SYMMETRIES OF EQUATIONS OF QUANTUM MECHANICS

TABLE OF CONTENTS

Chapter I. LOCAL SYMMETRY OF BASIS EQUATIONS OF RELATIVISTIC QUANTUM THEORY

1. Local Symmetry of the Klein-Gordon-Fock Equation
   1.1. Introduction ............................................ 1
   1.2. The IA of the KGF Equation ............................... 3
   1.3. Symmetry of the d’Alembert Equation ..................... 5
   1.4. Lorentz Transformations .................................... 6
   1.5. The Poincaré Group ....................................... 9
   1.6. The Conformal Transformations ............................ 12
   1.7. The Discrete Symmetry Transformations .................... 14

2. Local Symmetry of the Dirac Equation
   2.1. The Dirac Equation ....................................... 16
   2.2. Various Formulations of the Dirac equation ............... 17
   2.3. Algebra of the Dirac Matrices ............................ 19
   2.4. SOs and IAs ........................................... 19
   2.5. The IA of the Dirac Equation in the Class $M_1$ .......... 20
   2.6. The Operators of Mass and Spin .......................... 22
   2.7. Manifestly Hermitian Form of Poincaré Group Generators .... 23
   2.8. Symmetries of the Massless Dirac Equation ............... 24
   2.9. Lorentz and Conformal Transformations of Solutions of the Dirac
        Equation ................................................... 25
   2.10. $P$, $T$, and $C$-Transformations ........................ 27

3. Maxwell’s Equations
   3.1. Introduction ............................................. 28
   3.2. Various Formulations of Maxwell’s Equations ............ 30
### Chapter II. REPRESENTATIONS OF THE POINCARÉ ALGEBRA AND WAVE EQUATIONS FOR ARBITRARY SPIN

#### 4. IR of the Poincaré Algebra

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>44</td>
</tr>
<tr>
<td>4.2</td>
<td>Casimir Operators</td>
<td>45</td>
</tr>
<tr>
<td>4.3</td>
<td>Basis of an IR</td>
<td>46</td>
</tr>
<tr>
<td>4.4</td>
<td>The Explicit Form of the Lubanski-Pauli Vector</td>
<td>48</td>
</tr>
<tr>
<td>4.5</td>
<td>IR of the Algebra $A(c,n)$</td>
<td>50</td>
</tr>
<tr>
<td>4.6</td>
<td>Explicit Realizations of the Poincaré Algebra</td>
<td>53</td>
</tr>
<tr>
<td>4.7</td>
<td>Connections with the Canonical Realizations of Shirokov-Foldy-Lomont-Moses</td>
<td>55</td>
</tr>
<tr>
<td>4.8</td>
<td>Covariant Representations</td>
<td>58</td>
</tr>
</tbody>
</table>

#### 5. Representations of the Discrete Symmetry Transformations

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>60</td>
</tr>
<tr>
<td>5.2</td>
<td>Nonequivalent Multiplicators of the Group $G_8$</td>
<td>62</td>
</tr>
<tr>
<td>5.3</td>
<td>The General Form of the Discrete Symmetry Operators</td>
<td>64</td>
</tr>
<tr>
<td>5.4</td>
<td>The Operators $P$, $T$, and $C$ for Representation of Class $I$</td>
<td>67</td>
</tr>
<tr>
<td>5.5</td>
<td>Representations of Class $II$</td>
<td>70</td>
</tr>
<tr>
<td>5.6</td>
<td>Representations of Classes $III-IV$</td>
<td>71</td>
</tr>
<tr>
<td>5.7</td>
<td>Representations of Class $V$</td>
<td>73</td>
</tr>
<tr>
<td>5.8</td>
<td>Concluding Remarks</td>
<td>75</td>
</tr>
</tbody>
</table>

#### 6. Poincaré-Invariant Equations of First Order

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Introduction</td>
<td>75</td>
</tr>
<tr>
<td>6.2</td>
<td>The Poincaré Invariance Condition</td>
<td>77</td>
</tr>
<tr>
<td>6.3</td>
<td>The Explicit Form of the Matrices $\beta_p$</td>
<td>78</td>
</tr>
<tr>
<td>6.4</td>
<td>Additional Restrictions for the Matrices $\hat{\beta}_p$</td>
<td>79</td>
</tr>
</tbody>
</table>
10.7. Expansion in Power Series in $1/m$ ........................................ 124
10.8. Causality Principle and Wave Equations for Particles of Arbitrary
Spin ................................................................. 127
10.9. The Causal Equation for Spin-One Particles with Positive
Energies .............................................................. 128

Chapter III. REPRESENTATIONS OF THE GALILEI ALGEBRA AND
GALILEI-INvariant WAVE EQUATIONS

11. Symmetries of the Schrödinger Equation

11.1. The Schrödinger Equation ............................................ 131
11.2. Invariance Algebra of the Schrödinger Equation ................. 132
11.3. The Galilei and Generalized Galilei Algebras ..................... 134
11.4. The Schrödinger Equation Group ................................. 136
11.5. The Galilei Group ..................................................... 138
11.6. The Transformations $P$ and $T$ .................................... 139

12. Representations of the Lie Algebra of the Galilei Group

12.1. The Galilei Relativity Principle and Equations of Quantum
Mechanics ............................................................. 140
12.2. Classification of IRs ................................................ 141
12.3. The Explicit Form of Basis Elements of the Algebra $AG(1,3)$ 142
12.4. Connections with Other Realizations ............................. 144
12.5. Covariant Representations ......................................... 146
12.6. Representations of the Lie Algebra of the Homogeneous Galilei
Group ................................................................. 148

13. Galilei-Invariant Wave Equations

13.1. Introduction .......................................................... 154
13.2. Galilei-Invariance Conditions ................................... 154
13.3. Additional Restrictions for Matrices $\beta_{\mu}$ ......................... 155
13.4. General Form of Matrices $\beta_{\mu}$ in the Basis $|\lambda;l,m>$ ...... 157
13.5. Equations of Minimal Dimension ................................ 159
13.6. Equations for Representations with Arbitrary Nilpotency Indices 163
14. Galilei-Invariant Equations of the Schrödinger Type

14.1. Uniqueness of the Schrödinger equation ......................... 165
14.2. The Explicit Form of Hamiltonians of Arbitrary Spin Particles 167
14.3. Lagrangian Formulation ........................................ 170

15. Galilean Particle of Arbitrary Spin in an External Electromagnetic Field

15.1. Introduction of Minimal Interaction into First-Order Equations . 171
15.2. Magnetic Moment of a Galilei Particle of Arbitrary Spin ....... 173
15.3. Interaction with the Electric Field ............................. 175
15.4. Equations for a \( (2s+1) \)-Component Wave Function .......... 176
15.5. Introduction of the Minimal Interaction into Schrödinger-Type Equations ......................................................... 179
15.6. Anomalous Interaction ........................................... 180

Chapter IV. NONGEOMETRIC SYMMETRY

16. Higher Order SOs of the KGF and Schrödinger Equations

16.2. SOs of the KGF Equation ........................................ 185
16.3. Hidden Symmetries of the KGF Equation ..................... 188
16.4. Higher Order SOs of the d’Alembert Equation .............. 190
16.5. SOs of the Schrödinger Equation ............................. 191
16.6. Hidden Symmetries of the Schrödinger Equation ........... 193
16.7. Symmetries of the Quasi-Relativistic Evolution Equation .... 196

17. Nongeometric Symmetries of the Dirac Equation

17.1. The IA of the Dirac Equation in the Class \( M_1 \) ............... 198
17.2. Symmetries of the Dirac Equation in the Class of Integro-Differential Operators ................................................. 202
17.3. Symmetries of the Eight-Component Dirac Equation ....... 203
17.4. Symmetry Under Linear and Antilinear Transformations .... 206
17.5. Hidden Symmetries of the Massless Dirac Equation ....... 209
18. The Complete Set of SOs of the Dirac Equation

18.1. Introduction and Definitions ........................................... 211
18.2. The General Form of SOs of Order \( n \) ............................ 212
18.3. Algebraic Properties of the First-Order SOs ......................... 213
18.4. The Complete Set of SOs of Arbitrary Order ...................... 216
18.5. Examples and Discussion .............................................. 220
18.6. SOs of the Massless Dirac Equation .................................. 221

19. Symmetries of Equations for Arbitrary Spin Particles

19.1. Symmetries of the KDP Equation ...................................... 223
19.2. Arbitrary Order SOs of the KDP Equation ............................ 226
19.3. Symmetries of the Dirac-Like Equations for Arbitrary Spin Particles .............................................. 229
19.5. Symmetries of the Levi-Leblond Equation ........................... 234

20. Nongeometric Symmetries of Maxwell’s Equations

20.1. Invariance Under the Algebra \( AGL(2,C) \) .......................... 238
20.2. The Group of Nongeometric Symmetry of Maxwell’s Equations 241
20.3. Symmetries of Maxwell’s Equations in the Class \( M_2 \) ......... 243
20.4. Superalgebras of SOs of Maxwell’s Equations ....................... 247
20.5. Symmetries of Equations for the Vector-Potential .................. 249

21. Symmetries of the Schrödinger Equation with a Potential

21.1. Symmetries of the One-Dimension Schrödinger Equation ......... 251
21.2. The Potentials Admitting Third-Order Symmetries ............... 253
21.3. Time-Dependent Potentials ............................................ 257
21.4. Algebraic Properties of SOs ........................................... 257
21.5. Complete Sets of SOs for One- and Three-Dimensional Schrödinger equation .............................................. 259
21.6. SOs of the Supersymmetric Oscillator ............................... 262
22. Nongeometric Symmetries of Equations for Interacting Fields

22.1. The Dirac Equation for a Particle in an External Field .............. 263
22.2. The SO of Dirac Type for Vector Particles .......................... 267
22.3. The Dirac Type SOs for Particles of Any Spin ........................ 268
22.4. Other Symmetries of Equations for Arbitrary Spin Particles ....... 271
22.5. Symmetries of a Galilei Particle of Arbitrary Spin in the Constant
     Electromagnetic Field ........................................ 272
22.6. Symmetries of Maxwell’s Equations with Currents and Charges 273
22.7. Super- and Parasupersymmetries ................................. 276
22.8. Symmetries in Elasticity ................................... 278

23. Conservation Laws and Constants of Motion

23.1. Introduction .................................................. 281
23.2. Conservation Laws for the Dirac Field ................................ 284
23.3. Conservation Laws for the Massless Spinor Field .................... 285
23.4. The Problem of Definition of Constants of Motion for the
     Electromagnetic Field ........................................ 286
23.5. Classical Conservation Laws for the Electromagnetic Field . . . 289
23.6. The First Order Constants of Motion for the Electromagnetic
     Field ......................................................... 289
23.7. The Second Order Constants of Motion for the Electromagnetic
     Field ......................................................... 291
23.8. Constants of Motion for the Vector-Potential ....................... 295

Chapter V. GENERALIZED POINCARÉ GROUPS

24. The Group P(1,4)

24.1. Introduction .................................................. 297
24.2. The Algebra AP(1,n) ......................................... 298
24.3. Nonequivalent Realizations of the Tensor $W_{\mu\sigma}$ .................. 299
24.4. The Basis of an IR ........................................... 302
24.5. The Explicit Form of the Basis Elements of the algebra AP(1,4) . 303
24.6. Connection with Other Realizations ................................. 304

25. Representations of the Algebra AP(1,4) in the Poincaré-Basis

25.1. Subgroup Structure of the Group $P(1,4)$ .......................... 307
25.2. Poincaré-Basis ............................................... 307
25.3. Reduction $P(1,4) \rightarrow P(1,3)$ of IRs of Class $I$ ................. 308
25.4. Reduction $P(1,4) \rightarrow P(1,2)$ ........................................... 311
25.5. Reduction of IRs for the Case $c_1=0$ ................................. 312
25.6. Reduction of Representations of Class IV .......................... 315
25.7. Reduction $P(1,n) \rightarrow P(1,3)$ ........................................... 316

26. Representations of the Algebra $AP(1,4)$ in the $G(1,3)$- and $E(4)$-Basises

26.1. The $G(1,3)$-Basis ............................................................ 319
26.2. Representations with $P^\alpha P_\beta \neq 0$ ............................... 321
26.3. Representations of Classes II-IV ........................................ 324
26.4. Covariant Representations .................................................. 325
26.5. The $E(4)$-Basis .............................................................. 327
26.6. Representations of the Poincaré Algebra in the $G(1,2)$-Basis 328

27. Wave Equations Invariant Under Generalized Poincaré Groups

27.1. Preliminary Notes ............................................................ 330
27.2. Generalized Dirac Equations ............................................. 331
27.3. The Generalized Kemmer-Duffin-Petiau Equations ............... 334
27.4. Covariant Systems of Equations ......................................... 335

Chapter 6. EXACT SOLUTIONS OF LINEAR AND NONLINEAR
EQUATIONS OF MOTION

28. Exact Solutions of Relativistic Wave Equations for Particles of Arbitrary Spin

28.1. Introduction ................................................................. 338
28.2. Free Motion of Particles ................................................. 339
28.3. Relativistic Particle of Arbitrary Spin in Homogeneous Magnetic Field ......................................................... 341
28.4. A Particle of Arbitrary Spin in the Field of the Plane Electromagnetic Wave ................................................. 345

29. Relativistic Particles of Arbitrary Spin in the Coulomb Field

29.1. Separation of Variables in a Central Field ........................... 347
29.2. Solution of Equations for Radial Functions ........................... 349
29.3. Energy Levels of a Relativistic Particle of Arbitrary Spin in the Coulomb Field ......................................................... 351

viii
30. Exact Solutions of Galilei-Invariant Wave Equations

30.1. Preliminary Notes ............................................. 354
30.2. Nonrelativistic Particle in the Constant and Homogeneous Magnetic Field ............................................. 355
30.3. Nonrelativistic Particle of Arbitrary Spin in Crossed Electric and Magnetic Fields ...................................... 357
30.4. Nonrelativistic Particle of Arbitrary Spin in the Coulomb Field . 359

31. Nonlinear Equations Invariant Under the Poincaré and Galilei Groups

31.1. Introduction .............................................. 362
31.2. Symmetry Analysis and Exact Solutions of the Scalar Nonlinear Wave Equation ............................................. 362
31.3. Symmetries and Exact Solutions of the Nonlinear Dirac Equation 365
31.4. Equations of Schrödinger type Invariant Under the Galilei Group 368
31.5. Symmetries of Nonlinear Equations of Electrodynamics . . . . 371
31.7. Conditional Symmetry and Exact Solutions of the Boussinesq Equation ............................................. 377
31.8. Exact Solutions of Linear and Nonlinear Schrödinger equation . 381

Chapter 7. TWO-PARTICLE EQUATIONS

32. Two-Particle Equations Invariant Under the Galilei Group

32.1. Preliminary Notes ............................................. 384
32.2. Equations for Spinless Particles ................................. 385
32.3. Equations for Systems of Particles of Arbitrary Spin ............. 388
32.4. Two-Particle Equations of First Order .......................... 390
32.5. Equations for Interacting Particles of Arbitrary Spin ........... 392

33. Quasi-Relativistic and Poincaré-Invariant Two-Particle Equations

33.1. Preliminary Notes ............................................. 395
33.2. The Breit Equation ............................................ 396
33.3. Transformation to the Quasidiagonal Form ...................... 397
33.4. The Breit Equation for Particles of Equal Masses .............. 399
33.5. Two-Particle Equations Invariant Under the Group $P(1,6)$ .... 401
33.6. Additional Constants of Motion for Two- and Three-Particle Equations ............................................. 403
34. Exactly Solvable Models of Two-Particle Systems

34.1. The Nonrelativistic Model .......................................... 405
34.2. The Relativistic Two-Particle Model ................................. 405
34.3. Solutions of Two-Particle Equations ............................... 407
34.4. Discussing of Spectra of the Two-Particle Models ............. 409

Appendix 1. Lie Algebras, Superalgebras and Parasuperalgebras ...... 414

Appendix 2. Generalized Killing Tensors ................................. 416

Appendix 3. Matrix Elements of Scalar Operators in the Basis of Spherical Spinors ...................................................... 420

References ................................................................. 424

Additional List of References ............................................. 453

List of Abbreviations ....................................................... 457

Index ................................................................. 458
Preface to the English Edition

"In the beginning was the symmetry"* Hidden harmony is stronger then the explicit one
W. Heisenberg Heraclitus

The English version of our book is published on the initiative of Dr. Edward M. Michael, Vice-President of the Allerton Press Incorporated. It is with great pleasure that we thank him for his interest in our work.

The present edition of this book is an improved version of the Russian edition, and is greatly extended in some aspects. The main additions occur in Chapter 4, where the new results concerning complete sets of symmetry operators of arbitrary order for motion equations, symmetries in elasticity, super- and parasupersymmetry are presented. Moreover, Appendix II includes the explicit description of generalized Killing tensors of arbitrary rank and order; these play an important role in the study of higher order symmetries.

The main object of this book is symmetry. In contrast to Ovsiannikov’s term "group analysis" (of differential equations) [355] we use the term "symmetry analysis" [123] in order to emphasize the fact that it is not, in general, possible to formulate arbitrary symmetry in the group theoretical language. We also use the term "non-Lie symmetry" when speaking about such symmetries which can not be found using the classical Lie algorithm.

In order to deduce equations of motion we use the "non-Lagrangian" approach based on representations of the Poincaré and Galilei algebras. That is, we use for this purpose the principles of Galilei and Poincaré-Einstein relativity formulated in algebraic terms. Sometimes we use the usual term "relativistic equations" when speaking about Poincaré-invariant equations in spite of the fact that Galilei-invariant subjects are "relativistic" also in the sense that they satisfy Galilei relativity principle.

Our book continues the series of monographs [127, 157, 171, 10*, 11*] devoted to symmetries in mathematical physics. Moreover, we will edit "Journal of Nonlinear Mathematical Physics" which also will related to these problems.

We hope that our book will be useful for mathematicians and physicists in the English-speaking world, and that it will stimulate the development of new symmetry approaches in mathematical and theoretical physics.

Only finishing the contemplated work one understands how it was necessary to begin it
B. Pascal
Preface

Over a period of more than a hundred years, starting from Fedorov’s works on symmetry of crystals, there has been a continuous and accelerating growth in the number of researchers using methods of discrete and continuous groups, algebras and superalgebras in different branches of modern natural sciences. These methods have a universal nature and can serve as a basis for a deep understanding of the relativity principles of Galilei and Poincaré-Einstein, of Mendeleev’s periodic law, of principles of classification of elementary particles and biological structures, of conservation laws in classical and quantum mechanics etc.

The foundations of the theory of continuous groups were laid a century ago by Sofus Lie, who proposed effective algorithms to calculate symmetry groups for linear and nonlinear partial differential equations. Today the classical Lie methods (completed by theory of representations of Lie groups and algebras) are widely used in theoretical and mathematical physics.

Our book is devoted to the analysis of old (classical) and new (non-Lie) symmetries of the basic equations of quantum mechanics and classical field theory, classification and algebraic theoretical deduction of equations of motion of arbitrary spin particles in both Poincaré and Galilei-invariant approaches. We present detailed information about representations of the Galilei and Poincaré groups and their possible generalizations, and expound a new approach to investigation of symmetries of partial differential equations, which enables to find unknown before algebras and groups of invariance of the Dirac, Maxwell and other equations. We give solutions of a number of problems of motion of arbitrary spin particles in an external electromagnetic field. Most of the results are published for the first time in a monographic literature.

The book is based mainly on the author’s original works. The list of references does not have any pretensions to completeness and contains as a rule the papers immediately used by us.

We take this opportunity to express our deep gratitude to academicians N.N. Bogoliubov, Yu.A. Mitropolskii, our teacher O.S. Parasiuk, correspondent member of Russian Academy of Sciences V.G. Kadyshievskii, professors A.A. Borgardt and M.K. Polivanov for essential and constant support of our researches in developing the algebraic-theoretical methods in theoretical and mathematical physics. We are indebted to doctors L.F. Barannik, I.A. Egorchenko, N.I. Serov, Z.I. Simenoh, V.V. Tretynyk, R.Z. Zhdanov and A.S. Zhukovski for their help in the preparation of the manuscript.
Introduction

The symmetry principle plays an increasingly important role in modern researches in mathematical and theoretical physics. This is connected with the fact that the basis physical laws, mathematical models and equations of motion possess explicit or unexplicit, geometric or non-geometric, local or non-local symmetries. All the basic equations of mathematical physics, i.e. the equations of Newton, Laplace, d’Alembert, Euler-Lagrange, Lame, Hamilton-Jacobi, Maxwell, Schrodinger etc., have a very high symmetry. It is a high symmetry which is a property distinguishing these equations from other ones considered by mathematicians.

To construct a mathematical approach making it possible to distinguish various symmetries is one of the main problems of mathematical physics. There is a problem which is in some sense inverse to the one mentioned above but is no less important. We say about the problem of describing of mathematical models (equations) which have the given symmetry. Two such problems are discussed in detail in this book.

We believe that the symmetry principle has to play the role of a selection rule distinguishing such mathematical models which have certain invariance properties. This principle is used (in the explicit or implicit form) in a construction of modern physical theories, but unfortunately is not much used in applied mathematics.

The requirement of invariance of an equation under a group enables us in some cases to select this equation from a wide set of other admissible ones. Thus, for example, there is the only system of Poincaré-invariant partial differential equations of first order for two real vectors $E$ and $H$, and this is the system which reduces to Maxwell’s equations. It is possible to "deduce" the Dirac, Schrödinger and other equations in an analogous way.

The main subject of the present book is the symmetry analysis of the basic equations of quantum physics and deduction of equations for particles of arbitrary spin, admitting different symmetry groups. Moreover we consider two-particle equations for any spin particles and exactly solvable problems of such particles interaction with an external field.

The local invariance groups of the basic equations of quantum mechanics (equations of Schrodinger, of Dirac etc.) are well known, but the proofs that these groups are maximal (in the sense of Lie) are present only in specific journals due to their complexity. Our opinion is that these proofs have to be expounded in form easier to understand for a wide circle of readers. These results are undoubtedly useful for a deeper understanding of mathematical nature of the symmetry of the equations mentioned. We consider local symmetries mainly in Chapter 1.

It is well known that the classical Lie symmetries do not exhaust the
invariance properties of an equation, so we find it is necessary to expound the main results obtained in recent years in the study of non-Lie symmetries, super- and parasupersymmetries. Moreover we present new constants of motion of the basic equations of quantum physics, obtained by non-Lie methods. Of course it is interesting to demonstrate various applications of symmetry methods to solving concrete physical problems, so we present here a collection of examples of exactly solvable equations describing interacting particles of arbitrary spins.

The existence of the corresponding exact solutions is caused by the high symmetry of the models considered.

In accordance with the above, the main aims of the present book are:

1. To give a good description of symmetry properties of the basic equations of quantum mechanics. This description includes the classical Lie symmetry (we give simple proofs that the known invariance groups of the equations considered are maximally extensive) as well as the additional (non-Lie) symmetry.

2. To describe wide classes of equations having the same symmetry as the basic equations of quantum mechanics. In this way we find the Poincaré-invariant equations which do not lead to known contradictions with causality violation by describing of higher spin particles in an external field, and the Galilei-invariant wave equations for particles of any spin which give a correct description of these particle interactions with the electromagnetic field. The last equations describe the spin-orbit coupling which is usually interpreted as a purely relativistic effect.

3. To represent hidden (non-Lie) symmetries (including super- and parasupersymmetries) of the main equations of quantum and classical physics and to demonstrate existence of new constants of motion which can not be found using the classical Lie method.

4. To demonstrate the effectiveness of the symmetry methods in solving the problems of interaction of arbitrary spin particles with an external field and in solving of nonlinear equations.

Besides that we expound in details the theory of irreducible representations (IR) of the Lie algebras of the main groups of motion of four-dimensional space-time (i.e. groups of Poincaré and Galilei) and of generalized Poincaré groups $P(1,n)$. We find different realizations of these representations in the bases available to physical applications. We consider representations of the discrete symmetry operators $P$, $C$ and $T$, and find nonequivalent realizations of them in the spaces of representations of the Poincaré group.

The detailed list of contents gives a rather complete information about subject of the book so we restrict ourselves by the preliminary notes given above.

The main part of the book is based on the original papers of the authors. Moreover we elucidate (as much as we are able) contributions of other investigators in the branch considered.

We hope our book can serve as a kind of group-theoretical introduction to
quantum mechanics and will be interesting for mathematicians and physicists which use the group-theoretical approach and other symmetry methods in analysis and solution of partial differential equations.
1. LOCAL SYMMETRIES OF THE FUNDAMENTAL EQUATIONS OF RELATIVISTIC QUANTUM THEORY

In this chapter we study symmetries of the Klein-Gordon-Fock (KGF), Dirac and Maxwell equations. The maximal invariance algebras (IAs) of these equations in the class of first order differential operators are found, the representations of the corresponding symmetry groups and exact transformation laws for dependent and independent variables are given. Moreover we present with the aid of relatively simple examples, the main ideas of the algebraic-theoretical approach to partial differential equations and also, give a precise description of the symmetry properties of the fundamental equations of quantum physics.

1. LOCAL SYMMETRY OF THE KLEIN-GORDON-FOCK EQUATION

1.1. Introduction

One of the basic equations of relativistic quantum physics is the KGF equation which we write in the form

\[ L\psi = \left( p^\mu p_\mu - m^2 \right) \psi = 0 \]  \hspace{1cm} (1.1)

where \( p_\mu \) are differential operators: \( p_0 = \partial^0 = i\partial/\partial x_0 \), \( p_a = \partial^a = -i\partial/\partial x_a \). \( m^2 \) is a positive number. Here and in the following the covariant summation over repeated Greek indices is implied and Heaviside units are used in which \( \hbar = c = 1 \).

The equation (1.1) is a relativistic analog of the Schrödinger equation. In physics it is usually called the Klein-Gordon equation in spite of the fact that it was considered by Schrödinger [380] and then by Fock [102], Klein [253] and some other authors (see [9]). We shall use the term "KGF equation" or "wave equation".

In this section we study the symmetry of (1.1). The analysis of symmetry properties of the KGF equation enables us to proceed naturally to such important modern physical concepts as relativistic and conformal invariance and describe relativistic equations of motion for particles of arbitrary spin. We shall demonstrate also that the Poincaré (and when \( m=0 \) conformal) invariance represents in some sense...
the maximal symmetry of (1.1).

Let us formulate the problem of investigation of the symmetry of the KGF equation. The main concept used while considering the invariance of this equation (and other equations of quantum physics) is the concept of symmetry operator (SO).

In general a SO is any operator (linear, nonlinear, differential, integral etc.) $Q$ transforming solutions of (1.1) into solutions, i.e., satisfying the condition

$$L(Q\psi) = 0$$

for any $\psi$ satisfying (1.1). In order to find the concrete symmetries this intuitive definition needs to be made precise by defining the classes of solutions and of operators considered. Here we shall investigate the SOs which belong to the class of first-order linear differential operators and so can be interpreted as Lie derivatives or generators of continuous group transformations.

Let us go to definitions. We shall consider only solutions which are defined on an open set $D$ of the four-dimensional manifold $R$ consisting of points with coordinates $(x_0, x_1, x_2, x_3)$ and are analytic in the real variables $x_0, x_1, x_2, x_3$. The set of such solutions forms a complex vector space which will be denoted by $F_0$. If $\psi_1, \psi_2 \in F_0$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ then evidently $\alpha_1 \psi_1 + \alpha_2 \psi_2 \in F_0$. Fixing $D$ (e.g. supposing that $D$ coincides with $R$) we shall call $F$ the space of solutions of the KGF equation.

Let us denote by $F$ the vector space of all complex-valued functions which are defined on $D$ and are real-analytic, and by $L$ we denote the linear differential operator defined on $F$:

$$L = p^\mu p_\mu - m^2.$$  

Then $L \psi \in F$ if $\psi \in F$. Moreover $F_0$ is the subspace of the vector space $F$ which coincides with the zero-space (kernel) of the operator $L$ (1.3).

Let $M_1$ be the set (class) of first order differential operators defined on $F$. The concept of SO in the class $M_1$ can be formulated as follows.

**DEFINITION 1.1.** A linear differential operator of the first order

$$Q = A^\mu p_\mu + B,$$  

is a SO of the KGF equation in the class $M_1$ if

$$[Q, L] = \alpha_\psi L, \quad \alpha_\psi \in F$$

where $[Q, L] = QL - LQ$ is a commutator of the operators $Q$ and $L$.

The condition (1.5) is to be understood in the sense that the operator in the r.h.s. and l.h.s. give the same result when acting on an arbitrary function $\psi \in F$.

It can be seen easily that an operator $Q$ satisfying (1.5) also satisfies the condition (1.2) for any $\psi \in F_0$. Indeed, according to (1.5)

$$LQ\psi = (Q - \alpha_\psi) L\psi = 0, \quad \psi \in F_0.$$
Chapter 1. Symmetries of the Fundamental Equations

The converse statement is also true: if the operator (1.4) satisfies (1.2) for an arbitrary \( \psi \in F_0 \) then the condition (1.5) is satisfied for some \( \alpha \in F \).

Using the given definitions we will calculate all the SOs of the KGF equation. It happens that any SO of (1.4) can be represented as a linear combination of some basis elements. This fact follows from the following assertion

**THEOREM 1.1.** The set \( S \) of the SOs of the KGF equation in the class \( M_1 \) forms a complex Lie algebra, i.e., if \( Q_1, Q_2 \in S \) then

1) \( a_1 Q_1 + a_2 Q_2 \in S \) for any \( a_1, a_2 \in \mathbb{C} \),

2) \( [Q_1, Q_2] \in S \).

**PROOF.** By definition the operators \( Q_i \) (\( i = 1, 2 \)) satisfy the condition (1.5). By direct calculation we obtain that the operators \( Q_3 = \alpha_1 Q_1 + \alpha_2 Q_2 \) and \( Q_4 = [Q_1, Q_2] \) belong to \( M_1 \) and satisfy (1.5) with

\[
\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_i, [Q_1, \alpha_i], \alpha_0, \alpha_i \in F.
\]

So studying the symmetry of the KGF equation (or of other linear differential equations) in the class \( M_1 \) we always deal with a Lie algebra which can be finite dimensional (this is true for equation (1.1)) as well as infinite-dimensional. This is why speaking about such a symmetry we will use the term "invariance algebra" (IA).

**DEFINITION 1.2.** Let \( \{ Q_1 \} (A = 1, 2, \ldots) \) be a set of linear differential operators (1.4) forming a basis of a finite-dimensional Lie algebra \( G \). We say \( G \) is an IA of the KGF equation if any \( Q_A \in \{ Q_1 \} \) satisfies the condition (1.5).

According to Theorem 1.1 the problem of finding all the possible SOs of the KGF equation is equivalent to finding a basis of maximally extensive IA in the class \( M_1 \). As will be shown in the following (see Chapter 4) many of the equations of quantum mechanics possess IAs in the classes of second-, third-... order differential operators in spite of the fact that higher-order differential operators in general do not form a finite-dimensional Lie algebra.

### 1.2. The IA of the KGF Equation

In this section we find the IA of the KGF equation in the class \( M_1 \), i.e., in the class of first order differential operators. In this way it is possible with rather simple calculations to prove the Poincaré (and for \( m = 0 \) - conformal) invariance of the equation (1.1) and to demonstrate that this symmetry is maximal in some sense.

Let us prove the following assertion.

**THEOREM 1.2.** The KGF equation is invariant under the 10-dimensional Lie algebra whose basis elements are
Symmetries of Equations of Quantum Mechanics

The Lie algebra generated by the operators (1.6) is the maximally extensive IA of the KGF equation in the class $M_1$.

**PROOF.** It is convenient to write an unknown SO (1.4) in the following equivalent form

\[ Q = \frac{1}{2} [K^\mu, p_\mu] + C \]

where $[K^\mu, p_\mu] = [K_\mu^\prime p_\mu + p_\mu^\prime K_\mu, C-B+1/2[K_\mu, p_\mu]]$. Substituting (1.7) into (1.5) we come to the equation

\[ \frac{1}{2} \left[ (\partial^\nu K^\mu), p_\nu \right] + (\partial^\nu C), p_\nu \right] = \frac{1}{4} \left[ [\alpha_\mu^\nu p^\mu], p_\nu \right] + \frac{i}{2} \left[ (\partial^\mu \alpha_\mu), p_\nu \right] - m^2 \alpha_\mu = 0. \]  

(1.8)

We represent the r.h.s. of (1.5) in an equivalent form including anticommutators.

The equation (1.8) is to be understood in the sense that the operators in the l.h.s. and r.h.s. give the same result by action on an arbitrary function belonging to $F$. In other words, the necessary and sufficient condition of satisfying (1.8) is the equality of the coefficients of the same anticommutators:

\[ \partial^\nu K^\mu + \partial^\nu K^\mu = \frac{1}{2} g^{\mu \nu} \alpha_{\mu \nu}, \quad \partial^\mu C = \partial^\mu \alpha_{\mu}, \quad m^2 \alpha_{\mu} = 0. \]  

(1.9)

\[ g^{00} = -g^{11} = -g^{22} = -g^{33} = 1, \quad g^{\mu \nu} = 0, \quad \mu \neq \nu. \]  

(1.10)

For nonzero $m$ we obtain from (1.9) $\alpha_{\mu} = 0$ and

\[ \partial^\mu K^\nu + \partial^\nu K^\mu = 0, \quad \partial^\mu C = 0. \]  

(1.11)

The equations (1.11) are easily integrated. Indeed the first of them is the Killing equation [249] (see Appendix 1), the general solution of which is

\[ K^\mu = c^{[\mu]} x^\sigma - b^\mu \]

(1.12)

where $c^{[\mu]} = c^{[\mu]}$ and $b^\mu$ are arbitrary numbers. According to (1.11) $C$ does not depend on $x$.

Substituting (1.12) into (1.7) we obtain the general expression for a SO:

\[ Q = c^{[\mu]} p_\mu + b^\mu p_\mu + C. \]  

(1.13)

which is a linear combination of the operators (1.6) and trivial unit operator.

It is not easy to verify that the operators (1.6) form a basis of the Lie algebra, satisfying the relations

\[ P_0 = p_0 = \frac{\partial}{\partial x_0}, \quad P_\alpha = p_\alpha = \frac{\partial}{\partial x_\alpha}, \quad a = 1, 2, 3, \]

(1.6)
Chapter 1. Symmetries of the Fundamental Equations ...

\[[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\nu\lambda}] = i(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu), \quad (1.14)\]
\[[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\rho} J_{\nu\sigma} - g_{\nu\rho} J_{\mu\sigma} - g_{\mu\sigma} J_{\nu\rho} + g_{\nu\sigma} J_{\mu\rho}).\]

According to the above, the Lie algebra with the basis elements (1.6) is the maximally extended IA of the KGF equation.

The conditions (1.14) determine the Lie algebra of the Poincaré group, which is the group of motions of relativistic quantum mechanics. Below we will call this algebra "the Poincaré algebra" and denote it by \(AP(1,3)\).

The symmetry under the Poincaré algebra has very deep physical consequences and contains (in implicit form) the information about the fundamental laws of relativistic kinematics (Lorentz transformations, the relativistic law of summation of velocities etc.). These questions are discussed further in Subsections 1.4 and 1.5. The following subsection is devoted to description of the KGF equation symmetry in the special case \(m=0\).

1.3. Symmetry of the d’Alembert Equation

Earlier, we assumed the parameter \(m\) in (1.1) is nonzero. But in the case \(m=0\) this equation also has a precise physical meaning and describes a massless scalar field. The symmetry of the massless KGF equation (i.e., d’Alembert equation) turns out to be more extensive than in the case of nonzero mass.

**THEOREM 1.3.** The maximal invariance algebra of the d’Alembert equation

\[p_\mu p^\mu \psi = 0\]  
(1.15)

is a fifteen-dimensional Lie algebra. The basis elements of this algebra are given by formulae (1.6) and (1.16):

\[D = x^\mu p_\mu + 2i, \quad \hat{K}_\mu = 2x^\mu D - x_\mu x^\nu p_\nu,\]  
(1.16)

**PROOF.** Repeating the reasoning from the proof of Theorem 1.2 we come to the conclusion that the general form of the SO \(Q \in M_4\) for the equation (1.15) is given by in (1.7) where \(K^\mu, C\) are functions satisfying (1.9) with \(m=0\). We rewrite this equation in the following equivalent form

\[\partial^\mu K^\mu + \partial^\nu K^\nu - \frac{1}{2} g^{\mu\nu} \partial^\lambda K^\lambda = 0,\]

(1.17)

\[\lambda_0 = \frac{1}{2} \partial^\nu K^\nu.\]

Formula (1.17) defines the equation for the conformal Killing vector (see Appendix 1). The general solution of this equation is
Symmetries of Equations of Quantum Mechanics

\[ K^\mu = 2x^\mu x_\nu f^\nu - f^\mu x_\nu x^\nu + e^{[\mu \sigma]}x_\sigma + d X^\mu + e^\mu \]  

(1.18)

where \( f^\mu, e^{[\mu \sigma]}, d \) and \( e^\mu \) are arbitrary constants. Substituting (1.18) into (1.17) we obtain a linear combination of the operators (1.6), (1.16). These operators form a basis of a 15-dimensional Lie algebra, satisfying relations (1.14), (1.19):

\[ [J_{\mu \sigma}, \hat{K}_\lambda] = i(g_{\alpha \lambda} \hat{K}_\mu - g_{\mu \lambda} \hat{K}_\alpha), \quad [P_{\mu}, \hat{K}_\alpha] = 2i(g_{\mu \rho} D + J_{\rho \mu}), \]
\[ \hat{K}_\mu, \hat{K}_\nu] = 0, \quad [D, P_\mu] = -iP_\mu, \quad [D, \hat{K}_\mu] = i\hat{K}_\mu, \quad [D, J_{\mu \rho}] = 0. \]  

(1.19)

Relations (1.14), (1.19) characterize the Lie algebra of the conformal group \( C(1,3) \).

Thus we have made sure the massless KGF equation (1.15) is invariant under the 15-dimensional Lie algebra of the conformal group (called "conformal algebra" in the following). The conformal symmetry plays an important role in modern physics.

1.4. Lorentz Transformations

Thus we have found the maximal IA of the KGF equation in the class \( M_1 \). The following natural questions arise: why do we need to know this IA, and what information follows from this symmetry about properties of the equation and its solutions?

This information turns out to be extremely essential. First, knowledge of IA of a differential equation as a rule gives a possibility of finding the corresponding constants of motion without solving this equation. Secondly, it is possible with the IA to describe the coordinate systems in which the solutions in separated variables exist [305]. In addition, any IA in the class \( M_1 \) can be supplemented by the local symmetry group which can be used in order to construct new solutions starting from the known ones.

The main part of the problems connected with studying and using the symmetry of differential equations can be successively solved in terms of IAs without using the concept of the transformation group. For instance it will be the IA of the KGF equation which will be used as the main instrument in studying the relativistic equations of motion for arbitrary spin particles (see Chapter 2). But the knowledge of the symmetry group undoubtedly leads to a deeper understanding of the nature of the equation invariance properties.

Here we shall construct in explicit form the invariance group of the KGF equation corresponding to the IA found above. For this purpose we shall use one of the classical results of the group theory, established by Sophus Lie as long ago as the 19th century. The essence of this result may be formulated as follows: if an equation possesses an IA in the class \( M_1 \) then it is locally invariant under the continuous transformation group acting on dependent and independent variables (a rigorous
Chapter 1. Symmetries of the Fundamental Equations ...

formulation of this statement is given in many handbooks, see, e.g., [20, 379]).

The algorithm of reconstruction of the symmetry group corresponding to the
given IA is that any basis element of the IA corresponds to a one parameter
transformation group

\[ \begin{align*}
\theta & \mapsto x' = T(x, \theta) \\
\psi(x) & \mapsto \psi'(x') = \hat{D}(\theta, x)\psi(x)
\end{align*} \]

(1.20)

where \( \theta \) is a (generally speaking, complex) transformation parameter (it will be shown
in the following that for the KGF equation such parameters are real), \( g_\theta \) and \( \hat{D} \) are
analytic functions of \( \theta \) and \( x \), \( T(x, \theta) \) are linear operators defined on \( F \). The exact
expressions for \( g_\theta \) and \( \hat{D} \) can be obtained by integration of the Lie equations

\[ \frac{dx'^\mu}{d\theta} = K^\mu(x'), \quad x'^\mu \bigg|_{\theta=0} = x^\mu, \]

(1.21)

\[ \frac{d\psi'}{d\theta} = B(x')\psi', \quad \psi' \bigg|_{\theta=0} = \psi. \]

(1.22)

Here \( K^\mu \) and \( B \) are the functions from the definition (1.4) of a SO.

Each of the formulae (1.21), (1.22) gives a system of partial differential
equations with the given initial condition, i.e., the Cauchy problem which has a unique
solution. For the SOs (1.6) these equations are easily integrated. Comparing (1.4) and
(1.6) we conclude that for any operator \( P_\mu \) or \( J_\mu^\sigma \) \( B \equiv 0 \) and the solutions of (1.22) have
the form

\[ \psi'(x') = \hat{\psi}(x), \quad \psi'(x) = \hat{\psi}(g_\theta^{-1}(x)). \]

(1.23)

Solving equations (1.21) it is not difficult to find the transformation law for
the independent variables \( x'_\mu \). We obtain from (1.4), (1.6) that

\[ K^\mu = 1 \quad \text{if} \quad Q = P_\mu, \]

(1.24)

\[ A_\mu = x_\alpha g^{\alpha}_\mu - x_\mu g^{\alpha}_\alpha \quad \text{if} \quad Q = J_\mu^\alpha \]

(1.25)

where \( g^{\alpha}_\mu \) is the metric tensor (1.10). Denoting \( \theta = b_\mu \) for \( Q = P_\mu \) and substituting (1.24)
into (1.22) one comes to the equation

\[ \frac{dx'^\mu}{db_\mu} = 1, \quad x'^\mu \bigg|_{b=0} = x^\mu \]

(no sum over \( \mu \)), from which it follows that

\[ x'^\mu = x^\mu + b_\mu. \]

(1.26)

In a similar way using (1.25) one finds the transformations generated by \( J_\mu^\alpha \)
Symmetries of Equations of Quantum Mechanics

\[ x'^a = x^b \cos \theta_{ab} - x^b \sin \theta_{ab}, \]
\[ x'^b = x^b \cos \theta_{ab} - x^b \sin \theta_{ab}, \]  \hspace{1cm} (1.27)
\[ x'^\mu = x^\mu, \quad \mu \neq a, b, \quad a, b \neq 0, \]
\[ x'^a = x^0 \cosh \theta_{0a} - x^a \sinh \theta_{0a}, \]
\[ x'^0 = x^0 \cosh \theta_{0a} - x^a \sinh \theta_{0a}, \]  \hspace{1cm} (1.28)
\[ x'^\mu = x^\mu, \quad \mu \neq 0, a \]

where \( \theta_{ab}, \theta_{0a} \) are transformation parameters and there is no sum over \( a, b \).

So the KGF equation is invariant under the transformations (1.23), (1.26)-(1.28). The transformations (1.26)-(1.28) (which were first called Lorentz transformations by H. Poincaré) satisfy the group multiplication law and conserve the four-dimensional interval

\[ S(x^{(1)} - x^{(2)}) = S(x^{(1)} - x^{(2)}) \]  \hspace{1cm} (1.29)

where \( S(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \), and \( x^{(1)}, x^{(2)} \) are two arbitrary points of the space-time continuum.

The set of transformations satisfying (1.29) forms a group which is called the Poincaré group (the term suggested by Wigner).

The transformations (1.26)-(1.28) have a clear physical interpretation. Relations (1.26) and (1.27) define the displacement of the reference frame along the \( m \)-th coordinate and the rotation in the plane \( a-b \). As to (1.28) it can be interpreted as a transition to a new reference frame moving with velocity \( v \) relative to the original frame:

\[ x'_0 = (x_0 - v_a x_a), \]
\[ x'_a = (x_0 - v_a x_a) \beta, \quad \beta = \left[1 - v^2/c^2 \right]^{-1/2}, \]
\[ x'_\mu = x_\mu, \quad \mu \neq 0, a \]  \hspace{1cm} (1.30)

(no sum over \( a \)) where the parameter \( v_a \) is expressed through \( \theta_{0a} \) by the relation \( \theta_{0a} = \text{artanh}(v_a/c) \).

From (1.30) it is not difficult to obtain the relativistic law of summation of velocities

\[ V'_a = dx'_a/dx'_0 = (V_a - v_a)(1 - v_a V/c^2)^{-1}. \]  \hspace{1cm} (1.31)

We see that the IA of the simplest equation of motion of relativistic quantum physics (i.e., the KGF equation) possesses in an implicit form the information about the main laws of relativistic kinematics.

\* For clarity we give up the convention \( c=1 \) in (1.30), (1.31)
1.5. The Poincaré Group

Let us consider in more detail the procedure of reconstruction of the Lie group by the given Lie algebra presented in the above.

First we shall establish exactly the isomorphism of the algebra (1.6) and the Lie algebra of the Poincaré group.

The Poincaré group is formed by inhomogeneous linear transformations of coordinates $x_\mu$ conserving the interval (1.29), i.e., by transformations of the following type

$$x_\mu \rightarrow x'_\mu = a_{\mu\sigma} x^\sigma + b_\mu$$

(1.32)

where $a_{\mu\sigma}$, $b_\mu$ are real parameters satisfying the condition

$$a_{\mu\sigma} a_{\mu\lambda} = g_{\sigma\lambda}.$$

(1.33)

It follows from (1.33) that

$$(\det a_{\mu\sigma})^2 = 1, \quad a_{00} \geq 1$$

or

$$\det a_{\mu\sigma} = \pm 1, \quad |a_{00}| \geq 1.$$  

(1.34)

The group of linear transformations (1.32) satisfying (1.33) will be called the complete Poincaré group and denoted by $P_{c}(1,3)$. It is possible to select in the group $P_{c}(1,3)$ the subgroup $P(1,3)$ for which

$$\det a_{\mu\sigma} = 1, \quad a_{00} \geq 1.$$  

(1.35)

The set of transformations (1.32) satisfying (1.33) and (1.35) is called the proper orthochronous Poincaré group (or the proper Poincaré group). The group $P(1,3)$ is a Lie group but the group $P_{c}(1,3)$ is not, because for the latter, the determinant of the transformation matrix $a_{\mu\sigma}$ is not a continuous function and can change suddenly from -1 to 1.

It is convenient to write the transformations of the group $P(1,3)$ in the matrix form

$$\hat{x} \rightarrow \hat{x}' = A \hat{x}$$

(1.36)

where

$$\hat{x} = \text{column}(x_0, x_1, x_2, x_3, 1), \quad \hat{x}' = \text{column}(x'_0, x'_1, x'_2, x'_3, 1).$$

(1.37)
the symbols \( a \) and \( b \) denote the 4×4 matrix \( A_{\mu\sigma} \) and the vector column with components \( b_\mu \). The last coordinate 1 is introduced for convenience and is invariant under the transformations.

Inasmuch as any transformation (1.30) can be represented in the form (1.36)-(1.38) the group \( P(1,3) \) is isomorphic to the group of matrices (1.38) (denoted in the following by \( P_m(1,3) \)) The group multiplication in the group \( P_m(1,3) \) is represented by the matrix multiplication moreover

\[
A(a_1,b_1)A(a_2,b_2) = A(a_1a_2+b_1b_2) .
\]

The unit element of this group is the unit 5×5 matrix, the inverse element to \( A(a,b) \) has

\[
A(a^{-1},-a^{-1}b) .
\]

The general solution of (1.33), (1.35) can be represented in the following form

\[
a_{00} = \cos z \cos^2 \phi - \cosh y \sin^2 \phi - \lambda_2^2 (\cos z - \cosh y) ,
\]

\[
a_{00} = \frac{1}{R} \left[ \sin z \cos y (\lambda_2^2 \cos \phi - \theta_2 \sin \phi) - \cos z \sin y (\lambda_2^2 \sin \phi + \theta_2 \cos \phi) + \theta_2 \cos \phi \right] + \frac{1}{R} \epsilon_{abc} \lambda_2 \phi (\cos z - \cosh y) ,
\]

\[
a_{00} = a_{00} - \frac{2}{R} \epsilon_{abc} \lambda_2 \phi (\cos z - \cosh y) ,
\]

\[
a_{bc} = \frac{1}{R} \epsilon_{abc} \theta \left( \sin z \cosh y \cos \phi - \cos z \sin y \sin \phi \right) - \\
\left( \lambda_2 \lambda_2 + \theta \phi \delta_{bc} \right) (\cos z - \cosh y) + \delta_{bc} (\cos z \cosh y \cos \phi - \cos z \sin y \sin \phi) ,
\]

\[
\theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2} , \quad \lambda_2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2} , \quad \phi = \arctan \frac{\lambda_2 \theta}{\theta^2 - \lambda_2^2} .
\]

\[
z = R \cos \phi , \quad y = R \sin \phi , \quad R = [(\theta^2 - \lambda_2^2)^2 + 4(\lambda_2 \phi)^2]^{1/4}
\]

where \( \theta_m \) and \( \lambda_2 \) (\( m=1,2,3 \)) are arbitrary real parameters, \( \delta_{ab} \) is the Kronecker symbol.
Chapter 1. Symmetries of the Fundamental Equations ...

It follows from (1.37) that any matrix (1.38) depends continuously on ten real parameters \(b_\mu\), \(\theta_a\) and \(\lambda_a\). In other words, the group \(P_m(1,3)\) is a ten-parametric Lie group.

Let us determine the Lie algebra of the group \(P_m(1,3)\). Basis elements of this algebra by definition (see e.g. [20]) can be chosen in the form

\[
\hat{P}_\mu = \left( \begin{array}{cc} 0 & K\mu \\ \tilde{0} & 0 \end{array} \right), \quad \hat{j}_{\mu\sigma} = \left( \begin{array}{cc} S_{\mu\sigma} \tilde{0}^\dagger & 0 \\ 0 & 0 \end{array} \right)
\]

where \(\tilde{0}, 0\) and \(0\)' are the 4×4, 1×4 and 4×1 zero matrices,

\[
S_{12} = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad S_{23} = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{array} \right), \quad S_{31} = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{array} \right)
\]

\[
S_{01} = \left( \begin{array}{cccc} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad S_{02} = \left( \begin{array}{cccc} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad S_{03} = \left( \begin{array}{cccc} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{array} \right)
\]

\(K_0\) = \text{column}(0000), \(K_1\) = \text{column}(0i00), \(K_2\) = \text{column}(000i), \(K_3\) = \text{column}(0000i).

It is not difficult to verify that the matrices (1.41) satisfy the conditions (1.14). These conditions are satisfied also by the basis elements of the IA of the KGF equation, so this IA is isomorphic to the matrix algebra generated by the basis (1.41). Any matrix from the group \(P_m(1,3)\) can be constructed from the basis elements (1.41) by the exponential mapping

\[
A(a,b) = \exp\left( \frac{i}{2} \theta_a \hat{j}^{\mu\sigma} \right) \exp(i\hat{P}_\mu b^\mu)
\]

where
(1.44) \[ \exp B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n \]

(B is any 5x5 matrix, I is the unit matrix), \( \theta_{ab} = \theta_{cb} \theta_{a}/2 \), \( \theta_{ab} = \lambda_{ab} \); \( \theta_+ \), \( b_\mu \), \( \lambda_\mu \) are parameters present in (1.38), (1.35).

The IA of the KGF equation realizes a representation of this Lie algebra of the matrix group \( P_{sl}(1,3) \) in the vector space \( F \). This representation can be extended to local representation of the group \( P_{sl}(1,3) \) given by the relations (1.23), (1.26)-(1.28). In analogy with (1.43) these relations can be represented as an exponential mapping of the IA basis elements

\[ \psi \rightarrow \psi'(x) = T(a,b)\psi(x) \exp(\frac{1}{2} \int_{-\infty}^{x} \theta_{\nu} \nu) \exp(iP_\mu b_\mu) \psi(x) = \psi(A^{+}(a,b)\psi) \]

where the only parameter \( \theta_{\mu\sigma} \) or \( b_\mu \) does not vanish. The exponentials in (1.45) are defined according to

\[ \exp(\theta Q) \psi = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} Q^n \psi, \quad \psi \in F \]

where \( Q \) is an arbitrary basis element of the IA, \( \theta \) is the corresponding parameter.

The transformations (1.45) are defined also for the case of arbitrary parameters \( b_\mu \), \( \theta_{\mu\sigma} \). Moreover for \( T(a,b) \) the following conditions hold

\[ T(a,b)T(a',b') = T(aa', b+ab') \]

If \( Q \) belongs to the IA of the KGF equation in the class \( M \), then \( Q^n \) transforms solutions into solutions for any \( n=1,2,3,... \). The operator \( \exp(\theta Q) \) also has this property according to (1.46). One concludes from the above that if \( \psi(x) \) is an analytical solution of (1.1) then \( \psi'(x) \) (1.45) is also an analytical solution on \( F \). That is why we call the group of transformations (1.45) the symmetry group of the KGF equation.

Thus starting from the IA of the KGF equation we have constructed the symmetry group of this equation which is called the Poincaré group. This group includes the transformations (1.23), (1.32), (1.33), (1.35), i.e., such transformations which do not change wave function but include rotations and translations of the reference frame for independent variables. The requirement of invariance under the Poincaré group is the main postulate of relativistic quantum theory.

1.6. The Conformal Transformations

Let us find the explicit form of transformations from the symmetry group of the massless KGF equation. The IA of this equation is formed by the SO (1.6) and (1.16).
It is clearly sufficient to restrict ourselves to the construction of the transformations generated by the operators (1.16) inasmuch as the remaining transformations have already been considered in Subsections 1.4 and 1.5.

In order to find the one-parameter subgroups generated by \( K_\sigma \) and \( D \) we will solve the corresponding Lie equations. Comparing (1.4) and (1.16) we conclude that for the operator \( D A_\mu = x^\mu, B = 1, \) so the equations (1.21), (1.22) take the form

\[
\frac{dx^\mu}{d\theta} = x^\mu, \quad x^\mu \bigg|_{\theta=0} = x^\mu, \\
\frac{d\psi}{d\theta} = -\psi, \quad \psi \bigg|_{\theta=0} = \psi.
\]

(1.47)

The solutions of (1.47) have the form

\[
\psi = \exp(-\theta)\psi, \quad x^\mu = \exp(\theta)x^\mu.
\]

(1.48)

The transformations (1.48) are called dilatation transformations and reduce to a change of scale (any independent variable is multiplied by the same number).

For the operators \( K_\sigma \) the Lie equations are

\[
\frac{dx^\mu}{db_\sigma} = 2x^\mu x_\sigma - x_\sigma x^\mu g_\sigma, \quad x^\mu \bigg|_{b_\sigma=0} = x^\mu, \\
\frac{d\psi}{db_\sigma} = -2x^\mu \psi, \quad \psi \bigg|_{b_\sigma=0} = \psi.
\]

(1.49)

where \( b_\sigma \) are the transformations parameters. It is not difficult to verify that solutions of the Cauchy problems formulated in (1.49) are given by the formulae

\[
x^\mu = x^\mu - g_\sigma b_\sigma x_\lambda x^\lambda, \\
\psi = (1 - 2x^\mu b_\sigma b_\sigma x_\lambda x^\lambda)\psi
\]

(1.50)

(no sum over \( \sigma \)).

Formulae (1.50) give a family of transformations depending on a parameter \( b_\sigma \) (with a fixed value of \( \sigma \)). Using these transformations successively for different \( \sigma \) we come to the general transformation generated by \( K_\sigma \), which also has the form (1.50) where the summation over \( \sigma \) is assumed.

The transformations (1.50) are called conformal transformations and can be represented as a composition of the following transformations: the inversion

\[
x^\mu \rightarrow x^\mu = \frac{x^\mu}{x_\lambda x^\lambda},
\]

the displacement

\[x^\mu \rightarrow x^\mu = x^\mu - b^\mu,
\]
and the second inversion
\[ x^\mu \rightarrow x^\mu = \frac{x^\mu}{x^\lambda x^\mu x^\lambda}. \]

We see that the massless KGF equation is invariant under the scale and conformal transformations besides the symmetry with respect to Lorentz transformations. The set of transformations (1.30), (1.45), (1.48), (1.50) for \( x^\mu \) forms a 15-parameter Lie group called the conformal group. As is demonstrated in Section 3 conformal invariance occurs for any relativistic wave equation describing a massless field.

It is necessary to note that the transformations found above can be considered only as a local representation of the group \( C(1,3) \) since in addition to the problem of defining the domain of the transformed function it is necessary to take into account that the expression (1.50) for \( x^\mu \) becomes nonsense if \( 1-2b_\sigma x^\sigma + b_\sigma b_\sigma x^\mu x^\mu = 0 \).

1.7. The Discrete Symmetry Transformations

Although the IA of the KGF equation found above is in some sense maximally extensive, the invariance under this algebra and the corresponding Lie group does not exhaust symmetries of this equation. Moreover the KGF equation is invariant under the following discrete transformations

\[ x_0 \rightarrow x'_0 = x_0, \quad x_a \rightarrow x'_a = -x_a, \] (1.51)
\[ \psi(x) \rightarrow P\psi(x) = r_1 \psi(x'), \] (1.51)
\[ x_0 \rightarrow x'_0 = -x_0, \quad x_a \rightarrow x'_a = x_a, \] (1.52)
\[ \psi(x) \rightarrow T\psi(x) = r_2 \psi(x''), \] (1.52)
\[ x \rightarrow x, \quad \psi(x) \rightarrow C\psi(x) = r_3 \psi^*(x) \] (1.53)

where \( r_\sigma = \pm 1 \). The invariance under the transformations (1.51) (space inversion), (1.52) (time reflection) and (1.53) (charge conjugation) can be easily verified by direct calculations.

The determinants of the matrices of the coordinate transformations of (1.51) and (1.52) are equal to -1. So these transformations do not belong to the group \( P(1,3) \) but are contained in the complete Poincaré group \( P(1,3) \). As to the transformation of charge conjugation, it has nothing to do with the Poincaré group and represents the symmetry of the KGF equation under the complex conjugation.

The operators \( P, C, T \) satisfy the following commutation and anticommutation

14
Chapter 1. Symmetries of the Fundamental Equations ...

relations together with the Poincaré generators

\[ [P, P_i] = [P, P_{\mu}] = [P, J_{ab}] = [P, J_{\alpha\beta}] = 0, \]
\[ [T, P_0] = [T, P_{\mu}] = [T, J_{ab}] = [T, J_{\alpha\beta}] = 0, \]
\[ [C, P_{\mu}] = [C, J_{\mu\sigma}] = 0, \]
\[ P^2 = T^2 = C^2 = 1, \quad [P, T] = [C, T] = [C, P] = 0. \]

Conditions (1.54) can serve as a abstract definition of the operators \( P, C \) and \( T \).

So the IA of the KGF equation found in Section 1.3 can be fulfilled to the set of the symmetry operators \{ \( P, J_{ab}, C, P, T \) \}. These operators satisfy the invariance condition (1.5) and algebra (1.14), (1.54) (which, of course, is not a Lie algebra).

We note that the discrete symmetry transformations can be used to construct a group of hidden symmetry of the KGF equation. Actually, the KGF equation is transparently invariant under the transformation

\[ x \rightarrow x, \quad \psi \rightarrow R \psi(x) \rightarrow \psi(x) \sqrt{-1}. \]  

Combining this transformation with (1.53), we can select the set of symmetries \{ \( C, R, CR \) \} which satisfy the following commutation relations


since \( C^2 = R^2 = 1 \) and \( CR = -RC \).

In accordance with (1.56) the SOs \( C, R \) and \( CR \) form a Lie algebra which is isomorphic to the algebra \( AO(1,2) \), i.e., the Lorentz group in (1+2)-dimensional Minkowski space. This circumstance enables us to find exactly the corresponding symmetry group which is generated by the following one-parameter transformations

\[ \psi \rightarrow \cos \theta \psi, \quad \psi \rightarrow i \sin \theta \psi, \]
\[ \psi \rightarrow \cosh \theta \psi, \quad \psi \rightarrow i \sinh \theta \psi, \quad \psi \rightarrow \cosh \theta \psi, \quad \psi \rightarrow i \sinh \theta \psi, \]

where \( \theta_0, \theta_1 \) and \( \theta_2 \) are real parameters.

It is possible to point out the other sets of symmetries forming a representation of the algebra \( AO(1,2) \), i.e.,

\[ \{ T, R, TR \}, \quad \{ PT, R, PTR \}, \quad \{ CP, R, CPR \}, \]

or to select more extended IAs including more then three basis elements (for instance, the sets \{ \( C, R, CR, PC, PR, PCR \) \} and \{ \( C, R, CR, PC, PR, PCR, PCT, PCTR, PTR, TC, TR \) \).
Symmetries of Equations of Quantum Mechanics

TCR\} form representations of the algebras $AO(2,2)$ and $AO(2,2) \oplus AO(2,2)$. We will not analyze these algebraic structures but formulate a general statement valid for a wide class of linear differential equations.

**Lemma 1.1.** Let a linear partial differential equation is invariant under an antilinear transformation $Q$, satisfying the condition $Q^2=1$. Than this equation is invariant under the algebra $AO(1,2)$.

The proof is almost evident from the above, since any linear equation is invariant under the transformation $R$ of (1.55). Then such an equation admits the IA with the basis elements \{Q,R,QR\} which realize a representation of the algebra $AO(1,2)$.

We will see in the following that Lemma 1.1 enables to find hidden symmetries for great many of equations of quantum mechanics. The corresponding symmetry groups reduce to matrix transformation involving a wave function and a complex conjugated wave function.

Other hidden symmetries of the KGF equation are considered in Section 16.

### 2. LOCAL SYMMETRY OF THE DIRAC EQUATION

#### 2.1. The Dirac Equation

In 1928 Dirac found the relativistic equation for an electron, which can be written in the form

$$L\psi \equiv (\gamma^\mu p_\mu - m)\psi = 0 \quad (2.1)$$

where $\psi$ is a four-component wave function

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad (2.2)$$

$\gamma_\mu$ are 4×4 matrices satisfying the Clifford algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}. \quad (2.3)$$

For most of our needs the explicit form of the matrices $\gamma_\mu$ is not essential inasmuch as the conditions (2.3) determine them up to unitary equivalence. We will use, for the sake of concreteness, the following representation

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix} \quad (2.4)$$

where 0 and $I$ are the zero and unit 2×2 matrices, $\sigma_a$ are the Pauli matrices.
Chapter 1. Symmetries of the Fundamental Equations ...

The equation (2.1) is the simplest quantum mechanical equation describing a noninteracting particle with spin. The study of this equation symmetry does not differ in principle from the analysis of the KGF equation given above. Nevertheless taking into account the outstanding role of the Dirac equation in physics and special features connected with the fact that the function \( \psi \) has four components we will consider the symmetries of the Dirac equation in detail.

Let us note that any component of the function \( \psi \) satisfies the KGF equation. Indeed, multiplying (2.1) on the left by \( \gamma_\mu p_\mu + m \) and using (2.3) we obtain

\[
(p_\mu p^\mu - m^2)\psi = 0.
\]  

We see that the KGF equation is a consequence of the Dirac equation. The inverse statement is not true of course inasmuch as there is an infinite number of first order partial differential equations whose solutions satisfy (2.6) componentwise. The Dirac equation is the simplest example of such a system.

### 2.2. Various Formulations of the Dirac Equation

Let us consider other (different from (2.1)) representations of the Dirac equation to be found in the literature. All these representations are equivalent but give a possibility of obtaining different generalizations of (2.1) to the case of a field with arbitrary spin.

Starting from (2.1) it is not difficult to obtain the equation for a complex conjugated function \( \psi^\dagger \). Denoting

\[
\psi^\dagger = (\psi_1^\dagger \psi_2^\dagger \psi_3^\dagger \psi_4^\dagger), \quad \bar{\psi} \sim \psi^\dagger \gamma_0,
\]  

and making complex conjugation of (2.1) we obtain, using (2.3)

\[
\bar{\psi}(\gamma_\mu p^\mu - m) = 0
\]  

where it is implied that \( p^\mu \) act on \( \psi \). Using the representation (2.4) it is possible to write (2.8) in the following equivalent form

\[
(\gamma_\mu p^\mu - m)\psi^C = 0, \quad \psi^C = \gamma_2 \psi.
\]  

Indeed, the equation (2.8) and (2.9) coincide when written componentwise.

The Dirac equation in the form (2.9) is widely used in quantum field theory.

Multiplying (2.1) from the left by \( \gamma_\mu \) and using (2.3) we obtain the equation in the Schrödinger form
where the Hamiltonian $H$ has the form

$$H = \psi_0^* \gamma_0 \psi_0 + \gamma_2^* \gamma_2 \psi_2^* \psi_2 + \psi_1^* \gamma_1 \psi_1$$

(2.10)

It was in the form (2.10), (2.11) that the equation considered was found by Dirac [77] for the first time. And it is the formulation (2.10), (2.11) which will serve as a base for generalization of the Dirac equation for the case of arbitrary spin, see Chapter 2.

The other (so called covariant) formulation of the Dirac equation can be obtained by multiplication (2.1) from the left by an arbitrary matrix $\gamma_\mu$

$$p_\mu \psi = \hat{P}_\mu \psi = \frac{1}{2} [\gamma_\mu, \gamma_\nu] p_\nu \gamma_\mu \psi$$

(2.12)

In the equation (2.12) as in (2.1) all the variables play equal roles in contrast to (2.10) where the time variable is picked out. The equations in the form (2.12) also admit very interesting generalizations [11, 135]. In particular the infinite-component Dirac equation for positive energy particles [81] has this form.

In conclusion let us note, following Majorana [292] that the matrices $\gamma_\mu$ can be chosen in such a form that all the coefficients of the equation (2.1) are real. Namely setting

$$\gamma'_0 = \gamma_0 \gamma_2, \quad \gamma'_1 = -\gamma_1 \gamma_2, \quad \gamma'_2 = \gamma_1 \gamma_2, \quad \gamma'_3 = -\gamma_3$$

(2.13)

where $\gamma_\mu$ are the matrices (2.4), we can write the Dirac equation in the form

$$(\gamma'^\mu p_\mu - m) \psi' = 0,$$

(2.14)

where $\gamma'_\mu$ and $\psi'$ are connected with $\gamma_\mu$ and $\psi$ by the equivalence transformation

$$\psi' = U \psi, \quad \gamma'_\mu = U \gamma_\mu U^{-1}, \quad U^{-1} = (1 - \gamma_3) \gamma_2 \sqrt{2}.$$  

(2.15)

Using (2.4) it is not difficult to verify that the equation (2.14) includes real coefficients only and so can be reduced to two noncoupled systems of equations for the real and imaginary parts of the function $\psi'$.

Using other (distinct from $U$) nondegenerated matrices for the transformation (2.15) we can obtain infinitely many other realizations of the Dirac equation, which are equivalent to (2.1).
2.3. Algebra of the Dirac Matrices

As was noted in the above these are relations (2.3) (but not an explicit realization of the γ-matrices) which are used by solving concrete problems with the help of the Dirac equation. Here we present some useful relations following from (2.3).

First let us note that there exists just one more matrix satisfying (2.3). This matrix has the form

\[ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4. \]  

(2.16)

In the representation (2.4) we have

\[
\gamma_i = \begin{pmatrix}
-I & 0 \\
0 & I
\end{pmatrix},
\]

(2.17)

Furthermore it is not difficult to obtain from (2.3) the following relations

\[ S_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \gamma^\rho \gamma^\sigma, \quad S_{\mu \nu} = \frac{1}{4} \left[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} \right] = \frac{1}{4} \left[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} \right]. \]

Solving (2.6) it is not difficult to show the matrices (2.19) are linearly independent and hence any 4×4 matrix can be represented as a linear combination of the basis elements (2.19).

2.4. SOs and IAs

The main property of the Dirac equation is the relativistic invariance, i.e., symmetry under the Poincaré group transformations. Here we will prove the existence of this symmetry and demonstrate that it is the most extensive one, i.e., that there is no wider symmetry group leaving the Dirac equation invariant.

As in the case of the KGF equation we will describe symmetries of the Dirac equation using the language of Lie algebras, which first gives a possibility of clarity and rigor interpretation using relatively simple computations and, secondly, is suitable
for the description of hidden (non-geometrical) symmetries not connected with space-time transformations (see Chapter 4).

The problem of investigation of the Dirac equation symmetry in the class $M_1$ can be formulated in complete analogy with the corresponding problem for the KGF equation. However it is necessary to generalize the corresponding definitions for the case of a system of partial differential equations.

Let us denote by $F^4$ the vector space of complex valued functions (2.2) which are defined on some open and connected set $D$ of the real four-dimensional space $R$ and are real-analytic. In other words $\psi \in F^4$ if any component $\psi_i \in F$ (see Subsection 1.1). Then the linear differential operator $L$ of (2.1) defined on $D$ has the following property: $L \psi \in F^4$ if $\psi \in F^4$. Finally the symbol $G^4$ will denote the space of 4x4 matrices whose matrix elements belong to $F$.

The following definition is a natural generalization of Definition 1.1 (see Subsection 1.1):

**DEFINITION 2.1.** A linear first order differential operator

$$Q \equiv F^\mu p_\mu + D, \quad F^\mu \in F, \quad D \in G^4$$

is a SO of the Dirac equation if

$$[Q, L] \beta_0 L, \quad \beta_0 \in G^4. \quad (2.21)$$

The equation (2.21) is to be understood in the sense that the operators on the l.h.s. and r.h.s. give the same result acting on an arbitrary function $\psi \in F^4$.

As in the case of the KGF equation a SO transforms solutions of (2.1) into solutions and the complete set of SOs forms a Lie algebra. So, while speaking about the Dirac equation SOs we will use the term "invariance algebra" (IA).

### 2.5. The IA of the Dirac Equation in the Class $M_1$

Let us formulate and prove the main assertion about symmetries of the Dirac equation. As it will be shown further on this statement includes all the information about the kinematics of a particle described by the evolution equation (2.1).

**THEOREM 2.1.** The Dirac equation is invariant under the ten-dimensional Lie algebra which is isomorphic to the Lie algebra of the Poincaré group. The basis elements of this IA can be chosen in the following form

$$P_\mu = p_\mu \equiv i \frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = \gamma_\mu x_\nu - \gamma_\nu x_\mu + S_{\mu\nu} \quad (2.22)$$

where

$$S_{\mu\nu} = \frac{i}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \quad (2.23)$$
Chapter 1. Symmetries of the Fundamental Equations...

The Lie algebra defined by the basis elements (2.22) is the maximal IA of the Dirac equation in the class $M_1$.

**Proof.** The first statement of the theorem can be easily verified by the direct calculation of commutators of $P_\mu$ and $J_{\mu\nu}$ with $L$ of (2.1), which are equal to zero. The operators (2.22) satisfy relations (1.14) and hence form a basis of the Lie algebra isomorphic to $AP(1,3)$.

A little more effort is needed to prove the algebra (2.22) is the maximally extensive IA of (2.1).

Let us represent an arbitrary operator $Q$ and the matrix $\beta_0$ from (2.21) as a linear combination of the basis elements (2.19)

$$Q = (K^\mu p_\mu + a^0) + i\gamma^\alpha b^\mu + S_{\nu\sigma} f^{[\nu\sigma]} + \gamma_\mu d^\mu,$$

$$\beta_0 = i e^0 + i\gamma^\alpha e^\mu + i\gamma^\alpha h^\mu + S_{\nu\sigma} k^{[\nu\sigma]}$$

(2.24)

where the Latin letters denote unknown functions belonging to $F$. The problem is to find the general form of these functions using the conditions (2.21).

Calculating the commutator of the operators $Q$ (2.24) and $L$ (2.1) we obtain with (2.18):

$$[Q,L] = -i\gamma^\alpha K_\mu^\alpha p_\mu - i\gamma^\alpha a^0 - 2i\gamma^\alpha \gamma^\beta A_\beta^\mu p_\mu - i\gamma^\alpha d^\mu = 4i S_{\nu\sigma} f^{[\nu\sigma]} + 2i\gamma^\alpha f^{[\nu\sigma]} p_\mu + \gamma^\alpha f^{[\nu\sigma]} + e^{\sigma\nu}_{\mu\nu} f^{[\nu\sigma]} l_\gamma$$

(2.25)

where the bottom indices denote derivatives with respect to the corresponding variables: $B_\mu = \partial B/\partial x_\mu$. On the other hand it is not difficult to calculate that

$$\beta_0 L = e^0 q^\mu p_\mu - i(\gamma^\alpha e^\mu + q^\mu p_\mu - i\gamma^\alpha h^\mu) - 2i S_{\nu\sigma} f^{[\nu\sigma]} + 2i\gamma^\alpha e^\mu f^{[\nu\sigma]}$$

(2.26)

Substituting (2.25) and (2.26) into (2.21) and equating the coefficients of linearly independent matrices and differential operators we obtain the following system

$$K_\mu^\alpha - 2g_{\mu\sigma} f^{[\sigma\nu]} = 0,$$

(2.27)

$$a^0 + 8g_{\mu\sigma} f^{[\sigma\nu]} = 0,$$

(2.28)

$$a^\alpha + i e^{\nu}_{\mu\nu} f^{[\sigma\nu]} = 0,$$

(2.29)

$$e^\mu = 2a^\mu, \quad me^0 = 0, \quad me^0 = 0,$$

(2.30)

$$f^{\mu\nu} - b^{\mu\nu} = k^{[\mu\nu]} d^{[\mu\nu]} = 0$$

(2.31)

where $g_{\mu\nu}$ is the metric tensor (1.10).

The system (2.27)-(2.31) is easily integrated. The equations (2.27) can be rewritten in the form
The equivalence of (2.27) and (2.32), (2.33) follows from the symmetry of $g^{\mu \sigma}$ and antisymmetry of $f^{[\mu \sigma]}$ under the permutation of indices.

The equation (2.32) coincides with the conformal Killing equation (1.17), its solutions are given in (1.18). Substituting (1.18) into (2.33) and bearing in mind that in accordance with (2.30) for a nonzero $m \epsilon_0 = e^1 = a^1 = 0$ we obtain

$$K^{\mu} = c^{[\nu]}x_{\nu} + \Phi^{\nu}u_\nu, \quad f^{(\nu \nu)} = \frac{1}{2} c^{[\nu \nu]} S_{\nu \nu}, \quad \beta_0 = 0.$$  

(2.34)

It follows from the above that the general expression (2.24) for $Q$ and $\beta_0$ is reduced to the form

$$Q = (a_0 + c^{[\nu]}x_{\nu} + \Phi^{\nu}u_\nu) + \frac{1}{2} c^{[\nu \nu]} S_{\nu \nu}, \quad \beta_0 = 0.$$  

(2.35)

where $c^{[\nu \nu]}$, $\Phi^{\nu}$, $a^0$ are arbitrary complex numbers. The operator (2.35) is a linear combination of the operators (2.22) and trivial identity operator, so the operators (2.22) form a basis of the maximally extensive IA of the Dirac equation in the class $M_1$. We see the IA of the Dirac equation is isomorphic to the IA of the KGF equation considered in Section 1. The essentially new point is the presence of the matrix terms in the SOs (2.22). These terms correspond to an additional (spin) degree of freedom possessed by the field described by the Dirac equation. We shall see in the following that due to the existence of the spin degree of freedom the Dirac equation has additional symmetries in the classes of higher order differential operators.

2.6. The Operators of Mass and Spin

It is well known that the Dirac equation describes a relativistic particle of mass $m$ and spin $s$. Such an interpretation of this equation admits a clear formulation in the language of the Lie algebras representation theory.

The Dirac equation IA determined by the basis elements (2.22) has two main invariant (Casimir) operators (see Section 4)

$$C_1 = P_{\mu} P^{\mu}, \quad C_2 = W_{\mu} W^{\mu}.$$  

(2.36)

where $W_{\mu}$ is the Lubanski-Pauli vector.
Chapter 1. Symmetries of the Fundamental Equations ...

Let us recall that a Casimir operator is an operator belonging to the enveloping algebra of the Lie algebra, which commutes with any element of this algebra. One of the main results of Lie algebra representation theory says that in the space of an irreducible representation (IR) the Casimir operators are multiples of the unit operator. Moreover eigenvalues of invariant operators can be used for labelling of IRs inasmuch as different eigenvalues correspond to nonequivalent representations.

Thus, to label the representation of the Poincaré algebra, which is realized on the set of solutions of the Dirac equation, it is necessary to find eigenvalues of the operators (2.36). Substituting (2.22) into (2.36) and using (2.3), (2.6), (2.18) we obtain

\[ C_1 \psi = P_\mu P_\nu \psi = m^2 \psi, \]

\[ C_2 \psi = W_\mu W_\nu \psi = -\frac{1}{2} p_\mu S_{ab} S_{ab} \psi = -m^2 s(s+1) \psi, \quad s = \frac{1}{2}. \]

In relativistic quantum theory the space of states of a particle with mass \( m \) and spin \( s \) is set in correspondence with the space of the representation of the Poincaré algebra corresponding to the eigenvalues \( m^2 \) and \(-m^2(s+1)\) of the Casimir operators \( C_1 \) and \( C_2 \). So it follows from (2.37) that the Dirac equation can be interpreted as an equation of motion for a particle of spin 1/2 and mass \( m \).

2.7. Manifestly Hermitian Form of Poincaré Group Generators

Before we considered only such solutions of the Dirac equation which belong to the space \( F_s \). But the operator \( L \) (2.1) and the SOs (2.22) can be defined also on the set of finite functions \( (C_0^\infty)^4 \) everywhere dense in the Hilbert space \( L_2 \) of the square integrable functions with the scalar product

\[ \langle \psi^{(1)}, \psi^{(2)} \rangle = \int d^4x \psi^{(1)} \psi^{(2)} \]

where according to the definitions (2.2), (2.7)

\[ \psi^{(1)} = \psi_1 \psi_2 \psi_3 \psi_4, \quad \psi^{(2)} = \psi_1^* \psi_2^* \psi_3^* \psi_4^* \]

It is not difficult to verify that the operators (2.22) are Hermitian in respect to the scalar product (2.39) where \( \psi_\alpha \) satisfy the Dirac equation. To show that, it is sufficient to represent these operators in the following form

\[ P_0 = H = \gamma_\mu p_\mu + \gamma_0 m \quad p_\mu = \gamma_\mu \frac{\partial}{\partial x_\mu} \]

\[ J_{ab} = \sum_{a} x_\mu p_\mu - x_b p_a + S_{ab}, \quad J_{a0} = x_0 p_a - \frac{1}{2} [x_\alpha, H]. \]
Symmetries of Equations of Quantum Mechanics

Throughout on the set of solutions of the Dirac equation the operators (2.22) and (2.40) coincide inasmuch as \([x_{\mu}H]_a = 2(x_{\mu}H - S_{\mu a})\).

The operators (2.40) satisfy the commutation relations (1.14) and (in contrast to (2.22)) are written in a transparently Hermitian form. So the operators (2.22) also are Hermitian with respect to the scalar product (2.39). It follows from the above that the Poincaré group transformations generated by the operators (2.22) (see Subsection 2.9) are unitary, i.e., do not change the value of the scalar product (2.39).

2.8. Symmetries of the Massless Dirac Equation

The equation (2.1) has clear physical meaning also in the case \(m = 0\), describing a massless field with helicity \(\pm \frac{1}{2}\). The symmetry of the Dirac equation with \(m = 0\) is wider than in the case of nonzero mass.

THEOREM 2.2. The maximal IA in the class \(M_1\) of the equation

\[
\gamma^\mu p_\mu \psi = 0
\]  

(2.41)
is a 16-dimensional Lie algebra whose basis elements are given by formulae (2.22) and (2.42)

\[
D = x^{\mu} p_\mu + iK, \quad \Sigma = i\gamma^\mu \\
K_\mu = 2x_\mu D - x^\nu x_\mu p_\nu + 2S_{\mu \nu} x^\nu
\]  

where \(K = \frac{3}{2}\).

The proof can be carried out in complete analogy to that of Theorem 2.1 (see Subsection 2.5). The general form of a SO is given in (2.24), the equations determining the corresponding operators coefficients are given by relations (2.27)-(2.31) with \(m = 0\). So it is not difficult to find the general solution for a SO in the form

\[
Q = (2f^{\alpha \beta} x_{\alpha} p_{\beta} - f^{\mu \nu} x_\mu p_\nu) + c^{[\mu}_{\nu} x_{\rho} p_{\sigma} + dx^{\mu} p_{\mu} + \psi^\mu p_{\mu} + 3j^{\lambda}[x_{\lambda} a^0 + \dot{\gamma}^4 a^1 - S_{\alpha \beta}(\frac{1}{2} c^{[\mu \nu] - 2f^{\mu \nu} x^{\nu})].
\]  

(2.43)
The operator (2.43) is a linear combination of the generators (2.22), (2.42) and the unit operator which give the basis of IA of the equation (2.41) the class \(M_1\).

The operators (2.22), (2.42) satisfy the commutation relations (1.14), (1.19) which determine the Lie algebra of the conformal group. As to the operator \(\Sigma\), it commutes with any basis element of the IA. In other words the IA of the equation (2.42) consists of the 15-dimensional Lie algebra which isomorphic to the IA of the massless KGF equation, and an additional matrix operator \(\Sigma\) which is the center of the IA of the massless Dirac equation.

Let us notice that the massless Dirac equation with the matrices (2.4) reduces to two noncoupled equations:
where $\sigma_a$ are the Pauli matrices, $\varphi_\pm = (1 \pm i \gamma^4 \psi)/2$.

Relations (2.44) are called the Weyl equations. The SOs of the massless Dirac equation can be decomposed into the direct sum of operators defined on $\varphi_\pm$

The explicit form of the basis elements of the conformal algebra on the set of the Weyl equation solutions can be obtained from (2.22), (2.42) by the change $S_{\mu a} \rightarrow i[\sigma_a, \sigma_\mu]/4, S_0 \rightarrow \pm i \sigma_\mu/2$. As to the operator $\Sigma$ it is equal to the unit matrix on the subspaces $\varphi_\pm$.

2.9. Lorentz and Conformal Transformations of Solutions of the Dirac Equation

As was mentioned in Section 1 the main consequence of a symmetry of a differential equation under an IA in the class $M_1$ is that this equation turns out to be invariant under the Lie group whose generators form the basis of this IA. In other words proving the invariance of the Dirac equation under the algebra $AP(1,3)$ and (for $m=0$) conformal algebra we have actually established its invariance under Lorentz and conformal transformations.

Here we shall find an explicit form of the group transformations of solutions of the Dirac equation with zero and nonzero masses.

The general transformation of the symmetry group of (2.1) can be written in the following form (compare with (1.45))

$$\psi(x) \rightarrow \psi'(x) = \exp(i J_{\mu \nu} \theta^{\mu \nu}) \exp(i P_\mu b_\mu) \psi(x)$$

where $J_{\mu \nu}$ and $P_\mu$ are the operators (2.22), $\theta^{\mu \nu}$ and $b_\mu$ are real parameters. Using the commutativity of $S_{\mu \nu}$ with $x_\mu P_\nu - x_\nu P_\mu$ we can represent this transformation in the form

$$\psi(x) \rightarrow \psi'(x) = \exp(i J'_{\mu \nu} \theta^{\mu \nu}) \psi''(x)$$

(2.45)

where

$$\psi''(x) = \exp(i L_{\mu \nu} \theta^{\mu \nu}) \exp(ip_\mu b_\mu) \psi(x),$$

(2.46)

and $J'_{\mu \nu}$ are the operators (1.3).

Now, the transformations (2.46) have already been found in Section 1.5. In fact the operator in the r.h.s. of (2.46) is a multiple of the unit matrix (i.e., has the same action on any component of the wave function), hence, in accordance with (1.45)

$$\psi''(x) - \psi(x)$$

(2.47)
Symmetries of Equations of Quantum Mechanics

where $x''$ is connected with $x$ by the Lorentz transformation inverse to (1.32). Substituting (2.46), (2.47) into (2.45) we obtain finally

\[
\psi(x) \rightarrow \psi'(x') = \exp \left( i \frac{S_{\mu\sigma}}{2} \theta^{\mu\sigma} \right) \psi(x''),
\]

or

\[
\psi(x) \rightarrow \psi'(x') = \exp \left( i \frac{S_{\mu\sigma}}{2} \theta^{\mu\sigma} \right) \psi(x).
\]

Formula (2.49) (together with the relations (1.32), (1.39) determining transformations of independent variables) gives the general form of Lorentz transformations of solutions of the Dirac equation. In contrast to the transformation law (1.45) of a scalar field formula (2.49) contains the matrix multiple $\exp(i S_{\mu\sigma}/2)$ mixing the components of the wave function (2.2).

For the sake of convenience, in using formula (2.49) it is desirable to represent $\exp(i S_{\mu\sigma}/2)$ as a polynomial in $S_{\mu\sigma}$. Starting from (1.44) and using (2.3), (2.18) we can prove the identity

\[
\exp \left( i \frac{S_{\mu\sigma}}{2} \theta^{\mu\sigma} \right) = \Lambda \left( \cos \theta + i \frac{\gamma^a \gamma_b \theta^{ab}}{\sqrt{2}} \sin \theta \right) + \Lambda \left( \cos \theta^* + i \frac{\gamma^a \gamma_b \theta^{ab}}{\sqrt{2}} \sin \theta^* \right),
\]

where

\[
\Lambda_a = \frac{1}{2}(1 \pm i \gamma_a), \quad \theta^* = 2[(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2]^{1/2}, \quad \theta^a = \frac{1}{2} \epsilon_{abc} \gamma^a \theta_b \pm i \theta_c.
\]

So we have obtained a transformation law for solutions of the Dirac equation. Taking particular values of the parameters $\theta_{\mu\sigma}$, it is not difficult to obtain the corresponding Lorentz transformation for the wave function. If e.g. the only nonzero parameter is $\theta_{12}$ we obtain from (2.50)

\[
\exp \left( i \frac{S_{\mu\sigma}}{2} \theta^{\mu\sigma} \right) = \cos \left( \frac{1}{2} \theta_{12} \right) - \gamma_1 \gamma_2 \sin \left( \frac{1}{2} \theta_{12} \right).
\]

For nonzero $\theta_{01}$, $\theta_{ab} = \theta_{02} = \theta_{03} = 0$ we have

\[
\exp \left( i \frac{S_{\mu\sigma}}{2} \theta^{\mu\sigma} \right) = \cosh \left( \frac{1}{2} \theta_{01} \right) - \gamma_0 \gamma_1 \sinh \left( \frac{1}{2} \theta_{01} \right).
\]

The transformations (2.49), (2.51) and (2.49), (2.52) correspond to rotations of the reference frame in the plane 1-2 and to Lorentz transformation (1.28) for $a=1$.

Let us adduce the explicit form of the dilatation and conformal transformations of solutions of the massless Dirac equation:

\[
\psi(x) \rightarrow \psi'(x') = \exp \left( - \frac{3}{2} \theta \right) \psi(x),
\]

(2.53)
Chapter 1. Symmetries of the Fundamental Equations ...

\[ \psi(x) \rightarrow \psi'(x') = (1 - 2 b^\mu x^\mu + b^\mu b^\nu x^\mu x^\nu)(1 - \gamma^\lambda \gamma^\sigma b^\lambda b^\sigma) \psi(x) \]  

(2.54)

where \( x' \) are given by relations (1.48).

Formulae (2.53) (and (2.54)) can be obtained by solving the Lie equations corresponding to the generators \( D \) and \( K'_\mu \) (2.45). Later in Subsections 3.5, 3.9, we present solutions of these equations and the explicit form of the conformal transformations for fields with arbitrary spin.

2.10. \( P, T, C \)-Transformations

Let us study the symmetry of the Dirac equation under the space inversion and time reflection. In analogy with (2.48) we will seek these transformations operators in the form

\[ \psi(x,0,x) \rightarrow P \psi(x,0,x) = r_1 \psi(x,0,x), \]
\[ \psi(x,0,x) \rightarrow T \psi(x,0,x) = r_2 \psi(-x,0,x) \]  

(2.55)

where \( r_1 \) and \( r_2 \) are some numerical matrices.

The operators defined in this way satisfy the following evident relations

\[ Pp_0 = p_0 P, \quad Pp_a = p_a P, \quad Tp_0 = -p_0 T, \quad Tp_a = p_a T, \]

hence the invariance condition (2.21) for the transformations (2.55) can be written in the form

\[ r_1 L(p_0, p) - L(p_0, -p) r_1 = \alpha_1 L(p_0, p), \]
\[ r_2 L(p_0, p) - L(-p_0, p) r_2 = \alpha_2 L(p_0, p) \]  

(2.56)

where \( L(p_0, p) \equiv L \) is the Dirac operator (2.1). The relations (2.56) reduce to the following equations for \( r_1, r_2, \alpha_1, \alpha_2; \)

\[ [r_1, \gamma^0] = [r_1, \gamma^a] = [r_2, \gamma^0] = [r_2, \gamma^a] = 0, \quad \alpha_1 = \alpha_2 = 0. \]  

(2.57)

The general solution of (2.57) is

\[ r_1 = \tau_1 \gamma^0, \quad r_2 = \tau_2 \gamma^0 \gamma^i \]  

(2.58)

where \( \tau_1 \) and \( \tau_2 \) are complex parameters. The requirement of unitarity of the transformations (2.55) reduces these parameters to phase multipliers

\[ \tau_1 = \exp(i \varphi_1), \quad \tau_2 = \exp(i \varphi_2), \quad \varphi_1, \varphi_2 \in \mathbb{R} \]  

(2.59)

So the Dirac equation is invariant under the discrete transformations (2.55), (2.57) (2.58) which complete the representation of the proper orthochronous Poincaré group to a representation of the complete Poincaré group. One more symmetry transformation of the Dirac equation can be given by the antiunitary operator \( C \):
where $r_3$ is a matrix satisfying the conditions (compare with (2.56))

$$r_3 L(p_{ij}, p) = L(p_{ij}, p) r_3 \alpha_3 L(p_{ij}, p),$$

where the asterisk denotes that all terms in the corresponding operator should be changed to complex conjugated ones.

Using (2.1), (2.4), (2.61) it is not difficult to find that

$$a_3 \equiv 0, \quad r_3 = i \tau_3 \gamma_2, \quad \tau_3 = \exp(i \varphi_3).$$

Moreover without loss of generality we can set $\tau_3 = 1$.

The transformation (2.60) is called a charge conjugation. The sense of this name can be understood by considering the Dirac equation for a particle interacting with an external electromagnetic field, where the transformation (2.60) is accompanied by a change of the electric charge of a particle.

Let us require that the charge-conjugated function (2.60) have the same behavior under the transformations $P$ and $T$ as non-conjugated wave function. This requirement imposes the following conditions on $\tau_1, \tau_2$:

$$\text{Re} \tau_1 = \text{Im} \tau_2 = 0,$$

so we have from (2.58), (2.59)

$$r_1 = i \gamma_0, \quad r_2 = -i \gamma_0 \gamma_4, \quad r_3 = 1.$$  

Thus the Dirac equation is invariant under the $P$, $T$- and $C$-transformations just as the KGF equation. The transformations (2.55), (2.60), (2.63) together with the Poincaré group transformations found in Subsection 2.9 form a symmetry group of (2.1), which we denote by $P(1,3)$. The projective representations of the group $P(1,3)$ are considered in Section 4.

### 3. MAXWELL’S EQUATIONS

#### 3.1. Introduction

Maxwell’s equations are one of the main foundations of modern physics. Describing a very extensive branch of physical phenomena, these equations are distinguished by their extremely simple and elegant form. But the source of this simplicity and elegance lies in the remarkably rich symmetry of Maxwell’s equations.

The investigation of the symmetry of Maxwell’s equations has a long and glorious history. In 1893, having written these equations in the vector notations, Heaviside [219] pointed out that they are invariant under the change

$$E \rightarrow H, \quad H \rightarrow -E.$$
Chapter 1. Symmetries of the Fundamental Equations ...

where \( E \) and \( H \) are the vectors of the electric and magnetic field strengths. Larmor [272] and Rainich [368] found that this symmetry can be generalized to the family of one-parameter transformations (we will call them Heaviside-Rainich transformations)

\[
E \to E \cos \theta + H \sin \theta, \\
H \to H \cos \theta - E \sin \theta .
\]

(3.1)

Larmor [288], Poincaré [361, 362] and Einstein [90] obtained the most fundamental result connected with the symmetry of Maxwell’s equations, which paid a revolutionary role in physics. It was Lorentz who first found all possible linear transformations of space and time variables (and the corresponding transformations for \( E \) and \( H \)) leaving Maxwell’s equations invariant.

Augmenting and generalizing Lorentz’s results, Poincaré showed that in the presence of charges and currents Maxwell’s equations are invariant under Lorentz transformations. Poincaré first established and studied the main property of these transformations, i.e., their group structure, and he showed that “the Lorentz transformations represent a rotation in a space of four dimensions whose points have coordinates \((x, y, z, \sqrt{-1} t)\)” [361]. Thus, Poincaré combined space and time into a single four-dimensional space-time at least three years before Minkowski [306].

In Einstein’s famous work [90] which played an outstanding role in the history of modern science it was also established that Maxwell’s equations with currents and charges are invariant under Lorentz transformations. On the basis of their study of the symmetries of Maxwell’s equations, Lorentz, Poincaré and Einstein created the foundations of new relativity theory. Moreover new relativity principle (differing from the Galilei relativity principle) was created in physics.

The next important step in studying the symmetry of Maxwell’s equations was made by Bateman [22,23] and Cuningham [70] who proved that these equations are invariant under the inversion transformation

\[
x_{\mu} \to x_{\mu} / x^{\nu} x_{\nu},
\]

(supplemented by the corresponding transformation of the dependent variables) from which follows invariance under the conformal transformations (1.50). In fact Bateman proved that the conformal group invariance determines the maximal symmetry of Maxwell’s equations with currents and charges [22].

Not long ago the group-theoretical analysis of Maxwell’s equations was done using the classical Lie approach [71, 226]. Incidentally it was proved rigorously that the maximal local invariance group of Maxwell’s equations for the electromagnetic field in vacuum is the 16-parameter group \( C(l,3) \otimes H \) where \( H \) is the one-parameter group of Heaviside-Larmor-Rainich transformation (3.1).

But the transformations mentioned above do not exhaust all the symmetries
of Maxwell's equations. These equations possess hidden (nongeometric) symmetries which are not connected with transformations of independent variables [120, 144]. The main property of this "new" symmetry is that the basis elements of the corresponding IA do not belong to the class $M_1$ (in contrast to the classical Lie approach) but are integro-differential operators (see Chapter 4).

The classical (Lie) symmetry of Maxwell's equations is discussed in the present section.

### 3.2. Various Formulations of Maxwell's Equations

Maxwell's equations for the electromagnetic field in vacuum are usually written in the following form

\begin{align}
\mathbf{p} \times \mathbf{E} &= -i \frac{\partial \mathbf{H}}{\partial t}, \\
\mathbf{p} \times \mathbf{H} &= -i \frac{\partial \mathbf{E}}{\partial t}, \\
\mathbf{p} \cdot \mathbf{E} &= 0, \\
\mathbf{p} \cdot \mathbf{H} &= 0
\end{align}

where $\mathbf{E}$ and $\mathbf{H}$ are the vectors of the electric and magnetic field strengths.

In the presence of currents and charges Maxwell's equations take the form

\begin{align}
\frac{i}{c} \frac{\partial \mathbf{E}}{\partial t} &= -\mathbf{p} \times \mathbf{H} - j, \\
\mathbf{p} \cdot \mathbf{E} &= -\sigma_0 j
\end{align}

\begin{align}
\frac{i}{c} \frac{\partial \mathbf{H}}{\partial t} &= \mathbf{p} \times \mathbf{E}, \\
\mathbf{p} \cdot \mathbf{H} &= 0
\end{align}

where $j=(j_0, j_1, j_2, j_3)$ is the four-vector of the electric current and the constant of electromagnetic interaction has been taken equal to one.

The vector formulation of Maxwell's equations given above was proposed by Hertz and Heaviside. Besides (3.2), (3.3) we shall consider other representations of these equations suitable for the investigation of symmetries.

Let us denote by $\Phi$ the following vector-function

\[
\Phi = \begin{pmatrix} E_1 & E_2 & E_3 & H_1 & H_2 & H_3 \end{pmatrix}
\]

where $E_a$ and $H_a$ are the components of the electric and magnetic field strengths. The equation (3.2) may be represented in the form

\[
\hat{L}_1 \Phi = 0, \quad \hat{L}_2 \Phi = 0
\]

where

\[\hat{L}_1 = i \frac{\partial}{\partial t} - \sigma_0 \sigma_2 \mathbf{p}, \quad \hat{L}_2 = (Z^{ab} + i e^{ab} \sigma_1) \mathbf{p}
\]
Chapter 1. Symmetries of the Fundamental Equations ...

\[ Z^{ab} - 28^{ab} - S_a S_b - S^a S^b, \]

\[ S_a = \begin{pmatrix} \hat{S}_a & 0 \\ 0 & \hat{S}_a \end{pmatrix}, \quad \hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

and \( \sigma_2 \) is the 6x6 Pauli matrix commuting with \( \hat{S}_a \) (i.e., the matrix obtained from (2.5) by change 1 and 0 by the 3x3 unit and zero matrices).

It is not difficult to make sure the equations (3.2) and (3.5) coincide componentwise for any \( a=1,2,3 \).

The formulation (3.4)-(3.6) is mainly used in the following by the analysis of hidden symmetries of Maxwell’s equations.

Maxwell’s equations can be written in the form of a first-order covariant equation also

\[ (\beta_\mu \rho^\mu - \beta_\mu \kappa) \psi = 0 \]  

(3.7)

where \( \beta_\mu \) are irreducible 10x10 matrices of Kemmer-Duffin-Petiau (KDP) in the representation (6.22),

\[ \psi = \text{column}(E_1, E_2, E_3, H_1, H_2, H_3, A_1, A_2, A_3). \]  

(3.8)

Substituting (3.8), (6.22), (6.24) into (3.7) we obtain the system

\[ \frac{\partial A_\mu}{\partial x_\mu} + \frac{\partial A_0}{\partial x^\mu} = -\kappa E_\mu, \quad \kappa H = -ip \times A, \]

\[ i\frac{\partial E}{\partial t} = -p \times H, \quad p \cdot E = 0, \]

from which the equations (3.2) follow immediately.

Maxwell’s equations in the form (3.7) were discussed by Fedorov [95] and Bludman [38].

Maxwell’s equations with currents and charges can also be written in the covariant form. Denoting

\[ \psi = \text{column}(E_1, E_2, E_3, H_1, H_2, H_3, J_1, J_2, J_3, J_0) \]  

(3.9)

it is possible to represent (3.3) as the following system [154, 157]

\[ L_1 \psi = 0, \quad L_1 = (1 - \beta_0) (\beta^\nu p_\nu + 1), \]

\[ L_2 \psi = 0, \quad L_2 = \beta^\nu p_\nu \beta_\nu, \]  

(3.10)

where \( \beta_\mu \) are the matrices (6.22). The other possibility is to use the 16x16 KDP matrices which make it possible to write Maxwell’s equations with currents and
3.3. The Equation for the Vector-Potential

Let us consider one more formulation of Maxwell’s equations which makes use of a four-component function (vector-potential) $A=(A_0,A_1,A_2,A_3)$ connected with $E$ and $H$ by the relations

$$\mathbf{H}=\mathbf{i}p\times\mathbf{A}, \quad \mathbf{E}=-\frac{\partial \mathbf{A}}{\partial t}-\mathbf{i}pA_0. \tag{3.11}$$

Substituting these expressions into (3.3) we obtain the following equations for $A_\mu$

$$p_\nu p^\nu A_\mu - p_\mu p^\nu A_\nu = 0. \tag{3.12}$$

So instead of eight equations for $E$, $H$ one may solve the system (3.12) for $A_\mu$ and then find the vectors of the magnetic and electric field strengths using formulae (3.11). Moreover the system (3.12) can be simplified using the freedom in the choice of $A_\mu$. Actually the relations (3.11) are invariant with respect to the substitution $A_\mu \rightarrow A_\mu - ip_\mu \Phi \tag{3.13}$

where $\Phi$ is an arbitrary function. This is why the additional constraint (Lorentz gauge) is imposed usually on $A_\nu$:

$$p_\mu A^\mu = 0 \tag{3.14}$$

which reduces (3.11) to the system of noncoupled equations

$$p_\nu p^\nu A_\mu = 0. \tag{3.15}$$

As before, the equations (3.14), (3.15) determine the vector-potential up to the transformation (3.13) where $\Phi$ is a function satisfying the equation $p_\mu p^\mu \Phi = 0$. Using such transformations (called a gauge transformations of the second kind) it is possible to arrange that $A_0 = 0, \quad \mathbf{p} \cdot \mathbf{A} = 0. \tag{3.16}$

The conditions (3.16) (which in contrast to (3.14) are not relativistically invariant and can be imposed in a fixed reference frame only) are called the Coulomb gauge.

To conclude this subsection we notice that by using the connection of the vectors $E$ an $H$ with the four-vector $A_\mu$ given by formulae (3.11) Maxwell’s equations can be written in the tensor form

$$p_\nu F^{\nu\mu} = 0, \quad p_\nu \tilde{F}^{\nu\mu} = 0 \tag{3.17}$$

where
Chapter 1. Symmetries of the Fundamental Equations ...

\[ F^{\mu\nu} = (p^\mu p^\nu - p^\nu p^\mu), \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}. \]

3.4. The IA of Maxwell’s Equations in the Class \( M_1 \)

To investigate the local symmetries of Maxwell’s equations, we shall use the covariant formulation of these equations given in (3.10).

The problem of finding the IA of the equations (3.10) in the class \( M_1 \) can be formulated in analogy with the corresponding problem for the Dirac equation considered in Section 2. The only distinction is that Maxwell’s equations (3.10) are represented as a result of action of two linear operators on a vector \( \psi \) while the Dirac equation is determined by the only operator \( L \) of (2.1).

In analogy with Section 2 let us formulate a definition of a SO \( Q \in M_1 \) for Maxwell’s equations.

**DEFINITION 3.1.** A linear differential operator

\[ Q = A^\mu p_\mu + B, \quad A^\mu \in F, \quad B \in G^{10} \]  \hspace{1cm} (3.18)

is a SO of Maxwell’s equations (3.10) in the class \( M_1 \) if

\[ [Q, L_1] = \beta_1 \rho L_1, \beta_2 L_2, \quad \beta_3^0 \in G^{10}, \]

\[ [Q, L_2] = \lambda_1 L_1, \lambda_2 L_2, \quad \lambda_3^0 \in G^{10}, \]  \hspace{1cm} (3.19)

where \( L_1, L_2 \) are the operators (3.10) and the symbol \( G^{10} \) denotes the linear space of 10x10 matrices whose matrix elements belong to \( F \) (see Subsection 1.1).

As in the case of the Dirac equation, SOs transform solutions into solutions and the set of SOs forms a Lie algebra.

The main assertion concerning the symmetry of Maxwell’s equations can be formulated as follows

**THEOREM 3.1.** Maxwell’s equations (3.10) are invariant under the 15-dimensional Lie algebra which is isomorphic to the algebra \( AC(1,3) \). The basis elements of this IA can be taken in the form of (2.22), (2.42) where

\[ S_{\mu\nu} = i(\beta_\mu \beta_\nu - \beta_\nu \beta_\mu), \quad K = 3 - \beta_4^2. \]  \hspace{1cm} (3.20)

The Lie algebra spanned on the basis (2.22), (2.42), (3.20) is the maximal IA of Maxwell’s equations in the class \( M_1 \).

**PROOF.** Using the relations

\[ \beta_4^2 = \beta_1, \quad (1 - \beta_4^2) \beta_4 = -\beta_4^2, \quad \beta_4^3 + 3 = -\beta_4^2, \quad [\beta_\mu, S_{\nu\lambda}] = i(g_{\mu\nu} \beta_\lambda - g_{\mu\lambda} \beta_\nu) \]  \hspace{1cm} (3.21)

(which follows from (6.20), (6.23)) one verifies that the operators (3.10), (2.22), (2.42), (3.20) satisfy the invariance conditions
Symmetries of Equations of Quantum Mechanics

\[ [P_\mu, L_\alpha] = [J_{\mu\nu}, L_\alpha] = [D_\mu, L_\alpha] = [K_\mu, L_\alpha] = 0 \]  \hspace{1cm} (3.22)

which coincide with (3.19) for \( \beta^\alpha_0 = \lambda^\alpha_0 = 0 \).

Using relations (3.21) it is not difficult to verify that the operators (2.22), (2.42), (3.20) satisfy the commutation relations (1.14), (1.19) which characterize the algebra \( AC(1,3) \).

We see that the operators (2.22), (2.42), (3.20) actually form the IA of Maxwell’s equations. The proof that this IA is maximal in the class \( M \), will be given in Section 20 as a part of the solution of a more complicated problem.

**COROLLARY 1.** Each of the equations (3.10) is invariant under the algebra \( AC(1,3) \).

This statement follows from the commutativity of each of the operators \( L_1 \) and \( L_2 \) with the basis elements of the algebra \( AC(1,3) \).

**COROLLARY 2.** Maxwell’s equations for the electromagnetic field in vacuum are invariant under the 16-dimensional Lie algebra whose basis elements are given in (2.22), (2.42) (where \( S_{\rho\sigma}, K \) are the matrices (3.20)) and in (3.23):

\[ F = \beta_4 \]  \hspace{1cm} (3.23)

Indeed, Maxwell’s equations without currents and charges can be represented in the form of the system (3.10) with the additional constraint

\[ L_4 \psi = (1 - \beta^4_0) \psi = 0 \]  \hspace{1cm} (3.24)

The matrix \( 1 - (\beta^4_0) \) commutes with any element of the algebra \( AC(1,3) \) and, moreover, the relations

\[ [L_1, F] = -iL_2, \quad [L_2, F] = L_1 - L_3 - FL_2, \quad [L_3, F] = 0 \]

are satisfied. So the generators of the conformal group and the operator \( F \) are the SOs of the system (3.10), (3.24).

Thus the symmetry of Maxwell’s equations for the electromagnetic field in vacuum turns out to be broader than in presence of currents and charges. It is connected with the fact that the equations (3.3) includes the current in a nonsymmetric way (due to the absence of the magnetic charge). As a result Maxwell’s equations with currents and charges are not invariant under the Heaviside-Larmor-Rainich transformations (which are generated by the operator \( F \), as will be shown in the following).

### 3.5. Lorentz and Conformal Transformations

A direct consequence of symmetry of Maxwell’s equations under the IA found above is the invariance under the group of conformal transformations. Here we will show the explicit form of these transformations and then generalize these
transformations to the case of arbitrary spin (see Subsection (3.9)).

The deduction of Lorentz and conformal transformations for the electromagnetic field is analogous to the one given in Subsection 2.9 where the corresponding transformations for the Dirac equation solutions are found. Therefore we will omit the details and mention only the essential points.

As in Section 2, it is not difficult to show that the transformations of independent variables generated by the operators (2.22), (2.42), (3.20) have the form

\[ x \to x' = x + a, \quad x_0 \to x_0' = x_0 + a_0, \quad (3.25) \]

\[ x \to x'' = \cos \theta x - \frac{\theta x_0}{\theta} \sin \theta + \frac{\theta (\theta' x)}{\theta'} (1 - \cos \theta), \quad x_0 \to x_0'' = x_0', \quad (3.26) \]

\[ x \to x''' = -\frac{\lambda x_0}{\lambda} \sinh \lambda - \frac{\lambda (\lambda \cdot x)}{\lambda^2} (\cosh \lambda - 1), \quad (3.27) \]

\[ x_0 \to x_0''' = x_0 \cosh \lambda - \frac{x \cdot \lambda}{\lambda} \sinh \lambda, \]

\[ x_\mu \to x_\mu''' = \exp (\frac{-\lambda_0}{\lambda}) x_\mu, \]

\[ x_\mu \to x_\mu''' = \frac{x_\mu - b_\mu v^\nu x_\nu}{1 - 2b_v v^\nu + b_v b^\nu x_\nu x^\nu}, \quad (3.29) \]

where

\[ \theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}, \quad \lambda = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}, \]

\[ \theta_0, b_\mu, \lambda_\nu \] and \( a_\mu \) are real parameters.

Formulae (3.25)-(3.29) give the displacements of the time and space variables, rotation by an angle \( \theta \) around the axis \( \theta/\theta' \), proper Lorentz transformation (3.27), scale transformations (3.28) and conformal transformations (3.29). Moreover (3.25)-(3.27) are particular cases of the general Lorentz transformation (1.36), (1.38), (1.39).

In order for Maxwell’s equations to be invariant under transformations (3.25)-(3.29) it is necessary to transform simultaneously the vector-function (3.9) in accordance with the following law [154, 157]

\[ \psi(x) \to \psi'(x') = \psi(x), \quad (3.30) \]

\[ \psi(x) \to \psi''(x'') = \exp \left( \frac{i}{2} \varepsilon_{abc} S_{abc} \theta \right) \psi(x), \quad (3.31) \]

\[ \psi(x) \to \psi'''(x''') = \exp (iS_{0\mu} \lambda_\mu) \psi(x), \quad (3.32) \]

\[ \psi(x) \to \psi''''(x''') = \exp (-iK \lambda_0) \psi(x), \quad (3.33) \]
Formulae (3.30)-(3.32) are distinct from the corresponding formulae giving the transformations of solutions of the Dirac equation only by the realization of the matrices $S_{\mu\sigma}$ (compare with (2.47), (2.49)). It is not difficult to make sure also that the dilatation and conformal transformations for the Dirac spinors given in (2.53), (2.54) also can be represented in the form (3.33), (3.34) where $K=3/2$, $S_{\mu\nu}=[\gamma_{\mu},\gamma_{\nu}]/4$, $\gamma_{\mu}$ are the Dirac matrices.

Using relations (3.30)-(3.34) it is not difficult to obtain the transformation law for the vectors $H$, $E$ and four-vector of current $j$. Indeed bearing in mind the identity

$$(S_{\mu\nu}d^{\mu\nu})^3-(d_{\lambda\sigma}d^{\lambda\sigma}d^{\lambda\sigma}d_{\sigma\lambda})=S_{\mu\nu}d^{\mu\nu}$$

As to the operator $[\varphi(x,b)]^K$ (where $K=3-(\beta_3)^2$ is a Hermitian matrix) it can be represented in the form

$$[\varphi(x,b)]^K=\sum [\varphi(x,b)]^\sigma \Lambda_{\sigma},$$

where $\sigma$ are eigenvalues of the matrix $K$ (equal to 2 or 3) and $\Lambda_{\sigma}$ are the projection operators corresponding to these eigenvalues

$$\Lambda_1=\beta_3^2, \quad \Lambda_2=\beta_3^2\Lambda_2, \quad \Lambda_3=1-\beta_3^2.$$
obtain

$$S_a \rightarrow i e_{abc} \hat{S}_a \hat{\beta}_b \hat{\beta}_c$$,

where $\hat{S}_a$ are the 3×3 matrices (3.6), $S_{\mu \sigma}$ are the 4×4 matrices (1.42), $I$ is the 3x3 unit matrix, the dots denote zero matrices of an appropriate dimension.

Substituting (3.9), (3.36)-(3.39) into (3.30)-(3.34) we obtain the following transformation laws

$$E \rightarrow E', \quad H \rightarrow H', \quad j \rightarrow j' = j,$n

$$j_0 \rightarrow j_0', \quad (E, H, j) \rightarrow (E', H', j') =$$

$$-(E, H, j) \cos \theta - \theta \times (E, H, j) \frac{\sin \theta}{\theta} + \theta \cdot (\theta \cdot (E, H, j)) \frac{1 - \cos \theta}{\theta^2}.$$

$$E \rightarrow E'' = E \cosh \lambda - \lambda \times E \frac{\sinh \lambda}{\lambda} + \lambda (\lambda \cdot E) \frac{1 - \cosh \lambda}{\lambda},$$

$$H \rightarrow H'' = H \cosh \lambda - \lambda \times H \frac{\sinh \lambda}{\lambda} + \lambda (\lambda \cdot H) \frac{1 - \cosh \lambda}{\lambda},$$

$$j_0 \rightarrow j_0'' = j_0 \cosh \lambda - \lambda j_0 \frac{\sinh \lambda}{\lambda}, \quad j \rightarrow j'' = j - \lambda j_0 \frac{\sinh \lambda}{\lambda} - \lambda (\lambda \cdot j) \frac{1 - \cosh \lambda}{\lambda},$$

$$((E, H) \rightarrow (E'', H'')) = \exp(-2\lambda_0) (E, H), \quad j \rightarrow j'' = \exp(-3\lambda_0) j.$$

$$E \rightarrow E^v = \varphi [(b^v x_0 - 1) E + 2(b^v x_0 - 1) b_x V \times H - x_0 b \times H - b \times E \times E + b \cdot E] + b \times x V (x_0 b - b \times H - b \times E - b \times x V \times E - b \times E - b \times x V \times E - b \times E + b \times E),$$

$$H \rightarrow H^v = \varphi [(b^v x_0 - 1) H + 2(b^v x_0 - 1) x_0 b \times E - b \times H - b \times E \cdot b \times H - b \times E \cdot b \times H - b \times E - b \times E - b \times H - b \times E - b \times H - b \times E] + b \times x V (b \times x V \times H + b \times x V \times E - x_0 b - b \cdot E - (b \times x V \times E - x_0 b - b \cdot E) - b \times x V \times E - x_0 b - b \cdot E),$$

$$j_0 \rightarrow j^v \varphi^2 \{ b_0 (1 - 2 x_0 b) + x_0 b \cdot b \} j^v + 2(x_0 b - b \times x V \times E) b_j^v.$$

37
Symmetries of Equations of Quantum Mechanics

Formulae (3.40) give the explicit form of the transformations of the conformal group for the vectors of the electric and magnetic field strengths and the current four-vector. The corresponding transformations for \( t \) and \( x \) are given in (3.25)-(3.29). These formulae are quite complicated but simplify considerably if only one of the parameters \( a_\mu, b_\mu, \theta_a, \lambda_a \) is nonzero. For instance setting \( b_1=b_2=b_3=0, b_0=b \) in the transformation law (3.40) for \( E \) we obtain

\[
E'= (1-2bx^\mu\sigma b^\mu x^\sigma)[(1-2bx_0)^2E-b\cdot x (x\cdot E)]+2b(1-2bx_0)x\times H. \tag{3.41}
\]

Analogous transformations for \( H \) can be obtained from (3.41) by the change \( E \rightarrow H, H \rightarrow -E \).

Formulae (3.40), (3.41) may be useful for various applications - e.g. for the construction of nonlinear generalizations of Maxwell’s equations, being invariant under the conformal group.

3.6. Symmetry Under the \( P-, T- \) and \( C- \)Transformations

Invariance under the transformations considered above does not exhaust all symmetry properties of Maxwell’s equations. We will see later that these equations are invariant also with respect to nonlocal transformations not connected with geometrical space-time symmetries.

But there exist discrete symmetry transformations of dependent and independent variables which we not considered in the previous subsection. There are the transformations of time reflection and space inversion. In fact it is not difficult to verify that Maxwell’s equations do not change their form under the transformation

\[
x_0 \rightarrow x_0, \quad x \rightarrow -x, \quad E \rightarrow -E, \quad H \rightarrow H, \quad j \rightarrow -j, \quad j_0 \rightarrow j_0. \tag{3.42}
\]

There is one more symmetry which is trivial for Maxwell’s equations describing real field, but is not admitted by some other equations for massless fields. This is the charge conjugation transformation

\[
E \rightarrow E^*, \quad H \rightarrow H^*, \quad j \rightarrow j^*. \tag{3.43}
\]

Using the vector-function (3.9) it is possible to represent the transformations (3.42), (3.43) in the form (2.55), (2.60), where \( r_1=1-2b_0^2, r_2=(1-2b_0^2)(1-2b_0^2), r_3=1 \). The invariance of (3.10) under these transformations follows from the relations

\[
[P, L_1] \rightarrow [P, L_1], \quad [T, L_1] \rightarrow [T, L_1], \quad [C, L_1] \rightarrow [C, L_1], \quad [C, L_2] \rightarrow 0
\]

where \( L_1 \) and \( L_2 \) are the operators (3.10). The operators \( P, T \) and \( C \) satisfy the relations (1.54) just as in the cases of the KGF and Dirac equations.

38
3.7. Representations of the Conformal Algebra Corresponding to a Field with Arbitrary Discrete Spin

We have shown in the above that the IA of Maxwell’s equations in the class $M_1$ is the 15-dimensional algebra $AC(1,3)$. This algebra and their representations play a fundamental role in modern theoretical physics.

In this section we continue discussing the conformal symmetry of Maxwell’s equations and other relativistic wave equations for massless fields. It will be shown that the conformal group generators $K_µ$ and $D$ are expressed via the Poincaré group generators $P_µ$ and $J_µσ$ on the set of solutions of such equations

**DEFINITION 3.2.** We say that an equation

$$Lψ=0 \quad (3.44)$$

where $L$ is some linear differential (or integro-differential) operator, is Poincaré-invariant and describes a massless field with discrete spin if on the set of its solutions a representation of the Poincaré algebra is realized corresponding to zero eigenvalues of the Casimir operators (2.36):

$$P_µP_µψ=0, \quad W_µW_µψ=0 \quad (3.45)$$

In other words if an IA of some equation is given by the operators $P_µ, J_µσ$ satisfying (1.14), (3.45) than we call it an equation for a massless field with discrete spin. It appears that any such equation is also invariant under the more extensive (conformal) algebra as it is stated in the following theorem.

**THEOREM 3.2** [143]. Any Poincaré invariant equation for a massless field with discrete spin is invariant under the conformal algebra whose basis elements are given by the operators $P_µ, J_µσ$, forming the algebra $AP(1,3)$, and the operators $D, K_µ$ expressed via $P_µ, J_µσ$ by the relations

$$D=1 \frac{1}{2} \left[ P_µP_µJ_0,0 \right], \quad K_0=1 \frac{1}{2} \left[ P_0, J_0,0 + Λ^2 - \frac{1}{2} \right], \quad K_µ=i[K_0, J_0,0], \quad (3.46)$$

where

$$Λ=\frac{JP}{P}, \quad P=\sqrt{P_1^2 + P_2^2 + P_3^2}.$$

**PROOF.** Since $P_µ$ and $J_µσ$ by definition form an IA of equation (3.44), the operators (3.46) expressed via $P_µ, J_µσ$ are also included in the IA of this equation. Further, by assumption, $P_µ$ and $J_µσ$ satisfy the commutation relations (1.14), and the proof of the theorem reduces to verifying the validity of the relations (1.19) for the operators (3.46). Verification of these relations requires straightforward but
cumbersome calculations which can be done using the relations
\[ P_0^2 - P^2 = \frac{P_0}{P} \Lambda P_\mu, \quad [A, J_{\mu \nu}] = [A, P_\mu] = 0, \]
\[ \left[ \frac{1}{P^2}, J_{ab} \right] = 0, \quad \left[ \frac{1}{P^2}, J_{0a} \right] = -2i P_0^a P_a. \]

(3.47)

Thus, formulae (3.46) give explicit expressions for basis elements of the algebra \( AC(1,3) \) in terms of the operators \( P_\mu, J_{\mu \sigma} \) contained in its subalgebra \( AP(1,3) \).

Theorem (3.2) has constructive character since it enables to find the explicit form of the generators \( D \) and \( K_\mu \) starting from given basis elements of the Poincaré algebra. Thus, proceeding from the generators \( P_\mu, J_{\mu \sigma} \) in the Lomont-Moses representation (see (4.50) for \( n_1=n_2=0 \)) we obtain, by formulae (3.46), the generators of the conformal group in the Bose-Parker representation [50]. Other representations are considered in Section 4 and the next subsection.

In conclusion we notice that the algebra \( AC(1,3) \) is isomorphic to the Lie algebra of the group \( O(2,4) \) (the group of pseudoorthogonal matrices conserving the vector length in the (2+4) Minkowski space. This isomorphism can be established by the following relations
\[ J_{\mu \nu} \leftrightarrow S_{\mu \nu}, \quad P_\mu \leftrightarrow S_{55}, \quad K_\mu \leftrightarrow S_{55}, \quad D \leftrightarrow S_{45} \]

(3.48)

where \( P_\mu, J_{\mu \nu}, K_\mu, D \) are the basis elements of the algebra \( AC(1,3) \) satisfying the commutation relations (1.14), (1.19), and \( S_{\mu \nu} \) are the generators of the group \( O(2,4) \) satisfying the relations
\[ [S_{\mu \nu}, S_{\kappa \lambda}] = i(g_{\mu \kappa} S_{\nu \lambda} + g_{\nu \kappa} S_{\mu \lambda} - g_{\mu \lambda} S_{\nu \kappa} - g_{\nu \lambda} S_{\mu \kappa}), \]
\[ g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = -g_{55} = 0, \quad g_{0m} = 0, \quad m \neq n. \]

(3.49)

The existence of this isomorphism means that the problem of the description of the representations of the algebra \( AC(1,3) \) reduces to the description of the representations of the algebra \( AO(2,4) \).

3.8. Covariant Representations of the Algebras \( AP(1,3) \) and \( AC(1,3) \)

Of particular interest is the use of the algorithm given in Theorem 3.2 in the case when the Poincaré group generators have the covariant form (2.22) inasmuch as such representations are used for description of actual physical fields.

Here we will consider such representations, restricting ourselves to the case when the matrices \( S_{\mu \nu} \) realize the finite dimensional IR \( D(j \tau) \) of the algebra \( AO(1,3) \) (see Section 4). To simplify the discussion we will use the realization of (2.22) in the momentum representation where \( p_\mu \) are independent variables, \( x_\mu = i\partial \phi^\mu \).

Let \( \psi \) be an arbitrary solution of (3.44) describing a field with zero mass and
Chapter 1. *Symmetries of the Fundamental Equations...*

discrete spin, and the corresponding Poincaré generators have the form (2.22). Then by definition $\psi$ satisfies conditions (3.45) which take the following form

$$ (p_0^2 - p^2)\psi = 0 \quad (3.50) $$

$$ W^\mu \psi = S^\rho \rho^\nu p_\nu p^\rho \psi = 0 \quad (3.51) $$

It is convenient to rewrite (3.51) in the form (see (3.49))

$$ W^\mu \psi \equiv \frac{1}{2} \epsilon_{\nu \rho \sigma} p^\nu S^\rho \psi - \frac{p_0}{p} \Lambda_p^\mu \psi, \quad (3.52) $$

where $\Lambda$ is the helicity operator (3.46). When $\mu=0$, the equation (3.52) turns into identity according to (3.50), but when $\mu=a$, $a \neq 0$, this equation takes the form

$$ (p_0 S_\nu - p_\nu S_0) \psi = \epsilon_{abc} p_a^\nu, \quad (3.53) $$

Using relations (4.59), (7.19) given below for $S \rightarrow j$ and $S \rightarrow \tau$ one concludes that the equation (3.53) is equivalent to the following system

$$ S_0^a p_a \psi = i(j-\tau) p_0 \psi, \quad (3.54) $$

where $j$ and $\tau$ are integers or half integers enumerating the IR of the algebra $AO(1,3)$.

So if $\psi$ is a covariant massless field with discrete spin than it satisfies the equations (3.50), (3.54). In the case $j=0$, $\tau=1/2$ (or $j=1/2$, $\tau=0$) the equations (3.54) coincide and reduce to the Weyl equation, and for $j=1$, $\tau=0$ these equations are equivalent to Maxwell’s equations.

Let us find the operators $D$ and $K_\mu$ (3.46) corresponding to the generators (2.22). According to the reasons given below it is sufficient to restrict ourselves to the case when the matrices $S^\mu_\nu$ realize the representation $D(s 0)$ or $D(0 s)$.

Substituting (2.22) into (3.46) we obtain after a simple calculation

$$ D = \frac{1}{2} \left[ \frac{P_0}{P^2}, J_{0a}, J_0 \right] \equiv 2\gamma_0 p_\mu - x_\mu p_0 + i(s+1) + A(p_0^2 - p^2) + B(S_{0a} p_a - isp_0), \quad s=j+\tau, \quad (3.55a) $$

$$ K_0 = \frac{1}{2} \left[ \frac{P_0}{P^2}, J_{0a}, J_0 + \Lambda - \frac{1}{2} \right] \equiv 2\gamma_0 D - x_\mu x^\mu p_0 - 2S_{0a} x_a + C_0(p_0^2 - p^2) + E_0(S_{0a} p_a - isp_0), \quad (3.55b) $$

$$ K_0 \equiv [K_0, J_{0a}] = 2\gamma_0 D - x_\mu x^\mu p_a + 2S^a_\mu x^\mu p_0 + C_a(p_0^2 - p^2) + E_a(S_{0a} p_a - isp_0), $$

where $A, B, C_\mu, E_\mu$ are some functions of $p_\mu$, $x_\mu$ the exact form of which is not essential because the corresponding terms are equal to zero on the set of solutions of (3.50),

41
where $k = 1 + s$ according to $(3.50)$, $(3.54)$, $(3.55)$. 

In the case when $S_{\mu\sigma}$ belong to an arbitrary finite dimensional representation of the algebra $AO(1,3)$, formulae $(2.22)$, $(3.56)$ give an explicit form of the conformal group generators in the Mack-Salam [289] representation. For instance, the basis elements of the algebra $AC(1,3)$ have this form on the set of solutions of the massless Dirac, Weyl and Maxwell equations (compare with $(2.22)$, $(2.42)$).

Let us sum up. According to Theorem 2.3 any Poincaré-invariant equation for a massless field with discrete spin turns out to be invariant under the conformal algebra which, however, is generally realized in the class of nonlocal (integro-differential) operators.

In this section we have verified that if one starts from a covariant representation of the algebra $AP(1,3)$ then the algorithm given in Theorem 2.3 leads to the conformal algebra representation in the covariant form of Mack and Salam [289]. Thus it has been established that the operators $D$ and $K_\mu$ in the covariant realization $(3.56)$ can be expressed via the Poincaré group generators according to $(3.46)$. Of course this statement is valid only for the representations satisfying Definition 3.2, i.e., corresponding to zero mass and discrete spin. In particular it is valid for representations realized on the sets of solutions of the massless Dirac and Maxwell equations, i.e., the corresponding generators of dilatation $D$ and conformal transformations $K_\mu$ can be expressed via the Poincaré group generators $P_\mu$, $J_{\mu\sigma}$ according to relations $(3.46)$. This seems to be an explanation of the known fact that the conformal symmetry of Maxwell’s equations does not lead to new conservation laws in comparison with the Poincaré invariance, see [32, 358].

Let us explain why we restrict ourselves to considering such representations of the matrices $S_{\mu\sigma}$ of $(2.22)$ which have the type $D(0 \tau)$ or $D(j 0)$. As was established by Bracken [52] the operators $(2.22)$, $(3.56)$ form an IA of the d’Alembert equation $(3.50)$ only in the case when $j\tau = 0$ where $j$ and $\tau$ are indices labelling the representation of the group $O(1,3)$ realized by $S_{\mu\sigma}$. So only for such types representations it is possible to obtain a covariant realization of the generators $D$ and $K_\mu$ using Theorem 2.3 (another representations do not correspond to a field with zero mass [52]). If $j$ and $\tau$ are nonzero then formulae $(2.22)$, $(3.46)$ also define a representation of the algebra $AC(1,3)$ but the corresponding $D$ and $K_\mu$ do not belong to the class $M_\ell$ (for realizations in the coordinate representation).
Chapter 1. Symmetries of the Fundamental Equations ...

3.9. Conformal Transformations for Any Spin

In conclusion we should note that the conformal group transformations generated by the operators (2.22), (3.56), where $S_{\mu\sigma}$ are arbitrary matrices satisfying the algebra $AO(1,3)$, can be represented in the form (3.30)-(3.34). It is not difficult to make sure that the transformations (3.30)-(3.34) satisfy the Lie equations (1.47) for any matrices $S_{\mu\sigma}$ forming the algebra $AO(1,3)$, and so give the explicit form of the conformal group transformations for any representation of the Mack-Salam type. The other (but equivalent to (3.34)) realization of the conformal transformation matrix for arbitrary spin was given in [371].

For every particular representation $D(j, \tau)$ of the algebra $AO(1,3)$ the exponential of the matrices $S_{\mu\sigma} \in D(j, \tau)$ reduces to a finite sum of powers of these matrices, since \( \prod (S_{\mu\sigma} - \lambda) = 0 \) where $\lambda$ are the eigenvalues of $S_{\mu\sigma}$. Thus an explicit form of the transformations (3.30)-(3.34) is easily calculated for any $S_{\mu\sigma} \in D(j, \tau)$.

We notice that the transformations (3.30)-(3.34) can be considered as a local representation of the conformal group only, since we encounter not only the problem of defining the domain of the functions $\psi'(x')$, $\psi''(x'')$, ... but also the fact that the expressions (3.29), (3.34) become meaningless for $1 - 2b_\mu x^\mu + b_\mu x^\mu x^\nu = 0$. 

43
2. REPRESENTATIONS OF THE POINCARÉ
ALGEBRA AND WAVE EQUATIONS
FOR ARBITRARY SPIN

The two opening sections of this chapter contain a description of IRs of the algebra $AP(1,3)$ and operators $P, T,$ and $C.$ Incidentally the basis is used in which the Poincaré group generators have a common form for all the classes of IRs. The main elements of a theory of Poincaré-invariant equations for arbitrary spin particles are expounded in Sections 6-10.

4. IRREDUCIBLE REPRESENTATIONS OF THE
POINCARÉ ALGEBRA

4.1. Introduction

Representations of the Lie algebras of the main groups of relativistic and nonrelativistic physics, i.e., the Poincaré and Galilei groups, are one of the most important instruments of a symmetry analysis of equations of quantum mechanics. These representations are used for a classification and physical interpretation of known equations as well as for deduction of new motion equations satisfying relativity principles of Galilei or Lorentz, Poincaré and Einstein.

IRs of the Poincaré group were described mainly by Wigner as long ago as 1939 [413]. Then Wigner’s results were supplemented by Shirokov [386] who for the first time finds an explicit form of basis elements of the algebra $AP(1,3)$ for all the classes of IRs. In many publications appearing later, representations of the Poincaré algebra in various basis were obtained. (See the survey [29] and the references cited there.) Each of the realizations found of the algebra $AP(1,3)$ has its merits and drawbacks, each being more convenient for a particular class of physical problems.

The realization of IRs of the Poincaré algebra given below is remarkable for a simple and symmetric form of basis elements which is common for all the classes of IRs. In the following sections we will give the classification of IRs of the algebra $AP(1,3)$, find an explicit form of the Poincaré group generators, and then establish the connection of the representations found here with the canonical Shirokov-Foldy [386, 106] realization.
4.2. Casimir Operators

The commutation relations (1.14) can be used as an abstract definition of the algebra \( AP(1,3) \). Our task is to describe constructively all the nonequivalent realizations of these relations in terms of Hermitian operators (for definitions, see Appendix 1). According to Schur’s lemma classification of IRs of a Lie algebra \( L \) reduces to finding of a complete set of Casimir operators and calculating eigenvalues of these operators. Let \( C \) be a Casimir operator for the algebra \( L \) then only such vectors which correspond to the same eigenvalues of \( L \) can be included into the space of IRs. On the other hand if we find all the independent Casimir operators \( C_1, C_2, \ldots \) for an algebra \( L \) and define representation of \( L \) in the space of eigenvectors of operators \( C_1, C_2, \ldots \) belonging to one of eigenvalues of each of them, then such a representation will be irreducible. In this case all the operators commuting with any element of a representation of the algebra \( L \) are proportional to the unit operators. In other words only one IR corresponds to a set of eigenvalues of all the Casimir operators.

To find the Casimir operators of the algebra \( AP(1,3) \) we will use the method which admits a direct generalization to the case of the algebras \( AP(1,n) \), i.e., the Lie algebra of the generalized Poincaré group in \((1+n)\)-dimensional Minkowski space.

The Casimir operators of the algebra \( AP(1,3) \) have to commute with \( P_\mu \) as well as with \( J_\mu \). Quantities commuting with \( J_\mu \) are called scalars. Evidently there is not any scalar among basis elements of the algebra \( AP(1,3) \), that’s why we will look for Casimir operators in the enveloping algebra of the algebra \( AP(1,3) \), i.e., in the set of operators of a kind \( Q_\alpha \), \( Q_\alpha Q_\beta \), \( Q_\alpha Q_\beta Q_\gamma \), ..., \( Q_\alpha \in \{ P_\mu, J_\mu \} \). We will search for all possible scalars starting from vector and tensor quantities which can be defined as follows.

We say a set of operators \((A_\mu, A_1, A_2, A_3)\) is a vector if for any \( A_\mu (\mu = 0,1,2,3)\) the following condition is satisfied

\[ [J_\mu, A_\lambda] = (g_{\mu \lambda} A_\nu - g_{\nu \lambda} A_\mu), \]

where \( J_\mu \) are basis elements of the algebra \( AP(1,3) \). A set of operators \( A_\alpha \) which commute with \( J_\mu \) as a product of vector components \( A_\alpha A_\nu \) will be called a tensor of second rank. An arbitrary rank tensors are defined in analogous manner. The operators \( P_\mu \) and \( J_\mu \) are examples of a vector and second rank tensor.

It is well known that scalars can be obtained from tensors by the operation of index convolution. The example of scalar is the operator \( J_\mu J_\mu \). Our task is to find all independent scalars starting from \( P_\mu \) and \( J_\mu \). It is convenient to use for this purpose the vectors \( W_\mu \) and \( \Gamma_\mu \) defined by the relations

\[ W_\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} J^\nu P^\rho, \quad \Gamma_\mu = J_\mu P_\mu. \]
These operators satisfy the following relations

\[ W_\mu P^\mu = 0, \quad \Gamma_\mu P^\mu = 0, \]

\[ [P_\mu, W_\nu] = 0, \quad [W_\mu, W_\nu] = i \epsilon_{\mu \nu \sigma \rho} P^\sigma W^\rho, \]

\[ [W_\mu, \Gamma_\sigma] = -i P_\mu W_\sigma, \quad [\Gamma_\mu, P_\nu] = i (\delta_\mu P_\lambda - P_\mu P_\lambda), \]

\[ [\Gamma_\mu, \Gamma_\nu] = -i J_\mu J_\nu, \]

\[ P^\lambda J_\mu J_\sigma - P_\sigma (6 - \frac{1}{2} J^2) - \frac{1}{2} \epsilon_{\alpha \nu \rho \sigma} J^\alpha W^\nu + 2i \Gamma_\sigma, \quad J^2 = J_\mu J^\mu. \]

Using (4.3), (4.4) it is not difficult to show that all the independent scalars being constructed from \( P_\mu \) and \( J_\mu \) are exhausted by the set

\[ J^2, \quad B, \quad P_\mu P^\mu, \quad W_\mu W^\mu, \quad \Gamma_\mu \Gamma_\nu, \quad W_\mu \Gamma_\nu. \]

All other scalars (i.e., all the possible convolutions of the vectors \( P_\mu \) and tensors \( J_\mu \)) can be expressed via the operators (4.5) according to (4.3), (4.4).

Using (1.14) and (4.4), it is not difficult to show that only two of operators (4.5) commute with \( P_\mu \) as well as with \( J_\mu \). They are the operators

\[ C_1 = P_\mu P^\mu \quad \text{and} \quad C_2 = W_\mu W^\mu, \]

which are the main Casimir operators of the algebra \( A(1,3) \).

We note that for some classes of representations there are additional operators commuting with any element of the algebra \( A(1,3) \). For example, for \( P_\mu P^\mu \geq 0 \) this property is possessed by the energy sign operator \( C_3 = P_\mu / |P_\mu| \). To describe IRs of the Poincaré algebra it is necessary to use also these additional Casimir operators which will be enumerated in the following section.

4.2. Basis of an IR

In order to determine a representation of the algebra (1.14) effectively, it is necessary to set ourselves to some orthogonal basis in the representation space. As this basis eigenfunctions of a complete set of commuting selfadjoint operators can be chosen. All the Casimir operators are necessarily included in such a set and some elements of the algebra and enveloping algebra should be added to make this set complete.

We will choose the complete set of commuting operators in the following form
Chapter 2. Representations of the Poincaré algebra...

\[ P_0, P_1, P_2, P_3, W_0, C_1, C_2, \ldots \]  \hspace{1cm} (4.7)

where \( W_0 \) is a zero component of the Lubanski-Pauli vector (4.2) and the dots denote additional Casimir operators whose existence was mentioned above. We will denote the orthonormal set of eigenfunctions of the operators (4.3) by \( |c, \tilde{p}, \lambda \rangle \) where \( c = (c_1, c_2, \ldots) \) are eigenvalues of Casimir operators, \( \tilde{p} = (p_0, p_1, p_2, p_3) \) are eigenvalues of the operators \( P_\mu \), and \( \lambda \) is an eigenvalue of \( W_0 \), so that

\[ C_\alpha |c, \tilde{p}, \lambda \rangle = \lambda |c, \tilde{p}, \lambda \rangle, \]

\[ P_\mu |c, \tilde{p}, \lambda \rangle = p_\mu |c, \tilde{p}, \lambda \rangle, \]

\[ W_0 |c, \tilde{p}, \lambda \rangle = \lambda p_0 |c, \tilde{p}, \lambda \rangle, \quad p = (p_0^2 + p_1^2 + p_2^2 + p_3^2)^{1/2}. \]

Incidentally the eigenvalues of the operators \( P_\mu \) (which are the basis elements of the Abelian algebra) and \( C_1 \) lie in the interval

\[-\infty < p_0 < \infty, \quad -\infty < c_1 < \infty, \]  \hspace{1cm} (4.9)

the spectra of the other operators (4.7) will be determined further on.

The numbers \( c_1 \) and \( c_2 \) (the eigenvalues of the Casimir operators) assume fixed values in spaces of IRs. Following Wigner [413], we distinguish five qualitatively different classes of IRs corresponding to the following values of \( c_1, c_2, p_\mu \):

I. \( c_1 > 0 \).

II. \( c_1 = 0, \quad c_2 = 0, \quad p_\mu \neq 0 \).

III. \( c_1 = 0, \quad c_2 \neq 0 \).

IV. \( c_1 < 0 \).

V. \( p_\mu \equiv 0 \).

The main interest (from the physical point of view) is aroused by IRs of Classes I and II inasmuch as the space of such representations is compared to the state space of a non-interacting relativistic particle with mass \( m > 0 \) and zero mass. But IRs of Classes III and IV also find some applications - for instance, for describing hypothetical particles with infinite number of spin states [310], nonstable particles and tachyons [239, 240, 390]. As to representations of Class V, they are an integral part of any physical theory satisfying the principle of relativistic invariance.

Below we describe all the nonequivalent IRs of the algebra \( AP(1,3) \) belonging to the first four classes. IRs of Class V reduces to the representations of the Lie algebra of the homogeneous Lorentz group. The theory of such representations is expounded with exhaustive completeness in monographs [20, 197]. The necessary facts about finite dimensional representations of the algebra \( AO(1,3) \) are given in Subsection 4.8.
Let us enumerate the additional Casimir operators existing in any class of IRs and show the corresponding eigenvalues $c_1$ and $c_2$:

\[ c_1 \geq 0, \quad C_3 = \frac{P_0}{|P_0|} \]  
\[ c_1 = c_2 = 0, \quad C_4 = W_1/P_1 = W_2/P_2 = W_3/P_3 = W_4/P_4. \]
\[ c_1 < 0, \quad c_2 = c_1 l(l+1), \quad C_5 = W_0/W_0. \]

where $l$ are arbitrary integers or half-integers.

For IRs of Class $V$ there are two specific Casimir operators

\[ C_6 = J_{\mu \nu} J^{\mu \nu}, \quad C_7 = \frac{1}{8} \varepsilon_{\mu \nu \rho \sigma} J^{\nu \rho} J^{\mu \sigma}. \]

There is just one more universal Casimir operator for any class of IRs

\[ C_8 = \exp(2i\pi J_{12}) \exp(2i\pi J_{13}) \exp(2i\pi J_{23}). \]

whose eigenvalues are equal to $\pm 1$. The top sign corresponds to simple representations of the Poincaré group, the lower one corresponds to two-valued ones.

### 4.4. The Explicit Form of the Lubanski-Pauli Vector

Let us obtain an explicit form of the vector $W_\mu$ in the basis $|c,p,\lambda>$. Using (4.4),(4.8) it is not difficult to obtain the commutation relations for the components $W_\mu$ in the basis $|c,p,\lambda>$.\n
\[ [W_\mu, W_\nu] = i \varepsilon_{\mu \alpha \beta \sigma} (p_0 W_\beta - W_\beta p_0), \quad [W_\mu, W_\nu] = i \varepsilon_{\mu \alpha \beta \sigma} p_\sigma W_\nu. \]  
\[ \]  

For any fixed value of $p = (p_0, p_1, p_2, p_3)$ the conditions (4.14) determine some Lie algebra $A_\mu$. Incidentally nonequivalent algebras correspond to different values of $p$.

Our task is a constructive description of all the nonequivalent representations of the algebra (4.14). To simplify the relations (4.14) we use an invertible linear transformation

\[ W_\mu \rightarrow W'_\mu = \hat{\mathcal{R}}_p W_\mu, \quad p_\mu \rightarrow p'_{\mu} = \hat{\mathcal{R}}_{p_\mu} p_\mu. \]  

As a rule, this transformation is taken in a form which leads to the maximal simplification of the relations (4.14) for any specific class of the vectors $p_\mu$. For example, for $p_\mu p^\mu > 0$ a transformation is made to reduce $p_\mu$ to the form where $p_\mu = 0$, if $p_\mu p^\mu = 0$ then $p'_\mu = p_\mu$, $p'_\mu = p'_\mu = 0$, etc. (see, e.g., [386]). As a result we obtain such realizations of IRs of the algebra $AP(1,3)$ which have essentially different form for
different values of \( c \). Besides in the case \( p_\mu p^\mu \geq 0 \), one of the vector \( p_\mu \) components, for example, \( p_3 \), is always distinguished, although all \( p_\mu \) are included into the commutation relations (1.14) at an equal footing.

We choose the operator of transformation (4.15) in such a form so that it will maximally simplify the commutation relations (4.14) immediately for any type of the four-vector \( p_\mu \). Namely we set

\[
\hat{R}_{ab} = R_{ab}, \quad \hat{R}_{ab} = R_{ab}, \quad \hat{R}_{ab} = R_{ab},
\]

where \( R_{ab} \) are the matrix elements of the operator of transition to such a reference frame in which \( p'_a = n_\mu p \), \( n = (n_1, n_2, n_3) \) is an arbitrary unit vector, and

\[
\hat{R}_{ab} = \hat{R}_{ab} = R_{ab}, \quad \hat{R}_{ab} = R_{ab}, \quad \hat{R}_{ab} = R_{ab},
\]

As a result of the transformation (4.15)-(4.17), the commutation relations (4.14) take the form

\[
[W'_a, W'_b] = i e_{abc} (p'_a W'_c - n_c p W'_0), \quad [W'_0, W'_a] = i p e_{abc} n_b W'_c.
\]

Finally, by setting

\[
W'_0 = p \lambda_0, \quad W'_a = n_a \lambda_a, \quad W'_b = n_b \lambda_b,
\]

we obtain from (4.18) the commutation relations for the operators \( \lambda_\mu \): \( \lambda_\mu \)

\[
[\lambda_0, \lambda_a] = i e_{abc} n_b \lambda_c, \quad [\lambda_a, \lambda_b] = i c_1 e_{abc} n_c \lambda_0,
\]

where \( c_1 = p_0^\mu p^\mu \).

In a space of IRs of the algebra \( AP(1, 3) \), \( c_1 \) takes a fixed value (coinciding with the eigenvalue of the Casimir operator \( C_1 \)) and the commutation relations (4.20) determine some Lie algebra \( A(c_1, n) \) whose structure constants are dependent on \( c_1 \) and \( n \). The vector \( W_\mu \) can be taken into correspondence with any representation of these algebra. Indeed we can obtain by the transformation inverse to (4.15)-(4.17), bearing in mind the relations (4.19), the following:

\[
W_0 = W'_0 = p \lambda_0, \quad W_a = R_{ad} W'_d, \quad W'_b = \lambda_b + \hat{\rho}_{ab} \lambda_0 p_0 \frac{(\hat{\rho}_{ab} + n_a) \lambda_0 \hat{\rho}_{b}}{1 + n \cdot \hat{p}}.
\]

Thus, we have obtained an analytical expression of the Lubanski-Pauli vector via the components of the four-vector \( p_\mu \) and operators \( \lambda_\mu \) realizing a representation of the algebra (4.20). A description of representations of the vector \( W_\mu \) reduces to describing nonequivalent representations of the algebra \( A(c_1, n) \).
4.5. IRs of the Algebra \( A(c, n) \)

Let us study the structure of the algebra (4.20) and establish its connection with other well known Lie algebras.

It is not difficult to make sure the operators commute with \( \lambda \), i.e., they are the Casimir operators of the algebra (4.20). So, these operators are multiples of the unit operator in a space of an IR and their eigenvalues can be used to numerate IRs. It is not difficult to show that it is necessary to set \( n = 0 \) inasmuch as, according to (4.4), \( p_{0} \mathcal{W}' - n_{j} \mathcal{W}p_{0} = 0 \). It follows from this that the algebra (4.20) has three linearly independent elements only. Moreover its structure can be described as follows [157].

