^{2} z}, & z \gg 1,  \tag{3.49}\\
4 \pi^{-1 / 2} z^{-7 / 2} \mathrm{e}^{-1 / z}, & z \ll 1 .
\end{array}\right.
$$

Therefore, the probability to have atypically large or atypically small IPR is exponentially suppressed. The function $\mathscr{P}(z)$ is presented in Fig. 1.

The above fluctuations of IPR are due to fluctuations in the "central bump" of a localized eigenfunction. They should be distinguished from the fluctuations in the rate of exponential decay of eigenfunctions (Lyapunov exponent). The latter can be extracted from another important


Fig. 1. Distribution function $\mathscr{P}(z)$ of the normalized (dimensionless) inverse participation ratio $z=\left[\beta^{2} /(\beta+2)\right] \pi v D A^{2} P_{2}$ in a long $(L \gg \xi)$ quasi-1D sample. The average value is $\langle z\rangle=1 / 3$. From [18].
physical quantity - the distribution function $\mathscr{P}(v)$, where

$$
v=\left(2 \pi v D A^{2}\right)^{2}\left|\psi_{\alpha}^{2}\left(\boldsymbol{r}_{1}\right) \psi_{\alpha}^{2}\left(\boldsymbol{r}_{2}\right)\right|
$$

is the product of the eigenfunction intensity in the two points close to the opposite edges of the sample $r_{1} \rightarrow 0, r_{2} \rightarrow L$. The result is [23,18]

$$
\begin{align*}
& \mathscr{P}(-\ln v)=F[-(\beta \ln v) / 2 X] \frac{1}{2(2 \pi X / \beta)^{1 / 2}} \exp \left\{-\frac{\beta}{8 X}(2 X / \beta+\ln v)^{2}\right\}, \\
& F^{\mathrm{U}}(u)=u \frac{\Gamma^{2}[(3-u) / 2]}{\Gamma(u)}, \quad F^{\mathrm{O}}(u)=\frac{u \Gamma^{2}[(1-u) / 2]}{\pi \Gamma(u)} \tag{3.50}
\end{align*}
$$

Therefore, $\ln v$ is asymptotically distributed according to the Gaussian law with the mean value $\langle-\ln v\rangle=(2 \beta) X=L / \beta \pi v A D$ and the variance $\operatorname{var}(-\ln v)=2\langle-\ln v\rangle$. The same log-normal distribution is found for the conductance and for transmission coefficients of a quasi-1D sample from the Dorokhov-Mello-Pereyra-Kumar formalism [116,74] (see end of Section 6.2).

Note that the formula (3.50) is valid in the region of $v \ll 1$ (i.e. negative $\ln v$ ) only, which contains almost all normalization of the distribution function. In the region of still higher values of $v$ the log-normal form of $\mathscr{P}(v)$ changes into the much faster stretched-exponential fall-off $\propto \exp \left\{-2 \sqrt{2} \beta v^{1 / 4}\right\}$, as can be easily found from the exact solution given in [23,18]. The decay rate of all the moments $\left\langle v^{k}\right\rangle, k \geq 2$, is four times less than $\langle-\ln v\rangle$ and does not depend on $k$ : $\left\langle v^{k}\right\rangle \propto \mathrm{e}^{-X / 2 \beta}$. This is because the moments $\left\langle v^{k}\right\rangle, k \geq 2$, are determined by the probability to find an "anomalously delocalized state" with $v \sim 1$.

### 3.3. Arbitrary dimensionality: metallic regime

### 3.3.1. Distribution of eigenfunction amplitudes

In the case of arbitrary dimensionality $d$, deviations from the RMT distribution $\mathscr{P}(y)$ for not too large $y$ can be calculated [24,25] via the method described in Section 2. Applying this method to the moments (3.6), one gets

$$
\begin{align*}
& \left.\left.\langle | \psi(\boldsymbol{r})\right|^{2 q}\right\rangle=\frac{q!}{V^{q}}\left[1+\frac{1}{2} \kappa q(q-1)+\cdots\right] \quad(\mathrm{U})  \tag{3.51}\\
& \left.\left.\langle | \psi(\boldsymbol{r})\right|^{2 q}\right\rangle=\frac{(2 q-1)!!}{V^{q}}[1+\kappa q(q-1)+\cdots] \quad(\mathrm{O}), \tag{3.52}
\end{align*}
$$

where $\kappa=\Pi(\boldsymbol{r}, \boldsymbol{r})$. Correspondingly, the correction to the distribution function reads

$$
\begin{align*}
& \mathscr{P}(y)=\mathrm{e}^{-y}\left[1+\frac{\kappa}{2}\left(2-4 y+y^{2}\right)+\cdots\right] \quad(\mathrm{U}),  \tag{3.53}\\
& \mathscr{P}(y)=\frac{\mathrm{e}^{-y / 2}}{\sqrt{2 \pi y}}\left[1+\frac{\kappa}{2}\left(\frac{3}{2}-3 y+\frac{y^{2}}{2}\right)+\cdots\right] \quad(\mathrm{O}) . \tag{3.54}
\end{align*}
$$

Deviations of the eigenfunction distribution function $\mathscr{P}(y)$ from its RMT form are illustrated for the orthogonal symmetry case in Fig. 2. Numerical studies of the statistics of eigenfunction amplitudes


Fig. 2. Distribution $\mathscr{P}(y)$ of the normalized eigenfunction intensities $y=V\left|\psi^{2}(\boldsymbol{r})\right|$ in the orthogonal symmetry case. The dotted line shows the RMT result, Eq. (3.2), while the full line corresponds to Eq. (3.54) with $\kappa=0.4$.
in weak localization regime have been performed in Ref. [117] for the 2D and in Ref. [118] for the 3D case. The found deviations from RMT are well described by the above theoretical results. Experimentally, statistical properties of the eigenfunction intensity have been studied for microwaves in a disordered cavity [15]. For a weak disorder the found deviations are in good agreement with (3.54) as well.

In the quasi-one-dimensional case (with hard wall boundary conditions in the longitudinal direction), the one-diffuson loop $\Pi(r, r)$ is equal to

$$
\begin{equation*}
\kappa \equiv \Pi(\boldsymbol{r}, \boldsymbol{r})=\frac{2}{g}\left[\frac{1}{3}-\frac{x}{L}\left(1-\frac{x}{L}\right)\right], \quad 0 \leq x \leq L \tag{3.55}
\end{equation*}
$$

so that Eqs. (3.53) and (3.54) agree with results (3.40) and (3.41) obtained from the exact solution. For the periodic boundary conditions in the longitudinal direction (a ring) we have $\kappa=1 / 6 \mathrm{~g}$. In the case of 2D geometry,

$$
\begin{equation*}
\Pi(r, r)=\frac{1}{\pi g} \ln \frac{L}{l}, \tag{3.56}
\end{equation*}
$$

with $g=2 \pi v D$. Finally, in the 3D case the sum over the momenta $\Pi(r, r)=(\pi v V)^{-1} \sum_{q}\left(D \boldsymbol{q}^{2}\right)^{-1}$ diverges linearly at large $q$. The diffusion approximation is valid up to $q \sim l^{-1}$; the corresponding cut-off gives $\Pi(r, r) \sim 1 / 2 \pi v D l=g^{-1}(L / l)$. This divergency indicates that more accurate evaluation of $\Pi(r, r)$ requires taking into account also the contribution of the ballistic region $\left(q>l^{-1}\right)$ which depends on microscopic details of the random potential. We will return to this question in Section 3.3.4.

The formulas (3.53) and (3.54) are valid in the region of not too large amplitudes, where the perturbative correction is smaller than the RMT term, i.e. at $y \ll \kappa^{-1 / 2}$. In the region of large amplitudes, $y>\kappa^{-1 / 2}$ the distribution function was found by Fal'ko and Efetov [32,33], who applied to Eqs. (3.12) and (3.14) the saddle-point method suggested by Muzykantskii and Khmelnitskii [29]. We relegate the discussion of the method to Section 4 and only present the results here:

$$
\begin{align*}
& \mathscr{P}(y) \simeq \exp \left\{\frac{\beta}{2}\left(-y+\frac{\kappa y^{2}}{2}+\cdots\right)\right\} \times\left\{\begin{array}{ll}
1 & (\mathrm{U}) \\
\frac{1}{\sqrt{2 \pi y}} & (\mathrm{O})
\end{array}, \kappa^{-1 / 2} \lesssim y \lesssim \kappa^{-1},\right.  \tag{3.57}\\
& \mathscr{P}(y) \sim \exp \left\{-\frac{\beta}{4 \kappa} \ln ^{d}(\kappa y)\right\}, \quad y \gtrsim \kappa^{-1} . \tag{3.58}
\end{align*}
$$

Again, as in the quasi-one-dimensional case, there is an intermediate range where a correction in the exponent is large compared to unity, but small compared to the leading RMT term [Eq. (3.57)] and a far asymptotic region (3.58), where the decay of $\mathscr{P}(y)$ is much slower than in RMT. In the next section we will discuss the structure of anomalously localized eigenstates, which are responsible for the asymptotic behavior (3.44), (3.58).

### 3.3.2. $2 D$ : Weak multifractality of eigenfunctions

Since $d=2$ is the lower critical dimension for the Anderson localization problem, metallic 2D samples (with $g \gg 1$ ) share many common properties with systems at the critical point of the metal-insulator transition. Although the localization length $\xi$ in 2 D is not infinite (as for truly critical systems), it is exponentially large, and the criticality takes place in the very broad range of the system size $L \ll \xi$.
3.3.2.1. Multifractality: basic definitions. The criticality of eigenfunctions shows up via their multifractality. Multifractal structures first introduced by Mandelbrot [119] are characterized by an infinite set of critical exponents describing the scaling of the moments of a distribution of some quantity. Since then, this feature has been observed in various objects, such as the energy dissipating set in turbulence [120-122], strange attractors in chaotic dynamical systems [123-126], and the growth probability distribution in diffusion-limited aggregation [127-129]; see Ref. [130] for a review.

The fact that an eigenfunction at the mobility edge has the multifractal structure was noticed for the first time in [51], though the underlying renormalization group calculations were done by Wegner several years earlier [50]. For this problem, the probability distribution is just the eigenfunction intensity $\left|\psi^{2}(r)\right|$ and the corresponding moments are the inverse participation ratios,

$$
\begin{equation*}
P_{q}=\int \mathrm{d}^{d} r\left|\psi^{2 q}(\boldsymbol{r})\right| \tag{3.59}
\end{equation*}
$$

The multifractality is characterized by the anomalous scaling of $P_{q}$ with the system size $L$,

$$
\begin{equation*}
P_{q} \propto L^{-D_{q}(q-1)} \equiv L^{-\tau(q)} \tag{3.60}
\end{equation*}
$$

with $D_{q}$ different from the spatial dimensionality $d$ and dependent on $q$. Equivalently, the eigenfunctions are characterized by the singularity spectrum $f(\alpha)$ describing the measure $L^{f(\alpha)}$ of the set of those points $r$ where the eigenfunction takes the value $\left|\psi^{2}(\boldsymbol{r})\right| \propto L^{-\alpha}$. The two sets of exponents $\tau(q)$ and $f(\alpha)$ are related via the Legendre transformation,

$$
\begin{equation*}
\tau(q)=q \alpha-f(\alpha), \quad f^{\prime}(\alpha)=q, \quad \tau^{\prime}(q)=\alpha \tag{3.61}
\end{equation*}
$$

For a recent review on multifractality of critical eigenfunctions the reader is referred to $[55,131]$.
3.3.2.2. Multifractality in $2 D$. We note first that the formulas (3.51) and (3.52) for the IPRs with $q \leqq \kappa^{-1 / 2}$ can be rewritten in the 2D case (with (3.56) taken into account) as

$$
\begin{equation*}
\frac{\left\langle P_{q}\right\rangle}{P_{q}^{\text {RMT }}} \simeq\left(\frac{L}{l}\right)^{(1 / \beta \pi g) q(q-1)}, \tag{3.62}
\end{equation*}
$$

where $P_{q}^{\mathrm{RMT}}$ is the RMT value of $P_{q}$ equal to $q!L^{-2(q-1)}$ for GUE and $(2 q-1)!!L^{-2(q-1)}$ for GOE. We see that (3.62) has precisely the form (3.60) with

$$
\begin{equation*}
D_{q}=2-(q / \beta \pi g) . \tag{3.63}
\end{equation*}
$$

As was found in [32,33], the eigenfunction amplitude distribution (3.57), (3.58) leads to the same result (3.63) for all $q<2 \beta \pi g$. Since deviation of $D_{q}$ from the normal dimension 2 is proportional to the small parameter $1 / \pi g$, it can be termed "weak multifractality" (in analogy with weak localization). The result (3.63) was in fact obtained for the first time by Wegner [50] via the renormalization group calculations.

The limits of validity of Eq. (3.63) are not unambiguous and should be commented here. The singularity spectrum $f(\alpha)$ corresponding to (3.63) has the form

$$
\begin{equation*}
f(\alpha)=2-\frac{\beta \pi g}{4}\left(2+\frac{1}{\beta \pi g}-\alpha\right)^{2} \tag{3.64}
\end{equation*}
$$

so that $f\left(\alpha_{ \pm}=0\right)$ for

$$
\begin{equation*}
\alpha_{ \pm}=2\left[1 \pm \frac{1}{(2 \beta \pi g)^{1 / 2}}\right]^{2} \tag{3.65}
\end{equation*}
$$

If $\alpha$ lies outside the interval $\left(\alpha_{-}, \alpha_{+}\right)$, the corresponding $f(\alpha)<0$, which means that the most likely the singularity $\alpha$ will not be found for a given eigenfunction. However, if one considers the average $\left\langle P_{q}\right\rangle$ over a sufficiently large ensemble of eigenfunctions (corresponding to different realizations of disorder), a negative value of $f(\alpha)$ makes sense (see a related discussion in $[132,133]$ ). This is the definition which was assumed in [32,33] where Eq. (3.63) was obtained for all positive $q<2 \beta \pi g$.

In contrast, if one studies a typical value of $P_{q}$, the regions $\alpha>\alpha_{+}$and $\alpha<\alpha_{-}$will not contribute. In this case, Eq. (3.63) is valid only within the interval $q_{-} \leq q \leq q_{+}$with $q_{ \pm}= \pm(2 \beta \pi g)^{1 / 2}$; outside this region one finds [134,135]

$$
\tau(q) \equiv D_{q}(q-1)= \begin{cases}q \alpha_{-}, & q>q_{+},  \tag{3.66}\\ q \alpha_{+}, & q<q_{-} .\end{cases}
$$

Therefore, within this definition the multifractal dimensions $D_{q}$ saturate at the values $\alpha_{+}$and $\alpha_{-}$for $q \rightarrow+\infty$ and $q \rightarrow-\infty$, respectively. This is in agreement with results of numerical simulations [52-56].

### 3.3.3. Correlations of eigenfunction amplitudes and fluctuations of the inverse participation ratio

In this subsection, we study correlations of eigenfunctions in the regime of a good conductor [25-27,136,137]. The correlation function of amplitudes of one and the same eigenfunction with
energy $E$ can be formally defined as follows:

$$
\begin{equation*}
\left.\left.\alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)=\left.\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle\left._{E} \equiv \Delta\left\langle\sum_{k}\right| \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right)\right|^{2} \delta\left(E-\varepsilon_{k}\right)\right\rangle . \tag{3.67}
\end{equation*}
$$

An analogous correlation function for two different eigenfunctions is defined as

$$
\begin{align*}
& \left.\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)=\left.\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega} \\
& \left.\left.\quad \equiv \Delta^{2} R^{-1}(\omega)\left\langle\sum_{k \neq l}\right| \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2} \delta\left(E-\varepsilon_{k}\right) \delta\left(E+\omega-\varepsilon_{l}\right)\right\rangle, \tag{3.68}
\end{align*}
$$

where $R(\omega)$ is the two-level correlation function (2.1). To evaluate $\alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)$ and $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$, we employ an identity

$$
\begin{align*}
& 2 \pi^{2}\left[\Delta^{-1} \alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right) \delta(\omega)+\Delta^{-2} \tilde{R}(\omega) \sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)\right] \\
& \quad=\operatorname{Re}\left[\left\langle G^{\mathrm{R}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{1}, E\right) G^{\mathrm{A}}\left(\boldsymbol{r}_{2}, \boldsymbol{r}_{2}, E+\omega\right)-G^{\mathrm{R}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{1}, E\right) G^{\mathrm{R}}\left(\boldsymbol{r}_{2}, \boldsymbol{r}_{2}, E+\omega\right)\right\rangle\right], \tag{3.69}
\end{align*}
$$

where $G^{\mathrm{R}, \mathrm{A}}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E\right)$ are retarded and advanced Green's functions and $\tilde{R}(\omega)$ is the non-singular part of the level-level correlation function: $R(\omega)=\tilde{R}(\omega)+\delta(\omega / \Delta)$. A natural question, which arises at this point, is whether the r.h.s. of Eq. (3.69) cannot be simply found within the diffuson-Cooperon perturbation theory [9]. Such a calculation would, however, be justified only for $\omega \gg \Delta$ (more precisely, one has to introduce an imaginary part of frequency: $\omega \rightarrow \omega+\mathrm{i} \Gamma$, and require that $\Gamma \gg \Delta)$. Therefore, it would only allow to find a smooth in $\omega$ part of $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$ for $\omega \gg \Delta$. Evaluation of $\alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)$, as well as of $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$ at $\omega \sim \Delta$, cannot be done within such a calculation. For this reason the non-perturbative supersymmetry approach is to be used.

The r.h.s. of Eq. (3.69) can be expressed in terms of the supermatrix $\sigma$-model, yielding:

$$
\begin{align*}
& 2 \pi^{2}\left[\Delta^{-1} \alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right) \delta(\omega)+\Delta^{-2} \sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right) \tilde{R}(\omega)\right] \\
& \quad=(\pi v)^{2}\left[1-\operatorname{Re}\left\langle Q_{b b}^{11}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{22}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{S}}-k_{d}\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right) \operatorname{Re}\left\langle Q_{b b}^{12}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{21}\left(\boldsymbol{r}_{1}\right)\right\rangle_{\mathrm{s}}\right], \tag{3.70}
\end{align*}
$$

where $\langle\ldots\rangle_{\mathrm{S}}$ denotes the averaging with the sigma-model action and $k_{d}(\boldsymbol{r})=(\pi v)^{-2}\left\langle\operatorname{Im} G^{R}(\boldsymbol{r})\right\rangle^{2}$ is a short-range function explicitly given by

$$
k_{d}(r)=\exp (-r / l)\left\{\begin{array}{cc}
J_{0}^{2}\left(p_{\mathrm{F}} r\right), & 2 \mathrm{D},  \tag{3.71}\\
\left(p_{\mathrm{F}} r\right)^{-2} \sin ^{2} p_{\mathrm{F}} r, & 3 \mathrm{D} .
\end{array}\right.
$$

We consider the unitary ensemble first; results for the orthogonal symmetry will be presented in the end. Evaluating the $\sigma$-model correlation functions in the r.h.s. of Eq. (3.70) and separating the result into the singular the singular (proportional to $\delta(\omega)$ ) and regular at $\omega=0$ parts, one can obtain the correlation functions $\alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)$ and $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$. The two-level correlation function $R(\omega)$ entering Eq. (3.70) was studied in Section 2. We employ again the method of [10] described in Section 2 to calculate the sigma-model correlation functions $\left\langle Q_{b b}^{11}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{22}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}$ and $\left\langle Q_{b b}^{12}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{21}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}$ for relatively low frequencies $\omega \ll E_{c}$. First, we restrict ourselves to the terms of order $g^{-1}$. Then, the result for the first correlation function reads as

$$
\begin{equation*}
\left\langle Q_{b b}^{11}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{22}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}=-1-2 \mathrm{i} \frac{\exp (\mathrm{i} \pi s) \sin \pi s}{(\pi s)^{2}}-\frac{2 \mathrm{i}}{\pi s} \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \tag{3.72}
\end{equation*}
$$

where $s=\omega / \Delta+\mathrm{i} 0$. The first two terms in Eq. (3.72) represent the result of the zero-mode approximation; the last term is the correction of order $g^{-1}$. An analogous calculation for the second correlator yields:

$$
\begin{equation*}
\left\langle Q_{b b}^{12}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{21}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}=-2\left\{\frac{\mathrm{i}}{\pi s}+\left[1+\mathrm{i} \frac{\exp (\mathrm{i} \pi s) \sin \pi s}{(\pi s)^{2}}\right] \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\right\} . \tag{3.73}
\end{equation*}
$$

Now, separating regular and singular parts in r.h.s. of Eq. (3.70), we obtain the following result [27] for the autocorrelations of the same eigenfunction:

$$
\begin{equation*}
\left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E}-1=k_{d}(r)\left[1+\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{1}\right)\right]+\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \tag{3.74}
\end{equation*}
$$

and for the correlation of amplitudes of two different eigenfunctions

$$
\begin{equation*}
\left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega}-1=k_{d}(r) \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{1}\right), \quad k \neq l . \tag{3.75}
\end{equation*}
$$

In particular, for $\boldsymbol{r}_{1}=\boldsymbol{r}_{2}$ we have

$$
\begin{equation*}
\left.\left.V^{2}\langle | \psi_{k}(\boldsymbol{r}) \psi_{l} \boldsymbol{r}\right)\left.\right|^{2}\right\rangle_{E, \omega}-1=\delta_{k l}+\left(1+\delta_{k l}\right) \Pi(\boldsymbol{r}, \boldsymbol{r}) . \tag{3.76}
\end{equation*}
$$

Note that the result (3.74) for $\boldsymbol{r}_{1}=\boldsymbol{r}_{2}$ is the inverse participation ratio calculated above (Section 3.3.1); on the other hand, neglecting the terms with the diffusion propagator (i.e. making the zero-mode approximation), we reproduce the result of Refs. [110-112].

Eqs. (3.75) and (3.76) show that the correlations between different eigenfunctions are relatively small in the weak disorder regime. Indeed, they are proportional to the small parameter $\Pi(r, r)$. The correlations are enhanced by disorder; when the system approaches the strong localization regime, the relative magnitude of correlations, $\Pi(\boldsymbol{r}, \boldsymbol{r})$ ceases to be small. The correlations near the Anderson localization transition will be discussed in Section 5.

Another correlation function, generally used for the calculation of the linear response of the system,

$$
\begin{align*}
\gamma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right) & =\left\langle\psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right) \psi_{l}^{*}\left(\boldsymbol{r}_{2}\right)\right\rangle_{E, \omega} \\
& \equiv \Delta^{2} R^{-1}(\omega)\left\langle\sum_{k \neq l} \psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right) \psi_{l}^{*}\left(\boldsymbol{r}_{2}\right) \delta\left(E-\varepsilon_{k}\right) \delta\left(E+\omega-\varepsilon_{l}\right)\right\rangle \tag{3.77}
\end{align*}
$$

can be calculated in a similar way. The result reads

$$
\begin{equation*}
V^{2}\left\langle\psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right) \psi_{l}^{*}\left(\boldsymbol{r}_{2}\right)\right\rangle_{E, \omega}=k_{d}(r)+\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \quad k \neq l . \tag{3.78}
\end{equation*}
$$

As is seen from Eqs. (3.74), (3.75) and (3.78), in the $1 / g$ order the correlation functions $\alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)$ and $\gamma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$ survive for the large separation between the points, $r \gg l$, while $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$ decays exponentially for the distances larger than the mean free path $l$. This is, however, an artifact of the $g^{-1}$ approximation, and the investigation of the corresponding tails requires the extension of the above calculation to the terms proportional to $g^{-2}$. The correlator $\left\langle Q_{b b}^{11}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{22}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}$ gets the following correction in the $g^{-2}$ order:

$$
\begin{align*}
\delta\left\langle Q_{b b}^{11}\left(\boldsymbol{r}_{1}\right) Q_{b b}^{22}\left(\boldsymbol{r}_{2}\right)\right\rangle_{\mathrm{s}}= & -f_{1}+2 f_{4}+\exp (2 \mathrm{i} \pi s) f_{3}-2 \mathrm{i} \frac{\exp (2 \mathrm{i} \pi s)}{\pi s}\left(f_{2}-f_{3}\right) \\
& -\frac{\exp (2 \mathrm{i} \pi s)-1}{2(\pi s)^{2}}\left(f_{1}-4 f_{2}+3 f_{3}-2 f_{4}\right), \tag{3.79}
\end{align*}
$$

where we defined the functions

$$
\begin{align*}
& f_{1}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\Pi^{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \\
& f_{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=(2 V)^{-1} \int \mathrm{~d} \boldsymbol{r}\left[\Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}_{1}\right)+\Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}_{2}\right)\right], \\
& f_{3}=V^{-2} \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime} \Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right), \\
& f_{4}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=V^{-1} \int \mathrm{~d} \boldsymbol{r} \Pi\left(\boldsymbol{r}, \boldsymbol{r}_{1}\right) \Pi\left(\boldsymbol{r}, \boldsymbol{r}_{2}\right) . \tag{3.80}
\end{align*}
$$

Consequently, we obtain the following results for the correlations of different $(k \neq l)$ eigenfunctions at $r>l$ :

$$
\begin{align*}
\left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega}-1= & \frac{1}{2}\left(f_{1}-f_{3}-2 f_{4}\right) \\
& +2\left(f_{2}-f_{3}\right)\left(\frac{\sin ^{2} \pi s}{(\pi s)^{2}}-\frac{\sin 2 \pi s}{2 \pi s}\right)\left(1-\frac{\sin ^{2} \pi s}{(\pi s)^{2}}\right)^{-1} \tag{3.81}
\end{align*}
$$

As it should be expected, the double integral over the both coordinates of this correlation function is equal to zero. This property is just the normalization condition and should hold in arbitrary order of expansion in $g^{-1}$.

The quantities $f_{2}, f_{3}$, and $f_{4}$ are proportional to $g^{-2}$, with some (geometry-dependent) prefactors of order unity. On the other hand, $f_{1}$ in 2D and 3D geometry depends essentially on the distance $r=\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right|$. In particular, for $l \ll r \ll L$

$$
f_{1}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\Pi^{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \approx \begin{cases}\frac{1}{(\pi g)^{2}} \ln ^{2} \frac{L}{r}, & 2 \mathrm{D} \\ \frac{1}{\left(4 \pi^{2} v D r\right)^{2}}, & 3 \mathrm{D}\end{cases}
$$

Thus, for $l<r \ll L$, the contributions proportional to $f_{1}$ dominate in Eq. (3.81), yielding

$$
\begin{equation*}
\left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega}-1=\frac{1}{2} \Pi^{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \quad k \neq l . \tag{3.82}
\end{equation*}
$$

On the other hand, for the case of the quasi-1D geometry (as well as in 2D and 3D for $r \sim L$ ), all quantities $f_{1}, f_{2}, f_{3}$, and $f_{4}$ are of order of $1 / g^{2}$. Thus, the correlator $\sigma\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E, \omega\right)$ acquires a non-trivial (oscillatory) frequency dependence on a scale $\omega \sim \Delta$ described by the second term in the r.h.s. of Eq. (3.81). In particular, in the quasi-1D case the function $f_{2}-f_{3}$ determining the spatial dependence of this term has the form

$$
\begin{equation*}
f_{2}-f_{3}=-\frac{2}{3 g^{2}}\left[B_{4}\left(\frac{r_{1}}{L}\right)+B_{4}\left(\frac{r_{2}}{L}\right)\right], \tag{3.83}
\end{equation*}
$$

where $B_{4}(x)=x^{4}-2 x^{3}+x^{2}-1 / 30$ is the Bernoulli polynomial.

Let us remind the reader that the above derivation is valid for $\omega \ll E_{c}$. In the range $\omega \gtrsim E_{c}$ the $\sigma$-model correlation functions entering Eqs. (3.70) can be calculated by means of the perturbation theory [9], yielding

$$
\begin{align*}
& \left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega}=1+\operatorname{Re}\left\{k_{d}(r) \Pi_{\omega}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)+\frac{1}{2}\left[\Pi_{\omega}^{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)-\frac{1}{V^{2}} \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime} \Pi_{\omega}^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)\right]\right\}, \\
& V^{2}\left\langle\psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right) \psi_{l}^{*}\left(\boldsymbol{r}_{2}\right)\right\rangle_{E, \omega}=k_{d}(r)+\operatorname{Re} \Pi_{\omega}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \tag{3.84}
\end{align*}
$$

where $\Pi_{\omega}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ is the finite-frequency diffusion propagator

$$
\begin{equation*}
\Pi_{\omega}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=(\pi v)^{-1} \sum_{q} \frac{\phi_{q}\left(\boldsymbol{r}_{1}\right) \phi_{q}\left(\boldsymbol{r}_{2}\right)}{D q^{2}-\mathrm{i} \omega}, \tag{3.85}
\end{equation*}
$$

and the summation in Eq. (3.85) now includes $\boldsymbol{q}=0$. As was mentioned, the perturbation theory should give correctly the non-oscillatory (in $\omega$ ) part of the correlation functions at $\omega \gg \Delta$. Indeed, it can be checked that Eqs. (3.84) match the results (3.75) and (3.78) of the non-perturbative calculation in this regime. Furthermore, in the $1 / g$ order [which means keeping only linear in $\Pi_{\omega}$ terms in (3.84) and neglecting $-\mathrm{i} \omega$ in denominator of Eq. (3.85)]. Eqs. (3.84) and (3.85) reproduce the exact results (3.75) and (3.78) even at small frequencies $\omega \sim \Delta$. We stress, however, that the perturbative calculation is not justified in this region and only the supersymmetry method provides a rigorous derivation of these results.

Generalization to a system with unbroken time reversal symmetry (orthogonal ensemble) is straightforward [138]; in the $1 / g$-order Eqs. (3.74), (3.75), and (3.78) are modified as follows:

$$
\begin{align*}
& \left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E}=\left[1+2 k_{d}(r)\right]\left[1+2 \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\right],  \tag{3.86}\\
& \left.\left.V^{2}\langle | \psi_{k}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{2}\right)\right|^{2}\right\rangle_{E, \omega}-1=2 k_{d}(r) \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right),  \tag{3.87}\\
& V^{2}\left\langle\psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{l}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right) \psi_{l}^{*}\left(\boldsymbol{r}_{2}\right)\right\rangle_{E, \omega}=k_{d}(r)+\left[1+k_{d}(r)\right] \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right), \quad k \neq l . \tag{3.88}
\end{align*}
$$

3.3.3.1. IPR fluctuations. Using the supersymmetry method, one can calculate also higher-order correlation functions of the eigenfunction amplitudes. In particular, the correlation function $\langle | \psi_{k}^{4}\left(\boldsymbol{r}_{1}\right) \| \psi_{k}^{4}\left(\boldsymbol{r}_{2}\right)| \rangle_{E}$ determines fluctuations of the inverse participation ratio (IPR) $P_{2}=\int \mathrm{d} \boldsymbol{r}\left|\psi^{4}(\boldsymbol{r})\right|$. Details of the corresponding calculation can be found in Ref. [25]; the result for the relative variance of $\operatorname{IPR}, \delta\left(P_{2}\right)=\operatorname{var}\left(P_{2}\right) /\left\langle P_{2}\right\rangle^{2}$ reads

$$
\begin{equation*}
\delta\left(P_{2}\right)=\frac{8}{\beta^{2}} \int \frac{\mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime}}{V^{2}} \Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=\frac{32 a_{d}}{\beta^{2} g^{2}}, \tag{3.89}
\end{equation*}
$$

with a numerical coefficient $a_{d}$ defined in Section 2 (see Eqs. (2.25) and (2.26)). The fluctuations (3.89) have the same relative magnitude ( $\sim 1 / g$ ) as the famous universal conductance fluctuations. Note also that extrapolating Eq. (3.89) to the Anderson transition point, where $g \sim 1$, we find $\delta\left(P_{2}\right) \sim 1$, so that the magnitude of IPR fluctuations is of the order of its mean value (which is, in turn, much larger than in the metallic regime; see Section 5).

Eq. (3.89) can be generalized onto higher IPRs $P_{q}$ with $q>2$,

$$
\begin{equation*}
\frac{\operatorname{var}\left(P_{q}\right)}{\left\langle P_{q}\right\rangle^{2}} \simeq \frac{2}{\beta^{2}} q^{2}(q-1)^{2} \int \frac{\mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime}}{V^{2}} \Pi^{2}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=\frac{8 q^{2}(q-1)^{2} a_{d}}{\beta^{2} g^{2}}, \tag{3.90}
\end{equation*}
$$

so that the relative magnitude of fluctuations of $P_{q}$ is $\sim q(q-1) / g$. Furthermore, the higher irreducible moments (cumulants) $\left\langle\left\langle P_{q}^{n}\right\rangle, n=2,3, \ldots\right.$, have the form

$$
\begin{align*}
\frac{\left.\left\langle P_{q}^{n}\right\rangle\right\rangle}{\left\langle P_{q}\right\rangle^{n}} & =\frac{(n-1)!}{2}\left[\frac{2}{\beta} q(q-1)\right]^{n} \int \frac{\mathrm{~d} \boldsymbol{r}_{1} \cdots \mathrm{~d} \boldsymbol{r}_{n}}{V^{n}} \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \ldots \Pi\left(\boldsymbol{r}_{n}, \boldsymbol{r}_{1}\right) \\
& =\frac{(n-1)!}{2} \operatorname{Tr}\left[\frac{2}{\beta} q(q-1) \Pi\right]^{n} \tag{3.91}
\end{align*}
$$

where $\Pi$ is the integral operator with the kernel $\Pi\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right) / V$. This is valid provided $q^{2} n \ll 2 \beta \pi g$. Prigodin and Altshuler [137] obtained Eq. (3.91) starting from the assumption that the eigenfunction statistics is described by the Liouville theory. According to (3.91), the distribution function $\mathscr{P}\left(P_{q}\right)$ of the IPR $P_{q}$ (with $q^{2} / \beta \pi g \ll 1$ ) decays exponentially in the region $q(q-1) / g \ll P_{q} /\left\langle P_{q}\right\rangle-1 \ll 1$,

$$
\begin{equation*}
\mathscr{P}\left(P_{q}\right) \sim \exp \left\{-\frac{\pi \beta}{2} \frac{\varepsilon_{1}}{\Delta} \frac{P_{q} /\left\langle P_{q}\right\rangle-1}{q(q-1)}\right\} \tag{3.92}
\end{equation*}
$$

where $\varepsilon_{1}$ is the lowest non-zero eigenvalue of the diffusion operator $-D \nabla^{2}$.
The perturbative calculations show that the cumulants of the IPRs are correctly reproduced (in the leading order in $1 / g$ ) if one assumes [137] that the statistics of the eigenfunction envelopes $\left|\psi^{2}(r)\right|_{\text {smooth }}=\mathrm{e}^{\theta(r)}$ is governed by the Liouville theory (see, e.g. $[139,140]$ ) defined by the functional integral

$$
\begin{equation*}
\int D \theta \delta\left(\int \frac{\mathrm{~d}^{d} r}{V} \mathrm{e}^{\theta}-1\right) \exp \left\{-\frac{\beta \pi v D}{4} \int \mathrm{~d}^{d} \boldsymbol{r}(\nabla \theta)^{2}\right\} \ldots \tag{3.93}
\end{equation*}
$$

We will return to this issue in Section 4 where the asymptotics of the IPR distribution function will be discussed. We will see that these "tails" governed by rare realizations of disorder are described by saddle-point solutions which can be also obtained from the Liouville theory description (3.93). The multifractal dimensions (3.63) can be found from the Liouville theory as well [139,140]. It should be stressed, however, that this agreement between the supermatrix $\sigma$-model governing the eigenfunctions statistics and the Liouville theory is not exact, but only holds in the leading order in $1 / g$.

Let us note that the correlations of eigenfunction amplitudes determine also fluctuations of matrix elements of an operator of some (say, Coulomb) interaction computed on eigenfunctions $\psi_{k}$ of the one-particle Hamiltonian in a random potential. Such a problem naturally arises, when one wishes to study the effect of interaction onto statistical properties of excitations in a mesoscopic sample (see Section 9).

### 3.3.4. Ballistic effects

3.3.4.1. Ballistic systems. The above consideration can be generalized to a ballistic chaotic system, by applying a recently developed ballistic generalization of the $\sigma$-model [75-77]. The results are
then expressed [78] in terms of the (averaged over the direction of velocity) kernel $g\left(\boldsymbol{r}_{1}, \boldsymbol{n}_{1} ; \boldsymbol{r}_{2}, \boldsymbol{n}_{2}\right)$ of the Liouville operator $\hat{K}=v_{\mathrm{F}} \boldsymbol{n} \nabla$ governing the classical dynamics in the system,

$$
\begin{align*}
& \Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\int \mathrm{d} \boldsymbol{n}_{1} \mathrm{~d} \boldsymbol{n}_{2} g\left(\boldsymbol{r}_{1}, \boldsymbol{n}_{1} ; \boldsymbol{r}_{2}, \boldsymbol{n}_{2}\right), \\
& \hat{K} g\left(\boldsymbol{r}_{1}, \boldsymbol{n}_{1} ; \boldsymbol{r}_{2}, \boldsymbol{n}_{2}\right)=(\pi v)^{-1}\left[\delta\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right) \delta\left(\boldsymbol{n}_{1}-\boldsymbol{n}_{2}\right)-V^{-1}\right] . \tag{3.94}
\end{align*}
$$

Here $\boldsymbol{n}$ is a unit vector determining the direction of momentum, and normalization $\int \mathrm{d} \boldsymbol{n}=1$ is used. Equivalently, the function $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ can be defined as

$$
\begin{equation*}
\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\int_{0}^{\infty} \mathrm{d} t \int \mathrm{~d} \boldsymbol{n}_{1} \tilde{g}\left(\boldsymbol{r}_{1}, \boldsymbol{n}_{1}, t ; \boldsymbol{r}_{2}\right), \tag{3.95}
\end{equation*}
$$

where $\tilde{g}$ is determined by the evolution equation

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+v_{\mathrm{F}} \boldsymbol{n}_{1} \nabla_{1}\right) \tilde{g}\left(\boldsymbol{r}_{1}, \boldsymbol{n}_{1}, t ; \boldsymbol{r}_{2}\right)=0, \quad t>0 \tag{3.96}
\end{equation*}
$$

with the boundary condition

$$
\begin{equation*}
\left.\tilde{g}\right|_{t=0}=(\pi v)^{-1}\left[\delta\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right)-V^{-1}\right] . \tag{3.97}
\end{equation*}
$$

Eq. (3.94) is a natural "ballistic" counterpart of Eq. (2.22). Note, however, that $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ contains a contribution $\Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ of the straight line motion from $\boldsymbol{r}_{2}$ to $\boldsymbol{r}_{1}$ (equal to $1 /\left(\pi p_{\mathrm{F}}\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right|\right)$ in 2 D and to $1 / 2\left(p_{\mathrm{F}}\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right|\right)^{2}$ in 3 D ), which is nothing else but the smoothed version of the function $k_{d}\left(\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right|\right)$. For this reason, $\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ in Eqs. (3.86)-(3.88) should be replaced in the ballistic case by $\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)-\Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$. At large distances $\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right| \gg \lambda_{\mathrm{F}}$ the (smoothed) correlation function takes in the leading approximation the form

$$
\begin{equation*}
V^{2} \alpha\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, E\right)=1+\frac{2}{\beta} \Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) . \tag{3.98}
\end{equation*}
$$

A formula for the variance of matrix elements closely related to Eq. (3.98) was obtained in the semiclassical approach in Ref. [141]. In a recent paper [142] a similar generalization of the Berry formula for $\left\langle\psi_{k}^{*}\left(\boldsymbol{r}_{1}\right) \psi_{k}\left(\boldsymbol{r}_{2}\right)\right\rangle$ was proposed.

Eq. (3.98) shows that correlations in eigenfunction amplitudes in remote points are determined by the classical dynamics in the system. It is closely related to the phenomenon of scarring of eigenfunctions by the classical orbits [143,144]. Indeed, if $\boldsymbol{r}_{1}$ and $\boldsymbol{r}_{2}$ belong to a short periodic orbit, the function $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ is positive, so that the amplitudes $\left|\psi_{k}\left(\boldsymbol{r}_{1}\right)\right|^{2}$ and $\left|\psi_{k}\left(\boldsymbol{r}_{2}\right)\right|^{2}$ are positively correlated. This is a reflection of the "scars" associated with this periodic orbits and a quantitative characterization of their strength in the coordinate space. Note that this effect gets smaller with increasing energy $E$ of eigenfunctions. Indeed, for a strongly chaotic system and for $\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right| \sim L(L$ being the system size), we have in the 2 D case $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \sim \lambda_{\mathrm{F}} / L$, so that the magnitude of correlations decreases as $E^{-1 / 2}$. The function $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ was explicitly calculated in Ref. [78] for a circular billiard with diffuse surface scattering (see Section 8).
3.3.4.2. Ballistic effects in diffusive systems. We return now to the question of deviations of the eigenfunction amplitude distribution from the RMT in a diffusive 3D sample. As was shown in

Section 3.3.1, such deviations are controlled by the parameter $\kappa=\Pi(\boldsymbol{r}, \boldsymbol{r})$, see Eqs. (3.53)-(3.56). The physical meaning of the parameter $\kappa$ is the time-integrated return probability, see Eq. (3.95) generalizing definition of $\Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ to the ballistic case. The contribution to this return probability from the times larger than the momentum relaxation time, $t>\tau$, is given by

$$
\Pi^{\mathrm{diff}}(r, r)=(\pi \nu V)^{-1} \sum_{|q| \leqslant 1 / l}\left(D q^{2}\right)^{-1}
$$

The sum over the momenta diverges on the ultraviolet bound in $d \geq 2$, so that the cut-off at $q \sim 1 / l$ is required. This results in Eq. (3.56) in 2D and in $\Pi^{\mathrm{diff}}(\boldsymbol{r}, \boldsymbol{r}) \sim 1 /\left(k_{\mathrm{F}} l\right)^{2}$ in the 3D case. There exists, however, an additional, ballistic, contribution to $\Pi(\boldsymbol{r}, \boldsymbol{r})$, which comes from the times $t$ shorter than the mean free time $\tau$. Diagrammatically, it is determined by the first term of the diffuson ladder contributing to $\Pi(r, r)$ (that with one impurity line), i.e. by the probability to hit an impurity and to be rejected back after a time $t \ll \tau$. Contrary to the diffusive contribution, which has a universal form and is determined by the value of the diffusion constant $D$ only, the ballistic one is strongly dependent on the microscopic structure of the disorder. In particular, in the case of the white-noise disorder we find

$$
\Pi^{\mathrm{ball}}(\boldsymbol{r}, \boldsymbol{r})=\left\{\begin{array}{cc}
\frac{1}{\pi v v_{\mathrm{F}}^{2} \tau} \int \frac{(\mathrm{~d} q)}{q^{2}}=\frac{1}{2 \pi g} \ln \frac{l}{\lambda_{\mathrm{F}}}, & 2 \mathrm{D}  \tag{3.99}\\
\frac{\pi}{4 v v_{\mathrm{F}}^{2} \tau} \int \frac{(\mathrm{~d} q)}{q^{2}} \sim \frac{1}{k_{\mathrm{F}} l}, & 3 \mathrm{D}
\end{array}\right.
$$

Note that the integrals over the momenta are again divergent at large $q$ - precisely in the same way as in the diffusive region, but with different numerical coefficients - and are now cut-off at $q \sim k_{\mathrm{F}}$.

