SYMMETRIES OF EQUATIONS OF QUANTUM MECHANICS

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Preface to the English Edition

"In the beginning was the symmetry"
W. Heisenberg

Hidden harmony is stronger
then the explicit one
Heraclitus

The English version of our book is published on the initiative of Dr. Edward M. Michael, Vice-President of the Allerton Press Incorporated. It is with great pleasure that we thank him for his interest in our work.

The present edition of this book is an improved version of the Russian edition, and is greatly extended in some aspects. The main additions occur in Chapter 4, where the new results concerning complete sets of symmetry operators of arbitrary order for motion equations, symmetries in elasticity, super- and parasupersymmetry are presented. Moreover, Appendix II includes the explicit description of generalized Killing tensors of arbitrary rank and order: these play an important role in the study of higher order symmetries.

The main object of this book is symmetry. In contrast to Ovsiannikov's term "group analysis" (of differential equations) [355] we use the term "symmetry analysis" [123] in order to emphasize the fact that it is not, in general, possible to formulate arbitrary symmetry in the group theoretical language. We also use the term "non-Lie symmetry" when speaking about such symmetries which can not be found using the classical Lie algorithm.

In order to deduce equations of motion we use the "non-Lagrangian" approach based on representations of the Poincaré and Galilei algebras. That is, we use for this purpose the principles of Galilei and Poincaré-Einstein relativity formulated in algebraic terms. Sometimes we use the usual term "relativistic equations" when speaking about Poincaré-invariant equations in spite of the fact that Galilei-invariant subjects are "relativistic" also in the sense that they satisfy Galilei relativity principle.

Our book continues the series of monographs [127, 157, 171, 10*, 11*] devoted to symmetries in mathematical physics. Moreover, we will edit "Journal of Nonlinear Mathematical Physics" which also will related to these problems.

We hope that our book will be useful for mathematicians and physicists in the English-speaking world, and that it will stimulate the development of new symmetry approaches in mathematical and theoretical physics.

Only finishing the contemplated work one
understands how it was necessary to begin it
B. Pascal

Preface

Over a period of more than a hundred years, starting from Fedorov's works on symmetry of crystals, there has been a continuous and accelerating growth in the number of researchers using methods of discrete and continuous groups, algebras and superalgebras in different branches of modern natural sciences. These methods have a universal nature and can serve as a basis for a deep understanding of the relativity principles of Galilei and Poincaré-Einstein, of Mendeleev's periodic law, of principles of classification of elementary particles and biological structures, of conservation laws in classical and quantum mechanics etc.

The foundations of the theory of continuous groups were laid a century ago by Sofus Lie, who proposed effective algorithms to calculate symmetry groups for linear and nonlinear partial differential equations. Today the classical Lie methods (completed by theory of representations of Lie groups and algebras) are widely used in theoretical and mathematical physics.

Our book is devoted to the analysis of old (classical) and new (non-Lie) symmetries of the basic equations of quantum mechanics and classical field theory, classification and algebraic theoretical deduction of equations of motion of arbitrary spin particles in both Poincaré and Galilei-invariant approaches. We present detailed information about representations of the Galilei and Poincaré groups and their possible generalizations, and expound a new approach to investigation of symmetries of partial differential equations, which enables to find unknown before algebras and groups of invariance of the Dirac, Maxwell and other equations. We give solutions of a number of problems of motion of arbitrary spin particles in an external electromagnetic field. Most of the results are published for the first time in a monographic literature.

The book is based mainly on the author's original works. The list of references does not have any pretensions to completeness and contains as a rule the papers immediately used by us.

We take this opportunity to express our deep gratitude to academicians N.N. Bogoliubov, Yu.A. Mitropolskii, our teacher O.S. Parasiuk, correspondent member of Russian Academy of Sciences V.G. Kadyshevskii, professors A.A. Borgardt and M.K. Polivanov for essential and constant support of our researches in developing the algebraic-theoretical methods in theoretical and mathematical physics. We are indebted to doctors L.F. Barannik, I.A. Egorchenko, N.I. Serov, Z.I. Simenoh, V.V. Tretynyk, R.Z. Zhdanov and A.S. Zhukovski for their help in the preparation of the manuscript.

Introduction

The symmetry principle plays an increasingly important role in modern researches in mathematical and theoretical physics. This is connected with the fact that the basis physical laws, mathematical models and equations of motion possess explicit or unexplicit, geometric or non-geometric, local or non-local symmetries. All the basic equations of mathematical physics, i.e. the equations of Newton, Laplace, d'Alembert, Euler-Lagrange, Lame, Hamilton-Jacobi, Maxwell, Schrodinger etc., have a very high symmetry. It is a high symmetry which is a property distinguishing these equations from other ones considered by mathematicians.

To construct a mathematical approach making it possible to distinguish various symmetries is one of the main problems of mathematical physics. There is a problem which is in some sense inverse to the one mentioned above but is no less important. We say about the problem of describing of mathematical models (equations) which have the given symmetry. Two such problems are discussed in detail in this book.

We believe that the symmetry principle has to play the role of a selection rule distinguishing such mathematical models which have certain invariance properties. This principle is used (in the explicit or implicit form) in a construction of modern physical theories, but unfortunately is not much used in applied mathematics.

The requirement of invariance of an equation under a group enables us in some cases to select this equation from a wide set of other admissible ones. Thus, for example, there is the only system of Poincaré-invariant partial differential equations of first order for two real vectors **E** and **H**, and this is the system which reduces to Maxwell's equations. It is possible to "deduce" the Dirac, Schrödinger and other equations in an analogous way.

The main subject of the present book is the symmetry analysis of the basic equations of quantum physics and deduction of equations for particles of arbitrary spin, admitting different symmetry groups. Moreover we consider two-particle equations for any spin particles and exactly solvable problems of such particles interaction with an external field.

The local invariance groups of the basic equations of quantum mechanics (equations of Schrodinger, of Dirac etc.) are well known, but the proofs that these groups are maximal (in the sense of Lie) are present only in specific journals due to their complexity. Our opinion is that these proofs have to be expounded in form easier to understand for a wide circle of readers. These results are undoubtedly useful for a deeper understanding of mathematical nature of the symmetry of the equations mentioned. We consider local symmetries mainly in Chapter 1.

It is well known that the classical Lie symmetries do not exhaust the invariance properties of an equation, so we find it is necessary to expound the main results obtained in recent years in the study of non-Lie symmetries, super- and parasupersymmetries. Moreover we present new constants of motion of the basic equations of quantum physics, obtained by non-Lie methods. Of course it is interesting to demonstrate various applications of symmetry methods to solving concrete physical problems, so we present here a collection of examples of exactly solvable equations describing interacting particles of arbitrary spins.

The existence of the corresponding exact solutions is caused by the high symmetry of the models considered.

In accordance with the above, the main aims of the present book are:

1. To give a good description of symmetry properties of the basic equations of quantum mechanics. This description includes the classical Lie symmetry (we give simple proofs that the known invariance groups of the equations considered are maximally extensive) as well as the additional (non-Lie) symmetry.

2. To describe wide classes of equations having the same symmetry as the basic equations of quantum mechanics. In this way we find the Poincaré-invariant equations which do not lead to known contradictions with causality violation by describing of higher spin particles in an external field, and the Galilei-invariant wave equations for particles of any spin which give a correct description of these particle interactions with the electromagnetic field. The last equations describe the spin-orbit coupling which is usually interpreted as a purely relativistic effect.

3. To represent hidden (non-Lie) symmetries (including super- and parasupersymmetries) of the main equations of quantum and classical physics and to demonstrate existence of new constants of motion which can not be found using the classical Lie method.

4. To demonstrate the effectiveness of the symmetry methods in solving the problems of interaction of arbitrary spin particles with an external field and in solving of nonlinear equations.

Besides that we expound in details the theory of irreducible representations (IR) of the Lie algebras of the main groups of motion of four-dimensional space-time (i.e. groups of Poincaré and Galilei) and of generalized Poincaré groups $P(1,n)$. We find different realizations of these representations in the basises available to physical applications. We consider representations of the discrete symmetry operators P, C and T, and find nonequivalent realizations of them in the spaces of representations of the Poincaré group.

The detailed list of contents gives a rather complete information about subject of the book so we restrict ourselves by the preliminary notes given above.

The main part of the book is based on the original papers of the authors. Moreover we elucidate (as much as we are able) contributions of other investigators in the branch considered.

We hope our book can serve as a kind of group-theoretical introduction to quantum mechanics and will be interesting for mathematicians and physicists which use the group-theoretical approach and other symmetry methods in analysis and solution of partial differential equations.

1. LOCAL SYMMETRIES OF THE FUNDAMENTAL EQUATIONS OF RELATIVISTIC QUANTUM THEORY

In this chapter we study symmetries of the Klein-Gordon-Fock (KGF), Dirac and Maxwell equations. The maximal invariance algebras (IAs) of these equations in the class of first order differential operators are found, the representations of the corresponding symmetry groups and exact transformation laws for dependent and independent variables are given. Moreover we present with the aid of relatively simple examples, the main ideas of the algebraic-theoretical approach to partial differential equations and also, give a precise description of the symmetry properties of the fundamental equations of quantum physics.

1. LOCAL SYMMETRY OF THE KLEIN-GORDON-FOCK EQUATION

1.1. Introduction

One of the basic equations of relativistic quantum physics is the KGF equation which we write in the form

$$L\psi\equiv(p^{\mu}p_{\mu}-m^{2})\psi=0 \tag{1.1}$$

where p_μ are differential operators: $p_0=p^0=i\partial/\partial x_0$, $p_a=-p^a=-i\partial/\partial x_a$, m^2 is a positive number. Here and in the following the covariant summation over repeated Greek indices is implied and Heaviside units are used in which $\hbar=c=1$.

The equation (1.1) is a relativistic analog of the Schrödinger equation. In physics it is usually called the Klein-Gordon equation in spite of the fact that it was considered by Schrödinger [380] and then by Fock [102], Klein [253] and some other authors (see [9]). We shall use the term "KGF equation" or "wave equation".

In this section we study the symmetry of (1.1). The analysis of symmetry properties of the KGF equation enables us to proceed naturally to such important modern physical concepts as relativistic and conformal invariance and describe relativistic equations of motion for particles of arbitrary spin. We shall demonstrate also that the Poincaré (and when m=0 conformal) invariance represents in some sense the maximal symmetry of (1.1).

Let us formulate the problem of investigation of the symmetry of the KGF equation. The main concept used while considering the invariance of this equation (and other equations of quantum physics) is the concept of symmetry operator (SO). In general a SO is any operator (linear, nonlinear, differential, integral etc.) Q transforming solutions of (1.1) into solutions, i.e., satisfying the condition

$$L(Q\psi)=0 \tag{1.2}$$

for any ψ satisfying (1.1). In order to find the concrete symmetries this intuitive definition needs to be made precise by defining the classes of solutions and of operators considered. Here we shall investigate the SOs which belong to the class of first-order linear differential operators and so can be interpreted as Lie derivatives or generators of continuous group transformations.

Let us go to definitions. We shall consider only solutions which are defined on an open set D of the four-dimensional manifold R consisting of points with coordinates (x_0,x_1,x_2,x_3) and are analytic in the real variables x_0, x_1, x_2, x_3 . The set of such solutions forms a complex vector space which will be denoted by F_0. If $\psi_1,\psi_2\in F_0$ and α_1, $\alpha_2\in\mathbb{C}$ then evidently $\alpha_1\psi_1+\alpha_2\psi_2\in F_0$. Fixing D (e.g. supposing that D coincides with R_4) we shall call F the space of solutions of the KGF equation.

Let us denote by F the vector space of all complex-valued functions which are defined on D and are real-analytic, and by L we denote the linear differential operator defined on F:

$$L=p^{\mu}p_{\mu}-m^2. \tag{1.3}$$

Then $L\psi\in F$ if $\psi\in F$. Moreover F_0 is the subspace of the vector space F which coincides with the zero-space (kernel) of the operator L (1.3).

Let M_1 be the set (class) of first order differential operators defined on F. The concept of SO in the class M_1 can be formulated as follows.

DEFINITION 1.1. A linear differential operator of the first order

$$Q=A^{\mu}p_{\mu}+B, \quad A^{\mu}, B\in F \tag{1.4}$$

is a SO of the KGF equation in the class M_1 if

$$[Q,L]=\alpha_Q L, \quad \alpha_Q\in F \tag{1.5}$$

where $[Q, L] = QL-LQ$ is a commutator of the operators Q and L.

The condition (1.5) is to be understood in the sense that the operator in the r.h.s. and l.h.s. give the same result when acting on an arbitrary function $\psi\in F$.

It can be seen easily that an operator Q satisfying (1.5) also satisfies the condition (1.2) for any $\psi\in F_0$. Indeed, according to (1.5)

$$LQ\psi=(Q-\alpha_Q)L\psi=0, \quad \psi\in F_0.$$

The converse statement is also true: if the operator (1.4) satisfies (1.2) for an arbitrary $\psi \in F_0$ then the condition (1.5) is satisfied for some $\alpha_Q \in F$.

Using the given definitions we will calculate all the SOs of the KGF equation. It happens that any SO of (1.4) can be represented as a linear combination of some basis elements. This fact follows from the following assertion

THEOREM 1.1. The set S of the SOs of the KGF equation in the class M_1 forms a complex Lie algebra, i.e., if $Q_1, Q_2 \in S$ then

1) $a_1Q_1 + a_2Q_2 \in S$ for any $a_1, a_2 \in \mathbb{C}$,

2) $[Q_1, Q_2] \in S$.

PROOF. By definition the operators Q_i (i=1,2) satisfy the condition (1.5). By direct calculation we obtain that the operators $Q3=\alpha_1Q_1+\alpha_2Q_2$ and $Q_4=[Q_1,Q_2]$ belong to M_1 and satisfy (1.5) with

$$\alpha_{Q_3} = \alpha_1\alpha_{Q_1} + \alpha_2\alpha_{Q_2}, \quad \alpha_{Q_4} = [Q_1, \alpha_{Q_2}] - [Q_2, \alpha_{Q_1}], \quad \alpha_{Q_3}, \ \alpha_{Q_4} \in F. \quad \blacksquare$$

So studying the symmetry of the KGF equation (or of other linear differential equations) in the class M_1 we always deal with a Lie algebra which can be finite dimensional (this is true for equation (1.1)) as well as infinite-dimensional. This is why speaking about such a symmetry we will use the term "invariance algebra" (IA).

DEFINITION 1.2. Let $\{Q_A\}$ (A=1,2,...) be a set of linear differential operators (1.4) forming a basis of a finite-dimensional Lie algebra G. We say G is an IA of the KGF equation if any $Q_A \in \{Q_A\}$ satisfies the condition (1.5).

According to Theorem 1.1 the problem of finding all the possible SOs of the KGF equation is equivalent to finding a basis of maximally extensive IA in the class M_1. As will be shown in the following (see Chapter 4) many of the equations of quantum mechanics possess IAs in the classes of second-, third- ... order differential operators in spite of the fact that higher-order differential operators in general do not form a finite-dimensional Lie algebra.

1.2. The IA of the KGF Equation

In this section we find the IA of the KGF equation in the class M_1, i.e., in the class of first order differential operators. In this way it is possible with rather simple calculations to prove the Poincaré (and for m=0 - conformal) invariance of the equation (1.1) and to demonstrate that this symmetry is maximal in some sense.

Let us prove the following assertion.

THEOREM 1.2. The KGF equation is invariant under the 10-dimensional Lie algebra whose basis elements are

$$P_0 = p_0 = i\frac{\partial}{\partial x_0}, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \tag{1.6}$$
$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad \mu, \nu = 0,1,2,3.$$

The Lie algebra generated by the operators (1.6) is the maximally extensive IA of the KGF equation in the class M_1.

PROOF. It is convenient to write an unknown SO (1.4) in the following equivalent form

$$Q = \frac{1}{2}[K^\mu, p_\mu]_+ + C \tag{1.7}$$

where $[K^\mu, p_\mu]_+ \equiv K^\mu p_\mu + p_\mu K^\mu$, $C = B + 1/2[K^\mu, p_\mu]$. Substituting (1.7) into (1.5) we come to the equation

$$\frac{1}{2}[[(\partial^\nu K^\mu), p_\mu]_+, p_\nu]_+ + [(\partial^\nu C), p_\nu]_+ = \frac{1}{4}[[\alpha_Q, p^\mu]_+, p_\mu]_+ + \frac{i}{2}[(\partial^\mu \alpha_Q), p_\mu]_+ - m^2 \alpha_Q. \tag{1.8}$$

We represent the r.h.s. of (1.5) in an equivalent form including anticommutators.

The equation (1.8) is to be understood in the sense that the operators in the l.h.s. and r.h.s. give the same result by action on an arbitrary function belonging to F. In other words, the necessary and sufficient condition of satisfying (1.8) is the equality of the coefficients of the same anticommutators:

$$\partial^\nu K^\mu + \partial^\mu K^\nu = \frac{1}{2} g^{\mu\nu} \alpha_Q, \quad \partial^\mu C = \partial^\mu \alpha_Q, \quad m^2 \alpha_Q = 0, \tag{1.9}$$

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1, \quad g^{\mu\nu} = 0, \ \mu \neq \nu. \tag{1.10}$$

For nonzero m we obtain from (1.9) $\alpha_Q = 0$ and

$$\partial^\mu K^\nu + \partial^\nu K^\mu = 0, \quad \partial^\mu C = 0. \tag{1.11}$$

The equations (1.11) are easily integrated. Indeed the first of them is the Killing equation [249] (see Appendix 1), the general solution of which is

$$K^\mu = c^{[\mu\sigma]} x_\sigma + b^\mu \tag{1.12}$$

where $c^{[\mu\sigma]} = -c^{[\sigma\mu]}$ and b^μ are arbitrary numbers. According to (1.11) C does not depend on x.

Substituting (1.12) into (1.7) we obtain the general expression for a SO:

$$Q = c^{[\mu\sigma]} x_\mu p_\sigma + b^\mu p_\mu + C, \tag{1.13}$$

which is a linear combination of the operators (1.6) and trivial unit operator.

It is not easy to verify that the operators (1.6) form a basis of the Lie algebra, satisfying the relations

$$[P_\mu, P_\nu]=0, \quad [P_\mu, J_{\nu\lambda}]=i(g_{\mu\nu}P_\lambda - g_{\mu\lambda}P_\nu),$$
$$[J_{\mu\nu}, J_{\lambda\sigma}]=i(g_{\mu\sigma}J_{\nu\lambda}+g_{\nu\lambda}J_{\mu\sigma}-g_{\mu\lambda}J_{\nu\sigma}-g_{\nu\sigma}J_{\mu\lambda}). \tag{1.14}$$

According to the above, the Lie algebra with the basis elements (1.6) is the maximally extended IA of the KGF equation.■

The conditions (1.14) determine the Lie algebra of the Poincaré group, which is the group of motions of relativistic quantum mechanics. Below we will call this algebra "the Poincaré algebra" and denote it by *AP(1,3)*.

The symmetry under the Poincaré algebra has very deep physical consequences and contains (in implicit form) the information about the fundamental laws of relativistic kinematics (Lorentz transformations, the relativistic law of summation of velocities etc.). These questions are discussed further in Subsections 1.4 and 1.5. The following subsection is devoted to description of the KGF equation symmetry in the special case m=0.

1.3. Symmetry of the d'Alembert Equation

Earlier, we assumed the parameter m in (1.1) is nonzero.But in the case m=0 this equation also has a precise physical meaning and describes a massless scalar field. The symmetry of the massless KGF equation (i.e., d'Alembert equation) turns out to be more extensive than in the case of nonzero mass.

THEOREM 1.3. The maximal invariance algebra of the d'Alembert equation

$$p_\mu p^\mu \psi=0 \tag{1.15}$$

is a fifteen-dimensional Lie algebra. The basis elements of this algebra are given by formulae (1.6) and (1.16):

$$D=x^\mu p_\mu+2i, \quad \hat{K}_\mu=2x_\mu D-x_\sigma x^\sigma p_\mu. \tag{1.16}$$

PROOF. Repeating the reasoning from the proof of Theorem 1.2 we come to the conclusion that the general form of the SO $Q\in M_1$ for the equation (1.15) is given by in (1.7) where K^μ, C are functions satisfying (1.9) with $m\equiv 0$. We rewrite this equation in the following equivalent form

$$\partial^\nu K^\mu+\partial^\mu K^\nu-\frac{1}{2}g^{\mu\nu}\partial^\lambda K^\lambda=0, \tag{1.17}$$

$$\lambda_Q=\frac{1}{2}\partial^\nu K^\nu.$$

Formula (1.17) defines the equation for the conformal Killing vector (see Appendix 1). The general solution of this equation is

$$K^{\mu}=2x^{\mu}x_{\nu}f^{\nu}-f^{\mu}x_{\nu}x^{\nu}+c^{[\mu\sigma]}x_{\sigma}+dx^{\mu}+e^{\mu} \tag{1.18}$$

where f^{μ}, $c^{[\mu\sigma]}$, d and e^{μ} are arbitrary constants. Substituting (1.18) into (1.17) we obtain a linear combination of the operators (1.6), (1.16). These operators form a basis of a 15-dimensional Lie algebra, satisfying relations (1.14), (1.19):

$$\begin{gathered}[J_{\mu\sigma},\hat{K}_{\lambda}]=i(g_{\sigma\lambda}\hat{K}_{\mu}-g_{\mu\lambda}\hat{K}_{\sigma}), \quad [P_{\mu},\hat{K}_{\sigma}]=2i(g_{\mu\sigma}D+J_{\mu\sigma}),\\ [\hat{K}_{\mu},\hat{K}_{\sigma}]=0, \quad [D,P_{\mu}]=-iP_{\mu}, \quad [D,\hat{K}_{\mu}]=i\hat{K}_{\mu}, \quad [D,J_{\mu\sigma}]=0.\end{gathered} \tag{1.19}$$

Relations (1.14), (1.19) characterize the Lie algebra of the conformal group *C(1,3)*.■

Thus we have made sure the massless KGF equation (1.15) is invariant under the 15-dimensional Lie algebra of the conformal group (called "conformal algebra" in the following). The conformal symmetry plays an important role in modern physics.

1.4. Lorentz Transformations

Thus we have found the maximal IA of the KGF equation in the class M_1. The following natural questions arise: why do we need to know this IA, and what information follows from this symmetry about properties of the equation and its solutions?

This information turns out to be extremely essential. First, knowledge of IA of a differential equation as a rule gives a possibility of finding the corresponding constants of motion without solving this equation. Secondly, it is possible with the IA to describe the coordinate systems in which the solutions in separated variables exist [305]. In addition, any IA in the class M_1 can be supplemented by the local symmetry group which can be used in order to construct new solutions starting from the known ones.

The main part of the problems connected with studying and using the symmetry of differential equations can be successively solved in terms of IAs without using the concept of the transformation group. For instance it will be the IA of the KGF equation which will be used as the main instrument in studying the relativistic equations of motion for arbitrary spin particles (see Chapter 2). But the knowledge of the symmetry group undoubtedly leads to a deeper understanding of the nature of the equation invariance properties.

Here we shall construct in explicit form the invariance group of the KGF equation corresponding to the IA found above. For this purpose we shall use one of the classical results of the group theory, established by Sophus Lie as long ago as the 19th century. The essence of this result may be formulated as follows: if an equation possesses an IA in the class M_1 then it is locally invariant under the continuous transformation group acting on dependent and independent variables (a rigorous formulation of this statement is given in many handbooks, see, e.g., [20, 379]).

The algorithm of reconstruction of the symmetry group corresponding to the given IA is that any basis element of the IA corresponds to a one parameter transformation group

$$\begin{aligned} &x \to x' = g_\theta(x), \\ &\psi(x) \to \psi'(x') = T_{g_\theta}(\psi(x)) = \hat{D}(\theta, x)\psi(x) \end{aligned} \tag{1.20}$$

where θ is a (generally speaking, complex) transformation parameter (it will be shown in the following that for the KGF equation such parameters are real), g_θ and $\hat{D}$ are analytic functions of θ and x, T_{g_θ} are linear operators defined on F. The exact expressions for g_θ and $\hat{D}$ can be obtained by integration of the Lie equations

$$\frac{dx'^{\mu}}{d\theta} = K^{\mu}(x'), \quad x'^{\mu}\big|_{\theta=0} = x^{\mu}, \tag{1.21}$$

$$\frac{d\psi'}{d\theta} = iB(x')\psi', \psi'\big|_{\theta=0} = \psi. \tag{1.22}$$

Here K^μ and B are the functions from the definition (1.4) of a SO.

Each of the formulae (1.21), (1.22) gives a system of partial differential equations with the given initial condition, i.e., the Cauchy problem which has a unique solution. For the SOs (1.6) these equations are easily integrated. Comparing (1.4) and (1.6) we conclude that for any operator P_μ or $J_{\mu\sigma}$ $B \equiv 0$ and the solutions of (1.22) have the form

$$\psi'(x') = \psi(x), \quad \psi'(x) = \psi(g_\theta^{-1}(x)). \tag{1.23}$$

Solving equations (1.21) it is not difficult to find the transformation law for the independent variables x_μ. We obtain from (1.4), (1.6) that

$$K^{\mu} = 1, \quad \textit{if} \quad Q = P_\mu, \tag{1.24}$$

$$A^{\mu} = x_\sigma g_\lambda^{\mu} - x_\lambda g_\sigma^{\mu}, \quad \textit{if} \quad Q = J_{\mu\sigma} \tag{1.25}$$

where $g_\mu^{\ \sigma}$ is the metric tensor (1.10). Denoting $\theta = b_\mu$ for $Q = P_\mu$ and substituting (1.24) into (1.22) one comes to the equation

$$\frac{dx'^{\mu}}{db^{\mu}} = 1, \quad x'^{\mu}\big|_{b_\mu=0} = x^{\mu}$$

(no sum over μ), from which it follows that

$$x'^{\mu} = x^{\mu} + b^{\mu}. \tag{1.26}$$

In a similar way using (1.25) one finds the transformations generated by $J_{\mu\sigma}$

$$x'^a = x^a \cos\theta_{ab} + x^b \sin\theta_{ab},$$
$$x'^b = x^b \cos\theta_{ab} - x^a \sin\theta_{ab}; \qquad (1.27)$$
$$x'^\mu = x^\mu, \quad \mu \neq a, b, \ a, b \neq 0,$$

$$x'^a = x^a \cosh\theta_{0a} - x^0 \sinh\theta_{0a},$$
$$x'^0 = x^0 \cosh\theta_{0a} - x^a \sinh\theta_{0a}, \qquad (1.28)$$
$$x'^\mu = x^\mu, \ \mu \neq 0, a$$

where θ_{ab}, θ_{0a} are transformation parameters and there is no sum over *a, b*.

So the KGF equation is invariant under the transformations (1.23), (1.26)-(1.28). The transformations (1.26)-(1.28) (which were first called Lorentz transformations by H. Poincaré) satisfy the group multiplication law and conserve the four-dimensional interval

$$S(x^{(1)} - x^{(2)}) = S(x'^{(1)} - x'^{(2)}) \qquad (1.29)$$

where $S(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$, and $x^{(1)}$, $x^{(2)}$ are two arbitrary points of the space-time continuum.

The set of transformations satisfying (1.29) forms a group which is called the Poincaré group (the term suggested by Wigner).

The transformations (1.26)-(1.28) have a clear physical interpretation. Relations (1.26) and (1.27) define the displacement of the reference frame along the m-th coordinate and the rotation in the plane *a-b*. As to (1.28) it can be interpreted as a transition to a new reference frame moving with velocity *v* relative to the original frame:

$$x'_a = (x_a - v_a x_0)\beta, \quad x'_0 = (x_0 - v_a x_a)\beta,$$
$$x'_\mu = x_\mu, \ \mu \neq 0, \ a; \quad \beta = \left(1 - v_a^2 / c^2\right)^{-1/2} \qquad (1.30)$$

(no sum over *a*) where the parameter v_a is expressed through θ_{0a} by the relation $\theta_{0a} = \operatorname{artanh}(v_a/c)$, *c* is the velocity of light*.

From (1.30) it is not difficult to obtain the relativistic law of summation of velocities

$$V'_a = dx'_a / dx'_0 = (V_a - v_a)(1 - v_a V_a / c^2)^{-1}. \qquad (1.31)$$

We see that the IA of the simplest equation of motion of relativistic quantum physics (i.e., the KGF equation) possesses in an implicit form the information about the main laws of relativistic kinematics.

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* For clarity we give up the convention $c=1$ in (1.30), (1.31)

1.5. The Poincaré Group

Let us consider in more detail the procedure of reconstruction of the Lie group by the given Lie algebra presented in the above.

First we shall establish exactly the isomorphism of the algebra (1.6) and the Lie algebra of the Poincaré group.

The Poincaré group is formed by inhomogeneous linear transformations of coordinates x_μ conserving the interval (1.29), i.e., by transformations of the following type

$$x_\mu \to x'_\mu = a_{\mu\sigma} x^\sigma + b_\mu \tag{1.32}$$

where $a_{\mu\sigma}$, b_μ are real parameters satisfying the condition

$$a_{\mu\sigma} a^{\lambda\mu} = g_\sigma^{\ \lambda}. \tag{1.33}$$

It follows from (1.33) that

$$(\det\|a_{\mu\sigma}\|)^2 = 1, \quad a_{00}^2 \geq 1$$

or

$$\det\|a_{\mu\sigma}\| = \pm 1, \quad |a_{00}| \geq 1. \tag{1.34}$$

The group of linear transformations (1.32) satisfying (1.33) will be called the complete Poincaré group and denoted by $P_c(1,3)$. It is possible to select in the group $P_c(1,3)$ the subgroup $P(1,3)$ for which

$$\det\|a_{\mu\sigma}\| = 1, \quad a_{00} \geq 1. \tag{1.35}$$

The set of transformations (1.32) satisfying (1.33) and (1.35) is called the proper orthochronous Poincaré group (or the proper Poincaré group). The group $P(1,3)$ is a Lie group but the group $P_c(1,3)$ is not, because for the latter, the determinant of the transformation matrix $\|a_{\mu\sigma}\|$ is not a continuous function and can change suddenly from -1 to 1.

It is convenient to write the transformations of the group $P(1,3)$ in the matrix form

$$\hat{x} \to \hat{x}' = A\hat{x} \tag{1.36}$$

where

$$\hat{x} = column(x_0, x_1, x_2, x_3, 1), \quad \hat{x}' = column(x_0', x_1', x_2', x_3', 1), \tag{1.37}$$

$$A=A(a,b)=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & b_0 \\ a_{10} & a_{11} & a_{12} & a_{13} & b_1 \\ a_{20} & a_{21} & a_{22} & a_{23} & b_2 \\ a_{30} & a_{31} & a_{32} & a_{33} & b_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.38}$$

the symbols a and b denote the 4×4 matrix $\|a_{\mu\sigma}\|$ and the vector column with components b_μ. The last coordinate 1 is introduced for convenience and is invariant under the transformations.

Inasmuch as any transformation (1.30) can be represented in the form (1.36)-(1.38) the group $P(1,3)$ is isomorphic to the group of matrices (1.38) (denoted in the following by $P_m(1,3)$) The group multiplication in the group $P_m(1,3)$ is represented by the matrix multiplication moreover

$$A(a_1,b_1)A(a_2,b_2)=A(a_1a_2,b_1+a_1b_2).$$

The unit element of this group is the unit 5×5 matrix, the inverse element to $A(a,b)$ has the form

$$[A(a,b)]^{-1}=A(a^{-1},-a^{-1}b).$$

The general solution of (1.33), (1.35) can be represented in the following form

$$a_{00}=\cos z\cos^2\varphi+\cosh y\sin^2\varphi-\frac{\lambda^2}{R^2}(\cos z-\cosh y),$$

$$a_{0b}=\frac{1}{R}[\sin z\cosh y(\lambda_b\cos\varphi-\theta_b\sin\varphi)-\cos z\sinh y(\lambda_b\sin\varphi+$$

$$+\theta_b\cos\varphi)]+\frac{1}{R^2}\varepsilon_{bcd}\lambda_c\theta_d(\cos z-\cosh y),$$

$$a_{b0}=a_{0b}-\frac{2}{R^2}\varepsilon_{bcd}\lambda_c\theta_d(\cos z-\cosh y), \tag{1.39}$$

$$a_{bc}=\frac{1}{R}\varepsilon_{abc}\theta_a(\sin z\cosh y\cos\varphi-\cos z\sinh y\sin\varphi)-$$

$$-(\lambda_b\lambda_c+\theta_b\theta_c-\theta^2\delta_{bc})(\cos z-\cosh y)+\delta_{bc}(\cos z\cos^2\varphi+\cosh y\sin^2\varphi),$$

$$\theta=(\theta_1^2+\theta_2^2+\theta_3^2)^{1/2},\ \lambda=(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{1/2},\ \varphi=\arctan\frac{\lambda_a\theta_a}{\theta^2-\lambda^2},$$

$$z=R\cos\varphi,\ y=R\sin\varphi,\ R=[(\theta^2-\lambda^2)^2+4(\lambda_a\theta_a)^2]^{1/4}$$

where θ_a and λ_a (a=1,2,3) are arbitrary real parameters, δ_{ab} is the Kronecker symbol.

It follows from (1.37) that any matrix (1.38) depends continuously on ten real parameters b_μ, θ_a and λ_a. In other words, the group $P_m(1,3)$ is a ten-parametric Lie group.

Let us determine the Lie algebra of the group $P_m(1,3)$. Basis elements of this algebra by definition (see e.g. [20]) can be chosen in the form

$$\hat{P}_\mu = i\frac{\partial A(a,b)}{\partial b_\mu}\Big|_{b_\mu=\theta_a=\lambda_a=0}, \quad \hat{J}_{mn} = i\varepsilon_{mnc}\frac{\partial A(a,b)}{\partial \theta_c}\Big|_{b_\mu=\theta_a=\lambda_a=0}, \tag{1.40}$$
$$\hat{J}_{0c} = -\hat{J}_{c0} = \frac{\partial A(a,b)}{\partial \lambda_c}\Big|_{b_\mu=\theta_a=\lambda_a=0}.$$

Differentiating the matrices (1.38) with respect to the corresponding parameters, we obtain from (1.40)

$$\hat{P}_\mu = \begin{pmatrix} \hat{0} & K_\mu \\ \tilde{0} & 0 \end{pmatrix}, \quad \hat{J}_{\mu\sigma} = \begin{pmatrix} S_{\mu\sigma} & \tilde{0}^\dagger \\ \tilde{0} & 0 \end{pmatrix} \tag{1.41}$$

where $\hat{0}$, $\tilde{0}$ and $\tilde{0}^\dagger$ are the 4×4, 1×4 and 4×1 zero matrices,

$$S_{12} = \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&i&0&0\\ 0&0&0&0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}, \quad S_{31} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix},$$

$$S_{01} = \begin{pmatrix} 0&i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad S_{02} = \begin{pmatrix} 0&0&i&0\\ 0&0&0&0\\ i&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad S_{03} = \begin{pmatrix} 0&0&0&i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{pmatrix}, \tag{1.42}$$

$$K_0 = column(i000)\ , K_1 = column(0i00),$$
$$K_2 = column(00i0), K_3 = column(000i).$$

It is not difficult to verify that the matrices (1.41) satisfy the conditions (1.14). These conditions are satisfied also by the basis elements of the IA of the KGF equation, so this IA is isomorphic to the matrix algebra generated by the basis (1.41). Any matrix from the group $P_m(1,3)$ can be constructed from the basis elements (1.41) by the exponential mapping

$$A(a,b) = \exp(\frac{i}{2}\theta_{\mu\sigma}\hat{J}^{\mu\sigma})\exp(i\hat{P}_\mu b^\mu) \tag{1.43}$$

where

$$\exp B=\sum_{n=0}^{\infty}\frac{1}{n!}B^{n}, \quad B^{0}=I \tag{1.44}$$

(B is any 5×5 matrix, I is the unit matrix), $\theta_{ab}=\varepsilon_{abc}\theta_c/2$, $\theta_{0a}=\lambda_a$; θ_c, b_μ, λ_a are parameters present in (1.38), (1.35).

The IA of the KGF equation realizes a representation of this Lie algebra of the matrix group $P_m(1,3)$ in the vector space F. This representation can be extended to local representation of the group $P_m(1,3)$ given by the relations (1.23), (1.26)-(1.28). In analogy with (1.43) these relations can be represented as an exponential mapping of the IA basis elements

$$\psi\to\psi'(x)=T(a,b)\psi(x)=\exp(\frac{i}{2}J_{\mu\sigma}\theta^{\mu\sigma})\exp(iP_\mu b^{\mu})\psi(x)=\psi(A^{-1}(a,b)\hat{x}) \tag{1.45}$$

where the only parameter $\theta_{\mu\sigma}$ or b_μ does not vanish. The exponentials in (1.45) are defined according to

$$\exp(\theta Q)\psi=\sum_{n=0}^{\infty}\frac{\theta^n}{n!}Q^{n}\psi, \quad \psi\in F \tag{1.46}$$

where Q is an arbitrary basis element of the IA, θ is the corresponding parameter.

The transformations (1.45) are defined also for the case of arbitrary parameters b_μ, $\theta^{\mu\sigma}$. Moreover for $T(a,b)$ the following conditions hold

$T(a,b)T(a',b')=T(aa',b+ab')$.

If Q belongs to the IA of the KGF equation in the class M_1 then Q^n transforms solutions into solutions for any $n=1,2,3,\ldots$. The operator $\exp(\theta Q)$ also has this property according to (1.46). One concludes from the above that if $\psi(x)$ is an analytical solution of (1.1) then $\psi'(x)$ (1.45) is also an analytical solution on F. That is why we call the group of transformations (1.45) the symmetry group of the KGF equation.

Thus starting from the IA of the KGF equation we have constructed the symmetry group of this equation which is called the Poincaré group. This group includes the transformations (1.23), (1.32), (1.33), (1.35), i.e., such transformations which do not change wave function but include rotations and translations of the reference frame for independent variables. The requirement of invariance under the Poincaré group is the main postulate of relativistic quantum theory.

1.6. The Conformal Transformations

Let us find the explicit form of transformations from the symmetry group of the massless KGF equation. The IA of this equation is formed by the SO (1.6) and (1.16).

It is clearly sufficient to restrict ourselves to the construction of the transformations generated by the operators (1.16) inasmuch as the remaining transformations have already been considered in Subsections 1.4 and 1.5.

In order to find the one-parameter subgroups generated by K_μ and D we will solve the corresponding Lie equations. Comparing (1.4) and (1.16) we conclude that for the operator D $A_\mu = x^\mu$, $B = 1$, so the equations (1.21), (1.22) take the form

$$\begin{aligned} &\frac{dx'^\mu}{d\theta} = x'^\mu, \quad x'^\mu\big|_{\theta=0} = x^\mu, \\ &\frac{d\psi'}{d\theta} = -\psi', \quad \psi'\big|_{\theta=0} = \psi. \end{aligned} \tag{1.47}$$

The solutions of (1.47) have the form

$$\psi' = \exp(-\theta)\psi, \quad x' = \exp(\theta)x. \tag{1.48}$$

The transformations (1.48) are called dilatation transformations and reduce to a change of scale (any independent variable is multiplied by the same number).

For the operators K_μ the Lie equations are

$$\begin{aligned} &\frac{dx'^\mu}{db^\sigma} = 2x'^\mu x'_\sigma - x'_\lambda x'^\lambda g^\mu{}_\sigma, \quad x'^\mu\big|_{b_\sigma=0} = x^\mu, \\ &\frac{d\psi'}{db^\sigma} = -2x'_\sigma\psi, \quad \psi'\big|_{b_\sigma=0} = \psi \end{aligned} \tag{1.49}$$

where b_σ are the transformations parameters. It is not difficult to verify that solutions of the Cauchy problems formulated in (1.49) are given by the formulae

$$\begin{aligned} &x'^\mu = \frac{x^\mu - g^\mu{}_\sigma b^\sigma x_\lambda x^\lambda}{1 - 2x^\sigma x_\sigma + b_\sigma b^\sigma x_\lambda x^\lambda}, \\ &\psi' = (1 - 2x_\sigma b^\sigma + b_\sigma b^\sigma x_\lambda x^\lambda)\psi \end{aligned} \tag{1.50}$$

(no sum over σ).

Formulae (1.50) give a family of transformations depending on a parameter b_σ (with a fixed value of σ). Using these transformations successively for different σ we come to the general transformation generated by K_σ, which also has the form (1.50) where the summation over σ is assumed.

The transformations (1.50) are called conformal transformations and can be represented as a composition of the following transformations: the inversion

$$x_\mu \to x'''^\mu = \frac{x^\mu}{x_\lambda x^\lambda},$$

the displacement

$$x'''^\mu \to x''^\mu = x'''^\mu - b^\mu,$$

and the second inversion

$$x^{//\mu} \to x^{/\mu} = \frac{x^{//\mu}}{x_\lambda^{//} x^{//\lambda}}.$$

We see that the massless KGF equation is invariant under the scale and conformal transformations besides the symmetry with respect to Lorentz transformations. The set of transformations (1.30), (1.45), (1.48), (1.50) for x^μ forms a 15-parameter Lie group called the conformal group. As is demonstrated in Section 3 conformal invariance occurs for any relativistic wave equation describing a massless field.

It is necessary to note that the transformations found above can be considered only as a local representation of the group *C(1,3)* since in addition to the problem of defining the domain of the transformed function it is necessary to take into account that the expression (1.50) for x^μ becomes nonsense if $1\text{-}2b_\mu x^\mu + b^\sigma b_\sigma x_\mu x^\mu = 0$.

1.7. The Discrete Symmetry Transformations

Although the IA of the KGF equation found above is in some sense maximally extensive, the invariance under this algebra and the corresponding Lie group does not exhaust symmetries of this equation. Moreover the KGF equation is invariant under the following discrete transformations

$$\begin{aligned} &x_0 \to x_0^{/} = x_0, \quad x_a \to x_a^{/} = -x_a, \\ &\psi(x) \to P\psi(x) = r_1 \psi(x^{/}), \end{aligned} \tag{1.51}$$

$$\begin{aligned} &x_0 \to x_0^{//} = -x_0, \quad x_a \to x_a^{//} = x_a, \\ &\psi(x) \to T\psi(x) = r_2 \psi(x^{//}), \end{aligned} \tag{1.52}$$

$$x \to x, \quad \psi(x) \to C\psi(x) = r_3 \psi^*(x) \tag{1.53}$$

where $r_a = \pm 1$. The invariance under the transformations (1.51) (space inversion), (1.52) (time reflection) and (1.53) (charge conjugation) can be easily verified by direct calculations.

The determinants of the matrices of the coordinate transformations of (1.51) and (1.52) are equal to -1. So these transformations do not belong to the group *P(1,3)* but are contained in the complete Poincaré group $P_c(1,3)$. As to the transformation of charge conjugation, it has nothing to do with the Poincaré group and represents the symmetry of the KGF equation under the complex conjugation.

The operators *P*, *C*, *T* satisfy the following commutation and anticommutation relations together with the Poincaré generators

$$[P,P_0]=[P,P_a]_+=[P,J_{ab}]=[P,J_{0a}]_+=0,$$
$$[T,P_0]_+=[T,P_a]=[T,J_{ab}]=[T,J_{0a}]_+=0, \tag{1.54}$$
$$[C,P_\mu]_+=[C,J_{\mu\sigma}]_+=0,$$

$$P^2=T^2=C^2=1, \quad [P,T]=[C,T]=[C,P]=0.$$

Conditions (1.54) can serve as a abstract definition of the operators P, C and T.

So the IA of the KGF equation found in Section 1.3 can be fulfilled to the set of the symmetry operators $\{P_\mu,J_{\mu\sigma},C,P,T\}$. These operators satisfy the invariance condition (1.5) and algebra (1.14), (1.54) (which, of course, is not a Lie algebra).

We note that the discrete symmetry transformations can be used to construct a group of hidden symmetry of the KGF equation. Actually, the KGF equation is transparently invariant under the transformation

$$x\rightarrow x, \quad \psi\rightarrow R\psi(x)=i\psi(x), i=\sqrt{-1}\,. \tag{1.55}$$

Combining this transformation with (1.53), we can select the set of symmetries $\{C,R,CR\}$ which satisfy the following commutation relations

$$[C,R]=2CR, \quad [CR,C]=-2R, \quad [CR,R]=2C, \tag{1.56}$$

since $C^2=-R^2=1$ and $CR=-RC$.

In accordance with (1.56) the SOs C, R and CR form a Lie algebra which is isomorphic to the algebra *AO(1,2)*, i.e., the Lie algebra of the Lorentz group in (1+2)-dimensional Minkowski space. This circumstance enables us to find exactly the corresponding symmetry group which is generated by the following one-parameter transformations

$$\psi\rightarrow\cos\theta_1\psi+i\sin\theta_1\psi,$$
$$\psi\rightarrow\cosh\theta_2\psi+\sinh\theta_2\psi^*, \tag{1.57}$$
$$\psi\rightarrow\cosh\theta_3\psi+i\sinh\theta_3\psi^*$$

where θ_1, θ_2 and θ_3 are real parameters.

It is possible to point out the other sets of symmetries forming a representation of the algebra *AO(1,2)*, i.e.,

$$\{T,R,TR\}, \{PT,R,PTR\}, \{CP,R,CPR\},$$

or to select more extended IAs including more then three basis elements (for instance, the sets $\{C,R,CR,PC,PR,PCR\}$ and $\{C,R,CR,PC,PR,PCR,PCT,PCTR,PTR,TC,TR,$

TCR} form representations of the algebras *AO(2,2)* and *AO(2,2)⊕AO(2,2)*). We will not analyze these algebraic structures but formulate a general statement valid for a wide class of linear differential equations.

LEMMA 1.1. Let a linear partial differential equation is invariant under an antilinear transformation Q, satisfying the condition $Q^2=1$. Than this equation is invariant under the algebra *AO(1,2)*.

The proof is almost evident from the above, since any linear equation is invariant under the transformation R of (1.55). Then such an equation admits the IA with the basis elements $\{Q,R,QR\}$ which realize a representation of the algebra *AO(1,2)*.

We will see in the following that Lemma 1.1 enables to find hidden symmetries for great many of equations of quantum mechanics. The corresponding symmetry groups reduce to matrix transformation involving a wave function and a complex conjugated wave function.

Other hidden symmetries of the KGF equation are considered in Section 16.

2. LOCAL SYMMETRY OF THE DIRAC EQUATION

2.1. The Dirac Equation

In 1928 Dirac found the relativistic equation for an electron, which can be written in the form

$$L\psi\equiv(\gamma^{\mu}p_{\mu}-m)\psi=0 \tag{2.1}$$

where ψ is a four-component wave function

$$\psi=column(\psi_1,\psi_2,\psi_3,\psi_4), \tag{2.2}$$

γ_μ are 4×4 matrices satisfying the Clifford algebra

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}. \tag{2.3}$$

For most of our needs the explicit form of the matrices γ_μ is not essential inasmuch as the conditions (2.3) determine them up to unitary equivalence. We will use, for the sake of concreteness, the following representation

$$\gamma_0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \gamma_a=\begin{pmatrix}0 & -\sigma_a\\ \sigma_a & 0\end{pmatrix} \tag{2.4}$$

where 0 and I are the zero and unit 2×2 matrices, σ_a are the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.5}$$

The equation (2.1) is the simplest quantum mechanical equation describing a noninteracting particle with spin. The study of this equation symmetry does not differ in principle from the analysis of the KGF equation given above. Nevertheless taking into account the outstanding role of the Dirac equation in physics and special features connected with the fact that the function ψ has four components we will consider the symmetries of the Dirac equation in detail.

Let us note that any component of the function ψ satisfies the KGF equation. Indeed, multiplying (2.1) on the left by $\gamma^\mu p_\mu + m$ and using (2.3) we obtain

$$(p_\mu p^\mu - m^2)\psi = 0. \tag{2.6}$$

We see that the KGF equation is a consequence of the Dirac equation. The inverse statement is not true of course inasmuch as there is an infinite number of first order partial differential equations whose solutions satisfy (2.6) componentwise. The Dirac equation is the simplest example of such a system.

2.2. Various Formulations of the Dirac Equation

Let us consider other (different from (2.1)) representations of the Dirac equation to be found in the literature. All these representations are equivalent but give a possibility of obtaining different generalizations of (2.1) to the case of a field with arbitrary spin.

Starting from (2.1) it is not difficult to obtain the equation for a complex conjugated function ψ^*. Denoting

$$\psi^\dagger = (\psi_1^*\psi_2^*\psi_3^*\psi_4^*), \quad \overline{\psi} = \psi^\dagger\gamma_0, \tag{2.7}$$

and making complex conjugation of (2.1) we obtain, using (2.3)

$$\overline{\psi}(\gamma_\mu p^\mu - m) = 0 \tag{2.8}$$

where it is implied that p^μ act on ψ. Using the representation (2.4) it is possible to write (2.8) in the following equivalent form

$$(\gamma_\mu p^\mu - m)\psi^C = 0, \quad \psi^C = i\gamma_2\psi^*. \tag{2.9}$$

Indeed, the equation (2.8) and (2.9) coincide when written componentwise.

The Dirac equation in the form (2.9) is widely used in quantum field theory.

Multiplying (2.1) from the left by γ_0 and using (2.3) we obtain the equation in the Schrödinger form

$$i\frac{\partial}{\partial t}\psi = H\psi, \quad t = x_0 \tag{2.10}$$

where the Hamiltonian H has the form

$$H = \gamma_0\gamma_a p_a + \gamma_0 m. \tag{2.11}$$

It was in the form (2.10), (2.11) that the equation considered was found by Dirac [77] for the first time. And it is the formulation (2.10), (2.11) which will serve as a base for generalization of the Dirac equation for the case of arbitrary spin, see Chapter 2.

The other (so called covariant) formulation of the Dirac equation can be obtained by multiplication (2.1) from the left by an arbitrary matrix γ_μ

$$p_\mu\psi = \hat{P}_\mu\psi \equiv \left(\frac{1}{2}[\gamma_\mu, \gamma_\nu]p^\nu + \gamma_\mu m\right)\psi. \tag{2.12}$$

In the equation (2.12) as in (2.1) all the variables play equal roles in contrast to (2.10) where the time variable is picked out. The equations in the form (2.12) also admit very interesting generalizations [11, 135]. In particular the infinite-component Dirac equation for positive energy particles [81] has this form.

In conclusion let us note, following Majorana [292] that the matrices γ_μ can be chosen in such a form that all the coefficients of the equation (2.1) are real. Namely setting

$$\gamma_0' = \gamma_0\gamma_2, \quad \gamma_1' = -\gamma_1\gamma_2, \quad \gamma_3' = \gamma_3\gamma_2, \quad \gamma_2' = -\gamma_2 \tag{2.13}$$

where γ_μ are the matrices (2.4), we can write the Dirac equation in the form

$$(\gamma'^\mu p_\mu - m)\psi' = 0, \tag{2.14}$$

where γ'_μ and ψ' are connected with γ_μ and ψ by the equivalence transformation

$$\psi' = U\psi, \quad \gamma_\mu' = U\gamma_\mu U^{-1}, \quad U = U^{-1} = (1 - \gamma_2)\gamma_0/\sqrt{2}. \tag{2.15}$$

Using (2.4) it is not difficult to verify that the equation (2.14) includes real coefficients only and so can be reduced to two noncoupled systems of equations for the real and imaginary parts of the function ψ'.

Using other (distinct from U) nondegenerated matrices for the transformation (2.15) we can obtain infinitely many other realizations of the Dirac equation, which are equivalent to (2.1).

2.3. Algebra of the Dirac Matrices

As was noted in the above these are relations (2.3) (but not an explicit realization of the γ-matrices) which are used by solving concrete problems with the help of the Dirac equation. Here we present some useful relations following from (2.3).

First let us note that there exists just one more matrix satisfying (2.3). This matrix has the form

$$\gamma_4=\gamma_0\gamma_1\gamma_2\gamma_3. \tag{2.16}$$

In the representation (2.4) we have

$$\gamma_4=i\begin{pmatrix} -I & \hat{0} \\ \hat{0} & I \end{pmatrix}. \tag{2.17}$$

Furthermore it is not difficult to obtain from (2.3) the following relations

$$\begin{gathered}\gamma_\mu\gamma_\nu=g_{\mu\nu}-2iS_{\mu\nu}, \quad S_{\mu\nu}=i[\gamma_\mu,\gamma_\nu]/4, \quad \gamma_\mu\gamma^\mu=4, \\ S_{\mu\nu}\gamma_4=\frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma}S^{\varrho\sigma}, \quad S_{\mu\nu}\gamma_\lambda=i(g_{\mu\sigma}S_{\nu\lambda}-g_{\mu\lambda}\gamma_\nu+\varepsilon_{\mu\nu\lambda\sigma}\gamma_4\gamma^\sigma)/2,\end{gathered} \tag{2.18a}$$

$$[S_{\mu\nu},S_{\lambda\sigma}]=i(g_{\mu\sigma}S_{\nu\lambda}+g_{\nu\lambda}S_{\mu\sigma}-g_{\mu\lambda}S_{\nu\sigma}-g_{\nu\sigma}S_{\mu\lambda}). \tag{2.18b}$$

Finally it is possible to show that all the nonequivalent products of the Dirac matrices form a basis in the space of 4×4 matrices. All such products are exhausted by the following 16 combinations

$$\begin{gathered}\hat{I}, \quad \gamma_0, \quad i\gamma_1, \quad i\gamma_2, \quad i\gamma_3, \quad i\gamma_4, \quad \gamma_4\gamma_0, \quad i\gamma_4\gamma_1, \quad i\gamma_4\gamma_2, \\ i\gamma_4\gamma_3, \quad \gamma_0\gamma_1, \quad \gamma_0\gamma_2, \quad \gamma_0\gamma_3, \quad i\gamma_1\gamma_2, \quad i\gamma_2\gamma_3, \quad i\gamma_3\gamma_1\end{gathered} \tag{2.19}$$

where $\hat{I}$ is the unit matrix.

Using (2.3) it is not difficult to show the matrices (2.19) are linearly independent and hence any 4×4 matrix can be represented as a linear combination of the basis elements (2.19).

2.4. SOs and IAs

The main property of the Dirac equation is the relativistic invariance, i.e., symmetry under the Poincaré group transformations. Here we will prove the existence of this symmetry and demonstrate that it is the most extensive one, i.e., that there is no wider symmetry group leaving the Dirac equation invariant.

As in the case of the KGF equation we will describe symmetries of the Dirac equation using the language of Lie algebras, which first gives a possibility of clarity and rigor interpretation using relatively simple computations and, secondly, is suitable for the description of hidden (non-geometrical) symmetries not connected with space-time transformations (see Chapter 4).

The problem of investigation of the Dirac equation symmetry in the class M_1 can be formulated in complete analogy with the corresponding problem for the KGF equation. However it is necessary to generalize the corresponding definitions for the case of a system of partial differential equations.

Let us denote by F^4 the vector space of complex valued functions (2.2) which are defined on some open and connected set D of the real four-dimensional space R and are real-analytic. In other words $\psi \in F^4$ if any component $\psi_k \in F$ (see Subsection 1.1). Then the linear differential operator L of (2.1) defined on D has the following property: $L\psi \in F^4$ if $\psi \in F^4$. Finally the symbol G^4 will denote the space of 4×4 matrices whose matrix elements belong to F.

The following definition is a natural generalization of Definition 1.1 (see Subsection 1.1):

DEFINITION 2.1. A linear first order differential operator

$$Q = F^{\mu} p_{\mu} + D, \qquad F^{\mu} \in F, \quad D \in G^{4} \tag{2.20}$$

is a SO of the Dirac equation if

$$[Q, L] = \beta_Q L, \quad \beta_Q \in G^4. \tag{2.21}$$

The equation (2.21) is to be understood in the sense that the operators on the l.h.s. and r.h.s. give the same result acting on an arbitrary function $\psi \in F^4$.

As in the case of the KGF equation a SO transforms solutions of (2.1) into solutions and the complete set of SOs forms a Lie algebra. So, while speaking about the Dirac equation SOs we will use the term "invariance algebra" (IA).

2.5. The IA of the Dirac Equation in the Class M_1

Let us formulate and prove the main assertion about symmetries of the Dirac equation. As it will be shown further on this statement includes all the information about the kinematics of a particle described by the evolution equation (2.1).

THEOREM 2.1. The Dirac equation is invariant under the ten-dimensional Lie algebra which is isomorphic to the Lie algebra of the Poincaré group. The basis elements of this IA can be chosen in the following form

$$P_{\mu} = p_{\mu} \equiv i \frac{\partial}{\partial x^{\mu}}, \quad J_{\mu\nu} = x_{\mu} p_{\nu} - x_{\nu} p_{\mu} + S_{\mu\nu} \tag{2.22}$$

where

$$S_{\mu\nu} = \frac{i}{4}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}). \tag{2.23}$$

The Lie algebra defined by the basis elements (2.22) is the maximal IA of the Dirac equation in the class M_1.

PROOF. The first statement of the theorem can be easily verified by the direct calculation of commutators of P_μ and $J_{\mu\sigma}$ with L of (2.1), which are equal to zero. The operators (2.22) satisfy relations (1.14) and hence form a basis of the Lie algebra isomorphic to *AP(1,3)*.

A little more effort is needed to prove the algebra (2.22) is the maximally extensive IA of (2.1).

Let us represent an arbitrary operator Q and the matrix β_Q from (2.21) as a linear combination of the basis elements (2.19)

$$\begin{aligned} &Q=I(K^\mu p_\mu+a^0)+i\gamma_4 a^1+\gamma_\mu b^\mu+S_{\mu\nu}f^{[\mu\nu]}+\gamma_4\gamma_\mu d^\mu,\\ &\beta_Q=Ie^0+i\gamma_4 e^1+\gamma_\mu q^\mu+\gamma_4\gamma_\mu h^\mu+S_{\mu\nu}k^{[\mu\nu]} \end{aligned} \tag{2.24}$$

where the Latin letters denote unknown functions belonging to F. The problem is to find the general form of these functions using the conditions (2.21).

Calculating the commutator of the operators Q (2.24) and L (2.1) we obtain with (2.18):

$$\begin{aligned} [Q,L]=&-i\gamma^\lambda K^\mu_\lambda p_\mu-i\gamma^\lambda a^0_\lambda+2i\gamma_4\gamma^\lambda a^1 p_\lambda-\gamma_4\gamma^\lambda a^1_\lambda-ib^\lambda_\lambda-4iS_\mu{}^\lambda b^\mu p_\lambda+2S_\mu{}^\lambda b^\mu_\lambda+\\ &+\gamma_4(2d^\lambda p_\lambda+id^\lambda_\lambda)+\varepsilon^\lambda{}_{\mu\nu\sigma}S^{\nu\sigma}d^\mu_\lambda+2i\gamma_\mu f^{[\mu\lambda]}p_\lambda+\gamma_\mu f^{[\mu\nu]}_\nu+\varepsilon^{\sigma\lambda}{}_{\mu\nu}f^{[\mu\nu]}_\lambda\gamma_4\gamma_\sigma \end{aligned} \tag{2.25}$$

where the bottom indices denote derivatives with respect to the corresponding variables: $B_\mu=\partial B/\partial x_\mu$. On the other hand it is not difficult to calculate that

$$\begin{aligned} \beta_Q L=&\,e^0\gamma^\mu p_\mu+I(-e^0m+q^\mu p_\mu)-m\gamma_\mu q^\mu-2iS_\mu{}^\lambda q^\mu p_\lambda-i\gamma_4 me^1+i\gamma_4\gamma^\mu e^1 p_\mu-\\ &-m\gamma_4\gamma_\mu h^\mu+\gamma_4 h^\mu p_\mu-i\varepsilon_{\mu\lambda\sigma}{}^\nu S^{\lambda\sigma}h^\mu p_\nu-\gamma_4 k^{[\mu\nu]}p_\nu+\varepsilon_{\sigma\mu\nu\lambda}\gamma_4\gamma^\sigma k^{[\mu\nu]}p^\lambda. \end{aligned} \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.21) and equating the coefficients of linearly independent matrices and differential operators we obtain the following system

$$K^\mu_\nu-2g_{\lambda\sigma}f^{[\sigma\mu]}=ig^\mu_\lambda e^0, \tag{2.27}$$

$$a^0_\lambda+ig_{\lambda\sigma}f^{[\sigma\nu]}_\nu=0, \tag{2.28}$$

$$a^1_\lambda+i\varepsilon_{\mu\nu\lambda}{}^\sigma f^{[\mu\nu]}_\sigma=0, \tag{2.29}$$

$$e^1=2a^1,\ \ me^1=0,\ \ me^0=0, \tag{2.30}$$

$$f^\mu=b^\mu=h^\mu=k^{[\mu\nu]}=d^\mu=0 \tag{2.31}$$

where $g_{\mu\sigma}$ is the metric tensor (1.10).

The system (2.27)-(2.31) is easily integrated. The equations (2.27) can be rewritten in the form

$$K_{\mu}^{\nu}+K_{\nu}^{\mu}-g_{\nu}^{\mu}K_{n}^{n}=0, \tag{2.32}$$

$$f^{\sigma\mu}=\frac{1}{2}g^{\sigma\lambda}K_{\lambda}^{\mu}, \quad e^{0}=-iK_{\nu}^{\nu}. \tag{2.33}$$

The equivalence of (2.27) and (2.32), (2.33) follows from the symmetry of $g_{\mu\sigma}$ and antisymmetry of $f^{[\mu\sigma]}$ under the permutation of indices.

The equation (2.32) coincides with the conformal Killing equation (1.17), its solutions are given in (1.18). Substituting (1.18) into (2.33) and bearing in mind that in accordance with (2.30) for a nonzero m $e0=e^1=a^1=0$ we obtain

$$K^{\mu}=c^{[\mu\nu]}x_{\nu}+\varphi^{\mu}u, \quad f^{[\mu\nu]}=\frac{1}{2}c^{[\mu\nu]}, \quad d=0. \tag{2.34}$$

It follows from the above that the general expression (2.24) for Q and $ß_Q$ is reduced to the form

$$Q=I(a_0+c^{[\mu\nu]}x_{\mu}p_{\nu}+\varphi^{\mu}p_{\mu})+\frac{1}{2}c^{[\mu\nu]}S_{\mu\nu}, \quad \beta_Q=0 \tag{2.35}$$

where $c^{[\mu\sigma]}$, φ^{μ}, a^0 are arbitrary complex numbers. The operator (2.35) is a linear combination of the operators (2.22) and trivial identity operator, so the operators (2.22) form a basis of the maximally extensive IA of the Dirac equation in the class M_1. ■

We see the IA of the Dirac equation is isomorphic to the IA of the KGF equation considered in Section 1. The essentially new point is the presence of the matrix terms in the SOs (2.22). These terms correspond to an additional (spin) degree of freedom possessed by the field described by the Dirac equation. We shall see in the following that due to the existence of the spin degree of freedom the Dirac equation has additional symmetries in the classes of higher order differential operators.

2.6. The Operators of Mass and Spin

It is well known that the Dirac equation describes a relativistic particle of mass m and spin s. Such an interpretation of this equation admits a clear formulation in the language of the Lie algebras representation theory.

The Dirac equation IA determined by the basis elements (2.22) has two main invariant (Casimir) operators (see Section 4)

$$C_1=P_{\mu}P^{\mu}, \quad C_2=W_{\mu}W^{\mu} \tag{2.36}$$

where W_{μ} is the Lubanski-Pauli vector

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} J^{\nu\varrho} P^\sigma. \tag{2.37}$$

Let us recall that a Casimir operator is an operator belonging to the enveloping algebra of the Lie algebra, which commutes with any element of this algebra.

One of the main results of Lie algebra representation theory says that in the space of an irreducible representation (IR) the Casimir operators are multiples of the unit operator. Moreover eigenvalues of invariant operators can be used for labelling of IRs inasmuch as different eigenvalues correspond to nonequivalent representations.

Thus, to label the representation of the Poincaré algebra, which is realized on the set of solutions of the Dirac equation, it is necessary to find eigenvalues of the operators (2.36). Substituting (2.22) into (2.36) and using (2.3), (2.6), (2.18) we obtain

$$\begin{gathered} C_1\psi \equiv P_\mu P^\mu \psi \equiv p_\mu p^\mu \psi = m^2\psi, \\ C_2\psi \equiv W_\mu W^\mu \psi \equiv -\frac{1}{2} p_\mu p^\mu S_{ab} S_{ab}\psi = -m^2 s(s+1)\psi, \quad s=\frac{1}{2}. \end{gathered} \tag{2.38}$$

In relativistic quantum theory the space of states of a particle with mass m and spin s is set in correspondence with the space of the representation of the Poincaré algebra corresponding to the eigenvalues m^2 and $-m^2s(s+1)$ of the Casimir operators C_1 and C_2. So it follows from (2.37) that the Dirac equation can be interpreted as an equation of motion for a particle of spin 1/2 and mass m.

2.7. Manifestly Hermitian Form of Poincaré Group Generators

Before we considered only such solutions of the Dirac equation which belong to the space F_4. But the operator L (2.1) and the SOs (2.22) can be defined also on the set of finite functions $(C_0^\infty)^4$ everywhere dense in the Hilbert space L_2 of the square integrable functions with the scalar product

$$(\psi^{(1)}, \psi^{(2)}) = \int d^3x \psi^{(1)\dagger}\psi^{(2)} \tag{2.39}$$

where according to the definitions (2.2), (2.7)

$$\psi^{(1)\dagger}\psi^{(2)} = \psi_1^{(1)*}\psi_1^{(2)} + \psi_2^{(1)*}\psi_2^{(2)} + \psi_3^{(1)*}\psi_3^{(2)} + \psi_4^{(1)*}\psi_4^{(2)}.$$

It is not difficult to verify that the operators (2.22) are Hermitian in respect to the scalar product (2.39) where ψ_α satisfy the Dirac equation. To show that, it is sufficient to represent these operators in the following form

$$\begin{gathered} P_0 = H = \gamma_0\gamma_a p_a + \gamma_0 m, \quad p_a = p_a = -i\frac{\partial}{\partial x_a}, \\ J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad J_{0a} = x_0 p_a - \frac{1}{2}[x_a, H]_+. \end{gathered} \tag{2.40}$$

Throughout on the set of solutions of the Dirac equation the operators (2.22) and (2.40) coincide inasmuch as $[x_a,H]_+\equiv 2(x_aH\text{-}S_{0a})$.

The operators (2.40) satisfy the commutation relations (1.14) and (in contrast to (2.22)) are written in a transparently Hermitian form. So the operators (2.22) also are Hermitian with respect to the scalar product (2.39). It follows from the above that the Poincaré group transformations generated by the operators (2.22) (see Subsection 2.9) are unitary, i.e., do not change the value of the scalar product (2.39).

2.8. Symmetries of the Massless Dirac Equation

The equation (2.1) has clear physical meaning also in the case m=0, describing a massless field with helicity ±½. The symmetry of the Dirac equation with m=0 is wider than in the case of nonzero mass.

THEOREM 2.2. The maximal IA in the class M_1 of the equation

$$\gamma^\mu p_\mu \psi=0 \tag{2.41}$$

is a 16-dimensional Lie algebra whose basis elements are given by formulae (2.22) and (2.42)

$$\begin{gathered} D=x^\mu p_\mu+iK, \quad \Sigma=i\gamma_4, \\ K_\mu=2x_\mu D-x^\nu x_\nu p_\mu+2S_{\mu\nu}x^\nu \end{gathered} \tag{2.42}$$

where K=3/2.

The proof can be carried out in complete analogy to that of Theorem 2.1 (see Subsection 2.5). The general form of a SO is given in (2.24), the equations determining the corresponding operators coefficients are given by relations (2.27)-(2.31) with m=0. So it is not difficult to find the general solution for a SO in the form

$$\begin{gathered} Q=I(2f^\nu x_\nu x^\mu p_\mu-f^\mu x_\nu x^\nu p_\mu)+c^{[\mu\nu]}x_\mu p_\nu+dx^\mu p_\mu \\ +\varphi^\mu p_\mu+3if^\lambda x_\lambda+a^0+i\gamma_4 a^1+S_{\mu\nu}(\frac{1}{2}c^{[\mu\nu]}+2f^\mu x^\nu). \end{gathered} \tag{2.43}$$

The operator (2.43) is a linear combination of the generators (2.22), (2.42) and the unit operator which give the basis of IA of the equation (2.41) the class M_1.■

The operators (2.22), (2.42) satisfy the commutation relations (1.14), (1.19) which determine the Lie algebra of the conformal group. As to the operator Σ, it commutes with any basis element of the IA. In other words the IA of the equation (2.42) consists of the 15-dimensional Lie algebra which isomorphic to the IA of the massless KGF equation, and an additional matrix operator Σ which is the center of the IA of the massless Dirac equation.

Let us notice that the massless Dirac equation with the matrices (2.4) reduces to two noncoupled equations:

$$p_0\varphi_\pm = \sigma_a p_a \varphi_\pm \tag{2.44}$$

where σ_a are the Pauli matrices, $\varphi_\pm=(1\mp i\gamma_4\psi)/2$.

Relations (2.44) are called the Weyl equations. The SOs of the massless Dirac equation can be decomposed into the direct sum of operators defined on $\varphi_\pm$.

The explicit form of the basis elements of the conformal algebra on the set of the Weyl equation solutions can be obtained from (2.22), (2.42) by the change $S_{ab}\to i[\sigma_a,\sigma_b]/4$, $S_{0a}\to\pm i\sigma_a/2$. As to the operator Σ it is equal to the unit matrix on the subspaces $\varphi_\pm$.

2.9. Lorentz and Conformal Transformations of Solutions of the Dirac Equation

As was mentioned in Section 1 the main consequence of a symmetry of a differential equation under an IA in the class M_1 is that this equation turns out to be invariant under the Lie group whose generators form the basis of this IA. In other words proving the invariance of the Dirac equation under the algebra *AP(1,3)* and (for $m=0$) conformal algebra we have actually established its invariance under Lorentz and conformal transformations.

Here we shall find an explicit form of the group transformations of solutions of the Dirac equation with zero and nonzero masses.

The general transformation of the symmetry group of (2.1) can be written in the following form (compare with (1.45))

$$\psi(x)\to\psi'(x)=\exp(\frac{i}{2}J_{\mu\sigma}\theta^{\mu\sigma})\exp(iP^{\mu}b_{\mu})\psi(x)$$

where $J_{\mu\sigma}$ and P_μ are the operators (2.22), $\theta_{\mu\sigma}$ and b_μ are real parameters. Using the commutativity of $S_{\mu\sigma}$ with $x_\mu p_\sigma - x_\sigma p_\mu$ we can represent this transformation in the form

$$\psi(x)\to\psi'(x)=\exp(\frac{i}{2}S_{\mu\nu}\theta^{\mu\nu})\psi''(x) \tag{2.45}$$

where

$$\psi''(x)=\exp(iJ'_{\mu\nu}\theta^{\mu\nu})\exp(ip_\mu b^{\mu})\psi(x), \tag{2.46}$$

and $J'_{\mu\nu}$ are the operators (1.3).

Now, the transformations (2.46) have already been found in Section 1.5. In fact the operator in the r.h.s. of (2.46) is a multiple of the unit matrix (i.e., has the same action on any component of the wave function), hence, in accordance with (1.45)

$$\psi''(x)=\psi(x) \tag{2.47}$$

where x'' is connected with x by the Lorentz transformation inverse to (1.32). Substituting (2.46), (2.47) into (2.45) we obtain finally

$$\psi(x)\to\psi'(x)=\exp(\frac{i}{2}S_{\mu\sigma}\theta^{\mu\sigma})\psi(x''), \tag{2.48}$$

or

$$\psi(x)\to\psi'(x')=\exp(\frac{i}{2}S_{\mu\sigma}\theta^{\mu\sigma})\psi(x). \tag{2.49}$$

Formula (2.49) (together with the relations (1.32), (1.39) determining transformations of independent variables) gives the general form of Lorentz transformations of solutions of the Dirac equation. In contrast to the transformation law (1.45) of a scalar field formula (2.49) contains the matrix multiple $\exp(iS_{\mu\sigma}\theta^{\mu\sigma}/2)$ mixing the components of the wave function (2.2).

For the sake of convenience, in using formula (2.49) it is desirable to represent $\exp(iS_{\mu\sigma}\theta^{\mu\sigma}/2)$ as a polynomial in $S_{\mu\sigma}\theta^{\mu\sigma}$. Staring from (1.44) and using (2.3), (2.18) we can prove the identity

$$\exp\left(\frac{i}{2}S_{\mu\sigma}\theta^{\mu\sigma}\right)=\Lambda_+\left(\cos\theta^+ +\frac{i}{2\theta^+}\gamma_\mu\gamma_\sigma\theta^{\mu\sigma}\sin\theta^+\right)+\Lambda_-\left(\cos\theta^- +\frac{i}{2\theta^-}\gamma_\mu\gamma_\sigma\theta^{\mu\sigma}\sin\theta^-\right) \tag{2.50}$$

where

$$\Lambda_\pm=\frac{1}{2}(1\pm i\gamma_4),\quad \theta^\pm=2[(\theta_1^\pm)^2+(\theta_2^\pm)^2+(\theta_3^\pm)^2]^{1/2},\quad \theta_a^\pm=\frac{1}{2}\varepsilon_{abc}\theta_{bc}\pm i\theta_{0a}.$$

So we have obtained a transformation law for solutions of the Dirac equation. Taking particular values of the parameters $\theta_{\mu\sigma}$, b_μ it is not difficult to obtain the corresponding Lorentz transformation for the wave function. If e.g. the only nonzero parameter is θ_{12} we obtain from (2.50)

$$\exp\left(\frac{i}{2}S_{\mu\sigma}\theta^{\mu\sigma}\right)=\cos\left(\frac{1}{2}\theta_{12}\right)-\gamma_1\gamma_2\sin\left(\frac{1}{2}\theta_{12}\right). \tag{2.51}$$

For nonzero θ_{01}, $\theta_{ab}=\theta_{02}=\theta_{03}=0$ we have

$$\exp\left(\frac{i}{2}S_{\mu\sigma}\theta^{\mu\sigma}\right)=\cosh\left(\frac{1}{2}\theta_{01}\right)-\gamma_0\gamma_1\sinh\left(\frac{1}{2}\theta_{01}\right). \tag{2.52}$$

The transformations (2.49), (2.51) and (2.49), (2.52) correspond to rotations of the reference frame in the plane 1-2 and to Lorentz transformation (1.28) for a=1.

Let us adduce the explicit form of the dilatation and conformal transformations of solutions of the massless Dirac equation:

$$\psi(x)\to\psi'(x')=\exp\left(-\frac{3}{2}\theta\right)\psi(x), \tag{2.53}$$

$$\psi(x)\to\psi'(x')=(1-2b_\mu x^\mu+b_\mu b^\mu x_\nu x^\nu)(1-\gamma_\lambda\gamma_\sigma b^\lambda x^\sigma)\psi(x) \qquad (2.54)$$

where x' are given by relations (1.48).

Formulae (2.53) (and (2.54)) can be obtained by solving the Lie equations corresponding to the generators D and K_μ (2.45). Later in Subsections 3.5, 3.9, we present solutions of these equations and the explicit form of the conformal transformations for fields with arbitrary spin.

2.10. *P*-, *T*- and *C*-Transformations

Let us study the symmetry of the Dirac equation under the space inversion and time reflection. In analogy with (2.48) we will seek these transformations operators in the form

$$\begin{aligned}&\psi(x_0,\boldsymbol{x})\to P\psi(x_0,\boldsymbol{x})=r_1\psi(x_0,-\boldsymbol{x}),\\&\psi(x_0,\boldsymbol{x})\to T\psi(x_0,\boldsymbol{x})=r_2\psi(-x_0,\boldsymbol{x})\end{aligned} \qquad (2.55)$$

where r_1 and r_2 are some numerical matrices.

The operators defined in this way satisfy the following evident relations

$$Pp_0=p_0P,\quad Pp_a=-p_a p,\quad Tp_0=-p_0T,\quad Tp_a=p_aT,$$

hence the invariance condition (2.21) for the transformations (2.55) can be written in the form

$$\begin{aligned}&r_1L(p_0,\boldsymbol{p})-L(p_0,-\boldsymbol{p})r_1=\alpha_1L(p_0,\boldsymbol{p}),\\&r_2L(p_0,\boldsymbol{p})-L(-p_0,\boldsymbol{p})r_2=\alpha_2L(p_0,\boldsymbol{p})\end{aligned} \qquad (2.56)$$

where $L(p_0,\boldsymbol{p})\equiv L$ is the Dirac operator (2.1). The relations (2.56) reduce to the following equations for r_1, r_2, α_1, α_2:

$$[r_1,\gamma_0]=[r_1,\gamma_a]=[r_2,\gamma_0]=[r_2,\gamma_a]=0,\quad \alpha_1=\alpha_2=0. \qquad (2.57)$$

The general solution of (2.57) is

$$r_1=\tau_1\gamma_0,\quad r_2=\tau_2\gamma_0\gamma_4 \qquad (2.58)$$

where τ_1 and τ_2 are complex parameters. The requirement of unitarity of the transformations (2.55) reduces these parameters to phase multipliers

$$\tau_1=\exp(i\varphi_1),\quad \tau_2=\exp(i\varphi_2),\quad \varphi_1,\varphi_2\in\mathbb{R} \qquad (2.59)$$

So the Dirac equation is invariant under the discrete transformations (2.55), (2.57) (2.58) which complete the representation of the proper orthochronous Poincaré group to a representation of the complete Poincaré group. One more symmetry transformation of the Dirac equation can be given by the antiunitary operator C:

$$\psi(x_0,\boldsymbol{x})\rightarrow C\psi(x_0,\boldsymbol{x})=r_3\psi^*(x_0,\boldsymbol{x}), \tag{2.60}$$

where r_3 is a matrix satisfying the conditions (compare with (2.56))

$$r_3L(p_0,\boldsymbol{p})-L^*(p_0,\boldsymbol{p})r_3=\alpha_3L(p_0,\boldsymbol{p}), \tag{2.61}$$

where the asterisk denotes that all terms in the corresponding operator should to be changed to complex conjugated ones.

Using (2.1), (2.4), (2.61) it is not difficult to find that

$$a_3\equiv0,\quad r_3=i\tau_3\gamma_2,\quad \tau_3=\exp(i\varphi_3). \tag{2.62}$$

Moreover without loss of generality we can set $\tau_3=1$.

The transformation (2.60) is called a charge conjugation. The sense of this name can be understood by considering the Dirac equation for a particle interacting with an external electromagnetic field, where the transformation (2.60) is accompanied by a change of the electric charge of a particle.

Let us require that the charge-conjugated function (2.60) have the same behavior under the transformations P and T as non-conjugated wave function. This requirement imposes the following conditions on τ_1, τ_2: $Re\tau_1=Im\tau_2=0$, so we have from (2.58), (2.59)

$$r_1=\pm\gamma_0,\quad r_2=\pm i\gamma_0\gamma_4,\quad r_3=1. \tag{2.63}$$

Thus the Dirac equation is invariant under the P-, T- and C-transformations just as the KGF equation. The transformations (2.55), (2.60), (2.63) together with the Poincaré group transformations found in Subsection 2.9 form a symmetry group of (2.1), which we denote by $P_c(1,3)$. The projective representations of the group $P_c(1,3)$ are considered in Section 4.

3. MAXWELL'S EQUATIONS

3.1. Introduction

Maxwell's equations are one of the main foundations of modern physics. Describing a very extensive branch of physical phenomena, these equations are distinguished by their extremely simple and elegant form. But the source of this simplicity and elegance lies in the remarkably rich symmetry of Maxwell's equations.

The investigation of the symmetry of Maxwell's equations has a long and glorious history. In 1893, having written these equations in the vector notations, Heaviside [219] pointed out that they are invariant under the change

$$\boldsymbol{E}\rightarrow\boldsymbol{H},\quad \boldsymbol{H}\rightarrow-\boldsymbol{E},$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the vectors of the electric and magnetic field strengths. Larmor [272] and Rainich [368] found that this symmetry can be generalized to the family of one-parameter transformations (we will call them Heaviside-Rainich transformations)

$$\begin{aligned} &\boldsymbol{E}\to\boldsymbol{E}\cos\theta+\boldsymbol{H}\sin\theta,\\ &\boldsymbol{H}\to\boldsymbol{H}\cos\theta-\boldsymbol{E}\sin\theta. \end{aligned} \tag{3.1}$$

Lorentz [288], Poincaré [361, 362] and Einstein [90] obtained the most fundamental result connected with the symmetry of Maxwell's equations, which paid a revolutionary role in physics. It was Lorentz who first found all possible linear transformations of space and time variables (and the corresponding transformations for $\boldsymbol{E}$ and $\boldsymbol{H}$) leaving Maxwell's equations invariant.

Augmenting and generalizing Lorentz's results, Poincaré showed that in the presence of charges and currents Maxwell's equations are invariant under Lorentz transformations. Poincaré first established and studied the main property of these transformations, i.e., their group structure, and he showed that "the Lorentz transformations represent a rotation in a space of four dimensions whose points have coordinates $(x, y, z, \sqrt{-1t})$" [361]. Thus, Poincaré combined space and time into a single four-dimensional space-time at least three years before Minkowski [306].

In Einstein's famous work [90] which played an outstanding role in the history of modern science it was also established that Maxwell's equations with currents and charges are invariant under Lorentz transformations. On the basis of their study of the symmetries of Maxwell's equations, Lorentz, Poincaré and Einstein created the foundations of new relativity theory. Moreover new relativity principle (differing from the Galilei relativity principle) was created in physics.

The next important step in studying the symmetry of Maxwell's equations was made by Bateman [22,23] and Cuningham [70] who proved that these equations are invariant under the inversion transformation

$$x_\mu\to\frac{x_\mu}{x_\nu x^\nu},$$

(suplemented by the corresponding transformation of the dependent variables) from which follows invariance under the conformal transformations (1.50). In fact Bateman proved that the conformal group invariance determines the maximal symmetry of Maxwell's equations with currents and charges [22].

Not long ago the group-theoretical analysis of Maxwell's equations was done using the classical Lie approach [71, 226]. Incidentally it was proved rigorously that the maximal local invariance group of Maxwell's equations for the electromagnetic field in vacuum is the 16-parameter group $C(1,3)\otimes H$ where H is the one-parameter group of Heaviside-Larmor-Rainich transformation (3.1).

But the transformations mentioned above do not exhaust all the symmetries of Maxwell's equations. These equations possess hidden (nongeometric) symmetries which are not connected with transformations of independent variables [120, 144]. The main property of this "new" symmetry is that the basis elements of the corresponding IA do not belong to the class M_1 (in contrast to the classical Lie approach) but are integro-differential operators (see Chapter 4).

The classical (Lie) symmetry of Maxwell's equations is discussed in the present section.

3.2. Various Formulations of Maxwell's Equations

Maxwell's equations for the electromagnetic field in vacuum are usually written in the following form

$$\boldsymbol{p}\times\boldsymbol{E}=i\frac{\partial \boldsymbol{H}}{\partial t},\quad \boldsymbol{p}\times\boldsymbol{H}=-i\frac{\partial \boldsymbol{E}}{\partial t}, \tag{3.2a}$$

$$\boldsymbol{p}\cdot\boldsymbol{E}=0,\quad \boldsymbol{p}\cdot\boldsymbol{H}=0 \tag{3.2b}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the vectors of the electric and magnetic field strengths.

In the presence of currents and charges Maxwell's equations take the form

$$i\frac{\partial \boldsymbol{E}}{\partial t}=-\boldsymbol{p}\times\boldsymbol{H}+i\boldsymbol{j},\quad \boldsymbol{p}\cdot\boldsymbol{E}=-ij_0, \tag{3.3a}$$

$$i\frac{\partial \boldsymbol{H}}{\partial t}=\boldsymbol{p}\times\boldsymbol{E},\quad \boldsymbol{p}\cdot\boldsymbol{H}=0 \tag{3.3b}$$

where $j=(\boldsymbol{j},j_0)$ is the four-vector of the electric current and the constant of electromagnetic interaction has been taken equal to one.

The vector formulation of Maxwell's equations given above was proposed by Hertz and Heaviside. Besides (3.2), (3.3) we shall consider other representations of these equations suitable for the investigation of symmetries.

Let us denote by Φ the following vector-function

$$\Phi=\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}=column(E_1,E_2,E_3,H_1,H_2,H_3) \tag{3.4}$$

where E_a and H_a are the components of the electric and magnetic field strengths. The equation (3.2) may be represented in the form

$$\begin{aligned}&\hat{L}_1\Phi=0,\quad \hat{L}_1=i\frac{\partial}{\partial t}-H\equiv i\frac{\partial}{\partial t}-\sigma_2\boldsymbol{S}\cdot\boldsymbol{p},\\ &\hat{L}_2^{\,a}\,\Phi=0,\quad \hat{L}_2^{\,a}=(Z^{ab}+i\varepsilon^{abc}S_c)p_b\end{aligned} \tag{3.5}$$

where

$$Z^{ab}=2\delta^{ab}-S_aS_b-S_bS_a,$$

$$S_a=\begin{pmatrix}\hat{S}_a & 0\\ 0 & \hat{S}_a\end{pmatrix},\quad \hat{S}_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \hat{S}_2=\begin{pmatrix}o&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad \hat{S}_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \tag{3.6}$$

and σ_2 is the 6×6 Pauli matrix commuting with S_a (i.e., the matrix obtained from (2.5) by change 1 and 0 by the 3×3 unit and zero matrices).

It is not difficult to make sure the equations (3.2) and (3.5) coincide componentwise for any a=1,2,3.

The formulation (3.4)-(3.6) is mainly used in the following by the analysis of hidden symmetries of Maxwell's equations.

Maxwell's equations can be written in the form of a first-order covariant equation also

$$(\beta_\mu p^\mu-\beta_4^2\kappa)\tilde{\psi}=0 \tag{3.7}$$

where $ß_\mu$ are irreducible 10×10 matrices of Kemmer-Duffin-Petiau (KDP) in the representation (6.22),

$$\hat{\psi}=column(E_1,E_2,E_3,H_1,H_2,H_3,A_1,A_2,A_3,A_0). \tag{3.8}$$

Substituting (3.8), (6.22), (6.24) into (3.7) we obtain the system

$$\frac{\partial A_b}{\partial x_0}+\frac{\partial A_0}{\partial x_b}=-\kappa E_b,\quad \kappa\boldsymbol{H}=-i\boldsymbol{p}\times\boldsymbol{A},$$

$$\frac{i\partial\boldsymbol{E}}{\partial t}=-\boldsymbol{p}\times\boldsymbol{H},\qquad \boldsymbol{p}\cdot\boldsymbol{E}=0,$$

from which the equations (3.2) follow immediately.

Maxwell's equations in the form (3.7) were discussed by Fedorov [95] and Bludman [38].

Maxwell's equations with currents and charges can also be written in the covariant form. Denoting

$$\psi=column(E_1,E_2,E_3,H_1,H_2,H_3,J_1,J_2,J_3,J_0) \tag{3.9}$$

it is possible to represent (3.3) as the following system [154, 157]

$$\begin{aligned}&L_1\psi=0,\quad L_1=(1-\beta_4^2)(\beta^\mu p_\mu+1),\\ &L_2\psi=0,\quad L_2=\beta^\mu p_\mu\beta_4,\end{aligned} \tag{3.10}$$

where $ß_\mu$ are the matrices (6.22). The other possibility is to use the 16×16 KDP matrices which make it possible to write Maxwell's equations with currents and charges in the form of a single covariant equation [154].

3.3. The Equation for the Vector-Potential

Let us consider one more formulation of Maxwell's equations which makes use of a four-component function (vector-potential) $A=(A_0,A_1,A_2,A_3)$ connected with $\boldsymbol{E}$ and $\boldsymbol{H}$ by the relations

$$\boldsymbol{H}=i\boldsymbol{p}\times\boldsymbol{A}, \quad \boldsymbol{E}=-\frac{\partial \boldsymbol{A}}{\partial t}-i\boldsymbol{p}A_0. \tag{3.11}$$

Substituting these expressions into (3.3) we obtain the following equations for A_μ

$$p_\nu p^\nu A_\mu - p_\mu p_\nu A^\nu = j_\mu. \tag{3.12}$$

So instead of eight equations for $\boldsymbol{E}, \boldsymbol{H}$ one may solve the system (3.12) for A_μ and then find the vectors of the magnetic and electric field strengths using formulae (3.11). Moreover the system (3.12) can be simplified using the freedom in the choice of A_μ. Actually the relations (3.11) are invariant with respect to the substitution

$$A_\mu \to A_\mu + ip_\mu \Phi \tag{3.13}$$

where Φ is an arbitrary function. This is why the additional constraint (Lorentz gauge) is imposed usually on A_μ:

$$p_\mu A^\mu = 0 \tag{3.14}$$

which reduces (3.11) to the system of noncoupled equations

$$p_\nu p^\nu A_\mu = j_\mu. \tag{3.15}$$

As before, the equations (3.14), (3.15) determine the vector-potential up to the transformation (3.13) where Φ is a function satisfying the equation $p_\mu p^\mu \Phi=0$. Using such transformations (called a gauge transformations of the second kind) it is possible to arrange that

$$A_0=0, \quad \boldsymbol{p}\cdot\boldsymbol{A}=0. \tag{3.16}$$

The conditions (3.16) (which in contrast to (3.14) are not relativistically invariant and can be imposed in a fixed reference frame only) are called the Coulomb gauge.

To conclude this subsection we notice that by using the connection of the vectors $\boldsymbol{E}$ an $\boldsymbol{H}$ with the four-vector A_μ given by formulae (3.11) Maxwell's equations can be written in the tensor form

$$p_\nu F^{\nu\mu} = ij^\mu, \quad p_\nu \tilde{F}^{\nu\mu} = 0 \tag{3.17}$$

where

$$F^{\mu\nu}=i(p^{\mu}A^{\nu}-p^{\nu}A^{\mu}),\quad \tilde{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}F_{\varrho\sigma}.$$

3.4. The IA of Maxwell's Equations in the Class M_1

To investigate the local symmetries of Maxwell's equations, we shall use the covariant formulation of these equations given in (3.10).

The problem of finding the IA of the equations (3.10) in the class M_1 can be formulated in analogy with the corresponding problem for the Dirac equation considered in Section 2. The only distinction is that Maxwell's equations (3.10) are represented as a result of action of two linear operators on a vector ψ while the Dirac equation is determined by the only operator L of (2.1).

In analogy with Section 2 let us formulate a definition of a SO $Q\in \mathrm{M}_1$ for Maxwell's equations.

DEFINITION 3.1. A linear differential operator

$$Q=A^{\mu}p_{\mu}+B,\quad A^{\mu}\in F,\quad B\in G^{10} \tag{3.18}$$

is a SO of Maxwell's equations (3.10) in the class M_1 if

$$\begin{aligned}&[Q,L_1]=\beta_Q^1 L_1+\beta_Q^2 L_2,\quad \beta_Q^{\alpha}\in G^{10};\\ &[Q,L_2]=\lambda_Q^1 L_1+\lambda_Q^2 L_2,\quad \lambda_Q^{\alpha}\in G^{10},\end{aligned} \tag{3.19}$$

where L_1, L_2 are the operators (3.10) and the symbol G^{10} denotes the linear space of 10×10 matrices whose matrix elements belong to F (see Subsection 1.1).

As in the case of the Dirac equation, SOs transform solutions into solutions and the set of SOs forms a Lie algebra.

The main assertion concerning the symmetry of Maxwell's equations can be formulated as follows

THEOREM 3.1. Maxwell's equations (3.10) are invariant under the 15-dimensional Lie algebra which is isomorphic to the algebra *AC(1.3)*. The basis elements of this IA can be taken in the form of (2.22), (2.42) where

$$S_{\mu\nu}=i(\beta_{\mu}\beta_{\nu}-\beta_{\nu}\beta_{\nu}),\quad K=3-\beta_4^2. \tag{3.20}$$

The Lie algebra spanned on the basis (2.22), (2.42), (3.20) is the maximal IA of Maxwell's equations in the class M_1.

PROOF. Using the relations

$$\beta_4^3=\beta_4,\quad (1-\beta_4^2)\beta_{\mu}=\beta_{\mu}\beta_4^2,\quad \beta_{\mu}\beta^{\mu}=3-\beta_4^2,\quad [\beta_{\mu},S_{\nu\lambda}]=i(g_{\mu\lambda}\beta_{\nu}-g_{\mu\nu}\beta_{\lambda}) \tag{3.21}$$

(which follows from (6.20), (6.23)) one verifies that the operators (3.10), (2.22), (2.42), (3.20) satisfy the invariance conditions

$$[P_\mu, L_\alpha] = [J_{\mu\nu}, L_\alpha] = [D, L_\mu] = [K_\mu, L_\alpha] = 0 \tag{3.22}$$

which coincide with (3.19) for $\beta_Q^\alpha = \lambda_Q^\alpha = 0$.

Using relations (3.21) it is not difficult to verify that the operators (2.22), (2.42), (3.20) satisfy the commutation relations (1.14), (1.19) which characterize the algebra *AC(1.3)*.

We see that the operators (2.22), (2.42), (3.20) actually form the IA of Maxwell's equations. The proof that this IA is maximal in the class M_1 will be given in Section 20 as a part of the solution of a more complicated problem.■

COROLLARY 1. Each of the equations (3.10) is invariant under the algebra *AC(1,3)*.

This statement follows from the commutativity of each of the operators L_1 and L_2 with the basis elements of the algebra *AC(1,3)*.

COROLLARY 2. Maxwell's equations for the electromagnetic field in vacuum are invariant under the 16-dimensional Lie algebra whose basis elements are given in (2.22), (2.42) (where $S_{\mu\sigma}$, K are the matrices (3.20)) and in (3.23):

$$F = \beta_4. \tag{3.23}$$

Indeed, Maxwell's equations without currents and charges can be represented in the form of the system (3.10) with the additional constraint

$$L_3\psi \equiv (1 - \beta_4^2)\psi = 0. \tag{3.24}$$

The matrix 1-$(ß_4)^2$ commutes with any element of the algebra *AC(1,3)* and, moreover, the relations

$$[L_1, F] = -iL_2, \quad [L_2, F] = L_1 - L_3 - FL_2, \quad [L_3, F] = 0$$

are satisfied. So the generators of the conformal group and the operator F are the SOs of the system (3.10), (3.24).

Thus the symmetry of Maxwell's equations for the electromagnetic field in vacuum turns out to be broader than in presence of currents and charges. It is connected with the fact that the equations (3.3) includes the current in a nonsymmetric way (due to the absence of the magnetic charge). As a result Maxwell's equations with currents and charges are not invariant under the Heaviside-Larmor-Rainich transformations (which are generated by the operator *F*, as will be shown in the following).

3.5. Lorentz and Conformal Transformations

A direct consequence of symmetry of Maxwell's equations under the IA found above is the invariance under the group of conformal transformations. Here we will show the explicit form of these transformations and then generalize these transformations to the case of arbitrary spin (see Subsection (3.9)).

The deduction of Lorentz and conformal transformations for the electromagnetic field is analogous to the one given in Subsection 2.9 where the corresponding transformations for the Dirac equation solutions are found. Therefore we will omit the details and mention only the essential points.

As in Section 2, it is not difficult to show that the transformations of independent variables generated by the operators (2.22), (2.42), (3.20) have the form

$$\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a}, \quad x_0 \to x_0' = x_0 + a_0, \tag{3.25}$$

$$\boldsymbol{x} \to \boldsymbol{x}'' = \boldsymbol{x}\cos\theta - \frac{\theta \times \boldsymbol{x}}{\theta}\sin\theta + \frac{\theta(\theta \cdot \boldsymbol{x})}{\theta^2}(1-\cos\theta), \quad x_0 \to x_0'' = x_0, \tag{3.26}$$

$$\begin{aligned} \boldsymbol{x} \to \boldsymbol{x}''' &= \boldsymbol{x} - \frac{\lambda x_0}{\lambda}\sinh\lambda + \frac{\lambda(\lambda \cdot \boldsymbol{x})}{\lambda^2}(\cosh\lambda - 1), \\ x_0 \to x_0''' &= x_0\cosh\lambda - \frac{\boldsymbol{x} \cdot \lambda}{\lambda}\sinh\lambda, \end{aligned} \tag{3.27}$$

$$x_\mu \to x_\mu^{IV} = \exp(-\lambda_0)x_\mu,$$

$$x_\mu \to x_\mu^{V} = \frac{x_\mu - b_\mu x_\nu x^\nu}{1 - 2b_\nu x^\nu + b_\nu b^\nu x_\lambda x^\lambda} \tag{3.29}$$

where

$$\theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}, \quad \lambda = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2},$$

θ_a, b_μ, λ_a and a_μ are real parameters.

Formulae (3.25)-(3.29) give the displacements of the time and space variables, rotation by an angle θ around the axis θ/θ, proper Lorentz transformation (3.27), scale transformations (3.28) and conformal transformations (3.29). Moreover (3.25)-(3.27) are particular cases of the general Lorentz transformation (1.36), (1.38), (1.39).

In order for Maxwell's equations to be invariant under transformations (3.25)-(3.29) it is necessary to transform simultaneously the vector-function (3.9) in accordance with the following law [154, 157]

$$\psi(x) \to \psi'(x') = \psi(x), \tag{3.30}$$

$$\psi(x) \to \psi''(x'') = \exp\left(\frac{i}{2}\varepsilon_{abc}S_{ab}\theta_c\right)\psi(x), \tag{3.31}$$

$$\psi(x) \to \psi'''(x''') = \exp(iS_{0a}\lambda_a)\psi(x), \tag{3.32}$$

$$\psi(x) \to \psi^{IV}(x^{IV}) = \exp(-iK\lambda_0)\psi(x), \tag{3.33}$$

$$\psi(x)\rightarrow\psi'^{V}(x'^{V})=[\varphi(x,b)]^{K}\exp\left(\frac{2iS_{\mu\nu}b^{\mu}x^{\nu}}{a}\arctan\frac{a}{b_{\mu}x^{\mu}-1}\right)\psi(x) \tag{3.34}$$

where $S_{\mu\sigma}$ and K are the matrices (3.20),

$$\varphi(x,b)=1-2b_{\mu}x^{\mu}+b_{\nu}b^{\nu}x_{\lambda}x^{\lambda},\quad a=[b_{\mu}b^{\mu}x_{\nu}x^{\nu}-(b_{\mu}x^{\mu})^{2}]^{1/2}. \tag{3.35}$$

Formulae (3.30)-(3.32) are distinct from the corresponding formulae giving the transformations of solutions of the Dirac equation only by the realization of the matrices $S_{\mu\sigma}$ (compare with (2.47), (2.49)). It is not difficult to make sure also that the dilatation and conformal transformations for the Dirac spinors given in (2.53), (2.54) also can be represented in the form (3.33), (3.34) where K=3/2, $S_{\mu\nu}=[\gamma_{\mu},\gamma_{\nu}]/4$, γ_{μ} are the Dirac matrices.

Using relations (3.30)-(3.34) it is not difficult to obtain the transformation law for the vectors ***H***, ***E*** and four-vector of current j. Indeed bearing in mind the identity (which follows from (3.20), (6.20))

$$(S_{\mu\nu}d^{\mu\nu})^{3}=(d_{\lambda\sigma}d^{\lambda\sigma}-d_{\lambda\sigma}d^{\sigma\lambda})S_{\mu\nu}d^{\mu\nu}$$

where $d_{\mu\sigma}$ are arbitrary parameters, one can represent each of the exponentials from (3.31), (3.32) as a sum in powers of the matrices $S_{\mu\sigma}$

$$\exp\left(\frac{i}{2}\varepsilon_{abc}S_{ab}\theta_{c}\right)=1+i\frac{\boldsymbol{S}\cdot\theta}{\theta}\sin\theta+\left(\frac{\boldsymbol{S}\cdot\theta}{\theta}\right)^{2}(\cos\theta-1),$$

$$\boldsymbol{S}=(S_{1},S_{2},S_{3}),\quad S_{a}=\frac{1}{2}\varepsilon_{abc}S_{bc};$$

$$\exp(iS_{0a}\lambda_{a})=1+i\frac{S_{0a}\lambda_{a}}{\lambda}\sinh\lambda+\left(\frac{S_{0a}\lambda_{a}}{\lambda}\right)^{2}(\cosh\lambda-1), \tag{3.36}$$

$$\exp\left(2i\frac{S_{\mu\nu}b^{\mu}x^{\nu}}{a}\arctan\frac{a}{b_{\mu}x^{\mu}-1}\right)=$$

$$=\varphi(x,b)^{-1}[\varphi(x,b)+2iS_{\mu\nu}x^{\mu}b^{\nu}(b_{\lambda}x^{\lambda}-1)-2(S_{\mu\nu}x^{\mu}b^{\nu})^{2}].$$

As to the operator $[\varphi(x,b)]^{K}$ (where $K=3-(\beta_{5})^{2}$ is a Hermitian matrix) it can be represented in the form

$$[\varphi(x,b)]^{K}=\sum[\varphi(x,b)]^{\sigma}\Lambda_{\sigma}, \tag{3.37}$$

where σ are eigenvalues of the matrix K (equal to 2 or 3) and Λ_{σ} are the projection operators corresponding to these eigenvalues

$$\Lambda_{3}=\beta_{5}^{2},\quad \Lambda_{2}=1-\beta_{5}^{2}. \tag{3.38}$$

Then, using the explicit expressions of $S_{\mu\sigma}$ and β_{μ} given in (3.20), (6.22), (6.24), we obtain

$$S_a = i\varepsilon_{abc}\beta_b\beta_c = \begin{pmatrix} \hat{S}_a & \cdot & \cdot & \cdot \\ \cdot & \hat{S}_a & \cdot & \cdot \\ \cdot & \cdot & \hat{S}_a & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad S_{0a} = i[\beta_0, \beta_a] = \begin{pmatrix} \cdot & -\hat{S}_a & \cdot & \cdot \\ \hat{S}_a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \eta_a \\ \cdot & \cdot & -\eta_a^\dagger & \cdot \end{pmatrix}, \tag{3.39}$$

$$K = 3 - \beta_5^2 = \begin{pmatrix} 3I & \cdot & \cdot & \cdot \\ \cdot & 3I & \cdot & \cdot \\ \cdot & \cdot & 2I & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix}, \qquad \beta_5^2 = \begin{pmatrix} I & \cdot & \cdot & \cdot \\ \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $\hat{S}_a$ are the 3×3 matrices (3.6), $S_{\mu\sigma}$ are the 4×4 matrices (1.42), I is the 3x3 unit matrix, the dots denote zero matrices of an appropriate dimension.

Substituting (3.9), (3.36)-(3.39) into (3.30)-(3.34) we obtain the following transformation laws

$$\boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{E}, \quad \boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{H}, \quad j \to j' = j,$$

$$j_0 \to j_0'' = j_0, \quad (\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j}) \to (\boldsymbol{E}'', \boldsymbol{H}'', \boldsymbol{j}'') =$$

$$= (\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J})\cos\theta - \theta \times (\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j})\frac{\sin\theta}{\theta} + \theta\cdot(\theta\cdot(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j}))\frac{1-\cos\theta}{\theta^2},$$

$$\boldsymbol{E} \to \boldsymbol{E}''' = \boldsymbol{E}\cosh\lambda - \lambda\times\boldsymbol{H}\frac{\sinh\lambda}{\lambda} + \lambda(\lambda\cdot\boldsymbol{E})\frac{1-\cosh\lambda}{\lambda}, \tag{3.40a}$$

$$\boldsymbol{H} \to \boldsymbol{H}''' = \boldsymbol{H}\cosh\lambda + \lambda\times\boldsymbol{E}\frac{\sinh\lambda}{\lambda} + \lambda(\lambda\cdot\boldsymbol{H})\frac{1-\cosh\lambda}{\lambda},$$

$$j_0 \to j_0''' = j_0\cosh\lambda - \lambda\cdot\boldsymbol{j}\frac{\sinh\lambda}{\lambda}, \quad \boldsymbol{j} \to \boldsymbol{j}''' = \boldsymbol{j} - \lambda j_0\frac{\sinh\lambda}{\lambda} - \lambda(\lambda\cdot\boldsymbol{j})\frac{1-\cosh\lambda}{\lambda},$$

$$((\boldsymbol{E}, \boldsymbol{H}) \to (\boldsymbol{E}^{IV}, \boldsymbol{H}^{IV}) = \exp(-2\lambda_0)(\boldsymbol{E}, \boldsymbol{H}), \quad j \to j^{IV} = \exp(-3\lambda_0)j,$$

$$\boldsymbol{E} \to \boldsymbol{E}^{V} = \varphi[(b^\mu x_\mu{}^V - 1)^2\boldsymbol{E} + 2(b^\mu x_\mu{}^V - 1)b_0\boldsymbol{x}^V\times\boldsymbol{H} - x_0{}^V\boldsymbol{b}\times\boldsymbol{H} - \boldsymbol{b}\boldsymbol{x}^V\cdot\boldsymbol{E} + \boldsymbol{x}^V\boldsymbol{b}\cdot\boldsymbol{E}) +$$

$$+\boldsymbol{b}\times\boldsymbol{x}^V(x_0{}^V\boldsymbol{b}\cdot\boldsymbol{H} - b_0\boldsymbol{x}^V\cdot\boldsymbol{H} + \boldsymbol{b}\times\boldsymbol{x}^V\cdot\boldsymbol{E}) + (\boldsymbol{b}x_0{}^V - \boldsymbol{x}^V b_0)(\boldsymbol{b}\cdot\boldsymbol{x}^V\times\boldsymbol{H} - x_0{}^V\boldsymbol{b}\cdot\boldsymbol{E} + b_0\boldsymbol{x}^V\cdot\boldsymbol{E})],$$

$$\boldsymbol{H} \to \boldsymbol{H}^{V} = \varphi[(b^\mu x_\mu{}^V - 1)^2\boldsymbol{H} + 2(b^\mu x_\mu{}^V - 1)(x_0{}^V\boldsymbol{b}\times\boldsymbol{E} - b_0\boldsymbol{x}^V\times\boldsymbol{E} - \boldsymbol{b}\boldsymbol{x}^V\cdot\boldsymbol{H} + \boldsymbol{x}^V\boldsymbol{b}\cdot\boldsymbol{H}) + \tag{3.40b}$$

$$+\boldsymbol{b}\times\boldsymbol{x}^V(\boldsymbol{b}\times\boldsymbol{x}^V\cdot\boldsymbol{H} + b_0\boldsymbol{x}^V\cdot\boldsymbol{E} - x_0{}^V\boldsymbol{b}\cdot\boldsymbol{E}) + (\boldsymbol{b}x_0{}^V - \boldsymbol{x}^V b_0)(b_0\boldsymbol{x}^V\cdot\boldsymbol{H} - x_0{}^V\boldsymbol{b}\cdot\boldsymbol{H} - \boldsymbol{b}\times\boldsymbol{x}^V\cdot\boldsymbol{E})],$$

$$j_\lambda \to j_\lambda{}^V = \varphi^2\{\varphi j_\lambda - 2[b_\lambda(1 - 2x_\lambda{}^V b^\lambda) + x_\lambda{}^V b^\mu b^\mu]x_\mu{}^V j^\mu + 2(x_\lambda{}^V - b_\lambda x_\nu{}^V x^{V\nu})b_\mu j^\mu\}.$$

Formulae (3.40) give the explicit form of the transformations of the conformal group for the vectors of the electric and magnetic field strengths and the current four-vector. The corresponding transformations for t and $\boldsymbol{x}$ are given in (3.25)-(3.29). These formulae are quite complicated but simplify considerably if only one of the parameters a_μ, b_μ, θ_a, λ_a is nonzero. For instance setting $b_1=b_2=b_3=0$, $b_0=b$ in the transformation law (3.40) for $\boldsymbol{E}^V$ we obtain

$$\boldsymbol{E}^{V}=(1-2bx_0+b^2x_\mu x^\mu)[(1-bx_0)^2\boldsymbol{E}-b^2\boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{E})+2b(1-bx_0)\boldsymbol{x}\times\boldsymbol{H}]. \tag{3.41}$$

Analogous transformations for $\boldsymbol{H}$ can be obtained from (3.41) by the change $\boldsymbol{E}\to\boldsymbol{H}$, $\boldsymbol{H}\to-\boldsymbol{E}$.