**THEOREM 4.1.** The algebra \( A(c, n) \) of (4.20) is isomorphic to the algebra \( AO(3) \) if \( c > 0 \), to the algebra \( AE(2) \) if \( c = 0 \), and to the algebra \( AO(1,2) \) if \( c < 0 \).

**PROOF.** The commutation relations (4.20) are invariant under the transformations \( \lambda_{a} \rightarrow \lambda'_{a} = r_{ab} \lambda_{b} \), \( n_{a} \rightarrow n'_{a} = r_{ab} n_{b} \), where \( r_{ab} \) are elements of an orthogonal matrix. Choosing \( r_{ab} \) so that \( n' = (0, 0, 1) \), we come to the following equivalent algebra:

\[
[\lambda_{\alpha}, \lambda'_{\beta}] = \eta c_{\alpha} \lambda_{\beta}, \quad [\lambda_{0}, \lambda'_{1}] = -i \lambda'_{1},
\]

\[
[\lambda_{1}, \lambda'_{2}] = i c_{1} \lambda_{0}, \quad \lambda_{3} = 0.
\]

By setting

\[
\lambda_{\alpha} = S_{\alpha}, \quad \lambda'_{\alpha} = m S_{\alpha}, \quad \lambda_{a} = m \lambda_{a} > 0,
\]

\[
\lambda_{0} = T_0, \quad \lambda'_{1} = T_1, \quad \lambda'_{2} = T_2, \quad c_{1} = 0,
\]

\[
\lambda_{0} = S_{12}, \quad \lambda_{1} = \eta S_{01}, \quad \lambda_{2} = \eta S_{02}, \quad c_{1} = -\eta < 0,
\]

we obtain from (4.23) the commutation relations for \( S_{\alpha}, T_{\alpha}, S_{\alpha \beta} \):

\[
[S_{\alpha}, S_{\beta}] = i \epsilon_{\alpha \beta \gamma} S_{\gamma},
\]

\[
[T_{\alpha}, T_{\beta}] = i T_{\gamma}, \quad [T_{\alpha}, T_{\beta}] = -i T_{\gamma}, \quad [T_{\alpha}, T_{\beta}] = 0.
\]

\[
[S_{01}, S_{02}] = -i S_{12}, \quad [S_{01}, S_{12}] = -i S_{02}, \quad [S_{02}, S_{12}] = i S_{01},
\]

characterizing the algebras \( AO(3), AE(2) \) and \( AO(1,2) \) accordingly.

* The symbols \( AO(3), AE(2) \), and \( AO(1,2) \) denote the Lie algebras of the group of orthogonal 3×3 matrices, the Euclidean group in two-dimensional space, and a group of pseudo-orthogonal matrices in (1+2)-dimensional Minkowski space.
Chapter 2. *Representations of the Poincaré algebra...*

So the isomorphism formulated in the theorem can be given by the relations

\[ \lambda_\mu = \tilde{\lambda}_\mu, \quad \lambda_\nu = R^{-1}_{ab} \lambda_b \]  

(4.28)

where \( \lambda_\mu \) are connected with the basis elements of the algebras \( AO(3), AE(2), AO(1,2) \) according to (4.24), \( R^{-1}_{ab} \) are matrix elements of the rotation operator connecting \( n' = (0,0,1) \) with \( n = (n_1,n_2,n_3) \). The explicit expression for \( R^{-1}_{ab} \) can be obtained from (4.17) by the substitution \( p_\alpha \to pn' \).

The theorem proved makes it possible to reduce the problem of description of the algebra \( A(c_1,n) \) representations to describing representations of the Lie algebras determined by the commutation relations (4.25)-(4.27). Representations of these algebras are well known (see, e.g., [386]). A short survey of the main results connecting these representations is given in the following.

a) The algebra \( AO(3) \). IRs of it are labelled by positive integers or half integers and are realized by square matrices of dimension \( (2s+1) \times (2s+1) \). This algebra has two Casimir operators

\[ I_1 \sim S_1^2 + S_2^2 + S_3^2, \quad I_2 \sim \exp(2i\pi S_3). \]  

(4.29)

Let \(|s,s_3>\) be an eigenfunction of a complete set of the commuting matrices \( I_1 \) and \( S_3 \), then

\[ I_1 |s,s_3> = s(s+1) |s,s_3>, \quad I_2 |s,s_3> = (-1)^{s_3} |s,s_3>. \]  

(4.30)

The explicit form of the matrices \( S_\mu \) in the basis \(|s,s_3>\) is defined by the relations

\[ S_1 |s,s_3> = s |s,s_3>, \quad S_3 = s + 1, \ldots, s, \]  

\[ (S_\mu \pm iS_\nu) |s,s_3> = \sqrt{s(s+1) - s_3(s_3 \pm 1)} |s,s_3 \pm 1>. \]  

(4.31)

b) The algebra \( AE(2) \) is characterized by the commutation relations (4.26) and has the two Casimir operators

\[ I_1 \sim T_1^2 + T_2^2, \quad I_2 \sim \exp(2i\pi T_0). \]  

(4.32)

There are two distinguished classes of IRs of the algebra \( AE(2) \) corresponding to \( I_1 = 0 \) and \( I_1 = r > 0 \). If \( I_1 = 0 \) then

\[ T_1 = T_2 = 0, \quad T_0 \sim \tilde{\lambda}, \quad I_2 = (1)^{3\tilde{\lambda}} \]  

(4.33)

where \( \tilde{\lambda} \) are integers or half integers. If \( I_1 = r > 0 \) then IRs of the algebra \( AE(2) \) are realized by infinite matrices of the kind

\[ T_0 |r,n> \sim n |r,n>, \quad (T_1 \pm iT_2) |r,n> \sim (1)^{3\tilde{\lambda}} |r,n \pm 1> \]  

(4.34)

where \(|r,n>\) are eigenfunctions of the commuting matrices \( I_1 \) and \( T_0 \). Besides

\[ I_1 |r,n> \sim r^2 |r,n>, \quad I_2 |r,n> = (1)^{3\tilde{\lambda}} |r,n>, \quad 0 < r < \infty, \]  

(4.35)
and $n$ runs over either all integer or half integer values.

c) The algebra $AO(1,2)$ (4.27) has two main Casimir operators

$$I_1 S_{12}^2 - S_{01}^2 - S_{12}^2, \quad I_2 \exp(2\pi i S_{12}). \quad (4.36)$$

Let us denote by $|\alpha, n, \varphi\rangle$ an eigenfunction of the commuting operators $I_1$, $I_2$ and $S_{12}$ so that

$$I_1 |\alpha, n, \varphi\rangle = \alpha |\alpha, n, \varphi\rangle,$$

$$I_2 |\alpha, n, \varphi\rangle = \exp(2\pi i \varphi) |\alpha, n, \varphi\rangle,$$

$$S_{12} |\alpha, n, \varphi\rangle = (n \varphi) |\alpha, n, \varphi\rangle.$$ 

All the possible combinations of values of $\alpha$, $\varphi$ and $n$, corresponding to unitary representations of the group $O(1,2)$, are given by the formulae

a) $\varphi = 0, \quad -\infty < \alpha < 0, \quad n = 0, \pm 1, \pm 2, \ldots$;

b) $\varphi = 0, \quad \alpha = (l+1), \quad l = 0, 1, 2, \ldots, \quad n = l+1, l+2, \ldots$;

c) $\varphi = 0, \quad \alpha = (l+1), \quad l = 0, 1, 2, \ldots, \quad n = -l-1, -l-2, \ldots$;

d) $\varphi = 1/2, \quad -\infty < \alpha < -1/4, \quad n = 0, \pm 1, \pm 2, \ldots$;

e) $\varphi = 1/2, \quad \alpha = (l+1), \quad l = -1/2, 1/2, 3/2, \ldots, \quad n = l+1/2, l+3/2, \ldots$;

f) $\varphi = 1/2, \quad \alpha = (l+1), \quad l = -1/2, 1/2, 3/2, \ldots, \quad n = -l-1/2, -l-3/2, \ldots$.

Therefore, it is possible to distinguish six classes of IRs of the algebra $AO(1,2)$ corresponding to the variants of sets of the Casimir operators eigenvalues given in (4.38). The representations of types b), c) and e), f) correspond to the same sets of eigenvalues of operators (4.36) but are distinguished by values of an additional invariant, i.e., the sign of $S_{12}$. The explicit form of $S_{12}$ for any representation is given by the last formula (4.37), of $S_{0i}$ - by the following relations:

$$(S_{01} \pm i S_{02}) |\alpha, n, \varphi\rangle = i\sqrt{\alpha - (n - \varphi)(n + \varphi + 1)} |\alpha, n \pm 1, \varphi\rangle. \quad (4.39)$$

Using the results given above and taking into account the isomorphism of the algebras (4.25)- (4.27) and (4.20) established in Theorem 4.1 it is not difficult to describe IRs of the algebra $A(c_1, n)$. As a basis of any such a representation we choose a complete set of eigenfunctions of the commuting operators $I_1$ and $\lambda_\varphi$. Then the explicit form of the Casimir operators and nonequivalent matrices $\lambda_\varphi$ realizing a Hermitian IRs of the algebra $A(c_1, n)$ can be given by the formula

$$I_1 |c_1, c_2, \lambda\rangle = c_2 |c_1, c_2, \lambda\rangle,$$

$$I_2 |c_1, c_2, \lambda\rangle = (-1)^{2\lambda} |c_1, c_2, \lambda\rangle, \quad (4.40a)$$
Chapter 2. Representations of the Poincaré algebra...

\[ \lambda_0 |c_1,c_2,\lambda> = \lambda |c_1,c_2,\lambda>, \]
\[ (\lambda_1 \pm \tilde{\lambda}_0) |c_1,c_2,\lambda> = \frac{1}{2} (n_1 + 1) \sqrt{c_2 - c_1 \lambda(\lambda \pm 1)} |c_1,c_2,\lambda \pm 1> + \]
\[ \frac{(n_2 \pm i n_1)^2}{2(n_1 + 1)} \sqrt{c_2 - c_1 \lambda(\lambda \pm 1)} |c_1,c_2,\lambda \pm 1>, \]
\[ n_3 |c_1,c_2,\lambda> = \mp(n_1 \lambda + n_2 \lambda) |c_1,c_2,\lambda>. \]

In an IR the parameters \( c_1 \) and \( c_2 \) take fixed values from the intervals given below, where the value intervals of \( \lambda \) are specified also:

\[ c_1 = m > 0, \quad c_2 = c(s+1), \quad \lambda = s, -s+1, \ldots, s, \quad (4.41) \]

\[ c_1 = 0, \quad c_2 = 0, \quad \lambda = \tilde{\lambda}, \quad (4.42) \]

\[ c_1 = 0, \quad c_2 = m > 0, \quad \lambda = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad \lambda = \pm 1/2, \pm 3/2, \ldots, \quad (4.43) \]

\[ c_1 = -\eta^2 < 0, \quad c_2 = \eta^2 \alpha, \quad \lambda = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad \lambda = \pm 1/2, \pm 3/2, \ldots, \quad \alpha > 0 < -1/4, \quad (4.44) \]

\[ c_1 = -\eta^2 < 0, \quad c_2 = \eta^2 l(l+1), \quad \lambda = l+1, l-2, \ldots, \]
\[ c_1 = -\eta^2 < 0, \quad c_2 = \eta^2 l(l+1), \quad \lambda = -l-1, -l-2, \ldots, \]

where \( s > 0 \) and \( \tilde{\lambda} \) are arbitrary integers or half integers, \( l \) are positive integers or half integers satisfying \(-1/2 \leq l < \infty\). The values of these numbers in IRs are fixed.

Formulae (4.40)-(4.44) give the explicit forms of all the possible (up to unital equivalence) Hermitian IRs of the commutation relations (4.20). Together with (4.21), these formulae determine all the nonequivalent realizations of the Lubanski-Pauli vector.

4.6. Explicit Realizations of the Poincaré Algebra

Thus we have obtained all the nonequivalent representations of the vector \( W_\mu \).

To describe all possible (up to equivalence) IRs of the algebra \( AP(1,3) \) it is sufficient to show the explicit form of the operators \( P_\mu \) and \( J_\mu \) corresponding to the representations of the Lubanski-Pauli vector found above. It is not difficult to verify such operators can be chosen in the form

\[ P_0 = p_0, \quad P_\mu = p_\mu, \quad J_\mu = x \mp p_\lambda \mp \frac{n \pm p}{1 + n \mp p}, \quad (4.45a) \]

53
Symmetries of Equations of Quantum Mechanics

where components of the vectors $J$ and $N$ are connected with $J_{\mu\sigma}$ as follows:

\begin{equation}
J_{\nu}=\frac{1}{2}\varepsilon_{abc}J_{bc},
\end{equation}

$N_{a}=J_{0a}$; $p_{0}$ and $p_{a}$ are real variables connected by the relation

\begin{equation}
p_{0}=\varepsilon\sqrt{p^{2}+c_{1}}, \quad \varepsilon=\pm1,
\end{equation}

c_{1} is an arbitrary real number, and $x_{a}=i\partial/\partial p_{a}$.

For IRs of Classes I-III (corresponding to $c_{1}\geq0$) $\varepsilon$ assumes a fixed value, but for $c_{1}<0$ the energy sign is not fixed.

It is not difficult to verify that the operators (4.45) satisfy the commutation relations (4.15) and correspond to the Lubanski-Pauli vector in the form (4.21). Inasmuch as the formula (4.21) gives a general form (up to equivalence) of this vector for Classes I-IV of IRs, the operators (4.45), (4.46) form a basis for any of these representations. Representations of the algebra $AP(1,3)$ being realized by the operators (4.45), (4.46) are irreducible inasmuch as the corresponding Casimir operators (4.11), (4.12) are multiples of the unit operator.

The operators (4.45) are Hermitian in respect to the scalar product

\begin{equation}
(\Phi_{1},\Phi_{2})=\sum_{\lambda}\int dp\Phi_{1}^{\dagger}(p,\lambda)\Phi_{2}(p,\lambda),
\end{equation}

where $\Phi_{i}$ belong to the space of functions decreasing sufficiently fast along the direction $p=-\mathbf{n}p$ and $\lambda$ takes all the possible values coinciding with the matrix $\lambda_{0}$ eigenvalues.

The results given above are formulated in the following form.

THEOREM 4.2. IRs of the algebra $AP(1,3)$ are labelled by the sets of eigenvalues $c_{1}, c_{2}, \ldots$ of the Casimir operators $C_{1}, C_{2}$ (4.6) and of $C_{3}$ (if $c_{1}\geq0$), $C_{4}$ (if $c_{1}=c_{2}=0$), $C_{5}$ (if $c_{1}<0, c_{2}<0$), and $C_{6}$ (4.11), (4.19). All the admissible combinations of the eigenvalues $c_{1}, c_{2}$ and $c_{i}=\lambda$ are given in formulae (4.41)-(4.44), but eigenvalues of the operators $C_{1}, C_{3}$ and $C_{6}$ are equal to $\pm1$. The explicit expression of the corresponding basis elements of the algebra $AP(1,3)$ can be chosen in the form (4.45) (up to unitary equivalence) where $\lambda_{a}$ are the matrices (4.40)-(4.44).

So we have calculated the explicit expressions of the basis elements of IRs of the algebra $AP(1,3)$. Operators (4.45) have a relatively simple form which is common for all classes of IRs. It differs favourably our realization from the others known already.
4.7. Connections With the Canonical Realizations
of Shirokov-Foldy-Lomont-Moses

Let consider each class of IRs more precisely.
1. Representations of Class I corresponding to \( P_\mu P^\mu > 0 \) have the additional Casimir operator, i.e., the energy sign operator with eigenvalues \( \pm 1 \). The corresponding representation space is split into two subspaces, each of them corresponding to the fixed sign of \( P_\mu \).

So IRs of Class I are labelled by three numbers: \( m^2, s, \) and \( \epsilon \), refer to (4.41), and are realized in the space of square integrable functions \( \Phi(p, \lambda) \) having dimension \( 2(s+1) \) with respect to the index \( \lambda \). Besides, the Casimir operator (4.13) takes the value +1 for integer and -1 for half integer \( s \). We denote such IRs by \( D(\epsilon m s) \).

With the help of the unitary transformation

\[
(P_\mu, J, N) \rightarrow (P'_\mu, J', N') = U(P_\mu, J, N) U^\dagger \tag{4.48}
\]

where

\[
U = \exp(-i\lambda \cdot p \times n \theta_1) \exp(i\lambda \cdot n \theta_2) \tag{4.49}
\]

\[
\theta_1 = \arccos \hat{p} \cdot n, \quad \theta_2 = \arccos n_3, \quad n' = (0,0,1),
\]

the operators (4.45) realizing an IR of Class I can be transformed to the canonical form of Shirokov-Foldy [386, 106]:

\[
P_0' = \epsilon E = \sqrt{p^2 + m^2}, \quad P_a' = p_a, \tag{4.50}
\]

\[
J' = \epsilon \mathbf{E} \cdot \mathbf{S}, \quad N' = -\frac{\epsilon}{2} [x, E] - \frac{p \times S}{E + m}.
\]

Here \( S \) are generators of the IR \( D(\epsilon s) \) of the group \( O(3) \) given in (4.31).

2. Let us consider the case \( c_1 = c_2 = 0 \). There exist two additional Casimir operators for the corresponding representations, i.e., the energy sign operator \( C_3 \) and helicity operator \( C_4 \) of (4.11). We obtain from (4.11), (4.21) \( C_4 = \epsilon \lambda \), and conclude from (4.42) that eigenvalues of the helicity operator are equal to \( \epsilon \lambda \) where \( \lambda \) is an arbitrary integer or half integer.

So IRs of Class II are labelled by two numbers, \( \epsilon \) and \( \lambda \) and are one-dimensional in respect with the index \( \lambda \). The Casimir operator (4.12) has the eigenvalue +1 for integer \( \lambda \) and -1 if \( \lambda \) are half integers. Besides, the basis elements of the Poincaré algebra (4.40), (4.42), (4.45) take the form

\[
P_0 = \epsilon p, \quad P_a = p_a. \tag{4.51a}
\]
The operators (4.51) have a symmetric and compact form which differs them favourably from other realizations known. Choosing different unit vectors $n$, we obtain from (4.51) different (but equivalent) realizations of IRs of the algebra $\text{AP}(1,3)$. Taking $n=(0,0,1)$, we come to a representation in the Lomont-Moses [287] form, but if $n=(1,1,1)/\sqrt{3}$ the above formulae give the realization proposed in [154,157].

The transition from a representation characterized by a vector $n$ to an equivalent representation corresponding to a vector $n', n'\neq n$, can be carried out by the unitary transformation (4.48) where

$$U=\exp\left\{2i\lambda_{n} \arctan \frac{p\cdot n \times n'}{1+n\cdot \hat{p} \cdot n' \cdot \hat{p} + n \cdot n'}\right\}.$$  

If $n'=-n$, then

$$U=\exp\left\{2i\lambda_{n} \arctan \frac{\hat{p} \cdot n \times n''}{\hat{p} \cdot n'' - (\hat{p} \cdot n)(n \cdot n'')}\right\}$$  

where $n'' \neq \pm n$ is an arbitrary unit vector.

We note that for representations of Class II the following relations are valid:

$$W_{\mu} = \frac{p_{\mu}}{p} P_{\mu}, \quad P = (p^{2} + p_{1}^{2} + p_{3}^{2})^{1/2},$$  

which take the following form for IRs:

$$W_{\mu} = e\lambda P_{\mu}.$$  

The condition (4.55) is necessary and sufficient for the corresponding representation to belong to Class II.

As was noted in Section 3.7, representations of the Poincaré algebra belonging to Class II can be extended to the representations of the algebra $\text{AC}(1,3)$. Substituting (4.51) into (3.48), we obtain the corresponding generators $K_{\mu}$ and $D$ in the form

$$D=\frac{1}{2}(p \cdot x \cdot x \cdot p), \quad K_{\mu} = \frac{e}{2} (p \cdot x \cdot x \cdot p) + 2\lambda e \frac{n \cdot \hat{p}}{1-n \cdot \hat{p}}, \quad K_{\mu} = [K_{\nu}, N_{\mu}].$$  

Formulae (4.51), (4.56) set basis elements of the representation of the

\footnote{The operators $N$ considered in [154] include the term $px$ instead of $[p,x]$, which is caused by differently choosing a scalar product.}
conformal algebra, which reduces to the IR of the Poincaré algebra by the reduction with respect to the subalgebra $AP(1,3)$. When $n=(0,0,1)$ these formulae determine an IR of the algebra $AC(1,3)$ in the Bose-Parker \cite{50} realization.

3. Representations of Class III have four Casimir operators, i.e., $C_1$, $C_2$, $C_3$ and $C_8$ given by formulae (4.6), (4.11), (4.13). Besides, the eigenvalues of $C_1$ are equal to zero and IRs are labelled by triplets of eigenvalues of the operators $C_2$, $C_3$ and $C_8$: $c_2=0$, $c_3=\pm 1$, $c_8=\pm 1$.

An explicit form of operators forming a basis of an IR of the algebra $AP(1,3)$ can be obtained from (4.45) by the substitution $p_0=\varepsilon p$ using the corresponding expressions (4.40), (4.42) for the matrices $\lambda_\alpha$. Besides, IRs with integer $n$ correspond to $c_8=1$, and with half integer $n$ to $c_8=-1$. The corresponding functions $\Phi(p,\lambda)$ forming the representation space are infinite dimensional with respect to the index $\lambda$ which takes denumerable values. When $n=(0,0,1)$ formulae (4.45), (4.46) give the basis elements of the algebra $AP(1,3)$ in the Lomont-Moses form \cite{287} (besides, $\lambda_0\rightarrow T_0$, $\lambda_1\rightarrow T_1$, $\lambda_2\rightarrow T_2$, $\lambda_3\rightarrow 0$, $T_0$, $T_1$ and $T_2$ being the matrices (4.34).

4. Representations of Class IV are labelled by a set of eigenvalues of three main Casimir operators $C_1$, $C_2$ and $C_8$. The eigenvalues of $C_8$ are equal to $\pm 1$ where the top (lower) sign corresponds to integer (half integer) $\lambda$. Possible values of $C_1$, $C_2$ and $\lambda$ are given in (4.44). In the case $C_1>0$ there exist the additional Casimir operator $C_5=W_0/|W_0|={\text{sign}}\lambda_0$ so the space of the corresponding representation is split into two invariant subspaces, each of them having all $\lambda$ (the matrix $\lambda_0$ eigenvalues) with the same sign.

All the Hermitian IRs of Class IV are infinite dimensional in respect with the index $\lambda$ taking denumerable values. With the help of the unitary transformation (4.48) where

$$U=\exp\left[i\frac{S_{\alpha\beta}p_\alpha}{|p|}\arctan\frac{p_0|p|}{(p_3^2+\eta)(p_0^2+\eta^2)}\right]U', \quad |p|=(p_1^2+p_2^2)^{1/2}, \quad \alpha=1,2, \quad (4.57)$$

(where $U'$ is the operator of (4.52) for $n'=(0,0,1)$, $S_{\alpha\beta}$ are matrices (4.24)), (4.39), the operators (4.45) realizing an IR of Class IV can be reduced to the canonical form of Shirokov \cite{386}.

So we established the connection of IRs in the form (4.45) with the well-known "canonical" realizations of IRs of the algebra $AP(1,3)$. Other realizations of IRs of the Poincaré algebra are discussed in survey \cite{29}. 

57

In the set of all the possible realizations of representations of the algebra $AP(1,3)$, an outstanding role is played by so-called covariant representations which are characterized by the form of basis elements given in (2.22) where $S_{\mu\sigma}$ are numeric matrices. The necessary and sufficient condition for the operators (2.22) to realize a representation of the Poincaré algebra (i.e., satisfy commutation relations (1.14)) is that the matrices $S_{\mu\sigma}$ should satisfy the relations (2.18b).

Formulae (2.22) determine a general form of basis elements of the algebra $AP(1,3)$ belonging to the class $M_1$. Particular examples of such realizations of the algebra $AP(1,3)$ are the IA of the KGF, Dirac and Maxwell equations considered in Chapter 1. The operators (2.22) with arbitrary matrices $S_{\mu\sigma}$ generate local transformations of the kind given in (2.49). It follows from the fact that the "spin" part of $J_{\mu\sigma}$ (i.e., the matrices $S_{\mu\sigma}$) commutes with the "orbital" part $x_{\mu}p_{\sigma} - x_{\sigma}p_{\mu}$ (refer to Section 2.9).

We note that the operators (2.22) (in contrast with (4.45)) realize a reducible representation of the algebra $AP(1,3)$ which is not generally speaking Hermitian for the case of finite dimensional matrices $S_{\mu\sigma}$.

If we are not interested in refinements connected with different choosing of functional spaces of representations, then, to describe all the covariant representations of the algebra $AP(1,3)$, it suffices to present all the nonequivalent realizations of the matrices $S_{\mu\sigma}$ satisfying relations (2.18b). We restrict ourselves to the case of finite dimensional matrices.

The algebra (2.18b) has the two Casimir operators (4.12) whose eigenvalues label IRs. To describe a domain of these eigenvalues we consider the following operators

\[
j_a = \frac{1}{2} \left( \frac{1}{2} e_{abc} S_{bc} + i S_{0a} \right), \quad \tau_a = \frac{1}{2} \left( \frac{1}{2} e_{abc} S_{bc} - i S_{0a} \right)
\]

which satisfy the following commutation relations:

\[
[j_a, j_b] = i e_{abc} j_c, \quad [\tau_a, \tau_b] = i e_{abc} \tau_c, \quad [j_a, \tau_b] = 0.
\]

We note that the algebra $AO(1,3)$ is split into two commuting subalgebras formed by the matrices $j_a$ and $\tau_a$. The commutation relations between $j_a$ (and between $\tau_a$) coincide with the commutation relations (4.25) determining the algebra $AO(3)$. We concludes from this that finite dimensional IRs of the algebra $AO(1,3)$ are labelled by positive integers or half integers $j$ and $\tau$. The basis of a space of an IRs is formed by $(2j+1)\times(2\tau+1)$ eigenfunctions of the complete set of the commuting operators $j^2$, $\tau^2$, $j_3$ and $\tau_3$. Using for these eigenfunctions the notation $|j_m;\tau,n>$, we can represent the
Chapter 2. Representations of the Poincaré algebra...

The action of the operators $j_a$ and $\tau_a$ in the following form (compare with (4.31)):

\[ j^2 |j;m;\tau,n> = (j+1) |j;m;\tau,n>, \]
\[ j^2 |j;m;\tau,n> = m |j;m;\tau,n>, \]
\[ (j \pm ij) |j;m;\tau,n> = \sqrt{j(j+1)-m(m+1)} |j;m;\tau,n>, \]
\[ \tau^2 |j;m;\tau,n> = \tau(\tau+1) |j;m;\tau,n>, \]
\[ \tau_1 |j;m;\tau,n> = m |j;m;\tau,n>, \]
\[ (\tau \pm i\tau) |j;m;\tau,n> = \sqrt{\tau(\tau+1)-n(n+1)} |j;m;\tau,n>, \]

where $j,m$ are simultaneously integers or half integers, and $-j \leq m \leq j$, $-\tau \leq n \leq \tau$.

So, finite dimensional IRs of the algebra $\mathfrak{ap}(1,3)$ are realized by square matrices of dimension $(2j+1)(2\tau+1) \times (2j+1)(2\tau+1)$, whose elements are given in (4.58)-(4.60). We denote these representations by $D(j,\tau)$.

Sometimes it is more convenient to use the $O(3)$-basis $|l_1,l_2;l,m>$ in which the matrix $S^2 = (1/2) S_{ab} S_{ab}$ is diagonal. The numbers $l_1$ and $l_0$ are connected with $j$ and $\tau$ by the relation

\[ \frac{1}{2} S_{ab} l_0 |l_1,l_0;l,m> = (l_1^2-l_0^2-1) |l_1,l_0;l,m>, \]

and set the eigenvalues of the Casimir operators according to the relations

\[ \frac{1}{2} e_{a b c} S^a S^b |l_0,l_1;l,m> = 2il_1 |l_0,l_1;l,m>. \]

The numbers $l$ and $m$ determine eigenvalues of $S^2$ and $S_{12}$ in the following way:

\[ S^2 |l_0,l_1;l,m> = (l+1) |l_0,l_1;l,m>, \]
\[ S_{12} |l_0,l_1;l,m> = m |l_0,l_1;l,m>, \]

where $l = l_0, l_0+1, \ldots, l_1$ and $m = -l,l+1,\ldots,l$. The explicit expressions of the matrices realizing an IR are given by the formula

\[ S_{ab} |l_0,l_1;l,m> = \epsilon_{abc} (S_a^{(l)})_{mn} |l_0,l_1;l,m>, \]
\[ S_{ab} |l_0,l_1;l,m> = \delta_{ab} A(l)_{mn} |l_0,l_1;l,m>, \]
\[ C_{ab} |l_0,l_1;l,m> = \delta_{ab} C(l)_{mn} |l_0,l_1;l,m>, \]
\[ A(l)_{mn} = i \sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}}, \]
\[ C(l)_{mn} = i \sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}}, \]

and the matrix elements $(S_a^{(l)})_{mn}$ and $(K_a^{(l)})_{mn}$ are defined by the following relations:
Chapter 2. Representations of the Poincaré algebra...

Nonequivalent representations of the finite group formed by these operators and their products are encountered, and the explicit form of these operators in the space of the "universal" representation (4.45) of the algebra $AP(1,3)$ is found.

We will start with the discrete symmetry operators defined on the set of the KGF equation solutions (refer to (1.51)-(1.53)). These operators and their products form the group $G_8$ including the following eight elements:

\begin{align*}
G_8 \in I, \; P, \; T, \; C, \; PT, \; PC, \; TC, \; CPT
\end{align*}

where $I$ denotes the identity transformation. The group multiplication law for the group $G_8$ can be obtained from (1.51)-(1.53). It is presented in the following table:

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The problem of description of nonequivalent operators $P, \; T, \; C,$ defined in a representation space of the algebra $AP(1,3)$, includes a description of representations of the group $G_8$. Besides these operators must satisfy the commutation and anticommutation relations (1.54). It follows from the above the operators $P, \; C, \; T,$ and the Casimir operators (4.11), (4.19) satisfy the relations

\begin{align*}
&PC_1=-C_1P, \quad PC_2=-C_2P, \quad PC_3=-C_3P, \quad PC_4=-C_4P, \\
&PC_5=-C_5P, \quad PC_6=-C_6P; \quad PC_7=-C_7P, \quad PC_8=-C_8P; \\
&TC_1=-C_1T, \quad TC_2=-C_2T, \quad TC_3=-C_3T, \quad TC_4=-C_4T, \\
&TC_5=-C_5T, \quad TC_6=-C_6T; \quad TC_7=-C_7T, \quad TC_8=-C_8T;
\end{align*}

61
According to (5.2)-(5.4), the operators $P, C, T$ can be defined in a space of reducible representation only (inasmuch as they change the signs of eigenvalues of the Casimir operators).

The description of nonequivalent realizations of the discrete symmetry operators reduces in our approach to finding all the representations of the group $G_8$. In addition, the operators $P, C, T$ must satisfy the relations (1.54) where $P_\mu$ and $J_{\mu\nu}$ are basis elements of a direct sum of IRs of the algebra $AP(1,3)$.

5.2. Nonequivalent Multiplicators of the Group $G_8$

As is well known the state vector of a physical system is represented in quantum mechanics as a ray of a Hilbert space. In other words this vector is defined up to a phase factor $\exp(i\alpha)$ where $\alpha$ is a $c$-number. This situation predetermines the fundamental role of a ray (projective) representations of symmetry groups, i.e., such representations for which the group law is valid up to a phase factor only.

Ray representations of the proper Poincaré group always can be reduced to exact representations, so no new possibilities exist here. But in the case of the complete Poincaré group including space-time reflections there exist ray representation which cannot be reduced to exact ones.

Here we consider ray representations of the group $G_8$. Specifically we consider projective unitary and antiunitary representations ($PUA$- representations) of this group which are defined as follows.

We denote by the symbol $\mathbb{C}$ the field of complex numbers, by $U(\mathbb{C})$ - the multiplicative group of complex numbers with the unit module, and by $\mathbb{R}$ the field of real numbers.

A mapping $T: G \to UA(H)$ of a group $G$ into multiplicative group $UA(H)$ of all unitary and antiunitary operators $T_g, g \in G$ of space $H$ is called $PUA$-representation of a group $G$ in a Hilbert space $H$, if $T_g$ satisfy the conditions

$$T_{g_1, g_2} \sim \lambda(g_1, g_2) T_{g_1 g_2},$$

(5.5)

where $g_1, g_2 \in G, \lambda(g_1, g_2) \in U(\mathbb{C})$, and, besides,

$$\lambda(g_2 g_3, g_4) \lambda(g_1, g_2 g_4) \sim \lambda(g_1, g_2, g_3) \lambda(g_2 g_3, g_4),$$

(5.6)

where $\bar{\lambda}(g_2, g_3)=\lambda(g_2, g_3)$ if $T_{g_1}$ is a unitary operator and $\bar{\lambda}(g_2, g_3)=\lambda(g_2, g_3)$ if $T_{g_1}$ is an antiunitary one.

A system of elements $\lambda(g_1, g_2)$ is called a multiplicator of a representation $T$.

In the case of usual (exact) representations $\lambda(g_1, g_2)=1$ for any $g_1, g_2 \in G$.  

62
Chapter 2. *Representations of the Poincaré algebra*

PUA-representation differs from the regular exact representations by two features:

a) They include antiunitary operators besides unitary ones. Antiunitary operator by definition transforms all quantities into complex conjugated ones (for two examples of antiunitary operator of charge conjugation, refer to (1.53), (2.60)).

b) The group multiplication law for elements of PUA-representation is defined up to a phase factor $\lambda(g_1, g_2)$ which do not change the norm of a vector from a Hilbert space $H$.

The definitions given above (which go back to Wigner) lead to a new concept of representations equivalence. Namely PUA-representations $T$ and $T'$ is said to be equivalent if there exists such a unitary or antiunitary operator $V$ and such a function $\alpha(g) \in U(\mathbb{C})$ that

$$ T'_g = \alpha(g)VT_gV^{-1}, $$

for any $g \in G$. If $\alpha(g) = 1$ then an equivalence is called regular, otherwise it is called projective. Below we will omit the term "regular" when speaking about a regular equivalence.

In the case, when an operator $V$ in (5.7) is unitary, the multiplicators $\lambda$ and $\lambda'$ of representations $T$ and $T'$ are connected by the relation

$$ \lambda'(g_1, g_2) = \frac{\alpha(g_1) \alpha(g_2)}{\alpha(g_2, g_1)} \lambda(g_1, g_2), $$

(5.8)

where $\alpha(g_1) = \alpha(g_2)$ if $T_g$ is unitary, and $\alpha(g_2) = \alpha'(g_2)$ if $T_g$ is antiunitary operator.

If the operator $V$ in (5.7) is antiunitary then the condition (5.8) is satisfied by multiplicators $\lambda'$ and $\lambda''$.

The relations (5.8) can be used to describe all the possible nonequivalent multiplicators characterizing representations of a group. For the group $G_8$ such a description is given by the following assertion.

**THEOREM 5.1.** All the nonequivalent multiplicators of the group $G_8$ are present in Table 5.2 (see the following page) where $\mu_{a\alpha} = \mu_a \mu_{\alpha}$, $\mu_{a\alpha \beta} = \mu_a \mu_{\alpha} \mu_{\beta}$, $\mu_{1234} = \mu_1 \mu_2 \mu_3 \mu_4$ and $\mu_a$ ($\alpha = 1, 2, 3, 4$) accept the values $\pm 1$ independently.

We do not present a proof (see [12]).

According to Theorem 5.1, the group $G_8$ has 16 nonequivalent multiplicators \( \lambda_{\mu_1 \mu_2 \mu_3} \) corresponding to the possible combinations of values of $\mu_a$. These multiplicators determine all the possible (up to equivalence) commutation and anticommutation relations for representations of the operators $P$, $C$, and $T$.

Let $\hat{P}$, $\hat{T}$ and $\hat{C}$ be basis elements of a PUA-representation of the group $G_8$ with the multiplicator $\lambda_{\mu_1 \mu_2 \mu_3}$. Then we obtain from (5.5) and Table 5.2 that
We see that up to equivalence the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ must commute or anticommute with each other. Squared operators of space and time inversions and of charge conjugation are equal to the unit operator $I$ or to $\pm I$.

It is not difficult to make sure that the conditions (5.9) can be considered as a definition of a multiplicator. In other words, starting from (5.9) and using (5.5) it is not difficult to find all the possible multiplicators given in Table 5.2.

A set of the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ and their possible products form a finite multiplicative group $G(\mu_1, \mu_2, \mu_3, \mu_4)$ of dimension 8 if $\mu_1=\mu_2=\mu_3=\mu_4=1$, and 16 in other cases. There are 16 such groups corresponding to different sets of the parameters $\mu_a$.

A description of $PUA$-representations of the group $G_8$ reduces to the description of regular representations of the group $G(\mu_1, \mu_2, \mu_3, \mu_4)$.

### 5.3. The General Form of the Discrete Symmetry Operators

In addition to relations (5.9), the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ must satisfy the following conditions (refer to (1.54)):

$$\hat{P}J = J\hat{P}, \quad \hat{P}P_0 = P_0\hat{P}, \quad \hat{P}P = -P\hat{P}, \quad \hat{P}N = -N\hat{P},$$

(5.10a)

where $P_0$, $P$, $J$ and $N$ are the basis elements of the Poincaré algebra.
Chapter 2. \textit{Representations of the Poincaré algebra...}

\[ \hat{T}P_0 = -P_0 \hat{T}, \quad \hat{T}P = -P \hat{T}, \quad \hat{T}J = -J \hat{T}, \quad \hat{T}N = -N \hat{T}, \]
\[ \hat{C}P_0 = -P_0 \hat{C}, \quad \hat{C}P = -P \hat{C}, \quad \hat{C}J = -J \hat{C}, \quad \hat{C}N = -N \hat{C}, \]  

(5.10b)

We will start with the universal realizations of the algebra $AP(1,3)$ given in (4.45). According to (5.2)-(5.4), it is sufficient to restrict ourselves to considering the representations corresponding to fixed eigenvalues to the Casimir operators $C_1$ and $C_2$, while the operators $C_3, C_4, C_5$ and $C_7$ not being chosen to be multiples of the unit operator. Restricting ourselves for the time being by considering the representations of Classes I-III we conclude that $P_\mu, J$, and $N$ can be chosen in the form

\[ P_0 = \hat{\varepsilon} E, \quad P = p, \quad J = x \times p + \lambda_0 \frac{n \cdot \hat{p}}{1 - n \cdot \hat{p}}, \quad \hat{p} = \frac{p}{p}, \]
\[ N = -\frac{1}{2} \hat{\varepsilon} [E, x] + \frac{\lambda x \times p}{p^2} \frac{\hat{\varepsilon} n \cdot (\hat{\varepsilon} E - \lambda \cdot \hat{p})}{p^2 + n \cdot p}, \]

where $\hat{\varepsilon}$ is the energy sign operator having the eigenvalues $\pm 1$ and $\lambda_0$ are the matrices realizing a reducible representation of the algebra (4.20) (corresponding to fixed values $c_1$ and $c_2$ in (4.41)-(4.45)). Moreover

\[ [\hat{\varepsilon}, \lambda_0] = 0, \quad \hat{\varepsilon}^2 = 1, \]

(5.12)

and the operator $\hat{\varepsilon}$ can be chosen in the form of a diagonal matrix, and $\lambda_0$ - in the form of a direct sum of irreducible matrices (4.40).

Let $\Psi(p)$ be a vector from the space of representation of the algebra $AP(1,3)$ belonging to one of three first classes ($P_\mu, P \geq 0$). The transformations $\hat{P}, \hat{T}$, and $\hat{C}$ without loss of generality can be represented in the form

\[ \hat{P} \Psi(p) = U_1 \Psi((-p)), \quad \hat{T} \Psi(p) = U_2 \Psi(p), \quad \hat{C} \Psi(p) = U_3 \Psi((-p)), \]

(5.13)

where $U_1, U_2, U_3$ are operators satisfying (according to (4.10)) the following conditions

\[ U_\alpha P \Psi = P U_\alpha, \quad \alpha = 1, 2, 3, \]
\[ U_1 J = J U_1, \quad U_2 J = J U_2, \quad U_3 J = -J U_3, \]
\[ U_1 N = -N^* U_1, \quad U_2 N = -N U_2, \quad U_3 N = -N^* U_3, \]

(5.14)

(5.15)

Here $J'$ and $N'$ are the operators obtained from $J$ and $N$ by the change $p \rightarrow -p, x \rightarrow -x$.

One concludes from (5.14) that $U_\alpha$ do not include operators of differentiation with respect to $p$ and so can be considered as matrices depending on $p$. Then we obtain from (5.9) that

So the general form of the operators $\hat{P}, \hat{T}$, and $\hat{C}$ defined in the spaces of the
algebra $AP(1,3)$ representations of Classes I-III is given by formulae (5.13) where $U_a$ are matrices satisfying (5.16). The problem of finding of all nonequivalent operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ reduces to finding the irreducible sets of matrices $U_a$.

For representations of Class IV the energy sign operator is not a Casimir operator so operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ have to be defined in another way. A vector from this representation space is an infinite component column with components $\Psi(\lambda)$ with $\lambda$ running over the values given in (4.44). Denoting this column by $\Psi(p,\epsilon)$ we can represent the corresponding operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ in the form

$$\hat{P}\Psi(p,\epsilon) = U_1(p,\epsilon) \Psi(p,\epsilon),$$

$$\hat{T}\Psi(p,\epsilon) = U_2(p,\epsilon) \Psi(p,\epsilon),$$

$$\hat{C}\Psi(p,\epsilon) = U_3(p,\epsilon) \Psi(p,\epsilon),$$

as a result we come to the relations

$$U_1(p,\epsilon) U_2(-p,\epsilon) = \mu_1 U_1(p,\epsilon) U_2(-p,\epsilon),$$

$$U_2(p,\epsilon) U_3(-p,\epsilon) = \mu_2 U_2(p,\epsilon) U_3(-p,\epsilon),$$

$$U_3(p,\epsilon) U_1(-p,\epsilon) = \mu_3 U_3(p,\epsilon) U_1(-p,\epsilon),$$

$$U_1(p,\epsilon) U_1(-p,\epsilon) = U_2(p,\epsilon) U_2(-p,\epsilon) = 1,$$

$$U_3(p,\epsilon) U_3(-p,\epsilon) = \mu_4.$$

So for representations of Class IV the general form of the operators $P$, $C$ and $T$ is given by relations (5.17) where $U_a(p,\epsilon)$ are infinite dimensional matrices satisfying (5.18).

We note for completeness that the discrete symmetry operators can be defined for the representations of the algebra $AP(1,3)$ belonging to Class V (where $\mu = 0$) also. Besides, the operators $J$ and $N$ reduce to finite or infinite dimensional matrices: $J_a \to 1/2 \epsilon_{abc} S_{bc}$, $N_a \to S_{ba}$ and the corresponding operators of discrete transformations can be represented as follows:

$$\hat{P} = \xi_1, \quad \hat{T} = \xi_2, \quad \hat{C} = \xi_3 A,$$

where $A$ is the operator of the complex conjugation, $\xi_1$, $\xi_2$, and $\xi_3$ are numerical matrices of appropriate dimension satisfying the relations
Chapter 2. *Representations of the Poincaré algebra...*

In the following sections we find the explicit form of the matrices $U_a$ (and $\xi_a$) determining the discrete symmetry transformations for every class of representations of the algebra $AP(1,3)$.

5.4. The Operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ for Representations of Class I

Representations of the Poincaré algebra belonging to Class I are labelled by the eigenvalues $c_1=m^2>0$, $c_2=-m^2s(s+1)$, and $c_3=e=\pm 1$ of the Casimir operators (4.6), (4.11). According to previous section, we can restrict ourselves to the representations for which $c_1$ and $c_2$ are fixed but $e$ can take two possible values. We choose the basis elements of such representation in the form given in (5.11) where $\lambda_\mu$ are finite dimensional matrices realizing the direct sum of equivalent IRs of the algebra (4.20) corresponding to $c_1=m^2>0$, $c_2=-m^2s(s+1)$ (refer to (4.40)-(4.43)).

We represent $U_a$ of (5.14) in the form

$$U\xi, V\xi, U\Delta,$$

where

$$U=\exp\left(\frac{i\lambda\cdot n\exp(\pi)}{m[p^2-(n\cdot p)^2]}\right), \quad V=\exp(i\lambda_0\hat{\pi}),$$

and $\Delta$ is a numerical matrix satisfying the relations

$$\Delta\lambda_\mu=-\lambda_\nu\Delta, \quad \Delta^2=1,$$

$\xi_a$ being the operators to be determined. Choosing $U_a$ in the form (5.22) we do not lose generality inasmuch as the operators $U$, $V$ and $\Delta$ are invertible.

Using the relations

$$UJ=J^tU, \quad UN=N^tU, \quad V\lambda=-\lambda V,$$

($J'$ and $N'$ are determined in (5.15)) we conclude from (5.15), (5.11) that $\xi_a$ must satisfy the following relations

$$\xi_2, \xi_3, \xi_2, \xi_3, \xi_2, \xi_3, \xi_2, \xi_3, \xi_2, \xi_3,$$

$$[\xi_\mu, P_{\nu}]=[\xi_\mu, \hat{P}_{\nu}]=[\xi_\mu, J_{\nu}]=[\xi_\mu, \hat{N}_{\nu}]=0.$$ 

But the operators $\{P_{\nu}, \hat{P}_{\nu}, J_{\nu}, \hat{N}_{\nu}\}$ realize the direct sum of the IRs $D'(m,s)$ of the algebra $AP(1,3)$ from which it follows according to Schur’s lemma that $\xi_a$ must be
Symmetries of Equations of Quantum Mechanics

numerical matrices.

We conclude from (5.15), (5.16) that the matrices \( \xi_n \) must commute with \( \lambda_{\mu} \),

\[
[\xi_n, \lambda_{\mu}] = 0,
\]

and to satisfy the equations (5.21) which (together with (5.12), (5.26)) determine these matrices up to unitary equivalence.

So the problem of finding all the admissible operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) reduces to finding all the nonequivalent matrices \( \lambda_{\mu}, \hat{\epsilon} \) and \( \xi_n \). The irreducible sets of these matrices are given in the following assertion.

**Theorem 5.2.** All possible (up to equivalence) irreducible sets of matrices satisfying the commutation and anticommutation relations (4.20), (5.20), (5.26), (5.27) can be labelled by five numbers \( s, \mu_1, \mu_2, \mu_3, \mu_4 \), where \( s = 0, 1/2, 1, \ldots \), and \( \mu_k = \pm 1 \). Dimension of these matrices is \( 2(2s+1) \times 2(2s+1) \) if \( \mu_1 \mu_4 = \mu_2 \mu_3 = 1 \) and \( 4(2s+1) \times 4(2s+1) \) otherwise. The explicit form of the corresponding matrices is given by the following formulae.

\[
\xi_1 = \begin{pmatrix} \delta_{\mu_3} & 0 \\ 0 & \mu \lambda_{\mu} \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & \sigma_2 \\ \mu \sigma_2 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & \sigma_1 \\ \mu \sigma_1 & 0 \end{pmatrix}, \quad \lambda_{\mu} = \lambda_{\mu}^{(2)}.
\]

\[
\hat{\epsilon} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \lambda_{\mu}^{(2)} = \begin{pmatrix} 0 & \lambda_{\mu} \\ \lambda_{\mu} & 0 \end{pmatrix}
\]

where the following \( 2(2s+1) \times 2(2s+1) \) matrices are included

\[
\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},
\]

\[
\delta_{\mu} = \begin{pmatrix} I & 0 \\ 0 & \mu I \end{pmatrix}, \quad \alpha_{\mu} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda_{\mu}^{(2)} = \begin{pmatrix} 0 & \lambda_{\mu} \\ \lambda_{\mu} & 0 \end{pmatrix}.
\]

\( \lambda_{\mu} \) are \( 2(2s+1) \times 2(2s+1) \)-dimensional matrices realizing the IR \( D(s) \) of the algebra \( A(c_I = m^2, n) \) (4.20) (which is isomorphic to the algebra \( AO(3) \)), \( I \) is the unit matrix, \( 0 \) is the zero matrix of appropriate dimension.

**Proof** reduces to going through possible sets of values of \( \mu \) and finding nonequivalent irreducible sets of the matrices \( \xi_n, \hat{\epsilon} \) and \( \lambda_{\mu} \), for any combinations of values \( \mu_\nu \). We do not represent the corresponding calculations here (see [12]), but note that the matrices (5.28), (5.29) are determined up to projective equivalence.
Chapter 2. Representations of the Poincaré algebra...

transformations (compare with (5.7)):

$$\xi_a \rightarrow \alpha \xi_a \hat{V}, \quad a = 1, 2, 3, \quad e \rightarrow VeV^{-1}$$  \hspace{1cm} (5.31)

where $V$ is an arbitrary nondegenerated matrix commuting with $\lambda_{\mu}$.

If we require $\alpha_1 = \alpha_2 = \alpha_3 = 1$ (i.e., restrict ourselves to regular equivalence) then the matrices $\hat{\xi}_a$ which are multiples of the unit matrix will be determined up to a sign.

So the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ can be defined in the space of the algebra $AP(1,3)$ representation of Class I by sixteen nonequivalent ways corresponding to various multiplicators of the group $G_8$ (or, which is the same, to various commutation and anticommutation relations (5.9)). Incidentally, in four cases the representation space $D$ of the algebra $AP(1,3)$ is reduced to the direct sum of two invariant subspaces $D = D^+(m,s) \oplus D^-(m,s)$ (if $\mu_1 \mu_4 = \mu_2 \mu_3 = 1$) and in the remaining twelve cases the operators $P_{\mu}$, $J$ and $N$ of (5.11) realize the direct sum of four IRs $D = D^+(m,s) \oplus D^-(m,s) \oplus D^+(m,s) \oplus D^-(m,s)$.

The explicit form of the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ for the representation (5.11) is given by formulae (5.13), (5.22), where $\hat{\xi}_a$ are the matrices given in (5.29), (5.30). For other realizations of the algebra $AP(1,3)$ being equivalent to (5.11) these operators can be obtained from (5.13), (5.22) by the change $\Psi \rightarrow V(p)\Psi$, $U_1 \rightarrow U_1' = V(p)U_1 V^{-1}(-p)$, $U_2 \rightarrow U_2' = V(p)U_2 V^{-1}(-p)$, $U_3 \rightarrow U_3' = V(p)U_3 V^{-1}(-p)$, where $V(p)$ is an operator of equivalence transformation. In particular for the Foldy-Shirokov representation (4.50) we obtain

$$U_1' = \hat{\xi}_1, \quad U_2' = \hat{\xi}_2, \quad U_3' = \hat{\xi}_3 \Delta.$$  \hspace{1cm} (5.32)

We represent also the explicit form of the matrix $\Delta$ determined by relations (5.24). If $\mu_1 = \mu_4 = 1$, then

$$\Delta = \Delta_2 = \begin{pmatrix} \Delta' & 0 \\ 0 & \Delta' \end{pmatrix}$$  \hspace{1cm} (5.33)

where $\Delta'$ is the matrix of dimension $2(2s+1) \times 2(2s+1)$ given by the formula

$$\Delta' = (i)^2 \exp[i(\lambda_2 + n_2)\pi].$$  \hspace{1cm} (5.34)

If $\mu_1 = -1$ or $\mu_4 = -1$ then $\Delta$ is a direct sum of two matrices $\Delta_2$. 

69
5.5. Representations of Class II

The representations of the algebra \( AP(1,3) \) corresponding to the zero eigenvalues of \( C_1 \) and \( C_2 \) have the two additional Casimir operators \( C_3 \) and \( C_4 \) (4.12). The operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) have to satisfy (5.2)-(5.4) together with \( C_3 \) and \( C_4 \). So these operators transform spaces of IRs of the \( AP(1,3) \) algebra according to the following diagram

\[
\begin{array}{c}
D^\prime(\lambda) & \rightarrow & D^\prime(-\lambda) \\
\hat{P} & \uparrow & \hat{P} \\
D^\prime(-\lambda) & \leftarrow & D^\prime(\lambda) \\
\hat{C} & \end{array}
\] (5.35)

One concludes from (5.35) the operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) can be defined in a space of a reducible representation of the algebra \( AP(1,3) \) belonging to Class II which can be reduced to a direct sum of the least four IRs \( D(\lambda), D^\prime(\lambda), D^\prime(-\lambda) \), and \( D^\prime(-\lambda) \). We will choose the elements of such a representation in the form (4.51) where \( \lambda_0 \) and \( \hat{e} \) are diagonal matrices having the eigenvalues \( \pm s \) and \( \pm 1 \), \( s \) being a fixed integer or half integer.

We represent the operators \( U_1, U_2, \) and \( U_3 \) from (5.13) in the following form

\[
U_1 \equiv \xi_1 U, \quad U_2 \equiv \xi_2, \quad U_3 \equiv \xi_3 U
\] (5.36)

where \( U \) is the operator (4.53), \( \xi_a \) are the operators to be found. Bearing in mind that the operator \( U \) changes the sign of the vector \( \mathbf{n} \) in the operators (4.51), it is not difficult to show that the conditions (5.14), (5.15) reduce to the following equations for \( \xi_a \):

\[
\xi_1 \lambda_0 = -\lambda_0 \xi_1, \quad \xi_2 \lambda_0 = \lambda_0 \xi_2, \quad \xi_3 \lambda_0 = \lambda_0 \xi_3,
\] (5.37)

\[
\xi_1 \hat{e} = -\hat{e} \xi_1, \quad \xi_2 \hat{e} = -\hat{e} \xi_2, \quad \xi_3 \hat{e} = -\hat{e} \xi_3, \quad [\xi_a \xi_b] = [\xi_a \hat{e}] = 0.
\] (5.38)

Thus the problem of description of nonequivalent operators \( \hat{P}, \hat{C} \) and \( \hat{T} \) for the representations of Class II reduces to solving the system of equations (5.21), (5.37), (5.38) for the matrices \( \lambda_0, \hat{e}, \) and \( \xi_a \). The irreducible sets of the matrices satisfying these equations are exhausted by the following combinations [12].

When \( \mu_1 \mu_2 = 1 \), dimension of matrices is 4×4 and their explicit form is given by the formulae
Chapter 2. Representations of the Poincaré algebra...

\[ \hat{e} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \lambda_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \]

\[ (5.39) \]

where \( \sigma_0, \sigma_1, \sigma_3 \), and \( \lambda_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \) are the matrices (5.30) of dimension 2×2.

If \( \mu \mu_4 = -1 \) then dimension of the corresponding matrices is 8×8. Moreover

\[ \xi_1 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \mu_\sigma \sigma_1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & \delta_\mu \\ 0 & \mu_\delta \mu_\sigma \end{pmatrix} \]

where \( \sigma_0, \sigma_1, \delta_\sigma, \) and \( \delta_\mu \) are the matrices (5.30) of dimension 2×2.

5.6. Representation of Classes III-IV

Thus we have determined all the possible (up to projective equivalence) operators of space inversion, charge conjugation and time reflection which can be defined in the spaces of representations of Class II. The space \( D \) of such a representation is reduced to the direct sum of the subspaces \( D'(s) \oplus D(s) \oplus D'(-s) \oplus D(-s) \) for \( \mu_4 = 1 \) or \( D'(s) \oplus D'(s) \oplus D(s) \oplus D(s) \oplus D'(-s) \oplus D'(-s) \oplus D'(-s) \oplus D(s) \) for \( \mu_4 = -1 \). The corresponding operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) are given by formulae (5.20), (5.36) where \( \xi_\sigma \) are the matrices (5.39) or (5.40).

Thus there exist 16 nonequivalent possibilities of definition of the operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) in the space of a representations of Class II. The space \( D \) of such a representation is reduced to the direct sum of the subspaces \( D'(s) \oplus D(s) \oplus D'(-s) \oplus D(-s) \) for \( \mu \mu_4 = 1 \) or \( D'(s) \oplus D'(s) \oplus D(s) \oplus D(s) \oplus D'(-s) \oplus D'(-s) \oplus D'(-s) \oplus D(s) \) for \( \mu_4 = -1 \). The corresponding operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) are given by formulae (5.20), (5.36) where \( \xi_\sigma \) are the matrices (5.39) or (5.40).

\[ \xi_1 = \begin{pmatrix} \gamma_0 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \Gamma_\mu \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & \Gamma_\mu \mu_\sigma \\ -\Gamma_\mu \mu_\sigma & 0 \end{pmatrix}, \]

and \( \gamma_\mu \) are the Dirac matrices (2.4).

Thus we have determined all the possible (up to projective equivalence) operators of space inversion, charge conjugation and time reflection which can be defined in the spaces of representations of the algebra \( AP(1,3) \) belonging to the most important for physical applications Classes I and II. In an analogous way we can find these operators for other representation classes. Here we consider briefly the corresponding possibilities.

The representations of the algebra \( AP(1,3) \) belonging to Class III (\( P_\mu P_\mu = 0, -W_\mu W_\mu = r^2 > 0 \)) have the only additional Casimir operator \( \mathcal{C}_s = \hat{P}_s / |P_s| \), so the description of the corresponding operators \( \hat{P}, \hat{T} \) and \( \hat{C} \) does not differ in principle from the case of representations of Class I. Let us set in (5.13)
Symmetries of Equations of Quantum Mechanics

where $U$ and $V$ are the operators (4.53) and (5.23) (with an appropriate matrix $\lambda_0$), $\lambda$ and $\Delta$ are unitary matrices satisfying the conditions (5.24) and

\[ \Lambda^* = \Lambda, \quad \Lambda \Lambda^* = \left( \begin{array}{cc}
-\lambda + \frac{2(nx^*y^*)^n}{1-(n\cdot n^*)^2} & 
\lambda \end{array} \right) \Lambda. \]  
(5.42)

As a result, by using the following properties of the operators $U\Lambda$ and $V$:

\[ U\Lambda^* = \Lambda U, \quad U\Lambda(J, N) = (J', N')U\Lambda, \quad U\Lambda(P, P) = (P, P)U\Lambda, \quad \forall \lambda = -\lambda V, \]  
(5.43)

(where $J'$ and $N'$ are the operators obtained from $J$ and $N$ by the change $p \rightarrow -p$), we come once again to the equations (5.21) for the matrices $\xi_a$ whose general solution is given by (5.28), (5.29) (where $I$ is the unit operator in the space of IRs of the algebra $AE(2)$). The corresponding representation of the algebra $AP(1,3)$ is expanded in the direct sum either of two IRs $D(r) \oplus D(-r)$ if $\mu_1\mu_4 = \mu_2\mu_3 = 1$ or of four IRs $D(r) \oplus D^*(r) \oplus D^*(r) \oplus D(-r)$ otherwise. The explicit expressions for the matrices $\Lambda$ and $\Delta$ in the basis (4.40) are given by the formulae

\[ \Lambda_1 |0, r, \lambda> = \frac{\lambda}{1 |\lambda|} |0, r, -\lambda>, \quad \Lambda_2 |0, r, \lambda> = (-1)^{|\lambda|} \text{sgn} \lambda |0, r, -\lambda>, \]  
(5.44)

$[\lambda]$ is the entire part of $\lambda$.

Formulae (5.13), (5.41), (5.28), (5.29), (5.44) give the explicit form of all the nonequivalent operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ defined in the space of representations of the algebra $AP(1,3)$ of Class III.

A more complicated situation arises for the representations of Class IV corresponding to zero eigenvalues of the Casimir operator $C_1$. Such representations can be subdivided into two subclasses. The first one, subclass IVA for which the eigenvalues of the operator $W_\mu W_\mu$ are equal to $\eta^2(l+1)$ (refer to (4.44)) has the additional Casimir operator $C_5=W_0/|W_0|$. The representations belonging to this class are denoted by $D(\eta, l, \mu)$ where $\mu = \pm 1$ denotes the eigenvalue of the operator $W_0/|W_0|$. Other subclass (denoted by IVB) corresponds to negative eigenvalues of $W_\mu W_\mu$ and has no additional Casimir operators. The IRs belonging to subclass IVB are denoted by $D(\eta, \alpha)$, the meaning of $\eta$ and $\alpha$ being clear from (4.44).

The operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined in the space of the reducible representation $D=D(\eta, l, \mu) \oplus D(\eta, l, -\mu)$ if $\mu, \mu_1, \mu_2, \mu_3 = 1$ and $D=D(\eta, l, \mu) \oplus D(\eta, l, \mu) \oplus D(\eta, l, \mu) \oplus D(\eta, l, -\mu)$ otherwise. The explicit form of these operators is up to
equivalence given by formulae (5.17), (5.36) where \( \xi_a \) are the matrices represented in (5.28), (5.29). The symbol \( I \) in (5.28), (5.29) defines the unit operator in the space of the IRs \( D(\eta, l, e\mu) \), \( e=\pm 1 \).

For the subclass IVB the operators \( \hat{P} \), \( \hat{T} \) and \( \hat{C} \) can be defined in the space of the IRs \( D(\eta, \alpha) \) or in the space of the direct sum of two IRs \( D(\eta, \alpha) \oplus D(\eta, \alpha) \) or of four IRs \( D(\eta, \alpha) \oplus D(\eta, \alpha) \oplus D(\eta, \alpha) \oplus D(\eta, \alpha) \). The general form of these operators is defined in (5.17), (5.41), (5.44) where \( \xi_a \) are matrices satisfying (5.21), (5.27). The explicit form of all the nonequivalent and indecomposable matrices \( \xi_a \) is given by the following formulae

\[
\begin{align*}
\mu_1=\mu_2=\mu_4=1, & \quad \xi_1=\xi_2=\xi_3=I, \\
\mu_3=1, \mu_2=1, \text{ or } \mu_1\mu_4=1, & \quad \xi_1=\delta_{\mu_1}, \quad \xi_2=\delta_{\mu_2}, \quad \xi_3=\alpha_{\mu_4}, \\
\mu_3=-1, \mu_2=1, \mu_4=1, & \quad \xi_1=\sigma_{\mu_1}, \quad \xi_2=\sigma_{\mu_2}, \quad \xi_3=\delta_{\mu_4},
\end{align*}
\]

(5.45)

and for other values of \( \mu_i \) - by formulae (5.28), (5.29) where \( I \) is the unit operator in the space of the IRs \( D(\eta, \alpha) \).

5.7. Representations of Class V

Let us consider also the finite dimensional representations of Class V corresponding to \( p_\mu \equiv 0 \). The description of all nonequivalent operators \( \hat{P} \), \( \hat{T} \) and \( \hat{C} \) defined in the space of such representations is of great interest because they are the operators which can be defined on the sets of solutions of relativistic wave equations.

The representations belonging to Class V have the two Casimir operators (4.12). According to (5.2)–(5.4) the operators \( \hat{P} \), \( \hat{T} \) and \( \hat{C} \) have to anticommute with \( C_7 \). We conclude from this fact that these operators transform a vector from the space of the IR \( D(l_0, l_1) \) into the vector belonging to the space of the IR \( D(-l_0, l_1) \). Hence the operators \( \hat{P} \), \( \hat{T} \) and \( \hat{C} \) can be defined in a space of at least two IRs \( D(l_0, l_1) \oplus D(-l_0, l_1) \) (except the case \( l_0=0 \) when it is possible in principle to restrict ourselves to using a single IR).

Let \( S_{\mu} \) be matrices which realize the direct sum \( D(l_0, l_1) \oplus D(-l_0, l_1) \) of IRs with fixed \( l_0 \) and \( l_1 \), \( l_0 \neq 0 \), each of IRs being included with some multiplicity. The corresponding operators \( \hat{P} \), \( \hat{T} \) and \( \hat{C} \) can be defined by formulae (5.19) where \( \xi_a \) are the matrices satisfying relations (5.20), (5.21). All the nonequivalent indecomposable matrices \( \xi_a \) are given by the formulae

\[
\begin{align*}
\xi_1=\xi\beta, & \quad \xi_2=\xi\beta, \\
\xi_3=\xi\beta,
\end{align*}
\]

(5.46)

the matrices from the r.h.s. of this formula are represented in Table 5.3, see the following page. Moreover \( \alpha\beta \), \( \delta\beta \) are the matrices (5.30) with \( I \) being the unit
Symmetries of Equations of Quantum Mechanics

Table 5.3

<table>
<thead>
<tr>
<th>Values of $\mu_k$</th>
<th>$\mu_1,\mu_2,\mu_3=1$, $\mu_4=1$</th>
<th>$\mu_1,\mu_2,\mu_3=-1$, $\mu_4=1$</th>
<th>$\mu_1,\mu_2,\mu_3=1$, $\mu_4=-1$</th>
<th>$\mu_1,\mu_2,\mu_3=-1$, $\mu_4=-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>representation realized by $S_{\mu\sigma}^\alpha$</td>
<td>$D(l_0,l_1) \oplus D(-l_0,l_1)$</td>
<td>$D(l_0,l_1) \oplus D(-l_0,l_1)$</td>
<td>$D(l_0,l_1) \oplus D(-l_0,l_1)$</td>
<td>$D(l_0,l_1) \oplus D(-l_0,l_1)$</td>
</tr>
<tr>
<td>explicit form of $\xi'$</td>
<td>$\sigma_1$</td>
<td>$\begin{pmatrix} \sigma_1 &amp; 0 \ 0 &amp; \sigma_1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} \sigma_1 &amp; 0 \ 0 &amp; \sigma_1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} \sigma_1 &amp; 0 \ 0 &amp; \sigma_1 \end{pmatrix}$</td>
</tr>
<tr>
<td>explicit form of $\xi''$</td>
<td>$\alpha_{\mu_3}$</td>
<td>$\begin{pmatrix} 0 &amp; \delta_{\mu_3} \ \delta_{\mu_3} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; \delta_{\mu_3} \ \delta_{\mu_3} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} \alpha_{\mu_3} &amp; 0 \ 0 &amp; \alpha_{\mu_3} \end{pmatrix}$</td>
</tr>
<tr>
<td>explicit form of $\xi'''$</td>
<td>$\delta_{\mu_2}$</td>
<td>$\begin{pmatrix} 0 &amp; \alpha_{\mu_3} \ \mu_1 \alpha_{\mu_3} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; \alpha_{\mu_3} \ \mu_1 \alpha_{\mu_3} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; -\delta_{\mu_2} \ \delta_{\mu_2} &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>explicit form of $\beta$</td>
<td>$p^{(2)} \begin{pmatrix} 0 &amp; p \ p &amp; 0 \end{pmatrix}$</td>
<td>$p^{(2)} \begin{pmatrix} 0 &amp; p \ p &amp; 0 \end{pmatrix}$</td>
<td>$p^{(2)} \begin{pmatrix} 0 &amp; p \ p &amp; 0 \end{pmatrix}$</td>
<td>$p^{(2)} \begin{pmatrix} 0 &amp; p \ p &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Matrices in the space of the IRs $D(l_0,l_1)$, and $p$ are the space inversion matrices for the IRs $D(l_0,l_1)$, as determined by the relation

$$p_{|l_0,l_1;l,m\rangle} = (-1)^l |l_0,l_1;l,m\rangle.$$  \hfill (5.47)

Formulae (5.46) are valid only for the representations of the algebra $AO(1,3)$ in the basis (4.66)-(4.68). For the representations of the type $D(0,l_1)$ the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ are also given by formulae (5.46) where $\xi'_a$ are the matrices (5.28), (5.29).

So we have described all the nonequivalent representations of the discrete symmetry operators which can be defined in a space of a representation of the homogeneous Lorentz group. As in the case of representations of Classes I-IV there are 16 nonequivalent possibilities in defining of such operators corresponding to possible
5.8. Concluding Remarks

We have described all the possible nonequivalent realizations of the discrete symmetry transformations in a space of a representation of the algebra \( A_P(1,3) \). Actually, we have also determined all the projective-nonequivalent local representations of the extended Poincaré group \( P(1,3) \) including the transformations \( P \), \( C \) and \( T \) in addition to proper inhomogeneous Lorentz transformations.

We note that in the case of the two-digit representations of the group \( P(1,3) \) it is possible to determine the operators \( P \), \( C \) and \( T \) in such a way that their squares be equal to \( \pm 1 \), inasmuch as a double reflection can be represented either as an identity transformation or as a rotation by the angle \( 2\pi \) (which reduces to the multiplication by \( -1 \) in two-digit representations). At this, all the possible products of the operators \( P \), \( C \) and \( T \) also form a finite group and the number of different groups is equal to 64. But IRs of these groups reduces to IRs of the group \( G_8 \) up to projective equivalence.

It follows from the above that the operators of \( P \)-, \( C \)- and \( T \)-transformations defined on the sets of solutions of the KGF, Dirac, and Maxwell equations, realize only some of possible representations of these transformations. In other words, only some of possible groups \( G_8 \) are realized here. There arises the natural question if there exist relativistic wave equations for scalar, spinor and vector fields fields which correspond to another representations of the operators \( P \), \( T \) and \( C \). This intriguing problem is discussed partly in Sections 6-9.

6. POINCARÉ-IN Variant EQUATIONS OF FIRST ORDER

6.1. Introduction

In this and the following sections we present some elements of the theory of Poincaré-invariant equations for particles of arbitrary spin.

Let \( \{ \psi(x) \} \) be a set of solutions of a motion equation of a relativistic particle with fixed mass and spin. In relativistic quantum theory the space of states of a free (noninteracting) particle is identified with the space of the IR of the Poincaré group. So \( \{ \psi(x) \} \) by definition has to be a representation space of the algebra \( A_P(1,3) \) and any \( \psi \in \{ \psi(x) \} \) has to satisfy the following conditions

\[
P_{\mu}(\gamma^\mu m^2 - s(s+1))\psi = 0
\]

where \( m \) and \( s \) are parameters determining the mass and spin of a particle.
The conditions (6.1) mean that the Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ have fixed eigenvalues on the set of solutions of a particle equation. If we choose a concrete realization of the algebra $AP(1,3)$ then the conditions (6.1) by themselves can be considered as a system of linear equations describing a particle of spin $s$ and mass $m$.

As a matter of fact, all the known Poincaré-invariant equations for particles of fixed mass and spin are nothing but the conditions (6.1) written in other forms. Thus, the relations (6.1) represent a system of partial differential equations of second order in all cases when $P_\mu$ and $J_{\mu\nu}$ realize a covariant representation of the algebra $AP(1,3)$. In the case when $s=1$ and $S_{\mu\nu}$ belong to the representation $D(1/2,1/2)$ these relations reduce to the Proca equation [404]. The Dirac equation is a formulation of the conditions (6.1) in the form of an equivalent system of first order partial differential equations, etc.

Here we present some elements of the theory of Poincaré-invariant equations of the form

$$\left(\beta_\mu p^\mu - \beta_4 m\right)\psi = 0 \quad (6.2)$$

where $\beta_\mu$, $\beta_4$ are numerical matrices and $\psi$ is a vector-function (a column matrix). The importance of the analysis of such equations is that equations including the derivatives of higher order can always be reduced to a system of partial differential equations of first order by introducing new dependent variables. For example, the KG equation (1.1) reduces to the form (6.2) by the substitution $p_\mu \Phi = m \Phi_\mu$, $\Phi = \psi_4$, where $\psi = \text{column}(\psi_1, \psi_2, \psi_3, \psi_4)$, $\beta_4 = 1$ and $\beta_\mu$ are the Kemmer-Duffin matrices of dimension 5x5 (see (6.17) below).

Poincaré-invariance of the system (6.2) means that this system admits 10 independent SOs forming the algebra $AP(1,3)$. Restricting ourselves to the case when such SOs belong to the class $M_1$ (i.e., has the covariant form (2.22) with appropriate matrices $S_{\mu\nu}$) we come naturally to the following definition.

**DEFINITION 6.1.** We say the equation (6.2) is Poincaré-invariant and describes a particle of mass $m$ and spin $s$ if a covariant representation of the algebra $AP(1,3)$ is realized on the set of its solutions and the Casimir operators of this algebra satisfy the conditions (6.1).

Relativistic wave equations of the kind (6.2) where $\beta_4$ is an invertible matrix are well studied. In the works of Bhabha [35], Harish-Chandra [216], Wild [416], Umedzawa [404] the general form of the matrices $\beta_\mu$ was found for Poincaré-invariant equation (6.2). Besides the additional conditions for $\beta_\mu$ had been formulated which follows from the requirements (6.1). In the Russia studying of such equations started with the works of Tamm and Ginzburg [202, 203], the important results in this direction were obtained by Fedorov [97], Schelepin [384], and especially by Gelfand and Yaglom [200] who described finite and infinite dimensional first order wave
Chapter 2. Representations of the Poincaré algebra...

equations for any spin particles in the frame of the united approach. Nevertheless the theory of Poincaré-invariant equations of the kind (6.1) is still far from complete. In particular the equations with a singular matrix $\beta_4$ and the equations corresponding to indecomposable representations of the Lorentz group are not completely studied yet.

6.2. The Poincaré-Invariance Condition

We restrict ourselves to the case when the representation of the algebra $AP(1,3)$ over the set of the equation (6.1) solutions is completely reducible. Then, the basis elements of this algebra are given by formulae (2.2) where $S_{\mu\nu}$ are matrices realizing a direct sum of IRs of the algebra $AO(1,3)$.

Let the number of equations in the system (6.2) be equal to the number of unknowns. In analogy with (2.21), the condition of invariance of such a system under the algebra $AP(1,3)$ can be written in the form

$$[Q_\lambda, \beta^\nu_\mu p_\mu - \beta_4 m] p_\nu - \beta^\nu_\mu m = 0$$

(6.3)

where $Q_\lambda$ is any operator from the set (2.2), $\beta^\nu_\mu (x)$ is a matrix of dimension $n \times n$ depending on $x$. Substituting (2.2) into (6.3) and equating coefficients of the same differential operators we can make sure that for $Q_\lambda \in \{ P_\mu \}$ the condition (6.3) is satisfied identically, but for $Q_\lambda \in \{ J_{\mu\nu} \}$ we obtain the following conditions for the matrices $\beta_\mu$, $\beta_4$:

$$[S_{\mu\nu} \beta_4, \beta_\mu] = 0,$$

$$[S_{\mu\nu} \beta_\lambda, \beta_{\mu\nu}] = (g_{\nu\lambda} \beta_\mu - g_{\mu\lambda} \beta_\nu) \beta_{\mu\nu}.$$  

or

$$\tilde{S}_{\mu\nu} \beta_4 = 0,$$

$$\tilde{S}_{\mu\nu} \beta_\lambda = 0,$$

(6.4)

Using the commutation relations (3.49) for $S_{\mu\nu}$, it is not difficult to deduce the following conditions from (6.4):

$$[\tilde{S}_{\mu\nu}, \tilde{S}_{\mu\nu}] \beta_4 = 0,$$

$$[\tilde{S}_{\mu\nu}, \tilde{S}_{\mu\nu}] \beta_\lambda = (g_{\nu\lambda} \tilde{S}_{\mu\nu} - g_{\mu\lambda} \tilde{S}_{\mu\nu}) \beta_{\mu\nu}.$$  

(6.5)

A sufficient condition of validity of (6.5) is the requirement that the matrices $\tilde{S}_{\mu\nu}$ realize a representation of the algebra $AP(1,3)$ (which in general can differ from the representation realized by $S_{\mu\nu}$). If the conditions (6.4) are fulfilled with $S_{\mu\nu}$, $\tilde{S}_{\mu\nu}$ being matrices satisfying the algebra $AP(1,3)$, then we say the equation (6.2) is invariant under the algebra $AP(1,3)$ in the strong sense, or S-invariant.

So we come to the following definition.

DEFINITION 6.2. The equation (6.2) is S-invariant under the Poincaré algebra if there exist such matrices $S_{\mu\nu}$, $\tilde{S}_{\mu\nu}$ realizing representations of the algebra...
AO(1,3) then the conditions (6.5) are satisfied for \( \beta^\mu \) and \( \beta_4 \).

We emphasize that Definition 6.2 gives only the sufficient conditions of the Poincaré-invariance of the equation (6.2) inasmuch as in general the algebra \( AP(1,3) \) can be realized by operators which do not belong to the class \( M \), and, besides, it is not necessary for \( S_{\mu\sigma} \) to satisfy the algebra \( AP(1,3) \) (the requirement (6.5) is less strong).

We note that Definition 6.2 is also valid for the case when the number \( n \) of components of the function \( \psi \) does not coincide with the number \( m \) of equations. In this case \( \beta^\mu \) and \( \beta_4 \) are matrices of dimension \( m \times n \), \( m \neq n \), \( S_{\mu\nu} \) and \( \tilde{S}_{\mu\nu} \) are square matrices of dimension \( m \times m \) and \( n \times n \).

Further on, we restrict ourselves to considering of Poincaré-invariant equations (6.2) with square matrices \( \beta^\mu \), \( \beta_4 \) where \( \beta_4 \) is nonsingular. The theory of relativistic wave equations with a singular matrix \( \beta_4 \) is still not completed (see, however [1, 95, 221]).

If \( \beta_4 \) is nonsingular then, without loss of generality, we can set \( \beta_4 = I \) where \( I \) is the unit matrix. Incidentally the system (6.2) reduces to the form

\[
(\beta^\mu p^\nu - m)\psi = 0.
\]  

According to (6.4), \( S_{\mu\nu} = \tilde{S}_{\mu\nu} \) if \( \beta^\mu = I \), and the conditions (6.5) reduce to the following form

\[
[S_{\mu\nu}, \beta^\lambda] = (\sigma_{\alpha\beta} \beta_\alpha - \sigma_{\alpha\beta}\beta_\beta) \beta^\lambda.
\]  

The problem of describing of Poincaré-invariant equations (6.6) reduces to finding the general solution of the equations (3.49), (6.7). In the following subsection we represent this solution for the case when the matrices \( S_{\mu\nu} \) are completely reducible.

6.3. The Explicit Form of the Matrices \( \beta^\mu \)

Solving the equations (6.7) includes generally speaking the following steps:

a) finding the matrices \( S_{\mu\nu} \) satisfying the algebra (3.49), i.e., describing that representations of this algebra;

b) selection of such representations for which the equations (6.7) have nontrivial solutions;

c) and, finally, determining explicit form of the matrices \( \beta^\mu \) satisfying (6.7).

We restrict ourselves to the case when \( S_{\mu\nu} \) realize a finite dimensional reducible representation of the algebra \( AO(1,3) \). Such a representation without loss of generality can be taken in the form of a direct sum of IRs given in Subsection 4.8.

Let us denote by \(|(j \tau; lm)\rangle\) a vector belonging to the space of the IR \( D(j \tau) \) and being an eigenfunction of the operators (4.12), (4.63) (the index \( \lambda \) is introduced in order to label the spaces of equivalent representations) and by
Chapter 2. Representations of the Poincaré algebra...

\[ <(j; \lambda \gamma; lm)_k \beta \lambda \mu (j'; \lambda' \gamma'; l'm')_k> - \text{the matrix elements of the matrices } \beta \mu \text{ from the invariant equation (6.6).} \]

Then the main assertion concerning the general form of matrices \( \beta \mu \) satisfying (6.7) can be formulated as follows [58].

**THEOREM 6.3.** Let \( S \mu \) be matrices realizing a direct sum of the IRs \( D(j \tau) \) of the algebra \( AP(1,3) \) and \( \beta \mu \) be matrices satisfying (6.7). Then elements of the matrix \( \beta \mu \) differ from zero only in those cases when \( j-j'=1/2 \) and \( \tau-\tau'=1/2 \), with the following relations being valid:

\[
\begin{align*}
(j; \lambda \gamma; lm)_k \beta \lambda \mu (j'; \lambda' \gamma'; l'm')_k & = C_{\alpha \beta} \delta_{n \lambda} \delta_{\mu \alpha} [j-l+1(j-l+1)]^{1/2}, \\
(j; \lambda \gamma; lm)_k \beta \lambda \mu (j'; \lambda' \gamma'; l'm')_k & = (-1)^{j-j'} C_{\alpha \beta} \delta_{n \lambda} \delta_{\mu \alpha} [j+l+1(j+l+1)]^{1/2}, \\
(j+1/2; \lambda \gamma; lm)_k \beta \lambda \mu (j+1/2; j'; \lambda' \gamma'; l'm')_k & = (-1)^{j-j'} C_{\alpha \beta} \delta_{n \lambda} \delta_{\mu \alpha},
\end{align*}
\]

(6.8)

where \( j > \tau, \ j \leq \tau \leq j + \tau, \) \([l] \) is the entire part of \( l \),

\[
<(j; \lambda \gamma; lm)_k \beta \lambda \mu (j; \lambda' \gamma'; l'm')_k> = (-1)^{j-j'}<(j'; \lambda' \gamma'; l'm')_k \beta \lambda \mu (j; \lambda \gamma; lm)_k>, \quad j \neq \tau, \quad j \neq \tau',
\]

(6.9)

Here \( C_{\alpha \beta} \) are arbitrary complex numbers. The matrix elements of \( \beta \lambda \), \( \beta \mu \) and \( \beta \nu \) can be obtained from (6.8), (6.9) by using relations (6.7).

Thus, the matrices \( \beta \mu \) are determined by the conditions (6.7) up to arbitrary complex coefficients \( C_{\alpha \beta} \). The number of these coefficients decreases if we require the \( P-, T- \) and \( C- \) invariance of the equations (6.6). Let \( \lambda, \lambda' \) and \( \mu, \mu' \) numerate two pairs of nonequivalent representations which are transformed one into another by the space inversion. Then for \( P- \) invariance of (6.6) it is necessary to require \( C_{\alpha \beta} = -C_{\mu \nu} \). Other restrictions for \( C_{\alpha \beta} \) are considered in the following subsection.

6.4. Additional Restrictions for the Matrices \( \beta \mu \)

If we restrict ourselves to completely reducible representations of the algebra \( AO(1,3) \) then formulae (6.8), (6.9) give all the possible matrices \( \beta \mu \) such that the equation (6.2) is \( S- \) invariant under the Poincaré algebra. But a solution for this equation does not satisfy in general the conditions (6.1), and so cannot a priori be interpreted as a motion equation for a particle with fixed mass and spin.

The equations (6.1) impose additional restrictions on the form of the matrices \( \beta \mu \). As was shown in [416] the necessary and sufficient condition of validity of the first of relations (6.1) is
where $\beta$ is the maximal spin value appearing by the reduction of the representation of the algebra of matrices $S_{\mu\nu}$ by the algebra $AO(3)$. According to $(6.11)$, the corresponding eigenvalues of the matrix $\beta^2$ can be equal to 0 or 1 only.

So the first of the conditions $(6.1)$ admits the simple formulation $(6.10)$. As to the second equation $(6.1)$, it reduces to the following relations for $\beta^0$:

$$S^2 P \equiv \frac{1}{2} S_{\mu\nu} S_{\alpha\beta} \beta^{2-1-[i]} \equiv (S+1) \beta^{2-1-[i]}$$

(6.11)

where $S_{\mu\nu}$ are the matrices $S_{\mu\nu}$ with $\nu \neq 0$, and $P = \beta^{2-1-[i]}$ is the projector on the subspace corresponding to nonzero eigenvalues of the matrices $\beta$.

One more restriction on the matrices $\beta$ can be obtained from the requirement that the equation (6.6) admits the Lagrangian formulation, i.e., can be deduced from the corresponding Lagrangian using the minimal action principle. This requirement means that a nonsingular Hermitizing matrix $\eta$ exists satisfying the conditions

$$\eta S_{\mu\nu} = S_{\mu\nu}^\dagger \eta,$$

(6.12)

$$\eta \beta = \beta^\dagger \eta.$$

(6.13)

The necessary and sufficient condition of the matrix $\eta$ existence is that any IR $D(j \tau)$ included into the representation realized by $S_{\mu\nu}$ be supplemented by the conjugated representation $D(\tau j)$ if $j \neq \tau$. At this, the conditions (6.12) determine the matrix $\eta$ up to a factor if any IRs $D(j \tau)$ has the unit multiple $[58]$. The explicit form of $\eta$ is given by the formulae:

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(6.14)

where $I$ and $0$ are the zero and unit matrices of dimension $(2j+1) \times (2j+1)$;

$$\eta = \begin{pmatrix} 0 & P_j \\ P_j & 0 \end{pmatrix}$$

(6.15)

where $P_j$ are the $(2j+1)(2\tau+1) \times (2j+1)(2\tau+1)$ matrices given in (5.47), (4.61) (they are denoted there by $p$);

$$\eta = P_{ji}$$

(6.16)

As a representation $D$ realized by $S_{\mu\nu}$ is a direct sum of the representations a), b), and c), then as a matter of fact formulae (6.14)-(6.16) set up the general form of $\eta$ for the case when this sum is nondegenerated. For the general form of $\eta$ in the case when $S_{\mu\nu}$ realize a degenerated direct sum of IRs refer, for example, to [197].

Thus, the general form of the matrices $\beta$ determining the Poincaré-invariant
Chapter 2. Representations of the Poincaré algebra...

equation (6.6) for a particle of spin \( s \) is given by formulae (6.8), (6.9) with the coefficients \( C_{\mu} \) having to satisfy additional conditions stemming from (6.10), (6.11), (6.13)-(6.16). In the following we consider examples of these equations for \( s \leq 3/2 \).

6.5. The Kemmer-Duffin-Petiau (KDP) Equation

The simplest example of a Poincaré-invariant equation of the kind (6.6) is the Dirac equation considered in Section 2. This equation can be obtained with formulae (6.8), (6.9) starting with the representation

\[
D = D(0 \ 1/2) \oplus D(1/2 \ 0)
\]

for the matrices \( S_{\mu \nu} \).

Let us consider more complicated examples. By choosing the representation

\[
D = D(0 \ 0) \oplus D(1/2 \ 1/2)
\]

we obtain from (4.58)-(4.60), (6.8), (6.9), (6.10)-(6.12), (6.16) the following expressions for the matrices \( \beta_\mu \) in the basis

\[
\psi \rightarrow \text{column} \begin{pmatrix} 00;00, \frac{1}{2} 1;0;1, \frac{1}{2} 2;1;1, \frac{1}{2} 2;1;0, \frac{1}{2} 1;1;0, \frac{1}{2} 1;1;1, \frac{1}{2} 1;1;2, \frac{1}{2} 1;2;1, \frac{1}{2} 1;2;2, \frac{1}{2} 2;2;2 \end{pmatrix}
\]

\[
\beta_0 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & \hat{0} & 0 \end{pmatrix}, \quad \beta_Q = \begin{pmatrix} 0 & 0 & \lambda_a \\ 0 & 0 & \lambda_a^\dagger \\ \hat{\lambda}_a & \hat{\lambda}_a^\dagger & 0 \end{pmatrix}
\]

(6.17)

where \( \hat{0} \) and \( \hat{\lambda}_a \) are the zero matrices of dimensions 4×4 and 1×3, \( \lambda_a \) are the row matrices

\[
\lambda_1 = (i \ 0 \ 0), \quad \lambda_2 = (0 \ i \ 0), \quad \lambda_3 = (0 \ 0 \ i).
\]

(6.18)

Without loss of generality, we set an arbitrary constant \( C_{10} \) in (6.8) be equal to 1.

The equation (6.6) with the matrices (6.17) describes a particle of mass \( m \) and spin \( s = 0 \). Furthermore, \( \beta_\mu \) satisfies the criteria (6.10), (6.11), (6.13) where \( s = 0 \) and

\[
S_{\mu \nu} = [\beta_\mu, \beta_\nu], \quad \eta = (1 + 2\beta_0^2)(1 + 2\beta_1^2)(1 + 2\beta_2^2).
\]

(6.19)

Formulae (6.6), (6.17) define the KDP equation for a scalar particle. The matrices (6.17) satisfy the algebra

\[
\beta_\mu \beta_\nu - \beta_\nu \beta_\mu = g_{\mu \nu} \beta_3, \quad \gamma_{12} \beta_3 = \sigma_{\mu \nu} \beta_3 + g_{\mu \nu} \beta_3.
\]

(6.20)

These relations define \( \beta_\mu \) up to unitary equivalence.

The KDP equation is invariant under the \( P \)-, \( T \)- and \( C \)-transformations (2.55), (2.60) where

\[
r_1 = (1 - i\beta_0^2), \quad r_2 = 1, \quad r_3 = \eta.
\]

(6.21)

Consider the KDP equation for a vector particles which may be taken in the form (6.6) with \( \beta_\mu \) being 10×10 irreducible matrices satisfying the algebra (6.20). The explicit realization of these matrices (which is defined by the relations (6.20) up to
unitary equivalence) can be chosen in the following form:

\[
\beta_a \stackrel{\text{notation}}{=} \begin{bmatrix}
\hat{0} & \hat{0} & I & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \hat{0} \\
-I & \hat{0} & \hat{0} & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \hat{0}
\end{bmatrix}, \quad \beta_a^{\dagger} = \begin{bmatrix}
\hat{0} & \hat{0} & \hat{0} & \hat{0} \\
\hat{0} & \hat{0} & -S_a & \hat{0} \\
\hat{0} & S_a & \hat{0} & \hat{0} \\
-\lambda_{\alpha} & \tilde{0} & \tilde{0} & 0
\end{bmatrix}.
\tag{6.22}
\]

Here \( \hat{0} \) and \( 0 \) are the zero matrices of dimension 3\times3 and 1\times3, \( I \) is the 3\times3 unit matrix, \( S_a \) and \( \lambda_{\alpha} \) are the 3\times3 and 1\times3 matrices (3.6), (6.18).

The matrices (6.22) satisfy relations (6.20), from which it follows that they satisfy also the conditions (6.7), (6.11). The matrices \( S_{\mu\nu} \) of (6.19), (6.22) realize the representation \( D=D(1,0) \oplus D(0,1) \oplus D(1/2,1/2) \) of the algebra \( AO(1,3) \). Therefore, the equation (6.6), (6.22) is \( S \)-invariant under the Poincaré algebra. This equation describes a particle of mass \( m \) and spin \( s=1 \) since the matrices \( \beta_{\mu} \) satisfy (6.10), (6.11) for \( s=1 \).

The KDP equation for a vector particle is \( P \)-, \( T \)- and \( C \)-invariant and admits the Lagrangian formulation. The corresponding SOs and the Hermitizing matrix \( \eta \) are given in (2.55), (2.60), (6.21), (6.19) where \( \beta_{\mu} \) are the 10\times10 KDP matrices.

We note that besides \( \beta_{\mu} \) there exists just one more matrix satisfying (6.20). This matrix is determined according to the following relation

\[
\beta_4 = \frac{1}{4!} \epsilon_{\mu
u\rho\sigma} \beta^\mu \beta^\nu \beta^\rho \beta^\sigma.
\tag{6.23}
\]

In the realization (6.22)

\[
\hat{\beta}_4 = \begin{bmatrix}
\hat{0} & I & \hat{0} & \hat{0} \\
-I & \hat{0} & \hat{0} & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \hat{0}
\end{bmatrix}.
\tag{6.24}
\]

### 6.6. The Dirac-Fierz-Pauli Equation for a Particle of Spin 3/2

As a last example, we consider the equation describing a particle with spin \( s=3/2 \).

We will start with the following representation of the algebra \( AO(1,3) \):

\[
\]
Chapter 2. Representations of the Poincaré algebra...

\[
D \rightarrow D \left( \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right) \oplus D \left( \begin{array}{c} 0 \frac{1}{2} \\ \frac{1}{2} \end{array} \right) \oplus D \left( \begin{array}{c} \frac{1}{2} \right). \tag{6.25}
\]

It is possible to show that formula (6.25) provides the simplest (i.e., realized
by the matrices of the minimal dimension) representation of the matrices \( S_{\mu} \) for which
the equations (6.7), (6.11)-(6.13) has nontrivial solutions corresponding to \( s=3/2 \).

The vectors from the space of the representation (6.25) can be chosen as a
column with 16 rows. We use the following notation for such vectors
\[
\psi \left( \Phi^{1/2}, \Phi^{3/2} \right), \tag{6.26}
\]
where \( \Phi^{1/2} \) are the \((2s+1)\)-component eigenfunctions of the Casimir operators (4.12),
and the operator \( S^2=S_{ab}S_{ab}/2: \)
\[
\Phi^{1/2} \in D(\tau), \quad S^2 \Phi^{1/2} \sim (s+1) \Phi^{1/2}. \tag{6.27}
\]

From (6.8), (6.9) we obtain the following expression for the matrix \( \beta_0: \)
\[
\beta_0 = \begin{pmatrix}
0 & 0 & 2C_2 \\
0 & \sqrt{3}C_1 & -C_2 & 0 \\
C_3 & \sqrt{3}C_4 & 0 \\
0 & \sqrt{3}C_4 & C_3 \\
0 & -C_2 & \sqrt{3}C_1 & 0 \\
2C_2 & 0 & 0
\end{pmatrix}, \tag{6.28}
\]
where \( C_1, ..., C_4 \) are arbitrary complex parameters. The values of these parameters can be
determined up to a nonessential common multiple using the conditions (6.11):
\[
C_2 - C_3 = \frac{1}{2}, \quad C_1 = C_4 = \frac{i}{2}\sqrt{3}. \tag{6.29}
\]
It is not difficult to make sure that thus defined matrix \( \beta_0 \) satisfies the condition (6.10)
which guarantees the eigenvalue of the mass operator to be fixed. Using the relations
(4.64)-(4.66), (6.7) we find the explicit form of \( \beta_a: \)
\[
\beta_a = \begin{pmatrix}
\hat{0} & B_a \\
-B_a^t & \hat{0}
\end{pmatrix}, \quad B_a = \frac{1}{3} \begin{pmatrix}
iK_a^t & K_a^t & 2S_a \\
i\Sigma_a & -2\Sigma_a & K_a \\
-3\Sigma_a & i\Sigma_a & iK_a
\end{pmatrix}, \tag{6.30}
\]
where \( S_a \) and \( \Sigma_a \) are the spin matrices for the spins \( s=3/2 \) and \( s=1/2 \) (see (4.31)), and \( K_a \)
are the matrices of dimension 2x4:
Symmetries of Equations of Quantum Mechanics

\[ K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 3 & 0 & 5 \\ 3 & 0 & 5 & 0 \end{pmatrix}, \quad K_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 3 & 0 & 5 \\ -3 & 0 & -5 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix}. \]

\( \hat{0} \) being zero matrices of dimension 8×8.

The equation (6.6) with the matrices (6.29)-(6.31) is invariant under the complete Poincaré group and describes a relativistic particle of mass \( m \) and spin \( s = \frac{3}{2} \).

This equation admits the Lagrangian formulation with the Hermitizing matrix \( \eta \) having the following form:

\[ \eta = \begin{bmatrix} P_{1/2,3/2} & I \\ I & I \\ P_{3/2,1/2} \end{bmatrix} \]

(6.32)

where \( I \) is the unit matrix of dimension 2×2, and \( P_{\mu} \) are the corresponding matrices \( p \) from (5.47), (4.61).

In [75, 369] there was proposed another formulation of the Dirac-Fierz-Pauli equation being suitable for a direct generalization to the case of arbitrary spin particles. In that formulation a vector-spinor wave function \( \Phi^\mu_\lambda \) is used (\( \lambda \) being a vector index, \( \mu \) spinor index) which is included in the space of the representation \( D(1/2 \ 1/2) \oplus D(1/2 \ 0) \oplus D(0 \ 1/2) \) of the Lorentz group. According to the Clebsh-Gordon theorem (see, e.g., [3]) such a representation is decomposed into the direct sum of the IRs given by formula (6.25).

The equation for a particle of spin \( 3/2 \) in the Rarita-Schwinger formulation has the form

\[ \left( \gamma_\nu \gamma_\mu \gamma^\nu - m \right) \gamma_\lambda \phi - \frac{1}{3} (\gamma_\lambda \gamma_\nu \gamma_\mu + \gamma_\nu \gamma_\lambda \gamma_\mu + \gamma_\mu \gamma_\lambda \gamma_\nu) + \frac{1}{3} \gamma_\mu (\gamma_\nu \gamma_\lambda p_\nu + m) \gamma_\lambda \phi = 0, \]

(6.33)

where \( \gamma_\mu \) are the Dirac matrices acting on the spinor index \( \lambda \) of the function \( \Phi = column(\Phi^1_\lambda, \Phi^2_\lambda, \Phi^3_\lambda, \Phi^4_\lambda) \). Multiplying the l.h.s. of (6.33) by \( p_\nu \) and \( \gamma^\nu \) and making summation over \( \mu \) we reduce this equation to the form

\[ (\gamma_\nu p_\nu - m) \phi = 0, \quad \gamma_\mu \phi = 0. \]

(6.34)

The equations (6.33) is equivalent to the Dirac-Fierz-Pauli equation. In fact, multiplying it by \( g_{\nu\mu} - \gamma_\nu \gamma_\mu \) and making summation over \( \mu \) (this operation is invertible) we come to the equation in the form (6.6) which differs from the Dirac-Fierz-Pauli equation only in the realization of the matrices \( \beta_\nu \).
6.7. Transition to the Schrödinger Form

Solutions of the considered equations include more components that it is necessary for a description of the internal degrees of freedom of a particle (and antiparticle) with spin \( s > 1/2 \) (i.e., more then \( 2(2s+1) \)). The only exception is the four-component Dirac equation for a particle of spin 1/2. Besides, by elimination of redundant components we can reduce these equations to the Schrödinger form (2.10) with \( \psi \) being a \( 2(2s+1) \)-component wave function, and \( H \) being a Hamiltonian which can be either differential or integro-differential operator.

For the Dirac equation the transition to the Schrödinger formulation is trivial (see Subsection 2.2), so we consider the KDP and Dirac-Fierz-Pauli equations only.

Using algebraic properties of the \( \beta \)-matrices we consider the KDP equations for \( s = 0 \) and \( s = 1 \) simultaneously.

Let the matrices \( \beta_\mu \) in (6.6) satisfy the KDP algebra. Multiplying (6.6) by \( \beta_0 \) and \( 1 - \beta_0 \) and taking into account that \( \beta_0 \beta_0 = \beta_0 \), \( (1 - \beta_0)\beta_0 = \beta_0 \beta_0 \), we obtain after simple calculations the following system:

\[
\begin{align*}
\text{(6.35a)} & \\
\text{(6.35b)} & 
\end{align*}
\]

The system (6.35) is completely equivalent to (6.6) and includes the equation in the Schrödinger form (6.35a) and the additional condition (6.35b) which expresses the "nonphysical" components \( (1 - \beta_0)\psi \) via \( 2(2s+1) \) essential components \( \beta_0 \psi \).

Let us transform the system (6.35) to a representation with the nonphysical components being equal to zero. Using for this purpose the operator

\[
V = 1 - \frac{1}{m} \beta_0 p \beta_0, \quad V^{-1} = 1 - \frac{1}{m} \beta_0 p \beta_0
\]

we obtain

\[
V \left( 1 - \beta_0 - \frac{1}{m} \beta_0 p \beta_0 \right) V^{-1} \equiv 1 - \beta_0
\]

\[
V (\beta_0 \beta_0 \beta_0 p - \beta_0) V^{-1} \equiv \beta_0 \left( \beta_0 \beta_0 p - m + \frac{(\beta_0 p)^2}{m} \right)
\]

According to (6.36), the transformed function \( \psi' = V \psi \) satisfies the condition

\[
(1 - \beta_0)\psi' = 0
\]

from which it follows that \( \beta_0 \beta_0 p \psi = 0 \), and (6.35a) reduces to the form
Choosing for $\beta_0$ the representation (6.17) or (6.22), we conclude from (6.38) that $(s+3)$ components of $\psi'$ are identically equal to zero and that the Hamiltonian $H$ can be written in the form

$$H = \sum_{s} \left( m \frac{p^2}{2m} - \frac{i}{2m} \sigma_1 [2(S \cdot p)^2 - p^2] \right).$$

Here $\sigma_1$ and $\sigma_2$ are the $2(2s+1)$-row Pauli matrices (see (5.30)), $S$ are generators of the direct sum $D(s) \oplus D(s)$ of the group $O(3)$, i.e., the matrices (3.6) for $s=1$ and zero matrices for $s=0$. By the additional transformation $H \rightarrow UHU^\dagger$, $U=(1-i\sigma_3)/\sqrt{2}$, we can change $\sigma_2 \rightarrow \sigma_1$, $\sigma_1 \rightarrow -\sigma_2$ and obtain the standard form of Hamiltonian discussed in the following subsection.

We see that the KDP equations reduces to the Schrödinger form by elimination of nonphysical components. The corresponding Hamiltonians are second order differential operators with matrix coefficients. The equation (2.10), (6.39) was proposed for the first time by Tamm [401], Sakata and Taketani [377].

Consider the Dirac-Fierz-Pauli equation for a particle of spin 3/2. Multiplying (6.6) from the left by the invertible matrix $C$,

$$C = \begin{pmatrix} \hat{C}_0 & 0 \\ 0 & \hat{C} \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} I & 0 & 0 \\ 0 & i1 & 1 \\ 0 & -il & 1 \end{pmatrix},$$

where $I$ and 1 are the $4\times4$ and $2\times2$ unit matrices, and denoting the components of the wave function (6.26) according to the formulae

$$\Phi^{1/2}_{1/2} \Phi_1, \quad \Phi^{1/2}_{1/2} = \frac{i}{2}(\Phi_3 - \Phi_2), \quad \Phi^{0/2}_{1/2} = \frac{1}{2}(\Phi_3 + \Phi_2),$$

$$\Phi^{1/2}_{1/2} = \frac{1}{2}(\Phi_3 + \Phi_2), \quad \Phi^{1/2}_{1/2} = \frac{i}{2}(\Phi_3 - \Phi_2), \quad \Phi^{1/2}_{1/2} = \Phi_0,$$

we come to the following system

$$p_0 \Phi_1 + \frac{1}{3} K \cdot p \Phi_3 + \frac{2}{3} S \cdot p \Phi_1 - m \Phi_0 = 0,$$

$$-4 \frac{i\Sigma \cdot p \Phi_5 + \frac{2}{3} iK' \cdot p \Phi_1 - m \Phi_2 = 0,$$

$$\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5,$$
Chapter 2. Representations of the Poincaré algebra...

\[
-\frac{2}{3}i\vec{K} \cdot \vec{p} \Phi_6 + \frac{4}{3} \Sigma \cdot \vec{p} \Phi_2 - im \Phi_3 = 0, \tag{6.41c}
\]

\[
p_0 \Phi_3 + \frac{2}{3} \Sigma \cdot \vec{p} (2 \Phi_4 - \Phi_3) + im \Phi_3 = 0, \tag{6.41d}
\]

\[
p_0 \Phi_2 + \frac{2}{3} \Sigma \cdot \vec{p} (2 \Phi_3 - \Phi_2) + im \Phi_4 = 0, \tag{6.41e}
\]

\[
p_0 \Phi_6 - \frac{2}{3} \Sigma \cdot \vec{p} \Phi_6 - \frac{i}{3} \vec{K} \cdot \vec{p} \Phi_3 - m \Phi_1 = 0. \tag{6.41f}
\]

Acting on \(\psi\) (6.26) by the projector \(P_s = \beta_0^2\) (see Subsection 6.4), we see that the physical components of the wave function are \(\Phi_1\) and \(\Phi_6\). Unlike the case of the KDP equation, the redundant components \(\Phi_2, \Phi_3, \Phi_4\) and \(\Phi_5\) cannot be expressed via \(\Phi_1\) and \(\Phi_6\) and their first derivatives. But the corresponding expressions may be obtained with the help of nonlocal (integral) operators.

Considering (6.41) in the momentum representation, using the identity \((\Sigma \cdot \vec{p})^2 = \vec{p}^2 / 4\) and relations (12.26), we express nonphysical components via \(\Phi_1\) and \(\Phi_6\):

\[
\Phi_2 = iF \left( \vec{K} \cdot \vec{p} \Phi_1 + \frac{4}{3m} \Sigma \cdot \vec{p} \Phi_6 \right), \quad F = \frac{2m}{3 \left( m^2 + \frac{9}{4} \vec{p}^2 \right)},
\]

\[
\Phi_3 = iF \left( \vec{K} \cdot \vec{p} \Phi_1 - \frac{4}{3m} \Sigma \cdot \vec{p} \Phi_6 \right), \tag{6.42}
\]

\[
\Phi_4 = F \left( -\vec{K} \cdot \vec{p} \Phi_6 + \frac{4}{3m} \Sigma \cdot \vec{p} \Phi_1 \right),
\]

\[
\Phi_5 = F \left( -\vec{K} \cdot \vec{p} \Phi_6 + \frac{4}{3m} \Sigma \cdot \vec{p} \Phi_1 \right).
\]

Substituting (6.41) into (6.41a), (6.41f), we come to the equation in the Schrödinger form (2.10) for the eight-component wave function \(\psi = \text{column}(\Phi_1, \Phi_6)\), where, according to [308],

\[
H = \sigma_1 \vec{S} \cdot \vec{p} \left[ 1 + \frac{3F}{2m} \left[ \vec{p}^2 - (\vec{S} \cdot \vec{p})^2 \right] \right] + \sigma_3 m \left[ \frac{3F}{4m} \left[ \vec{p}^2 - (\vec{S} \cdot \vec{p})^2 \right] \right], \quad \vec{S} = (2/3) \vec{S}.
\]

(6.43)

Here \(\sigma_1\) and \(\sigma_3\) are the 8x8 Pauli matrices commuting with \(\vec{S}\), \(\vec{S}\) being generators of the direct sum \(D(3/2) \oplus D(3/2)\) of IRs of the group \(O(3)\) (compare with (7.6)).

The Hamiltonian (6.43) had been obtained in [308] starting from another
realization of the Dirac-Fierz-Pauli equation.

The Hamiltonian (6.43) can be defined as a nonlocal (integral) operator in the \( x \)-representation. This signifies that the Dirac-Fierz-Pauli equation corresponds to the integral evolution equation for physical components of a wave function. In other words, it is essentially a nonlocal equation. This is the main reason for paradoxes with the causality violation which take place when using this equation to describe a particle of spin 3/2 in the external electromagnetic field.

We note that the nonlocal character of the Hamiltonian (6.43) is caused by nilpotency of the corresponding matrix \( \beta_0 \) in (6.6). As far as this matrix is nilpotent for any Poincaré- and \( P_- \), \( T_- \), and \( C \)-invariant equations (6.6) describing a particle with a fixed value of spin \( s \geq 1 \), then such an equation corresponds to an integral Hamiltonian and is nonlocal in this sense.

7. POINCARÉ-INVARIANT EQUATIONS WITHOUT SUPERFLUOUS COMPONENTS

7.1. Preliminary Discussion

As was noted in the Introduction, the covariant first order wave equations for particles with higher spins can be used neither for a description of particle interaction with an external field, nor for the constructing of a second quantized theory. The only relativistic equation which does not lead to contradictions when solving real physical problems is the Dirac equation.

What is the reason to distinguish the Dirac equation from other covariant equations? There are several reasons for this. The Dirac equation possessing the unique set of properties given below.

(1) The transparent relativistic invariance. The covariant representation of the Poincaré algebra can be realized on the set of solutions of the Dirac equation.

(2) The existence of the Schrödinger formulation (2.10) with \( H \) being a local (differential) operator.

(3) The wave function satisfying the Dirac equation has \( 4 = 2(2s+1) \) components which exactly corresponds to the number of the spin degrees of freedom for a particle and antiparticle of spin 1/2.

(4) The Poincaré group generators defined on the set of solutions of the Dirac equation are Hermitian under the usual scalar product (2.39).

Covariant equations of first order considered in the preceding section possess (for \( s \geq 1/2 \)) the property (1) and, as an exception, the property (2). As to the properties (3), (4), the only equation (6.6) possessing those is the Dirac equation.
is the existence of superfluous components (i.e., those whose number is more
than 2(2s+1)) of a wave function \( \psi \), which is the source of difficulties for the theory of
higher spin equations.

In connection with the above arises the natural desire to find such equations
for particles of arbitrary spin which have no superfluous components and possess the
other properties mentioned in (1)-(4). It happens that such equations do exist, but the
price to be paid for the absence of nonphysical components is the loss of a
transparent relativistic invariance (i.e., evident symmetry between spatial and time
variables).

In this section we deduce such equations for particles of any spin.

### 7.2. Formulation of the Problem

We will search for Poincaré-invariant equations for a particle of arbitrary
spin in the Schrödinger form

\[
i \frac{\partial \psi}{\partial x_0} = H_s \psi,
\]

where \( \psi = \psi(x_0, x) \) is a 2(2s+1)-component wave function,

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{2(2s+1)} \end{pmatrix},
\]

and \( H_s = H_s(p) \) is an unknown linear differential operator to be determined from the
condition of (7.1) being Poincaré-invariant.

**DEFINITION 7.1.** We say the equation (7.1) is Poincaré-invariant and
describes a particle of mass \( m \) and spin \( s \) if it is invariant under the algebra \( AP(1,3) \),
and besides, the Casimir operators \( C_1 = P_\mu P^\mu \) and \( C_2 = W_\mu W^\mu \) satisfy the conditions
(6.1).

According to the definition, the Hamiltonian \( H_s = P_0 \) has to satisfy the
following conditions:

\[
[H_s, P_\alpha] = 0, \quad (7.3a)
\]

\[
[H_s, J_\alpha] = 0, \quad (7.3b)
\]

\[
[H_s, N_\alpha] = iP_\alpha \quad (7.3c)
\]

where \( P_\alpha, J_\alpha \), and \( N_\alpha \) are the basis elements of the algebra \( AP(1,3) \) satisfying the relations

\[
[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_\beta] = i\epsilon_{abc} P_\gamma, 
\]

\[
[J_\alpha, J_\beta] = i\epsilon_{abc} J_\gamma, \quad [J_\alpha, N_\beta] = i\epsilon_{abc} N_\gamma.
\]
Symmetries of Equations of Quantum Mechanics

It is not difficult to see that formulae (7.3), (7.4) represent nothing but a new form of the commutation relations (1.14) characterizing the Poincaré algebra.

Thus, to describe Poincaré-invariant equations in the Schrödinger form (7.1) it is sufficient to choose a representation of the generators \( P_a, J^a \) and \( N_a \) as satisfying the commutation relations (7.4), and then to find all the possible operators \( H_s \) satisfying (7.3).

As is well known (see Subsection 4), the conditions (6.1) define the operators \( P_a, J^a \) and \( N_a \) up to unitary equivalence. It is natural in choosing a specific realization of \( P_a, J^a \) and \( N_a \) to require that the representation of these operators to be defined for arbitrary spin \( s \) will make it possible to obtain the Dirac equation for \( s=1/2 \). In other words, it is desirable to generalize the representation of the algebra \( AP(1,3) \), realized on the set of solutions of the Dirac equation, for the case of arbitrary spin \( s \). We consider three of such generalizations corresponding to three different approaches to description of Poincaré-invariant equations without superfluous components [147, 331].

We will start with the following realization of the Poincaré group generators:

\[
[P_0, N_a] = i\delta_{ab} H_s, \quad (7.4c)
\]

\[
[N_a, N_b] = -i\epsilon_{abc} J_c, \quad (7.4d)
\]

It is not difficult to see that formulae (7.3), (7.4) represent nothing but a new form of the commutation relations (1.14) characterizing the Poincaré algebra.

Thus, to describe Poincaré-invariant equations in the Schrödinger form (7.1) it is sufficient to choose a representation of the generators \( P_a, J^a \) and \( N_a \) as satisfying the commutation relations (7.4), and then to find all the possible operators \( H_s \) satisfying (7.3).

As is well known (see Subsection 4), the conditions (6.1) define the operators \( P_a, J^a \) and \( N_a \) up to unitary equivalence. It is natural in choosing a specific realization of \( P_a, J^a \) and \( N_a \) to require that the representation of these operators to be defined for arbitrary spin \( s \) will make it possible to obtain the Dirac equation for \( s=1/2 \). In other words, it is desirable to generalize the representation of the algebra \( AP(1,3) \), realized on the set of solutions of the Dirac equation, for the case of arbitrary spin \( s \). We consider three of such generalizations corresponding to three different approaches to description of Poincaré-invariant equations without superfluous components [147, 331].

We will start with the following realization of the Poincaré group generators:

\[
P_0 = H_s, \quad P^p = J^x \times p \cdot S, \quad (7.5)
\]

\[
N^x = p - \frac{1}{2}[x, H_s] \cdot \lambda_s, \quad (7.6)
\]

where \( S \) are matrices realizing a direct sum of two IRs \( D(s) \) of the algebra \( AO(3) \)

\[
S = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad s \in D(s);
\]

\( x \) and \( p \) are canonically conjugated variables defined by the relations

\[
[x_a, p_b] = i\delta_{ab}; \quad (7.7)
\]

\( H_s \) and \( \lambda_s \) are yet unknown operators whose form will be obtained by applying the requirement for the operators (7.5) to satisfy the commutation relations (7.3), (7.4).

Formulae (7.5) represent the general form of basis elements of the algebra \( AP(1,3) \), which generate local transformations corresponding to translations and spatial rotations of a frame of reference.

On the set of solutions of the Dirac equation the Poincaré group generators
have the form (7.5) where $\lambda_s = 0$. Thus we assume that for the first approach \(^*\) 
\[ \lambda_s' = 0. \] (7.8)

The distinguishing feature of the representation (7.5), (7.8) is that all the Poincaré group generators are Hermitian in respect to the scalar product (2.39), where $\psi_1$, $\psi_2$ are $2(2s+1)$-component wave functions satisfying (7.1).

In the second approach we choose $\lambda_s$ in the form

\[ (\lambda_{I',II'}^s, -\frac{1}{2}[H_{I',II'}^s, x_\mu], S_{I',II'0} = i\sigma_i S_{I',II'0}^i) \begin{pmatrix} s_1 & 0 \\ 0 & -s_0 \end{pmatrix}. \] (7.9)

Here $s_1$ are $(2s+1)\times(2s+1)$ matrices belonging to the IR $D(s)$ of the algebra $AO(3)$, $0$ are the zero matrices and $\sigma_3$ is the $2(2s+1)$-row Pauli matrices (5.30).

The merit of the representations (7.5), (7.9) is that they correspond to the local transformations of the wave function $\psi$ by the transition to a new frame of references. Actually, on the set of the equation (7.1) solutions the generators (7.5), (7.9) can be represented in the covariant form (2.22) where $S_\mu$ are matrices realizing the representation $D(0s)\oplus D(s0)$ of the algebra $AO(1,3)$. However, these generators are not Hermitian in respect to the scalar product (2.39). The only exception is the case $s=1/2$ when the operators (7.8) and (7.9) coincides.

We define the operators of space inversion $P$, time reflection $T$, and charge conjugation $C$ according to formulae (2.55), (2.60) where $\psi$ is a $2(2s+1)$-component wave function, $r_1$, $r_2$, and $r_3$ are Hermitian matrices which can be determined from the condition that the operators $P$, $T$, and $C$ satisfy relations (1.54) together with the Poincaré group generators. It can be shown that up to unitary equivalence it is sufficient to restrict ourselves to considering the matrices of the following form:

\[ r_1^I = \sigma_3, \quad r_1^II = \sigma_1, \quad r_2^I = \sigma_3, \quad r_2^II = \sigma_2, \] (7.10a)

\[ r_3^I = \sigma_3 \Delta, \quad r_3^II = \sigma_1 \Delta \quad \text{or} \quad r_3^II = \sigma_3 \Delta, \quad \Delta = \begin{pmatrix} \Delta' & 0 \\ 0 & \Delta' \end{pmatrix}, \] (7.10b)

where $\Delta'$ is a $(2s+1)\times(2s+1)$ matrix determined up to a sign by the relations (refer to (5.24), (5.34))

\[ \Delta's \rightarrow -\Delta', \quad \Delta'^2 = \text{I}. \] (7.11)

We require the equation (7.1) be invariant under the transformations

\(^*\) We will use the indices $I$, $II$, and $III$ to distinguish the operators considered in the first, second, and third approaches, respectively.

91
(2.55),(2.60) from which it follows that \( H_1 \) has to satisfy the following relations
\[
[P, H_1] = [T, H_1] = [C, H_1] = 0.
\]

(7.12)

The approaches \( I \) and \( II \) make it possible to describe a wide class of Poincaré-invariant equations without superfluous components which, however, does not include the Tamm-Sakata-Taketani equation considered in Subsection (6.5). The class of equations including the Dirac and Tamm-Sakata-Taketani equation can be obtained in the third approach for which the problem is formulated as follows: to find all the possible differential equations (7.1) invariant under the Poincaré algebra generated by the operators (7.5). Besides, we assume that the corresponding Hamiltonians belong to the class of second order differential operators. At the same time we do not impose any restrictions on the form of \( \lambda_s \) and do not require \( P-, T- \) and \( C- \) invariance of the equation (7.1). We will see that such a formulation makes it possible to determine Hamiltonian of an arbitrary spin particle up to equivalence transformations.

### 7.3. The Explicit Form of Hamiltonians \( H_I \) and \( H_{II} \)

Following [133], let us find all the possible (up to equivalence) Hamiltonians \( H_I \). It is not difficult to make sure that the equations (7.4) reduces to identities if the equations (7.3), (7.4d) are satisfied. Substituting (7.5), (7.8) into this equations we obtain

\[
(H_I)^2 = p^2 + m^2,
\]

\[
[H_s, J] = [H_s, p] = 0,
\]

\[
[H_s, x] \times [H_s, x] = -4iS.
\]

(7.13)

Relations (7.13) are the necessary and sufficient conditions for equations (7.3), (7.4) to be satisfied.

We represent \( H_I \) as an expansion in the complete set of orthoprojectors

\[
H_I = \sum_{\nu} \psi \Lambda_{\nu}.
\]

(7.14)

where

\[
\Lambda_{\nu} = \prod_{\nu \neq \nu'} \frac{S_{\nu} - V'}{V - V'}, \quad \psi = \frac{S_p}{p}.
\]

(7.15)

\( h_{\nu} \) are matrices commuting with \( S \) which can be decomposed by the complete set of the Pauli matrices (5.30)
Chapter 2. Representations of the Poincaré algebra...

\[ h_{\nu} \equiv a_{\nu} \sigma_{\rho}, \quad (7.16) \]

and \(a_{\nu}^{\nu}\) are unknown functions of \(p\).

It is not difficult to make sure the operators (7.15) are the projectors into eigenstates of the operator \(S p / p\) and satisfy the following relations

\[ \Lambda_{\nu} \Lambda_{\nu} = \delta_{\nu, \nu}, \quad \sum_{\nu=-s}^{s} \Lambda_{\nu} = 1, \quad \sum_{\nu=-s}^{s} \nu \Lambda_{\nu} = S_{\nu}, \quad (7.17) \]

The operators \(\Lambda_{\nu}\) commute with \(x\) as follows [133]:

\[ [x, \Lambda_{\nu}] = \frac{p \times S}{2p^2} (\Lambda_{\nu} - \Lambda_{\nu+1} - 2 \Lambda_{\nu}) - \frac{i}{2p} \left( S - \frac{p \times \sigma_{\rho}}{p} \right) (\Lambda_{\nu+1} - \Lambda_{\nu-1}). \quad (7.18) \]

Using (7.18), we can prove the following assertion: the vectors \(p \times S \Lambda_{\nu}, S \Lambda_{\nu}, p \Lambda_{\nu}\) are linearly independent if \(\nu \neq \pm s\). If \(\nu = \pm s\) then

\[ \frac{p \times S}{p} \Lambda_{\pm s} = \pm \left( S + \frac{p \times \sigma_{\rho}}{p} \right) \Lambda_{\pm s}. \quad (7.19) \]

Relations (7.14)-(7.19) enable us to reduce equations (7.13) to the system of algebraic equations for the coefficients \(a_{\nu}^{\nu}\). As a matter of fact, we obtain from (7.13a), (7.14), (7.17)

\[ h_{\nu} = m^2 - p^2, \quad (7.20) \]

and then from (7.13c), (7.18) we obtain

\[ \frac{1}{2} [h_{\nu}, h_{\nu+1}] = m^2 - p^2. \quad (7.21) \]

Relations (7.20), (7.21) are necessary and sufficient conditions for the Hamiltonian \(H'_{\nu}\) to satisfy equations (7.13). The general solution of these relations and relations (7.12) is given by the formula

\[ h_{\nu} \equiv \sigma_{\rho} E \cos \varphi_{\nu}, \quad \sigma_{\rho} E \sin \varphi_{\nu}, \quad (7.22) \]

where the possible values of \(\varphi_{\nu}\) are determined by the recurrence relations

\[ \varphi_{\nu+1} = \varphi_{\nu} \pm 2 \theta', \quad \theta' = \arctan \frac{p}{m}. \quad (7.33) \]

If the space inversion matrix \(r_{\nu}'\) is equal to \(\sigma_{i}\) then we have for any \(s\)

\[ \varphi_{0} = 0, \quad \varphi_{s/2} = \theta', \quad \varphi_{s} = -\varphi_{-s}, \quad (7.24) \]

but for \(r_{\nu}' = \sigma_{0}\) solutions of the equations (7.20), (7.21) compatible with (2.55), (2.60), (7.12) exist for integer \(s\) only. Moreover,

\[ \varphi_{0} = \varphi(p/m), \quad \varphi_{s} = -\varphi_{-s}. \quad (7.25) \]
where $\varphi$ is an arbitrary function of $p/m$.

According to (7.23) the number of possible Hamiltonians $H_s^I$ increases with increasing of $s$ because by calculating $\varphi_{v+1}$ we can choose any of possible signs on each step. It is not difficult to calculate that for $r_i=\sigma_i$ the total number of different $H_s^I$ is equal to $2^s$ where $[s]$ is the entire part of $s$.

We represent the simplest solutions of the recurrence relations (7.23) compatible with (7.24), (7.25):

$$(\varphi_{v})_1 \sim (-1)^{v}\theta^{i}, \quad (\varphi_{v})_2 \sim 2v\theta^{i}.$$  

(7.26)

By substituting (7.22), (7.26) into (7.15) we obtain

$$(H_s^I)_1 \sim \sigma_i m + \sigma_j p \sum_v (-1)^v \lambda_v,$$  

(7.27)

$$(H_s^I)_2 \sim E \left[ \sum_v \sigma_j \cos(2v\theta') + \sigma_i \sin(2v\theta') \right] \lambda_v.$$  

(7.28)

The operators (7.27) and (7.28) coincide in the case of $s=1/2$ and reduce to the Dirac Hamiltonian (2.11) where $\gamma_0 = \sigma_i$ and $\gamma_1 = 2\sigma_i S_p$.

We represent also the explicit expressions for the Hamiltonians (7.27), (7.28) in the terms of the helicity operators $S_p$ for $s \leq 3/2$. According to (7.15) we obtain

$$(H_{0}^I)_1 \sim \sigma_i E, \quad (H_{0}^I)_2 \sim \sigma_i m + \sigma_j p,$$  

$$(H_{1/2}^I)_1 \sim \sigma_i m + 2\sigma_3 S \cdot p,$$  

$$(H_{1}^I)_1 \sim \sigma_i m + \sigma_j p(1 - 2S_p^z),$$  

$$(H_{3/2}^I)_1 \sim \sigma_i [E^2 - 2(S \cdot p)^2] + 2\sigma_3 m S \cdot p E^{-1},$$  

$$(H_{3/2}^I)_2 \sim \sigma_i [2E^2 p^2 - 2(S \cdot p)^2] m + \sigma_j [(2E^2 p^2/12) S \cdot p - 4/3(S \cdot p)^3] E^{-2}.$$  

(7.29a)

As it can be seen from (7.19), the Hamiltonians $H_s^I$ for $s \neq 1/2$ are nonlocal (integro-differential) operators in $x$-representation. Let us recall that first order covariant wave equations (6.6) also lead to nonlocal Hamiltonians in general.

The problem of finding the exact form of the Hamiltonians $H_s^I$ can be solved in complete analogy with the above. The Poincaré-invariance conditions (7.3), (7.4) for the representation (7.8), (7.9) reduce to relations (7.3b), (7.3c), the last of them being of the form

$$- [H_s^I, x] H_s^I + iS [H_s^I, \sigma_i] + i[H_s^I, S] \sigma_i = ip.$$  

(7.30)

The general solution of the equations (7.3b), (7.3c) compatible with (7.12)
is given by the formulae [297, 410]

\begin{align}
(H^{I/2}_0) &= \sum_v [\sigma_v \sech(2v\theta^1) - \sigma_v \tanh(2v\theta^1)]I_v, \quad r_3 = \sigma_3 \Delta, \\
(H^{I/2}_2) &= \sum_v [i\sigma_v \cosech(2v\theta^1) - \sigma_v \coth(2v\theta^1)]I_v, \quad r_3 = \sigma_3 \Delta,
\end{align}

where

\[ \theta^1 = \text{arctanh}\left(\frac{p}{E}\right) \]

The Hamiltonians (7.31) are defined for arbitrary values of \(s\), but (7.32) are valid for \(s\leq 3/2\) only because \((H^{I/2}_2)\) does not satisfy (7.13a) for integer \(s\) [147, 331].

For \(s=1/2\) the operator (7.31) reduces to the Dirac Hamiltonian.

Let us present the explicit expressions for \(H^{I/2}_s\), \(s\leq 3/2\) in the terms of helicity operators:

\[ H_0^{I/2} = \sigma_I E, \]

\begin{align}
(H^{I/2}_{1/2})_1 &= \sigma_I E \frac{2ES \cdot p [\sigma_I S \cdot p - \sigma_I E] (E^2 + p^2)^{-1}}{2 - 2(S \cdot p)^3} m + \\
(H^{I/2}_{3/2})_1 &= \sigma_1 \left[ (2E^2 + 7p^2) / 2 - 2(S \cdot p)^3 \right] m + \\
&+ \sigma_3 \left[ S \cdot p (20p^2 + 6E^2) / 3 - 8(3S \cdot p)^3 \right] (E^2 + 3p^2)^{-3}. 
\end{align}

The operators (7.34) correspond to \( r_{1/2} = \sigma_I \Delta \). In the cases \(s=1/2\) and \(s=3/2\) there exist two more Hamiltonians, one for each value \(s=1/2, 3/2\):

\begin{align}
(H^{I/2}_{1/2})_2 &= 2ES \cdot p (i\sigma_1 m \cdot \sigma_I E) p^{-2}, \\
(H^{I/2}_{3/2})_2 &= \left[ i\sigma_1 \frac{mE}{3p^2} [(20E^2 + 7p^2)S \cdot p - 4(p^2 + 2E^2)(S \cdot p)^3] p^{-2} \right] + \\
&+ \sigma_3 \left[ E^2 / 3p^2 [(20E^2 + 6p^2)S \cdot p - 8E^2(S \cdot p)^3] p^{-2} \right] (p^2 + 3E^2)^{-1}.
\end{align}

These operators correspond to the choice \( r_{1/2} = \sigma_I \Delta \).

We see the operators \(H^{I/2}_s\) as well as \(H^{I}_0\) are nonlocal in \(x\)-representation. Besides, the boost generators \(N^{I/2}_I\) are non-Hermitian in the metric (2.39). The invariant bilinear form for the group transformations generated by \(N^{I/2}_I\) is as follows [298, 410]:

\[ (\psi_1, \psi_2) = \int \bar{\psi}_1 \chi \psi_2, \quad (7.35) \]

where
\[ M = \sigma^3 \sum_{\nu} \frac{E}{m} \cosh(2\nu\theta)\Lambda_{\nu}, \quad r_3^\nu = \sigma^3 \Delta. \]

7.4. Differential Equations of Motion for Spinning Particles

The equations without superfluous components considered above include integro-differential operators \( H_s \) if \( s \neq 1/2 \). Of course, this fact complicates the using of these equations for solution of specific physics problems. In this subsection we consider the third approach which leads to differential equations of motion.

We represent a Hamiltonian \( H_s^{III} \), which has to be found, as an expansion in terms of spin matrices and \( 2(2s+1) \)-row Pauli matrices (5.30)

\[ H_s^{III} = h_0^s m + h_1^s \sigma \cdot \sigma S \cdot p + \frac{1}{m} h_2^s, \]  

(7.36)

where

\[ h_0^s = a^s \sigma_y, \quad h_1^s = b^s \sigma_y S \cdot p, \quad h_2^s = c^s \sigma_y (S \cdot p)^2 + d^s \sigma_y p^2. \]

The operators (7.36) satisfy relations (7.3b) for any values of constant coefficients \( a^s, b^s, c^s, d^s \). The remaining Poincaré-invariant conditions (7.3a), (7.3c), (7.4) define these coefficients up to an arbitrary parameter. We give the explicit form of all nonequivalent Hamiltonians \( H_s^{III} \) (for the proof see [147, 331]):

\[ H_s^{III} = \sigma \cdot m + \sigma_2 k_4 S \cdot p + \frac{1}{2m} [(\sigma_1 - i\sigma_2) [p^2 - 4k_4 (S \cdot p)^2], \quad s = 0, \frac{1}{2}, 1, \ldots. \]  

(7.37a)

\[ H_1^{III} = \sigma \cdot m + (\sigma_1 - i\sigma_2) \frac{p^2}{2m} + \frac{k_2^2 + k_3^2 - 1}{m} (S \cdot p)^2, \]  

(7.37b)

\[ H_1^{III} = \sigma \cdot m + \sigma_2 k_4 S \cdot p + (\sigma_1 - i\sigma_2) \frac{p^2}{2m} [k_2^2 \sigma_1 + i(k_2^2 - 2)\sigma_2] (S \cdot p)^2. \]  

(7.37c)

\[ H_{1/2}^{III} = \sigma \cdot \left\{ m + \frac{p^2}{2m} \right\} \frac{k_1}{2m} \sigma_3 \left[ (S \cdot p)^2 - \frac{5}{4} p^2 \right] + \frac{\sigma_3}{2m} \sqrt{k_4^2 - 1} p^2, \]  

(7.37d)

where \( k_1, k_2, \ldots, k_5 \) are arbitrary complex parameters.
Chapter 2. *Representations of the Poincaré algebra...*

Formulae (7.37) give all the nonequivalent Hamiltonians $H_s^{III}$ defined up to equivalence transformations generated by numerical matrices. Besides the Hamiltonian (7.37a) defined for any $s$, there exist two pairs of additional Hamiltonians for particles of spins 1 and 3/2. Each of the operators (7.37) depends on arbitrary complex number $k_j$, $l=1, 2, ..., 5$.

It is not difficult to make sure the Hamiltonians (7.37) are non-Hermitian in respect to the scalar product (2.39). But it is possible to choose proper values of the coefficients $k_j$ for these operators to be Hermitian in the indefinite metric

$$(\psi_1, \psi_2) = \int d^3x \psi_1^\dagger \sigma_3 \psi_2.$$  

(7.38)

Namely, the operators $H_s^{III}$ are Hermitian in respect to this metric iff

$$k_1 = -k_1, \quad k_2 = k_2, \quad k_3 = -k_3, \quad 0 < k_2^2 < 1, \quad k_4 = k_4, \quad 0 < k_4^2 < 1.$$  

(7.39)

It is not difficult to show that any equation (7.1), (7.37) is invariant under the product of the transformations $TC$, but in general is not invariant under the $P$, $T$- and $C$-transformations. By requiring the symmetry under any of these transformations we reduce the class of the operators $H_s^{III}$ to the following representatives

$$H_s^{III} = \sigma_3 \left\{ m + \frac{p^2}{2m} \right\} - i \sigma_2 \frac{p^2}{m},$$  

(7.40a)

$$H_{1/2}^{III} = \sigma_3 \left\{ m + 2 \sigma_3 \sigma_1 S \sigma_3 \right\}.$$  

(7.40b)

$$H_1^{III} = \sigma_3 \left\{ m + \frac{p^2}{2m} \right\} + i \frac{2m}{2m} \left[ 2(S \cdot p)^2 - p^2 \right],$$  

(7.40c)

$$H_{1/2}^{III} = \sigma_3 \sigma_1 S \sigma_3.$$  

(7.40d)

The operators (7.40b) and (7.40a), (7.40c) coincide with the Dirac and TST Hamiltonians.

The Hamiltonian (7.40a) for $s \neq 0$ is not of great interest inasmuch as it does not depend on spin matrices and so do not possesses any information about a particle spin. We see that there exist only four Poincaré- and $P$, $T$, $C$-invariant equations (7.1) where $H_s$ is a second-order differential operator depending on spin matrices. There are the Dirac, TST equations and the equation (7.1), (7.40d) for a particle of spin 3/2. The corresponding $C$, $P$, and $T$-transformations are given by formulae (2.55), (2.60) where

$$r_1^{III} = \sigma_3, \quad r_2^{III} = \sigma_2, \quad r_3^{III} = \sigma_3 \Delta.$$  

(7.41)
So for particles of spin $s > 3/2$ there exist no Poincaré and $P$, $T$, $C$-invariant Hamiltonians others than the trivial operators (7.40a). Moreover, it is possible to show that such Hamiltonians also do not exist in the class of differential operators of arbitrary finite order. Thus, when describing a particle of spin $s > 3/2$, it is necessary to choose between nonlocal (integro-differential) equations (7.1) considered in the preceding sections and the equations being non-invariant under the space inversion.

### 7.5. Connection with the Shirokov-Foldy Representation

It is easy to note that the problem of finding of Poincaré-invariant motion equations for the approaches $I$-$III$ reduces to the description of some special realizations of the representations of the algebra $AP(1,3)$ belonging to the class $I$. The natural question arises about the relation of these realizations to the IRs of the algebra $AP(1,3)$ considered in Section 4.

Here we establish such a connection and demonstrate that all the representations considered in the present section are equivalent to the direct sum $D'(s) \oplus D(s)$ of the IRs.

To simplify calculations we start from the Shirokov-Foldy realization where the basis elements of the mentioned direct sum of IRs have the form (compare with (4.50))

$$
P_0 \sigma_1 E, \quad P_\alpha \sigma_\alpha, \quad J = x p + S, \quad N = x p - \frac{1}{2} \sigma_1 [x, E], \quad p x S = E + m.
$$

The operators (7.42) and (7.5), (7.8), (7.9) are connected by the relation

$$
P_\mu \sigma_\mu, \quad J = J_\mu, \quad N = N_\alpha (V^{\alpha})^{-1}, \quad \alpha = I, II, III,
$$

where $V^{\alpha}$ are invertible operators of the following form:

$$
V' = \exp \left( \frac{i}{2} \sigma_1 \sum \varphi \Lambda \right),
$$

$$
(V^{\alpha})_m = \frac{m}{E} \left( \cosh \left( \frac{S_p \theta^\mu}{p} \right) + i \sigma_1 \sinh \left( \frac{S_p \theta^\mu}{p} \right) \right),
$$

$$
V^{\alpha} = (E + \sigma_1 H^{m}) \left[ 2E \left( E + \frac{1}{2} \sigma_1 H^{m} \right) \right]^{-1/2},
$$

98
where $\phi^\nu, \theta^I$ are the parameters given by the relations (7.23)-(7.25), (7.33), and $(V^\mu)$, corresponds to the Hamiltonians $(H^\mu)$, (7.31).

The transformation (7.43) with $\alpha=III$ can be used to determine the explicit form of $\lambda^\mu$ in (7.5) which turn out to be integro-differential operators. The representation (7.5) with differential operator $\lambda^\mu$ also can be reduced to the canonical form (7.42) but the corresponding transformation operators have a very complicated form. For example, for the Hamiltonians (7.37a) we can choose for $k_2=1$ the following

$$V^{III} = \frac{1}{2m} \exp \left( \sigma_3 \frac{S \cdot p \arctan \frac{p}{m}}{m} \right) \left[ E \sigma_3 + P \sigma_3 (m - 2\sigma_3 S \cdot p) + P E S \cdot p \right],$$

$$P_\pm = \frac{1}{2} (1 \pm \sigma_3),$$

which corresponds to the representation (7.5) with $\lambda^{III}=0$.

Formulae (7.44) give the explicit form of the operators which transform the Poincaré algebra realizations used in approaches I-III into the Shirokov-Foldy realization (7.42). For the Hamiltonians (7.27) and (7.28) the transformation operator $V^I$ can be choosen in the following form

$$V^I (E + \sigma_1 H^I_s) [2E(E+m)]^{-1/2},$$

and

$$V^I = \exp \left( \sigma_2 \frac{S \cdot p \arctan \frac{p}{m}}{m} \right).$$

(7.46)

(7.47)

For $s=1/2$ these operators coincide and reduce to the Foldy-Wouthuysen (FW) [108] operators diagonalizing the Dirac equation.

Formulae (7.46), (7.47) present very natural generalizations of the FW operator to a case of an arbitrary spin.

Using the connection (7.43) it is possible to define the mean position and spin operators for a particle of arbitrary spin. In the canonical representation (7.42) such operators have the form [108]

$$X^i_a \sim x^i_a, \quad S_{ab} \sim S_{ab}.$$ 

In the representations (7.5), (7.8), (7.12) these operators have the following explicit forms

$$X^a \sim V^a_x [V^x]^a, \quad S_{ab} \sim V^a S_{ab} [V^x]^a, \quad \alpha=I,II,III.$$ 

We present the explicit form of these operators corresponding to the Hamiltonians (7.27), (7.28):
Symmetries of Equations of Quantum Mechanics

\[(X^I_a)_1 = x^a_a + \frac{1}{p^2 E} \left\{ S_{ab} p_b [E - \sigma_s(H^s_I)], \sigma_s(H^s_I) \right\}, \]

\[(S^I_{ab})_1 = S_{ab} + \frac{1}{p^2 E} \varepsilon_{abc} S_{cd} p_d [E - \sigma_s(H^s_I)], \]

\[(X^I_a)_2 = x^a_a - \frac{1}{E^2 (E + m)} (E S_{ab} p_b - i \sigma_2 p_a S \cdot p), \]

\[(S^I_{ab})_2 = S_{ab} + \frac{1}{E} \varepsilon_{abc} S_{cd} p_d + \frac{1}{E (E + m)} (p \cdot S \cdot p - S \cdot p^2) \varepsilon_{abc}. \]

where \( S_a = \varepsilon_{abc} S_b / 2 \).

Formulae (7.48) generalize the mean position and spin operators of the Dirac electron [108] to the case of arbitrary spin particles.

In conclusion we note that the equations found above can be generalized in such a way that they will describe "particles" with several spin and mass states. Equations for particles with variable spin and mass were considered in [140, 315].

8. EQUATIONS IN DIRAC’S FORM FOR ARBITRARY SPIN PARTICLES

8.1. Covariant Equations with Coefficients Forming the Clifford Algebra

All the relativistic motion equations described above can be considered as generalizations of the Dirac equation. Here we consider the most natural generalization of this equation in which reducible representations of the Clifford algebra (2.3) are used.

As in Section 6, we shall search for the motion equation of relativistic particle of arbitrary spin in the form

\[(\Gamma^\mu p^\mu - m) \psi = 0, \]

where \( \Gamma^\mu \) are square numeric matrices, and \( \psi \) is a multicomponent wave function.

According to (6.1), the function \( \psi \) has to satisfy the KGF equation componentwise. The simplest way to assure for (6.1) to be satisfied is to require the matrices \( \Gamma^\mu \) satisfy the relations

\[\Gamma^\mu \Gamma^\nu - \Gamma^\nu \Gamma^\mu = 2 g^\mu \nu. \]

Multiplying (8.1) by \( \Gamma^\mu p^\mu + m \) and using (8.2), we come to the condition (6.1) which is a differential consequence of (8.1), (8.2). An IR of the algebra (8.2) is realized by the 4x4 Dirac matrices. The corresponding equation (8.1) reduces to the Dirac
Chapter 2. Representations of the Poincaré algebra...

equation.

In this section we consider equations (8.1) with reducible matrices $\Gamma_\mu$. We show that such equations can be interpreted as equations of motion of a relativistic particle with arbitrary spin $s$. Then we find the additional conditions needed to select the subspace of solutions of the equation (8.1) corresponding to a fixed value of $s$.

It is not difficult to make sure that the equations (8.1) are invariant under the Poincaré algebra. Taking the basis elements of this algebra in the covariant form (2.22) and representing the matrices $S_{\mu\nu}$ in the form

$$S_{\mu\nu} \rightarrow \tau_{\mu\nu}, \quad j_{\mu\nu} \rightarrow \frac{i}{4}[\Gamma_\mu, \Gamma_\nu].$$

we find that the operators (2.22), (8.4) satisfy the invariance condition of the equation (8.1) (i.e., the relations (6.7), where $\beta_\mu \rightarrow \Gamma_\mu$) if the matrices $\tau_{\mu\nu}$ satisfy the following relations:

$$[\tau_{\mu\nu}, \tau_{\lambda\sigma}] = (g_{\mu\nu} \tau_{\lambda\sigma} + g_{\nu\lambda} \tau_{\mu\sigma} - g_{\mu\sigma} \tau_{\nu\lambda} - g_{\nu\lambda} \tau_{\mu\sigma}),$$

$$[\tau_{\mu\nu}, \Gamma_\lambda] = [\tau_{\mu\nu}, j_{\lambda\sigma}] = 0,$$

i.e., if the matrices $\tau_{\mu\nu}$ commute with $\Gamma_\lambda$ and realize a finite-dimensional representation of the algebra $AO(1,3)$.

Thus, we can set a correspondence between the Poincaré-invariant equations (8.1), (8.2) and any finite-dimensional representation of the algebra $AO(1,3)$. On the set of solutions of such equations generators of the Poincaré group have the form (2.22) with $S_{\mu\nu}$ being a sum of commuting matrices $j_{\mu\nu}$ and $\tau_{\mu\nu}$ being given by (8.3), (8.4).

8.2. Equations with the Minimal Number of Components

Poincaré-invariant equations in the Dirac form admit various interpretations as far as it is not possible to determine in an unique fashion the corresponding representation of the Poincaré algebra. The only exception is the case of the irreducible $\Gamma$-matrices of dimension 4×4 corresponding to the Dirac equation, but this equation has also an alternative interpretation as an equation for a zero-mass particle (refer to the following subsection).

To interpret the equation (8.1) it is necessary to choose a possible representations of the matrices $\tau_{\mu\nu}$ of (8.3). The corresponding representation of the Poincaré algebra (2.22) turns out to be reducible so it is necessary to impose on $\psi$ a supplementary condition of the type of (6.1) in order to select the subspace corresponding to the fixed value of spin.

Consider the case when $\tau_{\mu\nu}$ produce the representation $D(\tau 0)$ of the algebra $AO(1,3)$. This means that (see Subsection 4.8)
Substituting (8.5) into (8.3), we obtain

$$
\tau_{ab} = \varepsilon_{abc} \tau_c, \quad \tau_{ab} = i \varepsilon_{abc} \tau_c, \quad \tau_{a \tau d} = \tau(\tau + 1).
$$

(8.5)

where the matrices $\tau_c$ commute with $\Gamma_{\mu}$ by definition.

According to Schur’s lemma we conclude that the minimal dimension of the matrices (8.5) is $[4(2\tau + 1)]^2$. Thus, we can set

$$
\Gamma_{\mu} = i \gamma_\mu I, \quad \tau_{ab} = i \hat{\tau}_{ab},
$$

(8.7)

where the symbol $\otimes$ denotes the direct (Kronecker) product, $\gamma_\mu$ are the 4×4 Dirac matrices, $\hat{\tau}_{ab}$ are matrices of dimension $(2\tau + 1)\times(2\tau + 1)$ realizing the representation $D(\tau)$ of the algebra $A\!O(3)$, $I$ and 1 are the unit matrices of dimension $(2\tau + 1)\times(2\tau + 1)$ and 4×4, correspondingly.

To find the spin value of a particle described by the equation (8.1) it is necessary to calculate the eigenvalues of the corresponding Casimir operator $W_{\mu}W^\mu$. Using (2.22), (2.28), (6.1) we find, in the frame of reference where $\tilde{p} = (m, 0, 0, 0)$, that

$$
W_{\mu}W^\mu = m^2 S^2, \quad S_{ab} = \varepsilon_{abc} S_{bc},
$$

(8.8)

and, therefore, these eigenvalues can be found by reduction of the matrices (8.6) by the algebra $A\!O(1,3)$. Inasmuch as the matrices $\Gamma_{\mu}$ (8.3), (8.6) realize the representation $D(0 \! \! \! \! \! \! \! \! \! / 1/2) \oplus D(1/2 \! \! \! \! \! \! \! \! \! / 0)$ of the algebra $A\!O(1,3)$ then the matrices $S_{\mu \nu}$ (8.6) generate the representation

$$
\left[ D \left( \frac{1}{2} \! \! \! \! \! \! \! \! \! / 0 \right) \oplus D \left( 0 \! \! \! \! \! \! \! \! \! / \frac{1}{2} \right) \right] \oplus D(\tau \! \! \! \! \! \! \! \! \! / 0) = D \left( \tau + \frac{1}{2} \! \! \! \! \! \! \! \! \! / 0 \right) \oplus D \left( \tau - \frac{1}{2} \! \! \! \! \! \! \! \! \! / 0 \right) \oplus D \left( \tau \! \! \! \! \! \! \! \! \! / \frac{1}{2} \right).
$$

(8.9)

By the reduction $A\!O(1,3) \rightarrow A\!O(3)$ we obtain from (8.9) the following direct sum of IRs of the algebra $A\!O(3)$: $D(\tau + 1/2) \oplus D(\tau - 1/2) \oplus D(\tau + 1/2) \oplus D(\tau - 1/2)$ which corresponds to the following spin values:

$$
s_\tau = \tau + 1/2, \quad s_\tau = \tau - 1/2.
$$

(8.10)

At this, dimension of the matrices $\Gamma_{\mu}$ is equal to $8s \times 8$.s.