The total return probability is given by the sum of the short-scale (ballistic) and long-scale (diffusive) contributions. It is important to notice, however, that the single-scattering contribution (3.99) should be divided by 2 in the orthogonal symmetry case, because the corresponding trajectory is identical to its time reversal. Thus, $\kappa=\Pi^{\text {diff }}(\boldsymbol{r}, \boldsymbol{r})+\Pi^{\text {ball }}(\boldsymbol{r}, \boldsymbol{r})$ for $\beta=2$ and $\kappa=\Pi^{\text {diff }}(\boldsymbol{r}, \boldsymbol{r})+(1 / 2) \Pi^{\text {ball }}(\boldsymbol{r}, \boldsymbol{r})$ for $\beta=1$. We see that for the white-noise random potential in 3D the return probability is dominated by the ballistic contribution, yielding $\kappa \sim 1 / k_{\mathrm{F}} l$. In the 2D case, taking into account of the ballistic contribution modifies only the argument of the logarithm in (3.56). Furthermore, even in the quasi-1D geometry the non-universal short-scale effects can be important. Indeed, if we consider a 3D sample of the quasi-1D geometry ( $L_{x}, L_{y} \ll L_{z}$ ), the diffusion contribution will be given by Eq. (3.55), $\Pi^{\text {diff }}(r, r) \sim 1 / g$, while the ballistic one will be $\Pi^{\text {ball }}(\boldsymbol{r}, \boldsymbol{r}) \sim 1 / k_{\mathrm{F}} l$. Therefore, the diffusion contribution is dominant only provided $g \ll k_{\mathrm{F}} l$.

On the other hand, let us consider the opposite case of a smooth random potential with correlation length $d \gg \lambda_{\mathrm{F}}$. Then the scattering is of small-angle nature and the probability for a particle to return back in a time $t \ll \tau$ is exponentially small, so that $\Pi^{\text {ball }}(\boldsymbol{r}, \boldsymbol{r})$ can be neglected. Therefore, the return probability $\kappa$ in Eqs. (3.51)-(3.54) is correctly given by the diffusion contribution, see Eq. (3.56) for 2D and the estimate below it for 3D. Thus the corrections to the "body" of the distribution function are properly given by the $\sigma$-model in this case.

## 4. Asymptotic behavior of distribution functions and anomalously localized states

In this section, we discuss asymptotics of distribution functions of various quantities characterizing wave functions in a disordered system. Asymptotic behavior of these distribution functions is determined by rare realizations of the disorder producing the states, which show much stronger localization features than typical states in the system. We call such states "anomalously localized states" (ALS).

It was found by Altshuler, Kravtsov and Lerner (AKL) [28] that distribution functions of conductance, density of states, local density of states, and relaxation times have slowly decaying logarithmically normal ( LN ) asymptotics at large values of the arguments. These results were obtained within the renormalization group treatment of the $\sigma$-model. The validity of this RG approach is restricted to 2D and $2+\varepsilon$-dimensional systems, with $\varepsilon \ll 1$. On the other hand, the conductance, LDOS and relaxation times fluctuations in strictly 1D disordered chains, where all states are strongly localized, were studied with the use of Berezinski and Abrikosov-Ryzhkin techniques [113,145-148]. The corresponding distributions were found to be of the LN form, too. It was conjectured on the basis of this similarity $[28,113,149]$ that even in a metallic sample there is a finite probability to find "almost localized" eigenstates, and that these states govern the slow asymptotic decay of the distribution functions. Similar conclusion [25] is implied by the exact results for the statistics of the eigenfunction amplitude in the quasi-one-dimensional case, which shows the identical asymptotic behavior in the localized and metallic regimes, see Section 3, Eqs. (3.44) and (3.47).
A new boost to the activity in this direction was given by the paper of Muzykantskii and Khmelnitskii [29], who proposed to use the saddle-point method for the supersymmetric $\sigma$-model in order to calculate the long-time dispersion of the average conductance $G(t)$. Their idea was to reproduce the AKL result by means of a more direct calculation. However, they found a different, power-law decay of $G(t)$ in an intermediate range of times $t$ in 2D. As was shown by the author [30] (and then reproduced in [31] within the ballistic $\sigma$-model approach), the far asymptotic behavior is of log-normal form and is thus in agreement with AKL. The saddle-point method of Muzykantskii and Khmelnitskii allowed also to study the asymptotic behavior of distribution functions of other quantities: relaxation times [29-31], eigenfunction intensities [32,33], local density of states [34], inverse participation ratio [35,36], level curvatures [37], etc. The form of the saddle-point solution describes directly the spatial shape of the corresponding anomalously localized state $[29,36]$.

We will consider the unitary symmetry $(\beta=2)$ throughout this section; in the general case, the conductance $g$ in the exponent of the distribution functions is replaced by $(\beta / 2) g$ (we will sometimes do it explicitly in the end of the calculation).

### 4.1. Long-time relaxation

In this subsection we study (mainly following Refs. [29,30]) the asymptotic (long-time) behavior of the relaxation processes in an open disordered conductor. One possible formulation of the problem is to consider the time dependence of the average conductance $G(t)$ defined by the non-local (in time) current-voltage relation

$$
\begin{equation*}
I(t)=\int_{-\infty}^{t} \mathrm{~d} t^{\prime} G\left(t-t^{\prime}\right) V\left(t^{\prime}\right) \tag{4.1}
\end{equation*}
$$

Alternatively, one can study the decay law, i.e. the survival probability $P_{s}(t)$ for a particle injected into the sample at $t=0$ to be found there after a time $t$. Classically, $P_{s}(t)$ decays according to the exponential law, $P_{s}(t) \sim \mathrm{e}^{-t / t_{D}}$, where $t_{D}^{-1}$ is the lowest eigenvalue of the diffuson operator $-D \nabla^{2}$ with the proper boundary conditions. The time $t_{D}$ has the meaning of the time of diffusion through the sample, and $t_{D}^{-1}$ is the Thouless energy (see Section 2). The same exponential decay holds for the conductance $G(t)$, where it is induced by the weak-localization correction. The quantities of interest can be expressed in the form of the $\sigma$-model correlation function

$$
\begin{equation*}
G(t), P_{s}(t) \sim \int \frac{\mathrm{d} \omega}{2 \pi} \mathrm{e}^{-\mathrm{i} \omega t} \int D Q(r) A\{Q\} \mathrm{e}^{-S[\varrho]}, \tag{4.2}
\end{equation*}
$$

where $S[Q]$ is given by Eq. (3.7) with $\eta \rightarrow-2 \mathrm{i} \omega$. The preexponential factor $A\{Q\}$ depends on specific formulation of the problem, but is not important for the leading exponential behavior studied here.

Varying the exponent in Eq. (4.2) with respect to $Q$ and $\omega$, one gets the equations [29]

$$
\begin{align*}
& 2 D \nabla(Q \nabla Q)+\mathrm{i} \omega[\Lambda, Q]=0,  \tag{4.3}\\
& \frac{\pi v}{2} \int \mathrm{~d} \boldsymbol{r} \operatorname{Str}(\Lambda Q)=t . \tag{4.4}
\end{align*}
$$

[We assume unitary symmetry $(\beta=2)$; in the orthogonal symmetry case the calculation is applicable with minor modifications and we will present the result in the end.] Note that in fact $\omega$ plays in (4.2) a role of the Lagrange multiplier corresponding to the condition (4.4).

Therefore, it remains
(i) to find a solution $Q_{\omega}$ of Eq. (4.3) (which will depend on $\omega$ );
(ii) to substitute it into the self-consistency equation (4.4) and thus to fix $\omega$ as a function of $t$;
(iii) to substitute the found solution $Q_{t}$ into Eq. (4.2), which will yield

$$
\begin{equation*}
P_{s}(t) \sim \exp \left\{\frac{\pi v D}{4} \operatorname{Str} \int\left(\nabla Q_{t}\right)^{2}\right\} . \tag{4.5}
\end{equation*}
$$

Note that Eq. (4.3) is to be supplemented by the boundary conditions

$$
\begin{equation*}
\left.Q\right|_{\text {leads }}=\Lambda \tag{4.6}
\end{equation*}
$$

at the open part of the boundary, and

$$
\begin{equation*}
\left.\nabla_{n} Q\right|_{\text {insulator }}=0 \tag{4.7}
\end{equation*}
$$

at the insulating part of the boundary (if it exists); $\nabla_{n}$ denotes here the normal derivative.
It is not difficult to show [29] the solution of Eq. (4.3) has in the standard parametrization the only non-trivial variable - bosonic "non-compact angle" ${ }^{7} 0 \leq \theta_{1}<\infty$; all other coordinates being equal to zero. As a result, Eq. (4.3) reduces to an equation for $\theta_{1}(\boldsymbol{r})$ (we drop the subscript " 1 " below)

$$
\begin{equation*}
\nabla^{2} \theta+\frac{\mathrm{i} \omega}{D} \sinh \theta=0, \tag{4.8}
\end{equation*}
$$

[^6]the self-consistency condition (4.4) takes the form
\[

$$
\begin{equation*}
\pi v \int \mathrm{~d}^{d} r(\cosh \theta-1)=t, \tag{4.9}
\end{equation*}
$$

\]

and Eq. (4.5) can be rewritten as

$$
\begin{equation*}
\ln P_{s}(t)=-\frac{\pi v D}{2} \int \mathrm{~d}^{d} r(\nabla \theta)^{2} . \tag{4.10}
\end{equation*}
$$

For sufficiently small times, $\theta$ is small according to (4.9), so that Eqs. (4.8) and (4.9) can be linearized

$$
\begin{align*}
& \nabla^{2} \theta+2 \gamma \theta=0, \quad 2 \gamma=\mathrm{i} \omega / D,  \tag{4.11}\\
& \frac{\pi v}{2} \int \mathrm{~d}^{d} r \theta^{2}=t . \tag{4.12}
\end{align*}
$$

This yields

$$
\begin{equation*}
\theta=\left(\frac{2 t}{\pi v}\right)^{1 / 2} \phi_{1}(r), \tag{4.13}
\end{equation*}
$$

where $\phi_{1}$ is the eigenfunction of the Laplace operator corresponding to the lowest eigenvalue $2 \gamma_{1}=1 / D t_{D}$. The survival probability (4.10) reduces thus to

$$
\begin{equation*}
\ln P_{s}(t)=\frac{\pi v D}{2} \int \mathrm{~d}^{d} r \theta \nabla^{2} \theta=-\pi v D \gamma_{1} \int \mathrm{~d}^{d} r \theta^{2}=-t / t_{D} \tag{4.14}
\end{equation*}
$$

as expected. Eq. (4.14) is valid (up to relatively small corrections) as long as $\theta \ll 1$, i.e. for $t \Delta \ll 1$ ( $\Delta=1 / v V$ being the mean level spacing). To find the behavior at $t \gtrsim \Delta^{-1}$, as well corrections at $t<\Delta^{-1}$, one should consider the exact (non-linear) equation (4.8), solution of which depends on the sample geometry.

### 4.1.1. Quasi-1D geometry

We consider a wire of a length $L$ and a cross-section $A$ with open boundary conditions at both edges, $\theta(-L / 2)=\theta(L / 2)=0$. Eqs. (4.8) and (4.9) take the form

$$
\begin{align*}
& \theta^{\prime \prime}+2 \gamma \sinh \theta=0,  \tag{4.15}\\
& \int_{-L / 2}^{L / 2} \mathrm{~d} x(\cosh \theta-1)=t / \pi v A . \tag{4.16}
\end{align*}
$$

From the symmetry consideration $\theta(x)=\theta(-x)$ and $\theta^{\prime}(0)=0$, so that it is sufficient to consider the region $x>0$. The solution of Eq. (4.15) reads

$$
\begin{equation*}
x=\int_{\theta(x)}^{\theta_{0}} \frac{\mathrm{~d} \vartheta}{2 \sqrt{\gamma\left(\cosh \theta_{0}-\cosh \vartheta\right)}}, \tag{4.17}
\end{equation*}
$$

where $\theta_{0}$ is determined by the condition $\theta(L / 2)=0$ yielding

$$
\begin{equation*}
L=\int_{0}^{\theta_{0}} \frac{\mathrm{~d} \vartheta}{\sqrt{\gamma\left(\cosh \theta_{0}-\cosh \vartheta\right)}} . \tag{4.1}
\end{equation*}
$$

In the large- $t$ region $(t \Delta \gg 1)$ we will have $\theta_{0} \gg 1$, and Eqs. (4.17) and (4.18) can be simplified to give

$$
\begin{align*}
& \theta(x) \simeq \theta_{0}\left(1-\frac{2 x}{L}\right),  \tag{4.19}\\
& \theta_{0} \simeq \ln \frac{2 \theta_{0}^{2}}{\gamma L^{2}} . \tag{4.20}
\end{align*}
$$

Substituting this into condition (4.16) allows to relate $\theta_{0}$ to $t$,

$$
\begin{equation*}
\mathrm{e}^{\theta_{0}}=\frac{2}{\pi} t \Delta \theta_{0} . \tag{4.21}
\end{equation*}
$$

Finally, substitution of Eqs. (4.19) and (4.21) into (4.10) yields the log-normal asymptotic behavior of $P_{s}(t)$ [29]:

$$
\begin{equation*}
\ln P_{s}(t) \simeq-g \ln ^{2} \frac{t \Delta}{\ln (t \Delta)} ; \quad t \Delta \gg 1 \tag{4.22}
\end{equation*}
$$

with $g=2 \pi v A D / L$ being the dimensionless conductance. We remind that Eq. (4.22) has been derived for the unitary ensemble ( $\beta=2$ ); in the general case, its r.h.s. should be multiplied by $\beta / 2$.

Eq. (4.22) has essentially the same form as the asymptotic formula for $G(t)$ found by Altshuler and Prigodin [148] for the strictly 1D sample with a length much exceeding the localization length:

$$
\begin{equation*}
G(t) \sim \exp \left\{-\frac{l}{L} \ln ^{2}(t / \tau)\right\} . \tag{4.23}
\end{equation*}
$$

If we replace in Eq. (4.23) the 1D localization length $\xi=l$ by the quasi-1D localization length $\xi=\beta \pi v A D$, we reproduce the asymptotics (4.22) (up to a normalization of $t$ in the argument of $\ln ^{2}$, which does not affect the leading term in the exponent for $t \rightarrow \infty$ ). This is one more manifestation of the equivalence of statistical properties of smooth envelopes of the wave functions in 1D and quasi-1D samples [18] (see Section 3). Furthermore, agreement of the results for the metallic and the insulating samples demonstrates clearly that the asymptotic "tail" (4.22) in the metallic sample is indeed due to anomalously localized eigenstates.

As another manifestation of this fact, Eq. (4.22) can be represented as a superposition of the simple relaxation processes with mesoscopically distributed relaxation times [28]:

$$
\begin{equation*}
P_{s}(t) \sim \int \mathrm{d} t_{\phi} \mathrm{e}^{-t / t_{\phi}} \mathscr{P}\left(t_{\phi}\right) . \tag{4.24}
\end{equation*}
$$

The distribution function $\mathscr{P}\left(t_{\phi}\right)$ then behaves as follows:

$$
\begin{equation*}
\mathscr{P}\left(t_{\phi}\right) \sim \exp \left\{-g \ln ^{2}\left(g \Delta t_{\phi}\right)\right\} ; \quad t_{\phi} \gg 1 / g \Delta \equiv t_{D} . \tag{4.25}
\end{equation*}
$$

This can be easily checked by substituting Eq. (4.25) into Eq. (4.24) and calculating the integral via the stationary point method; the stationary point equation being

$$
\begin{equation*}
2 g t_{\phi} \ln \left(g \Delta t_{\phi}\right)=t . \tag{4.26}
\end{equation*}
$$

Note that Thouless energy $t_{D}^{-1}$ determines the typical width of a level of an open system. Therefore, formula (4.25) concerns indeed the states with anomalously small widths $t_{\phi}^{-1}$ in the energy space.

The saddle-point solution $\theta(r)$ provides a direct information on the spatial shape of the corresponding ALS. This was conjectured by Muzykantskii and Khmelnitskii [29] and was explicitly proven in [36] for the states determining the distribution of eigenfunction amplitudes, see Section 4.2.1 below. Specifically, the smoothed (over a scale larger than the Fermi wavelength) intensity of the ALS is $\left|\psi^{2}(\boldsymbol{r})\right|=\mathcal{N}^{-1} \mathrm{e}^{\theta}(\boldsymbol{r})$, where $\mathcal{N}$ is the normalization factor determined by the requirement $\int \mathrm{d}^{d} r\left|\psi^{2}(r)\right|=1$. We get thus from Eq. (4.19)

$$
\begin{equation*}
\left|\psi^{2}(\boldsymbol{r})\right|_{\text {smooth }}=\frac{\theta_{0}}{A L} \mathrm{e}^{-2 \theta_{0}|x| / L} \tag{4.27}
\end{equation*}
$$

with

$$
\begin{equation*}
\theta_{0} \simeq \ln \left(\frac{2}{\pi} t \Delta \ln (t \Delta)\right) \simeq \ln \left(\frac{4}{\pi} g \Delta t_{\phi} \ln ^{2}\left(g \Delta t_{\phi}\right)\right) . \tag{4.28}
\end{equation*}
$$

Thus, the ALS, which gives a minimum to the level width $t_{\phi}^{-1}$, has an exponential shape (4.27), (4.28).

The saddle-point method allows also to find the corrections to Eq. (4.14) in the intermediate region $t_{D} \ll t \ll \Delta^{-1}$, where $\theta_{0} \ll 1$ [150]. For this purpose, we expand $\cosh \theta_{0}$ and $\cosh \vartheta$ in Eq. (4.18) up to the 4th order terms, which leads to the following relation between $\gamma$ and $\theta_{0}$ :

$$
\begin{equation*}
\gamma=\frac{\pi^{2}}{2 L^{2}}\left(1-\frac{1}{8} \theta_{0}^{2}+\cdots\right) . \tag{4.29}
\end{equation*}
$$

Further, we substitute Eq. (4.17) into (4.16),

$$
\begin{equation*}
\int_{0}^{\theta_{0}} \frac{\mathrm{~d} \vartheta(\cosh \vartheta-1)}{2 \sqrt{\gamma\left(\cosh \theta_{0}-\cosh \vartheta\right)}}=\frac{t}{2 \pi v A} \tag{4.30}
\end{equation*}
$$

and expand $\cosh \vartheta$ in (4.30) up to the 4th order terms. This gives the relation

$$
\begin{equation*}
\frac{t}{2 \pi v A}=\theta_{0}^{2} \frac{\pi}{8 \sqrt{2 \gamma}}\left(1-\frac{1}{96} \theta_{0}^{2}\right) . \tag{4.31}
\end{equation*}
$$

Using Eqs. (4.15) and (4.16), we can rewrite the action in the form

$$
\begin{equation*}
S \equiv \frac{\pi v A D}{2} \int \mathrm{~d} x\left(\theta^{\prime}\right)^{2}=2 \pi v A D L \gamma\left(\cosh \theta_{0}-1\right)-2 D t \gamma . \tag{4.32}
\end{equation*}
$$

Expressing now $\theta_{0}$ and $\gamma$ through $t$ according to Eqs. (4.30) and (4.31), we find

$$
\begin{equation*}
-\ln P_{s}(t)=S=\frac{t}{t_{D}}\left(1-\frac{1}{2 \pi^{2} g} \frac{t}{t_{D}}+\cdots\right) \tag{4.33}
\end{equation*}
$$

with $t_{D}=L^{2} / \pi^{2} D$. In the general case, $g$ is replaced by $(\beta / 2) g$ here. Eq. (4.33) is completely analogous to the formula (3.42), (3.43) for the statistics of eigenfunction amplitudes. It shows that the correction to the leading term $-t / t_{D}$ in $\ln P_{S}$ becomes large compared to unity at $t \gtrsim \sqrt{g} t_{D}$, though it remains small compared to the leading term up to $t \sim g t_{D} \sim \Delta^{-1}$.

Result (4.33) was also obtained by Frahm [151] from rather involved calculations based on the equivalence between the 1D $\sigma$-model and the Fokker-Plank approach and employing the approximate solution of the Dorokhov-Mello-Pereyra-Kumar equations in the metallic limit. The fact that the logarithm of the quantum decay probability, $\ln P_{S}(t)$, starts to deviate strongly (compared to unity) from the classical law, $\ln P_{s}^{\mathrm{cl}}(t)=-t / t_{D}$ at $t \sim t_{D} \sqrt{g}$ was observed in numerical simulations by Casati et al. [152]. For related results in the framework of the random matrix model see Section 4.1.3.

### 4.1.2. 2D geometry

We consider now a 2D disk-shaped sample of a radius $R$ with an open boundary. If the problem is formulated in terms of the conductance, we can assume the two leads attached to the disk boundary to be of almost semicircular shape, with relatively narrow insulating intervals between them. Then we can approximate the boundary conditions by using Eq. (4.6) for all the boundary. In fact, in view of the logarithmic dependence of the saddle point action on $R$ (see below), the result does not depend to the leading approximation on the specific shape of the sample and the leads attached. With the rotationally invariant form of the boundary condition, the minimal action corresponds to the function $\theta$ depending on the radial coordinate $r$ only. We get therefore the radial equation

$$
\begin{equation*}
\theta^{\prime \prime}+\theta^{\prime} / r+2 \gamma \sinh \theta=0, \quad 0 \leq r \leq R \tag{4.34}
\end{equation*}
$$

(the prime denotes the derivative $\mathrm{d} / \mathrm{d} r$ ) with the boundary conditions:

$$
\begin{align*}
& \theta(R)=0,  \tag{4.35}\\
& \theta^{\prime}(0)=0 \tag{4.3}
\end{align*}
$$

Condition (4.36) follows from the requirement of analyticity of the field in the disk center.
Assuming that characteristic values of $\theta$ satisfy the condition $\theta \gg 1$ (which corresponds to $t \Delta \gg 1$ ), one can replace $\sinh \theta$ by e ${ }^{\theta} / 2$. Eq. (4.34) can be then easily integrated, and its general solution reads:

$$
\begin{equation*}
\mathrm{e}^{\theta(r)}=\frac{2 C_{1}^{2}}{\gamma} \frac{C_{2} r^{C_{1}-2}}{\left(C_{2} r^{C_{1}}+1\right)^{2}}, \tag{4.37}
\end{equation*}
$$

with two integration constants $C_{1}$ and $C_{2}$. To satisfy the boundary condition (4.36), we have to choose $C_{1}=2$. Furthermore, the above assumption $\theta(0) \gtrdot>1$ implies that $2 C_{2} / \gamma \gg 1$. Therefore, the second boundary condition (4.35) is satisfied if $C_{2} \simeq 8 / \gamma R^{4}$, and the solution can be written in the form

$$
\begin{equation*}
\mathrm{e}^{\theta(r)} \simeq\left[(r / R)^{2}+\gamma R^{2} / 8\right]^{-2} . \tag{4.38}
\end{equation*}
$$

Using now the self-consistency equation (4.9) one finds $\gamma=4 \pi^{2} v / t$. Finally, calculating the action on the saddle point (4.38), we find [29]

$$
\begin{equation*}
-\ln P_{s}(t)=S \simeq 8 \pi^{2} v D \ln (t \Delta), \tag{4.39}
\end{equation*}
$$

The above treatment is valid provided $\theta^{\prime}(r)<l^{-1}$ on the saddle point solution, which is the condition of the applicability of the diffusion approximation (here $l$ is the mean free path). In combination with the assumption $\theta(0) \gtrdot>1$ this means that $1 \ll t \Delta \ll(R /)^{2}$.

Now let us consider the region of still longer times, $t \gg \Delta^{-1}(R / l)^{2}$. In order to support the applicability of the diffusion approximation, we should search for a function $\theta(r)$ minimizing the action with an additional restriction $\theta^{\prime} \leq A l^{-1}$. Here $A$ is a parameter of the order of unity, which cannot be fixed within the diffusion approximation. We will see however that the saddle-point action depends on $l$ through $\ln (R / l)$ only, and thus does not depend on $A$ in the leading order, so that we can set $A=1$. Since the derivative has a tendency to increase in the vicinity of $r=0$, the restriction can be implemented [30] via replacing the boundary conditions (4.36) by $\theta^{\prime}\left(r_{*}\right)=0$, where the parameter $r_{*}$ will be specified below. The solution reads now:

$$
\begin{equation*}
\mathrm{e}^{\theta(r)}=\frac{(r / R)^{C-2}}{\left[(r / R)^{C}+\frac{C+2}{C-2}\left(r_{*} / R\right)^{c}\right]^{2}} ; \quad r_{*} \leq r \leq R . \tag{4.40}
\end{equation*}
$$

The function $\theta(r)$ is meant as being constant within the vicinity $|r| \leq r_{*}$ of the disk center. The condition $\theta^{\prime} \leq l^{-1}$ yields $r_{*} \sim l C$. It is important to note that the result does not depend on details of the cut-off procedure. For example, one gets the same results if one chooses the boundary condition in the form $\theta^{\prime}\left(r_{*}\right)=1 / l$. The crucial point is that the maximum derivative $\theta^{\prime}$ should not exceed $1 / l$. The constant $C$ is to be found from the self-consistency equation (4.9) which can be reduced to the following form:

$$
\begin{equation*}
\left(\frac{R}{r_{*}}\right)^{c}=\frac{2 t}{\pi^{2} v R^{2}} \frac{C^{2}}{C-2} . \tag{4.41}
\end{equation*}
$$

Neglecting corrections of the $\ln (\ln \cdot)$ form, we find

$$
\begin{equation*}
C \simeq \frac{\ln (t \Delta)}{\ln \left(R / r_{*}\right)} \simeq \frac{\ln (t \Delta)}{\ln (R / l)} . \tag{4.42}
\end{equation*}
$$

The action (4.10) is then equal to

$$
\begin{equation*}
S \simeq \pi^{2} v D(C+2)^{2} \ln \left(R / r_{*}\right) \simeq \pi^{2} v D \frac{\ln ^{2}\left[t \Delta(R / l)^{2}\right]}{\ln (R / l)} . \tag{4.43}
\end{equation*}
$$

Combining Eqs. (4.39) and (4.43) and introducing the factor $\beta / 2$ for generality, we get thus the following long-time asymptotics of $P_{s}(t)$ (or $G(t)$ ) [30]:

$$
\begin{align*}
& P_{s}(t) \sim(t \Delta)^{-2 \pi \beta g}, \quad 1 \ll t \Delta \ll(R / l)^{2},  \tag{4.44}\\
& P_{s}(t) \sim \exp \left\{-\frac{\pi \beta g}{4} \frac{\ln ^{2}(t / g \tau)}{\ln (R / l)}\right\}, \quad t \Delta \gtrdot(R / l)^{2}, \tag{4.45}
\end{align*}
$$

where $g=2 \pi v D$ is the dimensionless conductance per square in 2 D and $\tau$ is the mean free time.

The far asymptotic behavior (4.45) is of the log-normal form and very similar to that found by AKL (see Eq. (7.8) in Ref. [28]). It differs only by the factor $1 / g$ in the argument of $\ln ^{2}$. It is easy to see however that this difference disappears if one does the last step of the AKL calculation with a better accuracy. Let us consider for this purpose the intermediate expression of AKL (Ref. [28], Eq. (7.11)):

$$
\begin{equation*}
G(t) \propto-\frac{\sigma}{\tau} \int_{0}^{\infty} \mathrm{e}^{-t / t_{\phi}} \exp \left[-\frac{1}{4 u} \ln ^{2} \frac{t_{\phi}}{\tau}\right] \frac{\mathrm{d} t_{\phi}}{t_{\phi}} \tag{4.46}
\end{equation*}
$$

where

$$
u \simeq \frac{1}{2 \pi^{2} v D} \ln \frac{R}{l}
$$

in the weak localization region in 2D, which we are considering. Evaluating the integral (4.46) by the saddle point method, we find

$$
\begin{align*}
G(t) & \sim \exp \left\{-\frac{1}{4 u} \ln ^{2} \frac{2 u t}{\tau}\right\} \\
& \sim \exp \left\{-\frac{\pi g}{4} \frac{\ln ^{2}(t / g \tau)}{\ln (R / l)}\right\}, \tag{4.47}
\end{align*}
$$

where we have kept only the leading term in the exponent. Eq. (4.47) is in exact agreement with Eq. (4.45) for $\beta=1$ (AKL assumed the orthogonal symmetry of the ensemble). Therefore, the supersymmetric treatment confirms the AKL result and also establishes the region of its validity. It is instructive to represent the obtained results in terms of the superposition of simple relaxation processes with mesoscopically distributed relaxation times $t_{\phi}$ :

$$
\begin{equation*}
G(t), P_{s}(t) \sim \int \frac{\mathrm{d} t_{\phi}}{t_{\phi}} \mathrm{e}^{-t / t_{t}} \mathscr{P}\left(t_{\phi}\right) . \tag{4.48}
\end{equation*}
$$

Eqs. (4.44) and (4.45) lead then to the following result for the distribution function $\mathscr{P}\left(t_{\phi}\right)$ [30]:

$$
\mathscr{P}\left(t_{\phi}\right) \sim \begin{cases}\left(t_{\phi} / t_{D}\right)^{-2 \pi \beta g}, & t_{D} \ll t_{\phi} \ll t_{D}(R / l)^{2},  \tag{4.49}\\ \exp \left\{-\frac{\pi \beta g}{4} \frac{\ln ^{2}\left(t_{\phi} / \tau\right)}{\ln (R / l)}\right\}, & t_{\phi} \gg t_{D}(R / l)^{2},\end{cases}
$$

where $t_{D} \simeq R^{2} / D$ is the time of diffusion through the sample.
The smooth envelope of the ALS corresponding to the intermediate region $t_{D} \ll t_{\phi} \ll t_{D}\left(R / l^{2}\right.$ has according to Eq. (4.37) the following spatial structure:

$$
\begin{equation*}
\left|\psi^{2}(r)\right|_{\text {smooth }}=\mathscr{N}^{-1} \mathrm{e}^{\theta(r)}=\frac{1}{8 \beta \pi D t_{\phi}\left[(r / R)^{2}+R^{2} / 8 \beta D t_{\phi}\right]^{2}}, \tag{4.50}
\end{equation*}
$$

Thus, this ALS has an effective localization length $\xi_{\text {ef }} \sim R\left(t_{D} / t_{\phi}\right)^{1 / 2}$, with the intensity decreasing as $1 / r^{4}$ outside the region of the extent $\xi_{\text {ef }}$. As to the ultra-long-time region, Eqs. (4.40) and (4.42) indicate that now the effective localization length is given by $l_{*}^{(t)}$ defined as

$$
\begin{equation*}
l_{*}^{(t)}=\gamma_{t} l_{;} ; \quad \gamma_{t} \simeq \frac{\ln \left(t_{\phi} / t_{D}\right)}{\ln (R / l)} . \tag{4.51}
\end{equation*}
$$

The ALS intensity decays in a power-law manner for $r>l_{*}^{(t)}$, with an exponent depending on $t_{\phi}$ :

$$
\begin{equation*}
\left|\psi^{2}(r)\right| \sim \frac{1}{l_{*}^{(t)}}\left(\frac{r}{l_{*}^{(t)}}\right)^{-\gamma_{1}-2}, \quad l_{*}^{(t)} \leq r \leq R . \tag{4.52}
\end{equation*}
$$

### 4.1.3. Random matrix model

Here we mention briefly the results on the quantum decay law obtained by Savin and Sokolov [153] within the RMT model. This will allow us to see the similarities and the differences between the diffusive systems and the random matrix model. The model describes a Hamiltonian of an open chaotic system by a Gaussian random matrix coupled to $M$ external (decay) channels. The decay law found has the form

$$
\begin{equation*}
P_{s}(t) \sim\left(1+\Gamma_{\mathrm{w}} t / M\right)^{-M} \tag{4.53}
\end{equation*}
$$

where $\Gamma_{\mathrm{w}}=M T \Delta / 2 \pi$ is a typical width of the eigenstate, with $T$ characterizing the channel coupling ( $T=1$ for ideal coupling, see also Section 6 ). In this case, the product $M T$ plays a role of the dimensionless conductance $g$ (in contrast to the diffusive case where $g$ is governed by the bulk of the system, here it is determined by the number of decay channels and the strength of their coupling). For not too large $t(t \Delta T \ll 1)$, Eq. (4.53) yields the classical decay law, $P_{s}(t) \sim \mathrm{e}^{-t \Gamma_{\mathrm{w}}}$, with the corrections of the form

$$
\begin{equation*}
\ln P_{s}(t)=-t \Gamma_{\mathrm{w}}\left(1-\Gamma_{\mathrm{w}} t / 2 M+\cdots\right), \tag{4.54}
\end{equation*}
$$

which is similar to the results found for the diffusive systems (see, e.g. Eq. (4.33)). At large $t \gtrdot(\Delta T)^{-1}$, the decay take the power-law asymptotic form [154]

$$
\begin{equation*}
\ln P_{s}(t) \simeq-M \ln \left(\Gamma_{\mathrm{w}} t / M\right) \tag{4.55}
\end{equation*}
$$

### 4.1.4. Distribution of total density of states

Here we discuss the contribution of ALS to the asymptotic behavior of the distribution function $\mathscr{P}(v)$ of the total density of states (DOS),

$$
\begin{equation*}
v(E)=\frac{-1}{\pi V} \operatorname{Im} \int \mathrm{~d}^{d} r G_{\mathrm{R}}(\boldsymbol{r}, \boldsymbol{r} ; E) \tag{4.56}
\end{equation*}
$$

(in the present subsection we denote the average DOS as $v_{0}$ to distinguish it from the fluctuating quantity $v(E)$ ). A resonance state with an energy $E$ and width $t_{\phi}^{-1}$ gives a following contribution to $v(E)$ :

$$
\begin{equation*}
v_{\mathrm{ALS}}(E)=\frac{2}{\pi} \frac{t_{\phi}}{V}=\frac{2}{\pi} t_{\phi} \Delta v_{0} . \tag{4.57}
\end{equation*}
$$

We expect that the asymptotic behavior of $\mathscr{P}(v)$ at $v \gg g v_{0} \sim t_{D} / V$ is determined by a single anomalously localized states with $t_{\phi} \gg t_{D}$. This yields [36]

$$
\begin{equation*}
\mathscr{P}(v) \sim \mathscr{P}\left(t_{\phi}=\frac{\pi v}{2 \Delta v_{0}}\right) \sim \exp \left\{-g \ln ^{2}\left(g v / v_{0}\right)\right\} . \tag{4.58}
\end{equation*}
$$

in the quasi-1D case and

$$
\mathscr{P}(v) \sim \mathscr{P}\left(t_{\phi}=\frac{\pi v}{2 \Delta v_{0}}\right) \sim \begin{cases}\left(g v / v_{0}\right)^{-4 \pi g}, & \frac{v}{v_{0}} \ll \frac{1}{g}\left(\frac{L}{l}\right)^{2},  \tag{4.59}\\ \exp \left\{-\frac{\pi g}{2} \frac{\ln ^{2}\left(v / v_{0} \Delta \tau\right)}{\ln (L / l)}\right\}, & \frac{v}{v_{0}} \gg \frac{1}{g}\left(\frac{L}{l}\right)^{2},\end{cases}
$$

in the 2D geometry. The far LN asymptotic tail in Eq. (4.59) is in full agreement with the RG calculation by Altshuler et al. [28]. We find also an intermediate power-law behavior, which could not be obtained from the study of cumulants in Ref. [28]. We note, however, that this power-law form is fully consistent with the change of the behavior of cumulants $《 v^{n} 》$ at $n \sim \pi g$ discovered in [28].

### 4.2. Distribution of eigenfunction amplitudes

### 4.2.1. Quasi-1D geometry

The spatial shape of the ALS determining the asymptotics of the distribution function of eigenfunction intensities can be found [36] via the exact solution of the $\sigma$-model. We define

$$
\begin{equation*}
\langle | \psi^{2}(\boldsymbol{r})| \rangle_{u}=\frac{\mathscr{2}(u, \boldsymbol{r})}{\mathscr{P}(u)}, \tag{4.60}
\end{equation*}
$$

where

$$
\begin{equation*}
\left.\mathscr{2}(u, \boldsymbol{r})=\left.\frac{1}{v V}\left\langle\sum_{\alpha}\right| \psi_{\alpha}(\boldsymbol{r})\right|^{2} \delta\left(\left|\psi_{\alpha}(0)\right|^{2}-u\right) \delta\left(E-E_{\alpha}\right)\right\rangle, \tag{4.61}
\end{equation*}
$$

and $\mathscr{P}(u)$ is the distribution function of $u=\left|\psi^{2}(0)\right|$ defined formally by Eq. (3.3). According to Eq. (4.60), $\langle | \psi^{2}(r)| \rangle_{u}$ is the average intensity of an eigenstate, which has in the point $\boldsymbol{r}=0$ the intensity $u$ (which will be assumed to be atypically large). The exact result for $\mathscr{P}(u)$ is given in the form of the Lebedev-Kontorovich expansion by Eqs. (3.12), (3.21) and (3.24). Calculating the moments $\left.\left.\langle | \psi(0)\right|^{2}|\psi(\boldsymbol{r})|^{2 q}\right\rangle$, and restoring the function $\mathscr{2}(u, \boldsymbol{r})$, we find for $r>l$

$$
\begin{equation*}
\mathscr{2}(u, r)=-\frac{1}{V \xi A} \frac{\partial}{\partial u}\left[\frac{W^{(2)}\left(u \xi A ; \tau_{1}, \tau_{2}\right) W^{(1)}\left(u \xi A, \tau_{-}\right)}{u}\right], \tag{4.62}
\end{equation*}
$$

where the function $W^{(2)}\left(z ; \tau_{1}, \tau_{2}\right)$ satisfies the same equation, as $W^{(1)}$,

$$
\begin{equation*}
\frac{\partial W^{(2)}\left(z ; \tau_{1}, \tau_{2}\right)}{\partial \tau_{1}}=\left(z^{2} \frac{\partial^{2}}{\partial z^{2}}-z\right) W^{(2)}\left(z ; \tau_{1}, \tau_{2}\right) \tag{4.63}
\end{equation*}
$$

with the boundary condition

$$
\begin{equation*}
W^{(2)}\left(z ; 0, \tau_{2}\right)=z W^{(1)}\left(z, \tau_{2}\right) . \tag{4.64}
\end{equation*}
$$

The solution of Eqs. (4.63) and (4.64) is

$$
\begin{align*}
& W^{(2)}\left(z, \tau_{1}, \tau_{2}\right)=2 \sqrt{z} \int_{0}^{\infty} \mathrm{d} \mu b\left(\mu, \tau_{2}\right) K_{i \mu}(2 \sqrt{z}) \mathrm{e}^{-\left(\left(1+\mu^{2}\right) / 4\right) \tau_{1}}, \\
& b\left(\mu, \tau_{2}\right)=\frac{\mu \sinh (\pi \mu)}{2 \pi^{2}} \int_{0}^{\infty} \mathrm{d} t K_{i \mu}(t) W^{(1)}\left(t^{2} / 4, \tau_{2}\right) . \tag{4.65}
\end{align*}
$$

Substituting here the formula (3.24) for $W^{(1)}\left(z, \tau_{2}\right)$ and evaluating the integral over $z$, we can reduce Eq. (4.65) for $b\left(\mu, \tau_{2}\right)$ to the form

$$
\begin{align*}
b\left(\mu, \tau_{2}\right)= & \frac{\mu \sinh (\pi \mu)}{16 \pi^{2}}\left|\Gamma\left(\frac{1+\mathrm{i} \mu}{2}\right)\right|^{4}\left(1+\mu^{2}\right)+\frac{\mu \sinh (\pi \mu)}{2 \pi^{3}} \int \frac{\mathrm{~d} \mu_{1} \mu_{1}}{1+\mu_{1}^{2}} \\
& \times \sinh \frac{\pi \mu_{1}}{2}\left|\Gamma\left(1+\mathrm{i} \frac{\mu+\mu_{1}}{2}\right)\right|^{2}\left|\Gamma\left(1+\mathrm{i} \frac{\mu-\mu_{1}}{2}\right)\right|^{2} \mathrm{e}^{-\left(\left(1+\mu_{1}^{2}\right) / 4\right) \tau_{2}} . \tag{4.66}
\end{align*}
$$

In the opposite case $r<l$ we find

$$
\begin{equation*}
\mathscr{2}(u, \boldsymbol{r})=\frac{1}{V}\left\{k_{d}(r)\left(u \frac{\mathrm{~d}^{2}}{\mathrm{~d} u^{2}}+\frac{\mathrm{d}}{\mathrm{~d} u}\right)-\frac{\mathrm{d}}{\mathrm{~d} u}\right\} Y_{a}(u), \tag{4.67}
\end{equation*}
$$

where the function $Y_{a}(u)$ was defined in Eq. (3.9). This formula is valid for any sample, which is locally $d$-dimensional. In the case of the quasi-1D geometry we get

$$
\begin{equation*}
\mathscr{2}(u, \boldsymbol{r})=\frac{1}{V}\left\{k_{d}(r)\left(u \frac{\mathrm{~d}^{2}}{\mathrm{~d} u^{2}}+\frac{\mathrm{d}}{\mathrm{~d} u}\right)-\frac{\mathrm{d}}{\mathrm{~d} u}\right\}\left[W^{(1)}\left(u \xi A, \tau_{-}\right) W^{(1)}\left(u \xi A, \tau_{+}\right)\right], \quad r \ll l \tag{4.68}
\end{equation*}
$$

4.2.1.1. Insulating sample $(L \gg \xi)$. The distribution $\mathscr{P}(u)$ is given by Eqs. (3.45) and (3.46). The "tail", Eq. (3.47), at $u \gg 1 / A \xi$ corresponds to atypically large local amplitudes. Analyzing the general formula for $\langle | \psi^{2}(\boldsymbol{r})| \rangle$ in this case, we find [36] the following spatial structure of the ALS with $\left|\psi^{2}(0)\right|=u$ :

$$
\begin{align*}
& \langle | \psi^{2}(r)| \rangle_{u}=\frac{\pi^{3 / 2}}{16} u^{-1 / 2} A^{-3 / 2} r^{-3 / 2} \mathrm{e}^{-r / 4 \xi}, \quad r \gg \xi,  \tag{4.69}\\
& \langle | \psi^{2}(r)| \rangle_{u}=\frac{1}{2}\left(\frac{u}{\xi A}\right)^{1 / 2} \frac{1}{\left(1+r \sqrt{\frac{u A}{\xi}}\right)^{2}}, \quad l<r \ll \xi,  \tag{4.70}\\
& \langle | \psi^{2}(r)| \rangle_{u}=\frac{1}{2}\left(\frac{u}{\xi A}\right)^{1 / 2}\left[1+2 \sqrt{u A \xi} k_{d}(r)\right], \quad r<l . \tag{4.71}
\end{align*}
$$

We see from Eqs. (4.69), (4.70) and (4.71) that the eigenfunction normalization is dominated by the region $r \sim \xi_{\text {ef }}$, where $\xi_{\text {ef }} \sim \sqrt{\xi / u A} \ll \xi$ plays the role of an effective localization length. In the
region $\xi_{\text {ef }} \ll r \ll \xi$ the wave intensity falls down as $1 / r^{2}$, and crosses over to the conventional localization behavior at $r \gg \xi$. Therefore, the appearance of an anomalously high amplitude $\left|\psi^{2}(0)\right|=u \gg 1 / A \xi$ is not just a local fluctuation, but rather a kind of a cooperative phenomenon corresponding to existence of a whole region $r \lesssim \xi_{\text {ef }}$ with an unusually large amplitude $\left|\psi^{2}(r)\right|=\frac{1}{2} \sqrt{u / \xi A} \sim 1 / A \xi_{\mathrm{ef}}$.