Formulae (3.40), (3.41) may be useful for various applications - e.g. for the construction of nonlinear generalizations of Maxwell's equations, being invariant under the conformal group.

3.6. Symmetry Under the *P*-, *T*- and *C*-Transformations

Invariance under the transformations considered above does not exhaust all symmetry properties of Maxwell's equations. We will see later that these equations are invariant also with respect to nonlocal transformations not connected with geometrical space-time symmetries.

But there exist discrete symmetry transformations of dependent and independent variables which we not considered in the previous subsection. There are the transformations of time reflection and space inversion. In fact it is not difficult to verify that Maxwell's equations do not change their form under the transformation

$$\begin{aligned}&x_0\to x_0,\ \boldsymbol{x}\to-\boldsymbol{x},\ \boldsymbol{E}\to-\boldsymbol{E},\ \boldsymbol{H}\to\boldsymbol{H},\ \boldsymbol{j}\to-\boldsymbol{j},\ j_0\to j_0,\\&x_0\to-x_0,\ \boldsymbol{x}\to\boldsymbol{x},\ \boldsymbol{E}\to\boldsymbol{E},\ \boldsymbol{H}\to-\boldsymbol{H},\ \boldsymbol{j}\to-\boldsymbol{j},\ j_0\to j_0.\end{aligned} \tag{3.42}$$

There is one more symmetry which is trivial for Maxwell's equations describing real field, but is not admitted by some other equations for massless fields. This is the charge conjugation transformation

$$\boldsymbol{E}\to\boldsymbol{E}^*,\quad \boldsymbol{H}\to\boldsymbol{H}^*,\quad j\to j^*. \tag{3.43}$$

Using the vector-function (3.9) it is possible to represent the transformations (3.42), (3.43) in the form (2.55), (2.60), where $r_1=1-2ß_0^2$, $r_2=(1-2ß_0^2)(1-2ß_5^2)$, $r_3=1$. The invariance of (3.10) under these transformations follows from the relations

$$[P,L_1]=[P,L_2]_+=[T,L_1]_+=[T,L_2]=[C,L_1]_+=[C,L_2]_+=0$$

where L_1 and L_2 are the operators (3.10). The operators P, T and C satisfy the relations (1.54) just as in the cases of the KGF and Dirac equations.

3.7. Representations of the Conformal Algebra Corresponding to a Field with Arbitrary Discrete Spin

We have shown in the above that the IA of Maxwell's equations in the class M_1 is the 15-dimensional algebra $AC(1,3)$. This algebra and their representations play a fundamental role in modern theoretical physics.

In this section we continue discussing the conformal symmetry of Maxwell's equations and other relativistic wave equations for massless fields. It will be shown that the conformal group generators K_μ and D are expressed via the Poincaré group generators P_μ and $J_{\mu\sigma}$ on the set of solutions of such equations

DEFINITION 3.2. We say that an equation

$$L\psi=0 \tag{3.44}$$

where L is some linear differential (or integro-differential) operator, is Poincaré-invariant and describes a massless field with discrete spin if on the set of its solutions a representation of the Poincaré algebra is realized corresponding to zero eigenvalues of the Casimir operators (2.36):

$$P^\mu P_\mu\psi=0, \quad W^\mu W_\mu\psi=0 \tag{3.45}$$

In other words if an IA of some equation is given by the operators P_μ, $J_{\mu\sigma}$ satisfying (1.14), (3.45) than we call it an equation for a massless field with discrete spin. It appears that any such equation is also invariant under the more extensive (conformal) algebra as it is stated in the following theorem.

THEOREM 3.2 [143]. Any Poincaré invariant equation for a massless field with discrete spin is invariant under the conformal algebra whose basis elements are given by the operators P_μ, $J_{\mu\sigma}$, forming the algebra $AP(1,3)$, and the operators D, K_μ expressed via P_μ, $J_{\mu\sigma}$ by the relations

$$D=\frac{1}{2}\left[\frac{P_0P_a}{P^2},J_{0a}\right]_+, \quad K_0=\frac{1}{2}\left[\frac{P_0}{P^2},J_{0a}J_{0a}+\Lambda^2-\frac{1}{2}\right]_+, \quad K_a=i[K_0,J_{0a}], \tag{3.46}$$

where

$$\Lambda=\frac{\boldsymbol{J}\cdot\boldsymbol{P}}{P}, \quad P=\sqrt{P_1^2+P_2^2+P_3^2}\,.$$

PROOF. Since P_μ and $J_{\mu\sigma}$ by definition form an IA of equation (3.44), the operators (3.46) expressed via P_μ, $J_{\mu\sigma}$ are also included in the IA of this equation. Further, by assumption, P_μ and $J_{\mu\sigma}$ satisfy the commutation relations (1.14) , and the proof of the theorem reduces to verifying the validity of the relations (1.19) for the operators (3.46). Verification of these relations requires straightforward but cumbersome calculations which can be done using the relations

$$P_0^2=P^2,\quad W_\mu=\frac{P_0}{P}\Lambda P_\mu,\quad [\Lambda,J_{\mu\nu}]=[\Lambda,P_\mu]=0,$$
$$[\frac{1}{P^2},J_{ab}]=0,\quad [\frac{1}{P^2},J_{0a}]=-2i\frac{P_0P_a}{P^4}.\quad \blacksquare \tag{3.47}$$

Thus, formulae (3.46) give explicit expressions for basis elements of the algebra *AC(1,3)* in terms of the operators P_μ, $J_{\mu\sigma}$ contained in its subalgebra *AP(1,3).*

Theorem (3.2) has constructive character since it enables to find the explicit form of the generators D and K_μ starting from given basis elements of the Poincaré algebra. Thus, proceeding from the generators P_μ, $J_{\mu\sigma}$ in the Lomont-Moses representation (see (4.50) for n_1=n_2=0) we obtain, by formulae (3.46), the generators of the conformal group in the Bose-Parker representation [50]. Other representations are considered in Section 4 and the next subsection.

In conclusion we notice that the algebra *AC(1,3)* is isomorphic to the Lie algebra of the group *O(2,4)* (the group of pseudoorthogonal matrices conserving the vector length in the (2+4) Minkowski space. This isomorphism can be established by the following relations

$$J_{\mu\nu}\longleftrightarrow S_{\mu\nu},\quad p_\mu\longleftrightarrow S_{5\mu}+S_{4\mu},\quad K_\mu\longleftrightarrow S_{5\mu}-S_{4\mu},\quad D\longleftrightarrow S_{45} \tag{3.48}$$

where P_μ, $J_{\mu\sigma}$, K_μ, D are the basis elements of the algebra *AC(1,3)* satisfying the commutation relations (1.14), (1.19), and S_{mn} are the generators of the group *O(2,4)* satisfying the relations

$$[S_{kl},S_{mn}]=i(g_{kn}S_{lm}+g_{lm}S_{kn}-g_{km}S_{ln}-g_{ln}S_{km}),$$
$$g_{00}=-g_{11}=-g_{22}=-g_{33}=-g_{44}=g_{55}=0,\quad g_{mn}=0,\ m\neq n. \tag{3.49}$$

The existence of this isomorphism means that the problem of the description of the representations of the algebra *AC(1,3)* reduces to the description of the representations of the algebra *AO(2,4).*

3.8. Covariant Representations of the Algebras *AP(1,3)* and *AC(1,3)*

Of particular interest is the use of the algorithm given in Theorem 3.2 in the case when the Poincaré group generators have the covariant form (2.22) inasmuch as such representations are used for description of actual physical fields.

Here we will consider such representations, restricting ourselves to the case when the matrices $S_{\mu\sigma}$ realize the finite dimensional IR *D(j τ)* of the algebra *AO(1,3)* (see Section 4). To simplify the discussion we will use the realization of (2.22) in the momentum representation where p_μ are independent variables, x_μ=i$\partial/\partial p^\mu$.

Let ψ be an arbitrary solution of (3.44) describing a field with zero mass and discrete spin, and the corresponding Poincaré generators have the form (2.22). Then by definition ψ satisfies conditions (3.45) which take the following form

$$(p_0^2 - p^2)\psi = 0 \tag{3.50}$$

$$W_\mu W^\mu \psi = S_{\mu\nu} S^{\nu\lambda} p_\lambda p^\mu \psi = 0 \tag{3.51}$$

It is convenient to rewrite (3.51) in the form (see (3.49))

$$W_\mu \psi \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} p^\nu S^{\rho\sigma}\psi = \frac{p_0}{p}\Lambda p_\mu \psi, \tag{3.52}$$

where Λ is the helicity operator (3.46). When $\mu=0$, the equation (3.52) turns into identity according to (3.50), but when $\mu=a$, $a\neq 0$, this equation takes the form

$$(p_0 S_{bc} - p_b S_{0c} + p_c S_{0b})\psi = \varepsilon_{abc} p_a \frac{p_0}{p}\Lambda\psi, \tag{3.53}$$

Using relations (4.59), (7.19) given below for $\boldsymbol{S}\to \boldsymbol{j}$ and $\boldsymbol{S}\to\tau$ one concludes that the equation (3.53) is equivalent to the following system

$$\begin{aligned} &S_{0a} p_a \psi = i(j+\tau) p_0 \psi, \\ &\frac{1}{2}\varepsilon_{abc} S_{ab} p_c \psi = (j-\tau) p_0 \psi, \end{aligned} \tag{3.54}$$

where j and τ are integers or half integers enumerating the IR of the algebra *AO(1,3)*.

So if ψ is a covariant massless field with discrete spin than it satisfies the equations (3.50), (3.54). In the case j=0, τ=1/2 (or j=1/2, τ=0) the equations (3.54) coincide and reduce to the Weyl equation, and for j=1, τ=0 these equations are equivalent to Maxwell's equations.

Let as find the operators D and K_μ (3.46) corresponding to the generators (2.22). According to the reasons given below it is sufficient to restrict ourselves to the case when the matrices $S_{\mu\sigma}$ realize the representation *D(s 0)* or *D(0 s)*.

Substituting (2.22) into (3.46) we obtain after a simple calculation

$$D = \frac{1}{2}\left[\frac{P_0 P_a}{P^2}, J_{0a}\right]_+ \equiv x_0 p_a - x_a p_0 + i(s+1) + A(p_0^2 - p^2) + B(S_{0a}p_a - isp_0), \quad s = j+\tau, \tag{3.55a}$$

$$K_0 = \frac{1}{2}\left[\frac{P_0}{P_2}, J_{0a} J_{0a} + \Lambda^2 - \frac{1}{2}\right]_+ \equiv 2x_0 D - x_\mu x^\mu p_0 - 2S_{0a}x_a + C_0(p_0^2 - p^2) + E_0(S_{0a}p_a - isp_0), \tag{3.55b}$$

$$K_a = i[K_0, J_{0a}] = 2x_a D - x^\mu x_\mu p_a + 2S_{a\mu}x^\mu + C_a(p_0^2 - p^2) + E_a(S_{0a}p_a - isp_0),$$

where A, B, C_μ, E_μ are some functions of p_μ, x_μ the exact form of which is not essential because the corresponding terms are equal to zero on the set of solutions of (3.50), (3.54). Moreover

$$D\psi=(x_0p_0-x_ap_a+ik)\psi,\quad K_\mu\psi=(2x_\mu D-x_\nu x^\nu p_\mu+2S_{\mu\nu}x^\nu)\psi, \tag{3.56}$$

where k=1+s according to (3.50), (3.54), (3.55).

In the case when $S_{\mu\sigma}$ belong to an arbitrary finite dimensional representation of the algebra *AO(1,3)*, formulae (2.22), (3.56) give an explicit form of the conformal group generators in the Mack-Salam [289] representation. For instance, the basis elements of the algebra *AC(1,3)* have this form on the set of solutions of the massless Dirac, Weyl and Maxwell equations (compare with (2.22), (2.42)).

Let us sum up. According to Theorem 2.3 any Poincaré-invariant equation for a massless field with discrete spin turns out to be invariant under the conformal algebra which, however, is generally realized in the class of nonlocal (integro-differential) operators.

In this section we have verified that if one starts from a covariant representation of the algebra *AP(1,3)* then the algorithm given in Theorem 2.3 leads to the conformal algebra representation in the covariant form of Mack and Salam [289]. Thus it has been established that the operators D and K_μ in the covariant realization (3.56) can be expressed via the Poincaré group generators according to (3.46). Of course this statement is valid only for the representations satisfying Definition 3.2, i.e., corresponding to zero mass and discrete spin. In particular it is valid for representations realized on the sets of solutions of the massless Dirac and Maxwell equations, i.e., the corresponding generators of dilatation D and conformal transformations K_μ can be expressed via the Poincaré group generators P_μ, $J_{\mu\sigma}$ according to relations (3.46). This seems to be an explanation of the known fact that the conformal symmetry of Maxwell's equations does not lead to new conservation laws in comparison with the Poincaré invariance, see [32, 358].

Let us explain why we restrict ourselves to considering such representations of the matrices $S_{\mu\sigma}$ of (2.22) which have the type $D(0\ \tau)$ or $D(j\ 0)$. As was established by Bracken [52] the operators (2.22), (3.56) form an IA of the d'Alembert equation (3.50) only in the case when $j\tau$=0 where j and τ are indices labelling the representation of the group *O(1,3)* realized by $S_{\mu\sigma}$. So only for such types representations it is possible to obtain a covariant realization of the generators D and K_μ using Theorem 2.3 (another representations do not correspond to a field with zero mass [52]). If j and τ are nonzero then formulae (2.22), (3.46) also define a representation of the algebra *AC(1,3)* but the corresponding D and K_μ do not belong to the class M_1 (for realizations in the coordinate representation).

3.9. Conformal Transformations for Any Spin

In conclusion we should note that the conformal group transformations generated by the operators (2.22), (3.56), where $S_{\mu\sigma}$ are arbitrary matrices satisfying the algebra *AO(1,3)*, can be represented in the form (3.30)-(3.34). It is not difficult to make sure that the transformations (3.30)-(3.34) satisfy the Lie equations (1.47) for any matrices $S_{\mu\sigma}$ forming the algebra *AO(1,3)*, and so give the explicit form of the conformal group transformations for any representation of the Mack-Salam type. The other (but equivalent to (3.34)) realization of the conformal transformation matrix for arbitrary spin was given in [371].

For every particular representation $D(j\ \tau)$ of the algebra *AO(1,3)* the exponential of the matrices $S_{\mu\sigma}\in D(j\ \tau)$ reduces to a finite sum of powers of these matrices, since $\prod(S_{\mu\sigma}-\lambda)=0$ where λ are the eigenvalues of $S_{\mu\sigma}$, $-|j-\tau|\leq\lambda\leq j+\tau$. Thus an explicit form of the transformations (3.30)-(3.34) is easily calculated for any $S_{\mu\sigma}\in D(j\ \tau)$.

We notice that the transformations (3.30)-(3.34) can be considered as a local representation of the conformal group only, since we encounter not only the problem of defining the domain of the functions $\psi'(x')$, $\psi''(x'')$, ... but also the fact that the expressions (3.29), (3.34) become meaningless for $1-2b_\mu x^\mu+b_\mu b^\mu x_\sigma x^\sigma=0$.

2. REPRESENTATIONS OF THE POINCARÉ ALGEBRA AND WAVE EQUATIONS FOR ARBITRARY SPIN

The two opening sections of this chapter contain a description of IRs of the algebra *AP(1,3)* and operators *P*, *T*, and *C*. Incidentally the basis is used in which the Poincaré group generators have a common form for all the classes of IRs. The main elements of a theory of Poincaré-invariant equations for arbitrary spin particles are expounded in Sections 6-10.

4. IRREDUCIBLE REPRESENTATIONS OF THE POINCARÉ ALGEBRA

4.1. Introduction

Representations of the Lie algebras of the main groups of relativistic and nonrelativistic physics, i.e., the Poincaré and Galilei groups, are one of the most important instruments of a symmetry analysis of equations of quantum mechanics. These representations are used for a classification and physical interpretation of known equations as well as for deduction of new motion equations satisfying relativity principles of Galilei or Lorentz, Poincaré and Einstein.

IRs of the Poincaré group were described mainly by Wigner as long ago as 1939 [413]. Then Wigner's results were supplemented by Shirokov [386] who for the first time finds an explicit form of basis elements of the algebra *AP(1,3)* for all the classes of IRs. In many publications appearing later, representations of the Poincaré algebra in various basis were obtained. (See the survey [29] and the references cited there.) Each of the realizations found of the algebra *AP(1,3)* has its merits and drawbacks, each being more convenient for a particular class of physical problems.

The realization of IRs of the Poincaré algebra given below is remarkable for a simple and symmetric form of basis elements which is common for all the classes of IRs. In the following sections we will give the classification of IRs of the algebra *AP(1,3)*, find an explicit form of the Poincaré group generators, and then establish the connection of the representations found here with the canonical Shirokov-Foldy [386, 106] realization.

4.2. Casimir Operators

The commutation relations (1.14) can be used as an abstract definition of the algebra *AP(1,3)*. Our task is to describe constructively all the nonequivalent realizations of these relations in terms of Hermitian operators (for definitions, see Appendix 1). According to Schur's lemma classification of IRs of a Lie algebra L reduces to finding of a complete set of Casimir operators and calculating eigenvalues of these operators. Let C be a Casimir operator for the algebra L then only such vectors which correspond to the same eigenvalues of L can be included into the space of IRs. On the other hand if we find all the independent Casimir operators C_1, C_2, ... for an algebra L and define representation of L in the space of eigenvectors of operators C_1, C_2, ... belonging to one of eigenvalues of each of them, then such a representation will be irreducible. In this case all the operators commuting with any element of a representation of the algebra L are proportional to the unit operators. In other words only one IR corresponds to a set of eigenvalues of all the Casimir operators.

To find the Casimir operators of the algebra *AP(1,3)* we will use the method which admits a direct generalization to the case of the algebras *AP(1,n)*, i.e., the Lie algebra of the generalized Poincaré group in (1+n)-dimensional Minkowski space.

The Casimir operators of the algebra *AP(1,3)* have to commute with P_μ as well as with $J_{\mu\sigma}$. Quantities commuting with $J_{\mu\sigma}$ are called scalars. Evidently there is not any scalar among basis elements of the algebra *AP(1,3)*, that's why we will look for Casimir operators in the enveloping algebra of the algebra *AP(1,3)*, i.e., in the set of operators of a kind Q_A, Q_AQ_B, $Q_AQ_BQ_C$, ..., $Q_A \in \{P_\mu, J_{\mu\sigma}\}$. We will search for all possible scalars starting from vector and tensor quantities which can be defined as follows.

We say a set of operators (A_0, A_1, A_2, A_3) is a vector if for any A_μ (μ= 0,1,2,3) the following condition is satisfied

$$[J_{\mu\nu},A_\lambda]=i(g_{\nu\lambda}A_\mu-g_{\mu\lambda}A_\nu),$$

where $J_{\mu\sigma}$ are basis elements of the algebra *AP(1,3)*. A set of operators $A_{\alpha\lambda}$ which commute with $J_{\mu\sigma}$ as a product of vector components $A_\mu A_\sigma$ will be called a tensor of second rank. An arbitrary rank tensors are defined in analogous manner. The operators P_μ and $J_{\mu\sigma}$ are examples of a vector and second rank tensor.

It is well known that scalars can be obtained from tensors by the operation of index convolution. The example of scalar is the operator $J_{\mu\sigma}J^{\mu\sigma}$. Our task is to find all independent scalars starting from P_μ and $J_{\mu\sigma}$. It is convenient to use for this purpose the vectors W_μ and Γ_μ defined by the relations

$$W_\mu=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^{\sigma}, \quad \Gamma_\lambda=J_{\lambda\mu}P^{\mu}. \qquad (4.2)$$

These operators satisfy the following relations

$$
\begin{aligned}
&W_\mu P^\mu=0, \quad \Gamma_\mu P^\mu=0,\\
&[P_\mu,W_\nu]=0, \quad [W_\mu,W_\nu]=i\varepsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma,\\
&[W_\mu,\Gamma_\sigma]=-iP_\mu W_\sigma, \quad [\Gamma_\mu P_\nu]=i(\delta_{\mu\nu}P_\lambda P^\lambda-P_\mu P_\nu),\\
&[\Gamma_\mu,\Gamma_\nu]=-iJ_{\mu\nu}P_\lambda P^\lambda.
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&P^\lambda J_{\lambda\mu}J^\mu{}_\sigma=P_\sigma(6-\frac{1}{2}J^2)-\frac{1}{2}\varepsilon_{\sigma\mu\nu\rho}J^{\mu\nu}W^\rho+2i\Gamma_\sigma, \quad J^2=J_{\lambda\rho}J^{\lambda\rho},\\
&J_{\mu\lambda}J^{\lambda\sigma}J_{\sigma\nu}=\frac{1}{2}(ig_{\mu\nu}-J_{\mu\nu})J^2+\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}B+3iJ_{\mu\lambda}J^\lambda_\nu, \quad B=\frac{1}{8}\varepsilon_{\alpha\beta\lambda\gamma}J^{\alpha\beta}J^{\lambda\gamma}.
\end{aligned}
\tag{4.4}
$$

Using (4.3), (4.4) it is not difficult to show that all the independent scalars being constructed from P_μ and $J_{\mu\sigma}$ are exhausted by the set

$$
J^2, \quad B, \quad P_\mu P^\mu, \quad W_\mu W^\mu, \quad \Gamma_\lambda\Gamma^\lambda, \quad W_\lambda\Gamma^\lambda. \tag{4.5}
$$

All other scalars (i.e., all the possible convolutions of the vectors P_μ and tensors $J_{\mu\sigma}$) can be expressed via the operators (4.5) according to (4.3), (4.4).

Using (1.14) and (4.4), it is not difficult to show that only two of operators (4.5) commute with P_μ as well as with $J_{\mu\sigma}$. They are the operators

$$
C_1=P_\mu P^\mu \quad \text{and} \quad C_2=W_\mu W^\mu, \tag{4.6}
$$

which are the main Casimir operators of the algebra *AP(1,3)*.

We note that for some classes of representations there are additional operators commuting with any element of the algebra *AP(1,3)*. For example, for $P_\mu P^\mu\geq 0$ this property is possessed by the energy sign operator $C_3=P_0/|P_0|$. To describe IRs of the Poincaré algebra it is necessary to use also these additional Casimir operators which will be enumerated in the following section.

4.2. Basis of an IR

In order to determine a representation of the algebra (1.14) effectively, it is necessary to set ourselves to some orthogonal basis in the representation space. As this basis eigenfunctions of a complete set of commuting selfadjoint operators can be chosen. All the Casimir operators are necessarily included in such a set and some elements of the algebra and enveloping algebra should be added to make this set complete.

We will choose the complete set of commuting operators in the following form

$$P_0,\ P_1,\ P_2,\ P_3,\ W_0;\quad C_1,\ C_2,..., \tag{4.7}$$

where W_0 is a zero component of the Lubanski-Pauli vector (4.2) and the dots denote additional Casimir operators whose existence was mentioned above. We will denote the orthonormal set of eigenfunctions of the operators (4.3) by $|c,\tilde{p},\lambda>$ where $c=(c_1,c_2,...)$ are eigenvalues of Casimir operators, $\tilde{p}=(p_0,p_1,p_2,p_3)$ are eigenvalues of the operators P_μ, and λ is an eigenvalue of W_0, so that

$$\begin{aligned} &C_\alpha|c,\tilde{p},\lambda>=c_\alpha|c,\tilde{p},\lambda>,\\ &P_\mu|c,\tilde{p},\lambda>=p_\mu|c,\tilde{p},\lambda>,\\ &W_0|c,\tilde{p},\lambda>=\lambda p|c,\tilde{p},\lambda>,\quad p=(p_1^2+p_2^2+p_3^2)^{1/2}. \end{aligned} \tag{4.8}$$

Incidentally the eigenvalues of the operators P_μ (which are the basis elements of the Abelian algebra) and C_1 lie in the interval

$$-\infty<p_\mu<\infty,\quad -\infty<c_1<\infty, \tag{4.9}$$

the spectra of the other operators (4.7) will be determined further on.

The numbers c_1 and c_2 (the eigenvalues of the Casimir operators) assume fixed values in spaces of IRs. Following Wigner [413], we distinguish five qualitatively different classes of IRs corresponding to the following values of c_1, c_2, p_μ:

$$\begin{aligned} &I.\quad c_1>0.\\ &II.\quad c_1=0,\quad c_2=0,\quad p_\mu\neq 0,\\ &III.\quad c_1=0,\quad c_2\neq 0,\\ &IV.\quad c_1<0,\\ &V.\quad p_\mu\equiv 0. \end{aligned} \tag{4.10}$$

The main interest (from the physical point of view) is aroused by IRs of Classes *I* and *II* inasmuch as the space of such representations is compared to the state space of a non-interacting relativistic particle with mass $m>0$ and zero mass. But IRs of Classes *III* and *IV* also find some applications - for instance, for describing hypothetical particles with infinite number of spin states [310], nonstable particles and tachyons [239, 240, 390]. As to representations of Class *V*, they are an integral part of any physical theory satisfying the principle of relativistic invariance.

Below we describe all the nonequivalent IRs of the algebra *AP(1,3)* belonging to the first four classes. IRs of Class *V* reduces to the representations of the Lie algebra of the homogeneous Lorentz group. The theory of such representations is expounded with exhaustive completeness in monographs [20, 197]. The necessary facts about finite dimensional representations of the algebra *AO(1,3)* are given in Subsection 4.8.

Let us enumerate the additional Casimir operators existing in any class of IRs and show the corresponding eigenvalues c_1 and c_2 :

$$\begin{aligned} &c_1\geq 0, && C_3=\frac{P_0}{|P_0|}, \\ &c_1=c_2=0, && C_4=W_1/P_1=W_2/P_2=W_3/P_3=W_4/P_4, \\ &c_1<0,\ c_2=c_1 l(l+1), && C_5=W_0/|W_0|, \end{aligned} \tag{4.11}$$

where l are arbitrary integers or half-integers.

For IRs of Class V there are two specific Casimir operators

$$C_6=J_{\mu\nu}J^{\mu\nu},\quad C_7=\frac{1}{8}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}J^{\mu\sigma}. \tag{4.12}$$

There is just one more universal Casimir operator for any class of IRs

$$C_8=\exp(2i\pi J_{12})=\exp(2i\pi J_{13})=\exp(2i\pi J_{23}) \tag{4.13}$$

whose eigenvalues are equal to ±1. The top sign corresponds to simple representations of the Poincaré group, the lower one corresponds to two-valued ones.

4.4. The Explicit Form of the Lubanski-Pauli Vector

Let us obtain an explicit form of the vector W_μ in the basis $|c,\tilde{p},\lambda>$ for any class of IRs of the algebra *AP(1,3)* enumerated above. Using (4.4),(4.8) it is not difficult to obtain the commutation relations for the components W_μ in the basis $|c,\tilde{p},\lambda>$

$$[W_a, W_b]=i\varepsilon_{abc}(p_0 W_c - W_0 p_c),\quad [W_0, W_a]=i\varepsilon_{abc}p_b W_c. \tag{4.14}$$

For any fixed value of $\tilde{p}=(p_0,p_1,p_2,p_3)$ the conditions (4.14) determine some Lie algebra $A_{\tilde{p}}$. Incidentally nonequivalent algebras correspond to different values of $\tilde{p}$.

Our task is a constructive description of all the nonequivalent representations of the algebra (4.14). To simplify the relations (4.14) we use an invertible linear transformation

$$W_\mu\rightarrow W_\mu'=\hat{R}_{\mu\nu}W^\nu,\quad p_\mu\rightarrow p_\mu'=\hat{R}_{\mu\nu}p^\nu. \tag{4.15}$$

As a rule, this transformation is taken in a form which leads to the maximal simplification of the relations (4.14) for any specific class of the vectors p_μ. For example, for $p_\mu p^\mu>0$ a transformation is made to reduce p_μ to the form where $p_a=0$, if $p_\mu p^\mu=0$ then $p'_0=p'_3$, $p'_1=p'_2=0$, etc. (see, e.g., [386]). As a result we obtain such realizations of IRs of the algebra *AP(1,3)* which have essentially different form for different values of c_1. Besides in the case $p_\mu p^\mu \geq 0$, one of the vector p_μ components, for for example, p_3, is always distinguished, although all p_μ are included into the commutation relations (1.14) at an equal footing.

We choose the operator of transformation (4.15) in such a form so that it will maximally simplify the commutation relations (4.14) immediatelly for any type of the four-vector p_μ. Namely we set

$$\hat{R}_{0a}=\hat{R}_{a0}=0, \quad \hat{R}_{00}=1, \quad \hat{R}_{ab}=-R_{ab}, \tag{4.16}$$

where R_{ab} are the matrix elements of the operator of transition to such a reference frame in which $p'_a=n_a p$, $\boldsymbol{n}=(n_1,n_2,n_3)$ is an arbitrary unit vector, and

$$\begin{gathered} R_{ab}=\hat{\boldsymbol{p}}\cdot\boldsymbol{n}\delta_{ab}-\varepsilon_{abc}\theta_c+\theta_a\theta_b(1+\hat{\boldsymbol{p}}\cdot\boldsymbol{n})^{-1}, \\ \theta_a=\varepsilon_{abc}\hat{p}_b n_c, \quad p_a=p_a/p. \end{gathered} \tag{4.17}$$

As a result of the transformation (4.15)-(4.17), the commutation relations (4.14) take the form

$$[W_a', W_b']=i\varepsilon_{abc}(p_0 W_c'-n_c p W_0'), \quad [W_0', W_a']=ip\varepsilon_{abc}n_b W_c'. \tag{4.18}$$

Finally, by setting

$$W_0'=p\lambda_0, \quad W_a'=n_a\lambda_0 p_0+\lambda_a \tag{4.19}$$

we obtain from (4.18) the commutation relations for the operators λ_0, λ_a:

$$\begin{gathered} [\lambda_0,\lambda_a]=i\varepsilon_{abc}n_b\lambda_c, \\ [\lambda_a,\lambda_b]=ic_1\varepsilon_{abc}n_c\lambda_0, \end{gathered} \tag{4.20}$$

where $c_1=p^\mu p_\mu$.

In a space of IRs of the algebra *AP(1,3)* c_1 takes a fixed value (coinciding with the eigenvalue of the Casimir operator C_1) and the commutation relations (4.20) determine some Lie algebra $A(c_1,\boldsymbol{n})$ whose structure constants are dependent on c_1 and $\boldsymbol{n}$. The vector W_μ can be taken into correspondence with any representation of these algebra. Indeed we can obtain by the transformation inverse to (4.15)-(4.17), bearing in mind the relations (4.19), the following:

$$W_0=W_0'=p\lambda_0, \quad W_a=R_{ab}^{-1}W_b'=\lambda_a+\hat{p}_a\lambda_0 p_0-\frac{(\hat{p}_a+n_a)\lambda_b\hat{p}_b}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}}. \tag{4.21}$$

Thus, we have obtained an analytical expression of the Lubanski-Pauli vector via the components of the four-vector p_μ and operators λ_μ realizing a representation of the algebra (4.20). A description of representations of the vector W_μ reduces to describing nonequivalent representations of the algebra $A(c_1,\boldsymbol{n})$.

4.5. IRs of the Algebra $A(c_1,\boldsymbol{n})$

Let us study the structure of the algebra (4.20) and establish its connection with other well known Lie algebras.

It is not difficult to make sure the operators

$$I_1=\lambda_0^2c_1+\lambda_1^2+\lambda_2^2+\lambda_3^2,\quad I_2=\exp(2i\pi\lambda_0) \tag{4.22}$$

commute with λ_μ, i.e., they are the Casimir operators of the algebra (4.20). So, these operators are multiples of the unit operator in a space of an IR and their eigenvalues can be used to numerate IRs. It is not difficult to show that it is necessary to set $\boldsymbol{n}\,\lambda=0$ inasmuch as, according to (4.4), $p_0W'_0\text{-}n_aW'_ap=0$. It follows from this that the algebra (4.20) has three linearly independent elements only. Moreower its structure can be described as follows [157].

THEOREM 4.1. The algebra $A(c_1,\boldsymbol{n})$ of (4.20) is isomorphic to the algebra $AO(3)$ if $c_1>0$, to the algebra $AE(2)$ if $c_1=0$, and to the algebra $AO(1,2)$ if $c_1<0$*.

PROOF. The commutation relations (4.20) are invariant under the transformations $\lambda_a\rightarrow\lambda'_a=r_{ab}\lambda_b$, $n_a\rightarrow n'_a=r_{ab}n_b$, where r_{ab} are elements of an orthogonal matrix. Choosing r_{ab} so that $\boldsymbol{n}'=(0,0,1)$, we come to the following equivalent algebra:

$$\begin{gathered}[\lambda_0,\lambda'_1]=i\lambda'_2,\quad [\lambda_0,\lambda'_2]=-i\lambda'_1,\\ [\lambda_1,\lambda'_2]=ic_1\lambda'_0,\quad \lambda'_3\equiv 0.\end{gathered} \tag{4.23}$$

By setting

$$\begin{gathered}\lambda_0=S_3,\quad \lambda'_1=mS_1,\quad \lambda'_2=mS_2,\quad c_1^2=m^2>0,\\ \lambda_0=T_0,\quad \lambda'_1=T_1,\quad \lambda'_2=T_2,\quad c_1=0\quad ,\\ \lambda_0=S_{12},\quad \lambda_1=\eta S_{01},\quad \lambda_2=\eta S_{02},\quad c_1=-\eta^2<0,\end{gathered} \tag{4.24}$$

we obtain from (4.23) the commutation relations for S_a, T_α, $S_{\alpha\beta}$:

$$[S_a,S_b]=i\varepsilon_{abc}S_c, \tag{4.25}$$

$$[T_0,T_1]=iT_2,\quad [T_0,T_2]=-iT_1,\quad [T_1,T_2]=0, \tag{4.26}$$

$$[S_{01},S_{02}]=-iS_{12},\quad [S_{01},S_{12}]=-iS_{02},\quad [S_{02},S_{12}]=iS_{01}, \tag{4.27}$$

characterizing the algebras $AO(3)$, $AE(2)$ and $AO(1,2)$ accordingly.

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* The symbols $AO(3)$, $AE(2)$, and $AO(1,2)$ denote the Lie algebras of the group of orthogonal 3×3 matrices, the Euclidean group in two-dimensional space, and a group of pseudo-orthogonal matrices in (1+2)-dimensional Minkowski space.

So the isomorphism formulated in the theorem can be given by the relations

$$\lambda_0=\lambda_0^{/}, \quad \lambda_a=R_{ab}^{-1}\lambda_b^{/} \tag{4.28}$$

where λ_μ are connected with the basis elements of the algebras *AO(3), AE(2), AO(1,2)* according to (4.24), $R^{-1}{}_{ab}$ are matrix elements of the rotation operator connecting $\boldsymbol{n'}$=(0,0,1) with $\boldsymbol{n}=(n_1,n_2,n_3)$. The explicit expression for $R^{-1}{}_{ab}$ can be obtained from (4.17) by the substitution $p_a \to pn'_a$. ■

The theorem proved makes it possible to reduce the problem of description of the algebra $A(c_1,\boldsymbol{n})$ representations to describing representations of the Lie algebras determined by the commutation relations (4.25)-(4.27). Representations of these algebras are well known (see, e.g., [386]). A short survey of the main results connecting these representations is given in the following.

a) The algebra *AO(3)*. IRs of it are labelled by positive integers or half integers and are realized by square matrices of dimension $(2s+1)\times(2s+1)$. This algebra has two Casimir operators

$$I_1=S_1^2+S_2^2+S_3^2, \quad I_2=\exp(2i\pi S_3). \tag{4.29}$$

Let $|s,s_3>$ be an eigenfunction of a complete set of the commuting matrices I_1 and S_3, then

$$I_1|s,s_3>=s(s+1)|s,s_3>, \quad I_2|s,,s_3>=(-1)^{2s}|s,,s_3>. \tag{4.30}$$

The explicit form of the matrices S_a in the basis $|s,s_3>$ is defined by the relations

$$\begin{aligned} &S_3|s,s_3>=s_3|s,s_3>, \quad s_3=-s,-s+1,...,s, \\ &(S_1\pm iS_2)|s,s_3>=\sqrt{s(s+1)-s_3(s_3\pm1)}\,|s,s_3\pm1>. \end{aligned} \tag{4.31}$$

b) The algebra *AE(2)* is characterized by the commutation relations (4.26) and has the two Casimir operators

$$I_1=T_1^2+T_2^2, \quad I_2=\exp(2i\pi T_0). \tag{4.32}$$

There are two distinguished classes of IRs of the algebra *AE(2)* corresponding to $I_1=0$ and $I_1=r^2>0$. If $I_1=0$ then

$$T_1=T_2=0, \quad T_0=\tilde{\lambda}, \quad I_2=(-1)^{2\tilde{\lambda}} \tag{4.33}$$

where $\tilde{\lambda}$ are integers or half integers. If $I_1=r^2>0$ then IRs of the algebra *AE(2)* are realized by infinite matrices of the kind

$$T_0|r,n>=n|r,n>, \quad (T_1\pm iT_2)|r,n>=r|r,n\pm1> \tag{4.34}$$

where $|r,n>$ are eigenfunctions of the commuting matrices I_1 and T_0. Besides

$$I_1|r,n>=r^2|r,n>, \quad I_2|r,n>=(-1)^{2n}|r,n>, \quad 0<r<\infty, \tag{4.35}$$

and n runs over either all integer or half integer values.

c) The algebra $AO(1,2)$ (4.27) has two main Casimir operators

$$I_1=S_{12}^2-S_{01}^2-S_{02}^2, \quad I_2=\exp(2i\pi S_{12}). \tag{4.36}$$

Let us denote by $|\alpha,n,\varphi>$ an eigenfunction of the commuting operators I_1, I_2 and S_{12} so that

$$\begin{aligned} &I_1|\alpha,n,\varphi>=\alpha|\alpha,n,\varphi>,\\ &I_2|\alpha,n,\varphi>=\exp(2i\pi\varphi)|\alpha,n,\varphi>,\\ &S_{12}|\alpha,n,\varphi>=(n+\varphi)|\alpha,n,\varphi>. \end{aligned} \tag{4.37}$$

All the possible combinations of values of α, φ and n, corresponding to unitary representations of the group $O(1,2)$, are given by the formulae

$$\begin{aligned} &a)\ \ \varphi=0, \quad -\infty<\alpha<0, \quad n=0,\pm1,\pm2,...;\\ &b)\ \ \varphi=0, \quad \alpha=l(l+1), \quad l=0,1,2,..., \quad n=l+1,l+2,...;\\ &c)\ \ \varphi=0, \quad \alpha=l(l+1), \quad l=0,1,2,..., \quad n=-l-1,-l-2,...;\\ &d)\ \ \varphi=1/2, \quad -\infty<\alpha<-1/4, \quad n=0,\pm1,\pm2,...;\\ &e)\ \ \varphi=1/2, \quad \alpha=l(l+1), \quad l=-1/2,1/2,3/2,..., \quad n=l+1/2,l+3/2,...;\\ &f)\ \ \varphi=1/2, \quad \alpha=l(l+1), \quad l=-1/2,1/2,3/2,..., \quad n=-l-1/2,-l-3/2,... . \end{aligned} \tag{4.38}$$

Therefore, it is possible to distinguish six classes of IRs of the algebra $AO(1,2)$ corresponding to the variants of sets of the Casimir operators eigenvalues given in (4.38). The representations of types b), c) and e), f) correspond to the same sets of eigenvalues of operators (4.36) but are distinguished by values of an additional invariant, i.e., the sign of S_{12}. The explicit form of S_{12} for any representation is given by the last formula (4.37), of $S_{0\alpha}$ - by the following relations:

$$(S_{01}\pm iS_{02})|\alpha,n,\varphi>=i\sqrt{\alpha-(n+\varphi)(n+\varphi\pm1)}\,|\alpha,n\pm1,\varphi>. \tag{4.39}$$

Using the results given above and taking into account the isomorphism of the algebras (4.25)- (4.27) and (4.20) established in Theorem 4.1 it is not difficult to describe IRs of the algebra $A(c_1,\boldsymbol{n})$. As a basis of any such a representation we choose a complete set of eigenfunctions of the commuting operators I_1 and λ_0 . Then the explicit form of the Casimir operators and nonequivalent matrices λ_μ realizing a Hermitian IRs of the algebra $A(c_1,\boldsymbol{n})$ can be given by the formula

$$\begin{aligned} &I_1|c_1,c_2,\lambda>=-c_2|c_1,c_2,\lambda>,\\ &I_2|c_1,c_2,\lambda>=(-1)^{2\lambda}|c_1,c_2,\lambda>, \end{aligned} \tag{4.40a}$$

$$\lambda_0|c_1,c_2,\lambda>=\lambda|c_1,c_2,\lambda>,$$

$$(\lambda_1\pm i\lambda_2)|c_1,c_2,\lambda>=\frac{1}{2}(n_3+1)\sqrt{-c_2-c_1\lambda(\lambda\pm1)}\,|c_1,c_2,\lambda\pm1>+$$

$$+\frac{(n_2\mp in_1)^2}{2(n_3+1)}\sqrt{-c_2-c_1\lambda(\lambda\mp1)}\,|c_1,c_2,\lambda\mp1>, \tag{4.40b}$$

$$n_3\lambda_3|c_1,c_2,\lambda>=-(n_1\lambda_1+n_2\lambda_2)|c_1,c_2,\lambda>.$$

In an IR the parameters c_1 and c_2 take fixed values from the intervals given below, where the value intervals of λ are specified also:

$$c_1=m^2>0,\quad -c_2=c_1s(s+1),\quad \lambda=-s,-s+1,\ldots,s, \tag{4.41}$$

$$c_1=0,\quad c_2=0,\quad \lambda=\tilde{\lambda},$$

$$c_1=0,\quad -c_2=r^2>0,\quad \lambda=0,\pm1,\pm2,\ldots\quad or\quad \lambda=\pm1/2,\pm3/2,\ldots, \tag{4.43}$$

$$c_1=-\eta^2<0,\quad c_2=\eta^2\alpha,\quad \lambda=0,\pm1,\pm2,\ldots\quad or\quad \lambda=\pm1/2,\pm3/2,\ldots,\quad -\infty<\alpha<-1/4,$$

$$c_1=-\eta^2<0,\quad 0<-c_2<\frac{1}{4}\eta^2,\quad \lambda=0,\pm1,\pm2,\ldots, \tag{4.44}$$

$$c_1=-\eta^2<0,\quad c_2=\eta^2l(l+1),\quad \lambda=l+1,l+2,\ldots,$$

$$c_1=-\eta^2<0,\quad c_2=\eta^2l(l+1),\quad \lambda=-l-1,-l-2,\ldots,$$

where $s>0$ and $\tilde{\lambda}$ are arbitrary integers or half integers, l are positive integers or half integers satisfying $-1/2\leq l<\infty$. The values of these numbers in IRs are fixed.

Formulae (4.40)-(4.44) give the explicit forms of all the possible (up to unitary equivalence) Hermitian IRs of the commutation relations (4.20). Together with (4.21), these formulae determine all the nonequivalent realizations of the Lubanski-Pauli vector.

4.6. Explicit Realizations of the Poincaré Algebra

Thus we have obtained all the nonequivalent representations of the vector W_μ. To describe all possible (up to equivalence) IRs of the algebra *AP(1,3)* it is sufficient to show the explicit form of the operators P_μ and $J_{\mu\sigma}$ corresponding to the representations of the Lubanski-Pauli vector found above. It is not difficult to verify such operators can be chosen in the form

$$P_0=p_0,\quad P_a=p_a,\quad \boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\lambda_0\frac{\boldsymbol{n}+\hat{\boldsymbol{p}}}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}}, \tag{4.45a}$$

$$\boldsymbol{N}=-\frac{1}{2}[p_0,\boldsymbol{x}]_+ +\frac{\boldsymbol{\lambda}\times\boldsymbol{p}}{p^2}-\frac{\hat{\boldsymbol{p}}\times\boldsymbol{n}(\lambda_0 p_0-\boldsymbol{\lambda}\cdot\hat{\boldsymbol{p}})}{p+\boldsymbol{n}\cdot\boldsymbol{p}},\tag{4.45b}$$

where components of the vectors $\boldsymbol{J}$ and $\boldsymbol{N}$ are connected with $J_{\mu\sigma}$ as follows: $J_a=1/2\varepsilon_{abc}J_{bc}$, $N_a=J_{0a}$; p_0 and p_a are real variables connected by the relation

$$p_0=\varepsilon\sqrt{p^2+c_1}\,,\quad \varepsilon=\pm1,\tag{4.46}$$

c_1 is an arbitrary real number, and $x_a=i\partial/\partial p_a$.

For IRs of Classes *I-III* (corresponding to $c_1\geq0$) ε assumes a fixed value, but for $c_1<0$ the energy sign is not fixed.

It is not difficult to verify that the operators (4.45) satisfy the commutation relations (4.15) and correspond to the Lubanski-Pauli vector in the form (4.21). Inasmuch as the formula (4.21) gives a general form (up to equivalence) of this vector for Classes *I-IV* of IRs, the operators (4.45), (4.46) form a basis for any of these representations. Representations of the algebra *AP(1,3)* being realized by the operators (4.45), (4.46) are irreducible inasmuch as the corresponding Casimir operators (4.11), (4.12) are multiples of the unit operator.

The operators (4.45) are Hermitian in respect to the scalar product

$$(\Phi_1,\Phi_2)=\sum_\lambda\int d^3p\,\Phi_1^\dagger(\boldsymbol{p},\lambda)\Phi_2(\boldsymbol{p},\lambda),\tag{4.47}$$

where Φ_α belong to the space of functions decreasing sufficiently fast along the direction $\boldsymbol{p}=-\boldsymbol{n}p$ and λ takes all the possible values coinciding with the matrix λ_0 eigenvalues.

The results given above are formulated in the following form.

THEOREM 4.2. IRs of the algebra *AP(1,3)* are labelled by the sets of eigenvalues c_1, c_2, ... of the Casimir operators C_1, C_2 (4.6) and of C_3 (if $c_1\geq0$), C_4 (if $c_1=c_2=0$), C_5 (if $c_1<0$, $c_2<0$), and C_8 (4.11), (4.19). All the admissible combinations of the eigenvalues c_1, c_2 and $c_4=\lambda$ are given in formulae (4.41)-(4.44), but eigenvalues of the operators C_3, C_5 and C_8 are equal to ±1. The explicit expression of the corresponding basis elements of the algebra *AP(1,3)* can be chosen in the form (4.45) (up to unitary equivalence) where λ_μ are the matrices (4.40)-(4.44). ■

So we have calculated the explicit expressions of the basis elements of IRs of the algebra *AP(1,3)*. Operators (4.45) have a relatively simple form which is common for all classes of IRs. It differs favourably our realization from the others known already.

4.7. Connections With the Canonical Realizations of Shirokov-Foldy-Lomont-Moses

Let consider each class of IRs more precisely.

1. Representations of Class *I* corresponding to $P_\mu P^\mu > 0$ have the additional Casimir operator, i.e., the energy sign operator with eigenvalues ±1. The corresponding representation space is split into two subspaces, each of them corresponding to the fixed sign of p_0.

So IRs of Class *I* are labelled by three numbers: m^2, s, and ε, refer to (4.41), and are realized in the space of square integrable functions $\Phi(\boldsymbol{p},\lambda)$ having dimension $2(s+1)$ with respect to the index λ. Besides, the Casimir operator (4.13) takes the value +1 for integer and -1 for half integer s. We denote such IRs by $D^\varepsilon(ms)$.

With the help of the unitary transformation

$$(P_\mu, \boldsymbol{J}, \boldsymbol{N}) \rightarrow (P'_\mu, \boldsymbol{J}', \boldsymbol{N}') = U(P_\mu, \boldsymbol{J}, \boldsymbol{N})U^\dagger \tag{4.48}$$

where

$$\begin{gathered} U = \exp(-i\boldsymbol{\lambda}\cdot\boldsymbol{p}\times\boldsymbol{n}\theta_1)\exp(i\lambda_0\pi)\exp(i\boldsymbol{\lambda}\times\boldsymbol{n}\cdot\boldsymbol{n}'\theta_2), \\ \theta_1 = \arccos\hat{\boldsymbol{p}}\cdot\boldsymbol{n}, \quad \theta_2 = \arccos n_3, \quad \boldsymbol{n}' = (0,0,1), \end{gathered} \tag{4.49}$$

the operators (4.45) realizing an IR of Class *I* can be transformed to the canonical form of Shirokov-Foldy [386, 106]:

$$\begin{gathered} P'_0 = \varepsilon E = \varepsilon\sqrt{p^2+m^2}, \quad P'_a = p_a, \\ \boldsymbol{J}' = \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{S}, \quad \boldsymbol{N}' = -\frac{\varepsilon}{2}[\boldsymbol{x},E]_+ - \varepsilon\frac{\boldsymbol{p}\times\boldsymbol{S}}{E+m}. \end{gathered} \tag{4.50}$$

Here $\boldsymbol{S}$ are generators of the IR $D(s)$ of the group $O(3)$ given in (4.31).

2. Let us consider the case $c_1=c_2=0$. There exist two additional Casimir operators for the corresponding representations, i.e., the energy sign operator C_3 and helicity operator C_4 of (4.11). We obtain from (4.11), (4.21) $C_4=\varepsilon\lambda_0$, and conclude from (4.42) that eigenvalues of the helicity operator are equal to $\varepsilon\tilde{\lambda}$ where $\tilde{\lambda}$ is an arbitrary integer or half integer.

So IRs of Class *II* are labelled by two numbers, ε and $\tilde{\lambda}$ and are one -dimensional in respect with the index λ. The Casimir operator (4.12) has the eigenvalue +1 for integer $\tilde{\lambda}$ and -1 if $\tilde{\lambda}$ are half integers. Besides, the basis elements of the Poincaré algebra (4.40), (4.42), (4.45) take the form

$$P_0 = \varepsilon p, \quad P_a = p_a, \tag{4.51a}$$

$$J=\boldsymbol{x}\times\boldsymbol{p}+\tilde{\lambda}\frac{\hat{\boldsymbol{p}}+\boldsymbol{n}}{1+\boldsymbol{p}\cdot\boldsymbol{n}},$$
$$\boldsymbol{N}=-\frac{1}{2}\varepsilon[p,\boldsymbol{x}]_{+}-\varepsilon\hat{\lambda}\frac{\hat{\boldsymbol{p}}\times\boldsymbol{n}}{1+\hat{\boldsymbol{p}}\cdot\boldsymbol{n}}. \tag{4.51b}$$

The operators (4.51) have a symmetric and compact form which differs them favourably from other realizations known. Choosing different unit vectors $\boldsymbol{n}$, we obtain from (4.51) different (but equivalent) realizations of IRs of the algebra *AP(1,3)*. Taking $\boldsymbol{n}=(0,0,1)$, we come to a representation in the Lomont-Moses [287] form, but if $\boldsymbol{n}=(1,1,1)/\sqrt{3}$ the above formulae give the realization proposed in [154,157]*.

The transition from a representation characterized by a vector $\boldsymbol{n}$ to an a equivalent representation corresponding to a vector $\boldsymbol{n}'$, $\boldsymbol{n}'\neq-\boldsymbol{n}$, can be carried out by the unitary transformation (4.48) where

$$U=\exp\left(2i\lambda_0\arctan\frac{\boldsymbol{p}\cdot\boldsymbol{n}\times\boldsymbol{n}'}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}+\boldsymbol{n}'\cdot\hat{\boldsymbol{p}}+\boldsymbol{n}\cdot\boldsymbol{n}'}\right). \tag{4.52}$$

If $\boldsymbol{n}'=-\boldsymbol{n}$ then

$$U=\exp\left(2i\lambda_0\arctan\frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{n}\times\boldsymbol{n}''}{\hat{\boldsymbol{p}}\cdot\boldsymbol{n}''-(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}'')}\right) \tag{4.53}$$

where $\boldsymbol{n}''\neq\pm n$ is an arbitrary unit vector.

We note that for representations of Class *II* the following relations are valid:

$$W_\mu=\frac{P_0}{P}\frac{\boldsymbol{J}\cdot\boldsymbol{P}}{P}P_\mu,\quad P=(P_1^2+P_2^2+P_3^2)^{1/2}, \tag{4.54}$$

which take the following form for IRs:

$$W_\mu=\varepsilon\lambda P_\mu. \tag{4.55}$$

The condition (4.55) is necessary and sufficient for the corresponding representation to belong to Class *II*.

As was noted in Section 3.7, representations of the Poincaré algebra belonging to Class *II* can be extended to the representations of the algebra *AC(1,3)*. Substituting (4.51) into (3.48), we obtain the corresponding generators K_μ and D in the form

$$D=\frac{1}{2}(\boldsymbol{p}\cdot x+\boldsymbol{x}\cdot\boldsymbol{p}),\quad K_0=\frac{\varepsilon}{2}[p,\boldsymbol{x}^2]+2\lambda\varepsilon\frac{\boldsymbol{n}\cdot\boldsymbol{J}}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}},\quad K_a=i[K_0,N_a]. \tag{4.56}$$

Formulae (4.51), (4.56) set basis elements of the representation of the

*The operators $\boldsymbol{N}$ considered in [154] include the term $p\boldsymbol{x}$ instead of $[p,\boldsymbol{x}]_+$ which is caused by differently choosing a scalar product.

conformal algebra, which reduces to the IR of the Poincaré algebra by the reduction with respect to the subalgebra *AP(1,3)*. When $\boldsymbol{n}$=(0,0,1) these formulae determine an IR of the algebra *AC(1,3)* in the Bose-Parker [50] realization.

3. Representations of Class *III* have four Casimir operators, i.e., C_1, C_2, C_3 and C_8 given by formulae (4.6), (4.11), (4.13). Besides, the eigenvalues of C_1 are equal to zero and IRs are labelled by triplets of eigenvalues of the operators C_2, C_3 and C_8:

$$-c_2=r^2>0, \quad c_3=\varepsilon=\pm 1, \quad c_8=\pm 1.$$

An explicit form of operators forming a basis of an IR of the algebra *AP(1,3)* can be obtained from (4.45) by the substitution $p_0=\varepsilon p$ using the corresponding expressions (4.40), (4.42) for the matrices λ_μ. Besides, IRs with integer n correspond to c_8=1, and with half integer n to c_8=-1. The corresponding functions $\Phi(\boldsymbol{p},\lambda)$ forming the representation space are infinite dimensional with respect to the index λ which takes denumerable values. When $\boldsymbol{n}$=(0,0,1) formulae (4.45), (4.46) give the basis elements of the algebra *AP(1,3)* in the Lomont-Moses form [287] (besides, $\lambda_0\to T_0$, $\lambda_1\to T_1$, $\lambda_2\to T_2$, $\lambda_3\to 0$, T_0, T_1 and T_2 being the matrices (4.34).

4. Representations of Class *IV* are labelled by a set of eigenvalues of three main Casimir operators C_1, C_2 and C_8. The eigenvalues of C_8 are equal to ± 1 where the top (lower) sign corresponds to integer (half integer) λ. Possible values of C_1, C_2 and λ are given in (4.44). In the case $C_1>0$ there exist the additional Casimir operator $C_5=W_0/|W_0|=sign\lambda_0$ so the space of the corresponding representation is split into two invariant subspaces, each of them having all λ (the matrix λ_0 eigenvalues) with the same sign.

All the Hermitian IRs of Class *IV* are infinite dimensional in respect with the index λ taking denumerable values. With the help of the unitary transformation (4.48) where

$$U=\exp\left(i\frac{S_{0\alpha}p_\alpha}{|p|}\arctan\frac{p_0|p|}{(p_3+\eta)(p_0^2+\eta^2)}\right)U', \quad |p|=(p_1^2+p_2^2)^{1/2}, \quad \alpha=1,2, \tag{4.57}$$

(where U' is the operator of (4.52) for $\boldsymbol{n}'$=(0,0,1), $S_{\alpha\beta}$ are matrices (4.24)), (4.39), the operators (4.45) realizing an IR of Class *IV* can be reduced to the canonical form of Shirokov [386].

So we established the connection of IRs in the form (4.45) with the well-known "canonical" realizations of IRs of the algebra *AP(1,3)*. Other realizations of IRs of the Poincaré algebra are discussed in survey [29].

4.8. Covariant Representations.

In the set of all the possible realizations of representations of the algebra *AP(1,3),* an outstanding role is played by so-called covariant representations which are characterized by the form of basis elements given in (2.22) where $S_{\mu\sigma}$ are numeric matrices. The necessary and sufficient condition for the operators (2.22) to realize a representation of the Poincaré algebra (i.e., satisfy commutation relations (1.14)) is that the matrices $S_{\mu\sigma}$ should satisfy the relations (2.18b).

Formulae (2.22) determine a general form of basis elements of the algebra *AP(1,3)* belonging to the class M_1. Particular examples of such realizations of the algebra *AP(1,3)* are the IA of the KGF, Dirac and Maxwell equations considered in Chapter 1. The operators (2.22) with arbitrary matrices $S_{\mu\sigma}$ generate local transformations of the kind given in (2.49). It follows from the fact that the "spin" part of $J_{\mu\sigma}$ (i.e., the matrices $S_{\mu\sigma}$) commutes with the "orbital" part $x_\mu p_\sigma - x_\sigma p_\mu$ (refer to Section 2.9).

We note that the operators (2.22) (in contrast with (4.45)) realize a reducible representation of the algebra *AP(1,3)* which is not generally speaking Hermitian for the case of finite dimensional matrices $S_{\mu\sigma}$.

If we are not interested in refinements connected with different choosing of functional spaces of representations, then, to describe all the covariant representations of the algebra *AP(1,3)*, it suffices to present all the nonequivalent realizations of the matrices $S_{\mu\sigma}$ satisfying relations (2.18b). We restrict ourselves to the case of finite dimensional matrices.

The algebra (2.18b) has the two Casimir operators (4.12) whose eigenvalues label IRs. To describe a domain of these eigenvalues we consider the following operators

$$j_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} + iS_{0a}\right), \quad \tau_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} - iS_{0a}\right) \tag{4.58}$$

which satisfy the following commutation relations:

$$[j_a, j_b] = i\varepsilon_{abc}j_c, \quad [\tau_a, \tau_b] = i\varepsilon_{abc}\tau_c, \quad [j_a, \tau_b] = 0.$$

We note that the algebra *AO(1,3)* is split into two commuting subalgebras formed by the matrices j_a and τ_a. The commutation relations between j_a (and between τ_a) coincide with the commutation relations (4.25) determining the algebra *AO(3)*. We concludes from this that finite dimensional IRs of the algebra *AO(1,3)* are labelled by positive integers or half integers j and τ. The basis of a space of an IRs is formed by $(2j+1)\times(2\tau+1)$ eigenfunctions of the complete set of the commuting operators j^2, τ^2, j_3 and τ_3. Using for these eigenfunctions the notation $|j,m;\tau,n>$, we can represent the action of the operators j_a and τ_a in the following form (compare with (4.31)):

$$\begin{aligned}
&\boldsymbol{j}^2|j,m;\tau,n\rangle=j(j+1)|j,m;\tau,n\rangle,\\
&j_3|j,m;\tau,n\rangle=m|j,m;\tau,n\rangle, \qquad (4.59)\\
&(j_1\pm ij_2)|j,m;\tau,n\rangle=\sqrt{j(j+1)-m(m\pm1)}\,|j,m;\tau,n\rangle,
\end{aligned}$$

$$\begin{aligned}
&\tau^2|j,m;\tau,n\rangle=\tau(\tau+1)|j,m;\tau,n\rangle,\\
&\tau_3|j,m;\tau,n\rangle=n|j,m;\tau,n\rangle, \qquad (4.60)\\
&(\tau_1\pm i\tau_2)|j,m;\tau,n\rangle=\sqrt{\tau(\tau+1)-n(n\pm1)}\,|j,m;,\tau,n\rangle,
\end{aligned}$$

where j,m (τ,n) are simultaneously integers or half integers, and $-j\leq m\leq j$, $-\tau\leq n\leq\tau$.

So, finite dimensional IRs of the algebra $AP(1,3)$ are realized by square matrices of dimension $(2j+1)(2\tau+1)\times(2j+1)(2\tau+1)$, whose elements are given in (4.58)-(4.60). We denote these representations by $D(j\ \tau)$.

Sometimes it is more convenient to use the $O(3)$-basis $|l_1,l_2;l,m\rangle$ in which the matrix $\boldsymbol{S}^2=(1/2)S_{ab}S_{ab}$ is diagonal. The numbers l_1 and l_0 are connected with j and τ by the relation

$$\varepsilon j=(l_0+l_1-\varepsilon)/2, \quad \varepsilon\tau=(l_1-l_0-\varepsilon)/2, \quad \varepsilon=sign(j-\tau) \qquad (4.61)$$

and set the eigenvalues of the Casimir operators according to the relations

$$\begin{aligned}
&\frac{1}{2}S_{\mu\nu}S^{\mu\nu}|l_0,l_1;l,m\rangle=(l_0^2+l_1^2-1)|l_0,l_1;l,m\rangle,\\
&\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}|l_0,l_1;l,m\rangle=2il_0l_1|l_0,l_1;l,m\rangle.
\end{aligned} \qquad (4.62)$$

The numbers l and m determine eigenvalues of $\boldsymbol{S}^2$ and S_{12} in the following way:

$$\begin{aligned}
&\boldsymbol{S}^2|l_0,l_1;l,m\rangle=l(l+1)|l_0,l_1;,l,m\rangle,\\
&S_{12}|l_0,l_1;l,m\rangle=m|l_0,l_1;,l,m\rangle
\end{aligned} \qquad (4.63)$$

where $l=l_0,l_0+1,\ldots,\ |l_1|-1$ and $m=-l,-l+1,\ldots,l$. The explicit expressions of the matrices realizing an IR are given by the formulae

$$\begin{aligned}
&S_{ab}|l_0,l_1;l,m\rangle=\varepsilon_{abc}(S_c^l)_{mm'}|l_0,l_1;l,m'\rangle,\\
&S_{0a}|l_0,l_1;l,m\rangle=[\delta_{l-1l'}C_l(K_a^l)_{mm'}+\delta_{ll'}A_l(S_a^l)_{mm'}+\delta_{l+1l'}C_{l+1}(K_a^{l+1})^\dagger_{mm'}]|l_0,l_1;l',m'\rangle, \qquad (4.64)\\
&A_l=\frac{il_0l_1}{l(l+1)}, \quad C_l=\frac{i}{l}\sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}}\,,
\end{aligned}$$

and the matrix elements $(S_a^l)_{mm'}$ and $(K_a^l)_{mm'}$ are defined by the following relations:

Nonequivalent representations of the finite group formed by these operators and their products are encountered, and the explicit form of these operators in the space of the "universal" representation (4.45) of the algebra *AP(1,3)* is found.

We will start with the discrete symmetry operators defined on the set of the KGF equation solutions (refer to (1.51)-(1.53)). These operators and their products form the group G_8 including the following eight elements:

$$G_8 \in I,\ P,\ T,\ C,\ PT,\ PC,\ TC,\ CPT \tag{5.1}$$

where I denotes the identity transformation. The group multiplication law for the group G_8 can be obtained from (1.51)-(1.53). It is presented in the following table:

Table 5.1

	I	*P*	*T*	*C*	*PT*	*CT*	*CP*	*CPT*
I	*I*	*P*	*T*	*C*	*PT*	*CT*	*CP*	*CPT*
P	*P*	*I*	*PT*	*PC*	*T*	*CPT*	*C*	*CT*
T	*T*	*PT*	*I*	*CT*	*P*	*C*	*CPT*	*CP*
C	*C*	*CP*	*CT*	*I*	*CPT*	*T*	*P*	*PT*
PT	*PT*	*T*	*P*	*CPT*	*I*	*CP*	*CT*	*C*
CT	*CT*	*CPT*	*C*	*T*	*CP*	*I*	*PT*	*P*
CP	*CP*	*C*	*CPT*	*P*	*CT*	*PT*	*I*	*T*
CPT	*CPT*	*CT*	*CP*	*PT*	*C*	*P*	*T*	*I*

The problem of description of nonequivalent operators P, T, and C, defined in a representation space of the algebra *AP(1,3)*, includes a description of representations of the group G_8. Besides these operators must satisfy the commutation and anticommutation relations (1.54). It follows from the above the operators P, C, T, and the Casimir operators (4.11), (4.19) satisfy the relations

$$\begin{gathered} PC_1=C_1P, \quad PC_2=C_2P, \quad PC_3=C_3P, \quad PC_4=-C_4P, \\ PC_5=-C_5P, \quad PC_6=C_6P, \quad PC_7=-C_7P, \quad PC_8=C_8P; \end{gathered} \tag{5.2}$$

$$\begin{gathered} TC_1=C_1T, \quad TC_2=C_2T, \quad TC_3=-C_3T, \quad TC_4=C_4T, \\ TC_5=C_5T, \quad TC_6=C_6T, \quad TC_7=-C_7T, \quad TC_8=C_8T; \end{gathered} \tag{5.3}$$

$$CC_1=C_1C, \quad CC_2=C_2C, \quad CC_3=-C_3C, \quad CC_4=C_4C,$$
$$CC_5=C_5C, \quad CC_6=C_6C, \quad CC_7=C_7C, \quad CC_8=C_8C. \tag{5.4}$$

According to (5.2)-(5.4), the operators *P*, *C*, *T* can be defined in a space of reducible representation only (inasmuch as they change the signs of eigenvalues of the Casimir operators).

The description of nonequivalent realizations of the discrete symmetry operators reduces in our approach to finding all the representations of the group G_8. In addition, the operators *P*, *C*, *T* must satisfy the relations (1.54) where P_μ and $J_{\mu\sigma}$ are basis elements of a direct sum of IRs of the algebra *AP(1,3)*.

5.2. Nonequivalent Multiplicators of the Group G_8

As is well known the state vector of a physical system is represented in quantum mechanics as a ray of a Hilbert space. In other words this vector is defined up to a phase factor $exp(i\alpha)$ where α is a *c*-number. This situation predetermines the fundamental role of a ray (projective) representations of symmetry groups, i.e., such representations for which the group law is valid up to a phase factor only.

Ray representations of the proper Poincaré group always can be reduced to exact representations, so no new possibilities exist here. But in the case of the complete Poincaré group including space-time reflections there exist ray representation which cannot be reduced to exact ones.

Here we consider ray representations of the group G_8. Specifically we consider projective unitary and antiunitary representations (*PUA*- representations) of this group which are defined as follows.

We denote by the symbol $\mathbb{C}$ the field of complex numbers, by $U(\mathbb{C})$ - the multiplicative group of complex numbers with the unit module, and by $\mathbb{R}$ the field of real numbers.

A mapping $T: G \to UA(H)$ of a group G into multiplicative group $UA(H)$ of all unitary and antiunitary operators T_g, $g\in G$ of space H is called *PUA-representation of a group G in a Hilbert space H*, if T_g satisfy the conditions

$$T_{g_1}T_{g_2}=\lambda(g_1,g_2)T_{g_1g_2}, \tag{5.5}$$

where $g_1, g_2 \in G$, $\lambda(g_1,g_2) \in U(\mathbb{C})$, and, besides,

$$\lambda(g_1g_2,g_3)\lambda(g_1,g_2g_3)=\lambda(g_1,g_2g_3)\tilde{\lambda}(g_2,g_3), \tag{5.6}$$

where $\tilde{\lambda}(g_2,g_3)=\lambda(g_2,g_3)$ if T_{g_1} is a unitary operator and $\tilde{\lambda}(g_2,g_3)=-\lambda(g_2,g_3)$ if T_{g_1} is an antiunitary one.

A system of elements $\lambda(g_1,g_2)$ is called a multiplicator of a representation *T*. In the case of usual (exact) representations $\lambda(g_1,g_2)\equiv 1$ for any $g_1,g_2\in$ G.

PUA-representation differs from the regular exact representations by two features:

a) They include antiunitary operators besides unitary ones. Antiunitary operator by definition transforms all quantities into complex conjugated ones (for two examples of antiunitary operator of charge conjugation, refer to (1.53), (2.60)).

b) The group multiplication law for elements of *PUA*-representation is defined up to a phase factor $\lambda(g_1,g_2)$ which do not change the norm of a vector from a Hilbert space H.

The definitions given above (which go back to Wigner) lead to a new concept of representations equivalence. Namely *PUA*-representations T and T' is said to be equivalent if there exists such a unitary or antiunitary operator V and such a function $\alpha(g)\in U(\mathbb{C})$ that

$$T_g^{\prime}=\alpha(g)VT_gV^{-1} \tag{5.7}$$

for any $g\in G$. If $\alpha(g)\equiv 1$ then an equivalence is called regular, otherwise it is called projective. Below we will omit the term "regular" when speaking about a regular equivalence.

In the case, when an operator V in (5.7) is unitary, the multiplicators λ and λ' of representations T and T' are connected by the relation

$$\lambda^{\prime}(g_1,g_2)=\frac{\alpha(g_1)\tilde{\alpha}(g_2)}{\alpha(g_1g_2)}\lambda(g_1,g_2), \tag{5.8}$$

where $\tilde{\alpha}(g_2)=\alpha(g_2)$ if T_{g_1} is unitary, and $\tilde{\alpha}(g_2)=\alpha^*(g_2)$ if T_{g_1} is antiunitary operator.

If the operator V in (5.7) is antiunitary then the condition (5.8) is satisfied by multiplicators λ' and λ^*.

The relations (5.6), (5.8) can be used to describe all the possible nonequivalent multiplicators characterizing representations of a group. For the group G_8 such a description is given by the following assertion.

THEOREM 5.1. All the nonequivalent multiplicators of the group G_8 are present in Table 5.2 (see the following page) where $\mu_{ab}=\mu_a\mu_b$, $\mu_{abc}=\mu_a\mu_b\mu_c$, $\mu_{1234}=\mu_1\mu_2\mu_3\mu_4$ and μ_a (a=1,2,3,4) accept the values ±1 independently.

We do not present a proof (see [12]). ■

According to Theorem 5.1, the group G_8 has 16 nonequivalent multiplicators $\lambda_{\mu_1\mu_2\mu_3\mu_4}$ corresponding to the possible combinations of values of μ_a. These multiplicators determine all the possible (up to equivalence) commutation and anticommutation relations for representations of the operators P, C, and T.

Let $\hat{P}$, $\hat{T}$ and $\hat{C}$ be basis elements of a *PUA*-representation of the group G_8 with the multiplicator $\lambda_{\mu_1\mu_2\mu_3\mu_4}$. Then we obtain from (5.5) and Table 5.2 that

$$\hat{P}^2=I, \quad \hat{T}^2=I, \quad \hat{C}^2=\mu_4 I,$$
$$\hat{C}\hat{P}=\mu_2\hat{P}\hat{C}, \quad \hat{C}\hat{T}=\mu_1\hat{T}\hat{C}, \quad \hat{P}\hat{T}=\mu_3\hat{T}\hat{P}. \tag{5.9}$$

Table 5.2

	I	P	T	C	PT	CT	CP	CPT
I	1	1	1	1	1	1	1	1
P	1	1	1	μ_2	1	μ_2	μ_2	μ_2
T	1	μ_3	1	μ_1	μ_3	μ_1	μ_{13}	μ_{13}
C	1	1	1	μ_4	1	μ_4	μ_4	μ_4
PT	1	μ_3	1	μ_{12}	μ_3	μ_{12}	μ_{123}	μ_{123}
CT	1	μ_3	1	μ_{14}	μ_3	μ_{14}	μ_{134}	μ_{134}
CP	1	1	1	μ_{24}	1	μ_{24}	μ_{24}	μ_{24}
CPT	1	μ_3	1	μ_{124}	μ_3	μ_{124}	μ_{1234}	μ_{1234}

We see that up to equivalence the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ must commute or anticommute with each other. Squared operators of space and time inversions and of charge conjugation are equal to the unit operator I or to $\pm I$.

It is not difficult to make sure that the conditions (5.9) can be considered as a definition of a multiplicator. In other words, starting from (5.9) and using (5.5) it is not difficult to find all the possible multiplicators given in Table 5.2.

A set of the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ and their possible products form a finite multiplicative group $G(\mu_1,\mu_2,\mu_3,\mu_4)$ of dimension 8 if $\mu_1=\mu_2=\mu_3=\mu_4=1$, and 16 in other cases. There are 16 such groups corresponding to different sets of the parameters μ_a. A description of *PUA*-representations of the group G_8 reduces to the description of regular representations of the group $G(\mu_1,\mu_2,\mu_3,\mu_4)$.

5.3. The General Form of the Discrete Symmetry Operators

In addition to relations (5.9), the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ must satisfy the following conditions (refer to (1.54)):

$$\hat{P}\boldsymbol{J}=\boldsymbol{J}\hat{P}, \quad \hat{P}P_0=P_0\hat{P}, \quad \hat{P}\boldsymbol{P}=-\boldsymbol{P}\hat{P}, \quad \hat{P}\boldsymbol{N}=-\boldsymbol{N}\hat{P}, \tag{5.10a}$$

where P_0, $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{N}$ are the basis elements of the Poincaré algebra.

$$\hat{T}P_0=-P_0\hat{T},\quad \hat{T}\boldsymbol{P}=\boldsymbol{P}\hat{T},\quad \hat{T}\boldsymbol{J}=\boldsymbol{J}\hat{T},\quad \hat{T}\boldsymbol{N}=-\boldsymbol{N}\hat{T},$$
$$\hat{C}P_0=-P_0\hat{C},\quad \hat{C}\boldsymbol{P}=-\boldsymbol{P}\hat{C},\quad \hat{C}\boldsymbol{J}=-\boldsymbol{J}\hat{C},\quad \hat{C}\boldsymbol{N}=-\boldsymbol{N}\hat{C},\tag{5.10b}$$

We will start with the universal realizations of the algebra *AP(1,3)* given in (4.45). According to (5.2)-(5.4), it is sufficient to restrict ourselves to considering the representations corresponding to fixed eigenvalues to the Casimir operators C_1 and C_2, while the operators C_3, C_4, C_5 and C_7 not being chosen to be multiples of the unit operator. Restricting ourselves for the time being by considering the representations of Classes *I-III* we conclude that P_μ, $\boldsymbol{J}$, and $\boldsymbol{N}$ can be chosen in the form

$$P_0=\hat{\varepsilon}E,\quad \boldsymbol{P}=\boldsymbol{p},\quad \boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\lambda_0\frac{\boldsymbol{n}+\hat{\boldsymbol{p}}}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}},\quad \hat{\boldsymbol{p}}=\frac{\boldsymbol{p}}{p},$$
$$\boldsymbol{N}=-\frac{1}{2}\hat{\varepsilon}[E,\boldsymbol{x}]_++\frac{\lambda\times\boldsymbol{p}}{p^2}-\frac{\hat{\boldsymbol{p}}\times\boldsymbol{n}(\hat{\varepsilon}\lambda_0E-\lambda\cdot\hat{\boldsymbol{p}})}{p+\boldsymbol{n}\cdot\boldsymbol{p}},\tag{5.11}$$

where $\hat{\varepsilon}$ is the energy sign operator having the eigenvalues ±1 and λ_μ are the matrices realizing a reducible representation of the algebra (4.20) (corresponding to fixed values c_1 and c_2 in (4.41)-(4.45). Moreover

$$[\hat{\varepsilon},\lambda_\mu]=0,\qquad \hat{\varepsilon}^2=1,\tag{5.12}$$

and the operator $\hat{\varepsilon}$ can be chosen in the form of a diagonal matrix, and λ_μ - in the form of a direct sum of irreducible matrices (4.40).

Let $\Psi(\boldsymbol{p})$ be a vector from the space of representation of the algebra *AP(1,3)* belonging to one of three first classes ($P_\mu P^\mu\geq0$). The transformations $\hat{P}$, $\hat{T}$, and $\hat{C}$ without loss of generality can be represented in the form

$$\hat{P}\Psi(\boldsymbol{p})=U_1\Psi(-\boldsymbol{p}),\qquad \hat{T}\Psi(\boldsymbol{p})=U_2\Psi(\boldsymbol{p}),\quad \hat{C}\Psi(\boldsymbol{p})=U_3\Psi^*(-\boldsymbol{p}),\tag{5.13}$$

where U_1, U_2, U_3 are operators satisfying (according to (4.10)) the following conditions

$$U_a\boldsymbol{p}=\boldsymbol{p}U_a,\qquad a=1,2,3,\tag{5.14}$$
$$U_1P_0=P_0U_1,\qquad U_2P_0=-P_0U_2,\qquad U_3P_0=-P_0U_3,$$
$$U_1\boldsymbol{J}=\boldsymbol{J}'U_1,\qquad U_2\boldsymbol{J}=\boldsymbol{J}U_2,\qquad U_3\boldsymbol{J}=-\boldsymbol{J}'^*U_3,$$
$$U_1\boldsymbol{N}=-\boldsymbol{N}'U_1,\qquad U_2\boldsymbol{N}=-\boldsymbol{N}U_2,\qquad U_3\boldsymbol{N}=-\boldsymbol{N}'^*U_3,\tag{5.15}$$

Here $\boldsymbol{J}'$ and $\boldsymbol{N}'$ are the operators obtained from $\boldsymbol{J}$ and $\boldsymbol{N}$ by the change $\boldsymbol{p}\to-\boldsymbol{p}$, $\boldsymbol{x}\to-\boldsymbol{x}$.

One concludes from (5.14) that U_a do not include operators of differentiation with respect to $\boldsymbol{p}$ and so can be considered as matrices depending on $\boldsymbol{p}$. Then we obtain from (5.9) that

So the general form of the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ defined in the spaces of the

$$
\begin{aligned}
&U_1(\boldsymbol{p})U_2(-\boldsymbol{p})=\mu_3U_2(\boldsymbol{p})U_1(-\boldsymbol{p}),\\
&U_1(\boldsymbol{p})U_3(-\boldsymbol{p})=\mu_2U_3(\boldsymbol{p})U_1^*(-\boldsymbol{p}),\\
&U_2(\boldsymbol{p})U_3(\boldsymbol{p})=\mu_1U_3(\boldsymbol{p})U_2^*(-\boldsymbol{p}),\\
&U_1(\boldsymbol{p})U_1(-\boldsymbol{p})=U_2^2(\boldsymbol{p})=1,\quad U_3(\boldsymbol{p})U_3^*(-\boldsymbol{p})=\mu_4.
\end{aligned}
\tag{5.16}
$$

algebra *AP(1,3)* representations of Classes *I-III* is given by formulae (5.13) where U_a are matrices satisfying (5.16). The problem of finding of all nonequivalent operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ reduces to finding the irreducible sets of matrices U_a.

For representations of Class *IV* the energy sign operator is not a Casimir operator so operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ have to be defined in another way. A vector from this representation space is an infinite component column with components $\Psi_\lambda(\boldsymbol{p},\varepsilon)$ with λ running over the values given in (4.44). Denoting this column by $\Psi(\boldsymbol{p},\varepsilon)$ we can represent the corresponding operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ in the form

$$
\begin{aligned}
&\hat{P}\Psi(\boldsymbol{p},\varepsilon)=U_1(\boldsymbol{p},\varepsilon)\Psi(-\boldsymbol{p},\varepsilon),\\
&\hat{T}\Psi(\boldsymbol{p},\varepsilon)=U_2(\boldsymbol{p},\varepsilon)\Psi(\boldsymbol{p},-\varepsilon),\\
&\hat{C}\Psi(\boldsymbol{p},\varepsilon)=U_3(\boldsymbol{p},\varepsilon)\Psi^*(-\boldsymbol{p},-\varepsilon),
\end{aligned}
\tag{5.17}
$$

as a result we come to the relations

$$
\begin{aligned}
&U_1(\boldsymbol{p},\varepsilon)U_2(-\boldsymbol{p},\varepsilon)=\mu_3U_2(\boldsymbol{p},\varepsilon)U_1(-\boldsymbol{p},\varepsilon),\\
&U_1(\boldsymbol{p},\varepsilon)U_3(-\boldsymbol{p},\varepsilon)=\mu_2U_3(\boldsymbol{p},\varepsilon)U_1^*(-\boldsymbol{p},-\varepsilon),\\
&U_2(\boldsymbol{p},\varepsilon)U_3(\boldsymbol{p},-\varepsilon)=\mu_1U_3(\boldsymbol{p},\varepsilon)U_2^*(-\boldsymbol{p},-\varepsilon),\\
&U_1(\boldsymbol{p},\varepsilon)U_1(-\boldsymbol{p},\varepsilon)=U_2(\boldsymbol{p},\varepsilon)U_2(\boldsymbol{p},-\varepsilon)=1,\\
&U_3(\boldsymbol{p},\varepsilon)U_3^*(-\boldsymbol{p},-\varepsilon)=\mu_4.
\end{aligned}
\tag{5.18}
$$

So for representations of Class *IV* the general form of the operators *P*, *C* and *T* is given by relations (5.17) where $U_a(\boldsymbol{p},\varepsilon)$ are infinite dimensional matrices satisfying (5.18).

We note for completeness that the discrete symmetry operators can be defined for the representations of the algebra *AP(1,3)* belonging to Class *V* (where $p_\mu\equiv 0$) also. Besides, the operators $\boldsymbol{J}$ and $\boldsymbol{N}$ reduce to finite or infinite dimensional matrices: $J_a\rightarrow 1/2\varepsilon_{abc}S_{bc}$, $N_a\rightarrow S_{0a}$ and the corresponding operators of discrete transformations can be represented as follows:

$$
\hat{P}=\xi_1,\qquad \hat{T}=\xi_2,\qquad \hat{C}=\xi_3A,
\tag{5.19}
$$

where *A* is the operator of the complex conjugation, ξ_1, ξ_2 and ξ_3 are numerical matrices of appropriate dimension satisfying the relations

$$\xi_1 S_{ab}=S_{ab}\xi_1, \quad \xi_2 S_{ab}=S_{ab}\xi_2, \quad \xi_3 S_{ab}=-S^*_{ab}\xi_3,$$
$$\xi_1 S_{0a}=-S_{0a}\xi_1, \quad \xi_2 S_{0a}=-S_{0a}\xi_2, \quad \xi_3 S_{0a}=-S^*_{0a}\xi_3, \tag{5.20}$$

$$\xi_1\xi_2=\xi_2\xi_1\mu_3, \quad \xi_1\xi_3=\xi_3\xi^*_1\mu_2, \quad \xi_2\xi_3=\xi_3\xi^*_2\mu_1, \quad \xi_1^2=\xi_2^2=1, \quad \xi_3\xi_3^*=\mu_4. \tag{5.21}$$

In the following sections we find the explicit form of the matrices U_a (and ξ_a) determining the discrete symmetry transformations for every class of representations of the the algebra *AP(1,3)*.

5.4. The Operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ for Representations of Class *I*

Representations of the Poincaré algebra belonging to Class *I* are labelled by the eigenvalues $c_1=m^2>0$, $c_2=-m^2s(s+1)$, and $c_3=\varepsilon=\pm 1$ of the Casimir operators (4.6), (4.11). According to previous section, we can restrict ourselves to the representations for which c_1 and c_2 are fixed but ε can take two possible values. We choose the basis elements of such representation in the form given in (5.11) where λ_μ are finite dimensional matrices realizing the direct sum of equivalent IRs of the algebra (4.20) corresponding to $c_1=m^2>0$, $c_2=-m^2s(s+1)$ (refer to (4.40)-(4.43)).

We represent U_a of (5.14) in the form

$$U_1=\xi_1 U, \quad U_2=\xi_2 V, \quad U_3=\xi_3 U\Delta, \tag{5.22}$$

where

$$U=\exp\left(\frac{i\lambda\cdot \boldsymbol{n}\times\boldsymbol{p}}{m\sqrt{p^2-(\boldsymbol{n}\cdot\boldsymbol{p})^2}}\pi\right), \quad V=\exp(i\lambda_0\hat{\varepsilon}\pi), \tag{5.23}$$

and Δ is a numerical matrix satisfying the relations

$$\Delta\lambda_\mu=-\lambda^*_\mu\Delta, \quad \Delta^2=1, \tag{5.24}$$

ξ_a being the operators to be determined. Choosing U_a in the form (5.22) we do not lose generality inasmuch as the operators U, V and Δ are invertible.

Using the relations

$$U\boldsymbol{J}=\boldsymbol{J}'U, \quad U\boldsymbol{N}=-\boldsymbol{N}'U, \quad V\lambda=-\lambda V, \tag{5.25}$$

($\boldsymbol{J}'$ and $\boldsymbol{N}'$ are determined in (5.15)) we conclude from (5.15), (5.11) that ξ_a must satisfy the following relations

$$\xi_1\hat{\varepsilon}=\hat{\varepsilon}\xi_1, \quad \xi_2\hat{\varepsilon}=-\hat{\varepsilon}\xi_2, \quad \xi_3\hat{\varepsilon}=-\hat{\varepsilon}\xi_3,$$
$$[\xi_a,P_b]=[\xi_a,\hat{\varepsilon}P_0]=[\xi_a,J_b]=[\xi_a,\hat{\varepsilon}N_b]=0. \tag{5.26}$$

But the operators $\{P_a,\hat{\varepsilon}P_0,J_a,\hat{\varepsilon}N_a\}$ realize the direct sum of the IRs $D^\varepsilon(m,s)$ of the algebra *AP(1,3)* from which it follows according to Schur's lemma that ξ_a must be numerical matrices.

We conclude from (5.15), (5.16) that the matrices ξ_a must commute with λ_μ,

$$[\xi_a,\lambda_\mu]=0, \tag{5.27}$$

and to satisfy the equations (5.21) which (together with (5.12), (5.26)) determine these matrices up to unitary equivalence.

So the problem of finding all the admissible operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ reduces to finding all the nonequivalent matrices λ_μ, $\hat{\varepsilon}$ and ξ_a. The irreducible sets of these matrices are given in the following assertion.

THEOREM 5.2. All possible (up to equivalence) irreducible sets of matrices satisfying the commutation and anticommutation relations (4.20), (5.20), (5.21), (5.26), (5.27) can be labelled by five numbers s, μ_1, μ_2, μ_3, and μ_4, where s=0, 1/2, 1, ..., and $\mu_k=\pm1$. Dimension of these matrices is $2(2s+1)\times2(2s+1)$ if $\mu_1\mu_4=\mu_2\mu_3=1$ and $4(2s+1)\times4(2s+1)$ otherwise. The explicit form of the corresponding matrices is given by the following formulae.

$\mu_1\mu_4=\mu_2\mu_3=1$:

$$\xi_1=\delta_{\mu_3},\quad \xi_2=\sigma_2,\quad \xi_3=\alpha_{\mu_1},\quad \hat{\varepsilon}=\sigma_3,\quad \lambda_\mu=\lambda_\mu^{(2)}; \tag{5.28}$$

$\mu_1\mu_4$=-1 or $\mu_2\mu_3$=-1:

$$\begin{gathered}\xi_1=\begin{pmatrix}\delta_{\mu_3} & 0\\ 0 & \mu_2\mu_3\delta_{\mu_3}\end{pmatrix},\quad \xi_2=\begin{pmatrix}\sigma_2 & 0\\ 0 & \mu_1\sigma_2\end{pmatrix},\quad \xi_3=\begin{pmatrix}0 & \sigma_1\\ \mu_4\sigma_1 & 0\end{pmatrix},\\ \hat{\varepsilon}=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},\quad \lambda_\mu=\begin{pmatrix}\lambda_\mu^{(2)} & 0\\ 0 & \lambda_\mu^{(2)}\end{pmatrix}\end{gathered} \tag{5.29}$$

where the following $2(2s+1)\times2(2s+1)$ matrices are included

$$\begin{gathered}\sigma_1=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \sigma_2=i\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad \sigma_0=\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},\\ \delta_{\mu_i}=\begin{pmatrix}I & 0\\ 0 & \mu_i I\end{pmatrix},\quad \alpha_{\mu_i}=i^{\frac{\mu_i-1}{2}}\begin{pmatrix}0 & I\\ \mu_i I & 0\end{pmatrix},\quad \lambda_\mu^{(2)}=\begin{pmatrix}\lambda_\mu & 0\\ 0 & \lambda_\mu\end{pmatrix},\end{gathered} \tag{5.30}$$

λ_μ are $2(2s+1)\times2(2s+1)$-dimensional matrices realizing the IR *D(s)* of the algebra $A(c_1=m^2, \boldsymbol{n})$ (4.20) (which is isomorphic to the algebra *AO(3)*), *I* is the unit matrix, 0 is the zero matrix of appropriate dimension.

PROOF reduces to going through possible sets of values of μ_a and finding nonequivalent irreducible sets of the matrices ξ, $\hat{\varepsilon}$ and λ_μ for any combinations of values μ_a. We do not represent the corresponding calculations here (see [12]), but note that the matrices (5.28), (5.29) are determined up to projective equivalence transformations (compare with (5.7)):

$$\xi_a \to \alpha_{\xi_a} V \xi_a \tilde{V}^{-1}, \quad a=1,2,3, \quad \varepsilon \to V \varepsilon V^{-1} \tag{5.31}$$

where V is an arbitrary nondegenerated matrix commuting with λ_μ,

$\tilde{V}^{-1}=V^{-1}$ $a=1,2;$ $\tilde{V}^{-1}=(V^{-1})^*$, $a=3.$

If we require $\alpha_{\xi_1}=\alpha_{\xi_2}=\alpha_{\xi_3}=1$ (i.e., restrict ourselves to regular equivalence) then the matrices ξ_a which are multiples of the unit matrix will be determined up to a sign. ■

So the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ can be defined in the space of the algebra *AP(1,3)* representation of Class *I* by sixteen nonequivalent ways corresponding to various multiplicators of the group G_8 (or, which is the same, to various commutation and anticommutation relations (5.9)). Incidentally, in four cases the representation space *D* of the algebra *AP(1,3)* is reduced to the direct sum of two invariant subspaces $D=D^+(m,s)\oplus D^-(m,s)$ (if $\mu_2\mu_3=\mu_1\mu_4=1$) and in the remaining twelve cases the operators P_μ, $\boldsymbol{J}$ and $\boldsymbol{N}$ of (5.11) realize the direct sum of four IRs

$$D=D^+(m,s)\oplus D^-(m,s)\oplus D^+(m,s)\oplus D^-(m,s).$$

The explicit form of the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ for the representation (5.11) is given by formulae (5.13), (5.22), where ξ_a are the matrices given in (5.29), (5.30). For other realizations of the algebra *AP(1,3)* being equivalent to (5.11) these operators can be obtained from (5.13), (5.22) by the change $\Psi\to V(\boldsymbol{p})\Psi$, $U_1\to U_1'=V(\boldsymbol{p})U_1V^{-1}(-\boldsymbol{p})$, $U_2\to U_2'=V(\boldsymbol{p})U_2V^{-1}(\boldsymbol{p})$, $U_3\to U_3'=V(\boldsymbol{p})U_3[V^{-1}(-\boldsymbol{p})]^*$, where $V(\boldsymbol{p})$ is an operator of equivalence transformation. In particular for the Foldy-Shirokov representation (4.50) we obtain

$$U_1'=\xi_1, \quad U_2'=\xi_2, \quad U_3'=\xi_3\Delta. \tag{5.32}$$

We represent also the explicit form of the matrix Δ determined by relations (5.24). If $\mu_1\mu_4=\mu_2\mu_3$ then

$$\Delta=\Delta_2=\begin{pmatrix}\Delta' & 0\\ 0 & \Delta'\end{pmatrix} \tag{5.33}$$

where Δ' is the matrix of dimension $2(2s+1)\times2(2s+1)$ given by the formula

$$\Delta'=(i)^{2s}\exp[i(\lambda_2+n_2\lambda_0)\pi]. \tag{5.34}$$

If $\mu_1\mu_4=-1$ or $\mu_2\mu_3=-1$ then Δ is a direct sum of two matrices Δ_2.

5.5. Representations of Class *II*

The representations of the algebra *AP(1,3)* corresponding to the zero eigenvalues of C_1 and C_2 have the two additional Casimir operators C_3 and C_4 (4.12). The operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ have to satisfy (5.2)-(5.4) together with C_3 and C_4. So these operators transform spaces of IRs of the *AP(1,3)* algebra according to the following diagram

$$
\begin{array}{ccc}
 & \hat{C} & \\
D^{\varepsilon}(\lambda) & \rightarrow & D^{-\varepsilon}(\lambda) \\
\hat{P}\uparrow & & \hat{P}\downarrow \\
D^{\varepsilon}(-\lambda) & \leftarrow & D^{-\varepsilon}(-\lambda) \\
 & \hat{C} &
\end{array}
\tag{5.35}
$$

One concludes from (5.35) the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined in a space of a reducible representation of the algebra *AP(1,3)* belonging to Class *II* which can be reduced to a direct sum of the least four IRs $D^{\varepsilon}(\lambda)$, $D^{-\varepsilon}(\lambda)$, $D^{-\varepsilon}(-\lambda)$, and $D^{\varepsilon}(-\lambda)$. We will choose the elements of such a representation in the form (4.51) where λ_0 and $\hat{\varepsilon}$ are diagonal matrices having the eigenvalues $\pm s$ and ± 1, s being a fixed integer or half integer.

We represent the operators U_1, U_2, and U_3 from (5.13) in the following form

$$U_1=\xi_1 U, \quad U_2=\xi_2, \quad U_3=\xi_3 U \tag{5.36}$$

where U is the operator (4.53), ξ_a are the operators to be found. Bearing in mind that the operator U changes the sign of the vector $\boldsymbol{n}$ in the operators (4.51), it is not difficult to show that the conditions (5.14), (5.15) reduce to the following equations for ξ_a:

$$\xi_1\lambda_0=-\lambda_0\xi_1, \quad \xi_2\lambda_0=\lambda_0\xi_2, \quad \xi_3\lambda_0=\lambda_0^*\xi_3, \tag{5.37}$$

$$\xi_1\hat{\varepsilon}=\hat{\varepsilon}\xi_1, \quad \xi_2\hat{\varepsilon}=-\hat{\varepsilon}\xi_2, \quad \xi_3\hat{\varepsilon}=-\hat{\varepsilon}^*\xi_3, \quad [\xi_a,p_b]=[\xi_a,x_b]=0. \tag{5.38}$$

Thus the problem of description of nonequivalent operators $\hat{P}$, $\hat{C}$ and $\hat{T}$ for the representations of Class *II* reduces to solving the system of equations (5.21), (5.37), (5.38) for the matrices λ_0, $\hat{\varepsilon}$, and ξ_a. The irreducible sets of the matrices satisfying these equations are exhausted by the following combinations [12].

When $\mu_1\mu_4=1$, dimension of matrices is 4×4 and their explicit form is given by the formulae

$$\hat{\varepsilon} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \lambda_0 = s\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},$$

$$\xi_1 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \mu_3\sigma_1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} \delta_{\mu_1} & 0 \\ 0 & \mu_2\delta_{\mu_1} \end{pmatrix} \tag{5.39}$$

where σ_0, σ_1, σ_3, and δ_{μ_1} are the matrices (5.30) of dimension 2×2.

If $\mu_1\mu_4$=-1 then dimension of the corresponding matrices is 8×8. Moreover

$$\hat{\varepsilon} = i\begin{pmatrix} \gamma_1\gamma_2 & 0 \\ 0 & \gamma_1\gamma_2 \end{pmatrix}, \quad \lambda_0 = is\begin{pmatrix} \gamma_4 & 0 \\ 0 & -\gamma_4 \end{pmatrix},$$

$$\xi_1 = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \Gamma_{\mu_3} & 0 \\ 0 & \Gamma_{\mu_3} \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & \Gamma_{\mu_1\mu_2} \\ -\Gamma_{\mu_1\mu_2} & 0 \end{pmatrix}. \tag{5.40}$$

Here

$$\Gamma_{\mu_3} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \mu_3\sigma_3 \end{pmatrix}, \quad \Gamma_{\mu_1\mu_2} = \begin{pmatrix} \delta_{\mu_1} & 0 \\ 0 & \mu_2\delta_{\mu_1} \end{pmatrix},$$

and γ_μ are the Dirac matrices (2.4).

Thus there exist 16 nonequivalent possibilities of definition of the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ in the space of a representations of Class *II*. The space *D* of such a representation is reduced to the direct sum of the subspaces $D^+(s)\oplus D^-(s)\oplus D^+(-s)\oplus D^-(-s)$ for $\mu_1\mu_4$=1 or $D^+(s)\oplus D^+(s)\oplus D^-(s)\oplus D^-(s)\oplus D^+(-s)\oplus D^+(-s)\oplus D^-(-s)\oplus D^-(-s)$ for $\mu_1\mu_4$=-1. The corresponding operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ are given by formulae (5.20), (5.36) where ξ_a are the matrices (5.39) or (5.40).

5.6. Representation of Classes *III-IV*

Thus we have determined all the possible (up to projective equivalence) operators of space inversion, charge conjugation and time reflection which can be defined in the spaces of representations of the algebra *AP(1,3)* belonging to the most important for physical applications Classes *I* and *II*. In an analogous way we can find these operators for other representation classes. Here we consider briefly the corresponding possibilities.

The representations of the algebra *AP(1,3)* belonging to Class *III* ($P_\mu P^\mu$=0, $-W_\mu W^\mu=r^2>0$) have the only additional Casimir operator $C_3=P_0/|P_0|$, so the description of the corresponding operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ does not differ in principle from the case of representations of Class *I*. Let us set in (5.13)

$$U_1=\xi_1 U\Lambda, \quad U_2=\xi_2 V, \quad U_3=\xi_3\Lambda\Delta \tag{5.41}$$

where U and V are the operators (4.53) and (5.23) (with an appropriate matrix λ_0), λ and Δ are unitary matrices satisfying the conditions (5.24) and

$$\Lambda\lambda_0=-\lambda_0\Lambda, \quad \Lambda\lambda=\left(-\lambda+\frac{2(\boldsymbol{n}\times\boldsymbol{n}'')(\boldsymbol{n}\times\boldsymbol{n}''\cdot\lambda)}{1-(\boldsymbol{n}\cdot\boldsymbol{n}'')^2}\right)\Lambda. \tag{5.42}$$

As a result, by using the following properties of the operators $U\Lambda$ and V:

$$U\Lambda\hat{\varepsilon}=\hat{\varepsilon}U\Lambda, \quad U\Lambda(\boldsymbol{J},\boldsymbol{N})=(\boldsymbol{J}'\boldsymbol{N}')U\Lambda, \quad U\Lambda(P_0,\boldsymbol{P})=(P_0,\boldsymbol{P})U\Lambda, \quad V\lambda=-\lambda V, \tag{5.43}$$

(where $\boldsymbol{J}'$ and $\boldsymbol{N}'$ are the operators obtained from $\boldsymbol{J}$ and $\boldsymbol{N}$ by the change $\boldsymbol{p}\to-\boldsymbol{p}$), we come once again to the equations (5.21) for the matrices ξ_a whose general solution is given by (5.28), (5.29) (where I is the unit operator in the space of IRs of the algebra *AE(2)*). The corresponding representation of the algebra *AP(1,3)* is expanded in the direct sum either of two IRs $D^{\varepsilon}(r)\oplus D^{-\varepsilon}(r)$ if $\mu_1\mu_4=\mu_2\mu_3=1$ or of four IRs $D^{\varepsilon}(r)\oplus D^{-\varepsilon}(r)\oplus D^{\varepsilon}(r)\oplus D^{-\varepsilon}(r)$ otherwise. The explicit expressions for the matrices Λ and Δ in the basis (4.40) are given by the formulae

$$\Lambda=\Lambda_1\cos\phi+\Lambda_2\sin\phi, \quad \Delta=\Lambda_1,$$

$$\phi=arctg\frac{n_1''(1+n_3)-n_1(n_3''+\boldsymbol{n}\cdot\boldsymbol{n}'')}{n_2''(1+n_3)-n_2(n_3''+\boldsymbol{n}\cdot\boldsymbol{n}'')}, \tag{5.44}$$

$$\Lambda_1|0,r,\lambda>=(-1)^{[\lambda]}|0,r,-\lambda>, \quad \Lambda_2|0,r,\lambda>=i(-1)^{[\lambda]}sign\lambda|0,r,-\lambda>,$$

$[\lambda]$ is the entire part of λ.

Formulae (5.13), (5.41), (5.28), (5.29), (5.44) give the explicit form of all the nonequivalent operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ defined in the space of representations of the algebra *AP(1,3)* of Class *III*.

A more complicated situation arises for the representations of Class *IV* corresponding to zero eigenvalues of the Casimir operator C_1. Such representations can be subdivided into two subclasses. The first one, subclass *IVA* for which the eigenvalues of the operator $W_\mu W^\mu$ are equal to $\eta^2 l(l+1)$ (refer to (4.44)) has the additional Casimir operator $C_5=W_0/|W_0|$. The representations belonging to this class are denoted by $D(\eta,l,\mu)$ where $\mu=\pm1$ denotes the eigenvalue of the operator $W_0/|W_0|$. Other subclass (denoted by *IVB*) corresponds to negative eigenvalues of $W_\mu W^\mu$ and has no additional Casimir operators. The IRs belonging to subclass *IVB* are denoted by $D(\eta,\alpha)$, the meaning of η and α being clear from (4.44).

The operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined in the space of the reducible representation $D=D(\eta,l,\mu)\oplus D(\eta,l,-\mu)$ if $\mu_1\mu_4=\mu_2\mu_3=1$ and $D=D(\eta,l,\mu)\oplus D(\eta,l,\mu)\oplus D(\eta,l,-\mu)\oplus D(\eta,l,-\mu)$ otherwise. The explicit form of these operators is up to equivalence given by formulae (5.17), (5.36) where ξ_a are the matrices represented in (5.28), (5.29). The symbol I in (5.28), (5.29) defines the unit operator in the space of the IRs $D(\eta, l, \varepsilon\mu)$, $\varepsilon=\pm 1$.

For the subclass *IVB* the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined in the space of the IRs $D(\eta,\alpha)$ or in the space of the direct sum of two IRs $D(\eta,\alpha)\oplus D(\eta,\alpha)$ or of four IRs $D(\eta,\alpha)\oplus D(\eta,\alpha)\oplus D(\eta,\alpha)\oplus D(\eta\alpha)$. The general form of these operators is defined in (5.17), (5.41), (5.44) where ξ_a are matrices satisfying (5.21), (5.27). The explicit form of all the nonequivalent and indecomposable matrices ξ_a is given by the following formulae

$$
\begin{array}{ll}
\mu_1=\mu_2=\mu_3=\mu_4=1, & \xi_1=\xi_2=\xi_3=I, \\
\mu_3=1,\ \mu_2=-1,\ or\ \mu_1\mu_4=-1, & \xi_1=\delta_{\mu_2},\ \xi_2=\delta_{\mu_1},\ \xi_3=\alpha_{\mu_4} \\
\mu_3=-1,\ \mu_2=\mu_1=\mu_4=1, & \xi_1=\sigma_3,\ \xi_2=\sigma_2,\ \xi_3=\delta_{-\mu_1}
\end{array}
\tag{5.45}
$$

and for other values of μ_a - by formulae (5.28), (5.29) where I is the unit operator in the space of the IRs $D(\eta,\alpha)$.

5.7. Representations of Class *V*

Let us consider also the finite dimensional representations of Class *V* corresponding to $p_\mu\equiv 0$. The description of all nonequivalent operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ defined in the space of such representations is of great interest because they are the operators which can be defined on the sets of solutions of relativistic wave equations.