It can be shown that if the matrices $\tau_{\mu \nu}$ realise either IRs $D(\tau, \tau_j)$ with $\tau_j \neq 0$, $\tau_j \neq 0$, or reducible representations, then dimension of the matrices $\Gamma_{\mu}$ is larger than $8s \times 8s$ ($s$ is the largest value of spin appearing by the reduction $A\!O(1,3) \rightarrow A\!O(3)$).

Thus, the equation (8.1) is Poincaré-invariant. It describes a particle of spin $s$ and has the minimal number of components if matrices $\Gamma_{\mu}$ have dimension $8s \times 8s$ and the corresponding matrices $S_{\mu \nu}$ realize the representation (8.9) of the algebra...
Chapter 2. Representations of the Poincaré algebra...

AO(1,3).

Let us require the solutions of (8.1) satisfy the second condition of (6.1). Using the definitions (2.22), (2.28), (8.6) and the equation (8.1), we reduce this condition to the following form [331]:

\[ L_2 \psi = [(\Gamma_\mu p^\mu + m)\hat{\mathcal{S}} - 16ms] \psi = 0, \]

(8.11)

where

\[ \hat{\mathcal{S}} = (1 + i \Gamma_4) [S_\mu S^{\mu} - 4s(s - 1)], \quad \Gamma_4 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \]

Thus, in contrast to the first order wave equations considered in Section 6 the second condition (6.1) is not a consequence of equation (8.1) but has to be considered as a supplementary requirement which can be written in the form (8.11).

Let us formulate the obtained results.

**THEOREM 8.1** [331]. The system of equations (8.1), (8.11) is Poincaré-invariant and describes a particle of spin \( s \) and mass \( m \).

The equations (8.1), (8.11) have certain advantages in comparison with the other Poincaré-invariant equation considered above. These are a relatively simple form which does not become more complicated by increasing of spin value, and existence of the reasonable limit at \( m \to 0 \) (which is not the case for the equations considered in Section 6). Finally, the equations (8.1), (8.11) admit a non-contradictive generalization to the case of particles interacting with an external electromagnetic field.

Dirac-like wave equations for any spin particles were considered by Lomont and Moses [286]. But as proposed in [286] subsidiary condition, selecting the solutions corresponding to fixed \( s \), differs from (8.11) and is incompatible with (8.1) after introduction of minimal interaction with an external electromagnetic field.

8.3. Connection with Equations without Superfluous Components

The equations in the Dirac form are closely connected with differential motion equations without superfluous components considered in Section 7. Namely, equations (8.1), (8.11) reduce to the form (7.1) where \( H \) is present in (7.37a).

Let us write (8.1), (8.11) in the form

\[ i \frac{\partial}{\partial x_0} \psi = H \psi, \quad \hat{\mathcal{P}} \psi = \psi, \]

(8.12)

where
Symmetries of Equations of Quantum Mechanics

$H = \Gamma_0 \Gamma_\alpha P_\alpha \Gamma_0 m,$

$$\hat{P}_s = P_s \frac{1}{2m} \{1 - \Gamma_4\} [\Gamma^\mu p_\mu, P_s], \quad P_s = \frac{1}{2s} [S^2 - s(s-1)].$$

(8.13)

The first of the equations (8.12) is obtained from (8.1) by the multiplication by $\gamma_0$,

the second is equivalent to (8.1) according to the relation

$$8s(1+i\Gamma_4)P_s = (1+i\Gamma_4)[S_{\mu\nu} S^{\mu\nu} - 4s(s-1)].$$

It is not difficult to make sure the operators (8.13) satisfy the conditions

$$\hat{P}_s^2 = P_s, \quad \hat{P}_s = \hat{P}_s.$$  

(8.14)

The operator $\hat{P}_s$ is a projector into subspace corresponding to spin $s$. By means of the transformation

$$\psi \rightarrow \Phi = V\psi, \quad H \rightarrow H = VH V^*,$$

$$\hat{P}_s \rightarrow \hat{P}_s = V \hat{P}_s V^*,$$

(8.15)

where

$$V = [1 - i\Gamma_4](\Gamma \cdot p - k, \Gamma \cdot S \cdot p), \quad V^{-1} = V(-p),$$

we reduce the equations (8.12) to the following equivalent form:

$$i \frac{\partial \Phi}{\partial \chi_0} - H = \{1 - i\Gamma_4\}[\Gamma_0 m + 2k, \Gamma_\alpha S_\alpha \cdot p] + \frac{1}{2m} \Gamma_4(1 - i\Gamma_4)[p^2 - 4k^2(S \cdot p)^2]\Phi,$$

$$\hat{P}_s \Phi = P_s \Phi = 0, \quad S^2 \Phi = s(s+1) \Phi.$$  

(8.16)

Choosing $\Gamma_0, \Gamma_\alpha, S$ in the form of

$$\Gamma_0 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \Gamma_\alpha = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad S = \begin{pmatrix} S & 0 \\ 0 & \tilde{S} \end{pmatrix},$$

(8.17)

where $\sigma_1, \sigma_3, S$ are the matrices of (5.30), (7.6), $\tilde{\sigma}_1, \tilde{\sigma}_3, \tilde{S}$ are analogous matrices for spin $s' = s-1$, and 0 are zero matrices of an appropriate dimension, we come to the equations in the form (6.1), (7.37a) for a 2(2s+1)-component function $\Phi = P_s \Phi$. The equations (6.1), (7.37a) are differential consequences of the equations (8.1), (8.11).

8.4. Lagrangian Formulation

Let us demonstrate that the equations in the Dirac form can be deduced in frames of the minimal action principle starting from an appropriate Lagrangian.

We write the system (8.1), (8.11) as a single equation
where $\chi$ is an arbitrary parameter which can be chosen to be equal to 1 without loss of generality, and $B_i$ is the projector

$$B_i = \frac{1}{16m^2} (\Gamma^\mu p_\mu + m)(1 + i\Gamma_4)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)].$$

Actually, multiplying (8.18) by $B_i$ and $1 - B_i$, and using the identities

$$B_i B_j = B_j, \quad (1 + i\Gamma_4)B_i (\Gamma^\mu p_\mu - m)B_j \equiv 2(\Gamma^\mu 629 p_\mu - m)B_j,$$

we come to the system (8.1), (8.11).

Using the formulation (8.18) it is not difficult to find the Lagrangian corresponding to the Dirac-like equations for particles of arbitrary spin. Choosing the Lagrangian density in the form

$$L(x) = \left( m\bar{\psi}' + i\frac{\partial \bar{\psi}'}{\partial x^\mu}\hat{\Gamma}^\mu \right) \hat{F} \Gamma^\lambda \frac{\partial \psi'}{\partial x^\lambda} + i\frac{\partial \bar{\psi}'}{\partial x^\lambda}\hat{\Gamma}_\lambda \left( m\psi' + i\frac{\partial \psi'}{\partial x^\mu}\hat{\Gamma}^\mu \right) + 16m^2 \bar{\psi}'\psi',$$

where $\psi'$ and $\psi'$ are 16-component wave functions,

$$\psi' = \text{column}(\psi, \chi), \quad \bar{\psi}' = \psi'^\dagger \hat{F} \Gamma^\lambda \hat{\Gamma}_\lambda,$$

$$\psi, \chi$$ being 8-component functions, and $\hat{\Gamma}^\mu_, F$ being the matrices of dimension 16x16:

$$F = (1 + i\Gamma_4)(\hat{S}_{\mu\nu} \hat{S}^{\mu\nu} - 4s(s-1)),$$

where $\bar{\psi}'$ and $\psi'$ are 16-component wave functions, and $\hat{\Gamma}^\mu, F$ being the matrices of dimension 16x16:

$$\hat{\Gamma}^\mu = \begin{pmatrix} \Gamma^\mu & 0 \\ 0 & \Gamma^\mu \end{pmatrix}, \quad \hat{\Gamma}_4 = \begin{pmatrix} \Gamma_4 & 0 \\ 0 & -\Gamma_4 \end{pmatrix}, \quad \hat{\Gamma}_5 = \begin{pmatrix} 0 & \Gamma_4 \\ -\Gamma_4 & 0 \end{pmatrix}, \quad \hat{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & 0 \\ 0 & S_{\mu\nu} \end{pmatrix},$$

it is not difficult to make sure that the equation (8.18) is an Euler-Lagrange equation of the type

$$\frac{\partial L}{\partial \psi'} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\psi'/\partial x^\mu)} \right) = 0,$$

where $L$ is given in (8.21). From (8.24) the equation for a function $\chi$ follows also, which can be reduced to (8.18) by the substitution $\chi \rightarrow \psi, \Gamma_4 \rightarrow -\Gamma_4$.

Thus, the equations in the Dirac form admit a Lagrangian formulation which however needs doubling the number of components of the wave function $\psi$.

We note that the equations (8.24) for $\psi$ and $\chi$ are invariant under the transformations $P, T$ and $C$ in contrast to the equations (8.1), (8.11).
As a concluding remark for this section, we show that the Dirac-like formulation is applicable to a wide class of wave equations used in modern physics.

Let us consider, for example, the equations for particles of spin 0 and 1 in the formulations of KDP (see Subsection 6.5), of Stueckelberg [398] and Hurley [224,225]. We may show that any of them can be represented in the form of the equation (8.1) with a subsidiary condition

\[ P \psi = 0, \quad (8.25) \]

with \( \Gamma_\mu \) being Dirac matrices of dimension 16×16 and \( P \) being some numerical matrix.

We will start with the following representation of the algebra \( AO(1,3) \)

\[
D = [D(1/2,0) \otimes D(0,1/2)] \otimes [D(1/2,0) \otimes D(0,1/2)] = \\
- D(1/2,0) \otimes D(0,1/2) \otimes D(1/2,0) \otimes D(0,1/2).
\]

(8.26)

Basis elements of this representation can be chosen in the form

\[
\hat{S}_{\mu \nu}, \hat{S}_{\mu}^\dagger, S_{\nu} = \frac{1}{4} [\Gamma_\mu, \Gamma_\nu], \quad S_{\mu}^\dagger = \frac{1}{4} [\Gamma_\mu, \Gamma_\nu] ^\dagger, \quad (8.27)
\]

where \( \Gamma_\mu \) and \( \Gamma_\mu^\dagger \) are commuting sets of 16×16 Dirac matrices.

The equation (8.1) is transparently invariant under the Poincaré algebra in the covariant realization (2.22), (8.27). The subsidiary condition (8.25) is Poincaré-invariant if the matrix \( P \) commutes with \( \hat{S}_{\mu \nu} \) of (8.27).

Let us demonstrate that the system of equations (8.1), (8.25) with

\[
P = 1 - P_1 \equiv \frac{1}{16} (1 - \Gamma_4^\dagger \Gamma_4)^2, \quad k = 0, 1, 2, 3, 4, \quad (8.28)
\]

is equivalent to the KDP equation for a particle of spin 1. Being specific, we choose the following realization of the matrices \( \Gamma_\mu \) and \( \Gamma_\mu^\dagger \):

\[
\Gamma_0 = \begin{pmatrix} 0 & 0 & 0 & g \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ g & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{\mu}^\dagger = \begin{pmatrix} 0 & 0 & -S_4 & S_4^a \\ 0 & 0 & -S_4 & S_4^a \\ S_4 & S_4 & 0 & 0 \\ -gS_4 & -S_a & 0 & 0 \end{pmatrix},
\]

where \( g = 2 \tau_3 - 1/2 \), \( S_4 = j_a \kappa_a \), \( S_4 = j_a + \kappa_a \), \( I \) and 0 are unit and zero matrices of dimension 4×4, and \( j_a \), \( \kappa_a \) are 4×4 matrices satisfying the relations...
Chapter 2. *Representations of the Poincaré algebra...*

The explicit form of $j_a$ and $\tau_a$ is given in the following, see (9.20).

$$
\Gamma_0 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -g & 0 \\ 0 & -g & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma_a = \begin{pmatrix} 0 & 0 & -S_a & -S_{4a} \\ 0 & 0 & gS_{4a} & -S_a \\ S_a & gS_{4a} & 0 & 0 \\ S_{4a} & S_a & 0 & 0 \end{pmatrix},
$$

(8.29)

\[ [j_a, \tau_b] = 0, \quad j_a^2 = S_a^2 = \frac{1}{4}, \quad \text{(no sum over } a). \]

(8.30)

\[ [j_a, j_b] = i \varepsilon_{abc} j_c, \quad [\tau_a, \tau_b] = i \varepsilon_{abc} \tau_c. \]

The explicit form of $j_a$ and $\tau_a$ is given in the following, see (9.20).

The matrix (8.28) is diagonal in the realization chosen:

\[
P_0 = \begin{pmatrix} P_0 & \hat{0} \\ \hat{0} & I \end{pmatrix}, \\
P_1 = \frac{1}{2}(1 + P_0) \\
\beta_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(8.31)

From (8.31) it follows that the condition (8.25), (8.28) sets to zero six out of sixteen components of $\psi$.

The system (8.1), (8.25), (8.28) can be written in the form of a single equation

\[ [P(\Gamma^\mu p_\mu - m)P - m(1 - P)]\psi = (\beta^\mu p_\mu - m)\psi = 0, \]

(8.32)

where

\[ \beta_\mu = P\Gamma^\mu P, \quad P = 1 - P_1. \]

(8.33)

The equivalence of the equations (8.32) and (8.1), (8.25), (8.28) follows from the commutativity of the projector $P$ with $\Gamma^\mu p_\mu$ on the set of solutions of the equation (8.32).

Formula (8.32) presents the KDP equation (6.6), (6.22) for a particle of spin 1. In fact, the equation (8.32) coincides with (6.6), (6.22) componentwise (the extra components of $\psi$ in (8.32) and of matrices $\beta_\mu$ in (8.33) are equal to zero).

In an analogous way we make sure that the system of equations (8.1), (8.25) where $P = 1 - P_0$, $P_0$ being the projector (8.28), is equivalent to the KDP equation for a spinless particle. The corresponding matrices $\beta_\mu$ are given in (6.17).

We see the KDP equation admits the Dirac-like formulation (8.1) and (8.25). It is curious that the KDP matrices satisfying the algebra (6.20) can be obtained from the $16 \times 16$ Dirac matrices with the help of the projection (8.33) nullifying some rows and columns.

107
The eleven component equation of Stueckelberg [398] describing a "particle" with two spin states (corresponding to $s=1$ and $s=0$) can also be formulated in the form of the Dirac equation (8.1) with the subsidiary condition (8.25) where

$$P = \frac{1}{64}(1 + \Gamma^i_z - \Gamma^i_x \Gamma^i_y)(\Gamma^i \Gamma^{i'})^2.$$  \hspace{1cm} (8.34)

In the representation (8.29) the projector (8.34) reduces to the diagonal matrix with four nonzero elements (placed on four last rows). The explicit form of the corresponding matrices $\beta_\mu$ for the Stueckelberg equation can be obtained from $\Gamma_\mu$ of (8.29) by deleting four last columns and rows (and the first columns and rows which are zero).

Finally, setting in (8.25)

$$P = \frac{1}{64}(1 + \Gamma^i_z - \Gamma^i_x \Gamma^i_y)(\Gamma^i \Gamma^{i'})^2,$$  \hspace{1cm} (8.35)

we obtain from (8.1), (8.25) a wave equation for a particle of spin 1 in the Lomont-Moses [286] form. Rewriting this equation in the equivalent formulation (8.32) we obtain the seven-component wave equation considered in detail by Hagen and Hurley [215].

We note that the Hurley equations for particles of arbitrary spin [224] are nothing but the Lomont-Moses equations written in the form (8.32).

Let us summarize. We make sure that the multicomponent Dirac equation with a covariant additional condition is a very effective construction for describing particles of arbitrary spin. In this way it is possible to obtain new equations considered in Subsections 7.4-7.8 and well known equations of KDP, Stueckelberg, and Hurley, as well. The Rarita-Schwinger equation also can be represented in the form (8.1), (8.25) (refer to (6.34)). Finally, in the following section it will be shown that equations for massless fields also admit convenient formulations in the Dirac-like form.

9. EQUATIONS FOR MASSLESS PARTICLES

9.1. Basic Definitions

In this section we consider Poincaré-invariant equations for massless fields. The description of such equations is a specific problem since they cannot be obtained, in general, from field equations for non-zero mass particles by passing to the limit $m \rightarrow 0$ [38]. The equations considered further on should also be of interest since fields with zero mass are real physical objects.

The definition of a Poincaré-invariant equation for a massless field is given
in Subsection 3.7. According to the Theorem 1.7, such equations are also invariant under the conformal algebra $AC(1,3)$.

We assume hereafter that the representation of the algebra $AC(1,3)$ realized on the set of solutions of the considered equations has the covariant form (2.22), (3.56):

$$
\begin{align*}
\mathbf{P}_\mu & = \mathbf{p}_\mu + i x_\mu, \\
J_{\mu\nu} & = \mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu + S_{\mu\nu}, \\
D & = \mathbf{x}_\mu \mathbf{p}_\mu + i \mathbf{K}, \\
K & = 2 \mathbf{x}_\mu \mathbf{p}_\mu x^\mu + 2 S_{\mu\nu} x^\mu,
\end{align*}
$$

(9.1)

where $S_{\mu\nu}$ are matrices realizing a representation of the algebra $AO(1,3)$ and $\mathbf{K}$ is a matrix commuting with $S_{\mu\nu}$.

The representation space of the algebra (9.1) will be identified with a space of states of a covariant massless field. According to (3.45), (9.1), any solution of a Poincaré-invariant equation for a covariant massless field has to satisfy the d’Alembert equation

$$
p_\mu p^\mu \psi = 0.
$$

(9.2)

Hence, we state the problem of finding all the nonequivalent linear equations which are invariant under the Lie algebra generated by the operators (9.1). Moreover, we assume that solutions of these equations satisfy (9.2) componentwise.

9.2. A Group Theoretic Derivation of Maxwell’s Equations

Before considering wave equations for arbitrary spin fields we make a look at Maxwell’s equations and demonstrate that it is possible to derive them starting from the requirement of relativistic (or conformal) invariance and some other suppositions.

First, we demonstrate that Maxwell’s equations can be deduced by using the postulate of the conformal invariance. We will look for an equation for a vector field described by a three-component wave function. The corresponding matrices in (9.1) will have the following form:

$$
\begin{align*}
S_{ab} & = e_{ab} S_c, \\
S_{aa} & = \pm i S_c, \\
K & = k I,
\end{align*}
$$

(9.3)

where $S_c$ are matrices (3.6), $I$ is the 3×3 unit matrix, and $k$ is an arbitrary number.

**THEOREM 9.1.** Let $\psi$ be a covariant massless vector field. Then $\psi$ must satisfy Maxwell’s equations.

**PROOF.** By definition $\psi$ satisfies the d’Alembert equation (9.2). This equation has to be invariant under the conformal algebra whose basis elements are given by (9.1), (9.3).

The operators $P_\mu, J_{\mu\nu}$, and $D$ are evident SOs of the equation (9.2) since
they commute with $L=p^\mu p_\mu$. As to the operators $K_\mu$, the corresponding invariance condition reduces to the form

$$[K_\mu, p_\nu p^\nu]\psi = [i(k-1)p_\mu + S_\mu p^\nu]\psi = 0.$$  \hspace{1cm} (9.4)

It is possible to show the system (9.4) is compatible for $k=2$ only, and, furthermore, it coincides with Maxwell’s equations if we set $\psi = E + iH$. To verify this statement it is sufficient to write the system (9.4) componentwise for $\mu = 0, 1, 2, 3$ and to compare it with (3.2).

We see that Maxwell’s equations are determined uniquely by the conformal invariance postulate and the vector nature of the electromagnetic field.

Consider now Maxwell’s equations (3.3) with currents and charges. These equations could not be deduced in the way presented above inasmuch as a current does not satisfy the condition (9.2). But it is possible to point to such minimal subsystems of the equations (3.3) which lead to the complete system of Maxwell’s equations, if we impose the requirement of Poincaré invariance.

We present (without proof) two assertions illustrating the possibilities of group-theoretic deduction of Maxwell’s equations with currents and charges.

**THEOREM 9.2.** Suppose $L(E, H, j_0, j)$ is a system of partial differential equations including the subsystem

$$p \cdot E = -ij_0, \quad p \cdot H = 0.$$  \hspace{1cm} (9.5)

Then, for $L(E, H, j_0, j)$ to be Poincaré invariant it is necessary for this system to include the following equations:

$$i \frac{\partial E}{\partial t} = p \times H - ij, \quad \frac{\partial H}{\partial t} = p \times E.$$  \hspace{1cm} (9.6)

Proof is given in [154, 157].

Hence, Maxwell’s equations are a consequence of the system (9.3) and the relativistic invariance postulate.

The inverse theorem is also true: the necessary and sufficient condition for the system (9.6) to be Poincaré-invariant is the requirement for $E$ and $H$ to satisfy the additional conditions (9.5) [154, 157].

**9.3. Conformal Invariant Equations for Fields of Arbitrary Spin**

In analogy with Theorem 9.1 it is possible to deduce an equation for a massless field with arbitrary helicity. In fact, supposing that such a field satisfies the conformal invariance condition, we come to the system of equations (9.4) with $S_{\mu\nu}$ being matrices belonging to a representation of the algebra $AO(1,3)$, and $\psi$ being a wave function of corresponding dimension.
Chapter 2. \textit{Representations of the Poincaré algebra...}

Thus, a covariant massless field satisfies necessarily the equations (9.4) with the appropriate matrices $S_{\mu\nu}$ and $K$. We restrict ourselves to the case when $S_{\mu\nu}$ form a completely reducible representation of the algebra $AO(1,3)$. Then the equations (9.4) reduce to a set of noncoupled subsystems. In any such subsystem the matrices $S_{\mu\nu}$ are the basis elements of the IR $D(j\tau)$ of the algebra $AO(1,3)$, and $K$ is a multiple of the unit matrix.

But the equations (9.4) have to be invariant under the algebra $AC(1,3)$, so we come to the following conditions for the operators $L_{\mu}$:

\begin{equation}
[L_{\mu}, Q]\psi = 0,
\end{equation}

where $Q$ is any of the generators (9.1). By direct calculation we obtain

\begin{align*}
[L_{\mu}, p_{\mu}](\psi) &= 0, \\
[L_{\mu}, D](\psi) &= 0,
\end{align*}

from which it follows that $P_{\mu}$, $D$, and $J_{\mu\nu}$ satisfy the invariance condition of the equation (9.4). As to the operators $K_{\mu}$ we obtain from (9.7) (with $Q$ to be changed for $K_{\mu}$) the following system of equations [52]:

\begin{align*}
[S_{\mu}^a S_{\nu}^a - i S_{\mu\nu}^a (1 - k) g_{\mu\nu}] \psi &= 0, \\
[k (1 - k) - j (j + 1) - \tau (\tau + 1)] \psi &= 0, \\
(j + \tau + 1 - k) (j - \tau - k) (k + j + \tau + 1) p_{\lambda} \psi &= 0,
\end{align*}

which has a nontrivial solution for $j\tau \neq 0$ only. Moreover,

\begin{equation}
k - j + \tau + 1 = s + 1.
\end{equation}

Substituting (9.9) into (9.4), we come to the following system of conform invariant equations:

\begin{align*}
(s p_{\mu} S_{\mu}^a p_{\nu}) \psi &= 0,
\end{align*}

where $S_{\mu\nu}$ are matrices belonging to the representation $D(s0)$ (or $D(0s)$) of the algebra $AO(1,3)$, and $\psi$ is a $2(s+1)$-component wave function.

It is not difficult to show that such equations describe a massless field with helicity $\pm s$. Multiplying (9.10) by $p^\mu$ and summing up over $\mu$, we come to the equation (9.2). On the other hand, representing (9.10) in the form

\begin{equation}
W_{\mu} \psi = \varepsilon^{\prime} s P_{\mu} \psi = \varepsilon^{\prime} \lambda P_{\mu} \psi, \quad \lambda = \varepsilon^{\prime} s,
\end{equation}

where $W_{\mu}$ are components of Lubanski-Pauli vector, $\varepsilon^{\prime} = 1$ for $S_{\mu\nu} \subset D(0s)$, and $\varepsilon^{\prime} = -1$ for $S_{\mu\nu} \subset D(s0)$, and comparing (9.11) with (4.55), we conclude that the direct sum $D^{\prime}(\varepsilon^{\prime} s) \oplus D(\varepsilon^{\prime} s)$ of the Poincaré group representations is realized on the set of solutions of the equation (9.10).

Thus, we obtain the system of equations (9.10) describing a massless
Symmetries of Equations of Quantum Mechanics

covariant field of arbitrary spin. In the case \( s=1/2 \) this system is equivalent to Weyl's equation, and for \( s=1 \) it reduces to Maxwell's equations. Equations of the (9.10) type for an arbitrary spin were considered in [135].

We note that the equation (9.10) follows from the supposition of conformal invariance of a field \( \psi \) in a unique fashion. In other words, if \( \psi \) is a covariant massless field then it has to satisfy to (9.10) with necessity.

9.4. Equations of Weyl's Type

The equations (9.10) represent a set of four systems of partial differential equations that should be simultaneously satisfied by a wave function \( \psi \). But in the case \( s=1/2 \) we have actually only one system of equations, i.e., the Weyl system (2.44). Essentially, any of the equations (9.10) with \( s=1/2 \) reduces to the form of (2.44) by multiplication by the Pauli matrices \( \sigma \).

Poincaré-invariant equations of Weyl's type exist for a massless field of arbitrary spin. Instead of four equations (9.10) it is possible to consider a single system from which (9.10) follow as a mere consequence.

It is well known that the Weyl equation is equivalent to the massless Dirac equation with the additional condition \( (1-i\gamma_4)\psi=0 \). Equations for any spin may be obtained in analogous way, starting with the Dirac-like wave equations (see Section 8).

A system of Poincaré-invariant equations for an arbitrary spin particle (8.1), (8.11) may be written for the case of \( m=0 \) as follows:

\[
\begin{align*}
&i \frac{\partial}{\partial \tau} \psi = \Gamma_0 \Gamma_a \sigma_a \psi, \\
&\left( i \frac{\partial}{\partial t} - \Gamma_0 \Gamma_a \sigma_a \right) (1-i\gamma_4) S \psi = 0.
\end{align*}
\]

Imposing on \( \psi \) the Poincaré-invariant additional condition

\[
(1-i\gamma_4)\psi = 0,
\]

and choosing \( \Gamma_\mu \) in the form

\[
\begin{align*}
\Gamma_0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \Gamma_+ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, & \Gamma_\sigma &= \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix},
\end{align*}
\]

where \( I \) and 0 are the 4s-row unit and zero matrices, and \( \sigma \) are the Pauli matrices of dimension 4s×4s, we obtain

112
Chapter 2. Representations of the Poincaré algebra...

\[
\left\{ \frac{i}{\hbar} \frac{\partial}{\partial x_0} - \sigma^\mu p_\mu \right\} \varphi(x) = \sigma^\mu p_\mu \varphi(x) = 0, \tag{9.15}
\]

\[
\sigma^\mu p_\mu \hat{S}_{\nu\lambda} \hat{S}^{\nu\lambda} \varphi(x) = 0,
\]

with \( \varphi(x) \) being a 4s-component wave function connected with \( \psi \) by the relation

\[
\varphi = \frac{1}{2} (1 + i \Gamma_4) \psi. \tag{9.16}
\]

According to (8.6), (9.16), the matrices \( \hat{S}_{\mu\nu} \) belong to the representation \( D(1/2 \ 0) \oplus D(s-1/2 \ 0) \) of the algebra \( AO(1,3) \), i.e., they have the following structure:

\[
\hat{S}_{\alpha\omega} = (j_\alpha + \frac{1}{2} \sigma_\alpha), \quad \hat{S}_{ab} = -i \epsilon_{abc} S_{bc}.
\]

Moreover,

\[
[j_\alpha, \sigma] = 0, \quad [j_a, j_b] = i \epsilon_{abc} j_c, \quad j_a j_a = s(s-1). \tag{9.18}
\]

The equations (9.15) are transparently Poincaré- and conform invariant. It is not difficult to make sure that these equations describe a massless field with helicity \( \pm s \). In fact, multiplying the first of them by \( i \partial/\partial x_0 + \sigma^\mu p_\mu \), we come to (9.2) from which it follows that the mass of the described field is equal to zero. Denoting \( \hat{S}_{\mu\nu} \hat{S}^{\mu\nu} = 4(g^2 - s^2) \) and taking into account the identities

\[
[g, \sigma] = 2S \sigma, \quad g^2 - s^2, \quad [g, S \cdot p] = 0, \quad S_a = \frac{1}{2} \epsilon_{abc} S_{bc},
\]

we obtain from (9.15) that

\[
S \cdot p \varphi = \sigma \cdot p \varphi,
\]

or

\[
S \cdot p \varphi = g \sigma \cdot p \varphi. \tag{9.19}
\]

Comparing (9.19) with (4.55), we conclude that the helicity of the field \( \varphi \) coincides with eigenvalues of the matrix \( g \), i.e., with \( \pm s \). It is possible to demonstrate that the systems of equations (9.10) follow from (9.15).

So, in order to describe massless particles of arbitrary helicity, it is possible to use the generalized Weyl equations (9.15). Consider the examples of such equations for \( s \leq 2 \).

a) \( s = 1/2 \). In this case the matrices \( \sigma_\alpha \) are of dimension \( 2 \times 2 \), \( j_\alpha \) are zero matrices, \( \hat{S}_{\mu\nu} \hat{S}^{\mu\nu} = 3 \), and the equations (9.15) reduce to the usual Weyl equation (2.43).

b) \( s = 1 \). Without loss of generality, we may choose the matrices \( \sigma_\alpha \) and \( j_\alpha \)
in the form
\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \\
\end{align*}
\]
(9.20)

Then, for \( \psi = \text{column}(\psi_1, \psi_2, \psi_3, \psi_4) \), we obtain from (9.15), (9.17) the following equations:
\[
\begin{align*}
\sigma_1 \frac{1}{2} & \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \sigma_2 \frac{1}{2} & \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \sigma_3 \frac{1}{2} & \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \\
\end{align*}
\]
(9.21)

Then, for \( \psi = \text{column}(\psi_1, \psi_2, \psi_3, \psi_4) \), we obtain from (9.15), (9.17) the following equations:
\[
\frac{\partial \psi}{\partial x_0} - \mathbf{p} \times \psi, \quad \mathbf{p} \cdot \psi = 0, \quad \psi \text{-const},
\]
(9.22)

where the constant \( \psi_4 \) can be taken to be zero without loss of generality.

The equations (9.21) reduce to Maxwell’s equations if we denote \( \psi = H - iE \), \( H \) and \( E \) being real vectors.

c) \( s = 3/2 \). Choosing \( \sigma_a \) and \( j_a \) in the form
\[
\sigma_a = I_2 \otimes \sigma_a', \quad j_a = S_a \otimes I_2,
\]
(9.23)

where \( S_a \) and \( \sigma_a \) are the matrices (3.6), (2.5), \( I_2 \) and \( I_3 \) are the unit matrices of dimension 2x2 and 3x3, and representing the wave function \( \varphi \) in the form
\[
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \varphi_\alpha = \text{column}(\psi_\alpha^1, \psi_\alpha^2, \psi_\alpha^3), \quad \alpha = 1, 2
\]

we obtain the following system of equations:
\[
\frac{\partial \psi_\alpha}{\partial x_0} - \mathbf{p} \times \psi_\alpha, \quad (\sigma_\mu)_{\alpha \alpha} \mathbf{p} \cdot \psi_\alpha = 0.
\]
(9.24)

We see that the massless field of helicity \( |\lambda| = 3/2 \) satisfies Maxwell’s equations in respect with the vector index \( a \), and does the Weyl equation in respect with the spinor index \( \alpha \).

d) \( s = 2 \). Let us choose \( \sigma_a \) and \( j_a \) in the form
\[
\begin{pmatrix} \sigma_a & j_a \otimes I_2 \\ j_a & -j_a \otimes \sigma_a \end{pmatrix}, \quad \sigma_a = I_2 \otimes \sigma_a',
\]
(9.25)

where \( \sigma_a \) are the Pauli matrices from (2.5), and

In our case the function \( \varphi \) has eight components \( \varphi_\alpha^k, \alpha = 1, 2, \) \( k = 1, 2, 3, 4 \), the matrices
Chapter 2. Representations of the Poincaré algebra...

\[ \hat{j}_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \hat{j}_2 = \frac{1}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \hat{j}_3 = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \]

\( j_a \) and \( \sigma_a \) acting on the indices \( k \) and \( \alpha \) respectively. Then we obtain from (9.15), (9.17), and (9.24) the following system:

\[ (\sigma^\mu_{\alpha\beta})_{\mu\nu} p^\nu \phi^\alpha_{\alpha} = 0, \quad i \frac{\partial \phi^\alpha_{\alpha}}{\partial x^0} = \frac{2}{3} (j^\mu_{\alpha\beta})_{\mu\nu} p^\nu \phi^\alpha_{\alpha}. \]

(9.25)

These equations describe a massless field of spin 2.

9.5. Equations of Other Types for a Massless Field

Essentially, the systems (9.10), (9.15) exhaust all the nonequivalent formulations of conformal invariant equations for a massless field of arbitrary spin, if we assume that the corresponding generators of the conformal group have the covariant form (9.1). This does not mean that there no other equations exist, which are invariant under the algebra \( AC(1,3) \), for a massless field, because the basis elements of this algebra can be, in principle, chosen in a noncovariant realization.

As the examples of equations for massless fields, which are not equivalent either to (9.10), or to (9.15), may serve the equations obtained from (7.1), (7.27) by passing to the limit \( m \to 0 \).

All nonequivalent equations for relativistic massless fields can be enumerated as follows. Any representation of the Poincaré algebra belonging to class \( II \) corresponds to the class of equivalent equations, and, thus, it is sufficient to choose one representative from each of these classes. Supposing that the corresponding representatives may be reduced to a nondegenerated direct sum of the IRs \( D^s(\epsilon, \lambda) \), \( \epsilon, \epsilon = \pm 1 \), we obtain the following combinations for fixed \( \lambda \):
Representations described by (9.26d) are realized on sets of solutions for the equations (9.10), (9.15). Equations corresponding to other representations of (9.26) may be obtained by the method proposed in [132, 139]. Namely, starting with an equation corresponding to the representations of (9.26f), we can find all the nonequivalent Poincaré-invariant additional conditions to be imposed on a set of solutions for this equation in order to determine subspaces of the representations (9.26). In this manner, it is possible to describe all the nonequivalent equations for a massless vector field including the Maxwell’s equations as a particular case. We will not consider this possibilities in detail as they are elucidated in [154].

10. RELATIVISTIC PARTICLES OF ARBITRARY SPIN IN AN EXTERNAL ELECTROMAGNETIC FIELD

10.1. The Principle of Minimal Interaction

Solutions of Poincaré invariant wave equations considered above determine wave functions of arbitrary spin particles which can be used for solving different problems of quantum mechanics. But the main value of these equations lies in the fact that they can be used to describe an interaction of a particle with an external field.

In the case of electromagnetic interaction the corresponding motion equation can be obtained from an equation for a free particle by the following substitution:

\[ p_\mu \rightarrow \pi_\mu - eA_\mu \]  

(10.1)

where \( A_\mu \) is a vector potential of the electromagnetic field, and \( e \) is a charge of a particle.

The rule for introduction of interaction given in (10.1) has to be considered
Chapter 2. *Representations of the Poincaré algebra...*

as a postulate which is called the minimal interaction principle. We will not discuss the bounds of validity of this principle, but note that the prescription of introduction of interaction given in (10.1) is not the only possible one. A more general approach is to take into account the so-called anomalous interaction, examples of which will be considered in the following.

Equations of motion, obtained from Poincaré-invariant wave equations for a free particle by means of the substitution (10.1), preserve the Poincaré-invariance if a wave function transforms in accordance with the local covariance law (2.49). But, as it happens, an introduction of the minimal coupling into relativistic equations for particles of spin \( s \geq 1 \) leads to difficulties of the principal manner which, briefly speaking, may be stated as follows.

1. A system of partial differential equations describing a spinning particle becomes inconsistent as a result of the substitution (10.1) being made. Such a situation takes place, for example, for the Procá equation [412] written in the form of second-order equation with a subsidiary condition.

2. Inclusion of the minimal interaction into the equation for a free particle can lead to such an equation which cannot be interpreted as a motion equation for a particle of spin \( s \) since the corresponding wave function has superfluous (with respect to \( 2(2s+1) \)) components. Such a result is true for Dirac-like equations for arbitrary spin particles proposed in [286].

3. Equations for a particle of spin \( s > 1 \) are relativistically invariant but describe a faster-than-light wave propagation. Thus, e.g., the Rarita-Schwinger equation (see Subsection 6.6) becomes nonhyperbolic (i.e., having no wave solutions) as a result of the substitution (10.1) corresponding to large field strength. For small \( E \) and \( H \) this equation remains hyperbolic but describes a faster-than-light wave propagation [405].

4. Equations with minimal interaction are inconsistent while solving concrete physical problems, e.g. the Kepler problem. This situation takes place for the KDP equation [401].

As was shown in [194, 405], this situation is typical for the majority of relativistic wave equations for particles of spin \( s > 1/2 \). The difficulties mentioned in pars. 1 and 2 can be surmounted if the motion equations are of Euler-Lagrange type [100]. The contradictions related to causality violations are of principal matter and follow from the fact that the relativistic wave equations usually include superfluous components if \( s > 1/2 \), and, in fact, they are nonlocal (refer to Subsection 6.7).

In connection with the above the natural question arises of the possibility of using Poincaré-invariant equations without superfluous components and Dirac-like equations (considered in Sections 7, 8) in order to describe a charged particle of spin \( s \) in an external electromagnetic field. The positive answer to this question is given
10.2. Introduction of Minimal Interaction into First-Order Wave Equations

As a result of the substitution (10.1), the equations (6.6) take the following form:

\[ (\beta^\mu \pi_\mu - m)\psi = 0. \]  

(10.2)

If the starting system (6.6) is Poincaré-invariant and, at the same time, a transformation law for \( \psi \) has the form of (2.49) (with the corresponding matrices \( S_\mu \)) then the system (10.2) has the similar property of invariance inasmuch as Lorentz transformations for \( \pi_\mu \) and \( p_\mu \) are the same. But, in accordance with the reasons given above, equations of the (10.2) type for arbitrary spin particles are in general inconsistent.

Here we consider the equations (10.2) for particles with spin \( s = 0, \frac{1}{2}, 1 \).

We will define the corresponding Hamiltonians and discuss briefly the problems arising while using these equations to solve particular physical problems.

The simplest example for a first-order wave equation is the Dirac equation for an electron. Substituting \( \beta^\mu = \gamma^\mu \) into (10.2), and multiplying it at the left by \( \gamma^0 \), we come to the equation

\[ i \frac{\partial}{\partial x_0} \psi - H(A_0, \pi) \psi, \]  

(10.3)

where

\[ H(A_0, \pi) = \gamma^0 \gamma^a \pi_a - \gamma^0 m + eA_\nu, \]  

(10.4)

The Hamiltonian (10.4) corresponds for a charged particle with spin \( \frac{1}{2} \) interacting with the electromagnetic field. This is a first-order differential operator which is formally Hermitian with respect to the scalar product (2.39). The equation (10.3) satisfies the causality principle (refer to Subsection 10.7) and serves as an adequate mathematical model for a wide set of physical problems for which the concept of an external field makes sense.

The KDP equations for scalar and vector particles interacting with an external electromagnetic field can also be represented in the form of (10.2) with \( \beta^\mu \) being the KDP matrices of dimensions 5×5 and 10×10. These equations also reduce to the Schrödinger form (10.3) using the same procedure as described in the Subsection 6.7. In addition, \( \psi \) is a \( 2(2s+1) \)-component wave function, and the corresponding Hamiltonians \( H(A_0, \pi) \) have the form

\[ H = \sigma_2 m + (\sigma_2 + i\sigma_1) \frac{\pi^2}{2m} + eA_\nu, \]  

(10.5)
Chapter 2. Representations of the Poincaré algebra...

\( s=1, \quad H=\sigma \cdot m - (\sigma_2 + i \sigma_1) \frac{\pi^2 + eS \cdot H}{2m} - i \sigma_1 \frac{(S \cdot \pi)^2}{2m} + eA_\nu. \)  \hspace{1cm} (10.6)

Here \( H=ip \times A \) is the vector of the magnetic field strength, \( \sigma_1, \sigma_2, \) and \( S \) are the matrices (5.30), (7.6), and (3.6).

The operators (10.5), (10.6) are Hermitian with respect to the scalar product (7.35) where \( M=\sigma_2, \) and in the case of \( e \to 0 \) they reduce to the free particle Hamiltonians (7.40a), (7.40c) (up to equivalence transformation \( \sigma_1 \to \sigma_2, \sigma_2 \to -\sigma_1. \) The corresponding equations of motion satisfy the causality principle, such as the Dirac equation (refer to Subsection 10.7).

The Rarita-Schwinger equation loses the properties of a causal equation and is not considered hereafter.

So, starting from first-order wave equations, we obtain the equations in the Schrödinger form for particles with spin 0, 1/2, 1. The corresponding equations for arbitrary spin particles are considered in the following subsection.

### 10.3. Introduction of Interaction into Equations in Dirac’s Form

We can obtain equations for a charged particle with an arbitrary spin in an external field by starting with Dirac-like equations for a free particle (see Section 8).

One may make sure (by a direct verification) that the substitution of (10.1) into the equations (8.1) and (8.11) leads to a system which is consistent only for the zero tensor \( F^{\mu \nu} \) of the electromagnetic field. To overcome this difficulty, it is sufficient to introduce the minimal interaction into the Lagrangian (8.21), or into the equations (8.18). As a result, we come to the system

\[
[B_j(\pi)(\Gamma^\nu \pi^\mu - m) + \gamma(1 - B_j(\pi))]\psi = 0, \hspace{1cm} (10.7)
\]

where

\[
B_j(\pi) = \frac{1}{16ms} \Gamma^\nu \pi^\mu + m) (1 - i \Gamma_4)[S_{\mu \nu} S^{\mu \nu} - 4s(s - 1)]. \hspace{1cm} (10.8)
\]

Multiplying (10.7) by \( B_j(\pi) \) and \( (1 - B_j(\pi)) \), and taking into account the identities

\[
B_j(\pi)B_j(\pi) = B_j(\pi), \quad B_j(\pi)(\Gamma^\nu \pi^\mu - m)B_j(\pi) =
\]

\[
\left[ \Gamma^\nu \pi^\mu + m + \frac{e}{4m}(1 - i \Gamma_4) \left( \frac{1}{s} S_{\mu \nu} - i \Gamma_{\mu \nu} \right) \right] F^{\mu \nu} B_j(\pi), \hspace{1cm} (10.9)
\]

we come to the following system:

\[
\left[ \Gamma^\nu \pi^\mu - m + \frac{e}{4m}(1 - i \Gamma_4) \left( \frac{1}{s} S_{\mu \nu} - i \Gamma_{\mu \nu} \right) \right] F^{\mu \nu} \psi = 0, \hspace{1cm} (10.10a)
\]
Thus, the Dirac-like equations for a particle with an arbitrary spin in an external electromagnetic field have the form of (10.10). We see that introduction of minimal interaction into the Lagrangian (8.21) results in the appearing of terms to be proportional to the tensor of an electromagnetic field strength $F_{\mu\nu}^\sigma$ in the equation of motion.

Let us demonstrate that (10.10) reduce to equations of the Schrödinger type for a $2(2s+1)$-component wave function. Multiplying any of the equations (10.10) by $\Gamma_0$, we come, after simple calculations, to the system of the form (8.12) where

$$H = \Gamma_\mu \pi^\mu + \Gamma_\mu m + \frac{e}{4m} \Gamma_\mu (1 - i \Gamma_\mu) \left[ \frac{1}{s} \pi_{\mu \nu} - i \Gamma_{\mu \nu} \right] F^{\mu \nu},$$

$$\hat{P}_s = \frac{1}{2m} (1 - i \Gamma_2) [\Gamma_\mu \pi^\mu, P_s],$$

and $P_s$ is the matrix (8.13). The identities necessary to reduce (10.10) to the form of (8.12) are found in (8.14) and (10.10a).

The operators (10.11b) satisfy the condition $\hat{P}_s^2 = \hat{P}_s$ and are orthoprojectors into a subspace corresponding to the spin $s$. As in the case of free particle equations, these projectors reduce to the numerical matrices $P_s$ by the transformations (8.15) where $V$ is the operator obtained from (8.15) by the change $p \to \pi$. At the same time, the Hamiltonian (10.10a) is transformed into the form

$$H \to H' = VH V^{-1} m + 2k_1 \Gamma_4 S \cdot \pi + \frac{1}{2m} \Gamma_0 (1 - i \Gamma_4) \times$$

$$\times \left\{ \pi^2 - 4k_1^2 (S \cdot \pi)^2 - \frac{1}{s} \pi \cdot [H - i (1 - 2k_1 s) E] \right\} + eA_0,$$

or, choosing the representation (8.15),

$$H = \sigma_3 m + \sigma_3 2k_1 S \cdot \pi + \frac{1}{2m} (\sigma_3 - i \sigma_3) \left\{ \pi^2 - 4k_1^2 (S \cdot \pi)^2 - \frac{1}{s} S \cdot [H - i (1 - 2k_1 s) E] \right\} + eA_0.$$

Formula (10.13) generalizes the free particle Hamiltonian (7.37a) to the case of a charged particle in an external field. Thus, starting with the first-order wave equations (10.10), we have obtained the formula for introduction of interaction into non-manifestly covariant equations without superfluous components.

We note that the Hamiltonians (10.13) corresponding to different values of the parameter $k_1$ are equivalent. Let $H(k'_1)$ be the Hamiltonian (10.13) with $k_1 = k'_1$, and $H(k''_1)$ be the operator corresponding to $k_1 = k''_1$. Then $H(k'_1) = VH(k''_1) V^{-1}$, where $VH(k''_1) V^{-1}$.

The second note is related to the fact that there exists one-to-one correspondence between solutions of the equations (10.10) and those of (8.12).
Chapter 2. Representations of the Poincaré algebra...

\[ V = \exp \left( \frac{i\sigma_i - i\sigma_j}{m} k_i - k_j^\prime \right) S \bar{\pi} \]  
(10.13), as it is evident from the above.

10.4. A Four-Component Equation for Spinless Particles

Let us demonstrate that a spinless particle can also be described by a Dirac-like equation.

As it was noted in [118], the Dirac equation for a free electron can be in principle interpreted as an equation for a massless particle since it is possible to define (noncovariant) representation of the Poincaré algebra corresponding to the zero spin on a set of Dirac equation solutions. It turns out that such an interpretation is also possible in the case of charged particles interacting with an external field if the coupling is taken into account in the specific manner [147].

Consider the equation
\[ \gamma_\mu \pi^\mu - m + \frac{iek}{4m} (1 - i\gamma_4) \gamma_\mu F^{\mu\nu} \psi = 0, \]  
(10.14)
where \( \gamma_\mu, \gamma_\nu \) are the Dirac matrices, and \( k \) is an arbitrary parameter.

The equation (10.14) is explicitly covariant and coincides (in the case of \( k=0 \)) with the Dirac equation for a minimally interacting particle with spin 1/2. The term \( iek(1-i\gamma_4)\gamma_\mu F^{\mu\nu}/4m \) can be interpreted as a contribution due to the anomalous interaction of the Pauli type.

Let us now demonstrate that in the case of \( k=1 \) the equation (10.14) describes a motion of a spinless charged particle. We multiply (10.14) by \( \gamma_0 \) to obtain the equation in the form of (10.3) where
\[ H = \gamma_0 \gamma_\mu \pi^\mu + \gamma_0 m + eA_0 - \frac{iek}{4m} \gamma_0 (1 - i\gamma_4) \gamma_\mu F^{\mu\nu}. \]

Then, applying the transformation \( H \rightarrow H' = VHV^{-1} - iV^{-1} \partial V/\partial \chi_0 \) with \( V \) being equal to
\[ V = \exp \left( \frac{1}{2m} \gamma_0 (1 - i\gamma_4) \pi^2 \right), \]
we obtain the following:
\[ H' = \omega_0 m + \frac{1}{2m} \gamma_0 (1 - i\gamma_4) \pi^2 + eA_0. \]

Choosing the representation (2.4) for the gamma-matrices, we can rewrite \( H' \) in the form of (10.5) where \( \sigma_1 \rightarrow \sigma_2, \sigma_2 \rightarrow -\sigma_1, \sigma_1 \) and \( \sigma_2 \) are the 4×4 Pauli...
Symmetries of Equations of Quantum Mechanics

matrices (5.30). In other words, this Hamiltonian reduces to the direct sum of two operators (10.5) corresponding to a spinless particle in an external electromagnetic field.

10.5. Equations for Systems with Variable Spin

Consider equations (10.10) again, and show that they can be generalized to describe a "particle" with a variable spin. The equation (10.10a) has a clear physical sense even in absence of the additional condition (10.10b). In fact, it can be reduced to a direct sum of the equations (7.1), (10.13) for particles with spin \( s \) and \( s' = s-1 \).

The equation (10.10a) admits a reasonable interpretation for the case of matrices \( S_{\mu \nu} \) belonging to a more general representation than given in (8.9). Here we consider the following representation of \( S_{\mu \nu} \):

\[
D = \left[ D \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{2} & 0 \end{array} \right) \right] \oplus D \left( \begin{array}{cc} s & 0 \\ \frac{1}{2} & s' \end{array} \right) = \left[ D \left( \begin{array}{cc} 1 & \frac{1}{2} \\ 0 & 1 \end{array} \right) \right] \oplus D \left( \frac{s}{2} \right).
\]

Transforming the corresponding Hamiltonian (10.10a) in accordance with (10.12), we obtain for \( k_\perp = 0 \) that

\[
H' = \Gamma_0 \hat{m} + \frac{1}{2m} \Gamma_0 \left( 1 - i \Gamma_4 \right) \left( \pi^2 - \frac{1}{s} S_{\mu \nu} F_{\mu \nu} \right) + e A_\nu.
\]

The following identity is valid:

\[
(1 - i \Gamma_4) S_{\mu \nu} \equiv (1 - i \Gamma_4) \hat{S}_{\mu \nu},
\]

where \( S_{\mu \nu} \) belong to the representation

\[
D = D \left( \begin{array}{cc} s & s \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) \oplus D \left( \begin{array}{cc} s & s-1 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) \oplus D \left( \begin{array}{cc} s & s-1 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) \oplus D \left( \frac{s}{2} \right).
\]

Choosing \( \Gamma_0 \) and \( \Gamma_4 \) in the form of (8.17), where \( \sigma_a \) and \( \sigma'_a \) are the Pauli matrices of dimension \( 2(s+1)^2 \times 2(s+1)^2 \) and \( 2(s'-1)^2 \times 2(s'-1)^2 \), and \( \hat{S}_{\mu \nu} \) as a direct sum of the matrices

\[
S_{\mu \nu}^{(1)} = D \left( \begin{array}{cc} s & s \\ \frac{1}{2} & \frac{1}{2} \end{array} \right), \quad S_{\mu \nu}^{(2)} = D \left( \begin{array}{cc} s & s-2 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right),
\]

we obtain the Hamiltonian \( H' \) in the form of a direct sum of the operators \( H_1 \) and \( H_2 \) where
Chapter 2. *Representations of the Poincaré algebra*...

\[
H_1 = i\sigma_1 m + (\sigma_1 - i\sigma_2) \frac{1}{2m}\left(\pi^2 - \frac{1}{s} S^{(1)}_{\mu\nu} F^{\mu\nu}\right) + eA_\nu, \\
H_2 = \sigma_1 m + (\sigma_1 - i\sigma_2) \frac{1}{2m}\left(\pi^2 - \frac{1}{s} S^{(2)}_{\mu\nu} F^{\mu\nu}\right) + eA_\nu. \\
\tag{10.18}
\]

The corresponding Schrödinger equation reduces to the pair of noncoupled equations with the Hamiltonians \(H_1\) and \(H_2\). Let us write the first of them:

\[
i\frac{\partial}{\partial x_0} \psi = \left[\sigma_1 m + (\sigma_1 - i\sigma_2) \frac{1}{2m}\left(\pi^2 - \frac{1}{s} S^{(1)}_{\mu\nu} F^{\mu\nu}\right) + eA_\nu\right] \psi. \\
\tag{10.19}
\]

The equation (10.19) differs from (7.1), (10.13) only in the representation realized by the matrices \(S_{\mu\nu}\). The representation \(D(s/2)\) reduces to the following direct sum of IRs of the algebra \(AO(3)\):

\[D(s/2) \rightarrow D(s) \oplus D(s-1) \oplus ... \oplus D(0),\]

and we can interpret (10.19) as an equation for a quasiparticle which can be in different spin states corresponding to the spin values \(s, s-1, ... , 0\).

The equation (10.19) is invariant under the \(P\), \(C\), and \(T\)-transformations which can be chosen in the form of (2.55), (2.60) where

\[
\begin{pmatrix} \eta' \\ 0 \end{pmatrix}, \begin{pmatrix} \delta' \\ 0 \end{pmatrix}, \begin{pmatrix} \eta' 0 \\ 0 \eta' \end{pmatrix}, \begin{pmatrix} \delta' 0 \\ 0 \delta' \end{pmatrix}, \tag{10.21}
\]

with \(\eta'\) and \(\delta'\) being matrices defined up to a sign by the following relations:

\[
\eta' S_{ab} = \eta' \delta', \quad \eta' S_{ab} = -S_{ab} \eta', \quad (\eta')^2 = 1, \quad \delta' S_{\mu\nu} = -S_{\mu\nu} \delta', \quad (\delta')^2 = (-1)^s.
\]

The explicit expressions for \(\eta'\) and \(\delta'\) (which will not be used further on) can be easily obtained using the results of Subsection 5.6.

**10.6. Introduction of Minimal Interaction into Equations Without Superfluous Components**

Inasmuch as the main difficulties in description of particles in an external electromagnetic field are connected with superfluous components of relativistic wave equations, it is natural to try to introduce an interaction into wave equations with the correct number of components considered in Section 7. Such equations do not possess an explicit covariant form, and, generally speaking, the minimal interaction principle has to be used with an appropriate carefulness since the substitution (10.1) can violate the Poincaré invariance of the equations considered.

One way to introduce an interaction into motion equations without
Symmetries of Equations of Quantum Mechanics

superfluous components is to make the substitution (10.1) in the corresponding first-order wave equation which reduces to the given equation in the Schrödinger form by delating superfluous components. This way has been used to obtain the Hamiltonians (10.5)-(10.7), (10.13), and (10.18). It seems that there exist other possibilities to include an interaction into equations without superfluous components (refer, for example, to [211]).

Here we consider a description of arbitrary spin particles in an external electromagnetic field which is based on the nonlocal equations (7.1), (7.28), (7.23), and (7.31). We restrict ourselves to the class of problems corresponding to particle momentum being small in comparison with a particle rest mass, and represent the corresponding Hamiltonians as a series in powers of $1/m$:

$$H_{\alpha}^{s} = \sigma_{1} \left[ m + \frac{1}{2m} \delta_{0}^a p_a p^a \right] + \sigma_{2} \left[ 2S_{\alpha}^b p_b \right] + \frac{1}{m^2} \left[ \frac{h}{m} \right] + \alpha \omega_{\alpha},$$

where

$$d^{ab} = \delta_{ab} - 2(S_{a}^b S^b_{a}) + h^I - 2h^I - 2S_{a}^b d_{ab} p_a p_b, \quad \alpha = I, II.$$  

(10.23)

Of course, the equations (7.1), (10.22) are not Poincaré-invariant, and can be considered only as approximate quasirelativistic models.

Changing $p_\mu \rightarrow \pi_\mu$ we come to the following systems:

$$H^\alpha(\pi) \psi = \frac{\partial}{\partial x_0} \psi,$$

(10.24)

$$H^\alpha(\pi) = \sigma_{1} \left[ m + \frac{1}{2m} \delta_{0}^a \pi_a \right] - 2(S_{\pi}^a \pi^a) + eS_{\pi} H/m] + \sigma_{2} \left[ 2S_{\pi}^a \pi^a \right] + \frac{1}{m^2} \left[ \frac{h}{m} \right] + \alpha \omega_{\alpha}(1/m^2).$$

It will be shown further on that the equations (10.24) describe satisfactorily an arbitrary spin particle in the electromagnetic field, taking into account such the well-known physical effects as the dipole, spin-orbital, and Darwin couplings.

10.7. Reduction in Power Series in $1/m$

As is well known, relativistic wave equations admit a consistent interpretation only in terms of the second quantized theory which enables to overcome the difficulties in interpreting negative energies. But these equations can serve as satisfactory models for a wide class of problems where a particle momentum is small in comparison with its mass, since in this case it is possible to separate the positive energy solutions.

To select the positive energy states we transform our equations to a representation for which the energy sign operator $H^+/H$ is diagonal. In contrast to the free particle case, it can be done only approximately, by supposing that $\pi^2 \ll m^2$. 

124
Chapter 2. Representations of the Poincaré algebra...

Hereafter we make such diagonalization. We represent Hamiltonians for arbitrary spin particle as a power series in $1/m$ which is convenient for calculations within the frames of the perturbation theory.

Let us consider the Hamiltonians (10.13), (10.24) defined for arbitrary values of spin $s$. After the series of successive transformations

$$H^u \rightarrow W^u H^u (W^{-1})^u - i \frac{\partial W^u}{\partial x_0} (W^{-1})^u = (H^u)^{\prime \prime}, \quad \alpha = I, II, III, \quad W^u = V_1^u V_2^u V_1^u, \quad (10.25)$$

where

$$V_1^{\prime \prime} = \exp \left( -i \sigma_2 \frac{S \cdot \pi}{m} \right), \quad V_1^{\prime \prime} = \exp \left( i \sigma_1 \frac{e S \cdot E}{2m^2} \right),$$

$$V_2^{\prime \prime} = \exp \left\{ -i \frac{\sigma_2}{2sm} \left[ h^{\prime \prime}(\pi) - \frac{4}{3} (S \cdot \pi)^2 + \frac{1}{2} \frac{\partial}{\partial x_0} S \cdot E - \frac{1}{2} S \cdot \pi^2 - 2S \cdot H \right] \right\},$$

$$V_3^{\prime \prime} = \exp \left\{ -i \sigma_2 \frac{S \cdot \pi}{2sm} \exp \left[ \left( k_1 - \frac{1}{2s} \right) \left( \sigma_1 - i \sigma_2 \right) S \cdot \pi \right] \right\},$$

$$V_3^{\prime \prime} = \exp \left\{ -i \frac{\sigma_2}{4m^2} \left[ \pi^2 - \left( \frac{S \cdot \pi}{s} \right)^2 - \frac{1}{s} S \cdot H + i \left( \frac{S \cdot \pi}{s} \right) \left( S \cdot E \right) \right] \right\},$$

$$V_3^{\prime \prime} = \exp \left\{ \left[ s \cdot \pi^2 - \frac{1}{s} S \cdot H + i \left( \frac{S \cdot \pi}{s} \right) \left( S \cdot E \right) \right] \right\}, \quad \mu = I, II,$$

we obtain (ignoring the terms of order of $1/m^2$) the following

$$(H^{\prime \prime})^{\prime \prime} = -e_o \frac{\pi^2}{2m} + eA_0 + eB_0 S \cdot H + eD^2 \left[ -\frac{1}{2} S \cdot (\pi \times E - E \times \pi) + \frac{1}{6} Q_{ab} \frac{\partial E_a}{\partial x_b} + \frac{1}{3} (s + 1) \text{div} E \right] + \frac{eC_0}{m^2} \left[ S \cdot (\pi \times H - H \times \pi) - \frac{2}{3} Q_{ab} \frac{\partial H_a}{\partial x_b} \right] + o \left( \frac{1}{m^3} \right). \quad (10.26)$$

Here $Q_{ab}$ is a quadruple interaction tensor defined by the following formula

\footnote{Detailed calculations related to the transformation (10.25) are presented in [147, 315, 331].}
Symmetries of Equations of Quantum Mechanics

\[ Q_{ab} = 3[S_a, S_b] - 2\delta_{ab}s(s + 1). \]  
(10.27a)

\[ B = -\mu^\alpha/2, \quad D_2 = -(\mu^\alpha)^2/4, \quad C = -\delta_{ab}(2s - 1)/8s^2, \]  
(10.27b)

\[ \mu' = \mu^\alpha = 2, \quad \mu_{II} = 1/2, \quad \varepsilon_0 = m. \]

Formula (10.26) generalizes the Pauli Hamiltonian to the case of arbitrary spin. Actually this operator commutes with \( \sigma_i \), so we can consider solutions \( \Phi \) satisfying the relations \( \sigma_1 \Phi = \Phi \). The Hamiltonian (10.26) includes the terms corresponding to a dipole (\( \sim S H \)), quadruple (\( \sim Q_{ab}\partial E_a/\partial x_b \)), spin-orbit (\( \sim S (E \times p - p \times E) \)), and Darwin (\( \sim \text{div} E \)) interaction.

For the case \( \alpha = \text{III} \), there exist two additional terms (equal to zero if \( s = 1/2 \)). They have no clear physical interpretation and correspond to a quadruple magnetic and spin-orbit interactions in case of a magnetic monopole field.

Thus, starting with the equations without superfluous components, we have obtained the quasirelativistic Hamiltonians (10.26). In the approximation \( 1/m^2 \) the Hamiltonians \( (H_{\text{II}})'' \) and \( (H_{\text{III}})'' \) coincide. In the case \( s = 1/2 \) all the three Hamiltonians reduce to the Foldy-Wouthuysen Hamiltonian [108].

We see the approximate Hamiltonians of an arbitrary spin particle include the terms corresponding to the Foldy-Wouthuysen Hamiltonian and additional terms corresponding to the quadruple coupling. We note that the dipole momentum (i.e., the coefficient of the term \( \varepsilon S H \)), predicted by the Dirac-like equations, is equal to \( 1/s \) and hence is in accordance with Belifante’s conjecture [30].

Consider now equations for a particle with a variable spin which were deduced in Subsection 10.5. By analogy with the above, the Hamiltonian \( H' \) (10.18) reduces to the approximate form

\[ H'' = \sigma \left( \frac{m}{2m} - \frac{e}{2em} S \cdot H \right) - \frac{e}{16m^2s^2} S \cdot (E \times \pi - \pi \times E) - \]  
(10.28)

\[ \frac{e}{24m^2s^2} \left[ \frac{1}{2} Q_{ab} \frac{\partial E_a}{\partial x_b} - N^2 \text{div} E \right] + \frac{i(e(2s - 1))}{8m^2s^2} N \cdot (\pi \times H - H \times \pi) + \frac{e}{24m^2s^2} Q_{ab} \frac{\partial H_a}{\partial x_b}, \]

where

\[ Q_{ab}'' = -3[N_a, N_b] + 2\delta_{ab}N^2, \quad N_a = S_a^{(1)}, \]

\[ Q_{ab}'' = \frac{3}{2} \{ ([N_a, S_b], + [N_b, S_a]) - 2iS \cdot N \delta_{ab} \}, \quad S_a = \frac{1}{2} e_{abc} S_b^{(1)}, \]

\( S_{\alpha\beta} \) being matrices realizing the direct sum of two IRs \( D(s/2, s/2) \) of the algebra \( AO(1,3) \).

All the terms in the r.h.s. of (10.28) are \( P^- \), \( T^- \), and \( C^- \)-invariant. Neglecting
the terms of order $1/m^2$, we obtain from (10.28) the direct sum of the Pauli Hamiltonians for particles of spins $s$, $s-1$, $s-2$, ... , 0. This fact confirms our interpretation of (10.19) as an equation for a particle with a variable spin.

Consider the quasirelativistic approximation of the KDP equation. Applying to the Hamiltonians (10.5), (10.6) the procedure similar to (10.25), we obtain

$$H' = \sigma \left( m + \frac{\pi^2}{2m} - \frac{e}{2m} S \cdot H \right) + eA_\mu - \frac{\sigma_i}{2m^3} \left( \frac{\pi^2}{2} + (S \cdot \pi)^i - \frac{1}{2} \left[ \pi^i, (S \cdot \pi) \right] - \frac{e}{4} [S \cdot H, \pi^i] + e^2 [S \cdot H, (S \cdot \pi)^i] - e^2 (S \cdot H)^i \right) \right),$$

(10.29)

where $S$ is either the direct sum of two spin matrices (for $s=1$), or $S=0$ (for $s=0$).

We see that the approximate Hamiltonians obtained from the KDP equations do not include terms of order $1/m^2$. This means that the KDP equations do not describe the spin-orbit and Darwin couplings in the frames of the minimal interaction principle.

10.8. Causality Principle and Wave Equations for Particles with Arbitrary Spin

Let us demonstrate that the covariant wave equations considered here do not lead to paradoxes with the causality violation.

The causal character of the Dirac-like equations can be established by transferring to the equivalent system of second-order equations. Actually, multiplying any of the equations (10.10) by $\lambda_+ = (1+\Gamma_4)/2$ and $\lambda_- = (1-\Gamma_4)/2$, and expressing $\psi_- = \lambda_- \psi$ via $\psi_+ = \lambda_+ \psi$ we obtain

$$\left( \frac{\pi^\mu}{2} \pi^\nu \right) = \frac{e}{2} S^\mu_\nu \epsilon^{\mu \nu} \psi_0, \quad \psi_0 = 0,$$

(10.30a)

$$[S^\mu_\nu, S^\rho_\sigma - 4s(s+1)] \psi_- = 0,$$

(10.30b)

$$\psi_+ \psi_+ = \frac{1}{m} \Gamma^\mu \pi^\mu \psi_+, \quad \psi_0 = 0,$$

(10.30c)

with the following identity being used:

$$(\Gamma^\mu \pi^\nu) = \pi^\mu \pi^\nu + i e \Gamma^\mu \Gamma^\nu.$$
equal to zero, and 2s+1 remaining components of $\psi_s$ form a spinor within the space of the representation $D(s 0)$ of the algebra $AO(1,3)$. We denote nonzero components of $\psi_s$ by $\Phi_s$ and rewrite the equation (10.30a) in the form
\[
\left[ e_{\mu}^\nu \pi^\nu - m^2 s \frac{e}{s} S \cdot (H - iE) \right] \psi_s = 0,
\] (10.31)
where $S$ are matrices of dimension $(2s+1) \times (2s+1)$ realizing the IRs $D(s)$ of the algebra $AO(3)$.

To prove the system (10.31) is causal it is sufficient to replace operators of differentiations in respect with $x_\mu$ by the component of the characteristic four-vector $n_\mu$ and then equate to zero the determinant of the obtained system of algebraic equations for $n_\mu$, taking into account the highest order terms only. The analysed system of partial differential equations is causal if the corresponding characteristic equation for $n_\mu$ has light-like solutions only [412].

The characteristic equation for the system (10.31) has the form
\[
(n^\mu n_\mu)^{s+1} = 0,
\]
thus all the characteristic vectors are light-like. It means this system is causal. Since any solution of the equation (10.10) can be expressed via solutions of the system (10.31) in accordance with (10.30), we conclude that the system (10.10) is causal.

In the case $s=1/2$ formula (10.31) defines the Zaitsev-Feynman-Gell-Mann equation for an electron [98, 420], and for an arbitrary $s$ it represents the simplest generalization of this equation. So, formulae (10.10) define a system of first-order equations corresponding to the generalized Zaitsev-Feynman-Gell-Mann equation.

In a similar way we can prove the causal character of the equations (10.19).

10.9. The Causal Equation for Spin-One Particles with Positive Energies

In conclusion, we consider in more detail the most popular equation for spin-1 particles, i.e., the KDP equation.

It is well known that the KDP equation also is not completely satisfactory if we use it to describe interacting particles. In the case of minimal coupling this equation has not good solutions if the external field reduces to the Coulomb potential, as it was pointed by Tamm [401] and Corben and Schwinger [40*]. The reason is that this equation does not take into account the spin-orbit coupling, see Subsection 10.7. On the other hand the KDP equation with anomalous interaction predicts complex energies for a particle interacting with the constant and homogeneous magnetic field [403]. Moreover taking into account an anomalous interaction we come in general to a non-causal equation [28*].

Here we demonstrate that it is possible to introduce such the anomalous
interaction into the KDP equation that the causality principle will be satisfied and complex energies will not arise while solving the problem of interaction of spin-one particle with the constant magnetic field. To achieve this goal it is sufficient to take into account interactions which are bilinear with respect to external field strengths.

We will start with the following equation

\[
\left[ \beta^\mu \pi^\nu - m + \frac{1}{2m} (1 - \beta_0^2) \left( \frac{k_1 e}{2} S^{\mu \sigma} F_{\mu \nu} - \frac{k_2 e^2}{4m^2} F^{\mu \sigma} F_{\mu \nu} \right) \right] \psi = 0
\]

where \( S^{\mu \sigma} = i [\beta^\mu, \beta^\sigma] \), \( \beta^\mu \) are the 10×10 KDP matrices, \( e, k_1 \) and \( k_2 \) are coupling constants which we choose be equal to 1 for simplicity.

The equation (10.32) is Poincaré- and \( P_-, T-, C- \) invariant and includes the terms describing the anomalous interaction of a particle with an external field. The term proportional to \( F_{\mu \sigma} \) represents the general form of anomalous interaction which does not violate the causality principle \([40]^*\). The term proportional to \( F_{\mu \sigma} F^{\mu \sigma} \) is introduced in order to overcome difficulties with complex energies\(^*\).

Let us demonstrate that the equation (10.32) is causal. To do it we will delete superfluous components of the wave function \( \psi \) and analyze the corresponding equation including the physical components of \( \psi \) only.

We find it is convenient to use the concrete realization (6.22), (6.24) of the \( \beta \)-matrices. Expressing nonphysical components \( (1 - \beta_0^2) \psi \) via physical components \( \Psi = \beta_0^2 \psi \) we come to the following generalized TST equation

\[
\pi^0 \Psi - \hat{H} \Psi = 0
\]

where

\[
\hat{H} = \frac{1}{2} \left( i \sigma_3 + \sigma_2 \right) \left[ M + \frac{S \cdot \pi}{m} + \frac{(E, E)}{M^2} + \frac{i}{2 M} (1 + \sigma_3) (E, \pi) + \right. \\
\left. + \frac{i}{2 M} (1 - \sigma_3) (\pi, 1) E + \frac{1}{2} (\sigma_2 - \sigma_1) \left[ m + (\pi, \frac{1}{M}) \right] \right]
\]

\( M = m + F^{\mu \sigma} F_{\mu \sigma} / 8 \), \( S \) and \( \sigma_i \) are the spin-one and Pauli matrices (7.6) and (3.6), the symbol \( (A, B) \) denotes the 6×6 matrix:

---

\(^*\)This term is absent in paper \([40]^*\) where nonlinear in \( F^{\mu \sigma} \) interactions were analyzed.
Symmetries of Equations of Quantum Mechanics

\[
\begin{pmatrix}
A_1B_1 & A_2B_2 & A_3B_3 \\
A_1B_2 & A_2B_1 & A_3B_3 \\
A_1B_3 & A_2B_3 & A_3B_2
\end{pmatrix}
\]

\[(A,B) = \begin{pmatrix}
A_1B_1 & A_2B_2 & A_3B_3 \\
A_1B_2 & A_2B_1 & A_3B_3 \\
A_1B_3 & A_2B_3 & A_3B_2
\end{pmatrix}
\]

Multiplying the l.h.s. and r.h.s. of (10.33) by \(\pi\) and using the identities

\[
\begin{pmatrix}
A_1B_1 & A_2B_2 & A_3B_3 \\
A_1B_2 & A_2B_1 & A_3B_3 \\
A_1B_3 & A_2B_3 & A_3B_2
\end{pmatrix}
\]

\[
(S \pi)^2 \left\{ \frac{1}{M} \right\} = -S \cdot H \left\{ \frac{1}{M} \pi \right\} + \ldots + \frac{i}{M} (\pi \times H) + \ldots,
\]

\[
\left( \pi, \frac{1}{M} \right) = -\frac{1}{M} \pi^2 + (S \pi)^2 + \ldots,
\]

\[
(E,E) \left( \pi, \frac{1}{M} \right) = (E,\pi) \frac{1}{M} (E,\pi) + \ldots, \quad (E,\pi)(S \pi)^2 = \ldots,
\]

\[
(\pi, \frac{1}{M} \pi)(S \pi)^2 = \left( \pi, \frac{1}{M} \pi \right) S \cdot H + \ldots - \frac{i}{M} (\pi,\pi \times H) + \ldots,
\]

\[
\frac{1}{M} (E,E) = \left( \pi, \frac{1}{M} \pi \right) \frac{1}{M} (E,\pi) + \ldots,
\]

\[
\frac{1}{M} (E,\pi) = \left( \pi, \frac{1}{M} \pi \right) \frac{1}{M} \pi + \ldots,
\]

(where the dots denote the terms which does include second-order differential operators) we come to the second-order equation

\[
\left[ \pi^0, \pi^\mu + \frac{1}{2} (\sigma_1, + i \sigma_2) \left\{ \pi, \frac{M}{M^2} \pi \right\} + \ldots \right] \Psi = 0
\]

which corresponds to the following characteristic equation

\[
(n^\mu n_\mu)^2 = 0.
\]

In accordance with (10.37) the characteristic four-vector \(n_\mu\) for the equation (10.32) is light-like, and so this equation is causal.

In the case of the constant and homogeneous magnetic field directed along the third coordinate axis, (10.32) reduces to the following exact equation

\[
\pi^0 \Psi = \left\{ m^2 + \pi^2 - 2 SH \frac{1}{4} \frac{H^2}{m^2} \right\} \Psi.
\]

Using relations (30.7) it is not difficult to make sure that all the eigenvalues of the operator \(p_0\) are positive defined, and so all the corresponding energy values are real.

Thus the equation (10.32) with \(k_1 = k_2 = 1\) is causal and does not lead to complex energies in the case of constant and homogeneous magnetic field. The same is true for the case of an arbitrary \(k_1\) if \(k_2\) is sufficiently large.
3. REPRESENTATIONS OF THE GALILEI ALGEBRA AND GALILEI-INARIANT WAVE EQUATIONS

One of the main requirements imposed on equations of nonrelativistic physics is the invariance under the Galilei transformations. This circumstance predetermines a fundamental role performed by representations of the Galilei group.

The present chapter is devoted to description of representations of the Lie algebra of the Galilei group and of Galilei-invariant equations for particles of arbitrary spins. We will see that the concept of spin arises naturally in frames of nonrelativistic quantum mechanics and most spin related effects (i.e., dipole, spin-orbital, and other interactions) can be successfully described by equations satisfying the Galilei relativity principle.

Section 11 is devoted to studying symmetries of the basic equation of nonrelativistic quantum mechanics, i.e., the Schrödinger equation.

11. SYMMETRIES OF THE SCHRODINGER EQUATION

11.1. The Schrödinger Equation

In quantum mechanics, the states of a system with \( n \) degrees of freedom, as described by a set of coordinates \( x=(x_1, x_2, ..., x_n) \), are determined by the wave function \( \psi(t,x) \) being a vector in a Hilbert space. Moreover the evolution of a system is described by the Schrödinger equation

\[
\frac{\partial}{\partial t} \psi(t,x) = \mathcal{H} \psi(t,x)
\]

(11.1)

where \( \mathcal{H} \) is the Hamiltonian operator or Hamiltonian of a system.

The simplest quantum mechanical system is a free spinless particle. The wave function of such a particle depends on three spatial variables \( x=(x_1, x_2, x_3) \), and the corresponding Hamiltonian has the form

\[
\mathcal{H} = \frac{\mathcal{P}^2}{2m}
\]

(11.2)

where \( m \) is a parameter determining the mass of a particle, \( \mathcal{P}^2=p_1^2+p_2^2+p_3^2 \), \( p_a=-i\hbar\frac{\partial}{\partial x_a}, a=1,2,3 \). Substituting (11.2) into (11.1) we receive the Schrödinger equation for a free particle.
Symmetries of Equations of Quantum Mechanics

\[ L \psi(t,x) = \left( \frac{i}{\hbar} \frac{\partial}{\partial t} - \frac{p^2}{2m} \right) \psi(t,x) = 0, \]  
(11.3)

which is going to be the object of our study.

One of the main postulates of quantum mechanics is the Galilei relativity principle which can be formulated as a requirement of invariance of the Schrödinger equation under the Galilei transformations. Of course, this requirement is satisfied by the simplest evolution equation (11.3). Moreover, as it was stated relatively recently, the equation (11.3) possesses a more extensive symmetry, as being invariant under scale and conformal transformations.

Below we will study the symmetries of the Schrödinger equation in detail. It will allow to explain, using a relatively simple example, the meaning of the Galilei invariance concept for quantum mechanics equations. Moreover, we will demonstrate that the invariance group of the equations (11.3) determines the maximal symmetry of this equation in some sense.

The problem of investigation of symmetries of Schrödinger equation can be formulated in complete analogy with the corresponding problem for the KGF equation (refer to Section 1). As before, it is sufficient to restrict ourselves to considering only such solutions of the equations (11.3) which are defined on some open set \( D \) of the four-dimensional manifold \( \mathbb{R}^4 \) and belong to the vector space \( \mathcal{F} \) of complex-valued functions being analytical on \( D \). Then, the set \( \mathcal{F}_0 \) of solutions of (11.3) can be defined as a zero-space of the differential operator

\[ L \psi = \psi = 0, \]  
(11.4)

defined on \( \mathcal{F} \): \( \psi \in \mathcal{F}_0 \) if \( \psi \in \mathcal{F} \) and \( L \psi = 0 \). The definitions of a SO and IA for the Schrödinger equation coincide with Definitions 1.1 and 1.2, p.p. 2, 3, if the symbol \( L \) denotes the differential operator (11.4).

11.2. Invariance Algebra of the Schrödinger equation

The starting point of our studies of Schrödinger equation symmetries lies in determining of the IA for this equation in the class \( M_1 \), i.e., the class of first-order differential operators.

**THEOREM 11.1.** The maximal IA of the Schrödinger equation in class \( M_1 \) is the 13-dimension Lie algebra with the following basis elements

\[ P_0 = p_0 = \hbar \frac{\partial}{\partial t}, \quad P_a = p_a = -i \frac{\partial}{\partial x_a}, \quad M = \text{Im}, \]  
(11.5a)

132
Chapter 3. Representations of the Galilei Algebra...

\[ J_a = e_{abc} x_b p_c, \quad G_a = t p_a - m x_a, \]  
\[ D = 2 p_0 x^a p_a + \frac{3}{2} t, \quad A = t p_0 - t D + \frac{1}{2} m x^a x_a, \]  
where \( x^a = -x_a I \) is the identity operator, and the summation from 1 to 3 is imposed over the repeating indices.

**PROOF.** It is not difficult to verify that the operators (11.5) form an IA of the Schrödinger equation. We can make sure that any of the operators (11.5) satisfies the invariance condition (1.5) where \( L \) is the operator (11.4), \( \alpha_a = \alpha_a, \alpha_a = \alpha_a, \alpha_a = \alpha_a, \alpha_a = \alpha_a, \alpha_a = \alpha_a, \alpha_a = \alpha_a. \)

It is also easy to verify that the operators (11.5) form a basis of the Lie algebra, satisfying the following commutation relations:

\[ [P_a, P_b] = [P_a, P_0] = [M, P_a] = [M, P_0] = [M, G_0] = [P_0, J_a] = 0, \]  
\[ [P_a, J_b] = i e_{abc} P_c, \quad [P_0, G_0] = i P_a, \]  
\[ [P_a, G_b] = i \delta_{ab} M, \quad [J_a, J_b] = i e_{abc} J_c, \]  
\[ [G_a, J_b] = i e_{abc} G_c, \quad [G_0, G_0] = 0, \]  
\[ [D, J_a] = [D, M] = [A, G_0] = [A, M] = [A, J_a] = 0, \]  
\[ [D, P_a] = -i P_a, \quad [D, G_a] = i G_0, \quad [D, P_0] = -2i P_0, \]  
\[ [A, P_a] = i G_0, \quad [A, P_0] = i D, \quad [A, D] = -2i A, \]  
\[ (11.6a) \]
\[ (11.6b) \]
\[ (11.6c) \]
\[ (11.6d) \]
\[ (11.7) \]

In analogy with the Theorem 1.2 proof it is possible to demonstrate that relations (11.5) define a basis of the maximal IA of the Schrödinger equation in the class \( M_1 \). We do not provide a detailed proof but it should be noted that by substituting (1.4) and (11.4) into the invariance condition (1.5), we come to the following system of determining equations (compare with (1.9)):

\[ A_a = A_a, \quad A_a + A_a = 0, \quad b \neq a, \]  
\[ A_0 = 0, \quad A_0 = 2 A_a, \quad \alpha = 2i A_a, \]  
\[ (11.8) \]
\[ i \Delta + \frac{1}{2m} \Delta A_a + i B_0 = 0, \quad i B = \frac{1}{2m} \Delta B = 0, \]  
where the dots denote derivatives in respect with the time variable \( t \), \( \Delta = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \partial^2 / \partial x_3^2 \), \( A_a = \partial A / \partial x_a, B_0 = \partial B / \partial x_0 \), and there are no summing over repeated indices.

The system (11.8) is easily integrated (compare with (1.8)) and leads to the
following expressions for $A^\nu$ and $B$:

\begin{align}
A^0 &= \hbar^2 t^2 + 2 j t + c, \quad \alpha = 2 i (\hbar t + j), \\
A^a &= c^{-a b} \chi_b x^b, \quad h x^a + j x^b + n x^c + m = 0, \\
B &= m \left( \frac{\hbar}{2} x^a - n x^a + g \right) + \frac{3}{2} (\hbar t + j),
\end{align}

(11.9)

where $c^{a b} = -c^{b a}$, $h$, $j$, $c$, $g$, $n^a$, and $m^a$ are arbitrary constants. Substituting (11.9) into (11.4) we come to a linear combination of the SO (11.5).

11.3. The Galilei and Generalized Galilei Algebras

Thus, we have found the IA of the Schrödinger equation in the class $M_1$ which includes 13 linearly independent SO of (10.5). These operators generate a Lie algebra which we call the generalized Galilei algebra and denote as $AG_2(1,3)$. An abstract definition of this algebra is given by relations (11.6) and (11.7).

To understand the structure of the algebra $AG_2(1,3)$, we represent it as a chain of maximal ideals (let us recall that an ideal of a Lie algebra $G$ is a subalgebra $A$ such that $[a, b] \in A$ for any $b \in G$, $a \in A$, and an ideal $A$ is called maximal if there do not exist ideals $A' \subsetneq A$). Starting with (11.6), (11.7), it is not difficult to make sure that the maximal ideal $A^1$ of the algebra $AG(1,3)$ includes 11 elements, i.e., $P_0$, $P_0^a$, $J_0$, $G_0$, and $M$. This fact can be written in the following symbolic form:

$$AG_2(1,3) = A^1 \supseteq \mathcal{R}, \quad A^1 \supseteq P_0, P_0^a, G_0, M, \quad \mathcal{R} \supseteq D, A.$$  

(11.10)

The subalgebra $A^1$ contains, in its turn, the maximal ideal $A^2 \supseteq \{M, P_0, P_0^a, G_0\}$, and the subalgebra $A^2$ does the maximal ideal $T_0 \supseteq \{M, P_0, P_0^a\}$. This may be reflected by the following symbolic relation:

$$AG_2(1,3) = \{(T_0 \supseteq T_0) \supseteq AO(3)\} \supseteq \mathcal{R}$$  

(11.11)

where $T_0 \supseteq \{G_0\}$, $AO(3) \supseteq \{J_0\}$, and $\mathcal{R} \supseteq \{D, A\}$.

Formula (11.11) describes a structure of the generalized Galilei algebra by providing its main subalgebras. The subalgebras $T_0$ and $T_0$ are commutative (Abelian) algebras of dimensions 5 and 3, $AO(3)$ is the three-dimensional algebra which is isomorphic to the Lie algebra of the rotation group. Let us point also one more important subalgebra with the basis elements

$$S_{01} = \frac{1}{2} (A + P_0), \quad S_{02} = \frac{1}{2} (A - P_0), \quad S_{12} = \frac{1}{2} D.$$  

(11.12)

The operators (11.12) satisfy the commutation relations (4.27) characterising the
algebra $AO(1,2)$.

The most interesting (from the physical point of view) subalgebra of the algebra $AG_2(1,3)$ is its maximal ideal which will be called the Galilei algebra and denoted by $AG(1,3)$. The meaning of the name will be explained in the following subsection.

The algebra $AG(1,3)$ is a linear span of the basis elements (10.5) satisfying relations (11.6). It will be shown further on that it is the invariance under the algebra $AG(1,3)$ that serves as a mathematical expression of the Galilei relativity principle. As for the operators $D$ and $A$, which complete the Galilei algebra to the algebra $AG_2(1,3)$, they reduced to the combinations of $P_{\alpha\beta}G_{\alpha\beta}$ and $M$ on a set of the equation (11.3) solutions. Indeed, it is not difficult to make sure that

$$D\psi=(2M)^{-1}(P_{\alpha\beta}G_{\alpha\beta})\psi, \quad A\psi=(2M)^{-1}(G_{\alpha\beta}\psi).$$  \hfill (11.13)

In other words the symmetry of the Schrödinger equation under the operators $D$ and $A$ is nothing but a direct consequence of the invariance under the Galilei algebra.

The Galilei algebra has three main Casimir operators

$$C_1=M, \quad C_2=2MP_{\alpha\beta}P_{\alpha\beta}, \quad C_3=(M_{\alpha\beta\gamma}e_{\alpha\beta\gamma}P_{\alpha\beta}G_{\alpha\beta}), \quad (11.14)$$

In quantum mechanics, the eigenvalues of the operators (11.14) are associated with a mass, internal energy and spin of a particle. Substituting (11.5) into (11.14), it is not difficult to make sure that eigenvalues of $C_1$, $C_2$ and $C_3$ on a set of Schrödinger equation solutions are $c_1=m, c_2=c_3=0$. So, we can conclude that the Schrödinger equation describes a particle of mass $m$ and of zero spin and internal energy.

In conclusion we note that the connection between the basis elements of the Galilei and generalized Galilei algebras given by relations (11.13) takes place for arbitrary representations of these algebras. More precisely, the following assertion is true [333]:

**LEMMA 11.1.** Let $\{P_\alpha P_{\alpha\beta}G_{\alpha\beta}, M\}$ be a set of operators satisfying the Galilei algebra (11.6), and, besides, $M$ is an invertible operator. Then the operators $\{D, A, P_\alpha P_{\alpha\beta}G_{\alpha\beta}, P_{\alpha\beta}G_{\alpha\beta}, M\}$, where $P_\alpha=(2M)^{-1}C_2$ and $D, A$ are the operators (11.13), form a representation of the generalized Galilei algebra, i.e., they satisfy conditions of (11.6) and (11.7).

Proof can be carried out by direct verification. □

According to Lemma 11.1 any representation of the Galilei algebra (corresponding to $c_1\neq 0$) can be extended to a representation of the generalized Galilei algebra (in the same way as any representation of the Poincaré algebra of Class II can be extended to a representation of the conformal algebra, refer to Section 3). It means that any equation being invariant under the Galilei algebra and describing a
particle of a non-zero mass is also invariant under the generalized Galilei algebra (but the corresponding operators $D$ and $A$ do not, in general, belong to the class $M_1$).

11.4. The Schrödinger Equation Group

The symmetry of the Schrödinger equation under the algebra $AG_2(1,3)$ is a fundamental fact which can be used as a base for Galilean kinematics. In this subsection we consider one of the main consequences of such invariance and clarify its physical meaning.

In physical terms, this consequence can be formulated as follows: the Schrödinger equation satisfies the Galilei relativity principle.

As in Section 1, we use the fact that an IA in the class $M_1$ generates a local representation of a Lie group. To find the explicit form of the corresponding one-parametrical subgroups, we use the Lie equations (1.21) and (1.22) which are easily integrable for the SO of (11.5). Comparing (11.4) with (11.5), we conclude that $B \equiv 0$ for the operators $P_\mu$ and $J_a$ and, therefore, solutions of the corresponding equations (1.22) will have the following form:

$$\psi'(x') = \psi(x), \quad x = (x_0, x_1, x_2, x_3), \quad x_0 = t.$$  

The corresponding equations of (1.22) will have the following solutions:

$$x'_\mu = x_\mu + b_\mu$$  

for $Q = P_\mu$ and

$$x'_0 = x_0, \quad x'_\mu = x_\mu, \quad x'_a = x_a \cos \theta_a + x_b \sin \theta_a,$$

$$x'_b = x_b \cos \theta_a - x_a \sin \theta_a$$  

for $Q = J_a$. Here $b_\mu, \theta_a$ are real parameters, and $(a,b,c)$ is the cycle $(1,2,3)$.

Integrating the equations (1.21), (1.22) we obtain the corresponding transformations generated by the remaining basis elements of the generalized Galilei algebra:

$$\psi'(x') = \exp \left( i m v_\mu x_\mu + i \frac{m v_\mu v_a}{2} \right) \psi(x),$$

$$x'_0 = x_0, \quad x'_\mu = x_\mu + v_\mu t, \quad x'_a = x_a + v_a t, \quad b \neq a,$$

if $Q = G_a$;

$$\psi'(x') = \exp(i m c) \psi(x), \quad x' = x,$$

if $Q = M$;
Chapter 3. Representations of the Galilei Algebra ...

\[ \psi'(x') = \exp \left( -\frac{3}{2} \frac{\lambda}{2} \right) \psi(x), \]  
\[ x'_a = \exp(2\lambda)x'_a, \quad x'_a = \exp(\lambda)x'_a, \]  
if \( Q = D \): 
\[ \psi'(x') = (1 + \xi x'_0)^{3/2} \exp \left[ \frac{-imx'_a \xi}{2(1 - \xi x'_0)} \right] \psi(x) , \]  
\[ x'_0 = \frac{x_0}{1 + \xi x'_0}, \quad x'_a = \frac{x'_0}{1 + \xi x'_0}, \]  
if \( Q = A \). 

Here \( G_a, M, A, D \) are the operators from (11.5), and \( v_a, c, \lambda, \xi \) are real parameters of the corresponding transformations.

Formulae (11.16)-(11.21) define a family of one-parameter transformations forming the 13-parameter Lie group called the Schrödinger equation group. The transformations (11.16) are translations of time and spatial variables, formulae (11.17) define a rotation of a reference frame, and relations (11.18) can be interpreted as a transition to a new frame of reference moving with the velocity \( v_a \) along the axis \( a \). Finally, the transformations (11.20) and (11.21) describe the symmetry of the Schrödinger equation in respect to the scaling and specific nonlinear change of variables \( x'_a \to x_a/(1 + \xi x'_0) \).

Using (11.6), (11.7), (11.16)-(11.21) it is simple to obtain the general transformation from the Schrödinger equation group in the following form:
\[ \psi(x) \to \psi'(x') = \exp[i f(x)] \psi(x) , \]  
where
\[ f(x) = \frac{3}{2} \ln(1 + \xi x'_0) - \frac{imx'_a x'_0 \xi}{2(1 + \xi x'_0)} - \frac{mvx'_0 + \frac{1}{2} mvx'_0 + \frac{3}{2} \lambda + mc}{2}, \]  
\[ x'_a = \frac{\exp(\lambda) R_{ab} x_b + v_a x'_0 + b_a}{1 + \exp(2\lambda) \xi x'_0}, \quad x'_0 = \frac{x_0}{\xi x'_0 + \exp(2\lambda) + b'_0}, \]  
\( R_{ab} \) is the operator of spatial rotations,
\[ R_{ab} = \frac{\delta_{ab}}{\cos \theta} + \frac{\xi_{abc}}{\theta} \frac{\theta_a}{\theta} (1 - \cos \theta) + \frac{\theta_a}{\theta^2} (\theta^2 - \theta^2 \xi^2) \xi_{abc} + \frac{\theta_a}{\theta^2} (\theta^2 - \theta^2 \xi^2) (\theta^2 - \theta^2 \xi^2) \xi_{abc}. \]  
(11.24)

with \( \theta_a, v_a, b_a, \xi, \lambda, \) and \( c \) being arbitrary real numbers.

It is possible to ascertain by direct verification that the Schrödinger equation (11.3) is invariant under the transformation (11.22) since \( \psi'(x') \) satisfies the equation
Symmetries of Equations of Quantum Mechanics

We see that the transformed function \( \psi'(x') \) of (11.22) differs from the initial function \( \psi(x) \) by the phase multiplier \( \exp[i\phi(x)] \). However, it may be shown that the transformations (11.22)-(11.24) do not change the norm of a wave function

\[
|\psi|^2 = \int d^3x \psi^* \psi = \int d^3x' \psi'^* \psi'.
\] (11.26)

Let us note that the use of formulae (11.22)-(11.24) needs an appropriate caution since it may occur that \( x'_0 \) and \( x'_a \) do not belong to \( D \), though \( x_0, x_a \in D \). For fixed \( x_0, x_a \in D \), \( x'_0 \) and \( x'_a \) belong to the domain of a function \( \psi \in F_0 \) if the transformation (11.23) belongs to an infinitesimal neighbourhood of the identity transformation. Because of this (and bearing also in mind that the expressions (11.23) become meaningless if \( \xi = -x_0^{-1}\exp(-2\lambda) \)), we say that the relations (11.22)-(11.24) define only a local representation of the Schrödinger equation group.

11.5. The Galilei Group

Consider in greater detail the main subgroup of the Schrödinger equation group which corresponds to \( \xi = \lambda = 0 \). The corresponding transformations (11.22)-(11.24) take the form

\[
\psi(x) \rightarrow \psi'(x') = \exp[i\phi(x)]\psi(x),
\] (11.27)

where

\[
\phi(x) = mv_a x_a + \frac{1}{2}mv_a v_a x_0 + mc,
\] (11.28)

\[
x' = R_{ab} x_b + v_a x_0 + b_a, \quad x'_0 = x_0 + b_0
\] (11.29)

The transformations of space and time variables given in (11.29) are called the Galilei transformations.

Let us demonstrate that the transformations (11.27)-(11.29) form a group. Let \( (R, v, b, c) \) be a transformation (11.27)-(11.29) with \( R = R(\theta) \) denoting the matrix of spatial rotation (11.24). Then, the group composition law is defined as follows

\[
\begin{align*}
&\left[R^{(2)} v^{(2)}, b^{(2)}, b_0^{(2)}, c^{(2)} \right] \left[R^{(1)} v^{(1)}, b^{(1)}, b_0^{(1)}, c^{(1)} \right] = \\
&\left[R^{(2)} R^{(1)} v^{(1)}, c^{(1)}, -R^{(1)} v^{(2)} + R^{(1)} b^{(2)} - v^{(1)} b_0^{(2)},
\right. \\
&b_0^{(1)} - b_0^{(2)}, c^{(2)} + v_a^{(1)} R_{ab} b_0^{(2)} + \frac{1}{2} b_0^{(2)} (v^{(1)})^2
\end{align*}
\] (11.30)

The unit element of the group is represented by the identity transformation.
and the element inverse to $(R,v,b,b_0,c)$ has the form

$$\left(R^{-1},-R^{-1}v,-R^{-1}(b-vb_0),-b_0,-c+b v-\frac{1}{2}b_0v^2\right)$$

(11.32)

This group of transformations (11.27)-(11.29) is called the extended Galilei group.

We see that the extended Galilei group is an 11-parameter Lie group while formulae (11.29) representing the transformations of the space-time continuum include 10 parameters only. The set of the transformations (11.29) also forms the so-called Galilei group. The group law for the transformations (11.29) has the form

$$g(R^{(2)},v^{(2)},b^{(2)},b_0^{(2)},c^{(2)})g(R^{(1)},v^{(1)},b^{(1)},b_0^{(1)},c^{(1)}) =$$

$$g(R^{(2)}R^{(1)},v^{(3)}+R^{(1)}v^{(2)},b^{(3)}+R^{(1)}b^{(2)}+v^{(1)}b^{(2)}_0+b^{(2)}_0+b_0^{(2)}).$$

(11.33)

The transformations (11.27) comprise a representation of the Galilei group if we set $c=0$ in (11.28). It is not difficult to make sure that such a representation is not exact (but is only a projective one) because of

$$\exp(i\omega_{12})$$

$$\omega_{12}=\omega^{a}_{b',b}R^{(1)}a_{b'}b^{(2)}_1b^{(2)}_0(v^{(1)})^2,.$$  

(11.35)

We see that the transformations (11.27) with $c=0$ satisfy the group composition law (11.33) up to the phase multiplier $\exp(i\omega_{12})$ only, the latter not changing the norm (11.26).

Thus, we can consider either the exact representation of the extended Galilei group or the projective representation of the Galilei group. We note that the transformations (11.27)-(11.29) can be considered as a global representation of the Galilei group if the domain of $\psi$ coincides with the four-dimensional manifold $R_4$.

### 11.6. The Transformations $P$ and $T$

The symmetry group of the Schrödinger equation considered above describes invariance properties of this equation with respect to continuous transformations of dependent and independent variables. Now we briefly discuss the
Symmetries of Equations of Quantum Mechanics

We can make sure that these transformations do not belong to the class defined by relations (11.29) or (11.23). They are linear transformations whose matrices have negative determinants in contrast to the linear homogeneous part of the transformations (11.23). Nevertheless the Schrödinger equation is invariant under these transformations if a wave function is simultaneously transformed according to

\[ \psi(t, x) \rightarrow P\psi(t, x) = \eta_1 \psi(t, -x), \]
\[ \psi(t, x) \rightarrow T\psi(t, x) = \eta_2 \psi(-t, x) \]

where the asterisk denotes the complex conjugation, \( \eta_1 \) and \( \eta_2 \) are complex numbers such that \( \eta_1 = \eta_1^* = \eta_2 \eta_2^* = 1 \). Without loss of generality we may choose \( \eta_1 = \pm 1 \), \( \eta_2 = \pm 1 \).

The set of transformations \( P, T, PT \) and the identity transformation form a symmetry group of the Schrödinger equation. This discrete group can be added to the Schrödinger group to obtain the so-called complete Schrödinger group. Unitary ray representations of the complete Schrödinger group are considered in [59].

This ends the brief discussion of classical symmetries of the basic equation of quantum mechanics. The results represented in this section will be repeatedly used further on at the deducing of motion equations for arbitrary spin particles.

12. REPRESENTATIONS OF THE LIE ALGEBRA OF THE GALILEI GROUP

12.1. Galilei Relativity Principle and Equations of Quantum Mechanics

As was shown above, the Schrödinger equation is invariant under the coordinate transformations corresponding to a transition to a new inertial frame of reference.

The relativity principle establishing equal rights of all the inertial frames of reference was formulated by Galileo Galilei more than 350 years ago. Of course, Galilei did not show the explicit form of transformations of space and time coordinates (the concept of coordinates was introduced much later). Nevertheless, these transformations bear the name of Galilei with a good reason, and the set of such transformations is properly called the Galilei group.
It looks surprising that the structure of the Galilei group and its representations has begun to be studied relatively recently, and much later than the corresponding problems of the Poincaré group. In 1952 Inönü and Wigner [229] described exact representations of this group. Bargman [16] was the first to point to a fundamental role of ray representations of the group $G(1,3)$ in quantum mechanics.

After the fundamental works [16, 229], it was Levi-Leblond who made an essential contribution into understanding of problems of Galilean relativity in quantum mechanics. He deduced nonrelativistic analog of the Dirac equation which is invariant under the Galilei group [276]. Like the Dirac equation, the Levi-Leblond equation describes the Pauli interaction and predicts the correct value of the gyromagnetic ratio.

The Levi-Leblond equation and its possible generalizations are considered in detail in the subsequent sections. Here we only note that there exist Galilei-invariant wave equations which describe correctly such a fine effect as the spin-orbit coupling. This makes it possible to use these equations to solve practical physical problems, and sometimes they are more suitable than complicated relativistic wave equations.

To describe Galilei-invariant wave equations it is necessary to know representations of the Galilei group. The basic information about these representations is given in this and subsequent sections.

12.2. Classification of IRs

Let us consider representations of the Lie algebra of the Galilei group as defined by the commutation relations (11.6). In this section we consider completely reducible representations only, so it is sufficient to restrict ourselves to IRs. Following [151], we will find such a realization of all the nonequivalent IRs of this algebra that is distinguished by a common and simple form of the Galilei group generators for all the classes of IRs. Besides, the approach used is closely related with that presented in Chapter 2. This enables us to avoid superfluous details.

The basic Casimir operators of the algebra $AG(1,3)$ are given by (11.14). We will show further on that representations of this algebra differ qualitatively for zero and nonzero eigenvalues $c_1$, $c_2$ of the operators $C_1$, $C_2$. Namely, it is possible to select five classes of representations corresponding to the set of values $c_1$ and $c_2$ as shown in (4.10). Thus, the classification of IRs of the Galilei group is very similar to the corresponding classification for the Poincaré algebra (do not take the Casimir operators of the Galilei group for those of the Poincaré group though we use the same notation for operators and corresponding eigenvalues).

The first three classes I-III of IRs can be realized by Hermitian operators.
Representations of Class IV are non-Hermitian but they also find an appropriate usage in physics.

When \( P_{\mu} = 0 \) we come to the representations of the homogeneous Galilei group which is isomorphic to the group \( E(3) \). Finite dimensional and irreducible representations of such a type are considered in Subsection 13.6.

As a basis of an IR we choose a set of eigenfunctions \( |c, \tilde{p}, \lambda > \) of the commuting operators \( C_1, C_2, C_3 \), \( P_0, P_1, P_2, P_3 \), \( W_0 = \mathbf{J} \cdot \mathbf{P} \), and of additional Casimir operators \( C_4, C_5, \ldots \), which will be different for different classes of IRs. The relations (4.8) can serve as a formal definition of the vectors \( |c, \tilde{p}, \lambda > \) (remember that the symbols \( C_{\mu}, P_{\mu}, \) and \( W_0 \) denote now another operators belonging to the algebra \( AG(1,3) \)).

Description of representations of the algebra \( AG(1,3) \) in the basis \( |c, \tilde{p}, \lambda > \) reduces to finding of an explicit form of nonequivalent operators \( J_a \) and \( G_a \) satisfying the commutation relations (11.6) together with given operators \( P_{\mu} \) and \( M \). In the same way as was done in studying of IRs of the algebra \( AP(1,3) \) (refer to Section 4), it is convenient to pass from the vectors \( J \) and \( G \) to the following four-vectors

\[
\begin{align*}
W_0 &= P \cdot J, \quad W = M \mathbf{J} \cdot \mathbf{P} \times G, \\
\Gamma_0 &= P \cdot G, \quad \Gamma = MG - P \times J.
\end{align*}
\]

(12.1)

If the operator \( m^2 + P^2 \) is invertible then \( J \) and \( G \) can be expressed via the operators (12.1):

\[
\begin{align*}
J &= (M^2 + P^2)^{-1} (P \times \Gamma + MW + PW_0), \\
G &= (M^2 + P^2)^{-1} (P \times W + M \Gamma + P \Gamma),
\end{align*}
\]

(12.2)

and our problem reduces to finding all the nonequivalent \( \Gamma, \Gamma_0 \) and \( W, W_0 \).

### 12.3. The Explicit Form of Basis Elements of the Algebra \( AG(1,3) \)

According to (4.8), (11.6), the operators (12.1) satisfy the following relations:

\[
\begin{align*}
MW_0 &= P \cdot W = 0, \quad [W_0, P_{\alpha}] = 0, \\
[W_\alpha, W_\beta] &= ic_{\alpha \beta} W_\gamma, \quad [W_0, W_\alpha] = -ie_{\alpha \beta \gamma} P_\beta W_\gamma,
\end{align*}
\]

(12.3)

(12.4)

Here \( c_{\alpha \beta} \) and \( p_\beta \) are eigenvalues of the commuting operators \( C_\alpha \) and \( P_{\alpha} \).

We do not provide commutation relations for \( \Gamma_\mu \) as they turn out to be unnecessary for determining the explicit form of basis elements of the Galilei algebra.

It will be shown further on that to describe nonequivalent IRs of the algebra \( AG(1,3) \) it is sufficient to find all the possible (up to equivalence) realizations of the
vector $W'$, the latter being the nonrelativistic analog of the Lubanski-Pauli vector, and then to find the corresponding operators $J$ and $G$. Thus, our task is to describe representations of the algebra (12.4).

In the frame of reference for which $p = pn$, the commutation relations (12.4) reduces to the following form:

$$[W'_a, W'_b] = i c \epsilon_{abc} W'_c, \quad [W'_0, W'_a] = i p \epsilon_{abc} n_b W'_c. \quad (12.5)$$

The relations (12.5) define a Lie algebra whose structural constants depend on $\hat{p}$. Making a transfer to the new basis $\{\lambda_0, \lambda_a\}$ in accordance with (12.6):

$$W_0 \rightarrow W_0 \lambda_0, \quad W_a \rightarrow \lambda_a n_a \lambda_0, \quad (12.6)$$

we obtain the following commutation relations for the operators $\lambda_0$ and $\lambda_a$:

$$[\lambda_0, \lambda_a] = i c \epsilon_{abc} n_b \lambda_0, \quad [\lambda_0, \lambda_a] = i c \epsilon_{abc} n_b \lambda_0. \quad (12.7)$$

The Lie algebra defined by the commutation relations (12.7) and its representations have already been considered above in Subsection 4.5. Any IR of this algebra corresponds to the vector $W'_\mu$ from (12.6) in the reference frame for which $p_\mu = pn_\mu$. An explicit form of $W'_\mu$ in an arbitrary frame of reference can be obtained by the following transformation:

$$W'_\mu \rightarrow W_0 \lambda_0, \quad W_a \rightarrow \lambda_a n_a \lambda_0, \quad (12.8)$$

where $R_{ab}$ is the operator of rotation of a reference frame given in (4.17) and $\hat{p}_a = p_\mu n_a / p$. It is not difficult to note that the analytical expressions for the vectors of (4.21) and (12.8) are almost identical. Namely, substituting $p_0 \rightarrow c_1$ in (4.21) we come the conclusion that the explicit expressions for $J_a$ and $G_a$ can be obtained from (4.45) with the help of the following correspondence rule: $p_0 \rightarrow c_1, J_{ab} \rightarrow c_{abc} J_c, J_0 \rightarrow G_a n_a, G_a \rightarrow G_a \partial / \partial p_a$. As a result we come to the representation of the Galilei algebra in the following form:

$$P_0 = p_0, \quad P_a = p_a, \quad M = c_1 = m, \quad (12.9)$$

$$J = x \times p + \lambda_0 n_0, \quad G = i p \partial / \partial p_0 - m x + \lambda_0 p n_0 - \lambda_0 n_0. \quad (12.10)$$

Here $m$ is a fixed number and the variables $p_0, p_a$ are connected by the relation

$$2 m p_0 = p^2 = c_1. \quad (12.10)$$

So, each IR of the algebra (12.7) corresponds to a class of IRs of the
Galilei algebra with the fixed values of $c_1=m$ and $c_3=2mp_0-p^2$. This result can be formulated as the following assertion.

**THEOREM 12.1.** IRs of the algebra $AG(1,3)$ are labelled by sets of numbers $c_1$, $c_2$, ... (eigenvalues of the Casimir operators) with the following possible values:

**I.** $c_1^2-m^2>0$, $c_2=m^2(s+1)$, $-\infty<c_3<\infty$, $s=0,1/2,1,\ldots$,

**II.** $c_1=0$, $c_2=0$, $c_3=-k^2<0$, $c_4=0,1/2,1,\ldots$,

**III.** $c_1=0$, $c_2=-r^2>0$, $c_3=-k^2<0$.

An explicit form of the corresponding basis elements of the algebra $AG(1,3)$ is given by formulae (12.9) where $m=c_1$ is a fixed real number, $\lambda_\mu$ are the matrices (4.40)-(4.43), and $p_\mu$, $p_a$ are related by (12.10).

**PROOF.** The validity of the theorem follows directly from the reasoning given above. Note that it is possible to verify that the operators (11.9) satisfy the commutation relations (11.6). We can also make sure that the Casimir operators (11.14) for the representation (12.9) have the form

$$C_1=m, \quad C_2=m^2\lambda_0^2+\lambda_1^2+\lambda_2^2, \quad C_3=2mp_0-p^2.$$

The operator $C_3$ coincides with the Casimir operator from (4.22) whose eigenvalues are given by formulae (4.40)-(4.43). It follows from the above that eigenvalues of the operators $C_\lambda$ used for numbering IRs do satisfy relations (12.11). In the case $c_1=c_2=0$ there exists the additional Casimir operator $C=W_0/p_0=\lambda_0$ eigenvalues of which are integers or half integers (refer to (4.42)).

We see that the basis elements of IRs of the Galilei algebra can be chosen in the form (12.9) where $\lambda_\mu$ and $\lambda_\sigma$ are matrices realizing an IR of the algebra (12.7). IRs of the algebra (12.7) are defined by the relations (4.40)-(4.44).

The realization of an IR of the algebra $AG(1,3)$ given by relations (12.9) is distinguished by a relative simple form for basis operators, which is common for any class of IRs. In the subsection that follows we discuss briefly different classes of IRs of the algebra $AG(1,3)$ and show what connections exist between the realization (12.9) and other known representations.

### 12.4. Connections with Other Realizations

Let us consider sequentially each of the classes of IRs enumerated in (4.10).

1. IRs of Class I ($c_1=m^2>0$) are characterized by three numbers $m$, $e_0=c_3$, and $s$, $m$ and $e_0$ being arbitrary real numbers, and $s\geq0$ being an integer or half integer. Such representations are realized in a space of square integrable functions $\Phi(p,\lambda)$ where $\lambda=-s,-s+1,\ldots,s$, i.e., dimension of $\Phi(p,\lambda)$ in respect with the index $\lambda$
Chapter 3. Representations of the Galilei Algebra ...

is equal to 2s+1. The corresponding scalar product and additional requirements for $\Phi(p,\lambda)$ are presented in Subsection 4.6 (refer to (4.47)).

The space of an IR of the algebra $AG(1,3)$ belonging to the Class I is associated with the state space of a free nonrelativistic particle of mass $m$, spin $s$, and internal energy $e_s/2m$. The corresponding basis elements of the Galilei algebra can be chosen in the form (12.9) where $\lambda_0$, $\lambda_a$ are matrices of dimension $(2s+1)\times(2s+1)$, connected with generators $S_1$, $S_2$ and $S_3$ of the orthogonal group according to the following relations (refer to (4.24)):

$$
\begin{align*}
\lambda_0 &= S_3, & \lambda_a &= R^{-1}_{ab} \lambda_b', & \lambda_0' &= m S_1, & \lambda_a' &= m S_a, & \lambda_0' &= 0, \\
R^{-1}_{ab} &= n \cdot n' \delta_{ab} + n_p n_b' - n_p n_b + (n \times n')_b (n \times n')_b (1 + n \cdot n')^{-1}, & n' &= (0,0,1).
\end{align*}
$$

The explicit form of the matrices $\lambda_0$, $\lambda_a$ in the Gelfand-Zetlin basis is given in formulae (4.40), (4.41).

With the help of the transformation

$$
P_\mu \rightarrow UP_\mu U^{-1}, \quad J_a \rightarrow U J_a U^{-1}, \quad G_a \rightarrow U G_a U^{-1}
$$

with $U$ being the operator (4.49), the generators (12.9) reduce to the usual form used in the literature on physics

$$
P_0 = -e_a \frac{p_a^2}{2m}, \quad P_a = p_a, \quad M = -m,
$$

$$
J_a = -ie_{abc} p_b \frac{\partial}{\partial p_c} + S_a, \quad G_a = -m x_a x_a p_a.
$$

2. IRs of Class II ($c_1 = c_2 = 0$) are labelled by a pair of numbers $c_3 < 0$ and $c_4 = 0, 1/2, 1, \ldots$. These representations are one-dimensional and defined in the space of square integrable functions $\varphi(p_0, p)$. The corresponding form of the Galilei group generators simplifies considerably. Indeed, setting $m = 0$ and $\lambda_0 = 0$ in (12.9) (refer to (4.24), (4.33)), we obtain

$$
G = -i p \frac{\partial}{\partial p_0}.
$$

The remaining basis elements of the algebra $AG(1,3)$ (i.e., $P_\mu$ and $J$) have the form (12.9) where $\lambda_0 = c_4$ is an integer or half integer, and $p^2 = k^2 = \text{const}$.

By means of the transformation (12.14) where $U$ is the operator (4.52) or (4.53) and $n' = (0,0,1)$, the generators $P_\mu$, $J$ of (11.9) and $G$ of (12.16) can be transformed to the form found in [375]. We do not give here the explicit expressions of the corresponding basis elements of the algebra $AG(1,3)$ because of their complicated and nonsymmetric form as contrasted to (12.16), (12.9).

Representations of the Galilei algebra of Class II are realized on solutions of
3. Let us now consider IRs of Class III \((c_1=0, c_2<0)\) which are labelled by pairs of positive numbers \(r^2\) and \(k^2\) (see (12.11)). Such representations are realized in the space of square integrable functions \(\varphi(p_0, p, \lambda)\) where \(\lambda\) takes the infinite number of values given by one of formulae (4.43). The explicit expressions of basis elements of the IRs are given in (12.9) where \(m=0, p^2=k^2\) and \(\lambda_\alpha, \lambda_\beta\) are infinite-dimension matrices connected with the generators \(T_\alpha\) of the group \(E(2)\) by the following relations (refer also to (4.24)):

\[
\lambda_\alpha T_\alpha \quad \lambda_\beta R_{\alpha\beta}^{-1} \lambda_\beta', \quad \lambda_\gamma' T_1, \quad \lambda_\delta' T_2, \quad \lambda_\epsilon'.
\]

Here \(R_{\alpha\beta}^{-1}\) are the matrix elements given in (12.13). The explicit form of the matrices \(\lambda_\alpha, \lambda_\beta\) can be chosen, for example, according to (4.40), (4.43).

As far as we know, representations of the Galilei algebra of Class III have not found yet direct usage in physics. If (in analogy with representations of Classes I, II) we take a space state of a quantum mechanical particle in correspondence with a space of a representation of Class III, then such a particle would have an infinite number of spin states.

4. The classes of IRs considered above exhaust all the nonequivalent Hermitian representations of the algebra \(AG(1,3)\). We note that formulae (12.9) define the explicit form of basis elements of the Galilei algebra for the case of \(c_1^2<0\) also. The corresponding matrices \(\lambda_\alpha\) realize a representation of the algebra \(AO(1,2)\) (refer to (4.24), (4.27), (4.36)-(4.39)).

IRs of the algebra \(AG(1,3)\) corresponding to \(c_1^2<0\) cannot be realized by Hermitian operators since the generator \(M=C_1\) has purely imaginary eigenvalues. This circumstance restricts the usage of such representations in quantum mechanics. But representations of this type naturally arise in various problems of classical physics. In particular, representations of the Galilei algebra corresponding to \(c_1^2<0\) are realized on a set of solutions of the linear heat equation.

5. IRs of Class V reduce to representations of the Lie algebra of the Euclidean group \(E(3)\). IRs and a class of finite dimension representations of this algebra are considered in Subsection 12.6.

12.5. Covariant Representations

IRs of the Galilei algebra considered above can be defined in Gilbert spaces of square integrable functions depending on \(p_\mu\). But for many applications (e.g., for describing differential equations being invariant under the Galilei group) it is more convenient to deal with representations defined in a space of functions
Chapter 3. Representations of the Galilei Algebra ...

\[ \psi(t, x) \] where the generators \( P_\alpha, \ P_\alpha \) are represented by differential operators. We will call such representations covariant if the corresponding basis elements belong to the class \( M_1 \) (i.e., they are differential operators of the first order).

It is not difficult to show that a basis of a covariant representation is formed by the following operators:

\[ P_0 = \frac{\partial}{\partial t}, \quad P_\alpha = -i \frac{\partial}{\partial x_\alpha}, \]

\[ J_\alpha = -i \epsilon_{abc} x_\beta \frac{\partial}{\partial x_\gamma} S_{\alpha \beta \gamma}, \quad M = m, \]

where \( S_{\alpha \beta}, \eta_\alpha), \text{and} m \) are numeric matrices satisfying the commutation relations

\[ [m, S_\alpha] = [m, \eta_\alpha] = 0, \]

\[ [S_\alpha, S_\beta] = i \epsilon_{abc} S_{\alpha \beta \gamma}, \] \hspace{1cm} (12.19)

\[ [S_\alpha, \eta_\beta] = i \epsilon_{abc} \eta_{\alpha \beta \gamma}, \quad [\eta_\alpha, \eta_\beta] = 0. \] \hspace{1cm} (12.20c)

Formulae (12.18) determine the general form of operators \( P_\alpha, J_\alpha, G_\alpha, \) and \( M \) satisfying the commutation relations (10.6) and belonging to the class \( M_1 \).

Integrating the corresponding Lie equations (refer to (1.20)-(1.22)), we obtain finite transformations generated by the operators (12.18) in the form

\[ \psi(t, x) \rightarrow \psi(t', x') = D(v, \theta) \exp[i \phi(x)] \psi(t, x) \] \hspace{1cm} (12.21)

where \( t', x' \) and \( \phi(x) \) are given by relations (11.28), (11.29) while \( D(v, \theta) \) is the matrix depending on parameters of the transformation (11.29),

\[ D(v, \theta) = \exp(i \eta \cdot v) \exp(i S \cdot \theta). \] \hspace{1cm} (12.22)

In the case of \( \eta = \mathbb{S} = 0 \) we come to formula (11.27) which determines the transformation law for solutions of the Schrödinger equation.

According to (12.21), a transformed function \( \psi(t', x') \) is expressed via \( \psi(t, x) \) multiplied by the phase multiplier \( \exp[i \phi(x)] \) and a numeric matrix \( D(v, \theta) \). In other words, a value of \( \psi(t', x') \) is completely determined by the value of a non-transformed function in the point \( (t, x) \) and by transformation parameters. The transformations of this type are called locally covariant (or simply covariant) in contrast to nonlocal transformations for which \( \psi(t', x') \) depends on values of \( \psi(t, x) \) within some domain. Formulae (12.18)-(12.20) determine the general form of generators of a covariant representation of the Galilei group.

Thus, a description of covariant representations of the algebra \( AG(1, 3) \)
Symmetries of Equations of Quantum Mechanics

reduces to finding the nonequivalent matrices $S_a$, $\eta_a$ and $m$ to satisfy the commutation relations (12.19), (12.20). Representations of the algebra (12.19), (12.20) are considered in the following subsection.

Besides representations of the proper Galilei group determined by the continuous transformations (11.29), a certain interest was exhibited in representations of the complete group $G(1,3)$ which includes transformations $P$, $T$, and $C$. An analysis of such representations is beyond the scope of this book.

Representations of the group including the transformations $P$ and $T$ and (11.28) are described in [59], while representations of the complete Galilei group are considered in [151].

12.6. Representations of the Lie Algebra of the Homogeneous Galilei Group

The Lie algebra of the homogeneous Galilei group is formed by six generators $S_a$, $\eta_a$ satisfying the commutation relations (12.20). This algebra is isomorphic to the Lie algebra of Euclidian group $E(3)$ and is denoted as $AE(3)$.

It will be shown further on that the problem of description of nonequivalent indecomposable representations of the algebra $AE(3)$ reduces to the algebraic problem which cannot be solved by known methods. So we restrict ourselves to description of some class of finite dimension representations. This class is wide enough to deduce consistent motion equations of arbitrary spin particles.

We consider also IRs of the algebra $AE(3)$. The problem of description of such representations can be solved in a closed form.

The algebra $AE(3)$, as presented by (12.20), includes the subalgebra $AO(3)$ formed by the matrices $S_a$. Since IRs of the algebra $AO(3)$ are well known, it is convenient to use the $O(3)$-basis $|\lambda,l,m>\rangle$ formed by eigenvectors of the commuting matrices $S^2$ and $S_3$. Explicit expressions for $S^2$ and $S_3$ in this basis are given by formulae (4.63), (4.64), where $(l_0,l,1)\rightarrow \lambda$. In our case $\lambda$ is an additional index labelling eigenvectors of the matrix $S^2$ corresponding to degenerated values of $l$.

The general expression for matrices $\eta_a$ commuting with $S_a$ in accordance with the rule (11.20b) is given in our basis by the following formula:

$$\eta_{\lambda,l,m>} = \sum_{l',m',\lambda'} \left[ \delta^{l'}_{l_0} a^{l',i}_{\lambda\lambda'} (S_a^1)_{nm} + \delta^{l_1}_{l_0} b^{l_1,i}_{\lambda\lambda'} (K_a^1)_{nm} + \delta^{l_1}_{l_0} c^{l_1,i}_{\lambda\lambda'} (K_a^1)^{\dagger}_{nm} \right] |\lambda',l',m'>,$$  \hspace{1cm} (12.23)

where $a^{l',i}_{\lambda\lambda'}$, $b^{l_1,i}_{\lambda\lambda'}$, $c^{l_1,i}_{\lambda\lambda'}$ are arbitrary complex numbers, $(S_a^1)_{nm}$ and $(K_a^1)_{nm}$ are the matrix elements given by relations (4.65), (4.66). The possible values of $l$ corresponding to indecomposable matrices $S_a$ and $\eta_a$ are [197] as follows

$$l=l_0, \ l_0+1, ..., l_1.$$  \hspace{1cm} (12.24)
Chapter 3. Representations of the Galilei Algebra...

$l_0$ and $l_1$ being positive integers or half integers. By definition
\begin{align}
a^{l_0} = i^{l_0}, \quad b^{l_1} = i^{l_1}, \quad b^{-l_1} = i^{-l_1} = 0. \tag{12.25}
\end{align}

We have marked in (12.23) the matrix elements $(S^l_a)_{mn}$ and $(K^l_a)_{mn}$ of the matrices $S^l_a$ and $K^l_a$ which satisfy the following relations [223]:
\begin{align}
K^l_a S^l_b - S^l_b K^l_a &= i\epsilon_{abc} S^l_c, \\
S^l_a S^l_b + S^l_b S^l_a &= l^2 \delta_{ab}, \\
K^l_a S^l_b + S^l_b K^l_a &= (2l-1) \epsilon_{abc} S^l_c, \\
K^l_a K^l_b - K^l_b K^l_a &= (2l+1) \epsilon_{abc} S^l_c, \\
S^{l+1}_a S^l_b - S^{l-1}_a S^l_b &= (l+1) \epsilon_{abc} S^l_c, \\
K^{l+1}_a K^l_b - K^{l-1}_a K^l_b &= i(2l+1) \epsilon_{abc} S^l_c. \tag{12.26a}
\end{align}

The conditions (12.26a) together with (4.63), can serve as a definition of the matrices $S^l_a$ and $K^l_a$, while relations (12.26b) follow from (12.26a). They are the conditions (12.26a) which are used in the following to describe representations of the algebra $AE(3)$.

The algebra $AE(3)$ has the two Casimir operators
\begin{align}
C_1 = S^l_a \eta^a, \quad C_2 = \eta^b S^l_b, \tag{12.27}
\end{align}
which reduce to the following form in the basis $|\lambda;l,m>$:
\begin{align}
C_1 |\lambda;l,m> &= \lambda^l \lambda^m |\lambda;l,m>, \tag{12.28a} \\
C_2 |\lambda;l,m> &= (l+1)(2l+3) \epsilon_{\lambda^l \lambda^m} a^{l+1} a^{l+1} + l(l+1) a^{l-1} a^{l-1} + l(2l-1) b^l b^l |\lambda;l,m>. \tag{12.28b}
\end{align}

There is also the additional Casimir operator
\begin{align}
C_3 = \exp(2\pi i S^l_a) \exp(2\pi i S^l_b) \exp(2\pi i S^l_c) \tag{12.29}
\end{align}
with eigenvalues equal to $\pm 1$ (refer to (4.13)).

The Casimir operators do not play an essential role in description of indecomposable representations, but they are very useful in searching of IRs.

Let us show that the condition of commutativity of matrices $\eta^a$ reduces to the system of quadratic equations for matrices $a^l = |a^l_{\lambda^l \lambda^m}|$, $b^l = |b^l_{\lambda^l \lambda^m}|$ and $c^l = |c^l_{\lambda^l \lambda^m}|$. Essentially, using the representation (12.23), it is not difficult to calculate the commutator

149
\[
[\eta_a, \eta_b] |\lambda; l, m \rangle =
\sum_{\nu, \lambda', l', m'} \left( c^{(l+1)}_{\nu, \lambda'} a^{(l+1)}_{\lambda} (K^{(l+1)}_a S^{(l+1)}_b - K^{(l+1)}_b S^{(l+1)}_a) + a^{(l+1)}_{\nu, \lambda} (S^{(l+1)}_a S^{(l+1)}_b - S^{(l+1)}_b S^{(l+1)}_a) \right) b^{(l+1)}_{\lambda', l'} |l'; m' \rangle,
\]
the above expression suggesting a summation over the repeated indices \( \nu, \lambda', l' \) and \( m' \).

Equating (12.30) to zero and taking into account relations (12.26), we come to the following system of equations for the matrices \( a' \), \( b' \) and \( c' \):

\[(l+1)c' a' = -(l+1) a c', \quad (l+1)a' b' = -(l+1) b' a', \quad (2l+3)c' b' = -(2l+1) b' c' = (a')^2.\]

Formulae (12.31) present necessary and sufficient conditions of commutativity of the matrices \( \eta_a \) from (12.23).

Thus the problem of description of representations of the algebra \( AE(3) \) reduces to the finding of nonequivalent solutions of the system of quadratic equations (12.31), (12.25).

First we consider Hermitian IRs of the algebra \( AE(3) \). A description of such representation is a relatively simple problem. Indeed, the Hermiticity condition leads to the following relation:

\[ c^{(l+1)} = b^{(l+1)}. \]

Then, according to Schur’s Lemma, the corresponding Casimir operators (12.28) have to be multiples of the unit matrix so that

\[ a^{(l+1)} c_{\lambda, \lambda'} a^{(l+1)} - \delta_{\lambda, \lambda'} a^{(l+1)} = -a^{(l+1)} c_{\lambda, \lambda'} a^{(l+1)} = 0, \]

where \( \lambda, \lambda' = 1, 2, \ldots, n_p \).

\[ a^{(l+1)} = c_{\lambda}, \]

\[ (l+1)(2l+3) b_{\lambda, \lambda'}^{(l+1)} + (2l+1) b_{\lambda', \lambda}^{(l+1)} = (a')^2 I_p. \]

Here \( c_1 \) and \( c_2 \) are eigenvalues of the Casimir operators (12.27), \( a' \) is a real number, \( I_p \) is the unit matrix of dimension \( n_p \times n_p \), and

\[ b_{\lambda, \lambda'}^{(l+1)} = b'_{\lambda, \lambda'} a^{(l+1)} b^{(l+1)}; \]

(12.33a, 12.33b, 12.33c)

The equations (12.31a) are satisfied identically according to (12.33b), and the
equation (12.31b) takes the form

\[(2l+3)b_{l+1}^2-(2l+1)b_l^2=r^2l_l^2I_{n_l}^2.\]  

(12.35)

So, a description of IRs of the algebra \(AE(3)\) reduces to the finding of a general solution of the equations (12.33)-(12.35).

It is not difficult to demonstrate that without loss of generality we can set \(n_l=1\) (otherwise we obtain a reducible representation with degenerated eigenvalues of the Casimir operators). Thus \(a_l, b_l\) and \(b_{l+1}\) reduce to numbers which can be easily calculated.

The system (12.33)-(12.35) is not consistent for any fixed \(l\), and therefore it is necessary to solve an infinite numerable system corresponding to an infinite dimension representation of the algebra \(AE(3)\). The corresponding solutions are:

\[c_1=r_{l_0}^2, \quad c_2=r^2, \quad l_0=0,1/2,1,\ldots, \quad c_3=(-1)^{l},\]  

(12.36)

\[a_l=r_{l_0}^2, \quad b_l=\frac{r}{l+1}, \quad b_{l+1}^2=\left(\frac{l^2-l_0^2}{4l^2-1}\right), \quad l_0, l_0+1, \ldots.\]  

(12.37)

So IRs of the algebra \(AE(3)\) are labelled by two numbers, an arbitrary real number \(r\) and non-negative integer or half integer \(l_0\). These numbers define possible values of the Casimir operators (12.36). All the Hermitian IRs are infinite dimensional, the explicit form of basis elements being as follows:

\[S^r_{m_{l_0}l_0}, \quad l_0, l_0+1, \ldots, \quad S_l^l, \quad l_0, l_0+1, \ldots, \quad S^r_{m_{l_0}l_0}, \quad l_0, l_0+1, \ldots.\]  

(12.38)

Consider now the problem of description of finite dimension indecomposable representations of the algebra \(AE(3)\). This problem is more complicated then in the case of IRs. Moreover it is unsolvable in the sense that it is impossible to enumerate effectively all the nonequivalent representations.

Thus we will try to solve the system (12.31) for the case when the numbers \(l_0\) and \(l\) determining possible values of \(l\) are positive integers or half integers. It means that the chain of equations (12.31) can include an arbitrary (but finite) number of links corresponding to the given \(l_0\) and \(l\). Dimensions of matrices \(a_l, b_l, c_l\) and \(c_{l+1}\) are equal to \(n_l\times n_l, n_l\times n_{l+1}, n_{l+1}\times n_l,\) and \(n_{l+1}\times n_l\) respectively, where \(n_l\) and \(n_{l+1}\) are multiples of the eigenvalues \(l(l+1)\) and \(l(l-1)\) of the operator \(S^2\). In contrast to the case of IRs we have no additional equations like (12.33) inasmuch as we consider
Symmetries of Equations of Quantum Mechanics

indecomposable representations and the corresponding eigenvalues of the Casimir operators cannot be fixed.

Unfortunately, it is not possible to describe effectively all the nonequivalent solutions of the equations (12.31). Indeed, in the particular case of \((a')^2=0\) these equations reduce to the problem of a description of all the nonequivalent pairs of commuting matrices \(B\) and \(C\) defined in accordance with the following relations:

\[ B|\lambda;l,m> = b' |\lambda;l-1,m>, \quad C|\lambda;l,m> = c' |\lambda;l+1,m>. \]

Such a problem is "wild" and cannot be solved by the known methods [199].

Imposing some additional restrictions on \(d'\), it is possible to describe appropriate classes of solutions of equations (12.31). One such restriction which enables us to describe completely the corresponding class of solutions is a requirement for matrices \(d'\) to be indecomposable. Here we consider a more restricted class of solutions of equations (12.31) corresponding to the case when the possible values of \(l\) being equal to \(l=s,s-1\).

To denote this class of solutions and the corresponding representations of the algebra \(AE(3)\), we use the symbol \(D_2\).

Indecomposable representations of the class \(D_2\) may be described as follows.

**PROPOSITION 12.1** [151]. Nonequivalent solutions of equations (12.31), \(d'\) being indecomposable matrices and possible values of \(l\) being given by (12.39), are labelled by triplets of integers \((k,\alpha,n)\) where\n
\[ k \leq 4, \quad n \leq 4, \quad |k-n| \leq 2, \quad kn < 9; \quad \alpha = 1, \quad |k-n| \geq 2, \quad \alpha = 1, 2, 3, 4. \]  

(12.40)

The corresponding matrices \(a', a'^{-1}, b', \) and \(c'\) are of dimensions \(k \times k, n \times n, k \times n\) and \(n \times k\), and their elements are equal (for \(\alpha = 1\)) to

\[
\begin{align*}
\tilde{a}_{ij} &= \frac{\tilde{a}_{ij}}{s(s+1)}, \quad \tilde{a}_{ij}^{-1} = \frac{\tilde{a}_{ij}^{-1}}{s(s+1)}, \quad \tilde{b}_{ij} = \frac{\tilde{b}_{ij}}{2s-1}, \quad \tilde{c}_{ij} = \frac{\tilde{c}_{ij}}{2s+1}, \\
\tilde{a}_{ij}^{-1} &= \delta_{i,j}, \quad \tilde{a}_{ij}^{-1} = \delta_{i,j}, \quad \lambda, \lambda' \leq k, \quad \nu, \nu' \leq n, \\
\tilde{b}_{ij} &= \begin{cases} 
\sqrt{k} \delta_{i,j}, & k > n, \\
\sqrt{k} \delta_{i,j}, & k \leq n, \quad kn \neq 4, \\
\mu \delta_{i,j}, & k = n = 2,
\end{cases}
\end{align*}
\]

(12.41a)
where 

\[
\begin{cases}
\begin{align*}
\tilde{c}_{\lambda\nu}^s &= \left(\frac{2s+1}{s^2(s+1)}\right) \delta_{\lambda\lambda'} \quad k>n, \\
\tilde{c}_{\lambda\nu}^s &= \left(\frac{1}{s^2(s+1)}\right) \delta_{\lambda\lambda'} \quad k\leq n,
\end{align*}
\end{cases}
\]

(12.41b)

If \( \alpha = 2 \) then the corresponding matrix elements can be obtained from (12.41) by the substitution

\[
\begin{align*}
a'_{\lambda\lambda'} &\rightarrow a'_{\lambda\lambda'}, \\
b'_{\lambda\nu} &\rightarrow c'_{\nu\lambda}, \\
c'_{\nu\lambda} &\rightarrow b'_{\lambda\nu}.
\end{align*}
\]

For the proof see [319]. We only note that in the case considered it follows from (12.31) that \( (a')^2 = (a'^{-1})^2 = 0 \) and the maximal dimension of matrices \( a' \) is \( 4\times4 \). Choosing these matrices in the Jordan form we obtain the compatibility condition for the system (12.31) in the form of (12.39) by direct calculations. The corresponding solutions can be chosen in the form (12.41) with up to equivalence.

It is not difficult to calculate that there are 18 nonequivalent solutions for any \( s \neq 0,1/2, \) two of them being dependent on an arbitrary complex number \( \mu \).

Any solution of equations (12.31) can be set in one-to-one correspondence with a representation of the algebra \( AE(3) \). So, from Proposition 12.1 follows

**PROPOSITION 12.2.** Indecomposable representations of the algebra \( AE(3) \), belonging to the class \( D_2 \) are labelled with triplets of integers \( (k,\alpha,n) \) satisfying the conditions (12.40). The explicit form of the corresponding basis elements \( S_a \) and \( \eta_a \) in the basis \( |k;l,m> \) is given by the formulae (4.64)-(4.66), (12.23) and (12.41).

So we have described the class \( D_2 \) of indecomposable representations of the Lie algebra of the homogeneous Galilei group. These representations will be used in the next section to deduce wave equations which are invariant under the Galilei group.

We should like to note that the integers \( k \) and \( n \) used (together with \( \alpha \)) to label indecomposable representations of the class \( D_2 \) define the nilpotency indices of the Casimir operators (12.26). Indeed, according to (12.27) and (12.33)

\[
C_1^N = C_2^{N_2} = 0, \quad N = \text{max}(k,n).
\]

It can be easily seen that \( N \leq 4 \) for the representations of the class \( D_2 \).
13. GALILEI-IN Variant WAVE EQUATIONS

13.1. Introduction

From Bargman’s work [16] it is known that the concept of spin arises in nonrelativistic quantum mechanics as naturally as in frames of relativistic theory. One of the Casimir operators of the Galilei group is nothing but the operator of squared spin. Thus, it is surprising that the number of papers devoted to Galilei-invariant wave equations is extremely small. This circumstance seems to be particularly strange in view of the fact that Poincaré invariant wave equations attract attention of a great many of investigators.

An important contribution into the theory of Galilei invariant wave equations was made by Levi-Leblond [276, 277] who obtained for the first time such an equation for a particle of spin 1/2. The Levi-Leblond equation takes into account the Pauli interaction and predicts the correct value of the gyromagnetic ratio such as the relativistic Dirac equation. Unfortunately this equation (and its generalizations proposed by Hagen and Hurley [215, 223]) does not describe such an important physical effect as a spin-orbit coupling.

In this section we describe a class of Galilei-invariant equations of the first order corresponding to the indecomposable representations of the homogeneous Galilei group described above. This class includes as the Levi-Leblond and Hagen-Hurley (LHH) equations as the new equations describing spin-orbit and quadruple couplings of an arbitrary spin particle with an external field.

The Galilei-invariant wave equations in many cases are good alternatives to Poincaré-invariant equations for particles of higher spin inasmuch the latter lead to contradictions by a description of a particle interaction with an external field, see Section 10.

13.2. Galilei-Invariance Conditions

To describe first-order equations invariant under Galilei transformations is to answer the question: what kind of matrices $\beta_\mu$, $\beta_4$ guarantees the invariance of the system (6.2) with respect to the Galilei algebra?

For convenience we rewrite (6.2) in the form

$$L\psi=0, \quad L_\mu=\beta_\mu p_\mu+\beta_4 m.$$ \hspace{1cm} (13.1)

The invariance under the Galilei algebra means that the equation (13.1) admits 11 SOs satisfying relations (11.6). We restrict ourselves to the case when these SOs realize a covariant representation of the Galilei algebra and so have the
Chapter 3. Representations of the Galilei Algebra ...

form of (12.18). We assume also that the mass operator $M$ is a multiple of the unit operator and has a positive eigenvalue $m>0$.

By definition the operators (12.18) are SOs of the equation (13.1) if they satisfy relations (6.3). Substituting $Q_\alpha = P_\alpha, J_\alpha, G_\alpha$ into (6.3) we come to the following relations (compare with (6.4), (6.5))

$$
\tilde{S}_a \beta \beta S_a = 0, \quad \tilde{S}_a \beta \beta _\beta S_a = 0, \\
\tilde{\eta}_a \beta \beta \eta _\beta = -i \beta _\beta , \quad \tilde{\eta}_a \beta \eta _\beta = -i \delta _{ab} \beta _c,
$$

(13.2)

Here $\tilde{S}_a$, $\tilde{\eta}_a$ are matrices realizing a representation of the algebra $AE(3)$ (11.20), $\beta _\alpha, \beta _\beta, \beta _c$ are unknown numeric matrices.

Using relations (12.20) it is not difficult to obtain from (13.2) the following conditions

$$
[\tilde{S}_a, \tilde{S}_b] \beta _\alpha = 0, \quad [\tilde{\eta}_a, \tilde{\eta}_b] \beta _\alpha = 0, \quad [\tilde{S}_a, \tilde{\eta}_b] \beta _\alpha = 0.
$$

A sufficient condition of validity of relations (13.3) is the requirement that the matrices $\tilde{S}_a, \tilde{\eta}_a$ realize a representation of the algebra $AE(3)$.

Thus we come to the following definition.

**DEFINITION 13.1.** The equation (13.1) is S-invariant under the Galilei algebra if there exist matrices $S_a, \eta _a$ and $\tilde{S}_a, \tilde{\eta}_a$ realizing (generally speaking nonequivalent) representations of the algebra $AE(3)$ and satisfying relations (13.3).

As in the case of Poincaré-invariant equations we will use the sufficient conditions of the Galilei-invariance given in Definition 13.1. These conditions are not necessary since there exist equations being invariant under noncovariant realizations of the Galilei algebra [162, 163]. Moreover it is possible in principle to renounce the requirement the matrices $\tilde{S}_a, \tilde{\eta}_a$ realize a representation of the algebra $AE(3)$ since the conditions (13.3) are more weak.

We note that Definition 13.1 determines the invariance conditions for the case of square matrices $\beta _\alpha, \beta _\beta$ as for equations with rectangular $\beta$-matrices of dimension $m \times n$. In the latter case the matrices $\tilde{S}_a, \tilde{\eta}_a$ and $S_a, \eta _a$ have dimensions $m \times m$ and $n \times n$.

### 13.3. Additional Restrictions for Matrices $\beta _\alpha$

Let us define the conditions imposed on $\beta _\alpha$ by some additional physical requirements.

First we restrict ourselves to considering only such equations which admit
the Lagrangian formulation. As in the case of relativistic wave equations it means
that we assume that a nonsingular Hermitizing matrix \( \eta \) exists which satisfies the
conditions
\[
\eta \beta_\mu = \beta_\mu^\dagger \eta, \quad \eta \beta_4 = \beta_4^\dagger \eta.
\] (13.4)
It appears that without loss of generality we can impose a stronger requirement
\[
\beta_\mu^\dagger = \beta_\mu, \quad \beta_4^\dagger = \beta_4
\] (13.5)
as it follows from the following assertion.

**Lemma 13.2.** Let \( \{ \tilde{S}_a, \tilde{\eta}_a, S_a, \eta_a \} \) be a set of matrices satisfying (13.2),
(11.20), and \( \eta \) be an arbitrary nondegenerated matrix. Then the matrices
\[
\beta'_\mu = \eta \beta_\mu, \quad \eta'_a = \eta \eta^{-1}_a, \quad \tilde{S}'_a = \eta \tilde{S}_a \eta^{-1}, \quad S'_a = S_a, \quad \eta'_a = \eta_a,
\] (13.6)
also satisfy these relations.

**Proof** is almost evident. Multiplying any of relations (13.2) from the left
by \( \eta \) and putting the matrix \( \eta^{-1} \equiv I \) between \( \tilde{S}_a, \tilde{\eta}_a \) and \( \beta_\mu, \beta_4 \) we come to relations
(13.2) for the matrices (13.6).

So besides the equation (13.1) there exist a class of the Galilei-invariant
equations with primed matrices (13.6). Thus if a Hermitizing matrix exists then we
always can consider an equivalent Galilei-invariant equation with Hermitian matrices
\( \beta_\mu \). In other words we can suppose these matrices satisfy (13.5).

The Lagrangian corresponding to the equation (13.1) without loss of
generality can be chosen in the form
\[
L_0 = -i \frac{1}{2} (\gamma^\dagger \beta_\mu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \psi^\dagger}{\partial x_\mu} \beta_\mu \psi) + m \gamma \beta_4 \psi.
\] (13.7)
It is not difficult to make sure the requirement of the Galilei-invariance of this
Lagrangian reduces to the conditions (13.2) where
\[
\tilde{S}_a = S_a^\dagger, \quad \tilde{\eta}_a = \eta_a^\dagger.
\] (13.8)

Thus, the problem of description of Galilei-invariant equations (13.1)
admitting a Lagrangian formulation reduces to finding Hermitian matrices \( \beta_\mu, \beta_4 \)
satisfying the relations
\[
[S_a, \tilde{\beta}_\mu] = 0, \quad [S_a, \beta_4] = 0,
\] (13.9a)
\[
\eta \beta_\mu \beta_a \eta_a = -i \beta_a,
\] (13.9b)
\[
\eta \beta_\mu \beta_a \eta_a = -i \delta_{\mu a} \beta_4
\] (13.9c)
The last relation (13.2) (supplemented by (13.5) (13.8)) turns into identity if relations
Chapter 3. Representations of the Galilei Algebra...

(13.9) are satisfied.

We require also the representation of the Galilei algebra, realized on the set of solutions of the equation (13.1), be irreducible and belong to Class I. It means that the wave function has to satisfy the conditions (which have to be consequences of (13.1))

\[(p_0 - \frac{p^2}{2m})\psi = \varepsilon_0 \psi, \quad (S - \frac{1}{m} p \times \eta) \psi = s(s+1)\psi\]

where \(\varepsilon_0\) and \(s\) are fixed parameters.

If the conditions (13.10) are satisfied, the corresponding Casimir operators of the Galilei algebra have fixed eigenvalues. This enables us to interpret (13.1) as a quantum mechanical equation of motion of a particle of spin \(s\), mass \(m\) and internal energy \(\varepsilon_0\). In the following, we analyze the restrictions imposed on matrices \(\beta\) by the conditions (13.10).

13.4. General Form of Matrices \(\beta\) in the Basis \(|\lambda; l, m>\)

Summarizing given in the above we note that the problem of description of Galilei-invariant wave equations can be subdivided into the following stages:

1. to describe finite-dimension representations of the algebra \(AE(3)\), i.e., to find all the nonequivalent matrices \(S_a\) and \(\eta_a\) included into the equations (13.9);
2. to select the representations for which nontrivial solutions of (13.9) exist;
3. to find an explicit form of the corresponding \(\beta\)-matrices;
4. to select the matrices \(\beta\) satisfying the conditions (13.10).

We search for the matrices \(\beta\) in the basis \(|\lambda; l, m>\). The matrices \(\beta_0\) and \(\beta_4\) commute with \(S\) so according to Schur’s lemma

\[\beta_4 |\lambda; l, m> = (2l+1)x^{i}_{\sqrt{\lambda}} |\lambda'; l, m>, \quad \beta_0 |\lambda; l, m> = A_{\sqrt{\lambda}} |\lambda'; l, m>\] (13.11)

where \(x^{i}_{\sqrt{\lambda}}, A^{i}_{\sqrt{\lambda}}\) are unknown coefficients. We will denote matrices with matrix elements \(x^{i}_{\sqrt{\lambda}}\) and \(A^{i}_{\sqrt{\lambda}}\) by \(x_i\) and \(A_i\).