Eq. (4.71) describes a sharp drop of the amplitude from $\left|\psi^{2}(0)\right|=u$ to $\langle | \psi^{2}(r)| \rangle=\frac{1}{2} \sqrt{u / \xi} A$ at $r \sim l$. This "quasi-jump" happens on a short scale $r_{0} \sim k_{\mathrm{F}}^{-1}(u A \xi)^{1 /[2(d-1)]}$. To understand the reason for it, let us recall that the above formulas represent the average intensity $\left|\psi^{2}(r)\right|$ (under the condition $\left.\left|\psi^{2}(0)\right|=u\right)$. One can also study the fluctuations of the intensity. It turns out [36] that in the region $r_{0} \ll r<\xi$ the fluctuations are of usual GUE type superimposed on the envelope (4.70). It is not difficult to understand that the quasi-jump has the same origin as the GUE-like fluctuations at $r \gg r_{0}$. One can ask, of course, why this short-scale fluctuation happens exactly in the center of the smooth ALS "bump" with a probability close to unity. The answer is as follows. We are studying the states with an anomalously large local intensity $u$, which is an exponentially rare event. There are two sources which may favor the formation of such a high intensity: (i) formation of an ALS with a spatially non-uniform smooth envelope, and (ii) short-scale GUE-like fluctuations. Both these mechanisms have exponentially small probabilities to produce an enhancement of the intensity by a large factor. The found configuration of $\langle | \psi^{2}(r)| \rangle_{u}$ (short-scale quasi-jump (4.71) on top of the smooth configuration (4.70)) represents just the optimal combination of the two mechanisms:

$$
\begin{equation*}
u=\left|\psi^{2}(0)\right|_{\text {smooth }} J(u) \equiv \frac{1}{2}\left(\frac{u}{\xi A}\right)^{1 / 2} 2(u A \xi)^{1 / 2} . \tag{4.72}
\end{equation*}
$$

For arbitrary geometry of the sample, the magnitude of the "quasi-jump" $J(u)$ is given according to Eq. (4.67) by

$$
\begin{equation*}
J(u) \simeq-u \frac{\mathrm{~d}}{\mathrm{~d} u} \ln Y_{a}(u) \simeq-u \frac{\mathrm{~d}}{\mathrm{~d} u} \ln \mathscr{P}(u) \tag{4.73}
\end{equation*}
$$

(in the quasi-1D case this reduces to $J(u)=2(u A \xi)^{1 / 2}$ as stated above). The formula (4.73) can be reproduced (within the saddle-point approximation) by writing the quantity $u$ as a product $u=u_{s} J$ of the smooth part $u_{s}$ and the local fluctuating quantity $J$, with the latter distributed according to $\mathscr{P}(J)=\mathrm{e}^{-J}$.
4.2.1.2. Metallic sample $(L \ll \xi)$. The asymptotic behavior of the intensity distribution function has the same stretched-exponential form, as in the localized regime, see Eq. (3.44). More accurately (with the subleading factors included), this formula reads [36]

$$
\begin{align*}
\mathscr{P}(u)= & \frac{16}{\pi^{2}} \sqrt{\frac{A \xi}{u}} \frac{\sqrt{L_{+} L_{-}}}{L_{-}} \exp \left\{-4 \sqrt{u \xi A}+\frac{\pi^{2} \xi}{4 L_{+}}\left(1-\frac{\sqrt{\xi / u A}}{L_{+}}+\cdots\right)\right. \\
& \left.+\frac{\pi^{2} \xi}{4 L_{-}}\left(1-\frac{\sqrt{\xi / u A}}{L_{-}}+\cdots\right)\right\} . \tag{4.74}
\end{align*}
$$

Calculation of $\mathscr{Q}(u, r)$ shows [36] that the ALS intensity has for $l<r \ll L$ the same form (4.70), provided the condition $V\left|\psi^{2}(0)\right| \gg g$ is fulfilled. This condition, under which the asymptotic
behavior (3.44), (4.74) is valid, acquires now a very transparent meaning. This is just the condition that the effective localization length of an ALS, $\xi_{\mathrm{ef}}=\sqrt{\xi / u A}$ is much less than the sample size $L$. Indeed, $\xi_{\text {ef }} / L=\sqrt{\xi / u A L^{2}}=\sqrt{g / u V}$.

Near the sample edges, $r \sim L \gtrdot \xi_{\text {ef }}$, the form of the ALS intensity is slightly modified by the boundary of the sample, see [36]. Finally, the "quasi-jump" of $\langle | \psi^{2}(r)| \rangle_{u}$ at $r \ll l$ has the same form (4.71) as in the insulating regime.
4.2.1.3. Saddle-point method. The saddle-point method of Muzykantskii and Khmelnitskii can be also applied to the problem of the statistics of eigenfunction amplitudes, as was done by Fal'ko and Efetov [32,33]. In this case, one should look for the saddle-point of the functional integral (3.8) determining the function $Y_{a}(u)$ (which in turn determines the eigenfunction statistics, see Eqs. (3.12) and (3.14)). The saddle-point is again parametrized by the bosonic non-compact angle $\theta(\boldsymbol{r})$ only, and the corresponding saddle-point equation has the form

$$
\begin{equation*}
\pi v D \nabla^{2} \theta-u \mathrm{e}^{\theta}=0 . \tag{4.75}
\end{equation*}
$$

It is similar to Eq. (4.8) of the long-time relaxation problem, but with different sign of the second term. Also, the boundary conditions have now a different form:

$$
\begin{equation*}
\theta(0)=0 \tag{4.76}
\end{equation*}
$$

and condition (4.7) at the boundary (since we consider now a closed sample). Alternatively, one can write Eqs. (4.75) and (4.76) in a slightly different form by shifting the variable $\theta+\ln (u V) \rightarrow \theta$. Then $u$ is removed from the saddle-point equation and from the action, but appears in the boundary condition.

The action determining the distribution function $\mathscr{P}(u)$ is given by

$$
\begin{equation*}
-\ln \mathscr{P}(u)=S=\int \mathrm{d}^{d} r\left[\frac{\pi v D}{2}(\nabla \theta)^{2}+u \mathrm{e}^{\theta}\right] . \tag{4.77}
\end{equation*}
$$

The formula (4.77) acquires a very transparent meaning if we take into account what was written in Section 4.2.1 concerning the two factors contributing to the large amplitude $\left|\psi^{2}(0)\right|=u$. Firstly, this is the non-uniform smooth envelope $\propto \mathrm{e}^{\theta(r)}$ yielding

$$
\left|\psi^{2}(0)\right|_{\text {smooth }}=\frac{\mathrm{e}^{\theta(0)}}{\int \mathrm{d}^{d} r \mathrm{e}^{\theta(r)}}=\frac{1}{\int \mathrm{~d}^{d} r \mathrm{e}^{\theta(r)}},
$$

the corresponding weight is represented by the first term in action (4.77). Secondly, these are the local Gaussian (GUE-like) fluctuations of the wave function amplitude, which should provide the remaining factor ("quasi-jump")

$$
J=\frac{u}{\left|\psi^{2}(0)\right|_{\text {smooth }}}=u \int \mathrm{~d}^{d} r \mathrm{e}^{\theta(r)},
$$

the corresponding probability $\mathscr{P}(J)=\mathrm{e}^{-J}$ reproduces the second term in action (4.77).

In the quasi-1D case and under the condition $u>g V^{-1}$, the solution of Eq. (4.75) reads [33,34]

$$
\begin{equation*}
\mathrm{e}^{\theta(r)}=\frac{1}{\left(1+r \sqrt{\frac{u}{2 \pi v D}}\right)^{2}}, \quad 0<r \ll L_{+} \tag{4.78}
\end{equation*}
$$

Comparing Eq. (4.78) with Eqs. (4.70), we see that the saddle-point solution nicely reproduces the average intensity of the ALS, $\langle | \psi^{2}(r)| \rangle_{u}$ for $r>l$, up to an overall normalization factor. Also, the form of $\mathscr{P}(u)$ found in the quasi-1D case by the saddle-point method [33] is in a very good agreement with the exact results presented above. This agreement does credit to the saddle-point method and allows to use the saddle-point configuration for characterizing the shape of ALS in higher dimensions and for other distribution functions, where the exact solution is not available.

### 4.2.2. 2D geometry

For a 2D disk-shaped sample of a radius $L$ with the high amplitude point $\boldsymbol{r}=0$ in the center of the disk, the saddle-point solution of Eqs. (4.75) and (4.76) is found to have the form $[32,33]$

$$
\begin{align*}
\mathrm{e}^{\theta(r)} & =\left(\frac{r}{l_{*}}\right)^{-2 \mu}\left\{1-\frac{l_{*}^{2} u}{8(1-\mu)^{2} \pi v D}\left(\frac{r}{l_{*}}\right)^{2-2 \mu}\right\}^{-2}, \quad r \geq l_{*} \\
& \approx\left(\frac{r}{l_{*}}\right)^{-2 \mu} \text { for } l_{*} \leq r \ll L, \tag{4.79}
\end{align*}
$$

where the exponent $0<\mu<1$ depends on $u$ and satisfies the equation

$$
\begin{equation*}
\left(\frac{L}{l_{*}}\right)^{2 \mu}=\frac{2-\mu}{8 \mu(1-\mu)^{2}} \frac{L^{2} u}{\pi v D} . \tag{4.80}
\end{equation*}
$$

We are interested in the asymptotic region $u L^{2} \gg \tau \nu D \ln ^{-1}(L / l)$, where the distribution of the eigenfunction intensity is given by Eq. (3.58), and an ALS is formed. Then the exponent $\mu$ can be approximated as

$$
\begin{equation*}
\mu \simeq \frac{\ln \left(\frac{L^{2} u}{2 \pi v D} \ln \frac{L}{l_{*}}\right)}{2 \ln \left(L / l_{*}\right)} . \tag{4.81}
\end{equation*}
$$

The lower cut-off scale $l_{*}$ appears in Eq. (4.79) for the same reason as in the long-time relaxation problem (Section 4.1.2), i.e. because of the restriction of the diffusion approximation on the momenta $q$ of the $\sigma$-model field: $q<l^{-1}$. It is determined by the condition $\left.\theta\left({ }^{\prime} r\right)\right|_{r=l_{*}} \sim l^{-1}$, which yields $l_{*} \sim \mu l$. The corresponding asymptotic behavior of $Y_{a}(u)$ (and consequently of the intensity distribution function), which was already quoted in Section 3.3.1, is

$$
\begin{equation*}
Y_{a}(u), \mathscr{P}(u) \sim \exp \left\{-\pi^{2} v D \frac{\ln ^{2}\left(\frac{V u}{2 \pi^{2} v D} \ln \frac{L}{l_{*}}\right)}{\ln \left(L / l_{*}\right)}\right\} \tag{4.82}
\end{equation*}
$$

Normalizing the expression (4.79), we find that the average ALS density for $r>l_{*}$ is equal to

$$
\begin{equation*}
\langle | \psi^{2}(r)| \rangle_{u}=\frac{u}{4 \pi^{2} v D \mu}\left(\frac{r}{l_{*}}\right)^{-2 \mu}\left\{1-\frac{l_{*}^{2} u}{8(1-\mu)^{2} \pi \nu D}\left(\frac{r}{l_{*}}\right)^{2-2 \mu}\right\}^{-2}, \quad r \geq l_{*} . \tag{4.83}
\end{equation*}
$$

The saddle-point calculation assumes that $\theta(r)$ is constant for $r<l_{*}$, so that Eq. (4.83) gives

$$
\langle | \psi^{2}(r)| \rangle_{u} \simeq \frac{u}{4 \pi^{2} v D \mu}
$$

in this region. However, for very small $r<l$ the average intensity $\langle | \psi^{2}(r)| \rangle_{u}$ changes sharply, as we have seen in the quasi-1D case. Using Eq. (4.67) for $\mathscr{Z}(u, r)$ and Eq. (3.12) for $\mathscr{P}(u)$, we get

$$
\begin{equation*}
\langle | \psi^{2}(r)| \rangle_{u} \equiv \frac{\mathscr{Q}(u, r)}{\mathscr{P}(u)}=\left[1-k_{2}(r)+J(u) k_{2}(r)\right]\langle | \psi^{2}\left(r=l_{*}\right)| \rangle_{u}, \quad r<l_{*} \tag{4.84}
\end{equation*}
$$

According to Eqs. (4.73) and (4.82), the height of the quasi-jump is given by

$$
\begin{equation*}
J(u) \simeq-u \frac{\mathrm{~d}}{\mathrm{~d} u} \ln Y_{a}(u) \simeq \frac{2 \pi^{2} v D}{\ln \left(L / l_{*}\right)} \ln \left(\frac{V u}{2 \pi^{2} v D} \ln \frac{L}{l_{*}}\right) \simeq 4 \pi^{2} v D \mu \tag{4.85}
\end{equation*}
$$

which is precisely the factor by which the value of $\langle | \psi^{2}\left(r=l_{*}\right)| \rangle_{u}$ found above differs from $\langle | \psi^{2}(0)| \rangle_{u} \equiv u$. Combining Eqs. (4.83)-(4.85), we get

$$
\begin{equation*}
\langle | \psi^{2}(r)| \rangle_{u}=\frac{u}{J(u)}\left[1-k_{2}(r)+k_{2}(r) J(u)\right], \quad r<l_{*} \tag{4.86}
\end{equation*}
$$

Therefore, in the 2 D case the ALS determining the asymptotics of the amplitude distribution function has the power-law shape (4.83) with the short-scale bump (4.86).

### 4.2.3. States localized near the boundary

We assumed in the above calculations that the center of an ALS is located far enough from the sample edge. For a quasi-1D sample, this means that $\xi_{\text {ef }} \ll L_{+}, L_{-}$. In the 2 D case this implies that the distance from the observation point to the boundary is of the same order of magnitude in all directions, so that $\ln (L / l)$ is defined without ambiguity. Here, we will consider briefly the role of ALS situated close to the boundary, when these conditions are violated [36].

We first consider the quasi-1D geometry. Let us calculate the distribution function $\mathscr{P}(u)$ in a point located very close to one of the sample edges. Formally, this means that $L_{-} \ll \xi_{\text {ef }}$. Then the function $W^{(1)}\left(u A \xi, \tau_{-}\right)$in Eq. (3.21) can be approximated by unity, and we get

$$
\begin{equation*}
\mathscr{P}(u)=\frac{2}{\pi} \xi^{3 / 4} A^{1 / 4} L^{-1 / 2} u^{-3 / 4} \exp \left\{-2 \sqrt{u \xi A}+\frac{\pi^{2} \xi}{4 L_{+}}\left(1-\frac{\sqrt{\xi / u A}}{L_{+}}+\cdots\right)\right\} . \tag{4.87}
\end{equation*}
$$

We see therefore that close to the boundary the distribution $\mathscr{P}(u)$ has the asymptotic decay $\mathscr{P}(u) \sim \exp \{-2 \sqrt{u A \xi}\}$, which is slower than in the bulk of the sample, $\mathscr{P}(u) \sim \exp \{-4 \sqrt{u A \xi}\}$. This means that if we consider the distribution $\mathscr{P}(u)$ averaged over the position of the observation point, its asymptotic tail will be always dominated by contribution of the points located close to the
boundary, $\mathscr{P}(u) \sim \exp \{-2 \sqrt{u A \xi}\}$. This could be already anticipated from Eq. (4.74), where the factor

$$
\exp \left\{\frac{\pi^{2}}{4}\left(\frac{\xi}{L_{+}}+\frac{\xi}{L_{-}}\right)\right\}
$$

increases strongly with approaching one of the sample edges. The same tendency, but in a weaker form, is observed in Eqs. (3.40)-(3.43). Calculating the average intensity $\langle | \psi^{2}(r)| \rangle_{u}$ of the corresponding ALS, we find that at $r>l$ the ALS spatial shape retain the form (4.70), with an additional overall factor of 2. At small $r$, Eq. (4.71) is slightly modified:

$$
\begin{equation*}
\langle | \psi^{2}(r)| \rangle_{u}=\left(\frac{u}{\xi A}\right)^{1 / 2}\left[1+\sqrt{u A \xi} k_{d}(r)\right] . \tag{4.88}
\end{equation*}
$$

In 2D, we can consider a sample of the semicircular shape, with the observation point located in the center of the diameter serving as a boundary. The saddle-point solution then has exactly the same form (4.79), and the ALS intensity is still given by Eq. (4.83), with an additional factor 2. The asymptotic form of the distribution function $\mathscr{P}(u)$ contains an extra factor $1 / 2$ in the exponent:

$$
\begin{equation*}
\mathscr{P}(u) \sim \exp \left\{-\frac{\pi^{2} v D}{2} \frac{\ln ^{2}\left(\frac{V u}{2 \pi^{2} v D} \ln \frac{L}{l_{*}}\right)}{\ln \left(L / l_{*}\right)}\right\} \tag{4.89}
\end{equation*}
$$

This result is expected to be applicable to any 2D sample of a characteristic size $L$, with a smooth boundary and the observation point taken in the vicinity of the boundary.

We see therefore, that, very generally, the probability of formation of an ALS with the center in a given point is strongly enhanced (via an extra factor $1 / 2$ in the exponent), if this point lies close to the sample edge. This leads to the additional factor $1 / 2$ in the exponent in the asymptotic form of the distribution $\mathscr{P}(u)$ near the boundary.

### 4.3. Distribution of local density of states

We again assume the sample to be open, as in the problem of the distribution of relaxation times, Section 4.1. Then it is meaningful to speak about the statistics of the local density of states (LDOS) $\rho(E, \boldsymbol{r})=(-1 / \pi) \operatorname{Im} G_{\mathbf{R}}(\boldsymbol{r}, \boldsymbol{r} ; E)$. In a metallic sample, the LDOS is a weakly fluctuating quantity, whose distribution $\mathscr{P}(\rho)$ is mostly concentrated in a narrow Gaussian peak [28,155] with mean value $\langle\rho\rangle=v$ and the $\operatorname{variance} \operatorname{var}(\rho / v) \sim \kappa \ll 1$, where $\kappa$ is the usual parameter of the perturbation theory $\kappa=\Pi(\boldsymbol{r}, \boldsymbol{r})=\sum_{q} 1 / \pi v V D q^{2}$, which already appeared in Section 3.3.1. This close-toGaussian shape of $\mathscr{P}(\rho)$ holds however in the region $|\rho / v-1| \ll 1$ only. We will consider in contrast the "tails" of the distribution, $\rho / v \ll 1$ and $\rho / v \gg 1$, where a much slower decay of $\mathscr{P}(\rho)$ will be found.

We begin by discussing on a qualitative level a relation of the asymptotics of $\mathscr{P}(\rho)$ to the behavior of the distribution functions considered above. Typically, in an open metallic sample the LDOS $\rho(E, r)$ is given by a superposition of $\sim 1 / t_{D} \Delta=g$ adjacent levels, since their widths are of order of $1 / t_{D}$. However, we can expect that for $\rho$ much larger than its average value $v$, the asymptotic form
of $\mathscr{P}(\rho)$ is determined by a probability to have a single narrow resonance, which gives this value of LDOS $\rho(E, r)$. The most favorable situation happens when the resonance is located around the point $\boldsymbol{r}$ in the real space and around the energy $E$ in energy space. The LDOS provided by such a resonance is:

$$
\begin{equation*}
\rho_{\mathrm{ALS}}=\left|\psi^{2}(r)\right| 2 t_{\phi} / \pi, \tag{4.90}
\end{equation*}
$$

where $t_{\phi}^{-1}$ is the resonance width. Thus, the optimal fluctuation should provide now a maximum to the product of the local amplitude $u=\left|\psi^{2}(\boldsymbol{r})\right|$ and the inverse level width $t_{\phi}$, and the asymptotics of the distribution $\mathscr{P}(\rho)$ should be related to that of $\mathscr{P}(u)$ and $\mathscr{P}\left(t_{\phi}\right)$. In particular, in the quasi-1D case, where the distribution $\mathscr{P}\left(t_{\phi}\right)$, Eq. (4.25), decays much more slowly than $\mathscr{P}(u)$, Eq. (3.47), one should expect the asymptotic behavior of $\mathscr{P}(\rho)$ to be mainly determined by $\mathscr{P}\left(t_{\phi}\right)$. We will see below that this is indeed the case.

Now we turn to a formal calculation. The distribution function $\mathscr{P}(\rho)$ of LDOS can be expressed through the function $Y\left(\lambda_{1}, \lambda_{2}\right)$ introduced in Section 3.1 as follows [19,20,322,323].

$$
\begin{equation*}
\mathscr{P}(\rho)=\delta(\rho-1)+\frac{1}{4 \pi} \frac{\partial^{2}}{\partial \rho^{2}}\left\{\int_{\left(\rho^{2}+1\right) / 2 \rho}^{\infty} \mathrm{d} \lambda_{1} \bar{Y}\left(\lambda_{1}\right)\left(\frac{2 \rho}{\lambda_{1}-\frac{\rho^{2}+1}{2 \rho}}\right)^{1 / 2}\right\}, \tag{4.91}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{Y}\left(\lambda_{1}\right)=\int_{-1}^{1} \mathrm{~d} \lambda_{2} \frac{Y\left(\lambda_{1}, \lambda_{2}\right)}{\lambda_{1}-\lambda_{2}} \tag{4.92}
\end{equation*}
$$

and $\rho$ is normalized by its mean value: $\rho / v \rightarrow \rho$. Let us note the symmetry relation found in [19,20]

$$
\begin{equation*}
\mathscr{P}\left(\rho^{-1}\right)=\rho^{3} \mathscr{P}(\rho) . \tag{4.93}
\end{equation*}
$$

It follows from Eq. (4.91) and is completely independent of a particular form of the function $Y\left(\lambda_{1}, \lambda_{2}\right)$. Obviously, Eq. (4.93) relates the small- $\rho$ asymptotic behavior of the distribution $\mathscr{P}(\rho)$ to its large- $\rho$ asymptotics.

Asymptotic behavior of $\mathscr{P}(\rho)$ was studied in [34] via the saddle-point method supplemented in the quasi-1D case by the exact solution. The saddle-point equation has for this problem a very simple form,

$$
\nabla^{2} \theta=0,
$$

with the boundary conditions

$$
\left.\theta\right|_{\text {leads }}=0, \quad \theta(0)=\rho / 2 .
$$

### 4.3.1. Quasi-1D geometry

In the quasi-1D case, the solution reads

$$
\mathrm{e}^{\theta(r)} \simeq \begin{cases}(\rho / v)^{1-r / L_{+}}, & r>0,  \tag{4.94}\\ (\rho / v)^{1-|r| / L_{-}}, & r<0,\end{cases}
$$

where, as before, $L_{+}$and $L_{-}$are the distances from the observation point $r=0$ to the sample edges. This yields the asymptotics of the distribution function

$$
\begin{equation*}
\mathscr{P}(\rho) \sim \exp \left\{-\frac{\xi}{4}\left(\frac{1}{L_{+}}+\frac{1}{L_{-}}\right) \ln ^{2}(\rho / v)\right\} . \tag{4.95}
\end{equation*}
$$

Let us note that this asymptotic behavior of $\mathscr{P}(\rho)$ in an open sample is strongly different from the asymptotics of $\mathscr{P}(u)$ in a closed sample, Eq. (3.47). As was explained above, the difference originates from the fact that $\mathscr{P}(\rho)$ is essentially determined by $\mathscr{P}\left(t_{\phi}\right)$. To demonstrate this explicitly, we put the observation point in the middle of the sample, $L_{+}=L_{-}=L / 2$. The configuration (4.94) acquires then precisely the same form as the optimal configuration (4.19) for the relaxation time $t_{\phi}$, and the asymptotics (4.95) of $\mathscr{P}(\rho)$ is identical to that of $\mathscr{P}\left(t_{\phi}\right)$, Eq. (4.25).

The corresponding values of $t_{\phi}$ and $\rho$ are related as follows:

$$
\begin{equation*}
\frac{4}{\pi} g \Delta t_{\phi} \ln ^{2}\left(g \Delta t_{\phi}\right)=\rho / v . \tag{4.96}
\end{equation*}
$$

Now we calculate the value of the local amplitude $\left|\psi^{2}(0)\right|$ for an ALS corresponding to the configuration (4.94). First, its smoothed intensity is given by

$$
\begin{equation*}
\left|\psi^{2}(r)\right|_{\text {smooth }}=\mathcal{N}^{-1} \mathrm{e}^{\theta(r)}=\frac{\ln (\rho / v)}{V}\left(\frac{\rho}{v}\right)^{-2|r| / L} . \tag{4.97}
\end{equation*}
$$

Second, the quasi-jump induced by the GUE-type fluctuations gives an additional factor, which is found according to Eq. (4.73) to be

$$
\begin{equation*}
J(\rho)=-\rho \frac{\partial}{\partial \rho} \ln \mathscr{P}(\rho)=2 g \ln (\rho / v) . \tag{4.98}
\end{equation*}
$$

Combining Eqs. (4.96)-(4.98), we can compute the LDOS (4.90) determined by this resonance state:

$$
\begin{equation*}
\rho_{\mathrm{ALS}}(E, 0)=\left|\psi^{2}(0)\right|_{\text {smooth }} J(\rho) \frac{2 t_{\phi}}{\pi}=\frac{\ln (\rho / v)}{V} 2 g \ln (\rho / v) \frac{\rho V}{2 g \ln ^{2}(\rho / v)}=\rho . \tag{4.99}
\end{equation*}
$$

We have explicitly checked therefore that the LDOS $\rho$ is indeed determined by a single ALS, smoothed intensity of which is given by Eq. (4.97). There are three sources of the enhancement of LDOS: (i) amplitude of the smooth envelope of the wave function, (ii) the short-scale GUE "bump", and (iii) the inverse resonance width. They are represented by the three factors in Eq. (4.99), respectively.

The result (4.95) can be also obtained [34] from the exact solution of the $\sigma$-model. It was also shown in [34] that in the long wire limit $(L \gg \xi)$ the whole distribution function takes a log-normal form analogous to that found in [113] for the case of a strictly 1D sample by the Berezinskii technique.

### 4.3.2. 2D geometry

In the 2D case, we introduce again (as in Sections 4.1.2 and 4.2.2) the small- $r$ cutoff, $l_{*}^{(\rho)}$. The saddle-point solution reads

$$
\begin{equation*}
\mathrm{e}^{\theta(r)} \simeq \frac{\rho}{v}\left(\frac{l_{*}^{(\rho)}}{r}\right)^{\gamma_{\rho}} \tag{4.100}
\end{equation*}
$$

where $l_{*}^{(\rho)}=\gamma_{\rho} l$, and

$$
\gamma_{\rho}=\ln (\rho / v) / \ln \left(L / l_{*}^{(\rho)}\right) .
$$

The distribution $\mathscr{P}(\rho)$ has the following asymptotics [34]:

$$
\begin{equation*}
\mathscr{P}(\rho) \sim \exp \left\{-\frac{\pi^{2} v D \ln ^{2} \rho}{\ln \left(L / l_{*}^{(\rho)}\right)}\right\} . \tag{4.101}
\end{equation*}
$$

So, the LDOS distribution has (as in the quasi-1D case) the log-normal form, which is now similar to the distribution (4.82) of the eigenfunction intensities (while the distribution of relaxation times has an intermediate power-law regime). This result is in perfect agreement with the asymptotic behavior of $\mathscr{P}(\rho)$ found by the renormalization group method in [28].

As for all the other distribution functions studied, the relevant ALS decay in a power-law fashion, $\left|\psi^{2}(r)\right| \propto\left(r / l_{*}^{(\rho)}\right)^{-\gamma_{0}}$. Like in the quasi-1D case, one can explicitly verify [36] that a single state with the spatial shape determined by (4.100) indeed provides, by virtue of Eq. (4.90), the value of LDOS equal to $\rho$.

### 4.4. Distribution of inverse participation ratio

In this subsection, we study the asymptotics of the distribution function of the IPR $P_{2}$, Eq. (3.39). We have already considered the fluctuations of $P_{2}$ in Section 3.3.3. As was explained there, the relative magnitude of the fluctuations is [r.m.s. $\left.\left(P_{2}\right)\right] /\left\langle P_{2}\right\rangle \sim 1 / g$. At $1 / g \ll P_{2} /\left\langle P_{2}\right\rangle-1 \ll 1$ the distribution function is of the exponential form,

$$
\begin{equation*}
\mathscr{P}\left(P_{2}\right) \sim \exp \left\{-\frac{\pi \beta}{4} \frac{\varepsilon_{1}}{\Delta}\left(\frac{P_{2}}{\left\langle P_{2}\right\rangle}-1\right)\right\} . \tag{4.102}
\end{equation*}
$$

Note that for negative deviations $P_{2} /\left\langle P_{2}\right\rangle-1$ with $\left|P_{2} /\left\langle P_{2}\right\rangle-1\right| \gg 1 / g$ the distribution function decays much faster [137], so that the distribution is strongly asymmetric. As was mentioned in Section 3.3.2, the "body" of the distribution $\mathscr{P}\left(P_{2}\right)$ is described properly (in the leading order in $1 / g$ ) by the Liouville theory. We will see below that this is also true for the asymptotic "tail" of $\mathscr{P}\left(P_{2}\right)$.

Our consideration of asymptotics of the IPR distribution is based on unpublished results [35] (partially announced in [36]). We derive first a relation between $\mathscr{P}\left(P_{2}\right)$ and the distribution of level velocities $\mathscr{P}_{v}(v)$ [25]. To this end, we consider a Hamiltonian $\mathscr{H}+\alpha \mathscr{W}$, where $\mathscr{W}$ is a random perturbation. Specifically, the matrix elements $\mathscr{W}_{r_{1} r_{2}}$ are supposed to be independent Gaussiandistributed random variables with the mean value equal to zero and the variance

$$
\left\langle\mathscr{W}_{\boldsymbol{r}_{1} \boldsymbol{r}_{2}}^{*} \mathscr{W}_{r_{2}^{\prime} r_{1}^{\prime}}^{\prime}\right\rangle=\mathscr{W}_{0}\left(\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right|\right) \delta\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{1}^{\prime}\right) \delta\left(\boldsymbol{r}_{2}-\boldsymbol{r}_{2}^{\prime}\right) .
$$

We will assume that $\mathscr{W}_{0}(r)$ is a short-ranged function with some characteristic scale $\zeta$. The level velocity $v_{n}$ corresponding to an energy level $E_{n}$ is defined as $v_{n}=\mathrm{d} E_{n}(\alpha) / \mathrm{d} \alpha$ (where $E_{n}(\alpha)$ is the level of the perturbed Hamiltonian $\mathscr{H}+\alpha \mathscr{W})$ and can be found within the conventional perturbation theory as

$$
\begin{equation*}
v_{n}=\int \mathrm{d}^{d} r \int \mathrm{~d}^{d} r^{\prime} \mathscr{W}_{r r^{\prime}} \psi_{n}^{*}(\boldsymbol{r}) \psi_{n}\left(\boldsymbol{r}^{\prime}\right) \tag{4.103}
\end{equation*}
$$

Using (4.103), we find

$$
\begin{equation*}
\left\langle\overline{v_{n}^{2}}\right\rangle=w_{0} P_{2}, \quad w_{0}=c \int \mathrm{~d}^{d} r \mathscr{W}_{0}(\boldsymbol{r}), \tag{4.104}
\end{equation*}
$$

where $c=1(1 / 2)$ if $k_{f} \zeta \ll 1$ (resp. $k_{f} \zeta \gg 1$ ).
This consideration can be extended to higher moments of the level velocity $\left\langle\overline{v_{n}^{2 q}}\right\rangle$ as well. This leads to the following relation between the two distributions:

$$
\begin{align*}
\mathscr{P}_{v}(v) & \equiv\left\langle\overline{\delta\left(v-v_{n}\right)}\right\rangle \\
& =\int_{0}^{\infty} \frac{\mathrm{d} P_{2}}{\left[2 \pi w_{0} P_{2}\right]^{1 / 2}} \exp \left[-\frac{v^{2}}{2 w_{0} P_{2}}\right] \mathscr{P}_{I}\left(P_{2}\right) . \tag{4.105}
\end{align*}
$$

On the other hand, the level velocity distribution can be expressed through the $\sigma$-model correlation function in the following way [156]. According to the definition,

$$
\begin{align*}
\mathscr{P}_{v}(v) & =\left.\frac{1}{v V}\left\langle\sum_{n} \delta\left(E-E_{n}\right) \delta\left(v-\partial E_{n} / \partial \alpha\right)\right\rangle\right|_{\alpha \rightarrow 0} \\
& =\lim _{\alpha \rightarrow 0} \frac{\alpha}{v V}\left\langle\sum_{n} \delta\left(E-E_{n}(0)\right) \delta\left(E+\alpha v-E_{n}(\alpha)\right)\right\rangle, \tag{4.106}
\end{align*}
$$

so that $\mathscr{P}_{v}(v)$ is determined by the $\alpha \rightarrow 0$ limit of the parametric level correlation function. The latter can be represented [as a generalization of Eq. (2.10)] in terms of a correlation function of the $\sigma$-model [157-159]. This yields

$$
\begin{align*}
& \mathscr{P}_{v}(v)=\lim _{\eta \rightarrow 0} \frac{-1}{8 V^{2}} \frac{\eta}{v V v} \int D Q\left(\int \mathrm{~d}^{d} r \operatorname{Str} Q_{11} k\right)\left(\int \mathrm{d}^{d} r \operatorname{Str} Q_{22} k\right) \mathrm{e}^{-S_{v}[Q]} ; \\
& S_{v}[Q]=\int \mathrm{d}^{d} r \operatorname{Str}\left[-\frac{\pi v D}{4}(\nabla Q)^{2}-\frac{\mathrm{i} \pi v \eta}{2} Q \Lambda+\frac{w_{0}(\pi v)^{2} \eta^{2}}{4 v^{2}} Q \Lambda Q \Lambda\right] . \tag{4.107}
\end{align*}
$$

Combining Eqs. (4.105) and (4.107), we find the expression for the IPR distribution function in terms of the $\sigma$-model,

$$
\begin{align*}
\mathscr{P}_{I}\left(P_{2}\right)= & \frac{-1}{8 V^{2}} \frac{1}{2 \mathrm{i} \sqrt{2 \pi} P_{2}^{3 / 2}} \lim _{\eta \rightarrow 0} \eta v V \int_{c-\mathrm{i} \infty}^{c+\mathrm{i} \infty} \frac{\mathrm{~d} u}{\sqrt{u}} \\
& \times \int D Q\left(\int \mathrm{~d}^{d} r \operatorname{Str} Q_{11} k\right)\left(\int_{\mathrm{d}^{d} r} \operatorname{Str} Q_{22} k\right) \mathrm{e}^{-S_{u}[\varrho]} \tag{4.108}
\end{align*}
$$

where $S_{u}[Q]=-u / 2 P_{2}+\left.S_{v}[Q]\right|_{v^{2} / w_{0}=u}$. The saddle point configuration is again parametrized by the bosonic "angle" $\theta(\boldsymbol{r})$ only; the action on such a configuration is

$$
\begin{equation*}
S_{u}[\theta]=-\frac{u}{2 P_{2}}+\int \mathrm{d}^{d} r\left[\frac{\pi v D}{2}(\nabla \theta)^{2}-\frac{\mathrm{i} \pi v \eta}{2} \mathrm{e}^{\theta}+\frac{(\pi v \eta)^{2}}{4 u} \mathrm{e}^{2 \theta}\right] . \tag{4.109}
\end{equation*}
$$

The corresponding saddle-point equations can be readily obtained by varying $S_{u}[\theta]$ with respect to $\theta(\boldsymbol{r})$ and $u$ :

$$
\begin{align*}
& -D \nabla^{2} \theta-\frac{\mathrm{i} \eta}{2} \mathrm{e}^{\theta}+\frac{\pi v \eta^{2}}{2 u} \mathrm{e}^{2 \theta}=0, \\
& \frac{(\pi v \eta)^{2}}{4 u^{2}} \int \mathrm{~d}^{d} r \mathrm{e}^{2 \theta}+\frac{1}{2 P_{2}}=0 . \tag{4.110}
\end{align*}
$$

Making a shift $\theta=\tilde{\theta}+\ln (i u / \pi v \eta)$ and dropping the tilde, we can reduce them to the form

$$
\begin{align*}
& \nabla^{2} \theta-\gamma\left(\mathrm{e}^{\theta}-\mathrm{e}^{2 \theta}\right)=0,  \tag{4.111}\\
& P_{2}=2 / \int \mathrm{d}^{d} r \mathrm{e}^{2 \theta}, \tag{4.112}
\end{align*}
$$

where $\gamma=u / 2 \pi v D$. Integration of Eq. (4.111) with the Neumann boundary condition yields $\int \mathrm{d}^{d} r\left(\mathrm{e}^{\theta}-\mathrm{e}^{2 \theta}\right)=0$, so that Eq. (4.112) can be rewritten in the following form (invariant with respect to a shift of the variable $\theta$ ):

$$
\begin{equation*}
P_{2}=2 \frac{\int \mathrm{~d}^{d} r \mathrm{e}^{2 \theta}}{\left(\int \mathrm{~d}^{d} r \mathrm{e}^{\theta}\right)^{2}} . \tag{4.113}
\end{equation*}
$$

The meaning of Eq. (4.113) is completely transparent if we recall that $\left|\psi^{2}(r)\right|_{\text {smoth }} \propto \mathrm{e}^{\theta(r)}$. The factor 2 comes from the GUE-like short-scale fluctuations.

Taking into account Eqs. (4.111) and (4.112), we find the action (4.109) on the saddle-point configuration to be equal to

$$
\begin{equation*}
S_{u}[\theta]=\frac{\pi v D}{2} \int \mathrm{~d}^{d} r(\nabla \theta)^{2} . \tag{4.114}
\end{equation*}
$$

The problem that we are solving is easily seen to be equivalent to searching for the minimum of (4.114) under the conditions $\int \mathrm{e}^{\theta} \mathrm{d}^{d} r=1, \int \mathrm{e}^{2 \theta} \mathrm{~d}^{d} r=P_{2}$ (or, equivalently, under the condition (4.113) invariant with respect to normalization of $\mathrm{e}^{\theta}$ ). This is nothing else but the optimum fluctuation problem for $\mathscr{P}\left(P_{2}\right)$ within the Liouville theory (3.93). Therefore, the Liouville theory (3.93) describes properly the asymptotics of the IPR distribution.

### 4.4.1. Quasi-1D geometry

## Equation

$$
\begin{equation*}
\theta^{\prime \prime}-\gamma \mathrm{e}^{\theta}+\gamma \mathrm{e}^{2 \theta}=0 \tag{4.115}
\end{equation*}
$$

has the following general solution:

$$
\begin{equation*}
\mathrm{e}^{-\theta}=\frac{\gamma}{C_{1}}\left[1+\sqrt{1-C_{1} / \gamma} \sin \left(\sqrt{C_{1}} x-C_{2}\right)\right] . \tag{4.116}
\end{equation*}
$$

The constants $C_{1}$ and $C_{2}$ should be found from the boundary conditions

$$
\theta^{\prime}(-L / 2)=\theta^{\prime}(L / 2)=0,
$$

yielding

$$
\cos \left(\sqrt{C_{1}} \frac{L}{2} \pm C_{2}\right)=0
$$

The solution providing the minimum to the action corresponds to $C_{2}=0, \sqrt{C_{1}}=\pi / L$ and gives the wave function intensity

$$
\begin{equation*}
A\left|\psi^{2}(r)\right|_{\text {smooth }}=\frac{\mathrm{e}^{\theta(x)}}{\int \mathrm{e}^{\theta(x)} \mathrm{d} x}=\frac{\pi}{\gamma^{1 / 2} L^{2}} \frac{1}{1+\sqrt{1-\pi^{2} / L^{2} \gamma} \sin (\pi x / L)} \tag{4.117}
\end{equation*}
$$

(with $A$ being the sample transverse cross-section) and the IPR value

$$
\begin{equation*}
P_{2}=\frac{2 \gamma^{1 / 2}}{A \pi}=P_{2}^{G U E} \frac{\gamma^{1 / 2} L}{\pi} . \tag{4.118}
\end{equation*}
$$

Calculating action (4.114) we find the following asymptotic behavior of the IPR distribution function

$$
\begin{equation*}
-\ln \mathscr{P}\left(P_{2}\right) \simeq S_{u}=\frac{\pi^{3} v D A^{2}}{4}\left(P_{2}-\frac{2}{L A}\right)=\frac{\pi^{2}}{4} g\left(\frac{P_{2}}{P_{2}^{\mathrm{GUE}}}-1\right) . \tag{4.119}
\end{equation*}
$$

Eq. (4.119) is valid for $1 / g \ll P_{2} / P_{2}^{G U E}-1 \ll L / l$. Therefore, in the quasi-1D case the exponential behavior (4.102) is not restricted to the region of small deviations from the average value; there is no change of the behavior of $\mathscr{P}\left(P_{2}\right)$ at $P_{2} / P_{2}^{G U E}-1 \sim 1$ (we will find such a change below in the 2D geometry case).

The far asymptotics at $P_{2} \gg P_{2}^{\text {GUE }}$ is determined by the states, which have an effective localization length $\xi_{\text {ef }}$ much smaller than the sample length $L$. The shape of these states is according to (4.117)

$$
\begin{equation*}
A\left|\psi^{2}(\boldsymbol{r})\right|_{\text {smooth }}=\frac{2}{\pi} \frac{\xi_{\mathrm{ef}}}{\pi(x+L / 2)^{2}+\xi_{\mathrm{ef}}^{2}}, \quad \xi_{\mathrm{ef}}=\frac{2}{\pi A P_{2}} . \tag{4.120}
\end{equation*}
$$

Let us note that the density of such a state is concentrated near the edge ( $x=-L / 2$ ) of the sample (of course, there exists an equivalent solution located near the opposite edge, $x=L / 2$ ). Since the shape of a state with $\xi_{\text {ef }} \ll L$ and the corresponding action $S_{u} \simeq \pi^{3} v D A^{2} P_{2} / 4$ are only weakly affected by the sample length $L$, we may expect such states to determine the asymptotic behavior of $\mathscr{P}_{I}\left(P_{2}\right)$ also in the localized regime, $L \gg \xi$. However, if we compare the action found above with the asymptotics of $\mathscr{P}_{I}\left(P_{2}\right)$ at $P_{2} \gg 1 / \xi A$ in an infinitely long sample [first line of Eq. (3.49)], we find that the absolute value of the exponent in the latter case is larger by factor of 4 . The explanation is as follows. The states determining the large- $P_{2}$ asymptotics in the limit $L \rightarrow \infty$ can be obtained
from Eq. (4.116) by taking a solution which has a maximum in the middle of the sample ( $C_{2}=\pi / 2, \sqrt{C_{1}}=2 \pi / L$ ); the result is

$$
\begin{equation*}
A\left|\psi^{2}(\boldsymbol{r})\right|_{\text {smooth }}=\frac{1}{\pi} \frac{\xi_{\mathrm{ef}}}{x^{2}+\xi_{\mathrm{ef}}^{2}}, \quad \xi_{\mathrm{ef}}=\frac{1}{\pi A P_{2}} . \tag{4.121}
\end{equation*}
$$

The only difference between (4.120) and (4.121) is that ALS is now located in the bulk of the sample. This leads to an extra factor 4 in the action (see similar discussion in Section 4.2.3 for the case of the statistics of eigenfunction amplitudes). However, in the limit of a long sample, $L \xi \gg 1$ the contribution of the states located near the boundary is additionally suppressed by a factor $\sim 1 / L$ as compared to that of the bulk ALS (which may be located everywhere in the sample). Therefore, if one fixes $P_{2}$ and considers $\mathscr{P}_{I}\left(P_{2}\right)$ in the limit $L \rightarrow \infty$, only the contribution of the bulk ALS survives, despite the fact that it has the exponent 4 times larger than that of the ALS located near the boundary. In other words, the contribution of the states located near the boundary to the first line of Eq. (3.49) (large- $P_{2}$ asymptotics of $\mathscr{P}_{I}\left(P_{2}\right)$ at $\left.X=L / \xi \gg 1\right)$ is $\propto(1 / X) \mathrm{e}^{-\pi^{2} z / 4}$.

On the other hand, if a sample with periodic boundary conditions in the longitudinal direction (a ring) is considered, only the bulk solution ( $C_{2}=\pi / 2, \sqrt{C_{1}}=2 \pi / L$ ) will survive. Consequently, the asymptotic form of $\ln \mathscr{P}\left(P_{2}\right)$ will be different from (4.119) by an extra factor of 4.