The representations belonging to Class *V* have the two Casimir operators (4.12). According to (5.2)-(5.4) the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ have to anticommute with C_7. We conclude from this fact that these operators transform a vector from the space of the IR $D(l_0,l_1)$ into the vector belonging to the space of the IR $D(-l_0,l_1)$. Hence the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined in a space of at least two IRs $D(l_0,l_1)\oplus D(-l_0,l_1)$ (except the case $l_0=0$ when it is possible in principle to restrict ourselves to using a single IR).

Let $S_{\mu\nu}$ be matrices which realize the direct sum $D(l_0,l_1)\oplus D(-l_0.l_1)$ of IRs with fixed l_0 and l_1, $l_0\neq 0$, each of IRs being included with some multiplicity. The corresponding operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ can be defined by formulae (5.19) where ξ_a are the matrices satisfying relations (5.20), (5.21). All the nonequivalent indecomposable matrices ξ_a are given by the formulae

$$
\xi_1=\xi_1'\hat{p},\qquad \xi_2=\xi_2'\hat{p},\qquad \xi_3=\xi_3'\hat{p},
\tag{5.46}
$$

the matrices from the r.h.s. of this formula are represented in Table 5.3, see the following page. Moreover α_{μ_i}, δ_{μ_i} are the matrices (5.30) with I being the unit

Table 5.3

Values of μ_k	$\mu_1\mu_2\mu_3=1,\ \mu_4=1$	$\mu_1\mu_2\mu_3=-1,\ \mu_4=1$	$\mu_1\mu_2\mu_3=-1,\ \mu_4=-1$	$\mu_1\mu_2\mu_3=1,\ \mu_4=-1$
representation realized by $S_{\mu\sigma}$	$D(l_0,l_1)\oplus D(-l_0,l_1)$	$D(l_0,l_1)\oplus D(-l_0,l_1)\oplus D(l_0,l_1)\oplus D(-l_0,l_1)$	$D(l_0,l_1)\oplus D(-l_0,l_1)\oplus D(l_0,l_1)\oplus D(-l_0,l_1)$	$D(l_0,l_1)\oplus D(-l_0,l_1)\oplus D(l_0,l_1)\oplus D(-l_0,l_1)$
explicit form of ξ'_1	σ_1	$\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$	$\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$	$\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$
explicit form of ξ'_2	α_{μ_3}	$\begin{pmatrix}0 & \delta_{\mu_3}\\ \delta_{\mu_3} & 0\end{pmatrix}$	$\begin{pmatrix}0 & \delta_{\mu_3}\\ \delta_{\mu_3} & 0\end{pmatrix}$	$\begin{pmatrix}\alpha_{\mu_3} & 0\\ 0 & \alpha_{\mu_3}\end{pmatrix}$
explicit form of ξ'_3	δ_{μ_2}	$\begin{pmatrix}\delta_{\mu_2} & 0\\ 0 & \delta_{\mu_2}\end{pmatrix}$	$\begin{pmatrix}0 & \alpha_{-\mu_2}\\ \mu_1\alpha_{-\mu_2} & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\delta_{\mu_2}\\ \delta_{\mu_2} & 0\end{pmatrix}$
explicit form of $\hat{p}$	$p_{(2)}=\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}$	$\begin{pmatrix}p_{(2)} & 0\\ 0 & p_{(2)}\end{pmatrix}$	$\begin{pmatrix}p_{(2)} & 0\\ 0 & p_{(2)}\end{pmatrix}$	$\begin{pmatrix}p_{(2)} & 0\\ 0 & p_{(2)}\end{pmatrix}$

matrices in the space of the IRs $D(l_0,l_1)$, and p are the space inversion matrices for theIRs $D(l_0,l_1)$, as determined by the relation

$$p\,|l_0,l_1;l,m\rangle=(-1)^{[l]}\,|-l_0,l_1;l,m\rangle. \tag{5.47}$$

Formulae (5.46) are valid only for the representations of the algebra $AO(1,3)$ in the basis (4.66)-(4.68). For the representations of the type $D(0,l_1)$ the operators $\hat{P}$, $\hat{T}$ and $\hat{C}$ are also given by formulae (5.46) where ξ'_a are the matrices (5.28), (5.29).

So we have described all the nonequivalent representations of the discrete symmetry operators which can be defined in a space of a representation of the homogeneous Lorentz group. As in the case of representations of Classes *I-IV* there are 16 nonequivalent possibilities in defining of such operators corresponding to possible multiplicators of the group G_8.

5.8. Concluding Remarks

We have described all the possible nonequivalent realizations of the discrete symmetry transformations in a space of a representation of the algebra *AP(1,3)*. Actually, we have also determined all the projective-nonequivalent local representations of the extended Poincaré group *P(1,3)* including the transformations *P*, *C* and *T* in addition to proper inhomogeneous Lorentz transformations.

We note that in the case of the two-digit representations of the group P(1,3) it is possible to determine the operators *P*, *C* and *T* in such a way that their squares be equal to ±1, inasmuch as a double reflection can be represented either as an identity transformation or as a rotation by the angle 2π (which reduces to the multiplication by -1 in two-digit representations). At this, all the possible products of the operators *P*, *C* and *T* also form a finite group and the number of different groups is equal to 64. But IRs of these groups reduces to IRs of the group G_8 up to projective equivalence.

It follows from the above that the operators of *P*-, *C*- and *T*-transformations defined on the sets of solutions of the KGF, Dirac, and Maxwell equations, realize only some of possible representations of these transformations. In other words, only some of possible groups G_8 are realized here. There arises the natural question if there exist relativistic wave equations for scalar, spinor and vector fields fields which correspond to another representations of the operators *P*, *T* and *C*. This intriguing problem is discussed partly in Sections 6-9.

6. POINCARÉ-INVARIANT EQUATIONS OF FIRST ORDER

6.1. Introduction

In this and the following sections we present some elements of the theory of Poincaré-invariant equations for particles of arbitrary spin.

Let $\{\psi(x)\}$ be a set of solutions of a motion equation of a relativistic particle with fixed mass and spin. In relativistic quantum theory the space of states of a free (noninteracting) particle is identified with the space of the IR of the Poincaré group. So $\{\psi(x)\}$ by definition has to be a representation space of the algebra *AP(1,3)* and any $\psi\in\{\psi(x)\}$ has to satisfy the following conditions

$$P_\mu P^\mu\psi=m^2\psi,\quad W_\mu W^\mu\psi=-m^2 s(s+1)\psi \tag{6.1}$$

where m and s are parameters determining the mass and spin of a particle.

The conditions (6.1) mean that the Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ have fixed eigenvalues on the set of solutions of a particle equation. If we choose a concrete realization of the algebra *AP(1,3)* then the conditions (6.1) by themselves can be considered as a system of linear equations describing a particle of spin s and mass m.

As a matter of fact, all the known Poincaré-invariant equations for particles of fixed mass and spin are nothing but the conditions (6.1) written in other forms. Thus, the relations (6.1) represent a system of partial differential equations of second order in all cases when P_μ and $J_{\mu\nu}$ realize a covariant representation of the algebra *AP(1,3)*. In the case when s=1 and $S_{\mu\nu}$ belong to the representation *D(1/2 1/2)* these relations reduce to the Procá equation [404]. The Dirac equation is a formulation of the conditions (6.1) in the form of an equivalent system of first order partial differential equations, etc.

Here we present some elements of the theory of Poincaré-invariant equations of the form

$$(\beta_\mu p^\mu - \beta_4 m)\psi=0 \tag{6.2}$$

where β_μ, β_4 are numerical matrices and ψ is a vector-function (a column matrix). The importance of the analysis of such equations is that equations including the derivatives of higher order can always be reduced to a system of partial differential equations of first order by introducing new dependent variables. For example, the KGF equation (1.1) reduces to the form (6.2) by the substitution $p_\mu\Phi=m\Phi_\mu$, $\Phi=\psi_4$, where $\psi=column(\psi_4\ \psi_0\ \psi_1\ \psi_2\ \psi_3)$, $\beta_4=1$ and β_μ are the Kemmer-Duffin matrices of dimension 5×5 (see (6.17) below).

Poincaré-invariance of the system (6.2) means that this system admits 10 independent SOs forming the algebra *AP(1,3)*. Restricting ourselves to the case when such SOs belong to the class M_1 (i.e., has the covariant form (2.22) with appropriate matrices $S_{\mu\nu}$) we come naturally to the following definition.

DEFINITION 6.1. We say the equation (6.2) is Poincaré-invariant and describes a particle of mass m and spin s if a covariant representation of the algebra *AP(1,3)* is realized on the set of its solutions and the Casimir operators of this algebra satisfy the conditions (6.1).

Relativistic wave equations of the kind (6.2) where β_4 is an invertible matrix are well studied. In the works of Bhabha [35], Harish-Chandra [216], Wild [416], Umedzawa [404] the general form of the matrices β_μ was found for Poincaré-invariant equation (6.2). Besides the additional conditions for β_μ had been formulated which follows from the requirements (6.1). In the Russia studying of such equations started with the works of Tamm and Ginzburg [202, 203], the important results in this direction were obtained by Fedorov [97], Schelepin [384], and especially by Gelfand and Yaglom [200] who described finite and infinite dimensional first order wave equations for any spin particles in the frame of the united approach. Nevertheless the theory of Poincaré-invariant equations of the kind (6.1) is still far from complete. In particular the equations with a singular matrix β_4 and the equations corresponding to indecomposable representations of the Lorentz group are not completely studied yet.

6.2. The Poincaré-Invariance Condition

We restrict ourselves to the case when the representation of the algebra *AP(1,3)* over the set of the equation (6.1) solutions is completely reducible. Then, the basis elements of this algebra are given by formulae (2.2) where $S_{\mu\nu}$ are matrices realizing a direct sum of IRs of the algebra *AO(1,3)*.

Let the number of equations in the system (6.2) be equal to the number of unknowns. In analogy with (2.21), the condition of invariance of such a system under the algebra *AP(1,3)* can be written in the form

$$[Q_A,\beta^\mu p_\mu-\beta_4 m]=\beta_{Q_A}(x)(\beta^\mu p_\mu-\beta_4 m) \tag{6.3}$$

where Q_A is any operator from the set (2.2), $\beta_{Q_A}(x)$ is a matrix of dimension $n\times n$ depending on x. Substituting (2.2) into (6.3) and equating coefficients of the same differential operators we can make sure that for $Q_A\in\{P_\mu\}$ the condition (6.3) is satisfied identically, but for $Q_A\in\{J_{\mu\nu}\}$ we obtain the following conditions for the matrices β_μ, β_4, $\beta_{Q_a}=\beta_{\mu\nu}$

$$[S_{\mu\nu},\beta_\lambda]=i(g_{\nu\lambda}\beta_\mu-g_{\mu\lambda}\beta_\nu)+\beta_{\mu\nu}\beta_\lambda,\quad [S_{\mu\nu},\beta_4]=\beta_{\mu\nu}\beta_4$$

or

$$\begin{aligned}&\tilde S_{\mu\nu}\beta_\lambda-\beta_\lambda S_{\mu\nu}=i(g_{\nu\lambda}\beta_\mu-g_{\mu\lambda}\beta_\nu),\\ &\tilde S_{\mu\nu}\beta_4-\beta_4 S_{\mu\nu}=0,\quad \tilde S_{\mu\nu}=S_{\mu\nu}-\beta_{\mu\nu}.\end{aligned} \tag{6.4}$$

Using the commutation relations (3.49) for $S_{\mu\nu}$, it is not difficult to deduce the following conditions from (6.4):

$$[\tilde S_{\mu\nu},\tilde S_{\lambda\sigma}]\beta_k=i(g_{\mu\sigma}\tilde S_{\nu\lambda}+g_{\nu\lambda}\tilde S_{\mu\sigma}-g_{\nu\sigma}\tilde S_{\mu\lambda}-g_{\mu\lambda}\tilde S_{\nu\sigma})\beta_k. \tag{6.5}$$

A <u>sufficient</u> condition of validity of (6.5) is the requirement that the matrices $\tilde S_{\mu\nu}$ realize a representation of the algebra *AP(1,3)* (which in general can differ from the representation realized by $S_{\mu\nu}$). If the conditions (6.4) are fulfilled with $S_{\mu\nu}$, $\tilde S_{\mu\nu}$ being matrices satisfying the algebra *AP(1,3)*, then we say the equation (6.2) is invariant under the algebra *AP(1,3)* in the strong sense, or S-invariant.

So we come to the following definition.

DEFINITION 6.2. The equation (6.2) is S-invariant under the Poincaré algebra if there exist such matrices $S_{\mu\nu}$, $\tilde S_{\mu\nu}$ realizing representations of the algebra

AO(1,3) then the conditions (6.5) are satisfied for β_μ and β_4.

We emphasize that Definition 6.2 gives only the sufficient conditions of the Poincaré- invariance of the equation (6.2) inasmuch as in general the algebra *AP(1,3)* can be realized by operators which do not belong to the class M_1, and, besides, it is not necessary for $S_{\mu\sigma}$ to satisfy the algebra *AP(1,3)* (the requirement (6.5) is less strong).

We note that Definition 6.2 is also valid for the case when the number n of components of the function ψ does not coincide with the number m of equations. In this case β_μ and β_4 are matrices of dimension $m\times n$, $m\neq n$, $S_{\mu\nu}$ and $\tilde{S}_{\mu\nu}$ are square matrices of dimension $m\times m$ and $n\times n$.

Further on, we restrict ourselves to considering of Poincaré-invariant equations (6.2) with square matrices β_μ, β_4 where β_4 is nonsingular. The theory of relativistic wave equations with a singular matrix β_4 is still not completed (see, however [1, 95, 221]).

If β_4 is nonsingular then, without loss of generality, we can set $\beta_4=I$ where I is the unit matrix. Incidentally the system (6.2) reduces to the form

$$(\beta_\mu p^\mu - m)\psi = 0. \tag{6.6}$$

According to (6.4), $S_{\mu\nu}=\tilde{S}_{\mu\nu}$ if $\beta_\mu=I$, and the conditions (6.5) reduce to the following form

$$[S_{\mu\nu},\beta_\lambda]=i(g_{\nu\lambda}\beta_\mu-g_{\mu\lambda}\beta_\nu). \tag{6.7}$$

The problem of describing of Poincaré-invariant equations (6.6) reduces to finding the general solution of the equations (3.49), (6.7). In the following subsection we represent this solution for the case when the matrices $S_{\mu\nu}$ are completely reducible.

6.3. The Explicit Form of the Matrices β_μ

Solving the equations (6.7) includes generally speaking the following steps:

a) finding the matrices $S_{\mu\nu}$ satisfying the algebra (3.49), i.e., describing that representations of this algebra;

b) selection of such representations for which the equations (6.7) have nontrivial solutions;

c) and, finally, determining explicit form of the matrices β_μ satisfying (6.7).

We restrict ourselves to the case when $S_{\mu\sigma}$ realize a finite dimensional reducible representation of the algebra *AO(1,3)*. Such a representation without loss of generality can be taken in the form of a direct sum of IRs given in Subsection 4.8.

Let us denote by $|(j\ \tau; lm)_\lambda>$ a vector belonging to the space of the IR $D(j\ \tau)$ and being an eigenfunction of the operators (4.12), (4.63) (the index λ is introduced in order to label the spaces of equivalent representations) and by

$<(j\,\tau;\, lm)_\lambda|\beta_\mu|(j'\,\tau';\, l'm')_{\lambda'}>$ - the matrix elements of the matrices β_μ from the invariant equation (6.6). Then the main assertion concerning the general form of matrices β_μ satisfying (6.7) can be formulated as follows [58].

THEOREM 6.3. Let $S_{\mu\nu}$ be matrices realizing a direct sum of the IRs $D(j\,\tau)$ of the algebra *AP(1,3)* and β_μ be matrices satisfying (6.7). Then elements of the matrix β_μ differ from zero only in those cases when $|j\text{-}j'|=1/2$ and $|\tau\text{-}\tau'|=1/2$, with the following relations being valid:

$$\begin{aligned}
&\left\langle (j\tau;lm)_\lambda|\beta_0|(j-1/2\quad \tau+1/2;\, l'm')_{\lambda'}\right\rangle = C_{\lambda\lambda'}\delta_{mm'}\delta_{ll'}[(j+l-1)(\tau-j+l+1)]^{1/2},\\
&\left\langle (j\tau;lm)_\lambda|\beta_0|(j-1/2\quad \tau-1/2;\, l',m')_{\lambda'}\right\rangle = (-1)^{j+\tau+l}C_{\lambda\lambda'}\delta_{ll'}\delta_{mm'}[(j+\tau-l)(j+\tau+l+1)]^{1/2},\\
&\left\langle (jj+1/2;\, lm)_\lambda|\beta_0|(j+1/2\;\, j;\, l',m')_{\lambda'}\right\rangle = (-1)^{[l]+1}C_{\lambda\lambda'}(l+1/2)\delta_{ll'}\delta_{mm'}
\end{aligned}\tag{6.8}$$

where $j>\tau,\; |j\text{-}\tau|\le l\le j+\tau,\; [l]$ is the entire part of l,

$$\begin{aligned}
&<(j\tau;\, lm)_\lambda|\beta_0|(j'\tau';\, l'm')_{\lambda'}> = (-1)^{2j+2}<j'\tau';\, l'm')_\lambda|\beta_0|(j\tau;\, lm)_{\lambda'}>,\\
&<(j\tau;\, lm)_\lambda|\beta_0|(j'\tau';\, l'm')_{\lambda'}> = -<\tau j;\, lm)_\lambda|\beta_0|(\tau' j';\, lm)_{\lambda'}>,\quad j\neq\tau,\quad j'\neq\tau',\\
&\left\langle (j-1/2\quad j+1/2;\, lm)_\lambda|\beta_0|(jj;\, l',m')_{\lambda'}\right\rangle = (-1)^{2j+l-1}C_{\lambda\lambda'}[(s(s+1)]^{1/2}\delta_{ll'}\delta_{mm'}.
\end{aligned}\tag{6.9}$$

Here $C_{\lambda\lambda'}$ are arbitrary complex numbers. The matrix elements of β_1, β_2 and β_3 can be obtained from (6.8), (6.9) by using relations (6.7).

Thus, the matrices β_μ are determined by the conditions (6.7) up to arbitrary complex coefficients $C_{\lambda\lambda'}$. The number of these coefficients decreases if we require the *P*-, *T*- and *C*-invariance of the equations (6.6). Let λ, λ' and μ, μ' numerate two pairs of nonequivalent representations which are transformed one into another by the space inversion. Then for *P*-invariance of (6.6) it is necessary to require $C_{\lambda\lambda'}=-C_{\mu\mu'}$. Other restrictions for $C_{\lambda\lambda'}$ are considered in the following subsection.

6.4. Additional Restrictions for the Matrices β_μ

If we restrict ourselves to completely reducible representations of the algebra *AO(1,3)* then formulae (6.8), (6.9) give all the possible matrices β_μ such that the equation (6.2) is S-invariant under the Poincaré algebra. But a solution for this equation does not satisfy in general the conditions (6.1), and so cannot a priori be interpreted as a motion equation for a particle with fixed mass and spin.

The equations (6.1) impose additional restrictions on the form of the matrices β_μ. As was shown in [416] the necessary and sufficient condition of validity of the first of relations (6.1) is

$$\beta_0^{2s+1}=\beta_0^{2s-1} \tag{6.10}$$

where s is the maximal spin value appearing by the reduction of the representation of the algebra of matrices $S_{\mu\nu}$ by the algebra *AO(3)*. According to (6.11), the corresponding eigenvalues of the matrix β_0^2 can be equal to 0 or 1 only.

So the first of the conditions (6.1) admits the simple formulation (6.10). As to the second equation (6.1), it reduces to the following relations for β_0 [58]:

$$\boldsymbol{S}^2P\equiv\frac{1}{2}S_{ab}S_{ab}\beta_0^{2s-1+[s]}=s(s+1)\beta_0^{2s-1+[s]} \tag{6.11}$$

where S_{ab} are the matrices $S_{\mu\nu}$ with μ, $\nu \neq 0$, and $P=\beta_0^{2s-1+[s]}$ is the projector on the subspace corresponding to nonzero eigenvalues of the matrices β_0.

One more restriction on the matrices β_μ can be obtained from the requirement that the equation (6.6) admits the Lagrangian formulation, i.e., can be deduced from the corresponding Lagrangian using the minimal action principle. This requirement means that a nonsingular Hermitizing matrix η exists satisfying the conditions

$$\eta S_{\mu\nu}=S_{\mu\nu}^{\dagger}\eta, \tag{6.12}$$

$$\eta\beta_\mu=\beta_\mu^{\dagger}\eta. \tag{6.13}$$

The necessary and sufficient condition of the matrix η existence is that any IR $D(j\ \tau)$ included into the representation realized by $S_{\mu\nu}$ be supplemented by the conjugated representation $D(\tau\ j)$ if $j\neq\tau$. At this, the conditions (6.12) determine the matrix η up to a factor if any IRs $D(j\ \tau)$ has the unit multiple [58]. The explicit form of η is given by the formulae:

$$a)\quad D=D(0j)\oplus D(j0),\quad \eta=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{6.14}$$

where I and 0 are the zero and unit matrices of dimension $(2j+1)\times(2j+1)$;

$$b)\quad D=D(j\tau)\oplus D(\tau j),\quad j\neq\tau,\quad \eta=\begin{pmatrix}0 & P_{j\tau}\\ P_{\tau j} & 0\end{pmatrix} \tag{6.15}$$

where $P_{j\tau}$ are the $(2j+1)(2\tau+1)\times(2j+1)(2\tau+1)$ matrices given in (5.47), (4.61) (they are denoted there by p);

$$c)\quad D=D(jj),\quad \eta=P_{jj}. \tag{6.16}$$

As a representation D realized by $S_{\mu\nu}$ is a direct sum of the representations a), b), and c), then as a matter of fact formulae (6.14)-(6.16) set up the general form of η for the case when this sum is nondegenerated. For the general form of η in the case when $S_{\mu\nu}$ realize a degenerated direct sum of IRs refer, for example, to [197].

Thus, the general form of the matrices β_μ determining the Poincaré-invariant equation (6.6) for a particle of spin s is given by formulae (6.8), (6.9) with the coefficients $C_{ll'}$ having to satisfy additional conditions stemming from (6.10), (6.11), (6.13)-(6.16). In the following we consider examples of these equations for $s\leq 3/2$.

6.5. The Kemmer-Duffin-Petiau (KDP) Equation

The simplest example of a Poincaré-invariant equation of the kind (6.6) is the Dirac equation considered in Section 2. This equation can be obtained with formulae (6.8), (6.9) starting with the representation $D=D(0\ 1/2)\oplus D(1/2\ 0)$ for the matrices $S_{\mu\nu}$.

Let us consider more complicated examples. By choosing the representation D in the form $D=D(0\ 0)\oplus D(1/2\ 1/2)$, we obtain from (4.58)-(4.60), (6.8), (6.9), (6.10)-(6.12), (6.16) the following expressions for the matrices β_μ in the basis

$$\psi=column\left(|00;00>,\ \left|\frac{1}{2}\frac{1}{2};00\right\rangle,\ \left|\frac{1}{2}\frac{1}{2};11\right\rangle,\ \left|\frac{1}{2}\frac{1}{2};10\right\rangle\ \left|\frac{1}{2}\frac{1}{2};1-1\right\rangle\right):$$

$$\beta_0=\begin{pmatrix}0 & -i & \tilde{0}\\ i & & \\ \tilde{0}^\dagger & & \hat{0}\end{pmatrix},\quad \beta_Q=\begin{pmatrix}0 & 0 & \lambda_a\\ 0 & & \\ -\lambda_a^\dagger & & \hat{0}\end{pmatrix} \tag{6.17}$$

where $\hat{0}$ and $\tilde{0}$ are the zero matrices of dimensions 4×4 and 1×3, λ_a are the row matrices

$$\lambda_1=(i\ \ 0\ \ 0),\quad \lambda_2=(0\ \ i\ \ 0),\quad \lambda_3=(0\ \ 0\ \ i). \tag{6.18}$$

Without loss of generality, we set an arbitrary constant C_{10} in (6.8) be equal to 1.

The equation (6.6) with the matrices (6.17) describes a particle of mass m and spin s=0. Furthermore, β_0 satisfies the criteria (6.10), (6.11), (6.13) where s=0 and

$$S_{\mu\nu}=i[\beta_\mu,\beta_\nu],\quad \eta=(1+2\beta_1^2)(1+2\beta_2^2)(1+2\beta_3^2). \tag{6.19}$$

Formulae (6.6), (6.17) define the KDP equation for a scalar particle. The matrices (6.17) satisfy the algebra

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=g_{\mu\nu}\beta_\lambda+g_{\nu\lambda}\beta_\mu. \tag{6.20}$$

These relations define β_μ up to unitary equivalence.

The KDP equation is invariant under the *P*-, *T*- and *C*-transformations (2.55), (2.60) where

$$r_1=(1-2\beta_0^2),\quad r_2=1,\quad r_3=\eta. \tag{6.21}$$

Consider the KDP equation for a vector particles which may be taken in the form (6.6) with β_μ being 10×10 irreducible matrices satisfying the algebra (6.20). The explicit realization of these matrices (which is defined by the relations (6.20) up to unitary equivalence) can be chosen in the following form:

$$\beta_0 = i\begin{bmatrix} \tilde{0} & \tilde{0} & I & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & \tilde{0} \\ -I & \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0}^\dagger & \tilde{0}^\dagger & \tilde{0}^\dagger & 0 \end{bmatrix}, \quad \beta_a = \begin{bmatrix} \hat{0} & \hat{0} & \hat{0} & \lambda_a^\dagger \\ \hat{0} & \hat{0} & -S_a & \tilde{0} \\ \hat{0} & S_a & \hat{0} & \tilde{0} \\ -\lambda_a & \tilde{0}^\dagger & \tilde{0}^\dagger & 0 \end{bmatrix}. \tag{6.22}$$

Here $\hat{0}$ and $\tilde{0}$ are the zero matrices of dimension 3×3 and 1×3, I is the 3×3 unit matrix, S_a and λ_a are the 3×3 and 1×3 matrices (3.6), (6.18).

The matrices (6.22) satisfy relations (6.20), from which it follows that they satisfy also the conditions (6.7), (6.11). The matrices $S_{\mu\nu}$ of (6.19), (6.22) realize the representation $D=D(1\ 0)\oplus D(0\ 1)\oplus D(1/2\ 1/2)$ of the algebra $AO(1,3)$. Therefore, the equation (6.6), (6.22) is S-invariant under the Poincaré algebra. This equation describes a particle of mass m and spin s=1 since the matrices β_μ satisfy (6.10), (6.11) for s=1.

The KDP equation for a vector particle is *P*-, *T*- and *C*-invariant and admits the Lagrangian formulation. The corresponding SOs and the Hermitizing matrix η are given in (2.55), (2.60), (6.21), (6.19) where β_μ are the 10×10 KDP matrices.

We note that besides β_μ there exists just one more matrix satisfying (6.20). This matrix is determined according to the following relation

$$\beta_4 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\beta^\mu\beta^\nu\beta^\rho\beta^\sigma. \tag{6.23}$$

In the realization (6.22)

$$\beta_4 = i\begin{bmatrix} \hat{0} & I & \hat{0} & \tilde{0} \\ -I & \hat{0} & \hat{0} & \tilde{0} \\ \hat{0} & \hat{0} & \hat{0} & \tilde{0} \\ \tilde{0}^\dagger & \tilde{0}^\dagger & \tilde{0}^\dagger & 0 \end{bmatrix}. \tag{6.24}$$

6.6. The Dirac-Fierz-Pauli Equation for a Particle of Spin 3/2

As a last example, we consider the equation describing a particle with spin s=3/2.

We will start with the following representation of the algebra $AO(1,3)$:

$$D=D\left(1\frac{1}{2}\right)\oplus D\left(0\frac{1}{2}\right)\oplus D\left(\frac{1}{2}0\right)\oplus D\left(\frac{1}{2}1\right). \tag{6.25}$$

It is possible to show that formula (6.25) provides the simplest (i.e., realized by the matrices of the minimal dimension) representation of the matrices $S_{\mu\nu}$ for which the equations (6.7), (6.11)-(6.13) has nontrivial solutions corresponding to s=3/2.

The vectors from the space of the representation (6.25) can be chosen as a column with 16 rows. We use the following notation for such vectors

$$\psi=\left(\Phi_{3/2}^{1,1/2},\Phi_{1/2}^{1,1/2},\Phi_{1/2}^{0,1/2},\Phi_{1/2}^{1/2,0},\Phi_{1/2}^{1/2,1},\Phi_{3/2}^{1/2,1}\right), \tag{6.26}$$

where $\Phi_s^{j,\tau}$ are the $(2s+1)$-component eigenfunctions of the Casimir operators (4.12), and the operator $\boldsymbol{S}^2=S_{ab}S_{ab}/2$:

$$\Phi_s^{j,\tau}\in D(j\tau),\quad \boldsymbol{S}^2\Phi_s^{j,\tau}=s(s+1)\Phi_s^{j,\tau}. \tag{6.27}$$

From (6.8), (6.9) we obtain the following expression for the matrix β_0:

$$\beta_0=\begin{bmatrix} & & & 0 & 0 & 2C_2 \\ & 0 & & \sqrt{3}\,C_1 & -C_2 & 0 \\ & & & C_3 & \sqrt{3}\,C_4 & 0 \\ 0 & \sqrt{3}\,C_4 & C_3 & & & \\ 0 & -C_2 & \sqrt{3}\,C_1 & & 0 & \\ 2C_2 & 0 & 0 & & & \end{bmatrix}, \tag{6.28}$$

where $C_1, \dots ,C_4$ are arbitrary complex parameters. The values of these parameters can be determined up to a nonessential common multiple using the conditions (6.11):

$$C_2=C_3=1/2,\qquad C_1=C_4=i/2\sqrt{3}. \tag{6.29}$$

It is not difficult to make sure that thus defined matrix β_0 satisfies the condition (6.10) which guarantees the eigenvalue of the mass operator to be fixed. Using the relations (4.64)-(4.66), (6.7) we find the explicit form of β_a:

$$\beta_a=\begin{pmatrix} \hat{0} & B_a \\ -B_a^\dagger & \hat{0} \end{pmatrix},\quad B_a=\frac{1}{3}\begin{pmatrix} iK_a^\dagger & K_a^\dagger & 2S_a \\ i\Sigma_a & -2\Sigma_a & K_a \\ -3\Sigma_a & i\Sigma_a & iK_a \end{pmatrix}, \tag{6.30}$$

where S_a and Σ_a are the spin matrices for the spins s=3/2 and s=1/2 (see (4.31)), and K_a are the matrices of dimension 2×4:

$$K_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&3&0&5\\3&0&5&0\end{pmatrix},\quad K_2=\frac{i}{\sqrt{2}}\begin{pmatrix}0&3&0&5\\-3&0&-5&0\end{pmatrix},\quad K_3=\begin{pmatrix}\sqrt{2}&0&0&0\\0&\sqrt{2}&0&0\end{pmatrix},\tag{6.31}$$

$\hat{0}$ being zero matrices of dimension 8×8.

The equation (6.6) with the matrices (6.29)-(6.31) is invariant under the complete Poincaré group and describes a relativistic particle of mass m and spin s=3/2. This equation admits the Lagrangian formulation with the Hermitizing matrix η having the following form:

$$\eta=\begin{bmatrix} & & & P_{1/2\,3/2}\\ & & I & \\ & I & & \\ P_{3/2\,1\,2} & & & \end{bmatrix}\tag{6.32}$$

where I is the unit matrix of dimension 2×2, and $P_{j\tau}$ are the corresponding matrices p from (5.47), (4.61).

In [75, 369] there was proposed another formulation of the Dirac-Fierz-Pauli equation being suitable for a direct generalization to the case of arbitrary spin particles. In that formulation a vector-spinor wave function φ^{λ}_{μ} is used (λ being a vector index, μ spinor index) which is included in the space of the representation $D(1/2\ 1/2)\otimes$ $[D(1/2\ 0)\oplus D(0\ 1/2)]$ of the Lorentz group. According to the Clebsh-Gordon theorem (see, e.g., [3]) such a representation is decomposed into the direct sum of the IRs given by formula (6.25).

The equation for a particle of spin 3/2 in the Rarita-Schwinger formulation has the form

$$\left[(\gamma_\nu p^\nu-m)g_{\mu\nu}-\frac{1}{3}(\gamma_\mu p_\lambda+\gamma_\lambda p_\mu)+\frac{1}{3}\gamma_\mu(\gamma_\nu p^\nu+m)\gamma_\lambda\right]\varphi^\lambda=0,\tag{6.33}$$

where γ_μ are the Dirac matrices acting on the spinor index λ of the function $\varphi=column(\varphi^\mu_1,\varphi^\mu_2,\varphi^\mu_3,\varphi^\mu_4)$. Multiplying the l.h.s. of (6.33) by p^μ and γ^μ and making summation over μ we reduce this equation to the form

$$(\gamma_\nu p^\nu-m)\varphi^\lambda=0,\quad \gamma_\mu\varphi^\mu=0.\tag{6.34}$$

The equations (6.33) is equivalent to the Dirac-Fierz-Pauli equation. In fact, multiplying it by $g_{\sigma\mu}-\gamma_{\sigma\mu}$ and making summation over μ (this operation is invertible) we come to the equation in the form (6.6) which differs from the Dirac-Fierz-Pauli equation only in the realization of the matrices β_μ.

6.7. Transition to the Schrödinger Form

Solutions of the considered equations include more components that it is necessary for a description of the internal degrees of freedom of a particle (and antiparticle) with spin $s>1/2$ (i.e., more then $2(2s+1)$). The only exception is the four-component Dirac equation for a particle of spin 1/2. Besides, by elimination of redundant components we can reduce these equations to the Schrödinger form (2.10) with ψ being a $2(2s+1)$-component wave function, and H being a Hamiltonian which can be either differential or integro-differential operator.

For the Dirac equation the transition to the Schrödinger formulation is trivial (see Subsection 2.2), so we consider the KDP and Dirac-Fierz-Pauli equations only.

Using algebraic properties of the β-matrices we consider the KDP equations for $s=0$ and $s=1$ simultaneously.

Let the matrices β_μ in (6.6) satisfy the KDP algebra. Multiplying (6.6) by β_0^2 and $1-\beta_0^2$ and taking into account that $\beta_0^3=\beta_0$, $(1-\beta_0^2)\beta_a=\beta_a\beta_0^2$, we obtain after simple calculations the following system:

$$i\frac{\partial}{\partial x_0}\psi=H^K\psi,\quad H^K=[\beta_0,\beta_a]p_a+\beta_0 m, \tag{6.35a}$$

$$\left(1-\beta_0^2-\frac{1}{m}\beta_a p_a\beta_0^2\right)\psi=0. \tag{6.35b}$$

The system (6.35) is completely equivalent to (6.6) and includes the equation in the Schrödinger form (6.35a) and the additional condition (6.35b) which expresses the "nonphysical" components $(1-\beta_0^2)\psi$ via $2(2s+1)$ essential components $\beta_0^2\psi$.

Let us transform the system (6.35) to a representation with the nonphysical components being equal to zero. Using for this purpose the operator

$$V=1-\frac{1}{m}\beta_a p_a\beta_0^2,\quad V^{-1}=1+\frac{1}{m}\beta_a p_a\beta_0^2, \tag{6.36}$$

we obtain

$$\begin{gathered}V\left(1-\beta_0^2-\frac{1}{m}\beta_a p_a\beta_0^2\right)V^{-1}\equiv 1-\beta_0^2,\\ V([\beta_0,\beta_a]p_a-\beta_0 m)V^{-1}=\beta_0\left(\beta_a p_a-m+\frac{(\beta_a p_a)^2}{m}\right).\end{gathered} \tag{6.37}$$

According to (6.36), the transformed function $\psi'=V\psi$ satisfies the condition

$$(1-\beta_0^2)\psi'=0 \tag{6.38}$$

from which it follows that $\beta_0\beta_a p_a\psi\equiv 0$, and (6.35a) reduces to the form

$$i\frac{\partial}{\partial x_0}\psi'=H\psi'\equiv\left[-\beta_0 m+\beta_0\frac{(\beta_a p_a)^2}{m}\right]\psi'. \tag{6.39}$$

Choosing for β_μ the representation (6.17) or (6.22), we conclude from (6.38) that (s+3) components of ψ' are identically equal to zero and that the Hamiltonian H can be written in the form

$$H=\sigma_2\left(m+\frac{p^2}{2m}\right)-\frac{i}{2m}\sigma_1[2(\boldsymbol{S}\cdot\boldsymbol{p})^2-p^2]. \tag{6.40}$$

Here σ_1 and σ_2 are the 2(2s+1)-row Pauli matrices (see (5.30)), $\boldsymbol{S}$ are generators of the direct sum *D(s)*⊕*D(s)* of the group *O(3)*, i.e., the matrices (3.6) for s=1 and zero matrices for s=0. By the additional transformation $H\rightarrow UHU^\dagger$, $U=(1-i\sigma_3)/\sqrt{2}$, we can change $\sigma_2\rightarrow\sigma_1$, $\sigma_1\rightarrow-\sigma_2$ and obtain the standart form of Hamiltonian discussed in the following subsection.

We see that the KDP equations reduces to the Schrödinger form by elimination of nonphysical components. The corresponding Hamiltonians are second order differential operators with matrix coefficients. The equation (2.10), (6.39) was proposed for the first time by Tamm [401], Sakata and Taketani [377].

Consider the Dirac-Fierz-Pauli equation for a particle of spin 3/2. Multiplying (6.6) from the left by the invertible matrix C,

$$C=\begin{pmatrix}\hat{C} & 0\\ 0 & \hat{C}\end{pmatrix},\quad \hat{C}=\begin{pmatrix}I & 0 & 0\\ 0 & i1 & 1\\ 0 & -i1 & 1\end{pmatrix},$$

where I and 1 are the 4×4 and 2×2 unit matrices, and denoting the components of the wave function (6.26) according to the formulae

$$\Phi^{1,1/2}_{3/2}=\Phi_1,\quad \Phi^{1,1/2}_{1/2}=\frac{i}{2}(\Phi_3-\Phi_2),\quad \Phi^{0,1/2}_{1/2}=\frac{1}{2}(\Phi_3+\Phi_2),$$

$$\Phi^{1/2,0}_{1/2}=\frac{1}{2}(\Phi_4+\Phi_5),\quad \Phi^{1/2,1}_{1/2}=\frac{i}{2}(\Phi_4-\Phi_5),\quad \Phi^{1/2,1}_{3/2}=\Phi_6,$$

we come to the following system

$$p_0\Phi_1+\frac{1}{3}\boldsymbol{K}\cdot\boldsymbol{p}\Phi_4+\frac{2}{3}\boldsymbol{S}\cdot\boldsymbol{p}\Phi_1-m\Phi_6=0, \tag{6.41a}$$

$$-\frac{4}{3}i\Sigma\cdot\boldsymbol{p}\Phi_5+\frac{2}{3}i\boldsymbol{K}^\dagger\cdot\boldsymbol{p}\Phi_1-m\Phi_2=0, \tag{6.41b}$$

$$-\frac{2}{3}i\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_6+\frac{4}{3}\Sigma\cdot\boldsymbol{p}\Phi_2-im\Phi_5=0, \tag{6.41c}$$

$$p_0\Phi_5+\frac{2}{3}\Sigma\cdot\boldsymbol{p}(2\Phi_4+\Phi_5)+im\Phi_3=0, \tag{6.41d}$$

$$p_0\Phi_2+\frac{2}{3}\Sigma\cdot\boldsymbol{p}(2\Phi_3-\Phi_2)+im\Phi_4=0, \tag{6.41e}$$

$$p_0\Phi_6-\frac{2}{3}\boldsymbol{S}\cdot\boldsymbol{p}\Phi_6-\frac{i}{3}\boldsymbol{K}\cdot\boldsymbol{p}\Phi_3-m\Phi_1=0, \tag{6.41f}$$

Acting on ψ (6.26) by the projector $P_s=\beta_0^2$ (see Subsection 6.4), we see that the physical components of the wave function are Φ_1 and Φ_6. Unlike the case of the KDP equation, the redundant components Φ_2, Φ_3, Φ_4 and Φ_5 cannot be expressed via Φ_1 and Φ_6 and their first derivatives. But the corresponding expressions may be obtained with the help of nonlocal (integral) operators.

Considering (6.41) in the momentum representation, using the identity $(\Sigma\cdot\boldsymbol{p})^2=p^2/4$ and relations (12.26), we express nonphysical components via Φ_1 and Φ_6:

$$\begin{aligned}
&\Phi_2=iF\left(\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_1+\frac{4}{3m}\Sigma\cdot\boldsymbol{p}\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_6\right), \quad F=\frac{2m}{3\left(m^2+\frac{9}{4}p^2\right)},\\
&\Phi_3=iF\left(\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_1-\frac{4}{3m}\Sigma\cdot\boldsymbol{p}\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_6\right),\\
&\Phi_4=F\left(-\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_6+\frac{4}{3m}\Sigma\cdot\boldsymbol{p}\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_1\right),\\
&\Phi_5=F\left(-\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_6+\frac{4}{3m}\Sigma\cdot\boldsymbol{p}\boldsymbol{K}^{\dagger}\cdot\boldsymbol{p}\Phi_1\right).
\end{aligned} \tag{6.42}$$

Substituting (6.41) into (6.41a), (6.41f), we come to the equation in the Schrödinger form (2.10) for the eight-component wave function $\psi=column(\Phi_1,\Phi_6)$, where, according to [308],

$$H=\sigma_3\hat{\boldsymbol{S}}\cdot\boldsymbol{p}\left\{1+\frac{3F}{2m}[p^2-(\hat{\boldsymbol{S}}\cdot\boldsymbol{p})^2]\right\}+\sigma_1 m\left\{+\frac{3F}{4m}[p^2-(\hat{\boldsymbol{S}}\cdot\boldsymbol{p})^2]\right\}, \quad \hat{\boldsymbol{S}}=(2/3)\boldsymbol{S}. \tag{6.43}$$

Here σ_1 and σ_3 are the 8×8 Pauli matrices commuting with $\boldsymbol{S}$, $\boldsymbol{S}$ being generators of the direct sum *D(3/2)⊕D(3/2)* of IRs of the group *O(3)* (compare with (7.6)).

The Hamiltonian (6.43) had been obtained in [308] starting from another realization of the Dirac-Fierz-Pauli equation.

The Hamiltonian (6.43) can be defined as a nonlocal (integral) operator in the x-representation. This signifies that the Dirac-Fierz-Pauli equation corresponds to the integral evolution equation for physical components of a wave function. In other words, it is essentially a nonlocal equation. This is the main reason for paradoxes with the causality violation which take place when using this equation to describe a particle of spin 3/2 in the external electromagnetic field.

We note that the nonlocal character of the Hamiltonian (6.43) is caused by nilpotency of the corresponding matrix β_0 in (6.6). As far as this matrix is nilpotent for any Poincaré- and P-, T-, and C-invariant equations (6.6) describing a particle with a fixed value of spin $s>1$, then such an equation corresponds to an integral Hamiltonian and is nonlocal in this sense.

7. POINCARÉ-INVARIANT EQUATIONS WITHOUT SUPERFLUOUS COMPONENTS

7.1. Preliminary Discussion

As was noted in the Introduction, the covariant first order wave equations for particles with higher spins can be used neither for a description of particle interaction with an external field, nor for the constructing of a second quantized theory. The only relativistic equation which does not lead to contradictions when solving real physical problems is the Dirac equation.

What is the reason to distinguish the Dirac equation from other covariant equations? There are several reasons for this. The Dirac equation possessing the unique set of properties given below.

(1) The transparent relativistic invariance. The covariant representation of the Poincaré algebra can be realized on the set of solutions of the Dirac equation.

(2) The existence of the Schrödinger formulation (2.10) with H being a local (differential) operator.

(3) The wave function satisfying the Dirac equation has $4=2(2s+1)$ components which exactly corresponds to the number of the spin degrees of freedom for a particle and antiparticle of spin 1/2.

(4) The Poincaré group generators defined on the set of solutions of the Dirac equation are Hermitian under the usual scalar product (2.39).

Covariant equations of first order considered in the preceding section possess (for $s>1/2$) the property (1) and, as an exception, the property (2). As to the properties (3), (4), the only equation (6.6) possessing those is the Dirac equation. It is the existence of superfluous components (i.e., those whose number is more than $2(2s+1)$) of a wave function ψ, which is the source of difficulties for the theory of higher spinequations.

In connection with the above arises the natural desire to find such equations for particles of arbitrary spin which have no superfluous components and possess the other properties mentioned in (1)-(4). It happens that such equations do exist, but the price to be paid for the absence of nonphysical components is the loss of a transparent relativistic invariance (i.e., evident symmetry between spatial and time variables).

In this section we deduce such equations for particles of any spin.

7.2. Formulation of the Problem

We will search for Poincaré-invariant equations for a particle of arbitrary spin in the Schrödinger form

$$i\frac{\partial}{\partial x_0}\psi=H_s\psi, \tag{7.1}$$

where $\psi=\psi(x_0,\boldsymbol{x})$ is a $2(2s+1)$-component wave function,

$$\psi=column(\psi_1,\psi_2,\ldots,\psi_{2(2s+1)}), \tag{7.2}$$

and $H_s=H_s(\boldsymbol{p})$ is an unknown linear differential operator to be determined from the condition of (7.1) being Poincaré-invariant.

DEFINITION 7.1. We say the equation (7.1) is Poincaré-invariant and describes a particle of mass m and spin s if it is invariant under the algebra $AP(1,3)$, and, besides, the Casimir operators $C_1=P_\mu P^\mu$ and $C_2=W_\mu W^\mu$ satisfy the conditions (6.1).

According to the definition, the Hamiltonian $H_s=P_0$ has to satisfy the following conditions:

$$[H_s, P_a]=0, \tag{7.3a}$$

$$[H_s, J_a]=0, \tag{7.3b}$$

$$[H_s, N_a]=iP_a \tag{7.3c}$$

where P_a, J_a and N_a are the basis elements of the algebra $AP(1,3)$ satisfying the relations

$$[P_a,P_b]=0, \quad [P_a,J_b]=i\varepsilon_{abc}P_c, \tag{7.4a}$$

$$[J_a,J_b]=i\varepsilon_{abc}J_c, \quad [J_a,N_b]=i\varepsilon_{abc}N_c, \tag{7.4b}$$

$$[P_a,N_b]=i\delta_{ab}H_s, \tag{7.4c}$$

$$[N_a,N_b]=-i\varepsilon_{abc}J_c. \tag{7.4d}$$

It is not difficult to see that formulae (7.3), (7.4) represent nothing but a new form of the commutation relations (1.14) characterizing the Poincaré algebra.

Thus, to describe Poincaré-invariant equations in the Schrödinger form (7.1) it is sufficient to choose a representation of the generators P_a, J_a and N_a as satisfying the commutation relations (7.4), and then to find all the possible operators H_s satisfying (7.3).

As is well known (see Subsection 4), the conditions (6.1) define the operators P_a, J_a and N_a up to unitary equivalence. It is natural in choosing a specific realization of P_a, J_a and N_a to require that the representation of these operators to be defined for arbitrary spin s will make it possible to obtain the Dirac equation for s=1/2. In other words, it is desirable to generalize the representation of the algebra *AP(1,3)*, realized on the set of solutions of the Dirac equation, for the case of arbitrary spin s. We consider three of such generalizations corresponding to three different approaches to description of Poincaré-invariant equations without superfluous components [147, 331].

We will start with the following realization of the Poincaré group generators:

$$P_0=H_s,\quad \boldsymbol{P}=\boldsymbol{p},\quad \boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S},$$
$$\boldsymbol{N}=x_0\boldsymbol{p}-\frac{1}{2}[\boldsymbol{x},H_s]_++\boldsymbol{\lambda}_s, \tag{7.5}$$

where $\boldsymbol{S}$ are matrices realizing a direct sum of two IRs $D(s)$ of the algebra *AO(3)*

$$\boldsymbol{S}=\begin{pmatrix}\boldsymbol{s} & 0\\ 0 & \boldsymbol{s}\end{pmatrix},\quad \boldsymbol{s}\in D(s); \tag{7.6}$$

$\boldsymbol{x}$ and $\boldsymbol{p}$ are canonically conjugated variables defined by the relations

$$[x_a,p_b]=i\delta_{ab}; \tag{7.7}$$

H_s and λ_s are yet unknown operators whose form will be obtained by applying the requirement for the operators (7.5) to satisfy the commutation relations (7.3), (7.4).

Formulae (7.5) represent the general form of basis elements of the algebra *AP(1,3)*, which generate local transformations corresponding to translations and spatial rotations of a frame of reference.

On the set of solutions of the Dirac equation the Poincaré group generators have the form (7.5) where $\lambda_s \equiv 0$. Thus we assume that for the first approach *

$$\lambda_s^I = 0. \tag{7.8}$$

The distinguishing feature of the representation (7.5), (7.8) is that all the Poincaré group generators are Hermitian in respect to the scalar product (2.39), where ψ_1, ψ_2 are 2(2s+1)-component wave functions satisfying (7.1).

In the second approach we choose λ_s in the form

$$(\lambda_s^{II})_a = -\frac{1}{2}[H_s^{II}, x_a] + S_{0a}, \quad S_{0a} = i\sigma_3 S_a = i\begin{pmatrix} s_a & 0 \\ 0 & -s_a \end{pmatrix}. \tag{7.9}$$

Here s_a are (2s+1)×(2s+1) matrices belonging to the IR *D(s)* of the algebra *AO(3)*, 0 are the zero matrices and σ_3 is the 2(2s+1)-row Pauli matrices (5.30).

The merit of the representations (7.5), (7.9) is that they correspond to the local transformations of the wave function ψ by the transition to a new frame of references. Actually, on the set of the equation (7.1) solutions the generators (7.5), (7.9) can be represented in the covariant form (2.22) where $S_{\mu\nu}$ are matrices realizing the representation $D(0\ s)\oplus D(s\ 0)$ of the algebra *AO(1,3)*. However, these generators are not Hermitian in respect to the scalar product (2.39). The only exception is the case s=1/2 when the operators (7.8) and (7.9) coincides.

We define the operators of space inversion *P*, time reflection *T*, and charge conjugation *C* according to formulae (2.55), (2.60) where ψ is a 2(2s+1)-component wave function, r_1, r_2, and r_3 are Hermitian matrices which can be determined from the condition that the operators *P*, *T*, and *C* satisfy relations (1.54) together with the Poincaré group generators. It can be shown that up to unitary equivalence it is sufficient to restrict ourselves to considering the matrices of the following form:

$$r_1^I = \sigma_1 \quad or \quad r_1^I = \sigma_0, \quad r_1^{II} = \sigma_1, \qquad r_2^I = \sigma_2, \quad r_2^{II} = \sigma_2, \tag{7.10a}$$

$$r_3^I = \sigma_2\Delta, \quad r_3^{II} = \sigma_2\Delta \quad or \quad r_3^{II} = \sigma_1\Delta, \quad \Delta = \begin{pmatrix} \Delta' & 0 \\ 0 & \Delta' \end{pmatrix}, \tag{7.10b}$$

where Δ' is a (2s+1)×(2s+1) matrix determined up to a sign by the relations (refer to (5.24), (5.34))

$$\Delta' \boldsymbol{s} = -\boldsymbol{s}^* \Delta', \quad \Delta'^2 = 1. \tag{7.11}$$

We require the equation (7.1) be invariant under the transformations

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* We will use the indices *I, II,* and *III* to distinguish the operators considered in the first, second, and third approaches, respectively.

(2.55),(2.60) from which it follows that H_s has to satisfy the following relations

$$[P, H_s] = [T, H_s]_+ = [C, H_s]_+ = 0. \tag{7.12}$$

The approaches *I* and *II* make it possible to describe a wide class of Poincaré-invariant equations without superfluous components which, however, does not include the Tamm-Sakata-Taketani equation considered in Subsection (6.5). The class of equations including the Dirac and Tamm-Sakata-Taketani equation can be obtained in the third approach for which the problem is formulated as follows: to find all the possible differential equations (7.1) invariant under the Poincaré algebra generated by the operators (7.5). Besides, we assume that the corresponding Hamiltonians belong to the class of second order differential operators. At the same time we do not impose any restrictions on the form of λ_s^{III} and do not require *P*-, *T*- and *C*-invariance of the equation (7.1). We will see that such a formulation makes it possible to determine Hamiltonian of an arbitrary spin particle up to equivalence transformations.

7.3. The Explicit Form of Hamiltonians H_s^I and H_s^{II}

Following [133], let us find all the possible (up to equivalence) Hamiltonians H_s^I. It is not difficult to make sure that the equations (7.4) reduces to identities if the equations (7.3), (7.4d) are satisfied. Substituting (7.5), (7.8) into this equations we obtain

$$(H_s^I)^2 = p^2 + m^2, \tag{7.13a}$$

$$[H_s, \boldsymbol{J}] = [H_s, \boldsymbol{p}] = 0, \tag{7.13b}$$

$$[H_s, \boldsymbol{x}] \times [H_s, \boldsymbol{x}] = -4i\boldsymbol{S}. \tag{7.13c}$$

Relations (7.13) are the necessary and sufficient conditions for equations (7.3), (7.4) to be satisfied.

We represent H_s as an expansion in the complete set of orthoprojectors

$$H_s^I = \sum_{\nu=-s}^{s} h_\nu \Lambda_\nu, \tag{7.14}$$

where

$$\Lambda_\nu = \prod_{\nu' \neq \nu} \frac{S_p - \nu'}{\nu - \nu'}, \quad S_p = \frac{\boldsymbol{S} \cdot \boldsymbol{p}}{p}, \tag{7.15}$$

h_ν are matrices commuting with $\boldsymbol{S}$ which can be decomposed by the complete set of the Pauli matrices (5.30)

$$h_\nu = a_\nu^\mu \sigma_\mu, \tag{7.16}$$

and a_ν^μ are unknown functions of p.

It is not difficult to make sure the operators (7.15) are the projectors into eigenstates of the operator $\boldsymbol{S}\,\boldsymbol{p}/p$ and satisfy the following relations

$$\Lambda_\nu \Lambda_{\nu'} = \delta_{\nu\nu'} \Lambda_\nu, \quad \sum_{\nu=-s}^{s} \Lambda_\nu = 1, \quad \sum_{\nu=-s}^{s} \nu \Lambda_\nu = S_p. \tag{7.17}$$

The operators Λ_ν commute with $\boldsymbol{x}$ as follows [133]:

$$[\boldsymbol{x}, \Lambda_\nu] = \frac{\boldsymbol{p}\times\boldsymbol{S}}{2p^2}(\Lambda_{\nu-1} + \Lambda_{\nu+1} - 2\Lambda_\nu) - \frac{i}{2p}\left(\boldsymbol{S} - \frac{\boldsymbol{p}}{p}S_p\right)(\Lambda_{\nu-1} - \Lambda_{\nu+1}). \tag{7.18}$$

Using (7.18), we can prove the following assertion: the vectors $\boldsymbol{p}\times\boldsymbol{S}\Lambda_\nu$, $\boldsymbol{S}\Lambda_\nu$, $\boldsymbol{p}\Lambda_\nu$ are linearly independent if $\nu \neq \pm s$. If $\nu = \pm s$ then

$$\frac{\boldsymbol{p}\times\boldsymbol{S}}{p}\Lambda_{\pm s} = \mp\left(\boldsymbol{S} \mp s\frac{\boldsymbol{p}}{p}\right)\Lambda_{\pm s}. \tag{7.19}$$

Relations (7.14)-(7.19) enable us to reduce equations (7.13) to the system of algebraic equations for the coefficients a_ν^μ. As a matter of fact, we obtain from (7.13a), (7.14), (7.17)

$$h_\nu^2 = m^2 + p^2, \tag{7.20}$$

and then from (7.13c), (7.18) we obtain

$$\frac{1}{2}[h_\nu, h_{\nu+1}]_+ = m^2 - p^2. \tag{7.21}$$

Relations (7.20), (7.21) are necessary and sufficient conditions for the Hamiltonian H_s^I to satisfy equations (7.13). The general solution of these relations and relations (7.12) is given by the formula

$$h_\nu = \sigma_1 E\cos\varphi_\nu + \sigma_3 E\sin\varphi_\nu, \tag{7.22}$$

where the possible values of φ_ν are determined by the recurrence relations

$$\varphi_{\nu+1} = \varphi_\nu \pm 2\theta^I, \quad \theta^I = \arctan\frac{p}{m}. \tag{7.33}$$

If the space inversion matrix r_1^I is equal to σ_1 then we have for any s

$$\varphi_0 = 0, \quad \varphi_{1/2} = \theta^I, \quad \varphi_\nu = -\varphi_{-\nu}, \tag{7.24}$$

but for $r_1^I = \sigma_0$ solutions of the equations (7.20), (7.21) compatible with (2.55), (2.60), (7.12) exist for integer s only. Moreover,

$$\varphi_0 = \varphi(p/m), \quad \varphi_\nu = \varphi_{-\nu} \tag{7.25}$$

where φ is an arbitrary function of p/m.

According to (7.23) the number of possible Hamiltonians H_s^I increases with increasing of s because by calculating $\varphi_{\nu+1}$ we can choose any of possible signs on each step. It is not difficult to calculate that for $r_1=\sigma_1$ the total number of different H_s^I is equal to $2^{[s]}$ where $[s]$ is the entire part of s.

We represent the simplest solutions of the recurrence relations (7.23) compatible with (7.24), (7.25):

$$(\varphi_\nu)_1=(-1)^{[\nu]}\theta^I, \quad (\varphi_\nu)_2=2\nu\theta^I. \tag{7.26}$$

By substituting (7.22), (7.26) into (7.15) we obtain

$$(H_s^I)_1=\sigma_1 m+\sigma_3 p\sum_\nu (-1)^{[\nu]}\lambda_\nu, \tag{7.27}$$

$$(H_s^I)_2=E\left[\sum_\nu \sigma_1\cos(2\nu\theta^I)+\sigma_3\sin(2\nu\theta^I)\right]\Lambda_\nu. \tag{7.28}$$

The operators (7.27) and (7.28) coincide in the case of $s=1/2$ and reduce to the Dirac Hamiltonian (2.11) where $\gamma_0=\sigma_1$ and $\gamma_a=-2i\sigma_2 S_a$.

We represent also the explicit expressions for the Hamiltonians (7.27), (7.28) in the terms of the helicity operators S_p for $s\leq 3/2$. According to (7.15) we obtain

$$(H_0^I)_1=\sigma_1 E, \quad (H_0^I)_2=\sigma_1 m+\sigma_3 p,$$

$$(H_{1/2}^I)_1=(H_{1/2}^I)_2=\sigma_1 m+2\sigma_3 \boldsymbol{S}\cdot\boldsymbol{p},$$

$$(H_1^I)_1=\sigma_1 m+\sigma_3 p(1-2S_p^2), \tag{7.29a}$$

$$(H_1^I)_2=\left\{\sigma_1[E^2-2(\boldsymbol{S}\cdot\boldsymbol{p})^2]+2\sigma_3 m\boldsymbol{S}\cdot\boldsymbol{p}\right\}E^{-1},$$

$$(H_{3/2}^I)_1=\sigma_1 m+\frac{1}{3}\sigma_3 pS_p(7-4S_p^2),$$

$$(H_{3/2}^I)_2=\left\{\sigma_1[2E^2+p^2+2(\boldsymbol{S}\cdot\boldsymbol{p})^2]m+\sigma_3\left[(2E^2+p^2/12)\boldsymbol{S}\cdot\boldsymbol{p}-4/3(\boldsymbol{S}\cdot\boldsymbol{p})^3\right]\right\}E^{-2}. \tag{7.29b}$$

As it can be seen from (7.19), the Hamiltonians H_s^I for $s\neq 1/2$ are nonlocal (integro-differential) operators in x-representation. Let us recall that first order covariant wave equations (6.6) also lead to nonlocal Hamiltonians in general.

The problem of finding the exact form of the Hamiltonians H_s^{II} can be solved in complete analogy with the above. The Poincaré-invariance conditions (7.3), (7.4) for the representation (7.8), (7.9) reduce to relations (7.3b), (7.3c), the last of them being of the form

$$-[H_s^{II},\boldsymbol{x}]H_s^{II}+i\boldsymbol{S}[H_s^{II},\sigma_3]+i[H_s^{II},\boldsymbol{S}]\sigma_3=i\boldsymbol{p}. \tag{7.30}$$

The general solution of the equations (7.3b), (7.3c) compatible with (7.12) is given by the formulae [297, 410]

$$(H_s^{II})_1 = E\sum_\nu [\sigma_1 sech(2\nu\theta^{II}) + \sigma_3 \tanh(2\nu\theta^{II})]\Lambda_\nu, \quad r_3 = \sigma_1\Delta, \tag{7.31}$$

$$(H_s^{II})_2 = E\sum_\nu [i\sigma_1 cosech(2\nu\theta^{II}) + \sigma_3 \coth(2\nu\theta^{II})]\Lambda_\nu, \quad r_3 = \sigma_2\Delta, \tag{7.32}$$

where

$$\theta^{II} = arctanh\frac{p}{E}. \tag{7.33}$$

The Hamiltonians (7.31) are defined for arbitrary values of s, but (7.32) are valid for half integer spins only because $(H_s^{II})_2$ does not satisfy (7.13a) for integer s [147, 331].

For $s=1/2$ the operator (7.31) reduces to the Dirac Hamiltonian.

Let us present the explicit expressions for H_s^{II}, $s \leq 3/2$ in the terms of helicity operators:

$$\begin{aligned} &H_0^{II} = \sigma_1 E,\\ &(H_1^{II})_1 = \sigma_1 E + 2E\boldsymbol{S}\cdot\boldsymbol{p}[\sigma_1 \boldsymbol{S}\cdot\boldsymbol{p} - \sigma_3 E](E^2+p^2)^{-1},\\ &(H_{3/2}^{II})_1 = \Big\{\sigma_1[(2E^2+7p^2)/2 - 2(\boldsymbol{S}\cdot\boldsymbol{p})^3]m +\\ &\qquad +\sigma_3\left[\boldsymbol{S}\cdot\boldsymbol{p}(20p^2+6E^2)/3 - \frac{8}{3}(\boldsymbol{S}\cdot\boldsymbol{p})^3\right]\Big\}(E^2+3p^2)^{-1}. \end{aligned} \tag{7.34}$$

The operators (7.34) correspond to $r_3^{II} = \sigma_1\Delta$. In the cases $s=1/2$ and $s=3/2$ there exist two more Hamiltonians, one for each value $s=1/2,\ 3/2$:

$$(H_{1/2}^{II})_2 = 2E\boldsymbol{S}\cdot\boldsymbol{p}(i\sigma_1 m + \sigma_3 E)p^{-2},$$

$$\begin{aligned} (H_{3/2}^{II})_2 = \Bigg(& i\sigma_1\frac{mE}{3p^2}[(20E^2+7p^2)\boldsymbol{S}\cdot\boldsymbol{p} - 4(p^2+2E^2)(\boldsymbol{S}\cdot\boldsymbol{p})^3 p^{-2}] +\\ &+\sigma_3\frac{E^2}{3p^2}[(20E^2+6p^2)\boldsymbol{S}\cdot\boldsymbol{p} - 8E^2(\boldsymbol{S}\cdot\boldsymbol{p})^3 p^{-2}]\Bigg)(p^2+3E^2)^{-1}. \end{aligned}$$

These operators correspond to the choice $r_3^{II} = \sigma_2\Delta$.

We see the operators H_s^{II} as well as H_s^{I} are nonlocal in x-representation. Besides, the boost generators $\boldsymbol{N}^{II}$ are non-Hermitian in the metric (2.39). The invariant bilinear form for the group transformations generated by $\boldsymbol{N}^{II}$ is as follows [298, 410]:

$$(\psi_1, \psi_2) = \int d^3x \psi_1^\dagger M \psi_2, \tag{7.35}$$

where

$$M=\sigma_3\frac{E}{m}\sum_\nu cosech(2\nu\theta^{II})\Lambda_\nu,\quad r_3^{II}=\sigma_2\Delta,$$

$$M=\frac{E}{m}\sum_\nu sech(2\nu\theta^{II})\Lambda_\nu,\quad r_3^{II}=\sigma_1\Delta.$$

7.4. Differential Equations of Motion for Spinning Particles

The equations without superfluous components considered above include integro-differential operators H_s if $s\neq 1/2$. Of course, this fact complicates the using of these equations for solution of specific physics problems. In this subsection we consider the third approach which leads to differential equations of motion.

We represent a Hamiltonian H_s^{III}, which has to be found, as an expansion in terms of spin matrices and 2(2s+1)-row Pauli matrices (5.30)

$$H_s^{III}=h_0^s m+h_1^s+\frac{1}{m}h_2^s, \tag{7.36}$$

where

$$h_0^s=a_\mu^s\sigma_\mu,\quad h_1^s=b_\mu^s\sigma_\mu \boldsymbol{S}\cdot\boldsymbol{p},\quad h_2^s=c_\mu^s\sigma_\mu(\boldsymbol{S}\cdot\boldsymbol{p})^2+d_\mu^s\sigma_\mu p^2.$$

The operators (7.36) satisfy relations (7.3b) for any values of constant coefficients $a_\mu^s, b_\mu^s, c_\mu^s, d_\mu^s$. The remaining Poincaré-invariant conditions (7.3a), (7.3c), (7.4) define these coefficients up to an arbitrary parameter. We give the explicit form of all nonequivalent Hamiltonians H_s^{III} (for the proof see [147, 331]):

$$H_s^{III}=\sigma_1 m+\sigma_3 2k_1\boldsymbol{S}\cdot\boldsymbol{p}+\frac{1}{2m}(\sigma_1-i\sigma_2)[p^2-4k_1(\boldsymbol{S}\cdot\boldsymbol{p})^2],\quad s=0,\frac{1}{2},1,..., \tag{7.37a}$$

$$H_1^{III}=\sigma_1 m+(\sigma_1-i\sigma_2)\frac{p^2}{2m}+[i\sigma_2k_2+\sigma_3\sqrt{k_2(k_2-1)}\,]\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}, \tag{7.37b}$$

$$H_1^{III}=\sigma_1 m+\sigma_3k_3\boldsymbol{S}\cdot\boldsymbol{p}+(\sigma_1-i\sigma_2)\frac{p^2}{2m}-[k_3^2\sigma_1+i(k_3^2-2)\sigma_2]\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}, \tag{7.37c}$$

$$H_{3/2}^{III}=\sigma_1\left(m+\frac{p^2}{2m}\right)+\frac{ik_4}{2m}\sigma_2\left[(\boldsymbol{S}\cdot\boldsymbol{p})^2-\frac{5}{4}p^2\right]+\frac{\sigma_3}{2m}\sqrt{k_4^2-1}\,p^2, \tag{7.37d}$$

(7.37e)

where k_1, k_2, ... , k_5 are arbitrary complex parameters.

Formulae (7.37) give all the nonequivalent Hamiltonians H_s^{III} defined up to equivalence transformations generated by numerical matrices. Besides the Hamiltonian (7.37a) defined for any s, there exist two pairs of additional Hamiltonians for particles of spins 1 and 3/2. Each of the operators (7.37) depends on arbitrary complex number k_l, l=1, 2, ..., 5.

It is not difficult to make sure the Hamiltonians (7.37) are non-Hermitian in respect to the scalar product (2.39). But it is possible to choose proper values of the coefficients k_l for these operators to be Hermitian in the indefinite metric

$$(\psi_1,\psi_2)=\int d_3x\psi_1^\dagger\sigma_1\psi_2. \tag{7.38}$$

Namely, the operators H_s^{III} are Hermitian in respect to this metric iff

$$\begin{gathered} k_1^*=-k_1, \quad k_3^*=-k_3, \quad k_5^*=-k_5, \\ k_2^*=k_2, \quad 0<k_2^2<1, \quad k_4^*=k_4, \quad 0<k_4<1. \end{gathered} \tag{7.39}$$

It is not difficult to show that any equation (7.1), (7.37) is invariant under the product of the transformations TC, but in general is not invariant under the P-, T- and C-transformations. By requiring the symmetry under any of these transformations we reduce the class of the operators H_s^{III} to the following representatives

$$H_s^{III}=\sigma_1\left(m+\frac{p^2}{2m}\right)-i\sigma_2\frac{p^2}{m}, \tag{7.40a}$$

$$H_{1/2}^{III}=\sigma_1 m+2\sigma_3\boldsymbol{S}\cdot\boldsymbol{p}, \tag{7.40b}$$

$$H_1^{III}=\sigma_1\left(m+\frac{p^2}{2m}\right)+\frac{i}{2m}\sigma_2[2(\boldsymbol{S}\cdot\boldsymbol{p})^2-p^2], \tag{7.40c}$$

(7.40d)

The operators (7.40b) and (7.40a), (7.40c) coincide with the Dirac and TST Hamiltonians.

The Hamiltonian (7.40a) for $s\neq 0$ is not of great interest inasmuch as it does not depend on spin matrices and so do not possesses any information about a particle spin. We see that there exist only four Poincaré- and P-, T-, C-invariant equations (7.1) where H_s is a second-order differential operator depending on spin matrices. There are the Dirac, TST equations and the equation (7.1), (7.40d) for a particle of spin 3/2. The corresponding C-, P-, and T-transformations are given by formulae (2.55), (2.60) where

$$r_1^{III}=1, \quad r_2^{III}=\sigma_3, \quad r_1^{III}=\sigma_3\Delta, \tag{7.41}$$

So for particles of spin $s>3/2$ there exist no Poincaré and P-, T-, C-invariant Hamiltonians others then the trivial operators (7.40a). Moreover, it is possible to show that such Hamiltonians also do not exist in the class of differential operators of arbitrary finite order. Thus, when describing a particle of spin $s>3/2$, it is necessary to choose between nonlocal (integro-differential) equations (7.1) considered in the preceding sections and the equations being non-invariant under the space inversion.

7.5. Connection with the Shirokov-Foldy Representation

It is easy to note that the problem of finding of Poincaré-invariant motion equations for the approaches *I-III* reduces to the description of some special realizations of the representations of the algebra *AP(1,3)* belonging to the class *I*. The natural question arises about the relation of these realizations to the IRs of the algebra *AP(1,3)* considered in Section 4.

Here we establish such a connection and demonstrate that all the representations considered in the present section are equivalent to the direct sum $D^{+}(s)\oplus D^{-}(s)$ of the IRs.

To simplify calculations we start from the Shirokov-Foldy realization where the basis elements of the mentioned direct sum of IRs have the form (compare with (4.50))

$$\begin{gathered} P_0=\sigma_1 E, \quad P_a=p_a, \quad \boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S}, \\ \boldsymbol{N}=x_0\boldsymbol{p}-\frac{1}{2}\sigma_1[\boldsymbol{x},E]_+-\sigma_1\frac{\boldsymbol{p}\times\boldsymbol{S}}{E+m}. \end{gathered} \tag{7.42}$$

The operators (7.42) and (7.5), (7.8), (7.9) are connected by the relation

$$P_\mu^\alpha=P_\mu, \quad \boldsymbol{J}_\mu^\alpha=\boldsymbol{J}, \quad \boldsymbol{N}^\alpha=V^\alpha\boldsymbol{N}(V^\alpha)^{-1}, \quad \alpha=I,II,III, \tag{7.43}$$

where V^α are invertible operators of the following form:

$$\begin{gathered} V^I=\exp\left(\frac{i}{2}\sigma_2\sum_\nu\varphi_\nu\Lambda_\nu\right), \\ (V^{II})_1=\sqrt{\frac{m}{E}}\left\{\cosh\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\theta^{II}\right)+i\sigma_2\sinh\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\theta^{II}\right)\right\}, \\ V^{III}=(E+\sigma_1H_s^{III})\left[2E\left(E+\frac{1}{2}[\sigma_1,H_s^{III}]_+\right)\right]^{-1/2}, \end{gathered} \tag{7.44}$$

where φ_ν, θ^{II} are the parameters given by the relations (7.23)-(7.25), (7.33), and $(V^{II})_1$ corresponds to the Hamiltonians $(H^{II})_1$ (7.31).

The transformation (7.43) with $\alpha=III$ can be used to determine the explicit form of λ_s^{III} in (7.5) which turn out to be integro-differential operators. The representation (7.5) with differential operator λ_s^{III} also can be reduced to the canonical form (7.42) but the corresponding transformation operators have a very complicated form. For example, for the Hamiltonians (7.37a) we can choose for $k_1=1$ the following

$$V^{III}=\frac{1}{2m}\exp\left(\sigma_1\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\arctan\frac{p}{m}\right)\left[EP_+ +P_-(m-2\sigma_1\boldsymbol{S}\cdot\boldsymbol{p})+P_+\sigma_1 E\boldsymbol{S}\cdot\boldsymbol{p}\right],$$
$$P_\pm=\frac{1}{2}(1\pm\sigma_3), \qquad (7.45)$$

which corresponds to the representation (7.5) with $\lambda_s^{III}=0$.

Formulae (7.44) give the explicit form of the operators which transform the Poincaré algebra realizations used in approaches *I-III* into the Shirokov-Foldy realization (7.42). For the Hamiltonians (7.27) and (7.28) the transformation operator V^{I} can be choosen in the following form

$$V^I=(E+\sigma_1 H_s^I)[2E(E+m)]^{-1/2}, \qquad (7.46)$$

and

$$V^I=\exp\left(i\sigma_2\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\arctan\frac{p}{m}\right). \qquad (7.47)$$

For $s=1/2$ these operators coincide and reduce to the Foldy-Wouthuysen (FW) [108] operators diagonalizing the Dirac equation.

Formulae (7.46), (7.47) present very natural generalizations of the FW operator to a case of an arbitrary spin.

Using the connection (7.43) it is possible to define the mean position and spin operators for a particle of arbitrary spin. In the canonical representation (7.42) such operators have the form [108]

$$X_a^k=x_a, \quad S_{ab}^k=S_{ab}.$$

In the representations (7.5), (7.8), (7.12) these operators have the following explicit forms

$$X_a^\alpha=V^\alpha x_a[V^\alpha]^{-1}, \quad S_{ab}^\alpha=V^\alpha S_{ab}[V^\alpha]^{-1}, \quad \alpha=I,II,III.$$

We present the explicit form of these operators corresponding to the Hamiltonians (7.27), (7.28):

$$(X_a^I)_1 = x_a + \frac{1}{p^2E}\left\{S_{ab}p_b[E-\sigma_1(H_s^I)_1] + ip_a[\sigma_1(H_s^I)_1 - m]\right\},$$
$$(S_{ab}^I)_1 = S_{ab} + \frac{1}{p^2E}\varepsilon_{abc}S_{cd}p_d[E-\sigma_1(H_s^I)_1], \tag{7.48a}$$
$$(X_a^I)_2 = x_a + \frac{1}{E}\sigma_2 S_a + \frac{1}{E^2(E+m)}(ES_{ab}p_b - i\sigma_2 p_a \boldsymbol{S\cdot p}),$$
$$(S_{ab}^I)_2 = S_{ab} + \frac{1}{E}\sigma_2\varepsilon_{abc}S_{cd}p_d + \frac{1}{E(E+m)}(p_c\boldsymbol{S\cdot p} - S_c p^2)\varepsilon_{abc}. \tag{7.48b}$$

where $S_a = \varepsilon_{abc}S_{bc}/2$.

Formulae (7.48) generalize the mean position and spin operators of the Dirac electron [108] to the case of arbitrary spin particles.

In conclusion we note that the equations found above can be generalized in such a way that they will describe "particles" with several spin and mass states. Equations for particles with variable spin and mass were considered in [140, 315].

8. EQUATIONS IN DIRAC'S FORM FOR ARBITRARY SPIN PARTICLES

8.1. Covariant Equations with Coefficients Forming the Clifford Algebra

All the relativistic motion equations described above can be considered as generalizations of the Dirac equation. Here we consider the most natural generalization of this equation in which reducible representations of the Clifford algebra (2.3) are used.

As in Section 6, we shall search for the motion equation of relativistic particle of arbitrary spin in the form

$$(\Gamma_\mu p^\mu - m)\psi = 0, \tag{8.1}$$

where Γ_μ are square numeric matrices, and ψ is a multicomponent wave function.

According to (6.1), the function ψ has to satisfy the KGF equation componentwise. The simplest way to assure for (6.1) to be satisfied is to require the matrices Γ_μ satisfy the relations

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2g_{\mu\nu}. \tag{8.2}$$

Multiplying (8.1) by $\Gamma^\mu p_\mu$+m and using (8.2), we come to the condition (6.1) which is a differential consequence of (8.1), (8.2). An IR of the algebra (8.2) is realized by the 4×4 Dirac matrices. The corresponding equation (8.1) reduces to the Dirac equation.

In this section we consider equations (8.1) with reducible matrices Γ_μ. We show that such equations can be interpreted as equations of motion of a relativistic particle with arbitrary spin *s*. Then we find the additional conditions needed to select the subspace of solutions of the equation (8.1) corresponding to a fixed value of *s*.

It is not difficult to make sure that the equations (8.1) are invariant under the Poincaré algebra. Taking the basis elements of this algebra in the covariant from (2.22) and representing the matrices $S_{\mu\nu}$ in the form

$$S_{\mu\nu}=j_{\mu\nu}+\tau_{\mu\nu}, \quad j_{\mu\nu}=\frac{i}{4}[\Gamma_\mu,\Gamma_\nu], \tag{8.3}$$

we find that the operators (2.22), (8.4) satisfy the invariance condition of the equation (8.1) (i.e., the relations (6.7), where $\beta_\mu \to \Gamma_\mu$) if the matrices $\tau_{\mu\nu}$ satisfy the following relations:

$$\begin{aligned}&[\tau_{\mu\nu},\tau_{\lambda\sigma}]=i(g_{\mu\nu}\tau_{\mu\lambda}+g_{\nu\lambda}\tau_{\mu\sigma}-g_{\mu\lambda}\tau_{\nu\sigma}-g_{\nu\sigma}\tau_{\mu\lambda}),\\&[\tau_{\mu\nu},\Gamma_\lambda]=[\tau_{\mu\nu},j_{\lambda\sigma}]=0,\end{aligned} \tag{8.4}$$

i.e., if the matrices $\tau_{\mu\nu}$ commute with Γ_λ and realize a finite-dimensional representation of the algebra *AO(1,3)*.

Thus, we can set a correspondence between the Poincaré-invariant equations (8.1), (8.2) and any finite-dimensional representation of the algebra *AO(1,3)*. On the set of solutions of such equations generators of the Poincaré group have the form (2.22) with $S_{\mu\nu}$ being a sum of commuting matrices $j_{\mu\nu}$ and $\tau_{\mu\nu}$ being given by (8.3), (8.4).

8.2. Equations with the Minimal Number of Components

Poincaré-invariant equations in the Dirac form admit various interpretations as far as it is not possible to determine in an unique fashion the corresponding representation of the Poincaré algebra. The only exception is the case of the irreducible Γ-matrices of dimension 4×4 corresponding to the Dirac equation, but this equation has also an alternative interpretation as an equation for a zero-mass particle (refer to the following subsection).

To interpret the equation (8.1) it is necessary to choose a possible representations of the matrices $\tau_{\mu\nu}$ of (8.3). The corresponding representation of the Poincaré algebra (2.22) turns out to be reducible so it is necessary to impose on ψ a supplementary condition of the type of (6.1) in order to select the subspace corresponding to the fixed value of spin.

Consider the case when $\tau_{\mu\nu}$ produce the representation *D(τ 0)* of the algebra *AO(1,3)*. This means that (see Subsection 4.8)

$$\tau_{ab}=\varepsilon_{abc}\tau_c,\quad \tau_{0a}=i\tau_a,\quad [\tau_a,\tau_b]=i\varepsilon_{abc}\tau_c,\quad \tau_a\tau_a=\tau(\tau+1). \tag{8.5}$$

Substituting (8.5) into (8.3), we obtain

$$S_{ab}=\frac{i}{4}[\Gamma_a,\Gamma_b]+\varepsilon_{abc}\tau_c,\quad S_{0a}=\frac{i}{4}[\Gamma_0,\Gamma_a]+i\tau_a, \tag{8.6}$$

where the matrices τ_a commute with Γ_μ by definition.

According to Schur's lemma we conclude that the minimal dimension of the matrices (8.5) is $[4(2\tau+1)]^2$. Thus, we can set

$$\Gamma_\mu=\gamma_\mu\otimes I,\quad \tau_a=1\otimes\hat{\tau}_a, \tag{8.7}$$

where the symbol $\otimes$ denotes the direct (Kronecker) product, γ_μ are the 4×4 Dirac matrices, $\hat{\tau}_a$ are matrices of dimension $(2\tau+1)\times(2\tau+1)$ realizing the representation $D(\tau)$ of the algebra $AO(3)$, I and 1 are the unit matrices of dimension $(2\tau+1)\times(2\tau+1)$ and 4×4, correspondingly.

To find the spin value of a particle described by the equation (8.1) it is necessary to calculate the eigenvalues of the corresponding Casimir operator $W_\mu W^\mu$. Using (2.22), (2.28), (6.1) we find, in the frame of reference where $\tilde{p}=(m,0,0,0)$, that

$$W_\mu W^\mu=-m^2\boldsymbol{S}^2,\quad S_a=\varepsilon_{abc}S_{bc}, \tag{8.8}$$

and, therefore, these eigenvalues can be found by reduction of the matrices (8.6) by the algebra $AO(3)$. Inasmuch as the matrices $j_{\mu\nu}$ (8.3), (8.6) realize the representation $D(0\ 1/2)\oplus D(1/2\ 0)$ of the algebra $AO(1,3)$ then the matrices $S_{\mu\nu}$ (8.6) generate the representation

$$\left[D\left(\frac{1}{2}\ 0\right)\oplus D\left(0\ \frac{1}{2}\right)\right]\otimes D(\tau\ 0)=D\left(\tau+\frac{1}{2}\ 0\right)\oplus D\left(\tau-\frac{1}{2}\ 0\right)\oplus D\left(\tau\ \frac{1}{2}\right). \tag{8.9}$$

By the reduction $AO(1,3)\to AO(3)$ we obtain from (8.9) the following direct sum of IRs of the algebra AO(3): $D(\tau+1/2)\oplus D(\tau-1/2)\oplus D(\tau+1/2)\oplus D(\tau-1/2)$ which corresponds to the following spin values:

$$s_1=s=\tau+1/2,\quad s_2=s-1=\tau-1/2. \tag{8.10}$$

At this, dimension of the matrices Γ_μ is equal to $8s\times 8s$.

It can be shown that if the matrices $\tau_{\mu\nu}$ realise either IRs $D(\tau_1\ \tau_2)$ with $\tau_1\neq 0$, $\tau_2\neq 0$, or reducible representations, then dimension of the matrices Γ_μ is larger than $8s\times 8s$ (s is the largest value of spin appearing by the reduction $AO(1,3)\to AO(3)$).

Thus, the equation (8.1) is Poincaré-invariant. It describes a particle of spin s and has the minimal number of components if matrices Γ_μ have dimension $8s\times 8s$ and the corresponding matrices $S_{\mu\nu}$ realize the representation (8.9) of the algebra

AO(1,3).

Let us require the solutions of (8.1) satisfy the second condition of (6.1). Using the definitions (2.22), (2.28), (8.6) and the equation (8.1), we reduce this condition to the following form [331]:

$$L_2\psi\equiv[(\Gamma_\mu p^\mu+m)\hat{S}-16ms]\psi=0, \tag{8.11}$$

where

$$\hat{S}=(1+i\Gamma_4)[S_{\mu\nu}S^{\mu\nu}-4s(s-1)],\quad \Gamma_4=\Gamma_0\Gamma_1\Gamma_2\Gamma_3$$

Thus, in contrast to the first order wave equations considered in Section 6 the second condition (6.1) is not a consequence of equation (8.1) but has to be considered as a supplementary requirement which can be written in the form (8.11).

Let us formulate the obtained results.

THEOREM 8.1 [331]. The system of equations (8.1), (8.11) is Poincaré-invariant and describes a particle of spin s and mass m. ■

The equations (8.1), (8.11) have certain advantages in comparison with the other Poincaré-invariant equation considered above. These are a relatively simple form which does not become more complicated by increasing of spin value, and exience of the reasonable limit at $m\to 0$ (which is not the case for the equations considered in Section 6). Finally, the equations (8.1), (8.11) admit a non-contradictive generalization to the case of particles interacting with an external electromagnetic field.

Dirac-like wave equations for any spin particles were considered by Lomont and Moses [286]. But as proposed in [286] subsidiary condition, selecting the solutions corresponding to fixed s, differs from (8.11) and is incompatible with (8.1) after introduction of minimal interaction with an external electromagnetic field.

8.3. Connection with Equations without Superfluous Components

The equations in the Dirac form are closely connected with differential motion equations without superfluous components considered in Section 7. Namely, equations (8.1), (8.11) reduce to the form (7.1) where H is present in (7.37a).

Let us write (8.1), (8.11) in the form

$$i\frac{\partial}{\partial x_0}\psi=H\psi,\quad \hat{P}_s\psi=\psi, \tag{8.12}$$

where

$$H=\Gamma_0\Gamma_a P_a+\Gamma_0 m,$$
$$\hat{P}_s=P_s+\frac{1}{2m}(1-\Gamma_4)[\Gamma^\mu p_\mu, P_s],\quad P_s=\frac{1}{2s}[\boldsymbol{S}^2-s(s-1)]. \tag{8.13}$$

The first of the equations (8.12) is obtained from (8.1) by the multiplication by γ_0, the second is equivalent to (8.1) according to the relation

$$8s(1+i\Gamma_4)P_s=(1+i\Gamma_4)[S_{\mu\nu}S^{\mu\nu}-4s(s-1)].$$

It is not difficult to make sure the operators (8.13) satisfy the conditions

$$P_s^2=P_s,\quad \hat{P}_s^2=\hat{P}_s. \tag{8.14}$$

The operator $\hat{P}_s$ is a projector into subspace corresponding to spin s. By means of the transformation

$$\psi\to\Phi=V\psi,\quad H\to H'=VHV^{-1},\quad \hat{P}_s\to V\hat{P}_sV^{-1}=P_s, \tag{8.15}$$

where

$$V=1+(1-i\Gamma_4)(\Gamma\cdot\boldsymbol{p}-k_1\Gamma_0\boldsymbol{S}\cdot\boldsymbol{p}),\quad V^{-1}=V(-\boldsymbol{p}),$$

we reduce the equations (8.12) to the following equivalent form:

$$i\frac{\partial}{\partial x_0}\Phi=H'\Phi\equiv[\Gamma_0 m+2k_1\Gamma_4\boldsymbol{S}\cdot\boldsymbol{p}+\frac{1}{2m}\Gamma_0(1-i\Gamma_4)[p^2-4k^2(\boldsymbol{S}\cdot\boldsymbol{p})^2]\Phi, \tag{8.16}$$

$$\hat{P}_s\Phi\equiv P_s\Phi=0,\quad or\quad \boldsymbol{S}^2\Phi=s(s+1)\Phi.$$

Choosing Γ_0, Γ_4, $\boldsymbol{S}$ in the form of

$$\Gamma_0=\begin{pmatrix}\sigma_1 & 0\\ 0 & \tilde{\sigma}_1\end{pmatrix},\quad \Gamma_4=\begin{pmatrix}\sigma_3 & 0\\ 0 & \tilde{\sigma}_3\end{pmatrix},\quad \boldsymbol{S}=\begin{pmatrix}\boldsymbol{S} & 0\\ 0 & \tilde{\boldsymbol{S}}\end{pmatrix}, \tag{8.17}$$

where σ_1, σ_3, $\boldsymbol{S}$ are the matrices of (5.30), (7.6), $\tilde{\sigma}_1$, $\tilde{\sigma}_3$, $\tilde{\boldsymbol{S}}$ are analogous matrices for spin $s'=s$-1, and 0 are zero matrices of an appropriate dimension, we come to the equations in the form (6.1), (7.37a) for a 2(2s+1)-component function $\Phi_s=P_s\Phi$. The equations (6.1), (7.37a) are differential consequences of the equations (8.1), (8.11).

8.4. Lagrangian Formulation

Let us demonstrate that the equations in the Dirac form can be deduced in frames of the minimal action principle starting from an appropriate Lagrangian.

We write the system (8.1), (8.11) as a single equation

$$[B_s(\Gamma_\mu p^\mu - m) + \chi m(1 - B_s)]\psi = 0, \tag{8.18}$$

where χ is an arbitrary parameter which can be chosen to be equal to 1 without loss of generality, and B_s is the projector

$$B_s = \frac{1}{16ms}(\Gamma_\mu p^\mu + m)(1 + i\Gamma_4)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]. \tag{8.19}$$

Actually, multiplying (8.18) by B_s and 1-B_s and using the identities

$$B_s B_s = B_s, \quad (1 + i\Gamma_4)B_s(\Gamma_\mu p^\mu - m)B_s \equiv 2(\Gamma_\mu 629 p^\mu - m)B_s, \tag{8.20}$$

we come to the system (8.1), (8.11).

Using the formulation (8.18) it is not difficult to find the Lagrangian corresponding to the Dirac-like equations for particles of arbitrary spin. Choosing the Lagrangian density in the form

$$L(x) = i\left(m\bar{\psi}' + i\frac{\partial\bar{\psi}'}{\partial x_\mu}\hat{\Gamma}_\mu\right)F\hat{\Gamma}_\lambda\frac{\partial\psi'}{\partial x_\lambda} + i\frac{\partial\bar{\psi}'}{\partial x_\lambda}\hat{\Gamma}_\lambda F\left(m\psi' + i\hat{\Gamma}_\mu\frac{\partial\psi'}{\partial x_\mu}\right) + 16m^2 s\,\bar{\psi}'\psi' \tag{8.21}$$

where ψ' and $\bar{\psi}'$ are 16s-component wave functions,

$$\psi' = column(\psi, \chi), \quad \bar{\psi}' = \psi^* i\hat{\Gamma}_0\hat{\Gamma}_5\hat{\Gamma}_4, \tag{8.22}$$

ψ, χ being 8s-component functions, and $\hat{\Gamma}_\mu$, F being the matrices of dimension 16s×16s:

$$F = (1 + i\Gamma_4)(\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} - 4s(s-1)),$$

$$\hat{\Gamma}_\mu = \begin{pmatrix}\Gamma_\mu & 0\\ 0 & \Gamma_\mu\end{pmatrix}, \quad \hat{\Gamma}_4 = \begin{pmatrix}\Gamma_4 & 0\\ 0 & -\Gamma_4\end{pmatrix}, \quad \hat{\Gamma}_5 = i\begin{pmatrix}0 & \Gamma_4\\ -\Gamma_4 & 0\end{pmatrix}, \quad \hat{S}_{\mu\nu} = \begin{pmatrix}S_{\mu\nu} & 0\\ 0 & S_{\mu\nu}\end{pmatrix}, \tag{8.23}$$

it is not difficult to make sure that the equation (8.18) is an Euler-Lagrange equation of the type

$$\frac{\partial L}{\partial\bar{\psi}} - \frac{\partial}{\partial x_\lambda}\frac{\partial L}{\partial(\partial\bar{\psi}/\partial x_\lambda)} = 0 \tag{8.24}$$

where L is given in (8.21). From (8.24) the equation for a function χ follows also, which can be reduced to (8.18) by the substitution $\chi \to \psi$, $\Gamma_4 \to -\Gamma_4$.

Thus, the equations in the Dirac form admit a Lagrangian formulation which however needs doubling the number of components of the wave function ψ.

We note that the equations (8.24) for ψ and χ are invariant under the transformations P, T and C in contrast to the equations (8.1), (8.11).

8.5. Dirac-Like Wave Equations as a Universal Model of a Particle with Arbitrary Spin

As a concluding remark for this section, we show that the Dirac-like formulation is applicable to a wide class of wave equations used in modern physics.

Let us consider, for example, the equations for particles of spin 0 and 1 in the formulations of KDP (see Subsection 6.5), of Stueckelberg [398] and Hurley [224,225]. We may show that any of them can be represented in the form of the equation (8.1) with a subsidiary condition

$$P\psi=0, \tag{8.25}$$

with Γ_μ being Dirac matrices of dimension 16×16 and P being some numerical matrix.

We will start with the following representation of the algebra $AO(1,3)$

$$\begin{aligned} D&=[D(1/2\ \ 0)\oplus D(0\ \ 1/2)]\otimes[D(1/2\ \ 0)\oplus D(0\ \ 1/2)]=\\ &=D(1\ \ 0)\oplus D(0\ \ 0)\oplus D(1/2\ \ 1/2)\oplus D(1/2\ \ 1/2)\oplus D(0\ \ 0)\oplus D(0\ \ 1).\end{aligned} \tag{8.26}$$

Basis elements of this representation can be chosen in the form

$$\hat{S}_{\mu\nu}=S_{\mu\nu}+S'_{\mu\nu},\quad S_{\mu\nu}=\frac{i}{4}[\Gamma_\mu,\Gamma_\nu],\quad S'_{\mu\nu}=\frac{1}{4}[\Gamma'_\mu,\Gamma'_\nu], \tag{8.27}$$

where Γ_μ and Γ'_μ are commuting sets of 16×16 Dirac matrices.

The equation (8.1) is transparently invariant under the Poincaré algebra in the covariant realization (2.22), (8.27). The subsidiary condition (8.25) is Poincaré-invariant if the matrix P commutes with $\hat{S}_{\mu\sigma}$ of (8.27).

Let us demonstrate that the system of equations (8.1), (8.25) with

$$\begin{aligned} &P=1-P_1\equiv\frac{1}{16}(1-\Gamma'_k\Gamma^k)^2,\quad k=0,1,2,3,4,\\ &\Gamma_4=-i\Gamma_0\Gamma_1\Gamma_2\Gamma_3,\quad \Gamma'_4=-i\Gamma'_0\Gamma'_1\Gamma'_2\Gamma'_3,\end{aligned} \tag{8.28}$$

is equivalent to the KDP equation for a particle of spin 1. Being specific, we choose the following realization of the matrices Γ_μ and Γ'_μ:

$$\Gamma_0=\begin{pmatrix} 0 & 0 & 0 & g\\ 0 & 0 & I & 0\\ 0 & I & 0 & 0\\ g & 0 & 0 & 0\end{pmatrix},\quad \Gamma_a=\begin{pmatrix} 0 & 0 & -S_a & -gS_{4a}\\ 0 & 0 & -S_{4a} & S_a\\ S_a & S_{4a} & 0 & 0\\ -gS_{4a} & -S_a & 0 & 0\end{pmatrix},$$

where $g=2\boldsymbol{j}\tau$-1/2, $S_{4a}=j_a-\tau_a$, $S_a=j_a+\tau_a$, I and 0 are unit and zero matrices of dimension 4×4, and j_a , τ_a are 4×4 matrices satisfying the relations

$$\Gamma_0' = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -g & 0 \\ 0 & -g & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_a' = \begin{pmatrix} 0 & 0 & -S_a & -S_{4a} \\ 0 & 0 & gS_{4a} & -S_a \\ S_a & gS_{4a} & 0 & 0 \\ S_{4a} & S_a & 0 & 0 \end{pmatrix}, \tag{8.29}$$

$$\begin{gathered} [j_a,\tau_b]=0, \quad j_a^2=\tau_a^2=1/4 \quad (\textit{no sum over } a), \\ [j_a,j_b]=i\varepsilon_{abc}j_c, \quad [\tau_a,\tau_b]=i\varepsilon_{abc}\tau_c. \end{gathered} \tag{8.30}$$

The explicit form of j_a and τ_a is given in the following, see (9.20).

The matrix (8.28) is diagonal in the realization chosen:

$$P_0 = \begin{pmatrix} P_+ & & & \\ & P_+ & & \\ & & \hat{0} & \\ & & & I \end{pmatrix}, \quad P_+ = \frac{1}{2}(1+g)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.31}$$

From (8.31) it follows that the condition (8.25), (8.28) sets to zero six out of sixteen components of ψ.

The system (8.1), (8.25), (8.28) can be written in the form of a single equation

$$[P(\Gamma^\mu p_\mu - m)P + m(1-P)]\psi \equiv (\beta^\mu p_\mu - m)\psi = 0, \tag{8.32}$$

where

$$\beta_\mu = P\Gamma_\mu P, \quad P = 1 - P_1. \tag{8.33}$$

The equivalence of the equations (8.32) and (8.1), (8.25), (8.28) follows from the commutativity of the projector P with $\Gamma^\mu p_\mu$ on the set of solutions of the equation (8.32).

Formula (8.32) presents the KDP equation (6.6), (6.22) for a particle of spin 1. In fact, the equation (8.32) coincides with (6.6), (6.22) componentwise (the extra components of ψ in (8.32) and of matrices β_μ in (8.33) are equal to zero).

In an analogous way we make sure that the system of equations (8.1), (8.25) where P=1-P_0, P_0 being the projector (8.28), is equivalent to the KDP equation for a spinless particle. The corresponding matrices β_μ are given in (6.17).

We see the KDP equation admits the Dirac-like formulation (8.1) and (8.25). It is curious that the KDP matrices satisfying the algebra (6.20) can be obtained from the 16×16 Dirac matrices with the help of the projection (8.33) nullifying some rows and columns.

The eleven component equation of Stueckelberg [398] describing a "particle" with two spin states (corresponding to s=1 and s=0) can also be formulated in the form of the Dirac equation (8.1) with the subsidiary condition (8.25) where

$$P=\frac{1}{64}(1+\Gamma_4'-\Gamma_4-\Gamma_4\Gamma_4')(\Gamma_\mu\Gamma'^{\mu})^2. \tag{8.34}$$

In the representation (8.29) the projector (8.34) reduces to the diagonal matrix with four nonzero elements (placed on four last rows). The explicit form of the corresponding matrices β_μ for the Stueckelberg equation can be obtained from Γ_μ of (8.29) by deleting four last columns and rows (and the first columns and rows which are zero).

Finally, setting in (8.25)

$$P=\frac{1}{64}(1+\Gamma_4'-\Gamma_4-\Gamma_4\Gamma_4')(\Gamma_\mu\Gamma'^{\mu})^2, \tag{8.35}$$

we obtain from (8.1), (8.25) a wave equation for a particle of spin 1 in the Lomont-Moses [286] form. Rewriting this equation in the equivalent formulation (8.32) we obtain the seven-component wave equation considered in detail by Hagen and Hurley [215].

We note that the Hurley equations for particles of arbitrary spin [224] are nothing but the Lomont-Moses equations written in the form (8.32).

Let us summarize. We make sure that the multicomponent Dirac equation with a covariant additional condition is a very effective construction for describing pasticles of arbitrary spin. In this way it is possible to obtain new equations considered in Subsections 7.4-7.8 and well known equations of KDP, Stueckelberg, and Hurley, as well. The Rarita-Schwinger equation also can be represented in the form (8.1), (8.25) (refer to (6.34)). Finally, in the following section it will be shown that equations for massless fields also admit convenient formulations in the Dirac-like form.

9. EQUATIONS FOR MASSLESS PARTICLES

9.1. Basic Definitions

In this section we consider Poincaré-invariant equations for massless fields. The description of such equations is a specific problem since they cannot be obtained, in general, from field equations for non-zero mass particles by passing to the limit $m \to 0$ [38]. The equations considered further on should also be of interest since fields with zero mass are real physical objects.

The definition of a Poincaré-invariant equation for a massless field is given in Subsection 3.7. According to the Theorem 1.7, such equations are also invariant under the conformal algebra *AC(1,3)*.

We assume hereafter that the representation of the algebra *AC(1,3)* realized on the set of solutions of the considered equations has the covariant form (2.22), (3.56):

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{9.1}$$
$$D = x_\mu p^\mu + iK, \quad K_\mu = 2x_\mu D - p_\mu x_\nu x^\nu + 2S_{\mu\nu}x^\nu,$$

where $S_{\mu\nu}$ are matrices realizing a representation of the algebra *AO(1,3)* and K is a matrix commuting with $S_{\mu\nu}$.

The representation space of the algebra (9.1) will be identified with a space of states of a covariant massless field. According to (3.45), (9.1), any solution of a Poincaré -invariant equation for a covariant massless field has to satisfy the d′Alembert equation

$$p_\mu p^\mu \psi = 0. \tag{9.2}$$

Hence, we state the problem of finding all the nonequivalent linear equations which are invariant under the Lie algebra generated by the operators (9.1). Moreover, we assume that solutions of these equations satisfy (9.2) componentwise.

9.2. A Group Theoretic Derivation of Maxwell's Equations

Before considering wave equations for arbitrary spin fields we make a look at Maxwell's equations and demonstrate that it is possible to derive them starting from the requirement of relativistic (or conformal) invariance and some other suppositions.

First, we demonstrate that Maxwell's equations can be deduced by using the postulate of the conformal invariance. We will look for an equation for a vector field described by a three-component wave function. The corresponding matrices in (9.1) will have the following form:

$$S_{ab} = \varepsilon_{abc} S_c, \quad S_{0a} = \pm i S_a, \quad K = kI, \tag{9.3}$$

where S_a are matrices (3.6), I is the 3×3 unit matrix, and k is an arbitrary number.

THEOREM 9.1. Let ψ be a covariant massless vector field. Then ψ must satisfy Maxwell's equations.

PROOF. By definition ψ satisfies the d'Alembert equation (9.2). This equation has to be invariant under the conformal algebra whose basis elements are given by (9.1), (9.3).

The operators P_μ, $J_{\mu\nu}$, and D are evident SOs of the equation (9.2) since they commute with $L=p^{\mu}p_{\mu}$. As to the operators K_{μ}, the corresponding invariance condition reduces to the form

$$[K_{\mu},\ p_{\nu}p^{\nu}]\psi\equiv[i(k-1)p_{\mu}+S_{\mu\nu}p^{\nu}]\psi=0. \tag{9.4}$$

It is possible to show the system (9.4) is compatible for k=2 only, and, furthermore, it coincides with Maxwell's equations if we set $\psi=\boldsymbol{E}\mp i\boldsymbol{H}$. To verify this statement it is sufficient to write the system (9.4) componentwise for μ = 0, 1, 2, 3 and to compare it with (3.2).

We see that Maxwell's equations are determined uniquely by the conformal invariance postulate and the vector nature of the electromagnetic field.

Consider now Maxwell's equations (3.3) with currents and charges. These equations could not be deduced in the way presented above inasmuch as a current does not satisfy the condition (9.2). But it is possible to point to such minimal subsystems of the equations (3.3) which lead to the complete system of Maxwell's equations, if we impose the requirement of Poincaré invariance.

We present (without proof) two assertions illustrating the possibilities of group-theoretic deduction of Maxwell's equations with currents and charges.

THEOREM 9.2. Suppose $L(\boldsymbol{E},\boldsymbol{H},j_0,\boldsymbol{j})$ is a system of partial differential equations including the subsystem

$$\boldsymbol{p}\cdot\boldsymbol{E}=-ij_0,\quad \boldsymbol{p}\cdot\boldsymbol{H}=0. \tag{9.5}$$

Then, for $L(\boldsymbol{E},\boldsymbol{H},j_0,\boldsymbol{j})$ to be Poincaré invariant it is necessary for this system to include the following equations:

$$i\frac{\partial\boldsymbol{E}}{\partial t}=-\boldsymbol{p}\times\boldsymbol{H}+i\boldsymbol{j},\qquad \frac{\partial\boldsymbol{H}}{\partial t}=\boldsymbol{p}\times\boldsymbol{E}. \tag{9.6}$$

Proof is given in [154, 157].

Hence, Maxwell's equations are a consequence of the system (9.3) and the relativistic invariance postulate.

The inverse theorem is also true: the necessary and sufficient condition for the system (9.6) to be Poincaré-invariant is the requirement for $\boldsymbol{E}$ and $\boldsymbol{H}$ to satisfy the additional conditions (9.5) [154, 157].

9.3. Conformal Invariant Equations for Fields of Arbitrary Spin

In analogy with Theorem 9.1 it is possible to deduce an equation for a massless field with arbitrary helicity. In fact, supposing that such a field satisfies the conformal invariance condition, we come to the system of equations (9.4) with $S_{\mu\nu}$ being matrices belonging to a representation of the algebra *AO(1,3)*, and ψ being a wave function of corresponding dimension.

Thus, a covariant massless field satisfies necessarily the equations (9.4) with the appropriate matrices $S_{\mu\nu}$ and K. We restrict ourselves to the case when $S_{\mu\nu}$ form a completely reducible representation of the algebra *AO(1,3)*. Then the equations (9.4) reduce to a set of noncoupled subsystems. In any such subsystem the matrices $S_{\mu\nu}$ are the basis elements of the IR $D(j\ \tau)$ of the algebra *AO(1,3)*, and K is a multiple of the unit matrix.

But the equations (9.4) have to be invariant under the algebra *AC(1,3)*, so we come to the following conditions for the operators L_μ:

$$[L_\mu, Q]\psi=0, \tag{9.7}$$

where Q is any of the generators (9.1). By direct calculation we obtain

$$[L_\nu,P_\mu]=0, \quad [L_\nu,D]=iL_\nu, \quad [L_\mu,J_{\nu\lambda}]=i(g_{\mu\nu}L_\lambda-g_{\mu\lambda}L_\nu),$$

from which it follows that P_μ, D, and $J_{\mu\nu}$ satisfy the invariance condition of the equation (9.4). As to the operators K_μ, we obtain from (9.7) (with Q to be changed for K_μ) the following system of equations [52]:

$$\begin{aligned}&[S_\lambda^{\ \alpha}S_{\alpha\mu}-iS_{\lambda\mu}-k(1-k)g_{\lambda\mu}]\psi=0,\\ &[k(1-k)-j(j+1)-\tau(\tau+1)]\psi=0,\\ &(j+\tau+1-k)(j-\tau-k)(j-\tau+k)(k+j+\tau+1)p_\lambda\psi=0,\end{aligned} \tag{9.8}$$

which has a nontrivial solution for $j\tau\neq0$ only. Moreover,

$$k=j+\tau+1=s+1. \tag{9.9}$$

Substituting (9.9) into (9.4), we come to the following system of conform invariant equations:

$$(isp_\mu+S_{\mu\nu}p^{\nu})\psi=0, \tag{9.10}$$

where $S_{\mu\nu}$ are matrices belonging to the representation $D(s\ 0)$ (or $D(0\ s)$) of the algebra *AO(1,3)*, and ψ is a $2(s+1)$-component wave function.