Using (13.9b), (12.23) we obtain the general form of the matrices \(\beta\) \((a=1,2,3)\):

\[\beta_a |\lambda; l, m> = -iB_a^{i\lambda}(K_a^{i\lambda})_{lm} |\lambda'; l-1, m'> +
\[\quad -iD_a^{j\lambda}(S_a^{j\lambda})_{lm} |\lambda'; l, m'> + iB_a^{j\lambda}(K_a^{j\lambda})_{lm} |\lambda'; l+1, m'> \] (13.12)

where \(B_a^{i\lambda}\) and \(D_a^{j\lambda}\) are the matrix elements of the matrices \(B_i\) and \(D_i\).

\[D_{\frac{2l+1}{l(l+1)}} (\delta_{i,j} \lambda_i x_j - \lambda_j x_i), \quad B_{\frac{2l+1}{l(l+1)}} x_i - \lambda_i \delta_{i,j}.\] (13.13)
Let us require the matrices $\beta_a$ satisfy (13.9c). Using relations (12.23), (12.31) we come to the following equations for $x_i$ and $A_i$:

\[
\begin{align*}
&2l^2(2l+1)\tilde{\epsilon}^i_j x_i \tilde{\epsilon}^i_j \tilde{x}_j -(4l^2-1)(\tilde{a} \cdot x)_j \tilde{l}^2(2l-1)A_j, \\
&2l^2(2l-1)\tilde{b}^i_j x_i \tilde{b}^i_j \tilde{x}_j -(4l^2-1)(\tilde{a} \cdot x)_j \tilde{l}^2(2l+1)A_j, \\
&\tilde{b}^i_j [(\tilde{l} \tilde{a}^i_j x_i -(l-1)x_i \tilde{a}_i)] \tilde{l} [(l+1)\tilde{a}_i \tilde{x}_i - l \tilde{x}_i \tilde{a}_i] \tilde{c}^i_j 
\end{align*}
\]

where 

\[
(\tilde{a} \cdot x)_j \tilde{a}^i_j (\tilde{a}^i_j x_i - x_i \tilde{a}_i) -(\tilde{a}^i_j x_i - x_i \tilde{a}_i) \tilde{a}_i.
\]

Formulæ (13.14) present the necessary and sufficient conditions that the matrices (12.13), (13.11) satisfy relations (13.9). Thus the problem of description of Galilei-invariant wave equations reduces to solving the system of linear algebraic equations (13.14) for the matrices $x_i$ and $A_i$ where $a_i$, $b_i$ and $c_i$ are matrices satisfying the system of coupled quadratic equation of (12.31).

Let us consider the restrictions imposed on $x_i$ and $A_i$ by the conditions (13.10). Making the transformation

\[
\psi \rightarrow V \psi \Phi, \quad L \rightarrow L' = V^* L V^{-1},
\]

\[
V = \exp (\frac{i}{m} \eta \cdot p)
\]

and using relations (13.9) and the Campbell-Hausdorf formula

\[
\exp(A') B \exp(-A) = \sum_{n=0}^{\infty} \frac{1}{n!} [A, B]^n,
\]

we obtain from (13.1) the following equivalent equation

\[
L' \Phi = 0, \quad L' = \beta_0 (p_0 - \frac{p^2}{2m}) + \beta_0 m.
\]

The equation (13.17) is more convenient for analysis then (13.1) since it includes only two matrices. Inasmuch as the operator $V$ satisfies the conditions

\[
[V, p_0 - \frac{p^2}{2m}] = 0, \quad V(S - \frac{1}{m} p \times \eta)V^{-1} = S.
\]

then it follows from (13.10) that $\Phi$ has to satisfy the following relations

\[
(p_0 - \frac{p^2}{2m}) \Phi = e_0 \Phi, \quad S \Phi = (s + 1) \Phi.
\]

Relations (13.18) follow from (13.17) iff
Chapter 3. Representations of the Galilei Algebra ...

\[ \det(\alpha X^l + \beta A^l) = \begin{cases} c_r, & l \neq s, \\ \sum_{l \omega} k_i (\alpha \cdot x^l_i), & l = s \end{cases} \]  

(13.19)

where $\alpha$ and $\beta$ are arbitrary functions, $c_i$ and $k_i$ are constants (at least one of $k_i$ is nonzero), $n_l$ is the number of rows of the matrices $x_l$, $A_l$.

Indeed, if (13.19) is satisfied then all the solutions of the equation (13.17) are eigenfunctions of the operator $S^2$, corresponding to the eigenvalue $s(s+1)$. Besides these solutions satisfy the first of the equations (13.18).

13.5. Equations of Minimal Dimension

The above results are valid for arbitrary representation of the homogeneous Galilei group realized on a set of solutions of (13.1). Now we consider the simplest case of the system (13.14) when the index $l$ can take only two values $l = l_0, l_0 + 1$. This corresponds to Galilei-invariant equations with the minimal number of components.

The corresponding indecomposable representations of the algebra $AE(3)$ belong to the class $D_2$ and are described in Subsection (12.6). For the case considered the system (13.14) reduces to the following form (refer to (12.25))

(13.20)

As to the matrices $A_i$ and $A_{i,1}$ they are expressed via $x_i$ and $x_{i,1}$:

\[ A_i = \frac{2l+1}{(l-1)^2} (\bar{a} \cdot x)_i, \quad A_{i,1} = \frac{2l-1}{(l-1)^2} (\bar{a} \cdot x)_{i,1}. \]

All the possible (up to equivalence) nontrivial solutions of (13.20) (and the corresponding expressions for the matrices $a_p$, $b_p$, $c_p$, $A_p$, $D_i$ and $B_i$) are present in Table 13.1 where $\alpha$, $\alpha_1$ and $\alpha_2$ are arbitrary parameters.
### Table 13.1

<table>
<thead>
<tr>
<th>series of solutions</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{a}_i$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>0</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\tilde{a}_{i,1}$</td>
<td>0</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\tilde{b}_i$</td>
<td>$\begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
<td>$\sqrt{k_i}(1 \ 0)$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$\sqrt{k_{i,1}} \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\tilde{c}_i$</td>
<td>$i\sqrt{k_i}(1 \ 0)$</td>
<td>$\begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
<td>$i\sqrt{k_i} \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\sqrt{k_{i,1}} \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$x_i$</td>
<td>$\begin{pmatrix} 0 &amp; \alpha \ \alpha &amp; 1 \end{pmatrix}$</td>
<td>1</td>
<td>$\begin{pmatrix} 0 &amp; \alpha_1 &amp; \alpha_2 \ \alpha_1 &amp; \alpha_2 &amp; 1 \ \alpha_2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} \alpha_2 &amp; l+1 \ l+1 &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
Here $q_i^\pm = (2l+1)(l\pm 1)^2$, $g_i = (2l+1)/l(l+1)$, $k_i$ is given in (13.20).
We see that only four of eighteen classes of representations of the algebra $AE(3)$ enumerated in Subsection 12.6 correspond to nontrivial Galilei-invariant wave equations. These classes are $D(3,1,2), D(2,1,1), D(2,1,3)$ and $D(1,1,2)$.

The results given above are summarized in the following assertion.

**THEOREM 3.3.** Let the equation (13.1) is $S$-invariant under the Galilei algebra, admits a Lagrangian formulation and describes a particle of spin $s$ and mass $m$, and let the representation of the inhomogeneous Galilei group realized on the set of solutions of this equation belong to the class $D_2$. Then the explicit form of these matrices up to equivalence coincides with the given in (13.11), (13.12) and Table 13.1.!

So we obtain four classes of Galilei-invariant wave equations corresponding to the matrices enumerated in Table 13.1. Consider these classes successively.

Equations of class $R_1$ have $6l+1$ components and describe a particle of spin $s=l$, mass $m$ and internal energy $\frac{3}{2}m$. These equations are equivalent to the Hagen-Hurley [215, 223] equations, and in the case $s=1/2$ reduce to the Levi-Leblond [276] equation.

Equations of class $R_2$ have $6l-1$ components and describe a Galilean particle of spin $s=l-1$, mass $m$ and internal energy $\frac{3}{2}s'm/2$. We will see further on these equations predict the value of the constant of the dipole coupling which differs from the value prophesied by the LHH equations.

Equations of classes $R_3$ and $R_4$ are the most interesting. They can be interpreted as quantum mechanical equations of a nonrelativistic particle of spin $s=l$ (and $s=l-1$), mass $m$ and internal energy $(s+1)(s^2-2s)m/2$ (and $s(s^2-2s)m/2$). It will be shown below these equations can serve as a starting point in the Galilei-invariant description of the spin-orbit coupling.

We present the simplest of the found equations writing them componentwise.

Class $R_2$, $s=l-1=0$,

$\psi=column(\psi_0, \psi_1, \psi_2, \psi_3)$

$p_0\psi_0 - p\psi = 0$,

$2m\psi - p\psi_0 = 0$; \hspace{1cm} (13.21)

Class $R_1$, $s=l=1/2$,

$\psi=column(\phi_0, \phi_1, \chi_1, \chi_2)$,

$2p_0\phi + \frac{3}{2}(i\sigma \cdot p + \frac{3}{2}em)\chi = 0$,

$(\frac{3}{2}em - i\sigma \cdot p)\phi + \frac{3}{2}m\chi = 0$; \hspace{1cm} (13.22)

Class $R_3$, $s=l=1$,
Chapter 3. Representations of the Galilei Algebra ...

\[ \psi = \text{column}(\phi, \chi, \Phi, \Phi_0), \]
\[ \frac{1}{2} p_0 \psi - p_\phi \Phi_0 + 2e_\omega \nu \Phi + 2e_\omega \nu \Phi_0 = 0, \]
\[ \frac{1}{2} p_0 \psi - p_\phi \Phi_0 + 2e_\omega \nu \Phi + 2e_\omega \nu \Phi_0 = 0, \]
\[ i \sqrt{3} p_\chi - 2e_\omega m \phi_0 = 0, \]
\[ p_\times \psi - 2e_\omega m \phi - 2m \chi = 0, \]
where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices.

13.6. Equations for Representations with Arbitrary Nilpotency Indices

To conclude this section, we discuss an interesting connection between Galilei-invariant wave equations and equations invariant under the generalized Poincaré group \( P(1,4) \) (refer to Chapter 8).

Consider the generalized Bhabha equation [340]
\[ (\beta^k p_k - e) \psi(x) = 0, \quad k = 0, 1, 2, 3, 4 \]  \hspace{2cm} (13.24)
where \( \mu_k = S_{k4} \) are matrices realizing a representation of the algebra \( AO(1,5) \) together with \( S_{44} \) and \( S_{k4} = i[S_{k4}, S_{44}] \).

It is possible to show that the equation (13.24) is invariant under the Galilei group as well as under the group \( P(1,4) \), refer to Subsection 26. To make Galilei invariance of (13.24) obvious, we use the change of variables
\[ x_0 = \frac{1}{2} (2 \tilde{x}_0 - \tilde{x}_4), \quad x_4 = \frac{1}{2} (2 \tilde{x}_0 - \tilde{x}_4), \]  \hspace{2cm} (13.25)
so that
\[ \frac{\partial}{\partial x_0} = \frac{1}{2} \frac{\partial}{\partial \tilde{x}_0} + \frac{\partial}{\partial \tilde{x}_4}, \quad \frac{\partial}{\partial x_4} = \frac{1}{2} \frac{\partial}{\partial \tilde{x}_0} - \frac{\partial}{\partial \tilde{x}_4}. \]  \hspace{2cm} (13.26)
As a result we come to the equation
\[ \mathcal{L} \psi = (\overline{\beta}_0 p_0 - \overline{\beta}_4 p_4 - \overline{\beta}_a p_a - e) \psi(\tilde{x}_0, \tilde{x}_4, x) = 0 \]  \hspace{2cm} (13.27)
where
\[ \overline{\beta}_0 = \frac{i}{2} S_{04} S_{44}, \quad \overline{\beta}_0 = S_{04} S_{44}, \quad \overline{\beta}_0 = S_{04}. \]  \hspace{2cm} (13.28a)
\[ \overline{\beta}_a = \frac{1}{2} S_{04} S_{44}, \quad \overline{\beta}_4 = S_{04} S_{44}, \quad \overline{\beta}_0 = S_{04}. \]  \hspace{2cm} (13.28b)
Galilei invariance of (13.27) can be proved by direct verification using the following realization of the algebra $AG(1,3)$

$$
P \vec{0} = \vec{p}_0, \quad P_a = \vec{p}_a, \quad M = \vec{p}_4, \quad J_a = \varepsilon_{abc}(x_b p_c - \frac{1}{2} S_{bc}), \quad G_a = \vec{x}_a p_a - x_a M + \eta_a$$

(13.29)

where

$$\eta_a = \frac{1}{2}(S_{0a} - S_{4a}).$$

Indeed, these generators commute with the operator $L$ of (13.27).

Imposing the Galilei-invariant additional condition on $\psi$, we reduce (13.27) to a Galilei-invariant equation of the kind (13.1) where

$$\beta_0 = \tilde{\beta}_0, \quad \beta_4 = \tilde{\beta}_4 + \lambda^{-1} I, \quad m = \lambda \alpha,$$

(13.31)

$I$ is the unit matrix. The corresponding generators of the Galilei group have the covariant form (12.18) where

$$M = \lambda \alpha, \quad \eta_a = \frac{1}{2}(S_{0a} - S_{4a}), \quad S_a = -\frac{1}{2} \varepsilon_{abc} S_{bc}.$$  

(13.32)

Finite-dimension IRs of the algebra $AO(1,5)$ (which is isomorphic to the algebra $AO(6)$) are labelled by triplets $(n_1, n_2, n_3)$ where $n_a$ are integers or half integers [197]. If matrices $S_{ij}$ form the representation $D(1/2,1/2,1/2)$ then the equations (13.1), (13.28b) are equivalent to the Levi-Leblond equation [276] for a particle of spin 1/2. The representation $D(1,1,1)$ corresponds to the equations (13.23) for a particle of spin 1 (besides $\alpha_1 = -\alpha_4, \alpha_2 = \alpha_3$). The corresponding matrices in (13.1) are expressed via the 10×10 KDP matrices (marked by "\^" in the following formula)

$$\beta_0 = \frac{1}{2}(\tilde{\beta}_0 + \tilde{\beta}_4), \quad \beta_4 = \tilde{\beta}_0 - \tilde{\beta}_4 + \lambda^{-1} I, \quad \beta_a = \tilde{\beta}_a.$$  

(13.33)

In general the considered equations describe a multiplet of particles of spins $s_1, s_2, \ldots$ where the numbers $s_i$ characterize the IR of the algebra $AO(3)$ consisted in the given representation of the algebra $AO(1,5)$ realized by the matrices $S_{ij}$.

It is possible to demonstrate the matrices (13.32) satisfy the conditions

$$(\eta_a S_{aj})^{2s} \neq 0, \quad (\eta_a S_{aj})^{2s-1} = 0, \quad s = \max(s_i).$$

Thus the equations considered here correspond to such representations of the algebra $AE(3)$ which are characterized by the nilpotency index $n = 2s + 1$ of the Casimir operator $C_1$ of (12.27).
We note that the equations (13.1), (13.31), (13.28b) are defined also for the case when the corresponding matrices $S_{ij}$ realize an infinite dimension Hermitian representation of the algebra $\text{AO}(1,5)$. So we come naturally to infinite component wave equations which are invariant under the Galilei group [151]. Poincaré-invariant infinite component equations are well studied in contrast to the Galilei-invariant ones (see, however, [222]).

We will return to the discussion about connections between Galilei- and $P(1,4)$-invariant wave equations in Chapter 5.

14. GALILEI-INvariant EQUATIONS OF THE HAMILTONIAN TYPE

14.1. Uniqueness of the Schrödinger Equation

Consider an alternative possibility in Galilei-invariant description of arbitrary spin particles which expects to use equations of the Hamiltonian type. These equations can be more convenient then first order systems of the kind (13.1) since they include explicitly the evolution operator (Hamiltonian). Besides the distinguishing of the time variable is in a good accordance with Galilei transformations (10.39) in contrast to the case of Poincaré-invariant equations.

First let us "deduce" the Schrödinger equation (11.1) and demonstrate that this is the only Galilei invariant evolution equation for a scalar wave function.

Let us search for linear partial differential equations for a complex valued function $\psi(t,x)$

$$L\psi(t,x)=0$$

where $L$ is an unknown differential operator. We assume $L=p_0-H$ where $H$ includes space derivatives only.

The Galilei-invariance condition means that $L$ commutes with any basis element of the Galilei algebra. Without loss of generality these basis elements $P_\mu$, $J$, $G$ can be chosen in the form (11.5) (this is the general form of covariant representation of the algebra $\text{AG}(1,3)$, realized in the space of scalar functions, refer to Section 12). Calculating commutators of $L$ with the corresponding generators of (11.5) we find immediately that

$$H=p^2/2m+\varepsilon_0, \varepsilon_0=\text{const},$$

165
Symmetries of Equations of Quantum Mechanics

i.e., we come to the equation which coincides with the Schrödinger equation up to nonessential constant term $\varepsilon_0$ in the Hamiltonian.

Thus the Schrödinger equation is unique, i.e., this is the only evolution equation which involves scalar wave function and satisfies the requirement of Galilei invariance.

We now consider the problem of deducing of Schrödinger-like equations for Galilean particles of arbitrary spin. We will search for a motion equation of a particle of spin $s$ in the form (11.1) where $\psi$ is a $2(2s+1)$-component wave function, $H$ is a differential operator (particle Hamiltonian) we need to find. We will demonstrated that this number of components of wave function is minimal for a profound equation of motion if $s$ is nonzero.

By definition the equation (11.1) is invariant under the Galilei algebra if it admits 11 SOs satisfying relations (11.6). We restrict ourselves to the case when these operators have the covariant form (12.18) corresponding to local transformations of a wave function by passing to a new reference frame.

It is not difficult to make sure the invariance condition (1.5) reduces to the following equations for $H$

\[ [H, P_a] = [H, J_a] = 0, \quad [H, G_a] = -iP_a \]

(14.1)

where $P_a$, $J_a$ and $G_a$ are the operators (12.18).

We require the eigenvalues of the Casimir operators (11.14) on a set of solutions of the equation (11.1) be equal to

\[ c_1 = m, \quad c_2 = \varepsilon_0, \quad c_3 = m^2 s(s+1), \]

(14.2)

where $s$ is an integer or half integer, $m$ and $\varepsilon_0$ are arbitrary fixed numbers. This makes it possible to interpret (11.1) as a motion equation of a particle of spin $s$. The corresponding matrices $S_a$, $\eta_a$ and $M$ of (12.18) without loss of generality can be chosen in the form:

\[ S_a = \begin{pmatrix} s_a & 0 \\ 0 & s_a \end{pmatrix}, \quad \eta_a = k(\sigma_1 + i\sigma_2)S_a, \quad M = \sigma_0 m \]

(14.3)

where $s_a$ matrices of dimension $(2s+1)\times(2s+1)$ realizing the IR $D(s)$ of the algebra $AO(3)$, $\sigma_\mu$ are $2(2s+1)$-row Pauli matrices (5.30), $k$ is an arbitrary complex number.

If $k$ is not equal to zero then the matrices (14.3) can be transformed to the equivalent representation with $k=1$. We find it is more convenient to admit arbitrary values of $k$, see footnote after Theorem 14.1, Subsection 14.2.

We impose the additional requirement the equation (11.1) be invariant under the transformation of simultaneous reflection of time and space variables. In analogy with (11.37) we represent this transformation in the form

\[ \psi(t, x) \rightarrow \Theta \psi(t, x) \rightarrow \psi'(\text{-}t, \text{-}x) \]

(14.4)
Chapter 3. Representations of the Galilei Algebra ...

where \( r \) is a matrix needed to determine. This transformation has to satisfy the following relations together with generators of the Galilei group:

\[
P_\mu \Theta = \Theta P_\mu, \quad J_a \Theta = \Theta J_a, \quad G_\mu \Theta = \Theta G_\mu
\]

which reduce to the following equations for \( r \)

\[
r S_a = -S_a r, \quad r \eta_a = -\eta_a r.
\]

It follows from (14.3), (14.6) that the parameter \( k \) can be either real or imaginary according to the cases the matrix \( r \) commute or anticommute with \( \sigma_1 + i\sigma_2 \).

The corresponding matrix \( r \) without loss of generality can be chosen in the form

\[
r = \exp(i\varphi) \Delta, \quad \text{if} \quad k \neq k
\]

\[
r = \exp(i\varphi) \sigma_3 \Delta, \quad \text{if} \quad k = k
\]

where \( \Delta \) is the matrix (7.10), \( \varphi \) is a real number.

The condition of invariance of the equation (11.1) under the transformation of complete reflection can be written in the form

\[
[H, \Theta] = 0
\]

where \( \Theta \) is the operator (14.4), (14.7).

So we state the problem of finding all nonequivalent Hamiltonians \( H \) satisfying the commutation relations (14.1), (14.8). Such Hamiltonians correspond to Galilei-invariant Schrödinger equations for particles of arbitrary spins.

14.2. The Explicit Form of Hamiltonians of Arbitrary Spin Particles

Here we present the general solution of the problem formulated above.

**THEOREM 3.4.** All the possible (up to equivalence) Hamiltonians \( H \) satisfying relations (14.1), (14.8) (together with the Galilei group generators and the operator (14.4)) are given by the following formulae

\[
H = H^{(1)} + \sigma_3 a m \cdot \sigma_1 2iakS_a p_a + \frac{1}{2m} C_{ab} p_a p_b
\]

\[
H = H^{(2)} + \sigma_3 b m \cdot (\sigma_1 + i\sigma_2) 2bkS_a p_a + \frac{p^2}{2m}
\]

where

\[
C_{ab} = \delta_{ab} + 2ak (\sigma_1 + i\sigma_2) (S_a \cdot S_b + S_b \cdot S_a).
\]

\( a, b, k \) are parameters satisfying the conditions
PROOF. It is convenient to solve the equations (14.1) in the representation which is connected with (12.18) by the transformation

\[ P'_\mu \rightarrow U P'_{\mu} U^{-1}, \quad J' \rightarrow U J U^{-1}, \quad G' \rightarrow U G U^{-1} \quad (14.12) \]

where

\[ U = \exp \left( \frac{i}{m} \eta p_{a} \right) \quad (14.13) \]

It is not difficult to make sure that

\[ P'_{a} \rightarrow P_{a}, \quad P'_{0} \rightarrow H', \quad J'_{a} \rightarrow J_{a}, \quad G'_{a} \rightarrow G_{a} - m x_{a}, \quad (14.14) \]

i.e., \( G_{a}' \) has a very simple form. Requiring the operators (14.14) satisfy (14.1) we come to the following equations

\[ [H', J_{a}] = i \frac{1}{m} p_{a}, \quad [H', \eta p_{a}] = [H', S_{a}] = 0 \quad (14.15) \]

whose general solutions have the form

\[ H' = \frac{p_{a}^2}{2m} B_{m}, \quad B = \sigma_{\mu} a_{\mu}, \quad (14.16) \]

where \( a_{\mu} \) are arbitrary complex numbers, moreover without loss of generality we can set \( a_{0} = 0 \).

Using the transformation \( B \rightarrow WB^{-1} \) where \( W \) is an invertible numeric matrix satisfying the condition

\[ W^{-1} \eta_{a} W = \lambda \eta_{a}, \quad \lambda \in \mathbb{C}, \quad \lambda \neq 0 \quad (14.17) \]

we can reduce \( B \) to one of the following forms

\[ B = \sigma_{\mu} a_{\mu}, \quad a = \sqrt{a_{1}^{2} + a_{2}^{2} + a_{3}^{2}}, \quad (14.18a) \]

The condition (14.17) can be deduced requiring the corresponding Galilei group generators in starting (imprimed) representation be connected by the equivalence transformation \( H \rightarrow WHW^{-1} = H'' \), \( P_{a} \rightarrow WP_{a} W^{-1} = P_{a} \), \( J_{a} \rightarrow WJ_{a} W^{-1} = J_{a} \), \( G_{a} \rightarrow WG_{a} W^{-1} = G_{a}'' \) where \( G_{a}'' \) differs from \( G_{a} \) only by the value of the parameter \( k \) (refer to (12.18), (14.3)). Fixing \( k \) in (14.3) (e.g., setting \( k = 1 \)) we lose the possibility to simplify \( H' \) with the help of a matrix transformation, so it is convenient to consider the representations of the Galilei algebra given by (12.18), (14.3) where \( k \) is an arbitrary parameter.
\[ B = \sigma_1 \pm i \sigma_2. \] (14.18b)

The matrices (14.18) are nonequivalent in the sense that there is not any matrix \( W \) satisfying (14.17) and transforming them one into another. Besides the solutions (14.18b) have to be neglected since they correspond to non-Hermitian representations of the Galilei algebra with a nilpotent Casimir operator \( C_3 = 2mP_0 - P^2 \).

So the general form of \( H' \) is given in (14.16) where \( B \) is one of the matrices of (14.18a). With the help of the transformation inverse to (14.12) we find all the nonequivalent primed Hamiltonians in the form (14.9) and obtain the conditions (14.11) from relations (14.4)-(14.6).

Thus we obtain Galilei-invariant equations for particles of arbitrary spin in the form (11.1), (14.9). Galilei invariance of these equations is evident from the deduction. Nevertheless we can easily verify it directly. Using the identity

\[ \exp(i\eta \psi) = 1 - i\eta \psi \] (14.19)

we find that the operators (14.9) satisfy the relations

\[ \exp[i\varphi(x)] D(\theta, v) \left( \frac{\partial}{\partial t} - H(p) \right) D^{-1}(\theta, v) \exp[-i\varphi(x)] = i \frac{\partial}{\partial t} - H(p'), \] (14.20)

where \( H(p') \) are the operators obtained from (14.9) by the change \( p \rightarrow p' = -i \partial / \partial x' \), \( x_0', x', \varphi(x) \) and \( D(\theta, v) \) are given in (11.28), (11.29), (12.22).

It follows from (14.20) the transformed function \( \psi'(x') \) of (12.21) satisfies the same equation as \( \psi(x) \):

\[ i \frac{\partial}{\partial x_0'} \psi'(x') = H(p') \psi'(x'). \] (14.21)

In other words the equation (11.1) with any of the Hamiltonians (14.9) is invariant under Galilei transformations (11.29) if \( \psi(x) \) is transformed according to (12.21).

We can make sure that the Casimir operators (11.14) corresponding to the Hamiltonians (14.9) have the eigenvalues (14.2) where \( \varepsilon = \pm am \). So we conclude that equations (11.1), (14.9) can be interpreted as motion equations of a free particle of mass \( m \), spin \( s \) and internal energy \( \pm am \).

In the case \( s=1/2 \), \( a=ik=1 \) the equations (11.1), (14.9a) reduce to the following form

\[ (\gamma_\mu p^\mu - m)\psi = (1 - \gamma_0 \gamma_5) \frac{p^2/2m}{2m} \psi. \]

This equation has the form of the usual Dirac equation including the additional term \( (p^2/2m)\psi \) in the l.h.s. This term violates Poincaré-invariance of the Dirac equation but generates invariance under the Galilei group.

We note that the simplest Poincaré-invariant equation, i.e., the KGF
Symmetries of Equations of Quantum Mechanics

equation (1.1) also can be "modified" in such a way. Adding to the l.h.s. of (1.1) the term \(-p_0p^2/2m+p^4/4m^2\)\(\psi\) we obtain an equation which is not Poincaré-invariant but is invariant under Galilei transformations.

14.3. Lagrangian Formulation

So we have find Galilei-invariant equations in the Schrödinger form for a particle of arbitrary spin. As like as the Dirac Hamiltonian the evolution operators (14.9) depend on the spin matrices but include differential operators of higher (second) order.

The nonrelativistic Hamiltonians (14.9) depend on two parameters \(a\) and \(k\) which cannot be fixed by the requirement of Galilei and \(PT\)-invariance. These operators are Hermitian with respect to the scalar product (7.35) where \(M\) is a positive defined metric operator

\[
\hat{M}^{-1}U U^\dagger = \frac{1}{m} \left[ i k \, \sigma_1 - (k \cdot \sigma) \sigma_2 \right] S_a p_a + \frac{2k \cdot k}{m} (1 - \sigma_3) (S_a p_a)^2.
\]

Besides the operators (14.9a), (12.18) (with \(S_a, \eta_a\) given in (14.3)) are Hermitian in the following indefinite metric

\[
(\psi_1, \psi_2) = \int d^3x \psi_1^\dagger \xi \psi_2,
\]

where

\[
\xi = \begin{cases} \sigma_1, & \text{if } a^*=-a, \, k^*=-k \\ \sigma_2, & \text{if } a^*=-a, \, k^*=-k. \end{cases}
\]

\(\xi\) is a Hermitizing matrix for the Hamiltonian (14.9a) and the generators (12.18). As to the Hamiltonian (14.9b) the corresponding Hermitizing matrix does not exist.

The equations (11.1), (14.9a) can be deduced using the variational principle if we start from the Lagrangian

\[
L_0(x) = \frac{i}{2} \left( \frac{\partial \psi}{\partial t} - \frac{\partial \tilde{\psi}}{\partial t} \right) - am\tilde{\psi} \sigma_1 \psi - \frac{1}{2m} C_{ab} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_b} + \sigma k \left( \tilde{\psi} \sigma S_a \frac{\partial \psi}{\partial x_a} - \frac{\partial \tilde{\psi}}{\partial x_a} \sigma S_a \psi \right),
\]

where \(\tilde{\psi} = \psi^\dagger \xi\). It is not difficult to make sure that the Euler-Lagrange equations (8.24), (14.24) reduce to (11.1), (14.9a). Changing in (8.24) \(\tilde{\psi} \to \psi\) we obtain the equation for \(\tilde{\psi}\) which is equivalent to (11.1), (14.9a).

Thus equations (11.1), (14.9a) admit the Lagrangian formulation. This circumstance makes it possible to use the canonical Lagrangian formalism to
Chapter 3. Representations of the Galilei Algebra...

construct the operators of the main physical values (momenta, energy, angular momenta etc.) and to generalize these equations to the case of a particle interacting with an external fields using the minimal coupling principle.

We note that the equations (11.1) with the Hamiltonian (14.9b) cannot be represented as Euler-Lagrange equations, because the Hamiltonian (14.9b) is non-Hermitian and the corresponding Hermitizing matrix does not exist. Therefore we will not consider this Hamiltonian restricting ourselves to the equations (11.1), (14.9a).

Let us summarize. We have obtained the motion equations for a particle of arbitrary spin \( s \) in the Hamiltonian form (11.1) where \( H \) is the second order differential operator (14.9a). These equations are invariant under the Galilei and total reflection transformations and admit a Lagrangian formulation.

We present here also Galilei invariant equations for a particle with variable spin. Such equations can be written in the form (11.1) where

\[
H = \sigma_a m + \sigma_2 iak S_{a_4} P_a + \frac{1}{2m} \tilde{C}_{ab} p_a p_b.
\]

Here \( S_{a_4} \) are matrices belonging to the representation \( D(1/2 \ 1/2) \) of the algebra \( AO(4) \), \( \sigma_a \) are the Pauli matrices of dimension \( 2(s+1)^2 \times 2(s+1)^2 \), commuting with \( S_{a_4} \).

The operator (14.25) is a Galilei-invariant analog of the relativistic Hamiltonian \( H_1 \) of (10.18). The corresponding equation (11.1) is invariant under the \( P \)-, \( T \)- and \( C \)-transformations and describes particles with variable spin whose values are \( s, s-1, ..., 0 \).

In papers [162,163] other types of Galilei-invariant equations where considered. They are second order equations with a singular matrix by \( p_0 \) and Schrödinger-type equations corresponding to non-covariant representations of the algebra \( AG(1,3) \). We do not analyze these equations here but note that they can be successfully used to describe a particle of arbitrary spin in an external field.

15. GALILEAN PARTICLE OF ARBITRARY SPIN IN AN EXTERNAL ELECTROMAGNETIC FIELD

15.1. Introduction of Minimal Interaction into First-Order Equations

A motion equation of a free particle is of interest for physics only as a first step in describing interacting particles. Moreover, the critical point is a possibility
to use such an equation to describe an interaction of a charged particle with the electromagnetic field.

We show further on that Galilei-invariant equations can be successfully used to solve the mentioned problem inasmuch as they take into account all the physical effects predicted by the relativistic Dirac equation in the approximation $1/m^2$. Using the generalized Foldy-Wouthuysen reduction we calculate the constants of the dipole, quadruple and spin-orbit couplings predicted by these equations. The constant of the dipole interaction generated by first-order Galilei-invariant equations is equal to $1/s$ in accordance with the Belifante conjecture [30].

We start with the Galilei-invariant equations of first order derived in Section 13. Since these equations admit the Lagrangian formulation, this is naturally to introduce an interaction with the electromagnetic field in frames of the minimal interaction principle, i.e., to make the change $\partial / \partial x'^\mu \to \partial / \partial x'^\mu - ieA'_\mu$ in the corresponding Lagrangian. As a result we come to the equations

$$L(\pi)\psi = (\beta^\mu \pi^\mu_\beta \beta_{\beta\mu}) \psi = 0, \quad \pi^\mu_\beta - eA'_\mu.$$  \hspace{1cm} (15.1)

To substantiate the minimal interaction principle in application to Galilei-invariant wave equations we will show that the change $p_\mu \to \pi_\mu$ does not violate the Galilei and gauge symmetry of the starting equations and leads to reasonable results by solving the concrete physical problems. Moreover we will not use a concrete realization of $\beta$-matrices (at least until a certain moment) so our results will be valid for arbitrary wave equations invariant under the Galilei group.

The equation (15.1) is manifestly invariant under the gauge transformations

$$\psi(x) \to \exp[i\phi(x)]\psi(x), \quad A_\mu \to A_\mu + \frac{\partial \phi(x)}{\partial x^\mu},$$  \hspace{1cm} (15.2)

where $\phi$ is an arbitrary differentiable function. To establish invariance of this equation under the Galilei transformations we assume that the vector-potential is transformed simultaneously according to Galilean law [273]

$$A_0 \to A'_0 - \mathbf{v} \cdot A, \quad A_\mu \to A'_\mu - R_{\alpha\beta} A^\alpha_\beta.$$  \hspace{1cm} (15.3)

Using relations (13.9) and applying the Campbell-Hausdorf formula (13.16) we make sure the operator $L(\pi)$ satisfies the condition

$$exp[-i\phi(x)]D(\theta, \mathbf{v})L(\pi)D^{-1}(\theta, \mathbf{v})exp[-i\phi(x)] = L(\pi')$$  \hspace{1cm} (15.4)

where $L(\pi')$ is the operator obtained from $L(\pi)$ of (15.1) by the change $\pi_\mu \to \pi'_\mu = i\partial / \partial x'^\mu - eA'_\mu$, $D(\theta, \mathbf{v}) = [D^{-1}(\theta, \mathbf{v})] \dagger$; $x'_\mu$, $\phi(x)$, $D(\theta, \mathbf{v})$, $A'_\mu$ are defined in (11.28), (11.29), (12.22), (15.3).

It follows from (15.4) the transformed wave function $\psi'(x')$ of (12.21) satisfies the same equation as $\psi(x)$:
so the equation (15.1) is invariant under Galilei transformations.

### 15.2. Magnetic Moment of Galilei Particle of Arbitrary Spin

Let us show the Galilei-invariant equations (15.1) are good models of charged particles interacting with the electromagnetic field and describe the dipole, quadruple and spin-orbit interactions. To analyze these equations it is convenient to pass to the representation where the operator \( L(\pi) \) reduces to a series in powers \( 1/m \). For this purpose we make the transformation

\[
\psi \rightarrow V^{-1}\psi, \quad L(\pi) \rightarrow V'L(\pi)V \tag{15.6}
\]

where

\[
V = \left(\begin{array}{c}
\beta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3
\end{array}\right)
\]

is the operator obtained from (13.15b) by the change \( p \rightarrow \pi \). Using the Campbell-Hausdorff formula (13.16) and the commutation relations (13.9) for \( \beta_a \) and \( \eta_a \) and restricting ourselves to the case when the nilpotency index of the matrices \( \eta_a \) is less then 4 we come to the following equation

\[
L'(\pi)\psi'(x')=0 \tag{15.8}
\]

where

\[
L'(\pi) = \beta_0 \left( \frac{\pi_0 - \pi^2}{2m} \right) + \beta_0 m \cdot \frac{e}{m} \left( \beta_0 \eta \cdot E - \frac{1}{2} \beta_0 \eta \cdot H + \frac{e}{2m^2} \beta_0 \eta \cdot (\pi \times H - H \times \pi) \right) \tag{15.9}
\]

If the nilpotency index \( n_a \) of \( \eta_a \) is larger then 3 the corresponding operators \( L'(\pi) \) includes the additional terms

\[
\frac{e}{2m^2} \beta_0 \eta_a \eta_b \eta_c \frac{\partial E_a}{\partial x_b} \sum_{i\neq a} \frac{1}{m} B_i
\]

where \( B_i \) depend linearly on the electromagnetic field strengths and their derivatives. These terms can be neglected proceeding from reasonable suppositions about the intensity of an external field.

Thus formula (15.9) being exact for \( n_a \leq 3 \) can be treated as an approximate one for \( n_a > 3 \).

The operator (15.9) includes the terms corresponding to interaction of a point charged particle with an external electromagnetic field (\( \sim \pi_0, \pi^2/2m \)) and the additional terms proportional to the vectors of the electromagnetic field strengths. Here we consider in detail the term \( e\beta_0 \eta \cdot H \) corresponding to the interaction of a
Symmetries of Equations of Quantum Mechanics

particle spin with the magnetic field (i.e., the Pauli interaction).
Using the representation (12.23), (13.12) for $\beta$ and $\eta$ and bearing in mind relations (12.26) we obtain the following expression for the matrix $\beta \times \eta$ in the basis $|\lambda; l, m>$

$$
\beta \times \eta |\lambda; l, m> = E^{l,1}_l \cdot S^{l,1}_l |\lambda'; m' > + F^{l,1}_l \cdot K^{l,1}_l |\lambda'; l-1, m' > +

+ (F^{l+1})^{l+1}_l \cdot K^{l+1}_l |\lambda'; l+1, m' >

$$

(15.10)

where

$$
E^{l,1}_l \cdot A^{l,1}_l = \frac{(2l+1)(a \cdot x)_l}{l(l+1)},
F^{l,1}_l \cdot A^{l,1}_l = \frac{1}{l} (a^{l-1} x^{l-1} c^{l-1} - b^{l-1} x^{l-1}), l \neq 0,

$$

(15.11)

da', b', c', (a \cdot x)$, and $A_l$ are matrices satisfying the relations (12.31), (13.14).

Consider the equation (15.8) for the case $E=0$. Using the representation (13.11), (15.10) for $\beta_0, \beta_4$ and $\beta \times \eta$ we can write the system (15.8) as a chain of equations for the functions $\psi_i$(eigenfunctions of the matrix $S^2$)

$$
A^{l,1}_l \left( \begin{array}{c} \pi_0 - \pi^2 + \frac{e}{2lm} S^{l,1}_l \cdot H \\
(2l+1)x^l m \\
(2l+1)x^{l+1} m \\
(2l+1)x^{l+2} m
\end{array} \right) \psi_i + \frac{1}{2ml} F^{l,1}_l K^{l,1}_l H \psi_{i,l+1} = 0,

\frac{1}{2ml} F^{l,1}_l (K^{l,1}_l H) \psi_{i,l} + \left[ A^{l,1}_l \left( \begin{array}{c} \pi_0 - \frac{e}{2lm} \\
(2l+1)x^l m \\
(2l+1)x^{l+1} m \\
(2l+1)x^{l+2} m
\end{array} \right) +

+ \frac{1}{2ml} F^{l,1}_l K^{l,1}_l H \psi_{i,l+1} = 0,

\right.

\left.
\vdots
\right)

$$

(15.12)

We used the identities

$$
E^{l,1}_l = \frac{1}{l+1} A^{l,1}_l, \quad E^{l,1}_l = \frac{1}{l+1} A^{l,1}_l
$$

which follows from (15.11), (13.14a), (13.14b). Here $E^l$ and $F^l$ are the matrices whose elements are defined in (15.11).

Until now we have not made any supposition about a concrete realization of $\beta$-matrices and use only relations (13.9) following from the requirement of Galilei invariance. Further we assume the equation (15.1) describes a particle of a fixed spin $s$ i. e. the conditions (13.19) are fulfilled. Besides we assume that $s=l_i$ and call the corresponding equation (15.1) "equation for the highest spin".

174
If the conditions (13.19) are satisfied and \( s=l_1 \) then the operator in square brackets in the last line of (15.12) is invertible. So we can express \( \psi_{l_1} \) via \( \psi_{l_1+1} \):

\[
\psi_{l_1} = -V^{-1} \frac{eF_{l_1+1}}{2m(l_1+1)} (K_{l_1+1} \cdot H) \psi_{l_1+1}; \quad V = A \begin{pmatrix} \pi_0 - \frac{\pi^2}{2m} - \frac{e}{2m(l_0+1)} \end{pmatrix}^{-1} \chi \psi_{l_1+1}.
\]

Since the determinant of the matrix \( V \) does not depend on \( H \) then \( \det(V^{-1}) \) also has this property. Roughly speaking it means that \( \psi_{l_1} \sim eH \psi_{l_1+1} \).

In analogous way supposing the magnetic field strength is small enough in order to neglect terms quadratic in respect with \( eH \) we can express \( \psi_{l_1} \) via \( \psi_{l_2} \), \( \psi_{l_2+1} \) via \( \psi_{l_3} \) etc. As a result we come to the following representation for the first of the equations (15.12):

\[
A \begin{pmatrix} \pi_0 - \frac{\pi^2}{2m} - \frac{e}{2sm} S \cdot H \end{pmatrix}^{-1} (2s+1) \chi \psi_{l_1} = 0.
\]

where \( s=l_1 \), \( S=S' \), \( F(H) \) is a term of order \( (eH)^2 \) which can be neglected.

Since \( A \) and \( \chi \) by definition satisfy the condition (13.19) it follows from (15.13) that

\[
\begin{pmatrix} \pi_0 - \frac{\pi^2}{2m} - \frac{e}{2sm} S \cdot H + e_0 \end{pmatrix} \psi_{l_1} = 0.
\]

Thus introducing the minimal interaction into first-order Galilei-invariant wave equations we come to the Schrödinger-Pauli equation (15.14) for a \((2s+1)\)-component wave function. This equation includes the term describing the Pauli interaction of spin with the magnetic field. The coefficient of \( eSH/2m \) is called the gyromagnetic ratio, in our case it is equal to \( g=1/s \).

The Belinfante conjecture [30] \( g=1/s \) has been supported repeatedly using relativistic wave equations [196, 224]. We see that Galilei-invariant wave equations for a particle of highest spin also predict the correct value of the gyromagnetic ratio moreover this result does not depend on a choice of the concrete equation but has a universal nature.

### 15.3. Interaction with the Electric Field

We see that the Pauli interaction can be successfully interpreted in frames of Galilei-invariant theory since any equation of the kind (15.1) can serve as a base for its description. But an interaction of a spin with the electric field (e.g., the spin-orbit coupling) is a finer effect which can be described only by special classes of Galilei-invariant equations.

Starting with the equations of the form (15.9) it is possible to find the general restrictions imposed on the matrices \( \eta_a \) by requiring these equations be a
suitable model of a real physical situation. A simple analysis shows that if the nilpotency index of these matrices is less than 3 then the operator \( L'(\pi) \) does not include any term depending on the electric field strength. Indeed, in this case \( \eta, \eta = 0 \) and so according to (13.9) \( \beta, \eta = 0 \). The corresponding operator (15.9) reduces to the form

\[
L'(\pi) = \beta_0 + \frac{\pi^2}{2m} \beta, m - \frac{e}{2m} \beta \times \eta \cdot H. \tag{15.15}
\]

It can be seen easily that the equations (15.8), (15.5) cannot describe an interaction of a spin with the electric field. In the case \( H = 0 \), \( E \neq 0 \) the operator (15.15) does not depend on \( E \) and \( S \), being commutative with \( S \).

We formulate this result in the form of the following assertion.

**PROPOSITION 15.1.** The necessary condition a Galilei-invariant equation of the form (15.1) describes interaction of a particle spin with the electric field is

\[
C^2 = (S \eta)^2 \neq 0 \tag{15.16}
\]

where \( S \) and \( \eta \) are generators of the homogeneous Galilei group realized on a set of solutions of this equation, \( C \) is the Casimir operator (12.27).

As will be shown in the following the condition (15.16) is also sufficient. Thus to answer the question whether the given Galilei-invariant wave equation describes an interaction of a spin with the electric field, it is sufficient to calculate the square of the corresponding Casimir operator \( C \) and apply the criterium (15.16).

The representations of the algebra \( AE(3) \) realized on sets of solutions of the LHH equations do not satisfy the condition (15.16) and so these equations do not describe a spin-orbit coupling of a particle with an external field. In the following subsection we consider in detail the Galilei-invariant equations of the class \( D_2 \) and select such equations which describe the mentioned coupling.

**15.4. Equations for a \((2s+1)\)-Component Wave Function**

Let us analyze the equations (15.1) with the minimal number of components. The explicit form of the corresponding \( \beta \)-matrices is represented in Subsection 13.5.

We show these equations reduce to the equations of the Schrödinger type for a \((2s+1)\)-component wave function \( \Phi \) and additional conditions expressing superfluous components of \( \psi \) via \( \Phi \).

First we consider the LHH equations. The corresponding matrices \( \beta_i \) and \( \eta_k \) are given in (12.23), (13.11), (13.12) and the column \( R_i \) of Table 13.1. It is convenient to analyze these equations in the equivalent representation (15.8). Denoting
\[ \psi' = \text{column}(\Phi_s, \Phi'_s, \chi) \]  
\[ \text{(15.17)} \]

where \( \Phi_s, \Phi'_s \) are \((2s+1)\)-component, \( \chi \) is a \((2s-1)\)-component wave function, we come to the equation (15.14) for \( \Phi_s \), where \( \varepsilon_0 = ma^2(s+1)^2 \). As to \( \Phi'_s \) and \( \chi \) then according to (15.8) \( \chi = 0, \Phi'_s = a\Phi_s \).

So the LHH equations reduce to the Pauli equation (15.14) for the \((2s+1)\)-component wave function \( \Phi_s \) and predict the correct value \( 1/s \) for the gyromagnetic ratio. Unfortunately these equations do not take into account as important effect as a spin-orbit coupling.

In an analogous way, we make sure that the equations corresponding to the column \( \text{R}_2 \) of Table 13.1 reduce to the Pauli equation (15.14) for a \((2s'+1)\)-component wave function of a particle of spin \( s' = s-1 \) besides \( g = 1/(s' + 1) \). Thus such equations also do not describe a spin-orbit coupling.

The equations (15.1) are more interesting in the case when the corresponding \( \beta \)-matrices are given in column \( \text{R}_3 \) of Table 13.1 since in this case the criterium (15.16) is satisfied. Denoting

\[ \psi' = \text{column}(\phi_1, \phi_2, \phi_3, \chi_{s1}, \chi_{s2}) \]  
\[ \text{(15.18)} \]

and substituting (15.18) and the explicit form of the corresponding matrices into (15.8), (15.9) we come to the following equations

\[ i \frac{\partial}{\partial x_0} \phi_1' - H \phi_1' = \left( \varepsilon_0 - \frac{\pi^2}{2m} + eA_0 - \frac{e}{2ms} S \cdot H - \frac{e}{s(s+1)ae_m} S \cdot F \right) \phi_1' \]  
\[ \text{(15.19)} \]

\[ \varepsilon_0 = \left( \varepsilon_1 + \frac{1}{2} \varepsilon_2 \right) m(s+1)^2, \quad F = E + \frac{1}{2m}(\pi \times H - H \times \pi), \]  
\[ \phi_2' = -e_2 \phi_1', \quad \phi_3' = \frac{1}{2} \left( \varepsilon_2 + \frac{eS \cdot F}{s(s+1)ae_m} \right) \phi_1' \]  
\[ \text{(15.20)} \]

\[ \chi_{s1} = 0, \quad \chi_{s2} = \frac{i\sqrt{k_s}}{(s-1)(2s-1)m} \phi_1', \quad s \neq 1, \quad \chi_0 = 0. \]

So the considered equations reduce to the Hamiltonian equation (15.19) for the \((2s+1)\)-component wave function \( \phi_1' \), the remaining components of \( \psi' \) are expressed via this function according to (15.20).

To inquire into the physical sense of solutions of this equation, we transform (15.19) into a form where the Hamiltonian does not include the term \(-SE\) corresponding to the nonphysical electric dipole coupling. Using for this purpose the transformation

\[ \text{(15.21)} \]
Symmetries of Equations of Quantum Mechanics

\[ U = \exp \left( \frac{i S \cdot \pi}{s(s+1)\alpha \cdot m} \right) \phi_i \]  

(15.22)

and using the identities

\[ i[S \cdot F, S \cdot \pi] = \frac{1}{6} Q_{ab} \frac{\partial F}{\partial x_b} - \frac{1}{3} s(s+1) p \cdot F - \frac{1}{2} S \cdot (F \times p \times F), \]

(15.23)

\[ i[S \cdot \pi, \pi] = -S \cdot E, \quad i[S \cdot \pi, \pi^2] = e S \cdot (\pi \times H - H \times \pi), \]

where \( F \) is an arbitrary vector depending on \( x \), \( Q_{ab} \) is the tensor of the quadruple (10.27), we come to the Hamiltonian (10.26) where

\[ B = \frac{1}{s}, \quad D = \frac{1}{\alpha \cdot s(s+1)}, \quad C = \frac{1}{2} BD, \quad e_0 = m_0 = \left( \alpha_1 - \frac{1}{2} \alpha_2 \right)^{s(s+1)m}. \]  

(15.24)

We see that the considered Galilei-invariant wave equations generate the approximate Hamiltonian of a particle of a spin s, which has the same structure as the Hamiltonian of a relativistic particle. This means that our equations describe dipole, quadruple, spin-orbit and Darwin interactions as relativistic equations, furthermore, the accuracy of the Galilei-invariant description in the approximation \( 1/m^2 \) is not less than in the case of using of Poincaré-invariant equations.

The principal distinguishing from the relativistic case is that the coefficient of the term corresponding to the spin-orbit coupling is not fixed but is determined up to an arbitrary constant \( \alpha_2 \).

Thus the spin-orbit coupling which is usually interpreted as a purely relativistic effect can be successfully described in frames of the Galilei-invariant approach.

In a complete analogy with the above we can demonstrate that the equation (15.1) with the \( \beta \)-matrices corresponding to column \( R_4 \) of Table 13.1 reduces to the following equation for a \((2s+1)\)-component function \( \chi_{s'} \)

\[ i \frac{\partial}{\partial x_0} \chi_{s'} = H \chi_{s'} \equiv \left( \bar{m}_0 + \frac{\pi^2}{2m} + eA_0 + \frac{e}{2m(s'+1)} S \cdot H - \frac{e}{s'(s'+1)\alpha \cdot m} S \cdot F \right) \chi_{s'}, \]

\[ \bar{m}_0 = \left( \alpha_1 - \frac{1}{2} \alpha_2 \right)^{s'(s'+1)m}, \quad S \in D(s-1). \]

The latter with the help of the transformation

\[ \chi_{s'} \rightarrow \chi_{s'} = \exp \left( \frac{i S \cdot \pi}{s'(s'+1)\alpha \cdot m} \right) \chi_{s'} \]

reduces to the form

178
Chapter 3. Representations of the Galilei Algebra...

The explicit form of the Hamiltonian $H'$ can be obtained from (10.26) by the change 
\[ S \rightarrow S', \quad B \rightarrow 1/(s'+1), \quad C \rightarrow BD/2, \quad D \rightarrow 1/[\epsilon s'(s'+1)], \quad \epsilon \rightarrow \tilde{m}_0. \]

15.5. Introduction of Minimal Interaction into Schrödinger-Type Equations

The Galilei-invariant equations in Hamiltonian form considered in Section 14 also can serve as a basis for description of spinning particles in the electromagnetic field. Here we discuss briefly the special features arising by using these equations.

Making in (11.1), (14.9a) the change $p_\mu \rightarrow \pi_\mu$ we come to the following equation

\[ L(\pi)\psi(x) = 0, \quad L(\pi) = i \frac{\partial}{\partial x_0} - H(\pi, A_0), \]  

where

\[ H(\pi, A_0) = \sigma_0 a m + \frac{\pi^2}{2m} + eA_0 + 2iak\sigma_3S \cdot \pi + (\sigma_1 + i\sigma_2)2iak^2 \frac{2}{m} \left[ (S \cdot \pi)^2 + \frac{e^2}{2} S \cdot H \right]. \]  

Equations (15.25) are Euler-Lagrange equations and can be deduced starting from the Lagrangian obtained from (14.24) by the change $\partial/\partial x_\mu \rightarrow \partial/\partial x_\mu - ieA_\mu$. These equations are manifestly invariant under the gauge transformations (15.2). It is not difficult to make sure that the operator $L(\pi)$ satisfies the Galilei-invariance condition (15.4) also, besides $\tilde{D}(\theta, \nu) = D(\theta, \nu)$ is the matrix (12.22), (14.3) (compare with (14.20)).

Let us demonstrate that the motion equation of charged particle in the electromagnetic field, given in (15.25), (15.26) presents a good model of the considered physical situation and describes spin-orbit and Darwin couplings.

To give a physical interpretation of the equation considered it is convenient to pass to such a representation where the operator $L(\pi)$ has a quasidiagonal form (i.e., commutes with the matrix $\sigma_1$ standing by the mass term). As like as in the case of the relativistic Dirac equation we can make an approximate diagonalization only, using $1/m$ as a small parameter. It turns out the approximate Hamiltonian obtained from (15.26) also has the form (10.26) where

\[ \epsilon_0 \rightarrow \sigma_0 a m, \quad B \rightarrow \sigma_3 a k^2, \quad D \rightarrow \kappa, \quad C \rightarrow \frac{1}{2} BD. \]  

In other words Galilei-invariant equations of the Schrödinger type can serve as a
good mathematical model of a charged particle of arbitrary spin. The corresponding Hamiltonian (10.26), (15.27) includes all the terms present in the quasirelativistic Foldy-Wouthuysen Hamiltonian except the relativistic correction to kinetic energy $p^4/8m^2$. So the equations (15.25) describe dipole, quadruple and spin-orbit couplings of arbitrary spin particles with an external field.

The essentially new point in comparison with Poincaré-invariant equations and first-order Galilei-invariant equations is that the coefficient of the term $eS H/2$ (denoted by $B$ in (15.27)) is not fixed but expressed via arbitrary parameters $a$ and $k$. The values of these parameters can be chosen in accordance with experimental data. Setting, e.g., $a=1$, $k=1/s$, $s=1/2$ we come to the Hamiltonian whose first six terms coincide with the corresponding terms of the Foldy-Wouthuysen Hamiltonian (see (10.26) for $\alpha=III$, $s=1/2$) obtained from the Dirac equation.

In conclusion we note that the explicit form of the corresponding transformations diagonalizing the Hamiltonian (15.26) is given by formulae (10.25) (where the index $\alpha$ has to be omitted) besides

$$V_1 = \exp(\sigma_x kS \cdot \pi/m),$$
$$V_2 = \exp(B) = \exp\left[\sigma_x (keS \cdot H - 2k(S \cdot \pi)^2 - eS \cdot E/\alpha)/2m^2\right],$$
$$V_3 = \exp\left[\sigma_y k^2 \left(\frac{2k(S \cdot \pi)^3}{3} - ke \frac{2}{2} [S \cdot \pi, S \cdot H]\right) - \frac{1}{2m} [B, \pi_0] - \frac{ik}{2m^3} [S \cdot \pi, \pi^2]\right].$$

### 15.6. Anomalous Interaction

The minimal change $p_\mu \rightarrow \pi_\mu$ in motion equations is not the only possibility in description of particles interaction with an external field. A more general approach (proposed by Pauli) is to take into account so called anomalous coupling described by the terms linearly dependent on $E$ and $H$.

A classical example of an equation describing the anomalous interaction is the Pauli generalization of the Dirac equation

$$\left[\gamma_\mu \pi^\mu - m + \frac{ie}{4m} [\gamma_\mu, \gamma_\nu] F^{\mu\nu}\right] \psi = 0.$$  

The term proportional to the tensor of the electromagnetic field strength $F^{\mu\nu}$ makes it possible to take into account a deviation of the dipole momentum from the value $g=1/s$.

Here we discuss possibilities of introducing of anomalous interactions into Galilei-invariant wave equations. Namely we consider generalized equations (15.1), (15.25) of the kind
Chapter 3. Representations of the Galilei Algebra ...

\[ L\psi = \left[ i \frac{\partial}{\partial x^0} - \tilde{H}(\pi, A_{\mu}) \right] \psi, \quad \tilde{H}(\pi, A_{\mu}) = H(\pi, A_{\mu}) + \frac{e}{m} \varepsilon_{\mu \lambda} F^{\mu \lambda} \]  \hspace{1cm} (15.28)

and

\[ \dot{L}\psi = \left( \tilde{\beta}_\mu \pi^\mu + \tilde{\beta}_i m + \frac{e}{m} \varepsilon_{\mu \lambda} F^{\mu \lambda} \right) \psi = 0 \]  \hspace{1cm} (15.29)

where \( H(\pi, A_{\mu}) \) is the operator (15.26), \( \Sigma_{\mu \lambda} \) and \( \Sigma'_{\mu \lambda} \) are some matrices which have to be of that kind that the equations (15.28), (15.29) are Galilei-invariant.

Our task is to find the exact form of these matrices satisfying the Galilei-invariance condition and to analyze the contribution of the additional terms introduced into equations of motion.

The following statement is valid for the equation (15.28).

**PROPOSITION 15.2.** The equation (15.28) is Galilei-invariant iff

\[ \Sigma_{\alpha a} = \frac{k_1}{2} \eta_\alpha, \quad \Sigma_{ab} = \frac{1}{2} \varepsilon_{abc} (k_1 S_c \eta_\alpha) \]  \hspace{1cm} (15.30)

where \( S_a \) and \( \eta_\alpha \) are the matrices (14.3), \( k_1 \) and \( k_2 \) are arbitrary numbers.

For the proof see [320].

If we require the equation (15.28) be invariant under the complete reflection transformation (14.4) then the parameters \( k_1 \) and \( k_2 \) have to be real.

Now we consider the first-order equations (15.29). Restricting ourselves to the case of the LHH equations (i.e., where \( \beta \)-matrices have the form (13.11), (13.12), \( R_i \) in Table 13.1) we obtain [320, 321] the following general form of \( \Sigma_{\mu \sigma} \):

\[ \Sigma'_{\alpha a} = \frac{k_1}{2} \varepsilon_{abc} \beta_a \beta_b \beta_c, \]  \hspace{1cm} (15.31)

\[ \Sigma'_{ab} = \frac{i}{2} k (1 - 2 \beta_0) \varepsilon_{abc} \beta_c + \frac{k_2}{2} \beta_0 \beta_a \beta_b \beta_e \]  \hspace{1cm} (15.32)

where \( k_3 \) and \( k_4 \) are arbitrary numbers.

So we have defined the general form of the terms representing anomalous interaction, which can be included into motion equation without loss of their Galilei-invariance. To analyze the physical consequences of such an inclusion, we transform the operator \( L \) of (15.28) to the quasidiagonal form and reduce (15.29) to the equation for a \((2s+1)\)-component wave function in analogy with the procedure made in Subsections 15.4, 15.5. As a result, we obtain the equation [320, 321]

\[ \left\{ \frac{i}{\hbar} \frac{\partial}{\partial t} - H'''' - \frac{2k k_1}{3m^2} Q_{\alpha \beta} \frac{\partial H'''}{\partial x^\gamma} \right\} \psi = 0 \]  \hspace{1cm} (15.33)

where \( H'''' \) is given in (10.26) besides
Symmetries of Equations of Quantum Mechanics

\[ \varepsilon_0 = \sigma, \quad B = k_1 - \sigma_1(k_2 - 4ak^2), \quad D = 2kk_2 - 1, \quad C = \frac{1}{2}BD, \]

and

\[
\begin{pmatrix}
  i \frac{\partial}{\partial t} - H''' - \frac{e^2k_z^2}{4m^2} \mathbf{H}^2 \\
  \phi_y = 0
\end{pmatrix}
\]

where \( H''' \) is the operator (10.26) again but

\[ \varepsilon_0 = \sigma_1 m, \quad B = k_1 - \sigma_1(1 + k_3), \quad D = \frac{K_1}{2s}, \quad C = \frac{1}{2}BD. \]

We see that introduction of anomalous interaction into Schrödinger-like equations does not change the structure of the approximate Hamiltonian but makes it possible to correct in a Galilei-invariant way the values of coefficients of the terms representing dipole, spin-orbit, etc. interactions. As to the LHH equations the anomalous interaction leads to principal changes of the corresponding approximate Hamiltonian which now includes the terms corresponding to spin-orbit coupling while the minimal interaction leads to the Pauli Hamiltonian (15.14).

Summarizing, we can say that Galilei-invariant wave equations give a wide possibilities in description of arbitrary spin particles in an external field. In particular such equations give a correct description of Pauli and spin-orbit couplings and predict the correct value \( g = 1/s \) of the gyromagnetic ratio. In Chapters 6 and 7 we use Galilei-invariant equations to solve concrete physical problems of one- and two-particle interactions.
4. NONGEOMETRIC SYMMETRIES

We show the symmetry of the basic equations of quantum physics is wider than the classical symmetry described in Chapter 1. Using the approach proposed in [113, 116, 122] we find IAs of the Dirac, Maxwell, KGF, Schrödinger and other equations in the classes of higher order differential and integro-differential operators. The corresponding SOs form wide Lie algebras and superalgebras including the Poincaré or Galilei algebra as a subalgebra.

16. HIGHER ORDER SOs OF THE KGF AND SCHRÖDINGER EQUATIONS


In Chapter 1 and Section 11 we studied symmetries of the basic equations of quantum physics in respect to continuous groups of transformations. This symmetry can be defined as an invariance under a finite dimension Lie algebra whose basis elements belong to the class \( M_1 \) (i.e., the class of first order differential operators). It is evident these symmetries do not exhaust all the invariance properties of the equations considered since a priori we do not take into account IAs including differential operators of higher orders.

Here we study symmetries of the fundamental equations of quantum theory in frames of a more general approach than the classical Lie method. The basic idea of this approach (called non-Lie in the following) is that the class of SOs can be essentially extended by including differential operators of the second, third, .. etc. orders and even integro-differential operators. We call them higher-order SOs.

A well-known example of a symmetry under such an extended class of operators is the invariance of the Schrödinger equation for the hydrogen atom under the algebra \( AO(4) \) established by Fock [103, 104]. Other examples can be found in [6, 39, 228], a good treatment of higher symmetries is present in Olver’s book [350].

Higher order SOs give an information about hidden symmetries of equations including Lie-Bäclund symmetries [422] and supersymmetry [159, 327]. These operators can be used to construct new conservation laws which cannot be found in the classical Lie approach. A very interesting application of these operators is a description of coordinate systems in which solutions in separated variables exist [305].

An effective algorithm for finding higher order SOs and the corresponding IAs is suggested in [116,122]. Here we discuss this algorithm briefly and give basic
Consider an arbitrary linear system of partial differential equations

\[ L\psi(x) = 0 \] (16.1)

where \( L \) is a linear differential operator, \( \psi \) is a multicomponent function. As in Subsection 1.2 we say an operator \( Q \) is a SO of the system (16.1) if it transforms solutions into solutions. In other words a SO satisfies by definition the condition

\[ [Q, L]\psi = 0 \] (16.2)

where \( \psi \) is an arbitrary solution of the equation (16.1).

In Chapter 1 we suppose that SOs belong to the class \( M_1 \) and thus describe Lie symmetries. Here we decline this supposition and search for SOs of arbitrary order. Besides we impose two types of restrictions on the SOs considered.

(1) We require the SOs form a finite-dimensional Lie algebra\(^\ast\), i.e., satisfy the relations

\[ [Q_A, Q_B] = f_{ABC} Q_C \] (16.3)

where \( f_{ABC} \) are structure constants. It will be shown further on this way makes it possible to find new wide IAs of equations of quantum physics.

The condition (16.3) is a very substantial restriction selecting subsets having the structure of a Lie algebra from a set of SOs which is infinite in general.

(2) We consider SOs belonging to the class of differential operators of order \( n \) (class \( M_n \)) where \( n \) is fixed. Generally speaking such SOs do not form a finite-dimensional Lie algebra, but sometimes they have another interesting algebraic structures, e.g., they can form a basis of a superalgebra.

To investigate non-Lie symmetries we use various combinations of the restrictions (1), (2) (i.e., impose one of them or both them). In some cases we decline these restrictions and search for a complete set of SOs of arbitrary order.

The principal question arising while investigating symmetries in frames of non-Lie approach is the following: how are SOs of the equation considered to be constructively calculated? By generalizing results of calculations of IAs of equations of quantum mechanics it is possible to formulate the following algorithm:

(1) by means of a nondegenerated transformation the system of partial differential equations is reduced to diagonal or quasidiagonal form, i.e., the maximal splitting of this system into independent subsystems is carried out; (2) the IA of the

\(^\ast\) According to Theorem 1.1 SOs belonging to the class \( M_1 \) always form a Lie algebra which can be either finite or infinite dimensional. But even second order SO do not satisfy (16.3) in general
transformed equation is found and the kind of a representation of the corresponding Lie algebra is determined; (3) by means of the inverse transformation the explicit form of the basis elements of the IA of the original equation is found.

The algorithm is based on one of the most fruitful and effective ideas in the theory of differential equations, i.e., using of transformations of independent and dependent variables.

While realizing the algorithm an important role is played by the concept of the symbol of an operator $L$ which can be defined by means of the Fourier transform

$$\hat{L} \psi(x) = (2\pi)^{-3/2} \int_{D(p)} d^3p \exp(i p \cdot x) \psi(x_0, p)$$

where $\psi \in C_0^\infty(R_4)$, $\tilde{\psi}(x_0, p) = F \psi(x)$ is a Fourier transform of $\psi(x)$, $F$ is the unitary Fourier operator mapping a vector of the Hilbert space $\hat{H}$ into $\hat{H}$, $\psi(x_0, p) \in \hat{H}$, $D(p)$ is a domain of integration. A formal connection between the operator $L$ and its symbol $\hat{L}$ can be given by the following relation

$$\hat{L} = F^{-1} LF, \quad L = \hat{L} F^{-1}.$$  \hspace{1cm} (16.5)

Relations (16.4), (16.5) can be used to realize the first step of the algorithm. Indeed if the symbol of the operator $L$ is a matrix with variable coefficients (and this occurs for the majority of equations of mathematical physics) the system (16.4) can be reduced in principle to a system of noncoupled integral equations by transforming $L$ to diagonal or Jordan form.

It should be noted that a realization of the algorithm present above for concrete equations of physics or mechanics, as a rule, is not a simple problem. Sometimes it is easier to use other ways considered in the following.

We saw in Chapter 1 that a description of first order SOs is based on using the explicit form of the Killing vectors corresponding to the space of independent variables. To describe higher order SOs it is necessary to calculate more complicated structures which we call generalized Killing tensors of order $s$. In the following we present a definition of these tensors as complete sets of solutions of some overdetermined system of partial differential equations of order $s$. The explicit form of the generalized Killing tensors used in this book is given in Appendix 2.

16.2. SOs of the KGF Equation

Like in Chapter 1 we start with the KGF equation, the simplest equation of relativistic quantum theory. Using notations and definitions of Section 1 we formulate the problem of description of higher order SOs for this equation.

Let $M_n$ be the class of differential operators of order $n$ defined on $F$. Then any
operator $Q_n \in M_n$ can be represented in the form

$$Q_n = \sum_{j=1}^{n} Q_j, \quad Q_j h^{\alpha_1 \alpha_2 \ldots \alpha_j} \frac{\partial^j}{\partial x_{\alpha_1} \partial x_{\alpha_2} \ldots \partial x_{\alpha_j}} h^{\alpha_1 \alpha_2 \ldots \alpha_j} \in F, \quad a_j = 0,1,2,3. \quad (16.6)$$

In a complete analogy with Definition 1.1 we formulate the following

**DEFINITION 16.1.** A linear differential operator (16.6) is a SO of the KGF equation in the class $M_n$ (or a SO of order $n$) if

$$[Q_n, L] = \alpha Q, \quad \alpha \in M_{n+1} \quad (16.7)$$

where $L$ is the operator (1.3).

In the case $n=1$ such defined SOs reduce to the generators of the Lie group considered in Section 1. The SOs of order $n>1$ describe generalised (non-Lie) symmetries of the KGF equation. The problem of description of the complete set of SOs of order $n$ reduces to finding the general solution of the operator equation (16.7).

It is convenient to represent all the operators (16.7) as sums of $j$-multiple anticommutators [328,342]

$$Q_n = \sum_{j=1}^{n} Q_j, \quad \alpha = \sum_{j=1}^{n-1} \alpha_j, \quad (16.8)$$

$$\alpha Q = \frac{1}{4} \left[ \left[ \alpha, \frac{\partial}{\partial \alpha} \right], \frac{\partial}{\partial \alpha} \right] - \frac{1}{2} \left[ \left[ \alpha, \frac{\partial}{\partial \alpha} \right], \frac{\partial}{\partial \alpha} \right] + \frac{1}{4} \left( \frac{\partial^j}{\partial \alpha} \frac{\partial}{\partial \alpha} \right)$$

where

$$Q_j = \left[ \ldots \left[ K^{\alpha_1 \alpha_2 \ldots \alpha_j}, \frac{\partial}{\partial \alpha_j} \right], \ldots, \frac{\partial}{\partial \alpha_j} \right], \quad (16.9a)$$

$$\alpha_j = \left[ \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j}, \frac{\partial}{\partial \alpha_j} \right], \ldots, \frac{\partial}{\partial \alpha_j} \right], \quad (16.9b)$$

$K^\alpha$ and $\alpha^\alpha$ are unknown symmetric tensors of rank $j$. Transferring differential operators to the right we can reduce $Q_n$ of (16.8) to the equivalent form (16.6).

Substituting (16.8) into (16.7) and equating the coefficients of the same differential operators we come to the following equation for $\alpha \neq 0$:

$$\partial^{(j)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} = 0, \quad j = 1,2,\ldots,n, \quad (16.10)$$

$$\alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} = 0 \quad (16.11)$$

where symmetrization is imposed over the indices in parenthesis:

$$\partial^{(j)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} = \partial^{(1)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} + \partial^{(2)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} + \partial^{(3)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} + \partial^{(4)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j} \ldots + \partial^{(n)} \alpha^{\alpha_1 \alpha_2 \ldots \alpha_j}.$$

So the problem of describing SOs of order $n$ for the KGF equation reduces to solving the noncoupled system of partial differential equations (16.10). In the case $n=1$...
Chapter 4. Nongeometric Symmetries

this system reduces to the Killing equations (1.9).

We will see further on that formula (16.10) gives the principal equation appearing while investigating higher order SOs of the basic equations of quantum theory. The equations of the type (16.10) were considered in [408]. We call a symmetric tensor $K$ satisfying (16.10) a Killing tensor of valence $j$ and order 1, the sense of the last term will be clear in the following.

The general solution of (16.10) is given in Appendix 2. The number $N_j$ of independent solutions is

$$N_j \equiv {C_b \choose a_j}$$

where $C_b$ is a number of combinations from $b$ elements by $a$ ones.

To present the explicit form of $F$ it is convenient to define a special kind of tensors (which we call basic tensors) $\lambda_{a_1...a_c}^{b_1...b_n}$ having the following properties:

1. symmetry and zero traces in respect with the indices $a_1, a_2, ..., a_c$;
2. symmetry in respect with a permutation of pairs of indices $[a_{i,j}, b_{i,j}]$, $i, j = 1, 2, ..., n-c$;
3. skew-symmetry in respect with the indices $[a_{i,j}, b_{j,i}]$;
4. a convolution in respect with any triplet of indices with absolutely skew-symmetric tensor $\epsilon_{\mu \nu \rho}$ is equal to zero.

Basic tensors are reducible since in general they have nonzero traces over pairs of indices $a_l, a_m$ if $l>c$ and/or $m>c$.

**LEMMA 16.1.** The general solution of (16.10) can be represented in the form

$$K_{a_1...a_c}^{-b_1...b_n} = \sum_{[a,b]} \lambda_{a_1...a_c}^{b_1...b_n} \epsilon_{a_1b_1} \epsilon_{a_2b_2} \cdots \epsilon_{a_nb_n}$$

where $K_{a_1...a_c}^{-b_1...b_n}$ is the general solution of (16.10) for $j=2$, $\lambda^{-}$ are arbitrary basic tensors independent on $x$.

**PROOF** reduces to direct verification the fact that the tensor (16.13) satisfies (16.10) and to calculation the number of independent components of $\lambda^{-}$ which is equal to $N\cdot N^{1/2}$ [328,342].

The first term in the r.h.s. of (16.13) corresponds to a SO of order $j$ which reduces to a SO of order $j-2$ on the set of solutions of (1.1) and so can be neglected. Then substituting (16.13) into (16.9) we obtain

$$Q \sum_{[a,b]} \lambda_{a_1...a_c}^{b_1...b_n} P_{a_1} P_{a_2} \cdots P_{a_n} J_{b_1} J_{b_2} \cdots J_{b_n}$$

where $P_{a}, J_{b}$ are the Poincaré group generators (1.6).

Thus we have obtained a complete set of SOs of order $n$ for the KGF equation.
Symmetries of Equations of Quantum Mechanics

in the form (16.14) where \( j \leq n \). All such operators belong to the enveloping algebra of the algebra \( \mathfrak{MP}(1,3) \). The number \( N_j \) of linearly independent operators \( Q_j \) is

\[
N_j = N_{j+2} - N_{j+1} = \frac{1}{4!} (j+1)(j+2)(2j+3)(j^2+3j+4),
\]

(16.15)

and the total number of SOs of order \( j \leq n \) is

\[
N(n) = \sum_{j=0}^{n} N_j = \frac{2}{3!4!} (n+1)(n+2)(n+3)(n^2+4n+6).
\]

In conclusion we note that the SO (16.14) can be represented in such a form which includes irreducible arbitrary parameters only. Indeed, expanding \( \lambda^{\alpha \beta \gamma \delta} \) in irreducible tensors (i.e., tensors having the properties (1)-(4) and besides being traceless over any pair of indices) we obtain

\[
Q_j = \sum \lambda^{\alpha_{a_1} \alpha_{a_2} \ldots \alpha_{a_{2n}}} P_{\alpha_{a_1} \mu_1} P_{\alpha_{a_2} \mu_2} \ldots P_{\alpha_{a_{2n}} \mu_{2n}} \times
\]

\[
\times \left[ I_{a_1 \mu_1} \right] \left[ I_{a_2 \mu_2} \right] \ldots \left[ I_{a_{2n} \mu_{2n}} \right] (J_{ab} P_b) (J_{\mu_2 \nu_2}) (J_{\mu_3 \nu_3}) \ldots (J_{\mu_{2n} \nu_{2n}})
\]

where

\[
I_{ab} = J_{ab}^n, \quad \mu = \mu_1 + \epsilon_{\mu_2} + \ldots + \epsilon_{\mu_{2n}}, \quad \nu = j - \mu_2, \quad \epsilon_{\mu_2} = \frac{1}{2} [1 + (-1)^n].
\]

\{A\} is the entire part of \( A \), and the summation is imposed over all the values of \( \mu_n \) satisfying the condition

\[
0 \leq \mu_n \leq (j/2); \quad 0 \leq \mu_2 \leq (j-2\mu_j)/2; \quad 0 \leq \mu_3 \leq 2\mu_2 - \mu_2.
\]

16.3. Hidden Symmetries of the KGF Equation

Let us demonstrate the higher order SOs of the KGF equation possesses nontrivial algebraic structures. Restricting ourselves to the SOs of the second order we see that there are 49 such operators having the form

\[
P_{ab}, \quad J_{ab}, \quad P_{a}^2 = P_{a} P_{b} - \frac{1}{4} g_{ab} m^2, \quad F_a = J_{ab} P^b,
\]

(16.17a)

\[
J = J_{ab} J^{ab}, \quad K_{ac} = J_{ab} J^{b c} + J_{ab} J^{b a} - \frac{1}{2} g_{ac} J^2, \quad K_{[ab]} = J_{ab} J_{[ab]} - \frac{1}{4} g_{bc} g_{ad} J_{[ad]}.
\]

(16.17b)

where the Latin indices run from 0 to 3.

In contrast with the first order SOs of (1.6) the operators (16.17) do not form
Chapter 4. Nongeometric Symmetries

a Lie algebra. But the subset (16.17a) includes bases of some Lie algebras, satisfying the following commutation relations

\[ [P_{ab}, P_{cd}] = 0, \quad [P_{ab}, J_{cd}] = 0, \quad [F_a, F_b] = -\frac{3}{4} m \gamma_{ab}, \quad [F_a, P_{cd}] = i P_{ab} \gamma_{cd}. \]  

Using (16.18) and (1.14) we can indicate two finite-dimensional Lie algebras

\[ A_{10} = \{ J_{\mu \nu}, F_{\mu} \}, \quad A_{19} = \{ P_{\mu}, P_{\mu \nu}, \gamma_{\mu \nu} \}. \]

The algebra \( A_{10} \) is isomorphic to \( AO(2,3) \).

\[ \gamma_{\mu \nu} \gamma_{\rho \sigma} \]  

Let us consider still hidden symmetry of the KGF equation. Denoting

\[ \psi \approx \text{column} (\chi), \quad \psi = \text{column} (\chi) \]

and the infinite dimension Lie algebra

\[ \gamma_{\mu \nu} \gamma_{\rho \sigma} \]

The operators (16.21) generate the following finite group transformations

\[ \phi \left( x' \right) = \frac{1}{2} \left[ (1 + \sigma_3) \cosh \lambda + 1 - \sigma_3 + (\sigma_1 + i \sigma_2) \frac{\lambda \cdot \mathbf{p}}{m \lambda} \right] \phi \left( x \right) \]

where \( x' \) is given in (1.32), (1.39). These transformations represent the group of a hidden symmetry of the KGF equation, mixing wave function and its derivations, see
We remind that the KGF equation admits also the algebras and groups of hidden symmetries connected with antilinear transformations, refer to Subsection 1.7.

16.4. Higher Order SOs of the d’Alembert Equation

Consider the KGF equation in the special case $m=0$. The problem of description of higher order SOs of this equation can be solved in a complete analogy with Subsection 16.2. Substituting (16.8) into (16.7) and setting $m=0$ in $L$ of (1.3) we come to the following determining equations (instead of (16.10)) [328]

$$\partial^{(a_1, K^{a_2...a_j})} \frac{1}{j+1} \partial^{\mu} K^{b_1 a_1...a_j a_{j+1}} = 0, \tag{16.21}$$

$$\alpha^{a_1...a_{j+1}} \frac{1}{j+1} \partial^{\mu} K^{b_1 a_1...a_{j+1}}, \tag{16.22}$$

where $K^{\mu...}$ and $\alpha^{\mu...}$ are symmetric traceless tensors, the first is called a conformal Killing tensor of valence $j$ and order 1.

The general solution of (16.21) is given in Appendix 2. The number $N^j$ of linearly independent solutions is equal to

$$N^j = \frac{1}{12} (j+1)^2 (j+2)^2 (2j+3), \tag{16.23}$$

each solution corresponds to the SO of the d’Alembert equation. Substituting (A.2.10), (A.2.14) into (16.9) we obtain linearly independent SOs in the form

$$Q_j = \sum_{a_1...a_j} \lambda^{a_1...a_j} [a_{i_1}...a_{i_j}] \frac{1}{D} \lambda^{b_1...b_{j+1}} [b_{i_1}...b_{i_{j+1}}], \tag{16.24}$$

where $\lambda^{a...}$ are arbitrary basic tensors, $P, K, J, D$ are the generators of the conformal group given in (1.6), (1.16), $0 \leq c_1 + c_2 \leq j-1$, $c_3 = 0,1$.

So we have found a complete set of SOs of order $n$ for the d’Alembert equation in the form (16.24). All these operators belong to the enveloping algebra of the algebra $AC(1,3)$.

We note that higher order SOs of a system of partial differential equations do not belong in general to the enveloping algebra of the Lie algebra of this system symmetry group (see Section 17).

In conclusion we consider briefly the problem of description of higher order SOs for the generalized KGF equation in the pseudo-Euclidean space of dimension
Chapter 4. Nongeometric Symmetries

\[ p + q = M \]

where

\[
\begin{cases}
0, & \mu \neq \nu, \\
1, & \mu = \nu \leq p, \\
-1, & p < \mu = \nu \leq p + q.
\end{cases}
\]

The problem of description of SOs of the equation (16.25) is formulated in complete analogy with the above. The generalization the above results to the case of an arbitrary number of variables is almost trivial so we restrict ourselves to presenting the number of linearly independent SOs of order \( j \) for the equation (16.25) in \( M \)-dimensional space [328, 342]

\[
N_j = \frac{1}{M} C_{j, M-1} C_{j, M-1}.
\]  \hspace{1cm} (16.26)

The total number of SOs of order \( j, 0 \leq j \leq M \) is

\[
N(n, M) = \frac{2n^2 \cdot 2nM \cdot M(M-1)}{M(M-1)} C_{n, M-1} C_{n, M-1}.
\]  \hspace{1cm} (16.27)

In particular for \( M = 2, 3 \)

\[
\begin{align*}
N_2 & = 2j + 1, \quad N(n, 2) = (n + 1)^2, \\
N_3 & = \frac{1}{3}(j + 1)(2j^2 + 4j + 3), \quad N(n, 3) = \frac{1}{6}(n + 1)(n + 2)(n^2 + 3n + 3).
\end{align*}
\]  \hspace{1cm} (16.28a, b)

Formulæ (16.28a) and (16.28b) give the numbers of SOs for the Helmholtz and inhomogeneous Laplace equation. For the homogeneous Laplace equation we have

\[
\tilde{N}_2 = \frac{1}{3}(j + 1)(2j + 1)(2j + 3), \quad \tilde{N}(n, 3) = \frac{1}{6}(n + 1)(n + 2)(2n^2 + 6n + 3).
\]  \hspace{1cm} (16.29)

16.5. SOs of the Schrödinger Equation

Here we find a complete set of SOs of order \( n \) for the Schrödinger equation. Without loss of generality we represent such an operator in the form (16.9a) where indices values run from 1 to 3.

The SOs considered do not include \( p_0 \) because it can be expressed via \( p^2/2m \) on the set of solutions of the equation (11.3).

Substituting \( L \) (11.3) and \( Q \) (16.8) into (16.7) we come to the following
Symmetries of Equations of Quantum Mechanics

system of coupled equations [330]

\[
\partial^{(a_1...a_l)}K_{a_2...a_l} = -2m(j+1)\partial^{a_1...a_l}K_{a_2...a_l}, \quad j=0,1,...n-1, \tag{16.30a}
\]

\[
\partial^{(a_1...a_l)}K_{a_2...a_l} = 0, \quad \dot{K}=0, \quad j=0 \tag{16.30b}
\]

where the dots denote the time derivatives.

So to find all the SOs of order \(n\) for the Schrödinger equation it is necessary to obtain the general solution of (16.30). The system (16.30) is over-determined and can be solved by transition to the set of noncoupled differential consequences. We make it as follows.

Consider (16.30a) for \(j=n-1\):

\[
\partial^{(a_1...a_l)}K_{a_2...a_l} = -2mn\partial^{a_1...a_l}K_{a_2...a_l}. \tag{16.31}
\]

Using (16.30b) we can obtain the following differential consequence of (16.31):

\[
\partial^{(a_1...a_l)}K_{a_2...a_l} = 0. \tag{16.32}
\]

Then differentiating (16.32) in respect with \(t\) and considering (16.30a) for \(j=n-2\) we obtain

\[
\partial^{(a_1...a_l)}\partial^{a_1...a_l}K_{a_2...a_l} = 0,
\]

and for \(j=n-s+1\)

\[
\partial^{(a_1...a_l)}\partial^{(a_1...a_l)}...\partial^{(a_1...a_l)}K_{a_2...a_l} = 0, \quad s=n+1-j. \tag{16.33}
\]

It follows from (16.30), (16.33) that

\[
\frac{\partial^{(a_1...a_l)}K_{a_2...a_l}}{\partial \tau^{(a_1...a_l)}} = 0. \tag{16.34}
\]

So starting with (16.10) we come to the system of noncoupled equations (16.33), (16.34).

The equations (16.33) are direct generalizations of the first-order equations (16.10) determining the Killing tensor. We call solutions of (16.33) generalized Killing tensors of valence \(j\) and order \(s\) [328, 342]. The general solution of (16.33), (16.34) is

\[
K_{a_1...a_l} = \sum_{t=0}^{t=n-j} K_{a_1...a_l}^{a_1...a_l} = 0, \quad s=n+1-j \tag{16.35}
\]

where \(K_{a_1...a_l}^{a_1...a_l}\) is a Killing tensor of valence \(j\) and order \(s\). Substituting (16.35) into (16.30) we come to the equation

\[
\alpha K_{a_1...a_l}^{a_1...a_l} = 2mj\partial^{a_1...a_l}K_{a_1...a_l}^{a_1...a_l}, \quad \alpha\neq0, \quad s=n-j. \tag{16.36}
\]

Let we know the general solution of (16.35) for a fixed value of \(j=j_0-1\) then the relations (16.35), (16.36) define an explicit form of the general solution for \(j=j_0\) up

192
Chapter 4. Nongeometric Symmetries

to an arbitrary generalized Killing tensor \( K_{\alpha_1\ldots\alpha_s}^{\beta_1\ldots\beta_{n-j}} \) of order \( s = n-j+1 \). So the number \( N_n \) of linearly independent solutions of the system (16.30) coincides with the total number of independent components of generalized Killing tensor of valence \( j \) and order \( s = n-j+1, 0 \leq j \leq n \). According to (A.2.4)

\[
N_n = \sum_{j=0}^{n-1} N_n^j = \frac{1}{3!4!} (n+1)(n+2)^2(n+3)^2(n+4). \tag{16.37}
\]

Thus the Schrödinger equation admits \( N_n \) linearly independent SOs of orders \( j \leq n \). Subtracting from (16.37) the number of SOs of orders \( j < n \) we obtain the number of SOs of order \( n \)

\[
\tilde{N}_n = N_n \quad \tilde{N}_n = \frac{1}{4!} (n+1)^2(n+2)^2(n+3). \tag{16.38}
\]

The corresponding SOs can be represented in the form

\[
Q_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} \lambda^{a_{i1} a_{i2} \ldots a_{ij} b_{j1} b_{j2} \ldots b_{jk}} A_{a_{i1} a_{i2} \ldots a_{ij} b_{j1} b_{j2} \ldots b_{jk}} A_{a_{i1} a_{i2} \ldots a_{ij} b_{j1} b_{j2} \ldots b_{jk}} \tag{16.39}
\]

where \( P, G, J \) are generators of the Galilei group of (11.5), \( \lambda^{a_{i1} \ldots a_{ij} b_{j1} \ldots b_{jk}} \) are arbitrary tensors symmetric in respect to permutations \( a_i \leftrightarrow a_j \) and \( b_i \leftrightarrow b_j \), besides

\[
\delta_{a_{ij}, b_{jk}} = 0. \tag{16.40}
\]

Summing up the numbers of independent components of \( \lambda^{a_{i1} \ldots a_{ij} b_{j1} \ldots b_{jk}} \) we make sure that the number of linearly independent SOs of (16.39) coincides with (16.38).

16.6. Hidden Symmetries of the Schrödinger equation

So we have calculated all the linearly independent SOs of arbitrary order for the Schrödinger equation. All such operators belong to the enveloping algebra of the Galilei algebra.

Here we demonstrate this enveloping algebra has a very interesting structure and includes wide Lie algebras. But before this we consider the other hidden symmetries generated by antilinear transformations.

In Subsection 11.6 we were discussing symmetries of the Schrödinger equation in respect with the antilinear transformation \( T \) of (11.37). Then in accordance with Lemma 1.1 (refer to Subsection 1.7) this equation is invariant under the algebra \( AO(1,2) \) whose basis elements are \( \{T,R,TR\} \) where \( R \) is the evident symmetry transformation (1.55). Moreover this symmetry can be extended by including the combined transformations consisting of the space reflection \( P \) of (11.37). As a result we come to the following IA of the Schrödinger equation

\[
\{T,R,TR,PT,PR,PTR\}.
\]

193
These operators satisfy the following commutation relations

\[ [T,R] = 2TR, \quad [T,TR] = 2R, \quad [T,PT] = 0, \quad [T,PR] = 2TR, \quad [T,PR] = -2PR, \]
\[ [R,TR] = -2T, \quad [R,PT] = -2PTR, \quad [R,PR] = 0, \quad [R,PR] = -2PT, \]
\[ [TR,PT] = 2PR, \quad [TR,PR] = -2PT, \quad [TR,TR] = -2T, \]
\[ [PT,PR] = 2TR, \quad [PT,PR] = 2R, \quad [PR,PTR] = -2T, \]

and thus form the Lie algebra isomorphic to \( AO(2,2) \).

Taking into account that the squares of the considered SOs are proportional to the unit operator (up to a sign) it is not difficult to find the corresponding symmetry group of the Schrödinger equation. We restrict ourselves by presenting one parameter transformations generated by these symmetries and belonging to the group \( O(2,2) \):

\[ \psi(t, x) \rightarrow \cos \theta_1 \psi(t, x) + i \sin \theta_1 \psi(t, x), \]
\[ \psi(t, x) \rightarrow \cos \theta_2 \psi(t, x) + i \sin \theta_2 \psi(t, -x), \]
\[ \psi(t, x) \rightarrow \cosh \theta \psi(t, x) - \sinh \theta \psi(t, -x), \]
\[ \psi(t, x) \rightarrow \cosh \theta \psi(t, x) + \sinh \theta \psi(-t, x), \]
\[ \psi(t, x) \rightarrow \cosh \theta \psi(t, x) - \sinh \theta \psi(-t, -x), \]
\[ \psi(t, x) \rightarrow \cosh \theta \psi(t, x) + i \sinh \theta \psi(-t, -x) \]

where \( \theta \) are real parameters.

But let us return to linear SOs considered in the previous subsection. Restricting ourselves to the second order SOs we obtain from (16.39) the following complete set of them

\[ P_{ab} + P_a P_b, \quad G_{ab} = G_a G_b, \quad Q_{ab} = -\frac{1}{2} (G_a P_b + G_b P_a), \quad \text{(16.41a)} \]

\[ F_{ab} = J_a J_b + J_b J_a, \quad F_a = \epsilon_{abc} P_b J_c, \quad G_a = \epsilon_{abc} G_b J_c, \quad \text{(16.41b)} \]

\[ L_{ab} = P_a J_b + P_b J_a, \quad N_{ab} = G_a J_b + G_b J_a, \quad (a,b,c) \rightarrow (1,2,3), \quad \text{(16.41c)} \]

including 40 elements (compare with (16.38)). In contrast with \( G_{ab}, Q_{ab} \) and \( F_{ab} \) the tensors \( L_{ab} \) and \( N_{ab} \) are traceless.

The SOs (16.41) do not form a closed Lie algebra. But some subsets of these operators are closed in respect with commutation and so form the bases of Lie algebras. One of such subsets is given by the operators \( P_a, G_a \) and \( J_a \) inasmuch as they satisfy the following commutation relations.
Chapter 4. Nongeometric Symmetries

The relations (11.6b)-(11.6d), (16.42) define the 27-dimensional Lie algebra

\[ [P_{ab}, P_{cd}] = [G_{ab}, G_{cd}] = 0, \]

\[ [P_{ab}, G_{cd}] = \text{im}(\delta_{ac} Q_{bd} + \delta_{ad} Q_{bc} + \delta_{bc} Q_{ad} + \delta_{bd} Q_{ac}), \]

\[ [P_{ab}, Q_{cd}] = \text{im}(\delta_{ac} P_{bd} + \delta_{ad} P_{bc} + \delta_{bc} P_{ad} + \delta_{bd} P_{ac}), \]

\[ [G_{ab}, Q_{cd}] = \text{im}(\delta_{ac} G_{bd} + \delta_{ad} G_{bc} + \delta_{bc} G_{ad} + \delta_{bd} G_{ac}), \]

\[ [Q_{ab}, Q_{cd}] = \text{im}(\delta_{ac} Q_{bd} + \delta_{ad} Q_{bc} - \delta_{bc} Q_{ad} - \delta_{bd} Q_{ac}), \]

\[ [P_{a1}, P_{b2}] = 0, \quad [P_{a1}, Q_{b2}] = \text{im}(\delta_{ab} P_{d} + \delta_{ad} P_{b}), \]

\[ [P_{a1}, G_{b2}] = \text{im}(\delta_{ab} G_{d} + \delta_{ad} G_{b}), \]

\[ [J_{a}^{1}, R_{b}^{1}] = (\epsilon_{ab} R_{1}^{a} + \epsilon_{ab} R_{1}^{a}), \quad R_{ab} \sim (Q_{ab}, G_{ab}, P_{ab}). \]

The relations (11.6b)-(11.6d), (16.42) define the 27-dimensional Lie algebra \( A_{27} \). This algebra includes very interesting subalgebras listed in the following table.

<table>
<thead>
<tr>
<th>Main subalgebras of the algebra ( A_{27} )</th>
<th>Basis elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AG_{2}(1,3) )</td>
<td>( P_{0} = \frac{P^{2}}{2}, P_{a1}, J_{a}, G_{a}, P \cdot G, G^{2} )</td>
</tr>
<tr>
<td>( AO(1,2) )</td>
<td>( P^{2}, G^{2}, P \cdot G )</td>
</tr>
<tr>
<td>( AGL(3) )</td>
<td>( P_{a}, J_{a}, Q_{ab} )</td>
</tr>
<tr>
<td>( AP(1,2) )</td>
<td>( P_{a}, Q_{12}, Q_{13}, J_{1} )</td>
</tr>
</tbody>
</table>

Thus, the Schrödinger equation has very extensive hidden symmetries including the Lie algebras of the groups of general inhomogeneous linear transformations \( IGL(3) \) and of the generalized Poincaré group in three-dimensional space. These algebras are realized in the class of second-order differential operators and can be used for various purposes including separation of variables and construction of exact solutions of linear and non-linear problems based on the Schrödinger equation.

Considering SOs of order \( n>2 \) we can find more extensive IAs of the Schrödinger equation. Here we represent two of them

\[ L_{1} \supseteq \left\{ P_{a1}, J_{a}, G_{a}, Q_{ab}, G_{a_{1}a_{2}...a_{i}} = G_{a_{i}} G_{a_{i-1}}...G_{a_{1}} \right\}, \quad i \leq n, \]

\[ L_{2} \supseteq \left\{ P_{a1}, J_{a}, G_{a}, Q_{ab}, P_{a_{1}a_{2}...a_{i}} = P_{a_{i}} P_{a_{i-1}}...P_{a_{1}} \right\} \]

where \( Q_{ab} \) is the operator (16.41a), \( n \) is an arbitrary integer.

It is not difficult to verify the operators (16.43) form finite-dimensional Lie algebras for any fixed \( n \).
Symmetries of Equations of Quantum Mechanics

Symmetries of the Schrödinger equation under integral transformations are considered in [165, 338], where so exotic IAs are found as \(AO(1,4)\) and \(AC(1,3)\).

16.7. Symmetries of the Quasi-Relativistic Evolution Equation

Let us consider briefly symmetries of a four-order partial differential equation which can be interpreted as a quasi-relativistic generalization of the Schrödinger equation. Namely we consider the equation

\[
\begin{aligned}
& i \frac{\partial}{\partial x_0} \Psi = H \Psi , \\
& H = a_0 m + \frac{p^2}{2m} - a_4 \frac{p^4}{8m^3},
\end{aligned}
\]  

(16.44)

where \(a_0, a_4\), and \(m\) are real constants.

If \(a_0 = a_4 = 0\) then (16.44) coincides with the Schrödinger equation for a free particle. For \(a_0 = a_4 = 1\) the Hamiltonian \(H\) includes three the first terms of the Taylor series of the relativistic Hamiltonian \(H'=(m^2+p^2)^{1/2}\).

The maximal IA of the equation (16.44) in the class \(M_1\) is the eight-dimension Lie algebra \(A_8\) including the basis elements \(P_0, P_a, J_a\), and \(M\) of (11.5). Thus this equation is invariant under neither Galilei nor Lorentz transformations. This does not mean however that symmetries of (16.44) are exhausted by the algebra \(A_8\). This equation is invariant under the 20-dimensional Lie algebra including \(P_0, P_a, J_a\), and \(M\) of (11.5) and the following higher order SOs [145]

\[
V_a = i[H, x_a] = \frac{1}{m}(1 - a_4 \frac{p^2}{2m^2})p_a,
\]

(16.45)

\[
G_a = (x_0 V_a - x_a)m, \quad R_{ab} = -\frac{a_4}{m}(p_a p_b + \frac{1}{2}\delta_{ab} p^2).
\]

Using the identities \([H, V_a] = [P_0, x_a] = 0\) it is not difficult to make sure that the operators (16.45) satisfy the invariance conditions (1.5) (together with \(L = i\partial/\partial x_0 - H\)) and the following commutation relations

\[
[P_0, G_a] = imV_a, \quad [V_a, G_b] = im(R_{ab} - \delta_{ab} \frac{1}{m^2} M),
\]

\[
[J_a, R_{bc}] = i(\epsilon_{abc} R_{cd} + \epsilon_{acd} R_{bd}), \quad [P_0, R_{ab}] = 0,
\]

\[
[J_a, V_b] = i\epsilon_{abc} V_c, \quad [P_0, V_a] = [P_a, R_{bc}] = 0,
\]

\[
[G_a, R_{bc}] = \frac{i a_4}{m}(\delta_{ab} p_c + \delta_{ac} p_b + \delta_{bc} p_a).
\]

the commutation relations between \(P_0, P_a, J_a, M\) and \(G_a\) are given in (11.6).

We see the quasirelativistic evolution equation (16.44) is really invariant under the 20-dimensional Lie algebra generated by the SOs \(P_0, P_a, J_a, M\) of (11.5) and
Chapter 4. Nongeometric Symmetries

\( V_\alpha G^\alpha R_{\mu\nu} \) of (16.45). Besides the SOs (16.45) are third-order differential operators and so they do not generate local transformations of the type (11.15). But it is possible to demonstrate (see [387]) that \( G_\alpha \) generate nonlocal transformations of the following kind

\[
\psi'(x') = \exp\left[i m(x_\alpha v_\alpha + \frac{1}{2} x_\alpha v_\alpha^2 - \frac{1}{2}\frac{a_4}{m^3} x_\alpha v_\alpha p_\alpha p_\alpha^2)\right] \psi(x)
\]

(16.46)

where \( v_\alpha \) are transformation parameters and \( x' \) are connected with \( x \) by the Galilei transformations (11.18). For \( a_4 = 0 \) formula (16.46) presents usual Galilei transformations of (11.27).

Let us demonstrate that if we interpret (16.44) as a motion equation for a particle of rest mass \( m \) then the complete mass \( M' \) depends on the velocity. Indeed, the corresponding quantum mechanical operator of velocity is the operator \( V_\alpha \) of (16.45), and the classical analogues of the operators \( p_\alpha \) and \( V_\alpha \) are the momentum \( p_\alpha \) and velocity \( v_\alpha \). In accordance with (16.45) the velocity depends on momentum in the following manner

\[
v_\alpha = p_\alpha - \frac{1}{2}\frac{a_4}{m^3} p_\alpha p_\alpha^2.
\]

(16.47)

But on the other hand we have according to the classical definition of velocity that

\[
v_\alpha = \frac{p_\alpha}{M'}
\]

(16.48)

Substituting (16.48) into (16.47) we come to the cubic equation

\[
\frac{M'}{m} - \frac{1}{2} a_4 v^2 \left( \frac{M'}{m} \right)^3 - 1 = 0.
\]

(16.49)

Solutions of this equation have the form

\[
M' = \frac{3m}{W} \sin \left( \frac{1}{3} \arctan \frac{W}{\sqrt[3]{1 - W^2}} \right), \quad W = \left[ \frac{3m}{2} \right]^{\frac{1}{3}} |a_4| |v|.
\]

(16.49)

It follows from (16.49) that the mechanics based on the equation (16.44) leads to the limiting velocity

\[
v_{\text{lim}} = \left[ \frac{3m}{2} \right]^{\frac{1}{3}} |a_4|^{\frac{1}{2}}
\]

like relativistic mechanics. Besides if \( v \to v_{\text{lim}} \) then \( M' \to (3/2) m \).

In conclusion we note that analogous analysis can be carried out for the equation (16.44) with a more general Hamiltonian \( H = \sum_{n=0}^{N} a_{2n} p^{2n} \).

Approximate symmetries and solutions of wave equations are discussed in [389].
17. NONGEOMETRIC SYMMETRIES OF THE DIRAC EQUATION

17.1. The IA of the Dirac Equation in the Class \( M_1 \)

The hidden symmetries of the Dirac equation are more interesting and complicated than considered above owing to the matrix structure of the corresponding SOs.

In Section 2 we found the maximal IA of the Dirac equation in the class \( M_1 \), it was isomorphic to the Lie algebra of the Poincaré group. It turns out this IA can be extended by including the SOs belonging to the class of first order differential operators with matrix coefficients. We denote this class by \( M_1 \).

For convenience let us rewrite the Dirac equation in the form

\[
L\psi = 0, \quad L = \gamma^\mu p_\mu - m. \tag{17.1}
\]

Using the definitions (16.2) and the algebraic properties of the Dirac matrices (refer to (2.3)) it is not difficult to show that a linear differential operator

\[
Q_A = A^\mu p_\mu + B, \quad A^\mu, B \in G^4 \tag{17.2}
\]

to be a SO of the Dirac equation in the class \( M_1 \) if

\[
[Q_A, L] = (f_A^\mu p_\mu + q_A)L, \tag{17.3}
\]

where \( f_A^\mu, Q_A \) are 4×4 matrices in general depending on \( x \). We make the same suppositions about the wave function and matrices \( f_A^\mu, q_A \): \( \psi \in F^4, f_A^\mu, q_A \in G^4 \).

Let us formulate and prove the following assertion.

**Theorem 17.1** [142]. The Dirac equation is invariant under the eight-dimensional Lie algebra \( A_8 \) defined over the field of real numbers. Basis elements of this algebra can be chosen in the form

\[
\hat{\Sigma}_{\mu\nu} = \frac{m}{4}[\gamma_\mu, \gamma_\nu] + \frac{1}{2}(1 - i\gamma_4)(\gamma_\mu p_\nu - \gamma_\nu p_\mu), \tag{17.4}
\]

\[
\hat{\Sigma}_0 = i L, \quad \hat{\Sigma}_1 = m\gamma_4 - i(1 - i\gamma_4)\gamma_\mu p^\mu,
\]

where \( L \) is the unit matrix. For nonzero \( m \) this algebra is isomorphic to the Lie algebra of the group \( GL(2, \mathbb{C}) \), for \( m=0 \) the operators (17.4) form a commutative (Abelian) algebra.

**Proof** can be carried out by direct verification. Using (2.3) it is not difficult to obtain the following relations

\[
[\hat{\Sigma}_{\mu\nu}, L] = \frac{1}{2}(\gamma_\mu p_\nu - \gamma_\nu p_\mu)L, \quad [\hat{\Sigma}_0, L] = 0, \quad [\hat{\Sigma}_1, L] = -2\gamma_4 p^\mu L \tag{17.5}
\]

which have the form (17.3). If \( m=0 \) these operators commute. For nonzero \( m \) we denote
and obtain the following commutation relations

\[ [\Sigma_{\mu}, \Sigma_{\nu}] = g_{\mu\sigma} \Sigma_{\nu\sigma} + g_{\nu\sigma} \Sigma_{\mu\sigma} - g_{\mu\nu} \Sigma_{\sigma\sigma} - g_{\nu\mu} \Sigma_{\sigma\sigma}, \]

\[ [\Sigma_1, \Sigma_0] = [\Sigma_1, \Sigma_{\mu\nu}] = [\Sigma_{\mu\nu}, \Sigma_0] = 0. \]

Thus the operators (17.4) form an IA of the Dirac equation. For \( m = 0 \) this IA is commutative, for nonzero \( m \) it is isomorphic to the algebra \( AGL(2, \mathbb{C}) \). This isomorphism can be established by the following relations

\[
\begin{align*}
\lambda_{11} &\cdot \Sigma_{00} + \Sigma_{12}, & \lambda_{22} &\cdot \Sigma_{00} - \Sigma_{12}, & \lambda_{12} &\cdot \Sigma_{23} - \Sigma_{02}, \\
\lambda_{21} &\cdot \Sigma_{23} + \Sigma_{00}, & \lambda_{11} &\cdot \Sigma_{11} + \Sigma_{03}, & \lambda_{22} &\cdot \Sigma_{11} - \Sigma_{03}, \\
\tilde{\lambda}_{12} &\cdot \Sigma_{01} + \Sigma_{11}, & \tilde{\lambda}_{11} &\cdot \Sigma_{11} + \Sigma_{01}
\end{align*}
\]

where \( \lambda_{ab} \) and \( \tilde{\lambda}_{ab} \) are the basis elements of the algebra \( AGL(2, \mathbb{C}) \) satisfying the commutation relations

\[ [\lambda_{ab}, \lambda_{cd}] = [\tilde{\lambda}_{ab}, \tilde{\lambda}_{cd}] = \delta_{ac} \lambda_{bd} - \delta_{ad} \lambda_{bc} - \delta_{bc} \lambda_{ad} + \delta_{bd} \lambda_{ac}, \]

\[ [\lambda_{ab}, \tilde{\lambda}_{cd}] = \delta_{ac} \tilde{\lambda}_{bd} - \delta_{ad} \tilde{\lambda}_{bc} - \delta_{bc} \tilde{\lambda}_{ad} + \delta_{bd} \tilde{\lambda}_{ac}. \]

We give the other more constructive proof of Theorem 17.1 in order to explain the nature of the additional symmetry of the Dirac equation. It is well known that the system of four first-order equations (17.1) is equivalent to the system of two second-order equations. Multiplying (17.1) from the left by \( (1 \pm i \gamma_4)/2 \), denoting

\[ \psi_\pm = \frac{1}{2} (1 \pm i \gamma_4) \psi \]

and expressing \( \psi \) via \( \psi_\pm \) we come to the equations

\[ (p_\mu p^\mu - m^2) \psi_\pm = 0, \]

\[ \psi_\pm = \frac{1}{m} \gamma_\mu p^\mu \psi. \]

But the equations (17.10), (17.11a) have the evident symmetry under arbitrary matrix transformations commuting with \( \gamma^\nu_4 \). They are the transformations generating the hidden symmetry described in Theorem 17.1.

To describe effectively all nonequivalent transformations of this kind, we use the fact that the transition from (17.1) to (17.11) can be represented in the form
Symmetries of Equations of Quantum Mechanics

\[ \psi \rightarrow \psi' = V \psi, \quad L \rightarrow L' = WLW^{-1} \]

(17.12)

where

\[ V = \exp \left[ -\frac{1}{2m} (1 - i \gamma_\mu \gamma_\nu \gamma_{\mu \nu}) \right] = 1 - \frac{1}{2m} (1 - i \gamma_\mu \gamma_\nu \gamma_{\mu \nu}) \]

(17.13a)

\[ W = \exp \left[ -\frac{1}{2m} (1 + i \gamma_\mu \gamma_\nu \gamma_{\mu \nu}) \right] = 1 + \frac{1}{2m} (1 + i \gamma_\mu \gamma_\nu \gamma_{\mu \nu}) \]

(17.13b)

\[ L' = -\frac{1}{2} (1 - i \gamma_\mu \gamma_\nu \gamma_{\mu \nu} + \frac{1}{2m} (1 + i \gamma_\mu \gamma_\nu \gamma_{\mu \nu}) \psi \]

(17.13c)

Indeed, the equation

\[ L' \psi' = 0, \quad \psi' = V \psi \]

(17.14)

reduces to (17.11) moreover \( \psi' = V \psi \).

The equation (17.14) is manifestly invariant under arbitrary matrix transformations commuting with \( \gamma_\mu \). The corresponding transformation operator can be represented as a linear combination of the matrices

\[ \Sigma_{\mu \nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu], \quad \Sigma_i = i \gamma_i, \quad \Sigma_0 = I \]

(17.15)

satisfying the relations (17.7) and so forming a basis of the eight-dimension Lie algebra. It is not difficult to make sure these matrices are linearly independent over the field of real numbers while

\[ (\Sigma'_{ab} - i \varepsilon_{abc} \Sigma_0' c) \psi' = (\Sigma_1 + i \Sigma_0) \psi' = 0 \]

where \( \psi' \) is a solution of (17.1).

Using the operator (17.13a) we obtain from (17.15) the explicit form of basis elements of the IA of the Dirac equation: \( \Sigma_\mu = V^{-1} \Sigma_{\mu} V, \Sigma_{\mu \nu} = V^{-1} \Sigma_{\mu \nu} V \) where \( \Sigma_{\mu \nu} \) and \( \Sigma_\mu \) are the operators (17.4), (17.6). \( \blacksquare \)

It follows from the proof that the additional symmetry of the Dirac equation described in Theorem 17.1 is maximal in the sense it includes all the possible matrix transformations in the representation (17.14).

So besides the well-known Poincaré-invariance the Dirac equation is additionally invariant under the algebra \( A_8 \). Basis elements of this algebra do not belong to the class \( M_1 \) and thus cannot be considered as a Lie derivatives. In spite of this fact, they form a Lie algebra and satisfy the relations

\[ \Sigma_0^2 = -\Sigma_{ab}^2 = -\Sigma_0^2 = -1 \]
Chapter 4. Nongeometric Symmetries

and so generate the eight-parameter group of transformations

\[ \psi \rightarrow \psi' = \left( \cos \theta_{ab} \gamma_a \sin \theta_{ab} \right) \psi - \frac{i}{m} \sin \theta_{ab} (1 - \gamma_z \partial x^\mu - \gamma^\mu \partial x_z), \]

\[ \psi \rightarrow \psi'' = \left( \cosh \theta_{0a} \gamma_0 \sinh \theta_{0a} \right) \psi - \frac{i}{m} \sinh \theta_{0a} (1 - \gamma_0 \partial x^a - \gamma^a \partial x_0), \]

(no sum over repeated indices),

\[ \psi \rightarrow \psi''' = \exp \theta_1 \gamma_4 \sin \theta_1 \psi - \frac{1}{m} \sin \theta_1 (1 - \gamma_4 \partial x^m - \gamma^m \partial x_4), \]

where \( \theta_{ab}, \theta_{0a}, \theta_0 \) and \( \theta_1 \) are real parameters. Any of formulae (17.16), (17.17) defines a one-parameter transformation group and can be represented in the form \( \psi \rightarrow \exp(2 \Sigma_\alpha \theta_\alpha) \psi \), where \( \alpha = 0, 1, 01, 02, \ldots \).

The principal distinguishing feature of the transformations (17.16) with respect to Lorentz transformations (2.59) is that the transformed functions depend on derivatives of the wave function. Furthermore, in accordance with (17.16) independent variables are not transformed in contrast to Lorentz transformations. In other words the additional invariance of the Dirac equation has nothing to do with transformations of the space-time continuum, that is why we call this symmetry nongeometric.

The question arises of whether it is possible to combine the symmetry group of the Dirac equation given by the transformations (17.16), (17.17) and the Poincaré group. It turns out that such a unification is possible since the generators of these transformations and the Poincaré group generators form an 18-dimensional Lie algebra.

**THEOREM 17.2.** The Dirac equation is invariant under the 18-dimensional Lie algebra whose basis elements are given by formulae (2.22), (17.4), (17.6) and satisfy the commutation relations (1.14), (17.7) and (17.18):

\[ [P_\mu, \Sigma_{\lambda\nu}] = [P_\nu, \Sigma_{\lambda\mu}] = [P_\mu, \Sigma_\lambda] = 0, \]

\[ [J_{\mu\nu}, \Sigma_\sigma] = i(g_{\mu\sigma} \Sigma_{\nu\lambda} + g_{\nu\sigma} \Sigma_{\mu\lambda} - g_{\mu\lambda} \Sigma_{\nu\sigma} - g_{\nu\lambda} \Sigma_{\mu\sigma}). \]

The proof of the theorem reduces to direct verifying the validity of relations (17.18).

We conclude from the above that the Dirac equation is invariant under the 18-parametrical Lie group including Lorentz transformations (2.49) and the nongeometric transformations (17.16), (17.17). A general form of this group transformation is

\[ \psi \rightarrow \psi' = A \psi (x') + B_\mu \frac{\partial \psi (x')}{\partial x_\mu} \]

where \( x'_\mu \) are connected with \( x_\mu \) by Lorentz transformations, \( A \) and \( B_\mu \) are numerical
17.2. Symmetries of the Dirac Equation in the Class of Integro-Differential Operators

Let us denote by \( M_\infty \) the set of nonlocal (integral) operators \( Q \) of the kind

\[
\tilde{\psi}(p, x_0) = \frac{(2\pi)^{-3/2}}{2\pi^2} \int \exp(-ip \cdot x) \psi(x, x_0) d^3p,
\]

where \( \tilde{\psi} \) is a Fourier transform of \( \psi(x) \):

\[
Q^\psi(x) = (2\pi)^{-3/2} \int \exp(ip \cdot x) \psi(x, x_0) d^3p,
\]

\( Q \) is a 4\( \times \)4 matrix depending on \( p \).

We show the Dirac equation has an additional invariance under the transformations belonging to the class \( M_\infty \). Moreover, the corresponding IA is more extensive than established above.

**THEOREM 17.3** [142]. The Dirac equation is invariant under the algebra \( A_8 \) defined over the field of complex numbers. The symbols of basis elements of this algebra are given by the formulae

\[
\tilde{\Sigma}^a_0 = \frac{1}{4} [\gamma^0, \gamma^a], \quad \frac{1}{2m} (\gamma^a - \gamma^0) \left( 1 - \frac{\gamma^4}{E} \right), \quad \Sigma^0 = I, \quad \Sigma_i = \frac{H}{E},
\]

where \( H \cdot \gamma^0 \cdot \gamma^a \cdot \gamma^0 \cdot m, \quad E = \sqrt{p^2 + m^2} \).

Instead of the proof we present the explicit form of the operator \( V \) diagonalizing the symbol of the operator \( L \) of (17.1)

\[
V = P_+ - P_+ \frac{H}{E}, \quad V^{-1} = \frac{1}{m} (H \gamma^0 + P_+), \quad P_+ = (1 + \gamma^0)/2.
\]

It is easy to ascertain that

\[
\frac{\partial}{\partial x_0} - \gamma^0 E.
\]

Symmetry operators \( \tilde{\Sigma}_i \in M_\infty \) in the representation (17.23) are matrices commuting with \( \gamma^0 \). Any such a matrix is a linear combination of the following basis matrices

\[
\tilde{\Sigma}^a_0 = [\gamma^0, \gamma^a], \quad \frac{1}{4} \gamma^a, \quad \gamma^a - \gamma^0, \quad \gamma^0, \quad \gamma^0 - 1
\]

which are linearly independent over the field of complex numbers and satisfy the commutation relations (17.7). Using the transformation \( \tilde{\Sigma}_i \rightarrow \Sigma_i = V^\dagger \tilde{\Sigma}_i V \) we obtain the SOs (17.22) in the starting representation. 

So the Dirac equation possesses an additional invariance under the nonlocal
operators belonging to the class $M$. We note that the basis elements of the corresponding IA are similar to (17.4), (17.6) and possess the only nonlocal element, i.e., the energy sign operator $\hat{\epsilon}=H/E$. But in contrast to (17.4), (17.6) the operators (17.22) are linearly independent over the field of complex numbers.

It is not difficult to make sure the operators (17.22) form a Lie algebra together with the Poincaré group generators (2.40) and satisfy the commutation relations (17.18). Thus the symmetry of the Dirac equation in the class $M$ is represented by the 18-dimensional Lie algebra $A_{18}$ including the subalgebra $AP(1,3)$. It can be shown the algebra $A_{18}$ is isomorphic to $A P(1,3) \oplus A G L(2, C)$.

In conclusion we note that it follows from Theorem 17.3 the Dirac equation is invariant under the 16-parametric group of transformations $\tilde{\psi} \rightarrow \tilde{\psi}'=\exp(\tilde{\Sigma}, \theta_{i})\tilde{\psi}$ where $\theta_{i}$ are complex parameters. The explicit form of these transformations is given by the following formulae

\[
\psi \rightarrow (\cos \theta_{ab}+\gamma_{a}\gamma_{b}\sin \theta_{ab})\psi - \frac{i}{m} \sin \theta_{ab} \sum_{a} (1+i\epsilon \gamma_{a})\gamma_{a} \frac{\partial \psi}{\partial x_{a}} - \gamma_{a} \frac{\partial \psi}{\partial x_{a}},
\]

\[
\psi \rightarrow (\cosh \theta_{0b}+\gamma_{0}\gamma_{b}\sinh \theta_{0b})\psi - \frac{i}{m} \sinh \theta_{0b} \sum_{a} (\gamma_{0} \frac{\partial \psi}{\partial x_{a}} + \gamma_{0} \frac{\partial \psi}{\partial x_{a}}),
\]

\[
\psi \rightarrow \sum_{z} (\cosh \theta_{z}+\sinh \theta_{z})\psi_{z}, \quad \psi \rightarrow \exp(\epsilon \theta)\psi.
\]

Here

\[
\psi' = \frac{1}{2} (1 - i \frac{H}{E}) \psi, \quad \psi \rightarrow \psi' + (\psi' - \psi), \quad \epsilon = \pm.
\]

17.3. Symmetries of the Eight-Component Dirac Equation

Until now, we consider only linear symmetry transformations of the Dirac equation. Here we extend class of symmetries by considering antilinear transformations which are also admissible in quantum mechanics.

Consider the eight-component Dirac equation

\[
\hat{L}\psi = 0, \quad \hat{L}=\Gamma_{\mu}p^{\mu} - m, \quad (17.24)
\]

where $\Gamma_{\mu}$ are 8x8 matrices satisfying the Clifford algebra (8.2) together with $\Gamma_{4}, \Gamma_{5}, \Gamma_{6}$. Choosing $\Gamma_{\mu}$ and the wave function in the form

\[
\Gamma_{\mu} = \begin{pmatrix} \gamma_{\mu} & 0 \\ 0 & \gamma_{\mu} \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \gamma_2 \psi_1 \end{pmatrix}, \quad (17.25)
\]

203
where \( \gamma_\mu \) are the 4\times4 Dirac matrices in the realization (2.4), \( \psi \) is a four-component wave function, we obtain from (17.24) a system of equations including the Dirac equation and the adjoint equation (2.8).

The condition (17.25) can be represented in the form independent on realization of the \( \Gamma \)-matrices:

\[
(1 - i\Gamma_5 \Gamma_6 \text{C}) \psi = 0
\]

(17.26)

where \( \text{C} \) is the charge conjugation operator. In the representation (17.25) we can choose

\[
\Gamma_4 = \begin{pmatrix} 0 & \gamma_4 \\ \gamma_4 & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & -\gamma_4 \\ \gamma_4 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} \gamma_4 & 0 \\ 0 & -\gamma_4 \end{pmatrix}
\]

(17.27)

\( \text{C} \psi = i\Gamma_5 \psi^* \).

Thus the Dirac equation and the adjoint equation (2.8) can be represented in the form (17.24) with the additional condition (17.26). Besides linear transformations of solutions of (17.24), (17.27) correspond to linear or antilinear transformations of solutions of the four-component Dirac equation. Moreover such a correspondence is an isomorphism. So the problem of description of linear and antilinear SOs of the Dirac equation is equivalent to finding the IA of the equations (17.24), (17.26) in the class of linear operators.

We note that the equation (17.24) admits different interpretations including that of the equation of motion of particles with spin 1 (see Section 8) and 3/2 [233]. This makes the problem of investigation of symmetries of this equation even more interesting. Here we consider symmetries of (17.24) without additional conditions and then symmetries of the system (17.24), (17.26).

Thanks to the increase of the number of components of the wave function the equation (17.24) has more extended symmetry than the four-component Dirac equation. In addition to manifest invariance under the Poincaré group (whose generators are given by formulae (2.22) where \( S_{\mu\sigma} = [\Gamma_{\mu}, \Gamma_{\sigma}] \)) this equation admits matrix transformations commuting with \( \Gamma_{\mu} \). Furthermore, the eight-component Dirac equation has a hidden (nongeometric) symmetry including SOs in the classes \( M_1 \) and \( M_\infty \).

In analogy with Theorem 17.3 it is possible to prove the following assertion.

**THEOREM 17.4.** The eight-component Dirac equation is invariant under the 32-dimensional Lie algebra defined over the field of complex numbers. The basis elements of this algebra belong to the class \( M_1 \) and are given by the formulae

\[
Q_{\mu\nu\lambda}, Q_{\alpha\beta}, Q_{\alpha\mu\nu}, Q_{\alpha\beta\mu}, Q_{\alpha\beta
\gamma}, Q_{\alpha\beta\mu\nu}, Q_{\alpha\beta\mu\nu\lambda}, Q_{\alpha\beta\mu\nu\lambda\gamma}
\]

(17.28)
where
\[ \Sigma_{\mu} = \frac{1}{2} \left[ \Gamma_{\mu}, \Gamma_{\nu} \right], \quad \frac{i}{m} \left( \Gamma_{\mu} p_{\nu} - \Gamma_{\nu} p_{\mu} \right) (1 - \Gamma_{4} \Gamma_{5} \Gamma_{6} \epsilon). \] (17.29)

\[ \Sigma_0 = D_0 = I, \quad \Sigma_1 = \Gamma_{4} \Gamma_{5} \Gamma_{6}, \quad \Sigma_2 = \Gamma_{1} \Gamma_{2}. \] (17.30)

I is the unit matrix, \( \epsilon = 1 \) or \( \epsilon = -1 \).

The proof is similar to the proof of Theorem 17.3. The Lie algebra spanned on the basis (17.28) is isomorphic to \( AGL(4, \mathbb{C}) \).

The operators (17.28) form a closed algebra together with the Poincaré group generators (2.22) (where \( S_{\mu \sigma} = i [\Gamma_{\mu}, \Gamma_{\sigma}] \)) inasmuch as
\[ [P_{\mu}, Q_{\alpha \lambda}] = [P_{\mu}, Q_{\alpha \lambda}] = [J_{\mu \nu}, Q_{\alpha \lambda}] = 0, \]
\[ [J_{\mu \nu}, Q_{\alpha \lambda}] = i (g_{\mu \nu} Q_{\alpha \lambda} + g_{\alpha \lambda} Q_{\mu \nu} - g_{\mu \lambda} Q_{\alpha \nu} - g_{\alpha \nu} Q_{\mu \lambda}). \]

It is possible to show the symmetry formulated in Theorem 17.4 defines the maximal IA of the eight-component Dirac equation in the class \( M_{\infty} \).

In complete analogy with the results presented in the preceding section we can show that the nongeometric symmetry of the eight-component Dirac equation in the class \( M_{\infty} \) is more extensive and is determined by the 32-dimensional Lie algebra defined over the field of complex numbers. Basis elements of this algebra are defined by formulae (17.28)-(17.30) where \( \epsilon = \Sigma_0 = H/E, H = \Gamma_0 \Gamma_{\mu} p_{\mu} + \Gamma_0 p_{\nu} \).

Before we restrict ourselves to searching sets of SO of the Dirac equation, forming bases of Lie algebras. Now we will demonstrate the existence of a more complicated algebraic structure generated by these SO.

**THEOREM 17.5.** The eight-component Dirac equation admits an IA isomorphic to the Poincaré superalgebra. Basis elements of this IA have the form
\[ P_{\mu} = -i \frac{\partial}{\partial x_{\mu}}, \quad J_{\mu \nu} = x_{\mu} p_{\nu} - x_{\nu} p_{\mu} + \Sigma_{\mu \nu}, \]
\[ Q_1 = \sqrt{2} \Gamma_{1} \left[ \Gamma_{1} + \frac{1}{m} \left( 1 + i \Gamma_{4} \right) (p_{1} + ip_{2}) \right] / 2, \]
\[ Q_2 = \sqrt{2} \Gamma_{2} \left[ \Gamma_{3} + \frac{1}{m} \left( 1 + i \Gamma_{4} \right) (p_{0} + ip_{3}) \right] / 2, \]
\[ \bar{Q}_1 = \sqrt{2} \Gamma_{3} \left[ (p_{0} - p_{3}) (\Gamma_{1} - i \Gamma_{2}) + (p_{1} - ip_{2}) (\Gamma_{3} - \Gamma_{0}) \right] / 2, \]
\[ \bar{Q}_2 = \sqrt{2} \Gamma_{3} \left[ (\Gamma_{3} p_{0} - \Gamma_{0} p_{3} + i (\Gamma_{1} p_{2} - \Gamma_{2} p_{1})) - im \Gamma_{4} \right] / 2, \]

where
Symmetries of Equations of Quantum Mechanics

\[ \Sigma_{ab} = \frac{i}{8} [\Gamma_a \Gamma_b - \Gamma_b \Gamma_a - 2 \epsilon_{abc} \Gamma_c (1 - i \Gamma_4) (\Gamma_\mu p_\mu - \Gamma_\mu p_\mu)] \]

\[ \Sigma_{ac} = \frac{i}{2} \epsilon_{abc} \Sigma_{bc} \]

and \( \Gamma_i \) are the 8x8 Dirac matrices which can be chosen in the form (17.25), (17.27).

**PROOF** reduces to direct verification that the mentioned SOs do satisfy the invariance condition (17.3) and the commutation and anticommutation relations (1.14) and

\[ [\tilde{Q}_A, \tilde{Q}_B] = [Q_A, Q_B] = 0 \]

\[ [Q_{\alpha}, Q_{\beta}] = 2 (\sigma_{\alpha \beta})_{ab} P^a \]

\[ [J_{\mu \sigma}, Q_\lambda] = -\frac{1}{2i} (\sigma_{\mu \lambda})_{ab} Q_\sigma \]

\[ [J_{\mu \sigma}, \tilde{Q}_\lambda] = -\frac{1}{2i} (\sigma_{\mu \lambda})_{ab} \tilde{Q}_\sigma \]

\[ [P_{\mu}, Q_\lambda] = [P_{\mu}, \tilde{Q}_\lambda] = 0 \]

where \( \sigma \) are the Pauli matrices and \( A, B = 1, 2 \)

which characterize the Poincaré superalgebra [411], refer also to Appendix 1 for definitions.

A specific feature of our realization of the Poincaré superalgebra is that the generators \( J_{\mu \sigma} \) are the first-order differential operators with matrix coefficients.

17.4. Symmetry Under Linear and Anti-linear Transformations

Let us investigate symmetries of the eight-component Dirac equation with the additional condition (17.26). As was noted in the above linear symmetries of this system correspond to linear and anti-linear symmetries of the eight component Dirac equation. To denote the classes of SOs including linear and antilinear transformations we use the symbols \( \mathcal{M}_1^*, \mathcal{M}_2^*, \ldots \) where \( \mathcal{M}_1^* \) is the class of linear and antilinear differential operators of first order etc.

The SOs of the eight-component Dirac equation are not SOs of the additional condition (17.26) in general. We will see that the symmetry of the system (17.24), (17.26) is more restricted than the symmetry of the eight-component Dirac equation but is more extended than the symmetry of the four-component Dirac equation.

**THEOREM 17.6.** The system of equations (17.24), (17.26) is invariant under a 14-dimensional Lie algebra isomorphic to \( AP(1,3) \oplus AO(1,2) \oplus T_1 \). Basis elements of this algebra can be chosen in the form (2.22), (17.30) where \( S_{\mu \nu} = [\Gamma_\mu, \Gamma_\nu] \), \( \Gamma_k \) \((k=0,1,\ldots,6)\) are 8x8 Dirac matrices satisfying the relations \( CT_\alpha = \Gamma_\alpha C \), \( CT_\alpha = \Gamma_\alpha C \), \( n=0,1,\ldots,5 \). The real Lie algebra spanned on the basis (2.22), (17.30) is the maximal IA of the system (17.24), (17.26) in the class \( \mathcal{M}_1 \).

**PROOF.** It is easily verified the operators (2.22), (17.30) commute with \( L_3 = \Gamma_\mu p^\mu - m \) and \( L_3 = -i \Gamma_4 \Gamma_\mu C \) and so are the SOs of the eight-component Dirac equation with the additional condition (17.26). To prove these operators form the maximal IA of the system (17.24), (17.26) it is sufficient to show that any \( Q \in \mathcal{M}_1 \).
satisfying the conditions \([QL_1]=\beta Q_1 L_1+\alpha Q_2 L_2, [QL_2]=\beta Q_2 L_1+\alpha Q_1 L_2\), where \(\alpha, \beta\) are matrices depending on \(x\), is a linear combination of the basis elements (2.22), (17.30).

We do not represent here the corresponding calculations but show a complete set of matrices which can be used to expand all unknown quantities:

\[
\{Q_A, \Gamma^\mu Q_A, \Gamma^\nu Q_A, \Gamma^\mu \Gamma^\nu Q_A, \Gamma^\mu \Gamma^\nu \Gamma^\lambda Q_A\}.
\]

Here \(Q_A (A=1,2,3,4)\) are the matrices (17.30), \(\mu, \nu, \lambda=1,2,3,4, \mu \neq \lambda, \mu \neq \nu, \lambda \neq \nu\).

The operators \(P_\mu, J_\sigma\) commute with \(Q_a\) which satisfy the commutation relations characterizing the Lie algebra of the group \(O(1,2)\).

We see that besides the obvious symmetry under the Poincaré algebra the system (17.24), (17.26) is invariant in respect with the three-dimension matrix algebra \(AO(1,2)\) (the trivial identity symmetry operator is not discussed here). So this system is invariant under the three-parameter group of matrix transformations

\[
\psi \rightarrow \exp(\frac{i}{2} \theta_1) \psi,
\]

\[
\psi \rightarrow \cosh \frac{\theta_1}{2} \psi + i \gamma_2 \sinh \frac{\theta_1}{2} \psi^*,
\]

\[
\psi \rightarrow \cosh \frac{\theta_1}{2} \psi - i \gamma_2 \sinh \frac{\theta_1}{2} \psi^*.
\]

The invariance of the Dirac equation under the transformations (17.32) was apparently established for the first time by Plebanski (see [72, 227, 364]). We note that the existence of this symmetry follows from Lemma 1.1, refer to Subsection 1.7.

**THEOREM 17.7.** The eight-component Dirac equation with the additional condition (17.26) is invariant under the 16-dimensional Lie algebra defined over the field of real numbers. Basis elements of this algebra belong to the class \(M_1^*\) and are given by the formulae

\[
\Sigma_{mn} = \frac{1}{2} [\Gamma^\mu \Gamma^\nu]_{mn} + \frac{1}{m} (1 - i \Gamma_5) (\Gamma_m p_n - \Gamma_n p_m), \quad \Sigma_0 \equiv I, \quad m, n \equiv 0, 1, \ldots, 5.
\]

The proof is similar to the proof of Theorem 1.1. We note that by definition \(p_{3a} \psi(x) = -i \partial / \partial x_{3a} \psi(x) = 0\), hence the operator \(\Sigma_{4+} \equiv \Sigma_{45}\) reduces to the numerical matrix.

The generators (17.33) satisfy the commutation relations (17.7) where \(g_{mn} = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1)\) and form the Lie algebra isomorphic to \(AO(1,5) \oplus T_1 \oplus T_1 \supset \Sigma_o\).
The operators (17.33) form a closed algebra together with the Poincaré group generators, satisfying the following relations

\[
[P_\mu, \Sigma_{m'\nu}] = [J_{\mu\nu}, \Sigma_{m'\nu}] = 0, \quad [J_{\mu\nu}, \Sigma_{\lambda\rho}] = i(g_{\lambda\mu} \Sigma_{\nu\rho} - g_{\lambda\rho} \Sigma_{\nu\mu}),
\]

(17.34)

where \(m', n' > 3\), \(\mu, \nu, \rho, \lambda \leq 3\). Hence it follows the system (17.24), (17.26) is invariant under the 26-parameter group including the Poincaré group and the group of transformations \(\hat{\psi} \rightarrow \exp(\Sigma_{\lambda\rho} A^\lambda_\rho) \hat{\psi}\) where \(A^\lambda_\rho\) are arbitrary real parameters. The latter transformations have the following explicit form

We can collate a linear or antilinear transformation of solutions of the four-component Dirac equation to any transformation of the kind (17.35). Substituting (17.25), (17.27) into (17.35) we obtain for \(k, l \leq 3\) the transformations (17.16). If \(k > 0\) then the corresponding transformations have the form

\[
\psi \rightarrow (\cos \theta_{a_4} + \Gamma_4 \sin \theta_{a_4}) \psi - \frac{i}{m} (1 - i\Gamma_4) \sin \theta_{a_4} \left( \Gamma_i \frac{\partial \psi}{\partial x_i} - \Gamma_4 \frac{\partial \psi}{\partial x_4} \right) \sin \theta_{a_4},
\]

\[
\psi \rightarrow \exp(i \theta_{a_4}) \psi,
\]

(17.35a)

\[
\psi \rightarrow (\cos \theta_{a_5} - \Gamma_4 \sin \theta_{a_5}) \psi - \frac{i}{m} (1 - i\Gamma_4) \sin \theta_{a_5} \left( \Gamma_i \frac{\partial \psi}{\partial x_i} + \Gamma_4 \frac{\partial \psi}{\partial x_4} \right) \sin \theta_{a_5},
\]

(17.36)

Formulae (17.16), (17.36) define the sixteen one-parameter transformations which do not change the form of the Dirac equation. We see the symmetry group of this equation in the class \(M_1^*\) is more extensive than in the class \(M_1\) and includes the latter as a subgroup.
17.5. Hidden Symmetries of the Massless Dirac Equation

We saw in Chapter 1 that IAs of motion equations for massless fields are more extended than the corresponding symmetries of equations describing particles of nonzero masses.

Here we investigate nongeometric symmetries of the massless Dirac equation which also turn out to be more wide than in the case of nonzero mass.

It was noted in Subsection 2.8 that the massless Dirac equation is invariant under the 16-dimensional Lie algebra whose basis is formed by the generators of the conformal group and the operator \( F = -\gamma^4 \). Besides this equation is invariant under the eight-dimensional commutative (Abelian) algebra belonging to the class \( M_1 \). Basis elements of this IA are given in (17.4) where \( m = 0 \).

The SOs (17.4) form a 18-dimensional Lie algebra together with the generators of the Poincaré group. This algebra cannot be extended by including the conformal group generators (2.42) inasmuch as the corresponding set of SOs is not closed in respect with the Lie brackets.

The IA of the massless Dirac equation given in Theorem 1.3 can be extended with the help of SOs belonging to the class \( M_1^* \). Let us write this equation in the form (17.24), (17.26) where \( m = 0 \). Then any SO of this system in the class \( M_1 \) corresponds to the SO of the four-component Dirac equation in the class \( M_1^* \), since linear and antilinear transformations can be represented as linear transformations for the real and imaginary components of the wave function.

In analogy with Theorem 17.6 we can formulate and prove the following assertion.

**THEOREM 17.8.** The maximal IA of the system (17.24), (17.26) (with \( m = 0 \)) in the class \( M_1 \) is a linear span of the basis elements \( \{P_\mu, J_\sigma^\mu, D, K_\mu, D^\mu, D^4 K_k, D^4 + k\} \) where

\[
Q_{4+k} = \Gamma^\mu \Gamma^\sigma \Gamma_k Q_k, \quad k = 1,2,3,4,
\]

and \( P_\mu, J_\rho^\sigma, K_\mu, D, D^\mu \) are the operators (2.22), (2.25), (17.30) (where \( S_{\mu \alpha} = i\lbrack \Gamma_\mu, \Gamma_\alpha \rbrack \)).

The proof is similar to the proof of Theorem 17.6 and so is not given here. The operators \( D_4, D_{4+k} \) commute with \( P_\mu, J_\rho^\sigma, K_\mu, D \) and form an eight-dimensional subalgebra. The commutation relations between \( D_4 \) and \( D_{4+k} \) can be represented in the form (17.7) if we denote

\[
D_2 = \Sigma_1, \quad D_3 = \Sigma_0, \quad D_1 = \Sigma_2, \quad D_6 = \Sigma_3, \quad D_7 = \Sigma_4, \quad D_8 = \Sigma_12, \quad D_9 = \Sigma_{13}, \quad D_{10} = \Sigma_{103}, \quad D_{11} = \Sigma_{03}.
\]

We conclude from the above that the massless Dirac equation admits a 23-dimensional Lie algebra isomorphic to \( AC(1,3) \otimes AGL(2, C) \).

It follows from Theorem 17.6 that the massless Dirac equation is invariant under the 23-dimensional Lie group locally isomorphic to the group \( C(1,3) \otimes GL(2, C) \).
The corresponding transformations from the conformal group are given in Subsection 2.9. The transformations from the subgroup \( GL(2,\mathbb{C}) \) can easily be found with the formula (17.31). These transformations for \( A \leq 3 \) are given in (17.31), for \( a=4 \) we have a trivial multiplication of wave function by an arbitrary number. For \( A>4 \) we obtain using (17.25), (17.27)

\[
\psi \to (\cosh \frac{\theta}{2} - i \gamma_5 \sinh \frac{\theta}{2})\psi,
\]

\[
\psi \to \cos \frac{\theta}{2} \psi - i \gamma_5 \sin \frac{\theta}{2} \psi^*,
\]

\[
\psi \to \cos \frac{\theta}{2} \psi + i \gamma_5 \sin \frac{\theta}{2} \psi^*,
\]

\[
\psi \to (\cos \frac{\theta}{2} - i \gamma_5 \sin \frac{\theta}{2})\psi.
\]

The group of transformations (17.32), (17.37) includes the subgroup of Pauli-Gürsey transformations [213] generated by \( D_1, D_6, D_7 \) and \( D_8 \). The remaining transformations (17.32), (17.37) are hyperbolic rotations which are not unitary in the metric (2.39). The invariance of the massless Dirac equation under the transformations (17.32), (17.37) was established by Danilov [72] and Ibragimov [226].

We note that the considered symmetry is a mere consequence of Lemma 1.1 and the obvious symmetry of the massless Dirac equation under the transformations \( \psi \to i \gamma_4 \psi \).

If we restrict ourselves to linear transformations then Lie symmetries of the massless Dirac equation reduce to invariance under the algebra \( AC(1,3) \oplus AT \), whose basis elements are given in (2.22), (2.42). But symmetry of this equation in the class \( M_\infty \) is described by a very extended algebra isomorphic to \( AC(1,3) \oplus AGL(2,\mathbb{C}) \). A proof of this statement is given in [148].

Let us summarize. Symmetries of the Dirac equation are very rich and cannot be described if we restrict ourselves to the classical Lie approach. Besides Poincaré invariance this equation admits hidden symmetries in classes of higher order SOs and integro-differential SOs, which form Lie algebras but present non-Lie symmetries. In the following we will demonstrate these SOs include subsets generating hidden supersymmetry of the Dirac equation.

The maximal IAs of this equation in various classes of SOs are presented in the following table.
### 18. THE COMPLETE SET OF SOs OF THE DIRAC EQUATION

#### 18.1. Introduction and Definitions

Until now we consider only such SOs of the Dirac equation which form a finite-dimension Lie algebra, i.e., satisfy relations (16.3). This restriction is completely justified since in this way it is possible to find not only algebras but also groups of nongeometric symmetry. However for great many of applications (e.g., for constructing of constants of motion, for describing coordinate systems admitting separation of variables etc.) it is not essential that SOs belong to a finite-dimensional Lie algebra. In principle such operators can belong to infinite-dimensional Lie algebras or possess other algebraic structures. Therefore it is of interest to investigate symmetries of the Dirac equation in a more general approach without requiring that SOs satisfy (16.3).

Here we present the results of such an investigation. More precisely we present a complete set of SOs of any finite order \( n \) for the Dirac equation. We will see such SOs have very interesting algebraic structures forming bases of Lie superalgebras.

We recall a Lie superalgebra \( SA \) is a graded vector space closed under a binary operation \((x_\sigma, y_{\sigma'}) \rightarrow [x_\sigma, y_{\sigma'}]_{\sigma\sigma'}\) which generalize the Lie brackets. The simplest gradation is so-called \( \mathbb{Z}_2 \)-gradation when \( SA \) consists in elements of two kinds: even \((E)\) and odd \((O)\). Besides \( SA \) is closed under the commutation and anticommutation relations corresponding to the scheme

\[[E,E]\sim E, \quad [E,O]\sim O, \quad [O,O]\sim E. \quad \text{(18.1)}\]

For more details see Appendix 1.

Let us return to SOs. A formal definition of SOs of order \( n \) for the Dirac equation can be written in the form of (16.6), (16.7) where \( L \) is the Dirac operator \((17.1), \ h^{\alpha\beta}[a] \in G_+ \). In other words a SO of order \( n \) is a differential operator of order

<table>
<thead>
<tr>
<th>Value of ( m )</th>
<th>Classes of SOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m \neq 0 )</td>
<td>( AP(1,3) )</td>
</tr>
<tr>
<td>( m = 0 )</td>
<td>( AG(1,3) \odot T_1 )</td>
</tr>
</tbody>
</table>
with matrix coefficients, besides this operator transforms solutions of the Dirac equation into solutions. To denote the class of such operators we use the symbol $M_n$.

In [382, 243] a complete set of the Dirac equation SOs belonging to the classes $M_1$ and $M_2$ was obtained. The problem of finding such operators reduces to solving of a very complicated system of determining equations.

Following [329] we present a simple proof that all the SOs of the Dirac equation of any order $n$ belong to the enveloping algebra of the Poincaré algebra, and find explicitly all linearly independent SOs. The main idea of this proof is to use the fact that solutions of the Dirac equation have to satisfy the KGF equation and thus a SO of the Dirac equation have to be a SO of the equation (1.1).

### 18.2. The General Form of SOs of Order $n$

Instead of the Dirac equation we consider the equivalent system (17.14). Choosing the realization (2.4), (2.17) for the $\gamma$-matrices we conclude that the function $\psi'$ has two non-zero components only which satisfy the KGF equation (17.11a).

Our chief idea is to describe SOs of order $n$ of the system (17.14) and then to find the corresponding SOs of the Dirac equation. Indeed, we can establish the one-to-one correspondence between SOs $Q'$ of the system (17.14) and SOs of the Dirac equation with the following relation

$$Q = V^{-1}Q'V, \quad Q' = VQV^{-1}$$

where $V$ is the operator (17.13a).

An operator $Q'$ defined on the set of two-component functions $\psi'=\psi^a$ can be expanded in the complete set of four matrices

$$Q' = \sigma^\mu Q'^\mu, \quad \sigma^\mu = e_{abc}^a S_{bc}, \quad \sigma_0 = \frac{1}{3} \sigma^a \sigma_a$$

where $S_{bc}$ are the spin matrices (2.23). An operator $Q'$ is a SO of the system (17.14) iff $Q_0'$ are the SOs of the KGF equation.

Let $Q_\mu'$ in (18.3) be SOs of order $n$ of the KGF equation. Then, according to (16.14) they are polynomials on the generators (1.6) or (which is the same) on the operators $P_\mu, J_\mu$, where $P_\mu, J_\mu$ are the generators (2.22), $S_{\mu\sigma}$ are the matrices (2.23). But it is not difficult to make sure the matrices $S_{\mu\sigma}$ also can be expressed via $P_\mu$ and $J_\mu$ on the set of solutions of the equation (17.14):

$$2S_{\mu\sigma} \psi' = \frac{1}{m^2} (P_\mu W_{\sigma} - P_{\sigma} W_\mu + i e_{\mu\rho\sigma} W_\rho P^\gamma) \psi'$$

where $W_\mu$ is the Lubanski-Pauli vector (2.37). The equation (17.14) is Poincaré-
Chapter 4. Nongeometric Symmetries

invariant, moreover, the corresponding generators of the Poincaré group have the form (2.22). Substituting (2.22) into the l.h.s. of (18.4) and using the equation (17.14) we make sure that relation (18.4) is really satisfied.

It follows from the above the SOs $Q_{\mu}'$ as well as $Q'$ of (18.3) are polynomials on $P_\mu, J_\mu$ (2.22). Inasmuch as the operator $V$ of (17.13) commutes with $P_\mu$ and $J_\mu$ it follows from (18.2) that $Q'=Q$, and thus all the SOs of any finite order for the Dirac equation are polynomials of the generators of the Poincaré group.

Thus, we have proved that any finite order SO of the Dirac equation belongs to the enveloping algebra generated by the algebra $AP(1,3)$. This means the problem of description of a complete set of SOs reduces to going over all the independent linear combinations of products of the generators (2.22).

18.3. Algebraic Properties of the First-Order SOs

By describing SOs of arbitrary order $n$ the key role is played by the case $n=1$. According to the above the corresponding complete set of SOs can be obtained by going over polynomials on $P_\mu, J_\mu$ besides as it will be shown in the following it is sufficient to restrict ourselves to considering polynomials of order $n\leq 3$. As a result we obtain known [382] 26 linearly independent SOs including the Poincaré generators (2.22), the identity operator $I$ and the following fifteen operators

\[ W_{4\mu} = \frac{i}{2} \gamma_4 (p_\mu - m\gamma_\mu), \]
\[ W_{\mu\nu} = \frac{i}{2} \gamma_\mu (p_\nu - m\gamma_\nu), \]
\[ B = \gamma_\mu (D - m\gamma_\mu x^\mu), \]
\[ A_{\mu} = \frac{i}{2} \gamma_\mu \varepsilon^{\mu}_{\nu\rho\sigma} J_{\nu\rho} \gamma_\sigma + \frac{1}{2} \gamma_\mu, \]

where $D$ is the dilatation generator (2.42).