### 4.4.2. 2D geometry

Now we calculate the far asymptotics of $\mathscr{P}\left(P_{2}\right)$ at $P_{2} \gg P_{2}^{G U E}$. We assume first the periodic boundary conditions (so that the sample has no boundary) and search for a rotationally invariant solution, which should satisfy the differential equation

$$
\begin{equation*}
\theta^{\prime \prime}+\frac{1}{r} \theta^{\prime}-\gamma\left(\mathrm{e}^{\theta}-\mathrm{e}^{2 \theta}\right)=0 . \tag{4.122}
\end{equation*}
$$

(In fact, for the hard-wall boundary conditions the asymptotics is determined by the states located near the boundary; however, such states can be obtained from the symmetric solution by putting the center at the boundary and restricting the solution to the interior of the sample; see below.) From our experience in the quasi-1D case, we expect the solution to have a form of a bump concentrated in a region $r \leqq l_{p}$ and decreasing with $r$ outside this region. For $r \gg l_{p}$ the term $\mathrm{e}^{2 \theta}$ is thus expected to become irrelevant, so that the equation takes the form (4.75). Its solution is given by (4.79) and can be approximated at $l_{p} \ll r \ll L$ as $\mathrm{e}^{\theta(r)} \simeq A / r^{\alpha}$ with some coefficient $A$ and exponent $\alpha$. In order to have large $P_{2}$, we require $\alpha>1$. Using the condition (4.113), we find scale $l_{p}$ to be given by

$$
l_{p} \sim\left\{\begin{array}{cc}
P_{2}^{-1 / 2}, & \alpha>2,  \tag{4.123}\\
\left(L^{4-2 \alpha} P_{2}\right)^{-1 /[2(\alpha-1)]}, & \alpha<2,
\end{array}\right.
$$

up to a numerical coefficient of order unity. Therefore, the action (4.114) is equal to

$$
\begin{equation*}
S_{u}=\alpha^{2} \pi^{2} v D \ln \left(L / l_{p}\right)=\mathscr{F}(\alpha) \pi^{2} v D \ln \left(L^{2} P_{2}\right), \tag{4.124}
\end{equation*}
$$

where

$$
\mathscr{F}(\alpha)=\left\{\begin{array}{cc}
\alpha^{2} / 2, & \alpha \geq 2  \tag{4.125}\\
\frac{\alpha^{2}}{2(\alpha-1)}, & 1<\alpha \leq 2
\end{array}\right.
$$

Thus, the minimum of $S_{u}$ corresponds to $\alpha=2$, yielding

$$
\begin{equation*}
\mathscr{P}\left(P_{2}\right) \sim\left(\frac{P_{2}}{\left\langle P_{2}\right\rangle}\right)^{-\beta \pi g / 2} \tag{4.126}
\end{equation*}
$$

The upper border of validity of (4.126) is $P_{2} \sim P_{2}^{\mathrm{RMT}}(L / l)^{2} \sim 1 / l^{2}$.
For hard wall boundary conditions, the asymptotic behavior of $\mathscr{P}\left(P_{2}\right)$ will be, however, determined by configurations with a maximum located near the sample boundary. Assuming that the sample has a smooth boundary with the single characteristic scale $L$ (for example, it is of the circular form), we can get such a state from the rotationally invariant bulk state by putting its center on the boundary and removing that half of the state which is outside of the sample. Such a truncated state will have the twice larger IPR and twice smaller action compared to its parent bulk state. Consequently, the asymptotics of the distribution function for a sample with a boundary will be different from (4.126) by an extra factor $1 / 2$ in the exponent.

Using the Liouville theory description (3.93), one can generalize the above consideration to the distribution $\mathscr{P}\left(P_{q}\right)$ of higher IPR's $P_{q}(3.90)$ with $q>2$ [160]. We will assume that $q$ is not too large, $q^{2}<2 \beta \pi g$, so that the average value $\left\langle P_{q}\right\rangle$ is at the same time the typical value of $P_{q}$ (see Section 3.3.2). Then in the region $q^{2} / \beta \pi g \lesssim P_{q} /\left\langle P_{q}\right\rangle-1 \lesssim 1$ the distribution has the exponential form (3.92). At larger $P_{q}$ the optimal configuration is again of the form $\left|\psi^{2}(r)\right|_{\text {smooth }} \equiv \mathrm{e}^{\theta(r)}=A / r^{\alpha}$ for $r>l_{p}$; minimizing the action, we find $\alpha=q / 4$ and the distribution function

$$
\begin{equation*}
\mathscr{P}\left(P_{q}\right) \sim \frac{1}{P_{q}}\left(\frac{P_{q}}{\left\langle P_{q}\right\rangle}\right)^{-2 \beta \pi g / q^{2}} \tag{4.127}
\end{equation*}
$$

This is valid for $P_{q}<P_{q}^{\mathrm{RMT}}(L / l)^{2}$; for still larger $P_{q}$ the corresponding optimal fluctuation would violate the condition of the applicability of the diffusion approximation $\theta^{\prime} \lesssim 1 / l$. Incorporating this restriction (cf. similar situation for the distribution of relaxation times, Section 4.1.2) leads to

$$
\begin{equation*}
\alpha=\frac{1}{q}\left[\frac{\ln \left(P_{q} / P_{q}^{\mathrm{RMT}}\right)}{\ln (L / l)}-2\right] \tag{4.128}
\end{equation*}
$$

and to the log-normal far asymptotics

$$
\begin{equation*}
\mathscr{P}\left(P_{q}\right) \sim \frac{1}{P_{q}} \exp \left\{-\frac{\beta \pi g}{4 q^{2}} \frac{\ln ^{2}\left[\left(P_{q} / P_{q}^{\mathrm{RMT}}\right)(L / l)^{2}\right]}{\ln (L / l)}\right\} \tag{4.129}
\end{equation*}
$$

for $(L / l)^{2} \lesssim P_{q} / P_{q}^{\mathrm{RMT}} \lesssim(L / l)^{2 q-2}$.

## 4.5. $3 D$ systems

As we will see below, in the 3D case the states determining the asymptotics of the distribution functions have just a local short-scale spike on top of a homogeneous background. In this sense, no

ALS is formed, in contrast to the quasi-1D and 2D situations. As a closely related feature, we find that the results in 3D are strongly dependent on microscopic details of the random potential. We start from discussing what the $\sigma$-model calculation gives when applied to the 3D geometry. Then we compare this with the results of the direct optimal fluctuation method [161].

Let us consider, e.g. the long-time relaxation problem. The solution of the $\sigma$-model saddle-point equation has the form [30]

$$
\begin{equation*}
\theta(r) \simeq C\left(\frac{l_{*}}{r}-\frac{l_{*}}{R}\right), \quad l_{*} \leq r \leq R, \tag{4.130}
\end{equation*}
$$

where $R$ is the system size (which is, in fact, irrelevant here) and

$$
\begin{equation*}
l_{*} \sim l \ln \left[\frac{t}{\tau\left(k_{\mathrm{F}} l\right)^{2}}\right] ; \quad C \sim \ln \left[\frac{t}{\tau\left(k_{\mathrm{F}} l\right)^{2}}\right] . \tag{4.131}
\end{equation*}
$$

The reason for introduction of the short-scale cut-off length $l_{*}$ is the same as in 2D (see Sections 4.1.2 and 4.2.2): the gradient $\theta^{\prime}$ should not exceed $\sim 1 / l$. Calculating the action, we find

$$
\begin{equation*}
-\ln P_{s}(t) \sim\left(k_{\mathrm{F}} l\right)^{2} \ln ^{3}\left[\frac{t}{\tau\left(k_{\mathrm{F}} l\right)^{2}}\right] \tag{4.132}
\end{equation*}
$$

with an uncertainty in numerical prefactor. This uncertainty originates from the fact that the action is dominated by the ultraviolet (short-distance) region $r \sim l_{*}$. Similar result is obtained for the eigenfunction amplitude statistics $[33,36]$

$$
\begin{equation*}
-\ln \mathscr{P}(u) \sim\left(k_{\mathrm{F}} l\right)^{2} \ln ^{3} \frac{V u}{\left(k_{\mathrm{F}} l\right)^{2}} \tag{4.133}
\end{equation*}
$$

and for the LDOS distribution function [34]

$$
\begin{equation*}
-\ln \mathscr{P}(\rho) \sim\left(k_{\mathrm{F}} l\right)^{2} \ln ^{3} \frac{\rho}{v} . \tag{4.134}
\end{equation*}
$$

As we see, all these distribution functions are found to have the exponential-log-cube asymptotic form. The corresponding eigenstates have the shape

$$
\left|\psi^{2}(\boldsymbol{r})\right|_{\text {smooth }} \simeq \frac{1}{V} \exp \left\{C_{i}-\frac{l}{r} \ln ^{2} Z_{i}\right\},
$$

where $C_{i} \sim 1, Z_{t}=t /\left[\tau\left(k_{\mathrm{F}} l\right)^{2}\right], Z_{u}=V u /\left(k_{\mathrm{F}} l\right)^{2}$, and $Z_{\rho}=\rho / v$, i.e. they consist of the usual homogeneous background $\left|\psi^{2}(r)\right|_{\text {smooth }}=1 / V$ supplemented by a short-scale bump containing only a minor part of the eigenfunction normalization.

Muzykantskii and Khmelnitskii [31] applied the ballistic $\sigma$-model approach to the problem of the long-time relaxation. They reproduced the result (4.132) and found the numerical prefactor there to be equal to $\pi / 9 \sqrt{3}$. On the other hand, Smolyarenko and Altshuler [161] employed the direct optimal fluctuation method to study the asymptotics of the distribution function $\mathscr{P}(u)$. This approach is similar to that used many years ago $[162,163]$ to calculate the tails of the density of states of a disordered system. The authors of [161] did not use the $\sigma$-model description, but rather searched directly for the optimal realization of the random potential (hence their term
"direct optimal fluctuation method"). While having confirmed the exponential-log-cube form of the asymptotics in the 3D case, they found a prefactor smaller by $\sim k_{\mathrm{F}} l$ compared to the $\sigma$-model result, i.e. they obtained

$$
\begin{equation*}
\mathscr{P}(u) \sim k_{\mathrm{F}} l \ln ^{3}(u V) . \tag{4.135}
\end{equation*}
$$

The physical reason for this difference lies in the ballistic effects which have been already discussed in Section 3.3.3 in connection with perturbative corrections to the eigenfunction amplitude distribution. It was shown there that in the 3D case and for the white-noise random potential the parameter $\kappa$ governing these corrections is dominated by a non-universal (depending on the type of the disorder) ballistic contribution yielding $\kappa \sim 1 / k_{\mathrm{F}} l$, while the diffusive contribution is $\sim 1 /\left(k_{\mathrm{F}} l\right)^{2}$. This is in direct correspondence with Eqs. (4.133) and (4.135), which show precisely the same difference. Eq. (4.135) is again non-universal; its derivation by Smolyarenko and Altshuler [161] relies on the white-noise disorder assumption. The corresponding optimal configuration of the potential found in [161] is nothing else but a potential wall surrounding the observation point with the height several times larger than $E_{\mathrm{F}}$ and the thickness $\sim \lambda_{\mathrm{F}} \ln (u V)$. Such configurations are not included in the $\sigma$-model consideration, which assumes that the absolute value of the particle velocity does not change appreciably in space. If one would consider a smooth random potential, whose magnitude is limited from above by some value $U_{\max } \ll E_{\mathrm{F}}$, such configurations would not be allowed. Whether in this case the $\sigma$-model result would hold remains to be seen.

### 4.6. Discussion

In Section 4, we have studied the asymptotic behavior of various distribution functions characterizing the eigenfunction statistics in a disordered sample. For this purpose, we used two methods of treatment of the $\sigma$-model: exact solution (in the quasi-1D case) and the saddle-point method. Physically, the saddle-point solution describes the relevant optimum fluctuations of the wave function envelope; probability of formation of such a fluctuation is found to be governed by the Liouville theory (3.93). In the quasi-1D case, the results of the saddle-point method are in agreement with those of the exact solution of the $\sigma$-model.

The 2D geometry is of special interest, since the eigenfunctions show the features of criticality. In this case, a full agreement between the saddle-point calculation and the renormalization-group (RG) treatment of Altshuler et al. [28] was found for all the distributions, where such a comparison was possible, namely for $\mathscr{P}\left(t_{\phi}\right), \mathscr{P}(\rho)$ and $\mathscr{P}(v)$. This agreement is highly non-trivial, for the following reason. The RG treatment is based on a resummation of the perturbation theory expansion and can be equally well performed within the replica (bosonic or fermionic) or supersymmetric formalism. At the same time, the present approach based on the supersymmetric formalism relies heavily on the topology of the saddle-point manifold combining non-compact ( $\lambda_{1}$ ) and compact $\left(\lambda_{2}\right)$ degrees of freedom. The asymptotic behavior of the distribution functions considered is determined by the region $\lambda_{1} \gg 1$ which is very far from the "perturbative" region of the manifold $Q \simeq \Lambda$ (i.e. $\lambda_{1}, \lambda_{2} \simeq 1$ ). It is well known [164] that for the problem of energy level correlation, the replica approach fails, since it does not reflect properly the topology of the $\sigma$-model manifold. The success of the RG treatment of [28] seems to be determined by the fact that for the present problem (in contrast to that of level correlation) only the non-compact sector of the supersymmetric $\sigma$-model is essential, with compact one playing an auxiliary role. Let us note that
the same situation appears in the vicinity of the Anderson transition [90,19,20] where the function $Y\left(\lambda_{1}, \lambda_{2}\right)$ acquires a role of the order parameter function and depends on the non-compact variable $\lambda_{1}$ only. The above agreement found for the "tails" of the distributions in the metallic region provides therefore support to the results concerning the Anderson transition obtained with making use of the renormalization group approach and $2+\varepsilon$ expansion.

We have found that the spatial structure of ALS relevant to the asymptotic behavior of different distributions may be different. This is because an ALS constitutes an optimal fluctuation for one of the above quantities, and the form of this fluctuation depends on the specific characteristic, which is to be optimized. Finally, we have discussed interrelations between asymptotics of various distribution functions. In the quasi-1D and 2D cases, we thus presented a comprehensive picture which explains all the studied asymptotics as governed by exponentially rare events of formation of ALS.

## 5. Statistics of energy levels and eigenfunctions at the Anderson transition

In $d>2$ dimensions a disordered system undergoes, with increasing strength of the disorder, a transition from the phase of extended states to that of localized states (see, e.g. [39] for review). This transition changes drastically the statistics of energy levels and eigenfunctions. In particular, at the mobility edge these statistics acquire distinct features reflecting criticality of the theory. This is the subject of the present section. Section 5.1 is devoted to the level statistics and Section 5.2 to the eigenfunctions correlations at the mobility edge. In Section 5.3 we study the level and eigenfunction statistics in a quasi-one-dimensional model with long-range (power-law) hopping which undergoes the Anderson transition and shows at criticality all the features characteristic for a conventional metal-insulator transition point in $d>2$.

### 5.1. Level statistics. Level number variance

The problem of the energy level statistics at the mobility edge was addressed for the first time by Altshuler et al. [40], who considered the variance $\left\langle\delta N^{2}(E)\right\rangle=\left\langle N^{2}(E)\right\rangle-\langle N(E)\rangle^{2}$ of the number of levels within a band of a width $E$. This quantity is related to the two-level correlator (2.1) via

$$
\begin{equation*}
\left.\left\langle\delta N(E)^{2}\right\rangle=\int_{-\langle N(E)\rangle}^{\langle N(E)\rangle}(\langle N(E)\rangle)-|s|\right) R^{(c)}(s) \mathrm{d} s, \tag{5.1}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d}\langle N(E)\rangle}\left\langle\delta N(E)^{2}\right\rangle=\int_{-\langle N(E)\rangle}^{\langle N(E)\rangle} R^{(c)}(s) \mathrm{d} s, \tag{5.2}
\end{equation*}
$$

where $\langle N(E)\rangle=E / \Delta$ and $R^{(c)}(s)=R(s)-1$ is the connected part of the two-level correlation function. In RMT, the $1 / s^{2}$ behavior of $R^{(c)}(s)$ leads to the logarithmic behavior of the variance $\left\langle\delta N^{2}\right\rangle_{\mathrm{wD}} \simeq\left(2 / \pi^{2} \beta\right) \ln \langle N\rangle$ for $\langle N\rangle \gg 1$. In the opposite situation characteristic for the phase of localized states, when all energy levels are completely uncorrelated (known as the Poisson statistics), one gets $\left\langle\delta N^{2}\right\rangle_{\mathrm{P}}=\langle N\rangle$. Supported by their numerical simulations, Altshuler et al. [40]
put forward a conjecture that at the critical point

$$
\begin{equation*}
\left\langle\delta N^{2}\right\rangle \simeq \chi\langle N\rangle, \tag{5.3}
\end{equation*}
$$

where $0<\chi<1$ is a numerical coefficient (which is conventionally called now "spectral compressibility"). More recently, Shklovskii et al. [41] introduced the concept of new universal statistics at the mobility edge (see also Ref. [165]). In Ref. [42] the correlator $R(s)$ at the mobility edge was studied by means of perturbation theory combined with scaling assumptions about a form of the diffusion propagator. It was found that for $s \gg 1$,

$$
\begin{equation*}
R^{(c)}(s) \propto s^{-2+\gamma} \tag{5.4}
\end{equation*}
$$

where $\gamma<1$ is certain critical index. The consideration of Ref. [42] led to the conclusion that $\gamma=1-1 / v d$ (where $v$ is the critical exponent of the localization length), which was however questioned later [166] in view of the oversimplified treatment of the diffusion propagator at the transition point in [42]. At any rate, the behavior (5.4) with some $\gamma$ is what one expects to hold at the mobility edge; the condition $\gamma<1$ follows from the requirement of convergence of $\int R(s) \mathrm{d} s$.

Using Eqs. (5.1) and (5.4) and the sum rule

$$
\begin{equation*}
\int R(s) \mathrm{d} s=0 \tag{5.5}
\end{equation*}
$$

(implied by the conservation of the number of energy levels), the authors of [42] concluded that

$$
\begin{equation*}
\left\langle\delta N^{2}\right\rangle \propto N^{\gamma} \tag{5.6}
\end{equation*}
$$

with $\gamma<1$, in contradiction with Ref. [40]. This conclusion was critically reexamined in Refs. [43,44], where it was shown that the asymptotic behavior (5.4) of the correlator $R(s)$ at $s \gg 1$ does not imply the absence of the linear term (5.3). The flaw in reasoning of Ref. [42] was in the assumption that the universal part of the correlator $R(s)$ (which is the one surviving in the limit $E / \Delta=\langle N\rangle=$ const, $L \rightarrow \infty$ ) satisfies the sum rule (5.5). It turns out, however, that the sum rule is fulfilled only if all contributions are taken into account, including the non-universal contribution of the "ballistic" region $\omega \sim 1 / \tau$, where $\tau$ is the elastic mean free time. To demonstrate this, we estimate below (following Ref. [43]) the contributions to the sum rule from all regions of the variable $s$.

In fact, for the conventional model of a particle in a random potential definition (2.1) of the correlator and the sum rule relation (5.4) should be modified, when the vicinity of the critical point is considered. The reason is that the Anderson transition point corresponds to a strong disorder regime $E_{\mathrm{F}} \tau \sim 1$, so that the condition $\omega \sim 1 / \tau$ implies $\omega \sim E_{\mathrm{F}}$. On the other hand, the density of states $v\left(E_{\mathrm{F}}\right)$ can be considered as a constant only for small variations of energy $\omega \ll E_{\mathrm{F}}$. This means that variation of $v\left(E_{\mathrm{F}}\right)$ should be taken into account in (2.1) and (5.5). Besides, the condition $E_{\mathrm{F}} \tau \sim 1$ leads to a breakdown of the perturbation theory, that complicates the analysis of the "ballistic" region contribution. To get rid of these problems, we consider a different microscopic model which has exactly the same universal part of $R(s)$, but whose density of states does not change within the range of $\omega \sim 1 / \tau$. This is so-called $n$-orbital Wegner model [167], which can be considered as a system of metallic granules forming a $d$-dimensional lattice, each granule being coupled to its nearest neighbors. In the limit $n \gg 1$ this model can be mapped onto a supersymmetric $\sigma$-model defined on a lattice. The action of this $\sigma$-model reads [90] (we consider the unitary
symmetry for definiteness)

$$
\begin{equation*}
S\{Q\}=\frac{\Gamma}{2} \sum_{\langle i j\rangle} \operatorname{Str} Q_{i} Q_{j}+\varepsilon \sum_{i} \operatorname{Str} Q_{i} \Lambda, \tag{5.7}
\end{equation*}
$$

where the supermatrices $Q_{i}$ are defined on sites $i$ of a $d$-dimensional lattice with a lattice spacing $a$. Summation in the first term of Eq. (5.7) goes over the pairs $\langle i j\rangle$ of nearest neighbors. The parameters $\Gamma$ and $\varepsilon$ are related to the classical diffusion constant $D$, the density of states $v$, and the frequency $\omega$ as follows:

$$
\begin{equation*}
\Gamma=\pi v D a^{d-2}, \quad \varepsilon=-\mathrm{i} \omega \pi v a^{d} / 2 . \tag{5.8}
\end{equation*}
$$

The two-level correlator can be expressed through a correlation function of this $\sigma$-model via the discretized version of Eq. (2.10):

$$
\begin{equation*}
R(s)=\operatorname{Re} \int \prod_{j} D Q_{j}\left\{\left[\frac{a^{d}}{4 V} \sum_{i} \operatorname{Str} Q_{i} k \Lambda\right]^{2}-1\right\} \exp (-S\{Q\}), \tag{5.9}
\end{equation*}
$$

where $V$ is the system volume. It is not difficult to prove explicitly [43] that the two-level correlation function of this lattice $\sigma$-model, Eq. (5.9), satisfies exactly the sum rule (5.5). As will become clear below, the continuum version of the $\sigma$-model does not possess this property: there is a deficiency of the sum rule for it which is related with the contribution of the range of $\omega$ close to the ultraviolet cut-off.

Let us stress that in the region $\omega \ll D / a^{2} \equiv E_{c}(a)$ the correlator $R(s)$ is universal, i.e. does not depend on microscopical details of the model. The region $\omega \sim D / a^{2}$ plays a role analogous to that of the ballistic region, $\omega \sim 1 / \tau$, in the case of the usual model of particle in a random potential. Despite the non-universality of the correlation function $R_{\mathrm{nu}}(s)$ in this "ballistic" domain, the corresponding integral contribution $I_{\mathrm{nu}}$ to the sum rule is universal, because it determines, according to Eq. (5.5), the sum rule deficiency for the universal part $R_{\mathrm{u}}(\mathrm{s})$ :

$$
\begin{align*}
& I_{\mathrm{u}}+I_{\mathrm{nu}}=0, \\
& I_{\mathrm{u}}=\int R_{\mathrm{u}}^{(c)}(s) \mathrm{d} s, \quad I_{\mathrm{nu}}=\int R_{\mathrm{nu}}^{(c)}(s) \mathrm{d} s . \tag{5.10}
\end{align*}
$$

As will be seen below, when the system is close to the Anderson transition the two regions of the variable $s$ dominating the integrals $I_{\mathrm{u}}$ and $I_{\mathrm{nu}}$, respectively, are separated by a parametrically broad range of $s$ giving a negligible contribution to the sum rule. For energy band width $E$ lying in this range, we get from Eqs. (5.2) and (5.10)

$$
\begin{equation*}
\left\langle\delta N(E)^{2}\right\rangle \simeq I_{\mathrm{u}}\langle N(E)\rangle=-I_{\mathrm{nu}}\langle N(E)\rangle, \tag{5.11}
\end{equation*}
$$

i.e. just the linear term (5.3) with $\chi=I_{\mathrm{u}}=-I_{\mathrm{nu}}$. We turn now to the analysis of the correlator $R(s)$ and of the sum rule in various situation. For completeness, we start from the case of a good metal, then we consider the critical point and the critical region cases.
(1) Good metal. Here the following three regions with different behavior of $R(s)$ can be found:
(A) "Wigner-Dyson" (WD) region: $\omega \ll E_{c}$. The correlator $R(s)$ in this region was studied in Section 2.2, see Eqs. (2.27) and (2.30). The corresponding contribution to the sum rule can be
estimated as (we omit numerical factors of order unity)

$$
\begin{equation*}
I_{A} \simeq \int_{0}^{g} R(s) \propto+1 / g \tag{5.12}
\end{equation*}
$$

(B) "Altshuler-Shklovskii" (AS) region: $E_{c} \ll \omega \ll E_{c}(a)$. Here $E_{c}(a)=D / a^{2}$ is the Thouless energy at the scale of lattice spacing $a$, which plays a role of the ultraviolet cut-off for the diffusion theory. The level correlation function is given by Eq. (2.33), yielding

$$
\begin{equation*}
I_{B} \simeq \int_{g}^{D / a^{2} \Delta} R(s) \mathrm{d} s \propto g^{-d / 2}\left(\frac{D}{a^{2} \Delta}\right)^{d / 2-1} \propto+1 / g(a), \tag{5.13}
\end{equation*}
$$

where $g(a)=g(a / L)^{d-2}$ is the conductance at the scale $a$. Note that for the case of a particle in a random potential, the following substitutions should be done: $a \rightarrow l ; E_{c}(a) \rightarrow E_{c}(l)=1 / \tau$; $1 / g(a) \rightarrow 1 / g(l) \propto(\varepsilon \tau)^{d-1}$.
(C) "Ballistic" region: $\omega \gtrsim E_{c}(a)$. To find the correlator $R(s)$ in this range, we can neglect in the leading approximation the first term in Eq. (5.7), that gives

$$
\begin{align*}
& R^{(c)}(s) \propto-\left(\frac{L}{a}\right)^{d} \frac{1}{s^{2}}  \tag{5.14}\\
& I_{C} \simeq \int_{D / a^{2} \Delta}^{\infty} R^{(c)}(s) \mathrm{d} s \propto-1 / g(a) . \tag{5.15}
\end{align*}
$$

Contribution (5.12) to the sum rule is dominated by the region $\omega \sim \Delta$, whereas the contributions (5.13) and (5.15) are dominated by a vicinity of the ultraviolet cut-off $\omega \sim D / a^{2}$. In $d=3 g \gg g(a)$, and these non-universal contributions $I_{B}, I_{C}$ are much larger (by absolute value) than $I_{A}$, but they should cancel each other according to the sum rule. The situation is more interesting in $d=2$, where the contribution (2.33), (5.13) of the AS region is absent in view of special analytical properties of the diffusion propagator $[168,169,44]$. In this case $g(a)=g$, so that the contributions $I_{A}$ and $I_{C}$ are of the same order, in agreement with the sum rule which prescribes their sum to be zero. We can identify then $I_{A}$ with $I_{\mathrm{u}}$ and $I_{C}$ with $I_{\mathrm{nu}}$ in Eq. (5.10), so that $I_{\mathrm{u}}=-I_{\mathrm{nu}} \propto 1 / g$. According to Eq. (5.11), this leads to a linear behavior of the variance,

$$
\begin{equation*}
\left\langle\delta N^{2}\right\rangle \propto(1 / g)\langle N\rangle, \quad E_{c} \ll E \ll D / a^{2}, \tag{5.16}
\end{equation*}
$$

but the corresponding coefficient is of order $1 / g \ll 1$. The numerical coefficient in (5.16) can be calculated using the explicit form of $R^{(c)}(s)$, yielding

$$
\begin{equation*}
\chi \equiv I_{\mathrm{u}} \simeq \frac{1}{2 \beta \pi g}, \quad g \gg 1 \tag{5.17}
\end{equation*}
$$

With an increase in disorder strength, the coupling constant $\Gamma$ in Eq. (5.7) decreases. When it approaches a critical value $\Gamma_{\mathrm{c}}$, corresponding to the metal-insulator transition, the correlation length $\xi$ becomes large: $\xi \gg a$. Depending on the relation between $\xi$ and the system size $L$, the following two situations are to be distinguished.
(2) Metal in the critical region: $a \ll \xi \ll L$. In this case $g(a) \simeq g_{*}$, but $g \equiv g(L) \gtrdot g_{*}$, where $g_{*}$ is the critical value of the conductance. ${ }^{8}$ We find then as much as four different regions of $s$ for the correlator $R(s)$ :
(A) WD region: $\omega \ll E_{c}$. The correlator $R(s)$ and the contribution to the sum rule for this region are given by the same Eqs. (2.27), (2.30) and (5.12), as for a good metal.
(B) AS region: $E_{c} \ll \omega \ll E_{c}(\xi)=g_{*} \Delta_{\xi}$, where $E_{c}(\xi)$ and $\Delta_{\xi}$ are Thouless energy and level spacing for a piece of the sample with linear size $\xi$. The correlator $R^{(c)}(s)$ is given by Eq. (2.33), and we find

$$
\begin{equation*}
I_{B} \simeq \int_{g}^{g_{*} t_{\xi} / \Delta} R^{(c)}(s) \mathrm{d} s \propto g^{-d / 2} g_{*}\left(\frac{\Delta_{\xi}}{\Delta}\right)^{d / 2-1} \propto+1 / g_{*}, \tag{5.18}
\end{equation*}
$$

where we have used the relations $\Delta_{\xi} / \Delta=(L / \xi)^{d}$ and $g / g_{*} \propto(L / \xi)^{d-2}$.
(C) "Kravtsov-Lerner-Altshuler-Aronov" (KLAA) region: $E_{c}(\xi) \equiv g_{*} \Delta_{\xi} \ll \omega \ll g_{*} \Delta_{a} \equiv E_{c}(a)$. The correlator $R^{(c)}(s)$ in this range was studied in Ref. [170]; the result reads:

$$
\begin{equation*}
R^{(c)}(s) \simeq-g_{*}^{-\gamma}\left(\Delta_{\xi} / \Delta\right)^{1-\gamma_{S}} S^{-2+\gamma} \tag{5.19}
\end{equation*}
$$

and, consequently,

$$
\begin{equation*}
I_{C} \simeq \int_{g_{*} \Delta_{\xi} / \Delta}^{g_{*} A_{a} / \Delta} R^{(c)}(s) \mathrm{d} s \propto-1 / g_{*} \tag{5.20}
\end{equation*}
$$

(D) "Ballistic" region: $\omega \gtrsim E_{c}(a)$. Here Eqs. (5.14) and (5.15) hold, yielding

$$
\begin{equation*}
I_{D} \propto-1 / g_{*} \tag{5.21}
\end{equation*}
$$

The contribution $I_{A}$ is dominated by $\omega \sim \Delta$, the contributions $I_{B}, I_{C}$ by $\omega \sim \Delta_{\xi}$, and finally, the contribution $I_{D}$ comes from the "non-universal" region $\omega \sim E_{c}(a)$. Therefore, this last contribution determines $I_{\mathrm{nu}}$ in Eq. (5.10), leading according to Eq. (5.11) to the linear behavior of the variance

$$
\begin{equation*}
\left\langle\delta N^{2}\right\rangle \propto\left(1 / g_{*}\right)\langle N\rangle, \quad E_{c}(\xi) \ll E \ll E_{c}(a) \tag{5.22}
\end{equation*}
$$

with a coefficient $\sim 1 / g_{*}$ (which is of order unity in 3D).
(3) Finally, a system is at the critical point, when $\xi \gg L$. In this case, $g(L)=g(a)=g_{*}$. The AS region disappears, and we have, in full analogy with the previous estimates:
(A) WD region: $\omega \lesssim E_{c}=g_{*} \Delta$.

$$
\begin{equation*}
I_{A} \propto+1 / g_{*}, \tag{5.23}
\end{equation*}
$$

(B) KLAA region: $g_{*} \Delta \ll \omega \ll \Delta_{a} g_{*}$.

$$
\begin{equation*}
R(s) \propto-g_{*}^{-\gamma} S^{-2+\gamma}, \quad I_{B} \propto-1 / g_{*}, \tag{5.24}
\end{equation*}
$$

(C) "Ballistic" region: $\omega \gtrsim \Delta_{a} g_{*}$.

$$
\begin{equation*}
I_{C} \propto-1 / g_{*} . \tag{5.25}
\end{equation*}
$$

[^7]Again the same conclusion, as for the critical region case, can be drawn: the "ballistic" contribution (5.25) can be identified as $I_{\mathrm{nu}}$ in Eq. (5.10), yielding:

$$
\begin{equation*}
\left\langle\delta N^{2}\right\rangle \simeq \chi\langle N\rangle, \quad E_{c} \ll E \ll E_{c}(a), \tag{5.26}
\end{equation*}
$$

with $\chi \sim 1 / g_{*}$.
Recently, a relation between the spectral compressibility $\chi$ and the multifractal dimension $D_{2}$ (for discussion of multifractality and corresponding bibliographic references, see Sections 3.3.2 and 5.2) was proposed $[49,166]$ :

$$
\begin{equation*}
\chi=\left(d-D_{2}\right) / 2 d . \tag{5.27}
\end{equation*}
$$

The central idea of the derivation [49] is to consider the motion of energy levels, when the system is subject to a random perturbation. This allows to link the spectral statistics with the wave function correlations. In 2D, one can check, by comparing Eqs. (3.63) and (5.27), that Eq. (5.27) is indeed satisfied in leading order in $1 / \mathrm{g}$. The general derivation of Eq. (5.27) is, however, based on certain approximate decoupling of a higher-order correlation function [171,172,49], so that it is not completely clear whether this is indeed an exact formula as argued in [49] or only an approximation valid for $g_{*} \gg 1$.

The linear behavior (5.3) of the level number variance at the Anderson transition has been confirmed by now in numerical simulations by several groups [45-48]. The spectral compressibility $\chi$ is an important universal parameter characterizing the critical point of Anderson transition. Its universality is of the same sort as that of the critical indices, i.e. $\chi$ depends only on the spatial dimensionality and on the symmetry (universality) class. Let us note that the whole level correlation function $R^{(c)}(s)$ is not as universal, since it depends also on the shape of the sample and on the boundary conditions [48,173-175]. This can be expected already from the perturbative $1 / g^{2}$ correction, Eqs. (2.27) and (2.30), where the coefficient $a_{d}$ does depend on the shape and on the boundary conditions. Also, the shape dependence becomes evident if one considers the limit of an elongated sample with a length considerably larger than the transverse sample size. Indeed, let us consider (in 3D) a rectangular sample with $L_{y}=L_{z}=L_{x} / \alpha$, where $\alpha$ is a numerical factor [173]. If we fix $\alpha$ and consider the limit $L \rightarrow \infty$, the level correlation function has a limiting form, which, depends, however on $\alpha$. In particular, at $\alpha \gg 1$ the sample is of quasi-1D geometry with a ratio of the sample length to the localization length $\sim \alpha$, so that the level statistics will be close to Poissonnian.

Asymptotic behavior of the nearest-neighbor level spacing distribution function $P(s)$ for $s \gg 1$ at the mobility edge has been also a controversial issue. We remind that for the Poissonnian statistics $P(s)=\mathrm{e}^{-s}$, while in the RMT $P(s) \sim \mathrm{e}^{- \text {const } s^{2}}$. While Refs. [40,41] conjectured that $P(s) \sim \mathrm{e}^{- \text {const } s}$, the authors of Ref. [176] found $P(s) \sim \mathrm{e}^{- \text {const } s^{2-\gamma}}$ with the same index $\gamma$ as in Eq. (5.4). Recent numerical studies $[177,178]$ support the former result, $P(s) \sim \mathrm{e}^{- \text {const } s}$.

### 5.2. Strong correlations of eigenfunctions near the Anderson transition

In this section, we discuss correlations between the amplitudes of different, but close in energy, eigenstates at and near the critical point of the Anderson transition. Let us recall that in the metallic phase far from the transition point a typical eigenfunction covers essentially uniformly the sample volume. This is reflected in the inverse participation ratio $P_{2}$, as well as in the higher moments
$P_{q}=\int \mathrm{d}^{d} r\left|\psi^{2 q}(\boldsymbol{r})\right|$, which differ only weakly from their RMT values, see Eqs. (3.15) and (3.52). When the system approaches the point of the Anderson transition $E_{c}$, these extended eigenfunctions become less and less homogeneous in space, showing regions with larger and smaller amplitudes and eventually forming a multifractal structure in the vicinity of $E_{c}$, see Section 3.3.2. This multifractal behavior is characterized by the following behavior of the moments $P_{q}$ at the critical point:

$$
\begin{equation*}
P_{q} \propto L^{-D_{q}(q-1)}, \quad L<\xi, \tag{5.28}
\end{equation*}
$$

as well as in the conducting phase in the vicinity of the critical point:

$$
\begin{equation*}
P_{q} \propto \xi^{\left(d-D_{q}\right)(q-1)} L^{-d(q-1)}, \quad L>\xi . \tag{5.29}
\end{equation*}
$$

Here $\xi \propto\left|E-E_{c}\right|^{-v}$ is the correlation length and $D_{q}$ is the set of multifractality exponents, $d>D_{2}>D_{3}>\cdots$. As Eq. (5.29) indicates, the eigenfunctions become more and more sparse when the system approaches the critical point of the Anderson transition from the metallic phase (i.e. when $\xi$ increases). Just at the mobility edge the scaling of IPR with the system size $L$ becomes different, see Eq. (5.28), so that the eigenfunction effectively occupies a vanishing fraction of the system volume. At last, in the insulating phase any eigenstate is localized in a domain of finite extension $\xi$, and IPR remains finite in the limit of infinite system size $L \rightarrow \infty$.

This transparent picture serves as a basis for qualitative understanding of spectral properties of disordered conductors. Indeed, as long as eigenstates are well extended and cover the whole sample, they overlap substantially, and corresponding energy levels repel each other almost in the same way as in RMT. As a result, the Wigner-Dyson (WD) statistics describes well energy levels in a good metal, see Section 2. In contrast, in the insulating phase different eigenfunctions corresponding to levels close in energy are localized far apart from one another and their overlap is negligible. This is the reason for absence of correlations of energy levels in this regime, i.e. the Poisson statistics.

A naive extrapolation of this argument to the vicinity of the transition point would lead to a wrong conclusion. Indeed, one might expect that sparse (multifractal in the critical point) eigenstates fail to overlap, that would result in essential weakening of level correlations close to the mobility edge and vanishing level repulsion at $E=E_{c}$. However, numerical simulations show [40,41,179-181] that even at the mobility edge levels repel each other strongly (although the level statistics is different from RMT). As we are going to explain now, this apparent contradiction is resolved in the following way: The critical eigenstates for nearby levels are so strongly correlated that they overlap well in spite of their sparse structure.

To demonstrate the strong correlations, we will consider the relation between the overlap function $\sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega)$, Eq. (3.68), and the "self-overlap" $\alpha(\boldsymbol{r}, \boldsymbol{r}, E)$, Eq. (3.67) (the latter determining the IPR $P_{2}$ according to $P_{2}=\int \mathrm{d} \boldsymbol{r} \alpha(\boldsymbol{r}, \boldsymbol{r}, E)$. As was shown ${ }^{9}$ in [182], everywhere in the metallic phase

[^8](including the vicinity of the critical point)
\[

$$
\begin{equation*}
\sigma(r, r, E, \omega)=\frac{\beta}{\beta+2} \alpha(r, r, E) \tag{5.30}
\end{equation*}
$$

\]

for $\omega \ll \Delta_{\xi}$, where $\Delta_{\xi} \sim 1 / v \xi^{d}$ is the level spacing in the correlation volume. Eq. (5.30) implies the following structure of eigenfunctions within an energy interval $\omega<\Delta_{\xi}$. Each eigenstate can be represented as a product $\Psi_{i}(\boldsymbol{r})=\psi_{i}(\boldsymbol{r}) \Phi_{E}(\boldsymbol{r})$. Here the function $\Phi_{E}(\boldsymbol{r})$ is an eigenfunction envelope of "humps and dips". It is the same for all eigenstates around energy $E$, reflects the underlying gross (multifractal) spatial structure, and governs the divergence of the inverse participation ratios $P_{q}$ at the critical point. In contrast, $\psi_{i}(\boldsymbol{r})$ shows RMT-like fluctuations on the scale of the wave length. It fills the envelope function $\Phi_{E}(r)$ in an individual way for each eigenfunction, but is not critical, i.e. is not sensitive to the vicinity of the Anderson transition. These Gaussian fluctuations are responsible for the factor $\beta /(\beta+2)$ (which is the same as in the corresponding Gaussian Ensemble) in Eq. (5.30).

As was already mentioned, this picture is valid in the energy window $\omega \sim \Delta_{\xi}$ around the energy $E$; the number of levels in this window being large as $\Delta_{\xi} / \Delta \sim(L / \xi)^{d} \gg 1$ in the limit $L \gg \xi$. These states form a kind of Gaussian Ensemble on a spatially non-uniform (multifractal for $E \rightarrow E_{c}$ ) background $\Phi_{E}(r)$. Since the eigenfunction correlations are described by the formula (5.30), which has exactly the same form as in the Gaussian Ensemble, it is not surprising that the level statistics has the WD form everywhere in the extended phase [183,184]. For larger $\omega, \sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega)$ is expected to decrease as $\omega^{-\eta / d}$, where $\eta=d-D_{2}$, according to the scaling arguments [131,185-189] (see below), so that

$$
\begin{equation*}
\sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega) / \alpha(\boldsymbol{r}, \boldsymbol{r}, E) \sim\left(\omega / \Delta_{\xi}\right)^{-\eta / d}, \quad \omega>\Delta_{\xi}, \tag{5.31}
\end{equation*}
$$

up to a numerical coefficient of order unity.
The above formulas are valid in the metallic regime, i.e. for $L \gg \xi$. Exactly at the critical point $(\xi \gg L)$ they take the form

$$
\begin{equation*}
\sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega) / \alpha(\boldsymbol{r}, \boldsymbol{r}, E) \sim 1, \quad \omega<\Delta \tag{5.32}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega) / \alpha(\boldsymbol{r}, \boldsymbol{r}, E) \sim(\omega / \Delta)^{-\eta / d}, \quad \omega>\Delta . \tag{5.33}
\end{equation*}
$$

Of course, Eq. (5.32) is not sufficient to ensure the WD statistics at the critical point, since there is only of order of one level within its validity range $\omega \sim \Delta$. Indeed, the numerical simulations show that the level statistics on the mobility edge is different from the WD one [40,41,179-181].

However, Eq. (5.32) allows us to make an important conclusion concerning the behavior of $R(s)$ (or, which is essentially the same, of the nearest neighbor spacing distribution $P(s)$ ) at small $s=\omega / \Delta$. For this purpose, it is sufficient to consider only two neighboring levels. Let their energy difference be $\omega_{0} \sim \Delta$. Let us now perturb the system by a random potential $V(\boldsymbol{r})$ with $\langle V(\boldsymbol{r})\rangle=0$, $\left\langle V(\boldsymbol{r}) V\left(\boldsymbol{r}^{\prime}\right)\right\rangle=\Gamma \delta\left(\boldsymbol{r}-\boldsymbol{r}^{\prime}\right)$. For the two-level system it reduces to a $2 \times 2$ matrix $\left\{V_{i j}\right\}, i, j=1,2$, with elements $V_{i j}=\int \mathrm{d}^{d} \boldsymbol{r} V(\boldsymbol{r}) \Psi_{i}^{*}(\boldsymbol{r}) \Psi_{j}(\boldsymbol{r})$. The crucial point is that the variances of the diagonal and off-diagonal matrix elements are according to Eq. (5.32) equal to each other up to a factor of order of unity:

$$
\begin{equation*}
\left\langle V_{11}^{2}\right\rangle /\langle | V_{12}^{2}| \rangle=\sigma(\boldsymbol{r}, \boldsymbol{r}, E, \omega) / \alpha(\boldsymbol{r}, \boldsymbol{r}, E) \sim 1 . \tag{5.34}
\end{equation*}
$$

The distance between the perturbed levels is given by $\omega=\left[\left(V_{11}-V_{22}+\omega_{0}\right)^{2}+\left|V_{12}\right|^{2}\right]^{1 / 2}$. Choosing the amplitude of the potential in such a way that the typical energy shift $V_{11} \sim \Delta$ and using Eq. (5.34), we find $\left.\left.\langle | V_{12}\right|^{2}\right\rangle \sim \Delta$. As a result, the probability density for the level separation $\omega$ is for $\omega \ll \Delta$ of the form $\mathrm{d} P \sim(\omega / \Delta)^{\beta} \mathrm{d} \omega / \Delta$, with some prefactor of order of unity. We thus conclude that in the critical point

$$
\begin{equation*}
P(s) \simeq c_{\beta} s^{\beta}, \quad s \ll 1, \tag{5.35}
\end{equation*}
$$

with a coefficient $c_{\beta}$ of order of unity, in agreement with the numerical findings [41,179,181].