It is not difficult to show that such equations describe a massless field with helicity $\pm\ s$. Multiplying (9.10) by p^μ and summing up over μ, we come to the equation (9.2). On the other hand, representing (9.10) in the form

$$W_\mu\psi=\varepsilon' sP_\mu\psi=\varepsilon\lambda p_\mu\psi, \quad \lambda=\varepsilon\varepsilon' s, \tag{9.11}$$

where W_μ are components of Lubanski-Pauli vector, $\varepsilon'=1$ for $S_{\mu\nu}\subset D(0\ s)$, and $\varepsilon'=-1$ for $S_{\mu\nu}\subset D(s\ 0)$, and comparing (9.11) with (4.55), we conclude that the direct sum $D^+(\varepsilon' s)\oplus D^-(-\varepsilon' s)$ of the Poincaré group representations is realized on the set of solutions of the equation (9.10).

Thus, we obtain the system of equations (9.10) describing a massless covariant field of arbitrary spin. In the case s=1/2 this system is equivalent to Weyl's equation, and for s=1 it reduces to Maxwell's equations. Equations of the (9.10) type for an arbitrary spin were considered in [135].

We note that the equation (9.10) follows from the supposition of conformal invariance of a field ψ in a unique fashion. In other words, if ψ is a covariant massless field then it has to satisfy to (9.10) with necessity.

9.4. Equations of Weyl's Type

The equations (9.10) represent a set of four systems of partial differential equations that should be simultaneously satisfied by a wave function ψ. But in the case s=1/2 we have actually only one system of equations, i.e., the Weyl system (2.44). Essentially, any of the equations (9.10) with s=1/2 reduces to the form of (2.44) by multiplication by the Pauli matrices σ_μ.

Poincaré-invariant equations of Weyl's type exist for a massless field of arbitrary spin. Instead of four equations (9.10) it is possible to consider a single system from which (9.10) follow as a mere consequence.

It is well known that the Weyl equation is equivalent to the massless Dirac equation with the additional condition $(1-i\gamma_4)\psi=0$. Equations for any spin may be obtained in analogous way, starting with the Dirac-like wave equations (see Section 8).

A system of Poincaré-invariant equations for an arbitrary spin particle (8.1), (8.11) may be written for the case of m=0 as follows:

$$\begin{gathered} i\frac{\partial}{\partial x_0}\psi=\Gamma_0\Gamma_a p_a\psi, \\ (i\frac{\partial}{\partial t}-\Gamma_0\Gamma_a p_a)(1+i\Gamma_4)S_{\mu\nu}S^{\mu\nu}\psi=0. \end{gathered} \tag{9.12}$$

Imposing on ψ the Poincaré-invariant additional condition

$$(1-i\Gamma_4)\psi=0, \tag{9.13}$$

and choosing Γ_μ in the form

$$\Gamma_0=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_4=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma_a=\begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \tag{9.14}$$

where I and 0 are the $4s$-row unit and zero matrices, and σ_μ are the Pauli matrices of dimension $4s\times4s$, we obtain

$$\left(i\frac{\partial}{\partial x_0}-\sigma\cdot\boldsymbol{p}\right)\varphi(x)\equiv\sigma_\mu p^\mu\varphi(x)=0, \tag{9.15}$$
$$\sigma_\mu p^\mu\hat{S}_{\nu\lambda}\hat{S}^{\nu\lambda}\varphi(x)=0,$$

with $\varphi(x)$ being a $4s$-component wave function connected with ψ by the relation

$$\varphi=\frac{1}{2}(1+i\Gamma_4)\psi. \tag{9.16}$$

According to (8.6), (9.16), the matrices $\hat{S}_{\mu\nu}$ belong to the representation *D(1/2 0)*$\oplus$*D(s-1/2 0)* of the algebra *AO(1,3)*, i.e., they have the following structure:

$$\hat{S}_{0a}=i(j_a+\frac{1}{2}\sigma_a),\quad \hat{S}_{ab}=-i\varepsilon_{abc}S_{0c}. \tag{9.17}$$

Moreover,

$$[j_a,\sigma_b]=0,\quad [j_a,j_b]=i\varepsilon_{abc}j_c,\quad j_aj_a=s(s-1). \tag{9.18}$$

The equations (9.15) are transparently Poincaré- and conform invariant. It is not difficult to make sure that these equations describe a massless field with helicity $\pm s$. In fact, multiplying the first of them by $i\partial/\partial x_0+\sigma\cdot\boldsymbol{p}$, we come to (9.2) from which it follows that the mass of the described field is equal to zero. Denoting $\hat{S}_{\mu\nu}\hat{S}^{\mu\nu}=-4(g^2-s^2)$ and taking into account the identities

$$[g,\sigma\cdot\boldsymbol{p}]_+=2\boldsymbol{S}\cdot\boldsymbol{p},\quad g^2=s^2,\quad [g,\boldsymbol{S}\cdot\boldsymbol{p}]=0,\quad S_a=\frac{1}{2}\varepsilon_{abc}S_{bc},$$

we obtain from (9.15) that

$$\boldsymbol{S}\cdot\boldsymbol{p}\varphi=g\sigma\cdot\boldsymbol{p}\varphi,$$

or

$$\boldsymbol{S}\cdot\boldsymbol{p}\varphi=g\sigma\cdot\boldsymbol{p}\varphi. \tag{9.19}$$

Comparing (9.19) with (4.55), we conclude that the helicity of the field φ coincides with eigenvalues of the matrix g, i.e., with $\pm s$. It is possible to demonstrate that the systems of equations (9.10) follow from (9.15).

So, in order to describe massless particles of arbitrary helicity, it is possible to use the generalized Weyl equations (9.15). Consider the examples of such equations for $s\leq 2$.

a) s=1/2. In this case the matrices σ_μ are of dimension 2×2, j_a are zero matrices, $\hat{S}_{\mu\nu}\hat{S}^{\mu\nu}=\sigma_a\sigma_a=3$, and the equations (9.15) reduce to the usul Weyl equation (2.43).

b) s=1. Without loss of generality, we may choose the matrices σ_a and j_a

in the form

$$\sigma_1=\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{pmatrix},$$

$$j_1=\frac{1}{2}\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix},\quad j_2=\frac{1}{2}\begin{pmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{pmatrix},\quad j_3=\frac{1}{2}\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}. \tag{9.20}$$

Then, for $\varphi=column(\psi_1,\psi_2,\psi_3,\psi_4)$, we obtain from (9.15), (9.17) the following equations:

$$\frac{\partial\psi}{\partial x_0}=\boldsymbol{p}\times\psi,\quad \boldsymbol{p}\cdot\psi=0,\quad \psi_4=const, \tag{9.21}$$

where the constant ψ_4 can be taken to be zero without loss of generality.

The equations (9.21) reduce to Maxwell's equations if we denote $\psi=\boldsymbol{H}-i\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{E}$ being real vectors.

c) s=3/2. Choosing σ_a and j_a in the form

$$\sigma_a=I_3\otimes\sigma'_a,\quad j_a=S_a\otimes I_2, \tag{9.22}$$

where S_a and σ_a are the matrices (3.6), (2.5), I_2 and I_3 are the unit matrices of dimension 2×2 and 3×3, and representing the wave function φ in the form

$$\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad \varphi_\alpha=column(\psi^1_\alpha,\psi^2_\alpha,\psi^3_\alpha),\quad \alpha=1,2$$

we obtain the following system of equations:

$$\frac{\partial\psi_\alpha}{\partial x_0}=\boldsymbol{p}\times\psi_\alpha,\quad (\sigma_\mu)_{\alpha\alpha'}p^\mu\psi_{\alpha'}=0. \tag{9.23}$$

We see that the massless field of helicity $|\lambda|$=3/2 satisfies Maxwell's equations in respect with the vector index a, and does the Weyl equation in respect with the spinor index α.

d) s=2. Let us choose σ_a and j_a in the form

$$j_a=\hat{j}_a\otimes I_2,\quad \sigma_a=I_4\otimes\sigma'_a, \tag{9.24}$$

where σ_a are the Pauli matrices from (2.5), and
In our case the function φ has eight components $\varphi_\alpha^{\ k}$, α=1,2, k=1,2,3,4, the matrices

$$\hat{j}_1=\frac{1}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\ \sqrt{3}&0&2&0\\ 0&2&0&\sqrt{3}\\ 0&0&\sqrt{3}&0\end{pmatrix},\quad \hat{j}_2=-\frac{1}{2i}\begin{pmatrix}0&-\sqrt{3}&0&0\\ \sqrt{3}&0&-2&0\\ 0&2&0&-\sqrt{3}\\ 0&0&\sqrt{3}&0\end{pmatrix},\quad \hat{j}_3=\frac{1}{2}\begin{pmatrix}3&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-3\end{pmatrix}.$$

j_a and σ_a acting on the indices k and α respectively. Then we obtain from (9.15), (9.17), and (9.24) the following system:

$$(\sigma_\mu^{/})_{\alpha\alpha'}p^\mu\varphi^k_{\alpha'}=0,\quad i\frac{\partial\varphi^k_\alpha}{\partial x_0}=\frac{2}{3}(j_a)_{kk'}p_a\varphi^{k'}_\alpha. \tag{9.25}$$

These equations describe a massless field of spin 2.

9.5. Equations of Other Types for a Massless Field

Essentially, the systems (9.10), (9.15) exhaust all the nonequivalent formulations of conformal invariant equations for a massless field of arbitrary spin, if we assume that the corresponding generators of the conformal group have the covariant form (9.1). This does not mean that there no other equations exist, which are invariant under the algebra *AC(1,3)*, for a massless field, because the basis elements of this algebra can be, in principle, chosen in a noncovariant realization. As the examples of equations for massless fields, which are not equivalent either to (9.10), or to (9.15), may serve the equations obtained from (7.1), (7.27) by passing to the limit $m\to 0$.

All nonequivalent equations for relativistic massless fields can be enumerated as follows. Any representation of the Poincaré algebra belonging to class *II* corresponds to the class of equivalent equations, and, thus, it is sufficient to choose one representative from each of these classes. Supposing that the corresponding representatives may be reduced to a nondegenerated direct sum of the IRs $D^{\varepsilon_1}(\varepsilon_2\lambda)$, ε_1, $\varepsilon_2=\pm1$, we obtain the following combinations for fixed λ:

$$D^{\varepsilon_1}(\varepsilon_2\lambda), \tag{9.26a}$$

$$D^{\varepsilon_1}(\lambda)\oplus D^{\varepsilon_1}(-\lambda), \tag{9.26b}$$

$$D^{+}(\varepsilon_2\lambda)\oplus D^{-}(\varepsilon_2\lambda), \tag{9.26c}$$

$$D^{+}(\varepsilon_2\lambda)\oplus D^{-}(-\varepsilon_2\lambda), \tag{9.26d}$$

$$D^{\varepsilon_1}(\varepsilon_2\lambda)\oplus D^{\varepsilon_1}(-\varepsilon_2\lambda)\oplus D^{-\varepsilon_1}(\varepsilon_2\lambda), \tag{9.26e}$$

$$D^{\varepsilon_1}(\varepsilon_2\lambda)\oplus D^{\varepsilon_1}(-\varepsilon_2\lambda)\oplus D^{-\varepsilon_1}(\varepsilon_2\lambda)\oplus D^{-\varepsilon_1}(-\varepsilon_2\lambda). \tag{9.26f}$$

Representations described by (9.26d) are realized on sets of solutions for the equations (9.10), (9.15). Equations corresponding to other representations of (9.26) may be obtained by the method proposed in [132, 139]. Namely, starting with an equation corresponding to the representations of (9.26f), we can find all the nonequivalent Poincaré-invariant additional conditions to be imposed on a set of solutions for this equation in order to determine subspaces of the representations (9.26). In this manner, it is possible to describe all the nonequivalent equations for a massless vector field including the Maxwell's equations as a particular case. We will not consider this possibilities in detail as they are elucidated in [154].

10. RELATIVISTIC PARTICLES OF ARBITRARY SPIN IN AN EXTERNAL ELECTROMAGNETIC FIELD

10.1. The Principle of Minimal Interaction

Solutions of Poincaré invariant wave equations considered above determine wave functions of arbitrary spin particles which can be used for solving different problems of quantum mechanics. But the main value of these equations lies in the fact that they can be used to describe an interaction of a particle with an external field.

In the case of electromagnetic interaction the corresponding motion equation can be obtained from an equation for a free particle by the following substitution:

$$p_\mu\rightarrow\pi_\mu=p_\mu-eA_\mu \tag{10.1}$$

where A_μ is a vector potential of the electromagnetic field, and e is a charge of a particle.

The rule for introduction of interaction given in (10.1) has to be considered as a postulate which is called the minimal interaction principle. We will not discuss the bounds of validity of this principle, but note that the prescription of introduction of interaction given in (10.1) is not the only possible one. A more general approach is to take into account the so-called anomalous interaction, examples of which will be considered in the following.

Equations of motion, obtained from Poincaré-invariant wave equations for a free particle by means of the substitution (10.1), preserve the Poincaré-invariance if a wave function transforms in accordance with the local covariance law (2.49). But, as it happens, an introduction of the minimal coupling into relativistic equations for particles of spin $s \geq 1$ leads to difficulties of the principal manner which, briefly speaking, may be stated as follows.

1. A system of partial differential equations describing a spinning particle becomes inconsistent as a result of the substitution (10.1) being made. Such a situation takes place, for example, for the Procá equation [412] written in the form of second-order equation with a subsidiary condition.

2. Inclusion of the minimal interaction into the equation for a free particle can lead to such an equation which cannot be interpreted as a motion equation for a particle of spin s since the corresponding wave function has superfluous (with respect to $2(2s+1)$) components. Such a result is true for Dirac-like equations for arbitrary spin particles proposed in [286].

3. Equations for a particle of spin $s>1$ are relativistically invariant but describe a faster-then-light wave propagation. Thus, e.g., the Rarita-Schwinger equation (see Subsection 6.6) becomes nonhyperbolic (i.e., having no wave solutions) as a result of the substitution (10.1) corresponding to large field strength. For small $\boldsymbol{E}$ and $\boldsymbol{H}$ this equation remains hyperbolic but describes a faster-than-light wave propagation [405].

4. Eguations with minimal interaction are inconsistent while solving concrete physical problems, e.g. the Kepler problem. This situation takes place for the KDP equation [401].

As was shown in [194, 405], this situation is typical for the majority of relativistic wave equations for particles of spin $s>1/2$. The difficulties mentioned in pars. 1 and 2 can be surmounted if the motion equations are of Euler-Lagrange type [100]. The contradictions related to causality violations are of principal matter and follow from the fact that the relativistic wave equations usually include superfluous components if $s>1/2$, and, in fact, they are nonlocal (refer to Subsection 6.7).

In connection with the above the natural question arises of the possibility of using Poincaré-invariant equations without superfluous components and Dirac-like equations (considered in Sections 7, 8) in order to describe a charged particle of spin s in an external electromagnetic field. The positive answer to this question is given below.

10.2. Introduction of Minimal Interaction into First-Order Wave Equations

As a result of the substitution (10.1), the equations (6.6) take the following form:

$$(\beta^{\mu}\pi_{\mu}-m)\psi=0. \tag{10.2}$$

If the starting system (6.6) is Poincaré-invariant and, at the same time, a transformation law for ψ has the form of (2.49) (with the corresponding matrices $S_{\mu\nu}$) then the system (10.2) has the similar property of invariance inasmuch as Lorentz transformations for π_{μ} and p_{μ} are the same. But, in accordance with the reasons given above, equations of the (10.2) type for arbitrary spin particles are in general inconsistent.

Here we consider the equations (10.2) for particles with spin s=0, 1/2, 1. We will define the corresponding Hamiltonians and discuss briefly the problems arising while using these equations to solve particular physical problems.

The simplest example for a first-order wave equation is the Dirac equation for an electron. Substituting $\beta_{\mu}=\gamma_{\mu}$ into (10.2), and multiplying it at the left by γ_0, we come to the equation

$$i\frac{\partial}{\partial x_0}\psi=H(A_0,\pi)\psi, \tag{10.3}$$

where

$$H(A_0,\pi)=\gamma_0\gamma_a\pi_a+\gamma_0 m+eA_0. \tag{10.4}$$

The Hamiltonian (10.4) corresponds for a charged particle with spin 1/2 interacting with the electromagnetic field. This is a first-order differential operator which is formally Hermitian with respect to the scalar product (2.39). The equation (10.3) satisfies the causality principle (refer to Subsection 10.7) and serves as an adequate mathematical model for a wide set of physical problems for which the concept of an external field makes sense.

The KDP equations for scalar and vector particles interacting with an external electromagnetic field can also be represented in the form of (10.2) with β_{μ} being the KDP matrices of dimensions 5×5 and 10×10. These equations also reduce to the Schrödinger form (10.3) using the same procedure as described in the Subsection 6.7. In addition, ψ is a $2(2s+1)$-component wave function, and the corresponding Hamiltonians $H(A_0,\pi)$ have the form

$$s=0,\quad H=\sigma_2 m+(\sigma_2+i\sigma_1)\frac{\pi^2}{2m}+eA_0, \tag{10.5}$$

$$s=1, \quad H=\sigma_2 m+(\sigma_2+i\sigma_1)\frac{\pi^2+e\boldsymbol{S}\cdot\boldsymbol{H}}{2m}+i\sigma_1\frac{(\boldsymbol{S}\cdot\pi)^2}{2m}+eA_0. \tag{10.6}$$

Here $\boldsymbol{H}=i\boldsymbol{p}\times\boldsymbol{A}$ is the vector of the magnetic field strength, σ_1, σ_2, and $\boldsymbol{S}$ are the matrices (5.30), (7.6), and (3.6).

The operators (10.5), (10.6) are Hermitian with respect to the scalar product (7.35) where $M=\sigma_2$, and in the case of $e\rightarrow 0$ they reduce to the free particle Hamiltonians (7.40a), (7.40c) (up to equivalence transformation $\sigma_1 \rightarrow \sigma_2$, $\sigma_2 \rightarrow -\sigma_1$. The corresponding equations of motion satisfy the causality principle, such as the Dirac equation (refer to Subsection 10.7).

The Rarita-Schwinger equation loses the properties of a causal equation and is not considered hereafter.

So, starting from first-order wave equations, we obtain the equations in the Schrödinger form for particles with spin 0, 1/2, 1. The corresponding equations for arbitrary spin particles are considered in the following subsection.

10.3. Introduction of Interaction into Equations in Dirac's Form

We can obtain equations for a charged particle with an arbitrary spin in an external field by starting with Dirac-like equations for a free particle (see Section 8).

One may make sure (by a direct verification) that the substitution of (10.1) into the equations (8.1) and (8.11) leads to a system which is consistent only for the zero tensor $F^{\mu\sigma}$ of the electromagnetic field. To overcome this difficulty, it is sufficient to introduce the minimal interaction into the Lagrangian (8.21), or into the equations (8.18). As a result, we come to the system

$$[B_s(\pi)(\Gamma_\mu\pi^\mu-m)+\chi(1-B_s(\pi))]\psi=0, \tag{10.7}$$

where

$$B_s(\pi)=\frac{1}{16ms}(\Gamma_\mu\pi^\mu+m)(1+i\Gamma_4)[S_{\mu\nu}S^{\mu\nu}-4s(s-1)]. \tag{10.8}$$

Multiplying (10.7) by $B_s(\pi)$ and $(1-B_s(\pi))$, and taking into account the identities

$$\begin{gathered} B_s(\pi)B_s(\pi)=B_s(\pi), \quad B_s(\pi)(\Gamma_\mu\pi^\mu-m)B_s(\pi)= \\ =\left[\Gamma_\mu\pi^\mu+m+\frac{e}{4m}(1-i\Gamma_4)\left(\frac{1}{s}S_{\mu\nu}-i\Gamma_\mu\Gamma_\nu\right)F^{\mu\nu}\right]B_s(\pi), \end{gathered} \tag{10.9}$$

we come to the following system:

$$\left[\Gamma_\mu\pi^\mu-m+\frac{e}{4m}(1-i\Gamma_4)\left(\frac{1}{s}S_{\mu\nu}-i\Gamma_\mu\Gamma_\nu\right)F^{\mu\nu}\right]\psi=0, \tag{10.10a}$$

$$[(\Gamma_\mu\pi^\mu+m)(1+i\Gamma_4)[S_{\mu\nu}S^{\mu\nu}-4s(s-1)]\psi=16ms\psi. \tag{10.10b}$$

Thus, the Dirac-like equations for a particle with an arbitrary spin in an external electromagnetic field have the form of (10.10). We see that introduction of minimal interaction into the Lagrangian (8.21) results in the appearing of terms to be proportional to the tensor of an electromagnetic field strength $F^{\mu\sigma}$ in the equation of motion.

Let us demonstrate that (10.10) reduce to equations of the Schrödinger type for a $2(2s+1)$-component wave function. Multiplying any of the equations (10.10) by Γ_0, we come, after simple calculations, to the system of the form (8.12) where

$$H=\Gamma_0\Gamma_a\pi_a+\Gamma_0 m+\frac{e}{4m}\Gamma_0(1-i\Gamma_4)\left(\frac{1}{s}S_{\mu\nu}-i\Gamma_\mu\Gamma_\nu\right)F^{\mu\nu}, \tag{10.11a}$$

$$\hat{P}_s=P_s+\frac{1}{2m}(1-i\Gamma_4)[\Gamma_\mu\pi^\mu,P_s], \tag{10.11b}$$

and P_s is the matrix (8.13). The identities necessary to reduce (10.10) to the form of (8.12) are found in (8.14) and (10.10a).

The operators (10.11b) satisfy the condition $\hat{P}_s^2=\hat{P}_s$ and are orthoprojectors into a subspace corresponding to the spin s. As in the case of free particle equations, these projectors reduce to the numerical matrices P_s by the transformations (8.15) where V is the operator obtained from (8.15) by the change $\boldsymbol{p}\to\pi$. At the same time, the Hamiltonian (10.10a) is transformed into the form

$$\begin{aligned}H\to H'=VHV^{-1}=\Gamma_0 m+2k_1\Gamma_4\boldsymbol{S}\cdot\pi+\frac{1}{2m}\Gamma_0(1-i\Gamma_4)\times\\ \times\left\{\pi^2-4k_1^2(\boldsymbol{S}\cdot\pi)^2-\frac{1}{s}\boldsymbol{S}\cdot[\boldsymbol{H}-i(1-2k_1s)\boldsymbol{E}]\right\}+eA_0,\end{aligned} \tag{10.12}$$

or, choosing the representation (8.15),

$$H=\sigma_1 m+\sigma_3 2k_1\boldsymbol{S}\cdot\pi+\frac{1}{2m}(\sigma_1-i\sigma_2)\left\{\pi^2-4k_1^2(\boldsymbol{S}\cdot\pi)^2-\frac{1}{s}\boldsymbol{S}\cdot[\boldsymbol{H}-i(1-2k_1s)\boldsymbol{E}]\right\}+eA_0. \tag{10.13}$$

Formula (10.13) generalizes the free particle Hamiltonian (7.37a) to the case of a charged particle in an external field. Thus, starting with the first-order wave equations (10.10), we have obtained the formula for introduction of interaction into non-manifestly covariant equations without superfluous components.

We note that the Hamiltonians (10.13) corresponding to different values of the parameter k_1 are equivalent. Let $H(k'_1)$ be the Hamiltonian (10.13) with $k_1=k'_1$, and $H(k''_1)$ be the operator corresponding to $k_1=k''_1$. Then $H(k'_1)=VH(k''_1)V^{-1}$ where

The second note is related to the fact that there exists one-to-one correspondence between solutions of the equations (10.10) and those of (8.12),

$$V = \exp\left[(\sigma_1 - i\sigma_2)\frac{k_1^{/} - k_1^{//}}{m}\boldsymbol{S}\cdot\pi\right].$$

(10.13), as it is evident from the above.

10.4. A Four-Component Equation for Spinless Particles

Let us demonstrate that a spinless particle can also be described by a Dirac-like equation.

As it was noted in [118], the Dirac equation for a free electron can be in principle interpreted as an equation for a massless particle since it is possible to define (noncovariant) representation of the Poincaré algebra corresponding to the zero spin on a set of Dirac equation solutions. It turns out that such an interpretation is also possible in the case of charged particles interacting with an external field if the coupling is taken into account in the specific manner [147].

Consider the equation

$$\left[\gamma_\mu \pi^\mu - m + \frac{iek}{4m}(1 - i\gamma_4)\gamma_\mu\gamma_\nu F^{\mu\nu}\right]\psi = 0, \tag{10.14}$$

where γ_μ, γ_ν are the Dirac matrices, and k is an arbitrary parameter.

The equation (10.14) is explicitly covariant and coincides (in the case of k=0) with the Dirac equation for a minimally interacting particle with spin 1/2. The term $iek(1\text{-}i\gamma_4)\gamma_\mu\gamma_\nu F^{\mu\nu}/4\mathrm{m}$ can be interpreted as a contribution due to the anomalous interaction of the Pauli type.

Let us now demonstrate that in the case of k=1 the equation (10.14) describes a motion of a spinless charged particle. We multiply (10.14) by γ_0 to obtain the equation in the form of (10.3) where

$$H = \gamma_0\gamma_a\pi_a + \gamma_0 m + eA_0 - \frac{ie}{4m}\gamma_0(1 - i\gamma_4)\gamma_\mu\gamma_\nu F^{\mu\nu}.$$

Then, applying the transformation $H \to H' = VHV^{-1} - iV^{-1}\partial V/\partial x_0$ with V being equal to

$$V = \exp\left[(1 - i\gamma_4)\frac{\gamma_a\pi^a}{m}\right] \equiv 1 + \frac{1}{2m}(1 - i\gamma_4)\gamma_a\pi_a,$$

we obtain the following:

$$H^{/} = \gamma_0 m + \frac{1}{2m}\gamma_0(1 - i\gamma_4)\pi^2 + eA_0.$$

Choosing the representation (2.4) for the gamma-matrices, we can rewrite H' in the form of (10.5) where $\sigma_1 \to \sigma_2$, $\sigma_2 \to -\sigma_1$, σ_1 and σ_2 are the 4×4 Pauli matrices (5.30). In other words, this Hamiltonian reduces to the direct sum of two operators (10.5) corresponding to a spinless particle in an external electromagnetic field.

10.5. Equations for Systems with Variable Spin

Consider equations (10.10) again, and show that they can be generalized to describe a "particle" with a variable spin.The equation (10.10a) has a clear physical sense even in absence of the additional condition (10.10b). In fact, it can be reduced to a direct sum of the equations (7.1), (10.13) for particles with spin s and $s'=s$-1.

The equation (10.10a) admits a reasonable interpretation for the case of matrices $S_{\mu\nu}$ belonging to a more general representation then given in (8.9). Here we consider the following representation of $S_{\mu\nu}$:

$$\begin{aligned} D &= \left[D\left(0\ \frac{1}{2}\right)\oplus D\left(\frac{1}{2}\ 0\right)\right]\otimes D\left(\frac{s}{2}\ \frac{s-1}{2}\right)= \\ &= D\left(\frac{s+1}{2}\ \frac{s-1}{2}\right)\oplus D\left(\frac{s-1}{2}\ \frac{s-1}{2}\right)\oplus D\left(\frac{s}{2}\ \frac{s}{2}\right)\oplus D\left(\frac{s}{2}\ \frac{s-2}{2}\right). \end{aligned} \tag{10.16}$$

Transforming the corresponding Hamiltonian (10.10a) in accordance with (10.12), we obtain for k_1=0 that

$$H' = \Gamma_0 m + \frac{1}{2m}\Gamma_0(1-i\Gamma_4)\left(\pi^2 - \frac{1}{s}S_{\mu\nu}F^{\mu\nu}\right) + eA_0. \tag{10.17}$$

The following identity is valid:

$$(1-i\Gamma_4)S_{\mu\nu} \equiv (1-i\Gamma_4)\hat{S}_{\mu\nu},$$

where $S_{\mu\nu}$ belong to the representation

$$D = D\left(\frac{s}{2}\ \frac{s}{2}\right)\oplus D\left(\frac{s}{2}\ \frac{s-2}{2}\right)\oplus D\left(\frac{s}{2}\ \frac{s-2}{2}\right)\oplus D\left(\frac{s}{2}\ \frac{s}{2}\right).$$

Choosing Γ_0 and Γ_4 in the form of (8.17), where σ_a and σ'_a are the Pauli matrices of dimension $2(s+1)^2\times 2(s+1)^2$ and $2(s^2-1)\times 2(s^2-1)$, and $\hat{S}_{\mu\nu}$ as a direct sum of the matrices

$$S_{\mu\nu}^{(1)} \subset D\left(\frac{s}{2}\ \frac{s}{2}\right), \quad S_{\mu\nu}^{(2)} \subset D\left(\frac{s}{2}\ \frac{s-2}{2}\right),$$

we obtain the Hamiltonian H' in the form of a direct sum of the operators H_1 and H_2 where

$$H_1=\sigma_1 m+(\sigma_1-i\sigma_2)\frac{1}{2m}\left(\pi^2-\frac{1}{s}S^{(1)}_{\mu\nu}F^{\mu\nu}\right)+eA_0,$$
$$H_2=\tilde{\sigma}_1 m+(\tilde{\sigma}_1-i\tilde{\sigma}_2)\frac{1}{2m}\left(\pi^2-\frac{1}{s}S^{(2)}_{\mu\nu}F^{\mu\nu}\right)+eA_0. \qquad (10.18)$$

The corresponding Schrödinger equation reduces to the pair of noncoupled equations with the Hamiltonians H_1 and H_2. Let us write the first of them:

$$i\frac{\partial}{\partial x_0}\psi=\left[\sigma_1 m+(\sigma_1-i\sigma_2)\frac{1}{2m}\left(\pi^2-\frac{1}{s}S^{(1)}_{\mu\nu}F^{\mu\nu}\right)+eA_0\right]\psi. \qquad (10.19)$$

The equation (10.19) differs from (7.1), (10.13) only in the representation realized by the matrices $S_{\mu\nu}$. The representation *D(s/2 s/2)* reduces to the following direct sum of IRs of the algebra *AO(3)*:

$$D(s/2\ \ s/2)\rightarrow D(s)\oplus D(s-1)\oplus\ldots\oplus D(0),$$

and we can interpret (10.19) as an equation for a quasiparticle which can be in different spin states corresponding to the spin values *s, s*-1, ... ,0.

The equation (10.19) is invariant under the *P*-, *C*-, and *T*-transformations which can be chosen in the form of (2.55), (2.60) where

$$r_1=\sigma_1\eta,\quad r_2=\sigma_2\eta\delta,\quad r_3=\sigma_2\delta,\quad \eta=\begin{pmatrix}\eta' & 0\\ 0 & \eta'\end{pmatrix},\quad \delta=\begin{pmatrix}\delta' & 0\\ 0 & \delta'\end{pmatrix}, \qquad (10.21)$$

with η' and δ' being matrices defined up to a sign by the following relations:

$$\eta' S_{ab}=S_{ab}\eta',\quad \eta' S_{0a}=-S_{0a}\eta',\quad (\eta)^2=1,\quad \delta' S_{\mu\nu}=-S^*_{\mu\nu}\delta',\quad (\delta')^2=(-1)^{2s}.$$

The explicit expressions for η' and δ' (which will not be used further on) can be easily obtained using the results of Subsection 5.6.

10.6. Introduction of Minimal Interaction into Equations Without Superfluous Components

Inasmuch as the main difficulties in description of particles in an external electromagnetic field are connected with superfluous components of relativistic wave equations, it is natural to try to introduce an interaction into wave equations with the correct number of components considered in Section 7. Such equations do not possess an explicit covariant form, and, generally speaking, the minimal interaction principle has to be used with an appropriate carefulness since the substitution (10.1) can violate the Poincaré invariance of the equations considered.

One way to introduce an interaction into motion equations without superfluous components is to make the substitution (10.1) in the corresponding first-order wave equation which reduces to the given equation in the Schrödinger form by delating superfluous components. This way has been used to obtain the Hamiltonians (10.5)-(10.7), (10.13), and (10.18). It seems that there exist other possibilities to include an interaction into equations without superfluous components (refer, for example, to [211]).

Here we consider a description of arbitrary spin particles in an external electromagnetic field which is based on the nonlocal equations (7.1), (7.28), (7.23), and (7.31). We restrict ourselves to the class of problems corresponding to particle momentum being small in comparison with a particle rest mass, and represent the corresponding Hamiltonians as a series in powers of $1/m$:

$$H_s^\alpha=\sigma_1\left[m+\frac{1}{2m}d_{ab}p_a p_b\right]+\sigma_3\left(2S_a p_a+\frac{1}{m^2}h^\alpha\right)+o\left(\frac{1}{m^3}\right), \tag{10.22}$$

where

$$d^{ab}=\delta_{ab}-2(S_aS_b+S_bS_a),\ \ h^I=-2h^{II}=\frac{2}{3}S_a d_{bc}p_a p_b p_c,\ \ \alpha=I,II. \tag{10.23}$$

Of course, the equations (7.1), (10.22) are not Poincaré-invariant, and can be considered only as approximate quasirelativistic models.

Changing $p_\mu \to \pi_\mu$ we come to the following systems:

$$H^\alpha(\pi)\psi=i\frac{\partial}{\partial x_0}\psi, \tag{10.24}$$

$$H^\alpha(\pi)=\sigma_1[m+\pi^2/2m-2(\boldsymbol{S}\cdot\pi)^2/m+e\boldsymbol{S}\cdot\boldsymbol{H}/m]+\sigma_3[2\boldsymbol{S}\cdot\pi+h^\alpha(\pi)/m^2]+eA_0+o(1/m^3).$$

It will be shown further on that the equations (10.24) describe satisfactorily an arbitrary spin particle in the electromagnetic field, taking into account such the well-known physical effects as the dipole, spin-orbital, and Darwin couplings.

10.7. Reduction in Power Series in 1/*m*

As is well known, relativistic wave equations admit a consistent interpretation only in terms of the second quantized theory which enables to overcome the difficulties in interpreting negative energies. But these equations can serve as satisfactory models for a wide class of problems where a particle momentum is small in comparison with its mass, since in this case it is possible to separate the positive energy solutions.

To select the positive energy states we transform our equations to a representation for which the energy sign operator $H/|H|$ is diagonal. In contrast to the free particle case, it can be done only approximately, by supposing that $\pi_\mu^2 \ll m^2$.

Hereafter we make such diagonalization. We represent Hamiltonians for arbitrary spin particle as a power series in $1/m$ which is convenient for calculations within the frames of the perturbation theory.

Let us consider the Hamiltonians (10.13), (10.24) defined for arbitrary values of spin s. After the series of successive transformations

$$H^{\alpha}\to W^{\alpha}H^{\alpha}(W^{\alpha})^{-1}-i\frac{\partial W_1^{\alpha}}{\partial x_0}(W^{\alpha})^{-1}=(H^{\alpha})^{III}, \quad \alpha=I,II,III, \qquad W^{\alpha}=V_3^{\alpha}V_2^{\alpha}V_1^{\alpha}, \tag{10.25}$$

where

$$V_1^{II}=V_1^{I}=\exp\left(-i\sigma_2\frac{\boldsymbol{S}\cdot\pi}{m}\right), \quad V_2^{I}=V_2^{II}=\exp\left(i\sigma_3\frac{e\boldsymbol{S}\cdot\boldsymbol{E}}{2m^2}\right),$$

$$V_3^{\mu}=\exp\left\{-\frac{i}{2m^3}\sigma_2\left(h^{\mu}(\pi)-\frac{4}{3}(\boldsymbol{S}\cdot\pi)^2+\frac{1}{2}\frac{\partial}{\partial x_0}\boldsymbol{S}\cdot\boldsymbol{E}-\frac{1}{2}\left[\boldsymbol{S}\cdot\pi,\pi^2-2\boldsymbol{S}\cdot\boldsymbol{H}\right]_+\right)\right\},$$

$$V_1^{III}=\exp\left(-i\sigma_2\frac{\boldsymbol{S}\cdot\pi}{2sm}\right)\exp\left[\left(k_1-\frac{1}{2s}\right)(\sigma_1-i\sigma_2)\frac{\boldsymbol{S}\cdot\pi}{m}\right],$$

$$V_2^{III}=\exp\left\{-\frac{\sigma_3}{4m^2}\left[\pi^2-\left(\frac{\boldsymbol{S}\cdot\pi}{s}\right)^2-\frac{1}{s}\boldsymbol{S}\cdot\boldsymbol{H}+\frac{i}{s}\boldsymbol{S}\cdot\boldsymbol{E}\right]\right\},$$

$$V_3^{III}=\exp\left\{\frac{i\sigma_2}{8m^3}\left(\left[\frac{\boldsymbol{S}\cdot\pi}{s},\pi^2-\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{s}-\frac{1}{3}\left(\frac{\boldsymbol{S}\cdot\pi}{s}\right)^2\right]_+ -\right.\right.$$
$$\left.\left.-\left[\pi^2-\left(\frac{\boldsymbol{S}\cdot\pi}{s}\right)^2-\frac{1}{s}\boldsymbol{S}\cdot\boldsymbol{H}+\frac{i}{s}\boldsymbol{S}\cdot\boldsymbol{E},\pi_0\right]_-\right)\right\}, \quad \mu=I,II,$$

we obtain (ignoring the terms of order of $1/m^2$) the following*

$$(H^{\alpha})^{III}=\varepsilon_0+\frac{\pi^2}{2m}+eA_0+\frac{eB}{2m}\boldsymbol{S}\cdot\boldsymbol{H}+\frac{eD^2}{2m^2}\left[-\frac{1}{2}\boldsymbol{S}\cdot(\pi\times\boldsymbol{E}-\boldsymbol{E}\times\pi)+\right.$$
$$\left.+\frac{1}{6}Q_{ab}\frac{\partial Ea}{\partial x_b}+\frac{1}{3}s(s+1)div\boldsymbol{E}\right]+\frac{eC}{m^2}\left[\boldsymbol{S}\cdot(\pi\times\boldsymbol{H}-\boldsymbol{H}\times\pi)-\frac{2}{3}Q_{ab}\frac{\partial H_a}{\partial x_b}\right]+o\left(\frac{1}{m^3}\right). \tag{10.26}$$

Here Q_{ab} is a quadruple interaction tensor defined by the following formula

*Detailed calculations related to the transformation (10.25) are presented in [147, 315, 331].

$$Q_{ab}=3[S_a,S_b]_+-2\delta_{ab}s(s+1), \tag{10.27a}$$

$$B=-\mu^\alpha/2,\quad D^2=-(\mu^\alpha)^2/4,\quad C=-\delta_{\alpha III}(2s-1)/8s^2,$$
$$\mu^I=\mu^{II}=2,\quad \mu^{III}=\frac{1}{s},\quad \varepsilon_0=m. \tag{10.27b}$$

Formula (10.26) generalizes the Pauli Hamiltonian to the case of arbitrary spin. Actually this operator commutes with σ_1, so we can consider solutions Φ_+ satisfying the relations $\sigma_1\Phi_+=\Phi_+$. The Hamiltonian (10.26) includes the terms corresponding to a dipole ($\sim \boldsymbol{S}\cdot\boldsymbol{H}$), quadruple ($\sim Q_{ab}\partial E_a/\partial x_b$), spin-orbit ($\sim \boldsymbol{S}\cdot(\boldsymbol{E}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{E})$), and Darwin ($\sim div\boldsymbol{E}$) interaction.

For the case $\alpha=III$ there exist two additional terms (equal to zero if $s=1/2$). They have no clear physical interpretation and correspond to a quadruple magnetic and spin-orbit interactions in case of a magnetic monopole field.

Thus, starting with the equations without superfluous components, we have obtained the quasirelativistic Hamiltonians (10.26). In the approximation $1/m^2$ the Hamiltonians $(H^I)'''$ and $(H_{II})'''$ coincide. In the case $s=1/2$ all the three Hamiltonians reduce to the Foldy-Wouthuysen Hamiltonian [108].

We see the approximate Hamiltonians of an arbitrary spin particle include the terms corresponding to the Foldy-Wouthuysen Hamiltonian and additional terms corresponding to the quadruple coupling. We note that the dipole momentum (i.e., the coefficient of the term $e\boldsymbol{S}\cdot\boldsymbol{H}$), predicted by the Dirac-like equations, is equal to $1/s$ and hence is in accordance with Belifante's conjecture [30].

Consider now equations for a particle with a variable spin which were deduced in Subsection 10.5. By analogy with the above, the Hamiltonian H^1 (10.18) reduces to the approximate form

$$\begin{aligned}H'''=&\sigma_1\left(m+\frac{\pi^2}{2m}-\frac{e}{2sm}\boldsymbol{S}\cdot\boldsymbol{H}\right)+eA_0-\frac{e}{16m^2s^2}\boldsymbol{S}\cdot(\boldsymbol{E}\times\pi-\pi\times\boldsymbol{E})-\\&-\frac{e}{24m^2s^2}\left[\frac{1}{2}Q^{/}_{ab}\frac{\partial E_a}{\partial x_b}-\boldsymbol{N}^2div\boldsymbol{E}\right]+\frac{ie(2s-1)}{8m^2s^2}\boldsymbol{N}\cdot(\pi\times\boldsymbol{H}-\boldsymbol{H}\times\pi)+\frac{e}{24m^2s^2}Q^{//}_{ab}\frac{\partial Ha}{\partial x_b},\end{aligned} \tag{10.28}$$

where

$$Q^{//}_{ab}=-3[N_a,N_b]_++2\delta_{ab}\boldsymbol{N}^2,\quad N_a=S^{(1)}_{0a},$$

$$Q^{//}_{ab}=-\frac{3}{2}i([N_a,S_b]_++[N_b,S_a]_+)+2i\boldsymbol{S}\cdot\boldsymbol{N}\delta_{ab},\quad S_a=\frac{1}{2}\varepsilon_{abc}S^{(1)}_{bc},$$

$S_{\mu\nu}$ being matrices realizing the direct sum of two IRs *D(s/2 s/2)* of the algebra *AO(1,3)*.

All the terms in the r.h.s. of (10.28) are *P*-, *T*-, and *C*-invariant. Neglecting the terms of order $1/m^2$, we obtain from (10.28) the direct sum of the Pauli Hamiltonians for particles of spins s, s-1, s-2, ... , 0. This fact confirms our interpretation of (10.19) as an equation for a particle with a variable spin.

Consider the quasirelativistic approximation of the KDP equation. Applying to the Hamiltonians (10.5), (10.6) the procedure similar to (10.25), we obtain [37]

$$H' = \sigma_1\left(m + \frac{\pi^2}{2m} - \frac{e}{2m}\boldsymbol{S}\cdot\boldsymbol{H}\right) + eA_0 - \frac{\sigma_1}{2m^3}\left(\frac{\pi^2}{2} + (\boldsymbol{S}\cdot\pi)^4 - \frac{1}{2}[\pi^2,(\boldsymbol{S}\cdot\pi)^2]_+ - \right. \tag{10.29}$$
$$\left. - \frac{e}{4}[\boldsymbol{S}\cdot\boldsymbol{H},\pi^2]_+ + \frac{e}{2}[\boldsymbol{S}\cdot\boldsymbol{H},(\boldsymbol{S}\cdot\pi)^2]_+ + e^2(\boldsymbol{S}\cdot\boldsymbol{H})^2\right)$$

where $\boldsymbol{S}$ is either the direct sum of two spin matrices (for s=1), or $\boldsymbol{S}$=0 (for s=0).

We see that the approximate Hamiltonians obtained from the KDP equations do not include terms of order $1/m^2$. This means that the KDP equations do not describe the spin-orbit and Darwin couplings in the frames of the minimal interaction principle.

10.8. Causality Principle and Wave Equations for Particles with Arbitrary Spin

Let us demonstrate that the covariant wave equations considered here do not lead to paradoxes with the causality violation.

The causal character of the Dirac-like equations can be established by transferring to the equivalent system of second-order equations. Actually, multiplying any of the equations (10.10) by $\lambda_+=(1+\Gamma_4)/2$ and $\lambda^-=(1-\Gamma_4)/2$, and expressing $\psi_-=\lambda_-\psi$ via $\psi_+=\lambda_+\psi$ we obtain

$$\left(\pi_\mu\pi^\mu - m^2 + \frac{e}{2s}S_{\mu\nu}F^{\mu\nu}\right)\psi_+ = 0, \tag{10.30a}$$

$$[S_{\mu\nu}S^{\mu\nu} - 4s(s+1)]\psi_+ = 0, \tag{10.30b}$$

$$\psi_- = \frac{1}{m}\Gamma_\mu\pi^\mu\psi_+, \tag{10.30c}$$

with the following identity being used:

$$(\Gamma_\mu\pi^\mu)^2 \equiv \pi_\mu\pi^\mu + \frac{ie}{2}\Gamma_\mu\Gamma_\nu F^{\mu\nu}.$$

Thus, the equations (10.10) can be reduced to the equations (10.30a,b) for the $4s$-component function ψ_+, the remaining $4s$ components of ψ (i.e., ψ_-) being expressed via ψ_+ in accordance with (10.30c).

The equation (10.30b) means that $2s$-1 components of the function ψ_+ are equal to zero, and $2s+1$ remaining components of ψ_+ form a spinor within the space of the representation *D(s 0)* of the algebra *AO(1,3)*. We denote nonzero components of ψ_+ by Φ_s and rewrite the equation (10.30a) in the form

$$\left[\pi_\mu\pi^\mu - m^2 + \frac{e}{s}\boldsymbol{S}\cdot(\boldsymbol{H} - i\boldsymbol{E})\right]\psi_s = 0, \qquad (10.31)$$

where $\boldsymbol{S}$ are matrices of dimension $(2s+1)\times(2s+1)$ realizing the IRs *D(s)* of the algebra *AO(3)*.

To prove the system (10.31) is causal it is sufficient to replace operators of differentations in respect with x_μ by the component of the characteristic four-vector n_μ and then equate to zero the determinant of the obtained system of algebraic equations for n_μ, taking into account the highest order terms only. The analysed system of partial differential equations is causal if the corresponding characteristic equation for n_μ has light-like solutions only [412].

The characteristic equation for the system (10.31) has the form

$$(n^\mu n_\mu)^{2s+1} = 0,$$

thus all the characteristic vectors are light-like. It means this system is causal. Since any solution of the equation (10.10) can be expressed via solutions of the system (10.31) in accordance with (10.30), we conclude that the system (10.10) is causal.

In the case $s=1/2$ formula (10.31) defines the Zaitsev-Feynman-Gell-Mann equation for an electron [98, 420], and for an arbitrary s it represents the simplest generalization of this equation. So, formulae (10.10) define a system of first-order equations corresponding to the generalized Zaitsev-Feynman-Gell-Mann equation.

In a similar way we can prove the causal character of the equations (10.19).

10.9. The Causal Equation for Spin-One Particles with Positive Energies

In conclusion, we consider in more detail the most popular equation for spin-1 particles, i.e., the KDP equation.

It is well known that the KDP equation also is not completely satisfactory if we use it to describe interacting particles. In the case of minimal coupling this equation has not good solutions if the external field reduces to the Coulomb potential, as it was pointed by Tamm [401] and Corben and Schwinger [40*]. The reason is that this equation does not take into account the spin-orbit coupling, see Subsection 10.7. On the other hand the KDP equation with anomalous interaction predicts complex energies for a particle interacting with the constant and homogeneous magnetic field [403]. Moreover taking into account an anomalous interaction we come in general to a non-causal equation [28*].

Here we demonstrate that it is possible to introduce such the anomalous interaction into the KDP equation that the causality principle will be satisfied and complex energies will not arise while solving the problem of interaction of spin-one particle with the constant magnetic field. To achieve this goal it is sufficient to take into account interactions which are bilinear with respect to external field strengths.

We will start with the following equation

$$\left[\beta^{\mu}\pi_{\mu}-m+\frac{1}{2m}\left(1-\beta_4^2\right)\left(\frac{k_1 e}{2}S^{\mu\sigma}F_{\mu\sigma}-\frac{k_2 e^2}{4m^2}F^{\mu\sigma}F_{\mu\sigma}\right)\right]\psi=0 \tag{10.32}$$

where $S_{\mu\sigma}=i[\beta_{\mu},\beta_{\sigma}]$, β_{μ} are the 10×10 KDP matrices, e, k_1 and k_2 are coupling constants which we choose be equal to 1 for simplicity.

The equation (10.32) is Poincaré- and *P*-, *T*-, *C*-invariant and includes the terms describing the anomalous interaction of a particle with an external field. The term proportional to $F^{\mu\sigma}$ represents the general form of anomalous interaction which does not violate the causality principle [40*]. The term proportional to $F_{\mu\sigma}F^{\mu\sigma}$ is introduced in order to overcome difficulties with complex energies*.

Let us demonstrate that the equation (10.32) is causal. To do it we will delete superfluous components of the wave function ψ and analyze the corresponding equation including the physical components of ψ only.

We find it is convenient to use the concrete realization (6.22), (6.24) of the β-matrices. Expressing nonphysical components $(1-\beta_0^2)\psi$ via physical components $\Psi=\beta_0^2\psi$ we come to the following generalized TST equation

$$\pi_0\Psi=\hat{H}\Psi \tag{10.33}$$

where

$$\begin{aligned}\hat{H}=&\frac{1}{2}(i\sigma_1+\sigma_2)\left[M+\frac{(\boldsymbol{S}\cdot\pi)}{m}-\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{m}+\frac{(\boldsymbol{E},\boldsymbol{E})}{Mm^2}\right]-\frac{i}{2Mm}(1+\sigma_3)(\boldsymbol{E},\pi)+\\&+\frac{i}{2m}(1-\sigma_3)(\pi,\frac{1}{M}\boldsymbol{E})+\frac{1}{2}(\sigma_2-i\sigma_1)\left[m+(\pi,\frac{1}{\mu}\pi)\right],\end{aligned} \tag{10.34}$$

$M=m+F^{\mu\sigma}F_{\mu\sigma}/8$, $\boldsymbol{S}$ and σ_a are the spin-one and Pauli matrices (7.6) and (3.6), the symbol $(\boldsymbol{A},\boldsymbol{B})$ denotes the 6×6 matrix:

This term is absent in paper [40] where nonlinear in $F^{\mu\sigma}$ interactions where analyzed.

$$(\boldsymbol{A},\boldsymbol{B})=\begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}.$$

Multiplying the l.h.s. and r.h.s. of (10.33) by π_0 and using the identities

$$(\boldsymbol{S}\cdot\pi)^2\left(\pi,\frac{1}{M}\boldsymbol{p}\right)=\boldsymbol{S}\cdot\boldsymbol{H}\left(\pi,\frac{1}{M}\pi\right)+\ldots=-\frac{i}{M}(\pi\times\boldsymbol{H})+\ldots\ ,$$

$$\left(\pi,\frac{1}{M}\pi\right)=\frac{1}{M}\left[\pi^2-(\boldsymbol{S}\cdot\pi)^2\right]+\ldots\ ,$$

$$(\boldsymbol{E},\boldsymbol{E})\left(\pi,\frac{1}{M}\pi\right)=(\boldsymbol{E},\pi)\frac{1}{M}(\boldsymbol{E},\pi)+\ \ldots\ ,\quad (\boldsymbol{E},\pi)(\boldsymbol{S}\cdot\pi)^2=\ldots\ ,$$

$$(\pi,\frac{1}{M}\pi)(\boldsymbol{S}\cdot\pi)^2=\left(\pi,\frac{1}{M}\pi\right)\boldsymbol{S}\cdot\boldsymbol{H}+\ldots=\frac{i}{M}(\pi,\pi\times\boldsymbol{H})+\ \ldots\ ,$$

$$\left(\pi,\frac{1}{M}\pi\right)\frac{1}{M}(\boldsymbol{E},\boldsymbol{E})=\left(\pi,\frac{1}{M}\boldsymbol{E}\right)\left(\pi,\frac{1}{M}\boldsymbol{E}\right)+\ \ldots\ ,$$

$$\left(\pi,\frac{1}{M}\pi\right)\frac{1}{M}(\boldsymbol{E},\pi)=\left(\pi,\frac{1}{M}\boldsymbol{E}\right)\left(\pi,\frac{1}{M}\pi\right)+\ \ldots\ ,$$

(where the dots denote the terms which does include second-order differential operators) we come to the second-order equation

$$\left[\pi_\mu\pi^\mu+\frac{1}{2}(\sigma_1+i\sigma_2)\left(\pi,\frac{\dot{M}}{M^2}\pi\right)+\ldots\right]\Psi=0 \tag{10.35}$$

which corresponds to the follofing characteristic equation

$$(n^\mu n_\mu)^6=0. \tag{10.37}$$

In accordance with (10.37) the characteristic four-vector n_μ for the equation (10.32) is light-like, and so this equation is causal.

In the case of the constant and homogeneous magnetic field directed along the third coordinate axis, (10.32) reduces to the following exact equation

$$\pi_0^2\Psi=\left(m^2+\pi^2-2S_3H+\frac{1}{4}\frac{H^2}{m^2}\right)\Psi.$$

Using relations (30.7) it is not difficult to make sure that all the eigenvalues of the operator p_0^2 are positive defined, and so all the corresponding energy values are real.

Thus the equation (10.32) with k_1=k_2=1 is causal and does not lead to complex energies in the case of constant and homogeneous magnetic field. The same is true for the case of an arbitrary k_1 if k_2 is sufficiently large.

3. REPRESENTATIONS OF THE GALILEI ALGEBRA AND GALILEI-INVARIANT WAVE EQUATIONS

One of the main requirements imposed on equations of nonrelativistic physics is the invariance under the Galilei transformations. This circumstance predetermines a fundamental role performed by representations of the Galilei group.

The present chapter is devoted to description of representations of the Lie algebra of the Galilei group and of Galilei-invariant equations for particles of arbitrary spins. We will see that the concept of spin arises naturally in frames of nonrelativistic quantum mechanics and most spin related effects (i.e., dipole, spin-orbital, and other interactions) can be successfully described by equations satisfying the Galilei relativity principle.

Section 11 is devoted to studying symmetries of the basic equation of nonrelativistic quantum mechanics, i.e., the Schrödinger equation.

11. SYMMETRIES OF THE SCHRÖDINGER EQUATION

11.1. The Schrödinger Equation

In quantum mechanics, the states of a system with n degrees of freedom, as described by a set of coordinates $\boldsymbol{x}=(x_1,x_2,...,x_n)$, are determined by the wave function $\psi(t,\boldsymbol{x})$ being a vector in a Hilbert space. Moreover the evolution of a system is described by the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,\boldsymbol{x})=H\psi(t,\boldsymbol{x}) \tag{11.1}$$

where H is the Hamiltonian operator or Hamiltonian of a system.

The simplest quantum mechanical system is a free spinless particle. The wave function of such a particle depends on three spatial variables $\boldsymbol{x}=(x_1,x_2,x_3)$, and the corresponding Hamiltonian has the form

$$H=\frac{p^2}{2m} \tag{11.2}$$

where m is a parameter determining the mass of a particle, $p^2=p_1^2+p_2^2+p_3^2$, $p_a=-i\partial/\partial x_a$, a=1,2,3. Substituting (11.2) into (11.1) we receive the Schrödinger equation for a free particle

$$L\psi(t,\boldsymbol{x})\equiv\left(i\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\psi(t,\boldsymbol{x})=0, \tag{11.3}$$

which is going to be the object of our study.

One of the main postulates of quantum mechanics is the Galilei relativity principle which can be formulated as a requirement of invariance of the Schrödinger equation under the Galilei transformations. Of course, this requirement is satisfied by the simplest evolution equation (11.3). Moreover, as it was stated relatively recently, the equation (11.3) possesses a more extensive symmetry, as being invariant under scale and conformal transformations.

Below we will study the symmetries of the Schrödinger equation in detail. It will allow to explain, using a relatively simple example, the meaning of the Galilei invariance concept for quantum mechanics equations. Moreover, we will demonstrate that the invariance group of the equations (11.3) determines the maximal symmetry of this equation in some sense.

The problem of investigation of symmetries of Schrödinger equation can be formulated in complete analogy with the corresponding problem for the KGF equation (refer to Section 1). As before, it is sufficient to restrict ourselves to considering only such solutions of the equations (11.3) which are defined on some open set D of the four-dimensional manifold R_4 and belong to the vector space F of complex-valued functions being analytical on D. Then, the set F_0 of solutions of (11.3) can be defined as a zero-space of the differential operator

$$L=p_0-\frac{p^2}{2m}, \quad p_0=i\frac{\partial}{\partial t} \tag{11.4}$$

defined on F: $\psi\in F_0$ if $\psi\in F_4$ and $L\psi=0$. The definitions of a SO and IA for the Schrödinger equation coincide with Definitions 1.1 and 1.2, p.p. 2, 3, if the symbol L denotes the differential operator (11.4).

11.2. Invariance Algebra of the Schrödinger equation

The starting point of our studies of Schrödinger equation symmetries lies in determining of the IA for this equation in the class M_1, i.e., the class of first-order differential operators.

THEOREM 11.1. The maximal IA of the Schrödinger equation in class M_1 is the 13-dimension Lie algebra with the following basis elements

$$P_0=p_0=i\frac{\partial}{\partial t}, \quad P_a=p_a=-i\frac{\partial}{\partial x_a}, \qquad M=Im, \tag{11.5a}$$

$$J_a=\varepsilon_{abc}x_b p_c,\quad G_a=tp_a-mx_a,$$
$$D=2tp_0+x^a p_a+\frac{3}{2}i,\quad A=t^2p_0-tD+\frac{1}{2}mx_a x^a, \tag{11.5b}$$

where $x^a=-x_a$, I is the identity operator, and the summation from 1 to 3 is imposed over the repeating indices.

PROOF. It is not difficult to verify that the operators (11.5) form an IA of the Schrödinger equation. We can make sure that any of the operators (11.5) satisfies the invariance condition (1.5) where L is the operator (11.4),

$$\alpha_M=\alpha_{P_0}=\alpha_{P_a}=\alpha_{J_a}=\alpha_{G_a}=0,\ \alpha_D=-2i,\ \alpha_A=-2it.$$

It is also easy to verify that the operators (11.5) form a basis of the Lie algebra, satisfying the following commutation relations:

$$[P_a,P_b]=[P_a,P_0]=[M,P_a]=[M,P_0]=[M,J_a]=[M,G_a]=[P_0,J_a]=0, \tag{11.6a}$$

$$[P_a,J_b]=i\varepsilon_{abc}P_c,\ [P_0,G_a]=iP_a, \tag{11.6b}$$

$$[P_a,G_b]=i\delta_{ab}M,\ [J_a,J_b]=i\varepsilon_{abc}J_c, \tag{11.6c}$$

$$[G_a,J_b]=i\varepsilon_{abc}G_c,\ [G_a,G_b]=0, \tag{11.6d}$$

$$[D,J_a]=[D,M]=[A,G_a]=[A,M]=[A,J_a]=0,$$
$$[D,P_a]=-iP_a,\quad [D,G_a]=iG_a,\quad [D,P_0]=-2iP_0, \tag{11.7}$$
$$[A,P_a]=iG_a,\quad [A,P_0]=iD,\ [A,D]=-2iA,$$

In analogy with the Theorem 1.2 proof it is possible to demonstrate that relations (11.5) define a basis of the maximal IA of the Schrödinger equation in the class M_1. We do not provide a detailed proof but it should be noted that by substituting (1.4) and (11.4) into the invariance condition (1.5), we come to the following system of determining equations (compare with (1.9)):

$$A_a^a=A_b^b,\quad A_a^b+A_b^a=0,\quad b\neq a,$$
$$A_a^0=0,\quad \dot{A}^0=-2A_a^a,\quad \alpha=2iA_a^a, \tag{11.8}$$
$$i\dot{A}^a+\frac{1}{2m}\Delta A^a+\frac{i}{m}B_a=0,\quad i\dot{B}+\frac{1}{2m}\Delta B=0,$$

where the dots denote derivatives in respect with the time variable t, $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$, $A^a{}_b=\partial A_a/\partial x_b$, $B_a=\partial B/\partial x_a$, and there are no summing over repeated indices.

The system (11.8) is easily integrated (compare with (1.8)) and leads to the following expressions for A^μ and B:

$$\begin{aligned}&A^0=ht^2+2jt+c, \quad \alpha=2i(ht+j),\\&A^a=c^{ab}x_b+htx^a+jx^a+tn^a+m^a,\\&B=m\left(\frac{h}{2}x_bx^b-n^ax_a+g\right)+\frac{3}{2}i(ht+j),\end{aligned} \qquad (11.9)$$

where $c^{ab}=-c^{ba}$, h, j, c, g, n^a and m^a are arbitrary constants. Substituting (11.9) into (1.4) we come to a linear combination of the SO (11.5). ■

11.3. The Galilei and Generalized Galilei Algebras

Thus, we have found the IA of the Schrödinger equation in the class M_1 which includes 13 linearly independent SO of (10.5). These operators generate a Lie algebra which we call the generalized Galilei algebra and denoted as $AG_2(1,3)$. An abstract definition of this algebra is given by relations (11.6) and (11.7).

To understand the structure of the algebra $AG_2(1,3)$, we represent it as a chain of maximal ideals (let us recall that an ideal of a Lie algebra G is a subalgebra A such that $[a,b]\in A$ for any $b\in G$, $a\in A$, and an ideal A is called maximal if there do not exist ideals $A'\not\subset A$). Starting with (11.6), (11.7), it is not difficult to make sure that the maximal ideal A^1 of the algebra $A_2G(1,3)$ includes 11 elements, i.e., P_0, P_a, J_a, G_a, and M. This fact can be written in the following symbolic form:

$$AG_2(1,3)=A^1 \ni\!\!\!\!+ \Re, \quad A^1\supset P_0,P_a,G_a,M, \quad \Re\supset D, A. \qquad (11.10)$$

The subalgebra A^1 contains, in its turn, the maximal ideal $A^2\supset\{M,P_0,P_a,G_a\}$, and the subalgebra A^2 does the maximal ideal $T_5\supset\{M,P_0,P_a\}$. This may be reflected by the following symbolic relation:

$$AG_2(1,3)=[(T_5 \ni\!\!\!\!+ T_3) \ni\!\!\!\!+ AO(3)] \ni\!\!\!\!+ \Re \qquad (11.11)$$

where $T_3\supset\{G_a\}$, $AO(3)\supset\{J_a\}$, and $\Re\supset\{D,A\}$.

Formula (11.11) describes a structure of the generalized Galilei algebra by providing its main subalgebras. The subalgebras T_5 and T_3 are commutative (Abelian) algebras of dimensions 5 and 3, $AO(3)$ is the three-dimensional algebra which is isomorphic to the Lie algebra of the rotation group. Let us point also one more important subalgebra with the basis elements

$$S_{01}=\frac{1}{2}(A+P_0), \quad S_{02}=\frac{1}{2}(A-P_0), \quad S_{12}=-\frac{1}{2}D. \qquad (11.12)$$

The operators (11.12) satisfy the commutation relations (4.27) characterising the algebra *AO(1,2)*.

The most interesting (from the physical point of view) subalgebra of the algebra $AG_2(1,3)$ is its maximal ideal which will be called the Galilei algebra and denoted by *AG(1,3)*. The meaning of the name will be explained in the following subsection.

The algebra *AG(1,3)* is a linear span of the basis elements (10.5) satisfying relations (11.6). It will be shown further on that it is the invariance under the algebra *AG(1,3)* that serves as a mathematical expression of the Galilei relativity principle. As for the operators D and A, which complete the Galilei algebra to the algebra $AG_2(1,3)$, they reduced to the combinations of P_a, G_a and M on a set of the equation (11.3) solutions. Indeed, it is not difficult to make sure that

$$D\psi=(2M)^{-1}(P_aG_a+G_aP_a)\psi, \quad A\psi=(2M)^{-1}(G_aG_a)\psi. \tag{11.13}$$

In other words the symmetry of the Schrödinger equation under the operators D and A is nothing but a direct consequence of the invariance under the Galilei algebra.

The Galilei algebra has three main Casimir operators

$$C_1=M, \quad C_2=2MP_0-P_aP_a, \quad C_3=(MJ_a-\varepsilon_{abc}P_bG_c)(MJ_a-\varepsilon_{ade}P_dG_e). \tag{11.14}$$

In quantum mechanics, the eigenvalues of the operators (11.14) are associated with a mass, internal energy and spin of a particle. Substituting (11.5) into (11.14), it is not difficult to make sure that eigenvalues of C_1, C_2 and C_3 on a set of Schrödinger equation solutions are $c_1=m$, $c_2=c_3=0$. So, we can conclude that the Schrödinger equation describes a particle of mass m and of zero spin and internal energy.

In conclusion we note that the connection between the basis elements of the Galilei and generalized Galilei algebras given by relations (11.13) takes place for arbitrary representations of these algebras. More precisely, the following assertion is true [333]:

LEMMA 11.1. Let $\{P_0,P_a,G_a,J_a,M\}$ be a set of operators satisfying the Galilei algebra (11.6), and, besides, M is an invertible operator. Then the operators $\{D,A,P'_0,P_a,G_a,J_a,M\}$, where $P'_0=P_0-(2M)^{-1}C_2$ and D, A are the operators (11.13),form a representation of the generalized Galilei algebra, i.e., they satisfy conditions of (11.6) and (11.7).

Proof can be carried out by direct verification.■

According to Lemma 11.1 any representation of the Galilei algebra (corresponding to $c_1\neq0$) can be extended to a representation of the generalized Galilei algebra (in the same way as any representation of the Poincaré algebra of Class II can be extended to a representation of the conformal algebra, refer to Section 3). It means that any equation being invariant under the Galilei algebra and describing a particle of a non-zero mass is also invariant under the generalized Galilei algebra (but the corresponding operators D and A do not, in general, belong to the class M_1).

11.4. The Schrödinger Equation Group

The symmetry of the Schrödinger equation under the algebra $AG_2(1,3)$ is a fundamental fact which can be used as a base for Galilean kinematics. In this subsection we consider one of the main consequence of such invariance and clarify its physical meaning.

In physical terms, this consequence can be formulated as follows: the Schrödinger equation satisfies the Galilei relativity principle.

As in Section 1, we use the fact that an IA in the class M_1 generates a local representation of a Lie group. To find the explicit form of the corresponding one-parametrical subgroups, we use the Lie equations (1.21) and (1.22) which are easily integrable for the SO of (11.5). Comparing (11.4) with (11.5), we conclude that B≡0 for the operators P_μ and J_a and, therefore, solutions of the corresponding equations (1.22) will have the following form:

$$\psi'(x')=\psi(x), \quad x=(x_0,x_1,x_2,x_3), \; x_0=t. \tag{11.15}$$

The corresponding equations of (1.22) will have the following solutions:

$$x_\mu'=x_\mu+b_\mu \tag{11.16}$$

for $Q=P_\mu$ and

$$\begin{aligned} &x_0'=x_0, \quad x_a'=x_a,\\ &x_b'=x_b\cos\theta_a+x_c\sin\theta_a,\\ &x_c'=x_c\cos\theta_a-x_b\sin\theta_a \end{aligned} \tag{11.17}$$

for $Q=J_a$. Here b_μ, θ_a are real parameters, and (a,b,c) is the cycle $(1,2,3)$.

Integrating the equations (1.21), (1.22) we obtain the corresponding transformations generated by the remaining basis elements of the generalized Galilei algebra:

$$\begin{aligned} &\psi'(x')=\exp\left(imv_0x_a+i\frac{mv_av_a}{2}\right)\psi(x),\\ &x_0'=x_0, \quad x_a'=x_a+v_at, \quad x_b'=x_b, \quad b\neq a, \end{aligned} \tag{11.18}$$

if $Q=G_a$;

$$\psi'(x')=\exp(imc)\psi(x), \quad x'=x, \tag{11.19}$$

if $Q=M$;

$$\psi'(x')=\exp\left(-\frac{3}{2}\lambda\right)\psi(x),$$
$$x_0'=\exp(2\lambda)x_0,\quad x_a'=\exp(\lambda)x_a, \tag{11.20}$$

if $Q=D$;

$$\psi'(x')=(1+\xi x_0)^{3/2}\exp\left[\frac{-imx^2\xi}{2(1+\xi x_0)}\right]\psi(x),$$
$$x_0'=\frac{x_0}{1+\xi x_0},\quad x_a'=\frac{x_a}{1+\xi x_0}, \tag{11.21}$$

if $Q=A$.

Here G_a, M, A, D are the operators from (11.5), and v_a, c, λ, ξ are real parameters of the corresponding transformations.

Formulae (11.16)-(11.21) define a family of one-parameter transformations forming the 13-parameter Lie group called the Schrödinger equation group. The transformations (11.16) are translations of time and spatial variables, formulae (11.17) define a rotation of a reference frame, and relations (11.18) can be interpreted as a transition to a new frame of reference moving with the velocity v_a along the axis a. Finally, the transformations (11.20) and (11.21) describe the symmetry of the Schrödinger equation in respect to the scaling and specific nonlinear change of variables $x_\mu \to x_\mu/(1+\xi x_0)$.

Using (11.6), (11.7), (11.16)-(11.21) it is simple to obtain the general transformation from the Schrödinger equation group in the following form:

$$\psi(x)\to\psi'(x')=\exp[if(x)]\psi(x), \tag{11.22}$$

where

$$f(x)=\frac{3}{2}\ln(1+\xi x_0)-\frac{imx_ax_a\xi}{2(1+\xi x_0)}+mv_ax_a+\frac{1}{2}mvx_0+\frac{3}{2}i\lambda+mc,$$
$$x_a'=\frac{\exp(\lambda)R_{ab}x_b}{1+\exp(2\lambda)\xi x_0}+v_ax_0+b_a,\quad x_0'=\frac{x_0}{\xi x_0+\exp(2\lambda)}+b_0, \tag{11.23}$$

R_{ab} is the operator of spatial rotations,

$$R_{ab}=\delta_{ab}\cos\theta+\frac{\varepsilon_{abc}\theta_c}{\theta}\sin\theta+\frac{\theta_a\theta_b}{\theta^2}(1-\cos\theta),\quad \theta=(\theta_1^2+\theta_2^2+\theta_3^2)^{1/2}, \tag{11.24}$$

with θ_a, v_a, b_0, b_a, ξ, λ, and c being arbitrary real numbers.

It is possible to accertain by direct verification that the Schrödinger equation (11.3) is invariant under the transformation (11.22) since $\psi'(x')$ satisfies the equation

$$\left(i\frac{\partial}{\partial x_0'}-\frac{p'^2}{2m}\right)\psi'(x')=0,\quad p_\mu'=i\frac{\partial}{\partial x'^\mu}. \tag{11.25}$$

We see that the transformed function $\psi'(x')$ of (11.22) differs from the initial function $\psi(x)$ by the phase multiplier $\exp[if(x)]$. However, it may be shown that the transformations (11.22)-(11.24) do not change the norm of a wave function

$$|\psi|^2=(\psi,\psi)=\int d^3x\psi^\dagger\psi=(\psi',\psi')\equiv\int d^3x'\psi'^\dagger\psi'. \tag{11.26}$$

Let us note that the use of formulae (11.22)-(11.24) needs an appropriate caution since it may occur that x'_0 and x'_a do not belong to D, though $x_0, x_a\in D$. For fixed x_0, $x_a\in D$, x'_0 and x'_a belong to the domain of a function $\psi\in F_0$ if the transformation (11.23) belongs to an infinitesimal neighbourhood of the identity transformation. Because of this (and bearing also in mind that the expressions (11.23) become meaningless if $\xi=-x_0^{-1}\exp(-2\lambda)$), we say that the relations (11.22)-(11.24) define only a local representation of the Schrödinger equation group.

11.5. The Galilei Group

Consider in greater detail the main subgroup of the Schrödinger equation group which corresponds to $\xi=\lambda=0$. The corresponding transformations (11.22)-(11.24) take the form

$$\psi(x)\to\psi'(x')=\exp[i\varphi(x)]\psi(x), \tag{11.27}$$

where

$$\varphi(x)=mv_ax_a+\frac{1}{2}mv_av_ax_0+mc, \tag{11.28}$$

$$x_a'=R_{ab}x_b+v_ax_0+b_a,\quad x_0'=x_0+b_0. \tag{11.29}$$

The transformations of space and time variables given in (11.29) are called the Galilei transformations.

Let us demonstrate that the transformations (11.27)-(11.29) form a group. Let $(R,\boldsymbol{v},\boldsymbol{b},b_0,c)$ be a transformation (11.27)-(11.29) with $R=R(\theta)$ denoting the matrix of spatial rotation (11.24). Then, the group composition law is defined as follows

$$\begin{aligned}&\left(R^{(2)},\boldsymbol{v}^{(2)},\boldsymbol{b}^{(2)},b_0^{(2)},c^{(2)})(R^{(1)},\boldsymbol{v}^{(1)},\boldsymbol{b}^{(1)},b_0^{(1)},c^{(1)}\right)=\\&=\Big(R^{(2)}\cdot R^{(1)},\boldsymbol{v}^{(1)}+R^{(1)}\boldsymbol{v}^{(2)},\boldsymbol{b}^{(1)}+R^{(1)}\boldsymbol{b}^{(2)}+\boldsymbol{v}^{(1)}b_0^{(2)},\\&\quad b_0^{(1)}+b_0^{(2)},c^{(1)}+c^{(2)}+v_a^{(1)}R_{ab}^{(1)}b_b^{(2)}+\frac{1}{2}b_0^{(2)}(\boldsymbol{v}^{(1)})^2\Big).\end{aligned} \tag{11.30}$$

The unit element of the group is represented by the identity transformation

$$E=(I,0,0,0,0), \tag{11.31}$$

and the element inverse to $(R,\boldsymbol{v},\boldsymbol{b},b_0,c)$ has the form

$$(R,\boldsymbol{v},\boldsymbol{b},b_0,c)^{-1}=\left(R^{-1},-R^{-1}\boldsymbol{v},-R^{-1}(\boldsymbol{b}-\boldsymbol{v}b_0),-b_0,-c+\boldsymbol{b}\cdot\boldsymbol{v}-\frac{1}{2}b_0\boldsymbol{v}^2\right) \tag{11.32}$$

This group of transformations (11.27)-(11.29) is called the extended Galilei group.

We see that the extended Galilei group is an 11-parameter Lie group while formulae (11.29) representing the transformations of the space-time continuum include 10 parameters only. The set of the transformations (11.29) also forms the so-called Galilei group. The group law for the transformations (11.29) has the form

$$\begin{aligned}&g\left(R^{(2)},\boldsymbol{v}^{(2)},\boldsymbol{b}^{(2)},b_0^{(2)},c^{(2)}\right)g\left(R^{(1)},\boldsymbol{v}^{(1)},\boldsymbol{b}^{(1)},b_0^{(1)},c^{(1)}\right)=\\ &=g\left(R^{(2)}R^{(1)},\boldsymbol{v}^{(1)}+R^{(1)}\boldsymbol{v}^{(2)},\boldsymbol{b}^{(1)}+R^{(1)}\boldsymbol{b}^{(2)}+\boldsymbol{v}^{(1)}b_0^{(2)},b_0^{(1)}+b_0^{(2)}\right).\end{aligned} \tag{11.33}$$

The transformations (11.27) comprise a representation of the Galilei group if we set $c=0$ in (11.28). It is not difficult to make sure that such a representation is not exact (but is only a projective one) because of

$$\begin{aligned}&\left(R^{(2)},\boldsymbol{v}^{(2)},\boldsymbol{b}^{(2)},b_0^{(2)},0\right)\left(R^{(1)},\boldsymbol{v}^{(1)},\boldsymbol{b}^{(1)},b_0^{(1)},0\right)=\\ &=\exp(i\omega_{12})\left(R^{(2)}R^{(1)},\boldsymbol{v}^{(1)}+R^{(1)}\boldsymbol{v}^{(2)},\boldsymbol{b}^{(1)}+R^{(1)}\boldsymbol{b}^{(2)}+b_0^{(2)}\boldsymbol{v}^{(1)},b_0^{(1)}+b_0^{(2)},0\right),\end{aligned} \tag{11.34}$$

where $\exp(i\omega_{12})$ is a phase multiplier:

$$\omega_{12}=v_a^{(1)}R_{ab}b_b^{(2)}+\frac{1}{2}b_0^{(2)}(\boldsymbol{v}^{(1)})^2. \tag{11.35}$$

We see that the transformations (11.27) with $c=0$ satisfy the group composition law (11.33) up to the phase multiplier $\exp(i\omega_{12})$ only, the latter not changing the norm (11.26).

Thus, we can consider either the exact representation of the extended Galilei group or the projective representation of the Galilei group. We note that the transformations (11.27)-(11.29) can be considered as a global representation of the Galilei group if the domain of ψ coincides with the four-dimensional manifold R_4.

11.6. The Transformations *P* and *T*

The symmetry group of the Schrödinger equation considered above describes invariance properties of this equation with respect to continuous transformations of dependent and independent variables. Now we briefly discuss the symmetry of the Schrödinger equations under the discrete transformations

$$\boldsymbol{x}\to-\boldsymbol{x}, \quad t\to t, \tag{11.36a}$$

$$\boldsymbol{x}\to\boldsymbol{x}, \quad t\to-t. \tag{11.36b}$$

We can make sure that these transformations do not belong to the class defined by relations (11.29) or (11.23). They are linear transformations whose matrices have negative determinants in contrast to the linear homogeneous part of the transformations (11.23). Nevertheless the Schrödinger equation is invariant under these transformations if a wave function is simultaneously transformed according to

$$\begin{aligned} &\psi(t,\boldsymbol{x})\to P\psi(t,\boldsymbol{x})=\eta_1\psi(t,-\boldsymbol{x}),\\ &\psi(t,\boldsymbol{x})\to T\psi(t,\boldsymbol{x})=\eta_2\psi^*(-t,\boldsymbol{x}) \end{aligned} \tag{11.37}$$

where the asterisk denotes the complex conjugation, η_1 and η_2 are complex numbers such that $\eta_1=\eta_2^*=\eta_2\eta_2^*=1$. Without loss of generality we may choose

$$\eta_1=\pm1, \quad \eta_2=\pm1. \tag{11.39}$$

The set of transformations *P*, *T*, *PT* and the identity transformation form a symmetry group of the Schrödinger equation. This discrete group can be added to the Schrödinger group to obtain the so-called complete Schrödinger group. Unitary ray representations of the complete Schrödinger group are considered in [59].

This ends the brief discussion of classical symmetries of the basic equation of quantum mechanics. The results represented in this section will be repeatedly used further on at the deducing of motion equations for arbitrary spin particles.

12. REPRESENTATIONS OF THE LIE ALGEBRA OF THE GALILEI GROUP

12.1. Galilei Relativity Principle and Equations of Quantum Mechanics

As was shown above, the Schrödinger equation is invariant under the coordinate transformations corresponding to a transition to a new inertial frame of reference.

The relativity principle establishing equal rights of all the inertial frames of reference was formulated by Galileo Galilei more than 350 years ago. Of course, Galilei did not show the explicit form of transformations of space and time coordinates (the concept of coordinates was introduced much later). Nevertheless, these transformations bear the name of Galilei with a good reason, and the set of such transformations is properly called the Galilei group.

It looks surprising that the structure of the Galilei group and its representations has begun to be studied relatively recently, and much later than the corresponding problems of the Poincaré group. In 1952 Inönü and Wigner [229] described exact representations of this group. Bargman [16] was the first to point to a fundamental role of ray representations of the group *G(1,3)* in quantum mechanics.

After the fundamental works [16, 229], it was Levi-Leblond who made an essential contribution into understanding of problems of Galilean relativity in quantum mechanics. He deduced nonrelativistic analog of the Dirac equation which is invariant under the Galilei group [276]. Like the Dirac equation, the Levi-Leblond equation describes the Pauli interaction and predicts the correct value of the gyromagnetic ratio.

The Levi-Leblond equation and its possible generalizations are considered in detail in the subsequent sections. Here we only note that there exist Galilei-invariant wave equations which describe correctly such a fine effect as the spin-orbit coupling. This makes it possible to use these equations to solve practical physical problems, and sometimes they are more suitable then complicated relativistic wave equations.

To describe Galilei-invariant wave equations it is necessary to know representations of the Galilei group. The basic information about these representations is given in this and subsequent sections.

12.2. Classification of IRs

Let us consider representations of the Lie algebra of the Galilei group as defined by the commutation relations (11.6). In this section we consider completely reducible representations only, so it is sufficient to restrict ourselves to IRs. Following [151], we will find such a realization of all the nonequivalent IRs of this algebra that is distinguished by a common and simple form of the Galilei group generators for all the classes of IRs. Besides, the approach used is closely related with that presented in Chapter 2. This enables us to avoid superfluous details.

The basic Casimir operators of the algebra *AG(1,3)* are given by (11.14). We will show further on that representations of this algebra differ qualitatively for zero and nonzero eigenvalues c_1, c_2 of the operators C_1, C_2. Namely, it is possible to select five classes of representations corresponding to the set of values c_1 and c_2 as shown in (4.10). Thus, the classification of IRs of the Galilei group is very similar to the corresponding classification for the Poincaré algebra (do not take the Casimir operators of the Galilei group for those of the Poincaré group though we use the same notation for operators and corresponding eigenvalues).

The first three classes *I-III* of IRs can be realized by Hermitian operators.

Representations of Class *IV* are non-Hermitian but they also find an appropriate usage in physics.

When $P_\mu \equiv 0$ we come to the representations of the homogeneous Galilei group which is isomorphic to the group *E(3)*. Finite dimensional and irreducible representations of such a type are considered in Subsection 13.6.

As a basis of an IR we choose a set of eigenfunctions $|c,\tilde{p},\lambda>$ of the commuting operators C_1, C_2, C_3 (11.14), P_0, P_1, P_2, P_3, $W_0 = \boldsymbol{J}\cdot\boldsymbol{P}$, and of additional Casimir operators C_4, C_5, ... , which will be different for different classes of IRs. The relations (4.8) can serve as a formal definition of the vectors $|c,\tilde{p},\lambda>$ (remember that the symbols C_a, P_μ, and W_0 denote now another operators belonging to the algebra *AG(1,3)*).

Description of representations of the algebra *AG(1,3)* in the basis $|c,\tilde{p},\lambda>$ reduces to finding of an explicit form of nonequivalent operators J_a and G_a satisfying the commutation relations (11.6) together with given operators P_μ and M. In the same way as was done in studying of IRs of the algebra *AP(1,3)* (refer to Section 4), it is convenient to pass from the vectors $\boldsymbol{J}$ and $\boldsymbol{G}$ to the following four-vectors

$$\begin{aligned} &W_0 = \boldsymbol{P}\cdot\boldsymbol{J}, \quad \boldsymbol{W} = M\boldsymbol{J} - \boldsymbol{P}\times\boldsymbol{G}, \\ &\Gamma_0 = \boldsymbol{P}\cdot\boldsymbol{G}, \quad \Gamma = M\boldsymbol{G} - \boldsymbol{P}\times\boldsymbol{J}. \end{aligned} \tag{12.1}$$

If the operator $m^2 + P^2$ is invertible then $\boldsymbol{J}$ and $\boldsymbol{G}$ can be expressed via the operators (12.1):

$$\begin{aligned} &\boldsymbol{J} = (M^2 + P^2)^{-1}(\boldsymbol{P}\times\Gamma + M\boldsymbol{W} + \boldsymbol{P}W_0), \\ &\boldsymbol{G} = (M^2 + P^2)^{-1}(\boldsymbol{P}\times\boldsymbol{W} + M\Gamma + \boldsymbol{P}\Gamma_0), \end{aligned} \tag{12.2}$$

and our problem reduces to finding all the nonequivalent Γ, Γ_0 and $\boldsymbol{W}$, W_0.

12.3. The Explicit Form of Basis Elements of the Algebra *AG(1,3)*

According to (4.8), (11.6), the operators (12.1) satisfy the following relations:

$$MW_0 - \boldsymbol{P}\cdot\boldsymbol{W} = 0, \quad [W_\mu, P_\sigma] = 0, \tag{12.3}$$

$$[W_a, W_b] = ic_1\varepsilon_{abc}W_c, \quad [W_0, W_a] = -i\varepsilon_{abc}P_bW_c, \tag{12.4}$$

Here c_1 and p_μ are eigenvalues of the commuting operators C_1 and P_μ.

We do not provide commutation relations for Γ_μ as they turn out to be unnecessary for determining the explicit form of basis elements of the Galilei algebra.

It will be shown further on that to describe nonequivalent IRs of the algebra *AG(1,3)* it is sufficient to find all the possible (up to equivalence) realizations of the vector W_μ, the latter being the nonrelativistic analog of the Lubanski-Pauli vector, and then to find the corresponding operators $\boldsymbol{J}$ and $\boldsymbol{G}$. Thus, our task is to describe representations of the algebra (12.4).

In the frame of reference for which $\boldsymbol{p}=p\boldsymbol{n}$, $\boldsymbol{n}=(n_1,n_2,n_3)$ being a constant vector satisfying the condition $\boldsymbol{n}^2=1$, the commutation relations (12.4) reduces to the following form:

$$[W_a^{\prime},W_b^{\prime}]=ic_1\varepsilon_{abc}W_c^{\prime},\quad [W_0^{\prime},W_a^{\prime}]=-ip\varepsilon_{abc}n_bW_c^{\prime}. \tag{12.5}$$

The relations (12.5) define a Lie algebra whose structural constants depend on $\hat{p}$. Making a transfer to the new basis $\{\lambda_0,\lambda_a\}$ in accordance with (12.6):

$$W_0=p\lambda_0,\quad W_a=c_1\lambda_0n_a+\lambda_a, \tag{12.6}$$

we obtain the following commutation relations for the operators λ_0 and λ_a:

$$[\lambda_0,\lambda_a]=i\varepsilon_{abc}n_b\lambda_c,\quad [\lambda_a,\lambda_b]=ic_1\varepsilon_{abc}n_c\lambda_0. \tag{12.7}$$

The Lie algebra defined by the commutation relations (12.7) and its representations have already been considered above in Subsection 4.5. Any IR of this algebra corresponds to the vector W'_μ from (12.6) in the reference frame for which $p_a=pn_a$. An explicit form of W_μ in an arbitrary frame of reference can be obtained by the following transformation:

$$W_0^{\prime}\rightarrow W_0=p\lambda_0,\qquad W_a^{\prime}\rightarrow W_a=R_{ab}^{-1}W_b^{\prime}=\frac{c_1p_a\lambda_0}{p}+\lambda_a-\frac{(\hat{p}_a+n_a)\lambda\cdot\boldsymbol{p}}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}}, \tag{12.8}$$

where R_{ab} is the operator of rotation of a reference frame given in (4.17) and $\hat{p}_a=p_a/\mathrm{p}$. It is not difficult to note that the analytical expressions for the vectors of (4.21) and that of (12.8) are almost identical. Namely, substituting $p_0\rightarrow c_1$ in (4.21) we come the conclusion that the explicit expressions for J_a and G_a can be obtained from (4.45) with the help of the following correspondence rule: $p_0\rightarrow c_1$, $J_{ab}\rightarrow\varepsilon_{abc}J_c$, $J_{0a}\rightarrow G_a\text{-}ip_a\partial/\partial p_0$. As a result we come to the representation of the Galilei algebra in the following form:

$$P_0=p_0,\quad P_a=p_a,\quad M=c_1=m,$$

$$\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\lambda_0\frac{\hat{\boldsymbol{p}}+\boldsymbol{n}}{1+\boldsymbol{n}\cdot\hat{\boldsymbol{p}}}, \tag{12.9}$$

$$\boldsymbol{G}=-i\boldsymbol{p}\frac{\partial}{\partial p_0}-m\boldsymbol{x}+\frac{\lambda\times\boldsymbol{p}}{p^2}-\frac{\hat{\boldsymbol{p}}\times\boldsymbol{n}(\lambda_0m-\lambda\cdot\hat{\boldsymbol{p}})}{p+\boldsymbol{n}\cdot\boldsymbol{p}}.$$

Here m is a fixed number and the variables p_0, p_a are connected by the relation

$$2mp_0-p^2=c_3. \tag{12.10}$$

So, each IR of the algebra (12.7) corresponds to a class of IRs of the

Galilei algebra with the fixed values of $c_1=m$ and $c_3=2mp_0$-p^2. This result can be formulated as the following assertion.

THEOREM 12.1. IRs of the algebra *AG(1,3)* are labelled by sets of numbers c_1, c_2, ... (eigenvalues of the Casimir operators) with the following possible values:

$$
\begin{array}{llll}
I. & c_1^2=m^2>0, & c_2=m^2s(s+1), & -\infty<c_3<\infty, \quad s=0,1/2,1,..., \\
II. & c_1=0, & c_2=0, & c_3=-k^2<0, \quad c_4=0,1/2,1,..., \\
III. & c_1=0, & c_2=r^2>0, & c_3=-k^2<0.
\end{array}
\tag{12.11}
$$

An explicit form of the corresponding basis elements of the algebra AG(1.3) is given by formulae (12.9) where $m=c_1$ is a fixed real number, λ_μ are the matrices (4.40)-(4.43), and p_0, p_a are related by (12.10).

PROOF. The validity of the theorem follows directly from the reasoning given above. Note that it is possible to verify that the operators (11.9) satisfy the commutation relations (11.6). We can also make sure that the Casimir operators (11.14) for the representation (12.9) have the form

$$
C_1=m, \quad C_2=m^2\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2, \quad C_3=2mp_0-p^2. \tag{12.12}
$$

The operator C_2 coincides with the Casimir operator from (4.22) whose eigenvalues are given by formulae (4.40)-(4.43). It follows from the above that eigenvalues of the operators C_a used for numbering IRs do satisfy relations (12.11). In the case $c_1=c_2=0$ there exists the additional Casimir operator $C=W_0/p=\lambda_0$ eigenvalues of which are integers or half integers (refer to (4.42)).■

We see that the basis elements of IRs of the Galilei algebra can be chosen in the form (12.9) where λ_0 and λ_a are matrices realizing an IR of the algebra (12.7). IRs of the algebra (12.7) are defined by the relations (4.40)-(4.44).

The realization of an IR of the algebra *AG(1,3)* given by relations (12.9) is distinguished by a relative simple form for basis operators, which is common for any class of IRs. In the subsection that follows we discuss briefly different classes of IRs of the algebra *AG(1,3)* and show what connections exist between the realization (12.9) and other known representations.

12.4. Connections with Other Realizations

Let us consider sequentially each of the classes of IRs enumerated in (4.10).

1. IRs of Class *I* ($c_1=m^2>0$) are characterized by three numbers m, $\varepsilon_0=c_3$ and s, m and ε_0 being arbitrary real numbers, and $s\geq0$ being an integer or half integer. Such representations are realized in a space of square integrable functions $\Phi(\boldsymbol{p},\lambda)$ where $\lambda=-s,-s+1,...,s$, i.e., dimension of $\Phi(\boldsymbol{p},\lambda)$ in respect with the index λ

is equal to $2s+1$. The corresponding scalar product and additional requirements for $\Phi(\boldsymbol{p},\lambda)$ are presented in Subsection 4.6 (refer to (4.47)).

The space of an IR of the algebra *AG(1,3)* belonging to the Class *I* is associated with the state space of a free nonrelativistic particle of mass m, spin s, and internal energy $\varepsilon_0/2m$. The corresponding basis elements of the Galilei algebra can be chosen in the form (12.9) where λ_0, λ_a are matrices of dimension $(2s+1)\times(2s+1)$, connected with generators S_1, S_2 and S_3 of the orthogonal group according to the following relations (refer to (4.24)):

$$\lambda_0=S_3,\ \lambda_a=R_{ab}^{-1}\lambda_b',\ \lambda_1'=mS_1,\ \lambda_2'=mS_2,\ \lambda_3'=0,$$
$$R_{ab}^{-1}=\boldsymbol{n}\cdot\boldsymbol{n}'\delta_{ab}+n_an_b'-n_bn_a'+(\boldsymbol{n}\times\boldsymbol{n}')_a(\boldsymbol{n}\times\boldsymbol{n}')_b(1+\boldsymbol{n}\cdot\boldsymbol{n}')^{-1},\quad \boldsymbol{n}'=(0,0,1). \tag{12.13}$$

The explicit form of the matrices λ_0, λ_a in the Gelfand-Zetlin basis is given in formulae (4.40), (4.41).

With the help of the transformation

$$P_\mu\rightarrow UP_\mu U^{-1},\quad J_a\rightarrow UJ_aU^{-1},\quad G_a\rightarrow UG_aU^{-1} \tag{12.14}$$

with U being the operator (4.49), the generators (12.9) reduce to the usual form used in the literature on physics

$$P_0=\varepsilon_0+\frac{p^2}{2m},\quad P_a=p_a,\quad M=m,$$
$$J_a=-i\varepsilon_{abc}p_b\frac{\partial}{\partial p_c}+S_a,\quad G_a=-mx_a+x_0p_a. \tag{12.15}$$

2. IRs of Class *II* ($c_1=c_2=0$) are labelled by a pair of numbers $c_3<0$ and $c_4=0,1/2,1,....$ These representations are one-dimensional and defined in the space of square integrable functions $\varphi(p_0,\boldsymbol{p})$. The corresponding form of the Galilei group generators simplifies considerably. Indeed, setting $m=0$ and $\lambda_a=0$ in (12.9) (refer to (4.24), (4.33)), we obtain

$$\boldsymbol{G}=-i\boldsymbol{p}\frac{\partial}{\partial p_0}. \tag{12.16}$$

The remaining basis elements of the algebra *AG(1,3)* (i.e., P_μ and $\boldsymbol{J}$) have the form (12.9) where $\lambda_0=c_4$ is an integer or half integer, and $p^2=-k^2=const$.

By means of the transformation (12.14) where U is the operator (4.52) or (4.53) and $\boldsymbol{n}'=(0,0,1)$, the generators P_μ, $\boldsymbol{J}$ of (11.9) and $\boldsymbol{G}$ of (12.16) can be transformed to the form found in [375]. We do not give here the explicit expressions of the corresponding basis elements of the algebra *AG(1,3)* because of their complicated and nonsymmetric form as contrasted to (12.16), (12.9).

Representations of the Galilei algebra of Class *II* are realized on solutions of equations describing nonrelativistic fields with zero rest mass, e.g., Galilei-invariant electromagnetic field [273, 276].

3. Let us now consider IRs of Class *III* (c_1=0, c_2<0) which are labelled by pairs of positive numbers r^2 and k^2 (see (12.11)). Such representations are realized in the space of square integrable functions $\varphi(p_0,\boldsymbol{p},\lambda)$ where λ takes the infinite number of values given by one of formulae (4.43). The explicit expressions of basis elements of the IRs are given in (12.9) where m=0, p^2=k^2 and λ_0, λ_a are infinite-dimension matrices connected with the generators T_α of the group *E(2)* by the following relations (refer also to (4.24)):

$$\lambda_0=T_0,\ \lambda_a=R^{-1}_{ab}\lambda'_b,\ \lambda'_1=T_1,\ \lambda'_2=T_2,\ \lambda'_3=0. \tag{12.17}$$

Here R_{ab}^{-1} are the matrix elements given in (12.13). The explicit form of the matrices λ_0, λ_a can be chosen, for example, according to (4.40), (4.43).

As far as we know, representations of the Galilei algebra of Class *III* have not found yet direct usage in physics. If (in analogy with representations of Classes *I, II*) we take a space state of a quantum mechanical particle in correspondence with a space of a representation of Class *III,* then such a particle would have an infinite number of spin states.

4. The classes of IRs considered above exhaust all the nonequivalent Hermitian representations of the algebra *AG(1,3)*. We note that formulae (12.9) define the explicit form of basis elements of the Galilei algebra for the case of c_1^2<0 also. The corresponding matrices λ_μ realize a representation of the algebra *AO(1,2)* (refer to (4.24), (4.27), (4.36)-(4.39)).

IRs of the algebra *AG(1,3)* corresponding to c_1^2<0 cannnot be realized by Hermitian operators since the generator M=C_1 has purely imaginary eigenvalues. This circumstance restricts the usage of such representations in quantum mechanics. But representations of this type naturally arise in various problems of classical physics. In particular, representations of the Galilei algebra corresponding to c_1^2<0 are realized on a set of solutions of the linear heat equation.

5. IRs of Class *V* reduce to representations of the Lie algebra of the Euclidean group *E(3)*. IRs and a class of finite dimension representations of this algebra are considered in Subsection 12.6.

12.5. Covariant Representations

IRs of the Galilei algebra considered above can be defined in Gilbert spaces of square integrable functions depending on p_μ. But for many applications (e.g., for describing differential equations being invariant under the Galilei group) it is more convenient to deal with representations defined in a space of functions

$\psi(t,\boldsymbol{x})$ where the generators P_0, P_a are represented by differential operators. We will call such representations covariant if the corresponding basis elements belong to the class M_1 (i.e., they are differential operators of the first order).

It is not difficult to show that a basis of a covariant representation is formed by the following operators:

$$P_0=i\frac{\partial}{\partial t},\quad P_a=p_a=-i\frac{\partial}{\partial x_a},$$
$$J_a=-i\varepsilon_{abc}x_b\frac{\partial}{\partial x_c}+S_a,\quad M=m, \tag{12.18}$$
$$G_a=tp_a-mx_a+\eta_a,$$

where S_a, η_a, and m are numeric matrices satisfying the commutation relations

$$[m,S_a]=[m,\eta_a]=0, \tag{12.19}$$

$$[S_a,S_b]=i\varepsilon_{abc}S_c, \tag{12.20a}$$

$$[S_a,\eta_b]=i\varepsilon_{abc}\eta_c,\quad [\eta_a,\eta_b]=0. \tag{12.20c}$$

Formulae (12.18) determine the general form of operators P_μ, J_a, G_a, and M satisfying the commutation relations (10.6) and belonging to the class M_1.

Integrating the corresponding Lie equations (refer to (1.20)-(1.22)), we obtain finite transformations generated by the operators (12.18) in the form

$$\psi(t,\boldsymbol{x})\rightarrow\psi'(t',\boldsymbol{x}')=D(\boldsymbol{v},\theta)\exp[i\varphi(x)]\psi(t,\boldsymbol{x}) \tag{12.21}$$

where t', $\boldsymbol{x}'$ and $\varphi(x)$ are given by relations (11.28), (11.29) while $D(\boldsymbol{v},\theta)$ is the matrix depending on parameters of the transformation (11.29),

$$D(\boldsymbol{v},\theta)=\exp(i\eta\cdot\boldsymbol{v})\exp(i\boldsymbol{S}\cdot\theta). \tag{12.22}$$

In the case of $\eta=\boldsymbol{S}=0$ we come to formula (11.27) which determines the transformation law for solutions of the Schrödinger equation.

According to (12.21), a transformed function $\psi'(t',\boldsymbol{x}')$ is expressed via $\psi(t,\boldsymbol{x})$ multiplied by the phase multiplier $\exp[i\varphi(x)]$ and a numeric matrix $D(\boldsymbol{v},\theta)$. In other words, a value of $\psi'(t',\boldsymbol{x}')$ is completely determined by the value of a non-transformed function in the point $(t,\boldsymbol{x})$ and by transformation parameters. The transformations of this type are called locally covariant (or simply covariant) in contrast to nonlocal transformations for which $\psi'(t',\boldsymbol{x}')$ depends on values of $\psi(t,\boldsymbol{x})$ within some domain. Formulae (12.18)-(12.20) determine the general form of generators of a covariant representation of the Galilei group.

Thus, a description of covariant representations of the algebra *AG(1,3)* reduces to finding the nonequivalent matrices S_a, η_a and m to satisfy the commutation relations (12.19), (12.20). Representations of the algebra (12.19), (12.20) are considered in the following subsection.

Besides representations of the proper Galilei group determined by the continuous transformations (11.29), a certain interest was exhibited in representations of the complete group G*(1,3)* which includes transformations *P*, *T*, and *C*. An analysis of such representations is beyond the scope of this book.

Representations of the group including the transformations *P* and *T* and (11.28) are described in [59], while representations of the complete Galilei group are considered in [151].

12.6. Representations of the Lie Algebra of the Homogeneous Galilei Group

The Lie algebra of the homogeneous Galilei group is formed by six generators S_a, η_a satisfying the commutation relations (12.20). This algebra is isomorphic to the Lie algebra of Euclidian group *E(3)* and is denoted as *AE(3)*.

It will be shown further on that the problem of description of nonequivalent indecomposable representations of the algebra *AE(3)* reduces to the algebraic problem which cannot be solved by known methods. So we restrict ourselves to description of some class of finite dimension representations. This class is wide enough to deduce consistent motion equations of arbitrary spin particles.

We consider also IRs of the algebra *AE(3)*. The problem of description of such representations can be solved in a closed form.

The algebra *AE(3)*, as presented by (12.20), includes the subalgebra *AO(3)* formed by the matrices S_a. Since IRs of the algebra *AO(3)* are well known, it is convenient to use the *O(3)*-basis $|\lambda;l,m>$ formed by eigenvectors of the commuting matrices $\boldsymbol{S}^2$ and S_3. Explicit expressions for $\boldsymbol{S}^2$ and S_3 in this basis are given by formulae (4.63), (4.64), where $(l_0,l_1) \rightarrow \lambda$. In our case λ is an additional index labelling eigenvectors of the matrix $\boldsymbol{S}^2$ corresponding to degenerated values of *l*.

The general expression for matrices η_a commuting with S_a in accordance with the rule (11.20b) is given in our basis by the following formula:

$$\eta_a|\lambda;l,m>=\sum_{l',m',\lambda'}\Big[\delta_{ll'}a^l_{\lambda\lambda'}(S_a^l)_{mm'}+\delta_{l-1,l'}b^l_{\lambda\lambda'}(K_a^l)^{\dagger}_{mm'}+ \\ +\delta_{l+1,l'}C^{l+1}_{\lambda\lambda'}(K_a^{l+1})_{mm'}\Big]|\lambda';l',m'> \qquad (12.23)$$

where $a^l_{\lambda\lambda'}$, $b^l_{\lambda\lambda'}$, $c^l_{\lambda\lambda'}$ are arbitrary complex numbers, $(S_a^l)_{mm'}$ and $(K_a^l)_{mm'}$ are the matrix elements given by relations (4.65), (4.66). The possible values of *l* corresponding to indecomposable matrices S_a and η_a are [197] as follows

$$l=l_0,\ l_0+1,\ \dots,\ l_1, \qquad (12.24)$$

l_0 and l_1 being positive integers or half integers. By definition

$$a^{l_0-1}=a^{l_1+1}=0, \quad b^{l_0}=c^{l_0}=0, \quad b^{l_1+1}=c^{l_1+1}=0. \tag{12.25}$$

We have marked in (12.23) the matrix elements $(S_a^l)_{mm'}$ and $(K_a^l)_{mm'}$ of the matrices S_a^l and K_a^l which satisfy the following relations [223]:

$$\begin{aligned}
&K_a^l S_b^l - S_b^{l-1} K_a^l = i\varepsilon_{abc} K_c^l,\\
&S_a^l S_b^l + K_a^{l\dagger} K_b^l = il\varepsilon_{abc} S_c^l + l^2\delta_{ab},\\
&S_a^l S_b^l - S_b^l S_a^l = i\varepsilon_{abc} S_c^l;
\end{aligned} \tag{12.26a}$$

$$\begin{aligned}
&K_a^l K_b^{l\dagger} + S_a^{l-1} S_a^{l-1} = l^2\delta_{ab} - il\varepsilon_{abc} S_c^{l-1},\\
&K_a^{l\dagger} K_b^l - K_b^{l\dagger} K_a^l = i(2l-1)\varepsilon_{abc} S_c^l,\\
&K_a^l S_b^l - K_b^l S_a^l = i(l+1)\varepsilon_{abc} K_c^l,\\
&S_a^{l-1} K_b^l - S_b^{l-1} K_a^l = i(1-l)\varepsilon_{abc} K_c^l,\\
&K_a^l K_b^{l\dagger} - K_b^l K_a^{l\dagger} = -i(2l+1)\varepsilon_{abc} S_c^{l-1}.
\end{aligned} \tag{12.26b}$$

The conditions (12.26a), together with (4.63), can serve as a definition of the matrices S_a^l and K_a^l, while relations (12.26b) follow from (12.26a). They are the conditions (12.26a) which are used in the following to describe representations of the algebra *AE(3)*.

The algebra *AE(3)* has the two Casimir operators

$$C_1=S_a\eta_a, \quad C_2=\eta_a\eta_a \tag{12.27}$$

which reduce to the following form in the basis $|\lambda;l,m>$:

$$C_1|\lambda;l,m>=a^l_{\lambda\lambda'}|\lambda;l,m>, \tag{12.28a}$$

$$C_2|\lambda;l,m>=\left[(l+1)(2l+3)c^{l+1}_{\lambda\mu}b^{l+1}_{\mu\lambda'}+l(l+1)a^l_{\lambda\mu}a^l_{\mu\lambda'}+l(2l-1)b^l_{\lambda\mu}c^l_{\mu\lambda'}\right]|\lambda';l,m>. \tag{12.28b}$$

There is also the additional Casimir operator

$$C_3=\exp(2\pi iS_1)=\exp(2\pi iS_2)=\exp(2\pi iS_3) \tag{12.29}$$

with eigenvalues equal to ±1 (refer to (4.13)).