Higher order SOs can be expressed via products of the operators (2.22), (18.5) so it is extremely useful to investigate the algebraic properties of this set. These properties turn out to be very engaging so such an investigation is very interesting by itself.

By direct verification we obtain the following commutation relations

\[ [P_\mu, W_{4\sigma}] = [P_\mu, W_{\lambda\sigma}] = [J_{\mu\nu}, B] = 0 \quad (18.6a) \]
Symmetries of Equations of Quantum Mechanics

\begin{align}
[G_\lambda, J_\nu] &= i(g_{\lambda\nu} G_\sigma - g_{\lambda\nu} G_\sigma), \quad G_\lambda \rightarrow A_\lambda \text{ or } W_4, \\
[J_{\mu\nu}, W_{kl}] &= i(g_{\mu\kappa} W_{\kappa\nu} + g_{\nu\kappa} W_{\kappa\mu} - g_{\mu\nu} W_{\kappa\lambda} - g_{\nu\lambda} W_{\kappa\mu}), \quad (18.6b)
\end{align}

\begin{align}
[P_\mu, B] &= 2iW_\mu, \quad [P_\mu, A_\nu] = i\epsilon_{\rho\sigma\mu\nu} W^{\rho\sigma}, \\
[W_{4\mu}, B] &= \frac{i}{2} P_\mu \times i\alpha_\mu, \quad [W_{\mu\nu}, B] = -\frac{i}{2} ([P_\mu, A_\nu], -[P_\nu, A_\mu]), \\
[W_{4\mu}, A_\nu] &= i(g_{\mu\nu} B + [J_{\mu\nu}, W_{4\lambda}]), \\
[W_{\kappa\nu}, A_\mu] &= i(\epsilon_{\rho\sigma\mu\kappa} P^{\rho\sigma} i \frac{1}{2} (g_{\nu\kappa} P_\sigma - g_{\nu\kappa} P_\sigma, B) - [W_{4\mu}, J_{\kappa\nu}]), \quad (18.7)
\end{align}

\begin{align}
[W_{4\mu}, W_{\mu\nu}] &= -\frac{i}{4} (\epsilon_{\lambda\mu\nu\rho} P^{\rho\kappa} + \epsilon_{\kappa\mu\nu\rho} P^{\rho\lambda} - \epsilon_{\lambda\mu\nu\sigma} P^{\sigma\rho} - \epsilon_{\kappa\mu\nu\sigma} P^{\sigma\rho}) W^{\rho\kappa}, \\
[A_\mu, B] &= i\epsilon_{\nu\rho\sigma\mu} J^{\nu\rho} A_\sigma, \quad [A_\mu, A_\nu] = i\epsilon_{\nu\rho\sigma\mu} (J^{\rho\sigma} B - W^{\rho\sigma}) - J_{\nu\mu}, \\
\end{align}

where \( m, n, k, s, r = 0, 1, ..., 4 \) and \( P_\mu = m \).

Other relations for \( P_\mu, J_\nu, A_\mu \) are given in (1.14), (4.3).

According to (18.7) the complete set of the first order SOs of the Dirac equation does not form a basis of a Lie algebra in contrast to the SOs (2.22). But we recognized that this set includes a subset forming a basis of the 18-dimensional Lie algebra, see (17.4). Here we note that the operators (17.4) can be expressed via \( W_{\mu\nu} \) on the set of solutions of the Dirac equation:

\begin{align}
\Sigma_{\mu\nu} &= \frac{1}{m} (W_{\mu\nu} + i \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} W^{\rho\sigma}). \quad (18.8)
\end{align}

The essentially new point is that the operators (2.22), (18.5) include subsets which have a structure of a superalgebra. To select these subsets we calculate the anticommutation relations

\begin{align}
[W_{4\mu}, W_{\mu\nu}] &= -\frac{1}{2} [J_{\mu\nu}, P_\mu], \quad [W_{\mu\nu}, B] = -mJ_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} W^{\rho\sigma}, \quad (18.9b)
\end{align}

\begin{align}
[A_\sigma, W_{\mu\nu}] &= \frac{1}{2} (\epsilon_{\alpha\lambda\mu} P_\lambda - \epsilon_{\alpha\gamma\mu} P_\gamma) J^{\rho\lambda} - g_{\mu\rho} W_{4\nu} + g_{\nu\rho} W_{4\nu}, \\
[W_{\mu}, A_\nu] &= -\frac{m}{2} \epsilon_{\nu\rho\sigma\mu} J^{\rho\sigma} W_{\mu\nu}, \\
[A_\mu, A_\nu] &= -\frac{1}{2} \left\{ [J_{\mu\nu} J^{\rho\sigma} \frac{1}{2} g_{\mu\nu} - [J_{\mu\nu}, J_{\rho\sigma}], [A_\mu, B] = 0, \quad B^2 = \frac{3}{4} - \frac{1}{2} J_{\mu\nu} \right\}, \quad (18.9c)
\end{align}
where \( g_{kn} = \text{diag}(1, -1, -1, -1, 1) \), \( P_{kn} = P_k P_n \), \( p = m \).

Using (18.6), (18.7), (18.9) it is possible to select various sets of SOs forming superalgebras. The set including the algebra \( AP(1,3) \) and the maximal number of the first order SOs is [160]

\[
S_1 = \{ W_{4i}, P_{\mu}, J_{\mu\nu}, P_{\mu} P_{\nu}, I \}.
\]  

(18.10)

Here the odd operators stand to the left of semicolon, the remaining operators are even. It follows from (1.14), (4.3), (18.6), (18.9) the commutation and anticommutation relations for the set (18.10) correspond to the scheme (18.1), so these operators form a basis of a 30-dimensional superalgebra which we denote by \( SA(30) \).

The superalgebra \( SA(30) \) has a very interesting subalgebraic structure. First it includes the Lie algebra of the Poincaré group generated by \( P_{\mu} \) and \( J_{\mu\sigma} \), secondly it contains two Clifford algebras whose elements are \( C_{\pm a} \) where \( \psi \) is a solution of the Dirac equation, for any of two possible (fixed) values of the superscript.

In accordance with (18.9a) these operators satisfy the relations (compare with (2.3))

\[
[C_{\pm a}, C_{\pm b}] \psi = 2\delta_{ab} \psi
\]

where \( \psi \) is a solution of the Dirac equation, for any of two possible (fixed) values of the superscript.

But the most interesting algebraic structure included in the \( SA(30) \) is formed by the following linear combinations [1]

\[
Q_{\pm a} = 2W_{4a} \pm \epsilon_{alc} W^{bc}, \quad H_{ss} = P_{mm} + m^2 l.
\]

For any fixed sign these operators satisfy the relations

\[
[Q_{\pm a}, Q_{\pm b}] = 2\delta_{ab} H_{ss}, \quad [Q_{\pm a}, H_{ss}] = 0
\]

and so realize a representation of a superalgebra characterizing supersymmetric quantum mechanics with three supercharges [417]. Following [417] we denote this superalgebra by \( sqm(3) \).

Thus the extended first order symmetries of the Dirac equation have a reach algebraic structure. In particular they include the subsets realizing representations of the superalgebra \( sqm(3) \) and in this sense the Witten supersymmetry is generated by the SOs of the Dirac equation.

The commutation and anticommutation relations (18.7), (18.9) are used in the following to calculate a complete set of SOs of arbitrary order \( n \) for the Dirac equation. Besides we present some additional useful relations which are satisfied on the set of solutions of the Dirac equation:
Symmetries of Equations of Quantum Mechanics

\[ W_{\mu\nu} = \frac{1}{m} (P^\mu W_\nu - P^\nu W_\mu), \quad W_{4\mu} = -W_\mu \]

\[ B = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma}, \quad (18.11a) \]

\[ A_\mu = \frac{1}{m} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} W^\sigma - \frac{1}{2 m} P_\mu, \]

\[ P_\mu P^\nu = m^2, \quad P_\mu W^\nu = 0, \quad P_\mu A^\mu = m, \quad P_\mu W^\rho = m W^\rho, \]

\[ [J_{\mu\nu}, W^\gamma] = \frac{1}{2} [B, P_\mu], \quad [J_{\mu\nu}, W^\rho] = m B, \]

\[ \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} W^\sigma = \frac{3}{2} m, \quad \epsilon_{\mu\nu\rho\sigma} W^{\nu\rho} P_\sigma = 0, \quad [J_{\mu\nu}, A^\gamma] = 0, \]

\[ [J_{\mu\nu}, W_{\rho\sigma}], [J_{\rho\sigma}, W_{\mu\nu}] = \frac{1}{4} \left( \epsilon_{\mu\nu\rho\sigma} P_\lambda + \epsilon_{\nu\rho\sigma\lambda} P_\mu - \epsilon_{\rho\sigma\lambda\mu} P_\nu - \epsilon_{\sigma\lambda\mu\nu} P_\rho \right) A^\lambda \]  \quad (18.11b)

18.4. The Complete Set of SOs of Arbitrary Order

According to Subsection 18.2 describing of all the nonequivalent SOs of order \( n \) for the Dirac equation reduces to going through the linearly independent combinations of the kind

\[ Q^i = \eta^a_{\mu_1 \cdots \mu_n} P_{a_{\mu_1}} P_{a_{\mu_2}} \cdots P_{a_{\mu_n}} J_{a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_k}} \]  \quad (18.12)

where \( P_{a_{\mu}}, J_{a_{\lambda}} \) are the generators (2.22), \( \eta^a \) are arbitrary numbers. The index \( k \) can take any integer value from the interval \([0, n]\). Moreover, a priori we cannot exclude the possibilities \( k > n \). It will be demonstrated in the following that it is sufficient to set \( 0 \leq k \leq n + 2 \), \( 0 \leq i \leq k \).

According to (1.14), (2.37), (18.11a) the tensors \( \eta^a \) have to satisfy the conditions (1)-(3) formulated before Lemma 16.1 (see page 184) but do not have the property (4). In other words these tensors are not basic; the reason of this is that the corresponding Lubanski-Pauli vector (2.37), (2.22) is nonzero in contrast to the case of the KGF equation.

To describe effectively linearly independent SOs of (18.12), it is convenient to expand \( \eta^a \) in basic tensors. We present the first terms arising by this expansion

\[ \eta^{a_{\mu_1 \cdots \mu_n} a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_k}} = \lambda \eta^{a_{\mu_1 \cdots \mu_n} a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_k}} + \epsilon_{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_n} a_{\lambda_1}, a_{\lambda_2}, \ldots, a_{\lambda_k}} \]

\[ + \epsilon_{a_{\mu_2}, a_{\mu_3}, \ldots, a_{\mu_n} a_{\lambda_1}, a_{\lambda_2}, \ldots, a_{\lambda_k}} + \epsilon_{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_n} a_{\lambda_1}, a_{\lambda_2}, \ldots, a_{\lambda_k}} \]

216
Chapter 4. Nongeometric Symmetries

Here $\lambda$ are basic tensors, the dots in the last line denote products of three and more then three completely antisymmetric tensors $\varepsilon_{\mu_{\rho_{\sigma}}}$ and symmetrization is imposed over the indices $a_{i_1}, a_{i_2}, \ldots, a_{i_{k-c}}$ and over the pairs of indices $[a_{i_1} b_i]$ ($i=1,2,\ldots,k-c$).

Calculating various convolutions $\eta$ with $\varepsilon_{\rho_{\sigma}}$ we can convert formula (18.14), i.e., express $\lambda$ via $\eta$.

Let us substitute (18.14) into (18.12) and go through all the values of $k$ with increasing order.

The first term in the r.h.s. of (18.12) corresponds to the SO of the kind where $P_a$ and $J_{ab}$ are the Poincaré group generators of (2.22). The order of this operator (18.15)

\[ Q_k^{a_1 a_2 \ldots a_{k-c}} = P_{a_1} P_{a_2} \cdots P_{a_{k-c}} J_{a_1 b_1} \cdots J_{a_{k-c} b_{k-c}} \]

is equal to $k$, the general form of $Q_1$ coincides with (16.14) up to the change from $J_{ab}$ (1.6) to the corresponding generators of (2.22). The number of such operators of order $n$ is equal to the number of SOs of the KGF equation and can be obtained from (16.15) by setting $j=n$:

\[ N_1^{(n)} = \frac{1}{4!} (n+1)(n+2)(2n+3)(n^2+3n+4). \]  

Using relations (2.37), (18.11) we obtain the following representation for the SOs corresponding to the second, third, fourth and fifth-seventh terms in the r.h.s. of (18.14)

\[ Q_2^k \sim B Q_1^{k-2}, \quad Q_3^k \sim W_{\nu} Q_1^{k-2}, \quad Q_4^k \sim W_{\mu \nu} Q_1^{k-3}, \quad Q_5^k \sim A_{\mu} Q_1^{k-3} \]  

where $Q_1^{k-1}$ are the operators (18.14), $B$, $W_{\nu}$, $W_{\mu \nu}$ and $A_{\mu}$ are the operators (18.15). It is easy to see the order of $Q_2$ and $Q_3$ is equal to $k-1$, the order of $Q_4$ and $Q_5$ is $k-2$. Thus to obtain SOs of order $n$ we have to set $k=n+1$ for $Q_2$, $Q_4$ and $k=n+2$ for $Q_3$, $Q_5$, and $k=n+2$ is the maximal needed value of $k$. This circumstance has already been noted in the above.

The terms denoted by dots in (18.14) can be neglected without loss of generality. The corresponding SOs include products of the operators (18.5) and so
reduce to the form (18.15), (18.17) with a smaller value of \( k \) in accordance with relations (18.7), (18.9).

We see that any finite order SO of the Dirac equation can be represented as a product of a SO of (18.15) (where \( P_\mu \) and \( J_{\sigma\rho} \) are the Poincaré group generators (2.22) corresponding to the Dirac equation) with one of the operators (18.5).

Using relations (18.11b) we represent the SOs of (18.17) in a more precise form. For \( Q^i_2 \) we obtain easily, setting \( k=n+1 \) and omitting the top index

\[
Q^i_2 = B \sum_{c=1}^{n+1} \lambda^a_{a_1a_2...a_{n+1}b_1b_2...b_c} P_{a_1}...P_{a_c} J_{b_1}...J_{b_c}.
\]

where \( \lambda^a_{a_1a_2...a_{n+1}b_1b_2...b_c} \) are arbitrary basic tensors. The number of the linearly independent operators of (18.17) is given by formula (16.15) for \( j=n-1 \), i.e.,

\[
N^i_2 = \frac{1}{4!} n(n+1)(2n+1)(n^2+n+2).
\]

Using (18.11b) we obtain for \( Q^i_3', k=n+1: \)

\[
Q^i_3 = \sum_{c=1}^{n+1} \lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} W_{a_1} P_{a_2}...P_{a_c} J_{b_1}...J_{b_c}.
\]

where \( \lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} \) are basic tensors satisfying the conditions

\[
\lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} g_{b_b} g_{a_{a_1a_2}} = 0; \\
\lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} g_{a_{a_1a_2}} g_{a_{a_1a_2}} = 0; \\
\lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} g_{b_b} = 0.
\]

Calculating the number of independent components of \( \lambda^a_{a_1a_2...a_{n+2}b_1b_2...b_c} \) we find the number of linearly independent operators (18.20):

\[
N^i_3 = \frac{1}{6} n(n+1)(5n^2 - 3n + 13) - n.
\]

For \( Q_4 \) we have

\[
Q_4 \sum_{c=1}^{n+2} \lambda^a_{a_1a_2...a_{n+6}b_1b_2...b_c} W_{a_1} P_{a_2}...P_{a_c} J_{b_1}...J_{b_c}.
\]

where \( \lambda^a_{a_1a_2...a_{n+6}b_1b_2...b_c} \) are basic tensors satisfying the conditions

\[
\lambda^a_{a_1a_2...a_{n+6}b_1b_2...b_c} g_{b_b} g_{a_{a_1a_2}}...g_{a_{a_1a_2}} = 0, \quad c \geq \frac{f}{2}; \\
\lambda^a_{a_1a_2...a_{n+6}b_1b_2...b_c} g_{a_{a_1a_2}}...g_{a_{a_1a_2}} g_{b_b} g_{b_b}...g_{b_b} = 0.
\]

The number of linearly independent operators is
Eventually we obtain for the operator $Q_5$

$$Q_5 = \sum_{i=1}^{n-1} \lambda^i \cdot \sum_{\alpha=5,8} A_{\alpha a_1 a_2 \ldots a_{n-1}} P_{a_1} \ldots P_{a_{n-1}} J_{a_1, a_2 \ldots a_{n-1}} +$$

$$+ \sum_{i=2}^{(n-1)/2} \lambda^i \cdot \sum_{\alpha=5,8} A_{\alpha a_1 a_2 \ldots a_{n-1}} J_{a_1, a_2 \ldots a_{n-1}} +$$

$$+ \sum_{i=2}^{[n/2]-1} \lambda^i \cdot \sum_{\alpha=5,8} A_{\alpha a_1 a_2 \ldots a_{n-1}} J_{a_1, a_2 \ldots a_{n-1}} (J_{\mu \nu}) +$$

$$+ \sum_{i=0}^{n+2} \lambda^i \cdot \sum_{\alpha=5,8} A_{\alpha a_1 a_2 \ldots a_{n-1}} J_{a_1, a_2 \ldots a_{n-1}} (J_{\mu \nu}) +$$

Here $\lambda^i$, $\lambda^j$ are arbitrary irreducible tensors, $\lambda^i$ and $\lambda^j$ are basic tensors satisfying the conditions

$$\lambda^i_{a\mu a\nu a\rho a\sigma} g_{a\mu} g_{a\nu} \ldots g_{a\rho} = 0, \quad \alpha=5,8,$$

and the corresponding number of linearly independent operators $Q_5$ is

$$N_5^{(n)} = \frac{1}{6} n(n+1)(n+3)(n^2+n+1).$$

The general expression of a SO of order $n$ for the Dirac equation is

$$Q^n = Q_1 + Q_2 + Q_3 + Q_4 + Q_5$$

where $Q_i$ are the operators given in (18.25), (18.18), (18.20), (18.22), (18.24). The complete number of SOs of order $n$ can be obtained by summing up of (18.16), (18.19), (18.21), (18.23), (18.25):

$$N^{(n)} = \sum_{i=1}^{5} N_i^{(n)} = 5N_0^{(1)} + \frac{1}{6} (2n+1)(13n^2-19n+18) - \varepsilon_n.$$

In particular

$$N^{(0)} = 1, \quad N^{(1)} = 25, \quad N^{(2)} = 154, \quad N^{(3)} = 601.$$

Let us formulate the obtained results in the form of the following assertion.

**THEOREM 18.1** [329]. The Dirac equation admits $N^{(n)}$ linearly independent SOs of order $n$, moreover, $N^{(n)}$ and the explicit form of the corresponding SOs is given in (18.27), (18.26).
18.5. Examples and Discussion

We have calculated the number and explicit form of all the linearly independent SOs of arbitrary order \( n \) for the Dirac equation. These SOs are defined up to arbitrary basic and irreducible tensors satisfying some additional restrictions. Expanding basic tensors in irreducible ones we can obtain realizations of SOs depending on indecomposable sets of parameters, compare with (16.14), (16.16).

We present linearly independent SOs of the second and third orders (SOs of zero order reduce to the unit matrix, the first order SOs are given in (2.22), (18.15), (18.5)).

\[
\begin{align*}
\lambda_{1}^{a} p_{a}, J_{ab}, \tilde{\lambda}_{1}^{a} J_{ac}, \lambda_{1}^{(bc)} p_{a} J_{bc}, \lambda_{1}^{a} J_{ab} p_{b}, \\
\lambda_{1}^{(ab)(cd)} J_{ab} J_{cd}, \lambda_{1}^{a} J_{ab}, \lambda_{2}^{a} p_{a} B, \lambda_{2}^{a} B J_{ab} \\
\lambda_{3}^{a} p_{a} W_{b}, \lambda_{3}^{a(b)} p_{a} W_{bc}, \lambda_{4}^{a} W_{ac} J_{bc}, \lambda_{4}^{a} W_{ac} J_{b} c, \\
\lambda_{4}^{(ab)(cd)} W_{ab} J_{cd}, \lambda_{4}^{a} P_{a} J_{ab}, \lambda_{4}^{a} P_{a} J_{b} c, \\
\lambda_{5}^{a} p_{a} A_{b}, \lambda_{5}^{a} A_{b}, \lambda_{5}^{a} J_{ab} A_{b}, \lambda_{5}^{a} J_{b} c, \\
\lambda_{6}^{a} A_{a} J_{bc}, \lambda_{6}^{a} A_{a} J_{b} c, \lambda_{6}^{a} J_{ab} A_{b}, \lambda_{6}^{a} A_{b} J_{bc}. \\
\end{align*}
\]

Here \( P_{a}, J_{ab}, W_{a}, W_{a'b}, B, A_{b} \) are the operators (2.22), (18.5), the values of the Latin indices run from 0 to 3, \( \lambda^{a} \) are arbitrary irreducible tensors, \( \{Q^{2}\} \) is the set of the second-order operators (18.29). It is not difficult to calculate the numbers of the linearly independent operators (18.29) and (18.30) which coincide with given in (18.28).

We note that the set of second-order SOs of (18.29) differs from found in
Chapter 4. Non-geometric Symmetries

18.6. SOs of the Massless Dirac Equation

Here we present a complete set of first and second-order SOs of the massless Dirac equation.

**Theorem 18.2.** The massless Dirac equation has 52 linearly independent SOs in the class $M_1$. A basis in the space of such SOs can be chosen in the form

\[
\begin{align*}
P_{\mu} & , \quad J_{\mu \nu}, \quad K_{\mu}, \quad D, \quad F = i\gamma_4, \quad P_{\mu} = i\gamma_4 P_{\mu} I, \\
\bar{J}_{\mu \nu} & = i\gamma_4 J_{\mu \nu}, \quad \bar{K}_{\mu} = i\gamma_4 K_{\mu}, \quad \bar{D} = i\gamma_4 D;
\end{align*}
\]

\[
\begin{align*}
A_{\mu} & = (D - i)\gamma_\mu - \gamma_4 x^\nu p_\nu, \quad \omega_{\mu \nu} = \gamma_{\mu} p_\nu - \gamma_\nu p_\mu, \\
\bar{A}_{\mu} & = i\gamma_4 A_{\mu}, \quad Q_{\mu \nu} = i[(K_{\mu}, A_{\nu}) - (K_{\nu}, A_{\mu})]
\end{align*}
\]

where $P_{\mu}, J_{\mu \nu}, K_{\mu}$ and $D$ are the conformal group generators of (2.22), (2.42).

The proof is given in [156].

We see the SOs $\bar{Q} \in M_1$ of the massless Dirac equation include the conformal group generators, products of these generators (and the identity operator $I$) with $F = i\gamma_4$ and 20 additional operators (18.32). We emphasize the SOs (18.32) cannot be expressed via generators of the group $C(1,3)$ and the matrix $i\gamma_4$ and are essentially new in this sense. Thus the massless Dirac equation admits SOs which do not belong to the enveloping algebra of the algebra $AC(1,3)$ in contrast to the case of nonzero mass.

It is not difficult to make sure the operators (18.31), (18.32) do not form a Lie algebra. However we can select such subsets of the first order SOs which have a structure of a Lie algebra. The evident example of such a set is given by the conformal group generators, a more extended set is formed by the operators (18.31) which define a basis of the 32-dimensional Lie algebra. The structure constants of this algebra are easily calculated using the commutativity $i\gamma_4$ with $P_{\mu}, J_{\mu \nu}, K_{\mu}$, and $D$ and the relation $\gamma_4^2 = -1$.

We present the set of SOs forming a superalgebra. This set includes the following basis elements

\[
\{\omega_{\mu \nu}, F = i\gamma_4, FP_{\mu}, P_{\mu}, J_{\mu \nu}, D, P_{\mu} = p_\mu P_{\chi}\}.
\]

Indeed, $\omega_{\mu \nu}$ satisfies the anticommutation relations (18.9a) and commutation relations (18.6) with $P_{\mu}, J_{\mu \nu}$. Beside that

\[
\begin{align*}
[FP_{\mu}, F], & = 2P_{\mu} - [FP_{\mu}, FP_{\chi}], = 2P_{\mu \nu}, \quad [\omega_{\mu \nu}, F], = [\omega_{\mu \nu}, FP_{\chi}]. \quad 0.
\end{align*}
\]

The algebra (18.33) includes 36 elements nine of which, i.e., $P_{\mu \nu}$ belong to a
more wide class than $M_1$ (we recall that $P_{\mu\nu}g^{\mu\nu}=m^2I$).

We present the subset of SOs forming a basis of a 42-dimensional Lie algebra. This subset includes the operators (18.31) and the following ten operators

$$F_{\mu} = A_{\mu}^{\nu} A_{\nu}^{\sigma}, \quad Q_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} Q^{\rho\sigma}, \quad \omega_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \omega^{\rho\sigma}$$

(only three of the operators $Q_{\mu\nu}$ are linearly independent, the same is true for $\omega_{\mu\nu}$). The operators (18.35) form a commutative ideal of this algebra.

In analogy with Section 16 it is possible to find complete sets of SOs of arbitrary order $n$ for the massless Dirac equation. We will not do it here restricting ourselves to a remark that these sets are completely defined by two conformal Killing tensors of valence $n$ and by the generalized Killing tensor of valence $R_1+2R_2$ where $R_1=n-1, R_2=1$ (see Appendix 2 for definitions). The number $N^n$ of linearly independent SOs of order $n$ is

$$N^n = \frac{1}{3}(n+1)(n+2)(2n+3)(n^2+3n+1),$$

besides $\tilde{N}^n$ of these operators do not belong to the enveloping algebra of the algebra $A[C(I,3)\otimes H]$, where

$$\tilde{N}^n = \frac{1}{6}n(n+1)(n+2)(n+3)(2n+3).$$

Formulae (18.36), (18.37) do not include the number of independent SOs whose order is less then $n$.

In conclusion we discuss briefly SOs of the Weyl equation. In fact we have already described such SOs in the class $M_1$. Indeed, using the Majorana representation (2.13) for $\gamma$-matrices we conclude that the corresponding SOs (18.31), (18.32) transform real solutions into real ones. On the other hand setting in the Weyl equation (2.44)

$$\varphi_1 \psi_1 + i \psi_2$$

where $\psi_1$ and $\psi_2$ are real functions we come to the Dirac equation with $\gamma$-matrices realizing the Majorana representation, moreover, the corresponding wave function has the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + \varphi_2^* \\ -i(\varphi_1 - \varphi_2^*) \end{pmatrix}.$$
transformations of \( \varphi \), being a solution of the Weyl equation.

So the Weyl equation has exactly 52 linearly independent SOs in the class \( M_1 \). These operators can be chosen in the form (18.31), (18.32), (2.13), their action on solutions of the Weyl equation is easily calculated using (18.38), (18.39).

The SOs of the Weyl equation in the class \( M_2 \) are found in [326]. Here we present the principal assertion of paper [326] only.

**THEOREM 18.3.** The Weyl equation has 84 SOs belonging to the class \( M_2 \). These SOs have the form

\[
\lambda^\mu P_\mu, \quad \lambda^{[\mu[\rho|\sigma]} P_\mu J_{\rho|\sigma]}, \quad \lambda^{[\mu[\rho|\sigma]} J_{\mu|\rho|\sigma]} J_{\sigma]}, \quad \lambda^\mu J_\mu
\]

where the Greek indices denote arbitrary irreducible tensors, \( P_\mu, J_\mu, K_\mu, D \) are generators of the conformal group. ■

We see that all the SOs of the class \( M_2 \) for the Weyl equation belong to the enveloping algebra of the algebra \( AC(1,3) \) in contrast to the massless Dirac equation.

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19. SYMMETRIES OF EQUATIONS FOR ARBITRARY SPIN PARTICLES

19.1. Symmetries of the KDP Equation

In this section we investigate symmetry properties of relativistic wave equations for particles of higher spins, i.e., the equations of KDP, TCT, the Dirac-like equations etc. It turns out that besides the Poincaré invariance these equations have additional (hidden) symmetries which are more extensive then in the cases of the Dirac or KGF equations.

We write the KDP equation for a particle of spin 1 in the form

\[
L\psi \equiv (\beta^\mu p_\mu - m)\psi = 0
\]

where \( \psi \) is a ten-component wave function, \( \beta^\mu \) are 10x10 KDP matrices satisfying the algebra (6.20).

As it was noted in Section 6 the KDP equation is invariant under the 10-dimensional Lie algebra of the Poincaré group. Basis elements of this algebra can be chosen in the covariant form (2.22) where \( S_{[\mu|\nu,\sigma]} = [\beta_{\mu}, \beta_{\nu}] \). It is possible to show the Poincaré algebra is the maximal IA of the KDP equation in the class \( M_1 \).

Here we demonstrate the KDP equation has a wide nongeometric symmetry. The nature of this symmetry lies in special properties of \( \beta \)-matrices and is not
connected with transformations of independent variables.

Let us show the equation (19.1), like the Dirac equation, is invariant under the algebra $A_\omega$, defined over the field of real numbers. Basis elements of this IA belong to the class $M_1$ and are given by the formulae \[142, 154]\n
where \[19.2\]

\[\Sigma_{\mu\nu} \rightarrow i(S_{\mu\nu} - \frac{i}{m} (a_\mu p_\nu - a_\nu p_\mu)), \quad \Sigma_0 \rightarrow i, \quad \Sigma_1 \rightarrow \beta_4 + \frac{i}{m} \sigma^a p_a, \]

where \[19.3\]

\[a_\mu = S_{\mu 0} + iS_{\mu 4}, \quad S_{kl} = [\beta_k, \beta_l], \quad S_{kl} = i\beta_k; \quad k, l = 0, 1, 2, 3, 4.\]

Indeed, using the fact that $\beta_\mu$ satisfy the algebra (6.20) and $S_{\mu\nu}$ belong to the algebra $SO(2, 4)$ we obtain easily the following relations

\[19.4\]

\[\Sigma_{\mu\nu} L \rightarrow f_{\mu\nu} L, \quad f_{\mu\nu} = \frac{1}{2m^2} (L - 2m)(\beta^\mu p_\nu - \beta^\nu p_\mu).\]

\[19.5\]

\[[\Sigma_\alpha L] = 0, \quad \alpha = 0, 1.\]

The l.h.s. of (19.4) includes differential operators of order 2 meanwhile the r.h.s. looks like a third order differential operator. But there is no contradiction inasmuch as according to (6.20)

\[\beta^\mu p_\nu (\beta^\nu p_\mu - \beta^\mu p_\nu) = 0.\]

It follows from (19.4), (19.5) the operators (19.2) do are SOs of the KDP equation. These operators satisfy the commutation relations (17.7) characterizing the algebra $A_\omega$. The last statement can be easily verified by making the transformation

\[19.6\]

\[\Sigma_{\mu\nu} \rightarrow V \Sigma_{\mu\nu} V^{-1} - [\beta_\mu, \beta_\nu], \quad \Sigma_0 \rightarrow V \Sigma_0 V^{-1} - \beta_4, \quad \Sigma_1 \rightarrow V \Sigma_1 V^{-1} - \beta_4\]

where $V = \exp(ia_\mu p_\mu / m)$.

Thus the KDP equation possesses the same nongeometric symmetry in the class $M_1$ as the Dirac equation, compare with Theorem 17.1. It is possible to show the Lie algebra spanned on the basis (19.2) is the maximally extended IA of the KDP equation in this class. But in contrast to the Dirac equation the KDP equation has a wide symmetry in the class $M_2$ as it follows from the following assertion.

**THEOREM 19.1.** The KDP equation is invariant under the 18-dimensional Lie algebra defined over the field of complex numbers. Basis elements of this algebra belong to the class $M_2$ and are given by the following formulae

\[19.7\]

\[\lambda_{ab} \rightarrow C_a C_b, \quad \tilde{\lambda}_{ab} \rightarrow \frac{1}{2} (D_a C_b + C_a D_b), \quad \lambda_{1 \rightarrow 1}, \quad \lambda_0 \rightarrow \frac{1}{2} D_a C_a\]

where
Chapter 4. Nongeometric Symmetries

\[ C_a = \frac{1}{2} \varepsilon_{abc} \Sigma_{bc}, \quad D_a = \Sigma_{oa}, \quad a, b = 1, 2, 3. \]

**PROOF.** The fact that the operators (19.7) are SOs of the KDP equation follows immediately from (19.4), (19.5). Besides that, these operators satisfy the following commutation relations

\[
[\lambda_{ab}, \lambda_{cd}] = i f^{i j l}_{abc} \lambda_{ijl}, \quad [\lambda_{ab}, \lambda_{cd}] = j f^{kl i}_{abcd} \lambda_{kli},
\]

(19.8)

\[ [\lambda_1, \lambda_2] = [\lambda_\alpha, \lambda_{abc}] = 0, \quad \alpha = 0, 1, \]

where \( f^{i j l}_{abc} \) are the structure constants of the algebra \( ASU(3) \) in the Okubo basis (see, e.g., [374]). The validity of relations (19.8) is easily verified using the representation (19.6).

Thus besides the symmetry under the Poincaré algebra the KDP equation is invariant in respect with the 18-dimensional Lie algebra spanned on the basis (19.7). The algebra (19.7) includes \( A_8 \) as a subalgebra besides basis elements of \( A_8 \) are linear combinations of the operators (19.7). For instance,

\[ \Sigma_{ab} = i(\lambda_{ba} - \lambda_{ab}). \]

Is it possible to unite the Poincaré algebra and the algebra (19.7)? Such an unification is very natural because the commutators of the operators (19.7) with the generators of the Poincaré group are expressed via linear combinations of these operators:

\[
[J_{\mu}, \lambda_{ab}] = [P_{\mu}, \hat{\lambda}_{ab}] = [J_{\nu}, \hat{\lambda}_{ab}] = 0
\]

(19.8)

\[ [P_\mu, \hat{\lambda}_{ab}] = [J_\nu, \hat{\lambda}_{ab}] = 0 \]

where \( J_\mu = \varepsilon_{abcd} J_{\mu}^{abcd} / 2. \)

The relation (1.14), (19.8), (19.9) define a 28-dimensional Lie algebra being the IA of the KDP equation. Starting with this algebra it is not difficult to reconstruct the corresponding symmetry group. The transformations generated by \( P_\mu, J_{\nu} \) have already been considered in the above (see (3.30)-(3.32)). As to the transformations generated by the operators (19.7) they can be found explicitly using the formula

\[ \psi \rightarrow \psi(Q, \theta), \quad Q = [\lambda_{ab}, \hat{\lambda}_{ab}, \lambda_\alpha], \]

(19.9)

where \( \theta \) are real parameters. The corresponding exponentials are easily calculated:

\[ \exp(Q_1, \theta) = 1 + Q_1 \theta; \quad Q_1 = [\lambda_{ab}, \hat{\lambda}_{ab}, \alpha \neq b], \]

\[ \exp(\lambda_\alpha, \theta) = 1 + \lambda_\alpha (\exp \theta - 1); \quad \exp(\lambda_\alpha, \theta) = \exp \theta, \]

\[ \exp(Q_{\lambda}, \theta) = 1 + Q_{\lambda} \cosh \theta - 1 + Q_{\lambda} \sinh \theta, \quad Q_{\lambda} = \lambda_{ab}, \lambda_\alpha. \]

Using the identities \( a_a a_a = 0 \) we can show that the general transformation
including (19.10) and Lorentz transformations has the form
\[ \psi(x) \rightarrow A\psi(x) + B_{\mu} \frac{\partial \psi(x')}{\partial x_{\mu}} + D_{\mu\nu} \frac{\partial^2 \psi(x')}{\partial x_{\mu} \partial x_{\nu}} \]
where x' are related to x by Lorentz transformation, A, B and D are numeric matrices depending on transformation parameters.

Let us discuss briefly nongeometric symmetries of the Tamm-Sakata-Taketani (TST) equation (7.1), (7.40c), which also describes a particle of spin 1 but has not superfluous components. Inasmuch as solutions of the KDP and TST equations are connected by the transformation (6.37) there is one-to-one correspondence between symmetries of these equations. We give the explicit formulation of a symmetry interesting from the physical point of view.

**THEOREM 19.2.** The TST equation is invariant under the algebra ASU(3) whose basis elements have the form
\[ \lambda_{abc} \hat{S}_a \hat{S}_b \]
where
\[ \hat{S}_a = \left(1 + \frac{p^2}{2m^2} \right) S_a - \frac{p_a S_b p_{\beta}}{2m} + \frac{i}{m} \sigma_{abc} p_b S_c, \]
\( S_a \) are matrices realizing the direct sum of two IRs \( D(1) \) of the algebra \( AO(3) \).

We do not present a proof but note that the operators (19.11) satisfy (19.8) and commute with the TST Hamiltonian (7.40c), i.e., form an IA of the TST equation.

We note that the KDP and TST equations admit nongeometric IAs in the class \( M_{\infty} \) also. For the explicit form of the corresponding SOs see [345].

**19.2. Arbitrary Order SOs of the KDP equation**

Here we present a principle description of SOs for the KGF equation in classes of differential operators of arbitrary order \( n (n<\infty) \) with matrix coefficients. A formal definition of such operators can be written in the form (16.6), (16.7) where \( L \) is the operator (19.1), \( \alpha_\beta \) is a differential operator of order \( m \) (besides in general \( m\neq n \)), \( h^{\alpha} \) are matrices of dimension 10x10 depending on \( x \).

**THEOREM 19.3.** Any SO of arbitrary order \( n \) of the KDP equation belongs to the enveloping algebra of the algebra AP(1,3).

**PROOF.** As in the case of the Dirac equation (see Subsection 18.2) we transform the KDP equation into such an equivalent representation that the transformed wave function has \( 2s+1 \) components only. Using the transformation (17.12) where \( \psi \) is a ten-component function, \( L \) is the operator (19.1),
Chapter 4. Nongeometric Symmetries

\[ W^{-1}m^{-1}B_i \beta^u p_u - m^{-1}(\beta^u p_u)^i \beta_i, \quad W^{-1}m^{-1}B^i \beta^u p_u + m^{-1}(\beta^u p_u)^2 \beta_i, \]

\[ V^{-1}m^{-1}B^i \beta^u + 2m^{-1}2(\beta^u p_u)^2 - p^\lambda p_\lambda, \]

we come from (19.1) to the equivalent equation of the form

\[ (19.12) \]

\[ W^{-1}m^{-1}B^i \beta^u p_u + m^{-1}(\beta^u p_u)^2 \beta_i, \]

\[ V^{-1}m^{-1}B^i \beta^u + 2m^{-1}2(\beta^u p_u)^2 - p^\lambda p_\lambda, \]

we come from (19.1) to the equivalent equation of the form

\[ (19.13) \]

where

\[ P_1 = \frac{1}{2}(i\beta_4 - \beta_2^2), \quad P_2 = \frac{1}{2}(i\beta_4 - \beta_2^2). \]  

(19.14)

The transformations (17.12) enable us to establish one-to-one correspondence between SOs of the KDP equation and the equation (19.13) (refer to (18.2)). The last is much more adapted for investigating of symmetries than (19.1) inasmuch as it includes only two (besides that commuting) matrices.

It is convenient to expand SOs of (19.13) in a complete set of numeric matrices. Inasmuch as \( \beta_4 \equiv \beta_4 \) the matrices (19.14) are orthoprojectors. Choosing \( \beta_i \) in the diagonal form

\[ \beta_i \equiv \text{diag}(1,1,1,-1,-1,0,0,0,0) \]

we conclude that \( \psi' \) of (19.13) has only three nonzero components and so there are exactly eight linearly independent matrices defined on \( \{ \psi' \} \). We choose the set of such matrices in the form

\[ S_{ab} = i[\beta_a, \beta_b], \quad Z_{ab} = [S_{ab}, S_{ab}]. \]  

(19.15)

and represent SOs of (19.13) in the form

\[ Q' = S_{ab} \hat{Q}^{ab} + Z_{ab} \hat{Q}^{ab}. \]

(19.16)

The operator \( Q' \) is a SO of the equation (19.13) iff \( \hat{Q}^{ab} \) and \( \hat{Q}^{ab} \) are SOs of the KGF equation. Thus it is easy to show that \( Q' \) belongs to the enveloping algebra generated by the generators of the Poincaré group of (2.22), (6.19). Indeed, according to results present in Subsection 16.2 \( \hat{Q}^{ab} \) and \( \hat{Q}^{ab} \) are polynomials on \( P_\mu \) and \( J_{\mu\sigma} \). But \( S_{\mu\sigma} \) (and therefore the matrices (19.15)) are expressed via \( P_\mu \) and \( J_{\mu\sigma} \) according to (18.4), and so \( Q' \) is a polynomial on \( P_\mu \) and \( J_{\mu\sigma} \).

The operator \( V \) of (19.12) commutes with \( P_\mu \) and \( J_{\mu\sigma} \) so it follows from (18.2) that \( Q' = Q \). Thus any SO of the KDP equation is a polynomial on \( P_\mu \) and \( J_{\mu\sigma} \).

So to calculate SOs of order \( n \) for the KDP equation it is sufficient to go through linearly independent combinations of the kind (18.12). Let us present a complete set of the first order SOs.

227
For \( n = 1 \) we obtain a set including the Poincaré group generators of (2.22), (6.19) and the following 15 operators (compare with (18.5))

\[
B = i\beta_{\mu} D - S_{4\mu} x^\mu, \\
A_{\mu} = \epsilon_{\mu\nu\rho\sigma} J^\nu S_{\sigma} - \frac{2}{m}(1 + \beta^{2} p_{\mu})
\]

where \( D = x^\mu p_{\mu} + 2i\beta_{\mu} \), \( S_{4\mu} = i[\beta_{4}, \beta_{\mu}] \).

As was done for the Dirac equation (refer to Section 17) it is possible to show that a SO of any order \( n > 1 \) can be expressed via products of the SOs of the KGF equation (18.15) and one or two operators from (19.17).

In contrast with the SOs of the Dirac equation the operators (19.17) cannot be included into a superalgebra like (18.10). Indeed, anticommutators of the operators (19.17) cannot be expressed via \( P_{\mu} \) in contrast with (18.9a). However, these operators satisfy the following relations

\[
[W_{\mu\nu} W_{mn}] = \frac{i}{2}(e_{knmf} P_{km} + e_{kmnf} P_{ln} - e_{knmf} P_{lm} - e_{kmnf} P_{lk}) W_{sf}, \\
W_{mn} W_{sf} + W_{sf} W_{mn} = m^{-2}(P_{mk} s_{nl} + P_{mn} s_{kl} - P_{mn} s_{kl} - P_{mk} s_{nl}) W_{mn}, \\
[W_{\mu\nu} J_{\mu\nu}] = i(g_{mn} W_{\mu\nu} + g_{\mu\nu} W_{mn} - g_{\mu\nu} W_{mn} - g_{mn} W_{\mu\nu})
\]

where \( P_{\mu} = p_{\mu}, J_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} P_{\rho\sigma} \).

Relations (19.18b) enable us to include \( W_{\mu\nu}, P_{\mu} \) and the generators \( P_{\mu}, J_{\mu\nu} \) into the 30-dimensional parasuperalgebra \( \{W_{\mu\nu}, P_{\mu}; P_{\mu}, J_{\mu\nu}\} \) which we denote by \( \text{PSA}(30) \), besides the odd terms stand to the left of semicolon. The remaining operators are even (for definitions see Appendix 1). Indeed, relations (19.18b), (12.18c) are in accordance with the scheme (A.1.3) defining a parasuperalgebra.

We note that the basis elements (18.5) of the superalgebra \( \text{SA}(30) \) satisfy relations (19.18b) also, thus

\( \text{SA}(30) \Rightarrow \text{PSA}(30) \)

and the parasuperalgebra \( \text{PSA}(30) \) includes the superalgebra \( \text{SA}(30) \) realizing on the set of solutions of the Dirac equation; the converse is not true.

The parasuperalgebra \( \text{PSA}(30) \) includes very interesting subalgebras. Among them are the Lie algebra of the Poincaré group \( \text{AP}(1,3) \supseteq P_{\mu}, J_{\mu\nu} \), the KDP algebras generated by \( Q_{s} \) of (19.2) and the parasuperalgebras which we denote by \( \text{psqm}(3) \). The last includes the following elements:
Chapter 4. Nongeometric Symmetries

\[ Q_\mu^* W_{\mu} = \frac{1}{2} \varepsilon_{abc} W_{bc}, \quad H_{\mu\nu} = p_\mu p_\nu + m^2 \]

satisfying the relations

\[ Q_\mu^*[Q_\sigma^*, Q_\tau^*] + Q_\mu^*[Q_\sigma^*, Q_\tau^*] + Q_\mu^*[Q_\sigma^*, Q_\tau^*] = (\delta_{ab} Q_\mu^* + \delta_{ac} Q_\mu^* + \delta_{bc} Q_\mu^*) H_{\mu\nu} \]

\[ [H_{\mu\nu}, Q_\alpha^*] = 0. \]

These relations characterize the IA of parasupersymmetric quantum mechanics [372] including three parasupercharges. Thus SOs of the KGF equation include the subset realizing a representation of this IA. In this sense parasupersymmetry is generated by symmetries of the KGF equation.

19.3. Symmetries of Dirac-Like Equations for Arbitrary Spin Particles

Here we consider nongeometric symmetries of equations of Dirac type for any spin particles discussed in Section 8.

The equations (8.1), (8.11) have a symmetric form which does not become more complicated by increasing of spin value. This circumstance makes it possible to generalize the main results of Section 17 to the case of arbitrary spin.

The equations (8.1), (8.11) are manifestly invariant under the algebra \( \mathfrak{AP}(1,3) \).

It turns out they are invariant under the algebra \( \mathfrak{A}_8 \) also.

**THEOREM 19.4.** The system (8.1), (8.11) is invariant under the algebra \( \mathfrak{A}_8 \) realized in the class \( M_1 \) and have the form

\[ \Sigma_{\mu\nu} = i \frac{1}{4} [\Gamma_\mu, \Gamma_\nu], \quad \frac{1}{2m} (\Gamma_\mu p_\nu - \Gamma_\nu p_\mu)(1 - i \Gamma_4), \]

\[ \Sigma_0 = I, \quad \Sigma_i = \Gamma_i - \frac{1}{m} (i + \Gamma_0) \Gamma_\nu p_\nu. \]

The proof can be carried out in a complete analogy with the proof of Theorem 17.3. The commutators of the operators (19.19) with \( L_2 \) of (8.11) and \( L_3 \) of (8.1) are equal to zero or given by relations (17.5) where \( \gamma_\mu \to \Gamma_\mu \). The operators (19.19) satisfy (17.7) and thus form the algebra \( \mathfrak{A}_8 \). 

We see the Dirac-like equations for particles of arbitrary spin turn out to be invariant under the algebra \( \mathfrak{A}_8 \) realized in the class \( M_1 \). So this symmetry is not a specific property of the Dirac and KDP equations but is admissible by equations for arbitrary spin particles.

As in the case of the four-component Dirac equation (see Theorem 17.4) the operators (19.19) form a closed algebra together with the Poincaré group generators.
Symmetries of Equations of Quantum Mechanics

More precisely they satisfy the commutation relations (17.18). It means the system (18.1), (18.11) is invariant under a 18-dimensional Lie algebra including the subalgebras \( A(1,3) \) and \( A_8 \).

It is easily seen that for \( s > 1/2 \) the symmetry of (8.1), (8.11) in the class \( M_1 \) is more extensive than described in Theorem 19.4. Indeed, products of the operators (18.19) are SOs of the considered system and belong to the class \( M_1 \) also.

**THEOREM 19.5.** The system (8.1), (8.11) is invariant under the \([10+2(2s+1)^2]\)-dimensional Lie algebra isomorphic to \( A_P(1,3) \otimes GL(2s+1, \mathbb{C}) \). Basis elements of this algebra belong to the class \( M_1 \) and are given by formulae (2.22) (where \( S_{\mu\nu} \) are the matrices (8.6)) and (19.20):

\[
\begin{align*}
\tilde{\lambda}_{\mu n} &= \frac{1}{2} a_{0n} [\Sigma_{23} - \Sigma_{02}] P_{s+1} (1 - i \Sigma_1) + P_{s+1} (\Sigma_{23} + \Sigma_{02}) (1 + i \Sigma_1), \\
\lambda_{\mu n} &= \frac{1}{2} a_{1n} [\Sigma_{23} - \Sigma_{02}] P_{s+1} (1 + i \Sigma_1) + P_{s+1} (\Sigma_{23} + \Sigma_{02}) (1 - i \Sigma_1),
\end{align*}
\]

(19.20)

where \( \Sigma_{\mu\nu}, \Sigma_i \) are the operators (19.19),

\[
P_{s+1} \prod_{n=0}^{s+1} \frac{\Sigma_{ij} - s - 1 - n}{n - n'}, \quad n=1,2,...,2s+1; \quad k=0,1,...,2s+1 - n,
\]

(19.21)

\( a_{kn} \) are coefficients defined by the following recurrence relations

\[
a_{0n} = 1, \quad a_{1n} = \left[n(2s+1 - n)\right]^{1/2} = a_{0n} \lambda_{1n} \lambda_{1,n+1}, \quad \lambda = 2,3,...,2s - n.
\]

(19.22)

**PROOF.** The operators (19.20) evidently are SOs of the system (8.1), (8.11) inasmuch as they are products of the SOs present in (19.19).

To prove these operators form a basis of the algebra \( AGL(2s+1, \mathbb{C}) \) it is sufficient to make sure that they are linearly independent and satisfy relations (17.9) for \( a,b,c,d = 1,2,...,2s+1 \). The simplest way to verify these statements is to transform (19.20) into such a representation where they reduce to numeric matrices. Using the transformation

\[
\Sigma_{\mu\nu} \rightarrow V \Sigma_{\mu\nu} V^{-1} = S_{\mu\nu}, \quad \Sigma_1 \rightarrow V \Sigma_1 V^{-1} = \Gamma_4
\]

(19.23)

where \( V \) is the operator obtained from (17.13a) by the change \( \gamma_k \rightarrow \Gamma_k, S_{\mu\nu} \) are the matrices (8.6), we come to a matrix realization of the basis elements (19.20). Simultaneously the equations (8.1), (8.11) reduce to the form

\[
L_0 \psi' = 0, \quad L_i' \psi' = 0
\]

(19.24)

where \( \psi' = \psi V, L_i' \) has the form (17.13c) where \( \gamma_i \rightarrow \Gamma_4 \) and

\[
L_0' = W (L_2 - F L_1) V^{-1} \frac{1}{s} [S_{\mu\nu} S^{\mu\nu} - 4s(s+1)],
\]

230
Chapter 4. Non-Geometric Symmetries

W is the operator obtained from (17.13b) by the change $\gamma_k \rightarrow \Gamma_k$. $F$ is the operator given in (8.23).

An equivalent form of the equations (19.24) is

$$ (p_\mu p^\nu - m^2)\psi' = 0, \quad (1 - \Gamma_4)\psi' = 0, \quad (S_\mu S^\nu - 4s(s + 1))\psi' = 0. $$

According to (19.26) the function $\psi'$ has exactly $2s + 1$ independent components. The equations (19.25), (19.26) are manifestly invariant in respect with arbitrary matrix transformations commuting with $\Gamma_4$ and $S_\mu S^\nu$.

There are exactly $(2s+1)(2s+1)$ linearly independent complex matrices or twice more real matrices commuting with $\Gamma_4$ and $S_\mu S^\nu$. To find a complete set of these matrices we use the fact that $S_\mu S^\nu$ reduce to the basis elements of the IR $D(s \, 0)$ of the algebra $A(1,3)$ on the set $\{\psi'\}$. All the possible products of them includes $2(2s+1)(2s+1)$ matrices linearly independent over the field of real numbers. These independent matrices can be chosen in the form of the set $\{\lambda_{ab}, \tilde{\lambda}_{ab}\}$ where the primed matrices are obtained from (19.20) by the change (19.23). Indeed, using for $S_\mu S^\nu \subset D(s \, 0)$ the realization (4.63) we make sure that nonzero matrix elements of $\lambda_{ab}$ and $\tilde{\lambda}_{ab}$ are given by the following formulae:

$$ (\lambda_{ab})_{mn} = \delta_{am} \delta_{bn}, \quad (\tilde{\lambda}_{ab})_{mn} = i \delta_{am} \delta_{bn}, \quad a, b = 1, 2, \ldots, 2s + 1. $$

These matrices are linearly independent over the field of real numbers and satisfy the commutation relations (17.9).

To complete the proof it is sufficient to calculate commutation relations of the operators (19.20) with generators of the Poincaré group. It can be easily shown that $P_\mu$, $J_{\mu\nu}$ of (2.22), (8.6) satisfy relations (17.18) with the operators $\Sigma_\mu$, $\Sigma_\sigma$ of (19.19). Thus the linear combinations

$$ \hat{P}_\mu = iP_\mu, \quad \hat{J}_{\mu\nu} = iJ_{\mu\nu} + \Sigma_\mu \Sigma_\nu $$

satisfy the Poincaré algebra and commute with $\Sigma_\mu$, $\Sigma_\sigma$ (and therefore with the matrices (19.20)). Because $\Sigma_\mu$ are linear combinations of $\hat{\lambda}_{mn}$ and $\hat{\lambda}_{mn}$:

$$ \Sigma_1 = \sum_n (s + 1 - n)\lambda_{mn}, \quad \Sigma_2 = \sum_n \frac{1}{2d_{1n}}(\lambda_{mn+1} + \lambda_{mn-1}), \quad \Sigma_3 = \frac{1}{2d_{1n}}(\hat{\lambda}_{mn+1} + \hat{\lambda}_{mn-1}) (19.28) $$

(the remaining $\Sigma_\mu$ are expressed via commutators of the operators (19.20)) it follows from the above that the SOs $\{P_\mu, J_{\mu\nu}, \lambda_{ab}, \tilde{\lambda}_{ab}\}$ form a Lie algebra isomorphic to $A[P(1,3) \otimes GL(2s+1, C)]$.

Thus, the Dirac-like equations for a particle of arbitrary spin $s$ have a wide symmetry in the class $M_s$ which increases if the spin value increases. The corresponding basis of the IA is given in (2.22), (8.6), (19.20).

231
Inasmuch as the operators (19.20) satisfy the conditions
\[ \lambda_{nn}^2 \psi = 0, \quad \lambda_{mn} \psi = -\lambda_{mn}^2 \psi \]
(no sum over \( n \)) it follows from Theorem 19.5 the equations (8.1), (8.11) are invariant
under a \( \{10+2(2s+1)^2\} \)-parametric Lie group including inhomogeneous Lorentz
transformations and the following transformations
\[
\psi \rightarrow \exp(\lambda_{mn} \theta_{mn}) \psi \begin{cases}
(1+\lambda_{mn} \theta_{mn}) \psi, & m \neq n, \\
(1+\lambda_{nn}(\exp\theta_{nn} - 1)) \psi, & m = n,
\end{cases}
\]
\[
\tilde{\psi} \rightarrow \exp(\tilde{\lambda}_{mn} \tilde{\theta}_{mn}) \tilde{\psi} \begin{cases}
(1+\tilde{\lambda}_{mn} \theta_{mn}) \psi, & m \neq n, \\
[\tilde{\lambda}_{nn}(\sin\theta_{nn} + 1 - \lambda_{nn}(\cos\theta_{nn} - 1)) \psi, & m = n
\end{cases}
\]
(no sum over repeated indices). It is easily verified that the general transformation
belonging to this group can be represented in the form (17.19) where \( A, B \)
are matrices of dimension \( 8s \times 8s \) depending on transformation parameters.

19.4. Hidden Symmetries Admitted by Any Poincaré-Invariant Wave Equation

Nongeometric symmetry described in the above is inherent in any Poincaré-
invariant equation for a particle of arbitrary spin \( s > 0 \). To prove this assertion is the
main goal of this subsection.

Let \( \{ \psi \} \) be a set of solutions of a Poincaré-invariant equation for a particle of
spin \( s \) and mass \( m \neq 0 \). We write such an equation in the symbolic form (16.1) and do
not impose any restriction on the explicit form of a linear operator \( L \) - it can be either
a differential operator of arbitrary finite order or integro-differential operator. The only
requirement imposed is that the corresponding equation (16.1) be invariant under the
algebra \( AP(1,3) \), moreover, the representation of this algebra realized on \( \{ \psi \} \) has to
belong to the class \( I(1,3) \) and be irreducible in respect with spin and mass (refer
to Subsection 6.1). It means the equation (16.1) admits ten SOs \( P_{\mu}, J_{\mu \sigma} \) satisfying the
commutation relations (1.14) besides eigenvalues of the corresponding Casimir
operators \( P_{\mu}P^{\mu} \) and \( W_{\mu \nu}W^{\mu \nu} \) are fixed and given by formulae (6.1)\( ^* \).

Let us show the Poincaré-invariance of (16.1) implies an additional
(nongeometric) symmetry of this equation.

\* It will be shown in the following that the condition the eigenvalue of \( P_{\mu}P^{\mu} \) to
be fixed is not essential and can be replaced by the weaker requirement \( P_{\mu}P^{\mu} \psi = m^2 \psi, \)
\( \varepsilon \leq m^2 < \infty, \varepsilon > 0. \)
**THEOREM 19.6** [156]. Any Poincaré-invariant equation for a particle of spin $s$ and mass $m>0$ is additionally invariant under the algebra $AGL(2s+1,C)$.

**PROOF.** A Poincaré-invariant equation by definition admits ten SOs satisfying the algebra (1.14). Then the tensors

$$
\Sigma_{\mu}^\pm = \frac{1}{m^2} \left[ i e_{\mu \nu \rho \sigma} W^\nu P^\rho \pm (P_\nu W_{\rho} - P_{\rho} W_\nu) \right]
$$

(19.29)

where $W_\mu$ is the Lubanski-Pauli vector (2.37), are the SOs of (16.1) also.

Using relations (4.3) we sure that the operators (19.29) satisfy the conditions

$$
[S_{\mu \nu}, S_{\sigma \alpha}] = (g_{\mu \lambda} S_{\nu \alpha} + g_{\nu \alpha} S_{\mu \lambda} - g_{\mu \alpha} S_{\nu \lambda} - g_{\nu \lambda} S_{\mu \alpha}) \frac{1}{m^2} P_\rho P^\rho,
$$

$$
C_6 = \frac{1}{2} e_{\mu \nu \rho \sigma} S_{\mu \rho} S_{\nu \sigma} = \frac{1}{m^4} W_\lambda W^\lambda P^\mu P_\mu,
$$

$$
C_7 = \frac{1}{4} e_{\mu \nu \rho \sigma} S_{\mu \nu} S_{\rho \sigma} - \frac{2i}{m^4} W_\lambda W^\lambda P^\mu P_\mu.
$$

(19.30)

According to (19.30), (6.1) the operators $S_{\mu \sigma}$ satisfy relations (17.7) characterizing the algebra $AO(1,3)$ besides the eigenvalues of the corresponding Casimir operators $C_6$ and $C_7$ are

$$
C_6 \psi = \frac{i}{2} C_7 \psi = (s+1) \psi.
$$

It follows from the above the operators $S_{\mu \sigma}$ realize the representation $D(s 0) (D(0 s))$ of the algebra $AO(1,3)$.

But linearly independent products of the operators (19.29) form a basis of a more wide Lie algebra than $AO(1,3)$. Indeed, choosing such a basis in the form (19.20) where $\Sigma = C_f/2(s+1)$ we come to $(2s+1)^2$ SOs satisfying the commutation relations (17.9), i.e., realizing a representation of the algebra $AGL(2s+1,C)$. The verification of validity of these relations can be made by using a matrix realization of the operators $S_{\mu \sigma}$.

Thus any Poincaré-invariant equation for a particle of spin $s>0$ and mass $m>0$ is invariant under the algebra $AGL(2s+1,C)$ whose basis elements belong to the enveloping algebra of the algebra $AP(1,3)$ and are given in (19.29), (19.20).

The operators (19.29) satisfy relations (17.18) with the generators of the Poincaré group. Hence it follows that the set $\{P_\mu, J_{\mu \sigma}, \lambda_{m \nu}, \tilde{\lambda}_{m \nu}\}$ forms a Lie algebra isomorphic to $AP(1,3) \otimes GL(2s+1,C)$, see the end of the proof of Theorem 19.5.

In the above we did not make any supposition about the class of SOs of the equation (16.1). If we assume the generators of the Poincaré group have the covariant form of (2.22) (with the corresponding matrices $S_{\mu \sigma}$) then the SOs of (19.29) belong to
Symmetries of Equations of Quantum Mechanics

the class $M_2$ and have the form

$$\Sigma_{ab}^* = -iS_{ab} - \frac{1}{m^2} \{ \epsilon_{abc}(p^2F_c - p \cdot F) \pm ip_0(F_a + p_b - F_b) \},$$

$$\Sigma_{oa}^* = -iS_{oa} - \frac{1}{m^2} [p_0 \epsilon_{abc} F_b p_c \pm ip_a p \cdot F - p^2 F_a]$$

where $F_a = i\epsilon_{abc} S_b c / 2 \Sigma_{oa}$. The corresponding SOs of (19.20) are in general differential operators of order $4s$ with matrix coefficients.

We note that the operators (19.29) were found and analyzed by Beckers [25] (without connections with the algebra $AGL(2s+1, C)$ and hidden symmetries).

19.5. Symmetries of the Levi-Leblond Equation

Here we discuss hidden symmetries of the Levi-Leblond equation, the simplest Galilei-invariant equation for a particle of nonzero spin (refer to Section 13). We rewrite this equation in the form (13.1) where

$$\beta_0 = \frac{1}{2}(1 + i\gamma_4), \quad \beta_4 = 1 - i\gamma_4, \quad \beta_a = \gamma_a,$$

(19.31)

$\gamma_0, \gamma_4$ are the Dirac matrices.

The Levi-Leblond equation is invariant under the algebra $AG(1, 3)$. The basis elements of this algebra belong to the class $M_1$ and are given in (12.18) where

$$M = m, \quad S = \frac{i}{4} \gamma \times \gamma, \quad \eta = \frac{1}{2}(1 - i\gamma_4)\gamma.$$  

(19.32)

The problem of description of SOs of higher orders for the Levi-Leblond equation is formulated in complete analogy with the corresponding problem for the Dirac equation, thus we use all the definitions and notations of Section 18.

To find a complete set of SOs of arbitrary order $n$ we consider other equation equivalent to (13.1), (19.31). Namely making the transformation $\psi \rightarrow V \psi = \Phi$ where $V$ is the operator (13.15) we come to the equation (13.17). As in the case of the Dirac equation there is one-to-one correspondence between the SOs $Q'$ of the equation (13.17) and the SOs of the Levi-Leblond equation. This correspondence is given by relations (18.2) where $V$ is the operator (13.15). That is why we will investigate symmetries of the more simple equation (13.17) instead of the Levi-Leblond one.

The function $\Phi$ satisfying (13.17) has two nonzero components only so it is convenient to search for SOs of (13.17) in the form

$$Q \rightarrow S_a Q_a + IQ_0,$$

(19.33)

where $S_a = \epsilon_{abc} \gamma_c c / 4$ together with the unit matrix $I$ form a complete set of linearly

234
independent matrices defined on $\Phi$, $Q_a$ and $Q_0$ are unknown operators commuting with $\gamma_{\mu}$. The operator (19.33) is a SO of the equation (13.17) iff $Q_a$ and $Q_0$ are SOs of the Schrödinger equation.

It was shown in Subsection 16.5 that all the finite order SOs of the Schrödinger equation are polynomials of the generators of the Galilei group, see (16.39). The same is true for the equation (13.17) since the corresponding generators of the Galilei group have the form

$$P_{\mu} - p_{\mu}, \ J = x \times p \cdot S, \ G = G$$

(19.34)

where $P_{\mu}, J$ and $G$ are the operators (11.5). On the other hand

$$S\Phi - \frac{1}{m}(mJ - P \times G)'\Phi$$

(19.35)

where $\Phi$ is an arbitrary solution of (13.17). So representing $Q_a$ and $Q_0$ in the form of (19.35) and expressing $P_{\mu}, J, G, S$ via $P_{\mu}', J', G'$ in accordance with (19.34), (19.35) we come to the conclusion the operators (19.33) are polynomials on the Galilei group generators (19.34). Therefore according to (18.2) all the finite order SOs of the Levi-Leblond equation are polynomials of the generators (12.18), (19.32) inasmuch as

$$V^{-1}P_{\mu}V, \ V^{-1}J/VJ, \ V^{-1}G/VG$$

where $P_{\mu}', J$ and $G$ are the Galilei group generators of (12.18) realized on the set of solutions of the Levi-Leblond equation.

It follows from the above that all the SOs of finite order $n$ for the Levi-Leblond equation belong to the enveloping algebra generated by the Galilei group generators. Thus a description of a complete set of SOs of order $n$ for this equation reduces to going through linearly independent polynomials on the operators (12.18).

Let us present a complete set of the first order SOs. Besides the generators (12.18), (19.32) this set includes the following 23 operators [423]

$$D = 2p_\beta - x \cdot p + \frac{i}{2}(3 + \beta_\beta), \ A = t^2p_\beta - tD - \frac{1}{2}mx^2 + \frac{1}{2}x \cdot \eta - \frac{i}{4}t\beta_\beta,$$

$$W_0 = -p \cdot S, \ W = S \cdot \frac{1}{m}p \times \eta,$$

$$U = \frac{1}{2}p \times S - \frac{i}{2}p_\beta \cdot p + 2\eta \cdot p_\beta, \ V = \beta t^2p_\beta S - \frac{1}{2}(1 - i\beta_\beta)^0 \eta p,$$

$$F_0 = tW_0 \cdot mS \cdot x + \eta \cdot J, \ F = tU - mxxS + x \cdot \eta \cdot p - (x \cdot p - i)\eta,$$

$$N = x \cdot U + \frac{1}{2}m(\frac{3}{2} - \beta_\beta), \ N = 2tV - m(x \times U + xW_0) + \eta x \times U + \eta W_0,$$

$$235$$
Symmetries of Equations of Quantum Mechanics

The operators (12.18), (19.36) do not form a Lie algebra but include subsets forming bases of Lie algebras and superalgebras. We present some of them.

1. The algebra $AO(1,2)$ is formed by $P_\mu, A$ and $D$, refer to (11.12).

2. The algebra $AO(3)$ is formed by $W=(W_1, W_2, W_3)$. This algebra can be united with $AG^2(1,3)$ to obtain the 15-dimensional IA including the following operators $A_15\supset P_\mu, J, G, D, A, W$.

3. The superalgebra $SA_{22}$ is formed by the following set $SA_{22}\supset W, W_0, U; P_\mu, J, G, A, D$.

Besides that the Levi-Leblond equation has a wide symmetry in the class $M_2$. The matter is that all the symmetries of the Schrödinger equation (refer to Section 16) are valid for (13.1), (19.31). Moreover these symmetries can be extended. Indeed, the operators (16.41a) and the generators $P_\mu, J_\alpha, G_\alpha$ of (12.18), (19.32) form a 27-dimensional Lie algebra being an IA of the Levi-Leblond equation. This IA can be extended to a 30-dimensional Lie algebra by including the SOs $W$ of (19.36).

We see the simplest Galilei-invariant wave equation for a particle of spin 1/2 has extensive hidden symmetries in spite of the fact that all the finite-order SOs belong to the enveloping algebra of the algebra $AG(1,3)$.


Investigating of symmetries of Galilei-invariant equations for arbitrary spin particles (refer to Sections 13, 14) can be carried out in analogy with the above. We do not present the corresponding cumbersome calculations here but formulate and discuss the main results of this investigating.

1. A principal description of SOs of the first-order Galilei-invariant wave equations for particles of arbitrary spin is analogous to the description of SOs of the Levi-Leblond equation. Namely all the finite order SOs of the equations considered in Section 13 belong to the enveloping algebra of the Lie algebra of the Galilei group. To prove this assertion it is sufficient to transform from the equations given in (13.1), (13.11), (13.12) and Table 13.1 to the equivalent representation (13.17) where the transformed wave function $\Phi'$ has $2s+1$ independent components only, and to expand a SO in the complete set of the matrices $\lambda_{mn}$ of (19.20) where

$\Sigma_{\alpha\beta} = i S_{\alpha \beta} - \epsilon_{\alpha\beta \gamma} S_\gamma$.

$S_\gamma$ are the spin matrices expressed via the Galilei group generators according to (19.35).

In fact the almost evident hidden symmetry of the equations considered is
formulated in the previous subsection, see assertions (1) and (2) there. Moreover this symmetry is more extensive if \( s > 1/2 \).

**Assertion.** Described in Section 13 Galilei-invariant wave equations for particles of spin \( s \) are invariant under the \([27+(2s+1)^2]-\)dimensional Lie algebra whose basis elements are

\[
\{ P_\mu, G_\sigma, J_a, A, D, G_a G_b, P_\mu G_\sigma, \lambda_{mn} \}
\]

(19.38)

where \( P_\mu, G_\sigma, J_a, A, D \) are the generators of the Schrödinger group, \( \lambda_{mn} \) are the operators of (19.20), (19.37), (19.35).

The proof reduces to calculation of commutation relations of the operators (19.38) (which evidently are the SOs of the equations considered) in order to make sure they form a basis of a Lie algebra.

2. SOs of Galilei-invariant equations in the Hamiltonian form (refer to Section 14) do not belong to the enveloping algebra of the algebra \( AG(1,3) \) in general. More precisely such SOs can be represented in the form

\[
Q = Q_1 + q_1 Q_2
\]

where \( Q_1 \) and \( Q_2 \) are polynomials of the Galilei group generators, \( q_1 \) is the operator not belonging to the enveloping algebra of the algebra \( AG(1,3) \):

\[
q_1 = [\frac{2}{m} \sigma_3 S \cdot p + \frac{2k^2}{m^2} (\sigma_1 - \sigma_2) (S \cdot p)^2] \cos(atm) - \\
- [\sigma_3 + \frac{2i}{m} (\sigma_1 + i \sigma_2) k S \cdot p] \sin(atm).
\]

(19.39)

This assertion can be proved by transforming the Hamiltonian (14.9a) and the corresponding generators (12.18), (14.3) into the equivalent representation (14.14), (14.16) where \( B = \sigma_3 am \).

It is not difficult to verify the equations (11.1), (14.9a) for a particle of spin \( s \) are invariant under the \([27+(2s+1)^2+3]-\)dimensional Lie algebra whose basis elements are present in (19.38), (19.39) and (19.40):

\[
q_2 = \sigma_3 + \frac{2i}{m} \sigma_3 k S \cdot p + \frac{2}{m^2} (\sigma_1 + \sigma_2) (k S \cdot p)^2,
\]

\[
q_3 = (\sigma_3 + \frac{2}{m} \sigma_3 k S \cdot p) \cos(atm) + [\sigma_3 + \frac{2}{m} \sigma_3 k S \cdot p + \frac{2}{m^2} (i \sigma_1 - \sigma_2) (k S \cdot p)^2] \sin(atm).
\]

(19.40)

The operators \( q_a \) form the subalgebra of the IA, isomorphic to \( AO(3) \).

A more extensive IA of the equations considered is formed by the following sets of SOs: \( \{ Q, q_a Q \} \) where \( Q \) are the operators (19.38), \( q_a \) is any of the operators.
Symmetries of Equations of Quantum Mechanics

(19.39), (19.40). Dimension of these algebras is $2[27+(2s+1)^2]$ so for $s=1/2$ we have 62-dimensional Lie algebras.

Just now we finish our brief discussion of hidden symmetries of Galilei-invariant wave equations. Some of these symmetries are admitted by equations for interacting particles, which sometimes have even more extended symmetries than free particle equations, refer to Subsections 21-23.

20. NONGEOMETRIC SYMMETRIES OF MAXWELL’S EQUATIONS

20.1. Invariance Under the Algebra $AGL(2,C)$

It was shown in Chapter 1 that the maximal symmetry of Maxwell’s equations in the class $M_1$ is exhausted by the invariance under the Lie algebra of the conformal group. The problem of interest is to investigate nongeometric symmetries of these equations also because the electromagnetic field is a real and measurable physical object whose hidden symmetries can have consequences which can be verified.

In this subsection we consider the problem of finding of an IA of Maxwell’s equations in the class $M_\infty$. These symmetries are rather nontrivial, and it is evident they cannot be found in the classical Lie approach.

We proceed from the formulation of Maxwell’s equations given in (3.4), (3.5). Following the first step of the algorithm outlined in Subsection 16.1 we go to the equations in the momentum representation. Writing $E(x_0,x)$ and $H(x_0,x)$ in the form

$$E(x_0,x)=(2\pi)^{-3/2}\int \tilde{E}(x_0,p)\exp(ip\cdot x),$$

$$H(x_0,x)=(2\pi)^{-3/2}\int \tilde{H}(x_0,p)\exp(ip\cdot x),$$

we obtain the following system

$$L_1\varphi(x_0,p)=0,$$

$$L_2\varphi(x_0,p)=0,$$

where

$$\varphi(x_0,p) = column(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3).$$

$L_1$ and $L_2$ are symbols of the operators (3.5):

$$L_1 = i\frac{\partial}{\partial x_0} - \sigma_2 S p, \quad L_2 = (Z_{ab} + i\epsilon_{abc} S_c)p_b.$$

It is necessary to take into account the condition of reality of the vectors.
(20.1), which can be written in the form
\[ \phi(x_0, p) = \phi(x_0, -p). \]  

SOs of Maxwell’s equations belonging to the class \( M_n \) are 6×6 matrices depending on \( p \). To find a complete set of these matrices we transform the equations (20.2) to the representation where the Hamiltonian \( H \) has a diagonal form. Using for this purpose the following transformation operator
\[ W = U_3 U_2 U_1, \]
where
\[
U_1 = P_+ P_+ Q \frac{\hat{p}}{\hat{p}}, \\
U_2 = \exp \left[ i \left( \frac{S_p - S_p p_1 \arctan \frac{p_1 - p_2^2}{p_1 + p_2^2}}{p_3} \right) \right], \\
U_3 = \frac{1}{\sqrt{2}} \left[ S_3^2 + i (S_3 S_2^* S_2 S_1^*) + 1 - S_3^2 \right],
\]
(20.7)
\[ Q = 1 - 2(S \hat{p} \times \hat{n})^2 (1 - (p \cdot n)^2)^{-1}, \quad n \perp 1, \]
\( n \) is an arbitrary constant vector, we obtain by consequent calculations that
\[
U_1 L_1 U_1^* - L_1^* i \frac{\partial}{\partial x_0} - S_3 \hat{p}, \quad U_2 L_2 U_2^* - L_2^* i \frac{\partial}{\partial x_0} - S_3 \hat{p}, \\
U_3 L_3 U_3^* - L_3^* i \frac{\partial}{\partial x_0} - \Gamma_0 \hat{p}.
\]  
(20.8)
Here \( \Gamma_0 \) is the diagonal matrix,
\[ \Gamma_0 = -i (S_3 S_2^* S_2 S_1^*) S_1^* = \text{diag}(1, -1, 0, 1, -1, 0). \]  
(20.9)
According to (20.2), (20.9) the transformed function \( \phi' = W \phi \) satisfies the following system of equations
\[
L_1^\prime \phi' = i \frac{\partial}{\partial x_0} - \Gamma_0 \hat{p} \phi' = 0, \\
L_3^\prime \phi' = (1 - \Gamma_0^2) p^2 \phi' = 0,
\]  
(20.10a)
(20.10b)
since \( W p L_2^\prime W^* \equiv (1 - \Gamma_0^2) p^2 \).

In the representation (20.10) a SO \( Q_0 \in M_n \) reduces to matrix depending on \( p \) and satisfying the relations

239
Symmetries of Equations of Quantum Mechanics

\[ [Q_A, L^{(i)}_1] = [\Gamma_0, Q_A] p = -\alpha A L^{(i)}_2, \]
\[ [Q_A, L^{(i)}_2] = [Q_A, 1 - \Gamma_0] p^2 = -\beta A L^{(i)}_2 \]

where \( \alpha_A, \beta_A \) are matrices depending on \( p \). Using the explicit expression for \( \Gamma_0 \) given in (20.11)

\[ Q_A = \begin{pmatrix} a & o & o & b & o & o \\ c & 0 & 0 & d & o \\ 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & f & 0 \end{pmatrix} + F L^{(i)}_3, \]

where \( F \) is an arbitrary 6\times6 matrix which can be chosen zero without loss of generality (refer to (20.10b)), \( a, b, ..., h \) are arbitrary functions of \( p \).

So there are exactly eight linearly independent matrices satisfying (20.11). We choose these matrices in the form

\[ \Sigma_{\alpha \beta} = \epsilon_{\alpha \beta \gamma} \sigma_{\gamma}, \quad \Sigma_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma} \Gamma_0 \Sigma_{\gamma k}, \quad \Sigma_0 = I, \quad \Sigma' = i \Gamma_0. \]

The matrices (20.13) satisfy the commutation relations (17.7) characterizing the algebra \( A_p \). Using the transformation

\[ \Sigma_{\alpha \beta} \rightarrow W^{-1} \Sigma'_{\alpha \beta} W, \quad \Sigma_0 \rightarrow W^{-1} \Sigma'_0 W \]

(where \( W \) is the operator of (20.6)) we obtain the corresponding linearly independent SOs for the equation (20.2):

\[ Q_1 = \Sigma_{01} - i \sigma_2 Q_4, \quad Q_2 = \Sigma_{01} - i \sigma_2 Q, \]
\[ Q_3 = \Sigma_{02} - i \sigma_2 Q_4, \quad Q_4 = \Sigma_{01} - i \sigma_2 Q, \]
\[ Q_5 = \Sigma_{02} - i \sigma_2 Q_6, \quad Q_6 = \Sigma_{01} - i \sigma_2 Q, \]
\[ Q_7 = \Sigma_{03} = I, \quad Q_8 = -i \sigma_2 S \hat{p} \]

where \( Q \) is the matrix (20.7).

The fact that the operators (20.14) are SOs of the equation ((20.2) can be easily verified by direct calculation also, bearing in mind the relations

\[ [Q, S \hat{p}] = 0, \quad Q^2 = I, \quad L_2 Q = -L_2. \]

Thus we have obtained a basis of the IA of Maxwell’s equations in the class \( M_\omega \). Let us formulate this result in the form of the following assertion [153,157].
THEOREM 20.1. Maxwell’s equations are invariant under the eight-dimensional algebra $A_8$ isomorphic to $AGL(2, C)$. Basis elements of this algebra in the momentum representation can be chosen in the form (20.14).

We see that besides the conformal invariance, Maxwell’s equations have the hidden symmetry described by Theorem 20.1. The SOs (20.14) are defined on the set of functions $\varphi(x_0, p)$ which are the Fourier transforms of solutions of Maxwell’s equations in the realization (3.4), (3.5).

To each matrix $Q_\alpha$ of (20.9) it is possible to assign the integral operator $\hat{Q}_\alpha$ defined in the space of the functions $\varphi(x_0, x)$ of (3.4):

$$\hat{Q}_\alpha \varphi(x_0, x) = (2\pi)^{-3} \int d^3p \int d^3y Q_\alpha \varphi(x_0, y) \exp[i\langle p \cdot (x - y)\rangle].$$

(20.16)

The integral operators (20.16), (20.14) form an IA of Maxwell’s equations.

20.2. The Group of Nongeometric Symmetry of Maxwell’s Equations

The SOs (20.14) form a Lie algebra and satisfy the conditions

$$Q^2_\alpha \varphi(x_0, p) = -\varphi(x_0, p), \quad A \leq 3, A = 8,$$
$$Q^2_\alpha \varphi(x_0, p) = \varphi(x_0, p), \quad 4 \leq A \leq 7$$

where $\varphi(x_0, p)$ is an arbitrary solution of (20.2). Considering exponential mapping of these SOs we come to the conclusion that Maxwell’s equations are invariant under the eight-parameter group of transformations which are defined by the following relations

$$\varphi(x_0, p) \rightarrow \varphi'(x_0, p) = \exp(Q_\alpha \theta_\alpha) \varphi(x_0, p) =
\begin{cases}
\cos \theta_\alpha \varphi(x_0, p) + Q_\alpha \sin \theta_\alpha \varphi(x_0, p), & A \leq 3, A = 8, \\
\cosh \theta_\alpha + Q_\alpha \sinh \theta_\alpha \varphi(x_0, p), & 4 \leq A \leq 7
\end{cases}$$

(20.17)

where $\theta_\alpha$ are real parameters.

Substituting (20.3), (20.14) into (20.17) we obtain the transformation law for the Fourier transforms of the vectors of the electric and magnetic field strengths in the following explicit form

$$\begin{align*}
\left\{ & E \rightarrow \tilde{E} \cos \theta_1 + \tilde{E} \sin \theta_1, \\
& \hat{H} \rightarrow \tilde{H} \cos \theta_1, \tilde{H} \sin \theta_1
\end{align*}$$

(20.18a)

$$\begin{align*}
\left\{ & E \rightarrow \tilde{E} \cos \theta_2 + E \sin \theta_2, \\
& \hat{H} \rightarrow \tilde{H} \cos \theta_2
\end{align*}$$

(20.18b)
Symmetries of Equations of Quantum Mechanics

\[
\begin{align*}
\tilde{E} &\rightarrow \tilde{E} \cos \theta - i \tilde{p} \times \tilde{H} - 2 \tilde{p} \times n (n \cdot \tilde{H}) \lambda \sin \theta, \\
\tilde{H} &\rightarrow \tilde{H} \cos \theta - i \tilde{p} \times \tilde{E} - 2 \tilde{p} \times n (n \cdot \tilde{E}) \lambda \sin \theta;
\end{align*}
\]
(20.19a)

\[
\begin{align*}
\tilde{E} &\rightarrow \tilde{E} \cosh \theta + \frac{1}{2} \tilde{H} + 2 \frac{\tilde{p} \cdot n}{\tilde{p}} - n (n \cdot \tilde{H}) \lambda \sinh \theta, \\
\tilde{H} &\rightarrow \tilde{H} \cosh \theta + \frac{1}{2} \tilde{E} + 2 \frac{\tilde{p} \cdot n}{\tilde{p}} - n (n \cdot \tilde{E}) \lambda \sinh \theta;
\end{align*}
\]
(20.19b)

\[
\begin{align*}
\tilde{E} &\rightarrow \tilde{E} \cos \theta + i \tilde{p} \times \tilde{E} \sinh \theta, \\
\tilde{H} &\rightarrow \tilde{H} \cosh \theta + i \tilde{p} \times \tilde{H} \sinh \theta;
\end{align*}
\]

where \( \lambda = [1 - (\tilde{p} \cdot n)^2]^{-1} \), \( \tilde{p} = p / p \), \( n \) is an arbitrary unit vector.

Formulae (20.18b) present the Heaviside-Larmor-Rainich transformations (3.2). The remaining relations (20.19), (20.18a) define a set of one-parametric transformations extending (3.2) to eight-parameter group. To any of these transformations it is possible to assign the integral transformation of the vectors \( E(x_0, x) \) and \( H(x_0, x) \) of (20.1).

Thus the Heaviside-Larmor transformations are nothing but a subgroup of the eight-parameter group of hidden symmetry of Maxwell’s equations.

It is possible to make sure the SOs of (20.14) do not form a Lie algebra together with the conformal group generators of (2.22), (3.56). It is possible to unite the algebra (20.14) and the conformal algebra iff the basis elements of the latter are realized in the class of integro-differential operators. The explicit form of the corresponding basis elements of the algebra AC(1,3) is given in [148, 154].

Let us demonstrate that Maxwell’s equations are invariant under the specific combinations of transformations including space-time reflections and realizing an IR of the algebra \(# AO(2,2)\). The existence of this symmetry is caused by the fact that Maxwell’s equations can be represented as equations for the complex wave function \( \Psi = E - iH \), admitting antilinear SOs satisfying conditions of Lemma 1.1, refer to Subsection 1.7. We will not analyse this representation (see, e.g., the equation (9.1)) but present the explicit form of the corresponding symmetries.
**PROPOSITION 20.1.** Maxwell’s equations (3.4), (3.5) are invariant under the algebra $AO(2,2)$ realized by the following operators

\[ Q_1 = T, \quad Q_2 = i\sigma_2, \quad Q_3 = -i\sigma_3 T, \quad Q_4 = -PQ_3, \]

where $P$ and $T$ are space and time reflection operators:

\[ P\phi(t, x) = \sigma_3 \phi(t, -x); \quad T\phi(t, x) = -\sigma_3 \phi(-t, x). \]

**PROOF** reduces to direct verification which can be made easily using properties of $\sigma$-matrices.

It follows from the above that Maxwell’s equations are invariant under the six-parameter group of transformations, isomorphic to $O(2,2)$ and including the Heaviside-Larmor-Rainich transformations as a subgroup. Calculating exponential mappings of the SOs mentioned in Proposition 20.1 it is not difficult to find these group transformations explicitly. We present here as an example the transformation generated by $Q_4$:

\[
E(t, x) \rightarrow E(t, x) \cos \theta_4 - i H(t, -x) \sin \theta_4; \\
H(t, x) \rightarrow H(t, x) \cos \theta_4 - i E(t, -x) \sin \theta_4.
\]

### 20.3. Symmetries of Maxwell’s Equations in the Class $M_2$

We made sure ourselves that the symmetry of Maxwell’s equations is more extensive than the familiar symmetry in respect to the conformal group. Indeed, if we search for symmetries in the class $M_\infty$ then it is possible to find a hidden symmetry of these equations under the algebra $A_8$. Moreover this symmetry is maximal in the class considered.

A natural question arises: do Maxwell’s equations admit local (differential) SOs which cannot be found in the classical Lie approach? To answer this question we find complete sets of the first and second order SOs of these equations.

As before we use the formulation (3.4), (3.5) of Maxwell’s equations. The corresponding SO of arbitrary finite order $n$ can be represented in the form

\[
Q = E + O,
\]

where $A, B, C$ and $D$ are differential operators of order $n$ (with real matrix coefficients) commuting with $\sigma_\mu$.

We pick out the even ($E$) and odd ($O$) parts of SOs which are independent on the set of solutions of Maxwell’s equations. Such a terminology is in accordance with the fact that products $EE$ and $OO$ are even but $EO$ is odd.

Without loss of generality we assume that $Q$ includes operators of
differentiating in respect with spatial variables only since \( \partial / \partial x_a \) can be expressed via \( \partial / \partial x_0 \) according to (3.5).

A linear differential operator \( Q \) of order \( n \) is a SO of Maxwell’s equations iff

\[
[Q, \hat{L}_1] = \alpha_a^a \hat{L}_2^a, \quad [Q, \hat{L}_2^a] = \beta_{ab}^b \hat{L}_2^b
\]

where \( \alpha_a^a \) and \( \beta_{ab}^b \) are differential operators of order \( n \). Without loss of generality these operators can be represented in the following form

\[
\alpha_a^a = \sigma_i \alpha_i^a + \sigma_j \alpha_j^a, \quad \alpha_a^b = \sigma_i \alpha_i^b + i \sigma_j \alpha_j^b;
\]

\[
\beta_{ab}^b = \sigma_i \beta_i^{ab} + \sigma_j \beta_j^{ab}, \quad \beta_{ab}^b = \sigma_i \beta_i^{ab} + i \sigma_j \beta_j^{ab}
\]

where \( \alpha_i^a \) and \( \beta_i^{ab} \) are differential operators of order \( n \) commuting with \( \sigma_i \) and \( S_i \).

Substituting (20.22) into (20.21) and bearing in mind linear independence of the Pauli matrices we obtain the two noncoupled systems of equations

\[
[E, \hat{L}_1] = \alpha_a^a \hat{L}_2^a, \quad [E, \hat{L}_2^a] = \beta_{ab}^b \hat{L}_2^b
\]

and

\[
[O, \hat{L}_1] = \alpha_a^a \hat{L}_2^a, \quad [O, \hat{L}_2^a] = \beta_{ab}^b \hat{L}_2^b.
\]

According to (20.23), (20.24) the problem of description of SOs of Maxwell’s equations reduces to the two separate problems related to the even and odd SOs.

First we consider the first order SOs. It is possible to show the corresponding system (20.24) has trivial solutions only so Maxwell’s equations do not admit odd SOs in the class \( M_1 \). The even SOs are described in the following assertion.

**THEOREM 20.2.** The complete set of SOs of Maxwell’s equations in the class \( M_1 \) is represented by the operators

\[
P_\mu, J_{\mu \nu}, K_{\mu}, D, F = i \sigma_2, FP_\mu, FJ_{\mu \nu}, FK_\mu, FD, I
\]

where \( P_\mu, J_{\mu \nu}, K_\mu \) and \( D \) are the conformal group generators of (2.22), (2.42) including the following spin matrices

\[
S_{ab}^c = e_{abc} S_c, \quad S_{0a}^i = i \sigma_2 S_a^i
\]

where \( S_\mu \) are the matrices (3.6).

For the proof see [360]. The operators (20.25) form a 32-dimensional Lie algebra whose structure constants are easily calculated.

The following assertion gives a description of all the linearly independent odd SOs in the class \( M_2 \). Such SOs are essentially non-Lie inasmuch as they do not belong to the enveloping algebra of the algebra \( A[C(1,3) \otimes H] \) (\( H \) is the group of Heaviside-
Theorem 20.3. Maxwell’s equations admit 70 linearly independent odd SOs in the class M. These operators have the form

\[ Q^{(a)} = \sigma_i \left[ (Sx^a, Sx^b), +p_x p_y \delta_{ab} [p^2 - 2(Sx^2)] \right], \]

\[ Q^{(a)}_2 = \frac{1}{2} \left[ O_2^{(a)}, x^i \right], \quad O_2^{(a)} = \frac{1}{2} \left[ O_1^{(a)}, x^i \right], \]

\[ Q^{(2)} = -\frac{1}{2} [e^{abc} Q^{(a)}_0, x^j], -2x^j Q^{(a)}_0 + \frac{i}{2} \left[ (p \times S)^a, S^b \right] + i[(p \times S)^a, S^a] \sigma_i, \]

\[ Q^{(2)} = -\frac{1}{2} [e^{abc} Q^{(a)}_0, x^j], -2x^j Q^{(a)}_0 + \frac{i}{2} \left[ (p \times S)^a, S^b \right] + i[(p \times S)^a, S^a] \sigma_i, \]

\[ Q^{(2)} = -[p_x x^b, S^b, x^j], -2x^j Q^{(a)}_0 + \frac{i}{2} \left[ (p \times S)^a, S^b \right] + i[(p \times S)^a, S^a] \sigma_i, \]

\[ Q^{(3)} = -[Q^{(a)}_0, K^a], \quad Q^{(a)}_0 = -[Q^{(a)}_0, K^a] - [Q^{(a)}_0, K^a], \quad j = 3, 4 \]

where \( K^a J^a = \epsilon_{abc} e_{a'b'c'/2} \) are the generators (2.22), (2.42), (2.26).

The proof needs cumbersome calculations so we present its outline only. To find all the linearly independent odd operators belonging to the class M, it is necessary and sufficient to find a complete set of solutions of the equations (20.24). Let us denote

\[ O = O^H + O^A, \quad \alpha'^H = (\alpha'^a)^H + (\alpha'^a)^A \]

where the indices \( H \) and \( A \) denote the Hermitian and anti-Hermitian parts of the corresponding operators. Equating the Hermitian and anti-Hermitian terms in the first of the equations (20.24) we come to the following system

\[ 2 [O^H, L_1] = [((\alpha'^a)^H, L_2^a] + ([\alpha'^a]^a, L_2^a] \right], \]

Symmetries of Equations of Quantum Mechanics

\( \hat{L}_1 = \frac{\partial}{\partial x_0} S_a P_a, \quad \hat{L}_2 = \epsilon_{abc} P_b S_c, \quad -iZ_a P_b, \)

\( O^\mu = \sigma_a \left\{ Z_{ab} \left( \left[ D_\alpha^{ab,cd} P_{ja} \right] - D_\alpha^{ab} \right) + S_a \left[ B_\alpha^{ac} P_{jc} \right] \right\}, \) \hspace{1cm} (20.30)

\( O^\mu = i\sigma_a \left\{ S_a \left( \left[ B_\alpha^{ac,cd} P_{ja} \right] + K_\alpha^a \right) + Z_{ab} \left[ F_\alpha^{ab,c} P_{ja} \right] \right\}, \)

\( (\alpha_\mu^a)^\nu = \sigma_a \left\{ \left[ G_\alpha^{ab,cd} P_{ja} \right] + H_\alpha^a \right\} \) \hspace{1cm} (\alpha_\mu^a)^\nu = i\sigma_a \left[ N_\alpha^{ab} P_{ja} \right], \)

where \( D_\alpha^{ab,cd}, B_\alpha^{ac}, D_\alpha^{ab}, F_\alpha^{ab,c}, K_\alpha^a, G_\alpha^{ab,cd}, H_\alpha^a \) are unknown functions of \( x_0 \) and \( x \), summation is imposed over the repeated indices \( \alpha, \alpha = 1,3. \)

Substituting (20.30) into (20.29) and using the relations

\[ [S_a, S_b] = i\epsilon_{abc} S_c, \quad [S_a, S_b] = 2\delta_{ab} - Z_{ab}, \]

\[ [Z_{ab}, S_c] = i(\epsilon_{abc} Z_{dk} + \epsilon_{dcb} Z_{ak}), \]

\[ [Z_{ab}, S_c] = 2\delta_{ab} S_c - \delta_{ac} S_b - \delta_{bc} S_a \]

and equating the coefficients of linearly independent matrices and differential operators we come to the following relations for coefficients of a SO of Maxwell’s equations

\[ D_1^{ab,cd} = K_1^{ab}, \quad D_3^{ab,cd} = -\frac{1}{2} \epsilon_{abc} K_{dk}^{ab}, \quad B_3^{ac,cd} = G_\alpha^{ac,cd} - 0, \quad K_1^a = -\frac{3}{10} \partial^n e_{abc} K_{0}^{ab}, \]

\[ K_3^a = -\frac{3}{5} \partial^n K_0^0, \quad B_1^{ab} = \frac{7}{10} \epsilon_{abc} \partial^n K_0^0, \quad B_3^{ab} = -\frac{7}{10} \partial^n K_0^0, \]

\[ D_1^{ab} = \frac{1}{10} \partial^n (\partial^n K_{00} + \partial^n K_{00} - \frac{1}{3} \delta_{ab} \partial^n K_{00}), \]

where the dots denote derivatives in respect with \( t=x_0, K_0^\nu \) is a generalized Killing tensor, i.e., an irreducible tensor which is antisymmetric in respect with permutations \( \mu = \nu \) or \( \rho = \sigma \) and symmetric under permutations of the pairs \( [\mu, \nu] = [\rho, \sigma] \). Moreover, this tensor satisfies the equations

\[ \partial^\nu K_0^\nu - \frac{1}{5} \partial^\nu g_{\nu \rho} K_0^\rho + \partial^n g_{\nu \rho} K_0^\rho = 0 \] \hspace{1cm} (20.33)

where symmetrization is imposed over the indices in brackets.

According to (20.32) all the coefficients of the SOs can be expressed via solutions of the equation (20.33). These solutions depend on 70 parameters and are given in (A.2.20)-(A.2.23). Substituting these solutions into (20.30) we obtain a linear combination of the SOs of (20.27), which satisfy the second condition (20.24).

Thus we have found a complete set of the odd SOs of Maxwell’s equations.
in the class $M_2$. It is evident these SOs do not belong to the enveloping algebra of the algebra $A[C(1,3) \otimes H]$ since this enveloping algebra includes even operators only.

Finally we present the even SOs of Maxwell’s equations in the class $M_2$. A complete set of these operators can be calculated in analogy with the above and includes 170 terms present below:

$\{Q_A, i \sigma_3 Q_A, i \sigma_2, I\}$

where $Q_A$ are the operators (18.40) (where $P, K, D, J_{\mu}$ are operators given in (2.22), (2.42), (20.26)). All the operators (20.34) belong to the enveloping algebra of the algebra $A[C(1,3) \otimes H]$.

In an analogous way we can consider SOs of Maxwell’s equations, belonging to the classes of third-, fourth-, ... etc. differential operators. Moreover it is not too difficult to find a complete set of SOs of arbitrary fixed order $n$. We will not do it here restricting ourselves to formulating of the following assertions.

1. The SOs of order $n>2$ for Maxwell’s equations are polynomials of the group $C(1,3) \otimes H$ generators (refer to (2.22), (2.42), (3.20), (3.23)) and the second order SOs (20.27).

The number of linearly independent SOs of order $n$ is equal to

$N_n=(2n+3)[2n(n-1)(n+3)(n+4)+(n+1)^2(n+2)^2]/12, \ n\geq 2,$

besides there are $N^E$ even and $N^O$ odd operators among them where

$N^E=(n+1)^2(n+2)(2n+3)/12,$

$N^O=n(n-1)(n+3)(n+4)(2n+3)/6.$

To calculate these numbers we use formula (A.2.16) and take into account that the odd SOs of order $n$ are completely determined by the generalized Killing tensor of order 1 and valence $R_1+2R_2$, $R_1=n-2$, $R_2=2$, and the even SOs are determined by two generalized Killing tensors of order 1 and valence $R_1+2R_2$, $R_1=n$, $R_2=0$. These facts can be proved in analogy with (20.28)-(20.32).

20.4. Superalgebras of SOs of Maxwell’s Equations

Here we discuss algebraic properties of the most interesting SOs of Maxwell’s equations, i.e., the odd SOs which do not belong to the enveloping algebra of $A[C(1,3) \otimes H]$.

The operators (20.27) do not form a Lie algebra. But there exist such subsets
of these operators which can be extended to superalgebras. A superalgebra including the maximal numbers of the operators (20.27) is formed by the following set

\[ \{Q_{(0)}^{ab}, \tilde{Q}_{(0)}^{ab}; P_\mu, J_{\mu\nu}, D, \eta_{\mu\nu\rho\sigma} \} \tag{20.35} \]

where \( Q_{(0)}^{ab} \) and \( \tilde{Q}_{(0)}^{ab} \) are the SOs of (20.27), \( P_\mu, J_{\mu\nu}, D \) are generators of the conformal group and the symbol \([...]^T\) denotes the traceless part of the corresponding tensor.

Indeed, it is not difficult to make sure the operators \( Q^{(a_0b)} \) and \( \tilde{Q}^{(a_0b)} \) commute with \( P_\mu, \eta_{\mu\nu\rho\sigma} \) and satisfy the following relations (we denote \( Q_{(0)}^{ab} = Q_1^{ab}, \tilde{Q}_{(0)}^{ab} = Q_2^{ab} \))

\[ [Q^{(a_0b)}, D] = -2iQ^{(a_0b)}, \alpha = 1, 2. \]

\[ [Q^{(a_0b)}, J_{\alpha\gamma}] = i(-1)^\alpha(e^{a_0b}Q^{k\alpha b} + e_{a_0b}Q^{k\alpha b}), \alpha' \neq \alpha, \tag{20.36} \]

\[ [Q^{(a_0b)}, Q^{(c_1d_1)}] = 2f^{a(bc}_{k1234} \eta_{klmn}, \{Q^{(a_0b)}, Q^{(c_1d_1)}\} = f^{a(bc}_{klmn} \eta_{klmn} \]

where

\[ f^{abcd}_{k1234} = (\delta^a_{ek} - \delta^a_{k1})(\delta^b_{dk} - \delta^b_{l4})(\delta^c_{km} - \delta^c_{mn})(\delta^d_{ln} - \delta^d_{lm}) + (\delta^a_{ek} - \delta^a_{k1})(\delta^b_{dk} - \delta^b_{l4})(\delta^c_{lm} - \delta^c_{mn})(\delta^d_{kn} - \delta^d_{mn}) + (\delta^a_{ek} - \delta^a_{k1})(\delta^b_{dk} - \delta^b_{l4})(\delta^c_{mn} - \delta^c_{lm})(\delta^d_{kn} - \delta^d_{ln}) + (\delta^a_{ek} - \delta^a_{k1})(\delta^b_{dk} - \delta^b_{l4})(\delta^c_{mk} - \delta^c_{mn})(\delta^d_{ln} - \delta^d_{lm}). \]

The commutation relations for \( \eta_{\mu\nu\rho\sigma} \) with \( P_\mu, D \) and \( J_{\mu\nu} \) are evident. We emphasize that anticommutation relations (20.36) are satisfied on the set of solutions of Maxwell’s equations.

We see the commutation and anticommutation relations for the operators (20.35) are in accordance with the scheme (18.1) characterizing a superalgebra. Thus the set (20.35) forms a basis of the 36-dimensional superalgebra which is an IA of Maxwell’s equations and includes the subalgebra \( AP(1,3) \). This superalgebra has infinitely many extensions in the classes of higher order SOs. For example, we can add to (20.35) the odd operators \( i\sigma_\mu P_\nu, i\sigma_\nu P_\mu, \sigma_\mu P_\nu P_\sigma P_\alpha \) and the even operators \( P_\mu P_\nu, P_\mu P_\sigma, P_\nu P_\sigma P_\alpha \). In this way we come to the 81-dimensional invariance superalgebra in the class \( M_4 \).

We see the SOs of Maxwell’s equations in the class \( M_4 \) include nontrivial algebraic structures. It is possible to show the set of the odd SOs (20.27) is closed under the commutation with the conformal group generators. Moreover all these SOs can be represented as successive commutators of \( Q_{(0)} \) with the generators (22.22), (22.42), (22.26).
20.5. Symmetries of Equations for the Vector-Potential

To conclude this section we discuss briefly symmetries of the equations for the vector-potential of the electromagnetic field, given in (3.14), (3.15).

We note that the equations (3.14), (3.15) are not invariant under the conformal algebra, i.e., symmetry of these equations is less extensive than symmetry of Maxwell’s equations [101]. At first glance, the situation seems to be rather strange since each of equations (3.14), (3.15) is conformal invariant. The thing is that different representations of the algebra $AC(1,3)$ are realized on the set of solutions of these equations. In both cases, the basis elements of this algebra have the form (2.22), (2.42) where $S_0^{a}=i(j^a\sigma_a/2)$, $S_{ab}^{c}=e_{abc}(j^c+\sigma_c/2)$, $j_a$ and $\sigma_a$ are the matrices of (20.19). But the corresponding values of $K$ in (2.42) are different: $K=2$ for (3.14) and $K=3$ for (3.15).

There are two ways to overcome possible difficulties connecting with the conformal noninvariance of the equations for the vector-potential in the Lorentz gauge: to use another gauge (linear but including higher derivatives [25,101] or even nonlinear [101]) or to consider realizations of the algebra $AC(1,3)$ in a more extended class of SOs [154, 157]. We do not discuss these possibilities here but consider symmetries of the equation (3.15) with the Coulomb gauge (3.14), (3.16). For convenience we present these equations again setting $j_{\mu}=0$:

\[
\begin{align*}
L_1 A &\equiv p_{\mu} p_\mu A = 0, \\
L_2 A &\equiv (Z_{ab}^{i} + i e_{abc} \hat{S}^c) p_\mu A = 0
\end{align*}
\]

where $\hat{S}^c_i$ and $Z_{ab}^{i}$ are matrices (3.6), $A=column(A_1,A_2,A_3)$.

The Lie symmetry of the system (20.37) is very restricted and reduces to the following 7-dimensional IA:

\[
\tilde{P}_0 \equiv \frac{\partial}{\partial x_0}, \quad \tilde{P}_a \equiv i p_\mu \frac{\partial}{\partial x_a},
\]

\[
\tilde{J} = i x^\mu p_\mu, \quad \tilde{D} = i x_0 p_0.
\]

To "extend" this symmetry we can investigate invariance properties of (20.37a) considering (20.37b) as an additional condition which in general is not invariant under the symmetry transformations of the main equation (20.37a). Such an approach is natural enough since it is the equation (20.37a) which describes evolution of $A$ in time and so the SOs of this equation corresponds to constants of motion. As to (20.37b) it can be considered as an additional condition extending the symmetry of the evolution equation (20.37a).

Such a method of investigation of symmetries of the equation (20.37a) is an example of using of the concept of the conditional symmetry [124] in analysis of partial differential equations. This concept enables to find a wide classes of exact solutions of
nonlinear equations of mathematical physics, refer to Chapter 6.

First we consider symmetries of the equation (20.37a) without an additional condition. In a complete analogy with Subsection 1.3 it is possible to prove the following assertion.

**Theorem 20.4.** The maximal IA of the equation (20.37a) in the class \( M_1 \) is the 24-dimensional Lie algebra whose basis elements are

\[
\{ iP_\mu, iJ_{\mu\nu}, iD, iK_\mu, i\vec{S}, Z_{ab} \}
\]

(20.39)

where \( P_\mu, J_{\mu\nu}, K_\mu \) and \( D \) are the generators of the conformal group of (1.6), (1.16).

The concept of the conditional invariance in application to the system (20.37) reduces to searching symmetries of the first equation on the subset of solutions satisfying the condition (20.37b). In this way we prove the following assertion.

**Theorem 20.5.** The equation (20.37) is conditionally invariant under the 27-dimensional Lie algebra whose basis elements are given in ((20.39), (20.40):

\[
\eta^{-}(Z_{ab} + i\epsilon_{abc} \vec{S})_{x^b}.\quad (20.40)
\]

This algebra is the maximal conditional IA in the class \( M_1 \).

The proof is analogous to the proof given in Subsection 20.3.

We see the conditional symmetry of the equation (20.37a) is more extensive than its ordinary symmetry and essentially more extensive than the symmetry of the system (20.37). Moreover we should like to emphasize that the conditional symmetry has a clear physical sense since this is the symmetry which corresponds to constants of motion. Indeed, we can assign the following conserved quantity to any of the operators \( Q \) of (20.39), (20.40):

\[
I = \int d^3x(A^TQA - A^TQ\dot{A}), \quad t = 0
\]

(20.41)

where

\[
A = \frac{\partial A}{\partial x^0}, \quad A^T = (A_1, A_2, A_3).
\]

We see the concept of conditional invariance arises naturally by studying the conservation laws for systems of partial differential equations. This is the conditional symmetry (but not the ordinary symmetry of (20.37a) or of the system (20.37)) which generates conservation laws for the vector-potential. The general problems connected with conservation laws are considered in Section 23.

In conclusion we note that the symmetries of Maxwell’s equations considered in the above in fact are conditional also. It happens the conditional and ordinary symmetries of the equations (3.4), (3.5) coincide.
21. SYMMETRIES OF THE SCHRODINGER EQUATION WITH A POTENTIAL

21.1. Symmetries of the One-Dimension Schrödinger Equation

Let us consider the one-dimension Schrödinger equation with an arbitrary potential \( V(x) \)

\[
L\psi = \left( p_0 - \frac{1}{2}(p^2 + V(x)) \right) \psi = 0, \quad p = i \frac{d}{dx}.
\]  

(21.1)

An investigation of symmetries of this equation includes the problems which can be subdivided into two types:

1) the potential is given, symmetries are searched;
2) to find potentials admitting the given (or any) symmetry.

We consider both types of problems.

Let us search for SOs of (21.1), having the following structure

\[
Q_1 = (h_0^1 p)_0 + (h_1^1 p)_1
\]  

(21.2)

if first order operators only are considered or

\[
Q_2 = (h_0^2 p)_0 + (h_1^2 p)_1 + (h_2^2 p)_2
\]  

(21.3)

if the second order is required. We use the notation

\[
(h_n^1 p)_n \equiv [(h_n^1 p)_{n-1}, p], \quad (h_0^1 p)_0 = h_0
\]  

(21.4)

where \( h_n^1 \) (\( n=0,1,2,\ldots \)) are arbitrary functions of \( x \) and \( t \).

Substituting (21.1), (21.2) into the invariance condition (16.7) (where \( \alpha_0 = 0 \) without loss of generality) and equating coefficients of linearly independent differential operators we come to the following system of determining equations

\[
h_0^1 = 0, \quad 2h_1^1 + h_0^1 = 0, \quad h_0^1 - V' h_1^1 = 0
\]  

(21.5)

where the dot and prime denote derivatives in respect with \( t \) and \( x \) respectively.

The equations (21.5) can be used to find all the nontrivial symmetries corresponding to the given \( V \). But it happens that it is possible to find the general form of \( V \) admitting any symmetry and to calculate these symmetries explicitly. Using the condition \( h_1 \neq 0 \) we come to the following differential consequences of (21.5):

\[
h_0^1 = 0, \quad V'' = 0,
\]  

(21.6)

from which we obtain
Symmetries of Equations of Quantum Mechanics

\( V = V_0 + V_1 x - V_2 x^2 \), \hspace{1cm} (21.7)

\( V_0, V_1 \) and \( V_2 \) are arbitrary constants.

We see the requirement that the equation (21.1) admits a first-order SO is very restrictive and leads to the potentials (21.7) only. Substituting (21.7) into (21.5) it is not difficult to find the corresponding SOs explicitly.

Now let us assume the equation (21.1) admits the second-order SO of (21.3) (we note that such an operator reduces to the first order one if we take into account the equation (21.1)). In analogy with the above it is not difficult to recover the old results [7,51] and find the most general potential

\[ V(x) = \frac{V_2}{(c_0 + c_1 x)^2} + V_0 x + V_2 x^2. \] \hspace{1cm} (21.8)

Let us extend these developments to arbitrary \( n \)-th-order symmetries with

\[ Q_n = \sum_{k=0}^{n} (h_1 \cdot p)_k \] \hspace{1cm} (21.9)

and search for the corresponding classes of potentials admitting such \( n \)-th-order SOs.

For any order \( k \), we note the following operator identity containing \( k \) anticommutators

\[ (h_1 \cdot p)_k = (-i)^k \sum_{i=1}^{k} \frac{k!2^{i-1}}{(k-i)!i!} (\partial_i h_1) \partial_i p_1 \] \hspace{1cm} (21.10)

where \( \partial_i = \partial/\partial x_i \). This relation will be very useful in the following in order to obtain the determining equations for the coefficients of \( Q_n \).

For each \( k \)-term in the operators (21.9) we can evolve the following commutators

\[ [i \partial_1 (h_1 \cdot p)_1] = i(h_1 \cdot p)_1 \] \hspace{1cm} (21.11)

and

\[ [-\frac{1}{2} \partial_1^2 (h_1 \cdot p)_1] = \frac{i}{2} (h_1 \cdot p)_{k-1}. \] \hspace{1cm} (21.12)

Using these relations we immediately get for the free case \((V=0)\) the set of commutators

\[ [L_0, (h_1 \cdot p)] = i(h_1 \cdot p)_1 + \frac{i}{2} (h_1 \cdot p)_{k-1}. \] \hspace{1cm} (21.13)

where

\[ L_0 = i \partial_1 - \frac{1}{2} \partial_1^2 + i \partial_1 + \frac{1}{2} \partial_1^2. \] \hspace{1cm} (21.14)
For considering $Q_n$-problem with arbitrary potential we need the additional commutators $[V, (h_k p)]$, $k=0,1,...,n$. Consider separately even and odd $k$'s. By using (21.10) we obtain for $k \geq 1$

$$[V, (h^2_k p)^k_{2n-1}] = -i \sum_{m=0}^{k-1} (-1)^{m+k} \frac{2(2k)!}{(2k-2m-1)!(2m+1)!} (h^2_k \partial^{2k-2m-1}_k V p)_{2m+1},$$ (21.15)

and for $k \geq 0$

$$[V, (h^2_{2k+1} p)_{2k+1}] = -i \sum_{m=0}^{k} (-1)^{m+1} \frac{2(2k+1)!}{(2k-2m-1)!(2m)!} (h^2_{2k+1} \partial^{2k-2m-1}_k V p)_{2m}.$$ (21.16)

By superposing the results (21.9)-(21.16) it is easy to evolve the invariance condition (16.7) (where $\alpha_q = 0$ without loss of generality) and equate the coefficients of linearly independent differential operators. As a result, we come to the following system of determining equations

$$h_n' = 0,$$

$$2h_{2m}' + h_{2m+1}' + \sum_{l=1}^{[m-1/2]} (-1)^{m-1} \frac{2(2k+1)!}{(2k-2m-1)!(2m)!} h_{2k+1} \partial^{2k+1}_l V = 0,$$ (21.17)

$$2h_{2l}' + h_{2l+1}' + \sum_{k=1}^{[m/2]} (-1)^{k+1} \frac{2(2k)!}{(2k-2l)!} h_{2k} \partial^{2k-1}_l V = 0,$$

where $m=0,1,...,\{n/2\}$, $l=0,1,...,\{(n-1)/2\}$, $h_i \equiv 0$.

### 21.2. The Potentials Admitting Third-Order Symmetries

The equations (21.17) define all the possible potentials $V$ admitting nontrivial symmetries of arbitrary order $n$. Let us consider in more detail the first new nontrivial context, i.e., the third-order SO. We are asking for symmetries of the Schrödinger equation (21.1) of the form

$$Q_3 = \sum_{k=0}^{\lambda} (h_k p)^k_{2n} = (h_0 p)^{2n} + (h_1 p)^{2n-1},$$ (21.18)

The corresponding determining equations (21.17) are

$$h_1' = 0; \quad 2h_2' + h_3' = 0; \quad 2h_1' + h_2' - 6h_1 V' = 0,$$ (21.19a)

$$2h_1 + h_0' - 4h_1 V'' = 0; \quad h_0' - h_1 V' + h_1 V'' = 0.$$ (21.19b)

It is not difficult to find the general solution of the system (21.19a), which has the following form
Symmetries of Equations of Quantum Mechanics

\[ h_0 = a, \quad h_2 = b - 2a, \quad h_1 = g_1 + 6aV, \quad g_1 = 2aV - 2b + c \quad (21.20) \]

where \( a, b \) and \( c \) are arbitrary functions of \( t \). Excluding \( h_0 \) from (21.19b) and using (21.20) we come to the following nonlinear ordinary differential equation for \( V \)

\[ F(a,b,c;V,x) = aV''' - (2aV + 6aV + c - 2b) + 6(2aV + aV - b)V' - 12aV - 2(2\partial^2_a)x^2 - 2(\partial^2_b)x + c) - 0. \quad (21.20') \]

Integrating this equation twice in respect with \( x \) and denoting \( V = U' \) we obtain

\[ a(U'' - 3(U')^2) - (g_1U)' = \frac{1}{3}(\partial^2_a)x^4 - \frac{2}{3}(\partial^2_b)x^3 + c + dx + e \quad (21.21) \]

where \( g_1 \) is given in (21.20), \( d \) and \( e \) are arbitrary functions of \( t \).

The function \( U \) depends on \( x \) only while \( a, b, c, d, e \) are functions of \( t \). This circumstance enables us to separate variables in (21.21). Dividing the l.h.s and the r.h.s. of (21.21) by \( a \) and differentiating them in respect with \( t \) we conclude that there exist the two possibilities.

1. This equation reduces to one of the following forms

\[ V'' - 3V^2 - 2\omega_0V = \omega_4, \quad (21.22a) \]

\[ V'' - 3V^2 = \omega_4, \quad (21.22b) \]

\[ (V'' - 3V^2)' = 2\omega_4(xV' + 2V) = 0, \quad (21.22c) \]

\[ U'' - 3(U')^2 - 2\omega_4(x^2U)' = \frac{1}{3}\omega_4^2x^4 + \omega_5 \quad (21.22d) \]

where \( \omega_0, ..., \omega_5 \) are arbitrary constants.

Formulae (21.22) define classes of equivalent equations connected by transformations \( V \to V + C_1, \; U \to U + C_1 + C_2x, \; x \to x + C_2 \); \( C_1, C_2 \) are arbitrary constants. Moreover the functions \( a, b, ..., e \) included in (21.21) have to satisfy the following conditions

\[ a = \omega_a, \quad b = \omega_a, \quad c = \omega_a, \quad k = \omega_a, \quad e = \omega_a \quad (21.23) \]

besides that all the \( \omega_4 \) absent in (21.22) (say, \( \omega_1, \omega_2, \omega_3, \omega_4 \) for the equation (21.22a)) are equal to zero.

2. The second possibility is that the solution of (21.21) has the form

\[ \varphi = C_1x^3 + C_2x^2 + C_3x + C_4 \quad (21.24) \]

and the conditions (21.23) are not necessary valid. This possibility is not too interesting inasmuch as the function (21.24) is a linear combination of particular solutions of the equations (21.22).
Let us consider consequently any of the equations (21.22) and describe the corresponding classes of potentials.

Formula (21.22a) defines the Weierstrasse equation whose solutions are expressed via elliptic integrals. Multiplying both parts of (21.22a) by \( V' \) and integrating we obtain the first integral of this equation

\[
\frac{1}{2}(V')^2 - V^3 - \omega_3 V^2 - \omega_5 V = C \tag{21.25}
\]

where \( C \) is the integration constant. Then changing the roles of dependent and independent variables it is possible to integrate (21.25) and to find \( V \) as implicit functions of \( x \). Here we present a particular solution of (21.25) only:

\[
V = 2\sqrt[3]{\tan^3(\nu x) - 1}. \tag{21.26}
\]

The relation (21.22b) defines the first Painlevé transcendent. Its solutions are meromorphic on all the complex plane but can not be expressed via elementary or special functions.

The equation (21.22c) also can be reduced to the Painlevé form using the Miura [24*] ansatz. Indeed, as a result of the substitution

\[
V = -\sqrt{\frac{4\omega_2}{3} F - \frac{\omega_0}{6}}, \quad x = \sqrt{\frac{1}{6\omega_2}} y \tag{21.27}
\]

this equation takes the following form

\[
F''' + FF' - \frac{1}{3} x F' - \frac{2}{3} F = 0 \tag{21.28}
\]

(where \( F' = \partial F / \partial x \)), and the ansatz [24*]

\[
F = \frac{1}{6} W^2 \tag{21.29}
\]

reduces (21.28) to the following form

\[
(\partial_x - \frac{1}{3} W)(W''') - \frac{1}{6} W^2 W' - \frac{1}{3} y W' - \frac{1}{3} W = 0.
\]

Equating the expression in the second brackets to zero and integrating it we come to the second Painlevé transcendent:

\[
W''' = \frac{1}{18} W^3 + \frac{1}{3} y W + K \tag{21.30}
\]

where \( K \) is an arbitrary constant.

Thus any solution of the second Painlevé transcendent corresponds to the
potential \( V(21.27), (21.29) \) moreover the related Schrödinger equation admits a third-order \( \text{SO.} \) Of course the ansatz \((21.29)\) does not make it possible to find all the solutions of \((21.28)\), in particular the solutions \((21.24)\) are missed.

The last of the equations considered, i.e. the equation \((21.22d)\), with the help of the change \( U=2f\omega_1/3 \) reduces to the form

\[
F^{''} + 2fF' - 4f^2 F = 2\omega_1 + 1/2
\]

where

\[
F = f' - f^2 - x^2/4.
\]

Choosing \( \omega_1 = -1/4 \) we conclude that any solution of the Riccati equation

\[
f' = f^2 + x^2/4
\]

\((21.31)\)
corresponds to the solution of the equation \((21.22d)\).

The equation \((21.31)\) is much simpler than \((21.22d)\) nevertheless it cannot be integrated in radicals. Particular solutions of the equation \((21.22d)\) have the following form

\[
U = \frac{2}{x}, \quad U = \frac{\omega_1}{3} x^3.
\]

We note that with the help of twice differentiation and the consequent change of variables

\[
U^{' 3} = \sqrt[3]{\frac{2\omega_1^2}{3} G - \frac{1}{9} \omega_1 x^2}, \quad x = \frac{y}{\sqrt[3]{4\omega_1}}
\]

the equation \((21.22d)\) reduces to the form

\[
\delta^{4} G + G^{'''} G^{'} + G^{''} G^{'} - \frac{1}{3} (8G + x^2 G^{'''} + 7xG^{''}) = 0.
\]

\((21.32)\)

The last equation had being met in literature and is nothing but the reduced Boussinesq equation, see, e.g., \([157, 25^*] \) (refer to Subsection 31.7 also). The procedure outlined above admits to reduce it to the Riccati equation.

Thus the third-order \( \text{SO} \) are admitted by a very extended class of potentials described above. It is necessary to emphasize that all these potentials (excepting \((21.24), (21.26)\)) correspond to the Schrödinger equation which does not possess any non-trivial (distinct from time displacements) Lie symmetry.
21.3. Time-Dependent Potentials

In this subsection we exploit the following observation: all the equations (21.22) are equivalent to reduced versions of the Boussinesq equation

\[ \dddot{V} - V'V'' - VV'' + V''' = 0 \]  

(21.33)

Indeed, the equations (21.22a) and (21.32) (the last is equivalent to (21.22d) appear from similarity reducing of the equation (21.33) [26'], and the equations (21.22b), (21.28) (the last is equivalent to (21.22c)) appear as a result of the reduction of (21.33) using its conditional symmetry, refer to Subsection 31.7.

Thus there exists a deep connection between solutions of the Boussinesq equation and the Schrödinger equation admitting third-order symmetries. This connection became more straightforward in the case of time-dependent potentials \( V(x,t) \). Indeed, the determining equations (21.19) are valid in this case also, but the compatibility condition for the system (21.19), in contrast with (21.20), takes the form

\[ F(a,b,c;x,V) + 12aV - 4(b - 2a x)V' = 0 \]  

(21.34)

where \( F(a,b,c;x,V) \) is the expression defined in (21.20').

The equation (21.34) is more complicated than (21.20') due to time dependence of \( V \), which makes it impossible to separate variables. That is why we restrict ourselves to analysing of sufficient conditions of its integrability.

Let \( a = \text{const}, b = 0 \) then (21.34) reduces to the form

\[ 12\dddot{V} - 6(VV'' + V'V'') + V''' = 0. \]  

(21.35)

With the help of the change \( V \rightarrow V/6, t \rightarrow t/(2\sqrt{3}) \) the equation (21.35) reduces to the Boussinesq equation.

*Thus if the potential satisfies the Boussinesq equation then the corresponding Schrödinger equation admits a non-trivial SO of third order. For more on the Boussinesq equation see Subsection 31.7.*

21.4. Algebraic Properties of SOs

Thus we have described a class of potentials admitting SOs of third order. A natural question arises about possible applications of these hidden symmetries.

In the following we will demonstrate that SOs of third order possess a very important information about the energy spectrum of a system described by the equation (21.1) and can be used to generate solutions of this equation and even of the nonlinear Schrödinger and wave equations.

Let us investigate algebraic properties of SOs which do not depend on the exact form of the potential. Moreover these properties are completely defined by the
type of equation satisfied by the potential.

Let the potential satisfies the equation (21.22a). It means that in (21.23) \( \omega_1=\omega_2=\omega_3=\omega_4=0 \). Then we conclude from (21.18), (21.20) that the corresponding third order SO reduces to a linear combination of the Hamiltonian (21.1) and the following operator

\[
Q= p^3 + \frac{1}{4} [3V,p] = 2pH + \frac{1}{2} Vp + \frac{i}{4} V',
\]

(21.36)
i.e., in fact we have the only non-obvious SO (21.36). This SO commutes with the Hamiltonian,

\[ [Q,H]=0, \]

(21.37)
i.e. \( Q \) is a constant of motion. In Subsection 31.8 we will use the property (21.37) to integrate the equations of motion.

If the potential satisfies the equation (21.22b) then \( \omega_1=\omega_2=\omega_3=0, k=k_0+at \), and the corresponding SO has the form

\[
Q= p^3 + \frac{3}{4} [V,p] + t.
\]

(21.38)

Just two more SOs are represented by the Hamiltonian and the unit operator \( I \), moreover, they form the Heisenberg algebra together with \( Q \), satisfying the following relations:

\[ [Q,H]=-il, \quad [Q,I]=[H,I]=0. \]

(21.39)

It is well-known that up to unitary equivalence all irreducible sets of self-adjoint operators satisfying (21.39) are exhausted by \( p, -i\partial/\partial p \) and \( 1 \), from which it follows that for any potential satisfying the first Painleve transcendent (21.22b) the spectrum of the corresponding Hamiltonian is continuous.

In the case when the potential satisfies (21.22c) the corresponding SO of third order has the form

\[
Q= p^3 + \frac{3}{4} [V,p] + \frac{\omega_3}{2} tH
\]

(21.40)
and generated the following algebra together with the Hamiltonian \( H \)

\[ [Q,H]=-i\frac{\omega_3}{2} H. \]

(21.41)

It follows from (21.41) the spectrum of the Hamiltonian \( H \) is continuous. Indeed, let \( H\Psi_E=E\Psi_E \), then the function \( \Psi'=\exp(i\lambda Q)\Psi_E \) (where \( \lambda \) is a real parameter) also is an eigenvector of the Hamiltonian with the eigenvalue \( \lambda E \). In other words if \( H \) has at least one non-zero eigenvalue then using the third-order SO it is possible to
construct eigenfunctions corresponding to any real eigenvalue.

In the case when the potential is defined by the equation (3.2d) there exist two SOs of third order, \( Q_+ \) and \( Q_- \),

\[
Q_\pm = \frac{1}{\sqrt{2}} \left[ p^3 \mp \frac{i}{4} \omega \left[ x, [x, p] \right], p \right] - \frac{\omega^2 x^2 + 3\phi', p}{2} \exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_1}. \tag{21.42}
\]

which satisfy the following relations

\[
[H, Q_\pm] = \omega Q_\pm, \quad [Q_+ , Q_-] = \omega H^2. \tag{21.43}
\]

If \( \omega_1 < 0 \) then \( Q_\pm \) play a role of increasing and decreasing operators for the Hamiltonian eigenvalues, i.e if \( H\Psi_\lambda = \lambda \Psi_\lambda \) then \( H(Q_\pm \Psi_\lambda) = (\lambda \pm \omega) \Psi_\lambda \), and so \( H \) has a discrete spectrum.

We see that the third-order SOs enable to make very general predictions about the Hamiltonian spectra for a very wide class of potentials. Of course these predictions have a formal level inasmuch we ignore considering of domains of the operators discussed.

For application of the third-order SOs to construction of exact solutions and generation of solutions of the linear and non-linear Schrödinger equation refer to Subsection 31.8.

**21.5. Complete Sets of SOs for One- and Three-Dimensional Schrödinger equation**

Consider some problems of the type 1) (see Subsection 21.1) where the potential is treated as a known function. We restrict ourselves to analysis of the following cases:

\[
V = V_1, \tag{21.44a}
\]

\[
V = V_2 x, \tag{21.44b}
\]

\[
V = V_3 x^2 \tag{21.44c}
\]

where \( V_\alpha \) are arbitrary constants, and find all the nonequivalent SOs of arbitrary order \( n \) for the corresponding Schrödinger equations.

For the case (21.44a) the problem reduces to description of SOs of the free Schrödinger equation. The equations (21.17) take the form
By the consequent differentiation of (21.44) we obtain the relations
\[ \dot{h}_k = 0, \quad \ddot{h}_k = 0, \quad \dddot{h}_k = 0, \quad 0 < k < n. \]  
(21.45)

from which it follows that
\[ \dot{h}_k = \sum_{i=0}^{n-k} \sum_{i=0}^{k} C_{i,j} h^i t^j \]  
(21.47)

where \( C_{i,j} \) are constant coefficients whose number is equal to \((k+1)(n-k+1)\). From (21.24) we obtain the only restriction for these constants
\[ 2l(l+1)C^p_{l+1} + (p+1)C^p_{l+1} = 0, \quad k = 1, 2, ..., n; \]  
(21.48)

therefore the total number of independent parameters in (21.45) is
\[ N^n = \sum_{l=0}^{n} (k+1)(n-k+1) - \sum_{l=0}^{n} k(n-k+1) = \frac{1}{2} (n+1)(n+2). \]  
(21.49)

The corresponding SOs of order \( n \) (whose number is evidently equal to \( N^n \)) are defined by relations (21.9), (21.47), (21.48) (the latter can be interpreted as a recurrence formulae). It is not difficult to see that all these SOs are nothing but polynomials of order \( n \) in the first order SOs \( P = p + Vt \) and \( G = tP - mx \).

For the potentials (21.44b) and (21.44c) the equations (21.17) reduces to the following forms
\[ h'_n = 0, \quad h'_0 - 2V_j h_1 = 0, \quad 2\dot{h}_n + 2\dot{h}_{n-1} = 0, \quad 0 < k < n; \]  
(21.50)

and
\[ h'_n = 0, \quad 2\dot{h}_n + 2\dot{h}_{n-1} = 0, \quad \dot{h}_0 - 2V_j h_1 = 0, \quad 2\dot{h}_n + 2\dot{h}_{n-1} - 4(k+1)V_j h_{k-1} = 0, \quad 0 < k < n. \]  
(21.51)

The equations (21.50) can be solved in complete analogy with (21.45). We again come to the differential consequences (21.46) and the representation (21.47) but instead of (21.48) we obtain the following relations
\[ 2m(l+1)C^p_{l+1} + (p+1)C^p_{l+1} - 4(k+1)V_j C^p_{k} = 0, \quad k = 1, ..., n. \]  
(21.52)

Thus the equation (21.1), (21.44b) admits \( N^n \) SOs of order \( n \). The explicit form of these operators is given by formulae (21.9), (21.47), (21.52), \( N^n \) is given in (21.49). All these SOs are polynomials in the first order SOs \( P = p + Vt \) and \( G = tP - mx \).

For the equations (21.51) we have only the latter of the consequences (21.46) which allows to represent \( h_k \) in the form

260
where $a_{i,j}$ are arbitrary functions of $t$. Substituting (21.53) into (21.51) and equating coefficients of the same powers of $x$ we obtain $N^n$ ordinary differential equations for $N^n$ unknowns $a_{i,j}$. The general solution of such a system depends on $N^n$ arbitrary parameters and the corresponding SOs can be represented in the following form [27]

$$Q^n = \sum_{\omega} \sum_{\alpha} C_{\omega, \alpha} (p + i\alpha x)^{\omega} (p - i\alpha x)^{\alpha} \exp[i(2\alpha - k)\omega]$$

(21.54)

where $\omega = (V_0)^{1/2}$, $C_{\omega, \alpha}$ are arbitrary parameters whose number is equal to $N^n$ of (21.47).

We see that all the SOs of the Schrödinger equation with the potential of harmonic oscillator reduce to polynomials on the first order SOs $P_\pm = (p_\pm i\omega x) \exp(i\omega)$. In the case $n=2$ this conclusion reduces to the well-known results [7,51].

Searching of higher symmetries of the three-dimension Schrödinger equation, can be carried out in analogy with the scheme used above. The essential complication of the problem, connected with the necessity to consider partial derivatives in respect with spatial variables is overcome by using the generalized Killing tensors.

As in Subsection 16.5 we search for SOs of arbitrary order $n$ in the form (16.8), (16.9a) where indices values run from 1 to 3, $K^\alpha$ are unknown functions of $x$ and $t$. Substituting these expressions and $L$ of (21.55) into the invariance condition (16.7) and equating coefficients of linearly independent differential operators we come to the following system of determining equations (compare with (21.17))

$$\partial^{(a_1, a_2}) K^a_1 a_2...a_m = 0,$$

$$2K^{a_1 a_2...a_m} + \frac{1}{2m} \partial^{(a_1, a_2}) K^a_1 a_2...a_m = \sum_{l=1}^{(n/2)} (-1)^{m-l} \frac{2(2k+1)!}{(2k-2m-1)!(2m)!} U_{k}^{a_1 a_2...a_m},$$

$$2K^{a_1 a_2...a_m} + \frac{1}{2l+1} \partial^{(a_1, a_2}) K^a_1 a_2...a_m = \sum_{l=1}^{(n/2)} (-1)^{l-1} \frac{2(2k)!}{(2k-2l-1)!(2l+1)!} W_{k}^{a_1 a_2...a_m},$$

(21.56)

where

$m=0, 1, ..., (n/2)$, $l=0, 1, ..., (n-1)/2$,

$$U_{k}^{a_1 a_2...a_m} = K^{a_1 a_2...a_m} b_1...b_m \partial^{b_1} \partial^{b_2}...\partial^{b_m} V,$$

$$W_{k}^{a_1 a_2...a_m} = K^{a_1 a_2...a_m} b_1...b_m \partial^{b_1} \partial^{b_2}...\partial^{b_m} V,$$

and symmetrization is imposed over the indices in brackets.
The equations (21.56) define potentials \( V \) admitting nontrivial symmetries of order \( n \) and the coefficients \( K \) of the corresponding SOs. In Subsection 16.5 we obtain the general solution of these equations for the case \( V=0 \). It is not difficult to solve these equations for the case of the harmonic oscillator potential \( 2^* \). Searching of potentials admitting nontrivial symmetries also can be carried out using the equations (21.56).

21.6. SOs of the Supersymmetric Oscillator

The equation for the supersymmetric oscillator has the form \[ L \psi \equiv \left[ i \frac{\partial}{\partial t} - \frac{1}{2} (p^2 + \omega^2 + \sigma_i \omega) \right] \psi = 0 \] (21.57)

where \( \psi \) is a two-component function, \( \sigma_i \) is the Pauli matrix, \( \omega \) is a real parameter.

The equation (21.57) has a specific symmetry in the class \( M_1 \) which is defined by the superalgebra \( sqm(2) \) [417]. This algebra is formed by the operators \( Q_1, Q_2, Q_3 \) which satisfy the following commutation and anticommutation relations

\[ [Q_1, Q_2] = 0, \quad [Q_1, Q_3] = [Q_2, Q_3] = 0. \] (21.59)

Invariance under the algebra (21.59) is the main property of equations of supersymmetric quantum mechanics [417].

Investigating all the nonequivalent SOs of arbitrary order for the equation (21.57) reduces to investigation of symmetries of the Schrödinger equation (21.1). Making the transformation

\[ \psi \rightarrow \psi' = \exp(-i \omega \sigma / 2), \quad L \rightarrow L' = \exp(-i \omega \sigma / 2) L \exp(i \omega \sigma / 2) \] (21.83)

we come to the equation \( L' \psi' = 0 \), where \( L' = i \partial / \partial t - (p^2 + \omega^2) / 2 \) is nothing but a direct sum of two operators (21.1), (21.22c). The corresponding SOs can be represented in the form \( Q^n = \sigma Q^n \) where \( Q^n \) are the SOs of the equation (21.1), (21.22c) present in (21.54) (where \( C_{\alpha \beta} \rightarrow C_{\alpha \beta}^{\nu} \)). Returning with the help of the inverse transformation to the starting equation (21.57) we obtain a complete set of SOs of this equation in the form

\[ Q^n = \sigma Q^n, \]

where

\[ \sigma_0 = \sigma_0, \quad \sigma_1 = \sigma_1, \quad \sigma_2 = \sigma_2 \cos \frac{\omega t}{2} - \sigma_1 \sin \frac{\omega t}{2}, \quad \sigma_3 = \sigma_2 \sin \frac{\omega t}{2}. \]

The number of linearly independent SOs of order \( n \) is equal to \( 4N^n \). \( N^n \) is given in
In a complete analogy with the above we can investigate symmetries of three-
dimensional Schrödinger equation with potential of supersymmetric oscillator
\[ V(x) = \omega^2 x^2 + \omega \sigma. \]
The number of linearly independent SOs of order \( n \) is equal to \( 4N_n \) where \( N_n \) is the
number given in (16.37).

The approach used above can be extended in order to investigate symmetries
of super- and parasupersymmetric Schrödinger equation with arbitrary potential, refer
to [3*, 4*, 5*]. It can be used to search for infinite order symmetries, in this case the
determining equations also are given by relations (21.8) or (21.54) where the first lines
have to be omitted and summing up is changed by infinite series (i.e., the top
summation limit tends to infinity).

## 22. NONGEOMETRIC SYMMETRIES OF EQUATIONS
FOR INTERACTING FIELDS

### 22.1. The Dirac Equation for a Particle in an External Field

The symmetry of an equation describing a charged particle in an external field
as a rule is less extended than the symmetry of the corresponding equation describing
a free particle. For instance if the external field is the electric field directed along the
fixed axis then the corresponding equation admits the cylindrical symmetry but not the
spherical one etc. However for some classes of external fields the above described
nongeometric symmetry is preserved and even extended.

Here we consider some examples of external fields corresponding to nontrivial
symmetries of the Dirac equation. The assertions present in the following can be
verified by direct calculations (for details see [146, 154, 157]).

a) Consider the Dirac equation for a particle in the constant and homogeneous
electromagnetic field:
\[ \hat{L}\psi \equiv (\gamma^\mu \pi_\mu - m)\psi = 0, \quad \pi_\mu = p_\mu - eA_\mu \]  
(22.1)

where
\[ A_\mu = -\frac{1}{2} F_{\rho\nu} x^\nu. \]  
(22.2)

Here \( F_{\rho\nu} \) are constants determining the tensor of the electromagnetic field strengths.

The equation (22.1), (22.2) has not the symmetry in respect with the algebra
Symmetries of Equations of Quantum Mechanics

$A_0$ described by Theorem 17.1. But there exist two constants of motion closely connected with this symmetry:

$$F_1 = -\frac{1}{2} \hat{\Sigma}^{\mu \sigma} F_{\mu \sigma}, \quad F_2 = -\frac{1}{4} e_{\mu \nu \rho \sigma} \hat{\Sigma}^{\mu \nu} F^{\rho \sigma}$$  \hspace{1cm} (22.3)

where $\hat{\Sigma}_{\mu \sigma}$ are the operators obtained from (17.6) by the change $p_\mu \rightarrow \pi_\mu$:

$$\hat{\Sigma}_{\mu \sigma} = \frac{1}{4} \left[ \gamma_{\mu}, \gamma_{\sigma} \right] - \frac{i}{2m} (1 - i \gamma_5) (\gamma_{\mu} \pi_{\sigma} - \gamma_{\sigma} \pi_{\mu}).$$  \hspace{1cm} (22.4)

The operators (22.3) commute and satisfy the relations

$$[F_1, \hat{L}] = -\frac{1}{2m} (\gamma_\mu \pi_{\sigma} - \pi_{\sigma} \gamma_\mu) F^{\rho \sigma} \hat{L},$$

$$[F_2, \hat{L}] = -\frac{1}{2m} e_{\mu \nu \rho \sigma} (\gamma^\rho \pi^\sigma - \gamma^\sigma \pi^\rho) F^{\rho \sigma} \hat{L},$$

so they are SOs of the equation (22.1), (22.2).

b) In the case of the selfdual external electromagnetic field the nongeometric symmetry of the equation (22.1) is just more extensive. Indeed, using the selfduality condition for the vectors of the electric and magnetic field strengths

$$H \sim eE \sim 0$$

it is not difficult to make sure the operators (22.4) are the SOs of the corresponding equation (22.1). These operators satisfy the invariance condition

$$[\hat{\Sigma}_{\mu \sigma}, \hat{L}] = -\frac{1}{2m} (\gamma_{\mu} \pi_{\sigma} - \pi_{\sigma} \gamma_{\mu}) \hat{L}$$  \hspace{1cm} (22.5)

and the commutation relations (17.7) characterizing the algebra $AO(1,3)$.

c) Now we consider the Dirac equation with the Pauli-type interaction

$$L \Psi = \left[ \gamma^\mu \pi_{\mu} - m + \frac{i}{2m} (1 - i \gamma_5) \gamma_{\mu} F^{\mu \nu} \right] \Psi = 0$$  \hspace{1cm} (22.6)

where $F^{\mu \nu} = -i [\pi^\mu, \pi^\nu]$, $A_\mu$ is an arbitrary vector-potential.

The equation (22.6) is manifestly invariant under the Poincaré group such as the Dirac equation for a free particle. It turns out the nongeometric symmetry of (22.6) in the class $M_1$ coincides with the corresponding symmetry of the free Dirac equation. Indeed, the operators (22.4) (and $\Sigma_0$, $\Sigma_1$ of (17.6) where $p_\mu \rightarrow \pi_\mu$) are the SOs of the equation (22.6), satisfying the invariance condition (22.5).

d) The following example is the Dirac equation for a particle in the constant magnetic field depending on two spatial variables. We choose the corresponding vector-potential in the form

$$A_0 = A_3 = 0, \quad A_1 = A_1(x_1, x_2), \quad A_2 = A_2(x_1, x_2)$$  \hspace{1cm} (22.7)
which corresponds to the magnetic field directed along the third coordinate axis.

The equation (22.1), (22.7) is invariant under the following operators

\[ \Sigma_{23} = i \gamma_3, \quad \Sigma_{31} = i \gamma_1 \gamma_0 \gamma_3, \]

\[ \Sigma_{12} = i \gamma_1 \gamma_2, \quad \Sigma_{10} = i \frac{1}{2} \hat{H} e_{ab} \Sigma_{ac}, \quad \alpha = 1, 2 \]

(where \( \hat{H} = \gamma_0 \gamma_a \pi_a + \gamma_0 \mathbf{m} \)) which form a superalgebra together with \((\Sigma_{\mu \sigma})^2\) and \(i \hat{H} (\Sigma_{\mu \sigma})^2\), satisfying the following commutation and anticommutation relations

\[
[\Sigma_{\mu \sigma}, \Sigma_{\lambda \tau}] = 2 \left( g_{\mu \lambda} g_{\sigma \tau} + g_{\mu \tau} g_{\sigma \lambda} - \gamma_5 \delta_{\mu \lambda} \delta_{\sigma \tau} \right) - i e_{\mu \lambda \sigma \tau} \hat{H} \Sigma_{\nu \rho} (1 - \delta_{\mu \nu}) (1 - \delta_{\lambda \rho}) (1 - \delta_{\sigma \tau}).
\]

The considered equation is invariant under the algebra \(A_8\), also, besides this algebra is realized in the class of integro-differential operators [146, 157]. The analogous symmetry is admitted by the Dirac equation for a particle in the electric field depending on \(x_0\) and \(x_3\) [146].

e) In conclusion, we discuss symmetries of the Dirac equation for a charged particle interacting with the Coulomb field. The corresponding potential has the form

\[ A_a = 0, \quad A_0 = \frac{e^2}{x}, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. \] (22.8)

It was Dirac who showed the SOs of the equation (22.1), (22.8) for the first time [81]. In addition to the evident symmetry under the spatial rotations group whose generators are

\[ J = x \times p + S, \quad (S - \frac{i}{4} \gamma \gamma_5), \] (22.9)

this equation admits the following SO

\[ Q_1 = (2 S \cdot J - \frac{1}{2} \gamma_0) \gamma_5 \] (22.10)

which is called the Dirac constant of motion. This SO plays an important role while solving the equation (22.1), (22.8) [81]. It can be easily shown the operator (22.10) coincides with \(A_0\) of (18.5).

In addition to (22.9), (22.10) there exists one more SO of the Dirac equation with the Coulomb potential. This SO belongs to the class \(M_2\) and has the form [237]

\[ Q_2 = \frac{2 \gamma_5}{x} + \frac{i}{2 \gamma_5} \gamma_0 \left( 2 S \cdot p + \gamma_4 e \right). \] (22.11)

The operators (22.9)-(22.11) together with the following operators
Symmetries of Equations of Quantum Mechanics

Form a basis of a seven-dimensional superalgebra, satisfying the following relations

\[ Q^2_1 = J^2 + \frac{1}{4}, \quad Q^2_2 = 1 + \frac{Q^2_1}{m^2 e^2} \left( p^2 + \frac{e^2}{x^2} - \frac{m e^2}{x} \right) \]

[22.12]

\[ [J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma} J^\gamma, \quad [J_\alpha, Q^\alpha] = [J_\alpha, Q^2_\alpha] = 0, \]

\[ [Q^\alpha, Q^\beta] = -2\delta^\beta_\alpha Q^2_\gamma, \quad \alpha = 1, 2, \]

\[ [H, Q^\alpha] = [H, J_\alpha] = 0, \quad H = \gamma_0 \gamma_\alpha p^\alpha + \gamma_0 m + \frac{e}{x}. \]

Using the identity

\[ 2S \cdot J = J^2 - L^2 + S^2, \quad J_\alpha p^\alpha - \text{cross product} \]

it is not difficult to show that in the space of square integrable functions the spectrum of the operator (22.10) is discrete and can be defined by the following formula [81]

\[ Q \psi \mp \epsilon \psi = \epsilon (j + 1/2) \psi, \quad \epsilon = +1, -1, \quad j = 1/2, 3/2, \ldots . \]  

The existence of the SOs satisfying (22.12) is the cause of the degeneracy of the energy spectrum of an electron in the Coulomb field. Indeed, an energy level \( E \) can be expressed via the eigenvalues \( \epsilon \) of the operator \( Q_1 \) commuting with the Dirac Hamiltonian, so \( E(\epsilon) = E(-\epsilon) \) in accordance with anticommutativity \( Q_1 \) with \( Q_2 \), and there is a degeneracy in respect with the change \( \epsilon \rightarrow -\epsilon \).