### 5.2.1. Spatial correlations at the mobility edge

Another question that one can ask concerning the wave function correlations at the mobility edge is how the correlations decay with distance in real space. Let us consider correlations $\alpha\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E\right)$ of the eigenfunction amplitudes in two different points. Matching the behavior of $\alpha\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E\right)$ at $\boldsymbol{r}=\boldsymbol{r}^{\prime}$ with that at $\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right| \sim \min \{L, \xi\}$ (where no strong correlations is expected), we find in the metallic phase $(L \gg \xi \gg l)$

$$
\begin{equation*}
\alpha\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E\right) \sim L^{-2 d \xi^{\eta}}\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|^{-\eta}, \quad l \leqq\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right| \lesssim \xi \tag{5.36}
\end{equation*}
$$

and at the critical point $(\xi \gg L)$

$$
\begin{equation*}
\alpha\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E\right) \sim L^{-2 d+\eta}\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|^{-\eta}, \quad l \lesssim\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right| \leqslant L . \tag{5.37}
\end{equation*}
$$

The same consideration can be straightforwardly applied to the higher correlation functions $\langle | \psi^{2 q_{1}}(\boldsymbol{r}) \| \psi^{2 q_{2}}\left(\boldsymbol{r}^{\prime}\right)| \rangle$, leading to the conclusion that they decay as

$$
\langle | \psi^{2 q_{1}}(\boldsymbol{r}) \| \psi^{2 q_{2}}\left(\boldsymbol{r}^{\prime}\right)| \rangle \propto\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|^{-\left[d+\tau\left(q_{1}\right)+\tau\left(q_{2}\right)-\tau\left(q_{1}+q_{2}\right)\right]},
$$

where $\tau(q) \equiv(q-1) D_{q}$ has been already introduced in Section 3.3.2. These results were obtained for the first time by Wegner [190] from renormalization-group calculations. The above formulas for the spatial correlations at the mobility edge can be generalized to the correlator $\sigma\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E, \omega\right)$ of two different eigenfunctions by using the scaling assumptions that this is the length $L_{\omega}=L(\Delta / \omega)^{1 / d}=(1 / \omega v)^{1 / d}$ which will set the correlation range at finite $\omega$. This give for the correlations at criticality ( $l<\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|<L_{\omega}<L, \xi$ )

$$
\begin{equation*}
\sigma\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E, \omega\right) \sim L^{-2 d}(\omega v)^{-\eta / d}\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|^{-\eta} . \tag{5.38}
\end{equation*}
$$

Chalker and Daniell [185,186] conjectured that the diffusion propagator $\gamma\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}, E, \omega\right)$, Eq. (3.77), has the same behavior (5.38) at the mobility edge. Extensive numerical simulations [185,187-189] fully confirm this conjecture, which determines the form of the anomalous diffusion at the critical point of the Anderson transition.

### 5.3. Power-law random banded matrix ensemble: Anderson transition in $1 D$

The ensemble of random banded matrices (RBM) is defined as the set of matrices with elements

$$
\begin{equation*}
H_{i j}=G_{i j} a(|i-j|), \tag{5.39}
\end{equation*}
$$

where the matrix $G$ runs over the Gaussian ensemble, and $a(r)$ is some function satisfying the condition $\lim _{r \rightarrow \infty} a(r)=0$ and determining the shape of the band. In the most frequently considered
case of RBM the function $a(r)$ is considered to be (exponentially) fast decaying when $r$ exceeds some typical value $b$ called the bandwidth. Matrices of this sort were first introduced as an attempt to describe an intermediate level statistics for Hamiltonian systems in a transitional regime between complete integrability and fully developed chaos [191] and then appeared in various contexts ranging from atomic physics (see [192] and references therein) to solid state physics [18] and especially in the course of investigations of the quantum behavior of periodically driven Hamiltonian systems [193,194]. The most studied system of the latter type is the quantum kicked rotor [104] characterized by the Hamiltonian

$$
\begin{equation*}
\hat{H}=\frac{\hat{l}^{2}}{2 I}+V(\theta) \sum_{m=-\infty}^{\infty} \delta(t-m T), \tag{5.40}
\end{equation*}
$$

where $\hat{l}=-\mathrm{i} \hbar \partial / \partial \theta$ is the angular momentum operator conjugated to the angle $\theta$. The constants $T$ and $I$ are the period of kicks and the moment of inertia, correspondingly, and $V(\theta)$ is usually taken to be $V(\theta)=k \cos \theta$. Classically, the kicked rotor exhibits an unbound diffusion in the angular momentum space when the strength of kicks $k$ exceeds some critical value. It was found, however, that the quantum effects suppress the classical diffusion in close analogy with the localization of a quantum particle by a random potential [104,195].

It is natural to consider the evolution (Floquet) operator $\hat{U}$ that relates values of the wavefunction over one period of perturbation, $\psi(\theta, t+T)=\hat{U} \psi(\theta, t)$, in the "unperturbed" basis of eigenfunctions of the operator

$$
\hat{l}:|l\rangle=\frac{1}{(2 \pi)^{1 / 2}} \exp (\operatorname{in} \theta), \quad n= \pm 0, \pm 1, \ldots .
$$

The matrix elements $\langle m| U|n\rangle$ tend to zero when $|m-n| \rightarrow \infty$. In the case $V(\theta)=k \cos \theta$ this decay is faster than exponential when $|m-n|$ exceeds $b \approx k / \hbar$, whereas within the band of the effective width $b$ matrix elements prove to be pseudorandom [104].

Let us note, however, that the exponentially rapid decay of $\langle m| U|n\rangle$ in the above mentioned situation is due to the infinite differentiability of $V(\theta)=\cos \theta$. If we take a function $V(\theta)$ having a discontinuity in a derivative of some order, the corresponding matrix elements of the evolution operator would decay in a power-law fashion in the limit $|n-m| \rightarrow \infty$. In fact, there is an interesting example of a periodically driven system where the matrix elements of the evolution operator decay in a power-law way, namely the so-called Fermi accelerator [58]. The power-law (pseudo-)random banded matrices appear also in other models of the quantum chaos, such as a close-to-circular Bunimovich stadium [196-198] or a Coulomb center inside an integrable billiard [59].

One may also consider the random matrix (5.39) as the Hamiltonian of a one-dimensional tight-binding model with long-ranged off-diagonal disorder (random hopping). A closely related problem with non-random long-range hopping and diagonal disorder was studied numerically in Ref. [199]. Effect of weak long-range hopping on the localized states in a 3D Anderson insulator was discussed by Levitov [200,201,324]. Similar models with a power-law hopping appear also in other physical contexts [60-62,202].

As was shown in [18,102], the conventional RBM model can be mapped onto a 1D supermatrix non-linear $\sigma$-model, which allows for an exact analytical solution. The same $\sigma$-model was derived
initially $[6,101]$ for a particle moving in a quasi-1D system (a wire) and being subject to a random potential. All states are found to be asymptotically localized, with the localization length

$$
\begin{equation*}
\xi=\frac{B_{2}\left(2 B_{0}-E^{2}\right)}{8 B_{0}^{2}} \propto b^{2}, \quad B_{k}=\sum_{r=-\infty}^{\infty} a^{2}(r) r^{k} . \tag{5.41}
\end{equation*}
$$

However, for the case of a power-like shape of the band,

$$
\begin{equation*}
a(r) \propto r^{-\alpha} \text { for large } r, \tag{5.42}
\end{equation*}
$$

this derivation should be reconsidered. As explained below, this leads [57] to a more general 1D $\sigma$-model with long-range interaction, which is much richer than the conventional short-range one. In particular, it exhibits the Anderson localization transition at $\alpha=1$. We mainly follow Ref. [57] in our presentation.

### 5.3.1. Mapping onto the effective $\sigma$-model

Let us consider the ensemble of large $N \times N$ matrices ( $N \rightarrow \infty$ ) defined by Eq. (5.39) with a function $a(r)$ having the form

$$
a(r) \simeq\left\{\begin{array}{cc}
1, & r<b  \tag{5.43}\\
(r / b)^{-\alpha}, & r>b
\end{array}\right.
$$

The parameter $b$ will serve to label the critical models with $\alpha=1$. We will consider $b$ to be large: $b \gg 1$, in order to justify formally the derivation of the $\sigma$-model. We will argue later on that our conclusions are qualitatively valid for arbitrary $b$ as well. We will call the ensemble (5.39), (5.43) the power-law random banded matrix (PRBM) model. While considering localization properties, we will restrict ourselves by one-loop calculation, and for this reason, by the orthogonal ensemble.

The PRBM model can be mapped onto the $\sigma$-model on a lattice,

$$
\begin{equation*}
\mathscr{S}\{Q\}=-\frac{1}{4}\left(\pi v A_{0}\right)^{2} \operatorname{Str} \sum_{i j}\left[\left(A^{-1}\right)_{i j}-A_{0}^{-1} \delta_{i j}\right] Q(i) Q(j)-\frac{\mathrm{i} \pi v \omega}{4} \sum_{i} \operatorname{Str} Q(i) \Lambda . \tag{5.44}
\end{equation*}
$$

Here $Q(i)=T_{i}^{-1} \Lambda T_{i}$ satisfies the constraint $Q^{2}(i)=1, A$ is the matrix with elements $A_{i j}=a^{2}(|i-j|), A_{0}$ is given by ${ }^{10} A_{0}=\sum_{l} A_{k l} \approx \sum_{r=-\infty}^{\infty} a^{2}(r)$, and $v$ is the density of states:

$$
\begin{equation*}
v=\frac{1}{2 \pi A_{0}}\left(4 A_{0}-E^{2}\right)^{1 / 2} . \tag{5.45}
\end{equation*}
$$

The standard next step is to restrict oneself to the long wavelength fluctuations of the $Q$-field. For usual RBM characterized by a function $a(r)$ decreasing faster than any power of $r$ as $r \rightarrow \infty$, this is

[^9]achieved by the momentum expansion of the first term in the action (5.44):
\[

$$
\begin{align*}
& \sum_{i j}\left[\left(A^{-1}\right)_{i j}-A_{0}^{-1} \delta_{i j}\right] Q(i) Q(j) \equiv N^{-1} \sum_{q}\left[A_{q}^{-1}-A_{0}^{-1}\right] Q_{q} Q_{-q} \\
& \quad \approx \frac{B_{2}}{2 A_{0}^{2}} N^{-1} \sum_{q} q^{2} Q_{q} Q_{-q}=\frac{B_{2}}{2 A_{0}^{2}} \int \mathrm{~d} x\left(\partial_{x} Q\right)^{2}, \tag{5.46}
\end{align*}
$$
\]

where $Q_{q}$ is the Fourier transform of $Q(i)$, and $B_{2}=\sum_{l} A_{k l}(k-l)^{2}$, as defined in Eq. (5.41). This immediately leads to the standard continuous version of the non-linear $\sigma$-model:

$$
\begin{equation*}
\mathscr{S}\{Q\}=-\frac{\pi v}{4} \operatorname{Str} \int \mathrm{~d} x\left[\frac{1}{2} D_{0}\left(\partial_{x} Q\right)^{2}+\mathrm{i} \omega Q \Lambda\right] \tag{5.47}
\end{equation*}
$$

with the classical diffusion constant $D_{0}=\pi \vee B_{2}$, which implies the exponential localization of eigenstates with the localization length $\xi=\pi v D_{0} \propto b^{2}$.

Let us try to implement the same procedure for the present case of power-like bandshape (5.43). Restricting ourselves to the lowest order term in the momentum expansion, we arrive again at Eq. (5.47) as long as $\alpha \geq 3 / 2$. This suggests that for $\alpha \geq 3 / 2$ the eigenstates of the present model should be localized in the spatial domain of the extension $\xi \propto v D_{0}$. However, in contrast to the usual RBM model this localization is power-like rather than exponential: $|\psi(r)|^{2} \propto r^{-2 \alpha}$ at $r \gg \xi$. This is quite evident due to the possibility of direct hopping with the power-law amplitude. On a more formal level the appearance of power-law tails of wave functions is a consequence of the breakdown of the momentum expansion for the function $A_{q}^{-1}-A_{0}^{-1}$ in higher orders in $q^{2}$. The presence of power-law "tails" of the wave functions, with an exponent $\alpha$ determined by the decay of hopping elements, was found in numerical simulations in Refs. [197-199].

The most interesting region $1 / 2<\alpha<3 / 2$ requires a separate consideration. In this case, Eq. (5.46) loses its validity in view of the divergence of the coefficient $B_{2}$. We find instead

$$
\begin{align*}
A_{0}^{2}\left(A_{q}^{-1}-A_{0}^{-1}\right) & \approx A_{0}-A_{q}=2 \int_{0}^{\infty} \mathrm{d} r a^{2}(r)(1-\cos q r) \\
& =\frac{2}{|q|}\left\{\int_{0}^{||q|} \mathrm{d} x(1-\cos x)+(b \mid q)^{2 \alpha} \int_{b|q|}^{\infty} \frac{\mathrm{d} x}{2^{2 \alpha}}(1-\cos x)\right\} \\
& \approx c_{\alpha} b^{2 \alpha}|q|^{2 \alpha-1} \quad \text { for } 1 / 2<\alpha<3 / 2 \text { and }|q| \ll 1 / b, \tag{5.48}
\end{align*}
$$

where

$$
c_{\alpha}=2 \int_{0}^{\infty} \frac{\mathrm{d} x}{x^{2 \alpha}}(1-\cos x)
$$

is a numerical constant, $c_{\alpha}=[2 \Gamma(2-2 \alpha) /(2 \alpha-1)] \sin \pi \alpha$ for $\alpha \neq 1$ and $c_{1}=\pi$ for $\alpha=1$.
The corresponding long-wavelength part of the action,

$$
\begin{equation*}
\mathscr{S}_{0}\{Q\}=-\frac{1}{t} \operatorname{Str} \int(\mathrm{~d} q)|q|^{2 \alpha-1} Q_{q} Q_{-q}, \tag{5.49}
\end{equation*}
$$

where

$$
\int(\mathrm{d} q) \equiv \int \frac{\mathrm{d} q}{2 \pi} \equiv N^{-1} \sum_{q}
$$

cannot be reduced to the local-in-space form in the coordinate representation any longer. Here, $1 / t=\frac{1}{4}(\pi v)^{2} c_{\alpha} b^{2 \alpha} \propto b^{2 \alpha-1} \gg 1$ plays the role of coupling constant, justifying the perturbative and renormalization group treatment of the model described below.

Let us mention that if we consider the RBM model as a tight-binding Hamiltonian, the corresponding classical motion described by the master equation on the 1 D lattice is superdiffusive for $1 / 2<\alpha<3 / 2$ with a typical displacement in a time $t$ being $r \propto t^{1 /(2 \alpha-1)}$ (Levy law of index $2 \alpha-1$; see Ref. [203] for a review). As will be discussed in Section 5.3.3, this influences the asymptotic behavior of the spectral correlation function for the corresponding quantum system.
5.3.1.1. Perturbative treatment of the non-linear $\sigma$-model: General formulas. Here we derive oneloop perturbative corrections to the density-density correlation function and inverse participation ratios. Analysis of these expressions for various values of the power-law parameter $\alpha$ will be presented in Section 5.3.2.

The density-density correlation function (diffusion propagator)

$$
K\left(r_{1}, r_{2} ; \omega\right)=G_{\mathrm{R}}^{E+\omega / 2}\left(r_{1}, r_{2}\right) G_{\mathrm{A}}^{E-\omega / 2}\left(r_{2}, r_{1}\right)
$$

can be expressed in terms of the $\sigma$-model as follows [6]:

$$
\begin{equation*}
K\left(r_{1}, r_{2} ; \omega\right)=-(\pi v)^{2} \int D Q Q_{12, \alpha \beta}\left(r_{1}\right) k_{\beta \beta} Q_{21, \beta \alpha}\left(r_{2}\right) \mathrm{e}^{-S\{Q\}} \tag{5.50}
\end{equation*}
$$

Here the indices $p, p^{\prime}$ of the matrix $Q_{p p^{\prime}, \alpha \beta}$ correspond to its advanced-retarded block structure, whereas $\alpha, \beta$ discriminate between bosonic and fermionic degrees of freedom. The matrix $k_{\beta \beta}$ is equal to 1 for bosons and $(-1)$ for fermions. To calculate the correlation function (5.50) perturbatively, we use here the following parametrization [6] of the matrix $Q$ :

$$
\begin{equation*}
Q=\Lambda\left(W+\sqrt{1+W^{2}}\right)=\Lambda\left(1+W+\frac{W^{2}}{2}-\frac{W^{4}}{8}+\ldots\right) \tag{5.51}
\end{equation*}
$$

where $W$ is block-off-diagonal in the advanced-retarded representation. To get the perturbative expansion for $K\left(r_{1}, r_{2} ; \omega\right)$, one has to substitute Eq. (5.51) into (5.50), to separate the part quadratic in $W$ from the rest in the exponent and to apply the Wick theorem. In the usual case, when the action is given by Eq. (5.47), the leading order (tree level) result reads in the momentum space as follows:

$$
\begin{equation*}
K_{0}(q, \omega)=\frac{2 \pi v}{D_{0} q^{2}-\mathrm{i} \omega} \tag{5.52}
\end{equation*}
$$

The perturbative quantum corrections do not modify the general form (5.52), but change the value of the diffusion constant. In particular, in one-loop order one gets Eq. (5.52) with $D_{0}$ replaced
by [204]

This induces the standard weak-localization correction to the conductivity.
Now we implement an analogous procedure for the non-local $\sigma$-model of the type of Eqs. (5.44) and (5.49):

$$
\begin{equation*}
\mathscr{S}\{Q\}=\frac{1}{t} \operatorname{Str} \sum_{r, r^{\prime}} U\left(r-r^{\prime}\right) Q(r) Q\left(r^{\prime}\right)-\mathrm{i} \frac{\pi v}{4} \sum_{r} \operatorname{Str} \Lambda Q(r), \tag{5.54}
\end{equation*}
$$

with $U(r) \propto r^{-2 \alpha}$ as $r \rightarrow \infty$, so that the Fourier transform of $U(r)$ behaves at small momenta as

$$
\begin{equation*}
\tilde{U}(q)=-|q|^{\sigma}, \quad 1 / 2<\sigma<2 . \tag{5.55}
\end{equation*}
$$

The exponent $\sigma$ is related to the parameter $\alpha$ of the RBM model by $\sigma=2 \alpha-1$. In leading order, we keep in the action the terms quadratic in $W$ only, which yields

$$
\begin{equation*}
K_{0}(q, \omega)=\frac{2 \pi v}{8(\pi v t)^{-1}|q|^{\sigma}-\mathrm{i} \omega}, \tag{5.56}
\end{equation*}
$$

corresponding to a superdiffusive behavior.
To calculate the one-loop correction to $K_{0}(q)$ (we set $\omega=0$ for simplicity) we expand the kinetic term in $\mathscr{S}\{Q\}$ up to fourth order in $W$ :

$$
\begin{equation*}
\left.\sum_{r, r^{\prime}} \operatorname{Str} U\left(r-r^{\prime}\right) Q(r) Q\left(r^{\prime}\right)\right|_{4 \text { th order }}=\sum_{r, r^{\prime}} \frac{1}{4} \operatorname{Str} W^{2}(r) W^{2}\left(r^{\prime}\right) \tag{5.57}
\end{equation*}
$$

The contraction rules are given by Eqs. (8) and (16) of Ref. [25] (reproduced below as Eq. (2.21) for the unitary symmetry), with the propagator $\Pi(q)$ replaced by

$$
\begin{equation*}
\Pi(q)=t / 8|q|^{\sigma} . \tag{5.58}
\end{equation*}
$$

Evaluating the one-loop correction, we get the following expression for the density-density correlation function up to the one-loop order:

$$
\begin{equation*}
K^{-1}(q)=K_{0}^{-1}(q)-\frac{(\pi v)^{2}}{2} \int(\mathrm{~d} k) \frac{|q+k|^{\sigma}-|k|^{\sigma}}{|k|^{\sigma}} . \tag{5.59}
\end{equation*}
$$

Now we calculate the perturbative correction to the inverse participation ratios $P_{q}$. The results of Ref. [25] presented in Section 3.3.1 are straightforwardly applicable to the present case of power-law RBM, provided the appropriate modification of the diffusion propagator entering the contraction rules is made, see the text preceding Eq. (5.58). One finds

$$
\begin{equation*}
\left\langle P_{q}\right\rangle=\left\{1+\frac{1}{N} q(q-1) \sum_{r} \Pi(r, r)\right\} \frac{(2 q-1)!!}{N^{q-1}}, \tag{5.60}
\end{equation*}
$$

where $\Pi\left(r, r^{\prime}\right)=(1 / N) \sum_{q} \Pi(q) \exp \left[\mathrm{i} q\left(r-r^{\prime}\right)\right]$ and $\Pi(q)$ is given by Eq. (5.58).
5.3.1.2. Renormalization group treatment. The effective $\sigma$-model, Eq. (5.54), is actually of onedimensional nature. However, for the sake of generality, we find it convenient to consider it here to be defined in $d$-dimensional space with arbitrary $d$. The form (5.56) of the generalized diffusion propagator implies that $d=\sigma$ plays the role of the logarithmic dimension for the problem. In the vicinity of this critical value we can carry out a renormalization group $(\mathrm{RG})$ treatment of the model, following the procedure developed for general non-linear $\sigma$-models in [205-208]. We begin by expressing the action in terms of the renormalized coupling constant $t=Z_{1}^{-1} t_{\mathrm{B}} \mu^{d-\sigma}$, where $t_{\mathrm{B}}$ is the bare coupling constant, $Z_{1}$ is the renormalization constant, and $\mu^{-1}$ is the length scale governing the renormalization: ${ }^{11}$

$$
\begin{align*}
S= & \frac{\mu^{d-\sigma}}{2 t Z_{1}} \sum_{r r^{\prime}} U\left(\left|r-r^{\prime}\right|\right) \operatorname{Str}\left[-W(r) W\left(r^{\prime}\right)+\sqrt{1+W^{2}(r)} \sqrt{1+W^{2}\left(r^{\prime}\right)}\right] \\
& -\frac{\mathrm{i} \pi v \omega}{4} \sum_{r} \operatorname{Str} \sqrt{1+W^{2}(r)} . \tag{5.61}
\end{align*}
$$

Expanding the action in powers of $W(r)$ and keeping terms up to 4th order, we get

$$
\begin{align*}
& S=S_{0}+S_{1}+\mathrm{O}\left(W^{6}\right), \\
& S_{0}=\frac{\mu^{d-\sigma}}{4 t Z_{1}} \sum_{r r^{\prime}} U\left(\mid r-r^{\prime}\right) \operatorname{Str}\left(W(r)-W\left(r^{\prime}\right)\right)^{2}-\frac{\mathrm{i} \pi v \omega}{8} \sum_{r} \operatorname{Str} W^{2}(r), \\
& S_{1}=\frac{\mu^{d-\sigma}}{8 t Z_{1}} \sum_{r r^{\prime}} U\left(\left|r-r^{\prime}\right|\right) \operatorname{Str} W^{2}(r) W^{2}\left(r^{\prime}\right)+\frac{\mathrm{i} \pi v \omega}{32} \sum_{r} \operatorname{Str} W^{4}(r) . \tag{5.62}
\end{align*}
$$

We have restricted ourselves to 4th order terms, since they are sufficient for obtaining the renormalized quadratic part of the action in one-loop order. The calculation yields, after the cancelation of an $\int(\mathrm{d} k) \propto \delta(0)$ term with the contribution of the Jacobian:

$$
\begin{align*}
S_{\text {quad }} & =S_{0}+\left\langle S_{1}\right\rangle \\
& =\frac{1}{2} \int(\mathrm{~d} q) \operatorname{Str} W_{q} W_{-q}\left[-\frac{\mu^{d-\sigma}}{t Z_{1}} \tilde{U}(q)-\frac{1}{2} \int(\mathrm{~d} k) \frac{\tilde{U}(k)-\tilde{U}(k+q)}{-\frac{\mu^{d-\sigma}}{t Z_{1}} \tilde{U}(k)-\frac{\mathrm{i} \pi v \omega}{4}}\right] . \tag{5.63}
\end{align*}
$$

According to the renormalization group idea, one has to chose the constant $Z_{1}(t)=1+a t+\cdots$ so as to cancel the divergence in the coefficient in front of the leading $|q|^{\sigma}$ term.

### 5.3.2. Localization and eigenfunction statistics

In this subsection we analyze the model in different regions of the exponent $\sigma=2 \alpha-1$ using the general formulas derived in Section 5.3.2.

[^10]5.3.2.1. Localized regime: $1<\sigma<2(1<\alpha<3 / 2)$. To evaluate the one-loop correction (5.59) to the diffusion propagator, we use the expansion
\[

\frac{|\boldsymbol{q}+\boldsymbol{k}|^{\sigma}}{|\boldsymbol{k}|^{\sigma}}-1 \simeq $$
\begin{cases}\sigma \frac{\boldsymbol{q} \boldsymbol{k}}{k^{2}}+\frac{\sigma q^{2}}{2} \frac{k^{2}}{k^{2}}+\sigma\left(\frac{\sigma}{2}-1\right)\left(\frac{\boldsymbol{q} \boldsymbol{k}}{k^{2}}\right)^{2}+\cdots, & q \ll k  \tag{5.64}\\ \frac{|q|^{\sigma}}{|k|^{\sigma}}, & q \gg k\end{cases}
$$
\]

Thus, the integral in Eq. (5.59) can be estimated as

$$
\begin{equation*}
I \equiv \int(\mathrm{~d} k)\left[\frac{|\boldsymbol{q}+\boldsymbol{k}|^{\sigma}}{|\boldsymbol{k}|^{\sigma}}-1\right] \approx \int_{k>q}(\mathrm{~d} k) \sigma\left(1+\frac{\sigma-2}{d}\right) \frac{q^{2}}{2 k^{2}}+\int_{k<q}(\mathrm{~d} k) \frac{|q|^{\sigma}}{|k|^{\sigma}}, \tag{5.65}
\end{equation*}
$$

and we are interested in the particular case of the dimension $d=1$. For $\sigma>d=1$ the integral diverges at low $k$, and the second term in Eq. (5.65) dominates. This gives

$$
\begin{equation*}
I \sim L^{\sigma-1}|q|^{\sigma}, \tag{5.66}
\end{equation*}
$$

where $L$ is the system size determining the infrared cut-off (in the original RBM formulation it is just the matrix size $N$ ). This leads to the following one-loop expression for the diffusion correlator

$$
\begin{equation*}
K(q)=\frac{(\pi v)^{2} \tilde{t}}{4|q|^{\sigma}}, \quad \tilde{t}^{-1}=t^{-1}-\text { const } L^{\sigma-1} . \tag{5.67}
\end{equation*}
$$

Now we turn to the renormalization group analysis, as described in Section 5.3.1. In the one-loop order, the expression for the renormalization constant $Z_{1}$ is determined essentially by the same integral $I$, Eq. (5.65), with the RG scale $\mu$ playing the role of the infrared cut-off (analogous to that of the system size $L$ in Eq. (5.67)). This yields (in the minimal subtraction scheme):

$$
\begin{equation*}
Z_{1}(t)=1-\frac{1}{2 \pi} \frac{t}{\sigma-1}+\mathrm{O}\left(t^{2}\right), \tag{5.68}
\end{equation*}
$$

leading to the following relation between the bare and the renormalized coupling constant analogous to Eq. (5.67):

$$
\begin{equation*}
\frac{1}{t} \mu^{1-\sigma} \equiv \frac{Z_{1}}{t_{\mathrm{B}}}=\frac{1}{t_{\mathrm{B}}}-\frac{1}{2} \frac{1}{\sigma-1} \mu^{1-\sigma} . \tag{5.69}
\end{equation*}
$$

From Eq. (5.68), we get the expression for the $\beta$-function:

$$
\begin{equation*}
\beta(t)=\left.\frac{\partial t}{\partial \ln \mu}\right|_{t_{\mathrm{B}}}=\frac{(1-\sigma) t}{1+t \partial_{t} \ln Z_{1}(t)}=-(\sigma-1) t-\frac{t^{2}}{2 \pi}+\mathrm{O}\left(t^{3}\right) . \tag{5.70}
\end{equation*}
$$

Both Eqs. (5.67) and (5.70) show that the coupling constant $t$ increases with the system size $L$ (resp. scale $\mu$ ), which is analogous to the behavior found in the conventional scaling theory of localization in $d<2$ dimensions [167,209]. The RG flow reaches the strong coupling regime $t \sim 1$ at the scale $\mu \sim t_{\mathrm{B}}^{1 /(\sigma-1)}$. Remembering the relation of the bare coupling constant $t_{\mathrm{B}}$ and the index $\sigma$ to the
parameters of the original PRBM model: $t_{\mathrm{B}}^{-1} \propto b^{2 \alpha-1} ; \sigma \propto 2 \alpha-1$, we conclude that the length scale

$$
\begin{equation*}
\xi \sim t_{\mathrm{B}}^{-1 /(\sigma-1)} \sim b^{(2 \alpha-1) /(2 \alpha-2)} \tag{5.71}
\end{equation*}
$$

plays the role of the localization length for the PRBM model.
This conclusion is also supported by an inspection of the expression for the IPR, Eq. (5.60). Evaluation of the one-loop perturbative correction in Eq. (5.60) yields

$$
\begin{equation*}
\left\langle P_{q}\right\rangle=\frac{(2 q-1)!!}{N^{q-1}}\left\{1+q(q-1) \frac{t}{8 \pi^{\sigma}} \zeta(\sigma) N^{\sigma-1}\right\}, \tag{5.72}
\end{equation*}
$$

where $\zeta(\sigma)$ is Riemann's zeta-function; $\zeta(\sigma) \simeq 1 /(\sigma-1)$ for $\sigma$ close to unity. The correction term becomes comparable to the leading (GOE) contribution for the system size $N \sim t^{1 /(\sigma-1)}$, parametrically coinciding with the localization length $\xi$. For larger $N$ the perturbative expression (5.72) loses its validity, and the IPR is expected to saturate at a constant value $\left\langle P_{q}\right\rangle \sim \xi^{1-q}$ for $N \gg \xi$.

In concluding this subsection, let us stress once more that the localized eigenstates in the present model are expected to have integrable power-law tails: $\left|\psi^{2}(r)\right| \propto r^{-2 \alpha}=r^{-\sigma-1}$ at $r \gg \xi$.
5.3.2.2. Delocalized regime: $0<\sigma<1(1 / 2<\alpha<1)$. We begin again by considering the perturbative corrections to the diffusion propagator (5.59). The first term in the r.h.s. of Eq. (5.65) proportional to $\int \mathrm{d} k / k^{2}$ is determined by the vicinity of its lower cut-off (i.e. $k \sim q$ ), whereas the second one is proportional to $\int \mathrm{d} k / k^{\sigma}$ and thus determined by the vicinity of its upper cut-off (i.e. again $k \sim q$ ). Therefore, the integral (5.65) is now dominated by the region $k \sim q$, and is proportional to $|q|$. We get, therefore:

$$
\begin{equation*}
(\pi v)^{2} K^{-1}(q)=\frac{4|q|^{\sigma}}{t}-C_{\sigma}|q| \tag{5.73}
\end{equation*}
$$

with a numerical constant $C_{\sigma}$.
We see that the correction term is of higher order in $|q|$ as compared to the leading one. Thus, it does not lead to a renormalization of the coupling constant $t$. This is readily seen also in the framework of the RG scheme, where the one-loop integral in Eq. (5.63) does not give rise to terms of the form $|q|^{\sigma}$. One can check that this feature is not specific to the one-loop RG calculation, but holds in higher orders as well. For the case of a vector model with long-ranged interaction this conclusion was reached by Brezin et al. [210]. Thus, the renormalization constant $Z_{1}$ is equal to unity and the $\beta$-function is trivial:

$$
\begin{equation*}
\beta(t)=(1-\sigma) t . \tag{5.74}
\end{equation*}
$$

This means that the model does not possess a critical point, and, for $\sigma<1$, all states are delocalized, for any value of the bare coupling constant $t$. This property should be contrasted with the behavior of a $d$-dimensional conductor described by the conventional local non-linear $\sigma$-model and undergoing an Anderson transition at some critical coupling $t=t_{\mathrm{c}}$ [167].
Though all states of the model are delocalized, their statistical properties are different from GOE. In particular, calculating the variance of the inverse participation ratio $P_{2}$ (see Section 3.3.3,

Eq. (3.89)) we get

$$
\begin{equation*}
\delta\left(P_{2}\right)=\frac{\left\langle P_{2}^{2}\right\rangle-\left\langle P_{2}\right\rangle^{2}}{\left\langle P_{2}\right\rangle^{2}}=\frac{8}{N^{2}} \sum_{r r^{\prime}} \Pi^{2}\left(r, r^{\prime}\right)=\frac{8}{N^{2}}{ }_{q=\pi n / N ; n=1,2, \ldots} \Pi^{2}(q) . \tag{5.75}
\end{equation*}
$$

At $\sigma>1 / 2$ the sum over $q$ is convergent yielding

$$
\begin{equation*}
\delta\left(P_{2}\right)=\frac{t^{2}}{8 \pi^{2 \sigma}} \frac{\zeta(2 \sigma)}{N^{2-2 \sigma}} . \tag{5.76}
\end{equation*}
$$

Thus, in this regime the fluctuations of the IPR are much stronger than for the GOE where $\delta\left(P_{2}\right) \propto 1 / N$. Only for $\sigma<1 / 2(\alpha<3 / 4)$ the IPR fluctuations acquire the GOE character. Considering higher irreducible moments (cumulants) of the IPR, $\left\langle\left\langle P_{2}^{n}\right\rangle\right.$, one finds that the GOE behavior is restored at $\sigma<\sigma_{c}^{(n)} \equiv 1 / n$. In this sense, the model is analogous to a $d$-dimensional conductor at $d=2 / \sigma$. Therefore, only when $\sigma \rightarrow 0$ (correspondingly, $\alpha \rightarrow 1 / 2$ in the original PRBM formulation) all statistical properties become equivalent to those characteristic of GOE.
5.3.2.3. Critical regime: $\sigma=1(\alpha=1)$. As we have seen, $\sigma=1$ separates the regions of localized $(\sigma>1)$ and extended $(\sigma<1)$ states. It is then natural to expect some critical properties showing up just at $\sigma=1$. Let us again consider the generalized diffusion propagator, Eq. (5.59). At $\sigma=1$ the one-loop correction yields

$$
\begin{equation*}
\frac{1}{(\pi v)^{2}} K^{-1}(q)=4|q|\left[t^{-1}-\frac{1}{8 \pi} \ln (|q| L)\right] . \tag{5.77}
\end{equation*}
$$

As expected at the critical point, the correction to the coupling constant is of logarithmic nature. However, Eq. (5.77) differs essentially from that typical for a 2D disordered conductor:

$$
\begin{equation*}
t^{-1}=t_{\mathbf{B}}^{-1}-\ln (L / l), \tag{5.78}
\end{equation*}
$$

where the bare coupling constant $t_{\mathrm{B}}$ corresponds to scale $l$. Comparing the two formulas, we see that in Eq. (5.77) the mean free path $l$ is replaced by the inverse momentum $q^{-1}$. Therefore, the correction to the bare coupling constant is small for low momenta $q \sim 1 / L$, and the correlator $K(q)$ is not renormalized. This implies the absence of eigenstate localization, in contrast to the 2D diffusive conductor case, where Eq. (5.78) results in an exponentially large localization length $\xi \propto \exp t_{\mathrm{B}}^{-1}$. On a more formal level, the absence of essential corrections to the low- $q$ behavior of $K(q)$ is due to the fact that the region $k>q$ does not give a logarithmic contribution. This is intimately connected with the absence of $t$ renormalization at $\sigma<1$.

To study in more details the structure of critical eigenfunctions, let us consider the set of IPR $P_{q}$. The perturbative correction, Eq. (5.60), is evaluated at $\sigma=1$ as

$$
\begin{equation*}
\left\langle P_{q}\right\rangle=\left\{1+q(q-1) \frac{t}{8 \pi} \ln (N / b)\right\} \frac{(2 q-1)!!}{N^{q-1}}, \tag{5.79}
\end{equation*}
$$

where the microscopic scale $b$, Eq. (5.43), enters as the ultraviolet cut-off for the $\sigma$-model, the role usually played by the mean free path $l$. This formula is valid as long as the correction is small:

$$
q \ll\left[\frac{t}{8 \pi} \ln (N / b)\right]^{-1 / 2}
$$

For larger $q$ the perturbation theory breaks down, and one has to use the renormalization group approach. This requires $[28,50]$ introduction of higher vertices of the type $z_{q} \int \operatorname{Str}^{q}(Q k \Lambda) \mathrm{d} r$ into the action of the non-linear $\sigma$-model and their subsequent renormalization. The resulting RG equations for the charges $z_{q}$ read, in the one-loop order,

$$
\begin{equation*}
\frac{\mathrm{d} z_{q}}{\mathrm{~d} \ln \mu^{-1}}=q(q-1) \frac{t}{8 \pi} z_{q}, \tag{5.80}
\end{equation*}
$$

where $\mu^{-1}$ is the renormalization scale. Integrating Eq. (5.80), we find

$$
\begin{equation*}
\left\langle P_{q}\right\rangle=\frac{(2 q-1)!!}{N^{q-1}}\left(\frac{N}{b}\right)^{q(q-1) t / 8 \pi} . \tag{5.81}
\end{equation*}
$$

Note, that this formula is reduced to the perturbative expression, Eq. (5.79), in the regime

$$
q \ll\left[\frac{t}{8 \pi} \ln (N / b)\right]^{-1 / 2}
$$

The behavior described by Eq. (5.81) is characteristic of a multifractal structure of wave functions (see Sections 3.3.2 and 5.2). Comparing (5.81) with (3.60), we find the multifractality dimensions $D_{q}=1-q t / 8 \pi$. The general formula valid for both orthogonal $(\beta=1)$ and unitary ( $\beta=2$ ) universality classes is

$$
\begin{equation*}
D_{q}=1-q \frac{t}{8 \pi \beta} . \tag{5.82}
\end{equation*}
$$

This form of the fractal dimensions is reminiscent of that found in two and $2+\varepsilon$ dimensions for the usual diffusive conductor, see Eq. (3.63). The one-loop result (5.82) holds for $q \ll 8 \pi / t$.

Now we can understand the reason for the $q$-dependent logarithmic correction to the diffusion propagator $K(q)$, Eq. (5.77). As was mentioned in the end of Section 5.2 , the multifractality of eigenfunctions determines the momentum dependence of the diffusion propagator at high $q$ in the critical point [185,186]

$$
\begin{equation*}
\left.K^{-1}(q) \propto|q|| | q \mid L\right)^{-\eta}, \quad \eta=d-D_{2} \tag{5.83}
\end{equation*}
$$

(at finite frequency $\omega$ the system size $L$ is replaced by $L_{\omega} \sim(D / \omega)^{1 / 2}$ ). On the other hand, the logarithmic correction in Eq. (5.77) is the first term of the expansion

$$
\begin{align*}
\frac{1}{(\pi v)^{2}} K^{-1}(q) & =\frac{4}{t}|q|\left\{1-\frac{t}{8 \pi} \ln (|q| L)+C_{2}\left[\frac{t}{8 \pi} \ln (|q| L)\right]^{2}+\cdots\right\} \\
& =\frac{|q|}{t} F\left(\frac{t}{8 \pi} \ln (|q| L)\right) \tag{5.84}
\end{align*}
$$

where $F(x)$ is some parameterless function. Since $\eta=1-D_{2} \simeq t / 4 \pi$, Eq. (5.84) has precisely the form expected from (5.83), assuming that $F(x) \sim \mathrm{e}^{-2 x}$ at $x \gg 1$.

The set of fractal dimensions $D_{q}$ (as well as spectral properties at $\sigma=1$, see Section 5.3.3) is parametrized by the coupling constant $t$. Strictly speaking, the above $\sigma$-model derivation is justified for $t \ll 1$ (i.e. $b \gtrdot 1$ ). However, the opposite limiting case can be also studied, following Levitov [200,201,324]. It corresponds to a $d$-dimensional Anderson insulator, perturbed by a weak
long-range hopping with an amplitude decreasing with distance as $r^{-\sigma}$. The arguments of Levitov imply that the states delocalize at $\sigma \leq d$, carrying fractal properties at $\sigma=d$. The PRBM model in the limit $b \ll 1$ is just the 1 D version of this problem. This shows that the conclusion about localization (delocalization) of eigenstates for $\sigma>1$ (resp. $\sigma<1$ ), with $\sigma=1$ being a critical point holds irrespective of the particular value of the parameter $b$. The $b \ll 1$ limit of the PRBM model was also studied in Ref. [61] where the IPR $P_{2}$ was calculated, yielding the fractal dimension $D_{2} \propto b$. Alternatively, the regime of the Anderson insulator with weak power-law hopping can be described in the framework of the non-linear $\sigma$-model, Eq. (5.54), by considering the limit $t \gg 1$. Formally, the non-linear $\sigma$-model for arbitrary $t$ can be derived from a microscopic tight-binding model by allowing $n \gg 1$ "orbitals" per site [167].

To summarize, the PRBM model (5.39), (5.43) with $0<b<\infty$ or the $\sigma$-model (5.54) with a coupling constant $0<t<\infty$ represent at $\sigma=1$ a continuous family of critical theories parametrized by the value of $b$ (respectively, $t$ ).
5.3.2.4. Numerical simulations. Numerical simulations of the PRBM model were performed in Ref. [57] for values of $\alpha \in[0,2]$ and $b=1$. In Fig. 3 we present typical eigenfunctions for four


Fig. 3. Typical eigenfunctions for the matrix size $N=800$ and four different values of $\alpha: 0.375,0.875,1.250$, and 1.625 . From [57].
different regions of $\alpha$. In agreement with the theoretical picture presented above, the eigenstates corresponding to $\alpha=0.375$ and $\alpha=0.875$ are extended, whereas those corresponding to $\alpha=1.25$ and $\alpha=1.625$ are localized. At the same time, one can notice that the states with $\alpha=0.875$ and $\alpha=1.25$ exhibit a quite sparse structure, as opposed to the other two cases. This can be explained by the proximity of the former two values of $\alpha$ to the critical value $\alpha=1.0$ where eigenstates should show the multifractal behavior. To get a quantitative information about the properties of the eigenstates, the mean value of the IPR, $\left\langle P_{2}\right\rangle$, and its relative variance, $\delta=\left(\left\langle P_{2}^{2}\right\rangle-\left\langle P_{2}\right\rangle^{2}\right) /\left\langle P_{2}\right\rangle^{2}$, were calculated in [57]. At any given $\alpha$ the dependences of the quantities $\left\langle P_{2}\right\rangle$ and $\delta$ on the matrix size $N$ were approximated by the power-laws $\left\langle P_{2}\right\rangle \propto 1 / N^{v}, \delta \propto 1 / N^{\mu}$ for $N$ ranging from 100 to 2400. In Figs. 4 and 5 the values of the exponents $v$ and $\mu$ obtained in this way are plotted versus the PRBM parameter $\alpha$. The expected theoretical curves are presented as well. We see from Fig. 4 that the data show a crossover from the behavior typical for extended states $(v=1)$ to that typical for localized states $(v=0)$, centered approximately at the critical point $\alpha=1$. The deviations from the sharp step-like theoretical curve $v(\alpha)$ can be presumably attributed to the finite-size effects which are unusually pronounced in the PRBM model due to the long-range nature of the off-diagonal coupling. The data for the exponent $\mu$ (Fig. 5) also show a reasonable agreement with the expected linear crossover, $\mu=4(1-\alpha)$ for $3 / 4<\alpha<1$, see Eq. (5.76).

While the presented data are in good agreement with the above theoretical picture, a more detailed numerical investigation of the structure of eigenstates and of spectral statistics is certainly desirable. In particular, it would be especially interesting to study the critical manifold, $\alpha=1$, where the multifractal properties of eigenstates and intermediate level statistics are predicted by the theory.