The Casimir operators do not play an essential role in description of indecomposable representations, but they are very useful in searching of IRs.

Let us show that the condition of commutativity of matrices η_a reduces to the system of quadratic equations for matrices $a^l=\|a^l_{\lambda\lambda'}\|$, $b^l=\|b^l_{\lambda\lambda'}\|$ and $c^l=\|c^l_{\lambda\lambda'}\|$. Essentially, using the representation (12.23), it is not difficult to calculate the commutator

$[\eta_a,\eta_b]|\lambda;l,m>=$

$$\begin{aligned}&=\Big\{\Big[c^{l+1}_{\lambda\nu}a^{l+1}_{\nu\lambda'}(K^{l+1}_aS^{l+1}_b-K^{l+1}_bS^{l+1}_a)_{mm'}+a^{l}_{\lambda\nu}c^{l+1}_{\nu\lambda'}(S^{l}_aK^{l+1}_b-S^{l}_bK^{l+1}_a)_{mm'}\Big]\delta_{l'l+1}+\\&+\Big[c^{l+1}_{\lambda\nu}b^{l+1}_{\nu\lambda'}(K^{l+1}_aK^{l+1\dagger}_b-K^{l+1}_bK^{l+1\dagger}_a)_{mm'}+a^{l}_{\lambda\nu}a^{l}_{\nu\lambda'}(S^{l}_aS^{l}_b-S^{l}_bS^{l}_a)_{mm'}+\\&+b^{l}_{\lambda\nu}c^{l}_{\nu\lambda'}(K^{l\dagger}_aK^{l}_b-K^{l\dagger}_bK^{l}_a)_{mm'}\Big]\delta_{ll'}+\Big[a^{l}_{\lambda\nu}b^{l}_{\nu\lambda'}(S^{l}_aK^{l\dagger}_b-S^{l}_bK^{l\dagger}_a)_{mm'}+\\&+b^{l}_{\lambda\nu}a^{l-1}_{\nu\lambda'}(K^{l\dagger}_aS^{l-1}_b-K^{l\dagger}_bS^{l-1}_a)_{mm'}\Big]\delta_{l-1l'}\Big\}|\lambda';l',m'>,\end{aligned}\tag{12.30}$$

the above expression suggesting a summation over the repeated indices ν, λ', l' and m'.

Equating (12.30) to zero and taking into account relations (12.26), we come to the following system of equations for the matrices a^l, b^l and c^l:

$$(l+1)c^la^l=(l-1)a^{l-1}c^l;\quad (l+1)a^lb^l=(l-1)b^la^{l-1},\tag{12.31a}$$

$$(2l+3)c^{l+1}b^{l+1}-(2l-1)b^lc^l=(a^l)^2.\tag{12.31b}$$

Formulae (12.31) present necessary and sufficient conditions of commutativity of the matrices η_a from (12.23).

Thus the problem of description of representations of the algebra *AE(3)* reduces to the finding of nonequivalent solutions of the system of quadratic equations (12.31), (12.25).

First we consider Hermitian IRs of the algebra *AE(3)*. A description of such representation is a relatively simple problem. Indeed, the Hermiticity condition leads to the following relation:

$$c^{l+1}=b^{l+1\dagger}.\tag{12.32}$$

Then, according to Schur's Lemma, the corresponding Casimir operators (12.28) have to be multiples of the unit matrix so that

$$a^l_{\lambda\lambda'}=\delta_{\lambda\lambda'}a^l,\quad \lambda,\ \lambda'=1,2,...n_l,\tag{12.33a}$$

$$a^l(l+1)=c_1,\tag{13.33b}$$

$$(l+1)(2l+3)b^2_{l+1}+l(2l-1)b^2_l=[c_2-l(l+1)(a^l)^2]I_{n_l}.\tag{12.33c}$$

Here c_1 and c_2 are eigenvalues of the Casimir operators (12.27), a^l is a real number, I_{n_l} is the unit matrix of dimension $n_l\times n_l$, and

$$b^2_l=b^lb^{l\dagger},\quad b^2_{l+1}=b^{l+1\dagger}b^{l+1}.\tag{12.34}$$

The equations (12.31a) are satisfied identically according to (12.33b), and the equation (12.31b) takes the form

$$(2l+3)b_{l+1}^2-(2l-1)b_l^2=a_l^2I_{n_l}. \tag{12.35}$$

So, a description of IRs of the algebra *AE(3)* reduces to the finding of a general solution of the equations (12.33)-(12.35).

It is not difficult to demonstrate that without loss of generality we can set $n_l=1$ (otherwise we obtain a reducible representation with degenerated eigenvalues of the Casimir operators). Thus a_l, b_l, and b_{l+1} reduce to numbers which can be easily calculated.

The system (12.33)-(12.35) is not consistent for any fixed l_1, and therefore it is necessary to solve an infinite numerable system corresponding to an infinite dimension representation of the algebra *AE(3)*. The corresponding solutions are:

$$c_1=rl_0,\quad c_2=r^2,\quad l_0=0,1/2,1,...,\quad c_3=(-1)^{l_0}, \tag{12.36}$$

$$a_l=\frac{rl_0}{l(l+1)},\quad b_l=\frac{|r|}{l}\sqrt{\frac{l^2-l_0^2}{4l^2-1}},\quad l=l_0,l_0+1,\ \dots\ . \tag{12.37}$$

So IRs of the algebra *AE(3)* are labelled by two numbers, an arbitrary real number r and non-negative integer or half integer l_0. These numbers define possible values of the Casimir operators (12.36). All the Hermitian IRs are infinite dimensional, the explicit form of basis elements being as follows:

$$S_a|r,l_0;l,m>=\sum_{m'}(S_a^l)_{mm'}|r,l_0;l,m>;$$

$$\eta_a|r,l_0\ ,l,m>=\sum_{c',m'}[a^l\delta_{ll'}(S_a^l)_{mm'}+b^l\delta_{l-1l'}(K_a^l)^\dagger_{mm'}+ \tag{12.38}$$

$$+b^{l+1}\delta_{l+1l'}(K_a^{l+1})_{mm'}]|r,l_0;l',m'>$$

where a^l and b^l are the coefficients from (12.37), $(S_a^l)_{mm'}$ and $(K_a^l)_{mm'}$ are the matrix coefficients (4.65), (4.66).

Consider now the problem of description of finite dimension indecomposable representations of the algebra *AE(3)*. This problem is more complicated then in the case of IRs. Moreover it is unsolvable in the sense that it is impossible to enumerate effectively all the nonequivalent representations.

Thus we will try to solve the system (12.31) for the case when the numbers l_0 and l_1 determining possible values of l are positive integers or half integers. It means that the chain of equations (12.31) can include an arbitrary (but finite) number of links corresponding to the given l_0 and l_1. Dimensions of matrices a^l, b^l and c^l are equal to $n_l\times n_l$, $n_l\times n_{l-1}$ and $n_{l-1}\times n_l$ respectively, where n_l and n_{l-1} are multiples of the eigenvalues $l(l+1)$ and $l(l-1)$ of the operator $\boldsymbol{S}^2$. In contrast to the case of IRs we have no additional equations like (12.33) inasmuch as we consider indecomposable representations and the corresponding eigenvalues of the Casimir operators cannot be fixed.

Unfortunately, it is not possible to describe effectively all the nonequivalent solutions of the equations (12.31). Indeed, in the particular case of $(a^2)^2=0$ these equations reduce to the problem of a description of all the nonequivalent pairs of commuting matrices B and C defined in accordance with the following relations:

$$B|\lambda;l,m>=b^l|\lambda;l-1,m>,\quad C|\lambda;l,m>=c^l|\lambda;l+1,m>.$$

Such a problem is "wild" and cannot be solved by the known methods [199].

Imposing some additional restrictions on a^l, it is possible to describe appropriate classes of solutions of equations (12.31). One such restriction which enables us to describe completely the corresponding class of solutions is a requirement for matrices a^l to be indecomposable. Here we consider a more restricted class of solutions of equations (12.31) corresponding to the case when the possible values of l being equal to

$$l=s,s-1. \tag{12.39}$$

To denote this class of solutions and the corresponding representations of the algebra $AE(3)$, we use the symbol D_2.

Indecomposable representations of the class D_2 may be described as follows.

PROPOSITION 12.1 [151]. Nonequivalent solutions of equations (12.31), a^l being indecomposable matrices and possible values of l being given by (12.39), are labelled by triplets of integers (k,α,n) where

$$k\leq 4,\ n\leq 4,\ |k-n|\leq 2,\ kn<9;\quad \alpha=1,\ |k-n|=2,\ \alpha=1,2,|k-n|\neq 2. \tag{12.40}$$

The corresponding matrices a^s, a^{s-1}, b^s, and c^s are of dimensions $k\times k$, $n\times n$, $k\times n$ and $n\times k$, and their elements are equal (for $\alpha=1$) to

$$a^s_{\lambda\lambda'}=\frac{\tilde{a}^s_{\lambda\lambda'}}{s(s+1)},\quad a^{s-1}_{\nu\nu'}=\frac{\tilde{a}^{s-1}_{\nu\nu'}}{s(s-1)},\quad b^s_{\lambda\nu}=\frac{\tilde{b}^s_{\lambda\nu}}{2s-1},\quad c^s_{\lambda\nu}=\frac{\tilde{c}^s_{\lambda\nu}}{2s+1},$$

$$\tilde{a}^s_{\lambda\lambda'}=\delta_{\lambda-1\lambda'},\ \tilde{a}^{s-1}_{\nu\nu'}=\delta_{\nu-1\nu'},\ \lambda,\lambda'\leq k,\ \nu,\nu'\leq n,$$

$$\tilde{b}^s_{\lambda\nu}=\begin{cases} i\sqrt{k_s}\ \delta_{\lambda-2\nu}, & k>n,\\ i\sqrt{k_s}\ \delta_{\lambda\nu}, & k\leq n,\ kn\neq 4,\\ \mu\delta_{\lambda\nu+1}, & k=n=2,\end{cases} \tag{12.41a}$$

$$\tilde{c}^s_{\lambda\nu}=\begin{cases} i\sqrt{k_s}\ \delta_{\nu\lambda}, & k>n, \\ i\sqrt{k_s}\ \delta_{\nu\lambda+2}, & k\leq n \end{cases} \tag{12.41b}$$

where $k_s=(2s+1)/s^2(s+1)^2$ and μ is an arbitrary complex number. If $\alpha=2$ then the corresponding matrix elements can be obtained from (12.41) by the substitution $a^s_{\lambda\lambda'} \to a^s_{\lambda'\lambda}$, $b^s_{\lambda\nu} \to c^s_{\nu\lambda}$, $c^s_{\nu\lambda} \to b^s_{\lambda\nu}$.

For the proof see [319]. We only note that in the case considered it follows from (12.31) that $(a^s)^4=(a^{s-1})^4=0$ and the maximal dimension of matrices a^l is 4×4. Choosing these matrices in the Jordan form we obtain the compatibility condition for the system (12.31) in the form of (12.39) by direct calculations. The corresponding solutions can be chosen in the form (12.41) with up to equivalence.

It is not difficult to calculate that there are 18 nonequivalent solutions for any $s\neq 0,1/2$, two of them being dependent on an arbitrary complex number μ.

Any solution of equations (12.31) can be set in one-to-one correspondence with a representation of the algebra *AE(3)*. So, from Proposition 12.1 follows

PROPOSITION 12.2. Indecomposable representations of the algebra *AE(3)*, belonging to the class D_2 are labelled with triplets of integers (k,α,n) satisfying the conditions (12.40). The explicit form of the corresponding basis elements S_a and η_a in the basis $|\lambda;l,m>$ is given by the formulae (4.64)-(4.66), (12.23) and (12.41).

So we have described the class D_2 of indecomposable representations of the Lie algebra of the homogeneous Galilei group. These representations will be used in the next section to deduce wave equations which are invariant under the Galilei group.

We should like to note that the integers k and n used (together with α) to label indecomposable representations of the class D_2 define the nilpotency indices of the Casimir operators (12.26). Indeed, according to (12.27) and (12.33)

$$C_1^N=C_2^{N/2}=0, \quad N=\max(k,n).$$

It can be easily seen that $N\leq 4$ for the representations of the class D_2.

13. GALILEI-INVARIANT WAVE EQUATIONS

13.1. Introduction

From Bargman's work [16] it is known that the concept of spin arises in nonrelativistic quantum mechanics as naturally as in frames of relativistic theory. One of the Casimir operators of the Galilei group is nothing but the operator of squared spin. Thus, it is surprising that the number of papers devoted to Galilei-invariant wave equations is extremely small. This circumstance seems to be particularly strange in view of the fact that Poincaré invariant wave equations attract attention of a great many of investigators.

An important contribution into the theory of Galilei invariant wave equations was made by Levi-Leblond [276, 277] who obtained for the first time such an equation for a particle of spin 1/2. The Levi-Leblond equation takes into account the Pauli interaction and predicts the correct value of the gyromagnetic ratio such as the relativistic Dirac equation. Unfortunately this equation (and its generalizations proposed by Hagen and Hurley [215, 223]) does not describe such an important physical effect as a spin-orbit coupling.

In this section we describe a class of Galilei-invariant equations of the first order corresponding to the indecomposable representations of the homogeneous Galilei group described above. This class includes as the Levi-Leblond and Hagen-Hurley (LHH) equations as the new equations describing spin-orbit and quadruple couplings of an arbitrary spin particle with an external field.

The Galilei-invariant wave equations in many cases are good alternatives to Poincaré-invariant equations for particles of higher spin inasmuch as the latter lead to contradictions by a description of a particle interaction with an external field, see Section 10.

13.2. Galilei-Invariance Conditions

To describe first-order equations invariant under Galilei transformations is to answer the question: what kind of matrices β_μ, β_4 guarantees the invariance of the system (6.2) with respect to the Galilei algebra?

For convenience we rewrite (6.2) in the form

$$L\psi=0, \quad L=\beta^\mu p_\mu+\beta_4 m. \tag{13.1}$$

The invariance under the Galilei algebra means that the equation (13.1) admits 11 SOs satisfying relations (11.6). We restrict ourselves to the case when these SOs realize a covariant representation of the Galilei algebra and so have the form of (12.18). We assume also that the mass operator M is a multiple of the unit operator and has a positive eigenvalue $m>0$.

By definition the operators (12.18) are SOs of the equation (13.1) if they satisfy relations (6.3). Substituting $Q_A=P_a$, J_a, G_a into (6.3) we come to the following relations (compare with (6.4), (6.5))

$$
\begin{aligned}
&\tilde{S}_a\beta_0-\beta_0 S_a=0, \quad \tilde{S}_a\beta_4-\beta_4 S_a=0,\\
&\tilde{\eta}_a\beta_4-\beta_4\eta_a=-i\beta_a, \quad \tilde{\eta}_a\beta_b-\beta_b\eta_a=-i\delta_{ab}\beta_0,\\
&\tilde{\eta}_a\beta_0-\beta_0\eta_a=0, \quad a=1,2,3.
\end{aligned}
\tag{13.2}
$$

Here $\tilde{S}_a=S_a-\tilde{\beta}_{J_a}$, $\tilde{\eta}_a=\eta_a-\tilde{\beta}_{G_a}$, S_a and η_a are matrices realizing a representation of the algebra *AE(3)* (11.20), $\tilde{\beta}_{J_a}$, $\tilde{\beta}_{G_a}$ are unknown numeric matrices.

Using relations (12.20) it is not difficult to obtain from (13.2) the following conditions

$$
\begin{aligned}
&[\tilde{S}_a,\tilde{S}_b]\beta_k=i\varepsilon_{abc}\tilde{S}_c\beta_k, \quad k=0,1,2,3,4,\\
&[\tilde{\eta}_a,\tilde{S}_b]\beta_k=i\varepsilon_{abc}\tilde{\eta}_c\beta_k, \quad [\tilde{\eta}_a,\tilde{\eta}_b]\beta_k=0.
\end{aligned}
\tag{13.3}
$$

A *sufficient* condition of validity of relations (13.3) is the requirement that the matrices $\tilde{S}_a$, $\tilde{\eta}_a$ realize a representation of the algebra *AE(3)*.

Thus we come to the following definition.

DEFINITION 13.1. The equation (13.1) is S-invariant under the Galilei algebra if there exist matrices S_a,η_a and $\tilde{S}_a,\tilde{\eta}_a$ realizing (generally speaking nonequivalent) representations of the algebra *AE(3)* and satisfying relations (13.3).

As in the case of Poincaré-invariant equations we will use the sufficient conditions of the Galilei-invariance given in Definition 13.1. These conditions are not necessary since there exist equations being invariant under noncovariant realizations of the Galilei algebra [162, 163]. Moreover it is possible in principle to renounce the requirement the matrices $\tilde{S}_a$, $\tilde{\eta}_a$ realize a representation of the algebra *AE(3)* since the conditions (13.3) are more week.

We note that Definition 13.1 determines the invariance conditions for the case of square matrices β_μ, β_4 as for equations with rectangular β-matrices of dimension *mxn, m≠n*. In the latter case the matrices $\tilde{S}_a$, $\tilde{\eta}_a$ and S_a, η_a have dimensions $m\times m$ and $n\times n$.

13.3. Additional Restrictions for Matrices β_k

Let us define the conditions imposed on β_k by some additional physical requirements.

First we restrict ourselves to considering only such equations which admit the Lagrangian formulation. As in the case of relativistic wave equations it means that we assume that a nonsingular Hermitizing matrix η exists which satisfies the conditions

$$\eta\beta_\mu=\beta_\mu^\dagger\eta, \quad \eta\beta_4=\beta_4^\dagger\eta. \tag{13.4}$$

It appears that without loss of generality we can impose a stronger requirement

$$\beta_\mu^\dagger=\beta_\mu, \quad \beta_4^\dagger=\beta_4 \tag{13.5}$$

as it follows from the following assertion.

LEMMA 13.2. Let $\{\tilde{S}_a, \tilde{\eta}_a, S_a, \eta_a\}$ be a set of matrices satisfying (13.2), (11.20), and η be an arbitrary nondegenerated matrix. Then the matrices

$$\beta_k^{/}=\eta\beta_k, \quad \tilde{\eta}_a^{/}=\eta\tilde{n}_a\eta^{-1}, \quad \tilde{S}_a^{/}=\eta\tilde{S}_a\eta^{-1}, \quad S_a^{/}=S_a, \quad \eta_a^{/}=\eta_a, \tag{13.6}$$

also satisfy these relations.

PROOF is almost evident. Multiplying any of relations (13.2) from the left by η and putting the matrix $\eta^{-1}\eta\equiv I$ between $\tilde{S}_a$, $\tilde{\eta}_a$ and $ß_\mu$, $ß_4$ we come to relations (13.2) for the matrices (13.6).

So besides the equation (13.1) there exist a class of the Galilei-invariant equations with primed matrices (13.6). Thus if a Hermitizing matrix exists then we always can consider an equivalent Galilei-invariant equation with Hermitian matrices $ß_k$. In other words we can suppose these matrices satisfy (13.5).

The Lagrangian corresponding to the equation (13.1) without loss of generality can be chosen in the form

$$L_0=\frac{i}{2}(\psi^+\beta_\mu\frac{\partial\psi}{\partial x_\mu}-\frac{\partial\psi^+}{\partial x_\mu}\beta_\mu\psi)+m\psi^+\beta_4\psi. \tag{13.7}$$

It is not difficult to make sure the requirement of the Galilei-invariance of this Lagrangian reduces to the conditions (13.2) where

$$\tilde{S}_a=S_a^\dagger=S_a, \quad \tilde{\eta}_a=\eta_a^\dagger. \tag{13.8}$$

Thus, the problem of description of Galilei-invariant equations (13.1) admitting a Lagrangian formulation reduces to finding Hermitian matrices $ß^\mu$, $ß^4$ satisfying the relations

$$[S_a,\beta_0]=0, \quad [S_a,\beta_4]=0, \tag{13.9a}$$

$$\eta_a^\dagger\beta_4-\beta_4\eta_a=-i\beta_a, \tag{13.9b}$$

$$\eta_a^\dagger\beta_b-\beta_b\eta_a=-i\delta_{ab}\beta_0. \tag{13.9c}$$

The last relation (13.2) (supplemented by (13.5) (13.8)) turns into identity if relations

(13.9) are satisfied.

We require also the representation of the Galilei algebra, realized on the set of solutions of the equation (13.1), be irreducible and belong to Class *I*. It means that the wave function has to satisfy the conditions (which have to be consequences of (13.1))

$$(p_0-\frac{p^2}{2m})\psi=\varepsilon_0\psi, \quad (\boldsymbol{S}-\frac{1}{m}\boldsymbol{p}\times\eta)^2\psi=s(s+1)\psi \tag{13.10}$$

where ε_0 and s are fixed parameters.

If the conditions (13.10) are satisfied, the corresponding Casimir operators of the Galilei algebra have fixed eigenvalues. This enables us to interpret (13.1) as a quantum mechanical equation of motion of a particle of spin s, mass m and internal energy ε_0. In the following, we analyze the restrictions imposed on matrices β^k by the conditions (13.10).

13.4. General Form of Matrices ß^k in the Basis $|\lambda;l,m>$

Summarizing given in the above we note that the problem of description of Galilei-invariant wave equations can be subdivided into the following stages:

(1) to describe finite-dimension representations of the algebra *AE(3),* i.e., to find all the nonequivalent matrices S^a and η_a included into the equations (13.9);

(2) to select the representations for which nontrivial solutions of (13.9) exist;

(3) to find an explicit form of the corresponding ß-matrices;

(4) to select the matrices β^k satisfying the conditions (13.10).

We search for the matrices β_k in the basis $|\lambda;l,m>$. The matrices β_0 and β_4 commute with $\boldsymbol{S}$ so according to Schur's lemma

$$\beta_4|\lambda;l,m>=(2l+1)x^l_{\lambda\lambda'}|\lambda';l,m>, \quad \beta_0|\lambda;l,m>=A_{\lambda\lambda'}|\lambda';l,m> \tag{13.11}$$

where $x^l_{\lambda\lambda'}$, $A^l_{\lambda\lambda'}$ are unknown coefficients. We will denote matrices with matrix elements $x^l_{\lambda\lambda'}$ and $A^l_{\lambda\lambda'}$ by x_l and A_l.

Using (13.9b), (12.23) we obtain the general form of the matrices β_a (a=1,2,3):

$$\begin{aligned}\beta_a|\lambda;l,m>=&-iB^{l\dagger}_{\lambda\lambda'}(K^{l\dagger}_a)_{mm'}|\lambda';l-1,m'>+\\&+iD^l_{\lambda\lambda'}(S^l_a)_{mm'}|\lambda';l,m'>+iB^{l+1}_{\lambda\lambda'}(K^{l+1}_a)_{mm'}|\lambda';l+1,m'>\end{aligned} \tag{13.12}$$

where $B^l_{\lambda\lambda'}$ and $D^l_{\lambda\lambda'}$ are the matrix elements of the matrices B_l and D_l,

$$D_l=\frac{2l+1}{l(l+1)}(\tilde{a}_l^\dagger x_l-x_l\tilde{a}_l), \quad B_l=\tilde{c}_l^\dagger x_{l-1}-x_l\tilde{b}_l. \tag{13.13}$$

Let us require the matrices β_a satisfy (13.9c). Using relations (12.23), (12.31) we come to the following equations for x_l and A_l:

$$
\begin{aligned}
&2l^2(2l+1)\tilde{c}_l^{\dagger}x_{l-1}\tilde{c}_l=l^2(2l+1)(x_l\tilde{b}_l\tilde{c}_l+\tilde{c}_l^{\dagger}\tilde{b}_l^{\dagger}x_l)-(4l^2-1)(\tilde{a}\cdot x)_l+l^2(2l-1)A_l,\\
&2l^2(2l-1)\tilde{b}_l^{\dagger}x_l\tilde{b}_l=l^2(2l-1)(\tilde{b}_l^{\dagger}\tilde{c}_l^{\dagger}x_l+x_l\tilde{c}_l\tilde{b}_l)+(4l^2-1)(\tilde{a}\cdot x)_{l-1}-l^2(2l+1)A_{l-1},\\
&\tilde{b}_l^{\dagger}[l\tilde{a}_l^{\dagger}x_l-(l-1)x_l\tilde{a}_l]=[(l+1)\tilde{a}_{l-1}^{\dagger}x_{l-1}-lx_{l-1}\tilde{a}_{l-1}]\tilde{c}_l
\end{aligned}
\qquad (13.14)
$$

where

$$(\tilde{a}\cdot x)_l=\tilde{a}_l^{\dagger}(\tilde{a}_l^{\dagger}x_l-x_l\tilde{a}_l)-(\tilde{a}_l^{\dagger}x_l-x_l\tilde{a}_l)\tilde{a}_l.$$

Formulae (13.14) present the necessary and sufficient conditions that the matrices (12.13), (13.11) satisfy relations (13.9). Thus the problem of description of Galilei-invariant wave equations reduces to solving the system of linear algebraic equations (13.14) for the matrices x_l and A_l where a_l, b_l and c_l are matrices satisfying the system of coupled quadratic equation of (12.31).

Let us consider the restrictions imposed on x_l and A_l by the conditions (13.10). Making the transformation

$$\psi\rightarrow V\psi=\Phi,\quad L\rightarrow L'=V^{\dagger}LV^{-1}, \qquad (13.15a)$$

$$V=\exp\left(\frac{i}{m}\eta\cdot\boldsymbol{p}\right) \qquad (13.15b)$$

and using relations (13.9) and the Campbell-Hausdorf formula

$$
\begin{aligned}
&\exp(A^{\dagger})B\exp(-A)=\sum_{n=0}^{\infty}\frac{1}{n}\{A,B\}^n,\\
&\{A,B\}^n=A^{\dagger}\{A,B\}^{n-1}-\{A,B\}^{n-1}A,\quad \{A,B\}^0=B,
\end{aligned}
\qquad (13.16)
$$

we obtain from (13.1) the following equivalent equation

$$L'\Phi=0,\quad L'=\beta_0\left(p_0-\frac{p^2}{2m}\right)+\beta_4 m. \qquad (13.17)$$

The equation (13.17) is more convenient for analysis then (13.1) since it includes only two matrices. Inasmuch as the operator V satisfies the conditions

$$\left[V,p_0-\frac{p^2}{2m}\right]=0,\quad V\left(\boldsymbol{S}-\frac{1}{m}\boldsymbol{p}\times\eta\right)V^{-1}=\boldsymbol{S}.$$

then it follows from (13.10) that Φ has to satisfy the following relations

$$\left(p_0-\frac{p^2}{2m}\right)\Phi=\varepsilon_0\Phi,\quad \boldsymbol{S}\Phi=s(s+1)\Phi. \qquad (13.18)$$

Relations (13.18) follow from (13.17) iff

$$\det(\alpha X_l+\beta A_l)=\begin{cases} c_l, & l\neq s,\\ \sum\limits_{i=0}^{n_l} k_i(\alpha-\varepsilon_0)^i, & l=s \end{cases} \tag{13.19}$$

where α and ß are arbitrary functions, c_l and k_i are constants (at least one of k_i is nonzero), n_l is the number of rows of the matrices x_l, A_l.

Indeed, if (13.19) is satisfied then all the solutions of the equation (13.17) are eigenfunctions of the operator $\boldsymbol{S}^2$, corresponding to the eigenvalue $s(s+1)$. Besides these solutions satisfy the first of the equations (13.18).

13.5. Equations of Minimal Dimension

The above results are valid for arbitrary representation of the homogeneous Galilei group realized on a set of solutions of (13.1). Now we consider the simplest case of the system (13.14) when the index l can take only two values $l=l_0$, l_0+1. This corresponds to Galilei-invariant equations with the minimal number of components.

The corresponding indecomposable representations of the algebra *AE(3)* belong to the class D_2 and are described in Subsection (12.6). For the case considered the system (13.14) reduces to the following form (refer to (12.25))

$$\begin{gathered}\tilde{c}_l^\dagger x_{l-1}\tilde{c}_l=-k_l\tilde{a}_l^\dagger x_l\tilde{a}_l-lk_l(\tilde{a}\cdot x)_l,\\ \tilde{b}_l^\dagger x_l\tilde{b}_l=k_{l-1}\tilde{a}_{l-1}^\dagger x_{l-1}\tilde{a}_{l-1}-lk_{l-1}(\tilde{a}\cdot x)_{l-1},\\ \tilde{b}_l^\dagger[l\tilde{a}_l^\dagger x_l-(l-1)x_l\tilde{a}_l]=[(l+1)\tilde{a}_{l-1}^\dagger x_{l-1}-lx_{l-1}\tilde{a}_{l-1}]\tilde{c}_l,\\ k_l=\frac{2l+1}{l^2(l+1)^2}.\end{gathered} \tag{13.20}$$

As to the matrices A_l and A_{l-1} they are expressed via x_l and x_{l-1}:

$$A_l=\frac{2l+1}{(l+1)^2}(\tilde{a}\cdot x)_l,\quad A_{l-1}=\frac{2l-1}{(l-1)^2}(\tilde{a}\cdot x)_{l-1}.$$

All the possible (up to equivalence) nontrivial solutions of (13.20) (and the corresponding expressions for the matrices a_l, b_l, c_l, A_l, D_l and B_l) are present in Table 13.1 where *æ*, $æ_1$ and $æ_2$ are arbitrary parameters.

Table 13.1

series of solutions	R_1	R_2	R_3	R_4
$\tilde{a}_l$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	0	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\tilde{a}_{l-1}$	0	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\tilde{b}_l$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\sqrt{k_l}(1 \;\; 0)$	$i\sqrt{k_l}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\sqrt{k_{l-1}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\tilde{c}_l$	$i\sqrt{k_l}(1 \;\; 0)$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$i\sqrt{k_l}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\sqrt{k_{l-1}}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$
x_l	$\begin{pmatrix} 0 & æ \\ æ & 1 \end{pmatrix}$	1	$\begin{pmatrix} 0 & æ_1 & æ_2 \\ æ_1 & æ_2 & 1 \\ æ_2 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} æ_2 & l+1 \\ l+1 & 0 \end{pmatrix}$

$x_{l\text{-}1}$	$2l\text{-}1$	$\begin{pmatrix} 0 & æ \\ æ & 1 \end{pmatrix}$	$-\begin{pmatrix} æ_2 & l-1 \\ l-1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & æ_1 & æ_2 \\ æ_1 & æ_2 & 1 \\ æ_2 & 1 & 0 \end{pmatrix}$
A_l	$2q_l^{+}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	$q_l^{+}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$A_{l\text{-}1}$	0	$2q_1^{-}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$q_{l-1}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
D_l	$g_l\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	0	$g_l\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$D_{l\text{-}1}$	0	$g_{l-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$g_{l-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
B_l	$-i(2l-1)\times$ $\times\sqrt{k_l}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$-(2l+1)\times$ $\times\sqrt{k_{l-1}}\begin{pmatrix} 1 & 0 \end{pmatrix}$	$i\sqrt{k_l}\begin{pmatrix} 0 & 1-l \\ -l & 0 \\ 0 & 0 \end{pmatrix}$	$-\sqrt{k_{l-1}}\times$ $\times\begin{pmatrix} 0 & l & 0 \\ l+1 & 0 & 0 \end{pmatrix}$

Here $q_l^{\pm}=(2l\pm1)/(l\pm1)^2$, $g_l=(2l+1)/l(l+1)$, k_l is given in (13.20).

We see that only four of eighteen classes of representations of the algebra *AE(3)* enumerated in Subsection 12.6 correspond to nontrivial Galilei-invariant wave equations. These classes are $D_l(3,1,2)$, $D_l(2,1,1)$, $D_l(2,1,3)$ and $D_l(1,1,2)$.

The results given above are summarized in the following assertion.

THEOREM 3.3. Let the equation (13.1) is S-invariant under the Galilei algebra, admits a Lagrangian formulation and describes a particle of spin s and mass m, and let the representation of the inhomogeneous Galilei group realized on the set of solutions of this equation belong to the class D_2. Then the explicit form of these matrices up to equivalence coincides with the given in (13.11), (13.12) and Table 13.1.■

So we obtain four classes of Galilei-invariant wave equations corresponding to the matrices enumerated in Table 13.1. Consider these classes successively.

Equations of class R_1 have $6l+1$ components and describe a particle of spin s=l, mass m and internal energy $æ$. These equations are equivalent to the Hagen-Hurley [215, 223] equations, and in the case s=1/2 reduce to the Levi-Leblond [276] equation.

Equations of class R_2 have $6l$-1 components and describe a Galilean particle of spin $s=l-1$, mass m and internal energy $æ^2s^2m/2$. We will see further on these equations predict the value of the constant of the dipole coupling which differs from the value prophesied by the LHH equations.

Equations of classes R_3 and R_4 are the most interesting. They can be interpreted as quantum mechanical equations of a nonrelativistic particle of spin $s=l$ (and $s=l-1$), mass m and internal energy $(s+1)^2(æ_2^2-2æ_1)m/2$ (and $s^2(æ_2^2-2æ_1)m/2$). It will be shown below these equations can serve as a starting point in the Galilei-invariant description of the spin-orbit coupling.

We present the simplest of the found equations writing them componentwise.

Class R_2, $s=l-1=0$,

$$\begin{aligned} &\psi=column(\psi_0,\psi_1,\psi_2,\psi_3)\\ &p_0\psi_0+\boldsymbol{p}\cdot\boldsymbol{\psi}=0,\\ &2m\boldsymbol{\psi}+\boldsymbol{p}\psi_0=0; \end{aligned} \tag{13.21}$$

Class R_1, $s=l=1/2$,

$$\begin{aligned} &\psi=column(\varphi_1,\varphi_2,\chi_1,\chi_2),\\ &2p_0\varphi+\frac{3}{2}(i\sigma\cdot\boldsymbol{p}+\frac{3}{2}æm)\chi=0,\\ &(\frac{3}{2}æm-i\sigma\cdot\boldsymbol{p})\varphi+\frac{3}{2}m\chi=0; \end{aligned} \tag{13.22}$$

Class R_3, $s=l=1$,

$$\psi=column(\varphi,\chi,\Phi,\Phi_0),$$

$$\frac{1}{2}p_0\chi+\boldsymbol{p}\times\Phi+2æ_1m\chi+2æ_2m\phi=0,$$

$$\frac{1}{2}p_0\varphi-\boldsymbol{p}\,\phi_0+2æ_1m\varphi+2æ_2m\chi+2m\phi=0, \tag{13.23}$$

$$i\sqrt{3}\,\boldsymbol{p}\cdot\chi-2æ_2m\phi_0=0,$$

$$\boldsymbol{p}\times\phi-2æ_2m\varphi-2m\chi=0,$$

where $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices.

13.6. Equations for Representations with Arbitrary Nilpotency Indices

To conclude this section, we discuss an interesting connection between Galilei-invariant wave equations and equations invariant under the generalized Poincaré group *P(1,4)* (refer to Chapter 8).

Consider the generalized Bhabha equation [340]

$$(\beta^k p_k-æ)\psi(x)=0,\quad k=0,1,2,3,4 \tag{13.24}$$

where $\mu_k=S_{5k}$ are matrices realizing a representation of the algebra *AO(1,5)* together with S_{54} and $S_{kl}=i[S_{5k},S_{5l}]$.

It is possible to show that the equation (13.24) is invariant under the Galilei group as well as under the group *P(1,4)*, refer to Subsection 26. To make Galilei invariance of (13.24) obvious, we use the change of variables

$$x_0=\frac{1}{2}(2\tilde{x}_0+\tilde{x}_4),\quad x_4=\frac{1}{2}(2\tilde{x}_0-\tilde{x}_4), \tag{13.25}$$

so that

$$\frac{\partial}{\partial x_0}=\frac{1}{2}\frac{\partial}{\partial\tilde{x}_0}+\frac{\partial}{\partial\tilde{x}_4},\quad \frac{\partial}{\partial x_4}=\frac{1}{2}\frac{\partial}{\partial\tilde{x}_0}-\frac{\partial}{\partial\tilde{x}_4}. \tag{13.26}$$

As a result we come to the equation

$$L\psi\equiv(\tilde{\beta}_0\tilde{p}_0-\tilde{\beta}_4\tilde{p}_4-\tilde{\beta}_a\tilde{p}_a-æ)\psi(\tilde{x}_0,\tilde{x}_4,\boldsymbol{x})=0 \tag{13.27}$$

where

$$\tilde{p}_0=i\frac{\partial}{\partial\tilde{x}_0},\quad \tilde{p}_4=-i\frac{\partial}{\partial\tilde{x}_4},\quad p_a=-i\frac{\partial}{\partial x_a}, \tag{13.28a}$$

$$\tilde{\beta}_0=\frac{1}{2}(S_{50}+S_{54}),\quad \tilde{\beta}_4=S_{50}-S_{54},\quad \tilde{\beta}_a=S_{5a}. \tag{13.28b}$$

Galilei invariance of (13.27) can be proved by direct verification using the following realization of the algebra *AG(1,3)*

$$P_0=\tilde{p}_0,\quad P_a=\tilde{p}_a,\quad M=\tilde{p}_4,$$
$$J_a=\varepsilon_{abc}(x_b p_c+\frac{1}{2}S_{bc}),\quad G_a=\tilde{x}_0 p_a-x_a M+\eta_a \tag{13.29}$$

where

$$\eta_a=\frac{1}{2}(S_{0a}+S_{4a}).$$

Indeed, these generators commute with the operator L of (13.27).

Imposing the Galilei-invariant additional condition on ψ,

$$\tilde{p}_4\psi(\tilde{x}_0,\tilde{x}_4,\boldsymbol{x})=\lambda æ\,\psi(\tilde{x}_0,\tilde{x}_4,\boldsymbol{x}), \tag{13.30}$$

we reduce (13.27) to a Galilei-invariant equation of the kind (13.1) where

$$\beta_0=\tilde{\beta}_0,\quad \beta_4=-\tilde{\beta}_4+\lambda^{-1}I,\quad m=\lambda æ, \tag{13.31}$$

I is the unit matrix. The corresponding generators of the Galilei group have the covariant form (12.18) where

$$M=æ\lambda,\quad \eta_a=\frac{1}{2}(S_{4a}+S_{0a}),\quad S_a=\frac{1}{2}\varepsilon_{abc}S_{bc}. \tag{13.32}$$

Finite-dimension IRs of the algebra *AO(1,5)* (which is isomorphic to the algebra *AO(6)*) are labelled by triplets (n_1,n_2,n_3) where n_a are integers or half integers [197]. If matrices S_{kl} form the representation *D(1/2,1/2,1/2)* then the equations (13.1), (13.28b) are equivalent to the Levi-Leblond equation [276] for a particle of spin 1/2. The representation *D(1,1,1)* corresponds to the equations (13.23) for a particle of spin 1 (besides $æ_1=-æ\lambda$, $æ_2=æ$). The corresponding matrices in (13.1) are expressed via the 10×10 KDP matrices (marked by "^" in the following formula)

$$\beta_0=\frac{1}{2}(\hat{\beta}_0+\hat{\beta}_4),\quad \beta_4=\hat{\beta}_0-\hat{\beta}_4+\lambda^{-1}I,\quad \beta_a=\hat{\beta}_a. \tag{13.33}$$

In general the considered equations describe a multiplet of particles of spins s_1, s_2,... where the numbers s_i characterize the IR of the algebra *AO(3)* consisted in the given representation of the algebra *AO(1,5)* realized by the matrices S_{kl}.

It is possible to demonstrate the matrices (13.32) satisfy the conditions

$$(\eta_a S_a)^{2s}\neq 0,\quad (\eta_a S_a)^{2s+1}=0,\quad s=\max(s_i).$$

Thus the equations considered here correspond to such representations of the algebra *AE(3)* which are characterized by the nilpotency index $n=2s+1$ of the Casimir operator C_1 of (12.27).

We note that the equations (13.1), (13.31), (13.28b) are defined also for the case when the corresponding matrices S_{kl} realize an infinite dimension Hermitian representation of the algebra *AO(1,5).* So we come naturally to infinite component wave equations which are invariant under the Galilei group [151]. Poincaré-invariant infinite component equations are well studied in contrast to the Galilei-invariant ones (see, however, [222]).

We will return to the discussion about connections between Galilei- and *P(1,4)*-invariant wave equations in Chapter 5.

14. GALILEI-INVARIANT EQUATIONS OF THE HAMILTONIAN TYPE

14.1. Uniqueness of the Schrödinger Equation

Consider an alternative possibility in Galilei-invariant description of arbitrary spin particles which expects to use equations of the Hamiltonian type. These equations can be more convenient then first order systems of the kind (13.1) since they include explicitly the evolution operator (Hamiltonian). Besides the distinguishing of the time variable is in a good accordance with Galilei transformations (10.39) in contrast to the case of Poincaré-invariant equations.

First let us "deduce" the Schrödinger equation (11.1) and demonstrate that this is the only Galilei invariant evolution equation for a scalar wave function.

Let us search for linear partial differential equations for a complex valued function $\psi(t,\boldsymbol{x})$

$$L\psi(t,\boldsymbol{x})=0$$

where L is an unknown differential operator. We assume $L=p_0-H$ where H includes space derivatives only.

The Galilei-invariance condition means that L commutes with any basis element of the Galilei algebra. Without loss of generality these basis elements P_μ, $\boldsymbol{J}$, $\boldsymbol{G}$ can be chosen in the form (11.5) (this is the general form of covariant representation of the algebra AG(1,3), realized in the space of scalar functions, refer to Section 12). Calculating commutators of L with the corresponding generators of (11.5) we find immediately that

$$H=p^2/2m+\varepsilon_0,\ \varepsilon_0=const,$$

i.e., we come to the equation which coinsides with the Schrödinger equation up to nonessential constant term ε_0 in the Hamiltonian.

Thus the Schrödinger equation is unique, i.e., this is the only evolution equation which involves scalar wave function and satisfies the requirement of Galilei invariance.

We now consider the problem of deducing of Schrödinger-like equations for Galilean particles of arbitrary spin. We will search for a motion equation of a particle of spin s in the form (11.1) where ψ is a $2(2s+1)$-component wave function, H is a differential operator (particle Hamiltonian) we need to find. We will demonstrated that this number of components of wave function is minimal for a profound equation of motion if s is nonzero.

By definition the equation (11.1) is invariant under the Galilei algebra if it admits 11 SOs satisfying relations (11.6). We restrict ourselves to the case when these operators have the covariant form (12.18) corresponding to local transformations of a wave function by passing to a new reference frame.

It is not difficult to make sure the invariance condition (1.5) reduces to the following equations for H

$$[H,P_a]=[H,J_a]=0, \quad [H,G_a]=iP_a \tag{14.1}$$

where P_a, J_a and G_a are the operators (12.18).

We require the eigenvalues of the Casimir operators (11.14) on a set of solutions of the equation (11.1) be equal to

$$c_1=m, \quad c_2=\varepsilon_0, \quad c_3=m^2s(s+1), \tag{14.2}$$

where s is an integer or half integer, m and ε_0 are arbitrary fixed numbers. This makes it possible to interpret (11.1) as a motion equation of a particle of spin s. The corresponding matrices S_a, η_a and M of (12.18) without loss of generality can be chosen in the form:

$$S_a=\begin{pmatrix} s_a & 0 \\ 0 & s_a \end{pmatrix}, \quad \eta_a=k(\sigma_1+i\sigma_2)S_a, \quad M=\sigma_0 m \tag{14.3}$$

where s_a matrices of dimension $(2s+1)\times(2s+1)$ realizing the IR $D(s)$ of the algebra $AO(3)$, σ_μ are $2(2s+1)$-row Pauli matrices (5.30), k is an arbitrary complex number.

If k is not equal to zero then the matrices (14.3) can be transformed to the equivalent representation with k=1. We find it is more convenient to admit arbitrary values of k, see footnote after Theorem 14.1, Subsection 14.2.

We impose the additional requirement the equation (11.1) be invariant under the transformation of simultaneous reflection of time and space variables. In analogy with (11.37) we represent this transformation in the form

$$\psi(t,\boldsymbol{x})\rightarrow\Theta\psi(t,\boldsymbol{x})=r\psi^*(-t,-\boldsymbol{x}) \tag{14.4}$$

where r is a matrix needed to determine. This transformation has to satisfy the following relations together with generators of the Galilei group:

$$P_\mu\Theta=\Theta P_\mu,\quad J_a\Theta=-\Theta J_a,\quad G_a\Theta=-\Theta G_a \tag{14.5}$$

which reduce to the following equations for r

$$rS_a=-S_a^*r,\quad r\eta_a=-\eta_a^*r. \tag{14.6}$$

It follows from (14.3), (14.6) that the parameter k can be either real or imaginary according to the cases the matrix r commute or anticommute with $\sigma_1+i\sigma_2$. The corresponding matrix r without loss of generality can be chosen in the form

$$\begin{aligned} &r=\exp(i\varphi)\Delta, \quad\ \ \textit{if}\quad k^*=k\\ &r=\exp(i\varphi)\sigma_3\Delta,\quad \textit{if}\quad k^*=-k \end{aligned} \tag{14.7}$$

where Δ is the matrix (7.10), φ is a real number.

The condition of invariance of the equation (11.1) under the transformation of complete reflection can be written in the form

$$[H,\Theta]=0 \tag{14.8}$$

where Θ is the operator (14.4), (14.7).

So we state the problem of finding all nonequivalent Hamiltonians H satisfying the commutation relations (14.1), (14.8). Such Hamiltonians correspond to Galilei-invariant Schrödinger equations for particles of arbitrary spins.

14.2. The Explicit Form of Hamiltonians of Arbitrary Spin Particles

Here we present the general solution of the problem formulated above.

THEOREM 3.4. All the possible (up to equivalence) Hamiltonians H satisfying relations (14.1), (14.8) (together with the Galilei group generators and the operator (14.4)) are given by the following formulae

$$H=H^{(1)}=\sigma_1 am+\sigma_3 2iakS_a p_a+\frac{1}{2m}C_{ab}p_a p_b, \tag{14.9a}$$

$$H=H^{(2)}=\sigma_3 bm+(\sigma_1+i\sigma_2)2bkS_a p_a+\frac{p^2}{2m} \tag{14.9b}$$

where

$$C_{ab}=\delta_{ab}+2ak^2(\sigma_1+i\sigma_2)(S_aS_b+S_bS_a), \tag{14.10}$$

a, b, k are parameters satisfying the conditions

$$b^*=b, \quad k^*=\pm k, \quad (ak)^*=ak. \tag{14.11}$$

PROOF. It is convenient to solve the equations (14.1) in the representation which is connected with (12.18) by the transformation

$$P_\mu \to P'_\mu = UP_\mu U^{-1}, \quad \boldsymbol{J} \to \boldsymbol{J}' = U\boldsymbol{J}U^{-1}, \quad \boldsymbol{G} \to \boldsymbol{G}' = U\boldsymbol{G}U^{-1} \tag{14.12}$$

where

$$U = \exp\left(\frac{i}{m}\eta_a p_a\right) = 1 + \frac{i}{m}\eta_a p_a. \tag{14.13}$$

It is not difficult to make sure that

$$P'_a = P_a, \quad P'_0 = H'_s, \quad J'_a = J_a, \quad G'_a = tp_a - mx_a, \tag{14.14}$$

i.e., G'_a has a very simple form. Requiring the operators (14.14) satisfy (14.1) we come to the following equations

$$[H', x_a] = -\frac{i}{m}p_a, \quad [H', p_a] = [H', S_a] = 0 \tag{14.15}$$

whose general solutions have the form

$$H' = \frac{p^2}{2m} + Bm, \quad B = \sigma_\mu a^\mu, \tag{14.16}$$

where a_μ are arbitrary complex numbers, moreover without loss of generality we can set $a_0=0$.

Using the transformation $B \to WBW^{-1}$ where W is an invertible numeric matrix satisfying the condition*

$$W^{-1}\eta_a W = \lambda\eta_a, \quad \lambda \in \mathbb{C}, \quad \lambda \neq 0 \tag{14.17}$$

we can reduce B to one of the following forms

$$B = \sigma_1 a, \quad B = \sigma_3 a, \quad a = \sqrt{a_1^2 + a_2^2 + a_3^2}, \tag{14.18a}$$

---

*The condition (14.17) can be deduced requiring the corresponding Galilei group generators in starting (imprimed) representation be connected by the equivalence transformation $H \to WHW^{-1} = H''$, $P_a \to WP_aW^{-1} = P_a$, $J_a \to WJ_aW^{-1} = J_a$, $G_a \to WG_aW^{-1} = G_a''$ where G_a'' differs from G_a only by the value of the parameter k (refer to (12.18), (14.3)). Fixing k in (14.3) (e.g., setting $k=1$) we lose the possibility to simplify H' with the help of a matrix transformation, so it is convenient to consider the representations of the Galilei algebra given by (12.18), (14.3) where k is an arbitrary parameter.

$$B=\sigma_1 \pm i\sigma_2. \tag{14.18b}$$

The matrices (14.18) are nonequivalent in the sense that there is not any matrix W satisfying (14.17) and and transforming them one into another. Besides the solutions (14.18b) have to be neglected since they correspond to non-Hermitian representations of the Galilei algebra with a nilpotent Casimir operator $C_3=2mP_0-P^2$.

So the general form of H' is given in (14.16) where B is one of the matrices of (14.18a). With the help of the transformation inverse to (14.12) we find all the nonequivalent imprimed Hamiltonians in the form (14.9) and obtain the conditions (14.11) from relations (14.4)-(14.6). ■

Thus we obtain Galilei-invariant equations for particles of arbitrary spin in the form (11.1), (14.9). Galilei invariance of these equations is evident from the deduction. Nevertheless we can easily verify it directly. Using the identity

$$\exp(i\eta\cdot\boldsymbol{v})=1+i\eta\cdot\boldsymbol{v} \tag{14.19}$$

we find that the operators (14.9) satisfy the relations

$$\exp[i\varphi(x)]D(\theta,\boldsymbol{v})\left(i\frac{\partial}{\partial t}-H(\boldsymbol{p})\right)D^{-1}(\theta,\boldsymbol{v})\exp[-i\varphi(x)]=i\frac{\partial}{\partial t'}-H(\boldsymbol{p}'), \tag{14.20}$$

where $H(\boldsymbol{p}')$ are the operators obtained from (14.9) by the change $\boldsymbol{p}\to\boldsymbol{p}'=-i\partial/\partial\boldsymbol{x}'$, x_0' $\boldsymbol{x}'$, $\varphi(x)$ and $D(\theta,\boldsymbol{v})$ are given in (11.28), (11.29), (12.22).

It follows from (14.20) the transformed function $\psi'(x')$ of (12.21) satisfies the same equation as $\psi(x)$:

$$i\frac{\partial}{\partial x_0'}\psi'(x')=H(\boldsymbol{p}')\psi'(x'). \tag{14.21}$$

In other words the equation (11.1) with any of the Hamiltonians (14.9) is invariant under Galilei transformations (11.29) if $\psi(x)$ is transformed according to (12.21).

We can make sure that the Casimir operators (11.14) corresponding to the Hamiltonians (14.9) have the eigenvalues (14.2) where $\varepsilon_0=\pm am$. So we conclude that equations (11.1), (14.9) can be interpreted as motion equations of a free particle of mass m, spin s and internal energy $\pm am$.

In the case $s=1/2$, $a=ik=1$ the equations (11.1), (14.9a) reduce to the following form

$$(\gamma_\mu p^\mu-m)\psi=(1-\gamma_0-\gamma_4)p^2/2m\psi.$$

This equation has the form of the usual Dirac equation including the additional term $(p^2/2m)\psi$ in the l.h.s. . This term violates Poincaré-invariance of the Dirac equation but generates invariance under the Galilei group.

We note that the simplest Poincaré-invariant equation, i.e., the KGF equation (1.1) also can be "modified" in such a way. Adding to the l.h.s. of (1.1) the term $-(p_0p^2/2m+p4/4m^2)\psi$ we obtain an equation which is not Poincaré-invariant but is invariant under Galilei transformations.

14.3. Lagrangian Formulation

So we have find Galilei-invariant equations in the Schrödinger form for a particle of arbitrary spin. As like as the Dirac Hamiltonian the evolution operators (14.9) depend on the spin matrices but include differential operators of higher (second) order.

The nonrelativistic Hamiltonians (14.9) depend on two parameters a and k which cannnot be fixed by the requirement of Galilei and PT-invariance. These operators are Hermitian with respect to the scalar product (7.35) where M is a positive defined metric operator

$$\hat{M}=U^{\dagger}U=1+\frac{1}{m}[i(k^*-k)\sigma_1-(k^*+k)\sigma_2]S_ap_a+\frac{2k^*k}{m}(1-\sigma_3)(S_ap_a)^2.$$

Besides the operators (14.9a), (12.18) (with S_a, η_a given in (14.3)) are Hermitian in the following indefinite metric

$$(\psi_1,\psi_2)=\int d^3x\psi_1^{\dagger}\xi\psi_2, \tag{14.22}$$

where

$$\xi=\begin{cases}\sigma_1, & if \quad a^*=a, \quad k^*=k\\ \sigma_2, & if \quad a^*=-a, \quad k^*=-k.\end{cases} \tag{14.23}$$

ξ is a Hermitizing matrix for the Hamiltonian (14.9a) and the generators (12.18). As to the Hamiltonian (14.9b) the corresponding Hermitizing matrix does not exist.

The equations (11.1), (14.9a) can be deduced using the variational principle if we start from the Lagrangian

$$\begin{aligned} L_0(x)=&\frac{i}{2}\left(\bar{\psi}\frac{\partial\psi}{\partial t}-\frac{\partial\bar{\psi}}{\partial t}\psi\right)-am\bar{\psi}\sigma_1\psi-\\ &-\frac{1}{2m}\frac{\partial\bar{\psi}}{\partial x_a}C_{ab}\frac{\partial\psi}{\partial x_b}+ak\left(\bar{\psi}\sigma_3S_a\frac{\partial\psi}{\partial x_a}-\frac{\partial\bar{\psi}}{\partial x_a}\sigma_3S_a\psi\right)\end{aligned} \tag{14.24}$$

where $\bar{\psi}=\psi^*\xi$. It is not difficult to make sure that the Euler-Lagrange equations (8.24), (14.24) reduce to (11.1), (14.9a). Changing in (8.24) $\bar{\psi}\to\psi$ we obtain the equation for $\bar{\psi}$ which is equivalent to (11.1), (14.9a).

Thus equations (11.1), (14.9a) admit the Lagrangian formulation. This circumstance makes it possible to use the canonical Lagrangian formalism to construct the operators of the main physical values (momenta, energy, angular momenta etc.) and to generalize these equations to the case of a particle interacting with an external fields using the minimal coupling principle.

We note that the equations (11.1) with the Hamiltonian (14.9b) cannnot be represented as Euler-Lagrange equations, because the Hamiltonian (14.9b) is non-Hermitian and the corresponding Hermitizing matrix does not exist. Therefore we will not consider this Hamiltonian restricting ourselves to the equations (11.1), (14.9a).

Let us summarize. We have obtained the motion equations for a particle of arbitrary spin s in the Hamiltonian form (11.1) where H is the second order differential operator (14.9a). These equations are invariant under the Galilei and total reflection transformations and admit a Lagrangian formulation.

We present here also Galilei invariant equations for a particle with variable spin. Such equations can be written in the form (11.1) where

$$H=\sigma_1 am+\sigma_3 2iakS_{4a}p_a+\frac{1}{2m}\tilde{C}_{ab}p_a p_b, \tag{14.25a}$$

$$\tilde{C}_{ab}=\delta_{ab}+2ak^2(\sigma_1+i\sigma_2)(S_{4a}S_{4b}+S_{4b}S_{4a}). \tag{14.25b}$$

Here S_{4a} are matrices belonging to the representation *D(1/2 1/2)* of the algebra *AO(4)*, σ_a are the Pauli matrices of dimension $2(s+1)^2\times 2(s+1)^2$, commuting with S_{4a}.

The operator (14.25) is a Galilei-invariant analog of the relativistic Hamiltonian H_1 of (10.18). The corresponding equation (11.1) is invariant under the *P*-, *T*- and *C*-transformations and describes particles with variable spin whose values are s, s-1, ..., 0.

In papers [162,163] other types of Galilei-invariant equations where considered. They are second order equations with a singular matrix by p_0 and Schrödinger-type equations corresponding to non-covariant representations of the algebra *AG(1,3)*. We do not analyze these equations here but note that they can be successfully used to describe a particle of arbitrary spin in an external field.

15. GALILEAN PARTICLE OF ARBITRARY SPIN IN AN EXTERNAL ELECTROMAGNETIC FIELD

15.1. Introduction of Minimal Interaction into First-Order Equations

A motion equation of a free particle is of interest for physics only as a first step in describing interacting particles. Moreover, the critical point is a possibility to use such an equation to describe an interaction of a charged particle with the electromagnetic field.

We show further on that Galilei-invariant equations can be successfully used to solve the mentioned problem inasmuch as they take into account all the physical effects predicted by the relativistic Dirac equation in the approximation $1/m^2$. Using the generalized Foldy-Wouthuysen reduction we calculate the constants of the dipole, quadruple and spin-orbit couplings predicted by these equations. The constant of the dipole interaction generated by first-order Galilei-invariant equations is equal to $1/s$ in accordance with the Belifante conjecture [30].

We start with the Galilei-invariant equations of first order derived in Section 13. Since these equations admit the Lagrangian formulation, this is naturally to introduce an interaction with the electromagnetic field in frames of the minimal interaction principle, i.e., to make the change $\partial/\partial x^\mu \rightarrow \partial/\partial x^\mu - ieA_\mu$ in the corresponding Lagrangian. As a result we come to the equations

$$L(\pi)\psi \equiv (\beta^\mu \pi_\mu + \beta_4 m)\psi = 0, \quad \pi_\mu = p_\mu - eA_\mu. \tag{15.1}$$

To substantiate the minimal interaction principle in application to Galilei-invariant wave equations we will show that the change $p_\mu \rightarrow \pi_\mu$ does not violate the Galilei and gauge symmetry of the starting equations and leads to reasonable results by solving the concrete physical problems. Moreover we will not use a concrete realization of β-matrices (at least until a certain moment) so our results will be valid for arbitrary wave equations invariant under the Galilei group.

The equation (15.1) is manifestly invariant under the gauge transformations

$$\psi(x) \rightarrow \exp[ie\varphi(x)]\psi(x), \quad A_\mu \rightarrow A_\mu + \frac{\partial\varphi(x)}{\partial x^\mu} \tag{15.2}$$

where φ is an arbitrary differentiable function. To establish invariance of this equation under the Galilei transformations we assume that the vector-potential is transformed simultaneously according to Galilean law [273]

$$A_0 \rightarrow A_0' = A_0 + \boldsymbol{v}\cdot\boldsymbol{A}, \quad A_a \rightarrow A_a' = R_{ab}A_b. \tag{15.3}$$

Using relations (13.9) and applying the Campbell-Hausdorf formula (13.16) we make sure the operator $L(\pi)$ satisfies the condition

$$\exp[i\varphi(x)]\tilde{D}(\theta,\boldsymbol{v})L(\pi)D^{-1}(\theta,\boldsymbol{v})\exp[-i\varphi(x)] = L(\pi') \tag{15.4}$$

where $L(\pi')$ is the operator obtained from $L(\pi)$ of (15.1) by the change $\pi_\mu \rightarrow \pi_\mu' = i\partial/\partial x'^\mu - eA'_\mu$, $\tilde{D}(\theta,\boldsymbol{v}) = [D^{-1}(\theta,\boldsymbol{v})]^\dagger$; x_μ', $\varphi(x)$, $D(\theta,\boldsymbol{v})$, A_μ' are defined in (11.28), (11.29), (12.22), (15.3).

It follows from (15.4) the transformed wave function $\psi'(x')$ of (12.21) satisfies the same equation as $\psi(x)$:

$$L(\pi')\psi'(x')=0, \tag{15.5}$$

so the equation (15.1) is invariant under Galilei transformations.

15.2. Magnetic Moment of Galilei Particle of Arbitrary Spin

Let us show the Galilei-invariant equations (15.1) are good models of charged particles interacting with the electromagnetic field and describe the dipole, quadruple and spin-orbit interactions. To analyze these equations it is convenient to pass to the representation where the operator $L(\pi)$ reduces to a series in powers $1/m$. For this purpose we make the transformation

$$\psi\to\psi'=V^{-1}\psi, \quad L(\pi)\to L'(\pi)=V^{\dagger}L(\pi)V \tag{15.6}$$

where

(15.7)

is the operator obtained from (13.15b) by the change $\boldsymbol{p}\to\pi$. Using the Campbell-Hausdorf formula (13.16) and the commutation relations (13.9) for β_μ and η_a and restricting ourselves to the case when the nilpotency index of the matrices η_a is less then 4 we come to the following equation

$$L'(\pi)\psi'(x)=0 \tag{15.8}$$

where

$$L'(\pi)=\beta_0\left(\pi_0-\frac{\pi^2}{2m}\right)+\beta_4 m+\frac{e}{m}\left(\beta_0\eta\cdot\boldsymbol{E}-\frac{1}{2}\beta\times\eta\cdot\boldsymbol{H}\right)+\frac{e}{2m^2}\beta_0\eta\cdot(\pi\times\boldsymbol{H}-\boldsymbol{H}\times\pi). \tag{15.9}$$

If the nilpotency index n_η of η_a is larger then 3 the corresponding operators $L'(\pi)$ includes the additional terms

$$\frac{e}{2m^2}\beta_0\eta_a\eta_b\frac{\partial E_a}{\partial x_b}+\sum_{k=3}^{n}\frac{1}{m^k}B_k$$

where B_k depend linearly on the electromagnetic field strengths and their derivatives. These terms can be neglected proceeding from reasonable suppositions about the intensity of an external field.

Thus formula (15.9) being exact for $n_\eta\le 3$ can be treated as an approximate one for $n_\eta>3$.

The operator (15.9) includes the terms corresponding to interaction of a point charged particle with an external electromagnetic field ($\sim \pi_0-\pi^2/2m$) and the additional terms proportional to the vectors of the electromagnetic field strengths. Here we consider in detail the term $e\beta\times\eta\cdot\boldsymbol{H}$ corresponding to the interaction of a particle spin with the magnetic field (i.e., the Pauli interaction).

Using the representation (12.23), (13.12) for β and η and bearing in mind relations (12.26) we obtain the following expression for the matrix β×η in the basis $|\lambda;l,m>$

$$\begin{aligned}\beta\times\eta\,|\lambda;l,m>=&E^{l}_{\lambda\lambda'}\boldsymbol{S}^{\boldsymbol{l}}_{\boldsymbol{mm}'}|\lambda';m'l>+F^{l}_{\lambda\lambda'}\boldsymbol{K}^{\boldsymbol{l}}_{\boldsymbol{mm}'}|\lambda';l-1,m'>+\\&+(F^{l+1})^{\dagger}_{\lambda\lambda'}\boldsymbol{K}^{\boldsymbol{l+1}}_{\boldsymbol{mm}'}|\lambda';l+1,m'>\end{aligned}\tag{15.10}$$

where

$$E^{l}_{\lambda\lambda'}=A^{l}_{\lambda\lambda'}-\frac{(2l+1)[(a\cdot x)_{l}]_{\lambda\lambda'}}{l(l+1)},\quad F^{l}_{\lambda\lambda'}=\frac{1}{l}(a^{l-1\dagger}x^{l-1}c^{\,l}-b^{\,l\dagger}x^{\,l}a^{\,l}),\ l\neq0,\tag{15.11}$$

a^l, b^l, c^l, $(a\cdot x)_l$ and A_l are matrices satisfying the relations (12.31), (13.14).

Consider the equation (15.8) for the case $\boldsymbol{E}$=0. Using the representation (13.11), (15.10) for β_0, β_4 and β×η we can write the system (15.8) as a chain of equations for the functions ψ_l (eigenfunctions of the matrix $\boldsymbol{S}^2$)

$$\begin{aligned}&\left[A^{\,l_1}\left(\pi_0-\pi^2+\frac{e}{2l_1m}\boldsymbol{S}^{\,l_1}\cdot\boldsymbol{H}\right)+(2l_1+1)x^{\,l_1}m\right]\psi_{l_1}+\frac{l}{2ml_1}F^{\,l_1\dagger}\boldsymbol{K}^{\,l_1}\cdot\boldsymbol{H}\psi_{l_1-1}=0,\\&\frac{l}{2ml_1}F^{\,l_1\dagger}(\boldsymbol{K}^{\,l_1}\cdot\boldsymbol{H})^{\dagger}\psi_{l_1}+\left[A^{\,l_1-1}\left(\pi_0-\frac{\pi^2}{\boldsymbol{2m}}\right)+\right.\\&\qquad\left.+E^{\,l_1-1}\frac{e}{2m}\boldsymbol{S}^{\,\boldsymbol{l_1-1}}\cdot\boldsymbol{H}+(2l_1-1)x^{\,l_1-1}m\right]\psi_{l_1-1}+\frac{l}{2l_1-1}F^{\,l_1-1}\boldsymbol{K}^{\boldsymbol{l_1-1}}\cdot\boldsymbol{H}\psi_{l_1-2}=0,\\&\ldots\\&\frac{eF^{\,l_0+1\dagger}}{2m(l_0+1)}(\boldsymbol{K}^{\boldsymbol{l_0+1}}\cdot\boldsymbol{H})^{\dagger}\psi_{l_0+1}+\left[A^{\,l_0}\left(\pi_0-\frac{\pi^2}{\boldsymbol{2m}}-\frac{l}{2m(l_0+1)}\boldsymbol{S}^{\,l_0}\cdot\boldsymbol{H}\right)+(2l_0+1)x^{\,l_0}m\right]\psi_{l_0}=0.\end{aligned}\tag{15.12}$$

We used the identities

$$E^{\,l_1}=-\frac{1}{l_1}A^{\,l_1},\quad E^{\,l_0}=\frac{1}{l_0+1}A^{\,l_0}$$

which follows from (15.11), (13.14a), (13.14b). Here E^l and F^l are the matrices whose elements are defined in (15.11).

Until now we have not made any supposition about a concrete realization of β-matrices and use only relations (13.9) following from the requirement of Galilei invariance. Further we assume the equation (15.1) describes a particle of a fixed spin s i. e. the conditions (13.19) are fulfilled. Besides we assume that $s=l_1$ and call the corresponding equation (15.1) "equation for the highest spin".

If the conditions (13.19) are satisfied and $s=l_1$ then the operator in square brackets in the last line of (15.12) is invertible. So we can express ψ_{l_0} via ψ_{l_0+1}

$$\psi_{l_0}=-V^{-1}\frac{eF^{l_0+1\dagger}}{2m(l_0+1)}(\boldsymbol{K}^{l_0+1}\cdot\boldsymbol{H})\psi_{l_0+1};\quad V=A^{l^0}\left(\pi_0-\frac{\pi^2}{2m}-\frac{e}{2m(l_0+1)}\right)+(2l_0+1)x^{l_0}m.$$

Since the determinant of the matrix V does not depend on $\boldsymbol{H}$ then $det(V^{-1})$ also has this property. Roughly speaking it means that $\psi_{l_0}\sim e\boldsymbol{H}\psi_{l_0+1}$.

In analogous way supposing the magnetic field strength is small enough in order to neglect terms quadratic in respect with $e\boldsymbol{H}$ we can express ψ_{l_0+1} via ψ_{l_0+2}, ψ_{l_0+2} via ψ_{l_0+3} etc. As a result we come to the following representation for the first of the equations (15.12):

$$\left[A^s\left(\pi_0-\frac{\pi^2}{2m}-\frac{e}{2sm}\boldsymbol{S}\cdot\boldsymbol{H}\right)+(2s+1)mx^s+F(\boldsymbol{H}^2)\right]\psi_s=0 \tag{15.13}$$

where $s=l_1$, $\boldsymbol{S}=\boldsymbol{S}^s$, $F(\boldsymbol{H})$ is a term of order $(e\boldsymbol{H})^2$ which can be neglected.

Since A^s and x^s by definition satisfy the condition (13.19) it follows from (15.13) that

$$\left(\pi_0-\frac{\pi^2}{2m}-\frac{e}{2sm}\boldsymbol{S}\cdot\boldsymbol{H}+\varepsilon_0\right)\psi_s=0 \tag{15.14}$$

Thus introducing the minimal interaction into first-order Galilei-invariant wave equations we come to the Schrödinger-Pauli equation (15.14) for a $(2s+1)$-component wave function. This equation includes the term describing the Pauli interaction of spin with the magnetic field. The coefficient of $e\boldsymbol{S}\cdot\boldsymbol{H}/2m$ is called the gyromagnetic ratio, in our case it is equal to $g=1/s$.

The Belifante conjecture [30] $g=1/s$ has been supported repeatedly using relativistic wave equations [196, 224]. We see that Galilei-invariant wave equations for a particle of highest spin also predict the correct value of the gyromagnetic ratio moreover this result does not depend on a choice of the concrete equation but has a universal nature.

15.3. Interaction with the Electric Field

We see that the Pauli interaction can be successfully interpreted in frames of Galilei-invariant theory since any equation of the kind (15.1) can serve as a base for its description. But an interaction of a spin with the electric field (e.g., the spin-orbit coupling) is a finer effect which can be described only by special classes of Galilei-invariant equations.

Starting with the equations of the form (15.9) it is possible to find the general restrictions imposed on the matrices η_a by requiring these equations be a suitable model of a real physical situation. A simple analysis shows that if the nilpotency index of these matrices is less than 3 then the operator $L'(\pi)$ does not include any term depending on the electric field strength. Indeed, in this case $\eta_a\eta_b=0$ and so according to (13.9) $\beta_0\eta_a = 0$. The corresponding operator (15.9) reduces to the form

$$L'(\pi)=\beta_0\left(\pi_0-\frac{\pi^2}{2m}\right)+\beta_4 m-\frac{e}{2m}\beta\times\eta\cdot\boldsymbol{H}. \tag{15.15}$$

It can be seen easily that the equations (15.8), (15.5) cannnot describe an interaction of a spin with the electric field. In the case $\boldsymbol{H}=0$, $\boldsymbol{E}\neq 0$ the operator (15.15) does not depend on $\boldsymbol{E}$ and $\boldsymbol{S}$, being commutative with S_a.

We formulate this result in the form of the following assertion.

PROPOSITION 15.1. The necessary condition a Galilei-invariant equation of the form (15.1) describes interaction of a particle spin with the electric field is

$$C_1^2\equiv(\boldsymbol{S}\cdot\eta)^2\neq 0 \tag{15.16}$$

where $\boldsymbol{S}$ and η are generators of the homogeneous Galilei group realized on a set of solutions of this equation, C_1 is the Casimir operator (12.27).

As will be shown in the following the condition (15.16) is also sufficient. Thus to answer the question whether the given Galilei-invariant wave equation describes an interaction of a spin with the electric field, it is sufficient to calculate the square of the corresponding Casimir operator C_1 and apply the criterium (15.16).

The representations of the algebra *AE(3)* realized on sets of solutions of the LHH equations do not satisfy the condition (15.16) and so these equations do not describe a spin-orbit coupling of a particle with an external field. In the following subsection we consider in detail the Galilei-invariant equations of the class D_2 and select such equations which describe the mentioned coupling.

15.4. Equations for a (2*s*+1)-Component Wave Function

Let us analyze the equations (15.1) with the minimal number of components. The explicit form of the corresponding β-matrices is represented in Subsection 13.5.

We show these equations reduce to the equations of the Schrödinger type for a $(2s+1)$-component wave function Φ_s and additional conditions expressing superfluous components of ψ via Φ_s.

First we consider the LHH equations. The corresponding matrices β_k and η_k are given in (12.23), (13.11), (13.12) and the column R_1 of Table 13.1. It is convenient to analyze these equations in the equivalent representation (15.8). Denoting

$$\psi' = column(\Phi_s, \Phi_s', \chi) \tag{15.17}$$

where Φ_s, Φ_s' are $(2s+1)$-component, χ is a $(2s-1)$-component wave function, we come to the equation (15.14) for Φ_s, where $\varepsilon_0 = m\ae^2(s+1)^2$. As to Φ_s' and χ then according to (15.8) $\chi=0$, $\Phi_s' = \ae\Phi_s$.

So the LHH equations reduce to the Pauli equation (15.14) for the $(2s+1)$-component wave function Φ_s and predict the correct value $1/s$ for the gyromagnetic ratio. Unfortunately these equations do not take into account as important effect as a spin-orbit coupling.

In an analogous way, we make sure that the equations corresponding to the column R_2 of Table 13.1 reduce to the Pauli equation (15.14) for a $(2s'+1)$-component wave function of a particle of spin $s'=s-1$ besides $g=1/(s'+1)$. Thus such equations also do not describe a spin-orbit coupling.

The equations (15.1) are more interesting in the case when the corresponding β-matrices are given in column R_3 of Table 13.1 since in this case the criterium (15.16) is satisfied. Denoting

$$\psi' = column(\phi_1^s, \phi_2^s, \phi_3^s, \chi_1^{s-1}, \chi_2^{s-1}) \tag{15.18}$$

and substituting (15.18) and the explicit form of the corresponding matrices into (15.8), (15.9) we come to the following equations

$$\begin{gathered} i\frac{\partial}{\partial x_0}\phi_1^s = H\phi_1^s \equiv \left(\varepsilon_0 + \frac{\pi^2}{2m} + eA_0 + \frac{e}{2ms}\boldsymbol{S}\cdot\boldsymbol{H} - \frac{e}{s(s+1)\ae_2 m}\boldsymbol{S}\cdot\boldsymbol{F}\right)\phi_1^s, \\ \varepsilon_0 = -\left(\ae_1 - \frac{1}{2}\ae_2^2\right)m(s+1)^2, \quad \boldsymbol{F} = \boldsymbol{E} + \frac{1}{2m}(\pi\times\boldsymbol{H} - \boldsymbol{H}\times\pi), \end{gathered} \tag{15.19}$$

$$\begin{gathered} \phi_2^s = -\ae_2\phi_1^s, \quad \phi_3^s = \frac{1}{2}\left(\ae_2^2 + \frac{e\boldsymbol{S}\cdot\boldsymbol{F}}{s(s+1)\ae_2 m}\right)\phi_1^s, \\ \chi_1^{s-1} \equiv 0, \quad \chi_2^{s-1} = \frac{i\sqrt{k_s}\; e\boldsymbol{K}\cdot\boldsymbol{H}}{(s-1)(2s-1)m}\phi_1^s, \;\; s\neq 1, \quad \chi_2^0 \equiv 0. \end{gathered} \tag{15.20}$$

So the considered equations reduce to the Hamiltonian equation (15.19) for the $(2s+1)$-component wave function ϕ_1^s, the remaining components of ψ' are expressed via this function according to (15.20).

To inquire into the physical sense of solutions of this equation, we transform (15.19) into a form where the Hamiltonian does not include the term ~ $\boldsymbol{S}\cdot\boldsymbol{E}$ corresponding to the nonphysical electric dipole coupling. Using for this purpose the transformation

(15.21)

$$U=\exp\left(\frac{i\boldsymbol{S}\cdot\pi}{s(s+1)æ_2 m}\right)\phi_1^s \tag{15.22}$$

and using the identities

$$i[\boldsymbol{S}\cdot\boldsymbol{F},\boldsymbol{S}\cdot\pi]=\frac{1}{6}Q_{ab}\frac{\partial F_a}{\partial x_b}-\frac{1}{3}s(s+1)i\boldsymbol{p}\cdot\boldsymbol{F}-\frac{1}{2}\boldsymbol{S}\cdot(\boldsymbol{F}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{F}),$$
$$i[\boldsymbol{S}\cdot\pi,\pi_0]=-\boldsymbol{S}\cdot\boldsymbol{E},\quad i[\boldsymbol{S}\cdot\pi,\pi^2]=e\boldsymbol{S}\cdot(\pi\times\boldsymbol{H}-\boldsymbol{H}\times\pi), \tag{15.23}$$

where $\boldsymbol{F}$ is an arbitrary vector depending on $\boldsymbol{x}$, Q_{ab} is the tensor of the quadruple interaction (10.27), we come to the Hamiltonian (10.26) where

$$B=\frac{1}{s},\quad D=\frac{1}{æ_2^2 s(s+1)},\quad C=\frac{1}{2}BD,\quad \varepsilon_0=m_0=-\left(æ_1-\frac{1}{2}æ_2^2\right)(s+1)m. \tag{15.24}$$

We see that the considered Galilei-invariant wave equations generate the approximate Hamiltonian of a particle of a spin s, which has the same structure as the Hamiltonian of a relativistic particle. This means that our equations describe dipole, quadruple, spin-orbit and Darwin interactions as relativistic equations, furthermore, the accuracy of the Galilei-invariant description in the approximation $1/m^2$ is not less than in the case of using of Poincaré-invariant equations.

The principal distinguishing from the relativistic case is that the coefficient of the term corresponding to the spin-orbit coupling is not fixed but is determined up to an arbitrary constant $æ_2$.

Thus the spin-orbit coupling which is usually interpreted as a purely relativistic effect can be successfully described in frames of the Galilei-invariant approach.

In a complete analogy with the above we can demonstrate that the equation (15.1) with the β-matrices corresponding to column R_4 of Table 13.1 reduces to the following equation for a $(2s'+1)$-component function $\chi_{s'}$

$$i\frac{\partial}{\partial x_0}\chi_{s'}=H\chi_{s'}\equiv\left(\tilde{m}_0+\frac{\pi^2}{2m}+eA_0+\frac{e}{2m(s'+1)}\boldsymbol{S}'\cdot\boldsymbol{H}-\frac{e}{s'(s'+1)æ_2 m}\boldsymbol{S}'\cdot\boldsymbol{F}\right)\chi_{s'},$$
$$\tilde{m}_0=-\left(æ_1-\frac{1}{2}æ_2^2\right)s'^2 m,\quad \boldsymbol{S}'\in D(s-1).$$

The latter with the help of the transformation

$$\chi_{s'}\to\chi'_{s'}=\exp\left(i\frac{\boldsymbol{S}'\cdot\pi}{s'(s'+1)æ_2 m}\right)\chi_{s'}$$

reduces to the form

$$i\frac{\partial}{\partial x_0}\chi'_{s'}=H'\chi'_{s'}.$$

The explicit form of the Hamiltonian H' can be obtained from (10.26) by the change $\boldsymbol{S}\to\boldsymbol{S}'$, $B\to 1/(s'+1)$, $C\to BD/2$, $D\to 1/[æ_2 s'(s'+1)]$, $\varepsilon\to\tilde{m}_0$.

15.5. Introduction of Minimal Interaction into Schrödinger-Type Equations

The Galilei-invariant equations in Hamiltonian form considered in Section 14 also can serve as a basis for description of spinning particles in the electromagnetic field. Here we discuss briefly the special features arising by using these equations.

Making in (11.1), (14.9a) the change $p_\mu\to\pi_\mu$ we come to the following equation

$$L(\pi)\psi(x)=0,\quad L(\pi)=i\frac{\partial}{\partial x_0}-H(\pi,A_0), \tag{15.25}$$

where

$$H(\pi,A_0)=\sigma_1 am+\frac{\pi^2}{2m}+eA_0+2iak\sigma_3\boldsymbol{S}\cdot\pi+(\sigma_1+i\sigma_2)\frac{2iak^2}{m}\left[(\boldsymbol{S}\cdot\pi)^2+\frac{e}{2}\boldsymbol{S}\cdot\boldsymbol{H}\right]. \tag{15.26}$$

Equations (15.25) are Euler-Lagrange equations and can be deduced starting from the Lagrangian obtained from (14.24) by the change $\partial/\partial x_\mu\to\partial/\partial x_\mu-ieA_\mu$. These equations are manifestly invariant under the gauge transformations (15.2). It is not difficult to make sure that the operator $L(\pi)$ satisfies the Galilei-invariance condition (15.4) also,besides $\tilde{D}(\theta,\boldsymbol{v})=D(\theta,\boldsymbol{v})$ is the matrix (12.22), (14.3) (compare with (14.20)).

Let us demonstrate that the motion equation of charged particle in the electromagnetic field, given in (15.25), (15.26) presents a good model of the considered physical situation and describes spin-orbit and Darwin couplings.

To give a physical interpretation of the equation considered it is convenient to pass to such a representation where the operator $L(\pi)$ has a quasidiagonal form (i.e., commutes with the matrix σ_1 standing by the mass term). As like as in the case of the relativistic Dirac equation we can make an approximate diagonalization only, using $1/m$ as a small parameter. It turns out the approximate Hamiltonian obtained from (15.26) also has the form (10.26) where

$$\varepsilon_0=\sigma_1 am,\quad B=\sigma_1 ak^2,\quad D=k,\quad C=\frac{1}{2}BD. \tag{15.27}$$

In other words Galilei-invariant equations of the Schrödinger type can serve as a good mathematical model of a charged particle of arbitrary spin. The corresponding Hamiltonian (10.26), (15.27) includes all the terms present in the quasirelativistic Foldy-Wouthuysen Hamiltonian except the relativistic correction to kinetic energy $p^4/8m^3$. So the equations (15.25) describe dipole, quadruple and spin-orbit couplings of arbitrary spin particles with an external field.

The essentially new point in comparison with Poincaré-invariant equations and first-order Galilei-invariant equations is that the coefficient of the term $e\boldsymbol{S}\cdot\boldsymbol{H}/2$ (denoted by B in (15.27)) is not fixed but expressed via arbitrary parameters a and k. The values of these parameters can be chosen in accordance with experimental data. Setting, e.g., a=1, k=1/s, s=1/2 we come to the Hamiltonian whose first six terms coincide with the corresponding terms of the Foldy-Wouthuysen Hamiltonian (see (10.26) for α=III, s=1/2) obtained from the Dirac equation.

In conclusion we note that the explicit form of the corresponding transformations diagonalizing the Hamiltonian (15.26) is given by formulae (10.25) (where the index α has to be omitted) besides

$$V_1=\exp(\sigma_2 k\boldsymbol{S}\cdot\pi/m),$$

$$V_2=\exp(B)\equiv\exp[\sigma_2 k(ke\boldsymbol{S}\cdot\boldsymbol{H}-2k(\boldsymbol{S}\cdot\pi)^2-e\boldsymbol{S}\cdot\boldsymbol{E}/a)/2m^2],$$

$$V_3=\exp\left[\sigma_2\frac{k^2}{m^3}\left(\frac{2}{3}k(\boldsymbol{S}\cdot\pi)^3-\frac{ke}{2}[\boldsymbol{S}\cdot\pi,\boldsymbol{S}\cdot\boldsymbol{H}]_+\right)-\sigma_1\left(\frac{1}{2m}[B,\pi_0]-\frac{ik}{2m^3}[\boldsymbol{S}\cdot\pi,\pi^2]\right)\right].$$

15.6. Anomalous Interaction

The minimal change $p_\mu\rightarrow\pi_\mu$ in motion equations is not the only possibility in description of particles interaction with an external field. A more general approach (proposed by Pauli) is to take into account so called anomalous coupling described by the terms linearly dependent on $\boldsymbol{E}$ and $\boldsymbol{H}$.

A classical example of an equation describing the anomalous interaction is the Pauli generalization of the Dirac equation

$$\left[\gamma_\mu\pi^\mu-m+\frac{iqe}{4m}[\gamma_\mu,\gamma_\nu]F^{\mu\nu}\right]\psi=0.$$

The term proportional to the tensor of the electromagnetic field strength $F^{\mu\nu}$ makes it possible to take into account a deviation of the dipole momentum from the value g=1/s.

Here we discuss possibilities of introducing of anomalous interactions into Galilei-invariant wave equations. Namely we consider generalized equations (15.1), (15.25) of the kind

$$L\psi\equiv\left[i\frac{\partial}{\partial x_0}-\hat{H}(\pi,A_0)\right]\psi,\quad \hat{H}(\pi,A_0)=H(\pi,A_0)+\frac{e}{m}\Sigma_{\mu\lambda}F^{\mu\lambda} \tag{15.28}$$

and

$$\hat{L}\psi\equiv\left(\beta_\mu\pi^\mu+\beta_4 m+\frac{e}{m}\Sigma'_{\mu\lambda}F^{\mu\lambda}\right)\psi=0 \tag{15.29}$$

where $H(\pi,A_0)$ is the operator (15.26), $\Sigma_{\mu\lambda}$ and $\Sigma'_{\mu\lambda}$ are some matrices which have to be of that kind that the equations (15.28), (15.29) are Galilei-invariant.

Our task is to find the exact form of these matrices satisfying the Galilei-invariance condition and to analyze the contribution of the additional terms introduced into equations of motion.

The following statement is valid for the equation (15.28).

PROPOSITION 15.2. The equation (15.28) is Galilei-invariant iff

$$\Sigma_{0a}=\frac{k_1}{2}\eta_a,\qquad \Sigma_{ab}=\frac{1}{2}\varepsilon_{abc}(k_1S_c+k_2\eta_c) \tag{15.30}$$

where S_a and η_a are the matrices (14.3), k_1 and k_2 are arbitrary numbers.

For the proof see [320]. ■

If we require the equation (15.28) be invariant under the complete reflection transformation (14.4) then the parameters k_1 and k_2 have to be real.

Now we consider the first-order equations (15.29). Restricting ourselves to the case of the LHH equations (i.e., where β-matrices have the form (13.11), (13.12), R_1 in Table 13.1) we obtain [320, 321] the following general form of $\Sigma_{\mu\sigma}$:

$$\Sigma'_{0a}=\frac{k_3}{2}\varepsilon_{abc}\beta_0\beta_b\beta_c, \tag{15.31}$$

$$\Sigma'_{ab}=\frac{i}{2}k_3(1-2\beta_0)\varepsilon_{abc}\beta_c+\frac{k_4}{2}\beta_0\beta_b\beta_c \tag{15.32}$$

where k_3 and k_4 are arbitrary numbers.

So we have defined the general form of the terms representing anomalous interaction, which can be included into motion equation without loss of their Galilei-invariance. To analyze the physical consequences of such an inclusion, we transform the operator L of (15.28) to the quasidiagonal form and reduce (15.29) to the equation for a $(2s+1)$-component wave function in analogy with the procedure made in Subsections 15.4, 15.5. As a result, we obtain the equation [320, 321]

$$\left(i\frac{\partial}{\partial t}-H'''-\frac{2kk_2}{3m^2}Q_{ab}\frac{\partial H_a}{\partial x_b}\right)\psi=0 \tag{15.33}$$

where H''' is given in (10.26) besides

$$\varepsilon_0=\sigma_1 am,\quad B=k_1+\sigma_1(k_2-4ak^2),\quad D=2kk_2-1,\quad C=\frac{1}{2}BD, \tag{15.34}$$

and

$$\left(i\frac{\partial}{\partial t}-H^{\prime\prime\prime}-\frac{e^2k_3^2}{4m^2}\boldsymbol{H}^2\right)\phi_s=0, \tag{15.35}$$

where H''' is the operator (10.26) again but

$$\varepsilon_0=m\text{æ}^2(s+1)^2,\ B=\frac{1}{s}(1+k_4),\ D=\frac{K_3}{2s},\ C=\frac{1}{2}BD. \tag{15.35}$$

We see that introduction of anomalous interaction into Schrödinger-like equations does not change the structure of the approximate Hamiltonian but makes it possible to correct in a Galilei-invariant way the values of coefficients of the terms representing dipole, spin-orbit, etc. interactions. As to the LHH equations the anomalous interaction leads to principal changes of the corresponding approximate Hamiltonian which now includes the terms corresponding to spin-orbit coupling while the minimal interaction leads to the Pauli Hamiltonian (15.14).

Summarizing, we can say that Galilei-invariant wave equations give a wide possibilities in description of arbitrary spin particles in an external field. In particular such equations give a correct description of Pauli and spin-orbit couplings and predict the correct value $g=1/s$ of the gyromagnetic ratio. In Chapters 6 and 7 we use Galilei-invariant equations to solve concrete physical problems of one- and two-particle interactions.

4. NONGEOMETRIC SYMMETRIES

We show the symmetry of the basic equations of quantum physics is wider than the classical symmetry described in Chapter 1. Using the approach proposed in [113, 116, 122] we find IAs of the Dirac, Maxwell, KGF, Schrödinger and other equations in the classes of higher order differential and integro-differential operators. The corresponding SOs form wide Lie algebras and superalgebras including the Poincaré or Galilei algebra as a subalgebra.

16. HIGHER ORDER SOs OF THE KGF AND SCHRÖDINGER EQUATIONS

16.1. The Generalized Approach to Studying Symmetries of Partial Differential Equations

In Chapter 1 and Section 11 we studied symmetries of the basic equations of quantum physics in respect to continuous groups of transformations. This symmetry can be defined as an invariance under a finite dimension Lie algebra whose basis elements belong to the class M_1 (i.e., the class of first order differential operators). It is evident these symmetries do not exhaust all the invariance properties of the equations considered since a priori we do not take into account IAs including differential operators of higher orders.

Here we study symmetries of the fundamental equations of quantum theory in frames of a more general approach than the classical Lie method. The basic idea of this approach (called non-Lie in the following) is that the class of SOs can be essentially extended by including differential operators of the second, third, .. etc. orders and even integro-differential operators. We call them higher-order SOs.

A well-known example of a symmetry under such an extended class of operators is the invariance of the Schrödinger equation for the hydrogen atom under the algebra *AO(4)* established by Fock [103, 104]. Other examples can be found in [6, 39, 228], a good treatment of higher symmetries is present in Olver's book [350].

Higher order SOs give an information about hidden symmetries of equations including Lie-Bäclund symmetries [422] and supersymmetry [159, 327]. These operators can be used to construct new conservation laws which cannot be found in the classical Lie approach. A very interesting application of these operators is a description of coordinate systems in which solutions in separated variables exist [305].

An effective algorithm for finding higher order SOs and the corresponding IAs is suggested in [116,122]. Here we discuss this algorithm briefly and give basic definitions.

Consider an arbitrary linear system of partial differential equations

$$L\psi(x)=0 \tag{16.1}$$

where L is a linear differential operator, ψ is a multicomponent function. As in Subsection 1.2 we say an operator Q is a SO of the system (16.1) if it transforms solutions into solutions. In other words a SO satisfies by definition the condition

$$[Q, L]\psi=0 \tag{16.2}$$

where ψ is an arbitrary solution of the equation (16.1).

In Chapter 1 we suppose that SOs belong to the class M_1 and thus describe Lie symmetries. Here we decline this supposition and search for SOs of arbitrary order. Besides we impose two types of restrictions on the SOs considered.

(1) We require the SOs form a finite-dimensional Lie algebra*, i.e., satisfy the relations

$$[Q_A, Q_B]=f_{ABC}Q_C \tag{16.3}$$

where f_{ABC} are structure constants. It will be shown further on this way makes it possible to find new wide IAs of equations of quantum physics.

The condition (16.3) is a very substantial restriction selecting subsets having the structure of a Lie algebra from a set of SOs which is infinite in general.

(2) We consider SOs belonging to the class of differential operators of order n (class M_n) where n is fixed. Generally speaking such SOs do not form a finite-dimensional Lie algebra, but sometimes they have another interesting algebraic structures, e.g., they can form a basis of a superalgebra.

To investigate non-Lie symmetries we use various combinations of the restrictions (1), (2) (i.e., impose one of them or both them). In some cases we decline these restrictions and search for a complete set of SOs of arbitrary order.

The principal question arising while investigating symmetries in frames of non-Lie approach is the following: how are SOs of the equation considered to be constructively calculated? By generalizing results of calculations of IAs of equations of quantum mechanics it is possible to formulate the following algorithm:

(1) by means of a nondegenerated transformation the system of partial differential equations is reduced to diagonal or quasidiagonal form, i.e., the maximal splitting of this system into independent subsystems is carried out; (2) the IA of the

* According to Theorem 1.1 SOs belonging to the class M_1 always form a Lie algebra which can be either finite or infinite dimensional. But even second order SO do not satisfy (16.3) in general

transformed equation is found and the kind of a representation of the corresponding Lie algebra is determined; (3) by means of the inverse transformation the explicit form of the basis elements of the IA of the original equation is found.

The algorithm is based on one of the most fruitful and effective ideas in the theory of differential equations, i.e., using of transformations of independent and dependent variables.

While realizing the algorithm an important role is played by the concept of the symbol of an operator L which can be defined by means of the Fourier transform

$$\hat{L}\psi(x)=(2\pi)^{-3/2}\int\limits_{D(\boldsymbol{p})} d^3pL\exp(i\boldsymbol{p}\cdot\boldsymbol{x})\tilde{\psi}(x_0,\boldsymbol{p}) \tag{16.4}$$

where $\psi\in C_0^\infty(R_4)$, $\tilde{\psi}(x_0,\boldsymbol{p})=F\psi(x)$ is a Fourier transform of $\psi(x)$, F is the unitary Fourier operator mapping a vector of the Hilbert space H into $\tilde{H}$, $\tilde{\psi}(x_0,\boldsymbol{p})\in\tilde{H}$, $D(\boldsymbol{p})$ is a domain of integration. A formal connection between the operator L and its symbol $\tilde{L}$ can be given by the following relation

$$\hat{L}=F^{-1}LF,\quad L=F\hat{L}F^{-1}. \tag{16.5}$$

Relations (16.4), (16.5) can be used to realize the first step of the algorithm. Indeed if the symbol of the operator L is a matrix with variable coefficients (and this occurs for the majority of equations of mathematical physics) the system (16.4) can be reduced in principle to a system of noncoupled integral equations by transforming L to diagonal or Jordan form.

It should be noted that a realization of the algorithm present above for concrete equations of physics or mechanics, as a rule, is not a simple problem. Sometimes it is easier to use other ways considered in the following.

We saw in Chapter 1 that a description of first order SOs is based on using the explicit form of the Killing vectors corresponding to the space of independent variables. To describe higher order SOs it is necessary to calculate more complicated structures which we call generalized Killing tensors of order s. In the following we present a definition of these tensors as complete sets of solutions of some over-determined system of partial differential equations of order s. The explicit form of the generalized Killing tensors used in this book is given in Appendix 2.

16.2. SOs of the KGF Equation

Like in Chapter 1 we start with the KGF equation, the simplest equation of relativistic quantum theory. Using notations and definitions of Section 1 we formulate the problem of description of higher order SOs for this equation.

Let M_n be the class of differential operators of order n defined on F. Then any operator $Q_n \in M_n$ can be represented in the form

$$Q_n = \sum_{j=0}^{n} Q_j, \quad Q_j = h^{a_1 a_2 \ldots a_j} \frac{\partial^j}{\partial x_{a_1} \partial x_{a_2} \ldots \partial x_{a_j}}, \quad h^{a_1 a_2 \ldots a_j} \in F, \ a_i = 0,1,2,3. \tag{16.6}$$

In a complete analogy with Definition 1.1 we formulate the following

DEFINITION 16.1. A linear differential operator (16.6) is a SO of the KGF equation in the class M_n (or a SO of order n) if

$$[Q_n, L] = \alpha_Q L, \quad \alpha_Q \in M_{n-1} \tag{16.7}$$

where L is the operator (1.3).

In the case n=1 such defined SOs reduce to the generators of the Lie group considered in Section 1. The SOs of order n>1 describe generalised (non-Lie) symmetries of the KGF equation. The problem of description of the complete set of SOs of order n reduces to finding the general solution of the operator equation (16.7).

It is convenient to represent all the operators (16.7) as sums of j-multiple anticommutators [328,342]

$$\begin{gathered} Q_n = \sum_{j=0}^{n} Q_j, \quad \alpha_Q = \sum_{j=0}^{n-1} \alpha_{Q_j}, \\ \alpha_q L = \frac{1}{4}\left[\left[\alpha_Q, \partial_\mu\right]_+, \partial^\mu\right]_+ - \frac{1}{2}\left[\partial^\mu \alpha_Q, \partial_\mu\right]_+ + \frac{1}{4}(\partial^\mu \partial_\mu \alpha_Q) \end{gathered} \tag{16.8}$$

where

$$Q_j = \left[\left[\ldots\left[K^{a_1 a_1 \ldots a_j}, \partial_{a_1}\right]_+, \partial_{a_2}\right]_+, \ldots \partial_{a_j}\right]_+, \tag{16.9a}$$

$$\alpha_j = \left[\left[\ldots\left[\alpha^{a_1 a_2 \ldots a_j}, \partial_{a_1}\right]_+, \partial_{a_2}\right]_+, \ldots, \partial_{a_j}\right]_+, \tag{16.9b}$$

$K^{\cdots}$ and $\alpha^{\cdots}$ are unknown symmetric tensors of rank j. Transferring differential operators to the right we can reduce Q_n of (16.8) to the equivalent form (16.6).

Substituting (16.8) into (16.7) and equating the coefficients of the same differential operators we come to the following equation for æ≠0:

$$\partial^{(a_{j+1}} K^{a_1 a_2 \ldots a_j)} = 0, \quad j = 1,2,\ldots,n, \tag{16.10}$$

$$\alpha^{a_1 a_2 \ldots a_j} = 0 \tag{16.11}$$

where symmetrization is imposed over the indices in parenthesis:

$$\partial^{(a_{j+1}} K^{a_1 a_2 \ldots a_j)} = \partial^{a_{j+1}} K^{a_1 a_2 \ldots a_j} + \partial^{a_1} K^{a_{j+1} a_2 \ldots a_j} + \partial^{a_2} K^{a_1 a_{j+1} a_3 \ldots a_j} + \ldots + \partial^{a_j} K^{a_1 a_2 \ldots a_{j-1} a_{j+1}}.$$

So the problem of describing SOs of order n for the KGF equation reduces to solving the noncoupled system of partial differential equations (16.10). In the case n=1 this system reduces to the Killing equations (1.9).

We will see further on that formula (16.10) gives the principal equation appearing while investigating higher order SOs of the basic equations of quantum theory. The equations of the type (16.10) were considered in [408]. We call a symmetric tensor $K^{\cdot\cdot\cdot}$ satisfying (16.10) a Killing tensor of valence j and order 1, the sense of the last term will be clear in the following.

The general solution of (16.10) is given in Appendix 2. The number N^j of independent solutions is

$$N^j = \frac{1}{4} C^3_{j+3} C^3_{j+4} \tag{16.12}$$

where C^a_b is a number of combinations from b elements by a ones.

To present the explicit form of $F^{\cdot\cdot\cdot}$ it is convenient to define a special kind of tensors (which we call basic tensors) $\lambda^{a_1 \ldots a_c [a_{c+1} b_1] \ldots [a_j b_{j-l}]}$ having the following properties:

(1) symmetry and zero traces in respect with the indices $a_1, a_2, \ldots, a_c$;

(2) symmetry in respect with a permutation of pairs of indices $[a_{l+i} b_i]$ and $[a_{l+j} b_j]$, $i,j=1,2,\ldots,n-c$;

(3) skew-symmetry in respect with the indices $[a_{c+i} b_i]$;

(4) a convolution in respect with any triplet of indices with absolutely skew-symmetric tensor $\varepsilon_{\mu\lambda\sigma\rho}$ is equal to zero.

Basic tensors are reducible since in general they have nonzero traces over pairs of indices a_l, a_m if $l>c$ and (or) $m>c$.

LEMMA 16.1. The general solution of (16.10) can be represented in the form

$$K^{a_1 a_2 \ldots a_j} = g^{(a_{j-1} a_j} K^{a_1 a_2 \ldots a_{j-2})} + \sum_{l=0}^{j} \lambda^{a_1 a_2 \ldots a_l [a_{l+1} b_1] \ldots [a_j b_{j-l}]} x_{b_1} x_{b_2} \ldots x_{b_{j-l}} \tag{16.13}$$

where $K^{a_1 a_2 \ldots a_{j-2}}$ is the general solution of (16.10) for $j \to j$-2, $\lambda^{\cdot\cdot\cdot}$ are arbitrary basic tensors independent on x.

PROOF reduces to direct verification the fact that the tensor (16.13) satisfies (16.10) and to calculation the number of independent components of $\lambda^{\cdot\cdot\cdot}$ which is equal to N^j-N^{j-2} [328,342]. ■

The first term in the r.h.s. of (16.13) corresponds to a SO of order j which reduces to a SO of order j-2 on the set of solutions of (1.1) and so can be neglected. Then substituting (16.13) into (16.9) we obtain

$$Q^j = \sum_{l=0}^{j} \lambda^{a_1 a_2 \ldots a_l [a_{l+1} b_1] \ldots [a_j b_{j-l}]} P_{a_1} P_{a_2} \ldots P_{a_l} J_{a_{c+1} b_1} J_{a_{l+2} b_2} \ldots J_{a_j b_{j-l}} \tag{16.14}$$

where P_a, J_{ab} are the Poincaré group generators (1.6).

Thus we have obtained a complete set of SOs of order n for the KGF equation in the form (16.14) where $j \le n$. All such operators belong to the enveloping algebra of the algebra $AP(1,3)$. The number N^j of linearly independent operators Q^j is

$$N_j = \hat{N}_j - \hat{N}_{j-2} = \frac{1}{4!}(j+1)(j+2)(2j+3)(j^2+3j+4), \tag{16.15}$$

and the total number of SOs of order $j \le n$ is

$$N_{(n)} = \sum_{j=0}^{n} N_j = \frac{2}{3!4!}(n+1)(n+2)^2(n+3)(n^2+4n+6).$$

In conclusion we note that the SO (16.14) can be represented in such a form which includes irreducible arbitrary parameters only. Indeed, expanding $\lambda^{a_1 \ldots}$ in irreducible tensors (i.e., tensors having the properties (1)-(4) and besides being traceless over any pair of indices) we obtain

$$Q_j = \sum \lambda^{a_1 a_2 \ldots a_\mu [a_{\mu+1} b_1][a_{\mu+2} b_2] \ldots [a_{j'} b_{j'-\mu}]} P_{a_1} P_{a_2} \ldots P_{a_{\mu_1}} \times$$

$$\times I_{a_{\mu_1+1} a_{\mu_1+2}} I_{a_{\mu_1+3} a_{m_1+4}} \ldots I_{a_{\mu-2} a_{\mu-1}} (J_{a_\mu b} P^b)^{\varepsilon_{\mu_2}} (J_{ac} J^{ac})^{\{\mu_2/2\}} \times J_{a_{\mu+1} b_1} J_{a_{\mu+2} b_2} \ldots J_{a_{j'} b_{j'-\mu}}$$

where

$$I_{ab} = J_{an} J^n{}_b, \quad \mu = \mu_1 + \varepsilon_{\mu_2} + 2\mu_3, \quad j' = j - \mu_2, \quad \varepsilon_{\mu_2} = \frac{1}{2}[1 + (-1)^{\mu_2}],$$

$\{A\}$ is the entire part of A, and the summation is imposed over all the values of μ_a satisfying the condition

$0 \le \mu_3 \le \{j/2\}$; $0 \le \mu_2 \le 2\{(j-2\mu_3)/2\}$; $0 \le \mu_1 \le j - 2\mu_3 - \mu_2$.

16.3. Hidden Symmetries of the KGF Equation

Let us demonstrate the higher order SOs of the KGF equation possesses nontrivial algebraic structures. Restricting ourselves to the SOs of the second order we see that there are 49 such operators having the form

$$P_a, \quad J_{ab}, \quad P_{ab} = p_a p_b - \frac{1}{4} g_{ab} m^2, \quad F_a = J_{ab} P^b, \tag{16.17.a}$$

$$\begin{aligned} &J = J_{\mu\nu} J^{\mu\nu}, \quad K_{ac} = J_{ab} J^b{}_c + J_{cb} J^b{}_a - \frac{1}{2} g_{ac} J^2, \\ &K^{[ab][cd]} = [J_{ab}, J_{cd}]_+ - \frac{1}{4} g_{bc} K_{ad} - \frac{1}{16} g_{bc}\, g_{ad} J^2. \end{aligned} \tag{16.17b}$$

where the Latin indices run from 0 to 3.

In contrast with the first order SOs of (1.6) the operators (16.17) do not form a Lie algebra. But the subset (16.17a) includes bases of some Lie algebras, satisfying the following commutation relations

$$
\begin{aligned}
&[P_a, P_{ab}]=0, \quad [P_{ab}, P_{cd}]=0,\\
&[P_{ab}, J_{cd}]=i(g_{ac}P_{bd}+g_{bd}P_{ac}-g_{ad}P_{bc}-g_{bc}P_{ad}),\\
&[F_a, F_b]=-im^2J_{ab}, \quad [F_a, J_{bc}]=i(g_{ab}F_c-g_{ac}F_b],\\
&[F_a, P_b]=i(P_{ab}-\frac{3}{4}m^2g_{ab}),\\
&[F_a, P_{bc}]=2iP_aP_bP_c-im^2g_{ab}P_c+im^2g_{ac}P_b.
\end{aligned}
\tag{16.18}
$$

Using (16.18) and (1.14) we can indicate two finite-dimensional Lie algebras

$$A_{10}=\{J_{\mu\nu},\ F_\mu\}, \quad A_{19}=\{P_\mu,\ P_{\mu\nu},\ J_{\mu\nu}\}$$

and the infinite dimension Lie algebra

$$A_\infty=\{F_\mu,\ J_{\mu\nu},\ P_\mu,\ P_\mu P_\nu,\ P_\mu P_\nu P_\lambda,\ P_\mu P_\nu P_\lambda P_\sigma,\ \ldots\}.$$

The algebra A_{10} is isomorphic to *AO(2,3)*.

Let us consider still hidden symmetry of the KGF equation. Denoting

$$p_0\psi=m\chi, \quad \varphi=column(\psi,\chi) \tag{16.19}$$

we can rewrite it in the form

$$p_0\varphi=[\sigma_1 m+(\sigma_1-i\sigma_2)\frac{p^2}{2m}]\varphi. \tag{16.20}$$

Now we can investigate symmetries of (16.20). Any symmetry transformation for (16.20) can be considered as a hidden symmetry of the KGF equation [121] inasmuch as any solution of (16.20) satisfies (1.1).