It is not difficult to verify that the operator \( Q_1 \) satisfies the relations

\[ [Q_1, S \cdot p] = [Q_1, S \cdot x], \quad [Q_1, \gamma_\alpha x] = [Q_1, \gamma_\alpha] = 0 \]

from which it follows that the operator \( Q_1 \) is a constant of motion of the Dirac equation in the case of a more complicated potentials also. In particular the following assertion is valid [161]:

**THEOREM 22.1.** The general form of a spherically symmetric potential \( V = V(x) \), such that the equation

\[ i \frac{\partial}{\partial x_0} \gamma_\alpha p^\alpha - \gamma_\beta p^\beta - V \psi = 0 \]  

(22.14a)

admits the SO \( Q_1 \) of (22.10), is given by the following formula

\[ V = V_1 + V_2 \gamma_\beta + V_3 \gamma_\alpha x + V_4 \gamma_\beta x \]  

(22.14b)

where \( V_i \) are arbitrary functions of \( x \). ■

In the case \( V_1 = qe^2/x, \quad V_2 = e^2 qu/x^3, \quad V_3 = V_4 = 0 \) relation (22.14b) defines the potential of the anomalous Pauli interaction with the field of a point charge and for \( V_1 = V_2, \quad V_3 = V_4 = 0 \) this relation defines the general form of the confinement potential used...
in the quark models based on the one-particle Dirac equation [392].

We see the Dirac constant of motion serves as a SO for a wide class of equations presented in (22.14).

There are other types of external fields generating nontrivial symmetries of the corresponding Dirac equation. Some exotic examples of such fields are collected in [9] where exact solutions of the corresponding equations are present.

### 22.2. The SO of Dirac Type for Vector Particles

Let us demonstrate that the additional SO exists for vector particles interacting with the field of a point charge [334].

Consider the KDP equation with anomalous interaction for a particle of spin 1 in the Coulomb field:

\[ L\psi = (\beta^\mu \pi^\mu = m - e k S^\mu \pi^\mu) \psi = 0 \quad (22.15) \]

where

\[ \pi^\mu = \pi^\mu - e A^\mu, \quad F^\mu = i[\pi^\mu, \pi^\rho], \quad S^\mu = i[\beta^\mu, \beta^\rho]. \]

\( \beta_i \) are the 10x10 KDP matrices satisfying the algebra (6.20), \( A^\mu \) is the vector-potential of (22.8).

The equation (22.15) can be represented in the Hamiltonian form (6.35) where

\[ H = (\beta_0^2 \beta_0) [\beta_a p_a + \beta_0 p_0 + e \frac{\beta^2}{2} \frac{1}{m} \beta_a x_a x^3 - \beta_0 \sum \frac{i e k}{m^2} \beta_a x_a x^3] \quad (22.16) \]

The evident SOs of the equation (6.35), (22.16) are the generators of the rotation group, i.e., the components of the vector \( J = x \times p \) commuting with the Hamiltonian \( H \) of (22.16). But it is not all. Using the relations (6.20) it is not difficult to verify that the following operator [334]

\[ Q = (1 - 2 \beta_0^2) [2(S \cdot J)^2 - 2S \cdot J - J^2] \quad (22.17) \]

is a SO of the equations (22.15) and (6.35), (22.16). Indeed, this operator commutes with \( L \) of (22.15) as well as with \( H \) and \( P \) of (22.16). This assertion is valid for \( k = 0 \) also, i.e., if anomalous interaction is absent.

Like the Dirac SO, the operator (22.17) has a discrete spectrum

\[ Q \psi = \varepsilon j(j+1) \psi, \quad \varepsilon = -1, \quad j = 0, 1, 2, \ldots \quad (22.18) \]

inasmuch as \( Q^2 \equiv (J^2)^2 \).
An additional SO exists for the Stueckelberg equation [398] also. In presence of the anomalous interaction with the Coulomb field this equation can be written in the form (22.15) where \( \beta_\mu \) are the 11x11 matrices given in Subsection 8.5, \( S_{\alpha \sigma} \) are matrices realising the representation \( D(1/2 1/2) \otimes D(1 0) \otimes D(0 1) \otimes D(0 0) \) of the algebra \( AO(1,3) \). The corresponding SO is given by formula (22.17) where \( S_\alpha^a = e_a^a S_{\alpha c} e_c^c/2, S_\kappa \) and \( \beta_0 \) are the corresponding matrices of dimension 11x11.

In paper [318] nongeometric symmetries of the KDP equation with anomalous Pauli-type interaction (22.15) were described for constant electric and magnetic fields. The corresponding IA is the algebra \( A[SU(2) \otimes SU(2) \otimes A_4] \) where \( A_4 \) is a four-dimensional commutative subalgebra. We do not present the explicit form of cumbersome basis elements of this IA here.

### 22.3. The Dirac-Type SOs for Particles of Any Spin

The results given above can be generalized to the case of Poincaré- and Galilei-invariant equations for particles of arbitrary spin [334].

Consider an arbitrary Poincaré-invariant equation of the form (22.15) where \( S_{\alpha \sigma} \) are generators of a direct sum of IRs of the Lorentz group, \( A_\alpha \) is the vector-potential (22.8), \( \beta_\mu \) are matrices satisfying (6.7). We assume the equation in question be invariant under the transformation of space inversion of (2.55) where the matrix \( r_1 \) by definition satisfies the following relations

\[
\begin{align*}
 r_1 \beta_0 & = - \beta_0 r_1, \\
 r_1 S_{ab} & = - S_{ab} r_1, \\
 r_1 S_{0a} & = - S_{0a} r_1.
\end{align*}
\]

(22.20)

In analogy with (22.10), (22.17) we search for SOs of the considered equation in the form

\[
Q = r_1 d, \quad d = d(x, p, S_{\alpha \sigma}).
\]

(22.21)

Requiring \( Q \) commutes with \( L \) (22.15) and using relations (6.7), (22.20) we obtain the following conditions for \( d \)

\[
\begin{align*}
[x, d] & = [\beta_\mu, d] = 0, \\
[S_{0\alpha} x, d] & = [S_{0\alpha} p, d] = 0.
\end{align*}
\]

(22.22a, 22.22b)

It follows from (22.22a) that \( d \) depends on \( S_{a b}, a \neq 0, \) and \( L = x \times p \). The explicit form of \( d = d(S_{a b} L) \) can be obtained solving the equations (22.22b), moreover it is sufficient to solve these equations for \( S_{0a} \) belonging to an IR \( D(j \tau) \subset D \).
Chapter 4. Nongeometric Symmetries

To find a solution of (22.21b) it is convenient to use the spherical spinor basis in which the operators $S_0a x_a/x$ and $xS_0a p_a$ reduce to the numeric matrices (refer to Appendix 3). A transition into this basis is one of realisations of the algorithm described in Subsection 16.1. In accordance with this algorithm we reduce a problem of description of SOs to a purely matrix problem.

Omitting cumbersome calculations (for details see [334]) we present the explicit form of $d$ satisfying (22.22) for an arbitrary IR $D(j\tau)$.

$j+\tau$ is integer:

$$d = c F \sum_{i=0}^{j-1} \sum_{l=0}^{s} (-1)^{\lambda} B^\lambda \_i \_x, \lambda = 0, 1, 2, ... \ (22.23)$$

$j+\tau$ is half integer:

$$d = c F \sum_{i=0}^{j-1} \sum_{l=0}^{s} (-1)^{\lambda+1/2} A^\lambda \_i \_x \ (22.24)$$

where $c$ is an arbitrary constant.

$$F = \begin{cases} \prod_{\alpha=1}^{2(j-\tau)-1} (4J^2 + 1 - \alpha^2), & j+\tau \neq \frac{1}{2} \\ 1, & j+\tau = \frac{1}{2} \end{cases}$$

$B^\lambda \_i, A^\lambda \_i$ are operators satisfying the relations

$$\sum_{\nu} B^\nu \_i \_x - 1, \ \nu = 0, 1, 2, ..., \nu = 1/2 [1 + (-1)^{s}] \ (22.25)$$

$$B^\lambda \_i B^\nu \_i = 0, \ B^\lambda \_i A^\nu \_i = 0 \delta_{\nu \nu} A^\lambda \_i \ (22.26)$$

$$A^\lambda \_0 = 0 \ \text{and} \ \sum_{\nu} (\nu A^\nu \_i - \nu^2 B^\nu \_i) - P_j (2S \cdot J - S^2) = G \_i \ (22.27)$$

Here $P_j$ is the projector

$$P_j = \prod_{s+s'} \frac{S^2 - s' (s'/+1)}{s(s+1) - s' (s'+1)}, \ |m-n| \leq s', s' \leq m+n$$

$S$ is a vector whose components are $S_0 = e_{\mu \nu} S_{\mu \nu}/2, S_{\mu \nu} \in D(j\tau)$.

For any given value of $s$ the operators $B^\lambda \_i$ and $A^\lambda \_i$ can be expressed via $G \_i$. For this purpose it is sufficient to raise the l.h.s. and r.h.s. of (22.27) in powers $n=1,2,...,2s$ and then to solve obtained system of $2s+1$ linear algebraic equations for $2s+1$
unknowns $B^\nu_s$ and $A^\nu_s$. Moreover according to (22.26) the equations whose number is $n=2k$ and $n=2k+1$ have the form

$$G^{2k}_s = \sum_{\nu} \left[ \sum_{|m|=0}^k C_{2k+m}^{2k}(4J^2+1)^{k-m}B^\nu_s + \sum_{|m|=1}^{k-1} C_{2k+m}^{2k+1}(4J^2+1)^{k-m-1}A^\nu_s \right], \quad k \leq s,$$

$$G^{2k+1}_s = \sum_{\nu} \left[ \sum_{|m|=0}^k C_{2k+m}^{2k+1}(4J^2+1)^{k-m}B^\nu_s + \sum_{|m|=1}^{k-1} C_{2k+m}^{2k+2}(4J^2+1)^{k-m-1}A^\nu_s \right], \quad k < s. \quad (22.28)$$

where $C^\nu_s$ is the number of combinations from $b$ elements by $a$. The equation with the number $n=2s+1$ is given in (22.25).

Let $s=(m+n)_{\text{max}}$ be the maximal value of the quantum number $s$ appearing by reduction of the representation (22.19) by the group $O(3)$. We present solutions of the equations (22.23)-(22.25), (22.28) for $d=d_s, s \leq 2$:

$$d_{1/2} \cdot 2S \cdot J - \frac{1}{2};$$

$$d_{3/2} \cdot 2(S \cdot J)^2 - 2S \cdot J - J^2,$$

$$d_{3/2} = \frac{4}{3} \left[ 5g^2 - 2S^2 + (4S^2 - 6)J^2 \right] + 3, \quad g = 2S \cdot J - \frac{3}{2}; \quad (22.29)$$

$$d_s = \frac{2}{3} [(S \cdot J)^2 - 2S \cdot J - 4J^2] (S \cdot J - 1) (S \cdot J - 3) - J^2 (J^2 - 2) +$$

$$+ \frac{1}{3} (S^2 - 2) [(4 - 3J^2)(S \cdot J)^2 + (7J^2 - 4)S \cdot J - 4J^2 + \frac{3}{8} S^4 (4J^2 + 1)].$$

Formulae (22.21), (22.29) give SOs for any Poincaré- and $P$-invariant equation describing a particle of spin $s \leq 2$. Among them are the Rarita-Schwinger and Dirac-Fierz-Pauli equations for a particle of spin $3/2$ (refer to Section 6), and the Bhabha equations [35]. Moreover these formulae define SOs for the LHG equations being invariant under the Galilei group, see Sections 13, 15. The corresponding matrices $r_1$ have the form $r_1 = \delta_{s,2}^2$.

In conclusion we note that the spectrum of the operators (22.29) is given by formulae (22.13), (22.18) (where $Q \rightarrow d_{1/2}$ and $Q \rightarrow d_s$) and (22.30):

$$d_{1/2} \psi = \epsilon (2j-1)(2j+1)(2j+3)\psi, \quad \epsilon = \pm 1, \quad j=1/2, 3/2, ...,$$

$$d_s \psi = \epsilon (j-1)(j+1)(j+2)\psi, \quad j=0, 1, ... \quad (22.30)$$
22.4. Other Symmetries of Equations for Arbitrary Spin Particles

Here we consider other examples of external fields preserving nongeometric symmetries of equations for particles of arbitrary spin.

We start from the Dirac-type equations (10.10) corresponding to the vector-potential (22.2). In the case $F_{\mu \sigma} = 0$ these equations are invariant under the algebra $AGL(2s+1,C)$ realized in the class $M_1$. In the case $F_{\mu \sigma} \neq 0$ the symmetry of these equations is less extensive and reduces to the invariance under a $2(2s+1)$-dimensional commutative algebra defined over the field of real numbers. Basis elements of this algebra have the form

\[ Q_n = (\hat{\Sigma}^{\mu \nu} F_{\mu \nu})^n, \quad Q_{2r+1,n} = Q_{n-1} e_{\mu \rho \kappa \sigma} F^{\mu \rho} \hat{\Sigma}^{\kappa \sigma}, \quad n = 1,2,...,2s \]

where $\hat{\Sigma}^{\mu \nu}$ are operators obtained from (19.19) by the change $p_\mu \rightarrow \pi_\mu$. Since the following conditions are satisfied

\[ [\hat{\Sigma}^{\mu \nu} F_{\mu \nu}, L_1] = \frac{i}{m} (\Gamma^\mu \pi_\mu - \Gamma_\mu \pi^\mu) F^{\mu \nu} L_1, \quad [\hat{\Sigma}^{\mu \nu} F_{\mu \nu}, L_2] = 0 \]

where $L_1$ and $L_2$ are the operators of (10.10), the operators (22.31) are SOs of the system (10.10), (22.2).

Using the deep analogy of (10.10) with the Dirac equation it is possible to reformulate all the results a)-d) of the previous subsection to the case of arbitrary spin. Such a reformulation is almost trivial and reduces to the change $\gamma_\mu \rightarrow \Gamma_\mu$ in the corresponding formulae. Moreover the symmetry of the system (10.10) is more extensive because the squares of the corresponding SOs do not reduce to the unit matrix in contrast to the case of the Dirac equation. For example in the case of the self-dual electromagnetic field we find a $2(2s+1)^2$-dimensional invariance algebra of the system (10.10) formed by the SOs of (19.20) where $\Sigma^{\mu \nu}$ are again the SOs obtained from (19.19) by the substitution $p_\mu \rightarrow \pi_\mu$.

Let us consider also the Poincaré-invariant equations without superfluous components for a particle of arbitrary spin in an external field described by the specific potential

\[ (H^I + V)\psi = i \frac{\partial}{\partial x_0} \psi \]

where $H^I$ is the Hamiltonian of a free particle of arbitrary spin given in (7.27),

\[ V = (1 + \sigma_j) \varphi(x). \]

In the case $s=1/2$ formula (22.32) presents a well-known equation used for description of quarks in the field with an effective potential being the sum of the scalar and vector components [392]. We will not work out details of the explicit form of $\varphi(x)$.
which is not essential for our aims.

The equation (22.32) has a wide nongeometric symmetry described in the following assertion.

**THEOREM 22.2.** The equation (22.32) is invariant under the algebra $AGL(2s+1,R)$ whose basis elements $\lambda_{mn}$ are given by formulae (19.20) where

$$\Sigma_{ab} \rightarrow -i e_{abc} \Sigma_{ac}, \quad \Sigma_{1} \rightarrow i,$$

(22.34a)

$$\hat{S}_{a} \rightarrow \sigma_{a} \hat{S}_{a} + (1 - \sigma_{a})\mathcal{P} S \mathcal{P}^{-1},$$

(22.34b)

**PROOF.** The equation (22.32) and the operators (22.34) can be reduced to the form for which the statements of the theorem become obvious. Using the transformation operator

$$U = U^{-1} \frac{1}{2} \left[ 1 + \sigma_{1} + (1 - \sigma_{1}) \sum_{\nu} \left(-1\right)^{\nu} \Lambda_{\nu} \right],$$

where $\Lambda_{\nu}$ are the projectors (7.15), we obtain

$$U (H^{I} + V) U^{-1} \sigma_{m} + \sigma_{p} + \frac{1}{2} (1 + \sigma_{1}) \varphi(x),$$

(22.35)

$$U \hat{S}_{a} U^{-1} \rightarrow \sigma_{a} \hat{S}_{a}.$$

(22.36)

Matrices (22.36) commute with the Hamiltonian (22.35) and satisfy the relations (4.25), (4.30), (4.31). So we can construct a basis of the algebra $AGL(2s+1,R)$ by formulae (19.20).

We note that the analogous symmetries are admitted by the equations

$$(H^{III} + V) \psi = i \frac{\partial}{\partial x_{0}} \psi$$

where $V$ is the potential (22.33), $H^{III}$ is one of the Hamiltonians (7.40). A basis of the corresponding IA is given by formulae (19.20), (22.34). Thus the TST equation with the specific potential (22.33), describing a particle of spin 1, is invariant under the algebra $AGL(3,R)$.

22.5. Symmetries of Galilei Particle of Arbitrary Spin in the Constant Electromagnetic Field

Here we study symmetries of Galilei-invariant wave equations describing a particle in the field corresponding to the vector-potential (22.2). Namely we consider the LHG equations, refer to (15.1), (13.11), (13.12), Column $R_{1}$ in the Table 13.1.

**THEOREM 22.3.** The LHG equation for a particle of spin $s$ interacting with the constant electromagnetic field is invariant under the algebra $AGL(2s+1,R)$ whose
basis elements are given in (19.20), (22.34a) where
\[
S - \left( S - \frac{\eta \times p}{m} \cos \left( \frac{eH}{2m} \right) + S \times H - \frac{(\eta \times p) \times H}{m} \sin \left( \frac{eH}{2m} \right) \right).
\] (22.37)

\[H = (H_1^2 + H_2^2 + H_3^2)^{1/2},\]

\(S\) and \(\eta\) are the corresponding generators of the homogeneous Galilei group.

The proof reduces to direct verification of validity of the following assertions:

a) the operators (22.37) satisfy the condition \(L \hat{S} = \hat{S} \hat{L}\) where \(L\) is the operator of (15.1);

b) the operators (22.37) satisfy the commutation relations (12.20a) moreover \(\hat{S}^\dagger \psi = (s+1) \psi\) for any \(\psi\) satisfying (15.1).

These assertions are easily verified using relations (13.9), (13.10).

We see nongeometric symmetries of the LHG equations are very extensive. All the SOs (19.20), (22.37) belong to the class \(M_1\) since the matrices \(\eta_a\) corresponding to the LHG equations satisfy the relation \(\eta_a \eta_b = 0\).

Another Galilei-invariant equations considered in Chapter 3 do not admit the symmetry formulated in Theorem 21.3. But for the constant and homogeneous magnetic field we can find a class of equations being invariant under the algebra \(AGL(2s+1,\mathbb{R})\). There are the equations (15.26), (15.28) for \(k_1 = 1\). The corresponding SOs are given in (19.20), (22.34) where

\[
S_a \rightarrow VS_a V^{-1}, \quad V = \exp \left( \frac{i}{m} e S \cdot H \right) \exp \left( \frac{it}{2m} S - \frac{e^2}{m} S \cdot H \right).
\]

This symmetry makes it possible to obtain the exact solutions of the corresponding equations (see Chapter 6).

22.6. Symmetries of Maxwell’s Equations with Currents and Charges

Here we show that Maxwell’s equations with currents and charges also have an additional symmetry extending well-known Poincaré and conformal invariance. Using the covariant formulation (3.9), (3.10) of these equations we formulate and prove the following assertion.

THEOREM 22.4. There exist twenty SOs of Maxwell’s equations with currents and charges in the class of second order differential operators with matrix coefficients. These SOs do not belong to the enveloping algebra of the algebra \(AC(1,3)\) and have the following form

\[
Q_{\mu\nu\rho\sigma} = (1 - 2\hat{S})[Z \nu \rho \sigma + Z \nu \rho \sigma - Z \nu \rho \sigma - Z \nu \rho \sigma]
\] (22.38)

where
Proof reduces to direct verification (using relations (6.20)) the fact that $Q^{[\mu [\nu[\rho [\sigma]}$ commutes with $L_1$ and $L_2$ of (3.10). \[\square\]

The operator (22.38) is antisymmetric under permutations of indices inside of square brackets and symmetric under the permutation of the pairs of indices included in the first and second square brackets. In other words, it is a basic tensor having 20 independent components (refer to Subsection 16.2 for definitions).

We note that Maxwell’s equations with currents and charges have a wide additional symmetry. The operators (22.38) do not belong to the enveloping algebra of the algebra $AC(1,3)$ and thus are essentially new in comparison with the SOs which can be obtained in the classical Lie approach.

Calculating commutators of the operators (22.38) with the generators of the conformal group given in (2.22), (2.42), (3.20) it is possible to find a more wide class of the second-order SOs which in general depend on $x_{\mu}$. An example of such a SO is the operator (22.17) which is admitted by Maxwell’s equations with currents and charges.

It is not difficult to make sure the SOs (22.38) do not form a basis of a Lie algebra. But they include a subset of operators which can be extended to a superalgebra. Denoting

\[ Q^{\alpha} = \frac{1}{2} \xi^{\alpha}Q^{[\mu [\nu[\rho [\sigma]} \xi^{\mu \nu \rho \sigma} \equiv \eta_{\alpha}^{\mu \nu \rho \sigma} \xi^{\mu \nu \rho \sigma} \eta_{\alpha}, \]

where

\[ f_{\alpha \beta \rho \sigma} = \frac{1}{12} \left( g^{\alpha \beta} g_{\rho \sigma} - g^{\alpha \sigma} g_{\rho \beta} \right) \left( g^{\alpha \sigma} g_{\rho \beta} - g^{\alpha \beta} g_{\rho \sigma} \right), \]

\[ g^{\alpha \mu \nu \rho} = \frac{1}{12} \left( g^{\alpha \beta} g_{\rho \sigma} - g^{\alpha \sigma} g_{\rho \beta} \right) \left( g^{\alpha \sigma} g_{\rho \beta} - g^{\alpha \beta} g_{\rho \sigma} \right), \]

and symmetrization is imposed over the top indices in the r.h.s. of (22.42).

In accordance with (22.42) the following set of SOs of Maxwell’s equations
where $P_{\mu}$, $J_{\mu}$ are generators of the Poincaré group given in (2.20), (3.20) form a basis of a superalgebra. Moreover $Q_{\mu\sigma}$ are the odd and the remaining operators are even elements of this superalgebra.

Using the explicit expressions (6.22), (6.23) for the $\beta$-matrices it is not difficult to find the action of the SOs (22.38) to components of the wave function (3.9), i.e., to find the corresponding symmetry transformations of the strengths $E$, $H$ and current $j$. In this way, it is possible to verify directly the validity of Theorem 21.4.

In conclusion of this subsection we consider Maxwell’s equations in a conducting medium:

\begin{align}
 i\frac{\partial E}{\partial x_0} &= -ip \times H + i\sigma E, \quad p \cdot E = 0; \\
 i\frac{\partial H}{\partial x_0} &= p \times E, \quad p \cdot H = 0
\end{align}

\begin{align}
 \text{(22.43)}
\end{align}

where $\sigma$ is the coefficient of conductivity. We will show that, just as equations for the electromagnetic field in vacuum, the equations (22.43) are invariant under the algebra $AGL(2,\mathbb{C})$.

Using the notations (3.6), (20.1), (20.3) we write (22.43) in the momentum representation

\begin{align}
 L_1 \phi(x_0, p) &= 0, \quad L_1 = i\frac{\partial}{\partial x_0} - \sigma_2 S_3 p, \\
 L_2 \phi(x_0, p) &= 0, \quad L_2 = \sigma_2 S_3 p S_a.
\end{align}

\begin{align}
 \text{(22.44a)}
\end{align}

\begin{align}
 L_2^a \phi(x_0, p) &= 0, \quad L_2^a = p_a S \cdot p S_a.
\end{align}

\begin{align}
 \text{(22.44b)}
\end{align}

The equations (22.44) are invariant under an eight-dimension Lie algebra isomorphic to $AGL(2,\mathbb{C})$. The basis elements of this IA have the form

\begin{align}
 \Sigma_{23} &= \frac{i}{2} \hbar |h|^{-1} S \cdot \hat{p}, \quad \Sigma_{31} = \frac{i}{2} \hbar |h|^{-1} \sigma_3 Q, \\
 \Sigma_{12} &= \frac{1}{2} \sigma_2 S \cdot \hat{p} Q, \quad \Sigma_{01} = \frac{1}{2} S \cdot \hat{p}, \quad \Sigma_{02} = \frac{1}{2} \sigma_2 Q, \\
 \Sigma_{03} &= -\frac{i}{2} \hbar |h|^{-1} \sigma_3 S \cdot \hat{p} Q, \quad \Sigma_1 = i\hbar |h|^{-1}, \quad \Sigma_0 = I
\end{align}

where $Q$ is the matrix given in (20.7),

$h \cdot \sigma_S \cdot \hat{p} = i\sigma_3 \sigma / 2, \quad \hat{p} \cdot p \cdot \mu,$

275
Symmetries of Equations of Quantum Mechanics

\[ |\hbar|^2 = \left( p^2 - \frac{1}{4} \sigma^2 \right)^{1/2} \left( S \cdot \vec{p} \right)^2 - \frac{2i}{\sigma} [1 - (S \cdot \vec{p})^2]. \]

For the proof see [154].

22.7. Super- and Parasupersymmetries

Here we search for relativistic wave equations for charged particles in external fields, which admit exact super- or parasupersymmetries.

An attractive feature of Witten’s supersymmetric quantum mechanics is that it is supported by a realistic physical model. We say about the problem of interaction of spin 1/2 particle with magnetic field. Indeed, as the Pauli as the Dirac equation predict supersymmetric energy spectra for such a particle interacting with the magnetic field depending on two space variables only [417,7’. Moreover there exist the corresponding symmetries (supercharges) causing the specific degeneration of this spectra.

We demonstrate here that parasupersymmetric quantum mechanics [372] also is supported by a wide class of wave equations for particles interacting with external fields. In contrast with the supersymmetric case these equations have to include anomalous (Pauli) interactions.

First we return to the Dirac equation (22.1) including the external magnetic field described by the vector-potential (22.7). Setting for simplicity \( p_3 = 0 \) we immediately find that the following operators

\[ Q_1 = i \gamma_4 (\gamma_1 \pi_2 - \gamma_2 \pi_1), \quad Q_2 = i (\gamma_2 \gamma_1 \pi_1 + \gamma_1 \gamma_2 \pi_2), \]

\[ H_{ss} = \pi_1^2 + \pi_2^2 - ie \gamma_4 \gamma_1 H \quad (H = i[\pi_1, \pi_2]) \]

are SOs of the equation considered. Besides that these SOs satisfy relations (18.10’) for \( a, b = 1, 2 \) and so realize a representation of the superalgebra \( sgm(2) \).

Inasmuch as the operator \( H_{ss} \) coincides with \( (p_0)^2 - m^2 \) on the set of solutions of the Dirac equation, it generates the spectrum of squared energies of a spin 1/2 particle in the field considered. The specific degeneration of this spectra (any line except the one corresponding to the ground state is twice degenerated) is caused by the symmetries (22.45).

It is necessary to emphasize that the above symmetry exists for \( p_3 \neq 0 \) also. The corresponding supercharges have the form

\[ \hat{Q}_1 = \gamma_0 Q_1 - \gamma_4 (1 - i \gamma_4)(p_3 Q_1 - H_{ss})/m, \quad \hat{Q}_2 = Q_2 -(1 - i \gamma_4)p_3 Q_1/m. \]

Consider the KDP equation with anomalous interaction

\[ \left[ \partial^\nu \pi_\mu - m - \left( k_1 \beta \beta^* k_2 \right) \frac{e}{2m} S_{\mu \nu} F^{\nu \nu} \right] \psi = 0 \quad (22.46) \]

276
where $\beta_\mu$ are 10×10 KDP matrices, $k_1$ and $k_2$ are arbitrary real parameters.

The last term in the l.h.s. of (22.46) represents the most general form of anomalous (Pauli) interaction preserving Poincaré and $P, C, T$ invariance of the equation considered.

We restrict ourselves to the case of the constant and homogeneous magnetic field directed along the third coordinate axis and again set $p_3=0$. The corresponding strengths and potentials have the form $F_{0\alpha}=F_{3\alpha}=0$, $F_{1\alpha}=H$, $A_{\alpha}=A_3=0$, $A_1=-iHx_2$.

**PROPOSITION.** Let $\psi$ satisfies the equation (22.46) with the above potentials, where $k_1=k_2=1$. Then

$$p_0^2\psi=\left(m^2+\pi_1^2+\pi_2^2-2S_\pi eH\right)\psi, \quad S_3=i[\beta_1,\beta_2].$$

**Proof.** Using the representation (6.22) of the $\beta$-matrices and expressing "nonphysical" components $(1-\beta^2_0)\psi$ via $\beta^2_0\psi$ we come to the generalized TST equation

$$i\frac{\partial}{\partial x_0}\psi'\tilde{H}\psi',$$

$${\tilde{H}}=\sigma_2\left(m\frac{eH}{m}S_3\right)-i\sigma_1\left(S_1\pi\right)^2-\left[\sigma_1+\sigma_3\right]\frac{\pi^2}{2m},$$

The Hamiltonian $\tilde{H}$ satisfies the relation

$$\tilde{H}^2=m^2+\pi^2-2eS_\pi H\tilde{H}+\tilde{H}_{\text{RSS}}$$

from which it follows that (22.48) does is valid.

We see the squared energy operator generated by the considered equation has a typical parasupersymmetric structure discovered in [372]. Let us demonstrated that this structure is caused by existence of the SOs of the equation considered which form a parasuperalgebra.

To find the corresponding parasupercharges it is convenient to transform (22.49) to an equivalent representation including the minimal number of matrices. Using the transformation operator

$$W=V_1V_2, \quad V_2=1-S_3^2-i\sigma_2S_3, \quad V_2^{-1}=1-S_3^2+i\sigma_2S_3,$$

$$V_1=1-i\sigma_2\frac{S_\pi}{m}+(1-\sigma_3)\frac{(S_1\pi)}{2m^2}, \quad V_1^{-1}=V_1(-\pi)$$

we obtain

$$p_0\psi'=\tilde{H}'\psi', \quad \psi'=W\psi,$$

where
\[ \hat{H}' = W \hat{H} W^{-1} = \sigma J m + (\sigma_i + i\sigma_i)(\pi^2 - 2eS, \hat{H})/2m. \] (22.51)

In contrast with (22.49) the Hamiltonian (22.51) includes the only spin matrix \( S_3 \). This circumstance simplifies finding the corresponding SOs which are parasupercharges. Choosing

\[ Q_1 = S_1, \pi_2, \quad Q_2 = S_3, \pi_1 \] (22.52)

it is not difficult to make sure that these operators are the SOs of the equation (22.50) and satisfy the relations (19.18) together with \( H_{\text{PS}} \) of (22.49).

Thus we found a symmetry parasuperalgebra of the equation (22.49). Using the transformation inverse to (22.51) it is not difficult to find the corresponding parasupercharges for the starting TCT equation (22.49) and than to reconstruct these symmetries for the KDP equation (22.46). We do not present the corresponding cumbersome formulae here.

We note that the choice (22.47) of the coupling constants correspond to the causal KDP equation with an anomalous interaction, while for \( k_1 \neq k_2 \) we come to a non-causal equation [27]. Thus, only the casual equation for spin-one particles generates parasupersymmetries.

It is well-known that a parasupersymmetric Hamiltonian has as positive, as negative eigenvalues [372]. In our case this "Hamiltonian" coincides with \( P_0^2 \), thus the equation (22.46), (22.47) generates complex energy values. To overcome this difficulties it is sufficient to take into account the anomalous interaction bilinear in respect with \( F_{\mu\sigma} \), refer to Subsection 10.9.

It is possible to show that analogous parasupersymmetries exist for any spin-one equation considered in Sections 2, 3, if anomalous Pauli-type interaction is taken into account [28].

### 22.8. Symmetries in Elasticity

Let us study symmetries of the main equation of elasticity [271]

\[ \frac{\partial^2 U}{\partial t^2} = -\frac{\lambda}{\rho_0} p \cdot U - \frac{\mu}{\rho_0} pU. \] (22.53)

Here \( U = (U^1, U^2, U^3) \) is the displacement vector, \( \lambda > 0, \mu > 0 \) and \( \rho_0 > 0 \) are the Lame coefficients.

The equation (22.53) is not Poincaré-invariant but its differential consequences have this invariance [271]. Inasmuch as symmetries of this equation can have important and physically measurable consequences we decided to investigate them besides symmetries of the basic equations of quantum mechanics.
Let us rewrite (22.53) in the matrix form

\[ LU = Z_{ab} \left[ \left( \frac{\lambda}{\rho_0^2} - \mathbf{p} \cdot \mathbf{p} \right) \delta^{ab} + \frac{\lambda + \mathbf{u} \cdot \mathbf{p} \cdot \mathbf{p}}{\rho_0} \right] U = 0 \]  

(22.54)

where \( U \) is a column with components \( U^1, U^2, U^3 \), and \( Z_{ab} \) are the matrices (3.6).

The maximal IA of (22.54) in the classical Lie approach was found in [66].

The basis elements of this algebra have the form

\[ P_0 = \frac{\partial}{\partial t}, \quad P_a = \frac{\partial}{\partial x_a}, \]  

\[ J_a = \varepsilon_{abc} x_b p_c + i S_a, \quad D = i P_0 - x_a p_a \]  

(22.55)

where \( S_a \) are the matrices (3.6).

It is natural to investigate symmetries of the equation (22.54) in the class \( M_2 \), i.e., in the class which includes the operator \( L \). The corresponding SOs can be useful for description of systems of coordinates in which solutions in separated variables exist [365], and for finding new conservation laws in elasticity.

We will search for SOs for the equation (22.54) using a complete set of the matrices \( Z_{ab} \) and \( S^a \):

\[ Q = Z_{ab} b^{ab} + i S^a C^a \]  

(22.56)

where \( b^{ab} \) and \( C^a \) are differential operators of the second order with real coefficients depending on \( x \). By definition the operator (22.56) is a SO of (22.54) if it satisfies the condition (16.7) where \( L \) is the operator of (22.54), \( \alpha \) is an operator which belongs to the class \( M_2 \), and so can be represented in the form given in the r.h.s. of (22.56).

Substituting \( L \) of (22.54) and \( Q \) of (22.56) into (16.7) and calculating the necessary commutators and anticommutators with the help of relations (20.31) we obtain a system of determining equations for coefficients of the operator (22.56). We will not rewrite this cumbersome system, but instead present a consequence of its solution (see [158] for the detail).

**THEOREM 22.5.** The basic equation of elasticity (22.54) admits 61 linearly independent SOs in the class \( M_2 \). Among them are the operators (22.55), products of these operators and the following 9 operators which do not belong to the enveloping algebra generated by the algebra (22.55):
Symmetries of Equations of Quantum Mechanics

\[ Q_0 = 2S \cdot J(S \cdot J - 1) - J^2, \]
\[ Q_a = \varepsilon_{abc} S_b P_c S \cdot J - \frac{1}{2} \varepsilon_{abc} [J, P], \]
\[ Q_{ad} = \hat{Q}_{ad} - \frac{1}{3} \delta_{ad} \hat{Q}_{mn}. \]

where
\[ \hat{Q}_{ad} Z_{ad} p^2 + p_a Z_{de} p^e - p_d Z_{ae} p^e + \delta_{ad} (Z_{cd} p^e p^d - p^2). \]

So the equation (22.54) has 9 essentially non-Lie SOs in the class \( M_2 \). We note that \( Q_0 \) of (22.57) is nothing but a Dirac-type SO, compare with (22.17).

The operators (22.57) do not form a Lie algebra. But the set of the SOs
\[ \{ \hat{Q}_{ab}, J \cdot P = S \cdot P; \ P_\alpha, J_\alpha, D, \ \eta_{ab}, \ \eta_{abcd} \} \]
where
\[ \eta_{ab} = P_a P_b \ [P^2 - (S \cdot P)^2], \ \eta_{abcd} = P_a P_b P_c P_d, \]
and the other operators given in (22.55), (22.58), form a basis of the superalgebra. Moreover \( S \cdot P \) anticommutes with \( \hat{Q}_{ab} \), the anticommutation relations for \( \hat{Q}_{ab} \) have the form (22.41) where \( \nu \rightarrow a, \ \sigma \rightarrow b, \ g_{\nu \sigma} \rightarrow \delta_{ab}. \)

In conclusion we consider conditional symmetries in elasticity. Imposing on \( U \) the condition of transversality \( \text{div} U = 0 \) we come from (22.53) to the following system
\[ \frac{\partial^2}{\partial t^2} U + \frac{\lambda}{\rho_0} p^2 U = 0; \quad p \cdot U = 0. \]  

With the help of the change \( t = t'(\lambda/\rho)^{1/2} \) this system reduces to the form (20.37). Thus all the results connecting with the conditional symmetry of the equations (20.37) (refer to Theorems 20.4, 20.5) are valid for the equations (22.49) describing the transverse elastic waves.

Consider also symmetries of the equation (22.53) supplemented by the subsidiary condition \( p \times U = 0 \), or
\[ S \cdot P U = L \cdot U = 0. \]

In this case we come from (22.53) to the following equation (describing longitudinal waves)
\[ L_1 U = 0, \quad L_1 = \frac{\partial^2}{\partial t^2} + \frac{\mu}{\rho_0} p^2. \]

Investigations of symmetries of the system (22.60), (22.61) can be carried out in a complete analogy with Subsection 20.5, that is why we restrict ourselves to
formulating the final results.

**ASSERTION 1.** The maximal IA of the system (22.60), (22.61) in the class $M_1$ is the eight-dimensional Lie algebra whose basis elements are given in (22.55) where $t \rightarrow t' = (\rho_0/\mu)^{1/2}$.

**ASSERTION 2.** The maximal IA of the equation (22.61) without subsidiary condition (22.60) is the 24-dimensional Lie algebra whose basis elements are given in (20.39) where $t \rightarrow t' = (\rho_0/\mu)^{1/2}$.

**ASSERTION 3.** If relation (22.60) is satisfied then the equation (22.61) is invariant under the 25-dimensional Lie algebra whose basis elements are given in (21.50), (22.62):

$$\eta \rightarrow S \eta.$$

We see the conditional symmetry of the Lamé equation is rather extensive. In any case it is considerably more extensive than the usual Lie symmetry, compare with (22.55). We recall this is the conditional symmetry which generates conservation laws and thus has a clear physical sense.

The conserved quantities corresponding to non-Lie and conditional symmetries of the equation (22.53) are discussed in the following section, refer to Subsection 23.8.

The non-Lie symmetries of (22.53) were found in [122]. The explicit form of the integro-differential SOs of this equation is given in [138]. Symmetries of the stationary equations of elasticity in the class $M_1$ were studied in details by Olver, refer to [351].

### 23. CONSERVATION LAWS AND CONSTANTS OF MOTION

#### 23.1 Introduction

In connection with the above the following questions arise: what are physical consequences of nongeometric symmetries and is it possible to use these symmetries while solving concrete physical problems?

The very existence a nongeometric symmetry of a motion equation is a fundamental fact which reflects that there is an internal degree of freedom of the object described or can give other nontrivial information. Some consequences of that symmetry can be used for different purposes: to find constants of motion, to construct the density matrix, to expand solutions in a complete set of eigenfunctions of a SO and so on. In Subsection 31.8 we use nongeometric symmetries in order to construct exact solutions and generate new solutions starting from known ones. A discussion of some
Symmetries of Equations of Quantum Mechanics

applications of nongeometric symmetries presents the main contents of this section.

One of the most important consequences of symmetries of equations of motion is existence of constants of motion, i.e., some combinations (as a rule bilinear) of solutions which are conserved in time. The well-known examples of constants of motion are energy, momentum and angular momentum.

But what type of constants of motion corresponds to the nongeometric symmetries described above? In the traditional approach to searching constants of motion, we investigate symmetries of the Lagrangian of the described system and then find the corresponding conservation laws using the Noether theorem (see, e.g., [41,42]). The advantage of this approach is that so defined conservation laws as a rule have meaningful physical interpretations. But this approach has evident limitations since not every equation of mathematical physics admits the Lagrangian formulation and moreover Noether’s theorem does not allow all the constants of motion even for those equations which can be obtained using the variational principle.

According to the above, in describing the conservation laws corresponding to nongeometrical symmetries, it is preferable to use another more universal approach whose essence is the direct calculation of bilinear combinations of solutions for the motion equation, which are conserved in time by virtue of symmetries of these equations. Moreover in this way it is possible to find conserved quantities which do not correspond to any SO. Examples of such quantities are given in Subsections 22.7, 22.8.

Let us consider an arbitrary evolution equation of the form

\[ i \frac{\partial}{\partial x_0} \psi = i \frac{\partial}{\partial t} \psi = H(p) \psi \]  

(23.1)

where \( H(p) \) is a differential operator with matrix coefficients, \( \psi(\chi_0=t,x) \) is a real (or complex) multicomponent function belonging to the space \( L_2(R_4) \).

Let \( Q \) be a linear operator defined on a set everywhere dense in the space of vector-functions \( \psi \subset L_2(R_4) \). To each solution of (23.1) we assign a bilinear combination of the form

\[ L_0 = \int d^3 \chi \psi^\dagger Q \psi \]  

(23.2)

where \( \psi^\dagger \) is a transposed and conjugated function.

Differentiating (23.2) in respect with \( x_0 \) and using (23.1) we obtain the following sufficient condition of conserving \( L_0 \) in time

\[ \left[ i \frac{\partial}{\partial x_0} - H(p) \right] Q - Q \left[ i \frac{\partial}{\partial x_0} - H(p) \right] \psi = 0. \]  

(23.3)

If the operator \( H(p) \) is Hermitian then (23.3) reduces to the form

But the condition (23.4) is nothing but a definition of a SO of the equation

282
Chapter 4. Nongeometric Symmetries

\[
\left[ \frac{\partial}{\partial x^0} - H, Q \right] \psi = 0. \tag{23.4}
\]

(23.1), compare with (1.5). In other words if the evolution operator (Hamiltonian) is Hermitian then we can assign the conserved quantity to any SO of the equation (23.1).

We emphasize the relation (23.3) is not a necessary condition of conservation of (23.2) in the case when \( \psi \) satisfies some additional conditions besides (23.1). Maxwell’s equations are a good example of a system of equations including the subsystem of the sort (23.1). But besides that Maxwell’s equations include additional conditions not containing time derivatives, and this is why these equations generate such constants of motion of the form (23.2) that the equations (23.3), (23.4) are not satisfied.

Conserved quantities can be calculated also using a more traditional approach based on the concept of currents satisfying the continuity equation. Let us formulate the corresponding definitions valid for an arbitrary system of partial differential equation of the form

\[
F^A(x, U, U', \ldots) = 0, \quad A = 1, 2, \ldots, N. \tag{23.5}
\]

Here \( x = (x_0, x_1, \ldots, x_n) \) are independent variables, \( U = (U^1, U^2, \ldots, U^m) \) is a vector-function depending on \( x \), \( U' \) denotes derivatives in respect with \( x \), i.e., the set of the following quantities

\[
U_j^k = \frac{\partial U^k}{\partial x_j}, \quad j = 0, 1, \ldots, n; \quad k = 1, 2, \ldots, m,
\]

the dots denote derivatives of higher order which can be included into the system (23.5).

Following [228] we say that there exists a conservation law for the system (23.5) if it is possible to assign \((n+1)\)-dimensional vector \( j_\mu(x, U, U', \ldots) \) \((\mu = 0, 1, \ldots, n)\) for any solution of (23.5) moreover

\[
p^{\mu} j_\mu = 0. \tag{23.6}
\]

According to the Ostrogradskii-Gauss theorem, it follows from (23.6) that the quantity

\[
<j_\mu> = \int_{R^0} d^n x j_\mu(x, U, U', \ldots)
\]

does not depend on \( x_0 \). Here \( R^0 \) is a domain of integration, which we assume to coincide with the \( n \)-dimensional manifold \( \{ x \} \).

Until now there is no a constructive algorithm of describing all the possible conservation laws admitted by an arbitrary system of partial differential equations. Our
approach is to find all the conserved bilinear quantities of the kind (23.2) being conserved in time. Then starting from the found \( j_0 \) we can calculate the remaining components of the conserved current using the symmetry group of the equation of interest. Such an approach enables us to formulate the natural definition of currents equivalence: we say two currents \( j_\mu \) and \( j'_\mu \) are equivalent if

\[
\int_{\mathbb{R}^n} d^n x j_0 = \int_{\mathbb{R}^n} d^n x j'_0,
\]

(23.7)
a current is called trivial if \( <j_\mu> = 0 \).

23.2. Conservation Laws for the Dirac Field

Let \( Q \) be a SO of the Dirac equation. We define the corresponding conserved current as follows

\[
j'_\mu = \frac{1}{2}(\bar{\psi} \gamma^\mu Q \psi + \bar{Q} \psi \gamma^\mu \psi)
\]

(23.8)

where \( \bar{\psi} = \psi^\dagger \) and \( \psi \) belong to the set of solutions of the Dirac equation. Inasmuch as by definition \( Q \) satisfies (16.2), (17.1) then \( j'_\mu \) (23.8) satisfies the continuity equation (23.6).

We note that the correspondence "SO - conservation law" given above is isomorphism in the sense that to any SO of the Dirac equation we can assign the conserved quantity (23.8). On the other hand any bilinear conserved quantity corresponds to a SO; to find a complete set of such quantities is to find the general solution of the equation (23.4) defining SOs of the Dirac equation. So there is one-to-one correspondence between SOs and zero components of conserved currents, the other components are easily found using Lorentz transformations.

We present the explicit form of the currents corresponding to the SOs of the class \( M_1 \). Starting from generators of the Poincaré group we obtain from (2.22), (23.8) the well-known expressions for the tensors of energy-momentum and angular momentum

\[
T_{\mu\nu} = \frac{i}{2} \left[ \overline{\psi} \gamma_\nu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \bar{\psi}}{\partial x_\mu} \gamma_\nu \psi \right],
\]

\[
M_{\mu\sigma} x_\sigma T_{\lambda\nu} - x_\nu T_{\mu\lambda} + \frac{1}{2} \overline{\psi} [\gamma_\mu, S_] \cdot \psi
\]

(23.9)

which satisfy the continuity equation in respect with the index \( \sigma \).

The trivial identity SO generates the current of probability density

\[
j_\mu = \overline{\psi} \gamma_\mu \psi.
\]

(23.10)
For the remaining SOs of (18.5) we obtain the following expressions:

a) the operators \( W_\mu \) and \( W_\mu^\sigma \) correspond to the tensors of valences 2 and 3

\[
\begin{align*}
\omega_{\mu k} &= \frac{1}{4} \left( \nabla_\lambda \gamma_\lambda \frac{\partial \psi}{\partial x_\mu} - \nabla_\lambda \gamma_\lambda \frac{\partial \gamma_\lambda \psi}{\partial x_\mu} \right) + m \nabla_\lambda S_{\mu k} \psi, \\
\omega_{\mu \rho \sigma} &= \frac{i}{4} \left( \frac{\partial \gamma_\lambda \psi}{\partial x_\rho} S_{\lambda \mu} \frac{\partial \psi}{\partial x_\sigma} \frac{\partial \psi}{\partial x_\mu} - \nabla_\lambda \gamma_\lambda \frac{\partial \gamma_\lambda \psi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\sigma} \right) + \frac{1}{2} m \nabla_\lambda [S_{\mu \rho \sigma}, \gamma_\lambda] \psi;
\end{align*}
\]

(23.11)

b) the operators \( B \) and \( A_\mu \) correspond to the vector \( J_\lambda \) and tensor \( Z_{\mu \lambda} \)

\[
J_\lambda = 2 \gamma_\lambda \omega_\mu \psi, \quad Z_{\mu \lambda} = 2 \gamma_\mu \omega_\rho \psi.
\]

(23.12)

We can make sure that all the quantities (23.11), (23.12) satisfy the continuity equation in respect to the index \( \lambda \). Using the Ostrogradskii-Gauss theorem it is possible to find the related constants of motion in the following form

\[
\begin{align*}
\langle P_\mu \rangle &= \int d^3 \chi T_{0\mu}, \quad \langle J_\mu^\sigma \rangle = \int d^3 \chi M_{0\mu \sigma}, \quad \langle j_\mu \rangle = \int d^3 \chi \psi, \\
\langle W_\mu \rangle &= \int d^3 \chi \omega_\mu, \quad \langle W_\mu^\sigma \rangle = \int d^3 \chi \omega_\mu^\sigma, \quad \langle B \rangle = \int d^3 \chi J_\mu, \quad \langle A_\mu \rangle = \int d^3 \chi Z_{\mu 0}.
\end{align*}
\]

(23.13)

(23.14)

So we present the explicit form of the conserved currents and constants of motion corresponding to the SOs of the Dirac equation in the class \( M_1 \). Besides the well-known constants of motion (23.13) we show the "new" conserved quantities (23.14) corresponding to the SOs (18.5).

Formulæ (23.13), (23.14) give a complete set of the conserved quantities depending bilinearly on \( \psi \) and \( \partial \psi / \partial x_\mu \).

23.3. Conservation Laws for the Massless Spinor Field

Here we present conservation laws corresponding to symmetries of the massless Dirac equation.

The SOs from the class \( M_1 \), admitted by the Dirac equation with \( m = 0 \) are given by formulæ (18.31), (18.32). They correspond to the conserved currents (23.8).

As in the case of nonzero mass we obtain the tensors of energy-momentum and angular momentum (23.9) and the probability density current (23.10) which correspond to the SOs \( P_\mu, J_\mu^\sigma, \) and \( I \).

For the operators \( D \) and \( K_\mu \) we obtain from (2.42), (23.8)
Symmetries of Equations of Quantum Mechanics

\[ J_\mu^\nu = \partial^{\nu} T_{\mu \alpha} \quad J_\mu^\kappa = -x^{\lambda} M_{\alpha \mu} + x_{\nu} J_\mu^\nu \]  
(23.15)

where \( T_{\mu \alpha} \) and \( M_{\alpha \mu} \) are the tensors (23.9). Bearing in mind that for the case \( m=0 \) the tensor \( T_{\mu \alpha} \) is traceless, it is not difficult to make sure the currents (23.15) satisfy the continuity equation.

In analogous way we find the currents corresponding to the SOs of (18.31), (18.32):

\[ J_\mu^\rho = \frac{1}{2} \left( \mp \partial_{\mu} \gamma_{\rho} - \frac{1}{4} \epsilon_{\rho \sigma \tau \lambda} \partial_{\mu} \right), \]

(23.16)

\[ J_\kappa^\rho = -x_{\kappa} J_\kappa^{\rho}, \quad J_\mu^\kappa = x^{\lambda} J_\lambda^{\rho}, \]

(23.17)

Here \( \partial_{\mu \nu} \) and \( \gamma_{\rho \sigma} \) are the tensors (23.11), (23.12) with \( m=0 \).

So we obtain the conserved currents corresponding to symmetries of the massless Dirac equation in the class M_1. The currents (23.17) are essentially new (in comparison with obtained by the classical methods) inasmuch as they correspond to the SOs which do not belong to the enveloping algebra of the algebra \( A(C(1,3) \otimes H) \).

Formulae (23.9), (23.10), (23.15)-(23.17) give a complete set of conserved currents depending bilinearly on \( \psi \) and \( \partial \psi / \partial x^{\mu} \) for the massless Dirac equation.

23.4. The Problem of Definition of Constants of Motion for the Electromagnetic Field

Studying constants of motion of Maxwell’s equations is of particular interest since these constants are expressed via the physically measurable quantities, i.e., strengths of the electric and magnetic fields.

The description of constants of motion for Maxwell’s equations can be considered as a separate problem inasmuch as there exist such conserved quantities which do not correspond to any symmetry. On the other hand there exist conserved currents which correspond to trivial (zero) constants of motion. That is why our strategy is to give the complete description of some class of constants of motion and...
then reconstruct the corresponding conserved currents. More precisely we find all the conserved quantities of the form

\[ I = \int d^3 x F \left( E, H, \frac{\partial E}{\partial x^\alpha}, \frac{\partial H}{\partial x^\alpha}, x_0, x \right) \]  

(23.18)

where \( F \) denotes a bilinear combination of the vectors \( E, H \) and their derivatives besides in general \( F \) depends on \( x \) and \( x_0 \). Without loss of generality we can represent \( F \) in the form

\[ F = \phi^\dagger Q \phi \]  

(23.19)

where \( \phi \) is the vector-function (3.4), \( Q \) is a second-order differential operator with matrix coefficients moreover all the matrices are of dimension 6×6. Without loss of generality we suppose \( Q \) is Hermitian (symmetric) operator inasmuch as skew-symmetric operators correspond to zero quantities of (23.18), (23.19).

The problem of describing the conserved quantities defined above is closely related to the problem of describing the second-order SOs of Maxwell’s equations, refer to Section 20, but has a lot of distinguishing features. Indeed, substituting (20.19) into (20.18) and differentiating the latter in respect with \( x_0 \) we obtain using (3.5)

\[ \dot{I} = \int d^3 x \psi (\dot{Q} - i [Q, \sigma, S, p]) \psi . \]  

(23.20)

Equating this quantity to zero, using (3.5) and taking into account Hermiticity of the operator in paratheses we obtain the following equation for \( Q \)

\[ Q - i [Q, \sigma, S, p] = i \sigma^\mu (\alpha^\mu E_2, \alpha^\mu O_2) \]  

(23.21)

where \( \hat{L}_2^a \) is the operator (3.5), \( \alpha^\mu_a \) are unknown second-order differential operators (commuting with \( \sigma^\mu \)) which have to be determined, \( \sigma^\mu = \sigma^\mu, \mu \neq 2; \sigma^2 = i \sigma^2 \).

Further on we follow the proof of Theorem 20.3. Representing \( Q \) in the form (20.20) we come again to the noncoupled equations for the even and odd parts of the operator \( Q = E + O \)

\[ [E, \hat{L}_1] = \alpha^{\mu}_E \hat{L}^{\mu}_2 + (\alpha^{\mu}_E \hat{L}^{\mu}_2)^\dagger, \]  

(23.22a)

\[ [O, \hat{L}_1] = \alpha^{\mu}_O \hat{L}^{\mu}_2 + (\alpha^{\mu}_O \hat{L}^{\mu}_2)^\dagger. \]  

(23.22b)

Here \( E, O, \alpha^{\mu}_E, \alpha^{\mu}_O \) are unknown second order differential operators of the form (20.20) (20.22), \( \hat{L}_2^a \) and \( \hat{L}_1 \) are the operators (20.26).

Thus the problem of describing the conserved quantities of the form (23.18) reduces to solving the operator equations (23.22) There are two essentially new points in comparison with the equations (20.23), (20.24) for a SO:

1) there are no commutation relations of SOs with the operator \( \hat{L}_2^a \) in (23.22),
i.e., the second equations of (20.23), (20.24) are absent;
2) the anti-Hermitian terms of (20.23), (20.24) are absent in (23.22).

In other words the conditions imposed on the operator $Q$ of the conserved bilinear form (23.18), (23.19) are weaker than the corresponding equations (20.23), (20.24) for the SOs of Maxwell’s equations. That is why the number of conservation laws is larger than the number of SOs in the class considered.

23.5. Classical Conservation Laws for the Electromagnetic Field

First we restrict ourselves to considering such bilinear conserved quantities which depend on strengths of the electric and magnetic fields but do not include derivatives of these strengths. We will call them conserved quantities of zero order.

**THEOREM 23.1** [159]. There exist exactly 15 linearly independent constants of motion of the form

$$I = \int d^3x F(E, H, x_0, x)$$

where $F(E, H, x_0, x)$ is a bilinear combination of the vectors $E, H$ satisfying Maxwell’s equations. The general form of $F$ corresponding to $I=0$ is given by the formula

$$F(E, H, x_0, X) = \frac{1}{2} K^0 (E^2 + H^2) + K \cdot E \times H$$

where $K=(K^0, K)$ is a conformal Killing vector, i.e., an arbitrary solution of the equations (1.17).

**PROOF.** Let us represent a bilinear combination of $E$ and $H$ in the form (23.19) where $\phi$ is the vector-function (3.4), $Q$ is a symmetric real matrix of dimension $6 \times 6$ (skew-symmetric matrices make zero contribution into the integral (23.18) since $\phi$ is a real function). Substituting (23.19) into (23.18) and differentiating the latter in respect with $x_0$ we come to the equation (23.21) where $a_\mu^\nu$ are unknown functions of $x_0$, $x$ have to be determined. Expanding $Q$ in the complete set of matrices (3.6), (5.30) in accordance with (20.20) where

$$A = Z_{ab} a^{ab}, \quad B = Z_{ab} b^{ab}, \quad C = Z_{ab} c^{ab}, \quad D = i S_a K^a,$$

$$a^{ab}, b^{ab}, c^{ab} \text{ and } f^a \text{ are unknown functions of } x_0, x, \text{ and equating coefficients of linearly independent matrices and differential operators we come to the relations}$$

$$a^{ab} - b^{ab} = 0, \quad c^{ab} = \delta^{ab} K^0,$$

moreover, $K^0$ has to satisfy the equations (1.17) together with $K^a$ from (23.25).

The corresponding matrix $Q$ has the form
Chapter 4. Nongeometric Symmetries

Substituting (23.26), (3.4), (3.6) into (23.21) we come to formula (23.24).

The general solution of (1.17) depends on 15 arbitrary parameters and is given by formula (1.18) To obtain the corresponding constants of motion we substitute (1.18) into (23.20) and choose consequently only one of parameters be nonzero. As a result we come to the formulae

\[ E = \frac{1}{2} \int d^3x (E^2 - H^2) = \int d^3x \hat{P}_0, \quad P = \int d^3x E \times H = \int d^3x \hat{P}, \]

\[ L = \int d^3x \vec{x} \times (E \times H), \quad N = \int d^3x \left[ x^a E_a H_b - x^b E_b H_a \right], \]

\[ D = \int d^3x \mu \hat{P}_\mu, \quad K_\mu = \int d^3x (2 \delta_{\mu a} x^a \hat{P}_a - x^a \hat{P}_\mu). \]

Relations (23.27) give the well-known classical constants of motion of the electromagnetic field in vacuum, i.e., the energy \( E \), momentum \( P \), angular momentum \( L \) etc. These constants of motion can be obtained using the Lagrangian formalism and Noether’s theorem [32].

According to Theorem 23.1 there are no other constants of motion bilinearly depending on \( E \) and \( H \) for Maxwell’s equations.

It is not difficult to establish the correspondence between the constants of motion described in the theorem and the conserved currents. These currents have the form

\[ j_\mu \cdot T_{\mu \nu} K^\nu \]

(23.28)

where \( K^\nu \) is a Killing vector, \( T_{\mu \nu} \) is the energy-momentum tensor:

\[ T_{00} = \frac{1}{2} (E^2 + H^2), \quad T_{0a} = - T_{0a} = - \varepsilon_{abc} E_b H_c, \]

\[ T_{ab} = - E_a E_b - H_a H_b + \frac{1}{2} \delta_{ab} (E^2 + H^2). \]

(23.29)

Using the well-known properties of \( T_{\mu \nu} \)

\[ \partial^\nu T_{\mu \nu} = 0, \quad T_{\mu \nu} \delta^{\rho \sigma} = 0 \]

(23.30)

it is not difficult to make sure that the four-vector \( j_\mu \) satisfies the continuity equation iff \( K^\mu \) is a Killing vector satisfying (1.17). This is the zero component of the current (23.28) which is given in (23.24).

According to Theorem 23.1 all the nonequivalent conserved currents depending bilinearly on \( E \) and \( H \) have the form (23.28) and so are completely determined by the Killing vector.
23.6. The First Order Constants of Motion for the Electromagnetic Field

We now consider constants of motion of the more general form (23.18) which depend on the electromagnetic field strength and its derivatives.

The obvious set of conserved quantities of this kind can be obtained by choosing \( Q \) in the form of products of the conformal group generators and the matrices (23.26). Conservation of such quantities in time is caused by relativistic and conformal invariance of Maxwell’s equations. Conservation laws of such a kind were found by Lipkin [282] for the first time (see [248, 109, 303] also).

The SOs found in Subsection 20.4 do not belong to the enveloping algebra of the algebra \( A(C(1,3) \otimes \mathcal{H}) \) and so are not caused by conformal invariance of Maxwell’s equations. Substituting these SOs into (23.18), (23.19) instead of \( Q \) we obtain the motion constants which are new in principle and have nothing to do with the relativistic and conformal invariance of Maxwell’s equations.

However there exist more exotic conserved quantities for the electromagnetic field which cannot be assigned by any SO. Examples of such constants of motion are considered in the following.

First we consider the conserved quantities which are bilinear combinations of the form \( (F^\mu_\sigma, \partial^\lambda F^\rho_\alpha) \) where \( F^\mu_\lambda \) is the tensor of the electromagnetic field. More precisely we search for conserved in time bilinear forms

\[
I = \int d^3 \phi \quad Q\phi = \int d^3 x j_0
\]

where \( \phi \) is the vector-function (3.4), \( Q \) is a first-order differential operator with matrix coefficients. We will call such conserved quantities first order constants of motion. In particular this class of conserved quantities includes Lipkin’s “zilch” but is only a subclass of more general constants of motion of the type (23.18).

**THEOREM 23.2.** There exist exactly 84 first order constants of motion for Maxwell’s equations. The corresponding function \( j_0 \) of (23.31) is the zero component of the four-vector

\[
j_\mu = K^\sigma Z^\sigma_{\alpha \mu} + 2 \epsilon_{\rho \lambda \alpha \delta} (\partial^\rho K^\lambda_{\sigma \delta}) T^\sigma
\]

where \( T^\sigma_\alpha \) is the energy-momentum tensor (23.29), \( K^\sigma\nu \) is a conformal Killing tensor of valence 2, satisfying the equations

\[
\partial^\rho K^\sigma\nu = -\frac{1}{3} \partial^\rho K_{\lambda \mu \rho,}^\lambda \sigma^\nu, \\
K^\sigma\nu - K^\nu,\sigma, \\
K^\sigma_{\nu \rho} = 0,
\]

\( Z^\sigma_{\alpha \mu} \) is Lipkin’s zilch tensor:

The proof reduces to finding the general solution of the equations (23.22) for
Chapter 4. Nongeometric Symmetries

We consider first-order differential operators \( E, O, \alpha \), \( a \) of the form (20.20), (20.22). We do not present the corresponding calculations (which are analogous to the ones given in Subsection 20.3) but note that the odd part of \( Q \) (i.e., the operator \( O \)) reduces to zero and the even part \( (E) \) has the following general form

\[
Z_{\mu\nu} = E \frac{\partial}{\partial \Lambda} H_{\mu} \Lambda_{\nu} - H_{\mu} \frac{\partial}{\partial \Lambda} E_{\nu};
\]

\[
Z_{\nu\lambda} = Z_{\lambda\nu} = E_{\nu} \frac{\partial}{\partial \Lambda} H_{\lambda} + H_{\lambda} \frac{\partial}{\partial \Lambda} E_{\nu},
\]

\[
Z_{\alpha\beta} \delta_{\sigma\tau} = E_{\alpha} \frac{\partial}{\partial \Lambda} H_{\beta} + H_{\beta} \frac{\partial}{\partial \Lambda} E_{\alpha} + \frac{1}{2} \partial^{\alpha} K_{\alpha\beta} Z_{\alpha\beta}.
\]

Equation (23.34)

Equation (23.35)

where \( S_{\alpha}, Z_{\alpha\beta} \) are the matrices (3.6), \( K^{\alpha\beta} \) is a function satisfying the equations (23.33) and depending on 84 arbitrary parameters, see Appendix 2, formulae (A.2.9), (A.2.17). Substituting (3.4), (23.35) into (23.31) we obtain \( j_0 \) coinciding with the zero component of the vector (23.32).

Using the identities

\[
Z_{\mu\nu} \delta_{\nu\lambda} = 0, \quad Z_{\nu\lambda} \delta_{\nu\mu} = 0, \quad \partial_{\mu} Z_{\alpha\beta\gamma} = 0,
\]

\[
Z_{\mu\nu} + Z_{\nu\lambda} \delta_{\nu\mu} = \frac{1}{2} \partial^{\nu} K_{\nu\lambda} Z_{\nu\lambda},
\]

and the relations (23.33) it is not difficult to make sure that the four-vector (23.32) satisfies the continuity equation. Substituting the general expression for \( K^{\alpha\beta} \) (refer to (A.2.24)) into (23.32) we come to a linear combination of the conserved currents exhausting all the nonequivalent currents of the first order. Some of them are well-known including Lipkin’s zilch and its generalizations (see [282,248,109]). Formula (23.32) includes the ”new” currents also which are polynomials of \( x_{\mu} \) of orders 3 and 4 and correspond to the conformal Killing tensors represented by \( G^{\mu\nu}, G^{\alpha\beta} \) in (A.2.24), (A.2.25). Theorem 23.2 gives a complete list of the first order conserved currents.

We note that the operator (23.35) is nothing but the Hermitian part of the product of the matrix \( i\sigma_2 Q \) (where \( Q \) is the matrix (23.26)) and a linear combination of the generators of the conformal group. Thus we say the first order constants of motion for Maxwell’s equations correspond to products of the usual conformal symmetries.

### 23.7. The Second Order Constants of Motion for the Electromagnetic Field

We now consider constants of motion being bilinear combinations of the first derivatives of the electric and magnetic field strengths. In other words we study the quantities of the sort (23.18) which are conserved in time. The corresponding functions \( F \) can be represented in the form (23.19) where \( Q \) is a second-order differential operator with matrix coefficients. We call such conserved quantities the second order

291
The problem of description of second-order constants of motion reduces to solving the equations (23.22) for a second-order differential operator of the form (20.20), (20.22). It is possible to show that even operators \( Q=E \) satisfying (23.22a) reduces to polynomials of the conformal group generators and the matrix (23.26) and thus are not of great interest. We will not represent the corresponding calculations but note that a complete set of such operators is determined by a conformal Killing tensor of valence 3. The number of these operators (and the corresponding constants of motion) is equal to 300.

Here we consider odd operators only which do not belong to the enveloping algebra of the algebra \( A[C(1,3)\mathfrak{G}H] \). In other words we investigate conserved quantities of the following form

\[
I = \int d^3x \varphi^\dagger O \varphi - \int d^3xF
\]

where \( O \) is an odd operator belonging to the class \( M_2 \), \( \varphi \) is the function (3.4). The general expression for \( O \) is given in (20.28), (20.30) where without loss of generality \( O A \equiv 0 \) inasmuch as anti-Hermitian operators make zero contribution into the integral (23.36). Substituting (20.32a), (20.32c) into (23.22b) and calculating the necessary commutators and anticommutators using (20.31), we can equate the coefficients of linearly independent matrices and differential operators. Then we come to the following system of equations for coefficients of the operator of conserved bilinear form:

\[
D_{1}^{ab,cd} = K_{(ab)}^{0(cd)} - \frac{2}{3} \delta^{(ab)} \left( \frac{1}{6} g^{(cd)} \right) ; \\
D_{2}^{ab,cd} = -K_{(0d,a)}^{ab} - \frac{8}{3} \delta^{(ab)} \left( \frac{1}{6} g^{(cd)} \right) ; \\
B_{1}^{ab} = \frac{4}{27} \partial^{(m)} \left( a_{(m)}^{ab} \right) \kappa_{ab} ; \\
B_{2}^{ab} = \frac{8}{27} \partial^{(m)} \left( a_{(m)}^{ab} \right) ,
\]

\[
D_{1}^{ab} = \frac{3}{5} \delta^{(ab)} \left( \frac{1}{6} g^{(cd)} \right) ; \\
D_{3}^{ab} = \frac{3}{10} \delta^{(ab)} \left( \frac{1}{6} g^{(cd)} \right) ;
\]

where \( K_{\alpha\beta}^{\mu\nu} \) is a generalized Killing tensor, i.e., an irreducible tensor which is antisymmetric under the permutations \( \mu=\nu \) or \( \rho=\sigma \) and symmetric under the permutations \( (\mu,\nu)\leftrightarrow(\rho,\sigma) \). Besides this tensor satisfies the equations

\[
\partial^{\nu} K_{\nu\alpha}^{\mu\lambda} \equiv \frac{1}{6} \delta^{\nu} \left( K_{\nu(\sigma)}^{(\rho)} \delta^{\mu\lambda} + \frac{1}{2} \delta^{\mu\lambda} K_{\nu(\alpha)}^{(\rho)} \right) = 0
\]

We recall that symmetrization is imposed over the indices in the paratheses.

\[292\]
We see that the odd constants of motion of the second order for Maxwell’s equations are completely determined by a generalized Killing tensor in accordance with (23.36), (20.32a), (22.37). The general expression for this tensor is given in Appendix 2, see (A.2.24). In other words formulae (23.36), (20.32a), (23.37) (A.2.24) define a complete set of odd constants of motion of the second order for Maxwell’s equations. The number of these constants of motion is large enough and is equal to the number of independent components of the corresponding generalized Killing tensor, i.e., 378.

We see that there exist significantly more constants of motion in the class considered than the corresponding SOs, compare with Theorem 20.3. More precisely any generalized conformal Killing tensor $\mathcal{K}_\nu^\alpha$ with $\lambda \neq 0$ generates a constant of motion which does not correspond to any SO in the class $\mathcal{M}_2$.

Let us represent some of the found constants of motion in the explicit form. Restricting ourselves to those which are polynomials on $x_\mu$ of order 0, 1 and 2, we obtain the following set of independent operators $F$ for (23.36):

\[
\begin{align*}
F_{10}^{abc} & = E^a E^b H^c - H^a E^b H^c + (\partial_a E^b) \partial_b H^c, \\
F_{30}^{abc} & = E^a E^b H^c - H^a E^b H^c, \\
F_{10}^{abc} & = \varepsilon^{abc} E^a - E^a \varepsilon^{bc}, \\
F_{30}^{abc} & = \varepsilon^{abc} E^a - E^a \varepsilon^{bc}, \\
F_{a1}^{\mu} & = \frac{1}{2} \varepsilon^{\mu \nu \rho \beta} F_{\alpha \beta}^\mu x_\nu + 2 i \alpha_1 \gamma_\mu \gamma_0 F_{a1}^{\mu}, \\
F_{a1}^{\mu} & = \varepsilon^{\mu \nu \rho \beta} F_{\alpha \beta}^\mu x_\nu + 2 i \alpha_1 \gamma_\mu \gamma_0 F_{a1}^{\mu}, \\
F_{a1}^{\mu} & = \varepsilon^{\mu \nu \rho \beta} F_{\alpha \beta}^\mu x_\nu + 2 i \alpha_1 \gamma_\mu \gamma_0 F_{a1}^{\mu}. 
\end{align*}
\]

(23.39)

Here

\[
C_{\alpha \beta}^{\mu} = E^a E^b H^c - H^a E^b H^c, \\
C_{\alpha \beta}^{\mu} = E^{(\alpha} H^{b)}, \\
\alpha \neq \alpha', \alpha' \neq \alpha, \ k = 1, 2, 3,
\]

and the dots denote time derivatives.

The traces of the tensors (23.39a) make zero contributions into the integrals (23.36). Time independence of the integrals (23.36), (23.39) can be verified directly using Maxwell’s equations.

Formulae (23.36), (23.39) present the complete set of odd constants of motion of second order, depending on $x_\mu$ as polynomials of orders 0, 1 and 2. Using relations (23.36), (20.32a) (23.37), (A.2.24) it is not difficult to rewrite the explicit form of other odd constants of motion which in general are polynomials on $x$ of order 6. We do not
present the corresponding cumbersome formulae here.

We emphasize that the constants of motion found in this subsection have nothing to do with the conformal or Lorentz invariance of Maxwell’s equations in contrast to the classical and first-order constants of motion. The constants of motion given in (23.36), (23.39) either are generated by the odd SOs not belonging to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$ or do not correspond to any SO in the class considered.

We note that the constants of motion found above admit a covariant formulation. Let us denote as usual by $F_{\mu\nu}$ the tensor of the electromagnetic field,

$$
F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad F_{\mu\nu} = \frac{\partial}{\partial x^\mu} F_{\nu\rho}.
$$

Then the following tensors

$$
G_{\mu\nu}^{[\rho\|\sigma]} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad F_{\mu\nu} = \frac{\partial}{\partial x^\mu} F_{\nu\rho}.
$$

satisfy the continuity equation in respect with the indices $\alpha$ and $\lambda$ [248].

It is not difficult to verify that formulae (23.39a) define the complete set of independent components of $G_{\mu\nu}^{[\rho\|\sigma]}$ if we exclude terms making zero contributions into the integral

$$
I = \int d^3 x G_{\mu\nu}^{[\rho\|\sigma]}.
$$

As to the other quantities present in (23.39b)-(23.39d) they are expressed via convolutions of $G_{\mu\nu}^{[\rho\|\sigma]}$ with $x_\mu$ and the tensors $C_{\langle a\rangle}, \tilde{C}_{\langle a\rangle}$.

In conclusion we note that it is not too difficult to formulate and solve the problem of description of constants of motion of arbitrary finite order $n$ for Maxwell’s
equations. We will not do it here but note that the even constants of motion are determined by the conformal Killing tensor of valence \( n \), the corresponding number of independent motion constants is given in (A.2.9), where \( m=4 \), \( s=1 \), \( j=n+1 \). The odd constants of motion are defined by the generalized conformal Killing tensor of valence \( R_1+2R_2 \), where \( R_1=n-1 \), \( R_2=2 \). The number of these constants is defined by formula (A.2.16).

### 22.8. Constants of Motion for the Vector-Potential

The constants of motion discussed above can be represented in terms of the vector-potential \( A_\mu \). However, there exist additional conserved quantities depending bilinearly on \( A_\mu \) which extend the sets of constants of motion described in Subsections 22.4-22.7. That is why we discuss the constants of motion for the vector-potential as a separate problem.

Consider the vector-potential \( A_\mu \) in the Coulomb gauge, then by definition \( A_\mu \) satisfies the equations (20.35). We search for the constants of motion in the form

\[
I = \int A \cdot Q A
\]

where \( A=\text{column}(A_1,A_2,A_3) \), \( Q \) is a differential operator whose coefficients are \( 3\times3 \) matrices, see (22.56).

An evident set of constant of motion of (23.41) can be obtained choosing \( Q \) be a SO of the equations (20.35). Such SOs (including operators satisfying the requirement of conditional invariance) are discussed in Subsection 20.5.

It happens the class of constants of motion of the type (22.41) is more extensive inasmuch as the corresponding operators \( Q \) are not in general SOs of the equations for the vector-potential. Restricting ourselves to the case when \( Q\in M_1 \) and denoting the corresponding class of constants of motion by \( m_1 \) we can prove the following assertion.

**THEOREM 23.3.** There exist 53 constants of motion in the class \( m_1 \) for the equation (20.32), including 50 conserved quantities of the sort (23.41) where \( Q\in M_1 \) are the operators of conditional symmetry of the system (20.35), and the three additional constants:

\[
I = \int d^3x v_x x_y (\varepsilon_{\mu
u} \frac{\partial A_\nu}{\partial x_\mu} x_\nu + \varepsilon_{\mu
u} A_\nu).
\]

The proof is analogous to the proofs of Theorems 23.1, 23.2. Time independence of \( I_\mu \) can be verified directly using the equations (3.14)-(3.16) for \( j=0 \).

Considering the vector-potential in the Lorentz gauge (see (3.14), (3.15) for
We find the following six constants of motion:

\[ I_{\mu} = \int d^3x \frac{\partial A_\nu}{\partial x_\mu} \left( x_\alpha \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\alpha}{\partial x_\mu} x_\nu \right) + x_\mu A_\nu - x_\nu A_\mu \]  

(23.43)

We emphasize that the constants of motion (23.42) cannot be represented in the form (22.41) where \( Q \) is an operator of the conditional symmetry of the equation (20.37). In other words it is not possible to assign to these constants any symmetry of the equation (20.37), either usual or conditional. The same is true for (23.43).

The constants of motion (23.42), (23.43) extend the class of conserved quantities of the electromagnetic field, described in Subsections 23.4-23.7. We note these “new” constants of motion cannot be expressed via the electric and magnetic field strengths so their physical interpretation seems to be problematic.

A natural question arises if there exist such constants of motion for the vector-potential which do not correspond to any SO but can be expressed via the vectors of the electric and magnetic fields strengths. This question admits a positive answer because such constants of motion are included in the set described in Subsection 23.7.

We rewrite three of them

\[ I_a = \int d^3x \left( E_c x_c x_b E_{abk} - H_c x_c x_b H_{abk} \right) \]  

(23.44)

Formula (23.44) defines the second order constant of motion for \( E, H \) which is a third order constant of motion for the vector-potential. It is possible to show that one cannot represent (23.44) either in the form (23.31) (where \( Q \) is a second-order SO) or in the form (21.41) where \( Q \) is a SO for the vector-potential.

In conclusion we note that the Lame equation (22.53) also admits constants of motion which do not correspond to any SO. Thus Theorem 23.3 can be reformulated for the Lame equation if we impose the transversity condition \( \text{div} U = 0 \). Indeed, in this case the Lame equation reduces to the form (22.59) which coincides with (20.37) up to the change \( t = t'(\lambda p)^{1/2} \). As to the equations (22.60), (22.61) describing longitudinal waves they admit the following constants of motion

\[ I_{\mu} = \int d^3x U_t \left( \delta_{ab} x^2 - x_\mu x_\nu \right) \left( e_{aal} - e_{aal} U_t \right) \]  

which is not generated by any SO. Besides the Lame equation admits a number of conserved quantities generated by SOs:

\[ I_{\mu} = \int d^3x (U^T Q U - U^T Q U) \]  

where \( U^T = \text{column}(U_1, U_2, U_3) \), \( Q \) is a product of SOs (22.55) and (22.57) [158].
5. GENERALIZED POINCARÉ GROUPS

In this chapter we present the basic information about the generalized Poincaré groups \( P(1,n) \) which are determined as sets of transformations preserving the value of interval in \((1+n)\)-dimensional Minkowsky space. We describe IRs of the Lie algebras corresponding to such groups and make the reduction of IRs of these algebras by the Poincaré and Galilei algebras. We also consider the connection of \( P(1,n) \)-invariant equations with Poincaré- and Galilei-invariant equations for a particle of variable mass and spin.

24. THE GROUP \( P(1,4) \)

24.1. Introduction

As was shown in Chapter 1 the maximal (in the Lie sense) invariance group of the Dirac and KGF equations is the ten parameter Poincaré group \( P(1,3) \). Here we consider a natural generalization of this group to the case of a space with more dimensions. The generalized Poincaré group \( P(1,n) \) can be defined as a semidirect product of the groups \( SO(1,n) \) and \( T \) where \( T \) is the additive group of \((n+1)\)-dimensional vectors \( p_0, p_1, \ldots, p_n \) and \( SO(1,n) \) is a connected component of unity in the group of all linear transformations of \( T \) into \( T \) preserving the quadratic form \( p_0^2-p_1^2-\ldots-p_n^2 \).

The groups \( P(1,n), n>3 \) can be used to describe physical systems with variable masses and spins, e.g., the systems of two coupling relativistic particles. Thus in Chapter 7 we use \( P(1,6) \)-invariant wave equations to describe coupled states of two particles of arbitrary spin. Besides the generalized Poincaré groups have direct connection to the problem of the extension of \( S \)-matrix over the mass shell [82, 240] and to description of particles with internal structure [61, 136]. These groups are also closely connected with the modern multidimensional models appearing in the string theory.

The group \( P(1,n), n>3 \) includes important (from the physical viewpoint) subgroups, such as the Poincaré group \( P(1,3) \) and Galilei group \( G(1,3) \). Inasmuch as IRs of the group \( P(1,n) \) are reducible in respect to \( P(1,3) \) and \( G(1,3) \) it is natural to consider the problem of reduction of these representations by IRs of the Poincaré and Galilei group. Such a reduction makes it possible to answer the question what kind of relativistic (and nonrelativistic) particles is described by the equation invariant under the group \( P(1,n) \) and to find the energy spectrum and the spin states of such particles. These and related questions are considered in Sections 25-27. The present section is devoted to the description of IRs of the group \( P(1,4) \).
24.2. The Algebra $AP(1,n)$

The Lie algebra of the group $P(1,n)$ is defined by the following commutation relations

\[ [P_{m'}, J_{m}] = (g_{mn} P_{n'} - g_{m'n} P_{n'}), \]
\[ [J_{mn}, J_{m'n'}] = (g_{mn} J_{m'n'} + g_{m'n} J_{m'n'} - g_{m'm} J_{n'n} - g_{m'n} J_{m'm}), \]

where $m, n, m', n' = 0, 1, \ldots, n$, $g_{kl}$ is a metric tensor for $(1+n)$-dimensional Minkowsky space: $g_{kl} = \text{diag}(1, -1, -1, \ldots, -1)$.

The problem of construction of local representations of the group $P(1,n)$ reduces to the description of nonequivalent representations of the Lie algebra (24.1) in terms of selfadjoint operators. We restrict ourselves to investigation of IRs of this algebra. To classify IRs it is necessary to find independent Casimir operators of the algebra (24.1) and to determine their spectra.

Finding the Casimir operators of the algebra $AP(1,n)$ cannot be considered as a trivial generalization of the corresponding procedure for the algebra $AP(1,3)$. In particular for $n \neq 3$ there is not an $(n+1)$-dimensional analog of the Lubanski-Pauli vector instead of which we have to consider tensors of an appropriate rank.

In this section we find all the independent Casimir operators of the algebra $AP(1,4)$.

We define the fundamental antisymmetric tensor of rank 3

\[ V_{\lambda\rho\sigma} = P_{\lambda\rho\sigma} J_{\rho\sigma} + P_{\rho\sigma\lambda} J_{\sigma\lambda}. \]  

(24.2)

It is a tensor $V^{\lambda\rho\sigma}$ which is a natural generalization of the Lubanski-Pauli vector to the case of the generalized Poincaré groups. In the case when $J_{\mu\nu}, P_{\mu}$ belong to the algebra $AP(1,3)$ we can assign to (24.2) the vector $W_{\mu} = e_{\mu\rho\sigma} V^{\rho\sigma}/6$ (compare with (4.2)). For the algebra $AP(1,4)$ the tensor (24.2) is equivalent to the tensor of rank 2:

\[ W_{\mu\nu} = \frac{1}{6} e_{\mu\rho\sigma\lambda} V^{\rho\sigma\lambda} = \frac{1}{2} e_{\mu\rho\sigma\lambda} P_{\rho} J_{\sigma\lambda}. \]  

(24.3)

Here $e_{\mu\rho\sigma\lambda}$ is the unit antisymmetrical tensor, $e_{\mu\nu\lambda\sigma} = 1$.

Besides $W_{\mu\nu}$ we consider the vector $\Gamma_{\sigma} = J_{\sigma\mu} P^{\mu}$. It follows from (24.1)-(24.3) that the operator $W_{\mu\nu}$ satisfies the relations

\[ P_{\mu} W^{\mu\nu} = 0, \quad [P_{\mu}, W_{\nu}] = 0, \]
\[ [W_{\mu\nu}, J_{\lambda\alpha}] = (g_{\mu\alpha} J_{\nu\lambda} + g_{\nu\lambda} J_{\mu\alpha} - g_{\mu\lambda} J_{\nu\alpha} - g_{\nu\alpha} J_{\mu\lambda}), \]
\[ [W_{\mu\alpha}, W_{\nu\beta}] = (g_{\mu\alpha} e_{\nu\beta\rho\sigma\lambda} + g_{\nu\beta} e_{\mu\rho\sigma\lambda} - g_{\mu\beta} e_{\nu\rho\sigma\lambda} - g_{\nu\alpha} e_{\mu\beta\rho\sigma\lambda}) W^{\rho\sigma\lambda}. \]  

(24.4)

As to the vector $\Gamma_{\sigma}$ it satisfies the relations (4.4) with an appropriate possible values of subindices.

298
Chapter 5. Generalized Poincaré Groups

It follows from the above that the scalar operators

\[ C_1 = P_\mu P^\mu, \]  
\[ C_2 = \frac{1}{2} W_{\mu\nu} W^{\mu\nu} = \frac{1}{2} p_\mu P_\nu J^{\mu\nu} - P_\mu P_\nu J^{\mu\nu}, \]  
\[ C_3 = \frac{1}{4} J_{\mu\nu} W^{\mu\nu} - \frac{1}{8} \varepsilon_{\mu\nu\sigma\lambda} J^{\mu\nu} J^{\sigma\lambda}, \]

commute with any basis element of the algebra \( AP(1,4) \). Moreover using (24.4) and relations analogous to (4.4) it is not difficult to show that formulae (24.5) give all the independent Casimir operators belonging to the enveloping algebra of the algebra \( AP(1,4) \). The operators \( C_1 \) and \( C_2 \) are nothing but generalization of the corresponding Casimir operators (4.2) and (4.6) of the Poincaré algebra, but the operator \( C_3 \) has not any analog in the algebra \( AP(1,3) \).

We note that besides the operators (24.5) there exist additional Casimir operators for different classes of IRs. These Casimir operators do not belong to the enveloping algebra of the algebra \( AP(1,4) \), as in the case of the representations of the algebra \( AP(1,3) \).

24.3. Nonequivalent Realizations of the Tensor \( W_{\mu\nu} \)

As was done by the description of IRs of the Poincaré algebra (see Section 4) we find all the possible (up to equivalence) realizations of the tensor \( W_{\mu\nu} \) and then determine the explicit form of the corresponding operators \( P_\mu \) and \( J_{\mu\nu} \).

We will seek for representations of the tensor \( W_{\mu\nu} \) in the basis of eigenfunctions \( |\vec{p}, \lambda\rangle \) of the commuting operators \( P_\mu \): 

\[ P_\mu |\vec{p}, \lambda\rangle = p_\mu |\vec{p}, \lambda\rangle, \quad \vec{p} = (p_0, p_1, p_2, p_3, p_4), \]  
where the symbol \( \lambda \) denotes eigenvalues of commuting operators forming a complete set together with \( P_\mu \). In the basis \( |\vec{p}, \lambda\rangle \) the commutation relations for \( W_{\mu\nu} \) take the form

\[ [W_{\mu\nu}, W_{\nu\beta}] = \left( \frac{1}{2} g_{\mu\nu} e_{\beta\rho\lambda\sigma} + \frac{1}{2} g_{\nu\lambda} e_{\nu\rho\sigma\beta} - \frac{1}{2} g_{\nu\sigma} e_{\nu\rho\beta\lambda} - \frac{1}{2} g_{\nu\beta} e_{\nu\rho\lambda\sigma} \right) W_{\rho\sigma}, \]  
\[ [W_{\mu\nu}, p_\rho] = 0, \quad W_{\mu\nu} p^\nu = 0, \]  
where \( p_\rho \) are arbitrary real numbers.

Relations (24.7) define a Lie algebra \( A_\vec{p} \) whose structure constants depend on \( \vec{p} \). Besides it will be demonstrated that for all \( p_\mu \) satisfying one of the following conditions

\[ p_\mu p^\mu > 0, \]  

299
Symmetries of Equations of Quantum Mechanics

\[ p_\mu p^\nu = 0, \quad (24.10) \]

\[ p_\mu p^\nu < 0, \quad (24.11) \]

the algebras \( A_\mu \) turn out to be isomorphic.

To describe constructively the representations of the algebra \( A_\mu \) we transform the commutation relations (24.7) to such a form that the structure constants will depend on the single parameter \( c_1 = p_\mu p^\mu \). For this purpose we will use the linear transformation

\[ p_\mu \rightarrow p_\mu - R_\mu \nu p^\nu, \quad W_{\mu \nu} \rightarrow W_{\mu \nu} - R_{\mu \nu \rho} R_{\rho \sigma} W^{\lambda \sigma} \]

(24.12)

where

\[ R_{\mu \nu} \rightarrow R_{\mu \nu} + \Theta_{\mu \nu},\quad \Theta_{\mu \nu} = \frac{\theta_{\mu \nu}}{2p} - \frac{\theta_{\rho \nu} \theta_{\rho \mu}}{2p(2p+p_1^2+p_2^2+p_3^2+p_4^2)} \]

(24.13)

This transformation is invertible, moreover

\[ R_{\mu \nu}^{-1} \rightarrow R_{\mu \nu}^{-1}(-\Theta_{\mu \nu}) \]

(24.14)

and it is nothing but the rotation of a reference frame, corresponding to the transition to the basis in which

\[ p_0 \rightarrow p_0,\quad p_i \rightarrow p_i,\quad p_0 \rightarrow p_i,\quad p_i \rightarrow p_0/2. \]

(24.15)

Using (24.7), (24.12) and (24.15) it is not difficult to find the commutation relations for \( W_{\mu \nu} \). These relations can be simplified essentially by passing to the new variables \( W'_{\mu \nu} = (\eta_a, \lambda_a) \). Using the fact that only six out of ten components of \( W'_{\mu \nu} \) are linearly independent (see (24.8)), it is convenient to set

\[ W'_{0 \nu} = p_0/2 \eta_a, \quad W'_{0 \nu} = p_0/2 \rho \eta_a, \]

\[ W'_{a \nu} = e_{abc} [\xi_\alpha + \frac{1}{4} p_0 (\eta_a - \eta_b)], \quad \xi_\alpha = \frac{1}{2} (\lambda_a - \lambda_c), \]

(24.16)

where

\[ \eta = \eta_1 + \eta_2 + \eta_3, \quad \lambda_a = \lambda_1 + \lambda_2 + \lambda_3, \quad a, b = 1, 2, 3. \]

(24.17)

Substituting (24.12), (24.15) and (24.16) into (24.7) we come to the following algebra of the operators \( \eta_a \) and \( \lambda_a \):
Chapter 5. Generalized Poincaré Groups

The relations (24.16) define the isomorphism of the Lie algebra formed by the components of $W'_{\mu\nu}$ to the algebra of the operators $\lambda_a, \eta_a$ which are characterized by the commutation relations (24.18). The structure constants of the algebra (24.18) depend on a single parameter $c_1$ (i.e., on the eigenvalue of the Casimir operator $C_1$). In the space of an IR of the algebra $AP(1,4)$ this eigenvalue is fixed and relations (24.18) determine the Lie algebra.

If we choose a representation of the operators $\eta_a$ and $\lambda_a$ then formulae (24.16) and (24.12) determine the corresponding representation of the tensor $W'_{\mu\nu}$ in the frame of reference where components $p_\mu$ have the form (24.15). To obtain this tensor in an arbitrary frame of reference we use the transformation

$$W'_{\mu\nu} \rightarrow W'_{\mu\nu} \cdot UR_{\mu\rho}R_{\nu\nu}W'_{\rho\nu}U^\dagger$$

where $U$ is the unitary operator

$$U = \exp \left( \frac{\eta_a p_\mu}{\bar{\rho}} \arctan \frac{\bar{\rho}}{p_1 + p_2 + p_3 + p_4} \right), \quad k=1,2,3,4, \quad \eta_i = -\eta_i,$$

$$\bar{\rho} = \sqrt{\left(p_1 - p_2\right)^2 + \left(p_2 - p_3\right)^2 + \left(p_3 - p_4\right)^2 + \left(p_4 - p_1\right)^2},$$

and $R_{\mu\nu}$ are given in (24.13) and (24.14). We obtain

$$W_{04} = \Sigma_{ij} p_i, \quad W_{ij} = p_0 \Sigma_{ij} + \frac{1}{p_0^2} \left[ p_i (p_j W_{0i} - p_j W_{0i}) - p_j (p_i W_{0j} - p_i W_{0j}) \right],$$

where

$$\Sigma_{ab} = \epsilon_{abc} \eta_c, \quad \Sigma_{ia} = -\eta_a, \quad \lambda_{ab} = \epsilon_{abc} \lambda_c, \quad \lambda_{ia} = \lambda_a.$$ (24.22)

Formulae (24.21) and (24.22) give a representation of the tensor $W'_{\mu\nu}$ in terms of the operators $\lambda_a$ and $\eta_a$ satisfying the algebra (24.18). To find nonequivalent realizations of $W'_{\mu\nu}$ it is sufficient to describe nonequivalent representations of this algebra which turn out to be equivalent to the well-known Lie algebras of the group of orthogonal 4×4 matrices $O(4)$, Lorentz group $O(1,3)$ and Euclidean group $E(3)$, depending on the value of the parameter $c_1$.

**Theorem 24.1.** The Lie algebra (24.18) is isomorphic to the algebra $AO(4)$ if $c_1 > 0$, $AE(3)$ if $c_1 = 0$ and $AO(1,3)$ if $c_1 < 0$.

**Proof.** The formulated isomorphism can be established explicitly by the relations

$$\eta_a = \Sigma_{a}, \quad \lambda_a = -m \Sigma_{a}, \quad \text{if} \quad c_1 = m^2 > 0;$$

(24.23a)
Symmetries of Equations of Quantum Mechanics

\[ \eta_a = \Sigma_a, \quad \lambda_a = T_a, \quad \text{if} \quad c_1 = 0; \]
\[ \eta_a = \Sigma_a, \quad \lambda_a = k \xi_a, \quad \text{if} \quad c_1 = -k^2 < 0, \]  
where \( \{ \Sigma_a, \Sigma_a' \}, \{ \Sigma_a, T_a \} \) and \( \{ \Sigma_a, \xi_a \} \) are basis elements of the algebras \( AO(4), AE(3) \) and \( AO(1,3) \), which satisfy the commutation relations

\[ [\Sigma_a, \Sigma_b'] = i\epsilon_{abc} \Sigma_c, \]
\[ [\Sigma_a, \Sigma_b] = 0, \quad [T_a, T_b] = 0, \]
\[ [\Sigma_a, \xi_b] = i\epsilon_{abc} \xi_c, \quad [\xi_a, \xi_b] = -i\epsilon_{abc} \Sigma_c. \]

It is not difficult to make sure that the relations (24.24) follow from (24.18), (24.23), and vice versa (24.18) is a consequence of (24.23) and (24.24).

We restrict ourselves by IRs of the algebras \( AO(4), AE(3) \) and \( AO(1,3) \) which have been well studied and described (see, e.g., [20], [197]). The necessary information about these IRs is given in Sections 4 and 12.

24.4. The Basis of an IR.

To describe constructively the nonequivalent tensors \( W_{\mu \nu} \) and the corresponding basis elements of the algebra \( AP(1,3) \) it is necessary to define a basis in a space of the IR. It is convenient to choose such a basis in a form of eigenfunctions of a complete set of commuting operators.

We choose the following set of commuting operators

\[ P_0, \quad P_1, \quad P_2, \quad P_3, \quad P_4, \quad S_3, \quad \tilde{S}_3, \quad C, \]

where

\[ \tilde{S}_a = \epsilon_{abc} P_b W_0 - P_4 W_{0a} + P_a W_{04}, \]

and \( C \) denotes all the Casimir operators existing in the representation considered, i.e., \( C_1, C_2, C_3 \) (24.5) and additional invariant operators for the different classes of IRs.

It follows from (24.21) that the operators (24.26) are expressed via \( \Sigma_a \) and \( \rho \):

\[ \tilde{S}_a = \rho^2 \Sigma_a. \]

We will denote the common eigenfunctions of the commuting operators (24.25) by the symbol \( |c, \rho, l, m\rangle \) where \( c = (c_1, c_2, c_3, \ldots) \) are eigenvalues of the Casimir operators, \( \rho = (p_0, p_1, p_2, p_3, p_4) \) are eigenvalues of the operators \( P_\mu \), \( l \) and \( m \) characterize eigenvalues of the operators \( S_3 \) and \( \tilde{S}_3 \tilde{S}_a \) so that
Chapter 5. Generalized Poincaré Groups

were \( l \) are positive integers or half integers, \( m = \pm l, \pm l+1, \ldots \).

In a space of an IR of the algebra \( AP(1,4) \) \( c_a \) are fixed and possible values of \( p_\mu \) are restricted by the relation \( p_\mu^2 = c_1 \).

We impose here the following normalization conditions for \( c_1 \neq 0 \)

\[
\langle c,\tilde{p},l,m|c,\tilde{p},l,m\rangle = 2\hat{p}_0 \delta(p_4-p_4') \delta(p-p') \delta_{ii} \delta_{mm}, \tag{24.29}
\]

where \( \hat{p}_0 = \sqrt{p^2 - c_1} \). If \( c_1 < 0 \) then it is convenient to require

\[
\langle c,\tilde{p},l,m|c,\tilde{p},l,m\rangle = 2\hat{p}_0 \delta(p_0-p_0') \delta(p-p') \delta_{ii} \delta_{mm}, \tag{24.30}
\]

where \( \hat{p}_0 = \sqrt{p^2 - c_1} \).

The orthonormalized basis defined by the relations (24.28) - (24.30) will be used below to determine an explicit form of the basis elements of the algebra \( AP(1,4) \).

24.5. The Explicit Form of the Basis Elements of the Algebra \( AP(1,4) \)

Let us determine a general form of the operators \( P_\mu, J_{\mu\nu} \) corresponding to the tensors \( W_{\mu\nu} \) described above. It is not difficult to make sure that without loss of generality we can set

\[
P_\mu = p_\mu, \quad J_{ij} = x_i p_j - x_j p_i + \Sigma_{ij}, \tag{24.31}
\]

\[
J_{0k} = x_0 p_k - x_k p_0 + \frac{1}{p^2} (\lambda_{x_0} p_k - p_0 \Sigma_{k} p_i),
\]

where \( p_\mu \) are independent variables satisfying the relation \( p_\mu p^\mu = c_1 \), \( x_\mu = -i \partial/\partial p^\mu \), \( \lambda_{x_\mu} \) and \( \Sigma_{ij} \) are the matrices realizing an IR of the algebra (24.18), (24.22). The operators (24.31) satisfy the commutation relations (24.1) and therefore form a representation of the algebra \( AP(1,4) \). On the other hand we obtain the representation (24.21) for the tensor \( W_{\mu\nu} \) after the substitution (24.31) into (24.3). Finally substituting (24.21) and (24.31) into (24.5) we come to the following expressions for the Casimir operators

\[
C_1 = p_\mu p^\mu = c_1, \quad C_2 = -m^2 (\Sigma^2 + \Sigma'^2), \quad C_3 = m \Sigma \cdot \Sigma', \tag{24.32}
\]

According to (24.23) these operators reduce to one of the following forms:

\[
C_1 = m^2 > 0, \quad C_2 = -m^2 (\Sigma^2 + \Sigma'^2), \quad C_3 = m \Sigma \cdot \Sigma', \tag{24.33a}
\]

\[
C_1 = 0, \quad C_2 = -T^2, \quad C_3 = T \cdot \Sigma, \tag{24.33b}
\]

303
We can see that the basic Casimir operators of the algebra \( AP(1,4) \) in the realization (24.31) reduce to the invariant operators of the groups \( O(4), E(3) \) and \( O(1,3) \) [135]. Thus any possible set of the eigenvalues of the operators \( C_1, C_2 \) and \( C_3 \) can be assigned to the representation of the algebra \( AP(1,4) \) given by the operators (24.31).

The basis elements of the algebra \( AP(1,4) \) for any class of IRs can be represented in the form (24.31). The matrices \( \lambda_{ij} \) and \( \Sigma_{ij} \) (included into (24.31)) satisfy the algebra (24.18), (24.22) which is isomorphic to the algebras \( AO(4), AE(3) \) or \( AO(1,3) \) for timelike, lightlike and spacelike vectors \( p_\mu \).

The realization (24.31) is distinguished by the simple form which is common for all the classes of the IRs. So this realization differs favorable from another known realizations of the algebra \( AP(1,4) \).

24.6. Connection with Other Realizations

Let us consider IRs of the generalized Poincaré algebras in more detail and discuss connections of the realization (24.31) with other representations of this algebra.

a) \( P_\mu P_\nu c_\nu = m^2 > 0 \). In this case the algebra of the matrices \( \Sigma_{ij} \) and \( \lambda_{ij} \) is isomorphic to the algebra \( AO(4) \). The IRs of the last are finite dimensional and can be realized by the square matrices of dimension \( (2j+1)(2\tau+1) \times (2j+1)(2\tau+1) \), where \( j \) and \( \tau \) are positive integers or half of integers, which are connected with the eigenvalues of the Casimir operators \( C_1 \) and \( C_3 \) as follows:

\[
C_1 = -m^2 l_0^2 + l_1^2 - 2m^2 j(j+1) + \tau(\tau+1),
\]
\[
C_3 = 2ml_0 l_1 = 2m[j(j+1)] - \tau(\tau+1).
\]

The explicit form of the matrices \( \Sigma_{ij} \), \( \lambda_{ij} \) in the basis \( |c, \tilde{p}, l, m\rangle \) is given by the relations

\[
\Sigma_{ab} |c, \tilde{p}, s, \mu\rangle = e_{abc} \Sigma_{ac} |c, \tilde{p}, s, \mu\rangle,
\]
\[
\lambda_{ab} |c, \tilde{p}, s, \mu\rangle = e_{abc} \lambda_{ac} |c, \tilde{p}, s, \mu\rangle - \epsilon_{abc} S_0 |c, \tilde{p}, s, \mu\rangle,
\]

where \( S_{\alpha\beta} \) are the matrices realizing the representation \( D(l_\alpha l_\beta) \) of the algebra \( AO(1,3) \) and acting on the vector \( |c, \tilde{p}, l, m\rangle \) according to (4.64).

The considered representations have the additional (energy sign) Casimir operator:

\[
C_4 = \frac{P_0}{p_0} = \frac{P_0}{\tilde{p}_0} = \tilde{p}_0 = E - \sqrt{p^2 + m^2}.
\]

In the space of an IR \( C_4 = \pm 1 \) and \( p_0 = \mp E \).

Using (24.23), (24.36) one can represent the operators (24.31) in the following
Chapter 5. Generalized Poincaré Groups

form:

\[ P_0^c = \varepsilon E, \quad P_i^c = p_i, \quad J_{ki}^c = x_k p_i - x_i p_k + \Sigma_{kl}, \quad J_{0k}^c = x_0 p_k - x_k p_0 + \frac{1}{p^2} (m \lambda_{kl} p_i - \varepsilon E \Sigma_{kl} p_i), \]

(24.37)

where \( \lambda_{kl}, \Sigma_{kl} \) are the matrices (24.35), (4.64).

The operators (24.37) are Hermitian with respect to the scalar product

\[ (\Psi_1, \Psi_2) \sum_{lm} \int_{-\infty}^{\infty} \frac{d^4p}{2E} \Psi_1^\dagger(p, p_4, l, m) \Psi_2(p, p_4, l, m), \]  

(24.38)

where \( l = l_0, l_0 + 1, \ldots, l_1 - 1, m = -l, -l + 1, \ldots, l \).

Using the unitary transformation \( P_\mu \rightarrow P_\mu = UP_\mu U^\dagger, J_{\mu\nu} \rightarrow J_{\mu\nu} = UJ_{\mu\nu} U^\dagger \), where

\[ U = \exp \left[ \frac{i}{4} \int p \sigma_{ab}(\Sigma_{ab} - \varepsilon \lambda_{ab} p_4) \arctan \frac{2|p_4|}{p_0^2 - p_4^2} \right], \]

(24.39)

the generators (24.37) reduce to the form found in [135,136]:

\[ P_0^c = \varepsilon E, \quad P_i^c = p_i^c, \quad J_{ki}^c = x_k p_i - x_i p_k + S_{kl}, \quad J_{0k}^c = x_0 p_k - \varepsilon E x_k + \frac{S_{kl} p_l}{E + m}, \]

(24.40)

where \( S_{ab} = \Sigma_{ab} \) and \( S_{a0} = \lambda_{a0} \) are basis elements of the algebra \( AO(4) \). The representation (24.40) is a natural generalization of the canonical Shirokov-Foldy representation (4.50) and we will call it therefore canonical for the sake of brevity.

b) \( P_\mu P_\mu = \varepsilon_1 = 0 \). Such representations also have the additional Casimir operator (24.36) (with \( m = 0 \)). Thus an explicit realization of \( P_\mu, J_{\mu\nu} \) can be chosen in the form (see (24.23), (24.25), (24.28)):

\[ P_0^c = \varepsilon E, \quad P_i^c = p_i^c, \quad J_{ki}^c = x_k p_i - x_i p_k + \Sigma_{kl}, \quad J_{0k}^c = x_0 p_k - \varepsilon E x_k + (T_{kl} p_l - \varepsilon p \Sigma_{kl} p_l)/p^2, \]

(24.41)

where \( \Sigma_{ab} = \sigma_{ab}, \Sigma_{a0} = \lambda_{a0}, T_{ab} = \sigma_{ab} T_a, T_{a0} = \sigma_{a0} T_a, T_a, \Sigma_a \) and \( T_a \) are the matrices satisfying the algebra (24.24).

The algebra (24.24) for \( \Sigma_a, T_a \) is isomorphic to the Lie algebra of Euclidean group \( E(3) \) and has two Casimir operators given in (24.33b). The eigenvalues of these operators that corresponding to unitary representations of the group \( E(3) \) are equal to

\[ -c_2 = r^2 \geq 0, \quad c_2 = l_0 r, \quad l_0 = 0, 1, \frac{3}{2}, 1, \frac{3}{2}, \ldots. \]  

(24.42)

Representations of the algebra \(\epsilon E(3) \) qualitatively differ for the cases \( r^2 = 0 \) and \( r^2 > 0 \). If \( r^2 = 0 \) then \( T_a = 0, T_{a0} = 0 \) and the algebra \(\epsilon E(3) \) reduces to \( AO(3) \). The last algebra generates the additional Casimir operator

305
whose eigenvalues equal \( s(s+1) \), \( s \) are positive integers or half integers. The corresponding representations are of finite dimensions and realized by the \((2s+1)\times(2s+1)\) matrices given in (4.31) (where \( S_\alpha = \Sigma_\alpha \)). Besides this the basis elements of the algebra \( AP(1,4) \) have the form (24.41) where \( T_\alpha \equiv 0 \) [135,136]. These operators are Hermitian with respect to the scalar product

\[
(\Psi_1,\Psi_2) = \sum_\mu \int \frac{d^4p}{2p} \Psi_1^*(p_\mu p_\mu) \Psi_2(p_\mu p_\mu),
\]

where \( \mu = -s,-s+1,...,s \) are the eigenvalues of the matrix \( \Sigma_\alpha \), \( s \) is an integer or a half integer which determines an eigenvalue of the Casimir operator (24.43).

If \( -c_2=r^2>0 \) then unitary representations of the group \( E(3) \) are of infinite dimensions. The IRs are characterized by the numbers \( r, l_0 \) satisfying the conditions (24.42) and are denoted below by \( D(l_0, r) \). The corresponding matrices \( \Sigma_\alpha \) and \( T_\alpha \) have the following form:

\[
\Sigma_\alpha = S_\alpha, \quad T_\alpha = \eta_\alpha, \tag{24.45}
\]

where \( S_\alpha \) and \( \eta_\alpha \) are given in (12.38).

Thus the basis elements of the IRs of the algebra \( AP(1,4) \) which corresponds to \( c_1=0, c_2 \neq 0 \) can be chosen in the form (24.41) where \( T_\alpha \) and \( \Sigma_\alpha \) are the operators (24.45). In contrast with the case \( c_2=0 \) such representations are infinite dimensional in respect with the index \( l \) since the unitary IRs of the group \( E(3) \) are infinite dimensional.

c) \( P_\mu, P_\mu = c_1 k^2<0 \). In this case the matrices \( \lambda_{ij} \) and \( \Sigma_{ij} \) from (24.31) form a Lie algebra isomorphic to \( AO(1,3) \). The Hermitian IRs of this algebra are infinite dimensional.

Using the isomorphism (24.23) we obtain from (24.31) the following explicit expressions of the basis elements of the algebra \( AP(1,4) \):

\[
P_\mu \cdot p_\nu, \quad J_{kl} = x_k p_l - x_l p_k + \Sigma_{kl}, \quad J_{0l} = x_l p_0 - x_0 p_l - (\xi_{0l} p_l + \xi_{l0} p_l + \Sigma_{0l} p_l)p_0^{-2},
\]

where \( \xi_{ab} = e_{abc} S_{bc}, \xi_{ab} = S_{ab}, \Sigma_{ab} = S_{ab}, \Sigma_{0a} = e_{abc} S_{bc}/2, S_{ab} \) and \( S_{0a} \) are the basis elements of the algebra \( AO(1,3) \), satisfying relations (2.18b). The explicit form of the matrices \( S_{ab} \) and \( S_{0a} \), which belonging to the IR \( D(l_0, l_1) \) is given by formulae (4.64)-(4.66). Besides that for the Hermitian infinite-dimensional representations the parameter \( l_1 \) in (4.63) which defines the Casimir operators eigenvalues (4.61) is purely imaginary.
25. REPRESENTATIONS OF THE ALGEBRA $AP(1,4)$
IN THE POINCARÉ-BASIS

25.1. Subgroup Structure of the Group $P(1,4)$

Investigation of a subgroup structure of the basic symmetry groups is one of the main problems of a group-theoretic analysis of equations of quantum mechanics. This problem is of great interest both for the group theory and for many applications, e.g., for finding the exact solutions of linear and nonlinear equations.

The subgroup structure of the basic groups $E(1,3)$, $P(1,3)$, $O(1,3)$ has been described in papers [415,396]. The subgroups of the group $P(1,4)$ were studied in [92,93]. It turns out that this group includes more than 400 connected subgroups, the main ones of which are:

a) the Euclidean group $E(4)$ in the four-dimensional space;

b) the Poincaré group $P(1,3)$;

c) the Galilei group $G(1,3)$.

In other words it is the group $P(1,4)$ which unites naturally the groups of motions of the relativistic and nonrelativistic quantum mechanics and the symmetry group of the Euclidean field theory.

Generally speaking IRs of the group $P(1,4)$ are reducible with respect to its subgroups. So the problem of reduction of a representation of the generalized Poincaré group by its subgroups evokes a great interest.

We mean that the reduction of the group $P(1,4)$ by its subgroups is a transformation of the group generators (basis elements of the algebra $AP(1,4)$) to such a basis where the Casimir operators of the corresponding subgroups (subalgebras) are diagonal. Transition to such a basis allows to answer the questions what sort of representations of a subalgebra are included into given IR of the algebra $AP(1,4)$ and what is the multiplicity of these representations and also allows to find an explicit form of the basis elements of the algebra $AP(1,4)$ in such a realization where the Casimir operators of the considered subalgebras are diagonal.

25.2. Poincaré-Basis

The operators $P_\mu$, $J_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ form a subalgebra of the algebra $AP(1,4)$. This subalgebra is isomorphic to the Lie algebra of the Poincaré group, i.e., to the algebra $AP(1,3)$. If $P_\mu$ and $J_{\mu\nu}$ belong to an IR of the algebra $AP(1,4)$ then the representation of the subalgebra $AP(1,3)$ is reducible because the Casimir operators (4.6)
are not multiples of the unit operator. Here \( W_\mu \) is the Lubanski-Pauli vector (4.2); according to (24.3) \( W_\mu = W_{4\mu} \).

Since the subalgebra \( AP(1,3) \) evokes the greatest interest in the algebra \( AP(1,4) \) it is desirable to find such realizations of this algebra that representations of the subalgebra \( AP(1,3) \) reduce to direct sums of the IRs. The basis of the IRs having the above-mentioned property will be called the Poincaré-basis (or \( P(1,3) \)-basis).

Let us give the more exact definitions.

**DEFINITION 25.1.** We say that an IR of the algebra \( AP(1,4) \) is defined in the Poincaré-basis if

1) the Casimir operators of the subalgebra \( AP(1,4) \) are diagonal;
2) the space of an IR of the algebra \( AP(1,4) \) is decomposed into a direct sum of Hilbert spaces invariant under IRs of the algebra \( AP(1,3) \).

In the following subsections we define constructively the Poincaré-basis and find the explicit form of the corresponding operators \( P_\mu, J_{\mu\nu} \) for all the classes of IRs.

### 25.3. Reduction \( P(1,4) \rightarrow P(1,3) \) of IRs of Class \( I \)

Consider IRs of the algebra \( AP(1,4) \) corresponding to positive values of the Casimir operator \( C_1 \): \( P_\mu P^\mu = \kappa^2 > 0 \). We start from the canonical realizations of such representations given by formulae (24.40).

The operators (24.40) are defined in the basis \( |\kappa, l_0, l_1, e; p, m, s, s_3> \) formed by the eigenvectors of the Casimir operators (24.33a), (24.36) and the commuting operators \( P_1, P_2, P_3, P_4, \Sigma^i \) and \( \Sigma_3 \). We refer to that basis as the canonical one.

Evidently there exist many other bases for IRs of the algebra \( AP(1,4) \). Among them is the Poincaré basis which is very useful for physical applications. It can be defined as a set of eigenfunctions of the operators \( P_\mu, J_{\mu\nu} \) for all the classes of IRs.

We normalize the basis vectors according to

\[
\langle \kappa, l_0, l_1; e; p, m, s, s_3 | \kappa, l_0, l_1; e; p', m', s', s'_3 \rangle = 2E \delta(m - m') \delta(p - p') \delta_{s s'}, \delta_{s_3 s'_3}.
\]

This corresponds to the following scalar product
Chapter 5. Generalized Poincaré Groups

\[ (\varphi_1, \varphi_2) = \sum_s \int_{-\infty}^{\infty} \int \frac{d^3 p}{2E} \varphi_1(p, m, s, s_3) \varphi_2(p, m, s, s_3), \]  

(25.3)

where \( \varphi_1(p, m, s, s_3) \) and \( \varphi_2(p, m, s, s_3) \) are vectors from a space of an IR of the algebra \( \text{AP}(1,4) \) defined in the Poincaré basis. It is not difficult to see that the bilinear form (25.3) is nothing but a sum over the discrete variable \( s \) and the integral with respect to \( m \) of the scalar products defined in orthogonal subspaces of the IRs of the algebra \( \text{AP}(1,3) \) corresponding to the eigenvalues \( m^2 \) and \( -m^2 s(s+1) \) of the Casimir operators (4.6).

Our task is to determine the possible eigenvalues of the operators (25.1), to find an explicit form of the operators \( P_{\mu}, J_{\mu\nu} \) in the Poincaré-basis and to find the unitary operator connecting the basises \( |\kappa, l_0, l_1; p, p_4, s, \lambda> \) and \( |\kappa, l_0, l_1, e; p, m, s, s_3> \).

We will see further on that the operators \( P_{\mu}, J_{\mu\nu} \) of (24.40) with \( \mu, \nu \neq 4 \) (these operators form the subalgebra \( \text{AP}(1,3) \)) can be transformed to the canonical form (4.50), where \( m = (\kappa^2 + p_4^2)^{1/2} \) and \( S_{ab} \) are matrices belonging to the IR \( D(l_0, l_1) \) of the algebra \( \text{AO}(4) \). By the reduction of the representation \( D(l_0, l_1) \) by the algebra \( \text{AO}(3) \) we obtain a direct sum of the representations \( D(s), l_0 \leq s \leq l_1 - 1 \). It follows from the above that a Hilbert space \( H \) of the IR \( D^\prime(m, s) \) of the algebra \( \text{AP}(1,3) \), moreover

\[ \kappa^2 \leq m^2 < \infty, \quad l_0 \leq s \leq l_1 - 1. \]  

(25.4)

We will search for an operator \( V \) connecting the canonical and Poincaré bases in the form

\[ V = R \exp \left( \frac{S_{ab} p_a}{|p^2|} \theta \right), \]  

(25.5)

where \( R \) and \( \theta \) are unknown functions of \( |p^2| \) and \( p_a \). \( S_{ab} \) are matrices belonging to a representation of the algebra \( \text{AO}(4) \) (they are included into the generators (24.40)).

The operators (25.5) can be used to define a class of IRs equivalent to (24.40):

\[ P_{\mu}', \quad V P_{\mu}' V^{-1}, \quad J_{\mu\nu}', \quad V J_{\mu\nu}' V^{-1}. \]  

(25.6)

By definition the operator (25.5) has to diagonalize the Casimir operators (25.1). We impose the stronger conditions on \( V \), requiring that \( P_{\mu}', J_{\mu\nu}' \) \( (\mu, \nu \neq 4) \) reduce to the canonical Shirokov-Foldy form (4.50). Substituting (24.40), (4.50) into (25.6) we come to the following equations for \( V \)

\[ VP_0 V^{-1} = \sqrt{p^2 + m^2}, \quad VP_a V^{-1} = p_a \]  

\[ VJ_{ab} V^{-1} = J_{ab} \equiv x_a p_b - x_b p_a + S_{ab}, \]  

(25.7a)
The conditions (25.7a) for the operator (25.5) are obviously satisfied. As to (25.7b)

\[ V_{J_0} V^{-1} = V \left( \frac{tp_a - \epsilon E x_a}{E + \kappa} - \frac{S_{ab} p_b + S_{a4} p_4}{E + \kappa} \right) V^{-1} = \frac{tp_a - \epsilon E x_a}{E + m}, \]

(25.7b)

\[ m = \sqrt{\kappa^2 + p_4^2}, \quad p = |p| = \sqrt{p_1^2 + p_2^2 + p_3^2}. \]

The conditions (25.7a) for the operator (25.5) are obviously satisfied. As to the equation (25.7b), we use the Campbell-Hausdorf formula (13.16) and obtain for the operators in the l.h.s. of it:

\[
V_{x_4} V^{-1} \approx x_4 + i \frac{\partial R}{\partial p_a} R^{-1} \left( \frac{p_a S_{ab} p_b}{p^2} \left( \frac{\partial \theta}{\partial p} - \frac{\sin \theta}{p} \right) \right) + \frac{S_{ab} p_b}{p^2} \left( 1 - \cos \theta \right) + \frac{1}{p} S_{a4} \sin \theta,
\]

(25.8)

\[
V S_{a4} V^{-1} = S_{a4} \cos \theta \left( \frac{1}{p} S_{a4} p_b - p S_{a4} \right) \sin \theta,
\]

\[
V S_{ab} p_b V^{-1} = S_{ab} p_b \cos \theta + \left( \frac{1}{p} p_a S_{ab} p_b - p S_{a4} \right) \sin \theta.
\]

Substituting (25.8) into (25.7b) and equating coefficients of the linearly independent matrices we come to the following equations for \( \theta \) and \( R \):

\[
\frac{\partial R}{\partial p_a} = 0, \quad E(E + \kappa) \sin \theta + pp_{x} \cos \theta + p^2 \sin \theta = 0,
\]

\[
E(E + \kappa) \left( 1 - \cos \theta \right) - pp_{x} \sin \theta + p^2 \cos \theta = \frac{p^2(E + \kappa)}{E + m},
\]

\[
E(E + \kappa) \left( \frac{\partial \theta}{\partial p} - \frac{\sin \theta}{p} \right) = p_4 \left( 1 - \cos \theta \right) + p \sin \theta = 0.
\]

The general solutions of these equations are

\[ R \approx R(p_4), \quad \theta \approx 2 \arctan \frac{pp_4}{(E - m)(m + \kappa)}. \]

(25.9)

Setting \( R = (m/p_4)^{1/2} \) (which corresponds to the scalar product (25.3)) we obtain

\[ V \approx \sqrt{\frac{m}{p_4}} \exp \left\{ \frac{2i}{p_4} S_{a4} p_a \arctan \frac{pp_4}{(E - m)(m + \kappa)} \right\}, \]

(25.10)

The operator (25.10) transforms \( P_{\mu} \), \( J_{\mu
\nu} \) (\( \mu, \nu \neq 4 \)) to the canonical Shirokov-Foldy form (4.50). The explicit form of the remaining generators (i.e., \( J'_{44} \) and \( J'_{a4} \)) can be easily found using (25.6), (25.8), (25.9) and the relation

\[ V_{x_4} V^{-1} = x_4 - \frac{i \kappa}{2 m p_4} \left( \frac{S_{a4} p_a (\kappa E - p_4^2)}{Em^2(E + \kappa)} \right) \]

(25.11)

with the subsequent change \( p_4 \to e'(m^2 - \kappa^2), \quad e' = p_4/|p_4| = \pm 1. \)
The results given above allow to formulate the following assertion [336]:

**Theorem 25.1.** The space $H$ of the IR $D'(\kappa, l_0, l_1)$ of the algebra $AP(1,4)$ with $P^\alpha P^\alpha > 0$ reduces to the direct sum of the subspaces corresponding to the IRs $D'(m, s)$ of the algebra $AP(1,3)$ with the eigenvalues of the Casimir operators $P^\mu P^\mu$ and $W^\mu W^\mu$ given in (25.4). The operator connecting the canonical and Poincaré bases is given in (25.10), the basis elements of the algebra $AP(1,4)$ in the Poincaré-basis are

\[ P_0' = \sqrt{p^2 + m^2}, \quad P_4' = \sqrt{E^2 - \kappa^2}, \quad P_a' = p_a, \]

\[ J_{ab}' = x_a p_b - x_b p_a + S_{ab}, \quad J_{0a}' = x_0 p_a - i P_0' \frac{\partial}{\partial p_a} - \frac{\kappa}{E + m} S_{ab} p_b, \]

\[ J_{04}' = 2 x_0 p_4 - i P_0' \left[ E - \frac{\kappa^2}{m^2} \frac{\partial}{\partial m} \right] - \frac{\kappa}{m} S_{4a} p_a, \]

\[ J_{0i}' = \frac{i}{2} p_i \left[ E - \frac{\kappa^2}{m^2} \frac{\partial}{\partial m} \right] - i P_0' \frac{\partial}{\partial p_i} + \frac{\kappa P_{ab} p_b}{m(E + m)} + P_i' \frac{S_{ab} p_b}{m(E + m)} + \frac{\kappa}{m} S_{ai}. \]

The operators (25.12) are Hermitian with respect to the scalar product (25.3). They realize a representation of the algebra $AP(1,4)$ which corresponds to diagonal Casimir operators of the subalgebra $AP(1,3)$. 

**25.4. Reduction $P(1,4) \rightarrow P(1,2)$**

In some physical problems (where the Poincaré-invariance is broken but the symmetry under the subgroup $P(1,2)$ is preserved) it is more convenient to use the $P(1,2)$-basis in which the Casimir operators of the algebra $AP(1,2)$ are diagonal. So it is interesting to continue the reduction of the algebra $AP(1,4)$ up to the subalgebra $AP(1,2)$. That reduction can be made by the transition to a basis where the operators $P_0, J_{12}, P_\alpha, J_{01}, J_{02}$ have the following (canonical) form:

\[ P_0 = \sqrt{p^2 + p_1^2 + p_2^2 + m_1^2}, \quad m_1^2 - p_2^2 + p_3^2, \]

\[ P_\alpha = p_\alpha, \quad \alpha = 1, 2, \quad J_{12} = i \left[ p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2} \right], \quad S_{12}, \]

\[ J_{01} = x_0 p - i P_0 \frac{\partial}{\partial p_1} - \frac{S_{12} p_2}{E + m_1}, \quad J_{02} = x_0 p - i P_0 \frac{\partial}{\partial p_2} + \frac{S_{12} p_1}{E + m_1}. \]

Here $S_{12}$ is a matrix belonging to the IR $D(l_0, l_1)$ of the algebra $AO(4)$. To find an explicit form of the remaining basis elements of the algebra $AP(1,4)$ it is sufficient to

311
find an operator $\tilde{V}$ satisfying the conditions

$$
\tilde{V} P'_0 = e E, \quad \tilde{V} P'_\alpha = p'_\alpha, \quad \tilde{V} J'_{12} = J'_{12},
$$

$$
\tilde{V} = x_0 p'_\alpha - i e E \frac{\partial}{\partial p'_\alpha} - e \frac{S_{ab} p'_\beta}{E + m}.
$$

where $P'_\alpha, P'_\alpha, J'_{12}, J'_{12}$ are the generators (25.12). Representing $\tilde{V}$ in the form

$$
\tilde{V} = R \exp \left( \frac{S_{3\alpha} p'_\alpha}{|p'_3|} \right), \quad |p'_3| = \sqrt{|p'_1|^2 + |p'_2|^2},
$$

where $R$ and $\tilde{\theta}$ are functions of $p'_3, |p'_3|, m$, and making the calculations analogous to ones in (25.8)-(25.10) we obtain

$$
\tilde{\theta} = \frac{2 \arctan \left( \frac{p'_1 |p'_3|}{(E + m)(m + m'_1)} \right)}{p'_3}.
$$

This choice leads to the following scalar product

$$
(\phi_1, \phi_2) = \int \int \frac{d^2 m}{2 m} \frac{d^2 m}{2 m} \frac{d^3 p}{2 E} \phi_1 \phi_2.
$$

Using the transformation operator (25.15), (25.16) and changing the variables $p'_3 \rightarrow e''(m'_1^2 - m^2)^{\epsilon''},$ we can find the explicit form of the generators $J_{03}$ and $J_{43}$ in the $P(1,2)$-basis:

$$
J_{03} = x_0 m_1 \lambda_{m_1} - \frac{i}{2} e E \left[ \lambda_{m_1}, \frac{\partial}{\partial m_1} \right] \frac{m S_{3a} p'_a}{m'_1},
$$

$$
J_{43} = \frac{im}{2} \left[ \lambda_{m_1}, \lambda_{m_1}, \frac{\partial}{\partial m} \right] + \frac{\kappa m_1}{m_2^2} S_{43},
$$

where

$$
\lambda_{m_1} = e' \sqrt{1 - 2}m, \quad \lambda_{m_1} = e'' \sqrt{1 - m_1^2}, \quad e', e'' = \pm 1.
$$

The remaining generators of the group $P(1,4)$ can be obtained from (25.13), (25.18) using the commutation relations (25.1).

25.5. Reduction of IRs for the Case $e_1 = 0$

In this subsection we transform the IRs of the Classes II and III (corresponding to $P_\mu P^\mu = 0$) of the algebra $\mathcal{A}(1,4)$ into the Poincaré basis, starting from the realization (24.41).

First let us consider IRs corresponding to zero eigenvalues of the Casimir operator $C_4$ of (24.33), i.e., the IRs of the Class II. The corresponding operators $P_\alpha, J_{mn}$
have the following form according to (24.41) (with $T_\alpha \equiv 0$):

\[
\begin{align*}
  P_\alpha &= \epsilon E, \quad P_\mu = p_\mu, \quad E = \sqrt{p_\mu^2 + p_\nu^2 + p_\lambda^2 + p_\sigma^2}, \\
  J_{\alpha i} &= x_i p_\alpha - x_\alpha p_i + \Sigma_{\alpha i}, \quad J_{\alpha i} &= x_\alpha p_i - \epsilon p x_i - \epsilon \frac{\Sigma_{\alpha i} p_i}{p},
\end{align*}
\]

(25.19)

where $\Sigma_{\alpha i}$ form the IR $D(s 0)$ of the algebra $AO(4)$.

It is not difficult to make sure that the Casimir operators of the subalgebra $AP(1, 3)$ (this subalgebra is formed by the generators $P_\mu, J_{\mu \nu}$ ($\mu, \nu \neq 4$)) are diagonal in the representation (25.19):

\[
P_\mu P_\mu = \frac{1}{4} m^2, \quad -W_\mu W_\mu = \frac{1}{2} \Sigma_{\alpha} m^2 = m^2 s(s + 1),
\]

where $\Sigma_{\alpha} = \Sigma_{\alpha i} = \epsilon \Sigma_{\alpha i}$. Since that to obtain an IR in the Poincaré basis it is sufficient to make the substitution $p_4 \rightarrow \epsilon' m$, $\epsilon' = \pm 1$ in (25.19). As a result we can obtain the following realization

\[
\begin{align*}
  P_0 &= \epsilon E, \quad P_\mu = p_\mu, \quad P_4 = \epsilon' m, \quad E = \sqrt{p^2 + m^2}, \\
  J_{\alpha i} &= x_i p_\alpha - x_\alpha p_i + \Sigma_{\alpha i}, \quad J_{\alpha i} &= x_\alpha p_i - \epsilon p x_i - \epsilon \frac{\Sigma_{\alpha i} p_i}{E}, \quad J_{04} &= \epsilon' \frac{x_0 p_4 - \epsilon \sum_{\alpha} p_\alpha}{E}.
\end{align*}
\]

(25.20)

(25.21)

For any fixed value of $m$ the operators (25.20) form a basis of the IR $D'(m, s)$ of the algebra $AP(1, 3)$ in the realization (25.20) which had been considered in [136].

Using the transformation (25.5) where

\[
V = V' = \exp \left( \frac{\sum_{\alpha} p_\alpha}{|p|} \arctan \left( \frac{|p|}{m} \right) \right),
\]

(25.22)

we can reduce (25.20), (25.21) to the form (25.12) where $\kappa \equiv 0$ but $S_{\alpha i}$ are matrices belonging to the IR $D(s 0)$ of the algebra $AO(4)$. We conclude from this fact that the result, formulated in the Theorem 24.1 is valid for the IRs of Class II also (besides, for the last $\kappa = 0$, $l_0 = s$, $l_1 = s + 1$).

Just now we consider IRs of the Class III corresponding to $P_\mu P_\mu = 0$, $W_\mu W_\mu = 2 - s^2 < 0$. Basis elements of such a representation can be chosen in the form (24.41). With the help of the unitary transformation (25.5), where
\[ V' = \exp \left( -i \frac{T \cdot p}{\mathcal{P}} \right) V' \]

\( V' \) is the operator (25.22) with the corresponding infinite-component matrices \( \Sigma_a \), and the change \( p_4 \rightarrow \epsilon m \) we come to the following realization

\[ P_0' = \epsilon E, \quad P_2' = p_2, \quad P_4' = \epsilon m, \quad E = \sqrt{p^2 + m^2}, \]

\[ J'_{ab} = x_a p_b - x_b p_a + \Sigma'_{ab} \]

\[ J'_{0a} = x_0 p_a - \epsilon E x_a - \frac{\Sigma_{ab} p_b}{E + m} \]

\[ J'_{4a} = \epsilon \left( ip_a \frac{\partial}{\partial m} - i m \frac{\partial}{\partial p_a} + \frac{\Sigma_{ab} p_b}{E + m} + \frac{E}{m^2} T_a \right) + \frac{(2E + m)p_a T \cdot p}{m^2(E + m)}, \]

\[ J'_{04} = \epsilon \left( x_0 m - i \epsilon E \frac{\partial}{\partial m} \right) - \frac{T \cdot p}{m^2} \]

The operators (25.23) are defined in the Poincaré basis \(|r, \lambda, \epsilon; p, m, l, l_s\rangle\) which is formed by the eigenvectors of the complete set of the commuting operators \( C_1, C_2, C_3 (24.5), C_4 (24.36) \) and \( P_1, P_2, P_3, P_4, P^\mu = P_4^\mu, W^\mu W^\nu - m^2 \Sigma_{ab} \Sigma^{ab}, \Sigma_1, \Sigma_2 \). Moreover, the eigenvalues of the Casimir operators of the subalgebra \( AP(1,3) \) take the form (see (12.36)-(12.38))

\[ \begin{align*}
P_\mu P^\mu |r, \lambda, \epsilon; p, m, l, l_s\rangle & = m^2 |r, \lambda, \epsilon; p, m, l, l_s\rangle, \\
W_\mu W^\mu |r, \lambda, \epsilon; p, m, l, l_s\rangle & = -m^2 s(s+1) |r, \lambda, \epsilon; p, m, l, l_s\rangle
\end{align*} \]

where \( 0 \leq m^2 < \infty \), \( s = l_\epsilon + l_\lambda + 1 \), ... (25.25)

The operators (25.23) are Hermitian with respect to the scalar product (25.3), where \( \kappa = 0 \) and \( s \) takes the values given in (25.25).

Let us summarize the results presented [336]:

THEOREM 25.2. The space of the IR \( D^\nu(r, \lambda) \) of Class III of the algebra \( AP(1,4) \) is decomposed into the direct integral of subspaces corresponding to the IRs \( D^\nu(m, s) \) of the algebra \( AP(1,3) \); moreover, the possible eigenvalues of the Casimir operators \( P_\mu P^\mu \) and \( W_\mu W^\mu \) are given in (25.24). The generators \( P_\mu \) and \( J_{\mu \nu} \) are given in the Poincaré basis by formulae (25.23) where \( T_a \) and \( \Sigma_a \) are the infinite dimension matrices (24.45),(12.38). \( \blacksquare \)
25.6. Reduction of Representations of Class IV

Consider IRs of Class IV of the algebra $AP(1,4)$. These representations correspond to negative eigenvalues of the Casimir operator $C_1 = P_\mu P^\mu; C_1 = -k^2 < 0$.

A specific feature of transformation of such IRs to the Poincaré basis is the following: by reducing them by the subalgebra $AP(1,3)$ we come to representations belonging to the different classes depending on the eigenvalues of the operator $M^2 = P_4^2 - k^2$.

We consider the case when the domain of eigenvalues $p_4$ is restricted by the condition $p_4 > k^2$.

We start from the realization (24.46) of IRs of Class IV. It is not difficult to make sure that the corresponding Casimir operators of the subalgebra $AP(1,3)$ are not diagonal. But using the transformation (25.5), where

$$V = \exp \left[ \frac{S_{a0} P_a}{|p|} \tanh^{-1} \left( \frac{k |p|}{p_0 p_4} \right) \right] \exp \left[ \frac{S_{a0} P_a}{|p|} \arctan \left( \frac{|p|}{p_4} \right) + \frac{i \pi}{2} (1 - e^{-m}) \right],$$

where $E = \sqrt{p^2 + m^2}$, $m = p_i^2 - k^2$, $S_{a0} = \xi_a$, $S_a = \Sigma_a = \frac{1}{2} \epsilon_{abc} S_{bc}$, and changing $p_0 \rightarrow E$, $p_4 \rightarrow (m^2 + k^2)^{1/2}$, we reduce (24.46) to the following form

$$P_0 = \sqrt{p^2 + m^2}, \quad P_a = p_a, \quad P_4 = \sqrt{m^2 + k^2},$$

$$J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad J_{a0} = x_a p_0 - E x_a - E S_{a0} P_0 E + m, \quad J_{04} = i E \left[ \frac{1 + \frac{k^2}{m^2}}{2} \frac{\partial}{\partial m} \right] + x_0 P_4 + \frac{k S_{a0} P_a}{m},$$

$$J_{0a} = i E \left[ \frac{1 + \frac{k^2}{m^2}}{2} \frac{\partial}{\partial m} \right] + x_0 P_4 + \frac{k S_{0a}}{m} + \left( \frac{E}{m} + \frac{1 + \frac{k^2}{m^2}}{2} \right) S_{ab} P_b E + m.$$

The operators $P_\mu, J_{\mu\nu}$ of (25.26) have the canonical Wigner-Shirokov form, however, the matrices $S_{ab}$ belong to the IR $D(l_0, l_0)$ of the algebra $AO(1,3)$. Hence we conclude IRs of the Class IV of the algebra $AP(1,4)$ are decomposed into the direct sums of IRs $D^t(s, m)$ by the reduction $P(1,4) \rightarrow P(1,3)$. Moreover, $m^2 - p_i^2 - k^2 > 0$, $s = l_0^2 + 1, l_0^2 + 2, \ldots, |l_0| - 1$.

In analogous way it is possible to consider the cases $|p_4| = k$ and $|p_4| < k$. 

115
Symmetries of Equations of Quantum Mechanics

We will not write out the corresponding cumbersome formulae here.

25.7. Reduction \( P(1,n) \rightarrow P(1,3) \)

The results presented above can be generalized directly to the cases of the groups \( P(1,n) \) defined in \((n+1)\)-dimensional Minkowsky spaces. Here we consider IRs of the algebra \( AP(1,n) \) corresponding to the positive values of the main Casimir operator \( P^2=P_0^2+P_1^2+P_2^2+...+P_n^2 \) and find a realization of such IRs in the Poincaré basis.

First we will show that the IRs of the algebra \( AP(1,n) \) can be defined in the \( P(1,n-1) \)-basis. The canonical realization of an IR corresponding to \( P^2=\kappa^2>0 \) is defined by the following relations [136]:

\[
P_0 = -E - \frac{1}{\kappa^2} \sum_{k=1}^{n} p_k^2, \quad P_k = p_k, \quad k=1,2,3,...,n,
\]

\[
J_{ab} = -x_a p_b - x_b p_a + S_{ab}, \quad a,b=1,2,3,...,n-1,
\]

\[
J_{0a} = -x_0 p_a - E x_a - \frac{S_{0a}}{E + \kappa^2},
\]

\[
J_{0n} = -x_0 p_n - E x_n - \frac{S_{0n}}{E + \kappa^2}.
\]

Here \( S_{ab} \) are matrices which form the IR \( D(m_1,m_2,...m_{[n/2]}) \) of the algebra \( AO(n) \), \( m_1, m_2, \) ... are the Gelfand-Zetlin numbers. The operators (25.27), (25.28) are Hermitian with respect to the following scalar product

\[
(\Psi_1, \Psi_2) = \int d^n p \frac{\Psi_1^* \Psi_2}{2E},
\]

The subalgebra \( P(1,n-1) \) is a linear span of the basis elements (25.27). In the \( P(1,n-1) \)-basis these elements by definition have to have a form of a direct sum of the generators of the IRs of the group \( P(1,n-1) \). If \( S_{ij} \) are defined in the Gelfand-Zetlin basis \( O(n) \supset O(n-1) \supset O(n-2) \) ... then we can choose the transformed operators (25.27) in the form

\[
P_0' = -E - \frac{1}{\kappa^2} \sum_{a} p_a^2 + m_a^2, \quad m_a^2 = \kappa^2 + p_a^2, \quad P_a = p_a,
\]

\[
J_{ab}' = -x_a p_b - x_b p_a + S_{ab}, \quad J_{0a}' = -x_0 p_a - E x_a - \frac{S_{0a}}{E + m_a^2},
\]

The problem of finding of a realization of the generators \( P_a, J_{ab} \) in the \( P(1,n-1) \)-basis reduces to the construction of an isometric operator which connects the realizations (25.27) and (25.30). In analogy with (25.5)-(25.11) it is possible to show that this operator has the form
Chapter 5. Generalized Poincaré Groups

\[ V_n = \left(\frac{m_n}{p_n}\right) \exp \left( i \frac{S_m p_n \theta_n}{|p_n|} \right), \]  

(25.31)

where

\[ \theta_n = 2 \arctan \frac{p_n |p_n|}{(E + m_n)(m_n + \kappa)}, \quad |p_n| = \sqrt{p_1^2 + p_2^2 + \ldots + p_{n-1}^2}. \]  

(25.32)

Using the following identities (which are easily verified with the Cambell-Hausdorf formula (13.16))

\[ x_a^V = -V_n^a x_n V_n^{-1} - x_a - \frac{p_a p_b S_{ab} p_a}{E m_n (E - m_n)(E + \kappa)} + \frac{S_{ab} p_b (m_n - \kappa)}{m_n (E - m_n)(E + \kappa)} + \frac{p_a S_{ab} p_b}{m_n (E + \kappa)}, \]  

(25.33)

we obtain

\[ V_n^a p_b V_n^{-1} = p_b, \]

where \( P'_\mu, J'_\mu \nu \) are the operators (25.30). Analogously using the substitution \( p_n \rightarrow \eta \varepsilon_n (m_n^2 + \kappa^2)^{1/2}, \varepsilon_n = \pm 1 \) one can find the remaining basis elements of the algebra \( AP(1,n) \) in the following form:

\[ P_n = \varepsilon_n \sqrt{m_n^2 - \kappa^2}, \]  

\[ J_{0n} = x_0' P_{0n} - \frac{i}{2} \varepsilon_n \left[ P_{0n}, \frac{\partial}{\partial m_n} \right] - \frac{\kappa S_{0n} p_n}{m_n^2}, \]  

(25.34)

\[ J_{mn} = i \frac{p_a}{2} \left[ P_{mn} - \frac{\partial}{\partial m_n} \right] P_{0n} x_a - \frac{\kappa p_a S_{ab} p_a}{m_n^2 (E + m_n)} + \frac{p_a S_{ab} p_a}{m_n (E - m_n)} + \frac{\kappa S_{mn}}{m_n}. \]

So we have found the explicit expressions for the generators of the group \( P(1,n) \) in the basis \( P(1,n-1) \). These generators are Hermitian with respect to the following scalar product

\[ \langle \phi_1, \phi_2 \rangle = \sum_i \int d\eta \frac{d^{n-1}}{2E} \phi_1(\eta, m) \phi_2(\eta, m). \]  

(25.35)

Here \( \eta \) is a set of numbers characterizing the IRs of the algebra \( AO(n-1) \) appearing by the reduction \( AO(n) \rightarrow AO(n-1) \).

Let us now define a representation of the algebra \( AP(1,n) \) in the \( P(1,n-2) \)-basis. Using the above results we conclude that the operator
Symmetries of Equations of Quantum Mechanics

\[ V_{n-1} = \sqrt{\frac{m_{n-1}}{P_{n-1}} \exp \left( \frac{2S_{n-1} P_a}{i} \tan \frac{p_{n-1}}{E/m_n(m_n, m_{n-1})} \right)} \]

\[ m_{n-1} = \sqrt{\kappa + p_{n-1}^2} - p_{n-1} \]

transforms (25.30) to the following form

\[ V_{n-1} = \sqrt{\frac{m_{n-1}}{P_{n-1}} \exp \left( \frac{2S_{n-1} P_a}{i} \tan \frac{p_{n-1}}{E/m_n(m_n, m_{n-1})} \right)} \]

\[ m_{n-1} = \sqrt{\kappa + p_{n-1}^2} - p_{n-1} \]

transforms (25.30) to the following form

\[ P_0 = \frac{e^i \sqrt{p_{n-1}^2 + m_{n-1}^2}}{P_{n-1}} \]

\[ p_a = \frac{e^i \sqrt{m_a^2 - \kappa}}{P_{n-1}} \]

\[ J_{ab} = x_a p_b - x_b p_a + S_{ab} \]

\[ J_{0a} = x_a p_a - \frac{e^i E}{x_0} p_a - e^i \frac{S_{ab} p_b}{(E + m_a^2)} \]

\[ J_{0n-1} = x_0 p_{n-1} - \frac{i}{2} E \left[ \frac{P_{n-1}}{m_{n-1}} \frac{\partial}{\partial m_{n-1}} \right] - \frac{m_{n-1} S_{n-1} a}{m_{n-1}^2} \]

To define the basis elements of the algebra \( AP(1,n) \) in the \( P(1,n-2) \)-basis it is sufficient to find the explicit form of just one more operator, i.e., \( J_{mn-1} \), since the remaining generators can be obtained using the commutation relations (24.1). Using the identities

\[ V_{n-1} = \sqrt{\frac{m_{n-1}}{P_{n-1}} \exp \left( \frac{2S_{n-1} P_a}{i} \tan \frac{p_{n-1}}{E/m_n(m_n, m_{n-1})} \right)} \]

\[ m_{n-1} = \sqrt{\kappa + p_{n-1}^2} - p_{n-1} \]

and the last of the relations (25.33) with \( n \to n-1 \), \( \kappa \to m \) we obtain

\[ J_{mn-1} = \frac{1}{2} \left[ \frac{P_{n-1}^2}{m_{n-1}} \frac{\partial}{\partial m_{n-1}} \right] + \frac{\kappa m_{n-1}}{m_{n-1}} s_{n-1} \]

The operators (25.36), (25.37) are Hermitian with respect to the scalar product

\[ (\phi_1, \phi_2) = \int dm_1 \int dm_{n-1} \sum_{\alpha} \int d^{n-2} \phi_1 (m_{n-1}, \alpha) \phi_2 (m_{n-1}, \alpha) \]

where \( \alpha \) denotes the sets of numbers that numerating IRs of the algebra \( AO(n-2) \) appearing by the reduction \( AO(n) \to AO(n-1) \to AO(n-2) \).

A representation of the algebra \( AP(1,n) \) in the \( P(1,n-k) \)-bases can be determined in an analogous way. Starting from (25.36), (25.37) and making the consequent transformations \( P \to V_{n-1} P (V_{n-1})^{-1} \), \( J_{mk} \to V_{n-1} J_{mk} V_{n-1}^{-1} \), where

318
Chapter 5. Generalized Poincaré Groups

Using the above results we obtain

\[ V_{n,a} = \sqrt{\frac{m_{a,\ell}}{p_{a,\ell}}} \exp \left( \frac{2iS_{n,\ell} p_{a}}{|p|_{n,\ell}} \frac{p_{a,\ell}|p|_{n,\ell}}{(E + m_{a,\ell})(p_{a,\ell} + m_{a,\ell})} \right), \]

\[ |p|_{n,\ell} = \sqrt{p_{1,\ell}^2 + p_{2,\ell}^2 + \cdots + p_{\ell,\ell}^2}, \quad \ell = 2, 3, \ldots, k, \]

\[ m_{a,\ell} = \sqrt{\kappa^2 + p_{a,\ell}^2 + \cdots + p_{n,\ell}^2}, \quad a = 1, 2, \ldots, n-l-1 \]

and using the above results we obtain

\[ P_0 = \epsilon \epsilon \sqrt{|p|_{n,\ell}^2 + m_{a,\ell}^2}, \quad P_a = p_a, \]

\[ P_{a-n} = \epsilon \epsilon \sqrt{m_{a-n,\ell}^2 - m_{a-n-1,\ell}^2}, \quad J_{a_1} = c_{a_1} p_{a_1} - c_{a_1} p_{a_1} + S_{a_1}, \]

\[ J_{0,a_1} = \epsilon \epsilon \frac{S_{a_1} p_b}{E + m_{a_1-k}} \]

(25.38)

\[ J_{0,a-n} = \epsilon \epsilon \left[ \frac{P_{a-n} p_{a-n-1}}{m_{a-n,\ell}}, \frac{\partial}{\partial m_{a-n,\ell}} \right] \frac{\kappa m_{a-n,\ell} S_{a-n,\ell-1}}{m_{a-n}^2}, \]

The generators (25.38) are Hermitian with respect to the following scalar product

\[ (\phi_1, \phi_2) = \int \int \int \int \sum_{k} \frac{d^{n-k} p}{2E} \phi_1(m_{a-k,\ell} \lambda) \phi_2(m_{a-k,\ell} \lambda). \]

They form a representation of the algebra \( AP(1,n) \) in the basis \( P(1,n) \supset P(1,n-1) \supset \ldots \supset P(1,n-k) \). In the case \( n-k=3 \) formulae (25.38) defines an IR of the algebra \( AP(1,n) \) in the Poincaré-basis.