Fig. 4. Index $v$ characterizing the dependence of the inverse participation ratio $\left\langle P_{2}\right\rangle$ on the matrix size $N$ via $\left\langle P_{2}\right\rangle \propto 1 / N^{v}$, as a function of $\alpha$. Points refer to the best-fit values obtained from matrix sizes between $N=100$ and $N=1000$ (squares) or $N=2400$ (circles). The dashed line is the theoretical prediction for the transition from $v=1$, at small $\alpha$, to $v=0$, at large $\alpha$. From [57].

Fig. 5. The same as Fig. 4, but for the index $\mu$, derived from the $N$ dependence of the variance $\delta$ of the inverse participation ratio: $\delta \equiv\left(\left\langle P_{2}^{2}\right\rangle-\left\langle P_{2}\right\rangle^{2}\right) /\left\langle P_{2}\right\rangle^{2} \propto 1 / N^{\mu}$. The dashed line corresponds to the predicted linear crossover from $\mu=1$ at $\alpha<3 / 4$ to $\mu=0$ at $\alpha>1$. From [57].

### 5.3.3. Spectral properties

Let us consider now the spectral statistics of the PRBM model. In the "metallic" regime, the leading correction to the Wigner-Dyson form $R_{\text {WD }}(s)$ of the level correlation function is given by the properly modified Eq. (2.30), i.e.

$$
\begin{equation*}
R(s)=\left[1+\frac{1}{2 \beta} \mathscr{C} \frac{\mathrm{~d}^{2}}{\mathrm{~d} s^{2}} s^{2}\right] R_{\mathrm{WD}}(s), \tag{5.85}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{C} & =\frac{1}{N^{2}} \sum_{r r^{\prime}} \Pi^{2}\left(r, r^{\prime}\right)=\frac{1}{N^{2}}{ }_{q=\pi n / N ;} \sum_{n=1,2, \ldots} \Pi^{2}(q) \\
& =\left\{\begin{array}{cl}
\frac{t^{2}}{64 \pi^{2 \sigma}} N^{2 \sigma-2}, & \sigma>1 / 2 \\
\text { const } \frac{t^{2}}{64 \pi^{2 \sigma} b^{1-2 \sigma}} N^{-1}, & \sigma<1 / 2 .
\end{array}\right. \tag{5.86}
\end{align*}
$$

At $\sigma<1 / 2$ the sum divergent at high momenta is cut off at $q \sim \pi / b$; the procedure leaving undetermined a constant of order of unity. The correlation function $R(s)$ is close to its RMT value if $\sigma<1$ (the region of delocalized states), or else if $\sigma>1$ and the system size $N$ is much less than the localization length $\xi$, Eq. (5.71). Under these conditions, Eq. (5.85) holds as long as the correction term is small compared to the leading one. This requirement produces the following restriction on the frequency $s=\omega / \Delta:$

$$
s<s_{c} \sim\left\{\begin{array}{cl}
t^{-1} N^{1-\sigma}, & \sigma>1 / 2  \tag{5.87}\\
t^{-1} b^{1 / 2-\sigma} N^{1 / 2} \propto(N b)^{1 / 2}, & \sigma<1 / 2
\end{array}\right.
$$

At larger frequencies $\left(s>s_{c}\right)$, the form of the level correlation function changes from the $1 / s^{2}$ behavior typical for RMT to a different one, in full similarity with the Altshuler-Shklovskii regime (2.33) in the case of a conventional $d$-dimensional conductor. Extending Eq. (2.28) to the present case, we find

$$
\begin{align*}
R^{(c)}(s) & =\frac{\Delta^{2}}{\beta \pi^{2}} \operatorname{Re} \sum_{n=0,1, \ldots} \frac{1}{\left[\frac{8}{\pi v t}\left(\frac{\pi n}{N}\right)^{\sigma}-\mathrm{i} \omega\right]^{2}} \\
& \propto\left\{\begin{array}{cc}
N^{1-1 / \sigma} t^{1 / \sigma} S^{1 / \sigma-2}, & \sigma>1 / 2(\sigma \neq 1), \\
t^{2} N^{-1} b^{2 \sigma-1} \propto(N b)^{-1}, & \sigma<1 / 2 .
\end{array}\right. \tag{5.88}
\end{align*}
$$

At last, let us consider the level statistics at criticality, $\sigma=1$. In this case the coefficient of proportionality in the asymptotic expression (5.88) vanishes in view of analyticity:

$$
\begin{equation*}
R^{(c)}(s) \sim \frac{\Delta t}{16 \pi^{2} \beta} \int_{-\infty}^{\infty} \frac{\mathrm{d} x}{(x-\mathrm{i} \omega)^{2}}=0 \tag{5.89}
\end{equation*}
$$

This is similar to what happens in the case of a 2 D diffusive conductor (see Section 5.1). A more accurate consideration requires taking into account the high-momentum cut-off at $q \sim b^{-1}$. In full
analogy with the 2D situation mentioned we find then a linear term in the level number variance:

$$
\begin{equation*}
\left\langle\delta N^{2}(E)\right\rangle \simeq \chi\langle N(E)\rangle ; \quad \chi=\int R^{(c)}(s) \mathrm{d} s=\frac{t}{8 \pi \beta} . \tag{5.90}
\end{equation*}
$$

The presence of the linear term (5.90) (as well as the multifractality of eigenfunctions, Section 5.3.2) makes the case $\sigma=1$ similar to the situation on the mobility edge of a disordered conductor in $d>2$ discussed in Sections 5.1 and 5.2. Let us finally mention that the value of $\chi$, Eq. (5.90), is in agreement with the formula (5.27), where $D_{2}$ is given by Eq. (5.82) with $q=2$, and $d=1$ in the present case. We stress, however, that both Eqs. (5.82) and (5.90) have been derived in the leading order in $t / 8 \pi \ll 1$. As we discuss below, at $t / 8 \pi \gtrsim 1 \mathrm{Eq}$. (5.27) appears to be violated.

In fact, one can calculate the whole two-level correlation function using the results of Refs. [10,11] presented in Section 2.2. We consider the unitary symmetry $(\beta=2)$ here for simplicity. In this case, the level correlation function is described by a single formula $R(s)=R_{\text {pert }}(s)+R_{\text {osc }}(s)$ [with $R_{\text {pert }}(s)$ and $R_{\text {osc }}(s)$ given by Eqs. (2.38) and (2.34), respectively] in the whole range of $s$. The spectral determinant $D(s)$, Eq. (2.37) can be easily calculated [211], yielding

$$
\begin{equation*}
D(s)=\frac{1}{s^{2}} \frac{(\pi s t / 16)^{2}}{\sinh ^{2}(\pi s t / 16)} \tag{5.91}
\end{equation*}
$$

in the case of periodic boundary conditions and

$$
\begin{equation*}
D(s)=\frac{1}{s^{2}} \frac{\pi s t / 8}{\sinh (\pi s t / 8)} \tag{5.92}
\end{equation*}
$$

in the case of hard-wall boundary conditions. The resulting expressions for $R(s)$ are

$$
\begin{equation*}
R^{(c)}(s)=-\frac{\sin ^{2} \pi s}{(\pi s)^{2}} \frac{(\pi s t / 16)^{2}}{\sinh ^{2}(\pi s t / 16)} \tag{5.93}
\end{equation*}
$$

for periodic boundary conditions and

$$
\begin{equation*}
R^{(c)}(s)=-\frac{1}{4 \pi^{2} s^{2}}\left[1+\frac{(\pi s t / 8)^{2}}{\sinh ^{2}(\pi s t / 8)}-2 \cos (\pi s) \frac{\pi s t / 8}{\sinh (\pi s t / 8)}\right] \tag{5.94}
\end{equation*}
$$

for the hard-wall boundary conditions. Calculating the spectral compressibility $\chi=\int R^{(c)}(s) \mathrm{d} s$, we reproduce Eq. (5.90).

There exists a deep connection between the PRBM ensemble and the random matrix ensemble introduced by Moshe et al. in Ref. [212]. This latter ensemble is determined by the probability density

$$
\begin{equation*}
\mathscr{P}(H) \propto \int \mathrm{d} U \exp \left\{-\operatorname{Tr} H^{2}-h^{2} N^{2} \operatorname{Tr}\left([U, H][U, H]^{\dagger}\right)\right\}, \tag{5.95}
\end{equation*}
$$

where $\int \mathrm{d} U$ denotes the integration over the group of unitary matrices with the Haar measure. The parameter $h$ allows to interpolate between the limits of the Wigner-Dyson $(h=0)$ and the Poisson $(h \rightarrow \infty)$ statistics. The relation between the ensemble (5.95) and the PRBM ensemble is established as follows [211]. For any given unitary matrix $U$ the level statistics in the ensemble determined by the density (5.95) (without integration over $U$ ) is determined by the eigenvalues $\mathrm{e}^{\mathrm{i} \theta_{k}}(k=1, \ldots, N)$ of
$U$. For $N \gg 1$ a typical matrix $U$ has an essentially uniform density of eigenvalues, so that we can consider as a typical representative the matrix with equidistant eigenvalues, $\theta_{k}=2 \pi k / N$. Indeed, weak fluctuations of $\theta_{k}$ around these values do not affect essentially the behavior of $a(r)$ in Eq. (5.97) below. On the other hand, the matrices with strongly non-uniform density of eigenvalues represent a vanishingly small fraction of the whole group volume and can be neglected. Making a transformation to the eigenbasis of $U$, we can transform $\mathscr{P}(H)$ to

$$
\begin{equation*}
\mathscr{P}(H) \propto \exp \left\{-\sum_{i j}\left|H_{i j}\right|^{2}\left[1+(2 N h)^{2} \sin ^{2} \frac{\pi}{N}(i-j)\right]\right\}, \tag{5.96}
\end{equation*}
$$

so that the matrix $H$ has a form of the PRBM in the critical regime $\alpha=1$ with

$$
\begin{align*}
a^{2}(r) & =\frac{\beta}{4} \frac{1}{1+\frac{1}{b^{2}} \frac{\sin ^{2}(\pi r / N)}{(\pi / N)^{2}}} \\
& \simeq \frac{\beta}{4} \frac{1}{1+(r / b)^{2}} \quad \text { for } r \ll N, \tag{5.97}
\end{align*}
$$

and $b=1 / 2 \pi h$. Calculating (for $h \ll 1$ ) the coupling constant in the center of the band ( $E=0$ ) according to Section 5.3.1, we get $t=4 / b=8 \pi h$.

A nice feature of the ensemble (5.95) is that at $\beta=2$ the level correlation function can be calculated exactly for arbitrary value of $h$ [212] (we remind that the above $\sigma$-model study of the PRBM ensemble was restricted to the regime $1 / 2 \pi b=h \ll 1$ ). After the integration over the unitary group with making use of the Itzykson-Zuber formula, the joint probability distribution of the eigenvalues of $H$ is found to be equal to the probability density of coordinates of a system of non-interacting 1D fermions in a harmonic confinement potential at finite temperature. The parameter $h$ of the model (5.95) determines, in this formulation, the ratio of the temperature to the Fermi energy, $h=T / E_{\mathrm{F}}$, or, in other words, the degree of degeneracy of the Fermi gas. Calculating the two-particle correlation function for non-interacting fermions, one finds the sought level correlation function [212]. In the center of the band the result reads

$$
\begin{equation*}
\langle v(-\omega / 2) v(\omega / 2)\rangle-\langle v(0)\rangle^{2}=-\left[\frac{N h}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d} p}{1+C \mathrm{e}^{p^{2}}} \mathrm{e}^{2 i p h N \omega}\right]^{2}, \tag{5.98}
\end{equation*}
$$

where $C=\left[\mathrm{e}^{1 / h}-1\right]^{-1}$ and the mean level spacing $\Delta$ is given by

$$
\begin{equation*}
\Delta^{-1}=\langle v(0)\rangle=\frac{N h}{\pi} \int \frac{\mathrm{~d} p}{1+C \mathrm{e}^{p^{2}}} . \tag{5.99}
\end{equation*}
$$

The spectral compressibility $\chi$ is found from (5.98) and (5.99) to be

$$
\begin{equation*}
\chi=1-\frac{\int \mathrm{d} p\left(1+C \mathrm{e}^{p^{2}}\right)^{-2}}{\int \mathrm{~d} p\left(1+C \mathrm{e}^{\left.p^{p^{2}}\right)^{-1}}\right.} . \tag{5.100}
\end{equation*}
$$

The formulas (5.98) and (5.99) can be simplified in the limits $h \ll 1$ and $h \gtrdot>1$. For $h \ll 1$, when the gas is strongly degenerate, one gets (as usual, we introduce $s=\omega / \Delta$ )

$$
\begin{equation*}
R^{(c)}(s)=-\left(\frac{\sin \pi s}{\pi s}\right)^{2}\left(\frac{\pi^{2} h s / 2}{\sinh \pi^{2} h s / 2}\right)^{2} \tag{5.101}
\end{equation*}
$$

in precise agreement with the result (5.93) obtained directly for the PRBM ensemble (with the above identification of the parameters of the two models, $t=8 \pi h$ ).

In the opposite limit, $h \gg 1$, the gas is almost classical and the correlations are weak,

$$
\begin{equation*}
R^{(c)}(s)=-\mathrm{e}^{-2 \pi h^{2} s^{2}} \tag{5.102}
\end{equation*}
$$

Note that the corresponding spectral compressibility $\chi \simeq 1-1 / \sqrt{2} h$ approaches unity in the limt $h \rightarrow \infty$, whereas the formula (5.27) would imply $\chi \leq 1 / 2$. Therefore, at least for the PRBM model, Eq. (5.27) is not an exact relation, but rather an approximation valid in the close-to-RMT regime $1 / 2 \pi b=h \ll 1$.

## 6. Conductance fluctuations in quasi-one-dimensional wires

This section is devoted to a study of the conductance fluctuations of a quasi-one-dimensional disordered system [64,65]. As was already mentioned in Section 3, there exist different microscopic models which can be mapped onto the same 1D supersymmetric $\sigma$-model and thus belong to the same "quasi-one-dimensional universality class". Our treatment below will be based on the Iida-Weidenmüller-Zuk (IWZ) model [63] representing a wire as a sequence of coupled $N \times N$ random matrices, the first and the last of which are coupled to the states propagating in the leads. Using the multi-channel Büttiker-Landauer formula [213-216], the mean conductance and its variance can be expressed in terms of end-to-end correlation functions of the $\sigma$-model. Similarly to what was done in Section 3.2, where the eigenfunction statistics was studied, these correlation functions can be calculated exactly via the transfer-matrix method. This means that calculation of the functional integral can be reduced to solution of a "Schrödinger equation", which can be found in terms of an expansion in corresponding eigenfunctions. The results depend on a single parameter $L / \xi$ (where $L$ is the sample length and $\xi$ the localization length) and do not depend on details of the underlying microscopic model. On top of the IWZ-model and two other models from the quasi-1D class already discussed in Section 3.2 (the thick wire model of Efetov and Larkin and the random banded matrix model), we mention here a system of weakly coupled 2D layers in strong magnetic field [68,217-224]. In the quantum Hall regime the transport in each layer is due to edge states. Tunneling between the layers leads to appearance of the transport in the transverse direction. The coupled edge states propagating on a surface of the cylinder form a 2D chiral metal, which can be described by a directed network model introduced by Chalker and Dohmen [217]. Mapping of this problem onto the supersymmetric spin chain was done in [218,221]; further mapping onto the supersymmetric $\sigma$-model was presented in [222]. If the number of layers is sufficiently large, the system is of quasi-one-dimensional nature. Very recent numerical study of the directed network model [68] showed perfect agreement with the analytical results presented below. First experimental realization of the multilayer quantum Hall system has been also reported recently [224].

### 6.1. Modeling a disordered wire and mapping onto $1 D \sigma$-model

The presentation below is based on Ref. [65]. We consider a quasi one-dimensional disordered wire of length $L$, decomposed into $K$ boxes with linear dimension $\ell$. With each box $i(i=1, \ldots, K)$ we associate $N$ electronic states $|i \mu\rangle(\mu=1, \ldots, N)$. The boxes 1 and $K$ are coupled to ideal leads $\kappa$ on the left $(\kappa=L)$ and right $(\kappa=R)$ side of the disordered region. In each of these leads we have a number of modes $|E, a, \kappa\rangle(a=1, \ldots)$ with transverse energy $\varepsilon_{a}$ whose longitudinal momentum $k$ is defined by their total energy $E$ being equal to $\varepsilon_{a}+\hbar^{2} k^{2} /\left(2 m_{\mathrm{e}}^{*}\right)$. Here $m_{\mathrm{e}}^{*}$ is the effective mass of the electrons. We work at zero temperature. The Hamiltonian of the system reads

$$
\begin{align*}
\mathscr{H}= & \sum_{\kappa, a} \int_{\varepsilon_{a}} \mathrm{~d} E|E, a, \kappa\rangle E\langle E, a, \kappa|+\sum_{i, j, \mu, v}|i \mu\rangle H_{\mu \nu}^{i j}\langle j v| \\
& +\sum_{i=1, K} \sum_{\mu, a, \kappa} \int_{\varepsilon_{a}} \mathrm{~d} E\left(|E, a, \kappa\rangle W_{a \mu}^{i}(E, \kappa)\langle\mathrm{i} \mu|+\text { c.c. }\right) . \tag{6.1}
\end{align*}
$$

The $N \times N$-matrices ( $H_{\mu \nu}^{i i}$ ) are taken to be members of the Gaussian unitary (GUE), orthogonal (GOE) or symplectic (GSE) ensemble. For definiteness, we will consider the orthogonal symmetry case in the present section; the results for all the three symmetry classes will be presented in Section 6.2. In the GOE case, the elements $H_{\mu \nu}^{i i}$ are independent real Gaussian random variables with zero mean value and

$$
\begin{equation*}
\left\langle H_{\mu \nu}^{i i} H_{\mu^{\prime} v^{\prime}}^{i i}\right\rangle=\frac{\lambda^{2}}{N}\left(\delta_{\mu v^{\prime}} \delta_{v \mu^{\prime}}+\delta_{\mu \mu^{\prime}} \delta_{v v^{\prime}}\right) . \tag{6.2}
\end{equation*}
$$

States in adjacent boxes are coupled by another set of Gaussian random variables with zero mean values and

$$
\begin{equation*}
\left\langle H_{\mu \nu}^{i i+1} H_{\nu \mu}^{i+1 i}\right\rangle=v^{2} / N^{2} . \tag{6.3}
\end{equation*}
$$

All other matrix elements $H_{\mu \nu}^{i j}$ vanish. The coupling between channel and box states is effected by the matrix elements $W_{a \mu}^{i}(E, \kappa)$. We assume that they do not depend on $E$ and $\kappa$, that they obey the symmetry $W_{\mu a}^{i}=W_{a \mu}^{i *}=W_{a \mu}^{i}$ and that they fulfill the orthogonality relation

$$
\begin{equation*}
\pi \sum_{\mu} W_{a \mu}^{i} W_{\mu b}^{i}=\delta_{a b} x \quad(i=1, K) \tag{6.4}
\end{equation*}
$$

with $x$ a normalization constant. Eq. (6.4) is convenient but does not result in any loss of generality [8]. We note that $W_{a \mu}^{i}$ vanishes unless $i=1, K$.

The conductance of the system is given by the many-channel Landauer-Büttiker formula [213-216],

$$
\begin{equation*}
g=\sum_{a b}\left(\left|S_{a b}^{R L}\right|^{2}+\left|S_{b a}^{L R}\right|^{2}\right)=2 \sum_{a b}\left|S_{a b}^{R L}\right|^{2} \tag{6.5}
\end{equation*}
$$

The $S$-matrix $S_{a b}^{\kappa \kappa^{\prime}}$ of the IWZ-model reads [225]

$$
\begin{equation*}
S_{a b}^{\kappa \kappa^{\prime}}=\delta_{a b} \delta^{\kappa \kappa^{\prime}}-2 \pi \mathrm{i} \sum_{i, j, \mu, v} W_{a \mu}^{i}(\kappa)\left(D^{-1}\right)_{\mu v}^{i j} W_{v b}^{j}\left(\kappa^{\prime}\right), \tag{6.6}
\end{equation*}
$$

where

$$
\begin{align*}
D_{\mu \nu}^{i j} & =E \delta^{i j} \delta_{\mu \nu}-H_{\mu \nu}^{i j}+i \Omega_{\mu \nu}^{1} \delta^{i 1} \delta^{j 1}+i \Omega_{\mu \nu}^{K} \delta^{i K} \delta^{j K}  \tag{6.7}\\
\Omega_{\mu \nu}^{i} & =\pi \sum_{a=1}^{N_{\mathrm{ch}}} W_{\mu a}^{i} W_{a \nu}^{i} \quad(i=1, K) . \tag{6.8}
\end{align*}
$$

The $S$-matrix in Eq. (6.5) is taken at the Fermi energy $E=E_{\mathrm{F}}$, and $N_{\text {ch }}$ denotes the number of open channels at this energy. It follows that in order to obtain $\langle g\rangle$ and $\left\langle g^{2}\right\rangle$ we have to calculate the ensemble average of a product of two resp.four propagators $D^{-1}$. Further manipulations are completely analogous to those outlined in Section 2.1 for the case of the level correlation function. We define a supervector $\psi=\left(S_{1}^{1}, S_{1}^{2}, \chi_{1}, \chi_{1}^{*}, S_{2}^{1}, S_{2}^{2}, \chi_{2}, \chi_{2}^{*}\right)$ of real commuting $(S)$ and complex anticommuting $(\chi)$ variables, with the first four components corresponding to the retarded and the last four to the advanced sector. Then

$$
\begin{equation*}
\left(D^{-1}\right)_{\mu \nu}^{i j}=\int \mathrm{d}[\psi]\left(S_{1}^{1}\right)_{\nu}^{j}\left(S_{1}^{1}\right)_{\mu}^{i} \exp \left(\frac{\mathrm{i}}{2} \psi^{\dagger} \Lambda^{1 / 2} \hat{D} \Lambda^{1 / 2} \psi\right) \tag{6.9}
\end{equation*}
$$

where $\hat{D}=\operatorname{diag}\left(D, D, D, D, D^{\dagger}, D^{\dagger}, D^{\dagger}, D^{\dagger}\right), \Lambda=\operatorname{diag}(1,1,1,1,-1,-1,-1,-1)$. Products of two and four propagators can be expressed in similar fashion (see Eq. (6.14) below). After averaging of Eq. (6.9) over the Gaussian distribution of random variables entering $\hat{D}$, we perform the Hubbard-Stratonovitch transformation introducing $8 \times 8$ supermatrices $\mathscr{R}_{i}(i=1, \ldots, K)$ conjugate to the dyadic product $\psi_{i} \psi_{i}^{\dagger}$ and then take the large- $N$ limit. As a result, the integration over $\mathscr{R}_{i}$ is restricted to solutions of the saddle-point equation $\mathscr{R}_{i}=\lambda^{2}\left(E-\mathscr{R}_{i}\right)^{-1}$, which have the form

$$
\begin{equation*}
\mathscr{R}=\frac{E}{2}-\mathrm{i} \sqrt{\lambda^{2}-\frac{E^{2}}{4}} T \Lambda T^{-1} \equiv \sigma \cdot I-\mathrm{i} \delta \cdot Q, \tag{6.10}
\end{equation*}
$$

with $Q=T \Lambda T^{-1}, Q^{2}=1$. As a result, Eq. (6.9) takes the form

$$
\begin{align*}
\left\langle\left(D^{-1}\right)_{\mu \nu}^{i j}\right\rangle= & \int D[Q] \exp \left(\frac{-\delta^{2} v^{2}}{4 \lambda^{4}} \sum_{i} \operatorname{Str} Q_{i} Q_{i+1}\right) \\
& \times \int \mathrm{d}[\psi]\left(S_{1}^{1}\right)_{v}^{j}\left(S_{1}^{1}\right)_{\mu}^{i} \exp \left(\frac{\mathrm{i}}{2} \psi^{\dagger} \Lambda^{1 / 2}(\hat{E}+\mathrm{i} \widehat{\Omega} \Lambda+\mathrm{i} \delta \hat{Q}) \Lambda^{1 / 2} \psi\right) \equiv\left\langle\left(S_{1}^{1}\right)_{v}^{j}\left(S_{1}^{1}\right)_{\mu}^{i}\right\rangle_{Q} . \tag{6.11}
\end{align*}
$$

Here $(\hat{E})_{\mu \nu}^{i j}=E \delta_{\mu \nu} \delta^{i j},(\widehat{\Omega})_{\mu \nu}^{i j}=\left(\Omega^{1}\right)_{\mu \nu} \delta^{i 1} \delta^{j 1}+\left(\Omega^{K}\right)_{\mu \nu} \delta^{i K} \delta^{j K}$ and $(\hat{Q})_{\mu \nu}^{i j}=\delta_{\mu \nu} Q_{i} \delta^{i j}$. The last line of Eq. (6.11) introduces a short-hand notation $\langle\ldots\rangle_{Q}$.

According to Eqs. (6.5) and (6.6), the first two moments of the conductance are given by

$$
\begin{align*}
& \langle g\rangle=8 \sum_{\mu v \mu^{\prime} v^{\prime}} \Omega_{v v^{\prime}}^{1} \Omega_{\mu^{\prime} \mu}^{K}\left\langle\left(D^{-1}\right)_{\mu v}^{K 1}\left(D^{-1 \dagger}\right)_{v^{\prime} \mu^{\prime}}^{1 K}\right\rangle,  \tag{6.12}\\
& \left\langle g^{2}\right\rangle=64 \sum_{\mu v \mu^{\prime} v^{\prime} \rho \tau \rho^{\prime} \tau^{\prime}} \Omega_{v v^{\prime}}^{1} \Omega_{\mu^{\prime} \mu}^{K} \Omega_{\tau \tau^{\prime}}^{1} \Omega_{\rho^{\prime} \rho}^{K}\left\langle\left(D^{-1}\right)_{\mu \nu}^{K 1}\left(D^{-1 \dagger}\right)_{v^{\prime} \mu^{\prime}}^{1 K}\left(D^{-1}\right)_{\rho \tau}^{K 1}\left(D^{-1 \dagger}\right)_{\tau^{\prime} \rho^{\prime}}^{1 K}\right\rangle . \tag{6.13}
\end{align*}
$$

The averages of products of the Green's functions relevant for $\langle g\rangle$ and $\left\langle g^{2}\right\rangle$ can be expressed similarly to Eq. (6.11) as follows:

$$
\begin{align*}
& \left\langle\left(D^{-1}\right)_{\mu v}^{K 1}\left(D^{-1 \dagger}\right)_{v^{\prime} \mu^{\prime}}^{1 K}\right\rangle=\left\langle\left(S_{1}^{1}\right)_{v}^{1}\left(S_{1}^{1}\right)_{\mu}^{K}\left(S^{1}\right)_{\mu^{\prime}}^{K}\left(S_{2}^{1}\right)_{v^{\prime}}^{1}\right\rangle_{Q}, \\
& \left\langle\left(D^{-1}\right)_{\mu v}^{K 1}\left(D^{-1}{ }^{-1}\right)_{v^{\prime} \mu^{\prime}}^{K}\left(D^{-1}\right)_{\rho \tau}^{K 1}\left(D^{-1 \dagger}\right)_{\tau^{\prime} \rho^{\prime}}^{1 K}\right\rangle \\
& \quad=\left\langle\left(S_{1}^{1}\right)_{v}^{1}\left(S_{1}^{1}\right)_{\mu}^{K}\left(S_{2}^{1}\right)_{\mu^{\prime}}^{K}\left(S_{2}^{1}\right)_{v^{\prime}}^{1}\left(S_{1}^{2}\right)_{\tau}^{1}\left(S_{1}^{2}\right)_{\rho}^{K}\left(S_{2}^{2}\right)_{\rho^{\prime}}^{K}\left(S_{2}^{2}\right)_{\tau^{\prime}}^{1}\right\rangle_{Q} . \tag{6.14}
\end{align*}
$$

For simplicity, we set $E=0$ below. Evaluating the contractions in (6.14), performing the convolution with the projectors $\Omega^{1}, \Omega^{K}$ in (6.12), (6.13) and using $N_{\mathrm{ch}} \gg 1$, one comes after some algebraic manipulations to the following $\sigma$-model representation ${ }^{12}$ of $\langle g\rangle$ and $\left\langle g^{2}\right\rangle[64,65]$ :

$$
\begin{align*}
& \langle g\rangle=\frac{\tilde{N}_{\mathrm{ch}}^{2}}{2} \int D[Q] Q_{1}^{51} Q_{K}^{51} \exp \{-S[Q]\},  \tag{6.15}\\
& \left\langle g^{2}\right\rangle=\frac{\tilde{N}_{\mathrm{ch}}^{4}}{4} \int D[Q] Q_{1}^{51} Q_{1}^{62} Q_{K}^{51} Q_{K}^{62} \exp \{-S[Q]\},  \tag{6.16}\\
& S[Q]=\frac{\xi}{8 \ell} \sum_{i} \operatorname{Str} Q_{i} Q_{i+1}+\frac{\tilde{N}_{\mathrm{ch}}}{8} \operatorname{Str} \Lambda\left(Q_{1}+Q_{K}\right) . \tag{6.17}
\end{align*}
$$

Here $\tilde{N}_{\mathrm{ch}}=N_{\mathrm{ch}} T_{0}$, with $T_{0}=4 \delta x /(\delta+x)^{2}$ being the so-called transmission coefficient, and $\xi=\left(4 v^{2} / \lambda^{2}\right) \ell$. In the weak-disorder limit, $\xi \gg \ell$, the discrete sum $-(\xi / 8 \ell) \sum_{i} \operatorname{Str} Q_{i} Q_{i+1}$ can be replaced by an integral

$$
\begin{equation*}
\frac{\xi}{16} \int_{0}^{L} \mathrm{~d} z \operatorname{Str}\left(\frac{\partial Q}{\partial z}\right)^{2} \tag{6.18}
\end{equation*}
$$

which is precisely the continuous version of the 1D $\sigma$-model (see Sections 2 and 3 ) with $\xi$ being the localization length equal to $\xi=2 \pi v A D$ for a thick wire. The second term in Eq. (6.17) containing $Q_{1}$ and $Q_{K}$ describes the coupling of the wire to the leads.

Let us note that the duplication of the number of field components in the supervector $\psi$ (which is forced by the supersymmetric formulation) gave us enough flexibility to write down the expressions Eq. (6.14) for products of two as well as of four propagators. As a result, we were able to express both $\langle g\rangle$ and $\left\langle g^{2}\right\rangle$ in terms of correlation functions of the usual "minimal" $\sigma$-model. We did not have to introduce $Q$-matrices of larger dimension. This fact no longer holds true when higher moments of the conductance are to be calculated. For that purpose we would have to introduce supervectors of larger size to express higher-order products of propagators in a form analogous to (6.14). The increase in the number of supervector components would force us to deal with $Q$-matrices of larger dimension. This enlargement, while posing no difficulty to perturbative calculations [28], gives rise to substantial complications in the case of the exact treatment.

[^11]To introduce the transfer operator for the "partition sum" in Eq. (6.16), we define the function

$$
\begin{equation*}
W\left(Q_{K}, Q_{1} ; L / 2 \xi\right)=\int D\left[Q_{2}\right] \ldots D\left[Q_{K-1}\right] \exp \left(\frac{\xi}{32 \ell} \sum_{i=1}^{K-1} \operatorname{Str}\left(Q_{i+1}-Q_{i}\right)^{2}\right), \tag{6.19}
\end{equation*}
$$

which has the property

$$
\begin{align*}
& W\left(Q_{K+1}, Q_{1} ;(L+\ell) / 2 \xi\right)=\int D\left[Q_{K}\right] W\left(Q_{K+1}, Q_{K} ; \ell / 2 \xi\right) W\left(Q_{K}, Q_{1} ; L / 2 \xi\right) \\
& \quad=\int D\left[Q_{K}\right] \exp \left(\frac{\xi}{32 \ell} \operatorname{Str}\left(Q_{K+1}-Q_{K}\right)^{2}\right) W\left(Q_{K}, Q_{1} ; L / 2 \xi\right) . \tag{6.20}
\end{align*}
$$

In the continuum limit $\xi \gg \ell$, when $Q_{i}$ and $Q_{i+1}$ are close to each other, one can reduce the integral equation (6.20) to a differential one:

$$
\begin{align*}
& \partial_{t} W=\Delta_{Q} W  \tag{6.21}\\
& \lim _{t \rightarrow 0} W\left(Q, Q^{\prime} ; t\right)=\delta\left(Q, Q^{\prime}\right) \tag{6.22}
\end{align*}
$$

where $\Delta_{Q}$ is the Laplace operator on the non-linear space of $Q$-matrices. Eq. (6.21) has a form of the heat (or diffusion) equation on this space, with $W$ being the corresponding heat kernel.

### 6.2. Conductance fluctuations

Expressions (6.15) and (6.16) contain the effective number of open channels, $\tilde{N}_{\text {ch }}$, as a parameter. As was shown in [63], the existence of this parameter sets an additional length scale $L_{0}=\xi / \widetilde{N}_{\text {ch }}$. For a very short system $\left(L \ll L_{0}\right)$, the conductance $\langle g\rangle$ is controlled by the sample-leads coupling and is of order of $\widetilde{N}_{\text {ch. }}$. In this case, the zero-mode (i.e. spatially constant $Q$ ) approximation can be used. We are interested here in the opposite limit $L \gtrdot L_{0}$, where the resistance of the sample is dominated by the bulk of the system. In this case, $\tilde{N}_{\mathrm{ch}}$ becomes an irrelevant parameter and the heat kernel itself already contains all information about $\langle g\rangle$ and $\left\langle g^{2}\right\rangle$. Since we assume that $\tilde{N}_{\text {ch }} \gg 1$, the crossover length scale $L_{0}$ is much less than $\xi$, so that the ratio of the sample length to the localization length $L / \xi$ in the bulk-dominated regime can be arbitrary.

Using the generalized Fourier expansion of the heat kernel in the eigenfunctions of the Laplace operator $\Delta_{Q}$, one can find after very lengthy calculations [64,65] (there was a subtle error in the GSE case in these papers corrected in [66]) the following exact results for $\langle g\rangle$ and $\left\langle g^{2}\right\rangle$ for an arbitrary value of the parameter $L / \xi$ :

GOE:

$$
\begin{aligned}
\left\langle g^{n}\right\rangle(L)= & \frac{\pi}{2} \int_{0}^{\infty} \mathrm{d} \lambda \tanh ^{2}(\pi \lambda / 2)\left(\lambda^{2}+1\right)^{-1} p_{n}(1, \lambda, \lambda) \exp \left[-\frac{L}{2 \xi}\left(1+\lambda^{2}\right)\right] \\
& +2^{4} \sum_{l \in 2 N+1} \int_{0}^{\infty} \mathrm{d} \lambda_{1} \mathrm{~d} \lambda_{2} l\left(l^{2}-1\right) \lambda_{1} \tanh \left(\pi \lambda_{1} / 2\right) \lambda_{2} \tanh \left(\pi \lambda_{2} / 2\right) \\
& \times p_{n}\left(l, \lambda_{1}, \lambda_{2}\right) \prod_{\sigma, \sigma_{1}, \sigma_{2}= \pm 1}\left(-1+\sigma l+\mathrm{i} \sigma_{1} \lambda_{1}+\mathrm{i} \sigma_{2} \lambda_{2}\right)^{-1} \exp \left[-\frac{L}{4 \xi}\left(l^{2}+\lambda_{1}^{2}+\lambda_{2}^{2}+1\right)\right],
\end{aligned}
$$

## GUE:

$$
\begin{equation*}
\left\langle g^{n}\right\rangle(L)=2^{2} \sum_{l \in 2 N-1} \int_{0}^{\infty} \mathrm{d} \lambda \lambda \tanh (\pi \lambda / 2) l\left(\lambda^{2}+l^{2}\right)^{-2} p_{n}(l, \lambda) \exp \left[-\frac{L}{4 \xi}\left(l^{2}+\lambda^{2}\right)\right], \tag{6.23}
\end{equation*}
$$

## GSE:

$$
\begin{aligned}
\left\langle g^{n}\right\rangle(L)= & 2^{5} \sum_{\substack{l_{1}, l_{2} \in 2 N-1 \\
l_{1}+l_{2} \in 4 N-2}} \int_{0}^{\infty} \mathrm{d} \lambda \lambda\left(\lambda^{2}+1\right) \tanh (\pi \lambda / 2) l_{1} l_{2} p_{n}\left(\lambda, l_{2}, l_{2}\right) \\
& \times \prod_{\sigma, \sigma_{1}, \sigma_{2}= \pm 1}\left(-1+\mathrm{i} \sigma \lambda+\sigma_{1} l_{1}+\sigma_{2} l_{2}\right)^{-1} \exp \left[-\frac{L}{8 \xi}\left(l_{1}^{2}+l_{2}^{2}+\lambda^{2}-1\right)\right] .
\end{aligned}
$$

The polynomials $p_{n}$ in the above expressions are given by
GOE: $\quad p_{1}\left(l, \lambda_{1}, \lambda_{2}\right)=l^{2}+\lambda_{1}^{2}+\lambda_{2}^{2}+1$,

$$
p_{2}\left(l, \lambda_{1}, \lambda_{2}\right)=\frac{1}{2}\left(\lambda_{1}^{4}+\lambda_{2}^{4}+2 l^{4}+3 l^{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)+2 l^{2}-\lambda_{1}^{2}-\lambda_{2}^{2}-2\right),
$$

GUE: $\quad p_{1}(l, \lambda)=l^{2}+\lambda^{2}$,

$$
p_{2}(l, \lambda)=\frac{1}{2}\left(l^{2}+\lambda^{2}\right)^{2},
$$

GSE: $\quad p_{1}\left(\lambda, l_{1}, l_{2}\right)=\lambda^{2}+l_{1}^{2}+l_{2}^{2}-1$,

$$
p_{2}\left(\lambda, l_{1}, l_{2}\right)=\frac{1}{4}\left(l_{1}^{4}+l_{2}^{4}+2 \lambda^{4}+3 \lambda^{2}\left(l_{1}^{2}+l_{2}^{2}\right)-2 \lambda^{2}+l_{1}^{2}+l_{2}^{2}-2\right) .
$$

The results (6.23) are presented in Figs. 6 and 7 for all the three symmetry classes. The above GUE formula is written for the case of broken time-reversal symmetry, but preserved spin-rotation


Fig. 6. Average conductance $\langle g\rangle$ normalized to its Ohm's law value as a function of the sample length (measured in units of $\xi=2 \pi v A D$ ). The full, dashed and dot-dashed lines correspond to the orthogonal, unitary, and symplectic symmetry, respectively.

Fig. 7. The same as Fig. 6, but for the conductance variance $\operatorname{var}(g)$.


Fig. 8. Numerical data [67] for the average conductance in the case of symplectic symmetry. The box size $N$ and the number of channels $N_{\mathrm{ch}}$ are equal to 10 (squares), 20 (diamonds), 60 (triangles), and 100 (stars). Each data point corresponds to an average over 100 realizations of disorder. The full line is the theoretical prediction.

Fig. 9. The same as Fig. 8, but for the conductance variance.
invariance. The result for systems with broken time-reversal symmetry and with strong spin interactions (e.g. systems with magnetic impurities, or systems with strong spin-orbit interaction in magnetic field; to be denoted as GUE' below) is

$$
\begin{equation*}
\left\langle g^{n}\right\rangle^{\operatorname{GUE}^{\prime}}(L)=(1 / 2)^{n}\left\langle g^{n}\right\rangle^{\operatorname{GUE}}(L / 2) . \tag{6.25}
\end{equation*}
$$

The results (6.23) have been confirmed by numerical simulations of the IWZ-model [67] and of the directed network model [68]. In particular, we present in Figs. 8 and 9 the numerical results of [67] for the symplectic symmetry class.

Let us discuss the behavior of Eqs. (6.23) in the limits of short $(L \ll \xi)$ and long $(L \gg \xi)$ wire. Condition $L \ll \xi$ corresponds to the metallic (weak localization) region, $\langle g\rangle \gg 1$. We find in this case the following perturbative (in $L / \xi$ ) expansion for $\langle g\rangle,\left\langle g^{2}\right\rangle$ and $\operatorname{var}(g)$ :

$$
\begin{align*}
& \langle g\rangle(L)=\frac{2 \xi}{L}-\frac{2}{3}+\frac{2}{45} \frac{L}{\xi}+\frac{4}{945}\left(\frac{L}{\xi}\right)^{2}+\mathrm{O}\left(\frac{L}{\xi}\right)^{3} \\
& \left\langle g^{2}\right\rangle(L)=\left(\frac{2 \xi}{L}\right)^{2}-\frac{8}{3} \frac{\xi}{L}+\frac{52}{45}-\frac{136}{945} \frac{L}{\xi}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2}  \tag{6.26}\\
& \operatorname{var}(g(L))=\frac{8}{15}-\frac{32}{315} \frac{L}{\xi}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2}
\end{align*}
$$

The perturbative results for the symplectic case are related to those for the orthogonal class via the symmetry relations [226-228] $\left\langle g^{n}\right\rangle^{\mathrm{Sp}}(L)=(-1 / 2)^{n}\left\langle g^{n}\right\rangle^{\mathrm{O}}(-L / 2)$ and have the form

$$
\langle g\rangle(L)=\frac{2 \xi}{L}+\frac{1}{3}+\frac{1}{90} \frac{L}{\xi}-\frac{1}{1890}\left(\frac{L}{\xi}\right)^{2}+\mathrm{O}\left(\frac{L}{\xi}\right)^{3}
$$

$$
\begin{align*}
& \left\langle g^{2}\right\rangle(L)=\left(\frac{2 \xi}{L}\right)^{2}+\frac{4}{3} \frac{\xi}{L}+\frac{13}{45}+\frac{17}{945} \frac{L}{\xi}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2},  \tag{6.27}\\
& \operatorname{var}(g(L))=\frac{2}{15}+\frac{4}{315} \frac{L}{\xi}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2} .
\end{align*}
$$

Finally, for unitary symmetry we have

$$
\begin{align*}
& \langle g\rangle(L)=\frac{2 \xi}{L}-\frac{2}{45} \frac{L}{\xi}+\mathrm{O}\left(\frac{L}{\xi}\right)^{3} \\
& \left\langle g^{2}\right\rangle(L)=\left(\frac{2 \xi}{L}\right)^{2}+\frac{4}{45}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2},  \tag{6.28}\\
& \operatorname{var}(g(L))=\frac{4}{15}+\mathrm{O}\left(\frac{L}{\xi}\right)^{2}
\end{align*}
$$

The expressions for $\langle g\rangle$ start from the Ohm's law term $2 \xi / L=4 \pi \nu A D / L$ (we remind that $g$ is measured in units of $\mathrm{e}^{2} / h=\mathrm{e}^{2} / 2 \pi \hbar$ and includes factor 2 due to the spin, while $v$ is the density of states per spin projection). The other terms constitute the weak-localization corrections. The leading terms in $\operatorname{var}(g)$ are well-known values of the universal conductance fluctuations in the case of quasi-1D geometry.

The opposite condition $L \gtrdot \xi$ defines the region of strong localization. In this case, the asymptotic behavior of $\left\langle g^{n}\right\rangle, n=1,2$, is as follows:

GOE: $\left\langle g^{n}\right\rangle=2^{-3 / 2-n} \pi^{7 / 2}(\xi / L)^{3 / 2} \mathrm{e}^{-L / 2 \xi}$,
GUE: $\left\langle g^{n}\right\rangle=2^{3-n} \pi^{3 / 2}(\xi / L)^{3 / 2} \mathrm{e}^{-L / 4 \xi}$,
GSE: $\left\langle g^{n}\right\rangle=2^{15 / 2-n} \pi^{3 / 2}(\xi / L)^{3 / 2} \mathrm{e}^{-L / 8 \xi}$.
Let us recall that $\xi$ is defined here as $\xi=2 \pi v A D$ independently of the symmetry. The formulas (6.29) demonstrate therefore well-known dependence of the localization length on the symmetry of the ensemble, $L_{\text {loc }} \propto \beta$. Let us stress, however, that in the GUE' case (broken time-reversal and spin-rotation symmetries), the results for which can be obtained via the relation (6.25) the localization length is the same as in GSE [101,229]. This is because the transition GSE $\rightarrow \mathrm{GUE}^{\prime}$ not only changes $\beta$ from 4 to 2 , but also breaks the Kramers degeneracy, increasing by factor of 2 the number of coupled channels. These two effects compensate each other. More generally, localization length is proportional to $\beta / s$, where $s$ is the degeneracy factor.