LEMMA 16.2. The equation (16.20) is invariant under the algebra *AP(1,3)* whose basis elements are

$$P_\mu=p_\mu, \quad J_{ab}=x_a p_b-x_b p_a, \quad J_{0a}=x_0 p_a-x_a p_0+(\sigma_1-i\sigma_2)\frac{p_a}{m}.$$

Proof reduces to a direct verification. ■

The operators (16.21) generate the following finite group transformations [170]

$$\varphi'(x')=\frac{1}{2}\left[(1+\sigma_3)\cosh\lambda+1-\sigma_3+(\sigma_1+i\sigma_2)\sinh\lambda\frac{\lambda\cdot\boldsymbol{p}}{m\lambda}\right]\varphi(x)$$

where x' is given in (1.32), (1.39). These transformations represent the group of a hidden symmetry of the KGF equation, mixing wave function and its derivations, see

(16.19).

We remind that the KGF equation admits also the algebras and groups of hidden symmetries connected with antilinear transformations, refer to Subsecrion 1.7.

16.4. Higher Order SOs of the d'Alembert Equation

Consider the KGF equation in the special case m=0. The problem of description of higher order SOs of this equation can be solved in a complete analogy with Subsection 16.2. Substituting (16.8) into (16.7) and setting m=0 in L of (1.3) we come to the following determining equations (instead of (16.10)) [328]

$$\partial^{(a_{j+1}}K^{a_1a_2\dots a_j)} = \frac{1}{j+1}\partial^b K^{b(a_1a_2\dots a_{j-1}}g^{a_ja_{j+1})} = 0, \qquad (16.21)$$

$$\alpha^{a_1a_2\dots a_{j-1}} = \frac{1}{j+1}\partial^b K^{ba_1a_2\dots a_{j-1}} \qquad (16.22)$$

where $K^{a_1\dots}$ and $\alpha^{a_1\dots}$ are symmetric traceless tensors, the first is called a conformal Killing tensor of valence j and order 1.

The general solution of (16.21) is given in Appendix 2. The number N'_j of linearly independent solutions is equal to

$$N'_j = \frac{1}{12}(j+1)^2(j+2)^2(2j+3), \qquad (16.23)$$

each solution corresponds to the SO of the d'Alembert equation. Substituting (A.2.10), (A.2.14) into (16.9) we obtain linearly independent SOs in the form

$$Q_j = \sum_{c_1,c_2,c_3} \lambda^{a_1a_2\dots a_{c_1+c_2}[a_{c_1+c_2+1}b_1]\dots[a_{j-c_3}b_{j-c_1-c_2-c_3}]}\times$$
$$\times P_{a_1}P_{a_2}\dots P_{a_{c_1}}K_{a_{c_1+1}}K_{a_{c_1+2}}\dots K_{a_{c_1+c_2}}(D)^{c_3}J_{a_{c_1+c_2+1}b_1}J_{a_{c_1+c_2+2}b_2}\dots J_{a_{j-c_3}b_{j-c_1-c_2-c_3}}, \qquad (16.24)$$

where $\lambda^{a_1\dots}$ are arbitrary basic tensors, P_a, K_a, J_{ab}, D (a,b=1,2,3) are the generators of the conformal group given in (1.6), (1.16), $0\le c_1+c_2\le j-c_3$, c_3=0,1.

So we have found a complete set of SOs of order n for the d'Alembert equation in the form (16.24). All these operators belong to the enveloping algebra of the algebra *AC(1,3)*.

We note that higher order SOs of a system of partial differential equations do not belong in general to the enveloping algebra of the Lie algebra of this system symmetry group (see Section 17).

In conclusion we consider briefly the problem of description of higher order SOs for the generalized KGF equation in the pseudo-Euclidean space of dimension

$p+q=M$:

$$(p_\mu p_\nu g^{\mu\nu} - m^2)\psi = 0, \quad \mu,\nu = 1,2,...p+q, \tag{16.25}$$

where

$$g^{\mu\nu} = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu \leq p, \\ -1, & p < \mu = \nu \leq p+q. \end{cases}$$

The problem of description of SOs of the equation (16.25) is formulated in complete analogy with the above. The generalization the above results to the case of an arbitrary number of variables is almost trivial so we restrict ourselves to presenting the number of linearly independent SOs of order j for the equation (16.25) in M-dimensional space [328, 342]

$$N_j^{\ M} = \frac{1}{M} C_{j+M-1}^{M-1} C_{j+M}^{M-1}. \tag{16.26}$$

The total number of SOs of order j, $0 \leq j \leq M$ is

$$N(n,M) = \frac{2n^2 + 2nM + M(M-1)}{M(M-1)} C_{n+M-2}^{M-2} C_{n+M-1}^{M-2}. \tag{16.27}$$

In particular for $M=2,3$

$$N_j^2 = 2j+1, \quad N(n,2) = (n+1)^2, \tag{16.28a}$$

$$N_j^3 = \frac{1}{3}(j+1)(2j^2+4j+3), \quad N(n,3) = \frac{1}{6}(n+1)(n+2)(n^2+3n+3). \tag{16.28b}$$

Formulae (16.28a) and (16.28b) give the numbers of SOs for the Helmholtz and inhomogeneous Laplace equation. For the homogeneous Laplace equation we have

$$\tilde{N}_j^3 = \frac{1}{3}(j+1)(2j+1)(2j+3), \ \tilde{N}(n,3) = \frac{1}{6}(n+1)(n+2)(2n^2+6n+3). \tag{16.29}$$

16.5. SOs of the Schrödinger Equation

Here we find a complete set of SOs of order n for the Schrödinger equation. Without loss of generality we represent such an operator in the form (16.9a) where indices values run from 1 to 3.

The SOs considered do not include p_0 because it can be expressed via $p^2/2m$ on the set of solutions of the equation (11.3).

Substituting L (11.3) and Q (16.8) into (16.7) we come to the following system of coupled equations [330]

$$\partial^{(a_{j+1}}K^{a_1a_2\dots a_j)} = -2m(j+1)\dot{K}^{a_1a_2\dots a_{j+1}}, \quad j=0,1,\dots n-1, \tag{16.30a}$$

$$\partial^{(a_{n+1}}K^{a_1a_2\dots a_n)} = 0, \quad \dot{K}=0, \quad j=0 \tag{16.30b}$$

where the dots denote the time derivatives.

So to find all the SOs of order n for the Schrödinger equation it is necessary to obtain the general solution of (16.30). The system (16.30) is over-determined and can be solved by transition to the set of noncoupled differential consequences. We make it as follows.

Consider (16.30a) for $j=n$-1:

$$\partial^{a_n}K^{a_1a_2\dots a_{n-1}} = -2mn\dot{K}^{a_1a_2\dots a_n}. \tag{16.31}$$

Using (16.30b) we can obtain the following differential consequence of (16.31):

$$\partial^{(a_{n+1}}\partial^{a_n}K^{a_1a_2\dots a_{n-1})} = 0. \tag{16.32}$$

Then differentiating (16.32) in respect with t and considering (16.30a) for $j=n$-2 we obtain

$$\partial^{(a_{n+1}}\partial^{a_n}\partial^{a_{n-1}}K^{a_1a_2\dots a_{n-2})} = 0,$$

and for $j=n$-s+1

$$\partial^{(a_{j+1}}\partial^{a_{j+2}}\dots\partial^{a_{j+s}}K^{a_1a_2\dots a_j)} = 0, \quad s=n+1-j. \tag{16.33}$$

It follows from (16.30), (16.33) that

$$\frac{\partial^{j+1}}{(\partial t)^{j+1}}K^{a_1a_2\dots a_j} = 0. \tag{16.34}$$

So starting with (16.10) we come to the system of noncoupled equations (16.33), (16.34).

The equations (16.33) are direct generalizations of the first-order equations (16.10) determining the Killing tensor. We call solutions of (16.33) *generalized Killing tensors of valence j and order s* [328, 342]. The general solution of (16.33), (16.34) is

$$K^{a_1a_2\dots a_j} = \sum_{\alpha=0}^{j} K_{s\alpha}^{a_1a_2\dots a_j} t^{\alpha}, \quad s=n+1-j \tag{16.35}$$

where $K_{s\alpha}^{a_1a_2\dots a_j}$ is a Killing tensor of valence j and order s. Substituting (16.35) into (16.30) we come to the equation

$$\alpha K_{s\alpha}^{a_1a_2\dots a_j} = 2mj\partial^{(a_j}K_{s+1\alpha-1}^{a_1a_2\dots a_{j-1})}, \quad \alpha\neq 0, \quad s=n-j. \tag{16.36}$$

Let we know the general solution of (16.35) for a fixed value of $j=j_0$-1 then the relations (16.35), (16.36) define an explicit form of the general solution for $j=j_0$ up to an arbitrary generalized Killing tensor $K_{s0}^{a_1a_2...a_j}$ of order $s=n-j_0+1$. So the number N_n of linearly independent solutions of the system (16.30) coincides with the total number of independent components of generalized Killing tensor of valence j and order $s=n-j+1$, $0\le j\le n$. According to (A.2.4)

$$N_n=\sum_{j=0}^{n} N_{sj}^3=\frac{1}{3!4!}(n+1)(n+2)^2(n+3)^2(n+4). \tag{16.37}$$

Thus the Schrödinger equation admits N_n linearly independent SOs of orders $j\le n$. Subtracting from (16.37) the number of SOs of orders $j'\le n-1$ we obtain the number of SOs of order n

$$\tilde{N}_n=N_n-N_{n-1}=\frac{1}{4!}(n+1)(n+2)^3(n+3). \tag{16.38}$$

The corresponding SOs can be represented in the form

$$Q_n=\sum_{c=0}^{n}\sum_{k=0}^{n-c}\lambda^{a_1a_2...a_cb_1b_2...b_{n-c}}P_{a_1}P_{a_2}...P_{a_k}G_{a_{k+1}}G_{a_{k+2}}...G_{a_c}J_{b_1}J_{b_2}...J_{b_{n-c}} \tag{16.39}$$

where P_a, G_a and J_a are generators of the Galilei group of (11.5), $\lambda^{a_1...}$ are arbitrary tensors symmetric in respect to permutations $a_i\longleftrightarrow a_j$ and $b_k\longleftrightarrow b_l$, besides

$$\lambda^{a_1a_2...a_cb_1b_2...b_{n-c}}\delta_{a_1b_1}=0. \tag{16.40}$$

Summing up the numbers of independent components of $\lambda^{a_1...}$ we make sure that the number of linearly independent SOs of (16.39) coincides with (16.38).

16.6. Hidden Symmetries of the Schrödinger equation

So we have calculated all the linearly independent SOs of arbitrary order for the Schrödinger equation. All such operators belong to the enveloping algebra of the Galilei algebra.

Here we demonstrate this enveloping algebra has a very interesting structure and includes wide Lie algebras. But before this we consider the other hidden symmetries generated by *antilinear* transformations.

In Subsection 11.6 we were discussing symmetries of the Schrödinger equation in respect with the antilinear transformation T of (11.37). Then in accordance with Lemma 1.1 (refer to Subsection 1.7) this equation is invariant under the algebra *AO(1,2)* whose basis elements are $\{T,R,TR\}$ where R is the evident symmetry transformation (1.55). Moreover this symmetry can be extended by including the combined transformations consisting of the space reflection P of (11.37). As a result we come to the following IA of the Schrödinger equation

$$\{T,R,TR,PT,PR,PTR\}.$$

These operators satisfy the following commutation relations

$[T,R]=2TR$, $[T,TR]=2R$, $[T,PT]=0$, $[T,PR]=2TPR$, $[T,PTR]=-2PR$,

$[R,TR]=-2T$, $[R,PT]=-2PTR$, $[R,PR]=0$, $[R,PTR]=-2PT$,

$[TR,PT]=2PR$, $[TR,PR]=-2PT$, $[TR,PTR]=-2P$,

$[PT,PR]=2TR$, $[PT,PTR]=2R$, $[PR,PTR]=-2T$,

and thus form the Lie algebra isomorphic to *AO(2,2)*.

Taking into account that the squares of the considered SOs are proportional to the unit operator (up to a sign) it is not difficult to find the corresponding symmetry group of the Schrödinger equation. We restrict ourselves by presenting one parameter transformations generated by these symmetries and belonging to the group *O(2,2)*:

$$\psi(t,\boldsymbol{x})\to\cos\theta_1\psi(t,\boldsymbol{x})+i\sin\theta_1\psi(t,\boldsymbol{x}),$$

$$\psi(t,\boldsymbol{x})\to\cos\theta_2\psi(t,\boldsymbol{x})+i\sin\theta_2\psi(t,-\boldsymbol{x}),$$

$$\psi(t,\boldsymbol{x})\to\cosh\theta_3\psi(t,\boldsymbol{x})+\sinh\theta_3\psi(t,-\boldsymbol{x}),$$

$$\psi(t,\boldsymbol{x})\to\cosh\theta_4\psi(t,\boldsymbol{x})+i\sinh\theta_4\psi(-t,\boldsymbol{x}),$$

$$\psi(t,\boldsymbol{x})\to\cosh\theta_5\psi(t,\boldsymbol{x})+\cosh\theta_5\psi(-t,-\boldsymbol{x}),$$

$$\psi(t,\boldsymbol{x})\to\cosh\theta_6\psi(t,\boldsymbol{x})+i\sinh\theta_6\psi(-t,-\boldsymbol{x})$$

where θ_a are real parameters.

But let us return to linear SOs considered in the previous subsection. Restricting ourselves to the second order SOs we obtain from (16.39) the following complete set of them

$$P_{ab}=P_aP_b,\quad G_{ab}=G_aG_b,\quad Q_{ab}=\frac{1}{2}(P_aG_b+G_aP_b), \tag{16.41a}$$

$$F_{ab}=J_aJ_b+J_bJ_a,\quad F_a=\varepsilon_{abc}P_bJ_c,\quad G_a=\varepsilon_{abc}G_bJ_c, \tag{16.41b}$$

$$L_{ab}=P_aJ_b+P_bJ_a,\quad N_{ab}=G_aJ_b+G_bJ_a,\quad (a,b,c)=(1,2,3), \tag{16.41c}$$

including 40 elements (compare with (16.38)). In contrast with G_{ab}, Q_{ab} and F_{ab} the tensors L_{ab} and N_{ab} are traceless.

The SOs (16.41) do not form a closed Lie algebra. But some subsets of these operators are closed in respect with commutation and so form the bases of Lie algebras. One of such subsets is given by the operators $\boldsymbol{P}^2$, $\boldsymbol{G}^2$ and $\boldsymbol{P}\cdot\boldsymbol{G}$ forming a basis of the algebra *AO(1,2)*, see (11.12), (11.13). The more extended Lie algebra is formed by the operators (16.41a) together with P_a, G_a and J_a inasmuch as they satisfy the following commutation relations

$$
\begin{aligned}
&[P_{ab}, P_{cd}] = [G_{ab}, G_{cd}] = 0,\\
&[P_{ab}, G_{cd}] = im(\delta_{ac}Q_{bd} + \delta_{ad}Q_{bc} + \delta_{bc}Q_{ad} + \delta_{bd}Q_{ac}),\\
&[P_{ab}, Q_{cd}] = im(\delta_{ac}P_{bd} + \delta_{ad}P_{bc} + \delta_{bc}P_{ad} + \delta_{bd}P_{ac}),\\
&[G_{ab}, Q_{dc}] = -im(\delta_{ac}Q_{bd} + \delta_{ad}Q_{bc} + \delta_{bc}G_{ad} + \delta_{bd}G_{ac}),\\
&[Q_{ab}, Q_{cd}] = im(\delta_{ac}Q_{bd} + \delta_{bd}Q_{ac} - \delta_{ad}Q_{bc} - \delta_{bc}Q_{ad}),\\
&[P_a, P_{bd}] = 0, \quad [P_a, Q_{bd}] = im(\delta_{ab}P_d + \delta_{ad}P_b),\\
&[P_a, G_{bd}] = im(\delta_{ab}G_d + \delta_{ad}G_b),\\
&[J_a, R_{bd}] = i(\varepsilon_{abk}R_{kd} + \varepsilon_{adk}R_{kb}), \quad R_{ab} = (Q_{ab}, G_{ab}, P_{ab}).
\end{aligned}
\qquad (16.42)
$$

The relations (11.6b)-(11.6d), (16.42) define the 27-dimensional Lie algebra A_{27} . This algebra includes very interesting subalgebras listed in the following table.

Table 16.1

Main subalgebras of the algebra A_{27}	*Basis elements*
$AG_2(1,3)$	$P_0=P^2/2m$, P_a, J_a, G_a, $\boldsymbol{P}\cdot\boldsymbol{G}$, $\boldsymbol{G}^2$
$AO(1,2)$	$\boldsymbol{P}^2$, $\boldsymbol{G}^2$, $\boldsymbol{P}\cdot\boldsymbol{G}$
$AIGL(3)$	P_a, J_a, Q_{ab}
$AP(1,2)$	P_a, Q_{12}, Q_{13}, J_1

Thus, the Schrödinger equation has very extensive hidden symmetries including the Lie algebras of the groups of general inhomogeneous linear transformations *IGL(3)* and of the generalized Poincaré group in three-dimensional space. These algebras are realized in the class of second-order differential operators and can be used for various purposes including separation of variables and construction of exact solutions of linear and non-linear problems based on the Schrödinger equation.

Considering SOs of order $n>2$ we can find more extensive IAs of the Schrödinger equation. Here we represent two of them

$$
\begin{aligned}
&L_1 \supset \left\{P_a,\ J_a,\ G_a,\ Q_{ab},\ G_{a_1a_2\ldots a_i} = G_{a_1}G_{a_2}\ldots G_{a_i}\right\}, \quad i \le n,\\
&L_2 \supset \left\{P_a,\ J_a,\ G_a,\ Q_{ab},\ P_{a_1a_2\ldots a_i} = P_{a_1}P_{a_2}\ldots P_{a_i}\right\}
\end{aligned}
\qquad (16.43)
$$

where Q_{ab} is the operator (16.41a), n is an arbitrary integer.

It is not difficult to verify the operators (16.43) form finite-dimensional Lie algebras for any fixed n.

Symmetries of the Schrödinger equation under integral transformations are considered in [165, 338], where so exotic IAs are found as *AO(1,4)* and *AC(1,3)*.

16.7. Symmetries of the Quasi-Relativistic Evolution Equation

Let us consider briefly symmetries of a four-order partial differential equation which can be interpreted as a quasi-relativistic generalization of the Schrödinger equation. Namely we consider the equation

$$i\frac{\partial}{\partial x_0}\psi=H\psi,\quad H=a_0 m+\frac{p^2}{2m}-a_4\frac{p^4}{8m^3}, \tag{16.44}$$

where a_0, a_4, and m are real constants.

If $a_0=a_4=0$ then (16.44) coincides with the Schrödinger equation for a free particle. For $a_0=a_4=1$ the Hamiltonian H includes three the first terms of the Taylor series of the relativistic Hamiltonian $H'=(m^2+p^2)^{1/2}$.

The maximal IA of the equation (16.44) in the class M_1 is the eight-dimension Lie algebra A_8 including the basis elements P_0, P_a, J_a and M of (11.5). Thus this equation is invariant under neither Galilei nor Lorentz transformations. This does not mean however that symmetries of (16.44) are exhausted by the algebra A_8. This equation is invariant under the 20-dimensional Lie algebra including P_0, P_a, J_a and M of (11.5) and the following higher order SOs [145]

$$\begin{gathered}V_a=i[H,x_a]=\frac{1}{m}\left(1-a_4\frac{p^2}{2m^2}\right)p_a,\\ G_a=(x_0V_a-x_a)m,\quad R_{ab}=-\frac{a_4}{m}\left(p_ap_b+\frac{1}{2}\delta_{ab}p^2\right).\end{gathered} \tag{16.45}$$

Using the identities $[H,V_a]=[P_0,x_a]=0$ it is not difficult to make sure that the operators (16.45) satisfy the invariance conditions (1.5) (together with $L=i\partial/\partial x_0-H$) and the following commutation relations

$$[P_0,G_a]=imV_a,\quad [V_a,G_b]=im\left(R_{ab}-\delta_{ab}\frac{1}{m^2}M\right),$$

$$[J_a,R_{bc}]=i(\varepsilon_{abn}R_{cn}+\varepsilon_{acn}R_{bn}),\quad [P_0,R_{bc}]=0,$$

$$[J_a,V_b]=i\varepsilon_{abc}V_c,\quad [P_a,V_b]=[P_a,R_{bc}]=0,$$

$$[G_a,R_{bc}]=\frac{ia_4}{m}(\delta_{ab}P_c+\delta_{bc}P_a+\delta_{ac}P_b),$$

the commutation relations between P_0, P_a, J_a, M and G_a are given in (11.6).

We see the quasirelativistic evolution equation (16.44) is really invariant under the 20-dimensional Lie algebra generated by the SOs P_0, P_a, J_a, M of (11.5) and

V_a, G_a, R_{ab} of (16.45). Besides the SOs (16.45) are third-order differential operators and so they do not generate local transformations of the type (11.15). But it is possible to demonstrate (see [387]) that G_a generate nonlocal transformations of the following kind

$$\psi'(x')=\exp[im(x_a v_a+\frac{1}{2}x_0 v^2-\frac{1}{2}\frac{a_4}{m^3}x_0 v_a p_a p^2)]\psi(x) \tag{16.46}$$

where v_a are transformation parameters and x' are connected with x by the Galilei transformations (11.18). For a_4=0 formula (16.46) presents usual Galilei transformations of (11.27).

Let us demonstrate that if we interpret (16.44) as a motion equation for a particle of rest mass m then the complete mass M' depends on the velocity. Indeed, the corresponding quantum mechanical operator of velocity is the operator V_a of (16.45), and the classical analogues of the operators P_a and V_a are the momentum p_a and velocity v_a. In accordance with (16.45) the velocity depends on momentum in the following manner

$$v_a=\frac{p_a}{m}-\frac{1}{2}\frac{a_4}{m^3}p_a p^2. \tag{16.47}$$

But on the other hand we have according to the classical definition of velocity that

$$v_a=\frac{p_a}{M'}. \tag{16.48}$$

Substituting (16.48) into (16.47) we come to the cubic equation

$$\frac{M'}{m}-\frac{1}{2}a_4 v^2\left(\frac{M'}{m}\right)^3-1=0. \tag{16.49}$$

Solutions of this equation have the form

$$M'=\frac{3m}{W}\sin\left(\frac{1}{3}\arctan\frac{W}{\sqrt{1-W^2}}\right),\quad W=\sqrt{\left(\frac{3m}{2}\right)^3|a_4|}\ |v|. \tag{16.49}$$

It follows from (16.49) that the mechanics based on the equation (16.44) leads to the limiting velocity $v_{\lim}=\left[\left(\frac{3m}{2}\right)^3|a_4|\right]^{-1/2}$ like relativistic mechanics. Besides if $v\to v_{lim}$ then $M'\to(3/2)m$.

In conclusion we note that analogous analysis can be carried out for the equation (16.44) with a more general Hamiltonian $H=\sum_{n=0}^{N}a_{2n}p^{2n}$.

Approximate symmetries and solutions of wave equations are discussed in [389].

17. NONGEOMETRIC SYMMETRIES OF THE DIRAC EQUATION

17.1. The IA of the Dirac Equation in the Class M_1

The hidden symmetries of the Dirac equation are more interesting and complicated than considered above owing to the matrix structure of the corresponding SOs.

In Section 2 we found the maximal IA of the Dirac equation in the class M_1, it was isomorphic to the Lie algebra of the Poincaré group. It turns out this IA can be extended by including the SOs belonging to the class of first order differential operators with matrix coefficients. We denote this class by M_1.

For convenience let us rewrite the Dirac equation in the form

$$L\psi=0, \quad L=\gamma^{\mu} p_{\mu}-m. \tag{17.1}$$

Using the definitions (16.2) and the algebraic properties of the Dirac matrices (refer to (2.3)) it is not difficult to show that a linear differential operator

$$Q_A=A^{\mu} p_{\mu}+B, \quad A^{\mu},\ B\in G^4 \tag{17.2}$$

to be a SO of the Dirac equation in the class M_1 if

$$[Q_A, L]=(f_A^{\mu} p_{\mu}+q_A)L, \tag{17.3}$$

where $f_A{}^{\mu}$, Q_A are 4×4 matrices in general depending on x. We make the same suppositions about the wave function and matrices f_A^{μ}, q_A: $\psi\in F^4$, f_A^{μ}, $q_A\in G^4$.

Let us formulate and prove the following assertion.

THEOREM 17.1 [142]. The Dirac equation is invariant under the eight-dimensional Lie algebra A_8 defined over the field of real numbers. Basis elements of this algebra can be chosen in the form

$$\hat{\Sigma}_{\mu\nu}=\frac{m}{4}[\gamma_{\mu},\gamma_{\nu}]+\frac{1}{2}(1-i\gamma_4)(\gamma_m p_{\nu}-\gamma_{\nu} p_{\mu}),$$
$$\hat{\Sigma}_0=I, \quad \hat{\Sigma}_1=m\gamma_4-i(1-i\gamma_4)\gamma_{\mu} p^{\mu}, \tag{17.4}$$

where I is the unit matrix. For nonzero m this algebra is isomorphic to the Lie algebra of the group $GL(2,C)$, for m=0 the operators (17.4) form a commutative (Abelian) algebra.

PROOF can be carried out by direct verification. Using (2.3) it is not difficult to obtain the following relations

$$[\hat{\Sigma}_{\mu\nu},L]=\frac{1}{2}(\gamma_{\mu} p_{\nu}-\gamma_{\nu} p_{\mu})L, \quad [\hat{\Sigma}_0,L]=0, \quad [\hat{\Sigma}_1,L]=-2\gamma_4\gamma^{\nu}p_{\nu}L \tag{17.5}$$

which have the form (17.3). If m=0 these operators commute. For nonzero m we denote

$$\Sigma_{\mu\nu} = \frac{1}{m}\hat{\Sigma}_{\mu\nu}, \quad \Sigma_0 = \hat{\Sigma}_0, \quad \Sigma_1 = \frac{1}{m}\hat{\Sigma}_1 \tag{17.6}$$

and obtain the following commutation relations

$$\begin{gathered}[\Sigma_{\mu\nu}, \Sigma_{\lambda\sigma}] = g_{\mu\lambda}\Sigma_{\nu\sigma} + g_{\nu\sigma}\Sigma_{\mu\lambda} - g_{\mu\sigma}\Sigma_{\nu\lambda} - g_{\nu\lambda}\Sigma_{\mu\sigma}, \\ [\Sigma_1, \Sigma_0] = [\Sigma_1, \Sigma_{\mu\nu}] = [\Sigma_0, \Sigma_{\mu\nu}] = 0.\end{gathered} \tag{17.7}$$

Thus the operators (17.4) form an IA of the Dirac equation. For m=0 this IA is commutative, for nonzero m it is isomorphic to the algebra *AGL(2.C)*. This isomorphism can be established by the following relations

$$\begin{gathered}\lambda_{11} = \Sigma_0 + \Sigma_{12}, \quad \lambda_{22} = \Sigma_0 - \Sigma_{12}, \quad \lambda_{12} = \Sigma_{23} - \Sigma_{02}, \\ \lambda_{21} = \Sigma_{23} + \Sigma_{02}, \quad \tilde{\lambda}_{11} = \Sigma_1 + \Sigma_{03}, \quad \tilde{\lambda}_{22} = \Sigma_1 - \Sigma_{03}, \\ \tilde{\lambda}_{12} = -\Sigma_{01} + \Sigma_{31}, \quad \tilde{\lambda}_{21} = \Sigma_{31} + \Sigma_{01}\end{gathered} \tag{17.8}$$

where λ_{ab} and $\tilde{\lambda}_{ab}$ are the basis elements of the algebra *AGL(2,C)* satisfying the commutation relations

$$\begin{gathered}[\lambda_{ab}, \lambda_{cd}] = -[\tilde{\lambda}_{ab}, \tilde{\lambda}_{cd}] = \delta_{ac}\lambda_{bd} + \delta_{bc}\lambda_{ad} - \delta_{ad}\lambda_{bc} - \delta_{bd}\lambda_{ac}, \\ [\lambda_{ab}, \tilde{\lambda}_{cd}] = \delta_{ac}\tilde{\lambda}_{bd} + \delta_{bc}\tilde{\lambda}_{ad} - \delta_{ad}\tilde{\lambda}_{bc} - \delta_{bd}\tilde{\lambda}_{ac}.\end{gathered} \tag{17.9}$$

We give the other more constructive proof of Theorem 17.1 in order to explain the nature of the additional symmetry of the Dirac equation. It is well known that the system of four first-order equations (17.1) is equivalent to the system of two second-order equations. Multiplying (17.1) from the left by $(1\pm i\gamma_4)/2$, denoting

$$\psi_\pm = \frac{1}{2}(1 \pm i\gamma_4)\psi \tag{17.10}$$

and expressing ψ_- via ψ_+ we come to the equations

$$(p_\mu p^\mu - m^2)\psi_+ = 0, \tag{17.11a}$$

$$\psi_- = \frac{1}{m}\gamma_\mu p^\mu \psi_+. \tag{17.11b}$$

But the equations (17.10), (17.11a) have the evident symmetry under arbitrary matrix transformations commuting with γ_4. They are the transformations generating the hidden symmetry described in Theorem 17.1.

To describe effectively all nonequivalent transformations of this kind, we use the fact that the transition from (17.1) to (17.11) can be represented in the form

$$\psi \to \psi' = V\psi, \quad L \to L' = WLV^{-1} \tag{17.12}$$

where

$$V = \exp[-\frac{1}{2m}(1 - i\gamma_4)\gamma^\mu p_\mu] \equiv 1 - \frac{1}{2m}(1 - i\gamma_4)\gamma^\mu p_\mu, \tag{17.13a}$$

$$W = \exp[\frac{1}{2m}(1 + i\gamma_4)\gamma^\mu p_\mu] \equiv 1 + \frac{1}{2m}(1 + i\gamma_4)\gamma^\mu p_\mu, \tag{17.13b}$$

$$L' = -\frac{1}{2}(1 - i\gamma_4)m + \frac{1}{2m}(1 + i\gamma_4)(p_\mu p^\mu - m^2). \tag{17.13c}$$

Indeed, the equation

$$L'\psi' = 0, \quad \psi' = V\psi \tag{17.14}$$

reduces to (17.11) moreover $\psi' = \psi_+$.

The equation (17.14) is manifestly invariant under arbitrary matrix transformations commuting with γ_4. The corresponding transformation operator can be represented as a linear combination of the matrices

$$\Sigma'_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu], \quad \Sigma'_0 = I, \quad \Sigma'_1 = \gamma_4 \tag{17.15}$$

satisfying the relations (17.7) and so forming a basis of the eight-dimension Lie algebra. It is not difficult to make sure these matrices are linearly independent over the field of real numbers while

$$(\Sigma'_{ab} - i\varepsilon_{abc}\Sigma'_{0c})\psi' = (\Sigma_1 + i\Sigma_0)\psi' = 0$$

where ψ' is a solution of (17.1).

Using the operator (17.13a) we obtain from (17.15) the explicit form of basis elements of the IA of the Dirac equation: $\Sigma_a = V^{-1}\Sigma'_a V$, $\Sigma_{\mu\sigma} = V^{-1}\Sigma'_{\mu\sigma} V$ where $\Sigma_{\mu\sigma}$ and Σ_a are the operators (17.4), (17.6).■

It follows from the proof that the additional symmetry of the Dirac equation described in Theorem 17.1 is maximal in the sense it includes all the possible matrix transformations in the representation (17.14).

So besides the well-known Poincaré-invariance the Dirac equation is additionally invariant under the algebra A_8. Basis elements of this algebra do not belong to the class M_1 and thus cannot be considered as a Lie derivatives. In spite of this fact, they form a Lie algebra and satisfy the relations

$$\Sigma_{0a}^2 = -\Sigma_{ab}^2 = \Sigma_0^2 = -\Sigma_1^2 = 1$$

and so generate the eight-parameter group of transformations

$$\begin{aligned}
&\psi\to\psi' = (\cos\theta_{ab}+\gamma_a\gamma_b\sin\theta_{ab})\psi-\frac{i}{m}\sin\theta_{ab}(1-i\gamma_4)(\gamma_a\frac{\partial\psi}{\partial x_b}-\gamma_b\frac{\partial\psi}{\partial x_a}),\\
&\psi\to\psi'' = (\cosh\theta_{0a}+\gamma_0\gamma_a\sinh\theta_{0a})\psi-\frac{i}{m}\sinh\theta_{0a}(1-i\gamma_4)(\gamma_0\frac{\partial\psi}{\partial x_a}-\gamma_a\frac{\partial\psi}{\partial x_0})
\end{aligned} \tag{17.16}$$

(no sum over repeated indices),

$$\begin{aligned}
&\psi\to\psi''' = (\cos\theta_1+\gamma_4\sin\theta_1)\psi+\frac{1}{m}\sin\theta_1(1-i\gamma_4)\gamma_\mu\frac{\partial\psi}{\partial x_m},\\
&\psi\to\psi^{IV}=\exp\theta_0\psi
\end{aligned} \tag{17.17}$$

where θ_{ab}, θ_{0a}, θ_0 and θ_1 are real parameters. Any of formulae (17.16), (17.17) defines a one-parameter transformation group and can be represented in the form $\psi\to\exp(2\Sigma_A\theta_A)\psi$, where A=0,1,01,02,... .

The principal distinguishing feature of the transformations (17.16) with respect to Lorentz transformations (2.59) is that the transformed functions depend on derivatives of the wave function. Furthermore, in accordance with (17.16) independent variables are not transformed in contrast to Lorentz transformations. In other words the additional invariance of the Dirac equation has nothing to do with transformations of the space-time continuum, that is why we call this symmetry nongeometric.

The question arises of whether it is possible to combine the symmetry group of the Dirac equation given by the transformations (17.16), (17.17) and the Poincaré group. It turns out that such a unification is possible since the generators of these transformations and the Poincaré group generators form an 18-dimensional Lie algebra.

THEOREM 17.2. The Dirac equation is invariant under the 18-dimensional Lie algebra whose basis elements are given by formulae (2.22), (17.4), (17.6) and satisfy the commutation relations (1.14), (17.7) and (17.18):

$$\begin{aligned}
&[P_\mu,\Sigma_{\lambda\nu}]=[P_\mu,\Sigma_0]=[P_\mu,\Sigma_1]=0.\\
&[J_{\mu\nu},\Sigma_{\lambda\sigma}]=i(g_{\mu\sigma}\Sigma_{\nu\lambda}+g_{\nu\lambda}\Sigma_{\mu\sigma}-g_{\mu\lambda}\Sigma_{\nu\sigma}-g_{\nu\sigma}\Sigma_{\mu\lambda}).
\end{aligned} \tag{17.18}$$

The proof of the theorem reduces to direct verifying the validity of relations (17.18).■

We conclude from the above that the Dirac equation is invariant under the 18-parametrical Lie group including Lorentz transformations (2.49) and the nongeometric transformations (17.16), (17.17). A general form of this group transformation is

$$\psi\to A\psi(x')+B^\mu\frac{\partial\psi(x')}{\partial x_\mu} \tag{17.19}$$

where x'_μ are connected with x_μ by Lorentz transformations, A and B^μ are numerical matrices depending on 18 parameters [153].

17.2. Symmetries of the Dirac Equation in the Class of Integro-Differential Operators

Let us denote by M_∞ the set of nonlocal (integral) operators Q of the kind

$$Q\psi(x)=(2\pi)^{-3/2}\int \exp(i\boldsymbol{p}\cdot\boldsymbol{x})Q\tilde{\psi}(\boldsymbol{p},x_0)d^3p, \tag{17.20}$$

where $\tilde{\psi}(x_0,\boldsymbol{p})$ is a Fourier transform of $\psi(x)$:

$$\tilde{\psi}(\boldsymbol{p},x_0)=(2\pi)^{-3/2}\int \exp(-i\boldsymbol{p}\cdot\boldsymbol{x})\psi(\boldsymbol{x},x_0)d^3p, \tag{17.21}$$

Q is a 4×4 matrix depending on $\boldsymbol{p}$.

We show the Dirac equation has an additional invariance under the transformations belonging to the class M_∞. Moreover, the corresponding IA is more extensive than established above.

THEOREM 17.3 [142]. The Dirac equation is invariant under the algebra A_8 defined over the field of complex numbers. The symbols of basis elements of this algebra are given by the formulae

$$\tilde{\Sigma}_{\mu\nu}=\frac{1}{4}[\gamma_\mu,\gamma_\nu]+\frac{1}{2m}(\gamma_\mu p_\nu-\gamma_\nu p_\mu)\left(1-i\gamma_4\frac{H}{E}\right),\quad \tilde{\Sigma}_0=I,\quad \tilde{\Sigma}_1=\frac{H}{E} \tag{17.22}$$

where $H=\gamma_0\gamma_a+\gamma_0 m,\ E=\sqrt{p^2+m^2}$.

Instead of the proof we present the explicit form of the operator V diagonalizing the symbol of the operator L of (17.1)

$$V=P_++P_-\frac{H}{E},\quad V^{-1}=\frac{1}{m}(HP_++P_-E),\quad P_\pm=(1\pm\gamma_0)/2.$$

It is easy to ascertain that

$$V\gamma_0 LV^{-1}=L'=i\frac{\partial}{\partial x_0}-\gamma_0 E. \tag{17.23}$$

Symmetry operators $\tilde{\Sigma}'_A\in M_\infty$ in the representation (17.23) are matrices commuting with γ_0. Any such a matrix is a linear combination of the following basis matrices

$$\tilde{\Sigma}'_{ab}=[\gamma_a,\gamma_b]/2,\quad \tilde{\Sigma}'_{0a}=[\gamma_4,\gamma_a]/4,\quad \tilde{\Sigma}'_0=\gamma_0,\quad \tilde{\Sigma}'_1=I$$

which are linearly independent over the field of complex numbers and satisfy the commutation relations (17.7). Using the transformation $\tilde{\Sigma}_A'\rightarrow\Sigma_A=V^{-1}\tilde{\Sigma}_A'V$ we obtain the SOs (17.22) in the starting representation.■

So the Dirac equation possesses an additional invariance under the nonlocal (integral) operators belonging to the class M_∞. We note that the basis elements of the corresponding IA are similar to (17.4), (17.6) and possess the only nonlocal element, i.e., the energy sign operator $\hat{\varepsilon}=H/E$. But in contrast to (17.4), (17.6) the operators (17.22) are linearly independent over the field of complex numbers.

It is not difficult to make sure the operators (17.22) form a Lie algebra together with the Poincaré group generators (2.40) and satisfy the commutation relations (17.18). Thus the symmetry of the Dirac equation in the class M_∞ is represented by the 18-dimensional Lie algebra A_{18} including the subalgebra $AP(1,3)$. It can be shown the algebra A_{18} is isomorphic to $AP(1,3)\oplus AGL(2,C)$.

In conclusion we note that it follows from Theorem 17.3 the Dirac equation is invariant under the 16-parametric group of transformations $\tilde{\psi}\to\tilde{\psi}'=\exp(\tilde{\Sigma}_A\theta_A)\tilde{\psi}$ where θ_A are complex parameters. The explicit form of these transformations is given by the following formulae

$$\psi\to(\cos\theta_{ab}+\gamma_a\gamma_b\sin\theta_{ab})\psi-\frac{i}{m}\sin\theta_{ab}\sum_\varepsilon(1+i\varepsilon\gamma_4)(\gamma_a\frac{\partial\psi^\varepsilon}{\partial x_b}-\gamma_b\frac{\partial\psi^\varepsilon}{\partial x_a}),$$

$$\psi\to(\cosh\theta_{0b}+\gamma_0\gamma_b\sinh\theta_{0b})\psi-\frac{i}{m}\sinh\theta_{0b}\sum_\varepsilon(\gamma_0\frac{\partial\psi^\varepsilon}{\partial x_b}+\gamma_b\frac{\partial\psi^\varepsilon}{\partial x_0}),$$

$$\psi\to\sum_\varepsilon(\cosh\theta_1+\varepsilon\sinh\varepsilon\theta_1)\psi^\varepsilon,\quad \psi\to\exp(i\theta)\psi.$$

Here

$$\psi^\varepsilon=\frac{1}{2}(1-\varepsilon\frac{H}{E})\tilde{\psi},\ \ \tilde{\psi}=\psi^++\psi^-,\ \ \varepsilon=\pm.$$

17.3. Symmetries of the Eight-Component Dirac Equation

Until now, we consider only linear symmetry transformations of the Dirac equation. Here we extend class of symmetries by considering antilinear transformations which are also admissible in quantum mechanics.

Consider the eight-component Dirac equation

$$\hat{L}\hat{\psi}=0,\quad \hat{L}=\Gamma_\mu p^\mu-m, \tag{17.24}$$

where Γ_μ are 8×8 matrices satisfying the Clifford algebra (8.2) together with Γ_4, Γ_5, Γ_6.

Choosing Γ_μ and the wave function in the form

$$\Gamma_\mu=\begin{pmatrix}\gamma_\mu & 0\\ 0 & \gamma_\mu\end{pmatrix},\quad \tilde{\psi}=\begin{pmatrix}\psi\\ i\gamma_2\psi^*\end{pmatrix}, \tag{17.25}$$

where γ_μ are the 4×4 Dirac matrices in the realization (2.4), ψ is a four-component wave function, we obtain from (17.24) a system of equations including the Dirac equation and the adjoint equation (2.8).

The condition (17.25) can be represented in the form independent on realization of the Γ-matrices:

$$(1-i\Gamma_5\Gamma_6 C)\hat{\psi}=0 \tag{17.26}$$

where C is the charge conjugation operator. In the representation (17.25) we can choose

$$\Gamma_4=\begin{pmatrix}0 & \gamma_4\\ \gamma_4 & 0\end{pmatrix},\quad \Gamma_5=i\begin{pmatrix}0 & -\gamma_4\\ \gamma_4 & 0\end{pmatrix},\quad \Gamma_5=\begin{pmatrix}\gamma_4 & 0\\ 0 & -\gamma_4\end{pmatrix}, \tag{17.27}$$
$$C\hat{\psi}=i\Gamma_2\hat{\psi}^*.$$

Thus the Dirac equation and the adjoint equation (2.8) can be represented in the form (17.24) with the additional condition (17.26). Besides linear transformations of solutions of (17.24), (17.27) correspond to linear or antilinear transformations of solutions of the four-component Dirac equation. Moreover such a correspondence is an isomorphism. So the problem of description of linear and antilinear SOs of the Dirac equation is equivalent to finding the IA of the equations (17.24), (17.26) in the class of linear operators.

We note that the equation (17.24) admits different interpretations including that of the equation of motion of particles with spin 1 (see Section 8) and 3/2 [233]. This makes the problem of investigation of symmetries of this equation even more interesting. Here we consider symmetries of (17.24) without additional conditions and then symmetries of the system (17.24), (17.26).

Thanks to the increase of the number of components of the wave function the equation (17.24) has more extended symmetry than the four-component Dirac equation. In addition to manifest invariance under the Poincaré group (whose generators are given by formulae (2.22) where $S_{\mu\sigma}=i[\Gamma_\mu,\Gamma_\sigma]$) this equation admits matrix transformations commuting with Γ_μ. Furthermore, the eight-component Dirac equation has a hidden (nongeometric) symmetry including SOs in the classes M_1 and M_∞.

In analogy with Theorem 17.3 it is possible to prove the following assertion.

THEOREM 17.4. The eight-component Dirac equation is invariant under the 32-dimensional Lie algebra defined over the field of complex numbers. The basis elements of this algebra belong to the class M_1 and are given by the formulae

$$Q_{\mu\nu\lambda}=\Sigma_{\mu\nu}D_\lambda,\quad Q_{\alpha\lambda}=\Sigma_\alpha D_\lambda \tag{17.28}$$

where

$$\Sigma_{\mu\nu}=\frac{1}{2}[\Gamma_\mu,\Gamma_\nu]+\frac{1}{m}(\Gamma_\mu p_\nu-\Gamma_\nu p_\mu)(1+\Gamma_4\Gamma_5\Gamma_6\varepsilon),$$

$$\Sigma_0=D_0=I,\quad \Sigma_1=i\Gamma_4\Gamma_5\Gamma_6-\frac{i}{m}\Gamma_\mu p^\mu(1-\Gamma_4\Gamma_5\Gamma_6\varepsilon), \tag{17.29}$$

$\mu,\nu,\lambda=0,1,2,3,\quad \alpha=1,2,$

$$D_1=i\Gamma_4\Gamma_5,\quad D_2=i\Gamma_5\Gamma_6,\quad D_3=i\Gamma_4\Gamma_6,\quad D_4=I, \tag{17.30}$$

I is the unit matrix, $\varepsilon=1$ or $\varepsilon=-1$.

The proof is similar to the proof of Theorem 17.3. The Lie algebra spanned on the basis (17.28) is isomorphic to *AGL(4,C)* [154].

The operators (17.28) form a closed algebra together with the Poincaré group generators (2.22) (where $S_{\mu\sigma}=i[\Gamma_\mu,\Gamma_\sigma]$) inasmuch as

$$[P_\mu,Q_{\lambda\sigma\rho}]=[P_\mu,Q_{\alpha\lambda}]=[J_{\mu\nu},Q_{\alpha\lambda}]=0,$$

$$[J_{\mu\nu},Q_{\lambda\sigma\rho}]=i(g_{\mu\sigma}Q_{\nu\lambda\rho}+g_{\nu\lambda}Q_{\mu\sigma\rho}-g_{\mu\lambda}Q_{\nu\sigma\rho}-g_{\nu\sigma}Q_{\mu\lambda\rho}).$$

It is possible to show the symmetry formulated in Theorem 17.4 defines the maximal IA of the eight-component Dirac equation in the class M_1.■

In complete analogy with the results presented in the preceding section we can show that the nongeometric symmetry of the eight-component Dirac equation in the class M_∞ is more extensive and is determined by the 32-dimensional Lie algebra defined over the field of complex numbers. Basis elements of this algebra are defined by formulae (17.28)-(17.30) where $\varepsilon=\Sigma_1=H/E$, $H=\Gamma_0\Gamma_a p_a+\Gamma_0 m$.

Before we restrict ourselves to searching sets of SO of the Dirac equation, forming bases of Lie algebras. Now we will demonstrate the existence of a more complicated algebraic structure generated by these SO.

THEOREM 17.5. The eight-component Dirac equation admits an IA isomorphic to the Poincaré superalgebra. Basis elements of this IA have the form

$$P_\mu=p_\mu=-i\frac{\partial}{\partial x^\mu},\quad J_{\mu\sigma}=x_\mu p_\sigma-x_\sigma p_\mu+\Sigma_{\mu\sigma},$$

$$Q_1=\sqrt{2}\,\Gamma_5\left[\Gamma_1+i\Gamma_2+\frac{1}{m}(1+i\Gamma_4)(p_1+ip_2)\right]/2,$$

$$Q_2=\sqrt{2}\,\Gamma_5\left[\Gamma_3+\Gamma_0+\frac{1}{m}(1+i\Gamma_4)(p_0+p_3)\right]/2,$$

$$\bar{Q}_1=\sqrt{2}\,\Gamma_5[(p_0-p_3)(\Gamma_1-i\Gamma_2)+(p_1-ip_2)(\Gamma_3-\Gamma_0)]/2,$$

$$\bar{Q}_2=\sqrt{2}\,\Gamma_5[\Gamma_3 p_0-\Gamma_0 p_3+i(\Gamma_1 p_2-\Gamma_2 p_1)-im\Gamma_4]/2$$

where

$$\Sigma_{ab}=\frac{i}{8}\left[\Gamma_a\Gamma_b-\Gamma_b\Gamma_a-2\varepsilon_{abc}\Gamma_0\Gamma_c+(1-i\Gamma_4)(\Gamma_a p_b-\Gamma_b p_a)\right],\quad \Sigma_{0a}=\frac{i}{2}\varepsilon_{abc}\Sigma_{bc},$$

and Γ_k are the 8×8 Dirac matrices which can be chosen in the form (17.25), (17.27).

PROOF reduces to direct verification that the mentioned SOs do satisfies the invariance condition (17.3) and the commutation and *anticommutation* relations (1.14) and

$$[\bar{Q}_A,\bar{Q}_B]_+=[Q_A,Q_B]_+=0,\quad [Q_A,Q_B]_+=2(\sigma_\mu)_{AB}P^\mu,$$

$$[J_{\mu\sigma},Q_A]=-\frac{1}{2i}(\sigma_\mu\sigma_\sigma)_{AB}Q_B,\quad [J_{\mu\sigma},\bar{Q}_A]=-\frac{1}{2i}(\sigma_\mu\sigma_\sigma)^*_{AB}\bar{Q}_B,$$

$$[P_\mu,Q_A]=[P_\mu,\bar{Q}_A]=0,\quad A,B=1,2$$

which characterize the Poincaré superalgebra [411], refer also to Appendix 1 for definitions.

A specific feature of our realization of the Poincaré superalgebra is that the generators $J_{\mu\sigma}$ are the first-order differential operators with matrix coefficients.

17.4. Symmetry Under Linear and Antilinear Transformations

Let us investigate symmetries of the eight-component Dirac equation with the additional condition (17.26). As was noted in the above linear symmetries of this system correspond to linear and *antilinear* symmetries of the eight component Dirac equation. To denote the classes of SOs including linear and antilinear transformations we use the symbols M_1^*, M_1^* ... where M_1^* is the class of linear and antilinear differential operators of first order etc.

The SOs of the eight-component Dirac equation are not SOs of the additional condition (17.26) in general. We will see that the symmetry of the system (17.24), (17.26) is more restricted than the symmetry of the eight-component Dirac equation but is more extended than the symmetry of the four-component Dirac equation.

THEOREM 17.6. The system of equations (17.24), (17.26) is invariant under a 14-dimensional Lie algebra isomorphic to $AP(1,3)\oplus AO(1,2)\oplus T_1$. Basis elements of this algebra can be chosen in the form (2.22), (17.30) where $S_{\mu\sigma}$=i$[\Gamma_\mu,\Gamma_\sigma]$, Γ_k (k=0,1,...,6) are 8×8 Dirac matrices satisfying the relations $C\Gamma_6=\Gamma_6 C$, $C\Gamma_n=-\Gamma_n C$, n=0,1,...,5. The real Lie algebra spanned on the basis (2.22), (17.30) is the maximal IA of the system (17.24), (17.26) in the class M_1.

PROOF. It is easily verified the operators (2.22), (17.30) commute with $L_1=\Gamma_\mu p^\mu-m$ and $L_2=1-\mathrm{i}\Gamma_5\Gamma_6 C$ and so are the SOs of the eight-component Dirac equation with the additional condition (17.26). To prove these operators form the maximal IA of the system (17.24), (17.26) it is sufficient to show that any $Q\in M_1$

satisfying the conditions $[Q,L_1]=ß_Q^{1}L_1+\alpha_Q^{1}L_2$, $[Q,L_2]=ß_Q^{2}L_1+\alpha_Q^{2}L_2$, where α_Q^{σ}, $ß_Q^{\sigma}$ are matrices depending on x, is a linear combination of the basis elements (2.22), (17.30). We do not represent here the corresponding calculations but show a complete set of matrices which can be used to expand all unknown quantities:

$$\{Q_A,\ \Gamma_\mu Q_A,\ \Gamma_\mu\Gamma_\nu Q_A,\ \Gamma_\mu\Gamma_\nu\Gamma_\lambda Q_A,\ \Gamma_\mu\Gamma_\nu\Gamma_\lambda\Gamma_6 Q_A\}.$$

Here Q_A (A=1,2,3,4) are the matrices (17.30), $\mu, \nu, \lambda=1,2,3,4$, $\mu\neq\lambda$, $\mu\neq\nu$, $\lambda\neq\nu$.

The operators P_μ, $J_{\mu\sigma}$ commute with Q_a which satisfy the commutation relations characterizing the Lie algebra of the group *O(1,2)*.■

We see that besides the obvious symmetry under the Poincaré algebra the system (17.24), (17.26) is invariant in respect with the three-dimension matrix algebra *AO(1,2)* (the trivial identity symmetry operator is not discussed here). So this system is invariant under the three-parameter group of matrix transformations

$$\hat{\psi}\to(\exp\theta_A Q_A)\hat{\psi} \tag{17.31}$$

where θ_A are real numbers. Using the representation (17.25),(17.27) we find the corresponding transformations of solutions of the four-component Dirac equation

$$\begin{aligned}&\psi\to\exp(-\frac{i}{2}\theta_1)\psi,\\ &\psi\to\cosh\frac{\theta_2}{2}\psi+i\gamma_2\sinh\frac{\theta_2}{2}\psi^*,\\ &\psi\to\cosh\frac{\theta_3}{2}\psi-\gamma_2\sinh\frac{\theta_3}{2}\psi^*.\end{aligned} \tag{17.32}$$

The invariance of the Dirac equation under the transformations (17.32) was apparently established for the first time by Plebanski (see [72, 227, 364]). We note that the existence of this symmetry follows from Lemma 1.1, refer to Subsection 1.7.

The symmetry of the Dirac equation in the class M_1^* is more extended.

THEOREM 17.7. The eight-component Dirac equation with the additional condition (17.26) is invariant under the 16-dimensional Lie algebra defined over the field of real numbers. Basis elements of this algebra belong to the class M_1^* and are given by the formulae

$$\Sigma_{mn}=\frac{1}{2}[\Gamma_m,\Gamma_n]+\frac{1}{m}(1-i\Gamma_6)(\Gamma_m p_n-\Gamma_n p_m),\quad \Sigma_0=I,\quad m,n=0,1,...,5. \tag{17.33}$$

The proof is similar to the proof of Theorem 17.1. We note that by definition $p_{3+a}\tilde{\psi}(x)\equiv-i\partial/\partial x_{3+a}\tilde{\psi}(x)=0$, hence the operator $\Sigma_{54}=-\Sigma_{45}$ reduces to the numerical matrix.

The generators (17.33) satisfy the commutation relations (17.7) where $g_{mn}=diag(1,-1,-1,-1,-1,-1)$ and form the Lie algebra isomorphic to $AO(1,5)\oplus T_1$, $T_1\supset\Sigma_0$.

The operators (17.33) form a closed algebra together with the Poincaré group generators, satisfying the following relations

$$
\begin{aligned}
&[P_\mu,\Sigma_{mn}]=[J_{\mu\nu},\Sigma_{54}]=0,\quad [J_{\mu\nu},\Sigma_{m'\lambda}]=i(g_{\mu\lambda}\Sigma_{m'\nu}-g_{\nu\lambda}\Sigma_{m'\nu}),\\
&[J_{\mu\nu},\Sigma_{\rho\lambda}]=i(g_{\mu\rho}\Sigma_{\nu\lambda}+g_{\nu\lambda}\Sigma_{\mu\rho}-g_{\mu\lambda}\Sigma_{\nu\rho}-g_{\nu\rho}\Sigma_{\mu\lambda})
\end{aligned}
\tag{17.34}
$$

where $m',n'>3$, $\mu,\nu,\rho,\lambda\le 3$. Hence it follows the system (17.24), (17.26) is invariant under the 26-parameter group including the Poincaré group and the group of transformations $\hat{\psi}\to\exp(\Sigma_A\theta_A)\hat{\psi}$ where θ_A are arbitrary real parameters. The latter transformations have the following explicit form

$$
\begin{aligned}
&\hat{\psi}\to(\cos\theta_{kl}+\Gamma_k\Gamma_l\sin\theta_{kl})\hat{\psi}-\frac{i}{m}(1-i\Gamma_6)\sin\theta_{kl}\left(\Gamma_k\frac{\partial\hat{\psi}}{\partial x_l}-\Gamma_l\frac{\partial\hat{\psi}}{\partial x_k}\right),\quad k,l\neq 0,\ k\neq l,\\
&\hat{\psi}\to(\cosh\theta_{0k}+\Gamma_0\Gamma_k\sinh\theta_{0k})\hat{\psi}-\frac{i}{m}(1-i\Gamma_6)\left(\Gamma_0\frac{\partial\hat{\psi}}{\partial x_k}+\Gamma_k\frac{\partial\hat{\psi}}{\partial x_0}\right)\sin\theta_{0k},\\
&\hat{\psi}\to\exp(\theta_0)\hat{\psi}.
\end{aligned}
\tag{17.35a}
$$

We can collate a linear or antilinear transformation of solutions of the four-component Dirac equation to any transformation of the kind (17.35). Substituting (17.25), (17.27) into (17.35) we obtain for $k,l\le 3$ the transformations (17.16). If $k>0$ then the corresponding transformations have the form

$$
\begin{aligned}
&\psi\to(\cos\theta_{4a}\psi+i\gamma_4\gamma_a\gamma_2\sin\theta_{4a}\psi^*)-\frac{i}{m}(\gamma_4\gamma_2-i\gamma_2)\frac{\partial\psi^*}{\partial x_a}\sin\theta_{4a},\\
&\psi\to(\cos\theta_{5a}\psi+\gamma_4\gamma_a\gamma_2\sin\theta_{5a}\psi^*)-\frac{i}{m}(\gamma_4\gamma_2-i\gamma_2)\frac{\partial\psi^*}{\partial x_a}\sin\theta_{5a},\\
&\psi\to\exp(i\theta_{54})\psi,\\
&\psi\to\cosh\theta_{04}\psi+i\gamma_1\gamma_3\sinh_{04}\psi^*-\frac{1}{m}(i+\gamma_4)\gamma_2\frac{\partial\psi^*}{\partial x_0}\sinh\theta_{04},\\
&\psi\to\cosh\theta_{05}+\gamma_1\gamma_3\sinh\theta_{05}\psi^*+\frac{1}{m}(1-i\gamma_4)\gamma_2\frac{\partial\psi^*}{\partial x_0}\sinh\theta_{05}.
\end{aligned}
\tag{17.36}
$$

Formulae (17.16), (17.36) define the sixteen one-parameter transformations which do not change the form of the Dirac equation. We see the symmetry group of this equation in the class M_1^* is more extensive than in the class M_1 and includes the latter as a subgroup.

17.5. Hidden Symmetries of the Massless Dirac Equation

We saw in Chapter 1 that IAs of motion equations for massless fields are more extended than the corresponding symmetries of equations describing particles of nonzero masses.

Here we investigate nongeometric symmetries of the massless Dirac equation which also turn out to be more wide than in the case of nonzero mass.

It was noted in Subsection 2.8 that the massless Dirac equation is invariant under the 16-dimensional Lie algebra whose basis is formed by the generators of the conformal group and the operator $F=-\gamma_4$. Besides this equation is invariant under the eight-dimensional commutative (Abelian) algebra belonging to the class M_1. Basis elements of this IA are given in (17.4) where m=0.

The SOs (17.4) form a 18-dimensional Lie algebra together with the generators of the Poincaré group. This algebra cannot be extended by including the conformal group generators (2.42) inasmuch as the corresponding set of SOs is not closed in respect with the Lie brackets.

The IA of the massless Dirac equation given in Theorem 1.3 can be extended with the help of SOs belonging to the class M_1^*. Let us write this equation in the form (17.24), (17.26) where m=0. Then any SO of this system in the class M_1 corresponds to the SO of the four-component Dirac equation in the class M_1^*, since linear and antilinear transformations can be represented as linear transformations for the real and imaginary components of the wave function.

In analogy with Theorem 17.6 we can formulate and prove the following assertion.

THEOREM 17.8. The maximal IA of the system (17.24), (17.26) (with m=0) in the class M_1 is a linear span of the basis elements $\{P_\mu, J_{\sigma\mu}, D, K_\mu, D_k, D_{4+k}\}$ where

$$Q_{4+k}=\Gamma_4\Gamma_5\Gamma_6 Q_k, \quad k=1,2,3,4,$$

and P_μ, $J_{\mu\sigma}$, K_μ, D, D_μ are the operators (2.22), (2.25), (17.30) (where $S_{\mu\sigma}=i[\Gamma_\mu,\Gamma_\sigma]$).

The proof is similar to the proof of Theorem 17.6 and so is not given here. The operators D_k, D_{4+k} commute with P_μ, $J_{\mu\sigma}$, K_μ, D and form an eight-dimensional subalgebra. The commutation relations between D_k and D_{4+k} can be represented in the form (17.7) if we denote

$$D_2=\Sigma_1, \quad D_8=\Sigma_0, \quad D_1=\Sigma_{23}, \quad D_6=\Sigma_{31}, \quad D_7=\Sigma_{12}, \quad D_3=\Sigma_{01}, \quad D_4=\Sigma_{02}, \quad D_5=\Sigma_{03}.$$

We conclude from the above that the massless Dirac equation admits a 23-dimensional Lie algebra isomorphic to $AC(1,3)\oplus AGL(2,C)$.

It follows from Theorem 17.6 that the massless Dirac equation is invariant under the 23-dimensional Lie group locally isomorphic to the group $C(1,3)\otimes GL(2,C)$.

The corresponding transformations from the conformal group are given in Subsection 2.9. The transformations from the subgroup *GL(2,C)* can easily be found with the formula (17.31). These transformations for $A\leq 3$ are given in (17.31), for $a=4$ we have a trivial multiplication of wave function by an arbitrary number. For $A>4$ we obtain using (17.25), (17.27)

$$\begin{aligned}
&\psi\to(\cosh\frac{\theta_5}{2}-\gamma_4\sinh\frac{\theta_5}{2})\psi,\\
&\psi\to\cos\frac{\theta_6}{2}\psi-i\gamma_4\gamma_2\sin\frac{\theta_6}{2}\psi^*,\\
&\psi\to\cos\frac{\theta_7}{2}\psi+\gamma_4\gamma_2\sin\frac{\theta_7}{2}\psi^*,\\
&\psi\to(\cos\frac{\theta_8}{2}-i\gamma_4\sin\frac{\theta_8}{2})\psi.
\end{aligned}\tag{17.37}$$

The group of transformations (17.32), (17.37) includes the subgroup of Pauli-Gürsey transformations [213] generated by D_1, D_6, D_7 and D_8. The remaining transformations (17.32), (17.37) are hyperbolic rotations which are not unitary in the metric (2.39). The invariance of the massless Dirac equation under the transformations (17.32), (17.37) was established by Danilov [72] and Ibragimov [226].

We note that the considered symmetry is a mere consequence of Lemma 1.1 and the obvious symmetry of the massless Dirac equation under the transformations $\psi\to i\gamma_4\psi$.

If we restrict ourselves to linear transformations then Lie symmetries of the massless Dirac equation reduce to invariance under the algebra $AC(1,3)\oplus AT_1$ whose basis elements are given in (2.22), (2.42). But symmetry of this equation in the class M_∞ is described by a very extended algebra isomorphic to $AC(1,3)\oplus AGL(2C)$. A proof of this statement is given in [148].

Let us summarize. Symmetries of the Dirac equation are very reach and cannot be described if we restrict ourselves to the classical Lie approach. Besides Poincaré invariance this equation admits hidden symmetries in classes of higher order SOs and integro-differential SOs, which form Lie algebras but present non-Lie symmetries. In the following we will demonstrate these SOs include subsets generating hidden supersymmetry of the Dirac equation.

The maximal IAs of this equation in various classes of SOs are presented in following table.

value of m	Classes of SOs			
	M_1	M_1	M_1^*	M_∞
$m\neq 0$	$AP(1,3)$	$AP(1,3)\oplus AGL(2,C)$	$AP(1,3)\oplus AO(1,3)\oplus T_1$	$AP(1,3)\oplus AGL(2,c)\oplus AGL(2,C)$
$m=0$	$AG(1,3)\oplus T_1$	$AC(1,3)\oplus T_{10}$	$AC(1,3)\oplus AGL(2,C)$	$AC(1,3)\oplus AGL(2,C)\oplus AGL(2,C)$

18. THE COMPLETE SET OF SOs OF THE DIRAC EQUATION

18.1. Introduction and Definitions

Until now we consider only such SOs of the Dirac equation which form a finite-dimension Lie algebra, i.e., satisfy relations (16.3). This restriction is completely justified since in this way it is possible to find not only algebras but also groups of nongeometric symmetry. However for great many of applications (e.g., for constructing of constants of motion, for describing coordinate systems admitting separation of variables etc.) it is not essential that SOs belong to a finite-dimensional Lie algebra. In principle such operators can belong to infinite-dimensional Lie algebras or possess other algebraic structures. Therefore it is of interest to investigate symmetries of the Dirac equation in a more general approach without requiring that SOs satisfy (16.3).

Here we present the results of such an investigation. More precisely we present a complete set of SOs of any finite order n for the Dirac equation. We will see such SOs have very interesting algebraic structures forming bases of Lie superalgebras.

We recall a Lie superalgebra SA is a graded vector space closed under a binary operation $(x_\sigma, y_{\sigma'}) \to [x_\sigma, y_{\sigma'}]'_{f(\sigma\sigma')}$ which generalize the Lie brackets. The simplest gradation is so-called Z_2-gradation when SA consists in elements of two kinds: even (E) and odd (O). Besides SA is closed under the commutation and anticommutation relations corresponding to the scheme

$$[E,E]\sim E, \quad [E,O]\sim O, \quad [O,O]_+\sim E. \tag{18.1}$$

For more details see Appendix 1.

Let us return to SOs. A formal definition of SOs of order n for the Dirac equation can be written in the form of (16.6), (16.7) where L is the Dirac operator (17.1), $h^{a_1 a_2 \dots a_j}\in G_4$. In other words a SO of order n is a differential operator of order

n with matrix coefficients, besides this operator transforms solutions of the Dirac equation into solutions. To denote the class of such operators we use the symbol M_n.

In [382, 243] a complete set of the Dirac equation SOs belonging to the classes M_1 and M_2 was obtained. The problem of finding of such operators reduces to solving of a very complicated system of determining equations.

Following [329] we present a simple proof that all the SOs of the Dirac equation of any order n belong to the enveloping algebra of the Poincaré algebra, and find explicitly all linearly independent SOs. The main idea of this proof is to use the fact that solutions of the Dirac equation have to satisfy the KGF equation and thus a SO of the Dirac equation have to be a SO of the equation (1.1).

18.2. The General Form of SOs of Order *n*

Instead of the Dirac equation we consider the equivalent system (17.14). Choosing the realization (2.4), (2.17) for the γ-matrices we conclude that the function ψ' has two non-zero components only which satisfy the KGF equation (17.11a).

Our chief idea is to describe SOs of order n of the system (17.14) and then to find the corresponding SOs of the Dirac equation. Indeed, we can establish the one-to one correspondence between SOs Q' of the system (17.14) and SOs of the Dirac equation with the following relation

$$Q=V^{-1}Q'V, \quad Q'=VQV^{-1} \tag{18.2}$$

where V is the operator (17.13a).

An operator Q' defined on the set of two-component functions $\psi'=\psi^+$ can be expanded in the complete set of four matrices

$$Q'=\sigma^\mu Q'_\mu, \quad \sigma_a=\varepsilon_{abc}S_{bc}, \quad \sigma_0=\frac{1}{3}\sigma_a\sigma_a \tag{18.3}$$

where S_{bc} are the spin matrices (2.23). An operator Q′ is a SO of the system (17.14) iff Q_μ' are the SOs of the KGF equation.

Let Q_μ' in (18.3) be SOs of order n of the KGF equation. Then, according to (16.14) they are polynomials on the generators (1.6) or (which is the same) on the operators P_μ, $J_{\mu\sigma}$-$S_{\mu\sigma}$, where P_μ, $J_{\mu\sigma}$ are the generators (2.22), $S_{\mu\sigma}$ are the matrices (2.23). But it is not difficult to make sure the matrices $S_{\mu\sigma}$ also can be expressed via P_μ and $J_{\mu\sigma}$ on the set of solutions of the equation (17.14):

$$2S_{\mu\sigma}\psi'=\frac{1}{m^2}(P_\mu W_\sigma-P_\sigma W_\mu+i\varepsilon_{\mu\sigma\rho\nu}W^\rho P^\nu)\psi' \tag{18.4}$$

where W_μ is the Lubanski-Pauli vector (2.37). The equation (17.14) is Poincaré-invariant, moreover, the corresponding generators of the Poincaré group have the form (2.22). Substituting (2.22) into the l.h.s. of (18.4) and using the equation (17.14) we make sure that relation (18.4) is really satisfied.

It follows from the above the SOs Q_μ'as well as Q' of (18.3) are polynomials on P_μ, $J_{\mu\sigma}$ (2.22). Inasmuch as the operator V of (17.13a) commutes with P_μ and $J_{\mu\sigma}$ it follows from (18.2) that $Q'=Q$, and thus *all the SOs of any finite order for the Dirac equation are polynomials of the generators of the Poincaré group.*

Thus, we have proved that any finite order SO of the Dirac equation belongs to the enveloping algebra generated by the algebra *AP(1,3)*. This means the problem of description of a complete set of SOs reduces to going over all the independent linear combinations of products of the generators (2.22).

18.3. Algebraic Properties of the First-Order SOs

By describing SOs of arbitrary order n the key role is played by the case n=1. According to the above the corresponding complete set of SOs can be obtained by going over polynomials on P_μ, $J_{\mu\sigma}$ besides as it will be shown in the following it is sufficient to restrict ourselves to considering polynomials of order $n\leq 3$. As a result we obtain known [382] 26 linearly independent SOs including the Poincaré generators (2.22), the identity operator I and the following fifteen operators

$$\begin{aligned}
&W_{4\mu}\equiv W_\mu=\frac{i}{2}\gamma_4(p_\mu-m\gamma_\mu),\\
&W_{\mu\nu}=\frac{i}{2}\gamma_4(\gamma_\mu p_\nu-\gamma_\nu p_\mu),\\
&B=i\gamma_4(D-m\gamma_\mu x^\mu),\\
&A_\mu=\frac{i}{2}\gamma_4\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}\gamma^\sigma+\frac{1}{2}\gamma_\mu
\end{aligned} \tag{18.5}$$

where D is the dilatation generator (2.42).

Higher order SOs can be expressed via products of the operators (2.22), (18.5) so it is extremely useful to investigate the algebraic properties of this set. These properties turn out to be very engaging so such an investigation is very interesting by itself.

By direct verification we obtain the following commutation relations

$$[P_\mu,W_{4\nu}]=[P_\mu,W_{\lambda\sigma}]=[J_{\mu\nu},B]=0 \tag{18.6a}$$

$$[G_\lambda,J_{\mu\nu}]=i(g_{\lambda\mu}G_\nu-g_{\lambda\nu}G_\mu),\quad G_\lambda=A_\lambda\ or\ W_{4\lambda},$$
$$[J_{\mu\nu},W_{\lambda\sigma}]=i(g_{\mu\sigma}W_{\nu\lambda}+g_{\nu\lambda}W_{\mu\sigma}-g_{\mu\lambda}W_{\nu\sigma}-g_{\nu\sigma}W_{\mu\lambda}),\tag{18.6b}$$

$$[P_\mu,B]=2iW_\mu,\quad [P_\mu,A_\nu]=i\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma},$$
$$[W_{4\mu},B]=\frac{i}{2}P_\mu+imA_\mu,\quad [W_{\mu\nu},B]=\frac{i}{2}([P_\mu,A_\nu]_+-[P_\nu,A_\mu]_+),$$
$$[W_{4\mu},A_\nu]=i(g_{\mu\nu}B+[J_{\mu\lambda},W^\lambda{}_\nu]_+),$$
$$[W_{\lambda\sigma},A_\mu]=i(\varepsilon_{\mu\sigma\lambda\rho}P^\rho+\frac{1}{2}[(g_{\mu\lambda}P_\sigma-g_{\mu\sigma}P_\lambda),B]_+-[W_{4\mu},J_{\lambda\sigma}]_+),\tag{18.7}$$
$$[W_{kl},W_{mn}]=\frac{i}{4}(\varepsilon_{klmsr}P_n+\varepsilon_{mnlsr}P_k-\varepsilon_{klnsr}P_m-\varepsilon_{mnksr}P_l)W^{sr},$$
$$[A_\mu,B]=i\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}A^\sigma,\quad [A_\mu,A_\nu]=i[\varepsilon_{\mu\nu\rho\sigma}(J^{\rho\sigma}B-W^{\rho\sigma})-J_{\mu\nu}]$$

where $m,n,k,s,r=0,1,...,4$ and $P_4=m$.

Other relations for P_μ, $J_{\mu\sigma}$ and W_μ are given in (1.14), (4.3).

According to (18.7) the complete set of the first order SOs of the Dirac equation does not form a basis of a Lie algebra in contrast to the SOs (2.22). But we recognized that this set includes a subset forming a basis of the 18-dimensional Lie algebra, see (17.4). Here we note that the operators (17.4) can be expressed via $W_{\mu\sigma}$ on the set of solutions of the Dirac equation:

$$\Sigma_{\mu\nu}=\frac{1}{m}(W_{\mu\nu}+\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}).\tag{18.8}$$

The essentially new point is that the operators (2.22), (18.5) include subsets which have a structure of a superalgebra. To select these subsets we calculate the anticommutation relations

$$[W_{kl},W_{mn}]_+=\frac{1}{2}(g_{kn}P_{lm}+g_{lm}P_{kn}-g_{km}P_{ln}-g_{ln}P_{km})\tag{18.9a}$$

$$[W_{4\mu},B]_+=-\frac{1}{2}[J_{\mu\nu},P^\nu]_+,\quad [W_{\mu\nu},B]_+=-mJ_{\mu\nu}-\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma},$$
$$[A_\sigma,W_{\mu\nu}]_+=\frac{1}{2}(\varepsilon_{\sigma\mu\lambda\rho}P_\nu-\varepsilon_{\sigma\nu\lambda\rho}P_\mu)J^{\lambda\rho}-g_{\nu\sigma}W_{4\mu}+g_{\mu\sigma}W_{4\nu},\tag{18.9b}$$

$$[W_\mu,A_\nu]_+=-\frac{m}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}-W_{\mu\nu},$$

$$[A_\mu,A_\nu]_+=\frac{1}{2}\left\{\left(J_{\nu\rho}J^{\nu\rho}+\frac{1}{2}\right)g_{\mu\nu}-[J_{\mu\lambda},J^\lambda{}_\nu]_+\right\},\quad [A_\mu,B]_+=0,\quad B^2=\frac{3}{4}-\frac{1}{2}J_{\mu\nu}J^{\mu\nu}\tag{18.9c}$$

where $g_{kn}=diag(1,-1,-1,-1,1)$, $P_{kn}=p_k p_n$, $p_4=m$.

Using (18.6), (18.7), (18.9) it is possible to select various sets of SOs forming superalgebras. The set including the algebra *AP(1,3)* and the maximal number of the first order SOs is [160]

$$S_1=\left\{\, W_{kl};\ P_\mu, J_{\mu\nu}, P_{\mu\nu}=p_\mu p_\nu, I \,\right\}. \tag{18.10}$$

Here the odd operators stand to the left of semicolon, the remaining operators are even. It follows from (1.14), (4.3), (18.6), (18.9) the commutation and anticommutation relations for the set (18.10) correspond to the scheme (18.1), so these operators form a basis of a 30-dimensional superalgebra which we denote by $SA_{(30)}$.

The superalgebra $SA_{(30)}$ has a very interesting subalgebraic structure. First it includes the Lie algebra of the Poincaré group generated by P_μ and $J_{\mu\sigma}$, secondly it contains two Clifford algebras whose elements are

$$C_a^{\pm}=\frac{1}{m}(\varepsilon_{abc}W^{bc}\pm 2iW_{0a}),\quad a=1,2,3.$$

In accordance with (18.9a) these operators satisfy the relations (compare with (2.3))

$$[C_a^{\pm},C_b^{\pm}]_+\psi=2\delta_{ab}\psi$$

where ψ is a solution of the Dirac equation, for any of two possible (fixed) values of the superscript.

But the most interesting algebraic structure included in the *SA(30)* is formed by the following linear combinations [1*]

$$Q_a^{\pm}=2W_{4a}\pm\varepsilon_{abc}W^{bc},\quad H_{ss}=P_{nn}+m^2 I.$$

For any fixed sign these operators satisfy the relations

$$[Q_a^{\pm},Q_b^{\pm}]_+=2\delta_{ab}H_{ss},\quad [Q_a^{\pm},H_{ss}]=0$$

and so realize a representation of a superalgebra characterizing supersymmetric quantum mechanics with three supercharges [417]. Following [417] we denote this superalgebra by *sqm(3)*.

Thus the extended first order symmetries of the Dirac equation have a reach algebraic structure. In particular they include the subsets realizing representations of the superalgebra *sqm(3)* and in this sense *the Witten supersymmetry is generated by the SOs of the Dirac equation.*

The commutation and anticommutation relations (18.7), (18.9) are used in the following to calculate a complete set of SOs of arbitrary order n for the Dirac equation. Besides we present some additional useful relations which are satisfied on the set of solutions of the Dirac equation:

$$W_{\mu\nu}=\frac{1}{m}(P_\mu W_\nu - P_\nu W_\mu), \quad W_{4\mu}=W_\mu,$$

$$B=\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}, \tag{18.11a}$$

$$A_\mu=\frac{1}{m}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}W^{\sigma}-\frac{1}{2m}P_\mu;$$

$$P_\mu P^\mu=m^2, \quad P_\mu W^\mu=0, \quad P_\mu A^\mu=m, \quad P_\mu W^{\mu\nu}=mW^\nu,$$

$$[J_{\mu\nu},W^\nu]_+=\frac{1}{2}[B,P_\mu]_+, \quad [J_{\mu\nu},W^{\mu\nu}]_+=mB,$$

$$\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}W^{\rho\sigma}=\frac{3}{2}m, \quad \varepsilon_{\mu\nu\rho\sigma}W^{\nu\rho}P^{\sigma}=0, \quad [J_{\mu\nu},A^\nu]_+=0, \tag{18.11b}$$

$$[J_{\mu\nu},W_{\rho\sigma}]_+-[J_{\rho\sigma},W_{\mu\nu}]_+=\frac{1}{4}(\varepsilon_{\nu\rho\sigma\lambda}P_\mu+\varepsilon_{\rho\mu\nu\lambda}P_\sigma-\varepsilon_{\mu\rho\sigma\lambda}P_\nu-\varepsilon_{\sigma\mu\nu\lambda}P_\rho, A^\lambda]_+.$$

18.4. The Complete Set of SOs of Arbitrary Order

According to Subsection 18.2 describing of all the nonequivalent SOs of order n for the Dirac equation reduces to going through the linearly independent combinations of the kind

$$Q^{ck}=\eta^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_kb_{k-c}]}P_{a_1}P_{a_2}\dots P_{a_c}J_{a_{c+1}b_1}\dots J_{a_kb_{k-c}} \tag{18.12}$$

where P_a, J_{ab} are the generators (2.22), $\eta^{\cdots}$ are arbitrary numbers. The index k can take any integer value from the interval $[0,n]$. Moreover, a priori we cannot exclude the possibilities $k>n$. It will be demonstrated in the following that it is sufficient to set

$$0\le k\le n+2, \quad 0\le c\le k. \tag{18.13}$$

According to (1.14), (2.37), (18.11a) the tensors $\eta^{\cdots}$ have to satisfy the conditions (1)-(3) formulated before Lemma 16.1 (see page 184) but do not have the property (4). In other words these tensors are not basic; the reason of this is that the corresponding Lubanski-Pauli vector (2.37), (2.22) is nonzero in contrast to the case of the KGF equation.

To describe effectively linearly independent SOs of (18.12), it is convenient to expand $\eta^{\cdots}$ in basic tensors. We present the first terms arising by this expansion

$$\eta^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_kb_{k-c}]}=\lambda^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_kb_{k-c}]}+\varepsilon^{a_{k-1}b_{k-1-c}a_kb_{k-c}}\lambda^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{k-2}b_{k-2-c}]}+$$

$$+\varepsilon_{d_1}{}^{b_1a_{c+1}a_1}\lambda^{a_2a_3\dots a_{c-1}[a_cd_1][a_{c+2}b_2][a_{c+3}b_3]\dots[a_kb_{k-c}]}+\varepsilon_{d_1}{}^{b_1a_{c+1}a_1}\lambda^{d_1a_2a_3\dots a_c[a_{c+2}b_2]\dots[a_kb_{k-c}]}+$$

$$
\begin{aligned}
&+\varepsilon_{d_1}^{\ b_1a_{c+1}a_1}\lambda^{d_1a_2a_3\ldots a_c[a_{c+2}b_2]\ldots[a_kb_{k-c}]}+\varepsilon_{d_1}^{\ b_1a_{c+1}a_1}\varepsilon_{d_2}^{\ d_1a_{c+2}b_2}\lambda^{d_2a_2\ldots a_c[a_{c+3}b_3]\ldots[a_kb_{k-c}]}+\\
&+\varepsilon^{a_{c+1}b_1a_{c+2}b_2}\varepsilon^{a_{c+3}b_3a_{c+4}b_4}\lambda^{a_1a_2\ldots a_c[a_{c+5}b_5]\ldots[a_kb_{k-c}]}+\\
&+\varepsilon^{a_{c+1}b_1a_{c+2}b_2}\varepsilon_{d_1}^{\ b_3a_{c+3}a_1}\lambda^{d_1a_2\ldots a_c[a_{c+4}b_4]\ldots[a_kb_{k-c}]}+ \qquad (18.14)\\
&+\varepsilon^{a_{c+1}b_1a_{c+2}b_2}\varepsilon_{d_1}^{\ b_3a_{c+3}a_1}\lambda^{a_2a_3\ldots a_{c-1}[d_1a_c][a_{c+4}b_4]\ldots[a_kb_{k-c}]}+\\
&+\varepsilon_{d_1}^{\ b_1a_{c+1}a_1}\varepsilon_{d_2}^{\ d_1a_{c+2}b_2}\varepsilon_{d_3}^{\ d_2a_{c+3}b_3}\lambda^{d_3a_2a_3\ldots a_c[a_{c+4}b_4]\ldots[a_kb_{k-c}]}+\ldots\ .
\end{aligned}
$$

Here $\lambda^{\cdots}$ are basic tensors, the dots in the last line denote products of three and more then three completely antisymmetric tensors $\varepsilon_{\mu\nu\rho\sigma}$ and symmetrization is imposed over the indices a_1, a_2,...,a_c and over the pairs of indices $[a_{c+i}b_i]$ (i=1,2,...,k-c). Calculating various convolutions $\eta^{\cdots}$ with $\varepsilon_{\mu\sigma\lambda\rho}$ we can convert formula (18.14), i.e., express $\lambda^{\cdots}$ via $\eta^{\cdots}$.

Let us substitute (18.14) into (18.12) and go through all the values of k with increasing order.

The first term in the r.h.s. of (18.14) corresponds to the SO of the kind

$$
Q_1^{\ k}=\sum_{c=0}^{k}\lambda_1^{a_1a_2\ldots a_c[a_{c+1}b_1]\ldots[a_kb_{k-c}]}P_{a_1}P_{a_2}\ldots P_{a_c}J_{a_{c+1}b_1}\ldots J_{a_kb_{k-c}} \qquad (18.15)
$$

where P_a and J_{ab} are the Poincaré group generators of (2.22). The order of this operator is equal to k, the general form of Q_1 coincides with (16.14) up to the change from J_{ab} (1.6) to the corresponding generators of (2.22). The number of such operators of order n is equal to the number of SOs of the KGF equation and can be obtained from (16.15) by setting j=n:

$$
N_1^{(n)}=\frac{1}{4!}(n+1)(n+2)(2n+3)(n^2+3n+4). \qquad (18.16)
$$

Using relations (2.37), (18.11) we obtain the following representation for the SOs corresponding to the second, third, fourth and fifth-seventh terms in the r.h.s. of (18.14)

$$
Q_2^{\ k}\sim BQ_1^{\ k-2},\quad Q_3^{\ k}\sim W_\mu Q_1^{\ k-2},\quad Q_4^{\ k}\sim W_{\mu\nu}Q_1^{\ k-3},\quad Q_5^{\ k}\sim A_\mu Q_1^{\ k-3} \qquad (18.17)
$$

where $Q_1^{\ k-i}$ are the operators (18.14), B, W_μ, $W_{\mu\sigma}$ and A_μ are the operators (18.15). It is easy to see the order of Q_2 and $Q_3^{\ k}$ is equal to k-1, the order of $Q_4^{\ k}$ and $Q_5^{\ k}$ is k-2. Thus to obtain SOs of order n we have to set k=n+1 for $Q_2^{\ k}$, $Q_3^{\ k}$ and k=n+2 for $Q_4^{\ k}$, $Q_5^{\ k}$, and k=n+2 is the maximal needed value of k. This circumstance has already been noted in the above.

The terms denoted by dots in (18.14) can be neglected without loss of generality. The corresponding SOs include products of the operators (18.5) and so reduce to the form (18.15), (18.17) with a smaller value of k in accordance with relations (18.7), (18.9).

We see that any finite order SO of the Dirac equation can be represented as a product of a SO of (18.15) (where P_μ and $J_{\mu\sigma}$ are the Poincaré group generators (2.22) corresponding to the Dirac equation) with *one* of the operators (18.5).

Using relations (18.11b) we represent the SOs of (18.17) in a more precise form. For Q_2^k we obtain easily, setting $k=n+1$ and omitting the top index

$$Q_2=B\sum_{c=0}^{n-1}\lambda_2^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-1-c}]}P_{a_1}\dots P_{a_c}J_{a_{c+1}b_1}\dots J_{a_{n-1}b_{n-c-1}} \tag{18.18}$$

where $\lambda_2^{\cdots}$ are arbitrary basic tensors. The number of the linearly independent operators of (18.17) is given by formula (16.15) for $j=n$-1, i.e.,

$$N_2^{(n)}=\frac{1}{4!}n(n+1)(2n+1)(n^2+n+2). \tag{18.19}$$

Using (18.11b) we obtain for Q_3^k, $k=n+1$:

$$Q_3=\sum_{c=1}^{n-1}\lambda_3^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-1-c}]}W_{a_1}P_{a_2}\dots P_{a_c}J_{a_{c+1}b_1}\dots J_{a_{n-1}b_{n-1-c}} \tag{18.20}$$

where $\lambda^{\cdots}$ are basic tensors satisfying the conditions

$$\lambda_3^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-1-c}]}g_{b_1b_2}g_{a_{c+1}a_{c+2}}=0;$$

$$\lambda_3^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-1-c}]}g_{a_1a_{a_{c+1}}}g_{a_2a_{c+2}}=0;$$

$$\lambda_3^{a_1[a_2b_1][a_3b_2]\dots[a_{f+1}b_f]}g_{a_1b_1}=0.$$

Calculating the number of independent components of $\lambda_3^{\cdots}$ we find the number of linearly independent operators (18.20):

$$N_3^{(n)}=\frac{1}{6}n(n+1)(5n^2-3n+13)-n. \tag{18.21}$$

For Q_4 we have

$$Q_4=\sum_{c=0}^{n-2}\lambda_4^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-1-c}]}P_{a_1}P_{a_2}\dots P_{a_c}W_{a_{c+1}b_1}J_{a_{c+2}b_2}\dots J_{a_{n-1}b_{n-1-c}} \tag{18.22}$$

where $\lambda_4^{\cdots}$ are basic tensors satisfying the conditions

$$\lambda_4^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_fb_{f-c}]}g_{a_1b_1}g_{a_2b_2}\dots g_{a_{f-c}b_{f-c}}=0,\quad c\geq\frac{f}{2};$$

$$\lambda_4^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_fb_{f-c}]}g_{a_{c+1}a_{c+2}}\dots g_{a_{f-1}a_f}g_{b_1b_2}\dots g_{b_{f-c-1}b_{f-c}}=0.$$

The number of linearly independent operators is

$$N_4^{(n)}=N_1^{(n)}-\frac{n}{6}(2n^2+9n+13)-\varepsilon_n,\quad \varepsilon_n=\frac{1}{2}[1-(-1)^n]. \tag{18.23}$$

Eventually we obtain for the operator Q_5

$$\begin{aligned}Q_5=&\sum_{c=1}^{n-1}\lambda_5^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-1}b_{n-c-1}]}A_{a_1}P_{a_2}\dots P_{a_c}J_{a_{c+1}b_1}\dots J_{a_{n-1}b_{n-c-1}}+\\&+\sum_{i=0}^{\{(n-1)/2\}}\sum_{c=0}^{n-3-2i}\lambda_6^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-2-2i}b_{n-2-2i-c}]}P_{a_1}\dots P_{a_c}A_{a_{c+1}}P_{b_1}J_{a_{c+2}b_2}\dots J_{a_{n-2-2i}b_{n-2-2i-c}}(J_{\mu\nu}J^{\mu\nu})^i+\\&+\sum_{i=0}^{\{n/2-1\}}\lambda^{[a_1b_1][a_2b_2]\dots[a_{n-3-2i}b_{n-3-2i}]}A_{a_1}J_{b_1c}P^{c}J_{a_2b_2}J_{a_3b_3}\dots J_{a_{n-3-2i}b_{n-3-2i}}(J_{\mu\sigma}J^{\mu\sigma})^i+\\&+\sum_{c=1}^{n-2}\lambda_8^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_{n-2}b_{n-2-c}]}\varepsilon_{a_1\mu\nu\sigma}J^{\mu\nu}A^{\sigma}P_{a_2}P_{a_3}\dots P_{a_c}J_{a_{c+1}b_1}J_{a_{c+2}b_2}\dots J_{a_{n-2}b_{n-2-c}}.\end{aligned} \tag{18.24}$$

Here $\lambda_6^{\cdots}$, $\lambda_7^{\cdots}$ are arbitrary irreducible tensors, $\lambda_5^{\cdots}$ and $\lambda_8^{\cdots}$ are basic tensors satisfying the conditions

$$\lambda_\alpha^{a_1a_2\dots a_c[a_{c+1}b_1]\dots[a_fb_{f-c}]}g_{a_1b_1}g_{a_2b_2}\dots g_{a_cb_c}=0,\quad c\le f/2,\quad \alpha=5,8,$$

and the corresponding number of linearly independent operators Q_5 is

$$N_5^{(n)}=\frac{1}{6}n(n+1)(n+3)(n^2+n+1). \tag{18.25}$$

The general expression of a SO of order n for the Dirac equation is

$$Q^{\,n}=Q_1+Q_2+Q_3+Q_4+Q_5 \tag{18.26}$$

where Q_i are the operators given in (18.25), (18.18), (18.20), (18.22), (18.24). The complete number of SOs of order n can be obtained by summing up of (18.16), (18.19), (18.21), (18.23), (18.25):

$$N^{(n)}=\sum_{i=1}^{5}N_i^{(n)}=5N_n^{(1)}-\frac{1}{6}(2n+1)(13n^2+19n+18)-\varepsilon_n. \tag{18.27}$$

In particular

$$N^{(0)}=1,\quad N^{(1)}=25,\quad N^{(2)}=154,\quad N^{(3)}=601. \tag{18.28}$$

Let us formulate the obtained results in the form of the following assertion.

THEOREM 18.1 [329]. The Dirac equation admits $N^{(n)}$ linearly independent SOs of order n, moreover, $N^{(n)}$ and the explicit form of the corresponding SOs is given in (18.27), (18.26).

18.5. Examples and Discussion

We have calculated the number and explicit form of all the linearly independent SOs of arbitrary order n for the Dirac equation. These SOs are defined up to arbitrary basic and irreducible tensors satisfying some additional restrictions. Expanding basic tensors in irreducible ones we can obtain realizations of SOs depending on indecomposable sets of parameters, compare with (16.14), (16.16).

We present linearly independent SOs of the second and third orders (SOs of zero order reduce to the unit matrix, the first order SOs are given in (2.22), (18.15), (18.5)).

$$
\begin{aligned}
n=2:\quad & \lambda_1^{ab}P_aP_b,\ \tilde{\lambda}_1^{ab}J_{ac}J^{c}{}_{b},\ \lambda_1^{a[bc]}P_aJ_{bc},\ \lambda_1^{a}J_{ab}P^{b},\\
& \lambda_1^{[ab][cd]}J_{ab}J_{cd},\ \lambda_1J_{ab}J^{ab},\ \lambda_2^{a}P_aB,\ \lambda_2^{[ab]}BJ_{ab},\\
& \lambda_3^{ab}P_aW_b,\ \lambda_3^{a[bc]}W_aJ_{bc},\ \lambda_4^{a[bc]}P_aW_{bc},\ \lambda_4^{ab}W_{ac}J^{c}{}_{b},\\
& \lambda_4^{[ab][cd]}W_{ab}J_{cd},\ \lambda_5^{ab}P_aA_b,\ \lambda_5^{a[bc]}A_aJ_{bc},\ \lambda_5^{a}\varepsilon_{abcd}A^{b}J^{cd},\ \lambda_6^{[ab]}P_aA_b;
\end{aligned}
\tag{18.29}
$$

$$
\begin{aligned}
n=3:\quad & \lambda_1^{abc}P_aP_bP_c,\ \lambda_1^{ab[cd]}P_aP_bJ_{cd},\ \lambda_1^{a[bc][de]}P_aJ_{bc}J_{de},\\
& \lambda_1^{[ab][cd][ef]}J_{ab}J_{cd}J_{ef},\ \lambda_1^{ab}P_aJ_{bc}P^{c},\ \lambda_1^{a[bc]}J_{ad}P^{d}J_{bc},\\
& \lambda_1^{a}P_aJ_{bc}J^{bc},\ \lambda_1^{ab}J_{ab}J_{cd}J^{cd},\ \tilde{\lambda}_1^{abc}P_aJ_{bk}J^{k}{}_{c},\ BQ^{(2)},\\
& \lambda_3^{abc}W_aP_bP_c,\ \lambda_3^{ab[cd]}W_aP_bJ_{cd},\ \lambda_3^{a[bc][de]}W_aJ_{bc}J_{de},\ \lambda_3^{ab}W_aJ_{bc}P^{c},\\
& \tilde{\lambda}_3^{abc}W_aJ_{bk}J^{k}{}_{c},\ \lambda_4^{ab[cd]}P_aP_bW_{cd},\ \lambda_4^{a[bc][de]}P_aJ_{bc}W_{de}\\
& \lambda_4^{[ab][cd][ef]}W_{ab}J_{cd}J_{ef},\ \lambda_4^{a[bc]}J_{ad}P^{d}W_{bc},\ \lambda_4^{abc}P_aJ_{bk}W^{k}{}_{c},\\
& \lambda_4^{[ab]}W_{ab}J_{cd}J^{cd},\ \tilde{\lambda}_4^{ab[cd]}(W_{cd}J_{ak}+J_{cd}W_{ak})J^{k}{}_{b},\ \lambda_5^{abc}P_aP_bA_c,\\
& \lambda_5^{ab[cd]}P_aA_bJ_{cd},\ \lambda_5^{ab}A_aJ_{bc}P^{c},\ \lambda_6^{[ab][cd]}P_aA_bJ_{cd},\\
& \lambda_5^{a[bc][de]}A_aJ_{bc}J_{de},\ \lambda_5^{abc}A_aJ_{bk}J^{k}{}_{c},\ \lambda_5^{a}A_aJ_{bc}J^{bc},\\
& \lambda_7^{[ab]}A_aJ_{bc}P^{c},\ \lambda_8^{a[bc]}\varepsilon_{akln}J_{bc}J^{kl}A^{n},\ \lambda_8^{ab}\varepsilon_{akln}P_bJ^{kl}A^{n}.
\end{aligned}
\tag{18.30}
$$

Here P_a, J_{ab}, W_a, W_{ab}, B, A_b are the operators (2.22), (18.5), the values of the Latin indices run from 0 to 3, $\lambda_{\cdot}^{\cdot\cdot}$ are arbitrary irreducible tensors, $\{Q^{(2)}\}$ is the set of the second-order operators (18.29). It is not difficult to calculate the numbers of the linearly independent operators (18.29) and (18.30) which coincide with given in (18.28).

We note that the set of second-order SOs of (18.29) differs from found in

[243] where a part of SOs is linearly dependent on the set of solutions of the Dirac equation.

18.6. SOs of the Massless Dirac Equation

Here we present a complete sets of the first and second-order SOs of the massless Dirac equation.

THEOREM 18.2. The massless Dirac equation has 52 linearly independent SOs in the class M_1. A basis in the space of such SOs can be chosen in the form

$$\begin{aligned} &P_\mu,\ J_{\mu\nu},\ K_\mu,\ D,\ F=i\gamma_4,\ \tilde{P}_\mu=i\gamma_4 P_\mu,\ I,\\ &\tilde{J}_{\mu\nu}=i\gamma_4 J_{\mu\nu},\ \tilde{K}_\mu=i\gamma_4 K_\mu,\ \tilde{D}=i\gamma_4 D; \end{aligned} \tag{18.31}$$

$$\begin{aligned} &A_\mu=(D-i)\gamma_\mu-\gamma_\sigma x^\sigma p_\mu,\quad \omega_{\mu\nu}=\gamma_\mu p_\nu-\gamma_\nu p_\mu,\\ &\tilde{A}_\mu=i\gamma_4 A_\mu,\quad Q_{\mu\nu}=i([K_\mu,A_\nu]-[K_\nu,A_\mu]) \end{aligned} \tag{18.32}$$

where P_μ, $J_{\mu\sigma}$, K_μ and D are the conformal group generators of (2.22), (2.42).

The proof is given in [156]. ■

We see the SOs $Q\in M_1$ of the massless Dirac equation include the conformal group generators, products of these generators (and the identity operator I) with $F=i\gamma_4$ and 20 additional operators (18.32). We emphasize the SOs (18.32) cannot be expressed via generators of the group $C(1,3)$ and the matrix $i\gamma_4$ and are essentially new in this sense. Thus the massless Dirac equation admits SOs which do not belong to the enveloping algebra of the algebra $AC(1,3)$ in contrast to the case of nonzero mass.

It is not difficult to make sure the operators (18.31), (18.32) do not form a Lie algebra. However we can select such subsets of the first order SOs which have a structure of a Lie algebra. The evident example of such a set is given by the conformal group generators, a more extended set is formed by the operators (18.31) which define a basis of the 32-dimensional Lie algebra. The structure constants of this algebra are easily calculated using the commutativity $i\gamma_4$ with P_μ, $J_{\mu\sigma}$, K_μ, D and the relation $\gamma_4^2=-1$.

We present the set of SOs forming a superalgebra. This set includes the following basis elements

$$\{\omega_{\mu\nu}, F=i\gamma_4, FP_\mu;\ P_\mu, J_{\mu\nu}, D, P_{\mu\nu}=p_\mu p_\nu\}. \tag{18.33}$$

Indeed, $\omega_{\mu\sigma}$ satisfies the anticommutation relations (18.9a) and commutation relations (18.6) with P_μ, $J_{\mu\sigma}$. Besides that

$$[FP_\mu,F]_+=2P_\mu,\quad [FP_\mu,FP_\sigma]_+=2P_{\mu\sigma},\quad [\omega_{\mu\nu},F]_+=[\omega_{\mu\nu},FP_\lambda]_+=0. \tag{18.34}$$

The algebra (18.33) includes 36 elements nine of which, i.e., $P_{\mu\sigma}$ belong to a more wide class than M_1 (we recall that $P_{\mu\sigma} g^{\mu\sigma}=m^2 I$).

We present the subset of SOs forming a basis of a 42-dimensional Lie algebra. This subset includes the operators (18.31) and the following ten operators

$$F_\mu=\tilde{A}_\mu+A_\mu,\quad \hat{Q}_{\mu\nu}=Q_{\mu\nu}+\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}Q^{\rho\sigma},\quad \hat{\omega}_{\mu\nu}=\omega_{\mu\nu}+\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\omega^{\rho\sigma} \tag{18.35}$$

(only three of the operators $\hat{Q}_{\mu\nu}$ are linearly independent, the same is true for $\hat{\omega}_{\mu\nu}$). The operators (18.35) form a commutative ideal of this algebra.

In analogy with Section 16 it is possible to find complete sets of SOs of arbitrary order n for the massless Dirac equation. We will not do it here restricting ourselves to a remark that these sets are completely defined by two conformal Killing tensors of valence n and by the generalized Killing tensor of valence R_1+2R_2 where R_1=n-1, R_2=1 (see Appendix 2 for definitions). The number N^n of linearly independent SOs of order n is

$$N^n=\frac{1}{3}(n+1)(n+2)(2n+3)(n^2+3n+1), \tag{18.36}$$

besides $\hat{N}^n$ of these operators do not belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$, where

$$\hat{N}^n=\frac{1}{6}n(n+1)(n+2)(n+3)(2n+3). \tag{18.37}$$

Formulae (18.36), (18.37) do not include the number of independent SOs whose order is less then n.

In conclusion we discuss briefly SOs of the Weyl equation. In fact we have already described such SOs in the class M_1. Indeed, using the Majorana representation (2.13) for γ-matrices we conclude that the corresponding SOs (18.31), (18.32) transform real solutions into real ones. On the other hand setting in the Weyl equation (2.44)

$$\varphi_+=\psi_1+i\psi_2 \tag{18.38}$$

where ψ_1 and ψ_2 are real functions we come to the Dirac equation with γ-matrices realizing the Majorana representation, moreover, the corresponding wave function has the form

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\varphi_+ +\varphi_+^*\\ -i(\varphi_+-\varphi_+^*)\end{pmatrix}. \tag{18.39}$$

The SOs (18.31), (18.32) are valid for arbitrary representation of γ-matrices and thus they are SOs for the Majorana equation (17.1), (2.13). These operators generate linear transformations of ψ or, which is the same, linear and antilinear transformations of φ_+ being a solution of the Weyl equation.

So the Weyl equation has exactly 52 linearly independent SOs in the class M_1^*. These operators can be chosen in the form (18.31), (18.32), (2.13), their action on solutions of the Weyl equation is easily calculated using (18.38), (18.39).

The SOs of the Weyl equation in the class M_2 are found in [326]. Here we present the principal assertion of paper [326] only.

THEOREM 18.3. The Weyl equation has 84 SOs belonging to the class M_2. These SOs have the form

$$\begin{gathered}\lambda^{\mu\nu}P_\mu P_\nu,\quad \lambda^{\mu[\rho\sigma]}P_\mu J_{\rho\sigma},\quad \lambda^\mu J_{\mu\sigma}P^\sigma,\quad \lambda^{[\mu\nu][\rho\sigma]}J_{\mu\nu}J_{\rho\sigma},\quad \hat{\lambda}^{\mu\rho}J_{\mu\sigma}J^\sigma{}_\rho,\\ \lambda J_{\mu\sigma}J^{\mu\sigma},\quad \eta^{\mu\sigma}K_\mu K_\sigma,\quad \eta^{\mu[\rho\sigma]}K_\mu J_{\rho\sigma},\quad \eta^\mu J_{\mu\sigma}K^\sigma,\quad \zeta^{\mu\sigma}J_{\mu\sigma}D\end{gathered} \tag{18.40}$$

where the Greek indices denote arbitrary irreducible tensors, P_μ, $J_{\mu\sigma}$, K_μ, D are generators of the conformal group. ■

We see that all the SOs of the class M_2 for the Weyl equation belong to the enveloping algebra of the algebra *AC(1,3)* in contrast to the massless Dirac equation.

19. SYMMETRIES OF EQUATIONS FOR ARBITRARY SPIN PARTICLES

19.1. Symmetries of the KDP Equation

In this section we investigate symmetry properties of relativistic wave equations for particles of higher spins, i.e., the equations of KDP, TCT, the Dirac-like equations etc. It turns out that besides the Poincaré invariance these equations have additional (hidden) symmetries which are more extensive then in the cases of the Dirac or KGF equations.

We write the KDP equation for a particle of spin 1 in the form

$$L\psi\equiv(\beta^\mu p_\mu - m)\psi = 0 \tag{19.1}$$

where ψ is a ten-component wave function, β^μ are 10×10 KDP matrices satisfying the algebra (6.20).

As it was noted in Section 6 the KDP equation is invariant under the 10-dimensional Lie algebra of the Poincaré group. Basis elements of this algebra can be chosen in the covariant form (2.22) where $S_{\mu\sigma}=[\beta_\mu,\beta_\sigma]$. It is possible to show the Poincaré algebra is the maximal IA of the KDP equation in the class M_1.

Here we demonstrate the KDP equation has a wide nongeometric symmetry. The nature of this symmetry lies in special properties of ß-matrices and is not connected with transformations of independent variables.

Let us show the equation (19.1), like the Dirac equation, is invariant under the algebra A_8, defined over the field of real numbers. Basis elements of this IA belong to the class M_1 and are given by the formulae [142, 154]

$$\Sigma_{\mu\nu}=-iS_{\mu\nu}-\frac{i}{m}(a_\mu p_\nu-a_\nu p_\mu),\quad \Sigma_0=I,\quad \Sigma_1=\beta_4+\frac{i}{m}a^\mu p_\mu, \tag{19.2}$$

where

$$a_\mu=S_{5\mu}+iS_{4\mu},\quad S_{kl}=i[\beta_k,\beta_l],\quad S_{5k}=i\beta_k;\quad k,l=0,1,2,3,4. \tag{19.3}$$

Indeed, using the fact that β_μ satisfy the algebra (6.20) and S_{mn} belong to the algebra $AO(2,4)$ we obtain easily the following relations

$$[\Sigma_{\mu\nu},L]=f_{\mu\nu}L,\quad f_{\mu\nu}=\frac{1}{2m^2}(L+2m)(\beta_\mu p_\nu-\beta_\nu p_\mu), \tag{19.4}$$

$$[\Sigma_\alpha,L]=0,\quad \alpha=0,1. \tag{19.5}$$

The l.h.s. of (19.4) includes differential operators of order 2 meanwhile the r.h.s. looks like a third order differential operator. But there is no contradiction inasmuch as according to (6.20)

$$\beta^\mu p_\mu(\beta_\lambda p_\sigma-\beta_\sigma p_\lambda)\beta^\rho p_\rho\equiv 0.$$

It follows from (19.4), (19.5) the operators (19.2) do are SOs of the KDP equation. These operators satisfy the commutation relations (17.7) characterizing the algebra A_8. The last statement can be easily verified by making the transformation

$$\Sigma_{\mu\nu}\to V\Sigma_{\mu\nu}V^{-1}=[\beta_\mu,\beta_\nu],\quad \Sigma_0\to V\Sigma_0V^{-1}=I,\quad \Sigma_1\to V\Sigma_1V^{-1}=\beta_4 \tag{19.6}$$

where $V=\exp(ia_\mu p^\mu/\mathrm{m})$.