26. REPRESENTATIONS OF THE ALGEBRA \( AP(1,4) \)
IN THE \( G(1,3) \)- AND \( E(4) \)-BASISES

26.1. The \( G(1,3) \)-Basis

As was noted in Subsection 24.1 the algebra \( AP(1,4) \) includes the subalgebras \( AP(1,3), AG(1,3) \) and \( AE(4) \), i.e., the Lie algebras of the main groups of quantum physics.

In the previous section we have found the generators of the generalized Poincaré group in the Poincaré basis. However it is very interesting for physical
Symmetries of Equations of Quantum Mechanics

applications to describe representations of the algebra \( AP(1,4) \) in the Galilei basis characterized by a diagonal form of the Casimir operators of Galilean subalgebra.

Here\(^*\) we obtain an explicit form of basis elements of the algebra \( AP(1,4) \) in the \( G(1,3) \)-basis for any class of IRs. The unitary operator is found also which makes the reduction of IRs of the Poincaré group by the Galilei group in \((1+2)\)-dimensional space (i.e., the reduction \( P(1,3) \rightarrow G(1,2) \)). Such a reduction plays the central role in the null-plane formalism (see, e.g., [275]).

In Subsection 25.5 we find IRs of the algebra \( AP(1,4) \) in the \( E(4) \)-basis in which the Casimir operators of the four-dimensional Euclidean group are diagonal.

To select the Galilei subalgebra from the algebra \( AP(1,4) \) we should come to the basis

\[
\hat{P}_0 = \frac{1}{2}(P_0 - P_3), \quad M = P_3, \quad \hat{P}_a = P_a, \\
J_a = \frac{1}{2}e_{abc} J_{bc}, \quad G^*_a J_0 + J_a, \quad K = \epsilon M,
\]

where as usually \( a = 1, 2, 3 \).

It follows from (24.1) that the operators (26.1) satisfy the following commutation relations

\[
\begin{align*}
\{\hat{P}_0, \hat{P}_a\} &= \{\hat{P}_a, M\} = \{\hat{P}_a, \hat{P}_b\} = 0, \\
\{\hat{P}_0, J_a\} &= \{\hat{M}, J_a\} = \{G^*_a, G_b\} = \{M, G^*_a\} = 0, \\
\{\hat{P}_a, J_b\} &= i\epsilon_{abc} \hat{P}_c, \quad \{G^*_a, \hat{P}_b\} = i\delta_{ab} M, \\
\{J_a, J_b\} &= i\epsilon_{abc} J_c, \quad \{\hat{P}_0, G^*_a\} = -i\hat{P}_a, \\
\{\hat{P}_0, G^*_a\} &= \{G^*_a, G^*_b\} = 0, \quad \{G^*_a, M\} = -2i\hat{P}_a, \\
\{G^*_a, J_b\} &= i\epsilon_{abc} G^*_c, \quad \{G^*_a, \hat{P}_b\} = -2i\delta_{ab} \hat{P}_0, \\
\{G^*_a, G^*_b\} &= 2i(\epsilon_{abc} J_c + \delta_{ab} K), \quad \{\hat{P}_0, K\} = [J_a, K] = 0, \\
\{\hat{P}_0, K\} &= -i\hat{P}_0, \quad [M, K] = iM, \quad [G^*_a, K] = \pm iG^*_a.
\end{align*}
\]

The relations (26.2) characterize the Lie algebra of the Galilei group \( G(1,3) \) (compare with (11.6)). This algebra has three main Casimir operators presented in (11.14). Using (26.1) we represent these operators in the form

\(^{*}\)For more details see [150]
\[ \hat{C}_1 = 2M\hat{P}_0 - \hat{P}_a \hat{P}_a = C_1, \quad \hat{C}_3 = \mp \hat{P}_0 \hat{P}_4, \]

\[ \hat{C}_2 = \mp \hat{P}_a \hat{P}_a \hat{G} \gamma^{0} \equiv (W_{\omega a} + W_{\omega 0})(W_{\omega a} + W_{\omega 0}), \]

where \( C_1 \) is the Casimir operator of the algebra \( AP(1,4) \) of (24.5a), \( W_{\omega a}, W_{\omega 0} \) are the components of the tensor \( W_{\omega \mu} \) of (24.3).

Our task is to transform the realizations of the algebra \( AP(1,4) \) described in Section 24 to such a basis where the operators (26.4) are diagonal. This basis can be formed by a complete set of eigenfunctions of the commuting operators \( \hat{P}_0, \hat{P}_2, \hat{P}_3, \hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_3 \) and \( \Sigma = W_{\omega a} + W_{\omega 0} \) with eigenvalues \( p_1, p_2, p_3, 2\eta m, m^2, s(s+1), m, ms \). To denote such eigenfunctions we use the notation \( | p, m, s, m, s; c \rangle \), where \( c \) is a set of the eigenvalues of the Casimir operators of the algebra \( AP(1,4) \) characterizing the IR.

We normalize the basis vectors according to

\[ \langle p, m, s, m, s; c | p', m', s', m', s'; c \rangle = 2m \delta (m-m') \delta (s-s') \delta _a \delta _b \delta _c. \]  

This normalization leads to the following definition of the scalar product in the Hilbert space spanned on the basis \( | p, m, s, m, s; c \rangle \):

\[ (\Phi_1, \Phi_2) \sum \int dm \int d^3p \Phi_1^\dagger (m,p,s,s) \Phi_2 (m,p,s,s). \]  

The domains of \( m \) and \( s \) are different for the different classes of the IRs and will be defined further on.

### 26.2. Representations with \( P_n P_n > 0 \)

We start from the realizations (24.40) of these representations. Our aim is to find the explicit form of the generators \( \hat{P}_n, \hat{J}_{mn} \) in the Galilei basis and define the unitary operator connecting this basis with the canonical one.

Substituting (24.40) into (26.2)-(26.4) we obtain

\[ \hat{P}_0 = \frac{1}{2} (E - \varepsilon p), \quad M = \varepsilon E + \varepsilon x, \quad E = \sqrt{p^2 + p_0^2 + \kappa^2}, \]

\[ J_a = \varepsilon_{abc} \left( s_b p_c + \frac{1}{2} S_{bc} \right), \]

\[ G_a = (x_a + x_0)p_a - M x_a - \frac{e S_{ab} p_b - S_{ab} (E + \kappa + \varepsilon p_a)}{E + \kappa}, \]

\[ \hat{C}_1 = \kappa^2, \quad \hat{C}_3 = M, \]
Symmetries of Equations of Quantum Mechanics

\[ \hat{\mathcal{C}}_2 = \left[ S^i [M(E+\kappa) - \epsilon \mathbf{p} \cdot \mathbf{N}] + \frac{1}{2} \mathbf{p} \cdot \mathbf{N} + (\mathbf{p} \cdot \mathbf{S})^2 \right] (E+\kappa)^2, \]

\[ (\mathbf{p} \cdot \mathbf{S})^2 \left[ 2 \epsilon M(E+\kappa) - \mathbf{p} \cdot \mathbf{N} \right] (E+\kappa)^2, \]

\[ K = x_0 \mathbf{p} - \epsilon \mathbf{E} x_4 - \frac{S_{ab} p_a}{E+\kappa}, \]

\[ G_a = (x_0 - x_a) p_a - 2p_{0a} x_a - \frac{\epsilon S_{ab} p_b + S_{a4} (E+\kappa - \epsilon p_a)}{E+\kappa}, \]

where

\[ S_a = \frac{1}{2} \epsilon_{abc} S_{bc}, \quad N_a = S_{a4}, \quad x_a = -i \frac{\partial}{\partial \epsilon}. \]

The Casimir operators \( \hat{\mathcal{C}}_1, \hat{\mathcal{C}}_3 \) are diagonal in the canonical basis \( |p, m_a, m_s, 1, \kappa, l_0, l_1, e \rangle \), but \( \hat{\mathcal{C}}_2 \) is a matrix depending on \( \mathbf{p} \) and \( p_a \). To diagonalize this matrix we use an operator

\[ U = \exp \left( \frac{S_{ab} p_a}{\mathbf{p}^4} \theta \right), \]

where \( \theta \) is an unknown so far function of \( \mathbf{p} \) and \( p_a \).

Using the unitary operator (26.8) we can define a series of the representations equivalent to (26.7):

\[ \hat{\mathcal{P}}_\mu ' U_1 \hat{\mathcal{P}}_\mu U_1^\dagger = \hat{\mathcal{P}}_\mu, \quad J_a' = U_1 J_a U_1^\dagger = J_a, \quad M' = U_1 M U_1^\dagger = M, \]

\[ (G_a)' = U_1 G_a U_1^\dagger = (x_0 - x_a) p_a - \frac{\epsilon S_{ab} p_b + S_{a4} (E+\kappa - \epsilon p_a)}{E+\kappa}, \]

\[ (G_a)' = U_1 G_a U_1^\dagger = K U_1 K U_1^\dagger, \]

where \( x_a', x'_a, S'_{ab}, S'_{a4} \) are given by formulae (25.8) (with \( R=1 \)). Moreover \( (G_a)' \) has an extremely simple form if we choose

\[ \theta = 2 \arctan \left( \frac{|\mathbf{p}|}{E+\kappa - \epsilon p_a} \right) + \frac{1}{2} (1-\epsilon) \pi, \]

Indeed, we have in this case

\[ (G_a)' = (x_0 - x_a) p_a - M x_a, \]

Substituting (26.9) and (26.10) into (26.4) we make sure that the corresponding Casimir operator \( \hat{\mathcal{C}}_2 \) is diagonal:
\[ C_2' = M^2 S^2 \equiv M^2 (S_{12}^2 + S_{31}^2 + S_{23}^2) \]

where \( S^2 \) is a diagonal matrix with eigenvalues \( s(s+1) \) for \( l_0 \leq s \leq l_1 - 1 \).

To find the explicit form of the generators of the group \( P(1,4) \) in the Galilei basis \( p, m, s, m, s, \kappa, l_0, l_1, \epsilon > \) it is sufficient to substitute (25.8), (25.10), (26.8) into (26.11) and to make the change of variables \( (p_1, p_2, p_3, p_4) \rightarrow (p_1, p_2, p_3, m) \) where \( m = E + \epsilon p_4 \). Besides that

\[
\frac{\partial}{\partial p_a} \rightarrow \frac{E}{m} \frac{\partial}{\partial m}, \quad \frac{\partial}{\partial p_a} \rightarrow \frac{1}{E} \frac{\partial}{\partial m},
\]

and the operators (26.9) take the following form:

\[ \hat{P}_0' = m_0 + \epsilon \frac{p_2}{2m}, \quad \hat{P}_a' = p_a, \quad M' = \epsilon m, \]

\[ J_a' = \epsilon x_a p_t + S_a \quad (G_a')' x_0 p_a - i \epsilon m \frac{\partial}{\partial p_a}, \]

\[ K' = -i m \frac{\partial}{\partial m} + \epsilon x_0 \left( \frac{em}{2} - \hat{P}_0' \right), \]

\[ (G_a')' x_0 p_a - 2i \left[ \epsilon p_a \frac{\partial}{\partial m} + \hat{P}_0' \frac{\partial}{\partial p_a} \right] + \frac{2(S_{ab} \kappa - S_{ab} p_t) \epsilon}{m} \]

\[ \hat{C}_1' = \kappa^2, \quad \hat{C}_2' = m^2(s+1), \quad \hat{C}_3' = \epsilon m, \quad \hat{C}_4' = \epsilon m, \]

where

\[ \kappa \leq m < \infty, \quad m_0 = \epsilon \frac{\kappa^2}{2m}, \quad l_0 \leq s \leq |l_1| - 1. \]

So we come to the realization (26.12) of the IR \( D'(\kappa, l_0, l_1) \) of the algebra \( AP(1,4) \). The distinguishing feature of this realization is that the operators \( \hat{P}_0', J_a', (G_a')' \) and \( M' \) form an IR of the Galilei algebra in the canonical realization (12.15) for any fixed values of \( m \) and \( s \). The operators (26.12) are Hermitian with respect to the scalar product (26.6) where \( \lambda_i = \kappa, \lambda_2 \rightarrow \infty, l_0 \leq s \leq |l_1| - 1. \)

Let us formulate the above results in the form of the following assertion [150]:

**THEOREM 26.1.** The Hilbert space of the IR \( D'(\kappa, l_0, l_1) \) of the algebra \( AP(1,4) \) is decomposed into the direct integral of subspaces corresponding to the IRs of the algebra \( AG(1,3) \) labelled by the eigenvalues (26.13) of the Casimir operators.

The explicit form of the generators of the group \( P(1,4) \) in the Galilei basis and of the transition operator connecting the canonical and \( G(1,3) \)-bases are given in formulae (26.12), (26.8), (26.10).
26.3. The Representations of Classes II-IV

Let us show that these representations can be transformed into the Galilei basis also and find the corresponding realizations of basis elements of the algebra $AP(1,4)$ in the explicit form.

First we consider the IRs corresponding to $P_n P_n = 0$ and belonging to Classes II or III. The corresponding basis elements can be chosen in the form (24.41). So we have in the basis (26.1)

\[ \hat{J}_a = \epsilon_{abc} \hat{x}_b \hat{P}_c + \Sigma_a, \]

\[ G_a = \hat{x}_a \hat{P}_a - M \hat{P}_a + \frac{1}{p^2} (\epsilon_{abc} \hat{P}_b (T_0 - \epsilon \hat{P}_c) + \Sigma_a (p^2 + \epsilon p^2)), \]

\[ K = x_0 \hat{P}_4 - \epsilon p x_4 + \frac{1}{p^2} (T_0 \hat{P}_4 - \epsilon \hat{P}_4 \hat{P}_a), \]

\[ G_a = (x_0 - x_4) \hat{P}_a - 2 \hat{x}_0 \hat{P}_a + \frac{1}{p^2} (\epsilon_{abc} \hat{P}_b (T_0 - \epsilon \hat{P}_c) + \Sigma_a (\epsilon p^2 - p^2)). \]

To transform (26.14) to the Galilei basis we use the operator

\[ V = \exp \left[ -i \frac{\hat{T} \cdot \hat{p}}{p(p + \epsilon \hat{p}_4)} + i \frac{\Sigma \cdot \hat{p}}{\hat{p}_4} \left( \arctan \left( \frac{p}{p_4} \right) - \frac{1}{2} (1 - \epsilon) \pi \right) \right]. \] (26.15)

As a result of the transformation (26.9), (26.15) and change of variables $(p_1, p_2, p_3, p_4) \to (p_1, p_2, p_3, m)$, $m = p + \epsilon p_4$, we obtain

\[ \hat{P}_0 = \frac{\hat{p}_3^2}{2m}, \quad M = \epsilon m, \quad \hat{P}_a = \hat{p}_a, \]

\[ \hat{J}_a = \epsilon_{abc} \hat{x}_b \hat{P}_c + \Sigma_a, \quad G_a = \hat{x}_a \hat{P}_a - M \hat{P}_a - i \epsilon m \frac{\partial}{\partial \hat{p}_a}, \]

\[ K = x_0 \left( \frac{\epsilon m}{2} \hat{P}_0 - \hat{p}_3 \right) - i \epsilon m \frac{\partial}{\partial m} + \frac{(m^2 + \epsilon^2) \hat{T} \cdot \hat{p}}{m^2 + \epsilon^2}. \]

The realization (26.16) is defined in the Galilei basis $| p, m, s, m_s; r, \lambda >$ in which the Casimir operators of the subalgebra $AG(1,3)$ are diagonal. Indeed according to (26.4), (26.16) we have

\[ \hat{C}_0 \equiv 0, \quad \hat{C}_2 = m^2 \Sigma_a, \quad \hat{C}_3 \equiv m, \]

and thus the eigenvalues of these operators are 324.
Chapter 5. Generalized Poincaré Groups

If a representation of the algebra $AP(1,4)$ is characterized by the zero eigenvalue of $C_2$ (i.e., belonging to Class II) then $r^2=0$, $T_i=0$, and the form of the generators (26.16) is simplified essentially.

The operators (26.16) are Hermitian with respect to the scalar product (26.6) where $\lambda_1=0$, $\lambda_2 \to \infty$. Thus either the sum with respect to $s$ reduces to the single term (for $c_2\equiv 0$) or $s$ runs over all the values given in (26.17).

We see that the IRs of Classes II and III can be defined in the Galilei basis also. In comparison with the representation of Class I there are two specific features: the possibility of zero eigenvalues of the mass operator and the infinite number of spin states appearing by the reduction $P(1,4) \to G(1,3)$.

In conclusion we present the explicit form of the basis elements of an IR of the algebra $AP(1,4)$, which belong to Class IV, in the Galilei basis. This form is given in (26.12) and (26.18):

$$K' = x_0 \left( \frac{e m}{2} - \hat{p}_0 \right) - i m \frac{\partial}{\partial m},$$

$$G_a' = 2 \left[ i \left( e p_m \frac{\partial}{\partial m} - \hat{p}_0 \frac{\partial}{\partial p_m} \right) - \frac{S_{a\mu} p_\mu - k S_{a\mu}}{e m} \right].$$

where $S_{a\mu}$ are generators of an IR of the Lorentz group, $m_0=\sqrt{k^2/(2e m)}$, $-\eta^2 m<0$, $0<m<\infty$.

26.4. Covariant Representations

One of the most interesting problems appearing with the reduction $P(1,4) \to G(1,3)$ is the transformation to the Galilean basis of the covariant representations, which are characterized by the following form of the basis elements

$$P_n = p_n, \quad J_m = x_m p_n - x_n p_m + S_{mn},$$

where $S_{mn}$ are matrices realizing a representation of the algebra $AO(1,4)$; $x_m$ and $p_n$ are the canonically conjugated variables satisfying the relations $[p_m, x_n]=i\delta_{mn}$. We will not concretize the realization of $p_n$ and $x_n$, so the further results are available both for the $x$- and for $p$-representations.

The operators (26.19) belong to the class $M_1$, and therefore generate the local finite transformations of the group $P(1,4)$. The covariant representations of the algebra $AP(1,4)$ are used for the description of $P(1,4)$-invariant wave equations, see Section 27.

We restrict ourselves to the case when the spectrum of the Casimir operator

\[ c_1=m_0=0, \quad c_2=m^2(s+1), \quad c_3=m, \quad 0\leq m<\infty, \quad s=\lambda_1, \lambda_2+1, \ldots, \quad r^2\neq 0. \]
\( C_1 = \mathcal{P}_0 \mathcal{P}^\ast \) is positive. Such a situation is typical for the cases for which the space of the representation (26.19) is defined as a completion of a set of the solutions of the \( P(1,4) \)-invariant wave equation. If we go over to the basis (26.1) then we obtain according to (26.19)

\[
\hat{P}_0 = \frac{1}{2}(p_0 - p_4), \quad M = p_0 + p_4, \quad \hat{P}_a = p_a,
\]

\[
J_a = \epsilon_{abc} x_b p_c + S_a, \quad G_a^\ast = \tilde{x}_a M + \lambda_a^\ast,
\]

\[
K = \hat{x}_a M - \tilde{x}_0 \hat{P}_a - S_{0a}, \quad G_a^\ast = 2 \hat{x}_a p_a - 2 x_a \hat{P}_0 + \lambda_a^\ast,
\]

where

\[
S_a = -\frac{1}{2} \epsilon_{abc} S_{bc}, \quad \lambda^\ast = S_{0a} \pm S_{4a}, \quad \tilde{x}_a = x_a - x_4, \quad \tilde{x}_4 = \frac{1}{2} (x_0 + x_4).
\]

It is not difficult to make sure that the Casimir operators of the Galilei subalgebra (formed by the operators (26.20a)) are not diagonal. To transform the representation (26.19) to the Galilean basis we use the operator

\[
V = \exp \left( \frac{i \lambda_a p_a}{M} \right).
\]

After the transformation (26.9) (where \( U \rightarrow V \), \( V \) is the operator (26.21)) we come to the realization

\[
\hat{P}_0' = \frac{1}{2}(p_0 - p_4), \quad M' = p_0 + p_4, \quad \hat{P}_a' = p_a,
\]

\[
J_a' = \epsilon_{abc} x_b p_c + S_a, \quad (G_a')^\ast = \tilde{x}_a M + \lambda_a^\ast,
\]

\[
K' = \hat{x}_a M - \tilde{x}_0 \hat{P}_a - S_{0a},
\]

\[
(G_a')^\ast = 2 \hat{x}_a p_a - 2 x_a \hat{P}_0 + \lambda_a^\ast.
\]

Besides that, the corresponding Casimir operators (26.4) take the form

\[
\hat{C}_1 = p_0 p^\ast - 2 M \hat{P}_0 - p^2, \quad \hat{C}_2 = M' S_a S_a, \quad \hat{C}_3 = M.
\]

According to (26.23) the eigenvalues of \( \hat{C}_1 \) coincide with the eigenvalues of the operator \( C_1 = \mathcal{P}_0 \mathcal{P}^\ast \) (which are positive by definition), the eigenvalues of \( \hat{C}_2 \) coincide with the ones of the matrix \( S^2 \) multiplied by \( M^2 \) and the eigenvalues of \( \hat{C}_3 \) lie in the interval \((c_3)^{1/2} \leq c_3 < \infty\).

We see that a covariant representation of the algebra \( \mathcal{A}P(1,4) \) can be transformed to the form (26.22) for which the Casimir operators of the subalgebra \( \mathcal{A}G(1,3) \) turn out to be diagonal. The operator (26.21) can be used for a diagonalization
of the wave equations being invariant under the group $P(1,4)$, see Subsection 26.2.

### 26.5. The $E(4)$-Basis

Besides the subalgebras $AP(1,3)$ and $AG(1,3)$, the Lie algebra of Euclidean group in the four-dimensional space is very interesting from the point of view of physics. This group (denoted by $E(4)$) plays an important role in the quantum field theory and quantum statistics.

In this section we consider the reduction of an IR of the algebra $AP(1,4)$ by the algebra $AE(4)$. We will see such a reduction can be made in a relatively simple way without using any transformations of the kind (26.9).

The subalgebra $AE(4)$ is formed by ten generators $P_k$ and $J_{kl}$ $(k,l=1,2,3,4)$ which satisfy the following commutation relations according to (24.1):

$[P_i, P_j] = 0$, $[P_i, J_{kl}] = i(\delta_{ik} P_j - \delta_{jk} P_i)$, $[J_{kl}, J_{ij}] = i(\delta_{ki} J_{lj} - \delta_{kj} J_{li} - \delta_{li} J_{kj} + \delta_{lj} J_{ki})$.

This algebra has the two Casimir operators

$\hat{C}_1 = P_i^2 + P_j^2 + P_k^2 + P_l^2$, $\hat{C}_2 = W_1^2 + W_2^2 + W_3^2 + W_4^2$,

where $W_k = \sum_{i=1}^4 P_i J_{ik}$. Substituting (24.31) into (26.25) we obtain

$\hat{C}_1 = p_i^2 + p_j^2 + p_k^2 + p_l^2$, $\hat{C}_2 = p_i^2 \Sigma_a \Sigma_a$,

where $\Sigma_a$ are matrices which realize a representation of the algebra $AO(3)$ (24.24). We can see the realization (24.31) of the algebra $AP(1,4)$ corresponds to the diagonal Casimir operators of the subalgebra $AE(4)$ since the matrix $\Sigma_a \Sigma_a$ can be chosen diagonal without loss of generality.

Let us denote eigenvectors of the complete set of the commuting operators $P_1$, $P_2$, $P_3$, $P_4$, $\hat{C}_1$, $\hat{C}_2$, $\Sigma_3$ and the Casimir operators of the algebra $AP(1,4)$ by $|\tilde{p},s,s; c\rangle$. Thus

$\hat{C}_1 |\tilde{p},s,s; c\rangle = p_i |\tilde{p},s,s; c\rangle$, $\hat{C}_2 |\tilde{p},s,s; c\rangle = p_i |\tilde{p},s,s; c\rangle - s(s+1) |\tilde{p},s,s; c\rangle$,

$\Sigma_3 |\tilde{p},s,s; c\rangle = s_3 |\tilde{p},s,s; c\rangle$, $P_k |\tilde{p},s,s; c\rangle = p_k |\tilde{p},s,s; c\rangle$,

$C_a |\tilde{p},s,s; c\rangle = c_a |\tilde{p},s,s; c\rangle$,

where $s_3 = -s, -s+1, \ldots, s$ are positive integers or half-integers, $c_a$ are eigenvalues of the Casimir operators of the algebra $AP(1,4)$, $\tilde{p} = (p_1, p_2, p_3, p_4)$, $c = (c_1, c_2, \ldots)$.

We impose on $|\tilde{p},s,s; c\rangle$ the normalization condition

$\langle \tilde{p},s,s, l | \tilde{p}',s', l' ; c \rangle = M_{l,l'}^l \delta(l-l') \delta_{s,s'} \delta_{c,c'}$,

where $M_{l,2P_0}$ if $c_i \geq 0$ and $M_{l,2P_4}$ if $c_i < 0$. The vectors $|\tilde{p},s,s; c\rangle$ form the
orthonormalized basis called below $E(4)$-basis.

The explicit form of the generators of the group $P(1,4)$ is given by formulae (24.37), (24.41) and (24.46) for $c_1>0$, $c_1=0$ and $c_1<0$ respectively. Besides eigenvalues of the Casimir operators of the subalgebra $AE(4)$ are infinitely degenerated and given by formulae (26.27) where $0 \leq p_s < \infty$ and the values of $s$ coincide with the numbers labelling IRs of the group $O(3)$ which arise by the reduction $O(4) \rightarrow O(3)$, $E(3) \rightarrow O(3)$ and $O(1,3) \rightarrow O(3)$ for $c_1>0$, $c_1=0$ and $c_1<0$ correspondingly.

26.6 Representations of the Poincaré Algebra in the $G(1,2)$-Basis

Let us now consider the IRs of the Poincaré algebra and solve the problem of a transformation of these representations to such a basis where the Casimir operators of the subalgebra $AG(1,2)$ (i.e., the Lie algebra of the Galilei group in the space of two spatial dimensions) are diagonal. Such a basis turns out to be very important for different applications. It is this basis which is used implicitly in the null-plane formalism [275].

A transformation of an IR of Class $I (P_\mu P^\mu=\kappa^2>0)$ to the $G(1,2)$-basis can be made in a complete analogy with the transformation of one for the algebra $AP(1,4)$ considered in Subsection 25.2. We will start from the canonical realization (4.50) of this IR. The basis elements of the subalgebra $AG(1,2)$ can be expressed via $P_\mu, J_{\mu\nu}$ with the help of the following relations

\begin{align}
\hat{P}^0 &= \frac{1}{2}(P_0^2 - P_3^2), \quad M = P_0 P_3, \quad \hat{P}_a = P_a^a \\
\hat{J}_{\alpha\beta} &= J_{\alpha\beta} + J_{\alpha\beta}, \quad \alpha = 1,2.
\end{align}

(26.28)

We choose the remaining basis elements of the algebra $AP(1,3)$ in the form

\begin{align}
G^a_{\alpha} &= J_{\alpha\alpha} - J_{\alpha\beta}, \quad K = J_{03}.
\end{align}

(26.29)

The operators (26.28), (26.29), (4.50) realize an IR of the Poincaré algebra in the canonical Shirokov-Foldy basis. To come to the $G(1,2)$-basis we use the transformation (26.9) where

\begin{align}
U = & \exp \left[ 2iS_{\alpha\beta} P_{\alpha} \left( \frac{|P|}{|P|} \left[ \arctan \left( \frac{|P_0| + \varepsilon P_1 + \kappa}{4} \right) \right] \right) \right],
\end{align}

and make the change of the variables $(p_1, p_2, p_3) \rightarrow (p_1, p_2, m)$, $m = \varepsilon p_1 + (p_1^2 + p_2^2 + \kappa^2)$. As a result we obtain the realization of the algebra $AP(1,4)$ in a basis where the Casimir operators of the subalgebra $AG(1,2)$ are diagonal, since

328
Chapter 5. Generalized Poincaré Groups

We do not present exact calculations which are analogous to (26.7)-(26.13).

\[
\hat{P}_0' = \frac{e}{2m}(\kappa^2 + m^2), \quad \hat{P}_a' = p_a, \quad |p|_2 = (p_1^2 + p_2^2)^{1/2},
\]

\[
J_3' = i \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) + S_{12}, \quad M' = \varepsilon m,
\]

\[
(G_a)' = x_a p_a - ie m \frac{\partial}{\partial p_a}, \quad \kappa \leq m < \infty,
\]

\[
K' = x_o \left( \frac{\varepsilon m}{2} - \hat{P}_0' \right) - im \frac{\partial}{\partial m},
\]

\[
(G_3') = -2 \left[ \left( \frac{\varepsilon p_a}{dm} + \hat{P}_0' \frac{\partial}{\partial p_a} \right) + \varepsilon S_{ab} p_b + S_{3\alpha} \kappa \right],
\]

\[
\hat{C}_1' = 2M(\hat{P}_0' - \hat{P}_1'^2 - \hat{P}_2'^2) = \varepsilon \kappa^2, \quad \hat{C}_3' = M' - \varepsilon m,
\]

\[
\hat{C}_2' = (J_3)' M' = \hat{P}_3'(G_3') + \hat{P}_2'(G_1') + \frac{m^2 \kappa^2}{m}. \tag{26.32}
\]

We do not present exact calculations which are analogous to (26.7)-(26.13).

The operators (26.30) coincide with the generators of the kinematical group used in the null-plane formalism (see, e.g., [257]). So we have found the explicit connection of these generators with the Poincaré group generators in the Shirokov-Foldy representation.

Using the results given in Subsections 26.3, 26.4 it is not difficult to transform the IRs of Classes II-IV to the \(G(1,2)\)-basis. We do not present the correspondent calculations here but consider the class of covariant representations which is the most interesting from the point of view of physical applications.

Thus we start from the realization (2.22) of the algebra \(AP(1,3)\) where \(S_{\mu\nu}\) are matrices realizing an arbitrary representation of the algebra \(AO(1,3)\). The transformation operator for the case \(P_\mu, P_\nu > 0\) has the form (compare with (26.21))

\[
V = \exp \left[ \frac{(S_{o\alpha} - S_{3\alpha})p_\alpha}{M} \right], \quad M = p_o + p_3, \tag{26.33}
\]

As a result of the transformation (26.9), (26.33) the operators (26.28), (26.29), (2.22) reduce to the form

\[
\hat{P}_0 = \frac{1}{2}(p_0 - p_3), \quad M = p_o + p_3, \quad \hat{P}_a = p_a,
\]

\[
J_3 = \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) + S_{12}, \quad G_a = x_\alpha p_a - x_\alpha M + \lambda_\alpha, \tag{26.34}
\]

329
It is not difficult to make sure the Casimir operators of the subalgebra \( \mathfrak{ag}(1,2) \) being a linear span of the operators (26.34) are diagonal.

The operator (26.33) can be used by solving of different problems in the null-plane formalism. Namely using this operator we can diagonalize the systems of the Poincaré-invariant motion equations for the particles of arbitrary spins interacting with the same types of external fields (e.g., the field of a plane wave, the homogeneous magnetic field and others). More precisely the corresponding transformation operator can be obtained from (26.33) by the change \( p_\mu \rightarrow p_\mu - eA_\mu \) where \( A_\mu \) is a vector-potential of an external field. Examples of the equations admitting such a diagonalization are present in Section 28.

27. WAVE EQUATIONS INVARIANT UNDER GENERALIZED POINCARÉ GROUPS

27.1. Preliminary Notes

In this section we discuss the wave equations having the symmetry under the generalized Poincaré group (mainly under the group \( \mathfrak{ap}(1,4) \)). Such equations are of great interest for physics inasmuch as they are invariant under both the Poincaré and the Galilei groups and can be interpreted as motion equations of a relativistic (or Galilean) particle with variable mass \([136,333]\).

Until now the theory of \( \mathfrak{P}(1,4) \)-invariant equations is far from the complete. Strictly speaking, only certain classes of such equations were completely described (see, e.g., \([136,115]\)). So we restrict ourselves by considering of the simplest (and the most important!) equations of the Dirac, Kemmer-Duffin type and some others.

Let us consider a system of first order partial differential equations:

\[
L \Psi(x) \equiv (\Gamma_\mu P^\mu - \kappa) \Psi(x) = 0
\]  

(27.1)

where \( x = (x_0, x_1, \ldots, x_n) \), \( P^\mu = i \partial / \partial x^\mu \), \( \Gamma_\mu \) are numeric matrices. The equation (27.1) by definition is invariant under the algebra \( \mathfrak{ap}(1,n) \) if the operator \( L \) satisfies the relations
where \(P_m, J_{mk}\) are the generators of the group \(P(1,4)\) in the covariant realization (26.19).

Substituting (26.19), (27.1) into (27.2) we come to the following equations for \(\Gamma_m\):

\[
[L,P_m]=L[J_{mk}]=0 \quad (27.2)
\]

Here \(S_{mn}\) are matrices which realize a representation of the algebra \(AO(1,n)\), \(g_{mn} = \text{diag}(1,-1,-1,...)\).

Thus the problem of the description of the \(P(1,n)\)-invariant equations of the kind (27.1) reduces to the solution of the equations (27.3). We will not search for a general solution of these relations restricting ourselves by considering of some particular (but important) examples.

**27.2. The Generalized Dirac Equation**

In the case \(n=4\) the simplest (i.e., realized by matrices of minimal dimension) solution of (27.3) has the form

\[
\Gamma_m = \gamma_m, \quad S_{ln} = \frac{1}{4} [\gamma_l, \gamma_n] \quad (27.4)
\]

where \(\gamma_m\) are the Dirac matrices of dimension 4x4, satisfying the Clifford algebra (2.3).

Substituting (27.4) into (27.1) we come to the generalized Dirac equation in the \((1+4)\)-dimensional Minkowsky space:

\[
(\gamma_m \Gamma^m - \kappa) \psi(x) = 0, \quad m = 0,1,2,3,4 \quad (27.5)
\]

This equation has a manifest symmetry under the algebra \(AP(1,4)\). But in contrast to the corresponding equation (2.1) the generalized Dirac equation turns out not to be invariant under the transformation \(x_a \rightarrow -x_a, a=1,2,3\) [135]. An equation being invariant under the complete (i.e. including all possible reflections of independent variables) group \(\hat{P}(1,4)\) can be obtained from (27.5) by doubling a number of the components of the wave function and by making the change [136]

\[
\gamma_m \rightarrow \Gamma_m = \begin{pmatrix}
\gamma_m & 0 \\
0 & -\gamma_m
\end{pmatrix} \quad (27.6)
\]

Let us discuss symmetries of the equation (27.5) and its possible interpretations.

In a complete analogy with Section 2 it is possible to show the algebra \(AP(1,4)\) is the maximal IA of the equation (27.5) in the class \(M_1\). To answer the question what kind of representations of this algebra is realized on the set of solutions of (27.5) we
use the fact that the corresponding generators (26.19) can be represented in the form

\[ P_0 \Psi = H \Psi, \quad H = \gamma_0 \gamma_k p_k + \gamma_0 m, \quad P_k \Psi = p_k \Psi, \]

\[ J_{\mu k} \Psi = (x_\mu p_k - x_k p_\mu + S_{\mu k}) \Psi, \]

\[ J_{0 k} \Psi = \left( x_0 p_k - \frac{1}{2} [x_\mu H] \right) \Psi, \tag{27.7} \]

where \( \Psi \) is a solution of (27.5), \( k,l = 1,2,3,4 \). The Casimir operators (24.5), (24.36) reduce to the following form in accordance with (27.7):

\[ C_1 = \gamma_\mu p^\nu \gamma^\mu = \kappa^2, \]

\[ C_2 = \frac{1}{2} W^m W_m = -\frac{1}{2} \kappa^2 S_{0i} S_{i0}, \]

\[ C_3 = -\frac{1}{4} J^{mn} W_m W_n = -\frac{1}{4} \kappa S_{ik} S_{j0} S_{i0} S_{j0}, \]

\[ C_4 = H/E, \quad E = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2 + \kappa^2}. \tag{27.8} \]

The eigenvalues of \( C_4 \) are equal to \( \pm 1 \). In order to determine the eigenvalues of \( C_2 \) and \( C_4 \) we choose the realization of \( \gamma \)-matrices given by the relations (2.4), (2.17). Then \( S_{0i} S_{i0} = \gamma_0 \gamma_{kln} S_{k0} S_{l0}/4 = 3 \), and \( C_4 \Psi = -2 \kappa C_4 \Psi = 3 \kappa^2 \Psi \), from which it follows that the operators (27.7) realize the representation \( D^+ \kappa(1/2 0) \oplus D^- \kappa(0 1/2) \) of the algebra \( AP(1,4) \).

Since the algebra \( AP(1,4) \) includes the subalgebras \( AP(1,3) \) and \( AG(1,3) \) the generalized Dirac equation (27.5) turns out to be invariant under the Poincaré and Galilei groups. Let us demonstrate the connection of this equation with the Dirac equation for a particle with a variable mass. Representing \( \Psi(x) \) in the form

\[ \Psi(x) = \int \exp(i p \cdot x_\mu) \Psi_{\mu \nu} (x_\mu, x) dp_\mu, \tag{27.9} \]

we obtain from (27.5) the following equation

\[ [\gamma_\mu p_\nu - \gamma^\mu (\kappa + \gamma_4 p_4)] \Psi_{\mu \nu} = 0. \]

Multiplying this equation by \( (\kappa - \gamma_4 p_4)m^{-1} \) where \( m = (\kappa^2 + p_4^2)^{1/2} \) we obtain

\[ (\gamma_\mu p_\nu - m) \Psi_{\mu \nu} = 0, \]

where \( \gamma_\mu \) are matrices satisfying the Clifford algebra (2.3) moreover \( \gamma_\mu = (\kappa - \gamma_4 p_4)m^{-1} \gamma_\mu \).

We see that for any fixed value of \( p_4 \) the equation (27.5) is equivalent to the Dirac equation for a particle of mass \( m = (\kappa^2 + p_4^2)^{1/2} \). Therefore it is possible to interpret (27.5) as a motion equation for a relativistic particle with a variable mass.

The other possibility of an interpretation of the equation (27.5) is connected with its symmetry under the Galilei group. Using the new variables (26.20c) we can
Chapter 5. Generalized Poincaré Groups

rewrite this equation in the form

\[
\left( \beta_0 i \frac{\partial}{\partial \tilde{x}_0} - \beta_a p_a + 2i\beta_5 \frac{\partial}{\partial \tilde{x}_4} - \kappa \right) \Psi(\tilde{x}_0, x, \tilde{x}_4) = 0
\]

(27.10)

where

\[
\beta_0 \gamma_0 + \gamma_4, \quad \beta_5 \gamma_0 - \gamma_4, \quad \beta_a \gamma_a.
\]

(27.11)

The equation (27.10) is invariant under the Galilei transformations of the variables \((\tilde{x}_0, x, \tilde{x}_4)\) (see Subsection 11.4). If we impose on \(\Psi\) the following Galilei-invariant additional condition

\[
i \frac{\partial}{\partial \tilde{x}_0} \Psi(\tilde{x}_0, x, \tilde{x}_4) = m \Psi(\tilde{x}_0, x, \tilde{x}_4),
\]

(27.12)

then for any fixed value of \(m\) the equation (27.10) reduces to the form

\[
L \Psi = \left( \beta_0 i \frac{\partial}{\partial \tilde{x}_0} - \beta_5 p_a + 2\beta_5 m - \kappa \right) \Psi = 0.
\]

(27.13)

In the case \(\kappa = 0\) formulae (27.11), (27.13) define the Levi-Leblond [275] equation for a Galilean particle of spin 1/2.

We note that using the operator (26.21), we can transform (27.13) to the canonical diagonal form because

\[
V L V^{-1} = \beta_0 \left( i \frac{\partial}{\partial \tilde{x}_0} - \frac{p^2}{2m} \right) + \beta_5 m - \kappa.
\]

(27.14)

According to (27.14) the function \(\Phi_+ = \frac{1}{2}(1+\gamma_0 \gamma_4)V \Psi\) satisfies the Schrödinger equation

\[
i \frac{\partial}{\partial \tilde{x}_0} \Phi_+ - \frac{p^2}{2m} - \frac{\kappa^2}{2m} \Phi_+ = 0
\]

and the function \(\Phi_- = \frac{1}{2}(1-\gamma_0 \gamma_4)V \Psi\) is expressed via \(\Phi_+; \Phi_- = \gamma_4(\kappa/2m) \Phi_+\).

We see the generalized Dirac equation (27.5) can serve as a base of a description of a relativistic particle with a variable mass or of a Galilean particle. Such an interpretation is admissible also for the generalized Dirac equation which includes potentials of an external field. For example replacing \(p_4\) by \(p_4 - eA_4\) in the equation (27.5), where \(A_4 = A_4(n_1 x_4) (n_0 = n_4 = 1, n_1 = n_2 = n_3 = 0)\) is a plane wave potential, we come to the equation preserving the symmetry under Galilei transformations. To solve such an equation it is convenient to use the variables (26.20c) and to decompose the wave function by the complete set of the functions satisfying (27.12). However if \(A_4 = A_4(x_4)\) then the corresponding equation can be used as a model of relativistic particles which possess the mass spectrum.

The Dirac equation in \((1+4)\)-dimensional space is considered in more detail
Symmetries of Equations of Quantum Mechanics

In conclusion we have discussed briefly the Dirac-type equations being invariant under the groups $P(1,n)$. By this are meant the equations of the form (27.5) where the summation over $m$ is extended to the case when $0 \leq m \leq n$, and $\gamma_m$ are matrices realizing a representation of the Clifford algebra of dimension $n+1$.

The $P(1,n)$-invariant equations (27.5) exist for any $n > 0$. Minimal dimension of the matrices $\gamma_m$ is equal to $2^{(n+1)/2} \times 2^{(n+1)/2}$ for any given $n$ where $[a]$ is the entire part of $a$. In the case of half integer $n$ the corresponding equations are invariant under the complete group $\tilde{P}(1,n)$ including all the possible reflections of coordinates. However if $n$ is integer then it is necessary to double the number of the components of the wave function in order to satisfy the requirement of $P$-invariance [115,136].

The Dirac equations being invariant under the groups $P(1,n), n=5,6,...$ can also serve as a mathematical models of the particles with a variable mass. Furthermore, these equations can be used to describe many-particle systems (see [136] and Subsection 32.5 of the present book).

26.3. The Generalized Kemmer-Duffin-Petiau Equation

The KDP equation also admits a direct generalization to the case of (1+4)-dimensional Minkowsky space since the equations (27.4) are obviously satisfied by the matrices

$$\Gamma_\mu = \beta_\mu, \quad \Gamma_4 = \beta_4, \quad S_{mn} = i[\beta_m, \beta_n]. \quad (27.15)$$

Here $\beta_\mu, \beta_4$ are the 10×10 KDP matrices which can be chosen, e.g., in the form (6.22), (6.24).

The equation (27.3), (27.15) is invariant under the algebra $AP(1,4)$ and thus under the Poincaré and Galilei algebras. Like the five-dimensional analog of the Dirac equation (27.5), this equation is non-invariant under the complete group $\tilde{P}(1,4)$ including space-time reflections.

The generalized KDP equation can be interpreted as an equation of motion of a relativistic particle with a variable mass and spin $s=1$. To make such an interpretation more clear we represent solutions of this equation in the form (27.9). In this way we come to the following equation

$$[\bar{\beta}_\mu p_\mu - \bar{\beta}_4 p_4 - \kappa - \bar{\beta}_4 p_4] \Psi_\nu = 0. \quad (27.16)$$

For any fixed value of $p_4$ the matrix $\kappa + \bar{\beta}_4 p_4$ is invertible; moreover

$$M = m(\kappa + \bar{\beta}_4 p_4)^{-1} = m + \frac{p_4^2}{\kappa m} \beta_4 - \frac{p_4}{m} \beta_4,$$

where $m = (\kappa^2 + p_4^2)^{1/2}$. Multiplying (27.16) by $M$ we come to the equivalent equation

334
The KDP equation in (1+4)-dimensional Minkowsky space can be interpreted as a motion equation of a Galilean particle of the spin 1 and a variable mass. To make such an interpretation it is convenient to come to the new variables (26.20c) and to impose the Galilei-invariant additional condition (27.12) on $\Psi$. As a result we come to the equation (27.13) where

$$\beta_0 - \beta_0 + \beta_s - \beta_0 - \beta_s, \quad \beta_a - \beta_0.$$  

(27.18)

For any fixed value of $m$ the equation (27.13), (27.18) coincides with the Galilei-invariant equation for a particle of spin 1, see Subsection 13.3.

Thus the generalized KDP equation can be used for the description of relativistic and Galilean particles with variable masses. Besides, the corresponding Galilei-invariant model of a spin-one particle takes into account the spin-orbit coupling in the frames of the minimal interaction principle, see Subsection 13.4.

We note that the KDP equation admits a generalization to the case of the Minkowsky space of arbitrary dimension 1+n. The corresponding representations of the $\beta$-matrices were described in [265].

The almost evident generalizations to the case of an (1+n)-dimensional Minkowsky space is admitted by the Bhabha equation also, i.e., by the equations of the form (6.2) where $\beta_s=1$, and $\beta_s=S_{sp}$ are matrices realizing a representation of the algebra $AO(1,5)$ together with $S_{pn}=[\beta_p,\beta_s]$. The corresponding equation being invariant under the group $P(1,4)$ has the form

$$(S_{sp} p^n - S_{sp} p^s + \kappa) \Psi = 0.$$  

(27.19)

This formula defines generalized Bhabha equations in the (1+4)-dimensional space, in particular these equations include the Dirac and KDP ones. In the case when $S_{mn}$ realize an arbitrary representation of the algebra $AO(1,5)$ the equations (27.19) can be interpreted as mathematical models of relativistic or Galilean particles with variable masses like the generalized Dirac and KDP equations.

### 27.4. Covariant Systems of Equations

One of possible formulations of wave equations invariant under the group $P(1,4)$ presupposes using the covariant systems of the form [11, 395]

$$p_n \Psi(x) = P_n \Psi(x), \quad P_n = L_{nn} p^n + \kappa L_n,$$  

(27.20)

where $L_n$ and $L_{nn} = L_{nn}$ are numerical matrices.
We have already seen in Subsection 2.2 that the Dirac equation also can be reduced to the form (27.20), refer to (2.12). In contrast to the standard formulation (2.1) none of the equations (2.12) is invariant under Lorentz transformations but is transformed into a linear combination of these equations with the different values of \( n \).

Covariant systems of equations in (1+4)-dimensional Minkowsky space are discussed in papers [11,395]. Here we consider a class of such equations corresponding to the choice \( L_{mn} = S_{mn} / id \), \( L_n = S_n / id \) where \( S_{mn} \) are matrices belonging to the algebra \( AO(1, 5) \). In other words we consider the following systems of the covariant equations

\[
L_n \Psi(x) = 0, \quad L_n = p_n - \frac{1}{id} (S_{mn} p^n + k S_{nm}),
\]

(27.21)

These equations have a manifestly covariant form since the operators \( L_n \) evidently satisfy the relations

\[
[P_m, L_n] = 0, \quad [J_{mn}, L_l] = i (g_{mn} L_m - g_{ml} L_n),
\]

where \( J_{mn} \) are the operators of (26.19).

Let us require that \( \Psi(x) \) should satisfy the KGF equation componentwise. Multiplying (27.21) by \( p^m \) and summing up over \( m \) we obtain the following additional condition for \( \Psi(x) \):

\[
(\frac{1}{id} S_{mn} p^n - k) \Psi(x) = 0.
\]

(27.22)

It is possible to show [11] that this equation is a necessary condition of a consistency of the system (27.21) if the matrices \( S_{mn} \) are finite-dimensional.

Thus let \( S_{mn} \) be finite-dimensional matrices realizing an IR of the algebra \( AO(1, 4) \) and (27.21) be the system of the covariant equations corresponding to this IR. It turns out that this system is consistent only in the following exceptional cases [352]:

(a) \( S_{\mu \nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu] \), \( S_{5\mu} = \gamma_\mu \) \( \mu, \nu = 0, 1, 2, 3, 4 \)

(27.23)

(b), (c) \( S_{\mu \nu} = i [\beta_\mu, \beta_\nu] \), \( S_{5\mu} = \beta_\mu \)

where \( \gamma_\mu \) are 4x4 Dirac matrices, \( \beta_\mu \) are KDP matrices of dimension 10x10 (case b)) or 6x6 (case c)).

In case a) it is necessary to set \( d = 1/2 \) (otherwise the system (27.21) is inconsistent). This leads to the system which is equivalent to the generalized Dirac equation (see Subsection 27.2). In cases b) and c) the system (27.21) possesses nontrivial solutions when \( d = 1 \) only. In case b) the corresponding system turns out to be equivalent to the generalized KDP equation for particles of spin 1 (see Subsection 26.3). In the case c) we have the covariant system of equations describing particles of
Chapter 5. Generalized Poincaré Groups

spin 0.

However if the matrices $S_{mn}$ realize an IR of the algebra $AO(1,5)$ which is not equivalent to the representations enumerated in (27.23) then the corresponding equations (27.22) are inconsistent. The proof of this statement is given in [352]. We see the class of covariant finite dimensional systems of the form (27.21) is exhausted by the representatives corresponding to the matrices (27.23). The analogous result is correct in respect to the covariant systems of equations in the frames of the Poincaré group [352].

Covariant systems of equations are with better prospects in the case when the corresponding matrices $S_{mn}$ realize an infinite dimensional Hermitian representations of the algebra $AO(1,4)$. A well-known example of such an equation in the (1+3)-dimensional Minkowsky space is the Dirac equation with positive energies [82].

The other types of the wave equations invariant under the generalized Poincaré groups have been investigated in papers [115,136,181]. There are the equations of the Bargman-Wigner type, equations with a proper time, equations invariant under the representations of the Classes $II-IV$ etc. An analysis of such equations is beyond the scope of the present book.

\[\text{In papers [352] the covariant infinite-component Dirac equation is generalized to the case of particles of arbitrary spin.}\]
6. EXACT SOLUTIONS OF LINEAR AND NONLINEAR EQUATIONS OF MOTION

We obtain exact solutions of Poincaré- and Galilei-invariant equations of motion of a particle of arbitrary spin $s$ interacting with some particular classes of external fields. Besides we present solutions of a number of nonlinear equations of modern theoretical physics. The key to finding these solutions is using symmetries of the equation considered.

28. EXACT SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS FOR PARTICLES OF ARBITRARY SPIN

28.1. Introduction

The main purpose of this and the following sections is the construction of solutions of motion equations of arbitrary spin particles in the external electromagnetic field.

The number of known exact solutions of relativistic wave equations is very small even for the cases $s=0$ and $s=1/2$. The most interesting of them (from the physical point of view) are the following problems: a particle in the Coulomb field [73, 77], a particle in the plane-wave field [407], a particle in the constant magnetic field [365] and a particle in the Redmond field [370] (the last is a combination of the constant magnetic and plane-wave fields). Finally we mention the problem of interaction of a charged particle with the magnetic monopole field which also admits exact solutions for $s=0, 1/2$ [217, 402] and for arbitrary spin [164].

The construction of exact solutions for particles of spin $s>1/2$ is complicated in accordance with increasing of number of components of the corresponding wave function. Besides there appear additional difficulties, i.e., causality violation [405], the absence of stable solutions in the Coulomb problem [401] and many others. In spite of that a number of exact solutions for the vector particles have been obtained in the papers [46, 47, 266, 267, 270].

The problems enumerated in the above have physically reasonable solutions for arbitrary spin particles if we start from the motion equations (10.10). The most important of these solutions are considered in the following.
28.2. Free Motion of Particles

The knowledge of the explicit form of wave functions of non-interacting particles is necessary for many problems of theoretical physics [41,42]. That is why we present the explicit solutions of (10.10) for the case of absence of an external field (i.e., for $A_\mu=F_{\mu\nu}=0$).

Instead of the equations (10.10) it is more convenient to consider the equivalent system (10.30) which takes the following form

\[(p_\mu p^\mu - m^2)\psi^{(-)}=0,\]  \hspace{1cm} (28.1a)

\[P_\mu \psi^{(-)}=\frac{1}{4s}[S_{ab}S_{ab} - 2s(s-1)],\]  \hspace{1cm} (28.1b)

\[\psi^{(-)}=\frac{1}{m}P_\mu \psi^*, \quad \psi^{(+)}=\frac{1}{2}(1\pm i\Gamma_\mu)\psi.\]  \hspace{1cm} (28.1c)

We will use the representation (9.14) for the matrices $\Gamma_\mu$ and choose $S_{\mu\nu}$ in the following form

\[
\begin{pmatrix}
\tilde{S}_{ab} & 0 \\
0 & \tilde{S}_{ab}
\end{pmatrix},
\begin{pmatrix}
S'_{ab} & 0 \\
0 & S''_{ab}
\end{pmatrix}
\]

where \{\tilde{S}_{ab}, S'_{ab}\} and \{\tilde{S}_{ab}, S''_{ab}\} are matrices realizing the representations $D(s\,0)\oplus D(s\,1\,0)$ and $D(s\,1/2\,1/2)$ of the algebra $AO(1,3)$. These matrices can be written in the form (compare with (4.64))

\[
\tilde{S}_{ab} = \varepsilon_{abc}S_c = \begin{pmatrix} S_e & 0 \\ 0 & S_e \end{pmatrix}, \quad S'_{oa} = -i\tilde{S}_{oa}, \quad S''_{oa} = -i\begin{pmatrix} (s-1)S_a & K_a \\ K_a & (s+1)S_a \end{pmatrix}
\]

where $S_a$ and $S'_a$ are generators of the IRs $D(s)$ and $D(s\,-1)$ of the group $O(3)$, $K_a = K_a^\dagger$ are matrices of dimension $(2s-1)(2s+1)$ given in (4.66). Then we obtain the following realization for the matrices $\sigma_a$ of (9.14)

\[
\sigma_a = \frac{1}{2} \varepsilon_{abc}S_{bc} - iS''_{oa} = \begin{pmatrix} S_a & -K_a^\dagger \\ K_a & -S_a \end{pmatrix}
\]

Let us find the general solution of the system (28.1) in the representation (9.14), (28.2)-(28.4). It is convenient to represent this solution in the form of the Fourier integral

\[
\psi(x_0, x) = \frac{1}{2E} \left\{ \psi_+ \exp \left[ i(p\cdot x - Ex_0) \right] - \psi_- \exp \left[ i(p\cdot x + Ex_0) \right] \right\}
\]

339
where $E=(p^2+m^2)^{1/2}$ and the Fourier transforms $\tilde{\psi}_\pm$ satisfy according to (28.1) the following conditions

\[ (\varepsilon \Gamma_0 E - \Gamma^\sigma p)\tilde{\psi}_\pm^{(\alpha)} - m\tilde{\psi}_\pm^{(\alpha)} = \frac{1}{2} (1 \pm \varepsilon i \Gamma_4)\tilde{\psi}_\pm, \tag{28.6} \]

\[ P\tilde{\psi}_\pm^{(\alpha)} = \tilde{\psi}_\pm^{(\alpha)}, \quad \varepsilon = \pm 1. \]

In the representation (9.14), (28.2)-(28.4) $\tilde{\psi}_\pm^{(\alpha)}$ are columns of the following form

\[ \tilde{\psi}_\pm^{(\alpha)} = \begin{pmatrix} \Phi_\pm^{(\alpha)} \\ 0 \end{pmatrix}, \quad \tilde{\psi}_\pm^{(\beta)} = \begin{pmatrix} 0 \\ \Phi_\pm^{(\beta)} \end{pmatrix} \tag{28.7} \]

where $\Phi_\pm^{(\alpha)}$ are 4-component spinors, $0$ are zero columns having 4 rows. From (28.6), (9.14), (28.2)-(28.4) we conclude that

\[ \Phi_\pm^{(\alpha)} = \begin{pmatrix} \tilde{\varphi}_\pm^{(\alpha)} \\ 0 \end{pmatrix}, \quad \Phi_\pm^{(\beta)} = \begin{pmatrix} \frac{1}{m} (\varepsilon E + \sigma \cdot p) \tilde{\varphi}_\pm^{(\beta)} \\ \frac{1}{m} \frac{1}{s} K \cdot p \tilde{\varphi}_\pm^{(\beta)} \end{pmatrix}. \tag{28.8} \]

Here $\varphi_\pm^{(\alpha)}$ is an arbitrary $(2s+1)$-component spinor, $0$ is the zero column having 4 rows.

Thus the general solution of the system (10.10) in the case $A_\mu = F^\lambda \sigma = 0$ can be represented in the form (28.5) where

\[ \tilde{\psi}_\pm = \tilde{\psi}_\pm^{(\alpha)} + \tilde{\psi}_\pm^{(\beta)} \tag{28.9} \]

and $\varphi_\pm^{(\alpha)}$ are the spinors given in (28.7), (28.8). This solution is defined up to arbitrary functions $\varphi_\pm^{(\alpha)}$ each having 2s+1 components. These functions can be represented in the form

\[ \varphi_\pm^{(\alpha)} = \sum_{i=0}^{m} b_i^{(\alpha)}(p) \eta_i^{(\alpha)} \tag{28.10} \]

where $\eta_i^{(\alpha)}$ are normalized eigenvectors of the matrix $S_\pm$:

\[ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \eta_i^{(\alpha)} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \eta_i^{(\alpha)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{28.11} \]

\[ \frac{m}{2E(E + \frac{\sqrt{s}}{s} p_3)} \]

is a normalizing multiplier, $b_i^{(\alpha)}(p)$ are arbitrary square integrable.
functions.

Substituting (28.10) into (28.5), (28.7), (28.8) and using the explicit form (4.65), (4.66) of the matrices $S$ and $K$, we obtained the corresponding general solutions in the form

$$\psi = \sum_{\nu} \int \frac{d^3p}{2E} b^*_{\nu}(p) \psi^\nu \exp[i(p \cdot x - \epsilon x_0/E)]$$

where

$$\psi^\nu = \frac{m}{\sqrt{2E(E^2 - Vp_3)}} \begin{pmatrix} \eta^\nu \cr \hat{\delta} \cr \chi^\nu \cr \tilde{\chi}^\nu \end{pmatrix},$$

$$\chi^\nu = (\epsilon E + \frac{V}{s} p_3) \eta^\nu + \frac{1}{2s} \sqrt{s(s+1) - V(s+1)} (p_1 - ip_2) \eta^\nu_{\nu-1} +$$

$$+ \frac{1}{2s} \sqrt{s(s+1) - V(s+1)} (p_1 + ip_2) \eta^\nu_{\nu+1},$$

$$\tilde{\chi}^\nu = \frac{1}{s} \sqrt{s^2 - V^2 + p_3} \eta^\nu_{\nu+1} - \frac{1}{2s} \sqrt{s(s+1) - V(s+1) + 2V} \eta^\nu_{\nu+1} -$$

$$-ip_2 \eta^\nu_{\nu-1} - \frac{1}{2s} \sqrt{s(s+1) - V(s+1) + 2V} \eta^\nu_{\nu+1},$$

$\hat{\delta}$ is a column containing 2s-1 zeros.

The spinors $\psi_{\nu}^{(z)}$ satisfy the normalization condition

$$\bar{\psi}_{\nu}^{(z)\nu} \psi_{\nu}^{(z)\nu} \delta_{\nu\nu},$$

and form a basis of solutions of the Dirac-type equations for a particle of arbitrary spin.

### 28.3. Relativistic Particle of Arbitrary Spin in the Homogeneous Magnetic Field

Consider the movement of a charged particle in the constant and homogeneous magnetic field. Without loss of generality, we choose the vector of the magnetic field strength be parallel to the third projection of the particle momentum, i.e.,

$$F_{0\alpha} = F_{23} = F_{31} = 0, \quad F_{12} = H_3 = H$$

where $F_{\mu\nu}$ is the tensor of the electromagnetic field. The corresponding four-vector $\pi$ can be chosen in the form

$$\pi_{\mu} = p_{\mu}, \quad \pi_1 = p_1 - eHx_2, \quad \pi_2 = p_2, \quad \pi_3 = p_3.$$
As in the case of a free particle we will solve the equivalent system (10.30) instead of the equations (10.10). Using the representation (9.14), (28.2)-(28.4) we conclude that the general form of solutions corresponding to the energy \( \varepsilon \) is given by the following formula
\[
\psi = \begin{pmatrix} \Phi_s \\ \hat{0} \\ \frac{1}{m} (\varepsilon + \frac{1}{s} S \cdot \pi) \Phi_s \\ -\frac{1}{ms} K \cdot \pi \Phi_s \end{pmatrix}
\]
(28.16)
where \( \hat{0} \) is a column including \( 2s-1 \) zeros, \( \Phi_s \) is a \((2s+1)\)-component spinor satisfying the equation
\[
\left[ p^2 + e^2 H \hat{x}_s^2 - eH \left( \frac{1}{s} S_y + 2x_s p_s \right) \right] \Phi_s = (\varepsilon^2 - m^2) \Phi_s.
\]
(28.17)

It is convenient to expand \( \Phi_s \) in a complete set of eigenvectors of the matrix \( S_3 \):
\[
\Phi_s = \sum \varphi_i \eta_i
\]
(28.18)
where \( \eta_i \) are the spinors of (28.11), \( \varphi_i \) are unknown functions which must satisfy the following equations according to (28.13):
\[
\left[ p^2 + e^2 H \hat{x}_s^2 - eH \left( \frac{\nu}{s} + 2x_s p_s \right) \right] \varphi_i = (\varepsilon^2 - m^2) \varphi_i.
\]
(28.19)

We search for solutions of (28.19) in the form
\[
\varphi_i = \exp\left[ i(p_1 x_s + p_3 x_3) \right] f_i(x_2)
\]
(28.20)
where \( p_1 \) and \( p_3 \) are constants. As a result we obtain the following equation for \( f_i(x_2) \):
\[
-\frac{d^2}{dx_2^2} + \left( eHx_2 - p_3 \right)^2 - \frac{\nu H}{s} f_i(x_2) = (\varepsilon^2 - m^2 - p_3^2) f_i(x_2).
\]
(28.21)

With the help of the change of variables
\[
x_2 = \frac{1}{\sqrt{eH}} (p_1 + \sqrt{eH} y), \quad \varepsilon^2 - m^2 - p_3^2 + e \frac{\nu}{s} H = \xi H
\]
we reduce (28.21) to the equation for the harmonic oscillator...
Chapter 6. Exact Solutions of Linear and Nonlinear ...

\[ \left( \frac{d^2}{dy^2} + y^2 \right) f_\nu(y) = \xi f_\nu(y) \tag{28.22} \]

Requiring, that \( f_\nu(y) \) tends to zero for \( y \to \pm \infty \), we obtain the well-known expression for \( \xi \) [3]

\[ \xi - 2n + 1, \quad n = 0, 1, ..., \tag{27.23} \]

from which it follows that

\[ \epsilon^2 = m^2 + p_s^2 + eH(2n + 1 - s/2) \tag{28.24} \]

The relation (28.24) generalizes the well-known formula [3,31] for energy levels of an electron in the constant and homogeneous magnetic field to the case of a particle of arbitrary spin.

To clarify the physical meaning of the expression (28.24) for particle energies, we consider the non-relativistic approximation

\[ |\epsilon| = \sqrt{m^2 + p_s^2 + eH(2n + 1 - s/2)} \equiv m + \frac{p_s^2}{2m} + \Omega \left( n + \frac{s - \nu}{2s} \right) \tag{28.25} \]

where \( \Omega = eH/m \) is the cyclotronic frequency.

We see that \( |\epsilon| \) includes the kinetic energy of the particle movement along the magnetic field, and the quantized part of energy depending on two discrete parameters, i.e., \( n \) and \( \nu \). When \( n = 1/2 \), \( \nu \) can take two values; \( \nu = \pm 1/2 \). The corresponding quantized part of energy determines the Landau levels and turns out to be proportional to an integer. Besides any level (excepting the ground one) is twice generated.

For the case \( s = 0 \) the Landau levels are nondegenerated and proportional to half integers. For \( s = 1 \) these levels are proportional to either integers or half integers besides all the integer levels are twice degenerated (the ground level and half integer ones are nondegenerated). For \( s > 1 \) the energy spectrum is more complicated but also includes generated levels corresponding to \( \nu = \pm s \). These levels are proportional to integers.

Let find the explicit form of the wave function satisfying the equation (10.10). The solutions of (28.22), (28.23) are given by the formula [3]

\[ f_\nu = C_\nu U_n(y), \tag{28.26} \]

where \( C_\nu \) is a normalizing multiplier, \( U_n(y) \) is the Hermite function

\[ U_n(y) = \frac{(eH)^n}{(2^n n! \pi^{1/2})^{1/2}} e^{-y^2/2} H_n(y), \tag{28.27} \]

\( H_n(y) \) is the Hermite polynomial which can be written in the form

343
Symmetries of Equations of Quantum Mechanics

\[ H_n(y) = (-1)^n e^{\frac{\partial}{\partial y}} e^{-y}, \quad n = 0, 1, \ldots \]  \hspace{1cm} (28.28)

Using formulae (4.65), (4.66) defining the matrix elements of \( S \) and \( K \), bearing in mind the recurrence relations

\[ \left( \frac{\partial}{\partial y} \pm y \right) U_n = \pm (2n + 1 \pm 1)^{1/2} U_n, \]  \hspace{1cm} (28.29)

and the following identity

\[ A \cdot \Phi_n = \left( A_1 P_3 + \frac{\sqrt{eH}}{2} \left( (A_1 - iA_2) \left( y + \frac{\partial}{\partial y} \right) + (A_1 + A_2) \left( y - \frac{\partial}{\partial y} \right) \right) \right) \Phi_n \]

(A = S or A = K), we obtain

\[ \Psi_{vn} = \frac{\exp[i(p_x x_1 + p_y y)_{1}]}{\sqrt{2L_1 L_3 \epsilon \left( e + \frac{\sqrt{eH}}{p_3} \right) \Phi_{vn}} \chi_{vn}} \]  \hspace{1cm} (28.30)

where \( \hat{0} \) is a \((2s-1)\)-component zero column, \( L_1 \) and \( L_2 \) are normalization constants,

\[ \varphi_{vn} = \left( e + \frac{\sqrt{eH}}{p_3} \right) U_n(y) \eta_{i1}^{\dagger} + \frac{1}{2s} \sqrt{[s(s-1)-\nu(\nu+1)]2eH(n+1)} U_{n+1}(y) \eta_{i1}^{\dagger} + \]

\[ -\frac{1}{2s} \sqrt{[s(s-1)-\nu(\nu+1)]2eH} U_{n-1}(y) \eta_{i1}^{\dagger} \]

\[ \chi_{vn} = -\frac{1}{3} \sqrt{1} \nu^2 \nu^2 \sqrt{eH} U_{n+1}(y) \eta_{i1}^{\dagger} \]

\[ + \frac{1}{2s} \sqrt{[s(s-1)-\nu(\nu+1)+2\nu s]2eH(n+1)} U_{n+1}(y) \eta_{i1}^{\dagger} + \]

\[ + \frac{1}{2s} \sqrt{[s(s-1)-\nu(\nu+1)-2\nu s]2eH} U_{n-1}(y) \eta_{i1}^{\dagger}. \]

Here \( U_n \) is the Hermit function (28.27), \( \eta_{i1} \) are the spinors (28.11), \( \gamma = (eH)^{1/2}(x_2, p_x)/eH \).

The wave functions (28.30) are normalized in accordance with the following relation

\[ \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} \Psi_{vn}^* \Psi_{vn} \, dx_1 \, dx_2 = 1 \]

where \( L_1 \) and \( L_2 \) are arbitrary numbers included into the normalization constant in (28.30).
So using the equations of the Dirac type (10.10), we have obtained exact solution of the problem of interaction of charged spinning particle with the constant and homogeneous magnetic field for any value of spin.

The equations (10.10) admit exact solutions also for the cases of the constant homogeneous electric field, the combination of the electric and magnetic fields mentioned above [400] and the field of the magnetic monopole [164]. We do not represent here the corresponding cumbersome formulae but consider the problem of motion of any spin particle in the constant electric field in Section 30.

28.4. A Particle of Arbitrary Spin in the Field of the Plane Electromagnetic Wave

The equation (10.10) admits exact solutions in the important case of the external field reducing to the plane wave [322].

The plane wave field characterized by the wave vector \( k_\mu \) \( (k_\mu k^\mu = 0) \) is determined by the following vector-potential

\[
A_\mu = A_\mu (\phi), \quad \phi = k x^\mu
\]

satisfying the Lorentz gauge condition

\[
p_\mu A^\mu = -ik_\mu A^\mu = 0,
\]

where the prime denote the derivative with respect to \( \phi \).

As in previous subsection we can represent solutions of the corresponding equations (10.10) in the form (28.16) where \( \Phi_s \) is a \((2s+1)\)-component spinor satisfying the second order equation (10.30a). In our case this equation takes the form

\[
\left(p_\mu p^\mu - 2e A_\mu p^\mu + e^2 A_\mu A^\mu - m^2 - \frac{e S \cdot F}{s}\right) \Phi_s = 0
\]

where \( F = k \times A' - i(k_\mu A' - k A'_\mu) \).

Like in [407], we look for the solution of (28.33) in the form

\[
\Phi_s = \exp(-\tilde{p}_\mu x^\mu) \psi(\phi),
\]

where \( \tilde{p}_\mu \) is a constant four-vector besides without loss of generality \( \tilde{p}^\mu \tilde{p}_\mu = m^2 \). Then using the relations

\[
p_\mu \psi(\phi) = -ik_\mu \psi'(\phi), \quad p_\mu p^\mu \psi(\phi) = k_\mu k^\mu \psi''(\phi)
\]

we obtain from (28.33) the following equation for \( \psi(\epsilon) \):

\[
2ik_\mu \tilde{p}_\mu \psi' \left[ -2e \tilde{p}_\mu A^\mu + e^2 A_\mu A^\mu - \frac{e S \cdot F}{s} \right] \psi = 0.
\]
This equation is easily integrated

$$\psi = \exp \left( -i \int_0^{k \cdot \tilde{p}} \left( \frac{e}{k \cdot \tilde{p}^\mu \tilde{p}^\nu} A^\mu - \frac{e^2}{2k \cdot \tilde{p}^\mu} A_\sigma A^\sigma \right) d\varphi - \frac{i e S \cdot F}{2s k \cdot \tilde{p}^\mu} \right) U_\mu$$

(28.35)

where $U_\mu$ is an arbitrary constant spinor.

The matrices $S \cdot F$ of (28.35) satisfy the relations

$$\prod_\lambda ((S \cdot F)^2 - \chi^2 F) = 0, \quad \chi = 1/2, 3/2, \ldots, s$$

besides the first formula is available for half integer $s$ and the second formula is valid for integer $s$. Inasmuch as $F^2 = k_\mu k_\nu A_\nu A^\nu$ and $k_\mu k_\nu = 0$ the conditions (28.36) reduce to the form $(S \cdot F)^{2s+1} = 0$, and we have from (28.34), (28.35)

$$\Phi = \exp(iS) \sum_{n=0}^{2s} \frac{1}{n!} \left( \frac{i e S \cdot F}{2s k \cdot \tilde{p}^\mu} \right)^n,$$

(28.37)

where $S$ is the classical action of the particle moving in the plane wave field [271]

$$S = -\tilde{p} \cdot \chi^\nu - \int_0^{k \cdot \tilde{p}} \left( \frac{e}{k \cdot \tilde{p}^\mu} \tilde{p}^\nu A^\nu - \frac{e^2}{2k \cdot \tilde{p}^\mu} A_\sigma A^\sigma \right) d\varphi.$$

We can choose an arbitrary spinor $U_\mu$ of (28.36) in accordance with the requirement that the corresponding wave function (28.16) reduces to the plane wave solution for a free particle if $A_\mu \to 0$. As a result, we obtain the following solutions

$$\tilde{\psi} = \sum_{\sigma=0}^{2s} \frac{1}{n!} \left( \frac{i e S \cdot F}{2s k \cdot \tilde{p}^\mu} \right)^n \exp(iS) \tilde{\psi}_\nu$$

where $\tilde{\psi}_\nu$ are the spinors (28.13).

We see that in contrast to the Volkov solution for an electron in the field of the plane electromagnetic wave [407] the solutions of relativistic equations for particles of arbitrary spin depend on the fields strength non-linearly (as a polynomial of order $2s$).

We note that the equations (10.10) can be solved exactly also for the case of the Redmond field which is a combination of the constant magnetic field and the field of the plane wave [400].
29. RELATIVISTIC PARTICLE OF ARBITRARY SPIN IN THE COULOMB FIELD

29.1. Separation of Variables in a Central Field

The problem of description of a spin particle in a central field is one of the basic problems of quantum mechanics. In the case of an electron in the Coulomb field, it is the problem of the hydrogen atom.

The problem of description of the hydrogen-type system in which the spin of the orbital particle is more than 1/2 is not of theoretical interest only, since such relatively stable particles as $W^-$-boson, $\Omega ^-$-hyperon in principle can play a role of orbital particles in exotic atoms.

Following [322] we present exact solutions of the equations (10.10) for a charged particle of arbitrary spin in the Coulomb field. It will be shown these equations do not lead to the difficulties connected with a possibility of particle falling to the center (such a situation is non-admissible from quantum mechanics point of view; nevertheless it is predicted e.g. by the KDP equation [401]).

Consider equations (10.30) for the case of the Coulomb field where the vector-potential reduces to the form (22.8). Such equations admit solutions in separated variables besides the general scheme of obtaining these solutions is valid for any central potential $A_0(x)$, $x = |x|$.

We use the representation (10.31) for the equation (10.30a). Solutions corresponding to a state with energy $\varepsilon$ are represented in the form

$$\Phi^\varepsilon = \exp(-i\varepsilon t)\Phi^\varepsilon(x),$$

where $\Phi^\varepsilon(x)$ is a $(2s+1)$-component spinor satisfying the equation

$$\left[ \begin{pmatrix} \varepsilon - \frac{Z e^2}{x} & -p^2 - m^2 & -ie \frac{S \cdot x}{s} \frac{x}{x^3} \\ -p^2 - m^2 & \varepsilon - \frac{Z e^2}{x} & ie \frac{S \cdot x}{s} \frac{x}{x^3} \end{pmatrix} \Phi^\varepsilon(x) \right] = 0.$$  \hspace{1cm} (29.2)

This equation admits solutions in separated variables. Indeed, bearing in mind the symmetry of (29.2) in respect with the group $O(3)$ we represent $\Phi^\varepsilon(x)$ as a linear combination of spherical spinors

$$\Phi^\varepsilon(x) = \sum_k \phi^\varepsilon_k(x) \Omega^\varepsilon_{j\lambda\mu},$$  \hspace{1cm} (29.3)

where $\Omega^\varepsilon_{j\lambda\mu}$ is a complete set of eigenfunctions of the commuting operators $J^2, L^2, J_3$ and $S^2$ ($J = x \times p + S, L = x \times p$), so that
Symmetries of Equations of Quantum Mechanics

Here \( j \) are arbitrary positive integers or half integers,

\[
\begin{align*}
J^{(2)} &\Omega_{jj,\lambda m} = -j(j+1)\Omega_{jj,\lambda m}, \\
L^{(2)} &\Omega_{jj,\lambda m} = -(j-\lambda)(j-\lambda-1)\Omega_{jj,\lambda m}, \\
S^{(2)} &\Omega_{jj,\lambda m} = s(s+1)\Omega_{jj,\lambda m}, \\
J^{(3)} &\Omega_{jj,\lambda m} = m\Omega_{jj,\lambda m}.
\end{align*}
\]

(29.4)

We find it is more convenient to use the quantum number \( \lambda = j-l \) instead of the usual orbital quantum number \( l \). The explicit expressions for the \( \Omega_{jj,\lambda m} \) are given in Appendix 3.

The representation (29.3) makes it possible to reduce (29.2) to the system of ordinary differential equations for the function \( \phi \). To find this system, it is necessary to know the action of the operator \( S \cdot x \) on spherical spinors. It is not difficult to make sure that

\[
S \cdot \hat{x} \Omega_{jj,\lambda m} = d^{(s)}_{\lambda,\lambda'} \Omega_{jj,\lambda' m}
\]

(29.6)

where \( d^{(s)}_{\lambda,\lambda'} \) are numeric coefficients, \( \hat{x} = x/x \), besides the summation is imposed over the repeated indices \( \lambda' \). Indeed, \( S \cdot x/x \) commutes with \( J^2 \) and hence with \( J^2 \) and \( J^3 \). This is why in the l.h.s. and r.h.s. of (29.6) we have an eigenfunction of \( J^2 \) and \( J^3 \) corresponding to the same eigenvalues. Expressing such an eigenfunction via the complete set of the spherical spinor we come to the relation (29.6).

The exact values of the coefficients \( d^{(s)}_{\lambda,\lambda'} \) are calculated in Appendix 3 and can be represented in the form

\[
d^{(s)}_{\lambda,\lambda'} = \frac{1}{2} \left\{ \delta_{\lambda-1,\lambda'} a^{(s)}_{\lambda,\lambda'} + \delta_{\lambda+1,\lambda'} a^{(s)}_{\lambda,\lambda'} \right\}.
\]

(29.7)

where

\[
a^{(s)}_{\lambda} = \mu(2j+1-\mu)(2s+1-\mu)(2j+2s+2-\mu) \right)^{1/2}
\]

(29.8)

besides the possible values of \( \lambda, \lambda'+1, \lambda'-1 \) are given in (29.5).

Substituting (29.3) into (29.2) and using (29.6) and the following identity

\[
p^2 = \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial}{\partial x} \right) - \frac{L^2}{x^2}
\]

(29.9)

we come to the following equations for radial functions
Chapter 6. Exact Solutions of Linear and Nonlinear

where

\[ D = \left( e + \frac{\alpha}{x} \right) - m^2 + \frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}, \]

\[ b_{\lambda\lambda}^{(j)} = \lambda^2 - \lambda(2j+1) \delta_{\lambda\lambda} - i \frac{\alpha}{s} d_{\lambda\lambda}^{(j)}, \quad \alpha = e \epsilon z. \]

Thus a problem of description of a charged particle of arbitrary spin in the Coulomb field reduces to solving the system of ordinary differential equation (29.10).

29.2. Solution of Equations for Radial Functions

The matrix of (29.10) commutes with the operator \( D \) and is normal, i.e.,

\[ \{ b_{\lambda\lambda}^{(j)} \}_x b_{\lambda\lambda}^{(j)} - b_{\lambda\lambda}^{(j)} (b_{\lambda\lambda}^{(j)})^*. \]

It means this matrix can be diagonalized and so the system (29.10) can be reduced to the following chain of noncoupled equations

\[ D\phi = x^{-2} b_{\lambda\lambda}^{(j)} \phi \]

where \( D \) is the operator (29.11), \( b_{\lambda\lambda}^{(j)} \) are eigenvalues of the matrix \( b_{\lambda\lambda}^{(j)} \). Any of the equations (29.12) in its turn reduces to the well-known equation [105]

\[ z \frac{d^2 y}{dz^2} + \frac{dy}{dz} \left[ \beta - \frac{z}{4} k^2 \right] y = 0 \]

where

\[ \beta = \frac{e \alpha}{(m^2 - \alpha^2)^{1/2}}, \quad k^2 = (2j+1)^2 + 4(b_{\lambda\lambda}^{(j)})^2 - 4 \alpha^2. \]

The equations (29.13) arise in the problem of the hydrogen atom (besides in this case \( s = 1/2, b_{\lambda\lambda}^{(j)} = 1/4 \pm (2j+1 - 4 \alpha^2)^{1/2} \)). In the case of an arbitrary spin particle we have the only new feature that the parameter \( k \) can take another values depending on \( s \).

We represent the solutions of (29.13) corresponding to coupled states \( (m^2 > e^2) \) in the form

\[ y = \zeta^{1/2} \exp(-\zeta/2) F \]

and obtain the following equation for \( F \)

349
 Formula (29.16) defines the equation for a degenerated hypergeometric function, thus the corresponding solutions have the form
\[
 F = C \mathcal{G} \left( \frac{k_\lambda + 1}{2} - \beta, k_\lambda + 1, z \right) \tag{29.17}
\]
where \( C \) is an arbitrary constant, \( \mathcal{G} \) is a degenerated hypergeometric function. The analytical expression for \( \mathcal{G} \) \((a,b,z)\) is given by the following formula
\[
 \mathcal{G} \cdot \Gamma(b) \sum_{n=0} \frac{\Gamma(a+n)z^n}{\Gamma(b+n)n!}, \tag{29.18}
\]
where \( \Gamma(a) \) is the \( \Gamma \)-function of Euler.

It follows from the boundary condition for the solutions (29.15) at infinity [31] that the argument \( a=(k_\lambda+1)/2-\beta \) has to be a negative integer or zero, i.e.,
\[
 \beta = (k_\lambda + 1)/2 + n', \quad n' = 0, 1, \ldots \tag{29.19}
\]

So we have obtained the solutions of the equations (29.12) in the form
\[
 \Phi_\lambda = C(m^2 - \varepsilon^2)^{\frac{k_\lambda - 1}{4}} e^{-\varepsilon x} \mathcal{G} \left( -n', k_\lambda + 1, 2(m^2 - \varepsilon^2)^{1/2}x \right) \tag{29.20}
\]
where the index \( \lambda \) labels the solutions corresponding to possible values of \( k_\lambda \). A solution of (29.2) can be written in the form
\[
 \Phi' = U_{\lambda\lambda} \Phi, \quad \Omega_{\lambda j} \psi_j \tag{29.21}
\]
where \( U_{\lambda\lambda} \) are the matrix elements of the unitary matrix diagonalizing \( b_{\lambda\lambda}^{ij} \) of (29.11) so that
\[
 \sum_{\lambda'} U_{\lambda\lambda'} \Phi_{\lambda'} = \Phi_{\lambda} \tag{29.22}
\]

The corresponding solutions of (10.10) in the representation (9.14), (28.2)-(28.4) have the form
\[
 \psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \quad \psi' = \left( \begin{array}{c} \Phi' \\ 0 \end{array} \right), \quad \psi_2 = \frac{1}{m} (\sigma \cdot \mathbf{p}) \psi_1, \tag{29.23}
\]
where \( \hat{0} \) is the zero column including \( 2s-1 \) rows, \( \sigma \) are the \( 4s \times 4s \) matrices of (28.4), \( \sigma_0 \) is the unity matrix of dimension \( 4s \times 4s \).

Substituting the matrix \((\sigma \cdot \mathbf{x})(\sigma \cdot \mathbf{x})/\lambda^2 = \sigma_0\) between \( \sigma \cdot \mathbf{p} \) and \( \psi_1 \), we obtain
\[
 \sigma \cdot \mathbf{p} \psi_1 = \left[ -i \partial \frac{\sigma \cdot \hat{x}}{\partial x} + \frac{2}{x} \hat{x} \right] \psi_1, \tag{29.24}
\]
Chapter 6. Exact Solutions of Linear and Nonlinear...

The action of $\sigma \cdot x/\alpha$ and $\sigma \cdot x \cdot p$ on spherical spinors is described in Appendix 3 (see (A.3.10)) thus we obtain finally

$$\psi_2 = U_{\lambda \lambda}^{\prime} \left\{ \varpi \varphi_\lambda \Omega^{\prime \lambda m}_{\lambda m} + \right.$$  
$$+ \left[ - \frac{i}{s} d_{\lambda \lambda}^{(j)} \left( \frac{d}{dx} + \frac{2}{x} \right) - \frac{i}{2s} \frac{2}{x} \left( d_{\lambda \lambda}^{(j)} f^{(i)}(\lambda) + ib_{\lambda \lambda}^{(j)} g^{(i)}(\lambda) \right) \right] \varphi_\lambda \Omega^{\prime \lambda m}_{\lambda m} +$$  
$$+ \left. \left[ \frac{1}{s} d_{\lambda \lambda}^{(j)} \left( \frac{\partial}{\partial x} + \frac{2}{x} \right) + \frac{1}{2s} \frac{2}{x} \left( d_{\lambda \lambda}^{(j)} g^{(i)}(\lambda) + ib_{\lambda \lambda}^{(j)} f^{(i)}(\lambda) \right) \right] \varphi_\lambda \Omega^{\prime \lambda m}_{\lambda m} \right\}. \tag{29.22}$$

where $d^{(j)}_{\lambda \lambda}$, $b^{(j)}_{\lambda \lambda}$, $f^{(i)}(\lambda)$ and $g^{(i)}(\lambda)$ are the coefficients given in (29.7), (A 3.4).

Formulae (29.19)-(29.22) define solutions of the Dirac-type equations for a particle of arbitrary spin in the Coulomb field. For any value of spin $s$ we can determine the exact value of an arbitrary constant $C$ starting from a normalization condition for the corresponding solution.

29.3. Energy Levels of a Relativistic Particle of Arbitrary Spin in the Coulomb Field

Starting from the condition (29.18) it is not difficult to find possible values of the parameters $\varepsilon$ defining the energy of the particle described. Indeed, according to (29.18) we have

$$\varepsilon = m \left[ 1 + \frac{\alpha^2}{\left( n + 1/2 + (j + 1/2)^2 - \frac{\alpha^2}{2s} + b^{(j)}_{\lambda \lambda} \right)^{1/2}} \right]. \tag{29.23}$$

Formula (29.23) defines the energy levels of arbitrary spin particle in the Coulomb field and so can be considered as a generalization of the Zommerfeld formula for an electron. The parameters $b^{(j)}_{\lambda \lambda}$ of (29.23) take the values coinciding with the roots of the characteristic equation for the matrix $b^{(j)}_{\lambda \lambda}$ of (29.11)

$$\det \left[ \lambda^2 - \lambda(2j + 1) - b^{(j)}_{\lambda \lambda} \delta_{\lambda \lambda} + \frac{\alpha^2}{s} d^{(j)}_{\lambda \lambda} \right] = 0 \tag{29.24}$$

where $d^{(j)}_{\lambda \lambda}$ are the coefficients (29.7).

Thus we have found energy spectrum of a particle of spin $s$ in the field of a point charge. However the practical using of (29.23) is restricted by the fact that to find all the possible values of $d^{(j)}_{\lambda \lambda}$ it is necessary to solve the algebraic equation (29.4) of order $2s + 1$ (for $j > s$) or of order $2j + 1$ (for $s \leq j$). This equation can be solved in radicals in the cases $s \leq 3/2$ or $j \leq 3/2$ only.
To analyze the spectrum (29.23) for arbitrary \( s \) and \( j \) it is sufficient to consider approximate solutions of (29.24) defined up to the terms of order \( \alpha^2 \). Such an approximation is reasonable for the cases when the charge \( z \) of a particle generating the Coulomb field satisfies the condition \( z \ll 137 \).

Representing \( b_{\lambda}^{(s)} \) in the form

\[
b_{\lambda}^{(s)} = -\lambda^2 - (2j + 1)\lambda + \tilde{b}_{\lambda}^{(s)} \alpha^2 + o(\alpha^4)
\]

(29.25)

where \( \tilde{b}_{\lambda}^{(s)} \) are unknown coefficients we obtain from (29.24)

\[
\tilde{b}_{\lambda}^{(s)} = \frac{1}{8s^2 \lambda} \left[ \begin{pmatrix} a_{\lambda,\lambda}^{(s)} \end{pmatrix}^2 - \begin{pmatrix} a_{\lambda,\lambda+1}^{(s)} \end{pmatrix}^2 \right].
\]

(29.26)

Here \( a_{\sigma}^{(s)} \) (\( \sigma = \lambda + s \) or \( \sigma = \lambda + s + 1 \)) are the coefficients (29.8).

Using (29.25) and representing the r.h.s. of (29.23) as series in respect with \( \alpha^2 \) we obtain with accuracy to \( \alpha^4 \):

\[
e = m \left( 1 - \frac{\alpha^2}{2n^2} + \frac{2\alpha^4}{n^3(2l+1)} - \frac{3\alpha^4}{8n^2} \right),
\]

(29.27)

where

\[
n = n + j - \lambda = 1, 2, \ldots; \quad l = j - \lambda = 0, 1, \ldots, n - 1.
\]

(29.28)

The relations (29.27), (29.28) define the fine structure of energy spectrum of arbitrary spin particle in the Coulomb field. The corresponding values of \( \tilde{b}_{\lambda}^{(s)} \) are easily calculated using (29.26), (29.8).

In addition to the nonessential constant term \( m \) formula (29.27) includes Balmer’s term \(-m\alpha^2/2n^2\), and the additional terms which are proportional to \( \alpha^4 \) and represent the contribution of the spin-orbit coupling of a particle with an external field.

The energy levels are labelled by three quantum numbers \( n, l \) and \( \lambda \) (the number \( j \) is expressed via \( l \) and \( \lambda \): \( j = \lambda + l \)). The possible values of \( n \) and \( l \) are given in (29.28), and admissible values of \( \lambda \) for fixed \( l \) are

\[
\lambda = -s, -s+1, \ldots, s-2m_{\lambda}, \quad m_{\lambda} = \min(s, l).
\]

(29.29)

We see that any energy level corresponding to the quantum number \( n \) is split to sublevels of the fine structure corresponding to possible values of \( l \) and \( \lambda \). It is not difficult to calculate the number \( N_n \) of such sublevels which equals to

\[
N_n = (2s+1)n - [(s+1/2)^2], \quad n \geq s + 1; \quad N_n = n^2, \quad n \leq s.
\]

(29.30)

where \([s+1/2]^2\) is the entire part of \((s+1/2)^2\).

We note that the energy levels corresponding to different sets of \( l \) and \( \lambda \) can to coincide in general. For instance for \( s = 1/2 \) we have \( \tilde{b}_{\lambda}^{(s)} = 2\lambda/(2j+1) \), and the energy corresponding to the same \( l \) but different \( \lambda \) are equal one to another. The corresponding
number of the fine structure sublevels is equal to \( n^2 \) besides any sublevel with \( j \neq n-1/2 \) is twice degenerated.

Let consider more precisely the cases \( s=0, 1/2, 3/2 \). The corresponding values of \( \tilde{b}^{(s)}_\lambda \) (calculated in accordance with (29.26), (29.8)) are

\[
\begin{align*}
\tilde{b}^{(s)}_\lambda &= 0, \quad \lambda = 0; \\
\tilde{b}^{1/2}_\lambda &= \frac{2\lambda}{d_j}, \quad \lambda = \pm \frac{1}{2}, \quad d_j = 2l+1; \\
\tilde{b}^{3/2}_\lambda &= \lambda \frac{d_j + \lambda}{2d_j(d_j - \lambda)} + \frac{2(1 - \lambda^2)}{d_j(d_j - \lambda)}, \quad \lambda = \begin{cases} 1,0, -1, & j \neq 0, \\ 1, j = 0; \end{cases} \\
\tilde{b}^{3/2}_{\pm 1/2} &= \frac{(d_j \pm 1)(5 \pm 1)}{18(d_j \mp 1)(d_j \pm 2)} = \frac{2(d_j^2 - 4)}{9d_j(d_j^2 - 1)}, \quad j \neq 1/2; \\
\tilde{b}^{3/2}_{-1/2} &= -\tilde{b}^{3/2}_{1/2} = -\frac{1}{54}.
\end{align*}
\]

We conclude from (29.31) that for the case \( s=0 \) formula (29.27) gives the known energy spectrum of a scalar particle (described by the KGF equation) in the Coulomb field, but when \( s=1/2 \), the relation (29.27) reduces to the Zommerfeld formula for the spectrum of the hydrogen atom [31].

Formulæ (29.27), (29.31) define the fine structure of the energy spectrum for particles of spins \( s=1 \) and \( 3/2 \) also. In contrast to the case \( s=1/2 \) the spectrum is nondegenerated and the number of sublevels of the fine structure is defined in (29.30).

The energy levels corresponding to the ground state \( n=1 \) are nondegenerated. But if \( n=2 \) then we have two sublevels for \( s=0 \) (and \( s=1/2 \)) and four sublevels for \( s=1 \) (and \( s=3/2 \)). It is not difficult to calculate that the energy splitting \( \Delta E \) (i.e., divergence between the highest and lowest sublevels) for \( n=2 \) is equal to the data given in the table

<table>
<thead>
<tr>
<th>( s )</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
<th>3/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta E )</td>
<td>( m\alpha^4/12 )</td>
<td>( m\alpha^4/32 )</td>
<td>( m\alpha^4/24 )</td>
<td>( 5m\alpha^4/72 )</td>
</tr>
</tbody>
</table>

Let us return to exact formula (29.23). For the cases \( s \leq 3/2 \) or \( j \leq 3/2 \) it is possible to solve (29.24) in radicals and find the exact values of \( b^{(s)}_\lambda \) for (29.23). We present the corresponding solutions for \( s \leq 1 \) and \( j \leq 1 \):

\[
\begin{align*}
\tilde{b}^{(0)}_\lambda &= 0, \quad \tilde{b}^{1/2}_\lambda = 1/4 + \lambda \sqrt{d_j^2 + 4\alpha^2}, \quad \lambda = \pm 1/2; \\
\tilde{b}^{3/2}_\lambda &= \frac{6}{3} + 2\sqrt{-c} \cos \left[ \frac{1}{3} \left( \gamma + \lambda \frac{\pi}{2} \right) \right], \quad \lambda = 0, \pm 1, \quad j \neq 0; \quad \tilde{b}^{10}_\lambda = 0; \quad (29.33)
\end{align*}
\]

353
Symmetries of Equations of Quantum Mechanics

\[
\begin{align*}
\lambda \neq 0, & \quad b_s = \frac{1}{4}(d_s^2 - 3) \pm \frac{1}{2} \sqrt{d_s^2 + \left(\frac{\alpha}{s}\right)^2}, & s \neq 0, & \quad d_s = 2s+1, \\
\lambda = 0, & \quad b_s = s(s+1) - 2s d_s^2 + 2\sqrt{-d_s} \cos \left[\frac{1}{3} \left(\frac{\xi + \lambda \pi}{2}\right)\right], & \lambda = 0, & \quad b_s = s(s+1) - 2s d_s^2 + 2\sqrt{-d_s} \cos \left[\frac{1}{3} \left(\frac{\xi + \lambda \pi}{2}\right)\right],
\end{align*}
\]  

where

\[
\begin{align*}
\cos \gamma &= \frac{b}{\sqrt{-c}} , \\
\frac{3}{2} \alpha^2 + \frac{1}{3} d_s^2 - \frac{1}{27} , \\
c &= -\alpha^2 - b + \frac{4}{27}.
\end{align*}
\]

Substituting (29.32) into (29.23) we come to the exact Zommerfeld formula for the hydrogen atom. The relations (29.23), (29.33) generalize this formula to the cases of a particle of spin 1 and for particles of any spin (but for \(j \leq 1\) only). Exact formulae (29.23), (29.33) can be useful for the case of large \(z\), i.e., for \(ze^2 \sim 1\).

Equation (29.24) can be solved in radicals for the cases \(s = 3/2\) (\(j\) is arbitrary) and \(j = 3/2\) (s is arbitrary). We do not present the corresponding cumbersome formulae here, see [322].

Summarizing we note that the Dirac-type equations (10.10) enable to solve the problem of motion of a spinning particle in the Coulomb field immediately for any value of a spin.

The movement of a spin-one particle in the Coulomb field was studied by Tamm [401] for the first time. Interesting contributions in solving the Coulomb problem for \(s = 1/2\) were made in papers [47, 195, 274]. In the above, we follow papers [322, 164].

354
30. EXACT SOLUTIONS OF GALILEI-IN Variant Wave Equations

30.1. Preliminary notes

In this section we use Galilei-invariant wave equations to solve the problems of motion of charged particles of any spin in external fields.

In accordance with the result of Section 3 it is possible to describe a charged particle of arbitrary spin using different Galilei-invariant equations which are equivalent in the physically reasonable approximation \(1/m^2\). This is why we will use the most convenient way in solving of any concrete problem.

We consider all the types of external fields studied in two previous sections. It makes it possible to compare the results obtained in the Poincaré- and Galilei-invariant approaches and to estimate the adequateness of Galilei-invariant models of spinning particles. Besides we solve the problem of motion of a particle of arbitrary spin in the combined field including the constant electric and magnetic components.

30.2. Nonrelativistic Particle in the Constant and Homogeneous Magnetic Field

To solve this problem we use equations in the Hamiltonian form (15.26), (15.28). We choose the vector of the magnetic field strength be parallel to \(p_3\) and so \(π_μ\) and \(F^ν_μ\) are of the form given in (28.14),(28.15). As a result choosing for convenience \(k_1=-1\) we come to the equation (15.28) where

\[
\mathcal{H}(\pi, A_0) = \sigma_3 a m + \frac{\pi^2}{2m} + \frac{e\mathbf{S} \cdot \mathbf{H}}{m} + \frac{1}{m}(\sigma_1 + i\sigma_2)[2ak^{-2}(\mathbf{S} \cdot \mathbf{π})^2 - ek_0] \mathbf{S} \cdot \mathbf{H}. \tag{30.1}
\]

Here \(k_0=ak^2-k_3\) is an arbitrary parameter and \(A_0\neq 0\).

We restrict ourselves to the case when the field strength is sufficiently small and satisfy the condition

\[
H < \frac{am^2}{2ek_0}. \tag{30.2}
\]

Such a restriction is physically reasonable inasmuch as for a very large \(H\) Galilei invariant approach is not available a priori.

Let us transform \(\mathcal{H}(\pi, A_0)\) to the representation in which it includes commuting terms only. It enables us to find eigenvalues of this operator without solving the motion equations. We will make the transformation in two steps.

Using the operator
where η are matrices of (14.3), and taking into account the identity

\[ [\pi^2 + 2eS \cdot H, \eta \cdot \pi] = 0 \]

we obtain

\[ H' = V_1 \hat{H} V_1^{-1} = \sigma, am + \frac{\pi^2}{2m} + \frac{eS \cdot H}{m} + \frac{ek_0}{mk} \eta \cdot H. \]  

The following transformation will be made using the operator

\[ V_2 = \frac{1}{2} \left( 1 - \frac{\sigma}{\sqrt{\hbar^2}} \right), \quad V_2^{-1} = \frac{1}{2} \left( 1 + \frac{\sigma}{\sqrt{\hbar^2}} \right), \]

which leads to the resulting Hamiltonian

\[ H'' = V_2 \hat{H} V_2^{-1} = \frac{\pi^2}{2m} - \frac{e}{m} S_3 H + \sigma, (a^2m^2 - 2eak_0 \nu H)^{-1} \Lambda, \]

All the terms of the last Hamiltonian commute and thus have a common
system of eigenfunctions. The corresponding eigenvalues are (see Subsection 28.3)

\[ \frac{\pi^2}{2m} \Phi = \frac{1}{2m} (2n+1)eH + p^2 \]  

\[ S_3 \Phi = \nu \Phi, \quad \nu = -s, -s+1, \ldots, s; \quad \sigma, \Phi = \epsilon \Phi, \quad \epsilon = \pm 1. \]  

Using (30.7) it is not difficult to find the eigenvalues of the Hamiltonian (30.6)

\[ E_{n\nu} = (2n+1-2\nu) \frac{eH}{2m} + \frac{p^2}{2m} + \epsilon (a^2m^2 - 2eak_0 \nu H)^{1/2}. \]  

Formula (30.8) gives the energy spectrum of an arbitrary spin particle moving
in the constant magnetic field directed along the third coordinate axis. This spectrum
lies on the real axis and includes as the continuous ingredient \((p)^2/2m\) as the discrete
terms labelled by the numbers \(n, \nu\) and \(\epsilon\).

It is interesting to compare (30.8) with the spectrum of energies of a
relativistic particle found in Subsection 28.3. Representing the last term of (30.8) as a
series in powers of \(1/m\) and restricting ourselves to the terms of order \(1/m^2\) we obtain

\[ E_{n\nu \epsilon} = \epsilon m + \frac{p^2}{2m} + \omega \left( n + \frac{s \nu}{2s} + \nu f(k_0) \right), \]

where \(\omega = eH/m\) is the cyclotronic frequency, \(f(k_0) = (2s+1+\epsilon k_0)/2s\) is the parameter
defining the deviation of the dipole moment of a particle (i.e., of the coefficient by the term $e\mathbf{S} \cdot \mathbf{H}/2m$ in the approximate Hamiltonian (10.26), (15.33)) from the value $q=1/s$.

Comparing (30.9) with (28.25) we come to the conclusion that in the case $f(k_0)=0$ the energy spectrum of a particle of arbitrary spin interacting with the constant and homogeneous magnetic field, predicted by the Galilei-invariant wave equations, coincides with the corresponding relativistic spectrum in the approximation $1/m^2$. If, however $f(k_0)\neq0$ the relation (30.9) can be interpreted as energy spectrum of a particle with anomalous (distinct from $1/s$) magnetic moment.

Let us present the explicit form of the eigenfunction of the Hamiltonian (30.6). In analogy with (28.17)-(28.27) we set

\[
\Phi_{n}^{\pm} = \exp[i(p_1 x_1 + p_3 x_3)] \left( C_+^\dagger U_n(y) \eta^\dagger \right) \begin{cases} 1, & \text{if } p_1 = 0, \\ 0, & \text{if } p_1 \neq 0 \end{cases}
\]

Here $U_n(y)$ is the Hermit function (28.27), $Y=(eH)^{1/2}(x_2^2p_1/eH)$, $\eta^\dagger$ are the spinors of (28.11), 0 is a $(2s+1)$-component zero column, $C_+$ are normalization constants. The set of eigenfunctions of the starting Hamiltonian (30.1) can be obtained by the transformation

\[
\Phi_{n}^{\pm} \rightarrow \psi_{n}^{\pm} = V_1^{-1}V_1^{-1}\Phi_{n}^{\pm},
\]

where $V_1$ and $V_2$ are the operators (30.3),(30.5).

### 30.3 Nonrelativistic Particle of Arbitrary Spin in Crossed Electric and Magnetic Fields

Let us consider a more complicated problem when the constant and homogeneous magnetic field is supplemented by the constant and homogeneous electric field being perpendicular to the magnetic one. We again start from the equations (15.26), (15.28) where $k_1=-1$. The corresponding vector-potential $A_\mu$ can be chosen in the form

\[
A_0 = -x_2 E, \quad A_1 = -x_2 H, \quad A_2 = A_3 = 0,
\]

then

\[
\mathbf{E} = (0,E,0), \quad \mathbf{H} = (0,0,H).
\]

The equations (15.26), (15.28) with the chosen potentials can be solved exactly. With the help of the transformation (30.4), (30.6) the corresponding
Symmetries of Equations of Quantum Mechanics

Hamiltonian (15.26) is reduced to the following form

\[ H'' = \frac{\pi^2}{2m} + eA_\theta - \frac{e}{m} S \cdot \mathbf{H} + \sigma_3 (a^2 m^2 - 2aek_\theta S \cdot \mathbf{H})^{1/2} \]  

(30.11)

or

\[ H'' = \frac{\pi^2}{2m} - eEx_2 + \sum_{\nu=0}^{s} \left[ -\frac{e}{m} \mathbf{H} + \sigma_3 (a^2 m^2 - 2aek_\theta \mathbf{H})^{1/2} \right] \Lambda_\nu \]  

(30.12)

where \( \Lambda_\nu \) are the projectors of (30.5).

The Hamiltonian \( H'' \) commutes with the matrices \( S_3, \sigma_3 \) and the operators \( P_1, P_3 \). This is why it is convenient to look for the corresponding eigenfunctions in the form

\[ \psi_{\lambda\nu} \sim \exp \left[ it(p_1 x_1 + p_2 x_2) \right] \chi_{\lambda\nu} \]  

(30.13)

where \( \chi_{\lambda\nu} \) is an eigenvector of the commuting matrices \( \sigma_3 \) and \( S_3 \) (corresponding to the eigenvalues \( \lambda = \pm 1 \) and \( \nu = -s, -s+1, ..., s \)), \( p_1 \) and \( p_3 \) are eigenvalues of the operators \( P_1 \) and \( P_3 \).

Substituting (30.12), (30.13) into the Schrödinger equation for stationary states we come to the following system

\[ H'' \psi_{\lambda\nu} = \tilde{E} \psi_{\lambda\nu} \]

which reduces to the following sequence of the noncoupled equations for \( \chi_{\lambda\nu} \):

\[ \left[ -\frac{d^2}{dx_2^2} + (p_1 + eHx_2)^2 + p_3^2 - 2evH + 2m\lambda(a^2 m^2 - 2aek_\theta \mathbf{H})^{1/2} - 2m(\tilde{E} - eEx_2) \right] \chi_{\lambda\nu} = 0. \]  

(30.14)

Using the change of variables

\[ y = \sqrt{|eH|} x_2 + (mE - p_1 H)/(eH^2) \]

(30.15)

(30.14) reduces to the equations for the harmonic oscillator (compare with (28.22))

\[ \left( -\frac{d^2}{dy^2} + y^2 \right) \chi_{\lambda\nu} = K_{\lambda\nu} \chi_{\lambda\nu}, \]  

(30.16)

besides \( K_{\lambda\nu} \) are connected with the eigenvalues \( \tilde{E} \) of the Hamiltonian \( H'' \) as follows

\[ eH K_{\lambda\nu} = (H(p_1 - mE))^2 H^{-2} + 2mE - p_3^2 + 2evH - 2m\lambda(a^2 m^2 - 2aevk_\theta \mathbf{H})^{1/2}. \]  

(30.17)

Solutions of (30.16) are proportional to the Hermite functions (28.27) and the corresponding eigenvalues \( K_{\lambda\nu} \) are equal to \( 2n+1, n=0,1,2, \ldots \). Hence it follows the possible values of energy are

\[ \tilde{E} = (2m)^{-1} \left[ (2n+1 - 2\nu)eH + p_3^2 \right] + (p_1 E)/(2H) - m/2 [E/H] + \lambda(a^2 m^2 - 2aevk_\theta \mathbf{H})^{1/2}. \]  

(30.18)

In the case \( E=0 \) formula (30.18) reduces to (30.8).

In the approximation \( 1/m^2 \tilde{E} \) reduces to the form
\[ E \cdot \lambda \alpha m + p_z^2 + \omega \left( n + \frac{s-v}{2s} - \sqrt{v} f(k_0) \right) + \frac{p_x e}{2H} - \frac{m}{2} \left( \frac{E}{H} \right)^2 \]  

(30.19)

where \( f(k_0) = (2s+1+\lambda k_0)/2s \). In the case \( k_0 = 0 \), \( s = 1/2 \) the energy levels (30.19) coincide with the levels predicted by the Dirac equation for the case of fields of the considered configuration.

We note that the equations (15.26), (15.28) can be solved exactly also for the case of parallel electric and magnetic fields not depending on \( x_\mu \). Then we can obtain solutions for arbitrary directed fields using Lorentz transformation.

### 30.4. Nonrelativistic Particle of Arbitrary Spin in the Coulomb Field

This problem also can be treated successively using Galilei-invariant wave equations.

We start with the first-order equations (15.1) with \( \beta \)-matrices corresponding to column \( R_i \) of the Table 13.1 (pp.160-161). Choosing the vector-potential in the form \( A_0 = 0, \) \( A_i = \frac{ze}{x} \) and going with the help of the transformation (15.6), (15.7) to the representation (15.19), (15.20) we obtain the following system

\[
\left\{ \begin{array}{l}
\frac{p_0}{x} - \frac{p^2}{2m} - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x - \frac{\alpha}{s(s+1)\kappa_2 m} S \cdot x
\end{array} \right\} \Phi' = 0,
\]

(30.20)

\[
\Phi'_2 = - \Phi', \quad \Phi'_1 = \left\{ \frac{\kappa^2}{2s} + \frac{\alpha S \cdot x}{2s^2 m(s+1)\kappa_2} \right\} \Phi',
\]

(30.21)

\[
\chi'_1 = - \chi'_2 = 0, \quad \Phi' = \Phi_1, \quad \alpha = ze^2.
\]

We see that our problem reduces to solving the Schrödinger-type equation (25.20) for (2s+1)-component wave function \( \Phi' \). The remaining components of the vector-function (15.18) are expressed via \( \Phi' \) according to (30.21). Further on, we set \( e_\nu = m \) without loss of generality.

Solutions of (30.20) can be found in accordance with the relativistic Coulomb problem. Representing \( \Phi' \) in the form of (29.1) where \( e \to \tilde{E} \) we come to the stationary Schrödinger equation

\[
\left\{ \tilde{E} - \frac{\alpha}{x} - m - \frac{p_0^2}{2m} - \frac{\alpha g}{2sm} S \cdot x \right\} \Phi'(x) = 0,
\]

(30.22)

where

\[
g = \frac{2}{(s+1)\kappa_2}.
\]

(30.23)

The equation (30.22) is invariant under the group \( O(3) \) and so admits solutions
in separated variables. Expanding $\Phi^s$ in the spherical spinors according to (29.3) and using the representation (29.9) for $\rho^2$, we come to the following system of ordinary differential equations for radial functions

$$D^\prime \Phi^s = x^{-2}b_{\lambda \lambda} \Phi^s.$$  \hfill (30.24)

where

$$D^\prime = 2m(\tilde{E} + \alpha x) \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{j(j+1)}{x^2} - 2m^2,$$

$$b_{\lambda \lambda} = [\lambda^2 - \lambda(2j+1)]\delta_{\lambda \lambda} + \frac{g\alpha}{s} d_{\lambda \lambda}^{s'},$$  \hfill (30.25)

$\lambda, \lambda' = s, -s+1, \ldots, -s+2n_s, \quad n_s = \text{min}(s, f)$

and $d_{\lambda \lambda}^{s'}$ are the matrix elements of the operator $Sx/x$ in the basis of spherical spinors, see (29.7).

The matrix $|b_{\lambda \lambda}|$ is diagonalizable so the equations (30.24) reduces to the chain of the following noncoupled equations

$$D^\prime \Phi = x^{-2}b^\prime \Phi$$

where $D^\prime$ is the operator (30.25), $b^\prime$ are eigenvalues of the matrix $|b_{\lambda \lambda}|$.

Until now we have repeated reasoning from Section 29, where the analogous but relativistic problem was considered. Further on, bearing in mind differences between the operators $D^\prime$ (30.25) and $D$ (29.11), we use a new (in comparison with (29.14)) change of variables

$$y = \sqrt{z} \Phi, \quad z = 2\sqrt{-2m(\tilde{E} - m)} \frac{x}{z}, \quad \beta = \left(\frac{-m}{2(\tilde{E} - m)}\right)^{\frac{1}{2}} \frac{\alpha}{k^2} = (2j+1)^2 + 4b^\prime,$$  \hfill (30.26)

and again come to the equations (29.13) for $y$.

We see that the Galilei-invariant equations for a particle of an arbitrary spin in the Coulomb field generate the equation for eigenvalues $\beta$ which is the same as in the case of relativistic Coulomb problem. But the energy spectrum of a Galilei particle differs from the relativistic one in accordance with another dependence of $\tilde{E}$ on $\beta$, compare (30.26) and (29.14). We obtain from (29.18), (30.26) the following energy eigenvalues:

$$\tilde{E} = m - \frac{m\alpha^2}{\left(\frac{j + \frac{1}{2}}{2} + 2b^\prime + n^\prime + \frac{1}{2}\right)^2}, \quad n^\prime = 0, 1, \ldots.$$  \hfill (30.27)

Formula (30.27) defines energy levels of a nonrelativistic particle of arbitrary
spin $s$ in the Coulomb field. Here $j$ takes integer (for integer $s$) or half integer (for half integer $s$) values defining the total angular momentum of a particle, besides the possible values of $b^j$ coincide with the roots of the characteristic equation (29.24) for the matrix $[b_{j,j'}]$ of (30.25).

To analyze the spectrum (30.27) and to compare it with the spectrum predicted by relativistic motion equations we consider approximate solutions of the characteristic equation (29.24), (30.25). Representing the r.h.s. of (30.27) as series in powers of $\alpha^2$ we obtain

$$E = m - \frac{m\alpha^2}{2n^2} - \frac{mg^2\alpha^4}{2n^4(l+1/2)(l+\lambda+1/2)} - o(\alpha^6),$$

(n \sim n' + l + 1, 2, \ldots; \quad l \sim j - \lambda = 0, 1, \ldots, n - 1).$$

Formula (30.28) defines a fine structure of the energy spectrum of an arbitrary spin particle in the Coulomb field. The values of the corresponding parameters $b^j_{\lambda}$ are easily calculated by formulae (29.8), (29.26).

Besides the constant term $m$ the energy of a particle is defined by the Balmer term $-m\alpha^2/2n^2$ and by the correcting term of order $\alpha^4$ connected with the spin. Further on it will be shown that this correcting term is caused by the spin-orbit, Darwin and quadruple couplings.

According to (30.28) any energy level corresponding to the given main quantum number $n$ is split into $N_n$ sublevels of the fine structure corresponding to the admissible values of $l$ and $\lambda$ given in (30.28). The number $N_n$ is given in (29.30). In the case $s=1/2$ formula (30.28) reduces to the form

$$E = m - \frac{m\alpha^2}{2n^2} + \frac{\lambda mg^2\alpha^4}{2n^4(l+1/2)(l+\lambda+1/2)}, \quad \lambda = \pm 1/2.$$

In contrast with the energy spectrum predicted by the Dirac equation (see (29.27), (29.31) for $s=1/2$) the energy levels (30.29) are nondegenerated.

Comparing formula (30.28) with (29.27) obtained by the solution of the relativistic Coulomb problem we come to the conclusion that for $g^2=-1$

$$E = \varepsilon + \Delta \varepsilon, \quad \Delta \varepsilon = \left(\frac{3}{4} - \frac{n}{l+1/2}\right)m\alpha^2$$

where $\varepsilon$ is energy of a relativistic particle given in (29.27).

We see the energy levels of a particle of arbitrary spin in the Coulomb field, obtained from the Galilei invariant wave equations, are shifted by the value $\Delta \varepsilon$ in comparison with the levels predicted by relativistic equations. This result is quite natural inasmuch as $\Delta \varepsilon$ coincides with the mean value of the relativistic correction to kinetic energy [18]: $\Delta \varepsilon = \langle p^2/8m^2 \rangle$, where averaging is made by Schrödinger’s wave functions.
So Galilei invariant wave equations take into account all the effects predicted by the Poincaré-invariant equations (i.e., the spin-orbit, Darwin and quadruple couplings) except relativistic correction to the kinetic energy. This conclusion is in good accordance with the result of the analysis of the corresponding approximate Hamiltonians of (10.26).

We repeat that the spin-orbit, Darwin and quadruple couplings can be described in frames of a Galilei-invariant approach, and therefore it is not necessary to interpret these couplings as purely relativistic effects.

In conclusion, we note that the Galilei invariant wave equations can be solved exactly also for some other types of external fields, e.g., for the field of the magnetic monopole. The last problem is formulated and solved in analogy with [164].

31. NONLINEAR EQUATIONS INVARIANT UNDER THE POINCARÉ AND GALILEI GROUPS

31.1. Introduction

In this section we present a brief survey of the main results of papers [128-131, 166-180, 183-192, 6', 10', 11', 14'-25'] where nonlinear equations are investigated which have the same symmetries as the linear equations of D’Alembert, Dirac, Schrödinger and Maxwell.

The superposition principle is not valid for nonlinear differential equations so it is important to select equations being invariant under wide groups from the class, e.g., first and second order partial differential equations. The symmetry properties of such equations enable us to construct wide classes of solutions starting from a known particular solution.

Later, we will consider symmetry properties and exact solutions of a number of nonlinear equations of modern theoretical and mathematical physics. We hope it will be a useful demonstration of power of symmetry methods in application to practically interesting problems.

31.2. Symmetry Analysis and Exact Solution of Scalar Nonlinear Wave Equations

Consider the nonlinear d’Alembert equation

\[ p_\mu p^\mu U + F(U) = 0, \]  

(31.1)

where \( F(U) \) is a smooth function, \( U = U(x_0, x_1, ..., x_n) \) is a real scalar function.

Let us denote by \( \tilde{P}(I,n) \) the extended generalized Poincaré group, i.e., the
group $P(1,n)$ completed by the one-parametric group of the scale transformations (1.48).

**THEOREM 31.1** [175]. The equation (31.1) is invariant under the group $\tilde{P}(1,n)$ in the two following cases only

\[ F(U) \Rightarrow F_1(U) \Rightarrow \lambda_1 U^r, \quad r \neq 1, \]

\[ F(U) \Rightarrow F_2(U) \Rightarrow \lambda_2 \exp(U), \]

where $\lambda_1, \lambda_2, r$ are arbitrary numbers. The corresponding generators of the group $P(1,n)$ have the form (1.6) (where $\mu, \sigma = 0, 1, \ldots, n$), and the dilatation generator is given by the following formulae

\[ D = -D_1 \cdot x_\mu p^\mu - \frac{2iU}{1-r} \frac{\partial}{\partial U}, \]

\[ D = -D_2 \cdot x_\mu p^\mu - 2i \frac{\partial}{\partial U}. \]

**Corollary 1.** The equation (31.1) with the nonlinearity (31.3) in the two-dimensional space $R(1,1)$ is invariant under the infinite-dimensional Lie algebra including the subalgebra $\tilde{A} \tilde{P}(1,n)$. Such symmetry makes it possible to construct the Liouville solution [175]

\[ U(x_0, x_1) = \ln \left( \frac{-8f_1(x_0 + x_1)f_2(x_0 - x_1)}{\lambda_2[f_1(x_0 + x_1) + f_2(x_0 - x_1)]^2} \right) \]

where $f_1$ and $f_2$ are arbitrary smooth functions, and the dots denote the derivations in respect with the corresponding arguments.

**Corollary 2.** The solutions (31.6) have singularity in the point $\lambda_2 = 0$, so it is impossible to apply the standard method of a small parameter to the equation (31.1), (31.3). Singular solutions of the two-dimensional Liouville equations are investigated in paper [363].

**Corollary 3.** If $r = (n+3)/(n-1)$ then the equation (31.1) with the nonlinearity (31.2) is invariant under the conformal algebra $AC(1,n)$ whose basis element have the form (1.6), (31.4), and $K_\mu$ of (1.16).

To construct exact solutions of the equation (31.1) we use the ansatz

\[ U(x) = f(x) \varphi(\omega) + g(x) \]

proposed in [124] and realized effectively for a number of nonlinear equations of mathematical physics [125, 166-168, 173-176]. Here $\varphi(\omega)$ is a function has to be determined which depends on invariant variables $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. The explicit form of $f(x)$ and $g(x)$ is determined from the condition of "separation of variables", i.e., from
the requirement the variables \( x=(x_0, x_1, \ldots, x_n) \) have not to be present explicitly in the equation for \( \varphi(\omega) \) obtained from (31.1), (31.7). The invariant variables \( \omega(x) \) are the first integrals of the corresponding Euler-Lagrange equation [124,171,186].

Let us represent one of possible ansätze (31.7) for the equation (31.1), which reduces it to the equation with three independent variables:

\[
U(x) = \left( (c \cdot x)^2 + (d \cdot x)^2 \right)^{1/2} \varphi(\omega_1, \omega_2, \omega_3), \quad r \neq 1,
\]

(31.8)

where

\[
a = (a_0, a_1, a_2, a_3), \quad b = (b_0, b_1, b_2, b_3), \quad c = (c_0, c_1, c_2, c_3), \quad d = (d_0, d_1, d_2, d_3)
\]

are parameters satisfying the conditions

\[
a^2 - b^2 - c^2 - d^2 = 1, \quad a \cdot x \neq 0, \quad b \cdot x \neq 0, \quad c \cdot x \neq 0,
\]

\[
a \cdot b = a \cdot c = a \cdot d = b \cdot d = c \cdot d = 0, \quad a^2 = a^2, \quad a \cdot b = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3,
\]

\[
\theta \text{ is an arbitrary parameter.}
\]

The ansatz (31.8) reduces the equations (31.1) (31.2) to the linear equation with variable coefficients:

\[
\frac{1}{(1 + \omega_1^2)} \frac{\partial^2 \varphi}{\partial \omega_1^2} - 4(1 + \theta^2) \frac{\partial^2 \varphi}{\partial \omega_1 \partial \omega_2} - \omega_3 \frac{\partial^2 \varphi}{\partial \omega_1 \partial \omega_3} + \omega_2 \left[ \omega_1 (\omega_1 + \omega_3^2) - 4 \right] \frac{\partial^2 \varphi}{\partial \omega_2^2} - 2 \omega_2 (1 + \omega_1^2) \frac{\partial^2 \varphi}{\partial \omega_1 \partial \omega_2} - 2 \omega_1 \frac{\partial^2 \varphi}{\partial \omega_2^2} + 4 k \frac{\partial \varphi}{\partial \omega_3} - k \varphi + \lambda \varphi' = 0, \quad k = \frac{2}{r - 1}.
\]

(31.9)

To find exact solutions of (31.9) is not an easy thing so it is convenient to reduce this equation to the equation with two independent variables, the last can be reduced to an ordinary differential equation (ODE). In some cases the obtained ODE can be solved analytically.

We represent two multiparametric families of solutions obtained in accordance with the above scheme

\[
U = [(a \cdot x)^2 + b \cdot x \cdot c \cdot x]^{1/(r-1)},
\]

(31.10)

\[
a \cdot b = a \cdot c = b^2 = c^2 = 0, \quad 2a^2 = b \cdot c = \frac{\lambda (r - 1)^2}{r - 3}, \quad r \neq 3;
\]

\[
U = [\Phi(a \cdot x) + b \cdot x]^2/(1 - r),
\]

where \( \Phi \) is an arbitrary smooth function,

\[
a \cdot a = a \cdot b = 0, \quad b \cdot b = -\frac{1}{2} \lambda (1 - r^2)(1 + r)^{-1}, \quad r \neq -1.
\]

Formula (31.10) gives solutions of (31.1) expressed via an arbitrary function. Such a solution is available for the corresponding initial or boundary values problems.
Chapter 6. Exact Solutions of Linear and Nonlinear ...
under the groups $P(1,3)$, $\tilde{P}(1,n)$ or $C(1,3)$ is given in the following assertion.

**THEOREM 31.4** [172]. The equation (31.12) is invariant under one of the following groups: a) the Poincaré group, b) extended Poincaré group, c) conformal group iff

\[
F_1, F_2, F_3, F_4 \text{ are arbitrary scalar functions of } \overline{\psi}\psi, \overline{\psi}\gamma_\mu \psi; \\
a) F(\overline{\psi}\psi) = F_1 + F_2 \gamma_\mu + F_3 \gamma_\mu \overline{\psi}\gamma_\nu \psi + F_4 \gamma^\mu \gamma_\nu \psi_\alpha \psi, \tag{31.13}
\]

where $F_1, F_2, F_3, F_4$ are arbitrary scalar functions of $\overline{\psi}\psi, \overline{\psi}\gamma_\mu \psi$;

b) $F$ has form (31.13) where

\[
F_i = (\overline{\psi}\psi)^{1/2} G_i, \quad i=1,2; \quad F_j = (\overline{\psi}\psi)^{(1-2k)/2} G_j, \quad j=3,4, \tag{31.14}
\]

$G_i, G_j$ are arbitrary functions of $\overline{\psi}\psi, \overline{\psi}\gamma_\mu \psi, k$ is an arbitrary constant;

c) $F$ has the form (31.13) where

\[
F_i = (\overline{\psi}\psi)^{1/3} G_i, \quad i=1,2; \quad F_j = (\overline{\psi}\psi)^{(1-2k)/3} G_j, \quad j=3,4. \tag{31.15}
\]

The simplest conform-invariant equations of the class (31.12), (31.13), (31.15) have the form

\[
[\gamma_\mu p^\mu + \lambda(\overline{\psi}\psi)^{1/3}]\psi = 0, \tag{31.16}
\]

\[
[\gamma_\mu p^\mu + \lambda(\overline{\psi}\psi)^{(1-2k)/3}]\psi = 0, \tag{31.17}
\]

The equation (31.16) was obtained by Gürsey [212] for the first time.

**Corollary.** There exist first order equations for the spinor field, which have a more extended symmetry than the Dirac equation (31.12). An example of such an equation is the following system [125,126]

\[
\overline{\psi}\gamma_\mu \psi = 0
\]

which is invariant under the infinite-dimensional algebra.

Consider the $\tilde{P}(1,n)$-invariant nonlinear equation

\[
[\gamma_\mu p^\mu + \lambda(\overline{\psi}\psi)^{1/2}]\psi = 0, \tag{31.18}
\]

where $\lambda$ and $k\neq 0$ are arbitrary constants. We search for solutions of (31.18) in the form [124,125]

\[
\psi = A(x) \phi(\omega) \tag{31.19}
\]

where $A(x)$ is a $4\times4$ matrix, $\phi(\omega)$ is a four-component function depending on three new variables $\omega = \{\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)\}$.

The ansatz (31.19) leads to the equation for $\phi(\omega)$, which depends on $\omega$ only if $A(x)$ and $\omega$ satisfy the equations [186-192]
Chapter 6. Exact Solutions of Linear and Nonlinear ... 

\[
\left( \xi^\mu(x) \frac{\partial}{\partial x^\mu} + \eta(x) \right) A(x) = 0, \quad \xi^\mu(x) \frac{\partial \Theta_i}{\partial x^\mu} = 0, \quad i=1,2,3,
\]

where \( \xi^\mu(x), \eta(x) \) are coefficients of infinitesimal operators of the group \( \tilde{P}(l,n) \).

Without going into details we present two solutions [186]:

\[
A(x) = (x_0-x_2)^{-a} \exp \left[ \frac{1}{2a} \chi \gamma_{1,2,3} \ln(x_0-x_2) \right],
\]

\[
\omega_1 = (x_0^2-x_1^2-x_2^2)x_3^2, \quad \omega_2 = (x_0-x_2)x_3^2, \quad \omega_3 = ax_1(x_0-x_2)^{-1} - \ln(x_0-x_2), \quad a \neq 0.
\]

If \( a = 0 \) then

\[
A(x) = (2x_0 + 2x_1 + \beta)^{-a^2} \exp \left[ \frac{1}{4} \chi \gamma_{1,2,3} \ln(2x_0 + 2x_1 + \beta) - \frac{1}{2} \chi \gamma_{1,2,3} \arctan \frac{x_2}{x_3} \right], \quad \beta \neq 0,
\]

\[
\omega_1 = (2x_0 + 2x_1 + \beta) \exp \left( 2(x_1 - x_0)\beta^{-1} \right), \quad \omega_2 = (2x_0 + 2x_1 + \beta)(x_2^2 + x_3^2)^{-1},
\]

\[
\omega_3 = b \ln(x_2^2 + x_3^2) + 2 \arctan \frac{x_2}{x_3}.
\]

Here \( \beta \) and \( b \) are arbitrary parameters.

Let us represent the explicit form of solutions of (31.18) for three values of the parameter \( k \) [186].

The case \( k = 1/2 \):

\[
\psi = [(a-z)^2 + (b-z)^2]^{-1/2} \exp \left[ -\frac{1}{2} \gamma \cdot a \gamma \cdot b \arctan \frac{a \cdot z}{b \cdot z} \right] \exp \left( \frac{\partial \chi}{2(1+\gamma^2)} \right),
\]

\[
+ \Omega \gamma \cdot a \left[ \ln((a-z)^2 + (b-z)^2) + 2 \theta \arctan \frac{a \cdot z}{b \cdot z} \right] \chi
\]

where \( z = x + \theta \), \( \theta = (\theta, \theta')^{1/2} \), \( a, b, \theta \) are arbitrary parameters satisfying the conditions \( a \cdot a = a, a^2 = -1, b \cdot b = -1, a \cdot b = 0 \), \( \chi \) is a constant spinor.

The case \( k \neq 1/2 \):

\[
\psi = [(a-z)^2 + (b-z)^2]^{-1/2} \exp \left[ -\frac{1}{2} \gamma \cdot a \gamma \cdot b \arctan \frac{a \cdot z}{b \cdot z} \right] \times (2ik\lambda \gamma \cdot b \chi \gamma \chi)^{[1/(2-k)](1/(2-k))} \chi.
\]

The case \( k = 1/3 \):

\[
\psi = \gamma \chi (x \alpha)^{-2} \exp[i \lambda k \gamma \cdot \beta \chi \alpha (x \alpha)^{-1}] \chi, \quad \beta \cdot \beta > 0, \quad x \cdot x \neq 0.
\]

The relations (31.20)-(31.24) define the multiparameter families of exact solutions of the equation (31.18).
If $k = 1/3$ then the equation (31.18) is conform-invariant. It means that if we know a particular solution of (31.18) then it is possible to generate other solution using conformal transformations.

Using the given ansätze, families of exact solutions of the equations of classical electrodynamics

\[
(f^\mu p_\mu + e^{-\gamma^\mu A_\mu + m})\psi = 0,
\]

\[
p_\mu p^\nu A_\mu - p_\nu p^\mu A_\nu - e\bar{\psi}y_\mu \psi
\]

and many other nonlinear equations have been obtained in [168,188].

A new conformal invariant equation for the spinor field

\[
\left\{y^\mu p^\nu - \gamma^{\mu\nu} W_1^{1/4} - \sqrt{2}(\bar{\psi}\psi)^{1/2}\right\}\psi = 0
\]

(where $C_2$ is the invariant of the electromagnetic field (31.34)) was proposed in [30\*].

We do not consider two-dimension nonlinear integrable equations which were investigated by great many of authors (see [350] and the references cited there in).

**NOTE.** In this book we consider only linear representations of algebras $AP(1,3)$ and $AC(1,3)$. But there exist nonlinear equations invariant under nonlinear realizations of the conformal algebra. Thus on the solutions of the eikonal equation

\[
\frac{\partial U}{\partial x^\mu} \frac{\partial U}{\partial x^\nu} = 1
\]

the following nonlinear representation of the algebra $AC(1,4)$ is realized [36\*]

\[
P_\mu = p_\mu - i \frac{\partial}{\partial x^\mu}, \quad J_{\mu\sigma} = x_\mu p_\sigma - x_\sigma p_\mu, \quad P_\mu = p_\mu - i \frac{\partial}{\partial U},
\]

\[
J_{04} = x_0 p_4 - Up_0, \quad J_{4\nu} = U p_\nu - x_\nu p_U,
\]

\[
D = x_\mu p^\nu - Up_\nu, \quad K_\mu = 2x_\mu D - \left(x_\mu x^\nu - U^2\right) p_\nu.
\]

Nonlinear representations of the algebras $AP(1,1)$, $AP(1,2)$, $AP(2,2)$, $AC(2,2)$ are found in [37\*-39\*]. Moreover, the nonlinear representation of the algebras $AP(2,2)$ and $AC (2,2)$ is realized by the operators $P_\mu$, $J_{\mu\sigma}$, $K_\mu$, $D$, which have the form (2.22), (2.42) where the Greek indices run over the values 1, 2, 3, 4,

\[
S_{ab} = \epsilon_{abc} \Sigma_c, \quad S_{4a} = \epsilon \Sigma_a, \quad e = \pm 1; \quad K = 0,
\]

\[
\Sigma_1 = -i \sin U \partial_U, \quad \Sigma_2 = i \cos U \partial_U, \quad \Sigma_3 = -i \partial_U,
\]

and the covariant summation over the repeating indices is imposed using the metric tensor $g_{\mu\sigma} = 1, \mu = \sigma = 1, 2; g_{\mu\sigma} = -1, \mu = \sigma = 3, 4; g_{\mu\sigma} = 0, \mu \neq \sigma$.  

368
31.4. Equations of Schrödinger Type Invariant Under the Galilei Group

Here we consider the following nonlinear generalization of the Schrödinger equation

\[
\left( p_0^2 - \frac{p^2}{2m} \right) U + F(x,U,U') = 0
\]  \hspace{1cm} (31.25)

where \( F \) is an arbitrary differentiable function.

The maximal IA of (31.25) for \( F=0 \) is the algebra \( AG(1,3) \) whose basis elements are given in (11.5). We denote by \( AG_f(1,3) \) the subalgebra including the basis elements \( P_\alpha, P_\sigma, J_\alpha, G_\alpha, D \) of (11.5).

Of course symmetry of the equation (31.25) depends on the structure of \( F \).

**THEOREM 31.5** [124,176]. The equation (31.25) is invariant under the following algebras:

- \( AG(1,3) \), iff \( F=\Phi(U)U \), \( \Phi \) is an arbitrary smooth function;
- \( AG_f(1,3) \), iff \( F=\lambda U^k U \), \( \lambda \) and \( k \neq 0 \) are arbitrary parameters besides the corresponding generator of scale transformations has the form \( D=2x_0 p_0 - x p + 2i/\kappa; \)
- \( AG_f(1,3) \), iff \( F=\lambda U^{1+2i} U \).

More general (than (31.25)) equations of Schrödinger type are investigated in [129]. One such equation has the form

\[
\left( p_0^2 - \frac{p^2}{2m} \right) U + \lambda U \frac{\partial(UU')}{\partial x^\sigma} \frac{\partial(UU')}{\partial x^\nu} (UU')^{-2} = 0.
\]  \hspace{1cm} (31.26)

This equation is invariant under the algebra \( AG_f(1,3) \). A wide classes of exact solutions of (31.26) have been constructed in [14] using continuous subgroups of the Galilei group [127].

In conclusion, we represent three ansätze and families of solutions of the equation [176]

\[
\left( p_0^2 - \frac{p^2}{2m} \right) U + \lambda |U|^{2i} U = 0.
\]  \hspace{1cm} (31.27)

We search for solutions of (31.27) in the form (31.7) where \( \varphi \) is a function of invariant variables \( \omega=(\omega_1,\omega_2,\omega_3) \). The explicit form of \( f(x) \) and \( \omega \) which enables us to reduce (31.27) to the equation with three independent variables is given by the following formulae
Symmetries of Equations of Quantum Mechanics

where \( a, b, c \) are constant vectors satisfying the conditions
\[
\begin{align*}
1, \\
(a \cdot a) = (b \cdot b) = (c \cdot c)
\end{align*}
\]

(31.28)

\[
\begin{align*}
f(x) &= (1 - x^2)^{3/4} \exp \left[ \frac{imx^2}{2(1 - x^2)} \right], \\
\omega_1 &= a \cdot x(1 - x^2)^{3/4}, \\
\omega_2 &= x^2(1 - x^2)^{3/4}, \\
\omega_3 &= \arctan x_0 + \arctan \frac{b \cdot x}{c \cdot x}.
\end{align*}
\]

(31.29)

Exact solutions of (31.27) are constructed using the ansätze (31.28)-(31.30). The simplest solutions are

\[
\begin{align*}
f(x) &= x_0^{3/2} \exp \left[ -\frac{imx^2}{2x_0} \right], \\
\omega_1 &= a \cdot xx_0^{-1}, \\
\omega_2 &= x^2 x_0^{-2}, \\
\omega_3 &= x_0^{-1} + \arctan \frac{b \cdot x}{c \cdot x}.
\end{align*}
\]

where \( a, b, c \) are constant vectors satisfying the conditions \( a \cdot a = b \cdot b = c \cdot c = 1, a \cdot b = a \cdot c = b \cdot c = 0 \).

Exact solutions of (31.27) are constructed using the ansätze (31.28)-(31.30). The simplest solutions are

\[
\begin{align*}
U &= (1 - x_0^2)^{3/4} \exp \left[ \frac{imx^2}{2(1 - x_0^2)} \right], \\
\lambda &= \frac{3}{2} i, \\
U &= x_0^{-3/2} \exp \left[ -\frac{im(x^2 - r \cdot x)x_0^{-3}}{2} \right], \\
r^2 &= \frac{8\lambda m}{\mu}, \\
U &= x_0^{-3/2} \varphi(\omega_1) \exp \left[ \frac{imx^2}{2x_0} \right], \\
\omega_1 &= a \cdot xx_0^{-1}
\end{align*}
\]

The function \( \varphi(\omega_1) \) is determined by the elliptic integral

\[
\int_0^{\varphi} \frac{d\tau}{(k_1 + \tau)^{3/2}} = \left( \frac{6\lambda m}{5} \right)^{1/2} \left( \omega_1 + k_2 \right),
\]

\( k_1 \) and \( k_2 \) are arbitrary constants.

Formulae (31.30) present multiparametrical families of exact solutions of the nonlinear equation (31.26).

Let \( U = U(x_0, x) \) is a solution of (31.26). To obtain other solutions we can use the following formulae

\[
U = U(x_0, x + v x_0) \exp \left[ im \left( \frac{v^2 x_0}{2} + v \cdot x \right) \right]
\]

370
where \(a, \nu\) are arbitrary constants. These formulae reflect the fact that the equation \(U_{2x_0} = \frac{\nu}{1-ax_0^2} \left(1-aax_0^2\right)^{\nu/2} \exp \left[ \frac{i\nu x^2}{2(1-ax_0^2)} \right] \)

is invariant under the Galilei group. For more details see [171, 176].

Symmetry properties and component reduction of the equation (31.25) with the logarithmic nonlinearity

\[ F(x, U, U') = (\lambda_1 + i\lambda_2) U \ln(U') U, \quad \lambda_2 \neq 0 \]

was investigated in papers [32 *, 33 *]. Besides the generators of the Galilei group this equation admits the additional SO

\[ Q = \exp(2i\lambda_2 x_0) \left[ U \left( \frac{\partial}{\partial U} - U - \frac{i\lambda_1}{\lambda_2} \left( \frac{\partial}{\partial U} - U \right) \right] \right] \]

This SO generates similarity transformations of coordinates \(x_0 \rightarrow x_0', x_a \rightarrow x_a'\), and nontrivial one-parametrical transformation for \(U\):

\[ U \rightarrow U' = \exp \left[ \frac{\theta}{1-i\frac{\lambda_1}{\lambda_2}} \exp(2\lambda_2 x_0) U \right]. \]

Lie symmetry and reduction of multidimensional systems of the Schrödinger type was investigated in papers [129, 14 *] where, in particular, one-dimension Galilei-invariant equations (31.25) with nonlinearity

\[ F = U|U|^4 h(U, |U'|) \]

(where \(h\) is an arbitrary smooth function) was considered.

### 31.5. Symmetries of Nonlinear Equations of Electrodynamics

The electromagnetic field in a medium is described by Maxwell’s equations

\[ \frac{i}{c} \frac{\partial D}{\partial x_0} = -p \times H, \quad \frac{i}{c} \frac{\partial B}{\partial x_0} = -p \times E, \quad p \cdot D = 0, \quad p \cdot B = 0. \]  

(31.31)

where \(D\) and \(B\) are vectors of induction, \(E\) and \(H\) are vectors of strengths. The underdetermined system (31.31) has to be completed by constitutive equations (equations connecting \(D, B, E\) and \(H\)) which reflect the properties of the medium.

Following [182] we present here some results connecting symmetries of equations (31.31) supplemented by constitutive equations.

First we note that the system (31.31) without additional conditions is invariant under an infinite-dimensional Lie algebra including the subalgebra \(AIGL(4,R)\) [157].

Let us represent constitutive equations in the form of the following functional
Symmetries of Equations of Quantum Mechanics

relations
\[ E = \Phi(D, H), \quad B = F(D, H), \]  
(31.32)

where \( \Phi \) and \( F \) are arbitrary smooth functions of \( D \) and \( H \).

**THEOREM 31.6.** The system of equations (31.31), (31.32) is invariant under the group \( P(1,3) \) iff

\[ D = ME + NB, \quad H = MB - NE \]  
(31.33)

where \( M = M(C_1, C_2), \quad N = N(C_1, C_2) \) are arbitrary functions of the invariants of the electromagnetic field,

\[ C_1 = E^2 - B^2, \quad C_2 = B \cdot E. \]  
(31.34)

Proof is given in [182].

In the case \( M = L^{-1}, \quad N = B \cdot E \cdot L^{-1}, \quad L = (1 - B^2 - E^2 - (B \cdot E))^1/2 \) the system (31.31), (31.33) coincides with the Born-Infeld equations [48].

If we set in (31.33) \( M = \varepsilon, \quad N = -\mu B \cdot E, \quad \varepsilon, \mu \) are constants, then the constitutive equations reduce to the form

\[ D = \varepsilon \left[ 1 + \frac{\mu^2 (E \cdot H)^2}{\varepsilon (\varepsilon + \mu E^2)^2} \right] E - \frac{\mu E \cdot H}{\varepsilon (\varepsilon + \mu E^2)} H, \quad B = \frac{1}{\varepsilon} H - \frac{\mu}{\varepsilon} \frac{E \cdot H}{\varepsilon (\varepsilon + \mu E^2)} E. \]

A popular form of constitutive equations is

\[ B = \mu (E, H) H, \quad D = \varepsilon (E, H) E. \]

It follows from Theorem 31.6 that the condition of Poincaré-invariance generates the following restrictions for \( \mu \) and \( \varepsilon \):

\[ \varepsilon \mu = 1. \]

If \( B = \Phi(H), \quad D = F(E, H) \) then in accordance with Theorem 31.6

\[ D = \mu E, \quad B = \mu^{-1} H \quad (\mu = const). \]

The restrictions imposed by the requirement of the conformal invariance are formulated in the following assertion.

**THEOREM 31.7.** The system of equations (31.31), (31.33) is invariant under the conformal group if
Chapter 6. Exact Solutions of Linear and Nonlinear ... M=M(C_1/C_2), \ N=N(C_1/C_2).

For the proof see [182].

An example of the conformal-invariant constitutive equations is given by the following formulae

\[ D = \left( \frac{\mu H^2}{\mu E^2 - E \cdot H} \right) E, \quad B = \left( \frac{\mu E^2 - E \cdot H}{\mu H^2} \right) H. \]

**Corollary 1.** The nonlinear Born-Infeld equations are not invariant under the group \( C(1,3) \).

**Corollary 2.** A class of nonlinear equations for the electromagnetic field was proposed in [124], which we write in the form

\[
D^\mu F_{\mu \nu} = j_\nu, \quad D^{(\alpha F^\mu_\nu)} = 0,
\]

\[
j_\nu = A_{\nu \rho} F^{\rho \mu}, \quad D^\nu = A_1 \frac{\partial}{\partial x^\nu} + A_2 F^{\nu \sigma} + A_3 \frac{\partial F^{\mu \nu}}{\partial x^\nu},
\]

where \( A_\rho, A_{\rho \mu} \) are arbitrary functions of invariants of the electromagnetic field and of \((\partial C_\nu/\partial x^\mu)(\partial C_\sigma/\partial x^\nu), \alpha=1,2,3\).

Let us show one more nonlinear generalization of Maxwell’s equations [29]:

\[
\begin{align*}
\frac{i}{\mu} \frac{\partial E}{\partial t} &= -p \times H, \quad \frac{i}{\mu} \frac{\partial H}{\partial t} = p \times E, \\
p \cdot E &= 0, \quad p \cdot H = 0,
\end{align*}
\]

where \( v = \sqrt{H^2 + E^2} \) is modula of the velocity of propagation of the electromagnetic field. The density \( \rho \) and velocity \( v \) can be defined as follows [29]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x^\nu} = 0, \quad \rho = \rho(E \cdot H, E^2, H^2),
\]

\[
\rho v_\nu = a(E^2 + H^2, E \cdot H) e_\mu E^\mu H^\nu, \quad k, l, n = 1, 2, 3,
\]

where \( a \) is a smooth function of the invariants of the electromagnetic field. The last relation coincides with the Pointing formula, if \( \rho = (E^2 + H^2)/2 \). In this case (31.36) coincides with the continuity equation.

The essential difference of the system (31.35)-(31.37) from the classical Maxwell equations is that this system is nonlinear.

By studying symmetries of the nonlinear equations for the vector-potential

\[
p_\rho p^\sigma A_\rho - p_\rho (p^\sigma A_\rho) = 0,
\]

\[
(p_\rho - i e A_\rho) A^\rho = 0
\]

373
Symmetries of Equations of Quantum Mechanics

A new vector representation of the algebra AC(1,3) was found [10]. The corresponding basis elements are $P_\mu, J_{\mu\sigma}, D$ of (2.22), (2.42) where

$$S_{\mu\sigma} = -i \left( A_\mu \frac{\partial}{\partial A^\sigma} - A_\sigma \frac{\partial}{\partial A^\mu} \right), \quad K_{iA} = i A_\mu \frac{\partial}{\partial A_\mu}.$$  

But the remaining generators $K_\mu$ are singular in respect with the coupling constant $\epsilon$:

$$K_\mu = 2x_\mu D - x^\sigma x_\sigma P_\mu + 2x^\sigma S_{\mu\sigma} - 2i \frac{\partial}{\epsilon \partial A_\mu}.$$  

In [29] nonlinear generalizations of different wave equations (d’Alembert, KGF, Dirac) was proposed. In particular, the equation

$$\frac{\partial^2 E}{\partial t^2} + \nu^2 p^2 E = 0, \quad \frac{\partial^2 H}{\partial t^2} + \nu^2 p^2 H = 0$$

was considered. Moreover, the velocity of propagation of the electromagnetic field is defined by the equation

$$\lambda_1 \frac{\partial^2 v}{\partial x^\mu} + \lambda_2 \frac{\partial^2 v}{\partial x^\mu} = 0$$

where $\lambda_1, \lambda_2$ are some parameters.