### 6.2.1. DMPK equations

As has been already mentioned, calculation of higher moments $\left\langle g^{n}\right\rangle$ with $\left.n\right\rangle 2$ and thus of the whole distribution function $\mathscr{P}(g)$ has not been achieved in the supersymmetry approach because of technical difficulties (necessity to increase the size of the $Q$-matrix with increasing $n$ ). The conductance distribution in the localized regime $L \gg \xi$ can be approximately calculated from the Dorokhov-Mello-Pereyra-Kumar (DMPK) approach. Within this approach, pioneered by Dorokhov [230] and developed by Mello et al. [231], one derives a Fokker-Planck (diffusion)
equation for the distribution of transmission eigenvalues $T_{n}$ (the eigenvalues of the transmissionmatrix product $t t^{\dagger}$ ). This equation is conveniently written in terms of the distribution function $P\left(\lambda_{1}, \lambda_{2}, \ldots ; L\right)$, where $\lambda_{i}=\left(1-T_{i}\right) / T_{i}$ and has the form

$$
\begin{align*}
& l \frac{\partial P}{\partial L}=\frac{2}{\gamma} \sum_{n=1}^{N} \frac{\partial}{\partial \lambda_{n}} \lambda_{n}\left(1+\lambda_{n}\right) J \frac{\partial}{\partial \lambda_{n}} J^{-1} P, \\
& J=\prod_{i=1}^{N} \prod_{j=i+1}^{N}\left|\lambda_{i}-\lambda_{j}\right|^{\beta}, \tag{6.30}
\end{align*}
$$

where $l$ is the mean free path, $N$ is the number of transverse modes, and $\gamma=\beta N+2-\beta$. Overview of the results obtained within the DMPK approach can be found in the review article of Beenakker [88]. It has been shown recently [66] that in the limit $N \gg 1$ the DMPK approach is equivalent to the supersymmetric $\sigma$-model considered above. Let us note that the DMPK approach is restricted to calculation of transport properties of the quasi-1D conductor (which can be expressed through the transmission eigenvalues $T_{i}$ ), while the supersymmetry method allows to study all sorts of quantities which can be expressed through the Green's functions (e.g. statistics of levels, eigenfunction amplitudes, local density of states etc. - see other sections of this article).

In the localized regime $L \gtrdot N l$ the distribution function $P\left(x_{1}, \ldots, x_{N} ; L\right)$ with $\lambda_{i}=\sinh ^{2} x_{i}$ (so that $T_{i}=1 / \cosh ^{2} x_{i}$ ) takes the form [230,232,233]

$$
\begin{equation*}
P\left(x_{1}, \ldots, x_{l}\right)=\left(\frac{\gamma l}{2 \pi L}\right)^{N / 2} \prod_{n=1}^{N} \exp \left[-\frac{\gamma l}{2 L}\left(x_{n}-\frac{L}{\xi_{n}}\right)^{2}\right], \quad 1 \ll x_{1} \ll x_{2} \ll \cdots<x_{n} \tag{6.31}
\end{equation*}
$$

where $\xi_{n}=\gamma l /(1+\beta n-\beta)$ are the inverse Lyapunov exponents. In the limit $L \rightarrow \infty$ all $x_{n}$ tend to non-random values $L / \xi_{n}$, which is a manifestation of the Oseledec theorem (matrix generalization of the "law of large numbers") [234-236]. For $L / N l \gg 1$ the conductance

$$
\begin{equation*}
g=G /\left(\mathrm{e}^{2} / h\right)=2 \sum_{n=1}^{N} T_{n} \tag{6.32}
\end{equation*}
$$

is dominated by $x_{1}$, e.g. $g \simeq 2 / \cosh ^{2} x_{1} \simeq 8 \mathrm{e}^{-2 x_{1}}$, which implies the Gaussian distribution of $\ln g$,

$$
\begin{equation*}
P(\ln g) \simeq\left(\frac{\gamma l}{8 \pi L}\right)^{1 / 2} \exp \left[-\frac{\gamma l}{8 L}\left(\ln g+\frac{2 L}{\gamma l}\right)^{2}\right], \tag{6.33}
\end{equation*}
$$

with the average $\langle\ln g\rangle=-2 L / \gamma l$ and the variance $\operatorname{var}(g)=4 L / \gamma l=-2\langle\ln g\rangle$. We have already encountered the same log-normal distribution in Section 3.2.4, when we calculated the distribution of the product of the wave function intensities in two points located close to the opposite edges of the sample. The result (6.33) is fully confirmed by numerical simulations as illustrated in Fig. 10.

The log-normal form (6.33) of the conductance distribution holds for $g<1$ only (since one transmission eigenvalue cannot produce a conductance larger than unity); for $g>1$ the distribution $P(g)$ decays very fast. The moments $\left\langle g^{n}\right\rangle$ with $n \geq 1$ are thus determined by the probability to have $g \sim 1$, which yields (with exponential accuracy)

$$
\begin{equation*}
\left\langle g^{n}\right\rangle \sim \exp (-L / 2 \gamma l) \tag{6.34}
\end{equation*}
$$

in full agreement with the asymptotics of the first two moments, Eqs. (6.29), found from the supersymmetric approach. [Eqs. (6.29) contain also preexponential factors, for derivation


Fig. 10. The average and the variance of the conductance logarithm as a function of the sample length (for the symplectic symmetry class). The box size $N$ and the number of channels $N_{\text {ch }}$ are equal to 10. Each data point corresponds to an average over 100 realizations of disorder. The dashed line corresponds to the formula (6.33).
of which the approximation (6.31), (6.33) of the solution of the DMPK equations is not sufficient.]

Finally, we note that in the strictly 1D case Abrikosov [147] derived the exact result for the conductance distribution function $P(g)$ for arbitrary value of $L / l$. In the limit $L / l \gg 1$ it approaches the log-normal distribution (also found by Melnikov [145,146]), which can be obtained from Eq. (6.33) by setting $N=1$ (i.e. $\gamma=2$ ). In fact, essentially equivalent results were obtained much earlier in the context of the classical wave propagation, see Refs. [237-239].

## 7. Statistics of wave intensity in optics

As was already mentioned in the introduction, our statistical considerations are applicable not only to properties of eigenfunctions and energy levels of quantum particles (electrons), but also to those of intensities and eigenfrequencies of classical waves. This is related to similarity of the stationary Schrödinger equation and the wave equation. Propagation of the classical field $\psi_{\omega}$ with frequency $\omega$ in an inhomogeneous medium is described by the wave equation

$$
\begin{equation*}
\left[\nabla^{2}+k_{0}^{2}(1+\mu(\boldsymbol{r}))\right] \psi_{\omega}(\boldsymbol{r})=0 \tag{7.1}
\end{equation*}
$$

supplemented by appropriate sources. Here $k_{0}=\omega / c$, with $c$ the speed of propagation in the average medium, and $\mu(\boldsymbol{r})$ describes fluctuations of the refraction index. The field $\psi_{\omega}$ can describe a component of the electromagnetic or acoustic wave. The impurity diagrammatic technique [240-243] and the $\sigma$-model approach [244,245] can be developed in full analogy with the case of the Schrödinger equation in random potential.

Let us consider an open system with a permanently radiating source (for example, a point-like source would correspond to addition of the term $\propto \delta\left(\boldsymbol{r}-\boldsymbol{r}_{0}\right)$ in the r.h.s. of Eq. (7.1). The problem of fluctuations of the wave intensity $\psi_{\omega}^{2}(\boldsymbol{r})$ in such a situation has a very long history. Almost
a century ago Rayleigh proposed a distribution which bears his name:

$$
\begin{equation*}
P_{0}(\widetilde{I})=\exp (-\widetilde{I}), \tag{7.2}
\end{equation*}
$$

where $\tilde{I}$ is the intensity normalized to its average value, $\tilde{I}=I /\langle I\rangle$. A simple statistical argument leading to Eq. (7.2) is based on representing $\psi_{\omega}(\boldsymbol{r})$ as a sum of many random contributions (plane waves with random amplitudes and phases). This is essentially the same argument that was used by Berry to describe fluctuations of eigenfunctions $\psi_{i}$ in chaotic billiards and which leads to the RMT statistics of $\left|\psi_{i}^{2}(\boldsymbol{r})\right|$ (see Section 3). Let us mention, however, a difference between the two cases (emphasized by Pnini and Shapiro [246]). In the case of an open system, $\psi_{\omega}$ is a sum of traveling waves, while for the closed system $\psi_{i}$ is represented as a sum of standing waves. As a result, the Rayleigh statistics has the same form as the statistics of eigenfunction amplitudes in a closed system with broken time-reversal invariance (unitary class), where the eigenfunctions are complex.
Diagrammatic derivation of Eq. (7.2) is very simple [247] (see below); for the case of a smooth randomness an essentially equivalent derivation [248] using path-integral arguments was given. However, similarly to distributions of other quantities studied above (eigenfunction amplitude, local DOS, relaxation time, etc.), distribution of optical intensities show deviations from the Rayleigh law, which will be studied below. More specifically, we will consider fluctuations of the intensity $I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)$ at a point $\boldsymbol{r}$ induced by a point-like source at $\boldsymbol{r}_{0}$, with the both points $\boldsymbol{r}$ and $\boldsymbol{r}_{0}$ located in the bulk of the sample. We will assume quasi-1D geometry of the sample with length $L$ much larger than the transverse dimension $W$ (see Fig. 11).

Let us note that there was a considerable activity recently in studying statistics of the transmission coefficients $T_{a b}$ of a disordered waveguide [72-74,249,250]. In this formulation of the problem, a source and a detector of the radiation are located outside the sample. The source produces a plane wave injected in an incoming channel $a$, and the intensity in an outgoing channel $b$ is measured. The transmission coefficients are related to the transmission matrix $t$ (already mentioned in Section 6) as $T_{a b}=\left|t_{a b}\right|^{2}$. One can also define the transmittance $T_{a}=\sum_{a} T_{a b}$ summed over the outgoing channels. Finally, the total transmittance $T=\sum_{a, b} T_{a b}$ is the optical analogue of the conductance $g$ (fluctuations of which were studied in Section 6). Combining the diagrammatic approach with the results on distribution of transmission eigenvalues in the metallic regime (following from the DMPK equations), Nieuwenhuizen and van Rossum [72] calculated the distribution functions $P\left(T_{a}\right)$ and $P\left(T_{a b}\right)$ for $g \gg 1$. The results have the following form:

$$
\begin{equation*}
P\left(s_{a}\right)=\int_{-i \infty}^{\infty} \frac{\mathrm{d} x}{2 \pi i} \exp \left[x s_{a}-\Phi_{\mathrm{con}}(x)\right] ; \tag{7.3}
\end{equation*}
$$



Fig. 11. Geometry of the problem. Points $\boldsymbol{r}_{0}=\left(x_{0}, y_{0}, z_{0}\right)$ and $\boldsymbol{r}=(x, y, z)$ are the positions of the source and of the observation point (detector) respectively. From [71].

$$
\begin{align*}
& P\left(s_{a b}\right)=\int_{0}^{\infty} \frac{\mathrm{d} v}{v} \int_{-i \infty}^{\infty} \frac{\mathrm{d} x}{2 \pi i} \exp \left[-\frac{s_{a b}}{v}+x v-\Phi_{\mathrm{con}}(x)\right]  \tag{7.4}\\
& \Phi_{\mathrm{con}}(x)=g \ln ^{2}(\sqrt{1+x / g}+\sqrt{x / g}) \tag{7.5}
\end{align*}
$$

where $s_{a}=T_{a} /\left\langle T_{a}\right\rangle$ and $s_{a b}=T_{a b} /\left\langle T_{a b}\right\rangle$. For large $g \gg 1$ (i.e. in the "metallic" regime), the distribution $P\left(s_{a}\right)$, Eq. (7.3), has a Gaussian form,

$$
\begin{equation*}
P\left(s_{a}\right) \simeq \sqrt{\frac{3 g}{4 \pi}} \exp \left[-\frac{3 g}{4}\left(s_{a}-1\right)^{2}\right], \tag{7.6}
\end{equation*}
$$

with log-normal tails at small $s_{a} \ll 1$ and an exponential asymptotics at large $s_{a} \gg 1$ of the form $P\left(s_{a}\right) \sim \exp \left(-g s_{a}\right)$. The distribution of the channel-to-channel transmission coefficients $s_{a b}$ is a close relative of the distribution of the point-to-point transmitted intensity $I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)$ which was discussed above and will be studied in detail below. For not too large $s_{a b}$ it has the Rayleigh form, $P\left(s_{a b}\right) \sim \mathrm{e}^{-s_{a b}}$, with the leading perturbative correction given by

$$
\begin{equation*}
P\left(s_{a b}\right) \simeq \mathrm{e}^{-s_{a b}}\left[1+\frac{1}{3 g}\left(s_{a b}^{2}-4 s_{a b}+2\right)\right], \quad s_{a b} \ll \sqrt{g} \tag{7.7}
\end{equation*}
$$

(this form of the correction to the Rayleigh law for the intensity distribution was found in a number of papers, see Refs. [248,249,251,252]). In the intermediate region $\sqrt{g}<s_{a b}<g$ Eq. (7.4) yields

$$
\begin{equation*}
P\left(s_{a b}\right) \simeq \exp \left[-s_{a b}\left(1+\frac{1}{3 g} s_{a b}+\ldots\right)\right] \tag{7.8}
\end{equation*}
$$

(correction of this type was also obtained earlier, Refs. [252,253]). For large $s_{a b}>g$ the distribution acquires a stretched-exponential form

$$
\begin{equation*}
P\left(s_{a b}\right) \sim \exp \left(-2 \sqrt{g s_{a b}}\right) \tag{7.9}
\end{equation*}
$$

We will return to Eq. (7.9) below, when comparing the results for the distribution of $I\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)$ with $P\left(s_{a b}\right)$. In the localized regime both distributions $P\left(s_{a}\right)$ and $P\left(s_{a b}\right)$ are determined by the single (largest) transmission eigenvalue and acquire the same log-normal form as the conductance distribution, Eq. (6.33). We note also that van Langen et al. [74] were able to calculate $P\left(s_{a}\right)$ and $P\left(s_{a b}\right)$ in the case of the unitary symmetry, $\beta=2$, in the whole range of the parameter $L / \xi$ (from weak to strong localization). The distributions $P\left(s_{a}\right)$ and $P\left(s_{a b}\right)$ were studied experimentally by Garcia and Genack [254,255] and by Stoychev and Genack [256]; their findings are in good agreement with the theoretical results.

Now we return to the statistics of the intensity $I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)$. The field at the point $\boldsymbol{r}$ is given by the (retarded) Green's function $G_{R}\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)$ and the radiation intensity is $I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)=\left|G_{R}\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)\right|^{2}$. The average intensity $\left\langle I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)\right\rangle$ is given diagrammatically by a diffuson $T\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$, attached to two external vertices. The vertices are short-range objects and can be approximated by a $\delta$-function times $(\ell / 4 \pi)$, so that $\left\langle I\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)\right\rangle=(\ell / 4 \pi)^{2} T\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)$. For the quasi-one-dimensional geometry, the expression for the diffuson reads

$$
\begin{equation*}
T\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)=\left(\frac{4 \pi}{\ell}\right)^{2} \frac{3}{4 \pi} \frac{\left[z_{<}\left(L-z_{>}\right)\right]}{A \ell L} \tag{7.10}
\end{equation*}
$$

where $\ell$ is the elastic mean free path, $A$ is the cross-section of the tube, $z$-axis is directed along the sample, $z_{<}=\min \left(z, z_{0}\right)$ and $z_{>}=\max \left(z, z_{0}\right)$. We assume that $\left|z-z_{0}\right| \gg W$.

The intensity distribution $P(I)$ is obtained, in the diagrammatic approach, by calculating the moments $\left\langle I^{n}\right\rangle$ of the intensity. In the leading approximation [247], one should draw $n$ retarded and $n$ advanced Green's functions and insert ladders between pairs $\left\{G_{\mathrm{R}}, G_{\mathrm{A}}\right\}$ in all possible ways. This leads to $\left\langle I^{n}\right\rangle=n!\langle I\rangle^{n}$ and, thus, to Eq. (7.2).

Corrections to the Rayleigh result come from diagrams with intersecting ladders, which describe interaction between diffusons. The leading correction is due to pairwise interactions. The diagram in Fig. 12 represents a pair of "colliding" diffusons. The algebraic expression for this diagram is

$$
\begin{align*}
C\left(\boldsymbol{r}, \boldsymbol{r}_{0}\right)= & 2\left(\frac{\ell}{4 \pi}\right)^{4} \int\left(\prod_{l=1}^{4} \mathrm{~d}^{3} \boldsymbol{r}_{l}\right) T\left(\boldsymbol{r}, \boldsymbol{r}_{1}\right) T\left(\boldsymbol{r}, \boldsymbol{r}_{2}\right) T\left(\boldsymbol{r}_{3}, \boldsymbol{r}_{0}\right) T\left(\boldsymbol{r}_{4}, \boldsymbol{r}_{0}\right) \\
& \times\left\{\left(\frac{\ell^{5}}{48 \pi k_{0}^{2}}\right) \int \mathrm{d}^{3} \rho\left[\left(\nabla_{1}+\nabla_{2}\right) \cdot\left(\nabla_{3}+\nabla_{4}\right)+2\left(\nabla_{1} \cdot \nabla_{2}\right)+2\left(\nabla_{3} \cdot \nabla_{4}\right)\right] \prod_{l=1}^{4} \delta\left(\rho-\boldsymbol{r}_{l}\right)\right\}, \tag{7.11}
\end{align*}
$$

where $k_{0}$ is the wave number and $\nabla_{l}$ acts on $\boldsymbol{r}_{l}$. The factor $(\ell / 4 \pi)^{4}$ comes from the 4 external vertices of the diagram, the $T$ 's represent the two incoming and two outgoing diffusons and the expression in the curly brackets corresponds to the internal (interaction) vertex ("Hikami box") [257,258]. Finally, the factor 2 accounts for the two possibilities of inserting a pair of ladders between the outgoing Green's functions. Integrating by parts and employing the quasi one-dimensional


Fig. 12. Diagram for a pair of interacting diffusons. The external vertices contribute the factor $(\ell / 4 \pi)^{4}$. The shaded region denotes the internal interaction vertex, see Eq. (7.11). From [71].
geometry of the problem, we obtain (for $z_{0}<z$ ):

$$
\begin{equation*}
C\left(z, z_{0}\right) \simeq 2\left\langle I\left(z, z_{0}\right)\right\rangle^{2}\left(1+\frac{4}{3 \gamma}\right) \tag{7.12}
\end{equation*}
$$

where

$$
\left\langle I\left(z, z_{0}\right)\right\rangle=\frac{3}{4 \pi} \frac{z_{0}(L-z)}{A \ell L}
$$

is the average intensity,

$$
\begin{equation*}
\gamma=2 g \frac{L^{3}}{L^{2}\left(3 z+z_{0}\right)-2 L z\left(z+z_{0}\right)+2 z_{0}^{2}\left(z-z_{0}\right)} \gg 1 \tag{7.13}
\end{equation*}
$$

and $g=k_{0}^{2} \ell A / 3 \pi L \gg 1$ is the dimensionless conductance of the tube. For simplicity, we will assume that the source and the detector are located relatively close to each other, so that $\left|z-z_{0}\right| \ll L$, in which case Eq. (7.13) reduces to $\gamma=g L^{2} / 2 z(L-z)$. (All the results are found to be qualitatively the same in the generic situation $z_{0} \sim z-z_{0} \sim L-z \sim L$.)

In order to calculate $\left\langle I^{n}\right\rangle$ one has to compute a combinatorial factor which counts the number $N_{i}$ of diagrams with $i$ pairs of interacting diffusons. This number is $N_{i}=(n!)^{2} /\left[2^{2 i} i!(n-2 i)!\right] \simeq$ $(n!/ i!)(n / 2)^{2 i}$. For not too large $n \ll \gamma^{1 / 2}$ it is sufficient to keep the $i=1$ contribution, yielding the leading perturbative correction [71],

$$
\begin{equation*}
\left\langle\tilde{I}^{n}\right\rangle=n!\left[1+\frac{2}{3 \gamma} n(n-1)\right] \tag{7.14}
\end{equation*}
$$

or in terms of the distribution function,

$$
\begin{equation*}
P(\widetilde{I})=\mathrm{e}^{-\tilde{I}\left[1+\frac{2}{3 \gamma}\left(\tilde{I}^{2}-4 \tilde{I}+2\right)\right], \quad \tilde{I} \ll \gamma^{1 / 2} . . . ~ . ~} \tag{7.15}
\end{equation*}
$$

For larger $n$ (or, equivalently, $\widetilde{I}$ ) we have to sum up the series over $i$, yielding

$$
\begin{equation*}
\frac{\left\langle I^{n}\right\rangle}{\langle I\rangle^{n}}=n!\sum_{i=0}^{[n / 2]} \frac{1}{i!}\left(\frac{2 n^{2}}{3 \gamma}\right)^{i} \simeq n!\exp \left(2 n^{2} / 3 \gamma\right) \tag{7.16}
\end{equation*}
$$

Although $i$ cannot exceed $n / 2$, the sum in Eq. (7.16) can be extended to $\infty$, if the value of $n$ is restricted by the condition $n \ll \gamma$. Eq. (7.16) represents the leading exponential correction to the Rayleigh distribution. Let us discuss now effect of higher order "interactions" of diffusons. Diagrams with 3 intersecting diffusons will contribute a correction of $n^{3} / \gamma^{2}$ in the exponent of Eq. (7.16), which is small compared to the leading correction in the whole region $n \ll \gamma$, but becomes larger than unity for $n \gtrsim \gamma^{2 / 3}$. Likewise, diagrams with 4 intersecting diffusons produce a $n^{4} / \gamma^{3}$ correction, etc. Restoring the distribution $P(I)$, we find [71]

$$
\begin{equation*}
P(\widetilde{I}) \simeq \exp \left\{-\tilde{I}+\frac{2}{3 \gamma} \tilde{I}^{2}+\mathrm{O}\left(\frac{\tilde{I}^{3}}{\gamma^{2}}\right)+\ldots\right\} \tag{7.17}
\end{equation*}
$$

Eqs. (7.14), (7.15) and (7.17) are analogous to those found in the case of transmission coefficient statistics, (7.7), (7.8), with the only difference that the parameter $g$ is now replaced by $\gamma / 2$ (if we
consider the limit $z_{0} \rightarrow 0, z \rightarrow L$, then $\gamma \rightarrow 2 g$, so that both results are consistent). As has been already mentioned, deviations of this form from the Rayleigh distribution of intensities were found earlier by various authors [248,249,251-253]; a value of the parameter governing the strength of deviations (here $1 / \gamma$ ) depends, however, on the geometry of the problem. It should be realized that Eq. (7.17) is applicable only for $\tilde{I}<\gamma \sim g$ and, thus, does not determine the far asymptotics of $P(I)$. The latter is unaccessible by the perturbative diagram technique and is handled below by the supersymmetry method.

For technical simplicity, we will assume now that the time reversal symmetry is broken by some magnetooptical effects (unitary ensemble). The moments of the intensity at point $\boldsymbol{r}$ due to the source at $\boldsymbol{r}_{0}$ are given by

$$
\begin{equation*}
\left\langle I^{n}\right\rangle=\left(-\frac{k_{0}^{2}}{16 \pi^{2}}\right)^{n} \int[D Q]\left(Q_{12, b b}(z)\right)^{n}\left(Q_{21, b b}\left(z_{0}\right)\right)^{n} \mathrm{e}^{-S[Q]}, \tag{7.18}
\end{equation*}
$$

where $S[Q]$ is the zero-frequency $\sigma$-model action,

$$
\begin{equation*}
S[Q]=-\frac{\pi v D}{4} \int \mathrm{~d}^{3} \boldsymbol{r} \operatorname{Str}(\nabla Q)^{2}, \tag{7.19}
\end{equation*}
$$

which reduces in the quasi-1D geometry to

$$
S[Q]=-(g L / 8) \int \mathrm{d} z \operatorname{Str}(\mathrm{~d} Q / \mathrm{d} z)^{2} .
$$

Assuming again that the two points $\boldsymbol{r}$ and $\boldsymbol{r}_{0}$ are sufficiently close to each other, $\left|z-z_{0}\right| \ll L$ and taking into account slow variation of the $Q$-field along the sample, we can replace the product $Q_{12, b b}(z) Q_{21, b b}\left(z_{0}\right)$ by $Q_{12, b b}(z) Q_{21, b b}(z)$. We get then the following result for the distribution of the dimensionless intensity $y=\left(16 \pi^{2} / k_{0}^{2}\right) I$ :

$$
\begin{equation*}
P(y)=\int \mathrm{d} Q \delta\left(y+Q_{12, b b} Q_{21, b b}\right) Y(Q), \tag{7.20}
\end{equation*}
$$

where $Y(Q)$ is a function of a single supermatrix $Q$ defined by Eq. (3.8). Using the fact that the function $Y(Q)$ depends only on the eigenvalues $1 \leq \lambda_{1}<\infty,-1 \leq \lambda_{2} \leq 1$, we find

$$
\begin{equation*}
P(y)=\left(\frac{\mathrm{d}}{\mathrm{~d} y}+y \frac{\mathrm{~d}^{2}}{\mathrm{~d} y^{2}}\right) \int \mathrm{d} \lambda_{1} \mathrm{~d} \lambda_{2}\left(\frac{\lambda_{1}+\lambda_{2}}{\lambda_{1}-\lambda_{2}}\right) Y\left(\lambda_{1}, \lambda_{2}\right) \delta\left(y+1-\lambda_{1}^{2}\right) . \tag{7.21}
\end{equation*}
$$

The function $Y\left(\lambda_{1}, \lambda_{2}\right)$ can be evaluated at $g \gg 1$ via the saddle-point method (see [34] and Section 4.3) with the result

$$
\begin{equation*}
Y\left(\lambda_{1}, \lambda_{2}\right) \simeq \exp \left\{-\frac{\gamma}{2}\left[\theta_{1}^{2}+\theta_{2}^{2}\right]\right\} . \tag{7.22}
\end{equation*}
$$

where $\lambda_{1} \equiv \cosh \theta_{1}, \lambda_{2} \equiv \cos \theta_{2}\left(0 \leq \theta_{1}<\infty, 0 \leq \theta_{2} \leq \pi\right)$. In fact, the dependence of $Y$ on $\theta_{2}$ is not important, within the exponential accuracy, because it simply gives a prefactor after the integration in Eq. (7.21). Therefore, the distribution function $P(y)$ is given by

$$
\begin{equation*}
P(y) \sim Y\left(\lambda_{1}=\sqrt{1+y}, \lambda_{2}=1\right) \sim \exp \left(-\gamma \theta_{1}^{2} / 2\right), \tag{7.23}
\end{equation*}
$$

where $\theta_{1}=\ln (\sqrt{1+y}+\sqrt{y})$. Finally, after normalizing $y$ to its average value $\langle y\rangle=2 / \gamma$, we obtain [71]:

$$
\begin{equation*}
P(\tilde{I}) \simeq \exp \left\{-\frac{\gamma}{2}\left[\ln ^{2}(\sqrt{1+2 \tilde{I} / \gamma}+\sqrt{2 \tilde{I} / \gamma})\right]\right\} . \tag{7.24}
\end{equation*}
$$

For $\tilde{I}<\gamma$, Eq. (7.24) reproduces the perturbative expansion (7.17), while for $\tilde{I} \gg \gamma$ it implies the log-normal asymptotic behavior of the distribution $P(\widetilde{I})$ :

$$
\begin{equation*}
\ln P(\tilde{I}) \simeq-(\gamma / 8) \ln ^{2}(8 \tilde{I} / \gamma) \tag{7.25}
\end{equation*}
$$

The log-normal "tail" (7.25) should be contrasted with the stretched-exponential asymptotic behavior of the distribution of transmission coefficients, Eq. (7.9). As was found in [71], these two results match each other in the following way. When the points $z$ and $z_{0}$ approach the sample edges, $z_{0}=L-z \ll L$, an intermediate regime of a stretched-exponential behavior emerges:

$$
\ln P(\tilde{I}) \simeq\left\{\begin{array}{cl}
-\tilde{I}+\frac{1}{3 g} \tilde{I}^{2}+\cdots, & \tilde{I} \ll g,  \tag{7.26}\\
-2 \sqrt{g \tilde{I}}, & g \ll \tilde{I} \ll g\left(\frac{L}{z_{0}}\right)^{2}, \\
-\frac{g L}{8 z_{0}} \ln ^{2}\left[16\left(\frac{z_{0}}{L}\right)^{2} \frac{\tilde{I}}{g}\right], & \tilde{I} \gg g\left(\frac{L}{z_{0}}\right)^{2} .
\end{array}\right.
$$

Thus, when the source and the detector move toward the sample edges, the range of validity of the stretched-exponential behavior becomes broader, while the log-normal "tail" gets pushed further away. In contrast, when the source and the detector are located deep in the bulk, $z_{0} \sim L-z \sim L$, the stretched-exponential regime disappears, and the Rayleigh distribution crosses over directly to the log-normal one at $\tilde{I} \sim g$.

Let us now describe the physical mechanisms standing behind these different forms of $P(\widetilde{I})$ [71]. The Green's function $G^{\mathrm{R}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)$ can be expanded in eigenfunctions of a non-Hermitean (due to open boundaries) "Hamiltonian" as

$$
G^{\mathrm{R}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)=\sum_{i} \psi_{i}^{*}\left(\boldsymbol{r}_{0}\right) \psi_{i}(\boldsymbol{r})\left(k_{0}^{2}-E_{i}+\mathrm{i} \gamma_{i}\right)^{-1} .
$$

Since the level widths $\gamma_{i}$ are typically of order of the Thouless energy $E_{c} \sim D / L^{2}$, there is typically $\sim g$ levels contributing appreciably to the sum. In view of the random phases of the wave functions, this leads to a Gaussian distribution of $G^{\mathrm{R}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)$ with zero mean, and thus to the Rayleigh distribution of $I\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)=\left|G^{\mathrm{R}}\left(\boldsymbol{r}_{0}, \boldsymbol{r}\right)\right|^{2}$, with the moments $\left\langle\tilde{I}^{n}\right\rangle=n!$. The stretched-exponential behavior results from such disorder realizations where one of the states $\psi_{i}$ has large amplitudes in both the points $\boldsymbol{r}_{0}$ and $\boldsymbol{r}$. Considering both $\psi_{i}\left(\boldsymbol{r}_{0}\right)$ and $\psi_{i}(\boldsymbol{r})$ as independent random variables with Gaussian distribution and taking into account that only one (out of $g$ ) term contributes in this case to the sum for $G^{\mathrm{R}}$, we find $\left\langle\tilde{I}^{n}\right\rangle \sim n!n!/ g^{n}$, corresponding to the above stretched-exponential form of $P(\tilde{I})$. Finally, the log-normal asymptotic behavior corresponds to those disorder realizations where $G^{\mathrm{R}}$ is dominated by an anomalously localized state, which has an atypically small width $\gamma_{i}$
(the same mechanism determines the log-normal asymptotics of the distribution of local density of states, see Section 4.3).

## 8. Statistics of energy levels and eigenfunctions in a ballistic system with surface scattering

In the preceding part of this article we considered statistical properties of spectra of disordered diffusive systems. Using the supersymmetric $\sigma$-model approach, we were able to demonstrate the relevance of the random matrix theory (RMT) and to calculate deviations from its predictions both for the level and eigenfunction statistics. Generalization of these results to the case of a chaotic ballistic system (i.e. quantum billiard) has become a topic of great research interest. For ballistic disordered systems the $\sigma$-model has been proposed [75], with the Liouville operator replacing the diffusion operator in the action. It has also been conjectured that the same $\sigma$-model in the limit of vanishing disorder describes statistical properties of spectra of individual classically chaotic system. This conjecture was further developed in $[76,77,259]$ where the $\sigma$-model was obtained by means of energy averaging, and the Liouville operator was replaced by its regularization - the PerronFrobenius operator.

However, straightforward application of the results of Refs. [10,11,25,27] to the case of an individual chaotic system is complicated by the fact that the eigenvalues of the Perron-Frobenius operator are unknown, while its eigenfunctions are extremely singular. For this reason the $\sigma$-model approach has so far failed to provide explicit results for any particular ballistic system.

In this section, we consider a ballistic system with surface disorder leading to diffusive scattering of a particle in each collision with the boundary. This models behavior of a quantum particle in a box with a rough boundary which is irregular on the scale of the wave length. Since the particle loses memory of its direction of motion after a single collision, this model describes a limit of an "extremely chaotic" ballistic system, with typical relaxation time being of order of the flight time. (This should be contrasted with the case of a relatively slight distortion of an integrable billiard [196,260,261].) One might naively think that all results for such a model could be obtained by setting $l \approx L$ in a system with bulk disorder. In fact, the level statistics in a system with bulk disorder and arbitrary relation between mean free path $l$ and system size $L$ was studied in [168,169,262]. However, the results presented below show that systems with bulk and surface disorder are not equivalent.

To simplify the calculations, we will assume a circular geometry of the billiard. A similar problem was studied numerically in Ref. [263,264] for a square geometry. We consider only the case of unitary symmetry (broken time-reversal invariance); generalization to the orthogonal case is straightforward. We follow Ref. [78] in the presentation below. The level statistics for the same problem was independently studied in Ref. [79]. Very recently, the same approach was used [265] to calculate the persistent current in a ring with diffusive scattering.

Our starting point is the sigma-model for ballistic disordered systems [31,75]. The effective action for this model has the form

$$
\begin{equation*}
F[g(\boldsymbol{r}, \boldsymbol{n})]=\frac{\pi v}{4} \int \mathrm{~d} \boldsymbol{r} \operatorname{Str}\left[\mathrm{i} \omega \Lambda\langle g(\boldsymbol{r})\rangle-\frac{1}{2 \tau(\boldsymbol{r})}\langle g(\boldsymbol{r})\rangle^{2}-2 v_{\mathrm{F}}\left\langle\Lambda U^{-1} \boldsymbol{n} \nabla U\right\rangle\right] . \tag{8.1}
\end{equation*}
$$

Here a $8 \times 8$ supermatrix field $g$ is defined on the energy shell of the phase space, i.e. $g$ depends on the coordinate $\boldsymbol{r}$ and direction of the momentum $\boldsymbol{n}$. The momentum dependence of the field $g$ distinguishes the ballistic $\sigma$-model from the diffusive case where the supermatrix field $Q$ depends on $\boldsymbol{r}$ only. The angular brackets denote averaging over $\boldsymbol{n}:\langle\mathcal{O}(\boldsymbol{n})\rangle=\int \mathrm{d} \boldsymbol{n} \mathcal{O}(\boldsymbol{n})$ with the normalization $\int \mathrm{d} \boldsymbol{n}=1$. Like in the case of the diffusive $\sigma$-model, the matrix $g$ is constrained by the condition $g^{2}(\boldsymbol{r}, \boldsymbol{n})=1$, and can be represented as $g=U \Lambda U^{-1}$, with $\Lambda=\operatorname{diag}(1,1,1,1,-1,-1,-1,-1)$. Since we are interested in the clean limit with no disorder in the bulk, the second term in the action (8.1) containing the elastic mean free time $\tau$ is zero everywhere except at the boundary where it modifies the boundary condition (see below).

As was explained in Section 2, the statistical properties of energy levels are governed by the structure of the action in the vicinity of the homogeneous configuration of the $g$-field, $g(\boldsymbol{r}, \boldsymbol{n})=\Lambda$. Writing $U=1-W / 2+\cdots$, we find the action in the leading order in $W$,

$$
\begin{equation*}
F_{0}[W]=-\frac{\pi v}{4} \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{n} \operatorname{Str}\left[W_{21}(\hat{K}-\mathrm{i} \omega) W_{12}\right], \tag{8.2}
\end{equation*}
$$

where the indices 1,2 refer to the "advanced-retarded" decomposition of $W$, and $\hat{K}$ is the Liouville operator, $\hat{K} \equiv v_{\mathrm{F}} \boldsymbol{n} \nabla$. This "linearized" action has the same form as that of a diffusive system, with the diffusion operator being replaced by the Liouville operator. This enables us to use the results derived for the diffusive case by substituting the eigenvalues and eigenfunctions of the operator $\hat{K}$ for those of the diffusion operator.

The operator $\hat{K}$ should be supplemented by a boundary condition, which depends on the form of the surface roughness. As a model approximation we consider purely diffuse scattering [266,267] for which the distribution function $\varphi(\boldsymbol{r}, \boldsymbol{n})$ of the outgoing particles is constant and is fixed by flux conservation: ${ }^{13}$

$$
\begin{equation*}
\varphi(\boldsymbol{r}, \boldsymbol{n})=\pi \int_{\left(N n^{\prime}\right)>0}\left(\boldsymbol{N} \boldsymbol{n}^{\prime}\right) \varphi\left(\boldsymbol{r}, \boldsymbol{n}^{\prime}\right) \mathrm{d} \boldsymbol{n}^{\prime}, \quad(\boldsymbol{N} \boldsymbol{n})<0 . \tag{8.3}
\end{equation*}
$$

Here the point $\boldsymbol{r}$ lies at the surface, and $\boldsymbol{N}$ is an outward normal to the surface. This boundary condition should be satisfied by the eigenfunctions of $\hat{K}$.

The eigenvalues $\lambda$ of the operator $\hat{K}$ corresponding to angular momentum $l$ obey the equation

$$
\begin{equation*}
\tilde{J}_{l}(\xi) \equiv-1+\frac{1}{2} \int_{0}^{\pi} \mathrm{d} \theta \sin \theta \exp [2 \mathrm{i} l \theta+2 \xi \sin \theta]=0 \tag{8.4}
\end{equation*}
$$

where $\xi \equiv R \lambda / v_{\mathrm{F}}$, and $R$ is the radius of the circle. For each value of $l=0, \pm 1, \pm 2, \ldots$ Eq. (8.4) has a set of solutions $\xi_{l k}$ with $\xi_{l k}=\xi_{-l, k}=\xi_{l,-k}^{*}$, which can be labeled with $k=0, \pm 1, \pm 2, \ldots$ (even $l$ ) or $k= \pm 1 / 2, \pm 3 / 2, \ldots$ (odd $l$ ). For $l=k=0$ we have $\xi_{00}=0$, corresponding to the zero mode

[^12]

Fig. 13. First $11 \times 11(0 \leq k, l<11)$ eigenvalues of the Liouville operator $\hat{K}$ in units of $v_{F} / R$, as given by Eq. (8.4). From Ref. [78].
$\varphi(\boldsymbol{r}, \boldsymbol{n})=$ const. All other eigenvalues have positive real part $\operatorname{Re} \xi_{l k}>0$ and govern the relaxation of the corresponding classical system to the homogeneous distribution in the phase space.
The asymptotic form of the solutions of Eq. (8.4) for large $|k|$ and/or $|l|$ can be obtained by using the saddle-point method,

$$
\xi_{k l} \approx\left\{\begin{array}{cl}
0.66 l+0.14 \ln l+0.55 \pi \mathrm{i} k, & 0 \leq k \ll l,  \tag{8.5}\\
(\ln k) / 4+\pi \mathrm{i}(k+1 / 8), & 0 \leq l \ll k .
\end{array}\right.
$$

Note that for $k=0$ all eigenvalues are real, while for high values of $k$ they lie close to the imaginary axis and do not depend on $l$ (see Fig. 13).

### 8.1. Level statistics, low frequencies

As was explained in Section 2.2 [see Eq. (2.27)], in the range of relatively low frequencies (which for our problem means $\omega \ll v_{\mathrm{F}} / R$, see below) the level correlation function $R(s=\omega / \Delta)$ has the form

$$
\begin{equation*}
R(s)-1=\delta(s)-\frac{\sin ^{2} \pi s}{(\pi s)^{2}}+\mathscr{A}\left(\frac{R \Delta}{\pi v_{\mathrm{F}}}\right)^{2} \sin ^{2} \pi s . \tag{8.6}
\end{equation*}
$$

The first two terms correspond to the zero-mode approximation and are given by RMT, while the last one is the non-universal correction to the RMT results. The information about the operator $\hat{K}$ enters through the dimensionless constant $\mathscr{A}=\sum^{\prime} \xi_{k l}^{-2}$, where the prime indicates that the eigenvalue $\xi_{00}=0$ is excluded. The value of $\mathscr{A}$, as well as the high-frequency behavior of $R(s)$ (see below), can be extracted from the spectral function [9]

$$
\begin{equation*}
S(\omega)=\sum_{l} S_{l}(\omega) ; \quad S_{l}(\omega) \equiv \sum_{k}\left(\lambda_{k l}-\mathrm{i} \omega\right)^{-2} . \tag{8.7}
\end{equation*}
$$

According to the Cauchy theorem, $S_{l}$ can be represented as an integral in the complex plane,

$$
S_{l}(\omega)=\left(\frac{R}{v_{\mathrm{F}}}\right)^{2} \frac{1}{2 \pi \mathrm{i}} \oint_{C} \frac{1}{\left(z-\mathrm{i} \omega R / v_{\mathrm{F}}\right)^{2}} \frac{\widetilde{J}_{l}^{\prime}(z)}{\widetilde{J}_{l}(z)} \mathrm{d} z,
$$

where the contour $C$ encloses all zeroes of the function $\tilde{J}_{l}(z)$. Evaluating the residue at $z=\mathrm{i} \omega R / v_{\mathrm{F}}$, we find

$$
\begin{equation*}
S_{l}(\omega)=-\left.\left(R / v_{\mathrm{F}}\right)^{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} z^{2}}\right|_{z=\mathrm{i} \omega R / v_{\mathrm{F}}} \ln \tilde{J}_{l}(z) . \tag{8.8}
\end{equation*}
$$

Considering the limit $\omega \rightarrow 0$ and subtracting the contribution of $\lambda_{00}=0$, we get

$$
\begin{equation*}
\mathscr{A}=-19 / 27-175 \pi^{2} / 1152+64 /\left(9 \pi^{2}\right) \approx-1.48 \tag{8.9}
\end{equation*}
$$

In contrast to the diffusive case, this constant is negative: the level repulsion is enhanced with respect to result for RMT. Eq. (8.6) is valid as long as the correction is small compared to the RMT result, i.e. provided $\omega$ is below the inverse time of flight, $v_{\mathrm{F}} / R$.