Thus the KDP equation possesses the same nongeometric symmetry in the class M_1 as the Dirac equation, compare with Theorem 17.1. It is possible to show the Lie algebra spanned on the basis (19.2) is the maximally extended IA of the KDP equation in this class. But in contrast to the Dirac equation the KDP equation has a wide symmetry in the class M_2 as it follows from the following assertion.

THEOREM 19.1. The KDP equation is invariant under the 18-dimensional Lie algebra defined over the field of complex numbers. Basis elements of this algebra belong to the class M_2 and are given by the following formulae

$$\lambda_{ab}=C_aC_b,\quad \tilde{\lambda}_{ab}=\frac{1}{2}(D_aC_b+C_aD_b),\quad \lambda_1=1,\quad \lambda_0=\frac{1}{2}D_aC_a \tag{19.7}$$

where

$$C_a=\frac{1}{2}\varepsilon_{abc}\Sigma_{bc},\quad D_a=\Sigma_{0a},\quad a,b=1,2,3.$$

PROOF. The fact that the operators (19.7) are SOs of the KDP equation follows immediately from (19.4), (19.5). Besides that, these operators satisfy the following commutation relations

$$\begin{aligned}&[\lambda_{ab},\lambda_{cd}]=-[\hat{\lambda}_{ab},\hat{\lambda}_{cd}]=if^{kl}_{abcd}\lambda_{kl},\quad [\lambda_{ab},\hat{\lambda}_{cd}]=if^{kl}_{abcd}\hat{\lambda}_{kl},\\ &[\lambda_1,\lambda_2]=[\lambda_\alpha,\lambda_{cd}]=[\lambda_\alpha,\hat{\lambda}_{cd}]=0,\quad \alpha=0,1,\end{aligned}\tag{19.8}$$

where f^{kl}_{abmn} are the structure constants of the algebra *ASU(3)* in the Okubo basis (see, e.g., [374]). The validity of relations (19.8) is easily verified using the representation (19.6). ■

Thus besides the symmetry under the Poincaré algebra the KDP equation is invariant in respect with the 18-dimensional Lie algebra spanned on the basis (19.7). The algebra (19.7) includes A_8 as a subalgebra besides basis elements of A_8 are linear combinations of the operators (19.7). For instance, $\Sigma_{ab}=i(\lambda_{ba}-\lambda_{ab})$.

Is it possible to unite the Poincaré algebra and the algebra (19.7)? Such an unification is very natural because the commutators of the operators (19.7) with the generators of the Poincaré group are expressed via linear combinations of these operators:

$$\begin{aligned}&[J_a,\lambda_{bc}]=-[J_{0a},\hat{\lambda}_{bc}]=i(\varepsilon_{abd}\lambda_{dc}+\varepsilon_{acd}\lambda_{bd}),\\ &[J_a,\hat{\lambda}_{bc}]=[J_{0a},\lambda_{bc}]=i(\varepsilon_{abd}\hat{\lambda}_{dc}+\varepsilon_{acd}\hat{\lambda}_{bd}),\\ &[P_\mu,\lambda_{ab}]=[P_\mu,\hat{\lambda}_{ab}]=[P_\mu,\lambda_\alpha]=[J_{\mu\nu},\lambda_\alpha]=0\end{aligned}\tag{19.9}$$

where $J_a=\varepsilon_{abc}J_{bc}/2$.

The relation (1.14), (19.8), (19.9) define a 28-dimensional Lie algebra being the IA of the KDP equation. Starting with this algebra it is not difficult to reconstruct the corresponding symmetry group. The transformations generated by P_μ, $J_{\mu\sigma}$ have already been considered in the above (see (3.30)-(3.32)). As to the transformations generated by the operators (19.7) they can be found explicitly using the formula

$$\psi\rightarrow\psi'(Q_f\theta_f)\psi,\quad Q_f\subset\{\lambda_{ab},\tilde{\lambda}_{ab},\lambda_\alpha\},\tag{19.10}$$

where θ_f are real parameters. The corresponding exponentials are easily calculated:

$$\exp(Q_f\theta_f)=1+Q_f\theta_f,\quad Q_f\subset\{\lambda_{ab},\tilde{\lambda}_{ab},a\neq b\},$$

$$\exp(\lambda_{bb}\theta_b)=1+\lambda_{bb}(\exp\theta_b-1);\quad \exp(\lambda_0\theta_0)=\exp\theta_0,$$

$$\exp(Q_A\theta_A)=1+Q_A^2(\cosh\theta-1)+Q_A\sinh\theta_A,\quad Q_A=\lambda_{bb},\lambda_1.$$

Using the identities $a_\mu a_\lambda a_\alpha\equiv 0$ we can show that the general transformation including (19.10) and Lorentz transformations has the form

$$\psi(x)\rightarrow A\psi(x)+B_{\mu}\frac{\partial\psi(x')}{\partial x_{\mu}}+D_{\mu\nu}\frac{\partial^{2}\psi(x')}{\partial x_{\mu}\partial x_{\nu}}$$

where x' are related to x by Lorentz transformation, A, B_{μ} and $D_{\mu\sigma}$ are numeric matrices depending on transformation parameters.

Let us discuss briefly nongeometric symmetries of the Tamm-Sakata-Taketani (TST) equation (7.1), (7.40c), which also describes a particle of spin 1 but has not superfluous components. Inasmuch as solutions of the KDP and TST equations are connected by the transformation (6.37) there is one-to-one correspondence between symmetries of these equations. We give the explicit formulation of a symmetry interesting from the physical point of view.

THEOREM 19.2. The TST equation is invariant under the algebra *ASU(3)* whose basis elements have the form

$$\lambda_{ab}=\hat{S}_{a}\hat{S}_{b} \tag{19.11}$$

where

$$\hat{S}_{a}=S_{a}\left(1+\frac{p^{2}}{2m^{2}}\right)-\frac{p_{a}S_{b}p_{b}}{2m}+\frac{i}{m}\sigma_{1}\varepsilon_{abc}p_{b}S_{c},$$

S_a are matrices realizing the direct sum of two IRs *D(1)* of the algebra *AO(3)*.

We do not present a proof but note that the operators (19.11) satisfy (19.8) and commute with the TST Hamiltonian (7.40c), i.e., form an IA of the TST equation. ■

We note that the KDP and TST equations admit nongeometric IAs in the class M_{∞} also. For the explicit form of the corresponding SOs see [345].

19.2. Arbitrary Order SOs of the KDP equation

Here we present a principle description of SOs for the KGF equation in classes of differential operators of arbitrary order n ($n<\infty$) with matrix coefficients. A formal definition of such operators can be written in the form (16.6), (16.7) where L is the operator (19.1), α_{Q} is a differential operator of order m (besides in general $m\neq n$), $h^{\cdot\cdot}$ are matrices of dimension 10×10 depending on x.

THEOREM 19.3. Any SO of arbitrary order n of the KDP equation belongs to the enveloping algebra of the algebra AP(1,3).

PROOF. As in the case of the Dirac equation (see Subsection 18.2) we transform the KDP equation into such ane equivalent representation that the transformed wave function has $2s+1$ components only. Using the transformation (17.12) where ψ is a ten-component function, L is the operator (19.1),

$$W=1-m^{-1}\beta_4 2\beta^{\mu}p_{\mu}-m^{-2}(\beta^{\mu}p_{\mu})^2 i\beta_4, \quad W^{-1}=1+m^{-1}\beta_4^2\beta^{\mu}p_{\mu}+m^{-2}(\beta^{\mu}p_{\mu})^2 i\beta_4,$$

$$V=1-m^{-1}\beta^{\mu}p_{\mu}\beta_4^2+2m^{-2}(\beta_4^2+i\beta_4)[2(\beta^{\mu}p_{\mu})-p^{\lambda}p_{\lambda}], \tag{19.12}$$

$$V^{-1}=1+m^{-1}\beta^{\mu}p_{\mu}\beta_4^2+2m^{-2}[2(\beta^{\mu}p_{\mu})^2-p^{\lambda}p_{\lambda}(1-m^{-1}\beta^{\mu}p_{\mu})](i\beta_4-\beta_4^2),$$

we come from (19.1) to the equivalent equation of the form

$$L'\psi'\equiv\left[\frac{1}{m}P_1(p^{\mu}p_{\mu}-m^2)-P_2 m\right]\psi'=0 \tag{19.13}$$

where

$$P_1=\frac{1}{2}(i\beta_4-\beta_4^2), \quad P_2=1-\frac{1}{2}(i\beta_4-\beta_4^2). \tag{19.14}$$

The transformations (17.12), (19.12) enable us to establish one-to-one correspondence between SOs of the KDP equation and the equation (19.13) (refer to (18.2)). The last is much more adapted for investigating of symmetries than (19.1) inasmuch as it includes only two (besides that commuting) matrices.

It is convenient to expand SOs of (19.13) in a complete set of numeric matrices. Inasmuch as $\beta_4^3=-\beta_4$ the matrices (19.14) are orthoprojectors. Choosing β_4 in the diagonal form

$$\beta_4=diag(1,1,1,-1,-1,-1,0,0,0,0)$$

we conclude that ψ' of (19.13) has only three nonzero components and so there are exactly eight linearly independent matrices defined on $\{\psi'\}$. We choose the set of such matrices in the form

$$S_{ab}=i[\beta_a,\beta_b], \quad Z_{ab}=[S_{ac},S_{cb}]_+ \tag{19.15}$$

and represent SOs of (19.13) in the form

$$Q'=S_{ab}\tilde{Q}^{ab}+Z_{ab}\hat{Q}^{ab}. \tag{19.16}$$

The operator Q' is a SO of the equation (19.13) iff $\tilde{Q}^{ab}$ and $\hat{Q}^{ab}$ are SOs of the KGF equation. Thus it is easy to show that Q' belongs to the enveloping algebra generated by the generators of the Poincaré group of (2.22), (6.19). Indeed, according to results present in Subsection 16.2 $\tilde{Q}^{ab}$ and $\hat{Q}^{ab}$ are polynomials on P_{μ} and ($J_{\mu\sigma}$ -$S_{\mu\sigma}$). But $S_{\mu\sigma}$ (and therefore the matrices (19.15)) are expressed via P_{μ} and $J_{\mu\sigma}$ according to (18.4), and so Q' is a polynomial on P_{μ} and $J_{\mu\sigma}$.

The operator V of (19.12) commutes with P_{μ} and $J_{\mu\sigma}$ so it follows from (18.2) that $Q'=Q$. Thus any SO of the KDP equation is a polynomial on P_{μ} and $J_{\mu\sigma}$. ■

So to calculate SOs of order n for the KDP equation it is sufficient to go through linearly independent combinations of the kind (18.12). Let us present a complete set of the first order SOs.

For n=1 we obtain a set including the Poincaré group generators of (2.22), (6.19) and the following 15 operators (compare with (18.5))

$$W_{4\mu}=W_{\mu}=i\beta_4 p_{\mu}-mS_{4\mu},\quad W_{\mu\nu}=S_{4\mu}p_{\nu}-S_{4\nu}p_{\mu},$$

$$B=i\beta_4 D-S_{4\mu}x^{\mu}, \tag{19.17}$$

$$A_{\mu}=\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}S_4^{\ \sigma}+2\beta_4^2\beta_{\mu}-\frac{2}{m}(1+\beta_4^2)p_{\mu}$$

where $D=x^{\mu}p_{\mu}+2i$, $S_{4\mu}=i[ß_4,ß_{\mu}]$.

As was done for the Dirac equation (refer to Section 17) it is possible to show that a SO of any order n>1 can be expressed via products of the SOs of the KGF equation (18.15) and *one or two* operators from (19.17).

In contrast with the SOs of the Dirac equation the operators (19.17) cannot be included into a superalgebra like (18.10). Indeed, anticommutators of the operators (19.17) cannot be expressed via P_{μ} in contrast with (18.9a). However, these operators satisfy the following relations

$$[W_{kl},W_{mn}]=\frac{i}{2}(\varepsilon_{lnsf4}P_{km}+\varepsilon_{kmsf4}P_{ln}-\varepsilon_{knsf4}P_{lm}-\varepsilon_{lmsf4}P_{kn})W^{sf}, \tag{19.18a}$$

$$\begin{aligned}W_{mn}W_{kl}W_{sf}+W_{sf}W_{kl}W_{mn}=m^2[(P_{mk}g_{nl}+P_{nl}g_{mk}-P_{ml}g_{nk}-\\ -P_{nk}g_{ml})W_{sf}+(P_{ks}g_{lf}+P_{lf}g_{ks}-P_{kf}g_{ls}-P_{ls}g_{kf})W_{mn},\end{aligned} \tag{19.18b}$$

$$[W_{kl},P_{\mu}]=0,\quad [W_{kl},J_{\mu\nu}]=i(g_{k\nu}W_{l\mu}+g_{l\mu}W_{k\nu}-g_{k\mu}W_{l\nu}-g_{l\nu}W_{k\mu}) \tag{19.18c}$$

where $P_{kl}=p_kp_l$, $p_4=m$, $g_{kl}=diag(1,-1,-1,-1,1)$.

Relations (19.18b) enable us to include W_{kl}, P_{kl} and the generators P_{μ}, $J_{\mu\sigma}$ into the 30-dimensional *parasuperalgebra* $\{W_{4\mu},W_{\mu\sigma};\ P_{\mu},J_{\mu\sigma}\}$ which we denote by $PSA_{(30)}$, besides the odd terms stand to the left of semicolon. The remaining operators are even (for definitions see Appendix 1). Indeed, relations (19.18b), (12.18c) are in accordance with the scheme (A.1.3) defining a parasuperalgebra.

We note that the basis elements (18.5) of the superalgebra $SA_{(30)}$ satisfy relations (19.18b) also, thus

$$SA_{(30)}\Rightarrow PSA_{(30)},$$

and the parasuperalgebra $PSA_{(30)}$ includes the superalgebra $SA_{(30)}$ realizing on the set of solutions of the Dirac equation; the converse is not true.

The parasuperalgebra $PSA_{(30)}$ includes very interesting subalgebras. Among them are the Lie algebra of the Poincaré group $AP(1,3)\supset P_{\mu},J_{\mu\sigma}$, the KDP algebras generated by Q_a of (19.2) and the parasuperalgebras which we denote by $psqm_{\pm}(3)$. The last includes the following elements:

$$Q_a^{\pm}=W_{4a}\pm\frac{1}{2}\varepsilon_{abc}W_{bc},\quad H_{PSS}=p_a p_a+m^2$$

satisfying the relations

$$Q_a^{\pm}[Q_b^{\pm},Q_c^{\pm}]_+ + Q_b^{\pm}[Q_a^{\pm},Q_c^{\pm}]_+ + Q_c^{\pm}[Q_a^{\pm},Q_b^{\pm}]_+ = (\delta_{ab}Q_c^{\pm}+\delta_{ac}Q_b^{\pm}+\delta_{bc}Q_a^{\pm})H_{PSS},$$
$$[H_{PSS},Q_a^{\pm}]=0. \tag{19.18'}$$

These relations characterize the IA of parasupersymmetric quantum mechanics [372] including three parasupercharges. Thus SOs of the KGF equation include the subset realizing a representation of this IA. In this sense parasupersymmetry is generated by symmetries of the KGF equation.

19.3. Symmetries of Dirac-Like Equations for Arbitrary Spin Particles

Here we consider nongeometric symmetries of equations of Dirac type for any spin particles discussed in Section 8.

The equations (8.1), (8.11) have a symmetric form which does not become more complicated by increasing of spin value. This circumstance makes it possible to generalize the main results of Section 17 to the case of arbitrary spin.

The equations (8.1), (8.11) are manifestly invariant under the algebra *AP(1,3)*. It turns out they are invariant under the algebra A_8 also.

THEOREM 19.4. The system (8.1), (8.11) is invariant under the algebra A_8 defined over the field of real numbers. Basis elements of this algebra belong to the class M_1 and have the form

$$\Sigma_{\mu\nu}=\frac{i}{4}[\Gamma_m,\Gamma_\nu]+\frac{1}{2m}(\Gamma_\mu p_\nu-\Gamma_\nu p_\mu)(1-i\Gamma_4),$$
$$\Sigma_0=I,\quad \Sigma_1=\Gamma_4-\frac{1}{m}(i+\Gamma_4)\Gamma^\mu p_\mu. \tag{19.19}$$

The proof can be carried out in a complete analogy with the proof of Theorem 17.3. The commutators of the operators (19.19) with L_2 of (8.11) and L_1 of (8.1) are equal to zero or given by relations (17.5) where $\gamma_\mu\to\Gamma_\mu$. The operators (19.19) satisfy (17.7) and thus form the algebra A_8. ■

We see the Dirac-like equations for particles of arbitrary spin turn out to be invariant under the algebra A_8 realized in the class M_1. So this symmetry is not a specific property of the Dirac and KDP equations but is admissible by equations for arbitrary spin particles.

As in the case of the four-component Dirac equation (see Theorem 17.4) the operators (19.19) form a closed algebra together with the Poincaré group generators.

More precisely they satisfy the commutation relations (17.18). It means the system (18.1), (18.11) is invariant under a 18-dimensional Lie algebra including the subalgebras *AP(1,3)* and A_8.

It is easily seen that for s>1/2 the symmetry of (8.1), (8.11) in the class M_1 is more extensive than described in Theorem 19.4. Indeed, products of the operators (18.19) are SOs of the considered system and belong to the class M_1 also.

THEOREM 19.5. The system (8.1), (8.11) is invariant under the $[10+2(2s+1)^2]$-dimensional Lie algebra isomorphic to $A[P(1,3)\otimes GL(2s+1,C)]$. Basis elements of this algebra belong to the class M_1 and are given by formulae (2.22) (where $S_{\mu\sigma}$ are the matrices (8.6)) and (19.20):

$$\begin{aligned}
\lambda_{n+kn} &= \frac{1}{2}a_{kn}[(\Sigma_{23}-\Sigma_{02})^k P_{s-n+1}(1-i\Sigma_1)+P_{s-n+1}(\Sigma_{23}+\Sigma_{02})^k(1+i\Sigma_1)],\\
\lambda_{nn+k} &= \frac{1}{2}a_{kn}[(\Sigma_{23}-\Sigma_{02})^k P_{s-n+1}(1+i\Sigma_1)+P_{s-n+1}(\Sigma_{23}+\Sigma_{02})^k(1-i\Sigma_1)],\\
\tilde{\lambda}_{mn} &= \Sigma_1\lambda_{mn}
\end{aligned} \tag{19.20}$$

where $\Sigma_{\mu\sigma}$, Σ_1 are the operators (19.19),

$$P_{s-n+1}=\prod_{n'\neq n}\frac{\Sigma_{12}-s-1+n'}{n-n'},\quad n=1,2,...,2s+1;\quad k=0,1,...,2s+1-n, \tag{19.21}$$

a_{kn} are coefficients defined by the following recurrence relations

$$a_{0n}=1,\quad a_{1n}=[n(2s+1-n)]^{1/2},\quad a_{\lambda n}=a_{\lambda-1n}a_{\lambda-1n+\lambda-1},\quad \lambda=2,3,...,2s-n. \tag{19.22}$$

PROOF. The operators (19.20) evidently are SOs of the system (8.1), (8.11) inasmuch as they are products of the SOs present in (19.19).

To prove these operators form a basis of the algebra *AGL(2s+1,C)* it is sufficient to make sure that they are linearly independent and satisfy relations (17.9) for a,b,c,d=1,2,...,2s+1. The simplest way to verify these statements is to transform (19.20) into such a representation where they reduce to numeric matrices. Using the transformation

$$\Sigma_{\mu\nu}\rightarrow V\Sigma_{\mu\nu}V^{-1}=S_{\mu\nu},\quad \Sigma_1\rightarrow V\Sigma_1 V^{-1}=\Gamma_4 \tag{19.23}$$

where V is the operator obtained from (17.13a) by the change $\gamma_k\rightarrow\Gamma_k$, $S_{\mu\sigma}$ are the matrices (8.6), we come to a matrix realization of the basis elements (19.20). Simultaneously the equations (8.1), (8.11) reduce to the form

$$L_1'\psi'=0,\quad L_2'\psi'=0 \tag{19.24}$$

where $\psi'=V\psi$, L_1' has the form (17.13c) where $\gamma_4\rightarrow\Gamma_4$ and

$$L_2'=W(L_2-FL_1)V^{-1}=\frac{1}{s}[S_{\mu\nu}S^{\mu\nu}-4s(s+1)],$$

W is the operator obtained from (17.13b) by the change $\gamma_k \to \Gamma_k$, F is the operator given in (8.23).

An equivalent form of the equations (19.24) is

$$(p_\mu p^\mu - m^2)\psi' = 0, \quad (1 - i\Gamma_4)\psi' = 0, \tag{19.25}$$

$$[S_{\mu\nu}S^{\mu\nu} - 4s(s+1)]\psi' = 0. \tag{19.26}$$

According to (19.26) the function ψ' has exactly $2s+1$ independent components. The equations (19.25), (19.26) are manifestly invariant in respect with arbitrary matrix transformations commuting with Γ_4 and $S_{\mu\sigma}S^{\mu\sigma}$.

There are exactly $(2s+1)\times(2s+1)$ linearly independent complex matrices or twice more real matrices commuting with Γ_4 and $S_{\mu\sigma}S^{\mu\sigma}$. To find a complete set of these matrices we use the fact that $S_{\mu\sigma}$ reduce to the basis elements of the IR *D(s 0)* of the algebra *AO(1,3)* on the set $\{\psi'\}$. All the possible products of them includes $2(2s+1)(2s+1)$ matrices linearly independent over the field of real numbers. These independent matrices can be chosen in the form of the set $\{\lambda'_{ab}, \tilde{\lambda}'_{ab}\}$ where the primed matrices are obtained from (19.20) by the change (19.23). Indeed, using for $S_{\mu\sigma} \subset$ *D(s 0)* the realization (4.63) we make sure that nonzero matrix elements of λ'_{ab} and $\tilde{\lambda}'_{ab}$ are given by the following formulae

$$(\lambda'_{ab})_{mn} = \delta_{am}\delta_{bn}, \quad (\tilde{\lambda}'_{ab})_{mn} = i\delta_{am}\delta_{bn}, \quad a, b = 1, 2, \ldots, 2s+1.$$

These matrices are linearly independent over the field of real numbers and satisfy the commutation relations (17.9).

To complete the proof it is sufficient to calculate commutation relations of the operators (19.20) with generators of the Poincaré group. It can be easily shown that P_μ, $J_{\mu\sigma}$ of (2.22), (8.6) satisfy relations (17.18) with the operators $\Sigma_{\mu\sigma}$, Σ_α of (19.19). Thus the linear combinations

$$\hat{P}_\mu = iP_\mu, \quad \hat{J}_{\mu\sigma} = iJ_{\mu\sigma} + \Sigma_{\mu\sigma} \tag{19.27}$$

satisfy the Poincaré algebra and commute with $\Sigma_{\mu\sigma}$, Σ_α (and therefore with the matrices (19.20)). Because $\Sigma_{\mu\sigma}$ are linear combinations of $\tilde{\lambda}_{mn}$ and λ_{mn}:

$$\Sigma_{12} = \sum_n (s+1-n)\lambda_{nn}, \quad \Sigma_{23} = \frac{1}{2a_{1n}}(\lambda_{nn+1} + \lambda_{n+1n}), \quad \Sigma_{02} = \frac{1}{2a_{1n}}(\tilde{\lambda}_{nn+1} - \tilde{\lambda}_{n+1n}) \tag{19.28}$$

(the remaining $\Sigma_{\mu\sigma}$ are expressed via commutators of the operators (19.20)) it follows from the above that the SOs $\{P_\mu, J_{\mu\nu}, \lambda'_{ab}, \tilde{\lambda}'_{ab}\}$ form a Lie algebra isomorphic to *A[P(1,3)⊗GL(2s+1,C)]*. ■

Thus, the Dirac-like equations for a particle of arbitrary spin s have a wide symmetry in the class M_1 which increases if the spin value increases. The corresponding basis of the IA is given in (2.22), (8.6), (19.20).

Inasmuch as the operators (19.20) satisfy the conditions

$$\lambda_{mn}^2\psi=\delta_{mn}\lambda_{nn}\psi=-\tilde{\lambda}_{mn}^2\psi$$

(no sum over n) it follows from Theorem 19.5 the equations (8.1), (8.11) are invariant under a $[10+2(2s+1)^2]$-parametric Lie group including inhomogeneous Lorentz transformations and the following transformations

$$\psi\to\exp(\lambda_{mn}\theta_{mn})\psi=\begin{cases}(1+\lambda_{mn}\theta_{mn})\psi, & m\neq n,\\ [1+\lambda_{nn}(\exp\theta_{nn}-1)]\psi, & m=n,\end{cases}$$

$$\psi\to\exp(\tilde{\lambda}_{mn}\tilde{\theta}_{mn})\psi=\begin{cases}(1+\tilde{\lambda}_{mn}\theta_{mn})\psi, & m\neq n,\\ [\tilde{\lambda}_{mm}\sin\tilde{\theta}_{mm}+1+\lambda_{mm}(\cos\tilde{\theta}_{mm}-1)]\psi, & m=n\end{cases}$$

(no sum over repeated indices). It is easily verified that the general transformation belonging to this group can be represented in the form (17.19) where A, B_μ are matrices of dimension $8s\times 8s$ depending on transformation parameters.

19.4. Hidden Symmetries Admitted by Any Poincaré-Invariant Wave Equation

Nongeometric symmetry described in the above is inherent in any Poincaré-invariant equation for a particle of arbitrary spin $s>0$. To prove this assertion is the main goal of this subsection.

Let $\{\psi\}$ be a set of solutions of a Poincaré-invariant equation for a particle of spin s and mass $m\neq 0$. We write such an equation in the symbolic form (16.1) and do not impose any restriction on the explicit form of a linear operator L - it can be either a differential operator of arbitrary finite order or integro-differential operator. The only requirement imposed is that the corresponding equation (16.1) be invariant under the algebra $AP(1,3)$, moreover, the representation of this algebra realized on $\{\iota\}$ has to belong to the class I $(P_\mu P^\mu>0)$ and be irreducible in respect with spin and mass (refer to Subsection 6.1). It means the equation (16.1) admits ten SOs P_μ, $J_{\mu\sigma}$ satisfying the commutation relations (1.14) besides eigenvalues of the corresponding Casimir operators $P_\mu P^\mu$ and $W_{\mu\sigma}W^{\mu\sigma}$ are fixed and given by formulae (6.1)[*].

Let us show the Poincaré-invariance of (16.1) implies an additional (nongeometric) symmetry of this equation.

[*] It will be shown in the following that the condition the eigenvalue of $P_\mu P^\mu$ to be fixed is not essential and can be replaced by the weaker requirement $P_\mu P^\mu\psi=m^2\psi$, $\varepsilon\leq m^2<\infty$, $\varepsilon>0$.

THEOREM 19.6 [156]. Any Poincaré-invariant equation for a particle of spin s and mass m>0 is additionally invariant under the algebra *AGL(2s+1,C)*.

PROOF. A Poincaré-invariant equation by definition admits ten SOs satisfying the algebra (1.14). Then the tensors

$$\Sigma^{\pm}_{\mu\nu}=\frac{1}{m^2}[i\varepsilon_{\mu\nu\rho\sigma}W^{\rho}P^{\sigma}\pm(P_{\nu}W_{\mu}-P_{\mu}W_{\nu})] \tag{19.29}$$

where W_μ is the Lubanski-Pauli vector (2.37), are the SOs of (16.1) also.

Using relations (4.3) we sure that the operators (19.29) satisfy the conditions

$$[\Sigma^{\pm}_{\mu\nu},\Sigma^{\pm}_{\lambda\sigma}]=(g_{\mu\lambda}\Sigma_{\nu\sigma}+g_{\nu\sigma}\Sigma_{\mu\lambda}-g_{\mu\sigma}\Sigma_{\nu\lambda}-g_{\nu\lambda}\Sigma_{\mu\sigma})\frac{1}{m^2}P_{\rho}P^{\rho},$$

$$C_6=\frac{1}{2}\Sigma^{\pm}_{\mu\lambda}\Sigma^{\pm\mu\lambda}=\frac{1}{m^4}W_{\lambda}W^{\lambda}P_{\mu}P^{\mu}, \tag{19.30}$$

$$C_7=\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\Sigma^{\pm\mu\nu}\Sigma^{\pm\rho\sigma}=\mp\frac{2i}{m^4}W_{\lambda}W^{\lambda}P_{\mu}P^{\mu}.$$

According to (19.30), (6.1) the operators $\Sigma^{\pm}_{\mu\sigma}$ satisfy relations (17.7) characterizing the algebra *AO(1,3)* besides the eigenvalues of the corresponding Casimir operators C_6 and C_7 are

$$C_6\psi=\pm\frac{i}{2}C_7\psi=s(s+1)\psi.$$

It follows from the above the operators $\Sigma^{+}_{\mu\sigma}$ ($\Sigma^{-}_{\mu\sigma}$) realize the representation *D(s 0)* (*D(0 s)*) of the algebra *AO(1,3)*.

But linearly independent products of the operators (19.29) form a basis of a more wide Lie algebra than *AO(1,3)*. Indeed, choosing such a basis in the form (19.20) where $\Sigma_1=C_7/2s(s+1)$ we come to $(2s+1)^2$ SOs satisfying the commutation relations (17.9), i.e., realizing a representation of the algebra *AGL(2s+1,C)*. The verification of validity of these relations can be made by using a matrix realization of the operators $\Sigma^{\pm}_{\mu\sigma}$.

Thus any Poincaré-invariant equation for a particle of spin s>0 and mass m>0 is invariant under the algebra *AGL(2s+1,C)* whose basis elements belong to the enveloping algebra of the algebra *AP(1,3)* and are given in (19.29), (19.20). ∎

The operators (19.29) satisfy relations (17.18) with the generators of the Poincaré group. Hence it follows that the set $\{P_\mu,J_{\mu\sigma},\lambda_{mn},\tilde{\lambda}_{mn}\}$ forms a Lie algebra isomorphic to *A[P(1,3)⊗GL(2s+1,C)]*, see the end of the proof of Theorem 19.5.

In the above we did not make any supposition about the class of SOs of the equation (16.1). If we assume the generators of the Poincaré group have the covariant form of (2.22) (with the corresponding matrices $S_{\mu\sigma}$) then the SOs of (19.29) belong to the class M_2 and have the form

$$\Sigma_{ab}^{\pm}=-iS_{ab}-\frac{1}{m^2}[\varepsilon_{abc}(p^2F_c-p_c\boldsymbol{p}\cdot\boldsymbol{F})\pm ip_0(F_ap_b-F_bp_a)],$$

$$\Sigma_{0a}^{\pm}=-iS_{0a}-\frac{1}{m^2}[p_0\varepsilon_{abc}F_bp_c\pm ip_a\boldsymbol{p}\cdot\boldsymbol{F}-p^2F_a)]$$

where $F_a=i\varepsilon_{abc}S_{bc}/2\pm S_{0a}$. The corresponding SOs of (19.20) are in general differential operators of order $4s$ with matrix coefficients.

We note that the operators (19.29) where found and analyzed by Beckers [25] (without connections with the algebra $AGL(2s+1,C)$ and hidden symmetries).

19.5. Symmetries of the Levi-Leblond Equation

Here we discuss hidden symmetries of the Levi-Leblond equation, the simplest Galilei-invariant equation for a particle of nonzero spin (refer to Section 13). We rewrite this equation in the form (13.1) where

$$\beta_0=\frac{1}{2}(1+i\gamma_4),\quad \beta_4=1-i\gamma_4,\quad \beta_a=\gamma_a, \tag{19.31}$$

γ_a, γ_4 are the Dirac matrices.

The Levi-Leblond equation is invariant under the algebra $AG(1,3)$. The basis elements of this algebra belong to the class M_1 and are given in (12.18) where

$$M=m,\quad \boldsymbol{S}=\frac{i}{4}\gamma\times\gamma,\quad \eta=\frac{1}{2}(1-i\gamma_4)\gamma. \tag{19.32}$$

The problem of description of SOs of higher orders for the Levi-Leblond equation is formulated in complete analogy with the corresponding problem for the Dirac equation, thus we use all the definitions and notations of Section 18.

To find a complete set of SOs of arbitrary order n we consider other equation equivalent to (13.1), (19.31). Namely making the transformation $\psi\rightarrow V\psi=\Phi$ where V is the operator (13.15) we come to the equation (13.17). As in the case of the Dirac equation there is one-to-one correspondence between the SOs Q' of the equation (13.17) and the SOs of the Levi-Leblond equation. This correspondence is given by relations (18.2) where V is the operator (13.15). That is why we will investigate symmetries of the more simple equation (13.17) instead of the Levi-Leblond one.

The function Φ satisfying (13.17) has two nonzero components only so it is convenient to search for SOs of (13.17) in the form

$$Q=S_aQ_a+IQ_0 \tag{19.33}$$

where $S_a=\varepsilon_{abc}\gamma_b\gamma_c/4$ together with the unit matrix I form a complete set of linearly independent matrices defined on Φ, Q_a and Q_0 are unknown operators commuting with γ_μ. The operator (19.33) is a SO of the equation (13.17) iff Q_a and Q_0 are SOs of the Schrödinger equation.

It was shown in Subsection 16.5 that all the finite order SOs of the Schrödinger equation are polynomials of the generators of the Galilei group, see (16.39). The same is true for the equation (13.17) since the corresponding generators of the Galilei group have the form

$$P_\mu = P_\mu, \ \boldsymbol{J}' = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S}, \ \boldsymbol{G}' = \boldsymbol{G} \tag{19.34}$$

where P_μ, $\boldsymbol{J}$ and $\boldsymbol{G}$ are the operators (11.5). On the other hand

$$\boldsymbol{S}\Phi = \frac{1}{m}(m\boldsymbol{J}' - \boldsymbol{P}' \times \boldsymbol{G}')\Phi \tag{19.35}$$

where Φ is an arbitrary solution of (13.17). So representing Q_a and Q_0 in the form of (16.39) and expressing $P_\mu, \boldsymbol{J}, \boldsymbol{G}, \boldsymbol{S}$ via $P_\mu', \boldsymbol{J}', \boldsymbol{G}'$ in accordance with (19.34), (19.35) we come to the conclusion the operators (19.33) are polynomials on the Galilei group generators (19.34). Therefore according to (18.2) all the finite order SOs of the Levi-Leblond equation are polynomials of the generators (12.18), (19.32) inasmuch as

$$V^{-1}P_\mu' V = \hat{P}_\mu, \quad V^{-1}\boldsymbol{J}' V = \hat{\boldsymbol{J}}, \quad V^{-1}\boldsymbol{G}' V = \hat{\boldsymbol{G}}$$

where $\hat{P}_\mu$, $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{G}}$ are the Galilei group generators of (12.18) realized on the set of solutions of the Levi-Leblond equation.

It follows from the above that *all the SOs of finite order n for the Levi-Leblond equation belong to the enveloping algebra generated by the Galilei group generators.* Thus a description of a complete set of SOs of order n for this equation reduces to going through linearly independent polynomials on the operators (12.18).

Let us present a complete set of the first order SOs. Besides the generators (12.18), (19.32) this set includes the following 23 operators [423]

$$\begin{gathered}
D = 2tp_0 - \boldsymbol{x}\cdot\boldsymbol{p} + \frac{i}{2}(3+\beta_4), \quad A = t^2 p_0 - tD - \frac{1}{2}m\boldsymbol{x}^2 + \frac{1}{2}\boldsymbol{x}\cdot\eta - \frac{i}{4}t\beta_4, \\
W_0 = \boldsymbol{p}\cdot\boldsymbol{S}, \quad \boldsymbol{W} = \boldsymbol{S} - \frac{1}{m}\boldsymbol{p}\times\eta, \\
\boldsymbol{U} = \boldsymbol{p}\times\boldsymbol{S} - \frac{i}{2}\beta_4\boldsymbol{p} + 2\eta p_0, \quad V = i\gamma_4 p_0 \boldsymbol{S} - \frac{1}{2}(1 - i\gamma_4)\gamma_0\boldsymbol{p}, \\
F_0 = tW_0 - m\boldsymbol{S}\cdot\boldsymbol{x} + \eta\cdot\boldsymbol{J}, \quad \boldsymbol{F} = t\boldsymbol{U} - mx\times\boldsymbol{S} + \boldsymbol{x}\cdot\eta\,\boldsymbol{p} - (\boldsymbol{x}\cdot\boldsymbol{p} - i)\eta, \\
N_0 = \boldsymbol{x}\cdot\boldsymbol{U} + \frac{1}{2}m(\frac{3}{2} - \beta_4), \quad \boldsymbol{N} = 2t\boldsymbol{V} - m(\boldsymbol{x}\times\boldsymbol{U} + \boldsymbol{x}W_0) + \eta\times\boldsymbol{U} + \eta W_0,
\end{gathered} \tag{19.36a}$$

$$\boldsymbol{R}=2\boldsymbol{x}F_0+\boldsymbol{x}^2\boldsymbol{W}+t^2\boldsymbol{V}-2t\boldsymbol{S}+t\boldsymbol{x}\times\boldsymbol{U}+4\boldsymbol{x}\times\eta+3t\beta/2 \tag{19.36b}$$

The operators (12.18), (19.36) do not form a Lie algebra but include subsets forming basises of Lie algebras and superalgebras. We present some of them.

(1) The algebra *AO(1,2)* is formed by P_0, A and D, refer to (11.12).

(2) The algebra *AO(3)* is formed by $\boldsymbol{W}=(W_1,W_2,W_3)$. This algebra can be united with $AG_2(1,3)$ to obtain the 15-dimensional IA including the following operators
$A_{15}\supset\{P_\mu, \boldsymbol{J}, \boldsymbol{G}, D, A, \boldsymbol{W}\}$.

(3) The superalgebra SA_{22} is formed by the following set
$SA_{22}\supset\{\boldsymbol{W}, W_0, \boldsymbol{U};\ P_\mu, \boldsymbol{J}, \boldsymbol{G}, A, D\}$.

Besides that the Levi-Leblond equation has a wide symmetry in the class M_2. The matter is that all the symmetries of the Schrödinger equation (refer to Section 16) are valid for (13.1), (19.31). Moreover these symmetries can be extended. Indeed, the operators (16.41a) and the generators P_μ, J_a, G_a of (12.18), (19.32)) form a 27-dimensional Lie algebra being an IA of the Levi-Leblond equation. This IA can be extended to a 30-dimensional Lie algebra by including the SOs $\boldsymbol{W}$ of (19.36).

We see the simplest Galilei-invariant wave equation for a particle of spin 1/2 has extensive hidden symmetries in spite of the fact that all the finite-order SOs belong to the enveloping algebra of the algebra *AG(1,3)*.

19.6. Symmetries of Galilei-Invariant Equations for Arbitrary Spin Particles

Investigating of symmetries of Galilei-invariant equations for arbitrary spin particles (refer to Sections 13, 14) can be carried out in analogy with the above. We do not present the corresponding cumbersome calculations here but formulate and discuss the main results of this investigating.

1. A principal description of SOs of the first-order Galilei-invariant wave equations for particles of arbitrary spin is analogous to the description of SOs of the Levi-Leblond equation. Namely all the finite order SOs of the equations considered in Section 13 belong to the enveloping algebra of the Lie algebra of the Galilei group. To prove this assertion it is sufficient to transform from the equations given in (13.1), (13.11), (13.12) and Table 13.1 to the equivalent representation (13.17) where the transformed wave function Φ' has $2s+1$ independent components only, and to expand a SO in the complete set of the matrices λ_{mn} of (19.20) where

$$\Sigma_{0a}=iS_a,\ \ \Sigma_{ab}=\varepsilon_{abc}S_c, \tag{19.37}$$

S_a are the spin matrices expressed via the Galilei group generators according to (19.35).

In fact the almost evident hidden symmetry of the equations considered is formulated in the previous subsection, see assertions (1) and (2) there. Moreover this symmetry is more extensive if $s>1/2$.

ASSERTION. Described in Section 13 Galilei-invariant wave equations for particles of spin s are invariant under the $[27+(2s+1)^2]$-dimensional Lie algebra whose basis elements are

$$\{P_\mu, G_a, J_a, A, D, G_aG_b, P_aG_b+P_bG_a, \lambda_{mn}\} \tag{19.38}$$

where P_μ, G_a, J_a, A, D are the generators of the Schrödinger group, λ_{mn} are the operators of (19.20), (19.37), (19.35).

The proof reduces to calculation of commutation relations of the operators (19.38) (which evidently are the SOs of the equations considered) in order to make sure they form a basis of a Lie algebra. ■

2. SOs of Galilei-invariant equations in the Hamiltonian form (refer to Section 14) do not belong to the enveloping algebra of the algebra *AG(1,3)* in general. More precisely such SOs can be represented in the form

$$Q=Q_1+q_1Q_2$$

where Q_1 and Q_2 are polynomials of the Galilei group generators, q_1 is the operator not belonging to the enveloping algebra of the algebra *AG(1,3):*

$$\begin{aligned} q_1=&[\sigma_2+\frac{2}{m}k\sigma_3\boldsymbol{S}\cdot\boldsymbol{p}+\frac{2k^2}{m^2}(i\sigma_1-\sigma_2)(\boldsymbol{S}\cdot\boldsymbol{p})^2]\cos(amt)-\\ &-[\sigma_3+\frac{2i}{m}(\sigma_1+i\sigma_2)k\boldsymbol{S}\cdot\boldsymbol{p}]\sin(amt). \end{aligned} \tag{19.39}$$

This assertion can be proved by transforming the Hamiltonian (14.9a) and the corresponding generators (12.18), (14.3) into the equivalent representation (14.14), (14.16) where $B=\sigma_1 am$.

It is not difficult to verify the equations (11.1), (14.9a) for a particle of spin s are invariant under the $[27+(2s+1)^2+3]$-dimensional Lie algebra whose basis elements are present in (19.38), (19.39) and (19.40):

$$\begin{aligned} &q_2=\sigma_1+\frac{2i}{m}\sigma_3k\boldsymbol{S}\cdot\boldsymbol{p}+\frac{2}{m^2}(\sigma_1+\sigma_2)(k\boldsymbol{S}\cdot\boldsymbol{p})^2,\\ &q_3=(\sigma_3+\frac{2}{m}\sigma_3k\boldsymbol{S}\cdot\boldsymbol{p})\cos(amt)+[\sigma_2+\frac{2}{m}\sigma_3k\boldsymbol{S}\cdot\boldsymbol{p}+\frac{2}{m^2}(i\sigma_1-\sigma_2)(k\boldsymbol{S}\cdot\boldsymbol{p})^2]\sin(amt). \end{aligned} \tag{19.40}$$

The operators q_a form the subalgebra of the IA, isomorphic to *AO(3)*.

A more extensive IA of the equations considered is formed by the following sets of SOs: $\{Q_A, q_aQ_A\}$ where Q_A are the operators (19.38), q_a is any of the operators

(19.39), (19.40). Dimension of these algebras is $2[27+(2s+1)^2]$ so for $s=1/2$ we have 62-dimensional Lie algebras.

Just now we finish our brief discussion of hidden symmetries of Galilei-invariant wave equations. Some of these symmetries are admitted by equations for interacting particles, which sometimes have even more extended symmetries than free particle equations, refer to Subsections 21-23.

20. NONGEOMETRIC SYMMETRIES OF MAXWELL'S EQUATIONS

20.1. Invariance Under the Algebra *AGL(2,C)*

It was shown in Chapter 1 that the maximal symmetry of Maxwell's equations in the class M_1 is exhausted by the invariance under the Lie algebra of the conformal group. The problem of interest is to investigate nongeometric symmetries of these equations also because the electromagnetic field is a real and measurable physical object whose hidden symmetries can to have consequences which can be verified.

In this subsection we consider the problem of finding of an IA of Maxwell's equations in the class M_∞. These symmetries are rather nontrivial, and it is evident they cannot be found in the classical Lie approach.

We proceed from the formulation of Maxwell's equations given in (3.4), (3.5). Following the first step of the algorithm outlined in Subsection 16.1 we go to the equations in the momentum representation. Writing $\boldsymbol{E}(x_0,\boldsymbol{x})$ and $\boldsymbol{H}(x_0,\boldsymbol{x})$ in the form

$$\begin{aligned}\boldsymbol{E}(x_0,\boldsymbol{x})&=(2\pi)^{-3/2}\int d^3p\tilde{\boldsymbol{E}}(x_0,\boldsymbol{p})\exp(i\boldsymbol{p}\cdot\boldsymbol{x}),\\ \boldsymbol{H}(x_0,\boldsymbol{x})&=(2\pi)^{-3/2}\int d^3p\tilde{\boldsymbol{H}}(x_0,\boldsymbol{p})\exp(i\boldsymbol{p}\cdot\boldsymbol{x}),\end{aligned}\tag{20.1}$$

we obtain the following system

$$\begin{aligned}L_1\varphi(x_0,\boldsymbol{p})&=0\\ L_2^a\varphi(x_0,\boldsymbol{p})&=0,\end{aligned}\tag{20.2}$$

where

$$\varphi(x_0,\boldsymbol{p})=column(\tilde{E}_1,\tilde{E}_2,\tilde{E}_3,\tilde{H}_1,\tilde{H}_2,\tilde{H}_3),\tag{20.3}$$

L_1 and L_2^a are symbols of the operators (3.5):

$$L_1=i\frac{\partial}{\partial x_0}-\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}\equiv i\frac{\partial}{\partial x_0}-\hat{H},\quad L_2^a=(Z_{ab}+i\varepsilon_{abc}S_c)p_b.\tag{20.4}$$

It is necessary to take into account the condition of reality of the vectors

(20.1), which can be written in the form

$$\varphi^*(x_0,\boldsymbol{p})=\varphi(x_0,-\boldsymbol{p}). \tag{20.5}$$

SOs of Maxwell's equations belonging to the class M_∞ are 6×6 matrices depending on $\boldsymbol{p}$. To find a complete set of these matrices we transform the equations (20.2) to the representation where the Hamiltonian H has a diagonal form. Using for this purpose the following transformation operator

$$W=U_3U_2U_1 \tag{20.6}$$

where

$$U_1=P_+ + P_- Q\boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \quad P_{+-}=\frac{1}{2}(1\pm\sigma_2),$$

$$U_2=\exp\left[i\frac{S_1p_2-S_2p_1}{\sqrt{p_1^2+p_2^2}}\arctan\frac{\sqrt{p_1^2+p_2^2}}{p_3}\right],$$

$$U_3=\frac{1}{\sqrt{2}}[S_3^2-i(S_1S_2+S_2S_1)]+1-S_3^2, \tag{20.7}$$

$$Q=1+2(\boldsymbol{S}\cdot\hat{\boldsymbol{p}}\times\boldsymbol{n})^2[1-(\boldsymbol{p}\cdot\boldsymbol{n})^2]^{-1}, \quad \boldsymbol{n}^2=1,$$

$\boldsymbol{n}$ is an arbitrary constant vector, we obtain by consequent calculations that

$$U_1L_1U_1^\dagger=L_1^/=i\frac{\partial}{\partial x_0}-\boldsymbol{S}\cdot\boldsymbol{p}, \quad U_2L_1^/U_2^\dagger=L_1^{//}=i\frac{\partial}{\partial x_0}-S_3p,$$
$$U_3L_1^{//}U_3^\dagger=L_1^{///}=i\frac{\partial}{\partial x_0}-\Gamma_0p. \tag{20.8}$$

Here Γ_0 is the diagonal matrix,

$$\Gamma_0=-i(S_1S_2+S_2S_1)S_3=diag(1,-1,0,1,-1,0). \tag{20.9}$$

According to (20.2), (20.9) the transformed function $\varphi'=W\varphi$ satisfies the following system of equations

$$L_1^{///}\varphi'\equiv\left(i\frac{\partial}{\partial x_0}-\Gamma_0p\right)\varphi'=0, \tag{20.10a}$$

$$L_3^{///}\varphi'\equiv(1-\Gamma_0^2)p^2\varphi'=0, \tag{20.10b}$$

since $Wp_aL_2^aW^{-1}\equiv(1-\Gamma_0^2)p^2$.

In the representation (20.10) a SO $Q_A\in M_\infty$ reduces to matrix depending on $\boldsymbol{p}$ and satisfying the relations

$$[Q_A, L_1^{///}] = [\Gamma_0, Q_A]p = \alpha_A L_2^{///},$$
$$[Q_A, L_2^{///}] = [Q_A, 1-\Gamma_0^2]p^2 = \beta_A L_2^{///} \tag{20.11}$$

where α_A, $ß_A$ are matrices depending on $\boldsymbol{p}$. Using the explicit expression for Γ_0 given in (20.9) we obtain the general expression of Q_A in the following form

$$Q'_A = \begin{pmatrix} a & o & o & b & o & o \\ 0 & c & 0 & 0 & d & o \\ 0 & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & f & 0 & 0 \\ 0 & g & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + FL_3^{///}, \tag{20.12}$$

where F is an arbitrary 6×6 matrix which can be chosen zero without loss of generality (refer to (20.10b)), $a, b, ..., h$ are arbitrary functions of $\boldsymbol{p}$.

So there are exactly eight linearly independent matrices satisfying (20.11). We choose these matrices in the form

$$\Sigma'_{ab} = i\varepsilon_{abc}\sigma_c, \quad \Sigma'_{0a} = \frac{1}{2}\varepsilon_{abc}\Gamma_0\Sigma'_{bc}, \quad \Sigma'_0 = \boldsymbol{I}, \quad \Sigma'_1 = i\Gamma_0. \tag{20.13}$$

The matrices (20.13) satisfy the commutation relations (17.7) characterizing the algebra A_8. Using the transformation

$$\Sigma'_{\mu\nu} \to \Sigma_{\mu\nu} = W^{-1}\Sigma'_{\mu\nu}W, \quad \Sigma'_{\alpha} \to \Sigma_{\alpha} = W^{-1}\Sigma'_{\alpha}W$$

(where W is the operator of (20.6)) we obtain the corresponding linearly independent SOs for the equation (20.2):

$$Q_1 = \Sigma_{23} = \sigma_3 \boldsymbol{S}\cdot\hat{\boldsymbol{p}}Q, \qquad Q_2 = \Sigma_{31} = i\sigma_2,$$
$$Q_3 = \Sigma_{12} = \sigma_1 \boldsymbol{S}\cdot\hat{\boldsymbol{p}}Q, \quad Q_4 = \Sigma_{01} = -\sigma_1 Q, \tag{20.14a}$$

$$Q_5 = \Sigma_{02} = \boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \quad Q_6 = \Sigma_{03} = -\sigma_3 Q,$$
$$Q_7 = \Sigma_0 = \boldsymbol{I}, \qquad Q_8 - \Sigma_1 = i\sigma_2 \boldsymbol{S}\cdot\hat{\boldsymbol{p}} \tag{20.14b}$$

where Q is the matrix (20.7).

The fact that the operators (20.14) are SOs of the equation ((20.2) can be easily verified by direct calculation also, bearing in mind the relations

$$[Q, \boldsymbol{S}\cdot\boldsymbol{p}]_+ = [Q, \sigma_a] = 0, \quad Q^2 = 1, \quad L_2 Q = -L_2. \tag{20.15}$$

Thus we have obtained a basis of the IA of Maxwell's equations in the class M_∞. Let us formulate this result in the form of the following assertion [153,157].

THEOREM 20.1. Maxwell's equations are invariant under the eight-dimensional algebra A_8 isomorphic to $AGL(2,C)$. Basis elements of this algebra in the momentum representation can be chosen in the form (20.14). ■

We see that besides the conformal invariance, Maxwell's equations have the hidden symmetry described by Theorem 20.1. The SOs (20.14) are defined on the set of functions $\varphi(x_0,\boldsymbol{p})$ which are the Fourier transforms of solutions of Maxwell's equations in the realization (3.4), (3.5).

To each matrix Q_A of (20.9) it is possible to assign the integral operator $\hat{Q}_A$ defined in the space of the functions $\varphi(x_0,\boldsymbol{x})$ of (3.4):

$$\hat{Q}_A\varphi(x_0,\boldsymbol{x})=(2\pi)^{-3}\int d^3p\,d^3y\,Q_A\varphi(x_0,\boldsymbol{y})\exp[i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})]. \tag{20.16}$$

The integral operators (20.16), (20.14) form an IA of Maxwell's equations.

20.2. The Group of Nongeometric Symmetry of Maxwell's Equations

The SOs (20.14) form a Lie algebra and satisfy the conditions

$$Q_A^2\varphi(x_0,\boldsymbol{p})=-\varphi(x_0,\boldsymbol{p}),\quad A\leq 3, A=8,$$

$$Q_A^2\varphi(x_0,\boldsymbol{p})=\varphi(x_0,\boldsymbol{p}),\qquad 4\leq A\leq 7$$

where $\varphi(x_0,\boldsymbol{p})$ is an arbitrary solution of (20.2). Considering exponential mapping of these SOs we come to the conclusion that Maxwell's equations are invariant under the eight-parameter group of transformations which are defined by the following relations

$$\varphi(x_0,\boldsymbol{p})\to\varphi'(x_0,\boldsymbol{p})=\exp\left(Q_A\theta_A\right)\varphi(x_0,\boldsymbol{p})=$$
$$=\begin{cases}(\cos\theta_A+Q_A\sin\theta_A)\varphi(x_0,\boldsymbol{p}), & A\leq 3,\ A=8,\\ (\cosh\theta_A+Q_A\sinh\theta_A)\varphi(x_0,\boldsymbol{p}), & 4\leq A\leq 7\end{cases} \tag{20.17}$$

where θ_a are real parameters.

Substituting (20.3), (20.14) into (20.17) we obtain the transformation law for the Fourier transforms of the vectors of the electric and magnetic field strengths in the following explicit form

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cos\theta_1+i[\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{E}}-2\hat{\boldsymbol{p}}\times\boldsymbol{n}(\boldsymbol{n}\cdot\tilde{\boldsymbol{E}})\lambda]\sin\theta_1\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cos\theta_1-i[\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{H}}-2\hat{\boldsymbol{p}}\times\boldsymbol{n}(\boldsymbol{n}\cdot\tilde{\boldsymbol{H}})\lambda]\sin\theta_1;\end{cases} \tag{20.18a}$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cos\theta_2+\tilde{\boldsymbol{H}}\sin\theta_2,\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cos\theta_2-\tilde{\boldsymbol{E}}\sin\theta_2;\end{cases} \tag{20.18b}$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cos\theta_3-i[\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{H}}-2\hat{\boldsymbol{p}}\times\boldsymbol{n}(\boldsymbol{n}\cdot\tilde{\boldsymbol{H}})\lambda]\sin\theta_3,\\ \boldsymbol{H}\to\tilde{\boldsymbol{H}}\cos\theta_3-i[\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{E}}-2\hat{\boldsymbol{p}}\times\boldsymbol{n}(\boldsymbol{n}\cdot\tilde{\boldsymbol{E}})\lambda]\sin\theta_3;\end{cases} \tag{20.19a}$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cosh\theta_4+\{\tilde{\boldsymbol{H}}+2[\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})-\boldsymbol{n}](\boldsymbol{n}\cdot\tilde{\boldsymbol{H}})\lambda\}\sinh\theta_4,\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cosh\theta_4+\{\tilde{\boldsymbol{E}}+2[\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})-\boldsymbol{n}](\boldsymbol{n}\cdot\tilde{\boldsymbol{E}})\lambda\}\sinh\theta_4;\end{cases}$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cosh\theta_5+i\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{E}}\sinh\theta_5,\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cosh\theta_5+i\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{H}}\sinh\theta_5;\end{cases}$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cosh\theta_6+\{\tilde{\boldsymbol{E}}-2[\hat{\boldsymbol{p}}(\boldsymbol{n}\cdot\hat{\boldsymbol{p}})-\boldsymbol{n}](\boldsymbol{n}\cdot\tilde{\boldsymbol{E}})\lambda\}\sinh\theta_6,\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cosh\theta_6-\{\tilde{\boldsymbol{E}}-2[\hat{\boldsymbol{p}}(\boldsymbol{n}\cdot\hat{\boldsymbol{p}})-\boldsymbol{n}](\boldsymbol{n}\cdot\tilde{\boldsymbol{H}})\lambda\}\sinh\theta_6;\end{cases} \tag{20.19b}$$

$$\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\exp\theta_7,\qquad \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\exp\theta_7;$$

$$\begin{cases}\tilde{\boldsymbol{E}}\to\tilde{\boldsymbol{E}}\cos\theta_8+i\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{H}}\sin\theta_8,\\ \tilde{\boldsymbol{H}}\to\tilde{\boldsymbol{H}}\cos\theta_8-i\hat{\boldsymbol{p}}\times\tilde{\boldsymbol{E}}\sin\theta_8,\end{cases}$$

where $\lambda=[1-(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})^2]^{-1}$, $\hat{\boldsymbol{p}}=\boldsymbol{p}/\mathrm{p}$, $\boldsymbol{n}$ is an arbitrary unit vector.

Formulae (20.18b) present the Heaviside-Larmor-Rainich transformations (3.2). The remaining relations (20.19), (20.18a) define a set of one-parametric transformations extending (3.2) to eight-parameter group. To any of these transformations it is possible to assign the integral transformation of the vectors $\boldsymbol{E}(x_0,\boldsymbol{x})$ and $\boldsymbol{H}(x_0,\boldsymbol{x})$ of (20.1).

Thus the Heaviside-Larmor transformations are nothing but a subgroup of the eight-parameter group of hidden symmetry of Maxwell's equations.

It is possible to make sure the SOs of (20.14) do not form a Lie algebra together with the conformal group generators of (2.22), (3.56). It is possible to unite the algebra (20.14) and the conformal algebra iff the basis elements of the latter are realized in the class of integro-differential operators. The explicit form of the corresponding basis elements of the algebra AC(1,3) is given in [148, 154].

Let us demonstrate that Maxwell's equations are invariant under the specific combinations of transformations including space-time reflections and realizing an IR of the algebra *AO(2,2)*. The existence of this symmetry is caused by the fact that Maxwell's equations can be represented as equations for the complex wave function $\Psi=\boldsymbol{E}-i\boldsymbol{H}$, admitting antilinear SOs satisfying conditions of Lemma 1.1, refer to Subsection 1.7. We will not analyse this representation (see, e.g., the equation (9.1)) but present the explicit form of the corresponding symmetries.

PROPOSITION 20.1. Maxwell's equations (3.4), (3.5) are invariant under the algebra *AO(2,2)* realized by the following operators

$$Q_1=T, \quad Q_2=i\sigma_2, \quad O_3=i\sigma_2 T, \quad Q_{3+a}=PQ_a,$$

where P and T are space and time reflection operators:

$$P\varphi(t,\boldsymbol{x})=\sigma_3\varphi(t,-\boldsymbol{x}); \quad T\varphi(t,\boldsymbol{x})=-\sigma_3\varphi(-t,\boldsymbol{x}).$$

PROOF reduces to direct verification which can be made easily using properties of σ-matrices.

It follows from the above that Maxwell's equations are invariant under the six-parameter group of transformations, isomorphic to *O(2,2)* and including the Heaviside-Larmor-Rainich transformations as a subgroup. Calculating exponential mappings of the SOs mentioned in Proposition 20.1 it is not difficult to find these group transformations explicitly. We present here as an example the transformation generated by Q_4

$$\boldsymbol{E}(t,\boldsymbol{x})\to\boldsymbol{E}(t,\boldsymbol{x})\cos\theta_4-i\boldsymbol{H}(t,-\boldsymbol{x})\sin\theta_4,$$
$$\boldsymbol{H}(t,\boldsymbol{x})\to\boldsymbol{H}(t,\boldsymbol{x})\cos\theta_4-i\boldsymbol{E}(t,-\boldsymbol{x})\sin\theta_4.$$

20.3. Symmetries of Maxwell's Equations in the Class M_2

We made sure ourselves that the symmetry of Maxwell's equations is more extensive than the familiar symmetry in respect to the conformal group. Indeed, if we search for symmetries in the class M_∞ then it is possible to find a hidden symmetry of these equations under the algebra A_8. Moreover this symmetry is maximal in the class considered.

A natural question arises: do Maxwell's equations admit local (differential) SOs which cannot be found in the classical Lie approach? To answer this question we find complete sets of the first and second order SOs of these equations.

As before we use the formulation (3.4), (3.5) of Maxwell's equations. The corresponding SO of arbitrary finite order n can be represented in the form

$$Q=E+O,$$
$$E=\sigma_0 C+i\sigma_2 D, \quad O=\sigma_1 A+\sigma_3 B \tag{20.20}$$

where A, B, C and D are differential operators of order n (with real matrix coefficients) commuting with σ_μ.

We pick out the even (E) and odd (O) parts of SOs which are independent on the set of solutions of Maxwell's equations. Such a terminology is in accordance with the fact that products EE and OO are even but EO is odd.

Without loss of generality we assume that Q includes operators of differentiating in respect with spatial variables only since $\partial/\partial x_0$ can be expressed via $\partial/\partial x_a$ according to (3.5).

A linear differential operator Q of order n is a SO of Maxwell's equations iff

$$[Q,\hat{L}_1]=\alpha_Q^a\hat{L}_2^{\ a}, \quad [Q,\hat{L}_2^{\ a}]=\beta_Q^{ab}\hat{L}_2^{\ b} \tag{20.21}$$

where α_Q^a and β_Q^{ab} are differential operators of order n. Without loss of generality these operators can be represented in the following form

$$\begin{aligned}
&\alpha_Q^a=\alpha_E^a+\alpha_Q^a,\\
&\alpha_O^a=\sigma_1\alpha_1^a+\sigma_3 a_3^{\ a}, \quad \alpha_E^a=\sigma_0\alpha_0^a+i\sigma_2\alpha_2^a;\\
&\beta_Q^{ab}=\beta_E^{ab}+\beta_O^{ab},\\
&\beta_O^{ab}=\sigma_1\beta_1^{ab}+\sigma_3\beta_3^{ab}, \quad \beta_E^{ab}=\sigma_0^{ab}+i\sigma_2\beta_2^{ab}
\end{aligned} \tag{20.22}$$

where α_μ^a and β_μ^{ab} are differential operators of order n commuting with σ_μ and S_a.

Substituting (20.22) into (20.21) and bearing in mind linear independence of the Pauli matrices we obtain the two noncoupled systems of equations

$$[E,\hat{L}_1]=\alpha_E^a\hat{L}_2^{\ a}, \quad [E,\hat{L}_2^{\ a}]=\beta_E^{ab}\hat{L}_2^{\ b} \tag{20.23}$$

and

$$[O,\hat{L}_1]=\alpha_O^a\hat{L}_2^{\ a}, \quad [O,\hat{L}_2^{\ a}]=\beta_O^{ab}\hat{L}_2^{\ b}. \tag{20.24}$$

According to (20.23), (20.24) the problem of description of SOs of Maxwell's equations reduces to the two separate problems related to the even and odd SOs.

First we consider the first order SOs. It is possible to show the corresponding system (20.24) has trivial solutions only so Maxwell's equations do not admit odd SOs in the class M_1. The even SOs are described in the following assertion.

THEOREM 20.2. The complete set of SOs of Maxwell's equations in the class M_1 is represented by the operators

$$P_\mu,\ J_{\mu\nu},\ K_\mu,\ D,\ F=i\sigma_2,\ FP_\mu,\ FJ_{\mu\nu},\ FK_\mu,\ FD,\ I \tag{20.25}$$

where P_μ, $J_{\mu\nu}$, K_μ and D are the conformal group generators of (2.22), (2.42) including the following spin matrices

$$S_{ab}=\varepsilon_{abc}S_c,\ \ S_{0a}=i\sigma_2 S_a, \tag{20.26}$$

where S_a are the matrices (3.6).

For the proof see [360]. The operators (20.25) form a 32-dimensional Lie algebra whose structure constants are easily calculated.

The following assertion gives a description of all the linearly independent odd SOs in the class M_2. Such SOs are essentially non-Lie inasmuch as they do not belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$ (H is the group of Heaviside-

Larmor-Rainich transformations).

THEOREM 20.3. Maxwell's equations admit 70 linearly independent odd SOs in the class M_2. These operators have the form

$$Q^{ab}_{(0)}=\sigma_1\left\{[(\boldsymbol{S}\times\boldsymbol{p})^a,(\boldsymbol{S}\times\boldsymbol{p})^b]_+ +p_a p_b+\delta_{ab}[p^2-2(\boldsymbol{S}\cdot\boldsymbol{p})^2]\right\},$$

$$Q^{a}_{(1)}=[Q^{ab}_{(0)},x_b]_+,\quad Q_{(2)}=\frac{1}{2}[Q^{a}_{(1)},x_a]_+,$$

$$Q^{ab}_{(1)}=\frac{1}{2}[\varepsilon_{anc}Q^{bc}_{(0)}+\varepsilon_{bnc}Q^{ac}_{(0)},x_n]_+ -2x_0\tilde{Q}^{ab}_{(0)}+\frac{i}{2}\left\{[(\boldsymbol{p}\times\boldsymbol{S})^a,S^b]_+ +[(\boldsymbol{p}\times\boldsymbol{S})^b,S^a]_+\right\}\sigma_1,$$

$$Q^{a}_{(2)}=\frac{1}{2}\varepsilon_{abc}[Q^{b}_{(1)},x_c]_+ +x_0\tilde{Q}^{a}_{(1)}+\frac{1}{2}\sigma_1\left\{\frac{1}{2}S^a[p_b,x_b]_+ -\right.$$

$$\left.-[p_a,S^b x_b]_+ +i[(\boldsymbol{S}\times\boldsymbol{p})^a,S^b x_b]_+ +i[S^a,S^b J_b-1]_+\right\},\tag{20.27}$$

$$Q^{ab}_{(2)}=-[x_\mu x^\mu,Q^{ab}_{(0)}]_+ +\frac{1}{2}\{[x_a,Q^{b}_{(1)}]_+ +[x^b,Q^{a}_{(1)}]_+\}-2x_0\tilde{Q}^{ab}_{(1)}-$$

$$-\left\{\frac{7}{2}[\delta_{ab}x_\mu x^\mu+x^a x^b,p^2-(\boldsymbol{S}\cdot\boldsymbol{p})^2]_+ +\frac{1}{4}[S^a,S^b]_+ -\right.$$

$$\left.-[S^a p^b+S^b p^a,\boldsymbol{S}\cdot\boldsymbol{x}]_+ -\frac{i}{2}[S^a x^b+S^b x^a,\boldsymbol{S}\cdot\boldsymbol{p}]_+\right\}\sigma_1,$$

$$Q^{a}_{(3)}=i[Q^{ad}_{(2)},K_d],\quad Q^{ab}_{(j)}=[Q^{a}_{(j-1)},K^b]+[Q^{b}_{(j-1)},K^a],\quad j=3,4$$

where K_a, $J_a=\varepsilon_{abc}J_{bc}/2$ are the generators (2.22), (2.42), (20.26).

The proof needs cumbersome calculations so we present its outline only. To find all the linearly independent odd operators belonging to the class M_2 it is necessary and sufficient to find a complete set of solutions of the equations (20.24). Let us denote

$$O=O^H+O^A,\quad \alpha^a_O=(\alpha^a_O)^H+(\alpha^a_O)^A\tag{20.28}$$

where the indices H and A denote the Hermitian and anti-Hermitian parts of the corresponding operators. Equating the Hermitian and anti-Hermitian terms in the first of the equations (20.24) we come to the following system

$$\begin{aligned}2[O^H,\tilde{L}_1]&=[(\alpha^a_O)^H,\hat{L}_2^{\,a}]+[(\alpha^a_O)^A,\hat{L}_2^{\,a}]_+,\\ 2[O^A,\hat{L}_1]&=[(\alpha^a_O)^H,\hat{L}_2^{\,a}]_+ +[(\alpha^a_O)^A,\hat{L}_2^{\,a}].\end{aligned}\tag{20.29}$$

It is convenient to expand all the operators (20.29) in the complete set of the matrices (3.6):

$$\hat{L}_1 = i\frac{\partial}{\partial x_0} + S_a p_a, \quad \hat{L}_2^a = \varepsilon_{abc} p_b S_c - iZ_{ab} p_b,$$

$$O^H = \sigma_\alpha \left\{ Z_{ab} \left(\left[\left[D_\alpha^{ab,cd}, p_c \right]_+, p_d \right]_+ + D_\alpha^{ab} \right) + S_a \left[B_\alpha^{ac}, p_c \right]_+ \right\},$$

$$O^H = i\sigma_\alpha \left\{ S_a \left(\left[\left[B_\alpha^{a,cd}, p_c \right]_+, p_d \right]_+ + K_\alpha^a \right) + Z_{ab} \left[F_\alpha^{ab,c}, p_c \right]_+ \right\}, \tag{20.30}$$

$$(\alpha_O^a)^H = \sigma_\alpha \left(\left[\left[G_\alpha^{a,bc}, p_b \right]_+, p_c \right]_+ + H_\alpha^a \right), \quad (\alpha_O^a)^A = i\sigma_\alpha \left[N_\alpha^{ab}, p_b \right]_+$$

where $D_\alpha^{ab,cd}$, B_α^{ca}, D_α^{ab}, $B_\alpha^{a,cd}$, $F_\alpha^{ab,c}$, K_α^c, $G_\alpha^{a,bc}$, H_α^a are unknown functions of x_0 and $\boldsymbol{x}$, summation is imposed over the repeated indices α, α=1,3.

Substituting (20.30) into (20.29) and using the relations

$$[S_a, S_b] = i\varepsilon_{abc} S_c, \quad [S_a, S_b]_+ = 2\delta_{ab} - Z_{ab},$$

$$[Z_{ab}, S_c] = i(\varepsilon_{cak} Z_{bk} + \varepsilon_{cbk} Z_{ak}), \tag{20.31}$$

$$[Z_{ab}, S_c]_+ = 2\delta_{ab} S_c - \delta_{ac} S_b - \delta_{bc} S_a$$

and equating the coefficients of linearly independent matrices and differential operators we come to the following relations for coefficients of a SO of Maxwell's equations

$$D_1^{ab,cd} = K_{cd}^{ab}, \quad D_3^{ab,cd} = \frac{1}{2}\varepsilon_{ak(c} K_{d)k}^{0b}, \quad B_\alpha^{a,cd} = G_\alpha^{a,cd} = 0, \quad K_1^a = -\frac{3}{10}\partial^n \varepsilon_{dca} \dot{K}_{nc}^{0d},$$

$$K_3^a = -\frac{3}{5}\partial^n \dot{K}_{an}^{00}, \quad B_1^{ab} = \frac{7}{10}\varepsilon_{abc}\partial^n \dot{K}_{cn}^{00}, \quad B_3^{ab} = -\frac{7}{10}\partial^n \dot{K}_{bn}^{0a}, \tag{20.32}$$

$$D_1^{ab} = \frac{1}{10}\partial^n(\partial^a K_{00}^{bn} + \partial^b K_{00}^{an} - \frac{1}{3}\delta^{ab}\partial^m K_{00}^{mn}),$$

where the dots denote derivatives in respect with $t = x_0$, $K_{\nu\sigma}^{\mu\rho}$ is a generalized Killing tensor, i.e., an irreducible tensor which is antisymmetric in respect with permutations $\mu \rightleftarrows \nu$ or $\rho \rightleftarrows \sigma$ and symmetric under permutations of the pairs $[\mu,\nu] \rightleftarrows [\rho,\sigma]$. Moreover, this tensor satisfies the equations

$$\partial^{(\lambda} K_{\nu\sigma}^{\mu\rho)} - \frac{1}{5}(\partial^n g^{(\lambda}{}_{\nu} K_{\sigma n}^{\rho\mu)} + \partial_n g^{(\lambda\mu} K_{(\nu\sigma)}^{\rho)n}) = 0 \tag{20.33}$$

where symmetrization is imposed over the indices in brackets.

According to (20.32) all the coefficients of the SOs can be expressed via solutions of the equation (20.33). These solutions depend on 70 parameters and are given in (A.2.20)-(A.2.23). Substituting these solutions into (20.30) we obtain a linear combination of the SOs of (20.27), which satisfy the second condition (20.24). ■

Thus we have found a complete set of the odd SOs of Maxwell's equations in the class M_2. It is evident these SOs do not belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$ since this enveloping algebra includes even operators only.

Finally we present the even SOs of Maxwell's equations in the class M_2. A complete set of these operators can be calculated in analogy with the above and includes 170 terms present below:

$$\{Q_A,\ i\sigma_2 Q_A,\ i\sigma_2,\ I\} \tag{20.34}$$

where Q_A are the operators (18.40) (where P_μ, K_μ, D, $J_{\mu\sigma}$ are operators given in (2.22), (2.42), (20.26)). All the operators (20.34) belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$.

In an analogous way we can consider SOs of Maxwell's equations, belonging to the classes of third-, fourth-, ... etc. differential operators. Moreover it is not too difficult to find a complete set of SOs of arbitrary fixed order n. We will not do it here restricting ourselves to formulating of the following assertions.

1. The SOs of order $n>2$ for Maxwell's equations are polynomials of the group $C(1,3)\otimes H$ generators (refer to (2.22), (2.42), (3.20), (3.23)) and the second order SOs (20.27).

The number of linearly independent SOs of order n is equal to

$$N_n=(2n+3)[2n(n-1)(n+3)(n+4)+(n+1)^2(n+2)^2]/12,\ n>2,$$

besides there are N^E even and N^O odd operators among them where

$$N^E=(n+1)^2(n+2)^2(2n+3)/12,$$

$$N^O=n(n-1)(n+3)(n+4)(2n+3)/6.$$

To calculate these numbers we use formula (A.2.16) and take into account that the odd SOs of order n are completely determined by the generalized Killing tensor of order 1 and valence R_1+2R_2, $R_1=n-2$, $R_2=2$, and the even SOs are determined by two generalized Killing tensors of order $s=1$ and valence R_1+2R_2, $R_1=n$, $R_2=0$. These facts can be proved in analogy with (20.28)-(20.32).

20.4. Superalgebras of SOs of Maxwell's Equations

Here we discuss algebraic properties of the most interesting SOs of Maxwell's equations, i.e., the odd SOs which do not belong to the enveloping algebra of $A[C(1,3)\otimes H]$.

The operators (20.27) do not form a Lie algebra. But there exist such subsets of these operators which can be extended to superalgebras. A superalgebra including the maximal numbers of the operators (20.27) is formed by the following set

$$\left\{Q^{ab}_{(0)}, \tilde{Q}^{ab}_{(0)};\ P_\mu, J_{\mu\sigma}, D, \eta_{\mu\nu\rho\sigma}=[p_\mu p_\nu p_\rho p_\sigma]^{TL}\right\} \tag{20.35}$$

where $Q^{ab}_{(0)}$ and $\tilde{Q}^{ab}_{(0)}$ are the SOs of (20.27), P_μ, $J_{\mu\sigma}$, D are generators of the conformal group and the symbol $[...]^{TL}$ denotes the traceless part of the corresponding tensor. Indeed, it is not difficult to make sure the operators $Q^{ab}_{(0)}$ and $\tilde{Q}^{ab}_{(0)}$ commute with P_μ, $\eta_{\mu\nu\rho\sigma}$ and satisfy the following relations (we denote $Q^{ab}_{(0)}=Q_1^{ab}$, $\tilde{Q}^{ab}_{(0)}=Q_2^{ab}$)

$$\begin{aligned}
&[Q_\alpha^{ab}, D]=-2iQ_\alpha^{ab},\ \ \alpha=1,2,\\
&[Q_\alpha^{ab}, J_{0c}]=i(-1)^\alpha(\varepsilon_{cak}Q_{\alpha'}^{kb}+\varepsilon_{cbk}Q_{\alpha'}^{ka}),\ \ \alpha'\neq\alpha,\\
&[Q_\alpha^{ab}, J_{cd}]=i(\delta_{ad}Q_\alpha^{bc}+\delta_{bc}Q_\alpha^{ad}-\delta_{ac}Q_\alpha^{bd}-\delta_{bd}Q_\alpha^{ac}),\\
&[Q_\alpha^{ab}, Q_\alpha^{cd}]_+=2f^{abcd}_{klnm}\eta^{klnm},\quad [Q_1^{ab}, Q_2^{cd}]_+=f^{abcd}_{kln}\eta^{0kln}
\end{aligned} \tag{20.36}$$

where

$$\begin{aligned}
f^{abcd}_{klnm}=&(\delta_{ac}\delta_{kl}-\delta_{ak}\delta_{cl})(\delta_{bd}\delta_{nm}-\delta_{dn}\delta_{bm})-(\delta_{ab}\delta_{kl}-\delta_{ak}\delta_{bm})(\delta_{dc}\delta_{nm}-\delta_{cn}\delta_{dm})+\\
&+(\delta_{bc}\delta_{kl}-\delta_{bk}\delta_{al})(\delta_{ad}\delta_{nm}-\delta_{am}\delta_{nd}),\\
f^{abcd}_{kln}=&\varepsilon_{ack}(\delta_{bl}\delta_{dn}-\delta_{bd}\delta_{ln})+\varepsilon_{adk}(\delta_{bl}\delta_{cn}-\delta_{bc}\delta_{ln})+\\
&+\varepsilon_{bck}(\delta_{al}\delta_{dn}-\delta_{ad}\delta_{ln})+\varepsilon_{bdk}(\delta_{al}\delta_{cn}-\delta_{ac}\delta_{ln}).
\end{aligned}$$

The commutation relations for $\eta_{\mu\nu\rho\sigma}$ with P_μ, D and $J_{\mu\sigma}$ are evident. We emphasize that anticommutation relations (20.36) are satisfied on the set of solutions of Maxwell's equations.

We see the commutation and anticommutation relations for the operators (20.35) are in accordance with the scheme (18.1) characterizing a superalgebra. Thus the set (20.35) forms a basis of the 36-dimensional superalgebra which is an IA of Maxwell's equations and includes the subalgebra *AP(1,3)*. This superalgebra has infinitely many extensions in the classes of higher order SOs. For example, we can add to (20.35) the odd operators $i\sigma_2$, $i\sigma_2 P_\mu$, $i\sigma_2 P_\mu P_\sigma$ and the even operators $P_\mu P_\sigma$, $P_\mu P_\sigma P_\alpha$. In this way we come to the 81-dimensional invariance superalgebra in the class M_4.

We see the SOs of Maxwell's equations in the class M_2 include nontrivial algebraic structures. It is possible to show the set of the odd SOs (20.27) is closed under the commutation with the conformal group generators. Moreover all these SOs can be represented as successive commutators of $Q_{(0)}$ with the generators (22.22), (22.42), (20.26).

20.5. Symmetries of Equations for the Vector-Potential

To conclude this section we discuss briefly symmetries of the equations for the vector-potential of the electromagnetic field, given in (3.14), (3.15).

We note that the equations (3.14), (3.15) are not invariant under the conformal algebra, i.e., symmetry of these equations is less extensive than symmetry of Maxwell's equations [101]. At first glance, the situation seems to be rather strange since each of equations (3.14), (3.15) is conformal invariant. The thing is that different representations of the algebra *AC(1,3)* are realized on the set of solutions of these equations. In both cases, the basis elements of this algebra have the form (2.22), (2.42) where $S_{0a}=i(j_a-\sigma_a/2)$, $S_{ab}=\varepsilon_{abc}(j_c+\sigma_c/2)$, j_a and σ_a are the matrices of (20.19). But the corresponding values of K in (2.42) are different: $K=2$ for (3.14) and $K=3$ for (3.15).

There are two ways to overcome possible difficulties connecting with the conformal noninvariance of the equations for the vector-potential in the Lorentz gauge: to use another gauge (linear but including higher derivatives [25,101] or even nonlinear [101]) or to consider realizations of the algebra *AC(1,3)* in a more extended class of SOs [154, 157]. We do not discuss these possibilities here but consider symmetries of the equation (3.15) with the Coulomb gauge (3.14), (3.16). For convenience we present these equations again setting $j_\mu=0$:

$$L_1A\equiv p^\mu p_\mu A=0, \tag{20.37a}$$

$$L_2^aA\equiv(Z_{ab}+i\varepsilon_{abc}\hat{S}_c)p_bA=0 \tag{20.37b}$$

where $\hat{S}_c$ and Z_{ab} are matrices (3.6), $A=column(A_1,A_2,A_3)$.

The Lie symmetry of the system (20.37) is very restricted and reduces to the following 7-dimensional IA:

$$\begin{aligned}&\hat{P}_0=\frac{\partial}{\partial x_0},\quad \hat{P}_a=ip_a=\frac{\partial}{\partial x_a},\\ &\hat{\boldsymbol{J}}=i\boldsymbol{x}\times\boldsymbol{p}+i\hat{\boldsymbol{S}},\quad D=ix_\mu p^\mu.\end{aligned} \tag{20.38}$$

To "extend" this symmetry we can investigate invariance properties of (20.37a) considering (20.37b) as an additional condition which in general is not invariant under the symmetry transformations of the main equation (20.37a). Such an approach is natural enough since it is the equation (20.37a) which describes evolution of $\boldsymbol{A}$ in time and so the SOs of this equation corresponds to constants of motion. As to (20.37b) it can be considered as an additional condition extending the symmetry of the evolution equation (20.37a).

Such a method of investigation of symmetries of the equation (20.37a) is an example of using of the concept of the conditional symmetry [124] in analysis of partial differential equations. This concept enables to find a wide classes of exact solutions of nonlinear equations of mathematical physics, refer to Chapter 6.

First we consider symmetries of the equation (20.37a) without an additional condition. In a complete analogy with Subsection 1.3 it is possible to prove the following assertion.

THEOREM 20.4. The maximal IA of the equation (20.37a) in the class M_1 is the 24-dimensional Lie algebra whose basis elements are

$$\left\{iP_\mu, iJ_{\mu\nu}, iD, iK_\mu, i\hat{S}_a, Z_{ab}\right\} \tag{20.39}$$

where P_μ, $J_{\mu\sigma}$, K_μ and D are the generators of the conformal group of (1.6), (1.16). ■

The concept of the conditional invariance in application to the system (20.37) reduces to searching symmetries of the first equation on the subset of solutions satisfying the condition (20.37b). In this way we prove the following assertion.

THEOREM 20.5. The equation (20.37) is conditionally invariant under the 27-dimensional Lie algebra whose basis elements are given in ((20.39), (20.40):

$$\eta_a = (Z_{ab} + i\varepsilon_{abc}\hat{S}_c)x_b. \tag{20.40}$$

This algebra is the maximal conditional IA in the class M_1.

The proof is analogous to the proof given in Subsection 20.3. ■

We see the conditional symmetry of the equation (20.37a) is more extensive than its ordinary symmetry and essentially more extensive than the symmetry of the system (20.37). Moreover we should like to emphasize that the conditional symmetry has a clear physical sense since this is the symmetry which corresponds to constants of motion. Indeed, we can assign the following conserved quantity to any of the operators Q of (20.39), (20.40):

$$I = \int d^3x(\dot{A}^{\,T}QA - A^{\,T}Q\dot{A}), \quad \dot{I} = 0 \tag{20.41}$$

where

$$\dot{A} = \frac{\partial A}{\partial x_0}, \ \ A^{\,T} = (A_1, A_2, A_3).$$

We see the concept of conditional invariance arises naturally by studying the conservation laws for systems of partial differential equations. This is the conditional symmetry (but not the ordinary symmetry of (20.37a) or of the system (20.37)) which generates conservation laws for the vector-potential. The general problems connected with conservation laws are considered in Section 23.

In conclusion we note that the symmetries of Maxwell's equations considered in the above in fact are conditional also. It happens the conditional and ordinary symmetries of the equations (3.4), (3.5) coincide.

21. SYMMETRIES OF THE SCHRODINGER EQUATION WITH A POTENTIAL

21.1. Symmetries of the One-Dimension Schrödinger Equation

Let us consider the one-dimension Schrödinger equation with an arbitrary potential $V(x)$

$$L\psi \equiv \left(p_0 - \frac{1}{2}(p^2 + V(x))\right)\psi = 0, \quad p = i\frac{\partial}{\partial x}. \tag{21.1}$$

An investigation of symmetries of this equation includes the problems which can be subdivided into two types:

1) the potential is given, symmetries are searched;

2) to find potentials admitting the given (or any) symmetry.

We consider both types of problems.

Let us search for SOs of (21.1), having the following structure

$$Q_1 = (h_0 \cdot p)_0 + (h_1 \cdot p)_1 \tag{21.2}$$

if first order operators only are considered or

$$Q_2 = (h_0 \cdot p)_0 + (h_1 \cdot p)_1 + (h_2 \cdot p)_2 \tag{21.3}$$

if the second order is required. We use the notation

$$(h_n \cdot p)_n \equiv [(h_n \cdot p)_{n-1}, p]_+, \quad (h_0 \cdot p)_0 = h_0 \tag{21.4}$$

where h_n (n=0,1,2,...) are arbitrary functions of x and t.

Substituting (21.1), (21.2) into the invariance condition (16.7) (where α_Q=0 without loss of generality) and equating coefficients of linearly independent differential operators we come to the following system of determining equations

$$h_1' = 0, \quad 2\dot{h}_1 + h_0' = 0, \quad \dot{h}_0 - V' h_1 = 0 \tag{21.5}$$

where the dot and prime denote derivatives in respect with t and x respectively.

The equations (21.5) can be used to find all the nontrivial symmetries corresponding to the given V. But it happens that it is possible to find the general form of V admitting any symmetry and to calculate these symmetries explicitly. Using the condition $h_1 \neq 0$ we come to the following differential consequences of (21.5):

$$h_0'' = 0, \quad V''' = 0, \tag{21.6}$$

from which we obtain

$$V = V_0 + V_1 x + V_2 x^2, \tag{21.7}$$

V_0, V_1 and V_2 are arbitrary constants.

We see the requirement that the equation (21.1) admits a first-order SO is very restrictive and leads to the potentials (21.7) only. Substituting (21.7) into (21.5) it is not difficult to find the corresponding SOs explicitly.

Now let us assume the equation (21.1) admits the second-order SO of (21.3) (we note that such an operator reduces to the first order one if we take into account the equation (21.1)). In analogy with the above it is not difficult to recover the old results [7,51] and find the most general potential

$$V(x) = \frac{V_{-2}}{(c_0 + c_1 x)^2} + V_0 + V_1 x + V_2 x^2. \tag{21.8}$$

Let us extend these developments to arbitrary n-th-order symmetries with

$$Q_n = \sum_{k=0}^{n} (h_k \cdot p)_k \tag{21.9}$$

and search for the corresponding classes of potentials admitting such n-th-order SOs.

For any order k, we note the following operator identity containing k anticommutators

$$(h_k \cdot p)_k = (-i)^k \sum_{l=0}^{k} \frac{k! 2^{k-l}}{(k-l)! l!} (\partial_x^l h_k) \partial_x^{k-l} \tag{21.10}$$

where $\partial_x = \partial / \partial x$. This relation will be very useful in the following in order to obtain the determining equations for the coefficients of Q_n.

For each k-term in the operators (21.9) we can evolute the following commutators

$$[i\partial_t, (h_k \cdot p)_k] = i(\dot{h}_k \cdot p)_k \tag{21.11}$$

and

$$[-\frac{1}{2} p^2, (h_k \cdot p)_k] = \frac{i}{2} (h_k' \cdot p)_{k+1}. \tag{21.12}$$

Using these relations we immediately get for the free case (V=0) the set of commutators

$$[L_0, (h_k \cdot p)_k] = i(\dot{h}_k \cdot p)_k + \frac{i}{2} (h_k' \cdot p)_{k+1} \tag{21.13}$$

where

$$L_0 = = i\partial_t - \frac{1}{2} p^2 = i\partial_t + \frac{1}{2} \partial_x^2. \tag{21.14}$$

For considering Q_n-problem with arbitrary potential we need the additional commutators $[V,(h_k{\cdot}p)_k]$, k=0,1,...,n. Consider separately even and odd k's. By using (21.10) we obtain for $k\geq 1$

$$[V,(h_{2k}{\cdot}p)_{2k}]=-i\sum_{m=0}^{k-1}(-1)^{m+k}\frac{2(2k)!}{(2k-2m-1)!(2m+1)!}(h_{2k}\partial_x^{2k-2m-1}V{\cdot}p)_{2m+1}, \tag{21.15}$$

and for $k\geq 0$

$$[V,(h_{2k+1}{\cdot}p)_{2k+1}]=-i\sum_{m=o}^{k}(-1)^{m+k+1}\frac{2(2k+1)!}{(2k-2m-1)!(2m)!}(h_{2k+1}\partial_x^{2k-2m+1}V{\cdot}p)_{2m}. \tag{21.16}$$

By superposing the results (21.9)-(21.16) it is easy to evolve the invariance condition (16.7) (where α_Q=0 without loss of generality) and equate the coefficients of linearly independent differential operators. As a result, we come to the following system of determining equations

$$h_n^{/}=0,$$

$$2\dot{h}_{2m}+h_{2m-1}^{/}+\sum_{k=m}^{\{(n-1)/2\}}(-1)^{m+k+1}\frac{2(2k+1)!}{(2k-2m+1)!(2m)!}h_{2k+1}\partial_x^{2k-2m+1}V=0, \tag{21.17}$$

$$2\dot{h}_{2l+1}+h_{2l}^{/}+\sum_{k=l+1}^{\{n/2\}}(-1)^{k+l}\frac{2(2k)!}{(2k-2l-1)!(2l+1)!}h_{2k}\partial_x^{2k-2l+1}V=0.$$

where m=0,1,...,{n/2}, l=0,1,...,{(n-1)/2}, $h_{-1}\equiv 0$.

21.2. The Potentials Admissing Third-Order Symmetries

The equations (21.17) define all the possible potentials V admitting nontrivial symmetries of arbitrary order n. Let us consider in more detail the first new nontrivial context, i.e., the third-order SO. We are asking for symmetries of the Schrödinger equation (21.1) of the form

$$Q_3=\sum_{k=0}^{3}(h_k{\cdot}p)_k=(h_0{\cdot}p)_0+(h_1{\cdot}p)_1+(h_2{\cdot}p)_2+(h_3{\cdot}p)_3. \tag{21.18}$$

The corresponding determining equations (21.17) are

$$h_3^{/}=0;\quad 2\dot{h}_3+h_2^{/}=0,\quad 2\dot{h}_2+h_1^{/}-6h_3V^{/}=0, \tag{21.19a}$$

$$2\dot{h}_1+h_0^{/}-4h_2V^{/}=0,\quad \dot{h}_0-h_1V^{/}+h_3V^{///}=0. \tag{21.19b}$$

It is not difficult to find the general solution of the system (21.19a), which has the following form

$$h_3=a, \quad h_2=b-2\dot{a}x, \quad h_1=g_1+6aV, \quad g_1=2\ddot{a}x^2-2\dot{b}x+c \tag{21.20}$$

where a, b and c are arbitrary functions of t. Excluding h_0 from (21.19b) and using (21.20) we come to the following nonlinear ordinary differential equation for V

$$\begin{aligned}F(a,b,c;V,x)\equiv aV'''' &-(2\ddot{a}x^2+6aV+c-2\dot{b}x)V''-\\ &-6(2\ddot{a}x+aV'-\dot{b})V'-12\ddot{a}V-2(2(\partial_t^4a)x^2-2(\partial_t^3b)x+\ddot{c})=0.\end{aligned} \tag{21.20$'$}$$

Integrating this equation twice in respect with x and denoting $V=U'$ we obtain

$$a(U'''-3(U')^2)-(g_1U)'=\frac{1}{3}(\partial_t^4a)x^4-\frac{2}{3}(\partial_t^3b)x^3+\ddot{c}x^2+dx+e \tag{21.21}$$

where g_1 is given in (21.20), d and e are arbitrary functions of t.

The function U depends on x only while a, b, c, d, e are functions of t. This circumstance enables us to separate variables in (21.21). Dividing the l.h.s and the r.h.s. of (21.21) by a and differentiating them in respect with t we conclude that there exist the two possibilities.

1. This equation reduces to one of the following forms

$$V''-3V^2-2\omega_0V=\omega_5, \tag{21.22a}$$

$$V''-3V^2=\omega_4x, \tag{21.22b}$$

$$(V''-3V^2)'-2\omega_2(xV'+2V)=0, \tag{21.22c}$$

$$U'''-3(U')^2-2\omega_1(x^2U)'=\frac{1}{3}\omega_1^2x^4+\omega_5 \tag{21.22d}$$

where $\omega_0, ..., \omega_5$ are arbitrary constants.

Formulae (21.22) define classes of equivalent equations connected by transformations $V\to V+C_1$, $U\to U+C_2+C_3x$, $x\to x+C_4$, C_k are arbitrary constants. Moreover the functions $a, b, ..., e$ included in (21.21) have to satisfy the following conditions

$$\ddot{a}=\omega_1a, \ \ \dot{b}=\omega_2a, \ \ c=\omega_3a, \ \ k=\omega_4a, \ \ e=\omega_5a \tag{21.23}$$

besides that all the ω_k absent in (21.22) (say, ω_1, ω_2, ω_3, ω_4 for the equation (21.22a)) are equal to zero.

2. The second possibility is that the solution of (21.21) has the form

$$\varphi=C_1x^3+C_2x^2+C_3x+C_4 \tag{21.24}$$

and the conditions (21.23) are not necessary valid. This possibility is not too interesting inasmuch as the function (21.24) is a linear combination of particular solutions of the equations (21.22).

Let us consider consequently any of the equations (21.22) and describe the corresponding classes of potentials.

Formula (21.22a) defines the Weierstrasse equation whose solutions are expressed via elliptic integrals. Multiplying both parts of (21.22a) by V' and integrating we obtain the first integral of this equation

$$\frac{1}{2}(V')^2-V^3-\omega_0 V^2-\omega_5 V=C \tag{21.25}$$

where C is the integration constant. Then changing the roles of dependent and independent variables it is possible to integrate (21.25) and to find V as implicit functions of x. Here we present a particular solution of (21.25) only:

$$V=2\nu^2\tanh^2(\nu x)-1. \tag{21.26}$$

The relation (21.22b) defines the first Painlevé transcendent. Its solutions are meromorphyc on all the complex plane but can not be expressed via elementary or special functions.

The equation (21.22c) also can be reduced to the Painlevé form using the Miura [24*] ansatz. Indeed, as a result of the substitution

$$V=-\sqrt[3]{\frac{4\omega_2^2}{3}}F-\frac{\omega_0}{6},\quad x=\sqrt[3]{\frac{1}{6\omega_2}}y \tag{21.27}$$

this equation takes the following form

$$F'''+FF'-\frac{1}{3}xF'-\frac{2}{3}F=0 \tag{21.28}$$

(where $F'=\partial F/\partial x$), and the ansatz [24*]

$$F=W'-\frac{1}{6}W^2 \tag{21.29}$$

reduces (21.28) to the following form

$$(\partial_y-\frac{1}{3}W)(W'''-\frac{1}{6}W^2W'-\frac{1}{3}yW'-\frac{1}{3}W)=0.$$

Equating the expression in the second brackets to zero and integrating it we come to the second Painlevé transcendent:

$$W''=\frac{1}{18}W^3+\frac{1}{3}yW+K \tag{21.30}$$

where K is an arbitrary constant.

Thus any solution of the second Painlevé transcendent corresponds to the potential V (21.27), (21.29) moreover the related Schrödinger equation admits a third-order SO. Of course the ansatz (21.29) does not make it possible to find all the solutions of (21.28), in particular the solutions (21.24) are missed.

The last of the equations considered, i.e. the equation (21.22d), with the help of the change $U=2f-\omega_1/3$ reduces to the form

$$F'' + 2fF' - 4f'F = 2\omega_5 + 1/2$$

where

$$F = f' - f^2 - x^2/4.$$

Choosing $\omega_5=-1/4$ we conclude that any solution of the Riccati equation

$$f' = f^2 + x^2/4 \tag{21.31}$$

corresponds to the solution of the equation (21.22d).

The equation (21.31) is much simpler than (21.22d) nevertheless it cannot be integrated in radicals. Particular solutions of the equation (21.22d) have the following form

$$U = -\frac{2}{x}, \quad U = -\frac{\omega_1}{3}x^3.$$

We note that with the help of twice differentiation and the consequent change of variables

$$U' = -\sqrt[3]{\frac{2\omega_1^2}{3}}G - \frac{1}{9}\omega_1 x^2, \quad x = \frac{y}{\sqrt[3]{4\omega_1}}$$

the equation (21.22d) reduces to the form

$$\partial^4 G + G''G + G'G' - \frac{1}{3}\left(8G + x^2 G'' + 7xG'\right) = 0. \tag{21.32}$$

The last equation had being met in literature and is nothing but the reduced Boussinesq equation, see, e.g., [157, 25*] (refer to Subsection 31.7 also). The procedure outlined above admits to reduce it to the Riccati equation.

Thus the third-order SO are admitted by a very extended class of potentials described above. It is necessary to emphasize that all these potentials (excepting (21.24), (21.26)) correspond to the Schrödinger equation which does not possess any non-trivial (distinct from time displacements) Lie symmetry.

21.3. Time-Dependent Potentials

In this subsection we exploit the following observation: *all the equations (21.22) are equivalent to reduced versions of the Boussinesq equation*

$$\ddot{V} - V'V' - VV'' + V'''' = 0 \tag{21.33}$$

Indeed, the equations (21.22a) and (21.32) (the last is equivalent to (21.22d) appear from similarity reducing of the equation (21.33) [26*], and the equations (21.22b), (21.28) (the last is equivalent to (21.22c)) appear as a result of the reduction of (21.33) using its conditional symmetry, refer to Subsection 31.7.

Thus there exists a deep connection between solutions of the Boussinesq equation and the Schrödinger equation admitting third-order symmetries. This connection became more straightforward in the case of time-dependent potentials $V=V(x,t)$. Indeed, the determining equations (21.19) are valid in this case also, but the compatibility condition for the system (21.19), in contrast with (21.20), takes the form

$$F(a,b,c;x,V) + 12a\ddot{V} - 4(b-2\dot{a}x)\dot{V}' = 0 \tag{21.34}$$

where $F(a,b,c;x,V)$ is the expression defined in (21.20′).

The equation (21.34) is more complicated then (21.20') due to time dependence of V, which makes it impossible to separate variables. That is why we restrict ourselves to analysing of sufficient conditions of its integrability.

Let $a=const$, $b=0$ then (21.34) reduces to the form

$$12\ddot{V} - 6(VV'' + V'V') + V'''' - 2cV'' = 0. \tag{21.35}$$

With the help of the change $V \to V/6$, $t \to t/(2\sqrt{3})$ the equation (21.35) reduces to the Boussinesq equation.

Thus if the potential satisfies the Boussinesq equation then the corresponding Schrödinger equation admits a non-trivial SO of third order. For more on the Boussinesq equation see Subsection 31.7.

21.4. Algebraic Properties of SOs

Thus we have described a class of potentials admitting SOs of third order. A natural question arises about possible applications of these hidden symmetries.

In the following we will demonstrate that SOs of third order possess a very important information about the energy spectrum of a system described by the equation (21.1) and can be used to generate solutions of this equation and even of the nonlinear Schrödinger and wave equations.

Let us investigate algebraic properties of SOs which do not depend on the exact form of the potential. Moreover these properties are completely defined by the type of equation satisfied by the potential.

Let the potential satisfies the equation (21.22a). It means that in (21.23) $\omega_1=\omega_2=\omega_3=\omega_4=0$. Then we conclude from (21.18), (21.20) that the corresponding third order SO reduces to a linear combination of the Hamiltonian (21.1) and the following operator

$$Q=p^3+\frac{1}{4}[3V,p]_+\equiv 2pH+\frac{1}{2}Vp+\frac{i}{4}V', \tag{21.36}$$

i.e., in fact we have the only non-obvious SO (21.36). This SO commutes with the Hamiltonian,

$$[Q,H]=0, \tag{21.37}$$

i.e. Q is a constant of motion. In Subsection 31.8 we will use the property (21.37) to integrate the equations of motion.

If the potential satisfies the equation (21.22b) then $\omega_1=\omega_2=_3=\omega_5=0$, $k=k_0+at$, and the corresponding SO has the form

$$Q=p^3+\frac{3}{4}[V,p]_++t. \tag{21.38}$$

Just two more SOs are represented by the Hamiltonian and the unit operator I, moreover, they form the Heisenberg algebra together with Q, satisfying the following relations:

$$[Q,H]=-iI,\quad [Q,I]=[H,I]=0. \tag{21.39}$$

It is well-known that up to unitary equivalence all irreducible sets of self-adjoint operators satisfying (21.39) are exhausted by p, $-i\partial/\partial p$ and 1, from which it follows that for any potential satisfying the first Painleve transcendent (21.22b) the spectrum of the corresponding Hamiltonian is continuous.

In the case when the potential satisfies (21.22c) the corresponding SO of third order has the form

$$Q=p^3+\frac{3}{4}[V,p]+\frac{\omega_2}{2}tH \tag{21.40}$$

and generated the following algebra together with the Hamiltonian H

$$[Q,H]=-i\frac{\omega_2}{2}H. \tag{21.41}$$

It follows from (21.41) the spectrum of the Hamiltonian H is continuous. Indeed, let $H\Psi_E=E\Psi_E$, then the function $\Psi'=\exp(i\lambda Q)\Psi_E$ (where λ is a real parameter) also is an eigenvector of the Hamiltonian with the eigenvalue λE. In other words if H has at least one non-zero eigenvalue then using the third-order SO it is possible to construct eigenfunctions corresponding to any real eigenvalue.

In the case when the potential is defined by the equation (3.2d) there exist two SOs of third order, Q_+ and Q_-,

$$Q_\pm = \frac{1}{\sqrt{2}}[p^3 \mp \frac{i}{4}\omega[[x,p]_+,p]_+ - [\omega^2 x^2 + 3\varphi', p]_+ \mp \\ \mp \frac{i}{2}\omega(\varphi + 2x\varphi' - \frac{\omega^2}{3}x^4 + \frac{1}{2})]\exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_1}. \tag{21.42}$$

which satisfy the following relations

$$[H, Q_\pm] = \mp\omega Q_\pm, \quad [Q_+, Q_-] = \omega H^2. \tag{21.43}$$

If $\omega_1<0$ then $Q_\pm$ play a role of increasing and decreasing operators for the Hamiltonian eigenvalues, i.e if $H\Psi_\lambda=\lambda\Psi_\lambda$ then $H(Q_\pm\Psi_\lambda)=(\lambda\pm\omega)\Psi_\lambda$, and so H has a discrete spectrum.

We see that the third-order SOs enable to make very general predictions about the Hamiltonian spectra for a very wide class of potentials. Of course these predictions have a formal level inasmuch we ignore considering of domains of the operators discussed.

For application of the third-order SOs to construction of exact solutions and generation of solutions of the linear and non-linear Schrödinger equation refer to Subsection 31.8.

21.5. Complete Sets of SOs for One- and Three-Dimensional Schrödinger equation

Consider some problems of the type 1) (see Subsection 21.1) where the potential is treated as a known function. We restrict ourselves to analysis of the following cases:

$$V=V_1, \tag{21.44a}$$

$$V=V_2 x, \tag{21.44b}$$

$$V=V_3 x^2 \tag{21.44c}$$

where V_a are arbitrary constants, and find all the nonequivalent SOs of arbitrary order n for the corresponding Schrödinger equations.

For the case (21.44a) the problem reduces to description of SOs of the free Schrödinger equation. The equations (21.17) take the form

$$\dot{h}_0=0, \quad h_n^{/}=0, \quad 2\dot{h}_k-2h_{k-1}^{/}=0, \quad 0<k<n. \tag{21.45}$$

By the consequent differentiation of (21.44) we obtain the relations

$$\partial_t^{k+1}h_k=0, \quad \partial_x^{n-k+1}h_k=0 \tag{21.46}$$

from which it follows that

$$h_k=\sum_{p=0}^{n-k}\sum_{l=0}^{k}C_k^{p,l}x^{\,p}t^{\,l} \tag{21.47}$$

where $C_k^{p,l}$ are constant coefficients whose number is equal to $(k+1)(n-k+1)$. From (21.24) we obtain the only restriction for these constants

$$2l(l+1)C_k^{p,l+1}+(p+1)C_{k-1}^{p+1,l}=0, \quad k=1,2,...,n; \tag{21.48}$$

therefore the total number of independent parameters in (21.45) is

$$N^{\,n}=\sum_{k=0}^{n}(k+1)(n-k+1)-\sum_{k=1}^{n}k(n-k+1)=\frac{1}{2}(n+1)(n+2). \tag{21.49}$$

The corresponding SOs of order n (whose number is evidently equal to N^n) are defined by relations (21.9), (21.47), (21.48) (the latter can be interpreted as a recurrence formulae). It is not difficult to see that all these SOs are nothing but polynomials of order n in the first order SOs $P=p$ and $G=tp-mx$.