It was pointed out in [6] that any of the equations (31.37) is not invariant under Galilei transformations. Thus if the Galilei relativity principle is valid for heat processes, then it is necessary either to search for other equations or to look for such subsets of solutions of (31.37) which are invariant under Galilei transformations. The


It is generally accepted to think that nonlinear heat processes are described by the equation

$$\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x^\mu} \left[ C(U) \frac{\partial U}{\partial x^\mu} \right] = 0$$  

where $U = U(x_\sigma, x_\alpha, x_\beta, x_\gamma)$ and $C(U)$ are real functions.

Group properties of the corresponding one-dimensional equation ($a=1$, $C(U)=const$) were investigated by Sofus Lie. The group analysis of one-dimensional equation with arbitrary $C$ was carried out by Ovsiannikov [355], group properties of the three-dimensional equation (31.37) are described in [85].

It was pointed out in [6] that any of the equations (31.37) is not invariant under Galilei transformations. Thus if the Galilei relativity principle is valid for heat processes, then it is necessary either to search for other equations or to look for such subsets of solutions of (31.37) which are invariant under Galilei transformations. The
first ability was discussed in [6]. Here we consider the second possibility and demonstrate that it is possible to add such additional conditions to (31.37) that the corresponding system of equations be Galilei invariant.

Let us consider a nonlinear equation of the second order

\[ L(x, U, U_t, U_{x_t}, \ldots, U_{x^n}) = 0, \quad x \in \mathbb{R}^{1,n}, \]

\[ U_1 = \left( \frac{\partial U}{\partial x_0}, \frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_n} \right), \quad U_2 = \left( \frac{\partial^2 U}{\partial x_0^2}, \frac{\partial^2 U}{\partial x_0 \partial x_1}, \ldots, \frac{\partial^2 U}{\partial x_n^2} \right), \ldots \quad (31.38) \]

**DEFINITION 31.1** (S.Lie). The equation (31.38) is invariant under the operators

\[ X = \xi^\mu(x, U) \frac{\partial}{\partial x^\mu} + \eta(x, U) \frac{\partial}{\partial U}, \quad \mu = 0, 1, \ldots, n, \]

if

\[ \tilde{X}L_{|_{t \to 0}} = 0, \quad \text{or} \quad \tilde{X}L = \lambda(x, U, U_t, \ldots, U_{x^n})L \]

where \( \tilde{X} \) is the corresponding prolongation of the operator \( X \), \( \lambda \) is an arbitrary smooth function (for definition of prolongation of operators see [355]).

Let an operator \( Q \) does not belong to the IA of the equation (31.38) and its prolongation is given by the formulae

\[ \tilde{Q}L \rightarrow \lambda_1L + \lambda_1L, \]
\[ \tilde{Q}L \rightarrow \lambda_2L + \lambda_2L, \quad (31.39) \]

besides the following equation is satisfied

\[ L_1 \equiv L_1(x, U, U_t, U_{x_t}) \rightarrow 0. \quad (31.40) \]

**DEFINITION 31.2** [167, 6]. We say the equation (31.38) is *conditionally invariant* if it is invariant under the operator \( Q \) together with the additional condition (31.40), i.e., if the relations (31.39) hold.

The additional condition (31.40) selects such subsets of solutions of the equation (31.38), which have a more extended symmetry than the complete set of solutions. Of course we suppose that the system (31.38), (31.40) is compatible.

The main point of the approach connected with conditional symmetry is to find a way to select such additional conditions which extend the symmetry of the starting equation.

**DEFINITION 31.3** [183]. We say the equation (31.36) is *\( Q \)-invariant* if

\[ \tilde{Q}L \rightarrow \lambda_1L + \lambda_1(QU). \quad (31.41) \]

Let us formulate the assertion about the conditional symmetry of the equation (31.37).
THEOREM 31.8 [180]. The equation (31.37) is conditionally invariant under the generators of the Galilei group
\[ G_a = x^a \frac{\partial}{\partial x^a} + M(U)x^a \frac{\partial}{\partial U} \]
if (31.40) has the form
\[ L_1 \equiv \frac{\partial U}{\partial x_0} + \frac{1}{2} M^{-1}(U) \frac{\partial U}{\partial x^a} \frac{\partial U}{\partial x_a} = 0, \quad M(U) = \frac{1}{2} UC^{-1}(U). \]

THEOREM 31.9 [180]. The equation (31.37) is \( Q \)-invariant under the operators (31.42) if
\[ C(U) = \frac{1}{2m} U^r, \quad M(U) = 2m r^{-2} U^{-1} \]
where \( n \) is the number of space variables, \( m \neq 0, r \neq -2 n^{-1} \) are arbitrary constants.

**Corollary.** The Galilei relativity principle is valid for the overdetermined system including (31.37) and the following equation:
\[ \frac{\partial U}{\partial x_0} + \frac{1}{2} M^{-1}(U) \frac{\partial U}{\partial x^a} \frac{\partial U}{\partial x_a} = 0 \]
where \( M \) is a function defined in (31.43).

We note that the system (31.37), (31.44) reduces to the system of the Laplace and Hamilton-Jacobi equations
\[ \Delta W = 0, \quad \frac{\partial W}{\partial x_0} + \frac{1}{2m}(\nabla W)^2 = 0, \quad W = 2m \int C(U) U^{-1} dU. \]

THEOREM 31.10. The maximal (in Lie sense) IA of the equations (31.45) is \( AG_{1}(1,3) \). The two bases of this IA are (with \( i = 1 \) or \( i = 2 \)):
\[ P_0^{(i)} = \frac{\partial}{\partial x_0}, \quad P_a^{(i)} = \frac{\partial}{\partial x_a}, \quad J_{ab}^{(i)} = x_a P_b - x_b P_a, \]
\[ D^{(1)} = 2 x_0 P_0 + x_a P_a, \quad D^{(2)} = 2 x_0 \int x_a P_a, \]
\[ G_a^{(1)} = x_0 P_a - mx_a P_{n+1}, \quad G_a^{(2)} = WP_a + x_a P_0, \quad a, b = 1, 2, \ldots, n. \]

The proof reduces to using of the standard Lie algorithm. ■

We note that the operators \( G_a^{(1)} \) generate usual Galilei transformations. As to \( G_a^{(2)} \), they generate the following finite transformations [180]
\[ x'_0 = \frac{m}{2} \tau^2 U + mx_0 \tau_a x_a, \quad x'_a = \tau_a U x_a, \quad U'(x') = U(x), \]

376
where $\tau^2 = \tau_0 \tau_2$, $\tau_2$ are group parameters. We see that independent variables are transformed in nonusual manner.

Conditionally invariant systems of equation of the Schrödinger type were investigated in [171].

Now it is established that all the main equations of mathematical and theoretical physics (Maxwell, d’Alembert, Dirac, Schrödinger, KdV, Born-Infeld, Navier-Stokes, etc.) have non-trivial conditional symmetries [34*].

31.7. Conditional Symmetry and Exact Solutions of the Boussinesq Equation

It is known that the maximal IA of the Boussinesq equation

$$U_{00} + \frac{1}{2} \Delta U^2 + \Delta^2 U = 0, \quad U = U(x), \quad x \in R_{1,n}$$

(31.47)

(where $U_{00} = \partial^2 U / \partial x_0^2$) is the Lie algebra of the extended Euclid group with the following basis elements

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a},$$

$$J_{ab} x_a P_b - x_b P_a, \quad D = 2 x_0 P_0 + x_a P_a - 2 U \partial U.$$

All the nonequivalent ansätze reducing the two-dimensional ($n=1$) equation (31.47) to an ODE, which are generated by this IA, have the form

$$U = \phi(\omega), \quad \omega = t_0 x_0 + a_1 x_1, \quad a_0, a_1 = \text{const};$$

$$U = x_0^{-1} \phi(\omega), \quad \omega = x_1 x_0^{-1/2}. \quad \text{(31.49)}$$

Using the ansatz

$$U = \phi(\omega) - 4 \mu x_0^2, \quad \omega = x_1 + \mu x_0^2, \quad \mu = \text{const} \quad \text{(31.50)}$$

Olver and Rosenau [351] reduced the two-dimensional equation (31.47) to the ODE

$$\phi'' + \phi \phi' + 2 \mu \phi = 8 \mu^2 \omega + C. \quad \text{(31.51)}$$

The operator corresponding to this ansatz has the form

$$Q = P_0 - 2 \lambda x_0 P_1 - 8 \lambda^2 x_0 \partial U, \quad \lambda = -2 \mu$$

and does not belong to the IA (31.48).

It is natural to call the ansatz (31.50) non-Lie since it does not follow from the group properties of the equation (31.47).

Following [127] and using the concept of conditional invariance we will describe non-Lie ansätze reducing (31.48) to ODE.

Using the direct substitution method Clarkson and Kruskal [68] had described
ansätze of the form (31.50) reducing the two-dimensional Boussinesq equation to ODE. The distinguishing feature of our approach from the approach used in [68] is that the conditional invariance idea makes to be clear the reason of appearing of such ansätze and gives the regular procedure to find non-Lie ansätze. Moreover, the conditional invariance makes it possible to construct such ansätze which cannot be obtained in the way proposed in [68].

First we consider the two-dimensional equation (31.47)

\[ U_{00} + U_{11} + U_1^2 + U_{1111} = 0. \]  

**THEOREM 31.11.** The equation (31.51) is \( Q \)-invariant under the operator

\[ Q = A(x) \partial_0 + B(x) \partial_1 + [\alpha(x) U + \beta(x)] \partial_U \]  

if the functions \( A(x), B(x), \alpha(x), \beta(x) \) satisfy the following equations.

Case 1. \( A \) is non-zero (without loss of generality we set \( A=1 \)).

\[ \alpha = -2B, \quad \alpha = B, \quad \beta = -2B(B_0 - 2BB_1). \]

\[ \beta_1 = - \frac{1}{2} B_0 + (\alpha B)_0 + B_1 (B_0 - BB_1 + 4 \alpha B), \]

\[ \beta_0 = 4B_1 \beta_1 - B_0 \beta_1 + 4B_1 (B_0 - \alpha \beta_1) + 2 \alpha \beta \beta_0 - 0; \]

Case 2. \( A=0, B=1 \)

\[ \alpha = 0, \quad \alpha = 5 \alpha \beta + 2 \alpha^2, \]

\[ \beta_1 = 3 \alpha \beta + 4 \alpha^2 \beta + 5 \alpha \beta + 5 \alpha \alpha (\alpha^2 - \alpha_1) + 5 \alpha \alpha (\alpha_1 + 2 \alpha^2), \]

\[ \beta_0 = 4 \alpha \alpha_1 \beta + 6 \alpha \alpha (\alpha_1 + \alpha \beta) + 4 \alpha_1 (\alpha^2 + \alpha \beta) + (\beta_1 + \alpha \beta), \]

\[ \beta_0 + 3 \beta \beta_0 + 2 \alpha \beta + 2 \alpha \beta^2 = 0. \]

The proof is carried out using formulae (5.7.8) from [171].

In Case 1 there exist the general solution of (31.54) which leads to the following operator (31.53)

\[ Q = \partial_0 + (ax_1 + b) \partial_1 - 2(aU + a(a' + 2a^2)x_1^2 + (a' + b + ab' + 4a^2 b)x_1 + b(b' + 2ab)) \partial_U \]

where \( a=a(x_0) \) and \( b=b(x_0) \) are solutions of the differential equations

\[ a'' + 2aa' - 4a^3 = 0, \quad b'' + 2ab' - 4a^2 b = 0. \]

In accordance with the explicit form of \( a \) and \( b \) we have several operators

\[ Q_1 = \partial_0 + x_1 \partial_1 - 2x_0 \partial_U \quad (a=0, \quad b=x_0); \]

\[ Q_2 = x_0 \partial_0 - (x_1 + 6x_0^2) \partial_1 + 2[U + 3x_1^2 x_0^2 - 24x_0^6 + 2x_1 x_0^3 \partial_U \left( \frac{1}{x_0}, \quad b=6x_0 \right) \] (31.56a)
Chapter 6. Exact Solutions of Linear and Nonlinear ...

\[ Q_1 = 2x_0 \partial_i (x_i - 3x_i^2) \partial_i - 2(U - 3x_i + 9x_i^2) \partial_i \begin{pmatrix} a = -(2x_0)^{-2}, & b = -\frac{3}{2} x_0 \end{pmatrix}; \]

\[ Q_2 = 2\Omega \partial_i + W' [x_i \partial_i - (2U + Wx_i^2) \partial_i], \quad \begin{pmatrix} a = \frac{W'}{2W'}, & b = 0 \end{pmatrix}; \]

\[ Q_3 = 2W \partial_i + W' (x_i + \Omega) \partial_i - [2W' U + WW'(x_i + \Omega)^2 + x_i + \Omega] \partial_i, \]

where \( a = W'/2W, \quad b = a\Omega, \quad \Omega = \int W(W')^{-2} dx_0, \quad W = W(x_0) \) is the Weierstrasse function satisfying the equation \( W'' = W^2 \).

In Case 2 we obtain only a few particular solutions of (31.54) corresponding to the operators

\[ Q_6 = x_0 \partial_1 + (x_0^2 - 2x_i) \partial_i \begin{pmatrix} \alpha = 0, & \beta = x_0^3 - 2x_i, x_0^{-2} \end{pmatrix}; \]

\[ Q_7 = \partial_1 + (\Lambda - \frac{1}{3} W x_i) \partial_i \begin{pmatrix} \alpha = 0, & \beta = \Lambda - \frac{1}{3} W x_i \end{pmatrix}; \]

\[ Q_8 = x_1 \partial_1 + 2U \partial_1 \begin{pmatrix} \alpha = \frac{2}{x_1}, & \beta = 0 \end{pmatrix}; \]

\[ Q_9 = x_1 \partial_1 + 2(x_i^2 U + 24) \partial_i \begin{pmatrix} \alpha = x_1^4, & \beta = 48x_i^4 \end{pmatrix}\]

where \( \Lambda = \Lambda(x_0) \) is the Lame functions satisfying the equation \( \Lambda'' = W\Lambda \).

Using the operators (31.55), (31.56) we find the ansätze:

1. \( U = \phi(\omega) - 4x_0^2, \quad \omega = x_i + x_0^2 \);
2. \( U = x_0^2 \phi(\omega) - \left( \frac{x_0}{x_i} + 6x_0^4 \right)^2, \quad \omega = x_0(x_i + x_0^2) \);
3. \( U = x_0^2 \phi(\omega) + 2(x_i + x_0^2), \quad \omega = x_0^{1/2}(x_i + x_0^2) \);
4. \( U = W^{-1} \phi(\omega) - \frac{1}{6} W x_i^2, \quad \omega = W^{-1/2} x_i \); \hfill (31.58)
5. \( U = W^{-1} \phi(\omega) - \frac{1}{4} W^{-2} (W')^2 (x_i + \Omega)^2, \quad \omega = W^{-1/2} x_i - \frac{1}{2} \int W' \Omega dx_0 \);
6. \( U = \phi(\omega) - x_0^2 x_i^2 + x_0^4 x_i, \quad \omega = x_i^2 \);
7. \( U = \phi(\omega) - \frac{1}{6} x_i^2 W + \Lambda x_i, \quad \omega = x_i^2 \);
8. \( U = \phi(\omega)x_i^2, \quad \omega = x_i \);
9. \( U = \phi(\omega)x_i^2 - 12x_i^{-2}, \quad \omega = x_i^{-2} \);
Substituting them into (31.51) we come the following reduced equations:

1. Equation (31.51) with $\mu=1$,
2. Equation (31.51) with $\mu=15$,
3. The following equation with $\lambda=6$,

\[
\phi'''' + \phi'' + (\phi')^2 + \frac{\lambda}{6} (\phi''')^2 + 7\phi'' + 8\phi = 0,
\]

(31.59)

4. $\phi'''' + \phi'' + (\phi')^2 + \frac{\lambda}{2} (\phi''')^2 = 0$,
5. $\phi'''' - 2\omega^2 \phi + \phi = 0$,
6. $\phi'''' - \frac{1}{3} W\phi + \Lambda^2 = 0$,
7. $\phi'''' + 6\phi'' = 0$.

Solving (31.59) we obtain from (31.58) solutions of the equation (31.52). We present some of them:

\[
U = -\frac{1}{6} x^2 W, \quad U = -12 x^2, \quad U = -\frac{1}{6} x^2 W - 12 x^2,
\]

\[
U = 2(x_1 - x_0), \quad U = 2(x_1 - x_0) + 12(x_1 + x_0)^2.
\]

Let us consider also the multidimensional Boussinesq equation.

**THEOREM 31.12.** The equation (31.52) with $n=6$ is invariant under the conformal algebra $AC(1,6)$ whose basis elements have the form

\[
P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a P_b - x_b P_a, \quad D = x_b P_a - 4U\partial U,
\]

\[
K_a = 2x^2 D - x^2 P_a \quad (a, b = 1, 2, \ldots, 6)
\]

if $U$ satisfies the additional condition $\Delta U + U^2/2 = 0$.

One of ansätze obtained using the operator $K_a$ has the form

\[
U = (x_1^2 - \omega_2^2)^2 \phi(\omega_1, \omega_2), \quad \omega_1 \mapsto x_0, \quad \omega_2 \mapsto \frac{b x - b^2 x^2}{x^4},
\]

where $b_a$ are constant numbers. The corresponding reduced equations is

\[
\phi_{11} = 0, \quad 2\omega_2 \phi_{22} + 5\phi_{2} - \frac{1}{4b^2} \phi^2.
\]

(31.60)

A particular solution of the equations (31.60) is $\phi = -4b^2(\omega_2)^{-1}$. The corresponding solution of (31.53) has the form

\[
U = 4\left[ x^2 - (\alpha x)^2 \right]^{-1} \quad (\alpha = \text{const}; \quad \alpha^2 = 1).
\]

For more details about conditional symmetry of Boussinesq equation and
other nonlinear equations of mathematical physics, see [171].

### 31.8. Exact Solutions of the Linear and Nonlinear Schrödinger Equation

In this subsection we apply higher order SOs admitted by the Schrödinger equation to construct exact solutions of it. In this way it is possible to find solutions as of the linear Schrödinger equation with a wide class of potentials as of the nonlinear Schrödinger equation. In particular we find soliton solutions and propose a new method of their generation using higher order SO.

The higher order SOs of the Schrödinger equation were described in Section 21. We restrict ourselves to the potentials satisfying the equation (21.22a). The corresponding SO (21.36) commute with the Hamiltonian so it is convenient to search for solutions of the Schrödinger equation in the form

$$\Psi(t, x) = \exp(-iEt)\Psi(x)$$  \hspace{1cm} (31.61)

where $\Psi(x)$ are eigenfunctions of the commuting operators $H$ and $Q$:

$$H \Psi(x) = E \Psi(x),$$  \hspace{1cm} (31.62a)

$$Q \Psi(x) = \lambda \Psi(x).$$  \hspace{1cm} (31.62b)

If relation (31.62a) is satisfied than (31.62b) reduces to the first order equation

$$\left(E + \frac{U}{2} + \frac{\omega_0}{2}\right) \Psi' = \frac{1}{4} \left(U' + i\lambda\right) \Psi$$  \hspace{1cm} (31.63)

which is easily integrated:

$$\Psi = A \sqrt{U + 2E + \omega_0} \exp\left(2i\lambda \int \frac{dx}{U - 2E + \omega_0}\right)$$  \hspace{1cm} (31.64)

where $A$ is an arbitrary constant. On the other hand, substituting the expression for $\Psi'$ from (31.63) into (31.62a) we obtain the compatibility condition for the system (31.62) in the form of the following algebraic relation for $E$ and $\lambda$

$$\lambda^2 \cdot E^2 + cE^2 + \frac{1}{4} (c^2 + b)E + \frac{1}{4} (bc - C),$$  \hspace{1cm} (31.65)

$C$ is an arbitrary constant (the first integral of the Weierstrasse equation, see (21.25)).

Thus using the third order SO it is possible to find solutions of the Schrödinger equation solving the first-order ordinary differential equation (31.63) and the algebraic equation (31.65). This remarkable simple procedure is admissible thanks to hidden symmetry of the equation considered.

Moreover, it happens that the third-order SO of the linear Schrödinger
Symmetries of Equations of Quantum Mechanics

equation enables us to find exact solutions of the corresponding nonlinear equation

\[ i\partial_t \Psi = \frac{1}{2}\nabla^2 \Psi + \frac{1}{2A^2} (\Psi^* \Psi) \Psi. \]  

(31.66)

In accordance with (31.61), (31.64)

\[ \Psi^* \Psi = A^2 (U + 2E + \omega_0) \quad \text{or} \quad U = \frac{1}{A^2} \Psi^* \Psi - 2E - \omega_0, \]  

(31.67)

from which it follows that the functions

\[ \Psi = \exp \left( \frac{i}{2} ct \right) \Psi(x) \]  

(31.68)

(where \( \Psi(x) \) are functions defined in (31.64)) are exact solutions of the nonlinear Schrödinger equation (31.66).

The equation (31.66) is invariant under the Galilei transformations which enables us to generate a more extended family of solutions starting from (31.68):

\[ \Psi = A \left[ U(x - vt) + 2E + \omega_0 \right] \exp \left\{ i \left( (\omega_0 - v^2) \frac{t}{2} + v_x + 2\lambda \int_0^y \frac{dy}{U(y) + 2E + \omega_0} \right) \right\}. \]  

(31.69)

Here \( U \) is a Weierstrasse function i.e. an arbitrary solution of the equation (21.25), \( v, \omega_0, \omega_3, \lambda \) and \( E > 0 \) are arbitrary parameters satisfying the condition (31.65).

Thus using third order SOs of the linear Schrödinger equation we obtain a wide class of exact solutions of the nonlinear Schrödinger equation (31.66). In particular we find soliton solutions corresponding to the potential given in (21.26). Indeed, this potential satisfies (21.25) with the following values of arbitrary parameters: \( \omega_0 = -v^2, \omega_3 = -v^4, C = \varepsilon, \) thus relation (31.65) reduces to the form

\[ \lambda^2 - E^2 (E + v^2) = E \varepsilon. \]  

(31.70)

The corresponding integral included in (31.69) is easily calculated, which enables us to represent solutions in the following form

\[ \Psi = \frac{Av}{\cosh[V(x - vt)]} \exp \left\{ i \left[ \left( v^2 - \frac{v^2}{2} \right) t + v_x + \varphi_0 \right] \right\}, \quad E = 0; \]  

(31.71)

\[ \Psi = A \left\{ v \tanh[V(x - vt)] \pm i \varepsilon \right\} \exp \left\{ i \left[ \left( v^2 - \frac{v^2}{2} \right) t + (v + \sqrt{\varepsilon}) x + \varphi_0 \right] \right\}, \quad E \neq 0, \varepsilon \geq 0. \]  

(31.72)

Formula (31.71) presents a quickly decreasing one-soliton solution [31']. In the case \( v = \pm \sqrt{\varepsilon} \) the solution (31.72) describes solitons with a finite density [31'].

In general the linear Schrödinger equation with a potential satisfying (21.22a) does not possesses any non-trivial (different from time displacements) Lie symmetry.

382
Chapter 6. Exact Solutions of Linear and Nonlinear ... Nevertheless its solutions admit group generating in frames of the concept of conditional symmetry. Indeed, these solutions satisfy (31.67), and the equation (21.1) with the additional condition (31.67) is invariant under the Galilei transformations. This conditional symmetry enables us to generate new solutions:

\[ \Psi = A \sqrt{U(x-vt)+2E+i\omega_0} \exp \left\{ i\left[ (-E-v^2/2 +v_0 x +2\lambda \int_0^y \frac{dy}{U(y)+2E+i\omega_0} \right) \right\}. \]  

(31.73)

The functions (31.73) satisfy the Schrödinger equation with a potential \( U(x-vt) \) where \( U(x) \) is a solution of the equation (21.22a).

The next generation of new solutions can be made using the third order SO. Indeed, if \( U(x) \) satisfies (21.22a) then \( U(x-vt) \) satisfies the Boussinesq equation (21.33), so the corresponding SO does exist. Moreover, this SO can be represented in the form

\[ Q = p^2 + \frac{1}{4}[3U + \omega_0 + 6v^2, p], \ 
\frac{3}{2} v U \equiv 2pH + \frac{1}{2}(U + \omega_0 + 6v^2)p + \frac{3}{2} v U + \frac{i}{4} U^\prime. \]  

(31.74)

Acting by the operator (31.74) on \( \Psi \) of (31.73) we obtain a family of solutions

\[ \Psi' = Q\Psi = a\Psi + ib\Psi_1 \]  

(31.75)

where \( a = \lambda + 2E + \omega_0 v/2 - 4v^3 \) is a constant multiplier, \( \Psi \) is given in (31.73),

\[ \Psi_1 = \frac{U' + 4\lambda}{2(2E + U + c)}. \]  

(31.76)

The first term in the r.h.s. of (31.75) is an evident solution of the Schrödinger equation (inasmuch as \( \Psi \) is a solution), but (31.76) is the essentially new solution obtained using the third order SO.

It is interesting to note that if \( \Psi \) is the soliton solution

\[ \Psi = \frac{vA}{\cosh[v(x-vt)]} \exp \left\{ i\left[ \frac{v^2}{2} t + v_0 x + \phi_0 \right] \right\}, \]  

(31.77)

then the generated solution (31.76) has the form

\[ \Psi_1 = \frac{v' A}{\cosh^2[v(x-vt)]} \sqrt{\cosh^2[v(x-vt)] - 1} \exp \left\{ i\left[ \frac{v^2}{2} t + v_0 x + \phi_0 \right] \right\}, \]  

(31.78)

i.e., is a soliton solution also.

Consequently using the SO (31.74) we can generate new and new solutions.

We see higher order SOs are effective instruments for solving equations of motion and generating new solutions starting from known ones.

383
7. TWO-PARTICLE EQUATIONS

Here we consider equations describing systems of two interacting particles and being invariant under the Galilei or Poincaré group. We investigate non-Lie symmetries of such equations and find exact solutions of some of them.

32. TWO-PARTICLE EQUATIONS INVARIANT UNDER THE GALILEI GROUP

32.1. Preliminary notes

Equations describing a particle motion in the given external fields correspond to the idealized physical situation when we can neglect the particle influence to the field. But in many cases such an influence is very essential, e.g., it is the case for the Kepler problem if the mass of a particle generating the central field is comparable with the mass of an orbital or scattered particle. The last case is a typical example of a two-particle problem in which it is necessary to take into account motions of two interacting objects simultaneously.

A formulation of a two-particle problem in frames of relativistic quantum theory clashes with great difficulties of mathematical and logical nature. We can say with some provisos there is no satisfactory relativistic theory of two-particle systems until now. Meanwhile the need in developing of such a theory is very large since a wide class of really existing particles (mesons) is usually interpreted as a coupled states of two "more elementary" objects, i.e., quarks.

To describe a two-particle system it is usually to use the covariant Bethe-Solpeter equation [33] or quasi-relativistic equations of the type of Breit [54] or Kraichick-Foldy [260]. The alternative possibilities are to use the quasipotential approach of Logunov, Tavchelidze, Todorov and Kadyshesvky [238, 283] or the theory of the direct interaction which was proposed in the classical Dirac work [80] (about the current situation in this theory see, e.g., [69, 394]). We have had not a possibility to analyze special features of any of the mentioned approaches. Here we note only the solutions of the Bethe-Solpeter equation depend on the additional parameter ("proper time" of the system) whose physical meaning is unclear, and the theory of the direct interaction a priory is not available in the cases when it is necessary to take into account the finiteness of velocity of signal propagation. As to the Breit equation, it is invariant neither under Lorentz nor Galilei transformations. So this equation does not satisfy any relativity principle accepted in modern physics. Nevertheless this equation is a good mathematical model of quasirelativistic
two-particle systems and is used widely by describing as hydrogen type systems with electromagnetic interaction as strongly coupled systems of the quark-antiquark type [220, 65] (in the last case the Breit potential has to be modified of course).

In this section we consider two-particle equations for particles of arbitrary spins which satisfy the Galilei relativity principle [155]. It will be shown such equations can be used successively by describing two-particle coupled systems, and take into account various fine effects caused by the spin. In particular we present a Galilei-invariant analog of the Breit equation, which leads to a correct hyperfine structure of spectrum of hydrogen-type systems.

32.2. Equations for Spinless Particles

Consider the simplest version of a two-particle system when the particle spins are equal to zero and an interaction between particles is absent. In quantum mechanics such a system is described by the following equation

\[
\frac{\partial}{\partial x_0} \psi(x_0, x_1, x_2) = \left( \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \right) \psi(x_0, x_1, x_2)
\]  

(32.1)

where \( \psi(x_0, x_1, x_2) \) is the wave function of a system, depending on the coordinates of the first \((x_1)\) and second \((x_2)\) particles, \(m_1\) and \(m_2\) are masses of the first and second particles, \(p_\alpha = -i\frac{\partial}{\partial x_\alpha} \) in the following we use the indices \((1)\) and \((2)\) to denote all the quantities connected with the particles with numbers 1 and 2.

The equation (32.1) is invariant under the Galilei algebra* whose basis elements are

\[
P_0 = H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2}, \quad P = p_1^* + p_2^*, \quad \frac{J}{J_1} = x_1^* x_2 - x_2^* x_1, \quad \frac{G}{G_1} = x_0 (p_1^* - p_2^*) - m_1 x_1^* - m_2 x_2.
\]  

(32.2)

*In reality the symmetry of (30.1) is more extensive. The maximal IA of this equation in the class \(M_1\) for \(m_1 \neq m_2\) is defined by the relation \(A = [AG'(1, 3) \oplus AG'(1, 3)] \subset [P_0^* D] \) where \(AG'(1, 3)\) is the nine-dimensional Lie algebra including all the basis elements of the Galilei algebra except \(P_0^*\) \{\(P_0^* D\) is the two-dimensional subalgebra including \(P_0 = -i\partial / \partial x_0\) and \(D = 2x_0 p_0^* + x^*_1 p_1^* + x^*_2 p_2\). If however \(m_1 = m_2\) then \(A\) coincides with the Lie algebra of the Schrödinger group in the space of dimension \((1+6)\) variables
Symmetries of Equations of Quantum Mechanics

\[ x - x_{(1)} - x_{(2)}, \quad X - \frac{m_1 x_{(1)} + m_2 x_{(2)}}{m_1 + m_2} \]

or

\[ x_{(1)} = X + \frac{m_2 x}{m_1 + m_2}, \quad x_{(2)} = X - \frac{m_1 x}{m_1 + m_2}. \]  \hspace{1cm} (32.3)

For the momentum operators we obtain

\[ p_{(1)} = p + \frac{m_1 p}{m_1 + m_2}, \quad p_{(2)} = -p + \frac{m_2 p}{m_1 + m_2}. \]  \hspace{1cm} (32.4)

where

\[ p = -i \frac{\partial}{\partial x}, \quad P = -i \frac{\partial}{\partial X}. \]

Substituting (32.4) into (32.2) we come to the following realization

\[ \hat{p}_0 = H - \frac{p^2}{2M} + E, \quad \hat{p} = p, \quad \hat{J} = x \times p, \quad \hat{G} = x_0 P - M X, \]  \hspace{1cm} (32.5)

where

\[ M = m_1 + m_2, \quad E = \frac{p^2}{2\mu}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad S = x \times p. \]  \hspace{1cm} (32.6)

The operators (32.5) have the same form as one-particle generators, compare with (12.15). Besides the role of coordinate and momentum is played by c.m. coordinate and total momentum, instead of spin and internal energy we have internal angular momentum \( x \times p \) and \( p^2/2\mu \).

It is not difficult to make sure the Casimir operators (11.14) have the following form for the representation (32.5)

\[ C_3 = \frac{p^2}{2\mu}, \quad C_1 = M, \quad C_2 = M^2 (xxp)^2, \]

and are characterized by the following eigenvalues

\[ 0 \leq C_3 \leq \infty, \quad C_1 = M, \quad c_2 = M^2 l(l+1), \quad l=0,1,... . \]

So in non-relativistic quantum mechanics, the system of two noninteracting spinless particles corresponds to the IR of the algebra \( A[G(1,3) \otimes G(1,3)] \) which reduces to the direct integral in respect with \( c_3 \) and direct sum in respect with \( c_2 \) of IRs of the algebra \( AG(1,3) \). In other words the system under consideration can be interpreted as a quasiparticle whose mass is equal to the sum of masses of constituent particles, spin can take arbitrary integer values and internal energy can...
Chapter 7. Two-Particle Equations

take arbitrary positive value.

An equation for interacting particles can be obtained from (32.1) by adding an interaction potential to the Hamiltonian. As a result we have

$$i\frac{\partial}{\partial x_0}\psi(x_0,x,X) = \left(\frac{P^2}{2M} + \frac{p^2}{2\mu} + V\right)\psi(x_0,x,X).$$  \hspace{1cm} (32.7)

Requiring the symmetry of (32.7) under the Galilei algebra we obtain the following conditions for $V$:

$$[V,P] = [V,X] = [V,S] = 0.$$ \hspace{1cm} (32.8)

where $P, X, S$ are given in (32.3)-(32.6).

It follows from (32.8) a potential is a scalar function of the internal variables $x$ and $p$:

$$V = V(x^2, p^2, x \cdot p).$$ \hspace{1cm} (32.9)

Formulae (32.7), (32.9) define the general form of two-particle Schrödinger equation satisfying the Galilei relativity principle. The corresponding Galilei group generators are given by formulae (32.5) where $H = P^2/2M + p^2/2\mu + V$.

The equations (32.7), (32.9) have high symmetry, being invariant under the 13-dimensional Lie algebra isomorphic to $A[G(1,3) \otimes O(3)]$. Besides the Galilei generators (32.5), this IA includes the generators $S$ of (32.6).

The symmetry under the Galilei algebra makes it possible to separate the motion of the center of mass of the system. Representing the wave function in the form $\psi = \chi(X)\varphi(x)$ we come to the one-particle Schrödinger equation with reduced mass:

$$\left(\frac{P^2}{2\mu} + V\right)\varphi(x) = E\varphi(x);$$ \hspace{1cm} (32.10)

moreover, $\chi(X)$ satisfies the free Schrödinger equation in c.m. variables:

$$i\frac{\partial}{\partial x_0}\chi = \left(\frac{p^2}{2M} + E\right)\chi,$$

$E$ is a constant of separation of variables.

The equation (32.10) is invariant under the algebra $AO(3)$ whose basis elements coincide with $S$ of (32.6). This invariance enables to separate angular variables and so to reduce (32.10) to the system of ordinary differential equations for radial wave functions.

Thus a Galilei-invariant Schrödinger equation for a pair of interacting
spinless particles can be reduced always to ordinary differential equation for radial wave functions in the c.m. frame. This conclusion does not depend on an explicit form of the interaction potential but follows from the analysis of symmetries of the two-particle Schrödinger equation (32.7).

We note that the equation (32.1), like (32.7), can be represented as a system of first-order partial differential equations:

\[(\beta_0, p^j - \beta_j M)\psi = 0, \quad \lambda = 0, 1, \ldots, 6,\]

(32.11)

where we denote \( p_0 = \hat{\partial}/\partial x_0, \ p_a = (p_1, p_2, p_3), \ p_{(2)} = (p_4, p_5, p_6), \) \( \psi \) is a seven-component wave function, \( \beta_0, \ \beta_j \) are the 7×7 matrices of the following form

\[
\beta_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_a & 0 \\
0 & 0 & \lambda_a
\end{pmatrix}, \quad \beta_a = \begin{pmatrix}
0 & \tilde{0} & \tilde{0} \\
\tilde{0} & \lambda_a & \tilde{0} \\
\tilde{0} & \tilde{0} & \lambda_a
\end{pmatrix}, \quad \beta_{(2)} = \begin{pmatrix}
0 & \tilde{0} & \tilde{0} \\
\tilde{0} & m_1 f & \tilde{0} \\
\tilde{0} & \tilde{0} & m_2 f
\end{pmatrix},
\]

(32.12)

\( \hat{0} \) and \( \hat{1} \) are the zero and unit matrices of dimension 3×3, \( \tilde{0} \) is the zero row matrix of dimension 1×3, \( \lambda_a \) are the matrices (6.18). Multiplying (32.11) by \( \beta_0 \) and \((1 - \beta_0)\) and expressing the functions \( \psi_2 = (1 - \beta_0)\psi \) via \( \psi_1 = \beta_0 \psi \) we come to the Schrödinger equation (32.1) for \( \psi_1 \). The remaining components of \( \psi \) (i.e., \( \psi_2 \)) are expressed via \( \psi_1 \):

\[\psi_2 = -\beta_7 (\beta_a p_j + \beta_{(2)} p_{(2)}) \psi_1.\]

The equations (32.11) are Galilei-invariant besides the corresponding basis elements of the algebra \( AG(1,3) \) have the form

\[P_0 = \hat{\partial}/\partial x_0, \quad P_a = -i \hat{\partial}/\partial x_a,\]

(32.13)

\[J = x_0 P_{-1} x_0 P - S', \quad G = x_0 P - M X - \eta\]

where

\[S_a' = i\epsilon_{abc}(1 - \beta_0)(\beta_a \beta_b + \beta_{(2)} \beta_{(2)}), \]

\[\eta_a = (1 - \beta_0)(\beta_a - \beta_{(2)}), \quad a = 1, 2, 3.\]

It is possible to show formulae (32.11), (32.12) define the simplest (i.e., including the minimal number of equations) Galilei-invariant system of first-order equations which reduces to the two-particle Schrödinger equation for scalar particles.

### 32.3. Equations for Systems of Particles of Arbitrary Spin

In analogy with the two-particle equations for scalar particles we will construct motion equations for particles of arbitrary spins. In this subsection we consider equations in the Schrödinger form and then the systems of equations of the
Chapter 7. Two-Particle Equations

We start with the one-particle Schrödinger equations (11.1), (14.9a) for a particle of arbitrary spin. In analogy with (32.1) we write the equation for two noncoupled particles in the form

\[ i \frac{\partial}{\partial x_0} \psi(x_0, x_{(1)}, x_{(2)}) = (H_{s1} + H_{s2}) \psi(x_0, x_{(1)}, x_{(2)}) \]  

(32.14)

where \( \psi(x_0, x_{(1)}, x_{(2)}) \) is a \( 4(2s_1+1)(2s_2+1) \)-component wave function, \( H_{s1} \) and \( H_{s2} \) are Hamiltonians of the first and second particles, i.e., the operators (14.9a) where we set \( k=\alpha=1 \) for simplicity.

The equation (32.14) is evidently Galilei-invariant. The corresponding Galilei group generators have the form of a sum of the one-particle generators (12.18), (14.3), i.e.,

\[ P \sigma = H_{s1} + H_{s2}, \quad P = p_{(1)} + p_{(2)}, \]

\[ J = x_{(1)} \times p_{(1)} + x_{(2)} \times p_{(2)} + S_{(1)} + S_{(2)}, \]

\[ G = x_{(1)} p - m_{(1)} x_{(1)} + \eta_{(1)} + \eta_{(2)}, \]  

(32.15)

The operators (32.15) realize a reducible representation of the algebra \( AG(1,3) \). It is not difficult to make sure the transition to the c.m. variables does not reduce these operators to a direct sum of generators of IRs in contrast to the case of zero mass particles. The corresponding Casimir operator \( C_3 \) (11.14) is not diagonalized in this way, i.e. we have to transform (32.15) to the representation where the internal energy operator

\[ C_3 = E = H_{s1} + H_{s2} - \frac{p^2}{2(m_1 + m_2)} \]

does not depend on the total momentum [155]. Using the following transformation operator (compare with (14.13))

\[ U = \left( 1 + \frac{i}{M} \eta_{(1)} P \right) \left( 1 + \frac{i}{M} \eta_{(2)} P \right) \exp\left( \frac{i}{M} \eta P \right), \quad \eta = \eta_{(1)} + \eta_{(2)}, \quad M = m_1 + m_2, \]

we obtain from (32.14) the following equivalent equation

\[ i \frac{\partial}{\partial x_0} \Phi = \hat{H} \Phi, \quad \Phi = U \psi, \quad \hat{H} = U H U^{-1} - \frac{p^2}{2M} + \hat{E}, \]  

(32.16)

where \( \hat{E} \) is the transformed operator of internal energy

\[ \hat{E} = \sigma_1^{(1)} m_1 + \sigma_2^{(2)} m_2 + 2 \sigma_3^{(1)} S_{(1)}, \quad p - 2 \sigma_3^{(2)} S_{(2)}, p + \]

\[ + \frac{p^2}{2} \left[ \frac{1}{\mu} \left( \frac{1}{m_1} - \left( \frac{1}{m_1} - \frac{1}{m_2} \right) \right) \right], \]  

(32.17)
Symmetries of Equations of Quantum Mechanics

$p$ and $\mu$ are the relative momentum and reduced mass. Simultaneously the generators (32.15) are transformed to the form (32.5) where

\[ S = x \times p + S^{(1)} + S^{(2)} , \]

and $\hat{E}$ is defined in (32.17).

Thus we have obtained the Galilei-invariant equations for a system of two particles of arbitrary spins in the form (32.16). It follows from (32.18) that such a system can be considered as a quasiparticle with variable spin defined as a result of adding of three angular momenta:

\[ L = x \times p , \quad S^{(1)} \quad \text{and} \quad S^{(2)} . \]

The equation (32.16) and the above interpretation can be used as a base of constructing of equations for a pair of coupling particles of arbitrary spins. We search for such equations in the following form

\[ i \frac{\partial}{\partial x_0} \Phi = (\hat{H} + V) \Phi , \]

where $\hat{H}$ is the Hamiltonian of noninteracting particles (32.16), $V$ is an interaction potential. The condition of Galilei-invariance of (32.19) reduces to the requirement of commutativity of $V$ with the generators (32.5), (32.18). This requirement can be written in the form (32.8), (32.18).

It follows from the above that potential $V$ has to be a scalar function of variables $x$ and $p$; moreover, there is no restriction on the exact form of this function imposed by the condition of Galilei invariance.

We see the condition of Galilei invariance admits a very wide class of interaction potentials for arbitrary spin particles. The examples of physically interesting potentials are considered in Subsection 32.5.

### 32.4 Two-Particle Equations of First Order

It is known that partial differential equations of order $N > 1$ can be reduced to equivalent equations including first order derivatives only. So it is interesting to consider Galilei-invariant equations describing pairs of arbitrary spin particles and having the form (32.11).

The problem of describing of equations of the form (32.11) being invariant under the Galilei group can be formulated in analogy with the corresponding one-particle problem considered in Section 13. The general form of the Galilei group generators defined on the set of solutions of these equations is given by formulæ (32.13) where

\[ S' = S^{(1)} + S^{(2)} , \]

\[ S^{(1)} \quad \text{and} \quad S^{(2)} \]

are the spin matrices of the first and second particle, besides,
Chapter 7. Two-Particle Equations

\[ S^a_{(1)}, S^b_{(2)} = 0, \]
\[ \eta^a, S^b = \text{numerical matrices satisfying relations (12.20) together with } S'. \]

Requiring the equations (32.11) admit the Lagrangian formulation and be invariant under the algebra (32.13), (32.20), we come to the following equations for matrices \( \beta_{a}, \beta_{b} \) (compare with (13.9))

\[ \begin{align*}
[S'_{a}, \beta_{a}] &= 0, \\
[S'_{b}, \beta_{b}] &= 0,
\end{align*} \]  
(32.21a)

\[ \begin{align*}
\eta_{a, b} - \beta_{a} \eta_{a, b} &= -i \left( \frac{m_{1}}{M} \beta_{a} + \frac{m_{2}}{M} \beta_{b, a} \right), \\
\eta_{a, b} - \beta_{a} \eta_{a, b} &= -i \delta_{a, b} \beta_{a, b} \left[ S'_{a, b}, \beta_{a, b} \right] + i e_{a, b} \beta_{a, b},
\end{align*} \]  
(32.21b)

We do not present detailed calculations which are analogous to given in Subsections (32.21a).

Thus the problem of describing of Galilei-invariant and two-particle equations of the form (32.11) reduces to the purely algebraic problem of finding of all the non-equivalent matrices \( S'_{a, b}, \eta_{a, b}, \beta_{a, b} \) satisfying (12.20), (32.21). In comparison with the corresponding one-particle problem we have the additional matrices \( \beta_{a, b} \) which have to satisfy (32.32b). Besides we require the wave function \( \psi \) of (32.11) satisfies the two-particle Schrödinger equation (32.1) componentwise (the last has to be a consequence of (32.11)).

Even the simplest (i.e., realized by matrices of the minimal dimension) solutions of (32.21) for arbitrary values of spins \( s_{1} \) and \( s_{2} \) are very cumbersome. This is why we restrict ourselves to considering the cases \( s_{1} = s_{2} = 1/2 \) and \( s_{1} = 0, s_{2} = 1/2 \).

For \( s_{1} = s_{2} = 1/2 \) we obtain the solutions of (32.21) in the form

\[ \begin{align*}
\beta_{a} &= \frac{1}{2} (1 - \Gamma_{0}), \\
\beta_{a} &= (1 + \Gamma_{0}), \\
\beta_{a} &= \frac{M_{1}}{m_{1}} \Gamma_{a, 0}, \quad \beta_{a, b} = \frac{M_{2}}{m_{2}} \Gamma_{a, 0},
\end{align*} \]  
(32.22)

\[ \begin{align*}
\langle S_{(1)} \rangle &= \frac{i}{4} e_{a b c} \Gamma_{a} \Gamma_{b} \Gamma_{c}, \\
\langle S_{(2)} \rangle &= \frac{i}{4} e_{a b c} \Gamma_{a} \Gamma_{b} \Gamma_{c}, \\
\eta_{a} &= \frac{i}{2M} (1 + \Gamma_{0}) \left( \sqrt{m_{1}} \Gamma_{a} + \sqrt{m_{2}} \Gamma_{a, 0} \right)
\end{align*} \]

where \( \Gamma_{0}, \Gamma_{a, b}, \Gamma_{a, b, c} (= 1, 2, 3) \) are matrices of dimension 8×8 satisfying the Clifford algebra (8.2). The explicit realization of these matrices can be chosen say in the form (17.25), (17.27).

The equation (32.11) with the matrices (32.22) is Galilei-invariant and describes a pair of noninteracting particles of masses \( m_{1}, m_{2} \) and spins \( s_{1} = s_{2} = 1/2 \). Indeed, expressing \( \psi = (1/2)(1 - \beta_{0}) \psi \) via \( \psi = \beta_{0} \psi \) we come to the two-particle Schrödinger equation in the c.m. variables. Besides the corresponding operator of the
system spin is given by formulae (32.18), (32.22).

For \( s_1 = 0, s_2 = 1/2 \) we obtain the simplest solution of (32.21) in the form of the following 6x6 matrices

\[
\beta_0 = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta_a = \frac{M}{\sqrt{m_1}} \begin{pmatrix} 0 & \sigma_a & 0 \\ \sigma_a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta_{3a} = \frac{M}{\sqrt{m_2}} \begin{pmatrix} 0 & 0 & \sigma_a \\ 0 & 0 & 0 \end{pmatrix},
\]

(32.23)

\[
\beta_7 = 2(1-\beta_0), \quad \beta_7 = \frac{i}{2\sqrt{m_1}} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{m_1} \sigma_a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta_7 = \frac{i}{2\sqrt{m_2}} \begin{pmatrix} 0 & 0 & \sigma_a \\ 0 & \sigma_a & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where \( \sigma_a \) are the Pauli matrices, 0 are the zero matrices of dimension 2\times2. The corresponding equation (32.11) is Galilei-invariant and can be interpreted as a motion equation of a system of two noninteracting particles of spins 0,1 and masses \( m_1, m_2 \).

The equations present above admit an obvious generalization to the case of interacting particles by the change \( p_0 \to p_0 + V \) where \( V \) is an interaction potential being a scalar function of internal variables. These equations make it possible to take into account interaction of a system with an external field using the standard change \( p_0 \to e^1 A_0^0(x_0, x_0) e^2 A_0^0(x_0, x_0), \quad p_0 \to e^1 A_0^0(x_0, x_0) 
\]

(32.24)

where \( A_0^0, A \) is the vector-potential of the electromagnetic field. It is necessary to emphasize the change (32.24) preserves the Galilei-invariance of the corresponding equation (32.11) if we postulate Galilean transformation law (15.3) for the electromagnetic field.

Examples of two particle interaction potentials and external fields being interesting from the physical point of view are considered in the following subsection.

### 32.5. Equations for Interacting Particles of Arbitrary Spin

Let us analyze the possibilities arising by using the equations considered above for description of pairs of interacting particles. We assume the interaction potential depends on the interparticle distance only, i.e., \( V = V(x) \).

The first-order equations considered in 31.4 reduces to the two-particle Schrödinger equation if we make the change \( p_0 \to p_0 + V \) in (32.11) and then delete...
Equations in the form (32.19) are more interesting. In the c.m. frame these equations take the form
\[ H\psi = (E + V)\psi - \mu\psi, \]
where \( E \) is the operator (32.17).

It is convenient to analyze (32.25) in a representation where the Hamiltonian \( H \) has a quasidiagonal form (i.e., commutes with the matrices \( \sigma^{(1)}_1 \) and \( \sigma^{(2)}_1 \) standing near the mass terms). Using the standard Barker-Glover-Chraplivy procedure [18, 67] (see Subsection 32.3) we obtain according to (33.7) (in our case \( \gamma_0^{(1)} = \sigma^{(1)}_1 \), \( \gamma_0^{(2)} = \sigma^{(2)}_1 \) etc.)

\[ H' = \sigma^{(1)}_1 m_1 + \sigma^{(2)}_1 m_2 + \frac{p^2}{2\mu} + V + i \left( \sigma^{(1)}_1 \frac{1}{m_1} \frac{S^{(1)}_1 \cdot x}{x} - \sigma^{(2)}_1 \frac{1}{m_2} \frac{S^{(2)}_1 \cdot x}{x} \right) \frac{\partial V}{\partial x}, \]

To snake off the non-Hermitian terms in the second row of (32.26) we make an additional transformation

\[ H' \to H'' = VH' V^{-1}, \quad V = \exp \left( \sigma^{(1)}_1 S^{(1)}_1 p - \sigma^{(2)}_1 S^{(2)}_1 p \right) \]

and obtain the following Hamiltonian, neglecting terms of order \( 1/m_1 m_2 m_1 \):

\[ H'' = \frac{p^2}{2\mu} + V + \sum_{i=1,2} \left[ \sigma^{(1)}_i m_i \frac{\partial V}{\partial x} + \sigma^{(2)}_i m_i \frac{\partial V}{\partial x} \right] \left( \frac{S^{(1)}_i \cdot \hat{x}}{2m_i x} + \frac{S^{(2)}_i \cdot \hat{x}}{2m_i x} \right) + \frac{1}{2} \left( \frac{S^{(1)}_i \cdot \hat{x}}{2m_i x} + \frac{S^{(2)}_i \cdot \hat{x}}{2m_i x} \right) \frac{s_i (s_i + 1)}{m_i x} \frac{\partial V}{\partial x}. \]

Thus starting from the Hamiltonian of (32.19) with a central potential \( V = V(x) \) we come to the approximate quasidiagonal Hamiltonian \( H'' \) including the terms representing the spin-orbit and quadruple couplings (compare (32.28) with (10.26)).

If however the interaction potential depends on spins, then the Galilei-invariant equations (32.19) can describe more fine effects also, e.g., effects connected with the retardation of the electromagnetic waves. As an example, consider the equation (32.19) for particles of spins \( s_1 = s_2 = 1/2 \) where \( V = V_B \) is the Breit potential (33.3) (we use the temporary notation \( \gamma_4^{(i)} = \sigma^{(i)}_3 \), \( i = 1, 2 \)).

The equation (32.19) with the Breit potential is manifestly invariant under the Galilei group inasmuch as \( V_B \) of (33.3) satisfies the relations (32.8). Let us demonstrate this equation is a good model of a pair of interacting particles of
spins $s_1=s_2=1/2$. Considering this equation in the rest frame we come to the relation (32.25) where $\hat{E}$ is the operator (32.17), $V_B$ is the Breit potential (33.3). Transforming $H$ to the form (33.7) and then using the additional transformation (32.27) we obtain

$$H' = \sigma_1^{(1)} m_1 + \sigma_2^{(2)} m_2 + \frac{p^2}{2\mu} + \frac{e^2}{x^3} \left( \frac{1}{x} \frac{1}{x} \frac{1}{x} \frac{1}{x} \right) + \frac{e^2}{x^3} \left( \frac{1}{2m_1} \frac{1}{2m_2} \left( S_1 \cdot S_2 \right) + \frac{1}{2m_1^2} S_1 + \frac{1}{2m_2^2} S_2 \right) \cdot x \times \left( \frac{1}{m_1 m_2} \right) x \times \left( \frac{1}{x^3} \right) x \times \left( \frac{1}{x^3} \right) \delta(x) + \alpha(e^4).$$

All the terms of the Hamiltonian (32.29) have a clear physical interpretation and describe the well-known physical effects appearing in systems of two interacting particles. We postpone the discussion of this Hamiltonian to Subsection 33.4 but note that on the set of functions corresponding to the positive eigenvalues of $\sigma_1^{(1)}$ and $\sigma_2^{(2)}$ (i.e., on the subspace corresponding the positive rest energy of any of particles) this Hamiltonian can be represented in the following form

$$H' = H_B' + \left( \frac{1}{8m_1^2} \frac{1}{8m_2^2} \right) p^4,$$

where $H_B'$ is the approximate Breit Hamiltonian (33.8).

We see the approximate Hamiltonian (32.29), obtained by diagonalization of the Galilei-invariant two-particle equation with the Breit potential, includes all the terms of the approximate Breit Hamiltonian excepting the relativistic correction to the kinetic energy. This means that this equation takes into account all the physical effects predicted by the Breit equation, i.e., spin-orbit coupling, retardation of the electromagnetic field etc. The correction to the kinetic energy only ($\sim p^4$), which has essentially relativistic nature, is absent. This correction can be taken into account by adding an additional term (depending on $x$) to the Breit potential. An example of such a generalized Breit potential is given in [155].

Galilei-invariant equations of first order also can be used successively for description of pairs of interacting particles but the corresponding potential has to be more complicated. Thus, for example, starting from the equations (32.11), (32.22) and changing $p_0 \rightarrow p_0 V$, where
Chapter 7. Two-Particle Equations

where

\[
V = e_1 e_2 \left( \frac{1}{x} + \frac{1}{2m_1 m_2} \left( ap \cdot \frac{1}{x} p + b p \cdot \frac{1}{x} x \cdot p \right) + \right.
\]

\[
\left. \frac{i}{2} \left( \frac{S_1}{m_1} - \frac{S_2}{m_2} \right) \frac{x}{x^4} - \frac{1}{m_1 m_2} \left[ \frac{S \cdot L}{x_3} + \left( \frac{1}{2} S^2 - \frac{3}{4} \right) \delta(x) \right] \right) \right.
\]

where

\[
L \propto x \cdot p, \quad a = 1 - \frac{1}{4} \delta, \quad b = 1 + \frac{3}{2} \delta, \quad \delta = \frac{m_1^3 + m_2^3}{m_1 m_2 (m_1 + m_2)}
\]

we come to the Galilei-invariant equations which reduce to the two-particle Schrödinger equation whose Hamiltonian is equivalent to the Breit Hamiltonian in the approximation \( 1/m_1 m_2 \).

Let us summarize. Two-particle equations being invariant under the Galilei group can serve as good mathematical models of quantum mechanical systems consisting in two interacting particles of arbitrary spins. These equations give a correct description (not only qualitative but also quantitative) of these systems and take into account such fine effects as a spin-orbit coupling and even the retardation of the electromagnetic potentials.

The advantages of the Galilei-invariant approach in comparison with quasirelativistic ones are not only of methodological nature (because the corresponding theory satisfies (Galilean) relativity principle) but are connected with the relative simplicity of the corresponding mathematical means which make it possible to obtain solutions of many problems for any values of spins immediately. One such problem is considered in Section 34.

33. QUASI-RELATIVISTIC AND POINCARÉ-INVARIANT TWO-PARTICLE EQUATIONS

33.1 Preliminary notes

We demonstrated in the above the system of two interacting particles can be described with a good accuracy by Galilei-invariant wave equations. Since such equations do not have a universal reputation yet, we present alternative approaches to two-particle problems here. Besides familiarizing readers with these approaches, we will try to attain clear formulation of bounds of application of Galilei-invariant two-particle equations by solving concrete physical problems.
Symmetries of Equations of Quantum Mechanics

We will consider only such two-particle equations which do not include an additional parameter (i.e., the proper time). For the equations with two time variables (equations of the Bethe-Salpeter type), see, e.g., [33].

In this section we present the main information about the Dirac-Breit equation [54] and its generalizations [266-269] and propose our formulations of quasi-relativistic models of two-particle interaction. We deduce equations for the radial wave function of parapositronium and give a formulation of two-particle problem in frames of our constructive model of direct interaction.

32.2. The Breit Equation

A two-particle quasirelativistic equation for spin 1/2 particles was proposed by Breit [54] in 1929 for the first time. This equation has the form

\[ i \frac{\partial}{\partial x_0} \psi = (H^{(1)} + H^{(2)} + V_B) \psi, \]  

(33.1)

where \( \psi = \psi(x_0, x_{(1)}, x_{(2)}) \) is a 16-component wave function,

\[ H^{(i)} = \sum_{\mu} \gamma^0 (\gamma^\mu x^{(i)}) m_i, \]  

(33.2)

\[ \pi_x^{(i)} = -i \frac{\partial}{\partial x^{(i)}}, \quad i = 1, 2, \]

\[ \{ \gamma_x^{(1)} \} \text{ and } \{ \gamma_x^{(2)} \} \text{ are two commuting sets of the Dirac matrices, } A_p \text{ is the vector-potential of an external field. The symbol } V_B \text{ denotes in (33.1) the interaction (Breit) potential} \]

\[ V_B = \frac{e^{(1)} e^{(2)}}{x} \left[ 1 - 2I_0 \left( \frac{x}{2} \right) \left( S^{(1)} \cdot S^{(2)} + S^{(1)} \cdot \hat{x} S^{(2)} \cdot \hat{x} \right) \right], \]  

(33.3)

\[ x = x_{(1)} - x_{(2)}, \quad \hat{x} = \frac{x}{|x|}, \quad S^{(i)} = \frac{I}{4} e_{abc} \gamma^{(i)a} \gamma^{(i)b}. \]

The physical reasons used by Breit to deduce the equation (33.1) was the following. As it was shown by Darwin in 1920 (see [54]) the classical approximate Hamiltonian of a system of two charged particles in the external electromagnetic field has the form

\[ H = H^{(1)} + H^{(2)} - \frac{e^{(1)} e^{(2)}}{x} \left[ \frac{p^{(1)} p^{(2)}}{2 m_1 m_2} \right] - \frac{e^{(1)} e^{(2)}}{x} \left[ \frac{p^{(1)} \cdot x p^{(2)} \cdot x}{x^2} \right], \]  

(33.4)

where \( H^{(1)} \) and \( H^{(2)} \) are the Hamiltonians of the first and second particles, \( p_{(i)} \) are classical momenta. The last two terms of (33.4) arise by taking into account the retardation of the potentials of the electromagnetic field. Changing \( H^{(1)} \) and \( H^{(2)} \) by the Dirac Hamiltonians and the velocities \( v_{(i)} = p_{(i)}/m_i \) by the operators \( V_{(i)} = [H_{(i)}, \hat{x}]_0 \) we
come to the equation (33.4).\footnote{It was Dirac who informed Breit about the correspondence \( \mathbf{p}_0 \rightarrow m[H_0(x-x_0)] = im\gamma_0 \frac{\mathbf{p}_0}{\hbar} \)\cite{54}}

There were proposed other ways of deduction of the Breit equation using quantum electrodynamics approach. We will not discuss these ways (see, e.g., \cite{33}) but note that the main argument of validity of the equation (33.1) is the correspondence of the physical effects predicted by this equation to experimental data (of course, up to accuracy which can be expected from an approximate equation). But the principal defect of the Breit equation is the absence of symmetry under either the Poincaré or Galilei group.

In the following we discuss the Breit equation and its quasirelativistic limit and deduce the system of equations for radial functions by separating the angular variables.

### 33.3 Transformation to the Quasidiagonal Form

To simplify applying of the perturbation theory and to clarify the physical sense of the terms included in the Hamiltonian it is convenient to transform the Breit equation to the representation where the Hamiltonian commutes with the matrices \( \gamma_0^{(1)} \) and \( \gamma_0^{(2)} \), i.e., has a quasidiagonal form. As in the case of one-particle equations (see Section 10) such a diagonalization can be carried out approximately only, using series of consequent transformations.

In papers \cite{18, 67} there was proposed a general method of diagonalization of two-particle Hamiltonians of the kind

\[
H = \gamma_0^{(1)} m_1 \gamma_0^{(2)} m_2 + (EE) + (OE) + (EO) + (OO),
\]

(33.5)

where \( (EE) \) are terms commuting with \( \gamma_0^{(1)} \) and \( \gamma_0^{(2)} \), \( (OO) \) are terms anticommuting with \( \gamma_0^{(1)} \) and \( \gamma_0^{(2)} \), \( (EO) \) are the terms commuting with \( \gamma_0^{(1)} \) and anticommuting with \( \gamma_0^{(2)} \), and \( (OE) \) are terms anticommuting with \( \gamma_0^{(1)} \) and commuting with \( \gamma_0^{(2)} \), besides it is supposed that all these terms are "small" in comparison with \( m_1 \) and \( m_2 \). In the case of the Breit Hamiltonian

\[
(EE) = \gamma_0^{(1)} A_0(x^{(1)} - x_0) + e^{(2)} A_0(x^{(2)} - x_0) + e^{(1)} e^{(2)} \frac{e^{(1)} e^{(2)}}{x},
\]

\( (OE) = \gamma_0^{(1)} \gamma_0^{(2)} (p_a^{(1)} - e^{(2)} A_a(x^{(1)} - x_0)), \)

\[
(EO) = \gamma_0^{(2)} \gamma_0^{(1)} (p_a^{(2)} - e^{(1)} A_a(x^{(2)} - x_0)),
\]

(33.6)

\[
\text{(AA) } = V_x
\]

It is shown in \cite{18} the Hamiltonian (33.5) can be transformed to the following form being quasidiagonal up to the terms of order \( 1/m_1 m_2 \):
Symmetries of Equations of Quantum Mechanics

It is not difficult to make sure that the first three lines of (33.7) give a sum

\[ H \gamma^0 (1) 0 m_1 \gamma^0 (2) 0 m_2 (EE) \]

\[ - \frac{1}{8 m_1} \gamma^0 (EO)^2 + \frac{1}{8 m_2} [((OE),(EE))], (EO) + \]

\[ + \frac{1}{4 m_1 m_2} \gamma^0 ((OE),(OO)) (EO) \] \[ \frac{1}{2 (m_1 - m_2)} (\gamma^0 (1) m_1 + \gamma^0 (2) m_2) (OO) + \ldots . \]

Any term in (33.8) has exact physical sense. The terms in the first line define the kinetic energy of a two-particle system, the terms from the second line correspond to interaction of any of particles with the Coulomb field generated by another particle (i.e., interactions of point charged particles plus spin orbit and Darwin interactions). The remaining terms of (33.8) represent essentially two-particle interactions and correspond to the classical relativistic correction to the Coulomb interaction, caused by the retardation of the electromagnetic field and spin-orbit coupling of the total spin of the system with the field generated by this system. The last three terms of (33.8) correspond to an interaction between the spin dipole...
moments of the particles.

In conclusion we note that to describe real two-particle systems, the modified Breit equation is usually used which takes into account anomalous magnetic moments of particles. For two-quark systems, the Breit potentials has to be changed by another one guaranteeing the confinement of constituent parts of a system [65, 220]. We consider some examples of such generalized equations in Subsections 33.4 and 33.5.

33.4. The Breit Equation for Particles of Equal Masses

Consider the equation (33.1) for the case when an external field is absent and the particle masses are equal, \( m_1=m_2=m \). The total momentum operator \( P=p_1+p_2 \) is a constant of motion for such an equation, so without loss of generality we can restrict ourselves to considering the rest frame of references where \( p_1+p_2=0 \), \( p_1=-p_2=p \).

Let us analyze the generalized Breit equation [266]

\[
\frac{i}{\hbar} \partial_x \psi = \left\{ \gamma_0^{(1)} \gamma_0^{(2)} \right\} \left\{ \gamma_0^{(1)} m - \gamma_0^{(2)} a p_0 m + V \right\} \psi \tag{33.9}
\]

where \( V \) and \( V' \) are arbitrary functions of \( x \), \( \hat{x}=x/x \).

When \( V=V'=e^{(1)} e^{(2)}/x \), formula (33.9) defines the Breit equation in the above representation. Since the explicit form of \( V \) and \( V' \) is not essential for further reasoning, we do not concretize it for the time being.

Let us demonstrate that the equation (33.9) decomposes into two noncoupled subsystems corresponding to the values \( s=0 \) and \( s=1 \) of the total spin of a system described. This means the states corresponding to \( s=1 \) (i.e., orthostates) and \( s=0 \) (i.e., parastates) are independent and transitions between these states are forbidden.

Multiplying the operator in the brackets from the left and the right by \( \gamma_0^{(1)} \) we come to the following equivalent equation for \( \psi'=\gamma_1^{(1)} \psi \)

\[
\frac{i}{\hbar} \partial_x \psi' = \left\{ \gamma_0^{(1)} \gamma_0^{(2)} \right\} \left\{ \gamma_0^{(1)} m + V + \gamma_0^{(2)} m + V' \right\} \psi' + 2 \gamma_0^{(1)} \gamma_0^{(2)} \left( \gamma_0^{(1)} \cdot \gamma_0^{(2)} \right) \left( S^{(1)} \cdot \hat{x} + (S^{(1)} - \hat{S}^{(1)} \cdot \hat{x}) \right) [V'] \psi' = 0. \tag{33.10}
\]

Any of matrices in (33.10) can be expressed via the 16×16 KDP matrices which are connected with \( \gamma \)-matrices by the following relations:

The matrices \( \beta_\mu \) satisfy the KDP algebra (6.20) and realize a reducible representation
Symmetries of Equations of Quantum Mechanics

\[ \beta_{\mu} = \frac{1}{2}(\gamma_{\mu}^{(1)} + \gamma_{\mu}^{(2)}) \quad \mu=0,1,2,3,4. \]  

(33.11)

of this algebra, reducing into three IRs realized by 10×10, 5×5 and 1×1 matrices (the last is the trivial zero representation).

Let us denote

\[ S_{\alpha a} = [\beta_{\alpha}, \beta_{\alpha}], \]  

(33.12)

then

\[ 2\gamma_{(1)}^{(1)}(S^{(1)}, S^{(2)}) = \frac{1}{2} \gamma_{(1)}^{(1)} \gamma_{(2)}^{(1)} - \frac{3}{2} S_{\alpha a}, \]

and (33.10) takes the form

\[ i \frac{\partial}{\partial x_{\alpha}} \psi' = (-2iS_{\alpha a} P_{\alpha} + 2\beta_{\alpha} m + W') \psi' = 0. \]  

(33.13)

We note that the equation (33.13) reduces to the form

\[ i \frac{\partial}{\partial x_{\alpha}} \psi' = \left( H^{K} + \frac{1}{2} W' \right) \psi', \]

where \( x'_{\alpha} = 2x_{\alpha} \), \( H^{K} \) is the KDP Hamiltonian (6.35a). In other words, the Breit equation for equal mass particles in the rest frame is equivalent to the KDP equation in the Schrödinger form with the special potential \( W'/2 \), where \( W \) is given in (33.14).

Consider the equation (33.11) for the case of 5×5 matrices \( \beta_{\alpha} \). Choosing these matrices in the form (6.17) and representing the wave function in the form \( \psi = \text{column}(\phi, \phi, \chi) \) we come to the following system of equations for stationary states
Chapter 7. Two-Particle Equations

\[(E-V+2V')\varphi_1 - 2m\varphi_2 = 0,\]
\[(E-V-2V')\varphi_2 - 2m\varphi_1 - 2p\cdot\chi = 0,\]  \hspace{1cm} (33.15)
\[-2p\varphi_2 + (E+V'-V)\chi - V'(\hat{x}\cdot\chi) = 0.\]

The system (33.15) reduces to the following equation for \(\varphi_2\)

\[
\begin{pmatrix}
\frac{\partial V}{E-V} \frac{\partial}{\partial x} - \frac{(E-V)(E-V-2V')}{4} + \frac{m^2(E-V)}{E-V+2V'} \frac{V'}{x^2} \frac{L^2}{E-V+2V'} \varphi_2 = 0,
\end{pmatrix}
\hspace{1cm} (33.16)

where \(L^2 = (x\times p)^2\). As to \(\varphi_1\) and \(\chi\), they are expressed via \(\varphi_2\) according to the following relations

\[
\chi = \frac{2V'}{(E-V)(E-V+V')} [\hat{x}(\hat{x}\cdot p) - p] \varphi_2,
\]

\[
\varphi_1 = \frac{E-V-2V'}{2m} \varphi_2 - \frac{p\cdot\chi}{m}. \]

We see the Breit equation for parastates reduces to the equation (33.16) for the scalar function \(\varphi_2\). This equation admits solutions in separated variables. Indeed, expanding \(\varphi_2\) by the spherical functions

\[
\varphi_2 = \psi^m(x) Y^m_{lm}(\hat{x}),
\]

and using the representation (29.9) for \(p^2\) it is possible to reduce (33.16) to the ordinary differential equations for radial functions

\[
\left\{ \begin{array}{c}
\frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x} + \frac{\partial V}{E-V} \frac{\partial}{\partial x} + \frac{(E-V)(E-V-2V')}{4} \varphi_2 + \frac{m^2(E-V)}{E-V+2V'} \\
+ \frac{E-V'}{E-V+2V'} \frac{L^2}{x^2} \end{array} \right\} \varphi^m = 0. \]

(33.17)

So the problem reduces to solving equation (33.17) for any possible integer \(l\). When \(V\) and \(V'\) coincide with the Coulomb potential, the equation (33.17) can be solved exactly [266].

The equation (33.17) was obtained in [266] but we present a more simple way of its deduction.

Starting from (33.13) and using the representation (6.22) for the \(\beta_\mu\)-matrices it is not difficult to find the radial equations for orthostates. All the necessary formulae are given in Appendix 3. Using formulae (A.3.2) we can obtain radial equations corresponding to nonequal masses \(m_1 \neq m_2\) and to generalized Breit potential which is an arbitrary \(O(3)\)-invariant function of \(x\). Such equations were considered in [65, 220, 267-269].

401
33.5. Two-Particle Equations Invariant Under the Group $P(1,6)$

The number of the space-time variables needed to describe a two-particle system is equal to 7 (this variables are particle coordinates and time). Thus a natural candidate for the role of the symmetry group of a two-particle system is the generalized Poincaré group $P(1,6)$, i.e., the group of motions of the (1+6)-dimensional Minkowski space (see Chapter 5).

An analog of the KGF equation invariant under the group $P(1,6)$ has the form

$$\left(p_0^2-p_1^2-p_2^2-p_3^2-p_4^2-p_5^2-p_6^2-\kappa^2\right)\psi=0. \quad (33.18)$$

If we denote $P=(p_1,p_2,p_3)$, $p=(p_4,p_5,p_6)$, $\kappa=2m$ then the expression in the brackets is nothing but the correct relation between energy $p_0$, total momentum $P$ and internal momentum $p$ of a two-particle system with equal masses $m_1=m_2=m$ of particles [290]. If $m_1 \neq m_2$ then it is convenient to choose $k=(p_4,p_5,p_6)$ where $k$ is a vector parallel to $p$, defined by the following relation [238]

$$k^2=-m_1m_2+\frac{m_1m_2}{(m_1+m_2)^2}\left(\sqrt{m_1^2+p^2}+\sqrt{m_2^2+p^2}\right)^2. \quad (33.19)$$

The equation (33.18) is manifestly invariant under the algebra $AP(1,6)$ and its subalgebra $AP(1,3)$. But in order to interpret (33.18) as a two-particle equation it is necessary to make sure that the following product of the Poincaré group representations

$$D=D(m_1 s_1) \otimes D(m_2 s_2) \quad (32.20)$$

is realized on the set of solutions of this equation. It is the case when $\psi$ of (33.18) has $(2s_1+1)(2s_2+1)$ components, moreover the corresponding generators of the Poincaré group can be chosen in the form

$$\hat{P}_0=\epsilon \sqrt{p^2+M^2}, \quad \hat{P}_a=p_a,$$

$$J=x \times P+\hat{j}, \quad N=-\frac{1}{2}[X,P_0], \quad -\epsilon = \frac{p \times \hat{j}}{E+M}, \quad \text{where}$$

$$\hat{j}=x \times p+S, \quad S=S^{(1)}+S^{(2)}, \quad M^2=p^2+\kappa^2, \quad \kappa=m_1+m_2. \quad (33.22)$$

Here $S^{(1)}$ and $S^{(2)}$ are the commuting matrices of spin of the first and second particles, $X$ and $x$ are coordinates canonically conjugated with $P$ and $p$ so that

$$[P_\alpha, X_\beta]=-\delta_{\alpha\beta}\epsilon, \quad [P_\alpha, x_\beta]=-i\delta_{\alpha\beta}, \quad [P_\alpha, X_\beta]=[P_\alpha, x_\beta]=[P_\alpha, p_\beta]=[X_\alpha, x_\beta]=0.$$

The operators (33.21) commute with the operator in the brackets of (33.18)
and so form an IA of this equation. To make this assertion evident we rewrite (33.18) in the following form

\[(P_0^2 - P^2 - M^2)\psi = 0. \tag{33.23}\]

In relativistic quantum theory we assign the space of the representation (33.23) to the space of states of a system of noninteracting particles with spins \(s_1\) and \(s_2\). Thus it is possible to interpret (33.18) (where \(\psi\) is a \((2s_1+1)(2s_2+1)\) component wave function) as a motion equation for such a system.

The operator \(M^2\) commutes with \(P_0^2\) and \(P^2\) thus it is possible to consider the equations for eigenvectors of \(M^2\):

\[\psi_m \text{ are solutions of the following equation}\]

\[M^2\psi_m = (p^2 + \kappa^2)\psi_m = m^2\psi_m. \tag{33.25}\]

Formula (33.24) gives the KFG equation in the variables \(x_0, X\), but relation (33.25) defines the eigenvalue problem besides \(\kappa^2 \leq m^2 < \infty\). The equations (33.24), (33.25) are invariant under the algebra \(AP(1,3)\) whose basis elements have the form (33.21), and describe a system of two noninteracting particles with spins \(s_1\) and \(s_2\).

Thus starting with the simplest partial differential equation of the second order, which is invariant under the generalized Poincaré group \(P(1,6)\), we come to the system (33.24), (33.25) which can be interpreted as motion equations for a pair of noninteracting relativistic particles. Of course the adequacy of such equations is comparative only inasmuch as the main interest is attracted by mathematical models of interacting particles. In the following section, we will consider a generalization of (33.24), (33.25) to the case of interacting particles.

In [119] the \(P(1,6)\)-invariant equation of the Dirac type was proposed, which has the form

\[(\Gamma_0 p_0 - \Gamma_k p_k - \kappa)\psi = 0, \quad k = 1, 2, \ldots, 6,\]

where \(\Gamma_0, \Gamma_k\) are matrices of dimension 8×8, satisfying the Clifford algebra. This equation can be used for describing of two relativistic particles of spin 1/2. An analysis of this equation (and its possible generalizations in the case of arbitrary spin) lies out of frames of the present book.

33.6. Additional Constants of Motion for Two- and Three-Particle Equations

We show in Subsection 22.3 that there exist a SO of the Dirac type for any relativistic wave equation describing a particle with spin \(s>0\) in a central field. Here it is demonstrated that such a SO exists for a number of relativistic and
quasirelativistic two- and three-particle equations.

Consider the Breit equation (33.1). The obvious SO of this equation are

\[ P_0 - p_0, \quad P_a - p_a^{(1)} + p_a^{(2)}, \quad J_a = \epsilon_{abc}(X_a P_c + x_a p_c) + S_a \]  

where

\[ S_a = S_a^{(1)} + S_a^{(2)}, \quad S_a^{(1)} = i \gamma_{\alpha}^{(1)} \gamma^{(2)}_{\beta} \gamma^{(2)}_{\gamma}, \quad S_a^{(2)} = \frac{\gamma_{\alpha}^{(1)} \gamma^{(2)}_{\beta} \gamma^{(2)}_{\gamma}}{4}. \]  

It is shown in [341] that the operators (33.26) form a basis of the maximal IA of the Breit equation in the class M_1 (i.e., are the generators of the maximal Lie group admitted by this equation) if \( m_1 \neq m_2 \).

To find an additional SO we consider the Breit equation in the c.m. frame, setting in (33.1) \( p_1 = -p_2 = p \). It is not difficult to verify that the operator [341, 161]

\[ \hat{Q} = \gamma_{\alpha}^{(1)} \gamma_{\beta}^{(2)} \left[ 2(\hat{S} \hat{j})^2 - 2 \hat{S} \hat{j} \hat{j}^2 \right] \]  

where \( \hat{S} \) and \( \hat{j} \) are defined in (33.22), is a SO of this equation.

The operator (33.28) satisfies the condition

\[ \hat{Q}^2 = (\hat{j})^2 \]

and so its spectrum is given in (22.18). We emphasize that this operator does not belong to the enveloping algebra generated by the generators (33.26) and so is essentially non-Lie.

Like the Dirac SO for the Dirac equation (see (22.10)) the operator (33.28) can be used by solving the Breit equation in separable variables. We note that the SO (33.28) is admitted also by the generalized Breit equation with arbitrary \( O(3) \)- and \( P \)-invariant potential.

Apparently it is possible to continue the list of the equations for which the operator (33.28) is a motion constant. For instance this is the case for the relativistic Barut-Komi equation [19].

Following [161], we present new constants of motion for the equation describing two interacting particles with spins 1/2 and 1 [269] and for the three-particle equation of Krolikowsky [267]. These constants have the form

\[ \hat{Q} = d_{3/2} \]  

where \( d_{3/2} \) is the operator given in (22.29). Moreover for the two-particle equation proposed in [269] \( \Gamma = \gamma_{\alpha}^{(1)} - 2 \beta_{\alpha} \), \( S_a = i \epsilon_{abc}(\gamma_{\beta}^{(2)} \gamma_{\beta}^{(2)} + \beta_{\beta}) \), \( \gamma_{\alpha} \) and \( \beta_{\alpha} \) are commuting sets of the Dirac and KDP matrices.

For the three-particle equation [267]

\[ \Gamma = \gamma_{\alpha}^{(1)} \gamma_{\alpha}^{(2)} \gamma_{\alpha}^{(3)}, \quad S_a = \frac{i}{4} \epsilon_{abc} \left[ \gamma_{\beta}^{(1)} \gamma_{\beta}^{(2)} \gamma_{\beta}^{(3)} \right]. \]  

(33.30)
where \( \gamma_\mu(1), \gamma_\mu(2) \) and \( \gamma_\mu(3) \) are commuting sets of the Dirac matrices.

The spectrum of the operators (33.29), (33.30) coincides with the spectrum of the operator \( d_{\mu2} \) of (21.30).

### 34. EXACTLY SOLVABLE MODELS OF TWO-PARTICLE SYSTEMS

#### 34.1. Nonrelativistic Model

Consider the two-particle Schrödinger equation of the kind

\[
i \frac{\partial}{\partial t} \psi \equiv \left( \frac{p^2}{2M} + \frac{p^2}{2\mu} + \hat{V} \right) \psi,
\]

where \( \psi \) is a \((2s_1+1)(2s_2+1)\)-component wave function, \( \hat{V} \) is the interaction potential of the following form

\[
\hat{V} = \alpha \left( -1 + i \frac{k_{(1)} S_{(1)} \cdot x}{m_1 x^2} + i \frac{k_{(2)} S_{(2)} x}{m_2 x^2} \right),
\]

\( \alpha, k_{(1)} \) and \( k_{(2)} \) are dimensionless constants.

The equation (34.1) is manifestly invariant under Galilean transformations inasmuch as the potential (34.2) satisfies the conditions (32.8), (32.18). The corresponding generators of the group \( G(1,3) \) are given by formulae (32.5), (32.18).

The considered equation is a particular case of two-particle equations in the Schrödinger form discussed in Subsection 31.1. The Hamiltonian \( H \) of (34.1) in the c.m. frame can be obtained from (32.26) by a special choosing of \( V, e_1 \) and \( e_2 \) and restricting ourselves to the subset of eigenfunctions of the matrices \( \sigma_i^{(1)} \) and \( \sigma_i^{(2)} \), corresponding to the eigenvalue +1.

In accordance with the results of Subsection 32.5 the considered model is realistic enough and takes into account such fine effects as spin-orbit, Darwin and quadruple couplings. Other important merit of this model is that it is exactly solvable for any values of spins of constitutive particles.

In the following we find the corresponding exact solutions which can be obtained also for the more complicated (relativistic) model considered in the next subsection.
34.2. Relativistic Two-Particle Model

To describe a relativistic two-particle system we use the Bakamjian-Tomas model [10] whose essence is expand in the following.

Let \((P_\mu, J, N)\) be Poincaré group generators describing the kinematics of a system of two non-interacting particles. Then a system of interacting particles corresponds to the generators \((P'_\mu, J', N')\) obtained from \((P_\mu, J, N)\) by the change \(M \rightarrow M - V\) (34.3)

where \(M\) is the mass operator determined as a Casimir operator:

\[ M^2 - P_0^2 = P^2, \]

and \(V\) is a potential of "instantaneous" interaction satisfying the conditions

\[ [V, P_\mu] = [V, J] = [V, N] = 0. \] (34.34)

Using the representation (33.21) it is not difficult to show that \(V\) has to be a scalar function of internal variables only:

\[ V = V(x, p), \quad [V, xp] = S^{(1)} + S^{(2)} = 0. \] (34.5)

Here we consider a constructive model of a potential of instantaneous interaction. On one hand this model is based on equations invariant under the group \(P(1,6)\) (see Subsection 33.6); on the other hand, it includes manifestly covariant equations in c.m. variables.

Let us define the interaction potential of (34.3) by the following relation

\[ \left( \hat{M} + \frac{\alpha}{x} \right)^2 \equiv -k^2 - \mu^2 - i \frac{\alpha}{x^3} (k^{(1)} \cdot S^{(1)} - k^{(2)} \cdot S^{(2)}), \] (34.6)

where \(x\) are variables canonically conjugated with \(k\), \(k\) is a vector related to the internal momenta by the condition (33.19), so that

\[ k^2 + \mu^2 = \tilde{M}^2, \quad \tilde{M}^2 = \frac{\mu}{\kappa} M, \quad \kappa = m_{(1)} + m_{(2)}. \] (34.7)

Formulae (34.5)-(34.7) define a potential \(V\) as an implicit function of \(x, k, S^{(1)}\), and \(S^{(2)}\). It is not difficult to make sure that such defined potential satisfies the conditions (34.4).

The considered potential can be introduced into the \(P(1,6)\)-invariant equations (33.24), (33.25), moreover, the last takes the form

\[ \left( \hat{M} + \frac{\alpha}{x} \right)^2 \equiv -k^2 - \mu^2 - i \frac{\alpha}{x^3} (k^{(1)} \cdot S^{(1)} - k^{(2)} \cdot S^{(2)} - x) \psi_{m} = 0. \] (34.8)

and can be considered as eigenvalue equation for the operator \(\hat{M}\).
Chapter 7. Two-Particle Equations

So starting with the $P(1,6)$-invariant equations (33.24), (33.25) describing a system of two noninteracting particles we go over to the equations (33.25), (34.8) which can be interpreted as a mathematical model of interacting particles of arbitrary spins. The choice of an interaction potential is caused by an additional requirement that the motion equations have to admit a manifestly covariant formulation and describe the electromagnetic interaction of particles. Such a formulation is discussed further on.

Let us demonstrate that in spite of lack of manifestly covariance the equation (34.8) can be rewritten in the covariant notations. For this purpose we consider the following system of equations for a $(2s_1+1)(2s_2+1)$-component wave function:

$$
(\pi^\mu - \mu^2 - e\Sigma_{\mu\nu}F^{\mu\nu})\psi = 0,
$$

(34.9)

where $\pi^\mu = k^\mu - A^\mu(x_0, x)$, $A^\mu$ and $F^{\mu\nu}$ are the vector-potential and tensor of the electromagnetic field. The symbol $\Sigma_{\mu\nu}$ denotes a matrix tensor of valence 2 defined in the space of the IR $D(s_1, s_2)$ of the Lorentz group.

Formula (34.9) defines the general form of the equation obtained from the KGF equation by the "minimal" change $k^\mu \rightarrow \pi^\mu$ and by taking into account the "anomalous" interaction linear in respect with strengths of an external field. When $s_1=0$ or $s_2=0$, (34.9) reduces to the form (10.31) (besides the corresponding $\Sigma_{\mu\nu}$ are proportional to the Lorentz group generators $S_{\mu\nu}$. For arbitrary $s_1$ and $s_2$ we can set without loss of generality

$$
\Sigma_{\mu\nu} = k_1^{(1)} S_{\mu\nu}^{(1)} + k_2^{(2)} S_{\mu\nu}^{(2)},
$$

(34.10)

where $k_1^{(1)}$, and $k_2^{(2)}$ are arbitrary coefficients, $S_{\mu\nu}^{(1)}$ and $S_{\mu\nu}^{(2)}$ are commuting sets of matrices being generators of the IR $D(s_1, 0)$ and $D(0 s_2)$ of the Lorentz group.

The equations (34.9), (34.10) can be interpreted as motion equations of a charged quasiparticle with variable spin $s$, $s_1+s_2 \leq s \leq |s_1-s_2|$. When the vector-potential $A^\mu$ reduces to the Coulomb potential, these equations reduce to the form (34.8) if we write them for stationary states. So these equations can be interpreted as a two-particle equation for particles of spins $s_1$ and $s_2$.

Thus we come to a constructive model of the direct interaction for two-particles systems. This model is based on the manifestly covariant equation (34.9), (34.10) which is the simplest relativistic-invariant equation for a $(2s_1+1)(2s_2+1)$-component wave function $\psi$, taking into account the minimal electromagnetic interaction and the anomalous interaction of the Pauli type. One of the merits of this model is that it is exactly solvable for the cases of the Coulomb and magnetic monopole potentials.
34.3. Solutions of Two-Particle Equations

The operator in square brackets of (34.8) commutes with the total angular momentum operator $j$ of (33.22) so this equation admits solutions in separable variables. Such solutions can be found by the scheme used in Subsection 29.1 for one-particle equations.

To separate angular variables we represent $\tilde{\psi}_m$ as a linear combination of the spherical spinors

$$\psi_m = \sum_{s,\lambda} \phi^s_{\lambda,j_s,k_s,m},$$

(34.11)

where $\Omega^s_{\lambda,j_s,k_s,m}$ are eigenfunctions of the operators $j^2$, $S^2$, $(x\times p)^2$ and $j_3$ (33.22), see (29.4). In contrast with the corresponding formula (29.3) where $s$ was fixed we suppose summing over $s$ by the values $|s_1-s_2| \leq s \leq s_1+s_2$, the domain of $\lambda$ for any fixed $s$ coincides with the possible values of $\lambda$ of (29.5).

Substituting (34.11) into (34.8) and separating angular variables we come to the following system of ordinary differential equations for the radial functions

$$D\phi^s_{\lambda,j_s,k_s,m} = x^{-2} b^{\mu,s}_{\lambda,j_s,k_s} \phi^s_{\lambda,j_s,k_s,m},$$

(34.12)

where $D$ is the operator (29.11), $b^{\mu,s}_{\lambda,j_s,k_s}$ are the following coefficients

$$b^{\mu,s}_{\lambda,j_s,k_s} = [\lambda - (2j+1)] \delta_{\alpha \beta} \delta_{\lambda,j_s} + i\alpha k_1 P_{\alpha \beta,1}^{\lambda,j_s} - i\alpha k_2 B_{\alpha \beta,2}^{\lambda,j_s},$$

(34.13)

and $B_{\alpha \beta,2}^{\lambda,j_s}$ are the coefficients given in (A.3.7).

The system (34.12) is easily integrated. $[b^{\mu,s}_{\lambda,j_s,k_s}]$ is a diagonalizable (normal) matrix so (34.12) reduces to the series of noncoupled equations of the kind

$$D\phi^s_{\lambda,j_s,k_s,m} = x^{-2} b^{\mu,s}_{\lambda,j_s,k_s} \phi^s_{\lambda,j_s,k_s,m},$$

(34.14)

where $D$ is the operator (29.11), $b^{\mu,s}_{\lambda,j_s,k_s}$ are eigenvalues of the matrix $[b^{\mu,s}_{\lambda,j_s,k_s}]$.

Equations of the form (34.14) had already been considered in Subsection 29.2. Repeating reasoning present after formula (29.12) (but changing $b^{\mu,s}_{\lambda,j_s,k_s} \rightarrow b^{\mu,s}_{\lambda,j_s,m}$, $m \rightarrow \mu$) we come to the conclusion that the following values of $\mu = \tilde{m}$ correspond to coupled states (compare with (29.23)):

$$\epsilon = \mu \left[ 1 + \frac{\alpha^2}{n^2 + \left( j + \frac{1}{2} \right)^2 - \alpha^2 + b^{\mu,s}_{\lambda,j_s,k_s}} \right]^{1/2}, \quad n^s = 0, 1, 2, \ldots,$$

(34.15)

and the explicit form of the corresponding eigenfunctions is given by the following formula

408
Chapter 7. Two-Particle Equations

Here is a degenerated hypergeometric function, \( C \) is an arbitrary constant,

\[
\phi = C \left( \mu^2 - \varepsilon^2 \right)^{(\mu-1)/2} x^{\mu-1/2} \exp \left[ -\left( \mu^2 - \varepsilon^2 \right)^{1/2} x \right] \mathcal{F}(-n^{\prime}, k+1, 2(\mu^2 - \varepsilon^2)^{1/2} x).
\]

Here \( \mathcal{F} \) is a degenerated hypergeometric function, \( C \) is an arbitrary constant,

\[
k^2 = (2j+1)^2 + 4(b^{\mu,\nu})^2 - 4\alpha^2.
\]

So we have described solutions of (34.8) and find the corresponding eigenvalues of the operator (34.6). The analysis of the spectrum (34.15) is present in Subsection 34.4.

Solutions of the Galilei-invariant equations (34.1), (34.2) can be found in analogy with the corresponding one-particle problem considered in Subsection 30.4. Considering these equations for stationary states in the rest frame, we come to the following system

\[
\begin{bmatrix}
\tilde{E} - \frac{\alpha}{x} - \frac{p^2}{2\mu} - \frac{i\alpha}{2\mu} \left( \tilde{k}^{(1)} \cdot \mathbf{s}^{(1)} \cdot \mathbf{x} + \tilde{k}^{(2)} \cdot \mathbf{s}^{(2)} \cdot \mathbf{x} \right)
\end{bmatrix} \psi_E(x) = 0.
\]

Representing solutions in the form (34.11) we come to the equations (34.12) for radial functions where

\[
D = 2\mu \left( \tilde{E} + \frac{\alpha}{x} \right) + \frac{\partial^2}{\partial x^2} + 2 \frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}.
\]

\[b_{\lambda^{\mu,i}}^{\alpha} = \left( \lambda^2 - \lambda(j+1) \right) \delta_{\alpha i} \delta_{\lambda^{\mu,i}} - \frac{i\tilde{k}^{(1)} \mathbf{u}^{-\mu} \mathbf{B}^{\mu,\nu} \mathbf{B}^{\nu,\lambda^{\mu,i}}}{m_1} - \frac{i\tilde{k}^{(2)} \mathbf{u}^{-\mu} \mathbf{B}^{\mu,\nu} \mathbf{B}^{\nu,\lambda^{\mu,i}}}{m_2},
\]

\( \mathbf{B}^{\mu,\nu} \) are the coefficients (A.3.7). Diagonalizing the matrix \( \mathbf{b}_{\lambda^{\mu,i}}^{\alpha} \) we reduce these equations to the form (34.14) where \( D \) is the operator (34.17), \( b^{\mu,\nu} \rightarrow \mathbf{B}^{\mu,\nu} \), the last are the eigenvalues of this matrix.

Repeating the arguments present in Subsection 30.4 after formula (30.25) we obtain the possible values of energy of a pair of interacting particles of arbitrary spins in the form

\[
\tilde{E} = \frac{\mu \alpha^2}{\sqrt{j(j+1)^2 + b^{\mu,\nu}}} \mathbf{B}^{\mu,\nu} + n^{\prime} + 1/2, \quad n^{\prime} = 0, 1, \ldots; \quad j = 1/2, 3/2 \ldots \text{ or } j = 0, 1, \ldots.
\]

The discussion of (34.19) is given in the following subsection. Here we present the explicit form of solutions of (34.14), (34.17), (34.18):

\[
\phi = \sqrt{-2\mu \tilde{E}} x^{(\alpha-1)/2} \exp \left[ -\sqrt{-2\mu \tilde{E}} x \right] \mathcal{F}(-n^{\prime}, k+1, 2 \sqrt{-2\mu \tilde{E}} x),
\]

where \( \mathcal{F} \) is a degenerated hypergeometric function, \( C \) is an arbitrary number, \( k^2 = (2j+1)^2 + 4b^{\mu,\nu} \).
33.4. Discussing the Spectra of the Two-Particle Models

We have obtained formulae (34.15), (34.19) giving the possible energy values of the considered two-particle models. The analysis of these formulae is complicated by the circumstance that the parameters $b^r$ are defined as roots of algebraic equations of order $(2s_1+1)(2s_2+1)$ if $j \geq s_1+s_2$. Thus it is convenient to consider approximate solutions which can be represented in the form

$$b^{s_1}_{\lambda} - (2j+1)\lambda_{\lambda} = b^{s_1}_{\lambda} - b^{\beta}_{\lambda}(\alpha^2 + o(\alpha^4)), \quad \tilde{b}^{s_1}_{\lambda} - (2j+1)\lambda_{\lambda} = \tilde{b}^{s_1}_{\lambda} - \tilde{b}^{\beta}_{\lambda}(\alpha^2 + o(\alpha^4)), \quad (34.20)$$

where $b^{s_1}_{\lambda}$, $\tilde{b}^{s_1}_{\lambda}$ are numeric parameters whose values for given $s_1$, $s_2$ and $j$ can be found by equating coefficients near $\alpha^2$ in the characteristic equations.

Substituting (34.20) into (34.15), (34.19) and representing the obtained expressions as series in powers $\alpha^2$ we come to the following formulae for energy levels, valid up to $\alpha^4$:

$$\varepsilon = -\mu \left[ 1 - \frac{\alpha^2}{2n^2} + \frac{\alpha^4 (b^{s_1}_{\lambda} - 1)}{n^4(2l+1)} + \frac{3}{8} \alpha^4 ight], \quad (34.21)$$

$$\tilde{\varepsilon} = -\mu \left( \frac{\alpha^2}{2n^2} + \frac{\alpha^4 \tilde{b}^{s_1}_{\lambda}}{n^4(2l+1)} \right), \quad (34.22)$$

where

$$n = 1, 2, ..., s = s_{(1)} + s_{(2)} + s_{(1)} + s_{(2)} - 1, ..., |s_{(1)} - s_{(2)}|,$$

$$\lambda_{\lambda} = s + l, ..., s + 2\min(s, j), \quad l = \lambda_{\lambda}, 0, 1, ..., n - 1. \quad (34.23)$$

Formulae (34.21), (34.22) differ from (29.27), (30.28) by the change $m \to \mu$ besides the parameters $b^{s_1}_{\lambda}$, $\tilde{b}^{s_1}_{\lambda}$ are defined as coefficients of approximate solutions (34.20) of the characteristic equations for the matrices whose elements are given in (34.13), (34.18).

According to (34.21), (34.22) the spectra of energies of the considered two-particle models are defined by Balmer’s term $-\mu \alpha^2/2n^2$ and by the additional terms of order $\alpha^4$ besides the last describe the fine splitting of the energy levels. The analysis of possible values of the quantum numbers $n, l, s$ and $\lambda_{\lambda}$ given in (34.23) makes it possible to calculate the number of sublevels of the fine structure:

$$N^s_{\lambda} = \sum_{s=1}^{n} N^s_{\lambda},$$

where $N^s_{\lambda}$ are the numbers given in (29.30). We note that the energy levels corresponding to different $l, s$ and $\lambda_{\lambda}$ can coincide in general and be degenerated.

Let us consider in more detail the cases when the particle spins are: A.
Chapter 7. Two-Particle Equations

$s_1=0$, $s_2$ is arbitrary; B. $s_1=s_2=1/2.$

A. If $s_1=0$ then the matrices 
\[
\begin{pmatrix}
\hat{b}^{s's'}_{ss'} \\
\hat{b}^{s's'}_{ss'}
\end{pmatrix}
\]
reduce to the form obtained from (29.11) by the changes $1/s \rightarrow k(1)$ and $1/s \rightarrow k_2 \mu_2$. The corresponding parameters included into (34.21), (34.22) take the form

\[
b_{\lambda}^{s} = \frac{k_2^2}{m_2} \tilde{b}_{\lambda}^{s}, \quad \tilde{b}_{\lambda}^{s} = \frac{m_1^2}{m_1^2 + m_2^2} b_{\lambda}^{s}, \quad s=s_2,
\]

(34.24)

where $\tilde{b}_{\lambda}^{s}$ are the parameters given in (29.26). Substituting (34.24) into (34.21), (34.22) we obtain

\[
\epsilon = \epsilon_\mu + \frac{\mu c^2 \tilde{b}_{\lambda}^{s, \lambda, \alpha}}{n^2(2l+1)} \left( k_2^2 s^2 - 1 \right),
\]

(34.25)

\[
\tilde{E} = \tilde{E}_\mu + \mu \frac{c^2 \tilde{b}_{\lambda}^{s, \lambda, \alpha}}{n^2(2l+1)} \left( s^2 k_2^2 \mu^2 - 1 \right)
\]

(34.26)

where $\epsilon_\mu$ and $\tilde{E}_\mu$ are the energy values obtained from (29.27) and (30.28) by the change $m \rightarrow \mu$.

We see in the case $s_1=0$ the formulae for energy levels of the considered models include the terms $\epsilon_\mu$ and $\tilde{E}_\mu$ obtained by multiplying the corresponding one-particle levels (29.27) and (30.28) by $m_1/(m_1+m_2)$. Besides these formulae include additional terms of order $\alpha^2$ which are nullified if $k_2 = k_2 \mu_2 = \pm 1/s$. In other words for the last values of arbitrary parameters formulae (34.25), (33.26) reduce to the formulae for energy levels of a particle of arbitrary spin in the Coulomb field, multiplied by the coefficient $m_1/(m_1+m_2)$ in order to take into account finiteness of the mass of a particle generating the field. Besides that formula (34.26) includes nonessential constant term $-\mu$. The relation between relativistic and nonrelativistic formulae is the same as in the corresponding one-particle problems (see Subsection 29.6), i.e., for $k_2 = k_2 \mu_2$ the levels (34.25) and (33.26) differ by the value $\Delta \epsilon$ (30.31) giving the relativistic correction to the kinetic energy.

For other values of the parameters $k_2$ and $\tilde{k}_2$, formulae (34.25), (34.26) take into account the anomalous (Pauli) interaction of the orbital particle with the electromagnetic field.

B. For $s_1=s_2=1/2$ the characteristic equation for the matrix (34.13) takes the form

\[
\left[ b_{\lambda}^{s} \right]^4 - 2b_{\lambda}^{s} - 4(j+1) - \alpha^2 \left( c^2 + d^2 \right) \left[ b_{\lambda}^{s} \right]^2 - 2\alpha^2 d^2 b_{\lambda}^{s} + \alpha^2 c^2 d^2 = 0,
\]

(34.27)

where
Symmetries of Equations of Quantum Mechanics

Representing solutions of (34.27) in the form (34.20) and equating the coefficients for the lowest orders of $\alpha^2$ we obtain the following values of the parameters $b_{\lambda s}$:

$$b_0 = \frac{1}{2} \left( k_{(1)} + k_{(2)} \right), \quad d = \frac{1}{2} \left( k_{(1)} - k_{(2)} \right), \quad j \neq 0,$$

(34.28)

In the exceptional case $j=0$ the index $\lambda_s$ can take two values $\lambda_1 = -1$ and $\lambda_0 = 0$ (see (34.29)) and the corresponding characteristic equation has the form

$$\left( b_{\lambda s} \right)^2 - 2b_{\lambda s} + \alpha^2 \mu^2 = 0,$$

from which we obtain the coefficients of (34.20)

$$b_{0}^\infty = -b_{0}^\infty = \mu^2 \nu^2 / 2.$$

(34.30)

Formulae (34.25), (34.29), (34.30) define possible values of energies of the considered model of two interacting particles of spins 1/2. Due to existing of two arbitrary parameters $c$ and $d$ there are wide possibilities of modelling of spectra of hydrogen type systems. We consider in detail two cases: the system of a particle and antiparticle and the system including one heavy and one light particle.

In the case of the system of “particle + antiparticle” it is natural to set $k_{(1)} = -k_{(2)} = d$. Then $c = 0$ and formulae (34.29) take the form

$$b_0^{(1)} = 0, \quad b_0^{(2)} = -\frac{d^2 (l+1)}{2l(l+1)}, \quad b_1^{(1)} = \frac{d^2 (l+2)}{2l(l+1)(2l+3)}, \quad b_1^{(2)} = -\frac{d^2 (l-1)}{2l(2l-1)}.$$

(34.31)

Substituting (34.31) into (34.25) we obtain the corresponding energy levels:

$$\varepsilon = \frac{1}{2n} \left( l - \frac{\alpha^2}{2n^2} \right) + W(s,j,l), \quad j = l + \lambda_s,$$

(34.32)

where $W(s,j,l)$ are the corrections of order $\alpha^4$, defining the fine structure of the spectrum:

$$W(0,l,l) = \mu \varepsilon \left( \frac{3}{8n^2} - \frac{1}{n^2(l+1)} \right),$$

$$W(1,l,l) = W(0,l,l) - \frac{\mu \varepsilon d^2}{2n^2 l(l+1)(2l+1)}, \quad l \neq 0.$$

(34.33)

In the case $d^2 = 1$ formulae (34.33), (34.34) are in excellent accord with the
Chapter 7. Two-Particle Equations

\[
W(1,l,l+1) = W(0,l,l) + \frac{\mu \alpha^4 (l+2)d^2}{2n^2(l+1)(2l+1)(2l+3)},
\]

(34.34)

\[
W(1,l,l+1) = W(0,l,l) - \frac{\mu \alpha^4 (l-1)d^2}{2n^2(l)(2l+1)(2l-1)}.
\]

Corresponding corrections for the positronium energy levels calculated in frames of quantum electrodynamics [33]. Namely (34.33) differ from the corresponding corrections for positronium by the value \(\Delta W\) not depending on \(l\) and \(s\) \((l \neq 0)\)

\[\Delta W = \frac{\mu \alpha^4}{32n^2} \approx 0.03 \frac{\mu \alpha^4}{n^2}.\]

(34.35)

Formulae (34.34) are less similar with the known results. But if we choose arbitrary parameters \(c\) and \(d\) in (34.26) in the form \(c^2 = 2d^2 = 2\) then relations (34.21), (34.29) give the correct dependence of \(W\) on \(l, s\) and \(j\) up to the common displacement (34.35) (compare with the corresponding results for energy levels of positronium given in [3, 33]).

Thus the mathematical model of a pair of interacting particles, based on the equation (34.9), is consistent enough and makes it possible to obtain a correct dependence of a fine structure of the positronium spectrum on the quantum numbers \(l, s\) and \(j\). The displacement (34.35) is not large and is of the same value for any sublevel corresponding to a fixed value of the main quantum number \((l \neq 0)\). For \(l = 0\) the divergence of (34.32)-(34.34) with the known results reaches 17% of the value of fine splitting. Such a divergence is clear inasmuch as the considered model does not take into account the interchange interaction.

We note that in the case \(c = d = 1\) formulae (34.29) take the form

\[
b_{0}^{l} = b_{0}^{l-1} = \frac{1}{2(l+1)}, \quad b_{0}^{l} = b_{0}^{l-1} = \frac{d^2}{T}
\]

and the relation (34.25) reduces to the form obtained from (29.27) by changing \(m\) to the reduced mass \(\mu\). In other words, in the case \(k_0 \rightarrow 0\) we obtain the known formula of the fine structure of the spectrum of a hydrogen type system besides, it takes into account the finiteness of the mass of a central particle. So the relation (34.25) includes the case corresponding to a system of a heavy and light particle.

It follows from the above that the considered model of two-particle interaction is realistic enough to be used successively for a description of two-particle systems with arbitrary spins.

The analysis of the spectrum (34.22) can be made in a complete analogy with the above. For \(|k_0\mu| = |k_0m_1|\) the energy levels of (34.22) differ from the levels (34.21) by the value \(\Delta e - \mu\) where \(\Delta e\) is given in (30.30) and corresponds to the relativistic correction for the kinetic energy.
Here we recall the basic definitions related to Lie algebras, super- and parasuperalgebras (for more details, see [20, 26, 411]).

We denote by $A$ a finite-dimensional vector space over the field $K$ of real (complex) numbers. The vector space $A$ is called a Lie algebra over $K$ if it is closed with respect to the binary operation $(x, y) \rightarrow [x, y]$ which satisfies the following axioms for all $x, y, z \in A$.

\[(A.1.1)\]

\[[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z], \]
\[ [x, y] = -[y, x], \]
\[ [x, [y, z]] + [y, [z, x]] - [z, [x, y]] = 0 \]

for all $x, y, z \in A$.

The operation $[\ , \ ]$ is called the Lie multiplication (Lie brackets), and the last of the relations (A.1.1) is called the Jacobi identity. A Lie algebra is called commutative or Abelian if $[x, y] = 0$ for any $x, y \in A$. A subspace $N$ of the algebra $A$ is called a subalgebra if $[N, N] \subset N$ and an ideal if $[A, N] \subset N$.

Let $A$ and $B$ be two arbitrary Lie algebras and $\Phi$ be a mapping of $A$ into $B$. This mapping is called a homomorphism if

\[ \phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y), \quad x, y \in A, \quad \alpha, \beta \in K \]
\[ \phi([x, y]) = [\phi(x), \phi(y)], \quad x, y \in A. \]

A one-to-one homomorphism of one algebra onto another is called isomorphism, and it is said, that the corresponding algebras $A$ and $B$ are isomorphic.

A representation of a Lie algebra is a homomorphism $x \rightarrow T(x)$ of this algebra into a set of linear operators $T$ defined on some linear space $H$:

$\alpha x + \beta y \rightarrow \alpha T(x) + \beta T(y),$
\[ [x, y] \rightarrow ([T(x), T(y)] = T(x)T(y) - T(y)T(x). \]

The last condition guarantees validity of the Jacobi identity for the algebras of operators $T$. If the representation space $H$ is infinite-dimensional then it is assumed additionally that for all $x \in A$ the operators $T(x)$ have a common invariant domain $D$ dense in $H$. A representation is called irreducible if $H$ does not have a subspace invariant under the operator $T(x)$ for all $x \in A$. If $H$ has invariant subspaces the corresponding representation is called reducible (and if all such subspaces are mutually orthogonal it is called completely reducible).

A superalgebra $SA$ is a vector space over the field $K$ of complex (or real) numbers. Besides, this space is graded and closed with respect to the following
binary operation

\[(x_{\sigma}, y_{\sigma}) \rightarrow [x_{\sigma}, y_{\sigma}] \rightarrow [x_{\sigma}, y_{\sigma}] \text{'}\]  \hspace{1cm} (A.1.2)

whose explicit form depends on values of grading indices \(\sigma\) and \(\sigma\'). This binary operation has to satisfy the axioms

\[[\alpha x_{\sigma} + \beta y_{\sigma}, z_{\sigma}] + \gamma [x_{\sigma}, y_{\sigma}', z_{\sigma}'] = 0,\]

\[[x_{\sigma}, y_{\sigma}]' + [y_{\sigma}, z_{\sigma}]' + [z_{\sigma}, x_{\sigma}]' = 0,\]

where \(\alpha, \beta, \gamma \in \mathbb{K}, \ x_{\sigma}, y_{\sigma}, z_{\sigma} \in \mathcal{S}A\).

We consider the simplest version of gradation, i.e., the so-called \(\mathbb{Z}_2\)-gradation when any element of a \(\mathcal{S}A\) is labelled by one of two possible values 0, 1 of the index \(\sigma\) besides \(f_{\sigma\sigma} = \sigma\sigma\'). Then the binary operation (A.1.2) reduces to commutator for \(\sigma\sigma' = 0\) and anticommutator for \(\sigma\sigma' = 1\).

In other words a superalgebra is a vector space consisting in elements of two kinds: even (\(E\)) corresponding to \(\sigma = 0\) and odd (\(O\)) corresponding to \(\sigma = 1\). This space is closed under the binary operation (A.1.2) which is nothing but a commutator for even-even and odd-odd elements and anticommutator for odd-odd elements. Furthermore, commutators of even-even and anticommutators of odd-odd elements are even and commutators of even-odd elements are odd. Such "multiplication rules" are formulated in (18.1).

We will not enter into details and discuss the main properties of so defined algebraic objects but note that the principal results of the theory of Lie algebras are extended to the case of superalgebras.

A parasuperalgebra \(\mathcal{P}SA\) is a graduated vector space in which bi- and trilinear generalized Lie brackets are defined [26, 88, 372]. Thus in Rubakov-Spiridonov [372] formulation of the parasupersymmetric quantum mechanics the following relations for even and odd elements are postulated

\[[E, E] = E, \ [E, O] = O, \ [O, O, O] = EO\]  \hspace{1cm} (A.1.3)

where

\[[A, B, C] = AB + BA + BC + CB,\]

An alternative definition of parasuperalgebra (which is not equivalent to (A.1.3)) was proposed by Beckers and Debergh [26]. It is characterized by the double commutation relations instead of the double anticommutators.

Generalized Lie structures called \(\mathcal{P}SA\) arises in parasupersymmetric quantum mechanics [26, 372]. We notice such structures are weaker then \(\mathcal{S}A\) in the sense that a \(\mathcal{S}A\) can be always considered as a \(\mathcal{P}SA\) (e.g., relations (A.1.3) follow from (18.1))
but the converse is not true.

A simple example of PSA is the set of the KDP matrices satisfying (6.20), and matrices $S_{\mu\sigma}=i[\beta_\mu, \beta_\sigma]$, besides $\beta_\mu$ are odd and $S_{\mu\sigma}$ are even elements.

**APPENDIX 2**

**GENERALIZED KILLING TENSORS**

1. Let us consider the following system of overdetermined partial differential equations:

$$\partial^{(\nu_1, \nu_2, \ldots, \nu_s)} K_{\nu_1 \nu_2 \ldots \nu_s} = 0$$

(A.2.1)

where $K$ is a symmetric tensor of valence $j$, depending on $m$ variables $x_1, x_2, \ldots, x_m$, $\partial\nu=\partial/\partial x_\nu$, and symmetrization is imposed over the indices in paratheses.

The equation (A.2.1) arises naturally in problems of description of higher-order SO of partial differential equations, see, e.g., (16.33). In the cases $s=1, j$ is arbitrary and $s=1, j=1$ relation (A.2.1) reduces to the equation for Killing tensor [408] and Killing vector [249] correspondingly. We shall call a symmetric tensor $K$ satisfying (A.2.1) with arbitrary $s$ a generalized Killing tensor of valence $j$ and order $s$ [328, 342].

Here we present the explicit form of solutions of (A.2.1) for $m\leq 4$.

Consider the case $s=1$, when (A.2.1) reduces to the form (1.11). This system is overdetermined and includes $C_{j+1}^{(m)}$ equations for $C_{j+1}^{(m)}$ unknowns. The corresponding solutions have the form [328]

$$K_{j+1}^{(m)} \sim \sum_{|a|} \lambda_{j+1}^{(m)} \partial_1 \partial_2 \ldots \partial_j x_{a_1} x_{a_2} \ldots x_{a_j}$$

(A.2.2)

where $\lambda$ are arbitrary tensors symmetric under the permutations $a_k \leftrightarrow a_i$ ($i,k=1,2,\ldots,j$), antisymmetric under the permutations $a_i\nu_i$ with $b_i$ ($1\leq i\leq j-1$). Besides, any cyclic permutation of three indices nullifies this tensor. The number of linearly independent solutions $N_{j+1}^{(m)}$ is equal to

$$N_{j+1}^{(m)} = \frac{1}{m} C_{j+1}^{(m-1)} C_{j+1}^{(m-1)}$$

(A.2.3)

Expanding $\lambda$ in basic tensors (see definitions in Section 16) we come to the representation (16.13). Expanding $\lambda$ in irreducible tensors having zero traces over any pair of indices it is possible to represent $K_{j+1}^{(m)}$ in a form including irreducible parameters only [342].

Killing tensors of arbitrary valence $j$ and order $s$ are polynomials on $x_\mu$ of
order \( s+j-1 \), depending on \( N_{ij}^m \) arbitrary parameters besides
\[
N_{ij}^m = \frac{5}{m} C_{ij}^{m-1} C_{ij}^{m-1}.
\] (A.2.4)

The explicit expressions of these tensors can be obtained from the recurrence relations
\[
K_s^{a_1a_2...a_l}K_s^{b_1b_2...b_l}K_s^{c_1c_2...c_l}x_{a_1b_1c_1} = 0
\] (A.2.5)
where \( K_s^{a_1a_2...a_l} \) and \( K_s^{b_1b_2...b_l} \) are the Killing tensors of order \( s-1 \) and valences \( j \) and \( j+1 \) correspondingly.

2. Consider one more class of overdetermined equations
\[
(\partial_a^0...\partial_a^s(\tilde{K}^{a_1a_2...a_l}))^{T_L} = 0
\] (A.2.6)
where \( \tilde{K}^{a_1a_2...a_l} \) is a symmetric traceless tensor of valence \( j \), depending on \( m \) variables, and the symbol \([...]^{T_L}\) denotes the traceless part of the tensor in the square brackets (in our case this tensor is symmetric and has the valence \( R = j + s \)):
\[
[T^{a_1a_2...a_l}T^{a_1a_2...a_l}]^{T_L} = 0
\] (A.2.7)
\[
\sum_{\ell=0}^{R/2} (-1)^\ell K_\ell \sum_{a=1}^n g^{a_1a_{\ell+1}...a_{l+1}}T^{a_{\ell+1}a_{l+1}...a_l}b_{a_{\ell+1}}b_{a_{l+1}}b_{a_l} = 0
\] (A.2.8)
where
\[
K_\ell = \frac{1}{2^{(R-2\ell)}} \prod_{i=1}^{R/2} \frac{1}{i!(2(R-i)+m-2)}
\]
In the case \( s=1 \) (A.2.6) reduces to the equation for the conformal Killing tensor [408]. We call a symmetric traceless tensor \( \tilde{K}^{a_1a_2...a_l} \) satisfying (A.2.6) a generalized conformal Killing tensor of valence \( j \) and order \( s \).

The equations (A.2.6) were analyzed and solved in [328]. The numbers \( \tilde{N}_{ij}^m \) of linearly independent solutions are given in (A.2.9):
\[
m = 3, \quad \tilde{N}_{ij}^3 = (2j+1)(2j+2s+1)(2j+s+1)/6,
\] (A.2.9)
\[
m = 4, \quad \tilde{N}_{ij}^4 = (j+1)(j+s+1)(2j+2s)/12.
\]
If \( m = 2 \), then there exist an infinite number of solutions of the system (A.2.6).

First we represent solutions of (A.2.6) for the case \( s = 1, m > 2 \):
\[
\tilde{K}^{a_1a_2...a_l} = \sum_{i=0}^{\frac{j-4}{2}} \sum_{j=0}^{\frac{j-4}{2}} (-1)^i C_{ij}^{l-4} \tilde{X}^{a_{i+1}a_{i+2}...a_l} x_{a_{i+1}a_{i+2}...a_l} \times
\]
\[
\times x^{a_1a_2...a_{i+1}} x^{a_3a_4...a_l} x_{a_3a_4...a_l} (x^2)^{T_L}
\] (A.2.10)
where \( l+1 \leq j \), \( \tilde{X} \) are arbitrary basic tensors. Solutions of (A.2.6) corresponding to
arbitrary \( s \) can be expressed via (A.2.10):

\[
\tilde{K}^{\alpha_1 \cdots \alpha_n}_{(s)} = \sum_{\{a\}} \left[ \tilde{K}^{\alpha_1 \cdots \alpha_n}_{(s)}(X^2)^{a-1} + \sum_{n=0}^{\infty} f^{\alpha_1 \cdots \alpha}_{(s)}(X^2)^n \right].
\]  

(A.2.11)

Here \( \tilde{K}^{\alpha_1 \cdots \alpha_n}_{(s)} \) are conformal Killing tensors of valence \( j \) and order \( s=1 \) (see (A.2.10)), the index \( i \) labels independent solutions with different powers of \( X^2 \), \( f^{\alpha_1 \cdots \alpha}_{(s)} \) are the tensors described in the following.

For \( m=3 \) independent tensors \( f^{\alpha_1 \cdots \alpha}_{(i)} \) are labeled by integers \( c \), \( 0 \leq c \leq 2j \), besides

\[
f^{\alpha_1 \cdots \alpha}_{(i)} = \sum_{c} \left[ \hat{f}^{\alpha_1 \cdots \alpha}_{(i)} + (1-c) \hat{f}^{\alpha_1 \cdots \alpha}_{(i)} \right]^T_{\mathcal{L}},
\]

(A.2.12)

where

\[
\hat{f}^{\alpha_1 \cdots \alpha}_{(i)} = \sum_{\{a\}} (-2)^{a} C_n^{a} \hat{\lambda}^{b_1 b_2 \cdots b_{n-1}}(X^2)^{a} X^{a_{1} \cdots a_{n}}
\]

\[
\times X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} \epsilon_a - \frac{1}{2} [1 - (-1)^\ell],
\]

\(
\hat{\lambda}^{b_1 b_2 \cdots b_{n-1}} \)

are arbitrary symmetric and traceless tensors.

In the case \( m=4 \) the tensors \( f^{\alpha_1 \cdots \alpha}_{(i)} \) are labelled by pairs of integers \( c=(c_1, c_2) \), satisfying the conditions \( j \leq c_1 \leq j \), \( 0 \leq c_2 \leq (j+c_1)/2 \). The corresponding explicit expressions of \( f^{\alpha_1 \cdots \alpha}_{(i)} \) are

\[
f^{\alpha_1 \cdots \alpha}_{(i)} = \sum_{\{a\}} (-2)^{a} C_n^{a} \hat{\lambda}^{b_1 b_2 b_3 b_4}(X^2)^{a} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} \epsilon_a - \frac{1}{2} [1 - (-1)^\ell],
\]

\( \hat{\lambda}^{b_1 b_2 b_3 b_4} \)

\[
= (-2)^{a} C_n^{a} \hat{\lambda}^{b_1 b_2 \cdots b_{n-1}}(X^2)^{a} X^{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} X_{a_{1} \cdots a_{n}} \epsilon_a - \frac{1}{2} [1 - (-1)^\ell],
\]

where \( n=c_1 \), \( c_1 \leq 0 \) and \( n=0 \) for \( c_1 \geq 0 \).

Let \( K^{b_1 b_2 b_3 b_4}_{(s)} \) is an irreducible tensor of valence \( R_1 + R_2 \), symmetric under the permutations \( a \leftrightarrow b \), antisymmetric under the permutations \( a \leftrightarrow c \), \( \epsilon \leftrightarrow \mu \leq [R_1] \) and antisymmetric under the permutations \( a \leftrightarrow b \), \( \mu \leq [R_2] \). We say \( K^{b_1 b_2 b_3 b_4}_{(s)} \) is a generalized conformal Killing tensor of valence \( R_1 + 2R_2 \) and order \( s \) if it satisfies the following overdetermined system of PDE

\[
\left[ \partial^{\alpha_1} \partial^{\alpha_2} \cdots \partial^{\alpha_n} K^{b_1 b_2 b_3 b_4}_{(s)} \right] = 0
\]

(A.2.15)

where \( j=R_1 + R_2 \). In the case \( s=1 \) and \( R_2=0 \), \( s \) is any integer, (A.2.15) reduces to the equation for the conformal Killing tensor [418] and to the equation (A.2.6).

In this book, we use only generalized Killing tensors of order \( s=1 \). The number of linearly independent solutions of the equation (A.2.15) for \( s=1 \), \( m=4 \) is

Let us present the explicit form of some of the above defined tensors.
Denoting arbitrary irreducible tensors by Greek letters we obtain from (A.2.14) the conformal Killing tensor for four-dimension Minkowski space \((m=4, s=1, j=2)\):

\[ \tilde{F}^{ab} = \sum_{\alpha=0}^{4} (G^a_{\alpha} + G^b_{\alpha} - \frac{1}{2} G^{ab} G^c_{\alpha} G^d_{\alpha} \delta_{cd}) \]  

(A.2.17)

where

\[ G^a_{\alpha} = \tilde{\chi}^a_{\alpha \beta \gamma} + \chi^a_{\alpha \beta \gamma} + \tilde{\chi}^a_{\alpha \beta \nu \rho \sigma} \xi_{\delta \gamma} (x^\mu \eta^{\nu \rho \sigma} - x^\nu \eta^{\mu \rho \sigma}) x_\alpha - \frac{1}{2} \eta^{[\nu \rho \sigma]} x^2. \]  

(A.2.19)

We present also the explicit form of the generalized conformal Killing tensors for \(m=4, s=1, R_1=0,1, R_2=1,2\).

\[ K_{\mu \rho \nu} = \tilde{K}_{\nu \rho \mu}, \]  

(A.2.20)

\[ K_{\mu \rho \nu} = 2G^a_{\rho \alpha \beta} - G^a_{\rho \alpha} + G^a_{\rho \nu \lambda} + \frac{1}{2} \left( \eta^{\mu \rho \sigma} \eta^{\nu \lambda \sigma} \right) G^a_{\mu \nu \lambda}, \]  

(A.2.21)

where

\[ G^a_{\rho \alpha \beta} = G^a_{\alpha \rho \beta} + G^a_{\beta \rho \alpha}, \]  

(A.2.22)

\[ G^a_{\rho \alpha} = \lambda^{[\nu \mu]} \eta_{[\rho \sigma]} x^\nu \eta^{[\rho \sigma]} x^\mu + \lambda^{[\nu \mu]} \eta_{[\rho \sigma]} x^\nu \eta^{[\rho \sigma]} x^\mu + \lambda^{[\nu \mu]} \eta_{[\rho \sigma]} x^\nu \eta^{[\rho \sigma]} x^\mu + \lambda^{[\nu \mu]} \eta_{[\rho \sigma]} x^\nu \eta^{[\rho \sigma]} x^\mu - \frac{1}{2} \eta^{[\nu \rho \sigma]} x^2 + \frac{1}{4} \lambda^{[\nu \mu]} \eta x^4. \]  

\[ R_1=1, R_2=2: \]
Symmetries of Equations of Quantum Mechanics

\[ K_{\nu \lambda} - 2K_{\nu \lambda} + K_{\lambda \nu} - K_{\nu \rho} + K_{\lambda \sigma} + K_{\nu \lambda} - \]
\[ -g_{\nu \lambda} K_{\rho \mu} + g_{\nu \lambda} K_{\mu \rho} - g_{\nu \lambda} K_{\mu \rho} + g_{\nu \lambda} K_{\rho \mu}, \quad (A.2.24) \]

\[ K_{\nu \lambda} \text{ is defined in (A.2.21) where} \]
\[ g_{\nu \lambda} \text{ is given by the relation} \]
\[ g_{\nu \lambda} = -g_{\lambda \nu} + g_{\nu \rho} + g_{\lambda \sigma} - g_{\nu \rho} - g_{\lambda \sigma} + g_{\rho \sigma} - g_{\nu \rho} - g_{\lambda \sigma}. \]

\[ (A.2.25) \]

**APPENDIX 3**

**MATRIX ELEMENTS OF SCALAR OPERATORS IN THE BASIS OF SPHERICAL SPINORS**

Spherical spinors satisfying the relations (29.4) can be represented as columns whose \( m \)-th component is given by the relation

\[ (Q^*_{\mu \nu})_m^\nu = C_{j \lambda m-\mu \nu} Y_{j \lambda m-\mu}, \quad (A.3.1) \]

where \( Y_{jm} \) are the spherical functions.
\[
Y_{lm}(\hat{x}) = \frac{1}{2\pi} \exp(i m \varphi)(-1)^{m} \left\{ \frac{1}{2} \right\}^{(l+1/2)(l-m)} \left( \frac{(l+1/2)(l-m)}{l+1/2} \right)^{1/2} P_{l} |m| (\cos \theta),
\]
\[
P_{l} |m| (\cos \theta) = \frac{1}{2!} \sin |m| \theta \frac{d |m|}{(d \cos \theta) |m| (\cos^{2} \theta - 1)^{1/2}},
\]

\(\varphi\) and \(\theta\) are the polar and azimuthal angles of the vectors \(\hat{x}=\hat{x}/x\), \(C_{jm}^{lm}\) are the Wigner coefficients. Such defined spherical spinors satisfy the normalization condition

\[
\left( \sum_{j} |C_{jm}^{lm}|^{2} \right)^{1/2} \delta_{m,n}.
\]

Let us present the action of some scalar operators on the spherical spinors.

Following [323] we write

\[
\frac{d |m|}{|m|} \delta_{\lambda_{1} \lambda_{2}} \delta_{\mu_{1} \mu_{2}}.
\]

Such defined spherical spinors satisfy the normalization condition

\[
\int \Omega_{j}^{\lambda} \Omega_{j}^{\lambda} d\omega \delta_{\lambda_{1} \lambda_{2}} \delta_{\mu_{1} \mu_{2}}.
\]

Following [323] we write

\[
S_{\lambda} \cdot \Omega_{j}^{\lambda} = \sum_{k} d_{k}^{\lambda} \Omega_{j}^{k \lambda},
\]

\[
S_{\lambda} \cdot p \Omega_{j}^{\lambda} = \sum_{k} d_{k}^{\lambda} \Omega_{j}^{k \lambda},
\]

\[
K_{\lambda} \cdot \Omega_{j}^{\lambda} = \sum_{k} b_{k}^{\lambda} \Omega_{j}^{k \lambda},
\]

\[
K_{\lambda} \cdot \Omega_{j}^{\lambda} = \sum_{k} b_{k}^{\lambda} \Omega_{j}^{k \lambda},
\]

\[
(A.3.3)
\]

where \(S_{\lambda}\) are generators of the IR \(D(s)\) of the group \(O(3)\), \(K_{\lambda}\) are the matrices defined by relations (12.26) and given in the explicit form in (4.65).
Symmetries of Equations of Quantum Mechanics

\[ h_\lambda^{ij} = \frac{1}{2}(\lambda d_j - \lambda^2 - s(s+1)), \quad d_j = 2j+1, \]

\[ g_\lambda^{ij} = \frac{1}{2}[(s^2 - \lambda^2)(d_j + s - \lambda)(d_j - s - \lambda)]^{\frac{1}{2}}, \]

\[ b_{\lambda\lambda}^{ij} = \frac{1}{2}(\delta_{\lambda\lambda} - h_\lambda^{ij}), \]

\[ (A.3.4) \]

\[ g_{\lambda}^{ij} = \sqrt{\frac{(2j + s - \lambda + 1)(s - \lambda - 1)(s - \lambda)(2j + s - \lambda)}{(2j - 2\lambda - 1)(2j - 2\lambda + 1)}} \]

\[ f_{\lambda}^{ij} = \sqrt{\frac{(2j - s - \lambda + 1)(2j - s - \lambda + 2)(s + \lambda) - (s + \lambda)}{(2j - 2\lambda - 1)(2j - 2\lambda + 3)}} \]

and the coefficients \( d_{\lambda\lambda}^{ij} \) are given in (29.7), (29.8).

Relations (A.3.2) are present, e.g., in [409] where another notations are used. Relations (A.3.3) complete (A.3.2) and make it possible to find the result of action of an arbitrary scalar operator \( \eta \prod F \) (where \( \eta \) is a vector matrix satisfying (12.20b), \( F \) is one of the operators \( p, x \) or \( x \times p \)) on the spherical spinor. Using the representation (12.23) we can write

\[ \eta \cdot F \Omega_{ij\lambda m} = (a_{ij}^{s} S^{(a)} F + b_{ij}^{s} K^{(a)} F + c_{ij}^{s} K^{(s)} F) \Omega_{ij\lambda m} \]

where \( a_{ij}^{s}, \ b_{ij}^{s}, \ c_{ij}^{s} \) are parameters defining the matrix \( \eta \), the index \( i \) labels degenerated eigenvalues \( s(s+1) \). Using (A.3.2)-(A.3.4) we obtain the explicit expression of the r.h.s. of (A.3.5).

Consider two commuting spin matrices \( S_\alpha^{(1)} \) and \( S_\alpha^{(2)} \) satisfying the relations

\[ [S^{(a)}_\alpha, S^{(b)}_\beta] = i\delta_{\alpha\beta} e_{abc} S^{(c)}_\alpha, \quad \sum_\alpha S^{(a)}_\alpha S^{(a)}_\alpha = s(s+1), \quad \alpha = 1, 2. \]

(A.3.6)

Without loss of generality we can choose

\[ S^{(1)}_\alpha = \frac{1}{2} \left( \frac{1}{2} e_{abc} S^{(a)}_\alpha - iS^{(b)}_\alpha \right), \quad S^{(2)}_\alpha = \frac{1}{2} \left( \frac{1}{2} e_{abc} S^{(a)}_\alpha + iS^{(b)}_\alpha \right) \]

where \( S^{(a)}_\alpha \) are generators of the finite-dimensional IR \( D(l_\alpha, l_\alpha) = D(l_\lambda^2 l_{\lambda^2}, l_\lambda l_{\lambda+1}) \) of the group \( O(1,3) \), which are given in (4.65). Then according to (4.63), (A.3.2)-(A.3.5) we obtain

\[ S^{(a)} \cdot k \Omega_{ij\lambda m} = \sum_{s', l', m'} B^{(a)}_{(ij)} k^{(s')} \Omega_{ij\lambda m} = \sum_{l'} \left( B^{(a)}_{(ij)} d_{l k}^{(s)} \Omega_{ij\lambda m} + \right. \]

\[ + iB^{(a)}_{(ij)} b_{l k}^{(s)} \Omega^{(s')}_{ij\lambda m} - \left. B^{(a)}_{(ij)} b_{l k}^{(s')} \Omega^{(s)}_{ij\lambda m} \right), \]

where \( d_{l k}^{(s)}, \ b_{l k}^{(s)} \) are the coefficients (28.7), (A.3.4).

For the operator \( S^{(a)} \cdot x \times p \) we obtain...
Appendices

\[ B_{s}^{(a)} = \frac{1}{2s} \left( -1 \right)^{s} \frac{(s_{1} + s_{2} + 1)(s_{1} - s_{2})}{2s(s + 1)^{1/2}}, \quad \alpha = 1, 2, \]

\[ B_{s+1}^{(a)} = \frac{(-1)^{s+1} [(s_{1} + s_{2} + 1)^{2} - s]^{1/2}}{2s(4s^{2} - 1)^{1/2}}, \]  

\[ B_{s-1}^{(a)} = B_{s}^{(a)} + s = s_{1} + s_{2} + s_{1} - s_{2} + 1, \quad s' = s + 1. \]

\[ S_{a}^{\lambda} = \sigma_{a}^{2}/2, \quad \sigma_{a} \text{ are the } 4 \times 4 \text{ Pauli matrices (27.4). Then} \]

\[ S^{(a)} \hat{x} \Omega_{j_{1}j_{2}m}^{\lambda} = B_{s}^{(a)} h_{j_{1}j_{2}m}^{\lambda} = i B_{s-1}^{(a)} q_{j_{1}j_{2}m}^{\lambda} + i B_{s+1}^{(a)} \Omega_{j_{1}j_{2}m}^{\lambda}, \quad (A.3.9) \]

all the coefficients included in (A.3.9) are given in (A.3.4), (A.3.8).

Let \( s = 1/2, S_{a}^{(3)} = \sigma_{a}/2 \) where \( \sigma_{a} \) are the \( 4 \times 4 \) Pauli matrices (27.4). Then according to (A.3.7), (A.3.8)

\[ \sigma \hat{x} \Omega_{j_{1}j_{2}m}^{\lambda} = -\frac{1}{s} \sum_{k} \left( d_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} - ib_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} \right), \]

\[ \sigma \hat{x} \Omega_{j_{1}j_{2}m}^{\lambda} = -\frac{1}{s} \left( h_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} - iq_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} \right), \quad (A.3.10) \]

\[ \sigma \hat{x} \Omega_{j_{1}j_{2}m}^{\lambda} = -\frac{1}{s} \left( h_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} + iq_{k}^{j_{1}j_{2}} \Omega_{j_{1}j_{2}m}^{\lambda} \right). \]

Relations (A.3.2)-(A.3.5) can be used for separation of variables for a wide class of \( O(3) \)-invariant systems of partial differential equations. We use them (as well as (A.3.7)-(A.3.10)) in Subsections 29.2, 30.4, 34.3.
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442
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451
Symmetries of Equations of Quantum Mechanics


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1Fushchych=Fushchich (the first version is closer to the Ukrainian transcription)
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LIST OF ABBREVIATIONS

FW - Foldy-Wouthuysen
IA - invariance algebra,
IR - irreducible representation,
KDP - Kemmer-Duffin-Petiau (algebra, equation, matrices),
KGF - Klein-Gordon-Fock (equation),
LHH - Levi-Leblond-Hagen-Hurley (equations),
ODE - ordinary differential equations,
PSA - parasuperalgebra,
PUA - projective unitary and antiunitary (representation),
SA - superalgebra,
SO - symmetry operator,
TST - Tamm-Sakata-Taketani (equation).
Symmetries of Equations of Quantum Mechanics

SUBJECT INDEX

Abelian algebra 63
Algebraic-theoretical approach 1
Angular momentum 379, 383, 385, 389, 487, 516, 547
Born-Infeld equation 501
Boussinesq equation 506, 507, 510, 511
Breit equation 514, 528, 530, 532, 535, 537, 541, 542
Casimir operator 29, 60, 74, 97-99, 225, 230, 241, 242, 435, 439, 520, 544
Causality 121, 162-164, 175, 457
Cauchy problem 10
Center of mass 515, 518
Change of variables 186, 223, 437, 462, 483, 486
Charge conjugation 19, 37, 51, 82, 85, 87, 97, 126, 279
Commutator 3, 27, 205, 558, 559
Conditional symmetry 342, 377, 378, 399, 504, 506, 511
Conformal
  algebra 8, 32, 52, 53, 56, 77, 150, 151, 184, 332, 340, 491, 510
  invariance 2, 18, 151, 152, 154, 329, 357, 391, 501
  transformations 17, 18, 33, 35, 39, 46, 47, 48, 57
Classical conservation laws 388
Conserved currents 383-386, 390, 392, 393
Constants of motion 8, 288, 344, 379, 384
  first order 390, 391, 393
  for the Dirac field 341, 342, 344, 379-381
  for the electromagnetic field 386, 388, 389-391
  for the vector-potential 398, 399
for two- and three-particle equations 541, 542
  in elasticity 400
second order 395, 397, 400
Coulomb
  field 346-349, 457, 468, 475, 477, 478, 484, 486-488, 534, 552
  gauge 43, 340, 398
  problem 458, 478, 485, 486, 488
Covariant
  representation 56, 103, 104, 122, 201, 202, 212, 226
  system of equations 454, 456
massless field 55

458
References

Different ial consequence 138, 263
Dilatation 17, 35, 48, 57, 77, 290, 490
Dirac
  constant of motion 346, 348
  form equation for arbitrary spin 138, 139, 142, 144
  equation 22-28, 44, 46, 125, 235, 495, 542
  equation with positive energies 456
Elasticity 375-378
Energy levels 463, 474, 475, 477, 484, 487, 488, 551, 552, 554-556
Energy-momentum tensor 390, 392
Equation
  Boussinesq 506, 507, 510, 511
  continuity 381, 383-385, 390, 392, 397
  covariant 41, 42, 122, 547
  D’Alembert 7, 57, 150, 151, 260, 261, 490, 493
Dirac-Fierz-Pauli 116, 117, 121
  evolution 26, 121, 226, 267, 268, 341, 380
Hamilton-Jacobi 505
  heat 192, 200
Helmholtz 262
  Kemmer-Duffin-Petiau 451
  Klein-Gordon-Fock 1
  Laplace 262
  Levi-Leblond 192, 211, 224, 320-323
LHG 211, 222, 249, 250
massless field 149, 150, 153, 158
nonlinear Dirac 495
nonlinear Schrödinger 497, 498
  Schrödinger 169, 178-188, 323, 361, 524, 525, 529, 543
  Stueckelberg 148
  wave 250, 513-515, 517-520, 522-525, 528-530, 532
equations
  Euler-Lagrange 233, 245
  Galilei-invariant 178, 211, 214, 225, 231, 232, 549
  Maxwell’s 37-47, 51, 52, 55, 57, 150-152, 154, 157, 330-334,
    336-340, 390, 391, 393, 395-398, 500
  nonlinear heat 502
  reduced 37-47, 51, 52, 325, 326, 329, 330-334, 500
  two-particle 513-515, 517-520, 523-525, 528-530, 532, 551, 556

459
Symmetries of Equations of Quantum Mechanics

Euclidean
  group 200, 413, 415, 432, 441
Euler-Lagrange equations 233, 246
Exact solutions 267, 341, 348, 357, 415, 457, 458, 465, 497-499, 506, 513, 544
Field
  Coulomb 346-349, 457, 468, 471, 474, 484, 486-488, 534, 552
electric 42, 240-242, 343, 346, 465, 466, 482
external 121, 163, 212, 234, 237, 546
magnetic 174, 457
magnetic monopole 489
  plane wave 466, 467
  Redmond 457, 468
Foldy-Shirokov
  representation 94
Foldy-Wouthuysen operator 137
Formula
  Campbell-Hausdorf 217, 236, 237, 419
  Zommerfeld 474, 477
Function
  Weierstrasse 365, 508
Hermit 464, 465, 482
Galilei
  algebra 178, 182-185, 195-200, 212, 213, 215, 221
group 178, 188-193, 199, 201-203, 520, 522, 527, 529, 532
invariant wave equations 211, 488, 489
relativity principle 39, 178, 179, 183, 185, 191, 502, 503, 505, 514, 517
transformations 178, 179, 188, 212, 225, 231, 232, 236, 269, 449, 450, 503, 505
Group
  conformal 8, 18, 32, 39, 45, 50-53, 325, 331, 332, 494, 502
Euclidean 68, 200, 413, 415, 432, 441
finite 82, 102, 261
Galilei 178, 188-193, 265, 321, 322, 324, 325, 353, 443, 449, 497, 498, 513, 517, 520, 522, 527, 529
generalized Poincaré 60, 223, 267, 401, 415, 432, 446, 490, 538, 540
Lorentz 64, 101, 105, 116, 350, 407, 439, 546

460
References

one-parameter 17, 38, 39, 186, 274, 283
orthogonal 198
Poincaré 6, 12, 16, 17, 19, 25, 26, 30, 31, 36, 37, 494, 513, 538-540, 544
Hamilton-Jacobi equation 505
Harmonic oscillator 368, 369, 371, 463, 483
Heat equation 192, 200
Heaviside-Larmor-Rainich transformation 39
Helicity 31, 55, 75, 130, 131, 152, 153, 156, 157
Helmholtz equation 262
Hermit
function 464, 465, 482
polynomial 464
Hermitizing matrix 110, 113, 116, 213, 214, 233
Hilbert space 30, 84, 85, 178, 254, 418, 434, 436
Hydrogen atom 252, 468, 472, 477
Integral transformation 331
Integro-differential operator 117, 317
Interaction
anomalous 160, 167, 177, 247-250, 348, 349, 372, 547
electromagnetic 40, 160, 514, 546, 547
minimal 142, 160, 161, 170, 175, 235, 240, 453, 493
two-particle 530, 556
Invariance algebra 4, 7, 26, 44, 180, 354
Invariant equations 59, 103-105, 528, 529, 545, 549
Inversion 18, 19, 35, 39, 51, 82, 109, 126, 129, 135, 350
Isomorphism 12, 54, 68, 71, 272, 407, 410, 414, 557
Jacobi identity 557, 558
KDP
algebra 117, 452, 536
equation 112, 113, 120, 146, 455, 469, 537
matrices 42, 113, 148, 163, 224
Kepler problem 513
generalized 264, 302, 336, 338, 395, 560
Killing vector 7, 388, 390, 560
Levi-Leblond equation 192, 211, 224, 320-323
Lie
algorithm 505

461
Symmetries of Equations of Quantum Mechanics

superalgebra 287

Lie algebra
  of the conformal group 8, 32, 325
  of the Galilei group 178, 191, 193, 324, 433, 443

Lie-Bäcklund symmetry 252

Lie algebra of the Poincaré group 6, 12, 26, 270, 312, 416

Lie equations 9, 17, 35, 58, 185, 201

Liouville equation 491

Local symmetry group 8

Lubanski-Pauli vector 29, 63, 65, 67, 72, 73, 154, 416

Mass
  center of 115, 212, 438, 515, 518, 544
  operator 115, 212, 438, 544
  reduced 115, 212, 438, 518, 521, 544, 555
  rest 115, 171, 199, 212, 269, 438, 544

Matrix

Dirac 24, 48, 96
  hermitizing 110, 113, 116, 213, 214, 233
  orthogonal 68
  Pauli 41, 370
  spin 374

Maxwell’s equations 37-47, 51, 52, 55, 57, 150, 151, 152, 334, 336-340,
                      388, 390, 391, 393, 395-398, 500

Metric
  indefinite 134, 232
  operator 232

Momentum
  angular 379, 383, 385, 389, 487, 516, 547
  representation 54, 120, 325, 329, 360

Multiplicator 85, 87

Nilpotency index 225, 237, 241

Noether theorem 379


Non-Lie symmetry 253, 255, 286

Nonlocal transformations 51, 202, 269

Normalization condition 442, 461, 474, 568

Operator
  Casimir 29, 60, 65, 66, 74, 75, 433, 435, 439
  internal energy 520
  mass 115, 212, 438, 544

462
References

mean-position 137
mean-spin 137
symmetry 2, 281
Overdetermined system 255, 263
Parasuperalgebra 311, 312, 374, 559
Parasupersymmetry 312
Pauli interaction 192, 211, 238, 240, 348
Pauli matrices 21, 32, 132
Poincaré
algebra 6, 12, 16, 17, 125, 126, 127, 136, 139, 140
group 6, 12, 16, 17, 82, 84, 102, 103, 104, 116, 544
invariance 57, 152, 170, 286, 304
Potential
confinement 348
Coulomb 346, 538, 546
Projective representation 190, 192
Projector 110, 120, 143, 144, 148, 351
Prolongation 503, 504
Pseudo-Euclidean space 262
PsQuasiparticle 169, 517, 521, 546
Reduced equations 510, 511
Relativity principle
Galilei 39, 178, 179, 183, 185, 191, 502, 503, 505, 514, 517
Representation
canonical 137
indecomposable 208-211, 218
Scalar product 30, 31, 73, 76, 122, 125, 126, 434, 436,
Schrödinger equation
Galilei invariant 518
two particle 517-519, 523-525, 529, 543
space inversion 19, 35, 51, 82, 97, 100, 109, 126, 129, 135, 350
Spin
arbitrary 2, 9, 103, 116, 122-124
discrete 52, 53, 55-57
variable 138, 168, 174, 175, 234, 521, 546
Spinor
spherical 351, 470, 569
Superalgebra 253, 287, 291-293, 302, 311, 312, 323, 345, 346, 358,
359, 370, 372, 377, 558, 559

463
Symmetries of Equations of Quantum Mechanics

Symmetry
- conditional 342, 377, 378, 399, 504, 506, 511
- hidden 41, 252, 260, 265, 272, 324, 329, 331, 332

Taylor series 268

Tensor
- basic 358
- energy-momentum 390, 392
- generalized Killing 264, 302, 336, 338, 395, 560
- irreducible 336, 395, 564
- Killing 257, 261, 264, 338, 394, 398, 560, 562, 564,
- zilch 392

Time reflection 19, 35, 51, 97, 126

Transformation
- antilinear 283
- charge conjugation 51
- conformal 58, 77

discrete symmetry 81
- inversion 39

Galilei 178, 179, 231, 236, 451
- Lorentz 34, 35, 47, 309, 484

scale 17, 18, 47

Transformation group
- one-parameter 274

Variables
- angular 518, 532, 547
- internal 517, 525, 545
- invariant 491, 498

Vector
- Killing 7, 388, 390, 560
- Lubanski-Pauli 29, 63, 65, 67, 294, 318, 403, 416

Wave
- plane 445, 450, 466-468

wave equation 2, 18, 149, 162, 170, 323, 439, 490, 541
- Galilei-invariant 241, 323
- nonlinear 490
- Poincaré-invariant 317

Weyl equation 32, 55, 155, 156, 158, 303, 304

Zilch tensor 392

464