### 8.2. Level statistics, high frequencies

In the range $\omega \gg \Delta$ the connected part of the level correlation function $R(\omega)-1$ can be decomposed into the smooth Altshuler-Shklovskii (AS) part $R_{\mathrm{AS}}(\omega)=\left(\Delta^{2} / 2 \pi^{2}\right) \operatorname{Re} S(\omega)$ [9] and the part $R_{\text {osc }}$ which oscillates on the scale of the level spacing (see Section 2.2). Evaluating the asymptotic behavior of $S_{l}(\omega)$ from Eq. (8.8), we find in the high-frequency regime when $\omega \gg v_{\mathrm{F}} / R$ :

$$
\begin{equation*}
R_{\mathrm{As}}(\omega)=\left(\frac{\Delta R}{v_{\mathrm{F}}}\right)^{2}\left(\frac{v_{\mathrm{F}}}{2 \pi \omega R}\right)^{1 / 2} \cos \left(4 \frac{\omega R}{v_{\mathrm{F}}}-\frac{\pi}{4}\right) . \tag{8.10}
\end{equation*}
$$

The oscillating part of the level correlation function $R_{\mathrm{osc}}(s)$ for frequencies $\omega \gg \Delta$ is given by Eq. (2.34) with the spectral determinant

$$
D(s)=s^{-2} \prod_{k l \neq(00)}\left(1-\mathrm{i} s \Delta / \lambda_{k l}\right)^{-1}\left(1+\mathrm{i} s \Delta / \lambda_{k l}\right)^{-1} .
$$

Since $\Delta^{-2} \partial^{2} \ln D(s) / \partial s^{2}=-2 \operatorname{Re} S(\omega)$, we can restore $D(s)$ from Eqs. (8.7), (8.8) up to a factor of the form $\exp \left(c_{1}+c_{2} s\right)$, with $c_{1}$ and $c_{2}$ being arbitrary constants. These constants are fixed by the requirement that Eq. (2.34) in the range $\Delta \ll \omega \ll v_{\mathrm{F}} / R$ should reproduce the low-frequency behavior (8.6). As a result, we obtain

$$
\begin{equation*}
D(s)=\left(\frac{\pi}{2}\right)^{6} \frac{1}{N} \prod_{l} \frac{1}{\widetilde{J}_{l}\left(\mathrm{is} N^{-1 / 2}\right) \tilde{J}_{l}\left(-\mathrm{i} s N^{-1 / 2}\right)} . \tag{8.11}
\end{equation*}
$$

Here $N=\left(v_{\mathrm{F}} / R \Delta\right)^{2}=\left(p_{\mathrm{F}} R / 2\right)^{2}$ is the number of electrons below the Fermi level. For high frequencies $\omega \gg v_{\mathrm{F}} / R$ this yields the following expression for the oscillating part of the level correlation function:

$$
\begin{equation*}
R_{\mathrm{osc}}(\omega)=\frac{\pi^{4}}{128}\left(\frac{\Delta R}{v_{\mathrm{F}}}\right)^{2} \cos \left(\frac{2 \pi \omega}{\Delta}\right) . \tag{8.12}
\end{equation*}
$$

It is remarkable that the amplitude of the oscillating part does not depend on frequency. This is in contrast to the diffusive case, where in the AS regime ( $\omega$ above the Thouless energy) the oscillating part $R_{\text {osc }}(\omega)$ is exponentially small, see Eq. (2.39).

### 8.3. The level number variance

The smooth part of the level correlation function can be best illustrated by plotting the variance of the number of levels in an energy interval of width $E=s \Delta$,

$$
\begin{equation*}
\Sigma_{2}(s)=\int_{-s}^{s}(s-|\tilde{s}|) R(\tilde{s}) \mathrm{d} \tilde{s} \tag{8.13}
\end{equation*}
$$

A direct calculation gives for $s \ll N^{1 / 2}$

$$
\begin{equation*}
\pi^{2} \Sigma_{2}(s)=1+\gamma+\ln (2 \pi s)+\mathscr{A} s^{2} /(2 N) \tag{8.14}
\end{equation*}
$$

and for $s \gg N^{1 / 2}$

$$
\begin{equation*}
\pi^{2} \Sigma_{2}(s)=1+\gamma+\ln \frac{16 N^{1 / 2}}{\pi^{2}}-\frac{\pi^{2}}{16}\left(\frac{2 N^{1 / 2}}{\pi s}\right)^{1 / 2} \cos \left(\frac{4 s}{N^{1 / 2}}-\frac{\pi}{4}\right) . \tag{8.15}
\end{equation*}
$$

Here $\gamma \approx 0.577$ is Euler's constant, and $\mathscr{A}$ is defined by Eq. (8.9). The first three terms at the rhs of Eq. (8.14) represent the RMT contribution (curve 1 in Fig. 14).

As seen from Fig. 14, the two asymptotics (8.14) and (8.15) perfectly match in the intermediate regime, $s \sim N^{1 / 2}$. Taken together, they provide a complete description of $\Sigma_{2}(s)$. According to Eq. (8.15), the level number variance saturates at the value $\Sigma_{2}^{(0)}=\pi^{-2}\left(1+\gamma+\ln \left(16 N^{1 / 2} / \pi^{2}\right)\right)$, in


Fig. 14. Level number variance $\Sigma_{2}(E)$ as a function of energy; $s=E / \Delta$. Curve 1 shows the RMT result, while curves 2 and 3 correspond to asymptotic regimes of low (8.14) and high (8.15) frequencies. The saturation value $\Sigma_{2}^{(0)}$ is given in the text. From Ref. [78].
contrast to the behavior found for diffusive systems [9] or ballistic systems with weak bulk disorder [168,262]. The saturation occurs at energies $s \sim N^{1 / 2}$, or in conventional units $E \sim v_{\mathrm{F}} / R$. This saturation of $\Sigma_{2}(s)$, as well as its oscillations on the scale set by short periodic orbits, is expected for a generic chaotic billiard [80,81]. It is also in good agreement with the results for $\Sigma_{2}(s)$ found numerically for a tight-binding model with moderately strong disorder on boundary sites [263,264].

### 8.4. Eigenfunction statistics

Now we turn to correlations of the amplitudes of an eigenfunction in two different points defined by Eq. (3.67). As was discussed in Section 3.3.3, these correlations are governed by the ballistic propagator $\Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$ (see Eq. (3.98) and the text preceding it). Direct calculation gives:

$$
\begin{align*}
& \Pi_{\mathrm{B}}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\Pi_{1}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)+\Pi_{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right),  \tag{8.16}\\
& \Pi_{1}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right)-V^{-1} \int \mathrm{~d} \boldsymbol{r}_{1}^{\prime} \Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}^{\prime}-\boldsymbol{r}_{2}\right) \\
& \quad-V^{-1} \int \mathrm{~d} \boldsymbol{r}_{2}^{\prime} \Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}^{\prime}\right)+V^{-2} \int \mathrm{~d} \boldsymbol{r}_{1}^{\prime} \mathrm{d} \boldsymbol{r}_{2}^{\prime} \Pi_{\mathrm{B}}^{(0)}\left(\boldsymbol{r}_{1}^{\prime}-\boldsymbol{r}_{2}^{\prime}\right),  \tag{8.17}\\
& \Pi_{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\frac{1}{4 \pi p_{\mathrm{F}} R} \sum_{k=1}^{\infty} \frac{4 k^{2}-1}{4 k^{2}}\left(\frac{r_{1} r_{2}}{R^{2}}\right)^{k} \cos k\left(\theta_{1}-\theta_{2}\right)
\end{align*}
$$

where $\Pi_{\mathbf{B}}^{(0)}(\boldsymbol{r})=1 /\left(\pi p_{\mathrm{F}}|\boldsymbol{r}|\right)$, and $(r, \theta)$ are the polar coordinates. This formula has a clear interpretation. The function $\Pi_{\mathrm{B}}$ can be represented as a sum over all paths leading from $\boldsymbol{r}_{1}$ to $\boldsymbol{r}_{2}$, with possible surface scattering in between. In particular, $\Pi_{1}$ corresponds to direct trajectories from $\boldsymbol{r}_{1}$ to $\boldsymbol{r}_{2}$ with no reflection from the surface, while the contribution $\Pi_{2}$ is due to the surface scattering. The first term in the numerator $4 k^{2}-1$ comes from trajectories with only one surface reflection, while the second sums up contributions from multiple reflections.

Let us summarize the main results of this section. We have used the ballistic $\sigma$-model approach to study statistical properties of levels and eigenfunctions in a billiard with diffusive surface scattering, which exemplifies a ballistic system in the regime of strong chaos. It was found that the level repulsion and the spectral rigidity are enhanced compared to RMT. In particular, the level number variance saturates at the scale of the inverse time of flight, in agreement with Berry's prediction for a generic chaotic system [80,81]. As another manifestation of the strong spectral rigidity, the oscillating part of the level correlation function does not vanish at large level separation. We calculated also the ballistic analog of the diffusion propagator in this model, which governs correlations of eigenfunction amplitudes in different spatial points.

While we focused our attention on the statistics of levels and wave functions in a closed ballistic sample, the surface nature of scattering will also modify statistical properties of the transport characteristics for an open system. In this connection, we mention the recent papers [271,272] where quantum localization and fluctuations of the transmission coefficients and of the conductance were studied for a quasi-1D wire with a rough surface. A number of important differences compared to the case of bulk disorder was found.

Note that the motion in a quasi-1D wire with surface scattering is closely related to the PRBM ensemble of Section 5.3. According to (8.3), the probability density for a particle to leave the surface after a scattering event with an angle $\theta \ll 1$ with respect to the surface is $\mathscr{P}(\theta) \sim \theta$. Since the distance of the ballistic flight is $r \sim \theta^{-1}$ for small $\theta$, this yields $\mathscr{P}(r) \sim r^{-3}$ for $r$ larger than the transverse size of the wire. Thus, we get a power-law "tail" at large $r$ of the form (5.43) with $\alpha=3 / 2$. According to Section 5.2 , this is precisely the marginal value separating the regions of conventional $K_{0}^{-1}(q, 0) \propto q^{2}$ and unconventional $K_{0}^{-1}(q, 0) \propto q^{2 \alpha-1}$ behavior of the diffusion propagator $K_{0}(q, \omega)$. It is clear that at $\alpha=3 / 2$ the propagator acquires a logarithmic correction, $K_{0}^{-1}(q, 0) \propto q^{2} \ln q$, see Ref. [271] for details.

## 9. Electron-electron interaction in disordered mesoscopic systems

In the preceding sections we have considered statistical properties of energy levels and eigenfunctions of a single particle in a disorder potential. However, if an electronic system is considered, the Coulomb interaction between the electrons has to be taken into account. The influence of the electron-electron interaction on transport properties of disordered systems has been intensively studied during the last two decades, in particular in connection with such phenomena as weak localization [273] and universal conductance fluctuations [274]. The electron-electron interaction sets the length scale $l_{\phi}$ (phase breaking length) below which the electron wave function preserves its coherence. Also, interplay of the interaction and disorder leads to a singular (at Fermi energy or at zero temperature) correction to the density of states and to the conductivity [273].

More recently, another kind of problems has attracted the research interest: to what extent does the electron-electron interaction influence the properties of the electron spectrum in a disordered dot? This interest is largely motivated by a recent progress in nanotechnology which allows to observe experimentally discrete electronic levels in semiconductor quantum dots [93,275] and in small metallic grains [276,277]. In fact, there are two types of the quantum dot spectra studied experimentally via measuring their $I-V$ characteristics: (i) excitation spectrum, when excited levels are probed in a dot with given number of electrons by increasing the source-drain voltage, and (ii) addition spectrum, when electrons are added one by one by changing the gate voltage.

As concerns the excitation spectrum, it was found in [99,278] that the quasiparticle levels with energies below the Thouless energy $E_{c}$ (counted from the Fermi energy) have a width smaller than the mean level spacing and thus form a well-defined discrete spectrum. To show this, let us calculate the r.m.s. value of the matrix element $V_{i j k l}$ of the screened Coulomb interaction which describes a decay of the quasiparticle state $|i\rangle$ to a three-particle (more precisely, two particles + one hole) state $|j k l\rangle$. Using Eqs. (3.78), (3.88) for a diffusive dot, one finds [99] $\left.\left.\langle | V_{i j k l}\right|^{2}\right\rangle \sim(\Delta / g)^{2}$. On the other hand, the density of states of the three-particle states is $v_{3}(E) \sim E^{2} / \Delta^{3}$, so that the Golden Rule width of the one-particle states is $\left.\Gamma(E)=\left.2 \pi\langle | V\right|^{2}\right\rangle v_{3}(E) \sim E^{2} / \Delta g^{2}$. Comparing $\Gamma(E)$ with $\Delta$, we get the above threshold $E \sim g \Delta=E_{c}$. The number of the discrete excited levels is thus of order of the intrinsic dimensionless conductance $g$ of the dot (not to be confused with the tunneling conductance in the Coulomb blockade regime determined by the contacts). These findings are in agreement with experiment [93].

The fine structure of these excited single particle levels was studied theoretically in [100,279,280] (see also Refs. $[325,326]$ ) where the Hamiltonian of the many-body interacting problem was
considered as a tight-binding model in the Fock space, with matrix elements of the Coulomb interaction playing a role of the hopping terms. It was found that only for the energy above $E_{*} \sim \sqrt{g} \Delta$ is the Golden Rule applicable and the levels have the regular Lorentzian shape. For smaller energies the quasiparticle excitation consist essentially of a single peak with a small admixture of other many-particle exact eigenstates. This corresponds to the Anderson localization in the Fock space.

Though the properties of the single-particle excitations discussed above are of most interest, one can also discuss statistical properties of exact many-body levels. For excitation energy $E \gg \Delta$ a typical state consists of $\sim \sqrt{E / \Delta} \gg 1$ quasiparticles, each of them having an energy $\sim \sqrt{E \Delta}$. Comparing the density of those states to which any of such states is coupled with the matrix element of the interaction, one estimates the border for the complete delocalization of many-body states as $E_{c h} \sim \Delta g^{2 / 3}$ [279,281] (see also [282]). For energies $E \ll E_{c h}$ the many-body level statistics is Poissonnian, while for the energies above $E_{c h}$ it should acquire the Wigner-Dyson form. Numerical studies of statistical properties of spectra of many-body fermionic systems [283-287] indeed show such a crossover from the Poisson to RMT statistics with increasing excitation energy.

Statistical properties of the addition spectrum have been studied by several experimental groups recently [83,86,288-290]. We will address the theoretical aspects of this problem (following Refs. [84,136]) in Section 9.1.

Let us briefly mention another topic which has attracted a great deal of research interest recently. This is the problem of localization of two interacting particles raised by Shepelyansky [291] (see also an early paper by Dorokhov [292]). It was suggested in [291] that the effective localization length $\xi_{2}$ of two interacting particles in a 1D disordered sample can be much larger than the localization length $\xi_{1}$ of a non-interacting particle. This paper stimulated a considerable analytical and numerical activity. In the quasi-1D case, the problem can be mapped onto an ensemble of random banded matrices with strongly fluctuating diagonal elements, which was studied numerically in [293] and analytically (via the supersymmetric $\sigma$-model approach) in [294,295]. The results confirmed the original conjecture of Shepelyansky, $\xi_{2} \propto \xi_{1}^{2}$. In the case of a strictly 1D system, the situation is somewhat more complicated, since the corresponding random matrix ensemble was found [61] to be of the power-law random banded matrix type (see Section 5.3) in the critical regime $\alpha=1$. Still, the localization length enhancement was predicted, $\xi_{2} \propto \xi_{1}^{\mu}$, but with a non-universal exponent $\mu(1<\mu<2)$ depending on the interaction strength. Generalization of the Shepelyansky's idea to the vicinity of the Anderson transition point in higher-dimensional systems was proposed by Imry [296]. A detailed review of recent activity in this direction can be found in [89].

### 9.1. Coulomb blockade: fluctuations in the addition spectra of quantum dots

The Coulomb blockade effect shows up the most distinctly in the addition spectrum experiment, where one measures the conductance of an almost isolated quantum dot as a function of the gate voltage. Detailed description of the experimental setup can be found, e.g. in [82,297]. At low temperature $T$ one observes a sequence of almost equidistant peaks separated by "valleys" where the conductance is very small. The positions of the peaks are determined by the condition that the energies of the dot with $N$ and $N+1$ electrons are equal. We will assume below that the following
inequality is met: $\Gamma \ll k T \ll \Delta$, where $\Gamma=\Gamma_{l}+\Gamma_{r}$ is the sum of the tunneling rates to the leads. The peak width is then set by the temperature, while the height is given by [298]

$$
\begin{equation*}
g_{\max }=\frac{\mathrm{e}^{2}}{h} \frac{\pi}{2 k T} \frac{\Gamma_{l} \Gamma_{r}}{\Gamma_{l}+\Gamma_{r}} . \tag{9.1}
\end{equation*}
$$

The peak heights show strong fluctuations induced by RMT-like fluctuations of eigenfunction amplitudes, as predicted in [13,96] and observed experimentally in [94,95]. In the valleys between the peaks, it is energetically costly (of order of the charging energy $e^{2} / C$, where $C$ is the dot capacitance) to add an electron to (or to remove from) the dot, and the conductance is determined by virtual processes (so-called elastic cotunneling [299]) and is strongly suppressed.

The issue of the statistics of the peak spacings was addressed for the first time in [83]. Basing on numerical data for very small systems, the authors of [83] concluded that the r.m.s. deviation of the peak spacing $S_{N}$ is proportional to the charging energy $e^{2} / C$, with a coefficient $\approx 0.15$. We will show below (our consideration will closely follow Refs. $[84,136]$ ) that the fluctuations are in fact much smaller, of order of the mean level spacing $\Delta$. Let us note that in an analogous problem for classical particles the fluctuation magnitude $\operatorname{rms}\left(S_{N}\right)$ would indeed be proportional to the mean value $\left\langle S_{N}\right\rangle[300,301]$. The physical reason for smaller fluctuations in the quantum case is in the delocalized nature of the electronic wave functions, which are spread roughly uniformly over the system. These theoretical conclusions were confirmed recently by thorough experimental studies, as discussed in the end of the section.

The simplest theoretical model which may be used to study distribution of the peak spacings is as follows. One considers a dot as a fixed size diffusive mesoscopic sample and assumes that changing a gate voltage by an amount $\delta V_{g}$ simply reduces to a uniform change of the potential inside the dot by a constant $\gamma \delta V_{g}$, with certain numerical coefficient $\gamma$ ("lever arm"). Such a model was used for numerical simulations of the addition spectra in Refs. [83,302]. We start by considering the statistics of peak spacings within this model [84]; we will later return to the approximations involved and relax some of them. We will neglect the spin degree of freedom of electrons first; inclusion of the spin will be also discussed in the end of the section.

The distance between the two consecutive conductance peaks is given by (Fig. 15)

$$
\begin{align*}
S_{N} & =\left(\mathscr{E}_{N+2}-\mathscr{E}_{N+1}\right)-\left(\mathscr{E}_{N+1}-\mathscr{E}_{N}\right) \\
& =\mu_{N+1}^{N+2}-\mu_{N}^{N+1}, \tag{9.2}
\end{align*}
$$

where $\mathscr{E}_{N}$ is the ground state of a sample with $N$ electrons. In the second line of Eq. (9.2) we rewrote $S_{N}$ in terms of the Hartree-Fock single electron energy levels, with $\mu_{i}^{j}$ denoting the energy of the state $\# j$ in the dot containing $i$ electrons. It is convenient to decompose $S_{N}$ in the following way

$$
\begin{align*}
S_{N} & =\left(\mu_{N+1}^{N+2}-\mu_{N}^{N+2}\right)+\left(\mu_{N}^{N+2}-\mu_{N}^{N+1}\right) \\
& \equiv E_{1}+E_{2} . \tag{9.3}
\end{align*}
$$

The quantity $E_{2}$ is the distance between the two levels of the same one-particle (Hartree-Fock) Hamiltonian $\hat{H}_{N}$ (describing a dot with $N$ electrons) and is expected to obey RMT; in particular $\left\langle E_{2}\right\rangle=\Delta$ and r.m.s. $\left(E_{2}\right)=a \Delta$ with a numerical coefficient $a$ of order of unity $[a=0.52(0.42)$ for the orthogonal (resp. unitary) ensemble]. On the other hand, $E_{1}$ is a shift of the level $\#(N+2)$ due to the change of the Hamiltonian $\hat{H}_{N} \rightarrow \hat{H}_{N+1}$ accompanying addition of the electron $\#(N+1)$ to the system.


Fig. 15. Energy levels before and after the electron $\#(N+1)$ has been added to the dot. From Ref. [84].

The effective interaction $U\left(r, r^{\prime}\right)$ between the electrons \#( $N+1$ ) and \#( $N+2$ ) can be found from the RPA-type equation,

$$
\begin{equation*}
U\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=U_{0}\left(\boldsymbol{r}-\boldsymbol{r}^{\prime}\right)-\int \mathrm{d} \boldsymbol{r}_{1} \mathrm{~d} \boldsymbol{r}_{2} U_{0}\left(\boldsymbol{r}-\boldsymbol{r}_{1}\right) P_{0}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) U\left(\boldsymbol{r}_{2}, \boldsymbol{r}^{\prime}\right) \tag{9.4}
\end{equation*}
$$

where $U_{0}(r)=e^{2} / \varepsilon r, \varepsilon$ is the dielectric constant, and

$$
\begin{equation*}
P_{0}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=v\left[\delta\left(\boldsymbol{r}_{1}-\boldsymbol{r}_{2}\right)-V^{-1}\right] \tag{9.5}
\end{equation*}
$$

is the polarization operator. Its solution has the form [84]

$$
\begin{equation*}
U\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)=\bar{U}+\delta U(\boldsymbol{r})+\delta U\left(\boldsymbol{r}^{\prime}\right)+U_{\kappa}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right) . \tag{9.6}
\end{equation*}
$$

Here $\bar{U} \equiv e^{2} / C$ is a constant (charging energy), $\delta U(r)$ is the change of the self-consistent potential due to addition of one electron (i.e. difference in the self-consistent potential in the dots with $N$ and $N+1$ electrons), and $U_{\kappa}(\boldsymbol{r})$ is the screened Coulomb interaction. In particular, in the experimentally most relevant 2D case (which we will consider below) and assuming a circular form of the dot with radius $R$, we have

$$
\begin{equation*}
\delta U(\boldsymbol{r})=-\frac{e^{2}}{2 \varepsilon \kappa_{s} R}\left(R^{2}-r^{2}\right)^{-1 / 2}, \tag{9.7}
\end{equation*}
$$

while $U_{\kappa}$ is given in the Fourier space by $\widetilde{U}_{\kappa}(\boldsymbol{q})=2 \pi e^{2} / \varepsilon\left(q+\kappa_{s}\right)$ with the inverse screening length $\kappa_{s}=2 \pi e^{2} v / \varepsilon$.

According to Eq. (9.6), the term $E_{1}$ can be decomposed into the following three contributions:

$$
\begin{align*}
E_{1} & =e^{2} / C+\int \mathrm{d} \boldsymbol{r}\left(\left|\psi_{N+1}^{2}(\boldsymbol{r})\right|+\left|\psi_{N+2}^{2}(\boldsymbol{r})\right| \delta \delta U(\boldsymbol{r})+\int \mathrm{d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime}\left|\psi_{N+1}^{2}(\boldsymbol{r})\right| \psi_{N+2}^{2}\left(\boldsymbol{r}^{\prime}\right) \mid U_{\kappa}\left(\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|\right)\right. \\
& =E_{1}^{(0)}+E_{1}^{(1)}+E_{1}^{(2)} . \tag{9.8}
\end{align*}
$$

The first term in Eq. (9.8) (the charging energy) determines the average value $\left\langle E_{1}\right\rangle$ and thus the average peak spacing $\left\langle S_{N}\right\rangle$ (since $e^{2} / C \gtrdot \Delta$ for a large dot with $N \gg 1$ ). This is the only contribution to $E_{1}$ kept by the so-called constant interaction model, which in addition neglects fluctuations of the capacitance $C$. Consequently, fluctuations of $S_{N}$ within the constant interaction model are determined solely by fluctuations of the single-particle level spacing $E_{2}$ and thus follow RMT: r.m.s. $\left(S_{N}\right)=a \Delta$.

The term $E_{1}$ in Eq. (9.3) is however an additional source of fluctuations and is thus responsible for the enhancement of fluctuations in comparison with RMT. In principle, all the three terms $E_{1}^{(0)}$, $E_{1}^{(1)}$, and $E_{1}^{(2)}$ in Eq. (9.8) contribute to this enhancement. Fluctuations of the first one, $E_{1}^{(0)}=e^{2} / C$ are due to the fact that the capacitance is slightly different from its purely geometric value because of a finite value of the screening length. The corresponding correction to $C$ can be expressed in terms of the polarization operator $P\left(r, r^{\prime}\right)$. The latter is a fluctuating quantity (because of fluctuations of the eigenfunctions in the Fermi sea) and contains a random part $P_{\mathrm{r}}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)$ leading to the following expression for the random part of the charging energy:

$$
\begin{equation*}
\left(e^{2} / C\right)_{\mathrm{r}}=2 \int \mathrm{~d} \boldsymbol{r} \mathrm{~d} \boldsymbol{r}^{\prime} \delta U(\boldsymbol{r}) P_{\mathrm{r}}\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right) \delta U\left(\boldsymbol{r}^{\prime}\right) \tag{9.9}
\end{equation*}
$$

Evaluating the fluctuations of the polarization operator, we find [138]

$$
\begin{equation*}
\operatorname{var}\left(E_{1}^{(0)}\right)=\frac{48}{\beta} v^{2} \ln g\left[\frac{1}{V} \int \mathrm{~d} \boldsymbol{r}_{1} \mathrm{~d} \boldsymbol{r}_{2} \delta U\left(\boldsymbol{r}_{1}\right) \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \delta U\left(\boldsymbol{r}_{2}\right)\right]^{2} \propto \frac{1}{\beta} \ln g\left(\frac{\Delta}{g}\right)^{2} . \tag{9.10}
\end{equation*}
$$

Now we consider fluctuations of the last term, $E_{1}^{(2)}$, in Eq. (9.8). Using Eqs. (3.74) and (3.86) for the correlations of eigenfunction amplitudes in two remote points, the variance of $E_{1}^{(2)}$ is found to be

$$
\begin{align*}
\operatorname{var}\left(E_{1}^{(2)}\right) & =\frac{4}{\beta^{2} V^{4}} \int \mathrm{~d} \boldsymbol{r}_{1} \mathrm{~d} \boldsymbol{r}_{1}^{\prime} \mathrm{d} \boldsymbol{r}_{2} \mathrm{~d} \boldsymbol{r}_{2}^{\prime} U_{\kappa}\left(\left|\boldsymbol{r}_{1}-\boldsymbol{r}_{1}^{\prime}\right|\right) U_{\kappa}\left(\left|\boldsymbol{r}_{2}-\boldsymbol{r}_{2}^{\prime}\right|\right) \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \Pi\left(\boldsymbol{r}_{1}^{\prime}, \boldsymbol{r}_{2}^{\prime}\right) \\
& \approx \frac{4 \Delta^{2}}{\beta^{2} V^{2}} \int \mathrm{~d} \boldsymbol{r}_{1} \mathrm{~d} \boldsymbol{r}_{2} \Pi^{2}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \propto \frac{1}{\beta^{2}}\left(\frac{\Delta}{g}\right)^{2} . \tag{9.11}
\end{align*}
$$

Finally, fluctuations of the term $E_{1}^{(1)}$ can be also evaluated with a help of Eqs. (3.74), (3.86), yielding

$$
\begin{equation*}
\operatorname{var}\left(E_{1}^{(1)}\right)=\frac{4}{\beta V^{2}} \int \mathrm{~d} \boldsymbol{r}_{1} \mathrm{~d} \boldsymbol{r}_{2} \delta U\left(\boldsymbol{r}_{1}\right) \Pi\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right) \delta U\left(\boldsymbol{r}_{2}\right) \propto \frac{1}{\beta} \frac{\Delta^{2}}{g} . \tag{9.12}
\end{equation*}
$$

It is seen that for $g \gg 1$ all the contributions Eqs. (9.10)-(9.12) are parametrically small compared to the RMT fluctuations (which are $\sim \Delta$ ). Fluctuations of the term $E_{1}^{(1)}$ related to the change $\delta U(r)$ of the self-consistent potential represent parametrically leading contribution to the enhancement of the peak spacing fluctuations with respect to RMT.
Let us now discuss approximations made in the course of the above derivation:
(i) The dot was supposed to be diffusive in the calculation. For a ballistic dot one should replace $\Pi\left(\boldsymbol{r}, \boldsymbol{r}^{\prime}\right)$ by its ballistic counterpart. This would mean that the parameter $g$ is replaced by $\sim N^{1 / 2} \sim L / \lambda_{\mathrm{F}}$, where $N$ is the number of electrons in the dot and $L$ the characteristic linear dimension. The numerical coefficient would depend, however, on "how strongly chaotic" is the dot. The role of the eigenfunctions fluctuations and correlations ("scars") in enhancement of
the peak spacing fluctuations was studied in [303] via numerical simulations of a dot with $N \approx 100$ electrons.
(ii) It was assumed that changing the gate voltage results in a spatially uniform change of the potential in the sample. This has led us to the expression Eq. (9.7) for the change of the self-consistent potential $\delta U(\boldsymbol{r})$ accompanying the addition of one electron to the dot. This result would correspond to a gate located far enough from the sample. In a more realistic situation, when the gate is relatively narrow and located close to the sample, the potential change $\delta U(r)$ (as well as the additional electron density) will be concentrated on the side of the dot facing the gate. The change of the potential $\delta U(\boldsymbol{r})$ corresponds then to a slight deformation of the dot with adding each electron to it. To estimate $\delta U(\boldsymbol{r})$ in this case, we can consider a model problem of a point-like charged object (modeling the gate) located a distance $d$ from the edge of the dot. Assuming that the dot size is larger than $d$, we can approximate the dot [while calculating $\delta U(\boldsymbol{r})$ ] by a half plane. In one period of the Coulomb blockade oscillations the gate charge changes by $e$ (electron charge). The charge distribution induced in the dot is

$$
\begin{equation*}
\delta \rho(x, y)=-\frac{e}{\pi^{2}} \sqrt{\frac{d}{x}} \frac{1}{\sqrt{(d+x)^{2}+y^{2}}}, \tag{9.13}
\end{equation*}
$$

where the closest to the gate point of the dot is chosen as the coordinate origin and the $x$ axis is directed along the dot edge. The corresponding change of the potential is $\delta U(\boldsymbol{r})=\left(e^{2} v / \varepsilon\right)^{-1} \delta \rho(\boldsymbol{r})$. Substituting this into the first line of Eq. (9.12), we come to the same result

$$
\text { r.m.s. }\left(E_{1}^{(1)}\right) \propto \Delta / \sqrt{\beta g},
$$

in the diffusive limit $d>l$, and to

$$
\text { r.m.s. }\left(E_{1}^{(1)}\right) \sim \Delta / \sqrt{\beta k_{\mathrm{F}} d}
$$

in the case $d<l$.
Effect of the quantum dot deformation on the peak spacing fluctuations has been recently considered in Ref. [304]. The authors of [304] characterized the strength of the deformation by a phenomenological dimensionless parameter $x$ and assumed that it can be large ( $x \gtrsim 1$ ), strongly affecting the spacing distribution. However, the above estimates indicate that for a typical geometry this parameter is much less than unity, $x \sim 1 / \sqrt{g}$ or $x \sim 1 / \sqrt{k_{\mathrm{F}} d}$.
(iii) It was assumed that the dot energy and the measured gate voltage are related through a constant (or smoothly varying) coefficient $\gamma$. This "lever arm" $\gamma$ depends, however on the dot-gate capacitance, which is also a fluctuating quantity. If the gate size and the distance to the gate is of the order of the size of the dot, these fluctuations should be of the same order as fluctuations of the dot self-capacitance and thus lead to additional fluctuations parametrically small compared to $\Delta$, see Eq. (9.10).
(iv) The calculation was done within the random phase approximation, which assumes that the ratio of the interparticle Coulomb energy to the kinetic energy is small, $r_{s} \equiv \sqrt{2} e^{2} / \varepsilon \hbar v_{\mathrm{F}} \ll 1$ ( $v_{\mathrm{F}}$ is the Fermi velocity). However, most of the experimental realizations of semiconductor quantum dots correspond to $r_{s} \approx 1$. Since this value is still considerably lower than the Wigner crystallization threshold, the calculations should be still valid, up to a numerical factor $\alpha\left(r_{s}\right)$ [depending on $r_{s}$ only and such that $\alpha\left(r_{s} \ll 1\right)=1$ ].
(v) We considered the model of spinless electrons up to now. Let us briefly discuss the role of the spin degree of freedom. Within the constant interaction model, it would lead to a bimodal distribution [302] of peak spacings

$$
\begin{equation*}
\mathscr{P}\left(S_{N}\right)=\frac{1}{2}\left[\delta\left(S_{N}-e^{2} / C\right)+\frac{1}{2 \Delta} P_{\mathrm{wD}}\left(\frac{S_{N}-e^{2} / C}{2 \Delta}\right)\right], \tag{9.14}
\end{equation*}
$$

where $P_{\mathrm{WD}}(s)$ is the Wigner-Dyson distribution and $\Delta$ denotes the level spacing in the absence of spin degeneracy. The value of the coefficient $a$ in the relation r.m.s. $\left(S_{N}\right)=a \Delta$ is then increased (compared to the spinless case) and is equal to 1.24 (1.16) for the orthogonal (resp. unitary) ensemble. Taking into account fluctuations of eigenfunctions (and thus of $E_{1}$ ) however modifies the form of the distribution. The value of the term $E_{1}^{(2)}$ representing the interaction between two electrons is larger in the case when $\psi_{N+2}$ and $\psi_{N+1}$ correspond to two spin-degenerate states (i.e. have the same spatial dependence of the wave function), since

$$
\begin{align*}
& \left\langle\int \mathrm{d} \mathbf{r} \mathrm{~d} \mathbf{r}^{\prime}\right| \psi_{i}^{2}(\mathbf{r}) \| \psi_{i}^{2}\left(\mathbf{r}^{\prime}\right)\left|U_{\kappa}\left(\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right)\right\rangle-\left\langle\int \mathrm{d} \mathbf{r} \mathrm{~d} \mathbf{r}^{\prime}\right| \psi_{i}^{2}(\mathbf{r}) \| \psi_{j}^{2}\left(\mathbf{r}^{\prime}\right)\left|U_{\kappa}\left(\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right)\right\rangle \\
& \quad=\frac{2}{\beta V^{2}} \int \mathrm{~d} \mathbf{r} \mathrm{~d} \mathbf{r}^{\prime} k_{d}\left(\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right) U_{\kappa}\left(\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right) \sim \Delta \tag{9.15}
\end{align*}
$$

for $r_{s} \sim 1$ (the coefficient depends on $r_{s}$, see [84]). Therefore, filling a state $\psi_{i \uparrow}$ pushes up the level $\psi_{i \downarrow}$ (with respect to other eigenstates) by an amount of order of $\Delta$. This removes a bimodal structure of the distribution of peak spacings and slightly modifies the value of the coefficient $a$.
Basing on the above analysis, we can make the following general statement. Imagine that we fix $r_{s} \sim 1$ (i.e. fix the electron density and thus the Fermi wave length) and the system geometry, and then start to increase the linear dimension $L$ of the system. Then, while the average value of the peak spacing $S_{N}$ scales as $\left\langle S_{N}\right\rangle \approx e^{2} / C \propto 1 / L$, its fluctuations will scale differently: r.m.s. $\left(S_{N}\right) \sim \Delta \propto 1 / L^{2}$. This conclusion is also corroborated by diagrammatic calculations of Ref. [85].

As was mentioned in the beginning of this section, this result should be contrasted with that for classical particles [300,301], where the fluctuations are proportional to the mean value $\left\langle S_{N}\right\rangle$. The parametrically smaller fluctuations in the quantum case are due to the delocalized nature of the electronic wave functions, which are spread roughly uniformly over the system.

The above prediction was confirmed by recent experiments [86,87,290], where a thorough study of the peak spacing statistics was carried out. It was found that the low-temperature value of r.m.s. $\left(S_{N}\right)$, as well the typical temperature scale for its change are approximately given by the mean level spacing $\Delta$ (while in units of $E_{c}$ the magnitude of fluctuations was as small as $1-4 \%$ ).

Several recent papers studied statistical properties of the peak spacing numerically. Stopa [303] used the density functional theory and found the fluctuations of the addition energies to be approximately 0.74 , in agreement with the above results. Disappearance of the bimodal character of the peak spacing distribution with increasing $r_{s}$ was observed recently in numerical simulations by Berkovits [305] via exact diagonalization. However, the system size and the number of particles
in the exact diagonalization studies are very small, so that it is difficult to draw any quantitative conclusions concerning the dots with large number of electrons from the results of [305].

## 10. Summary and outlook

In this article we have reviewed the recent progress in the study of statistical properties of disordered electronic systems. We have discussed statistics of energy levels and eigenfunction amplitudes, as well as of several related quantities (local density of states, escape time, conductance, etc.) In most of the article the supersymmetric $\sigma$-model approach was used, as a unique and powerful tool allowing one to calculate various distribution and correlation functions. Within this approach, we have employed a number of complementary methods of treating the $\sigma$-model: exact solution (in particular, transfer-matrix method in 1D), perturbation theory, renormalization group, saddle-point method. The main emphasis has been put onto system-specific deviations from the universal predictions of the random matrix theory. The results presented constitute a detailed and, in many respects, complete description of fluctuations of spectra and wave functions of disordered systems.

Still, there is a number of directions in this field in which a more complete understanding is needed, so that the corresponding research remains active at present. Let us point out some of them.

- Statistical properties of energy levels and wave functions of ballistic systems whose classical counterparts are chaotic require more theoretical studies. It remains to be understood what are the conditions of applicability of the ballistic $\sigma$-model derived in [75-77]. Apparently, a certain amount of disorder present in a system (or, in other words, some ensemble averaging) is needed to justify the derivation in these papers, but the precise conditions have not been quantified yet (see Ref. [306] where this problem is discussed in the context of correlations of quantum maps). Also, a discrepancy between the results of the $\sigma$-model and of the semiclassical approach in treating repetitions of periodic orbits $[307,308]$ is to be resolved. It remains to be seen whether the ballistic $\sigma$-model approach can be developed to predict the weak-localization corrections (recent attempts [309,310] did not lead to any definite conclusions), the asymptotic "tails" of the distributions (like those discussed in Section 4 for diffusive systems), etc.
- Interplay of disorder and electron-electron interaction in mesoscopic systems continues attracting considerable research interest. In Section 9 of this article we discussed only the issue of fluctuations in an (almost) closed system. In the experiment, coupling of the dot to the outside world is controlled by the gate(s) and may be varied. There arise a rich variety of regimes depending on the coupling strength, magnetic field, and the strength of the Coulomb interaction (parameter $r_{s}$ ). Fluctuations in open quantum dots have been studied recently, both theoretically [311-313] and experimentally [289,290,314,315], but the issue is not fully understood yet. In Ref. [316] it was found that application of a strong magnetic field affects dramatically the addition spectrum of a quantum dot, leading to strong fluctuations and to bunching of the Coulomb blockade peaks. A consistent explanation of these experimental results is still missing.

Very recently, several groups [317-321] simultaneously performed self-consistent HartreeFock calculations of addition spectra of quantum dots. Though the method allows one to study
considerably larger systems as compared to the exact diagonalization method, the obtained results were not sufficient to extract unambiguously a scaling dependence of the spacing distribution function on the parameters of the problem. A drawback of the Hartree-Fock method with the bare Coulomb potential is that the exchange interaction does not get screened, whereas the screening was crucially important for the theoretical consideration in Section 9.1. More work in this direction may be expected in the nearest future.

- The physics of the Anderson metal-insulator transition, including fluctuations at the critical point, remains an actively studied field. Parameters of modern computers allow one to evaluate numerically critical indices and various distribution functions with a high accuracy. It would be very interesting to study numerically the critical point in higher dimensions (see, in this respect, recent paper [178]) as well as in an effectively infinite-dimensional tree-like sparse random matrix model and in a long-range 1D model (power-law random banded matrix ensemble considered in Section 5.3). It remains to be seen what is the status of the conjecture of Ref. [49] relating the spectral compressibility to the eigenfunction multifractality. Also, the form of the conductance distribution at the critical point remains an open problem. Ref. [28] predicted a log-normal form of the distribution function at $g \gg\langle g\rangle \sim 1$, similarly to the distribution functions of local density of states and of relaxation times. However, in contrast to the latter two quantities, an anomalously large conductance cannot be explained by a single anomalously localized state (since a single state cannot produce the conductance larger than 1 ). It seems much more probable that the conductance distribution falls off much more fast, in a Gaussian (or similar) fashion at large $g$, but no corresponding calculation has been available so far. Technically, the problem is that calculating higher moments of the conductance requires increasing of the size of the $Q$-matrix in the $\sigma$-model, which complicates tremendously a non-perturbative treatment of the distribution function. For recent numerical studies of the conductance distribution at criticality, see Refs. [327-329].


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## Appendix A. Abbreviations

AKL Altshuler, Kravtsov, and Lerner, Ref. [28]
ALS anomalously localized state
DMPK Dorokhov-Mello-Pereira-Kumar (equations)
DOS density of states

| GOE | Gaussian orthogonal ensemble |
| :--- | :--- |
| GSE | Gaussian symplectic ensemble |
| GUE | Gaussian unitary ensemble |
| IPR | inverse participation ratio |
| LDOS | local density of states |
| LN | logarithmically normal |
| PRBM | power-law random banded matrix |
| RBM | random banded matrix |
| RG | renormalization group |
| RPA | random phase approximation |
| RMT | random matrix theory |
| WD | Wigner-Dyson (level statistics) |
| 1D, 2D, 3D | one dimensional, two dimensional, three dimensional |

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[^1]:    ${ }^{2}$ The two-level correlation function is conventionally denoted [3,4] as $R_{2}(s)$. Since we will not consider higher-order correlation functions, we will omit the subscript " 2 ".

[^2]:    ${ }^{3}$ Strictly speaking, the level correlation functions (2.11)-(2.13) contain an additional term $\delta(s)$ corresponding to the "self-correlation" of an energy level. Furthermore, in the symplectic case all the levels are double degenerate (Kramers degeneracy). This degeneracy is disregarded in (2.13) which thus represents the correlation function of distinct levels only, normalized to the corresponding level spacing.

[^3]:    ${ }^{4}$ For all the ensembles, we denote by $g$ the conductance per one spin projection: $g=2 \pi v D L^{d-2}$, without multiplication by factor 2 due to the spin.

[^4]:    ${ }^{5}$ The first two indices of $Q$ correspond to the advanced-retarded and the last two to the boson-fermion decomposition.

[^5]:    ${ }^{6}$ Let us stress that we consider a sample with the hard-wall (not periodic) boundary conditions in the logitudinal direction, i.e. a wire with two ends (not a ring).

[^6]:    ${ }^{7} \theta_{1}$ is related to the eigenvalue $\lambda_{1}$ used in Section 3.1 as $\lambda_{1}=\cosh \theta_{1}$.

[^7]:    ${ }^{8}$ In 3D the value of $g_{*}$ is, of course, of order unity. However, if one considers formally $d=2+\varepsilon$ with $\varepsilon \ll 1$, then $g_{*} \sim 1 / \varepsilon$ is parametrically large. It is thus instructive to $k e e p g_{*}$ as a parameter in all estimates.

[^8]:    ${ }^{9}$ Eq. (5.30) was explicitly derived in [182] for the case of the sparse random matrix model [183,184] corresponding to the limit of infinite dimensionality, $d=\infty$. So, even at $d=\infty$, where the wave function sparsity (multifractality) takes its extreme form, nearby in energy eigenfunctions are fully correlated.

[^9]:    ${ }^{10}$ The expression for $A_{0}$ is valid for $\alpha>1 / 2$ and in the limit $N \rightarrow \infty$. When $\alpha<1 / 2, A_{0}$ starts to have a dependence on $N$ which can be removed by a proper rescaling of the matrix elements $H_{i j}$ in Eq. (5.39). Then, the properties of the model with $\alpha<1 / 2$ turn out to be equivalent to those of the GOE, so we will not consider this case any longer.

[^10]:    ${ }^{11}$ Note that the $W$-field renormalization is absent due to the supersymmetric character of the problem, which is physically related to the particle number conservation [6].

[^11]:    ${ }^{12}$ We use here a brief notation for the matrix elements of the $Q$-matrices. Each of the two upper indices runs from 1 to 8 according to ordering of components of a supervector introduced before Eq. (6.9).

[^12]:    ${ }^{13}$ Exact form of the boundary condition depends on the underlying microscopic model. In particular, the diffuse scattering can be modelled by surrounding the cavity by a disordered layer with a bulk mean free path $l$ and a thickness $d \gg l$. The corresponding boundary condition [268,269] differs from Eq. (8.3) by a parameterless function of order unity. For a review of the boundary conditions corresponding to various microscopic realizations of the rough surface see [270].