For the potentials (21.44b) and (21.44c) the equations (21.17) reduces to the following forms

$$\begin{gathered} h_n^{/}=0, \quad \dot{h}_0-2V_2h_1=0, \quad 2\dot{h}_n+\dot{h}_{n-1}=0, \\ 2\dot{h}_k+h_{k-1}-2(k+1)V_2h_{k+1}=0, \quad 0<k<n; \end{gathered} \tag{21.50}$$

and

$$\begin{gathered} h_n^{/}=0, \quad 2\dot{h}_n+h_{n-1}^{/}=0, \quad \dot{h}_0-2V_3xh_1=0, \\ 2\dot{h}_k+h_{k-1}^{/}-4(k+1)V_3xh_{k+1}=0, \quad 0<k<n. \end{gathered} \tag{21.51}$$

The equations (21.50) can be solved in complete analogy with (21.45). We again come to the differential consequences (21.46) and the representation (21.47) but instead of (21.48) we obtain the following relations

$$2m(l+1)C_k^{p,l+1}+(p+1)C_{k-1}^{p+1,l}-4(k+1)V_2C_k^{p,l}=0, \quad k=1,...,n. \tag{21.52}$$

Thus the equation (21.1), (21.44b) admits N^n SOs of order n. The explicit form of these operators is given by formulae (21.9), (21.47), (21.52), N^n is given in (21.49). All these SOs are polynomials in the first order SOs $P=p+Vt$ and $G=tP-mx$.

For the equations (21.51) we have only the latter of the consequences (21.46) which allows to represent h_k in the form

$$h_k = \sum_{l=0}^{n-k} a_{k,l}\, x^{\,l} \tag{21.53}$$

where $a_{k,l}$ are arbitrary functions of t. Substituting (21.53) into (21.51) and equating coefficients of the same powers of x we obtain N^n ordinary differential equations for N^n unknowns $a_{k,l}$. The general solution of such a system depends on N^n arbitrary parameters and the corresponding SOs can be represented in the following form [2*]

$$Q^{\,n} = \sum_{k=0}^{n} \sum_{\alpha=0}^{k} C_{k,\alpha}(p-i\omega x)^{\alpha}(p+i\omega x)^{k-\alpha}\exp[i(2a-k)\omega t] \tag{21.54}$$

where $\omega=(V_3)^{1/2}$, $C_{k,\alpha}$ are arbitrary parameters whose number is equal to N^n of (21.47).

We see that all the SOs of the Schrödinger equation with the potential of harmonic oscillator reduce to polynomials on the first order SOs $P_{\pm}=(p\pm i\omega x)\exp(\mp i\omega)$. In the case n=2 this conclusion reduces to the well-known results [7,51].

Searching of higher symmetries of the three-dimension Schrödinger equation,

$$L\psi \equiv \left[i\frac{\partial}{\partial t} - \frac{1}{2}\left(\boldsymbol{p}^{\,2} + V(\boldsymbol{x})\right)\right]\psi = 0, \tag{21.55}$$

can be carried out in analogy with the scheme used above. The essential complication of the problem, connected with the necessity to consider partial derivatives in respect with spatial variables is overcame by using the generalized Killing tensors.

As in Subsection 16.5 we search for SOs of arbitrary order n in the form (16.8), (16.9a) where indices values run from 1 to 3, $K^{\cdots}$ are unknown functions of $\boldsymbol{x}$ and t. Substituting these expressions and L of (21.55) into the invariance condition (16.7) and equating coefficients of linearly independent differential operators we come to the following system of determining equations (compare with (21.17))

$$\partial^{(a_{n+1}}K^{a_1a_2\ldots a_n)} = 0,$$

$$2\dot{K}^{a_1a_2\ldots 2_{2m}} + \frac{1}{2m}\partial^{(a_{2m}}K^{a_1a_2\ldots a_{2m-1})} + \sum_{k=m}^{\{(n-1)/2\}}(-1)^{m+k+1}\frac{2(2k+1)!}{(2k-2m+1)!(2m)!}U_k^{a_1a_2\ldots a_{2m}}, \tag{21.56}$$

$$2\dot{K}^{a_1a_2\ldots a_{2l+1}} + \frac{1}{2l+1}\partial^{(a_{2l+1}}K^{a_1a_2\ldots a_{2l})} + \sum_{k=l+1}^{\{n/2\}}(-1)^{k+l}\frac{2(2k)!}{(2k-2l-1)!(2l+1)!}W_k^{a_1a_2\ldots a_{2l+1}}$$

where

$$m = 0, 1, , \{n/2\}, \quad l = 0, 1, \ldots, \{(n-1)/2\},$$

$$U_k^{a_1a_2\ldots a_{2m}} = K^{a_1a_2\ldots a_{2m}b_1b_2\ldots b_{2k-2m+1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k-2m+1}}V,$$

$$W_k^{a_1a_2\ldots a_{2l+1}} = K^{a_1a_2\ldots a_{2l+1}b_1b_2\ldots b_{2k-2l-1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k-2l-1}}V,$$

and symmetrization is imposed over the indices in brackets.

The equations (21.56) define potentials V admitting nontrivial symmetries of order n and the coefficients $K^{\cdots}$ of the corresponding SOs. In Subsection 16.5 we obtain the general solution of these equations for the case V=0. It is not difficult to solve these equations for the case of the harmonic oscillator potential [2*]. Searching of potentials admitting nontrivial symmetries also can be carried out using the equations (21.56).

21.6. SOs of the Supersymmetric Oscillator

The equation for the supersymmetric oscillator has the form [417]

$$L\psi \equiv \left[i\frac{\partial}{\partial t} - \frac{1}{2}(p^2 + \omega^2 + \sigma_3\omega) \right]\psi = 0 \tag{21.57}$$

where ψ is a two-component function, σ_3 is the Pauli matrix, ω is a real parameter.

The equation (21.57) has a specific symmetry in the class M_1 which is defined by the superalgebra *sqm(2)* [417]. This algebra is formed by the operators

$$Q_1 = \frac{1}{\sqrt{2}}(\sigma_1 p + \sigma_2\omega x), \quad Q_2 = \frac{1}{\sqrt{2}}(\sigma_2 p - \sigma_1\omega x), \quad Q_3 = \frac{1}{2}(p^2 + \omega^2 x^2 + \sigma_3\omega) \tag{21.58}$$

which satisfy the following commutation and anticommutation relations

$$[Q_1, Q_2]_+ = 0, \quad Q_1^2 = Q_2^2 = Q_3, \quad [Q_1, Q_3] = [Q_2, Q_3] = 0. \tag{21.59}$$

Invariance under the algebra (21.59) is the main property of equations of supersymmetric quantum mechanics [417].

Investigating of all the nonequivalent SOs of arbitrary order for the equation (21.57) reduces to investigation of symmetries of the Schrödinger equation (21.1). Making the transformation

$$\psi \to \psi' = \exp(-i\omega t\sigma_3/2), \quad L \to L' = \exp(-i\omega t\sigma_3/2) L exp(i\omega t\sigma_3/2) \tag{21.83}$$

we come to the equation $L'\psi'$=0, where $L'=i\partial/\partial t$-$(p^2+\omega^2)/2$ is nothing but a direct sum of two operators (21.1), (21.22c). The corresponding SOs can be represented in the form $Q'^n=\sigma^\mu Q_\mu^n$ where Q_μ^n are the SOs of the equation (21.1), (21.22c) present in (21.54) (where $C_{k,\alpha} \to C_{k\alpha}^\mu$). Returning with the help of the inverse transformation to the starting equation (21.57) we obtain a complete set of SOs of this equation in the form

$$Q^n = \hat{\sigma}^\mu Q_\mu^n,$$

where

$$\hat{\sigma}_0 = \sigma_0, \quad \hat{\sigma}_3 = \sigma_3, \quad \hat{\sigma}_1 = \sigma_1\cos\frac{\omega t}{2} + \sigma_2\sin\frac{\omega t}{2}, \quad \hat{\sigma}_2 = \sigma_2\cos\frac{\omega t}{2} - \sigma_1\sin\frac{\omega t}{2}.$$

The number of linearly independent SOs of order n is equal to $4N^n$, N^n is given in

(21.49).

In a complete analogy with the above we can investigate symmetries of three-dimensional Schrödinger equation with potential of supersymmetric oscillator

$$V(\boldsymbol{x}) = \omega^2 \boldsymbol{x}^2 + \omega\sigma_3.$$

The number of linearly independent SOs of order n is equal to $4N_n$ where N_n is the number given in (16.37).

The approach used above can be extended in order to investigate symmetries of super- and parasupersymmetric Schrödinger equation with arbitrary potential, refer to [3*,4*,5*]. It can be used to search for infinite order symmetries, in this case the determining equations also are given by relations (21.8) or (21.54) where the first lines have to be omitted and summing up is changed by infinite series (i.e., the top summation limit tends to infinity).

22. NONGEOMETRIC SYMMETRIES OF EQUATIONS FOR INTERACTING FIELDS

22.1. The Dirac Equation for a Particle in an External Field

The symmetry of an equation describing a charged particle in an external field as a rule is less extended than the symmetry of the corresponding equation describing a free particle. For instance if the external field is the electric field directed along the fixed axis then the corresponding equation admits the cylindrical symmetry but not the spherical one etc. However for some classes of external fields the above described nongeometric symmetry is preserved and even extended.

Here we consider some examples of external fields corresponding to nontrivial symmetries of the Dirac equation. The assertions present in the following can be verified by direct calculations (for details see [146, 154, 157]).

a) Consider the Dirac equation for a particle in the constant and homogeneous electromagnetic field:

$$\hat{L}\psi \equiv (\gamma^\mu \pi_\mu - m)\psi = 0, \quad \pi_\mu = p_\mu - eA_\mu \tag{22.1}$$

where

$$A_\mu = \frac{1}{2} F_{\mu\nu} x^\nu. \tag{22.2}$$

Here $F_{\mu\nu}$ are constants determining the tensor of the electromagnetic field strengths.

The equation (22.1), (22.2) has not the symmetry in respect with the algebra

A_8 described by Theorem 17.1. But there exist two constants of motion closely connected with this symmetry:

$$F_1=\frac{1}{2}\hat{\Sigma}_{\mu\sigma}F^{\mu\sigma},\quad F_2=\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\hat{\Sigma}^{\mu\nu}F^{\rho\sigma} \tag{22.3}$$

where $\hat{\Sigma}_{\mu\sigma}$ are the operators obtained from (17.6) by the change $p_\mu \to \pi_\mu$:

$$\hat{\Sigma}_{\mu\sigma}=\frac{1}{4}[\gamma_\mu,\gamma_\sigma]-\frac{i}{2m}(1-i\gamma_4)(\gamma_\mu\pi_\sigma-\gamma_\sigma\pi_\mu). \tag{22.4}$$

The operators (22.3) commute and satisfy the relations

$$[F_1,\hat{L}]=\frac{1}{2m}(\gamma_\mu\pi_\sigma-\gamma_\sigma\pi_\mu)F^{\mu\sigma}\hat{L},$$

$$[F_2,\hat{L}]=\frac{1}{2m}\varepsilon_{\mu\nu\rho\sigma}(\gamma^\mu\pi^\nu-\gamma^\nu\pi^\mu)F^{\rho\sigma}\hat{L},$$

so they are SOs of the equation (22.1), (22.2).

b) In the case of the selfdual external electromagnetic field the nongeometric symmetry of the equation (22.1) is just more extensive. Indeed, using the selfduality condition for the vectors of the electric and magnetic field strengths

$$\boldsymbol{H}-i\boldsymbol{E}=0$$

it is not difficult to make sure the operators (22.4) are the SOs of the corresponding equation (22.1). These operators satisfy the invariance condition

$$[\hat{\Sigma}_{\mu\nu},\hat{L}]=\frac{1}{2m}(\gamma_\mu\pi_\nu-\gamma_\nu\pi_\mu)\hat{L} \tag{22.5}$$

and the commutation relations (17.7) characterizing the algebra *AO(1,3)*.

c) Now we consider the Dirac equation with the Pauli-type interaction

$$L\psi\equiv\left[\gamma_\mu\pi^\mu-m+\frac{i}{2m}(1-i\gamma_4)\gamma_\mu\gamma_\nu F^{\mu\nu}\right]\psi=0 \tag{22.6}$$

where $F^{\mu\nu}=-i[\pi^\mu,\pi^\nu]$, A_μ is an arbitrary vector-potential.

The equation (22.6) is manifestly invariant under the Poincaré group such as the Dirac equation for a free particle. It turns out the nongeometric symmetry of (22.6) in the class M_1 coincides with the corresponding symmetry of the free Dirac equation. Indeed, the operators (22.4) (and Σ_0, Σ_1 of (17.6) where $p_\mu \to \pi_\mu$) are the SOs of the equation (22.6), satisfying the invariance condition (22.5).

d) The following example is the Dirac equation for a particle in the constant magnetic field depending on two spatial variables. We choose the corresponding vector-potential in the form

$$A_0=A_3=0,\quad A_1=A_1(x_1,x_2),\quad A_2=A_2(x_1,x_2) \tag{22.7}$$

which corresponds to the magnetic field directed along the third coordinate axis.

The equation (22.1), (22.7) is invariant under the following operators

$$\Sigma_{23}=i\gamma_4(\gamma_3 m+p_3),\quad \Sigma_{31}=i\gamma_3\gamma_0\gamma_\alpha\pi_\alpha,$$

$$\Sigma_{12}=i\Sigma_{23}\Sigma_{31},\quad \Sigma_{0a}=\frac{i}{2}\hat{H}\varepsilon_{abc}\Sigma_{bc},\quad \alpha=1,2$$

(where $\hat{H}=\gamma_0\gamma_a\pi_a+\gamma_0 m$) which form a superalgebra together with $(\Sigma_{\mu\sigma})^2$ and $i\hat{H}(\Sigma_{\mu\sigma})^2$, satisfying the following commutation and anticommutation relations

$$[\Sigma_{\mu\nu},\Sigma_{\lambda\sigma}]_+=2(g_{\mu\lambda}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\lambda}\Sigma^2_{\mu\nu}+i\varepsilon_{\mu\nu\lambda\sigma}\hat{H}[\Sigma^2_{\mu\nu}(1-\delta_{\mu0})+\Sigma^2_{\lambda\sigma}(1-\delta_{\mu0})(1-\delta_{\nu\sigma})],$$

$$[\Sigma_{\mu\nu},\Sigma^2_{\lambda\sigma}]=[\Sigma_{\mu\nu},i\hat{H}\Sigma^2_{ab}]=[\Sigma^2_{\mu\nu},\Sigma^2_{\lambda\sigma}]=[\Sigma^2_{\mu\nu},i\hat{H}\Sigma^2_{ab}]=0.$$

The considered equation is invariant under the algebra A_8 also, besides this algebra is realized in the class of integro-differential operators [146,157]. The analogous symmetry is admitted by the Dirac equation for a particle in the electric field depending on x_0 and x_3 [146].

e) In conclusion, we discuss symmetries of the Dirac equation for a charged particle interacting with the Coulomb field. The corresponding potential has the form

$$A_a=0,\quad A_0=\frac{e^2}{x},\quad x=\sqrt{x_1^2+x_2^2+x_3^2}. \tag{22.8}$$

It was Dirac who showed the SOs of the equation (22.1), (22.8) for the first time [81]. In addition to the evident symmetry under the spatial rotations group whose generators are

$$\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S},\quad (\boldsymbol{S}=\frac{i}{4}\gamma\times\gamma), \tag{22.9}$$

this equation admits the following SO

$$Q_1=(2\boldsymbol{S}\cdot\boldsymbol{J}-\frac{1}{2})\gamma_0 \tag{22.10}$$

which is called the Dirac constant of motion. This SO plays an important role while solving the equation (22.1), (22.8) [81]. It can be easily shown the operator (22.10) coincides with A_0 of (18.5).

In addition to (22.9), (22.10) there exists one more SO of the Dirac equation with the Coulomb potential. This SO belongs to the class M_2 and has the form [237]

$$Q_2=\frac{2\boldsymbol{S}\cdot\boldsymbol{x}}{x}+\frac{i}{me^2}Q_1\left(2\boldsymbol{S}\cdot\boldsymbol{p}+i\gamma_4\frac{e}{x}\right). \tag{22.11}$$

The operators (22.9)-(22.11) together with the following operators

$$Q_1^2=\boldsymbol{J}^2+\frac{1}{4}, \quad Q_2^2=1+\frac{Q_1^2}{m^2e^4}\left(p^2+\frac{e^2}{x^2}-\gamma_0\frac{me^2}{x}\right)$$

form a basis of a seven-dimensional superalgebra, satisfying the following relations

$$[J_a,J_b]=i\varepsilon_{abc}J_c, \quad [J_a,Q_\alpha]=[J_a,Q_\alpha^2]=0,$$

$$[Q_\alpha,Q_{\alpha'}]_+=2\delta_{\alpha\alpha'}Q_\alpha^2, \quad \alpha=1,2, \tag{22.12}$$

$$[H,Q_\alpha]=[H,J_a]=0, \quad H=\gamma_0\gamma_a\pi_a+\gamma_0 m+\frac{e}{x}.$$

Using the identity

$$2\boldsymbol{S}\cdot\boldsymbol{J}=\boldsymbol{J}^2-\boldsymbol{L}^2+\boldsymbol{S}^2, \quad \boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p}$$

it is not difficult to show that in the space of square integrable functions the spectrum of the operator (22.10) is discrete and can be defined by the following formula [81]

$$Q_1\psi=æ\psi\equiv\varepsilon(j+1/2)\psi, \quad \varepsilon=+-1, \quad j=1/2,3/2,\ldots\ . \tag{22.13}$$

The existence of the SOs satisfying (22.12) is the cause of the degeneracy of the energy spectrum of an electron in the Coulomb field. Indeed, an energy level E can be expressed via the eigenvalues æ of the operator Q_1 commuting with the Dirac Hamiltonian, so $E(æ)=E(-æ)$ in accordance with anticommutativity Q_1 with Q_2, and there is a degeneracy in respect with the change æ → -æ.

It is not difficult to verify that the operator Q_1 satisfies the relations

$$[Q_1,\boldsymbol{S}\cdot\boldsymbol{p}]_+=[Q_1,\boldsymbol{S}\cdot\boldsymbol{x}]_+=[Q_1,\gamma\cdot\boldsymbol{x}]=[Q_1,\gamma_0]=0$$

from which it follows that the operator Q_1 is a constant of motion of the Dirac equation in the case of a more complicated potentials also. In particular the following assertion is valid [161]:

THEOREM 22.1. The general form of a spherically symmetric potential $V=V(\boldsymbol{x})$, such that the equation

$$\left(i\frac{\partial}{\partial x_0}-\gamma_0\gamma_a p_a-\gamma_0 m-V\right)\psi=0 \tag{22.14a}$$

admits the SO Q_1 of (22.10), is given by the following formula

$$V=V_1+V_2\gamma_0+V_3\gamma\cdot\boldsymbol{x}+V_4\gamma_0\gamma\cdot\boldsymbol{x} \tag{22.14b}$$

where V_k are arbitrary functions of x. ■

In the case $V_1=qe^2/x$, $V_3=e^2q\mu/x^3$, $V_2=V_4=0$ relation (22.14b) defines the potential of the anomalous Pauli interaction with the field of a point charge and for $V_1=V_2$, $V_3=V_4=0$ this relation defines the general form of the confinement potential used in the quark models based on the one-particle Dirac equation [392].

We see the Dirac constant of motion serves as a SO for a wide class of equations presented in (22.14).

There are other types of external fields generating nontrivial symmetries of the corresponding Dirac equation. Some exotic examples of such fields are collected in [9] where exact solutions of the corresponding equations are present.

22.2. The SO of Dirac Type for Vector Particles

Let us demonstrate that the additional SO exists for vector particles interacting with the field of a point charge [334].

Consider the KDP equation with anomalous interaction for a particle of spin 1 in the Coulomb field:

$$L\psi \equiv (\beta^{\mu}\pi_{\mu} - m - ekS^{\mu\nu}F_{\mu\nu})\psi = 0 \tag{22.15}$$

where

$$\pi_{\mu} = p_{\mu} - eA_{\mu}, \quad F_{\mu\nu} = i[\pi_{\mu}, \pi_{\nu}], \quad S_{\mu\nu} = i[\beta_{\mu}, \beta_{\nu}],$$

β_{μ} are the 10×10 KDP matrices satisfying the algebra (6.20), A_{μ} is the vector-potential of (22.8).

The equation (22.15) can be represented in the Hamiltonian form (6.35) where

$$H = [\beta_0, \beta_a]p_a + \beta_0 m + \frac{e^2}{x} + \frac{ie^2}{m}(k - 1 + \beta_0^2)\frac{\beta_a x_a}{x^3} + \frac{ike^2}{m^2}\left[\frac{\beta_a x_a}{x^3}, \beta_b p_b\right],$$

$$P = 1 - \beta_0^2 + \frac{1}{m}\beta_a p_a \beta_0^2 - \frac{ike^2}{m^2}\beta_a\beta_0\frac{x_a}{x^3}. \tag{22.16}$$

The evident SOs of the equation (6.35), (22.16) are the generators of the rotation group, i.e., the components of the vector $\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p}$ commuting with the Hamiltonian H of (22.16). But it is not all. Using the relations (6.20) it is not difficult to verify that the following operator [334]

$$Q = (1 - 2\beta_0^2)[2(\boldsymbol{S}\cdot\boldsymbol{J})^2 - 2\boldsymbol{S}\cdot\boldsymbol{J} - \boldsymbol{J}^2] \tag{22.17}$$

is a SO of the equations (22.15) and (6.35), (22.16). Indeed, this operator commutes with L of (22.15) as well as with H and P of (22.16). This assertion is valid for k=0 also, i.e., if anomalous interaction is absent.

Like the Dirac SO, the operator (22.17) has a discrete spectrum

$$Q\psi = \varepsilon j(j+1)\psi, \quad \varepsilon = +-1, \quad j = 0, 1, 2, \ldots \tag{22.18}$$

inasmuch as $Q^2 \equiv (\boldsymbol{J}^2)^2$.

An additional SO exists for the Stueckelberg equation [398] also. In presence of the anomalous interaction with the Coulomb field this equation can be written in the form (22.15) where β_μ are the 11×11 matrices given in Subsection 8.5, $S_{\mu\sigma}$ are matrices realising the representation $D(1/2\ 1/2)\oplus D(1\ 0)\oplus D(0\ 1)\oplus D(0\ 0)$ of the algebra $AO(1,3)$. The corresponding SO is given by formula (22.17) where $S_a=\varepsilon_{abc}S_{bc}/2$, S_{bc} and β_0 are the corresponding matrices of dimension 11×11.

In paper [318] nongeometric symmetries of the KDP equation with anomalous Pauli-type interaction (22.15) were described for constant electric and magnetic fields. The corresponding IA is the algebra $A[SU(2)\otimes SU(2)\otimes A_4]$ where A_4 is a four-dimensional commutative subalgebra. We do not present the explicit form of cumbersome basis elements of this IA here.

22.3. The Dirac-Type SOs for Particles of Any Spin

The results given above can be generalized to the case of Poincaré- and Galilei-invariant equations for particles of arbitrary spin [334].

Consider an arbitrary Poincaré-invariant equation of the form (22.15) where $S_{\mu\sigma}$ are generators of a direct sum

$$D=\sum\oplus D(j\ \tau) \tag{22.19}$$

of IRs of the Lorentz group, A_μ is the vector-potential (22.8), β_μ are matrices satisfying (6.7). We assume the equation in question be invariant under the transformation of space inversion of (2.55) where the matrix r_1 by definition satisfies the following relations

$$\begin{aligned} &r_1\beta_0=\beta_0 r_1, \quad r_1\beta_a=-\beta_a r_1,\\ &r_1S_{ab}=S_{ab}r_1, \quad r_1S_{0a}=-S_{0a}r_1. \end{aligned} \tag{22.20}$$

In analogy with (22.10), (22.17) we search for SOs of the considered equation in the form

$$Q=r_1 d, \quad d=d(\boldsymbol{x},\boldsymbol{p},S_{\mu\nu}). \tag{22.21}$$

Requiring Q commutes with L (22.15) and using relations (6.7), (22.20) we obtain the following conditions for d

$$[x,d]=[\beta_0,d]=0, \tag{22.22a}$$

$$[S_{0a}x_a,d]_+=[S_{0a}p_a,d]_+=0. \tag{22.22b}$$

It follows from (22.22a) that d depends on S_{ab}, $a,b\neq 0$, and $\boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p}$. The explicit form of $d=d(S_{ab},\boldsymbol{L})$ can be obtained solving the equations (22.22b), moreover it is sufficient to solve these equations for S_{0a} belonging to an IR $D(j\ \tau)\subset D$.

To find a solution of (22.21b) it is convenient to use the spherical spinor basis in which the operators $S_{0a}x_a/x$ and $xS_{0a}p_a$ reduce to the numeric matrices (refer to Appendix 3). A transition into this basis is one of realisations of the algorithm described in Subsection 16.1. In accordance with this algorithm we reduce a problem of description of SOs to a purely matrix problem.

Omitting cumbersome calculations (for details see [334]) we present the explicit form of d satisfying(22.22) for an arbitrary IR $D(j\ \tau)$.

$j+\tau$ is integer:

$$d=cF\sum_{s=|j-\tau|}^{j+\tau}\sum_{\lambda=0}^{s}(-1)^{\lambda}B_{\lambda}^{s},\quad \lambda=0,1,2,\dots\ . \tag{22.23}$$

$j+\tau$ is half integer:

$$d=cF\sum_{s=|j-\tau|}^{j+\tau}\sum_{\nu=1/2}^{s}(-1)^{s+1/2-\nu}A_{\nu}^{s} \tag{22.24}$$

where c is an arbitrary constant,

$$F=\begin{cases}\prod\limits_{\alpha=1}^{2(j+\tau)-1}(4\boldsymbol{J}^2+1-\alpha^2), & j+\tau\neq\frac{1}{2},\\ 1, & j+\tau=\frac{1}{2},\end{cases}$$

B_{λ}^{s}, A_{λ}^{s} are operators satisfying the relations

$$\sum_{\mu=\nu_0}^{s}B_{\nu}^{s}=1,\quad \nu=\nu_0,\ \nu_o+1,\ \nu_0+2,\ \dots,\quad \nu_0=\frac{1}{2}[1+(-1)^{2s}], \tag{22.25}$$

$$\begin{aligned}&B_{\nu}^{s}B_{\nu'}^{s}=B_{\nu}^{s},\quad B_{\nu}^{s}A_{\nu'}^{s}=\delta_{\nu\nu'}A_{\nu}^{s},\\ &A_{\nu}^{s}A_{\nu'}^{s}=\delta_{\nu\nu'}(4\boldsymbol{J}^2+1)B_{\nu}^{s},\quad A_{0}^{s}\equiv 0,\end{aligned} \tag{22.26}$$

$$\sum_{\nu=\nu_0}^{s}(\nu A_{\nu}^{s}-\nu^2B_{\nu}^{s})=P_s(2\boldsymbol{S}\cdot\boldsymbol{J}-\boldsymbol{S}^2)\equiv G_s. \tag{22.27}$$

Here P_s is the projector

$$P_s=\prod_{s'\neq s}\frac{\boldsymbol{S}^2-s'(s'+1)}{s(s+1)-s'(s'+1)},\quad |m-n|\leq s,s'\leq m+n,$$

$\boldsymbol{S}$ is a vector whose components are $S_a=\varepsilon_{abc}S_{bc}/2$, $S_{bc}\subset D(j\ \tau)$.

For any given value of s the operators B_{ν}^{s} and A_{ν}^{s} can be expressed via G_s. For this purpose it is sufficient to raise the l.h.s. and r.h.s. of (22.27) in powers n=1,2,...,2s and then to solve obtained system of 2s+1 linear algebraic equations for 2s+1 unknowns B_ν^s and A_ν^s. Moreover according to (22.26) the equations whose number is n=2k and n=2k+1 have the form

$$\begin{aligned} G_s^{2k} &= \sum_{\nu=\nu_0}^{s}\left[\sum_{m=0}^{k} C_{2k_{2m}}\nu^{2(k+m)}(4\boldsymbol{J}^2+1)^{k-m}B_\nu^{\ s} - \sum_{n=0}^{k-1} C_{2k}^{2n+1}\nu^{2(n+k)-1}(4\boldsymbol{J}^2+1)^{k-n-1}A_\nu^{\ s}\right], \ k\le s, \\ G_s^{2k+1} &= \sum_{\nu=\nu_0}^{s}\left[\sum_{m=0}^{k} C_{2k+1}^{2m}\nu^{2(k+m)+1}(4\boldsymbol{J}^2+1)^{k-m}A_\nu^{\ s} - \sum_{n=0}^{k-1} C_{2k+1}^{2n+1}\nu^{2(n+k)}(4\boldsymbol{J}^2+1)^{k-n}B_\nu^{\ s}\right], \ k<s. \end{aligned} \tag{22.28}$$

where C_b^a is the number of combinations from b elements by a. The equation with the number n=2s+1 is given in (22.25).

Let $s=(m+n)_{max}$ be the maximal value of the quantum number s appearing by reduction of the representation (22.19) by the group $O(3)$. We present solutions of the equations (22.23)-(22.25), (22.28) for $d=d_s$, $s\le 2$:

$$\begin{aligned} d_{1/2} &= 2\boldsymbol{S}\cdot\boldsymbol{J} - \frac{1}{2}, \\ d_1 &= 2(\boldsymbol{S}\cdot\boldsymbol{J})^2 - 2\boldsymbol{S}\cdot\boldsymbol{J} - \boldsymbol{J}^2, \\ d_{3/2} &= \frac{4}{3}[g^3 - g^2 - (7\boldsymbol{J}^2+\boldsymbol{S}^2)g + (4\boldsymbol{S}^2-6)\boldsymbol{J}^2] + 3, \quad g = 2\boldsymbol{S}\cdot\boldsymbol{J} - \frac{3}{2}; \\ d_2 &= \frac{2}{3}[(\boldsymbol{S}\cdot\boldsymbol{J})^2 - 2\boldsymbol{S}\cdot\boldsymbol{J} - 4\boldsymbol{J}^2](\boldsymbol{S}\cdot\boldsymbol{J}-1)(\boldsymbol{S}\cdot\boldsymbol{J}-3) - \boldsymbol{J}^2(\boldsymbol{J}^2-2) + \\ &+ \frac{1}{3}(\boldsymbol{S}^2-2)[(4-3\boldsymbol{J}^2)(\boldsymbol{S}\cdot\boldsymbol{J})^2 + (7\boldsymbol{J}^2-4)\boldsymbol{S}\cdot\boldsymbol{J} - 4\boldsymbol{J}^2 + \frac{3}{8}\boldsymbol{S}^2(4\boldsymbol{J}^2+1)]. \end{aligned} \tag{22.29}$$

Formulae (22.21), (22.29) give SOs for any Poincaré- and P-invariant equation describing a particle of spin $s\le 2$. Among them are the Rarita-Schwinger and Dirac-Fierz-Pauli equations for a particle of spin 3/2 (refer to Section 6), and the Bhabha equations [35]. Moreover these formulae define SOs for the LHG equations being invariant under the Galilei group, see Sections 13, 15. The corresponding matrices r_1 have the form r_1=ß$_0$-2.

In conclusion we note that the spectrum of the operators (22.29) is given by formulae (22.13), (22.18) (where $Q_1 \to d_{1/2}$ and $Q \to d_1$) and (22.30):

$$\begin{aligned} & d_{3/2}\psi = \varepsilon(2j-1)(2j+1)(2j+3)\psi, \quad \varepsilon = \pm 1, \quad j = 1/2, 3/2, ..., \\ & d_2\psi = \varepsilon(j-1)j(j+1)(j+2)\psi, \quad j = 0, 1, ... \ . \end{aligned} \tag{22.30}$$

22.4. Other Symmetries of Equations for Arbitrary Spin Particles

Here we consider other examples of external fields preserving nongeometric symmetries of equations for particles of arbitrary spin.

We start from the Dirac-type equations (10.10) corresponding to the vector-potential (22.2). In the case $F_{\mu\sigma}=0$ these equations are invariant under the algebra *AGL(2s+1,C)* realized in the class M_1. In the case $F_{\mu\sigma}\neq 0$ the symmetry of these equations is less extensive and reduces to the invariance under a 2(2s+1)-dimensional commutative algebra defined over the field of real numbers. Basis elements of this algebra have the form

$$Q_n=(\hat{\Sigma}_{\mu\nu}F^{\mu\nu})^n,\quad Q_{2s+1+n}=Q_{n-1}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\hat{\Sigma}^{\rho\sigma},\quad n=1,2,...,2s \tag{22.31}$$

where $\hat{\Sigma}_{\mu\nu}$ are operators obtained from (19.19) by the change $p_\mu\rightarrow\pi_\mu$. Since the following conditions are satisfied

$$[\hat{\Sigma}_{\mu\nu}F^{\mu\nu},L_1]=\frac{i}{m}(\Gamma_\mu\pi_\nu-\Gamma_\nu\pi_\mu)F^{\mu\nu}L_1,\quad [\hat{\Sigma}_{\mu\nu}F^{\mu\nu},L_2]=0$$

where L_1 and L_2 are the operators of (10.10), the operators (22.31) are SOs of the system (10.10), (22.2).

Using the deep analogy of (10.10) with the Dirac equation it is possible to reformulate all the results a)-d) of the previous subsection to the case of arbitrary spin. Such a reformulation is almost trivial and reduces to the change $\gamma_\mu\rightarrow\Gamma_\mu$ in the corresponding formulae. Moreover the symmetry of the system (10.10) is more extensive because the squares of the corresponding SOs do not reduce to the unit matrix in contrast to the case of the Dirac equation. For example in the case of the self-dual electromagnetic field we find a $2(2s+1)^2$-dimensional invariance algebra of the system (10.10) formed by the SOs of (19.20) where $\Sigma_{\mu\sigma}$ are again the SOs obtained from (19.19) by the substitution $p_\mu\rightarrow\pi_\mu$.

Let us consider also the Poincaré-invariant equations without superfluous components for a particle of arbitrary spin in an external field described by the specific potential

$$(H_s^I+V)\psi=i\frac{\partial}{\partial x_0}\psi \tag{22.32}$$

where H_s^I is the Hamiltonian of a free particle of arbitrary spin given in (7.27),

$$V=(1+\sigma_1)\varphi(x). \tag{22.33}$$

In the case s=1/2 formula (22.32) presents a well-known equation used for description of quarks in the field with an effective potential being the sum of the scalar and vector components [392]. We will not work out details of the explicit form of φ(x) which is not essential for our aims.

The equation (22.32) has a wide nongeometric symmetry described in the following assertion.

THEOREM 22.2. The equation (22.32) is invariant under the algebra *AGL(2s+1,R)* whose basis elements λ_{mn} are given by formulae (19.20) where

$$\Sigma_{ab}=-i\varepsilon_{abc}\Sigma_{0c}=-i\varepsilon_{abc}\hat{S}_c,\quad \Sigma_1=i, \tag{22.34a}$$

$$\hat{S}_a=\sigma_1 S_a+(1-\sigma_1)p_a\boldsymbol{S}\cdot\boldsymbol{p}p^{-2}. \tag{22.34b}$$

PROOF. The equation (22.32) and the operators (22.34) can be reduced to the form for which the statements of the theorem become obvious. Using the transformation operator

$$U=U^{-1}=\frac{1}{2}\left[1+\sigma_1+(1-\sigma_1)\sum_{\nu}(-1)^{\nu}\Lambda_{\nu}\right],$$

where Λ_ν are the projectors (7.15), we obtain

$$U(H_s^I+V)U^{\dagger}=\sigma_1 m+\sigma_3 p+\frac{1}{2}(1+\sigma_1)\varphi(x), \tag{22.35}$$

$$U\hat{S}_aU^{\dagger}=S_a. \tag{22.36}$$

Matrices (22.36) commute with the Hamiltonian (22.35) and satisfy the relations (4.25), (4.30), (4.31). So we can construct a basis of the algebra *AGL(2s+1,R)* by formulae (19.20). ■

We note that the analogous symmetries are admitted by the equations

$$(H_s^{III}+V)\psi=i\frac{\partial}{\partial x_0}\psi$$

where V is the potential (22.33), H_s^{III} is one of the Hamiltonians (7.40). A basis of the corresponding IA is given by formulae (19.20), (22.34). Thus the TST equation with the specific potential (22.33), describing a particle of spin 1, is invariant under the algebra *AGL(3,R)*.

22.5. Symmetries of Galilei Particle of Arbitrary Spin in the Constant Electromagnetic Field

Here we study symmetries of Galilei-invariant wave equations describing a particle in the field corresponding to the vector-potential (22.2). Namely we consider the LHG equations, refer to (15.1), (13.11), (13.12), Column R_1 in the Table 13.1.

THEOREM 22.3. The LHG equation for a particle of spin s interacting with the constant electromagnetic field is invariant under the algebra *AGL(2s+1,R)* whose basis elements are given in (19.20), (22.34a) where

$$\hat{\boldsymbol{S}}=\left(\boldsymbol{S}-\frac{\eta\times\boldsymbol{p}}{m}\right)\cos\left(\frac{eH}{2m}x_0\right)+\left[\boldsymbol{S}\times\boldsymbol{H}-\frac{(\eta\times\boldsymbol{p})\times\boldsymbol{H}}{m}\right]\sin\left(\frac{eH}{2m}x_0\right),\tag{22.37}$$

$H=(H_1^2+H_2^2+H_3^2)^{1/2}$,

$\boldsymbol{S}$ and η are the corresponding generators of the homogeneous Galilei group.

The proof reduces to direct verification of validity of the following assertions:

a) the operators (22.37) satisfy the condition $L\hat{\boldsymbol{S}}=\hat{\boldsymbol{S}}^{\dagger}L$ where L is the operator of (15.1);

b) the operators (22.37) satisfy the commutation relations (12.20a) moreover $\hat{\boldsymbol{S}}^2\psi=s(s+1)\psi$ for any ψ satisfying (15.1).

These assertions are easily verified using relations (13.9), (13.10).

We see nongeometric symmetries of the LHG equations are very extensive. All the SOs (19.20), (22.37) belong to the class M_1 since the matrices η_a corresponding to the LHG equations satisfy the relation $\eta_a\eta_b=0$.

Another Galilei-invariant equations considered in Chapter 3 do not admit the symmetry formulated in Theorem 21.3. But for the constant and homogeneous magnetic field we can find a class of equations being invariant under the algebra *AGL(2s+1,R).* There are the equations (15.26), (15.28) for $k_1=1$. The corresponding SOs are given in (19.20), (22.34) where

$$\hat{S}_a=VS_aV^{-1},\quad V=\exp\left(\frac{i}{m}te\boldsymbol{S}\cdot\boldsymbol{H}\right)\exp\left(it\left[H(\pi,A_0)-\frac{\pi^2}{2m}-\frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{H}\right]\right).$$

This symmetry makes it possible to obtain the exact solutions of the corresponding equations (see Chapter 6).

22.6. Symmetries of Maxwell's Equations with Currents and Charges

Here we show that Maxwell's equations with currents and charges also have an additional symmetry extending well-known Poincaré and conformal invariance. Using the covariant formulation (3.9), (3.10) of these equations we formulate and prove the following assertion.

THEOREM 22.4. There exist twenty SOs of Maxwell's equations with currents and charges in the class of second order differential operators with matrix coefficients. These SOs do not belong to the enveloping algebra of the algebra *AC(1,3)* and have the following form

$$Q^{[\mu\nu][\rho\sigma]}=(1+2\beta_4^2)[Z^{\mu\rho}p^{\nu}p^{\sigma}+Z^{\nu\sigma}p^{\mu}p^{\rho}-Z^{\mu\sigma}p^{\nu}p^{\rho}-Z^{\nu\rho}p^{\mu}p^{\sigma}]\tag{22.38}$$

where

$$Z^{\mu\nu}=g^{\mu\nu}-\beta^{\nu}\beta^{\mu}-\beta^{\mu}\beta^{\nu}. \qquad (22.39)$$

Proof reduces to direct verification (using relations (6.20)) the fact that $Q^{[\mu\nu][\rho\sigma]}$ commutes with L_1 and L_2 of (3.10). ■

The operator (22.38) is antisymmetric under permutations of indices inside of square brackets and symmetric under the permutation of the pairs of indices included in the first and second square brackets. In other words, it is a basic tensor having 20 independent components (refer to Subsection 16.2 for definitions).

We note that Maxwell's equations with currents and charges have a wide additional symmetry. The operators (22.38) do not belong to the enveloping algebra of the algebra *AC(1,3)* and thus are essentially new in comparison with the SOs which can be obtained in the classical Lie approach.

Calculating commutators of the operators (22.38) with the generators of the conformal group given in (2.22), (2.42), (3.20) it is possible to find a more wide class of the second-order SOs which in general depend on x_μ. An example of such a SO is the operator (22.17) which is admitted by Maxwell's equations with currents and charges.

It is not difficult to make sure the SOs (22.38) do not form a basis of a Lie algebra. But they include a subset of operators which can be extended to a superalgebra. Denoting

$$\begin{aligned}Q^{\nu\sigma}&=Q^{[\mu\nu][\rho\sigma]}g_{\mu\rho}-\frac{1}{2}g^{\nu\sigma}Q^{[\mu\lambda][\rho\alpha]}g_{\mu\rho}g_{\lambda\alpha}\equiv\\&\equiv(1+2\beta_4^2)\{Z^{\nu\sigma}p_\lambda p^{\lambda}-Z^{\nu\rho}p_\rho p^{\sigma}-Z^{\sigma\rho}p_\rho p^{\nu}-\\&-2(1+\beta_4^2)p^{\nu}p^{\sigma}+g^{\nu\sigma}[2(1+\beta_4^2)p_\lambda p^{\lambda}+Z^{\lambda\alpha}p_\lambda p_\alpha]\}\end{aligned} \qquad (22.40)$$

we select ten SOs satisfying the following anticommutation relations [327]

$$[Q^{\nu\sigma},Q^{\nu'\sigma'}]_+=f^{\nu\sigma\nu'\sigma'}_{\alpha\beta\lambda\rho}\eta^{\alpha\beta\lambda\rho}+g^{\nu\sigma\nu'\sigma'}_{\alpha\lambda}\eta^{\alpha\lambda} \qquad (22.41)$$

where

$$\begin{aligned}&\eta^{\alpha\beta\lambda\rho}=p^{\lambda}p^{\beta}p^{\lambda}p^{\rho},\quad \eta^{\alpha\lambda}=p^{\alpha}p^{\lambda}[p_\mu p^{\mu}-(\beta_\mu p^{\mu})^2],\\&f^{\nu\sigma\nu'\sigma'}_{\alpha\beta\lambda\rho}=\frac{1}{12}(g^{(\nu\nu'}g_{\alpha\beta}-g_\alpha{}^{\nu}g_\beta{}^{\nu'})(g^{\sigma\sigma'}g_{\lambda\rho}-g^{\sigma}{}_{\lambda}-g^{\sigma')}{}_{\rho})-\\&\qquad-\frac{1}{24}(g^{(\nu\sigma}g_{\alpha\beta}-g^{\nu}{}_{\alpha}g^{\sigma}{}_{\beta})(g^{\nu'\sigma'}g_{\lambda\rho}-g^{\nu'}{}_{\lambda}g^{\sigma')}{}_{\rho});\end{aligned} \qquad (22.42)$$

$$g^{\nu\sigma\nu'\sigma'}_{\alpha\lambda}=\frac{1}{12}(g^{(\nu\nu'}g^{\sigma}{}_{\alpha}g^{\sigma'}{}_{\lambda}-g^{\nu'\sigma'}g^{\nu}{}_{\alpha}g^{\sigma)}{}_{\lambda})+\frac{1}{24}(g^{(\nu\sigma}g^{\nu'\sigma'}-g^{\nu\sigma'}g^{\nu'\sigma)})g_{\alpha\lambda},$$

and symmetrization is imposed over the top indices in the r.h.s. of (22.42).

In accordance with (22.42) the following set of SOs of Maxwell's equations

$$\{Q_{\mu\sigma};\ P_\sigma, \eta_{\mu\sigma\alpha\rho}\}$$

(where P_μ, $J_{\mu\sigma}$ are generators of the Poincaré group given in (2.20), (3.20)) form a basis of a superalgebra. Moreover $Q_{\mu\sigma}$ are the odd and the remaining operators are even elements of this superalgebra.

Using the explicit expressions (6.22), (6.23) for the ß-matrices it is not difficult to find the action of the SOs (22.38) to components of the wave function (3.9), i.e., to find the corresponding symmetry transformations of the strengths $\boldsymbol{E}$, $\boldsymbol{H}$ and current j. In this way, it is possible to verify directly the validity of Theorem 21.4.

In conclusion of this subsection we consider Maxwell's equations in a conducting medium:

$$\begin{aligned} &i\frac{\partial \boldsymbol{E}}{\partial x_0} = -i\boldsymbol{p}\times\boldsymbol{H} + i\sigma\boldsymbol{E}, \quad \boldsymbol{p}\cdot\boldsymbol{E}=0; \\ &i\frac{\partial \boldsymbol{H}}{\partial x_0} = \boldsymbol{p}\times\boldsymbol{E}, \qquad\qquad\quad \boldsymbol{p}\cdot\boldsymbol{H}=0 \end{aligned} \tag{22.43}$$

where σ is the coefficient of conductivity. We will show that, just as equations for the electromagnetic field in vacuum, the equations (22.43) are invariant under the algebra *AGL(2,C)*.

Using the notations (3.6), (20.1), (20.3) we write (22.43) in the momentum representation

$$L_1\varphi(x_0,\boldsymbol{p})=0, \quad L_1=i\frac{\partial}{\partial x_0}-\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}+\frac{i}{2}(1+\sigma_3)\sigma, \tag{22.44a}$$

$$L_2^a\varphi(x_0,\boldsymbol{p})=0, \quad L_2^a=p_a-\boldsymbol{S}\cdot\boldsymbol{p}S_a. \tag{22.44b}$$

The equations (22.44) are invariant under an eight-dimension Lie algebra isomorphic to *AGL(2,C)*. The basis elements of this IA have the form

$$\Sigma_{23}=\frac{i}{2}h|h|^{-1}\boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \quad \Sigma_{31}=\frac{i}{2}h|h|^{-1}\sigma_3 Q,$$

$$\Sigma_{12}=\frac{1}{2}\sigma_3\boldsymbol{S}\cdot\hat{\boldsymbol{p}}Q, \quad \Sigma_{01}=\frac{1}{2}\boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \quad \Sigma_{02}=\frac{1}{2}\sigma_3 Q,$$

$$\Sigma_{03}=-\frac{i}{2}h|h|^{-1}\sigma_3\boldsymbol{S}\cdot\hat{\boldsymbol{p}}Q, \quad \Sigma_1=ih|h|^{-1}, \quad \Sigma_0=I$$

where Q is the matrix given in (20.7),

$$h=\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}-i\sigma_3\sigma/2, \quad \hat{\boldsymbol{p}}=\boldsymbol{p}/p,$$

$$|h|^{-1}=\left(p^2-\frac{1}{4}\sigma^2\right)^{-1/2}(\boldsymbol{S}\cdot\hat{\boldsymbol{p}})^2-\frac{2i}{\sigma}[1-(\boldsymbol{S}\cdot\hat{\boldsymbol{p}})^2].$$

For the proof see [154].

22.7. Super- and Parasupersymmetries

Here we search for relativistic wave equations for charged particles in external fields, which admit exact super- or parasupersymmetries.

An attractive feature of Witten's supersymmetric quantum mechanics is that it is supported by a realistic physical model. We say about the problem of interaction of spin 1/2 particle with magnetic field. Indeed, as the Pauli as the Dirac equation predict supersymmetric energy spectra for such a particle interacting with the magnetic field depending on two space variables only [417,7*]. Moreover there exist the corresponding symmetries (supercharges) causing the specific degeneration of this spectra.

We demonstrate here that parasupersymmetric quantum mechanics [372] also is supported by a wide class of wave equations for particles interacting with external fields. In contrast with the sypersymmetric case these equations have to include anomalous (Pauli) interactions.

First we return to the Dirac equation (22.1) including the external magnetic field described by the vector-potential (22.7). Setting for simplicity p_3=0 we immediately find that the following operators

$$\begin{aligned}&Q_1=i\gamma_4(\gamma_1\pi_2-\gamma_2\pi_1),\quad Q_2=i(\gamma_2\gamma_3\pi_1+\gamma_3\gamma_1\pi_2),\\&\quad H_{SS}=\pi_1^2+\pi_2^2-ie\gamma_1\gamma_2H\qquad (H=i[\pi_1,\pi_2])\end{aligned}\tag{22.45}$$

are SOs of the equation considered. Besides that these SOs satisfy relations (18.10′) for *a,b*=1,2 and so realize a representation of the superalgebra *sgm(2)*.

Inasmuch as the operator H_{SS} coincides with $(p_0)^2$-m^2 on the set of solutions of the Dirac equation, it generates the spectrum of squared energies of a spin 1/2 particle in the field considered. The specific degeneration of this spectra (any line except the one corresponding to the ground state is twice degenerated) is caused by the symmetries (22.45).

It is necessary to emphasize that the above symmetry exists for $p_3\neq0$ also. The corresponding supercharges have the form

$$\hat{Q}_1=\gamma_0Q_1-\gamma_0(1-i\gamma_4)(p_3Q_2-H_{SS})/m,\quad \hat{Q}_2=Q_2-(1-i\gamma_4)p_3Q_1/m.$$

Consider the KDP equation with anomalous interaction

$$\left[\beta^\mu\pi_\mu-m-\left(k_1+\beta_4^2k_2\right)\frac{e}{2m}S_{\mu\nu}F^{\mu\nu}\right]\psi=0\tag{22.46}$$

where β_μ are 10×10 KDP matrices, k_1 and k_2 are arbitrary real parameters.

The last term in the l.h.s. of (22.46) represents the most general form of anomalous (Pauli) interaction preserving Poincaré and P, C, T invariance of the equation considered.

We restrict ourselves to the case of the constant and homogeneous magnetic field directed along the third coordinate axis and again set p_3=0. The corresponding strengths and potentials have the form $F_{0a}=F_{31}=F_{23}=0$, F_{12}=H, $A_0=A_3=A_2=0$, $A_1=-iHx_2$.

PROPOSITION. Let ψ satisfies the equation (22.46) with the above potentials, where

$$k_1=k_2=1. \tag{22.47}$$

Then

$$p_0^2\psi=\left(m^2+\pi_1^2+\pi_2^2-2S_3eH\right)\psi,\quad S_3=i\left[\beta_1,\beta_2\right]. \tag{22.48}$$

PROOF. Using the representation (6.22) of the β-matrices and expressing "nonphysical" components $(1-\beta_0^2)\psi$ via $\beta_0^2\psi$ we come to the generalized TST equation

$$i\frac{\partial}{\partial x_0}\psi'=\hat{H}\psi',$$
$$\hat{H}=\sigma_2\left(m-\frac{eH}{m}S_3\right)-i\sigma_1\frac{(\boldsymbol{S}\cdot\pi)^2}{m}+(\sigma_1+\sigma_2)\frac{\pi^2}{2m}. \tag{22.49}$$

The Hamiltonian $\hat{H}$ satisfies the relation

$$\hat{H}^2\equiv m^2+\pi^2-2eS_3H\equiv m^2+\hat{H}_{PSS} \tag{22.48}$$

from which it follows that (22.48) does is valid. ■

We see the squared energy operator generated by the considered equation has a typical parasupersymmetric structure discovered in [372]. Let us demonstrated that this structure is caused by existence of the SOs of the equation considered which form a parasuperalgebra.

To find the corresponding parasupercharges it is convenient to transform (22.49) to an equivalent representation including the minimal number of matrices. Using the transformation operator

$$W=V_1V_2,\quad V_2=1-S_3^2-i\sigma_2S_3,\quad V_2^{-1}=1-S_3^2+i\sigma_2S_3,$$
$$V_1=1-i\sigma_2\frac{\boldsymbol{S}\cdot\pi}{m}+(1-\sigma_3)\frac{(\boldsymbol{S}\cdot\pi)}{2m^2},\quad V_1^{-1}=V_1(-\pi) \tag{22.49}$$

we obtain

$$p_0\psi'=\hat{H}'\psi',\quad \psi'=W\psi, \tag{22.50}$$

where

$$\hat{H}^{/}=W\hat{H}W^{-1}=\sigma_2 m+(\sigma_2+i\sigma_1)(\pi^2-2eS_3H)/2m. \tag{22.51}$$

In contrast with (22.49) the Hamiltonian (22.51) includes the only spin matrix S_3. This circumstance simplifies finding the corresponding SOs which are parasupercharges. Choosing

$$Q_1^{/}=S_1\pi_1+S_2\pi_2, \quad Q_2^{/}=S_1\pi_2-S_2\pi_1 \tag{22.52}$$

it is not difficult to make sure that these operators are the SOs of the equation (22.50) and satisfy the relations (19.18′) together with H_{PSS} of (22.49).

Thus we found a symmetry parasuperalgebra of the equation (22.49). Using the transformation inverse to (22.51) it is not difficult to find the corresponding parasupercharges for the starting TCT equation (22.49) and than to reconstruct these symmetries for the KDP equation (22.46). We do not present the corresponding cumbersome formulae here.

We note that the choice (22.47) of the coupling constants correspond to the causal KDP equation with an anomalous interaction, while for $k_1\neq k_2$ we come to a non-causal equation [27*]. Thus, only the casual equation for spin-one particles generates parasupersymmetries.

It is well-known that a parasupersimmetric Hamiltonian has as positive, as negative eigenvalues [372]. In our case this "Hamiltonian" coinsides with P_0^2, thus the equation (22.46), (22.47) generates complex energy values. To overcome this difficulties it is sufficient to take into account the anomalous interaction bilinear in respect with $F_{\mu\sigma}$, refer to Subsection 10.9.

It is possible to show that analogous parasupersymmetries exist for any spin-one equation considered in Sections 2, 3, if anomalous Pauli-type interaction is taken into account [28*].

22.8. Symmetries in Elasticity

Let us study symmetries of the main equation of elasticity [271]

$$\frac{\partial^2\boldsymbol{U}}{\partial t^2}=-\frac{\lambda}{\rho_0}p^2\boldsymbol{U}+\frac{\lambda+\mu}{\rho_0}\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{U}). \tag{22.53}$$

Here $\boldsymbol{U}=(U^1,U^2,U^3)$ is the displacement vector, $\lambda>o$, $\mu+\lambda>0$ and $\rho_0>0$ are the Lame coefficients.

The equation (22.53) is not Poincaré-invariant but its differential consequences have this invariance [271]. Inasmuch as symmetries of this equation can have important and physically measurable consequences we decided to investigate them besides symmetries of the basic equations of quantum mechanics.

Let us rewrite (22.53) in the matrix form

$$LU \equiv Z_{ab}[(p_0^2 - \frac{\lambda}{\rho_0}p^2)\delta^{ab} + \frac{\lambda+\mu}{\rho_0}p_a p_b]U=0 \tag{22.54}$$

where U is a column with components U^1, U^2, U^3, Z_{ab} are the matrices (3.6).

The maximal IA of (22.54) in the classical Lie approach was found in [66]. The basis elements of this algebra have the form

$$\begin{aligned} &P_0 = \frac{\partial}{\partial t}, \quad P_a = \frac{\partial}{\partial x_a}, \\ &J_a = \varepsilon_{abc} x_b p_c + iS_a, \quad D = tP_0 - x_a p_a \end{aligned} \tag{22.55}$$

where S_a are the matrices (3.6).

It is natural to investigate symmetries of the equation (22.54) in the class M_2, i.e., in the class which includes the operator L. The corresponding SOs can be useful for description of systems of coordinates in which solutions in separated variables exist [365], and for finding new conservation laws in elasticity.

We will search for SOs for the equation (22.54) using a complete set of the matrices Z^{ab} and S^a:

$$Q = Z_{ab} b^{ab} + iS_a c^a \tag{22.56}$$

where b^{ab} and C^a are differential operators of the second order with real coefficients depending on $\boldsymbol{x}$. By definition the operator (22.56) is a SO of (22.54) if it satisfies the condition (16.7) where L is the operator of (22.54), α_Q is an operator which belongs to the class M_2 and so can be represented in the form given in the r.h.s. of (22.56).

Substituting L of (22.54) and Q of (22.56) into (16.7) and calculating the necessary commutators and anticommutators with the help of relations (20.31) we obtain a system of determining equations for coefficients of the operator (22.56). We will not rewrite this cumbersome system, but instead present a consequence of its solution (see [158] for the detail).

THEOREM 22.5. The basic equation of elasticity (22.54) admits 61 linearly independent SOs in the class M_2. Among them are the operators (22.55), products of these operators and the following 9 operators which do not belong to the enveloping algebra generated by the algebra (22.55):

$$Q_0=2\boldsymbol{S}\cdot\boldsymbol{J}(\boldsymbol{S}\cdot\boldsymbol{J}-1)-\boldsymbol{J}^2,$$

$$Q_a=[\varepsilon_{abc}S_b p_c,\boldsymbol{S}\cdot\boldsymbol{J}-\frac{1}{2}]_+ +\frac{1}{2}\varepsilon_{abc}[J_b,P_c]_+, \tag{22.57}$$

$$Q_{ad}=\hat{Q}_{ad}-\frac{1}{3}\delta_{ad}\hat{Q}_{nn}$$

where

$$\hat{Q}_{ad}=Z_{ad}p^2+p_a p_d-p_a Z_{dc}p^c-p_d Z_{ac}p^c+\delta_{ad}(Z_{cd}p^c p^d-p^2). \tag{22.58}$$

So the equation (22.54) has 9 essentially non-Lie SOs in the class M_2. We note that Q_0 of (22.57) is nothing but a Dirac-type SO, compare with (22.17).

The operators (22.57) do not form a Lie algebra. But the set of the SOs

$$\left\{\hat{Q}_{ab},\ \boldsymbol{J}\cdot\boldsymbol{P}=\boldsymbol{S}\cdot\boldsymbol{P};\quad P_0,\ P_a,\ J_a,\ D,\ \eta_{ab},\ \eta_{abcd}\right\}$$

where

$$\eta_{ab}=P_a P_b[P^2-(\boldsymbol{S}\cdot\boldsymbol{P})^2],\quad \eta_{abcd}=P_a P_b P_c P_d$$

and the other operators given in (22.55), (22.58), form a basis of the superalgebra. Moreover $\boldsymbol{S}\cdot\boldsymbol{P}$ anticommutes with $\hat{Q}_{ab}$, the anticommutation relations for $\hat{Q}_{ab}$ have the form (22.41) where $\nu\to a$, $\sigma\to b$, $g_{\mu\nu}\to -\delta^{ab}$.

In conclusion we consider conditional symmetries in elasticity. Imposing on $\boldsymbol{U}$ the condition of transversality $div\boldsymbol{U}=0$ we come from (22.53) to the following system

$$\frac{\partial^2}{\partial t^2}\boldsymbol{U}+\frac{\lambda}{\rho_0}p^2\boldsymbol{U}=0;\quad \boldsymbol{p}\cdot\boldsymbol{U}=0. \tag{22.59}$$

With the help of the change $t=t'(\lambda/\rho)^{1/2}$ this system reduces to the form (20.37). Thus all the results connecting with the conditional symmetry of the equations (20.37) (refer to Theorems 20.4, 20.5) are valid for the equations (22.49) describing the transverse elastic waves.

Consider also symmetries of the equation (22.53) supplemented by the subsidiary condition $\boldsymbol{p}\times\boldsymbol{U}=0$, or

$$\boldsymbol{S}\cdot\boldsymbol{p}U\equiv L_2 U=0. \tag{22.60}$$

In this case we come from (22.53) to the following equation (describing longitudinal waves)

$$L_1 U=0,\quad L_1=\frac{\partial^2}{\partial t^2}+\frac{\mu}{\rho_0}p^2. \tag{22.61}$$

Investigations of symmetries of the system (22.60), (22.61) can be carried out in a complete analogy with Subsection 20.5, that is why we restrict ourselves to formulating the final results.

ASSERTION 1. The maximal IA of the system (22.60), (22.61) in the class M_1 is the eight-dimensional Lie algebra whose basis elements are given in (22.55) where $t \to t' = t(\rho_0/\mu)^{1/2}$.

ASSERTION 2. The maximal IA of the equation (22.61) without subsidiary condition (22.60) is the 24-dimensional Lie algebra whose basis elements are given in (20.39) where $t \to t' = t(\rho_0/\mu)^{1/2}$.

ASSERTION 3. If relation (22.60) is satisfied then the equation (22.61) is invariant under the 25-dimensional Lie algebra whose basis elements are given in (21.50), (22.62):

$$\eta = S_a x_a. \tag{22.62}$$

We see the conditional symmetry of the Lame equation is rather extensive. In any case it is considerably more extensive than the usual Lie symmetry, compare with (22.55). We recall this is the conditional symmetry which generates conservation laws and thus has a clear physical sense.

The conserved quantities corresponding to non-Lie and conditional symmetries of the equation (22.53) are discussed in the following section, refer to Subsection 23.8.

The non-Lie symmetries of (22.53) were found in [122]. The explicit form of the integro-differential SOs of this equation is given in [138]. Symmetries of the stationary equations of elasticity in the class M_1 were studied in details by Olver, refer to [351].

23. CONSERVATION LAWS AND CONSTANTS OF MOTION

23.1 Introduction

In connection with the above the following questions arise: what are physical consequences of nongeometric symmetries and is it possible to use these symmetries while solving concrete physical problems?

The very existence a nongeometric symmetry of a motion equation is a fundamental fact which reflects that there is an internal degree of freedom of the object described or can give other nontrivial information. Some consequences of that symmetry can be used for different purposes: to find constants of motion, to construct the density matrix, to expand solutions in a complete set of eigenfunctions of a SO and so on. In Subsection 31.8 we use nongeometric symmetries in order to construct exact solutions and generate new solutions starting from known ones. A discussion of some applications of nongeometric symmetries presents the main contents of this section.

One of the most important consequences of symmetries of equations of motion is existence of constants of motion, i.e., some combinations (as a rule bilinear) of solutions which are conserved in time. The well-known examples of constants of motion are energy, momentum and angular momentum.

But what type of constants of motion corresponds to the nongeometric symmetries described above? In the traditional approach to searching constants of motion, we investigate symmetries of the Lagrangian of the described system and then find the corresponding conservation laws using the Noether theorem (see, e.g., [41,42]). The advantage of this approach is that so defined conservation laws as a rule have meaningful physical interpretations. But this approach has evident limitations since not every equation of mathematical physics admits the Lagrangian formulation and moreover Noether's theorem does not allow all the constants of motion even for those equations which can be obtained using the variational principle.

According to the above, in describing the conservation laws corresponding to nongeometrical symmetries, it is preferable to use another more universal approach whose essence is the direct calculation of bilinear combinations of solutions for the motion equation, which are conserved in time by virtue of symmetries of these equations. Moreover in this way it is possible to find conserved quantities which do not correspond to any SO. Examples of such quantities are given in Subsections 22.7, 22.8.

Let us consider an arbitrary evolution equation of the form

$$i\frac{\partial}{\partial x_0}\psi\equiv i\frac{\partial}{\partial t}\psi=H(\boldsymbol{p})\psi \tag{23.1}$$

where $H(\boldsymbol{p})$ is a differential operator with matrix coefficients, $\psi=\psi(x_0\equiv t,\boldsymbol{x})$ is a real (or complex) multicomponent function belonging to the space $L_2(R_4)$.

Let Q be a linear operator defined on a set everywhere dense in the space of vector-functions $\psi\subset L_2(R_4)$. To each solution of (23.1) we assign a bilinear combination of the form

$$I_Q=\int d^3x\psi^\dagger Q\psi \tag{23.2}$$

where $\psi^\dagger$ is a transposed and conjugated function.

Differentiating (23.2) in respect with x_0 and using (23.1) we obtain the following sufficient condition of conserving I_Q in time

$$\left[\left(i\frac{\partial}{\partial x_0}-H^\dagger(\boldsymbol{p})\right)Q-Q\left(i\frac{\partial}{\partial x_0}-H(\boldsymbol{p})\right)\right]\psi=0. \tag{23.3}$$

If the operator $H(\boldsymbol{p})$ is Hermitian then (23.3) reduces to the form

But the condition (23.4) is nothing but a definition of a SO of the equation

$$\left[i\frac{\partial}{\partial x_0}-H,Q\right]\psi=0. \tag{23.4}$$

(23.1), compare with (1.5). In other words if the evolution operator (Hamiltonian) is Hermitian then we can assign the conserved quantity to any SO of the equation (23.1).

We emphasize the relation (23.3) is not a necessary condition of conservation of (23.2) in the case when ψ satisfies some additional conditions besides (23.1). Maxwell's equations are a good example of a system of equations including the subsystem of the sort (23.1). But besides that Maxwell's equations include additional conditions not containing time derivatives, and this is why these equations generate such constants of motion of the form (23.2) that the equations (23.3), (23.4) are not satisfied.

Conserved quantities can be calculated also using a more traditional approach based on the concept of currents satisfying the continuity equation. Let us formulate the corresponding definitions valid for an arbitrary system of partial differential equation of the form

$$F^A(x,U,U',...)=0,\quad A=1,2,...,N. \tag{23.5}$$

Here $x=(x_0,x_1,...,x_n)$ are independent variables, $U=(U^1,U^2,...,U^m)$ is a vector-function depending on x, U' denotes derivatives in respect with x, i.e., the set of the following quantities

$$U_j^k=\frac{\partial U^k}{\partial x_j},\quad j=0,1,...,n;\quad k=1,2,...,m,$$

the dots denote derivatives of higher order which can be included into the system (23.5).

Following [228] we say that there exists a conservation law for the system (23.5) if it is possible to assign $(n+1)$-dimensional vector $j_\mu(x,U,U',...,)$ $(\mu=0,1,...,n)$ for any solution of (23.5) moreover

$$p^\mu j_\mu=0. \tag{23.6}$$

According to the Ostrogradskii-Gauss theorem, it follows from (23.6) that the quantity

$$<j_0>=\int_{R_n} d^n x j_0(x,U,U',...)$$

does not depend on x_0. Here R_n is a domain of integration, which we assume to coincide with the n-dimensional manifold $\{x\}$.

Until now there is no a constructive algorithm of describing of all the possible conservation laws admitted by an arbitrary system of partial differential equations. Our approach is to find all the conserved bilinear quantities of the kind (23.2) being conserved in time. Then starting from the found j_0 we can calculate the remaining components of the conserved current using the symmetry group of the equation of interest. Such an approach enables us to formulate the natural definition of currents equivalence: we say two currents j_μ and j_μ' are equivalent if

$$\int_{R_n} d^n x j_0 = \int_{R_n} d^n x j_0', \tag{23.7}$$

a current is called trivial if $\langle j_0 \rangle = 0$.

23.2. Conservation Laws for the Dirac Field

Let Q be a SO of the Dirac equation. We define the corresponding conserved current as follows

$$j_\mu^Q = \frac{1}{2}(\overline{\psi}\gamma_\mu Q\psi + \overline{Q\psi}\gamma_\mu\psi) \tag{23.8}$$

where $\overline{\psi}=\psi^\dagger$ and ψ belong to the set of solutions of the Dirac equation. Inasmuch as by definition Q satisfies (16.2), (17.1) then j_μ^Q (23.8) satisfies the continuity equation (23.6).

We note that the correspondence "SO - conservation law" given above is isomorphism in the sense that to any SO of the Dirac equation we can assign the conserved quantity (23.8). On the other hand any bilinear conserved quantity corresponds to a SO; to find a complete set of such quantities is to find the general solution of the equation (23.4) defining SOs of the Dirac equation. So there is one-to-one correspondence between SOs and zero components of conserved currents, the other components are easily found using Lorentz transformations.

We present the explicit form of the currents corresponding to the SOs of the class M_1. Starting from generators of the Poincaré group we obtain from (2.22), (23.8) the well-known expressions for the tensors of energy-momentum and angular momentum

$$T_{\mu\sigma} = \frac{i}{2}\left(\overline{\psi}\gamma_\sigma\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\overline{\psi}}{\partial x_\mu}\gamma_\sigma\psi\right),$$
$$M_{\mu\lambda\sigma} = x_\mu T_{\lambda\sigma} - x_\lambda T_{\mu\sigma} + \frac{1}{2}\overline{\psi}[\gamma_\sigma, S_{\mu\lambda}]_+\psi \tag{23.9}$$

which satisfy the continuity equation in respect with the index σ.

The trivial identity SO generates the current of probability density

$$j_\mu = \overline{\psi}\gamma_\mu\psi. \tag{23.10}$$

For the remaining SOs of (18.5) we obtain the following expressions:

a) the operators W_μ and $W_{\mu\sigma}$ corresponds to the tensors of valences 2 and 3

$$\omega_{\mu\lambda}=\frac{1}{4}\left(\overline{\psi}\gamma_4\gamma_\lambda\frac{\partial\psi}{\partial x^\mu}-\frac{\partial\overline{\psi}}{\partial x^\mu}\gamma_\lambda\gamma_4\psi\right)+m\overline{\psi}\gamma_4 S_{\mu\lambda}\psi,$$
$$\omega_{\mu\lambda\rho}=\frac{i}{4}\left(\frac{\partial\overline{\psi}}{\partial x^\rho}S_{\lambda\mu}\psi+\overline{\psi}S_{\lambda\rho}\frac{\partial\psi}{\partial x^\mu}-\overline{\psi}S_{\lambda\mu}\frac{\partial\psi}{\partial x^\rho}-\frac{\partial\overline{\psi}}{\partial x_\mu}S_{\lambda\rho}\psi\right)+\frac{1}{2}m\overline{\psi}[S_{\mu\lambda},\gamma_\rho]_+\psi; \tag{23.11}$$

b) the operators B and A_μ correspond to the vector J_λ and tensor $Z_{\mu\lambda}$

$$J_\lambda=2x^\mu\omega_{\lambda\mu},\quad Z_{\mu\lambda}=2x^\sigma\omega_{\mu\sigma\lambda}. \tag{23.12}$$

We can make sure that all the quantities (23.11), (23.12) satisfy the continuity equation in respect to the index λ. Using the Ostrogradskii-Gauss theorem it is possible to find the related constants of motion in the following form

$$<P_\mu>=\int d^3xT_{0\mu},\quad <J_{\mu\sigma}>=\int d^3xM_{\mu\sigma0},\quad <j_0>=\int d^3x\psi^\dagger\psi, \tag{23.13}$$

$$<W_\mu>=\int d^3x\omega_{\mu0},\quad <W_{\mu\sigma}>=\int d^3x\omega_{\mu\sigma0},\quad <B>=\int d^3xJ_0,\quad <A_\mu>=\int d^3xZ_{\mu0}. \tag{23.14}$$

So we present the explicit form of the conserved currents and constants of motion corresponding to the SOs of the Dirac equation in the class M_1. Besides the well-known constants of motion (23.13) we show the "new" conserved quantities (23.14) corresponding to the SOs (18.5).

Formulae (23.13), (23.14) give a complete set of the conserved quantities depending bilinearly on ψ and $\partial\psi/\partial x_\mu$.

23.3. Conservation Laws for the Massless Spinor Field

Here we present conservation laws corresponding to symmetries of the massless Dirac equation.

The SOs from the class M_1 admitted by the Dirac equation with m=0 are given by formulae (18.31), (18.32). They correspond to the conserved currents (23.8).

As in the case of nonzero mass we obtain the tensors of energy-momentum and angular momentum (23.9) and the probability density current (23.10) which correspond to the SOs P_μ, $J_{\mu\sigma}$ and I.

For the operators D and K_μ we obtain from (2.42), (23.8)

$$J_{\mu}^{D}=x^{\sigma}T_{\sigma\mu},\quad J_{\mu}^{K_{\sigma}}=x^{\lambda}M_{\sigma\lambda\mu}+x_{\sigma}J_{\mu}^{D} \tag{23.15}$$

where $T_{\sigma\mu}$ and $M_{\sigma\lambda\mu}$ are the tensors (23.9). Bearing in mind that for the case m=0 the tensor $T_{\sigma\mu}$ is traceless, it is not difficult to make sure the currents (23.15) satisfy the continuity equation.

In analogous way we find the currents corresponding to the SOs of (18.31), (18.32):

$$J_{\mu}^{\tilde{P}_{\nu}}=\frac{1}{2}\left(\overline{\psi}\gamma_{\mu}\gamma_{4}\frac{\partial\psi}{\partial x_{\nu}}+\frac{\partial\overline{\psi}}{\partial x_{\nu}}+\frac{\partial\overline{\psi}}{\partial x_{\nu}}\gamma_{4}\gamma_{\mu}\psi\right),$$

$$J_{\lambda}^{\tilde{J}_{\mu\nu}}=x_{\mu}J_{\lambda}^{\tilde{P}_{\nu}}-x_{\nu}J_{\lambda}^{\tilde{P}_{\mu}}+\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\overline{\psi}[\gamma_{\lambda},S^{\rho\sigma}]_{+}\psi, \tag{23.16}$$

$$J_{\lambda}^{i\gamma_{4}}=i\overline{\psi}\gamma_{4}\gamma_{\lambda}\psi,\quad J_{\mu}^{\tilde{D}}=x^{\nu}J_{\mu}^{\tilde{P}_{\nu}},$$

$$J_{\nu}^{\tilde{K}_{\mu}}=x^{\lambda}J_{\nu}^{\tilde{J}^{\mu\lambda}}+x_{\mu}J_{\nu}^{\tilde{D}},$$

$$J_{\lambda}^{W_{\mu\nu}}=\tilde{\omega}_{\mu\nu\lambda},\quad J_{\nu}^{A_{\mu}}=\tilde{Z}_{\mu\nu},\quad J_{\mu}^{\tilde{A}_{\lambda}}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}x^{\nu}\tilde{\omega}^{\rho\sigma}{}_{\lambda},$$

$$J_{\lambda}^{Q_{\mu\nu}}=x_{\mu}\tilde{Z}_{\nu\lambda}-x_{\nu}\tilde{Z}_{\mu\lambda}+\frac{1}{2}x_{\sigma}x^{\sigma}\tilde{\omega}_{\mu\nu\lambda}. \tag{23.17}$$

Here $\tilde{\omega}_{\mu\nu\lambda}$ and $\tilde{Z}_{\mu\nu}$ are the tensors (23.11), (23.12) with m=0.

So we obtain the conserved currents corresponding to symmetries of the massless Dirac equation in the class M_1. The currents (23.17) are essentially new (in comparison with obtained by the classical methods) inasmuch as they correspond to the SOs which do not belong to the enveloping algebra of the algebra *A[C(1,3)⊗H]*.

Formulae (23.9), (23.10), (23.15)-(23.17) give a complete set of conserved currents depending bilinearly on ψ and $\partial\psi/\partial x_{\mu}$ for the massless Dirac equation.

23.4. The Problem of Definition of Constants of Motion for the Electromagnetic Field

Studying constants of motion of Maxwell's equations is of particular interest since these constants are expressed via the physically measurable quantities, i.e., strengths of the electric and magnetic fields.

The description of constants of motion for Maxwell's equations can be considered as a separate problem inasmuch as there exist such conserved quantities which do not correspond to any symmetry. On the other hand there exist conserved currents which correspond to trivial (zero) constants of motion. That is why our strategy is to give the complete description of some class of constants of motion and then reconstruct the corresponding conserved currents. More precisely we find all the conserved quantities of the form

$$I=\int d^3xF\left(\boldsymbol{E},\boldsymbol{H},\frac{\partial\boldsymbol{E}}{\partial x_a},\frac{\partial\boldsymbol{H}}{\partial x_a},x_0,\boldsymbol{x}\right) \tag{23.18}$$

where F denotes a bilinear combination of the vectors $\boldsymbol{E}$, $\boldsymbol{H}$ and their derivatives besides in general F depends on $\boldsymbol{x}$ and x_0. Without loss of generality we can represent F in the form

$$F=\varphi^\dagger Q\varphi \tag{23.19}$$

where φ is the vector-function (3.4), Q is a second-order differential operator with matrix coefficients moreover all the matrices are of dimension 6×6. Without loss of generality we suppose Q is Hermitian (symmetric) operator inasmuch as skew-symmetric operators correspond to zero quantities of (23.18), (23.19).

The problem of describing the conserved quantities defined above is closely related to the problem of describing the second-order SOs of Maxwell's equations, refer to Section 20, but has a lot of distinguishing features. Indeed, substituting (20.19) into (20.18) and differentiating the latter in respect with x_0 we obtain using (3.5)

$$\dot{I}=\int d^3x\psi^\dagger(\dot{Q}-i[Q,\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}])\psi. \tag{23.20}$$

Equating this quantity to zero, using (3.5) and taking into account Hermiticity of the operator in paratheses we obtain the following equation for Q

$$\dot{Q}-i[Q,\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}]=i\sigma'^{\mu}[\alpha^a_\mu\hat{L}_2{}^a+(\alpha^a_\mu\hat{L}_2{}^a)^\dagger] \tag{23.21}$$

where $\hat{L}^a_2$ is the operator (3.5), $a_\mu{}^a$ are unknown second-order differential operators (commuting with σ'^μ) which have to be determined, $\sigma'^\mu=\sigma^\mu$, $\mu\neq2$; $\sigma'^2=i\sigma^2$.

Further on we follow the proof of Theorem 20.3. Representing Q in the form (20.20) we come again to the noncoupled equations for the even and odd parts of the operator $Q=E+O$

$$[E,\hat{L}_1]=\alpha^a_E\hat{L}_2{}^a+(\alpha^a_E\hat{L}_2{}^a)^\dagger, \tag{23.22a}$$

$$[O,\hat{L}_1]=\alpha^a_O\hat{L}_2{}^a+(\alpha^a_O\hat{L}_2{}^a)^\dagger. \tag{23.22b}$$

Here E, O, α^a_E, α^a_O are unknown second order differential operators of the form (20.20) (20.22), $\hat{L}^a_2$ and $\hat{L}_1$ are the operators (20.26).

Thus the problem of describing the conserved quantities of the form (23.18) reduces to solving the operator equations (23.22) There are two essentially new points in comparison with the equations (20.23), (20.24) for a SO:

1) there are no commutation relations of SOs with the operator $\hat{L}^a_2$ in (23.22),

i.e., the second equations of (20.23), (20.24) are absent;

2) the anti-Hermitian terms of (20.23),(20.24) are absent in (23.22).

In other words the conditions imposed on the operator Q of the conserved bilinear form (23.18), (23.19) are weaker than the corresponding equations (20.23), (20.24) for the SOs of Maxwell's equations. That is why the number of conservation laws is larger than the number of SOs in the class considered.

23.5. Classical Conservation Laws for the Electromagnetic Field

First we restrict ourselves to considering such bilinear conserved quantities which depend on strengths of the electric and magnetic fields but do not include derivatives of these strengths. We will call them conserved quantities of zero order.

THEOREM 23.1 [159]. There exist exactly 15 linearly independent constants of motion of the form

$$I=\int d^3x F(\boldsymbol{E},\boldsymbol{H},x_0,\boldsymbol{x}) \tag{23.23}$$

where $F(\boldsymbol{E},\boldsymbol{H},x_0,\boldsymbol{x})$ is a bilinear combination of the vectors $\boldsymbol{E}$, $\boldsymbol{H}$ satisfying Maxwell's equations. The general form of F corresponding to $\dot{I}=0$ is given by the formula

$$F(\boldsymbol{E},\boldsymbol{H},x_0,\boldsymbol{X})=\frac{1}{2}K^0(\boldsymbol{E}^2+\boldsymbol{H}^2)+\boldsymbol{K}\cdot\boldsymbol{E}\times\boldsymbol{H} \tag{23.24}$$

where $K=(K^0,\boldsymbol{K})$ is a conformal Killing vector, i.e., an arbitrary solution of the equations (1.17).

PROOF. Let us represent a bilinear combination of $\boldsymbol{E}$ and $\boldsymbol{H}$ in the form (23.19) where φ is the vector-function (3.4), Q is a symmetric real matrix of dimension 6×6 (skew-symmetric matrices make zero contribution into the integral (23.18) since φ is a real function). Substituting (23.19) into (23.18) and differentiating the latter in respect with x_0 we come to the equation (23.21) where $a^a{}_\mu$ are unknown functions of x_0, $\boldsymbol{x}$ have to be determined. Expanding Q in the complete set of matrices (3.6), (5.30) in accordance with (20.20) where

$$A=Z_{ab}a^{ab},\quad B=Z_{ab}b^{ab},\quad C=Z_{ab}c^{ab},\quad D=iS_aK^a, \tag{23.25}$$

a^{ab}, b^{ab}, c^{ab} and f^a are unknown functions of x_0, $\boldsymbol{x}$, and equating coefficients of linearly independent matrices and differential operators we come to the relations

$$a^{ab}=b^{ab}=0,\quad c^{ab}=\delta^{ab}K^0,$$

moreover, K^0 has to satisfy the equations (1.17) together with K^a from (23.25).

The corresponding matrix Q has the form

$$Q=\sigma_0 K^0-\sigma_2 S_a K^a. \tag{23.26}$$

Substituting (23.26), (3.4), (3.6) into (23.21) we come to formula (23.24). ■

The general solution of (1.17) depends on 15 arbitrary parameters and is given by formula (1.18) To obtain the corresponding constants of motion we substitute (1.18) into (23.20) and choose consequently only one of parameters be nonzero. As a result we come to the formulae

$$\begin{aligned}
&E=\frac{1}{2}\int d^3x(\boldsymbol{E}^2+\boldsymbol{H}^2)\equiv\int d^3x\hat{P}_0, \quad \boldsymbol{P}=\int d^3x\boldsymbol{E}\times\boldsymbol{H}\equiv\int d^3x\hat{\boldsymbol{P}},\\
&\boldsymbol{L}=\int d^3x\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{H}), \quad \boldsymbol{N}=\int d^3x[x_0\boldsymbol{E}\times\boldsymbol{H}-\boldsymbol{x}(\boldsymbol{E}^2+\boldsymbol{H}^2)],\\
&D=\int d^3x x_\mu\hat{P}^\mu, \quad K_\mu=\int d^3x(2x_\mu x^\nu\hat{P}_\nu-x^\lambda x_\lambda\hat{P}_\mu).
\end{aligned} \tag{23.27}$$

Relations (23.27) give the well-known classical constants of motion of the electromagnetic field in vacuum, i.e., the energy E, momentum $\boldsymbol{P}$, angular momentum $\boldsymbol{L}$ etc. These constants of motion can be obtained using the Lagrangian formalism and Noether's theorem [32].

According to Theorem 23.1 *there are no other constants of motion bilinearly depending on* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *for Maxwell's equations.*

It is not difficult to establish the correspondence between the constants of motion described in the theorem and the conserved currents. These currents have the form

$$j_\mu=T_{\mu\nu}K^\nu \tag{23.28}$$

where K^ν is a Killing vector, $T_{\mu\nu}$ is the energy-momentum tensor:

$$\begin{aligned}
&T_{00}=\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{H}^2), \quad T_{0a}=T_{a0}=\varepsilon_{abc}E_bH_c,\\
&T_{ab}=-E_aE_b-H_aH_b+\frac{1}{2}\delta_{ab}(\boldsymbol{E}^2+\boldsymbol{H}^2).
\end{aligned} \tag{23.29}$$

Using the well-known properties of $T_{\mu\nu}$

$$\partial^\mu T_{\mu\nu}=0, \quad T_{\mu\nu}g^{\mu\nu}=0 \tag{23.30}$$

it is not difficult to make sure that the four-vector j_μ satisfies the continuity equation iff K^μ is a Killing vector satisfying (1.17). This is the zero component of the current (23.28) which is given in (23.24).

According to Theorem 23.1 all the nonequivalent conserved currents depending bilinearly on $\boldsymbol{E}$ and $\boldsymbol{H}$ have the form (23.28) and so are completely determined by the Killing vector.

23.6. The First Order Constants of Motion for the Electromagnetic Field

We now consider constants of motion of the more general form (23.18) which depend on the electromagnetic field strength and its derivatives.

The obvious set of conserved quantities of this kind can be obtained by choosing Q in the form of products of the conformal group generators and the matrices (23.26). Conservation of such quantities in time is caused by relativistic and conformal invariance of Maxwell's equations. Conservation laws of such a kind were found by Lipkin [282] for the first time (see [248, 109, 303] also).

The SOs found in Subsection 20.4 do not belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$ and so are not caused by conformal invariance of Maxwell's equations. Substituting these SOs into (23.18), (23.19) instead of Q we obtain the motion constants which are new in principle and have nothing to do with the relativistic and conformal invariance of Maxwell's equations.

However there exist more exotic conserved quantities for the electromagnetic field which cannot be assigned by any SO. Examples of such constants of motion are considered in the following.

First we consider the conserved quantities which are bilinear combinations of the form $(F_{\mu\sigma,}\partial_\lambda F_{\rho\alpha})$ where $F_{\mu\lambda}$ is the tensor of the electromagnetic field. More precisely we search for conserved in time bilinear forms

$$I=\int d^3x\varphi^\dagger Q\varphi=\int d^3xj_0 \tag{23.31}$$

where φ is the vector-function (3.4), Q is a first-order differential operator with matrix coefficients. We will call such conserved quantities first order constants of motion. In particular this class of conserved quantities includes Lipkin's "zilch" but is only a subclass of more general constants of motion of the type (23.18).

THEOREM 23.2. There exist exactly 84 first order constants of motion for Maxwell's equations. The corresponding function j_0 of (23.31) is the zero component of the four-vector

$$j_\mu=K^{\sigma\nu}Z_{\sigma\nu,\mu}+2\varepsilon_{\mu\nu\lambda\sigma}(\partial^\lambda K^{\rho\nu})T_\rho^{\ \sigma} \tag{23.32}$$

where $T^\rho_{\ \sigma}$ is the energy-momentum tensor (23.29), $K^{\sigma\nu}$ is a conformal Killing tensor of valence 2, satisfying the equations

$$\partial^{(\mu}K^{\sigma\nu)}=\frac{1}{3}\partial^\lambda K^{\lambda(\mu}g^{\sigma\nu)},$$

$$K^{\sigma\nu}=K^{\nu\sigma},\quad K^{\mu\sigma}g_{\mu\sigma}=0,$$

$Z_{\sigma\nu,\mu}$ is Lipkin's zilch tensor:

The proof reduces to finding the general solution of the equations (23.22) for

$$Z_{00,\lambda}=E_a\,\partial_\lambda H_a-H_a\,\partial_\lambda E_a;$$
$$Z_{0a,\lambda}=Z_{a0,\lambda}=-\varepsilon_{abc}(E^{\,b}\,\partial_\lambda H^{\,c}+H_b\,\partial_\lambda E^{\,c}), \tag{23.34}$$
$$Z_{ab,\lambda}=\delta_{ab}(E_c\partial_\lambda H_c-H_c\partial_\lambda E_c)+H^{(a}\partial_\lambda E^{b)}-E^{(a}\partial_\lambda H^{b)}.$$

first-order differential operators E, O, $\alpha^a{}_E$, $\alpha^a{}_O$ of the form (20.20), (20.22). We do not present the corresponding calculations (which are analogous to the ones given in Subsection 20.3) but note that the odd part of Q (i.e., the operator O) reduces to zero and the even part (E) has the following general form

$$Q=E=\sigma_2F^{\,0\mu}p_\mu+S_aF^{\,a\mu}p_\mu+\varepsilon_{abc}(\sigma_2\partial^cK^{\,0a}S_b+\frac{1}{2}\partial^bK^{\,ad}Z_{cd}) \tag{23.35}$$

where S_a, Z_{ab} are the matrices (3.6), $K^{\mu\sigma}$ is a function satisfying the equations (23.33) and depending on 84 arbitrary parameters, see Appendix 2, formulae (A.2.9), (A.2.17). Substituting (3.4), (23.35) into (23.31) we obtain j_0 coinciding with the zero component of the vector (23.32). ■

Using the identities

$$Z_{\mu\nu,\lambda}g^{\,\nu\lambda}=0,\quad Z_{\mu\nu,\lambda}g^{\,\mu\nu}=0,\quad \partial_\mu Z_{\lambda\sigma,\mu}=0,$$
$$Z_{\nu\lambda,}{}^{\mu}+Z_{\mu\lambda,}{}^{\nu}=\partial^\rho(\varepsilon_{\rho\nu\lambda\sigma}T^{\,\sigma\mu}+\varepsilon_{\rho\mu\lambda\sigma}T^{\,\sigma\nu})$$

and the relations (23.33) it is not difficult to make sure that the four-vector (23.32) satisfies the continuity equation. Substituting the general expression for $K^{\mu\sigma}$ (refer to (A.2.24)) into (23.32) we come to a linear combination of the conserved currents exhausting all the nonequivalent currents of the first order. Some of them are well-known including Lipkin's zilch and its generalizations (see [282,248,109]). Formula (23.32) includes the "new" currents also which are polynomials of x_μ of orders 3 and 4 and correspond to the conformal Killing tensors represented by $G_3{}^{ab}$, $G_4{}^{ab}$ in (A.2.24), (A.2.25). Theorem 23.2 gives a complete list of the first order conserved currents.

We note that the operator (23.35) is nothing but the Hermitian part of the product of the matrix $i\sigma_2Q$ (where Q is the matrix (23.26)) and a linear combination of the generators of the conformal group. Thus we say the first order constants of motion for Maxwell's equations correspond to products of the usual conformal symmetries.

23.7. The Second Order Constants of Motion for the Electromagnetic Field

We now consider constants of motion being bilinear combinations of the first derivatives of the electric and magnetic field strengths. In other words we study the quantities of the sort (23.18) which are conserved in time. The corresponding functions F can be represented in the form (23.19) where Q is a second-order differential operator with matrix coefficients.We call such conserved quantities the second order constants of motion.

The problem of description of second-order constants of motion reduces to solving the equations (23.22) for a second-order differential operator of the form (20.20), (20.22). It is possible to show that even operators $Q=E$ satisfying (23.22a) reduces to polynomials of the conformal group generators and the matrix (23.26) and thus are not of great interest. We will not represent the corresponding calculations but note that a complete set of such operators is determined by a conformal Killing tensor of valence 3. The number of these operators (and the corresponding constants of motion) is equal to 300.

Here we consider odd operators only which do not belong to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$. In other words we investigate conserved quantities of the following form

$$I=\int d^3x\varphi^\dagger O\varphi=\int d^3xF \tag{23.36}$$

where O is an odd operator belonging to the class M_2, φ is the function (3.4). The general expression for O is given in (20.28), (20.30) where without loss of generality $O^A\equiv0$ inasmuch as anti-Hermitian operators make zero contribution into the integral (23.36). Substituting (20.32a), (20.32c) into (23.22b) and calculating the necessary commutators and anticommutators using (20.31), we can equate the coefficients of linearly independent matrices and differential operators. Then we come to the following system of equations for coefficients of the operator of conserved bilinear form:

$$\begin{aligned}
&D_1^{ab,cd}=K_{(ab)}^{0(cd)}+\frac{2}{3}\delta^c{}_{(a}K_{b)}^{0d0};\\
&D_3^{ab,cd}=-K_{0(d}^{n0(a}\varepsilon^{b)}{}_{c)n}+\frac{8}{3}\delta^{ab}\varepsilon_{mn(c}K_{d)n}^{00m}-\frac{2}{3}\varepsilon_{mn}{}^{(b}\delta^{a)}{}_{(c}K_{d)n}^{00m};\\
&B_1^{ab}=\frac{4}{27}\partial^kK_{n0}^{m(ab)}\varepsilon^{kn}{}_m;\quad B_3^{ab}=\frac{8}{27}\partial^kK_{00}^{abk},
\end{aligned} \tag{23.37a}$$

$$\begin{aligned}
&D_1^{ab}=\frac{3}{5}\delta^{ab}\,\partial^m\partial^nK_{mn}^{000}+\frac{8}{5}\partial^n\partial^{(a}K_{0n}^{b)00}+\frac{2}{7}\partial^n\dot{K}_{00}^{(abn)};\\
&D_3^{ab}=\frac{3}{10}\delta^{ab}\,\partial^c\partial^d\varepsilon_{mnc}K_{d0}^{0mn}-\frac{2}{5}\partial^c(\partial^a\varepsilon_{mn(c}K_{b)0}^{0mn}+\partial^b\varepsilon_{mn(c}K_{a)0}^{0mn})-\frac{1}{7}\partial^k\dot{K}_{n0}^{m(ab}\varepsilon^{k)n}{}_m
\end{aligned} \tag{23.27b}$$

where $K_{\nu\sigma}^{\mu\rho\lambda}$ is a generalized Killing tensor, i.e., an irreducible tensor which is antisymmetric under the permutations $\mu\rightleftarrows\nu$ or $\rho\rightleftarrows\sigma$ and symmetric under the permutations $(\mu,\nu)\rightleftarrows(\rho\sigma)$. Besides this tensor satisfies the equations

$$\partial^{(\alpha}K_{\nu\sigma}^{\mu\rho\lambda)}-\frac{1}{6}\partial^\beta\left(K_{\beta(\sigma}^{(\mu\rho\lambda}g^{\alpha)}{}_{\nu)}+\frac{1}{2}g^{(\alpha\lambda}K_{\nu\sigma}^{\mu\rho)\beta}+K_{(\nu\sigma)}^{\beta(\mu\rho}g^{\alpha\lambda)}\right)=0 \tag{23.38}$$

We recall that symmetrization is imposed over the indices in the paratheses.

We see that the odd constants of motion of the second order for Maxwell's equations are completely determined by a generalized Killing tensor in accordance with (23.36), (20.32a), (22.37). The general expression for this tensor is given in Appendix 2, see (A.2.24). In other words formulae (23.36), (20.32a), (23.37) (A.2.24) define a complete set of odd constants of motion of the second order for Maxwell's equations. The number of these constants of motion is large enough and is equal to the number of independent components of the corresponding generalized Killing tensor, i.e., 378.

We see that there exist significantly more constants of motion in the class considered than the corresponding SOs, compare with Theorem 20.3. More precisely any generalized conformal Killing tensor $K^{\mu\rho\lambda}_{\nu\sigma}$ with $\lambda\neq 0$ generates a constant of motion which does not correspond to any SO in the class M_2.

Let us represent some of the found constants of motion in the explicit form. Restricting ourselves to those which are polynomials on x_μ of order 0,1 and 2, we obtain the following set of independent operators F for (23.36):

$$
\begin{aligned}
&F^{0ab}_{10}=\dot{E}^{(a}\dot{H}^{b)}+(\partial_k E^{(a)})\partial_k H^{b)},\\
&F^{0ab}_{30}=\dot{E}^{a}\dot{E}^{b}+(\partial_k E^{a})\partial_k E^{b}-\dot{H}^{a}\dot{H}^{b}-(\partial_k H^{a})\partial_k H^{b},\\
&F^{abc}_{10}=\dot{H}^{(a}\partial^b H^{c)}-\dot{E}^{(a}\partial^b E^{c)},\\
&F^{abc}_{30}=\dot{E}^{(a}\partial^b H^{c)}+\dot{H}^{(a}\partial^b E^{c)},\\
&F^{\mu bc}_{\alpha 1}=-\frac{1}{2}\varepsilon^{0nf(b}F^{\mu c)f}_{\alpha 0}x_n+2i^{\,\alpha-1}x_0F^{\mu bc}_{\alpha' 0},
\end{aligned}
\tag{23.39a}
$$

$$
\begin{aligned}
&F^{\mu b}_{\alpha 1}=F^{\mu b\nu}_{\alpha 0}x_\nu+i^{\,\alpha+1}(1-g_{\mu 0})\dot{C}^{\mu b}_{(\alpha)},\\
&\tilde{F}^{abc}_{\alpha 1}=-F^{0(bc}_{\alpha 0}x^{a)}+i^{\,\alpha-1}x_0F^{abc}_{\alpha' 0},\\
&F^{\mu}_{\alpha 2}=F^{\mu b}_{\alpha 1}x_b-g^{\mu 0}C^{nn}_{(\alpha)},
\end{aligned}
\tag{23.39b}
$$

Here

$$C^{ab}_{(1)}=E^aE^b-H^aH^b,\quad C^{ab}_{(3)}=E^{(a}H^{b)},\quad \alpha=1,3,\quad \alpha'=1,3,\quad \alpha'\neq\alpha,\quad k=1,2,3,$$

and the dots denote time derivatives.

The traces of the tensors (23.39a) make zero contributions into the integrals (23.36). Time independence of the integrals (23.36), (23.39) can be verified directly using Maxwell's equations.

Formulae (23.36), (23.39) present the complete set of odd constants of motion of second order, depending on x_μ as polynomials of orders 0, 1 and 2. Using relations (23.36), (20.32a) (23.37), (A.2.24) it is not difficult to rewrite the explicit form of other odd constants of motion which in general are polynomials on x of order 6. We do not

$$F^{0a}_{\alpha 2}=F^{0ab}_{\alpha 1}x_b+i^{\alpha+1}x_0F^{0a}_{\alpha' 1},$$

$$F^{ab}_{\alpha 2}=-\frac{1}{2}x^{\mu}x_{\mu}F^{0ab}_{\alpha 0}+\frac{1}{3}i^{\alpha-1}x_0(F^{ab}_{\alpha' 1}-F^{0ab}_{\alpha' 1}),$$

$$\tilde{F}^{ab}_{\alpha 2}=\frac{1}{2}F^{abc}_{\alpha 1}x_c+\frac{1}{3}i^{\alpha-1}(F^{ab}_{\alpha' 1}+2F^{0ab}_{\alpha' 1})x_0-4C^{ab}_{(\alpha)},$$

$$F^{0ab}_{\alpha 2}=-x_{\mu}x^{\mu}F^{0ab}_{\alpha 0}-\frac{1}{2}F^{0(a}_{\alpha 1}x^{b)}+i^{\alpha+1}x_0F^{0ab}_{\alpha' 1}, \tag{23.39c}$$

$$F^{abc}_{\alpha 2}=3F^{0(ab}_{\alpha 1}x^{c)}+i^{\alpha-1}x_0\left[\delta^{(ab}F^{c)}_{\alpha' 1}+F^{abc}_{\alpha' 1}+\tilde{F}^{abc}_{\alpha' 1}\right],$$

$$\tilde{F}^{abc}_{\alpha 2}=3x_{\mu}x^{\mu}F^{abc}_{\alpha 0}-3\tilde{F}^{(ab}_{\alpha 1}x^{c)}-3\dot{C}^{(ab}_{(\alpha)}x^{c)}+6\delta^{(ab}\dot{C}^{c)k}_{\alpha)}x_k+$$
$$+i^{\alpha-1}x_0\left[2f^{(a}_{\alpha' 1}\delta^{bc)}+\frac{2}{3}(F^{abc}_{\alpha' 1}+\tilde{F}^{abc}_{\alpha' 1})\right]$$

present the corresponding cumbersome formulae here.

We emphasize that the constants of motion found in this subsection have nothing to do with the conformal or Lorentz invariance of Maxwell's equations in contrast to the classical and first-order constants of motion. The constants of motion given in (23.36), (23.39) either are generated by the odd SOs not belonging to the enveloping algebra of the algebra $A[C(1,3)\otimes H]$ or do not correspond to any SO in the class considered.

We note that the constants of motion found above admit a covariant formulation. Let us denote as usual by $F_{\mu\sigma}$ the tensor of the electromagnetic field,

$$\tilde{F}^{\mu\sigma}=\frac{1}{2}\varepsilon^{\mu\sigma\rho\lambda}F_{\rho\lambda},\quad F^{\mu\sigma}_{\lambda}=\frac{\partial}{\partial x_{\lambda}}F^{\mu\sigma}.$$

Then the following tensors

$$G^{[\mu\nu][\rho\sigma]}_{\lambda\alpha}=F^{\mu\nu}_{\lambda}\tilde{F}^{\rho\sigma}_{\alpha}+F^{\rho\sigma}_{\alpha}\tilde{F}^{\mu\nu}_{\lambda}+F^{\mu\nu}_{\alpha}\tilde{F}^{\rho\sigma}_{\lambda}+F^{\rho\sigma}_{\lambda}\tilde{F}^{\mu\nu}_{\alpha}-g_{\lambda\alpha}(F^{\mu\nu}_{\mu}\tilde{F}^{\rho\sigma}_{\gamma}+F^{\rho\sigma}_{\gamma}\tilde{F}^{\mu\nu}_{\beta})g^{\beta\gamma} \tag{23.40}$$

satisfy the continuity equation in respect with the indices α and λ [248].

It is not difficult to verify that formulae (23.39a) define the complete set of independent components of $G^{[\mu\nu][\rho\sigma]}_{0\lambda}$ if we exclude terms making zero contributions into the integral

$$I=\int d^3xG^{[\mu\nu][\rho\sigma]}_{0\lambda}.$$

As to the other quantities present in (23.39b)-(23.39d) they are expressed via convolutions of $G^{[\mu\nu][\rho\sigma]}_{0\lambda}$ with x_{μ} and the tensors $C^{ab}_{(\alpha)}$, $\dot{C}^{ab}_{(\alpha)}$.

In conclusion we note that it is not too difficult to formulate and solve the problem of description of constants of motion of arbitrary finite order n for Maxwell's equations. We will not do it here but note that the even constants of motion are determined by the conformal Killing tensor of valence n, the corresponding number of independent motion constants is given in (A.2.9), where m=4, s=1, j=n+1. The odd constants of motion are defined by the generalized conformal Killing tensor of valence R_1+2R_2, where $R_1=n-1$, $R_2=2$. The number of these constants is defined by formula (A.2.16).

22.8. Constants of Motion for the Vector-Potential

The constants of motion discussed above can be represented in terms of the vector-potential A_μ. However, there exist additional conserved quantities depending bilinearly on A_μ which extend the sets of constants of motion described in Subsections 22.4-22.7. That is why we discuss the constants of motion for the vector-potential as a separate problem.

Consider the vector-potential A_μ in the Coulomb gauge, then by definition A_μ satisfies the equations (20.35). We search for the constants of motion in the form

$$I=\int A^{T}QA \tag{23.41}$$

where A=*column*(A_1,A_2,A_3), Q is a differential operator whose coefficients are 3×3 matrices, see (22.56).

An evident set of constant of motion of (23.41) can be obtained choosing Q be a SO of the equations (20.35). Such SOs (including operators satisfying the requirement of conditional invariance) are discussed in Subsection 20.5.

It happens the class of constants of motion of the type (22.41) is more extensive inasmuch as the corresponding operators Q are not in general SOs of the equations for the vector-potential. Restricting ourselves to the case when $Q\in M_1$ and denoting the corresponding class of constants of motion by M_1 we can prove the following assertion.

THEOREM 23.3. There exist 53 constants of motion in the class M_1 for the equation (20.32), including 50 conserved quantities of the sort (23.41) where $Q\in \mathrm{M}_1$ are the operators of conditional symmetry of the system (20.35), and the three additional constants:

$$I_a=\int d^3x\dot{A}_c x_c x_b\left(\varepsilon_{akl}\frac{\partial A_b}{\partial x_l}x_k+\varepsilon_{abk}A_k\right). \tag{23.42}$$

The proof is analogous to the proofs of Theorems 23.1, 23.2. Time independence of I_a can be verified directly using the equations (3.14)-(3.16) for j=0.■

Considering the vector-potential in the Lorentz gauge (see (3.14), (3.15) for

j_μ=0) we find the following six constants of motion:

$$I_{\mu\nu}=\int d^3x\dot{A}_\lambda x^\lambda\left[x_\sigma\left(\frac{\partial A_\sigma}{\partial x_\mu}x_\nu-\frac{\partial A_\sigma}{\partial x_\nu}x_\mu\right)+x_\mu A_\nu-x_\nu A_\mu\right] \tag{23.43}$$

We emphasize that the constants of motion (23.42) cannot be represented in the form (22.41) where Q is an operator of the conditional symmetry of the equation (20.37). In other words it is not possible to assign to these constants any symmetry of the equation (20.37), either usual or conditional. The same is true for (23.43).

The constants of motion (23.42), (23.43) extend the class of conserved quantities of the electromagnetic field, described in Subsections 23.4-23.7. We note these "new" constants of motion cannot be expressed via the electric and magnetic field strengths so their physical interpretation seems to be problematic.

A natural question arises if there exist such constants of motion for the vector-potential which do not correspond to any SO but can be expressed via the vectors of the electric and magnetic fields strengths. This question admits a positive answer because such constants of motion are included in the set described in Subsection 23.7. We rewrite three of them

$$I_a=\int d^3x\left[\dot{E}_c x_c x_b\left(\varepsilon_{akl}\frac{\partial E_b}{\partial x_l}x_k+\varepsilon_{abk}E_k\right)-\dot{H}_c x_c x_b\left(\varepsilon_{akl}\frac{\partial H_b}{\partial x_l}x_k+\varepsilon_{abk}H_k\right)\right] \tag{23.44}$$

Formula (23.44) defines the second order constant of motion for ***E, H*** which is a third order constant of motion for the vector-potential. It is possible to show that one cannot represent (23.44) either in the form (23.31) (where Q is a second-order SO) or in the form (21.41) where Q is a SO for the vector-potential.

In conclusion we note that the Lame equation (22.53) also admits constants of motion which do not correspond to any SO. Thus Theorem 23.3 can be reformulated for the Lame equation if we impose the transversity condition $div\boldsymbol{U}=0$. Indeed, in this case the Lame equation reduces to the form (22.59) which coincides with (20.37) up to the change $t=t'(\lambda/\rho)^{1/2}$. As to the equations (22.60), (22.61) describing longitudinal waves they admit the following constants of motion

$$I_a=\int d^3x\dot{U}_c(\delta_{cb}x^2-x_bx_c)\left(\varepsilon_{akl}\frac{\partial U_b}{\partial x_l}+\varepsilon_{abk}U_k\right)$$

which is not generated by any SO. Besides the Lame equation admits a number of conserved quantities generated by SOs:

$$I_Q=\int d^3x(\dot{U}^TQU-U^TQ\dot{U})$$

where $U^T=column(U_1,U_2,U_3)$, Q is a product of SOs (22.55) and (22.57) [158].

5. GENERALIZED POINCARÉ GROUPS

In this chapter we present the basic information about the generalized Poincaré groups *P(1,n)* which are determined as sets of transformations preserving the value of interval in $(1+n)$ -dimensional Minkowsky space. We describe IRs of the Lie algebras corresponding to such groups and make the reduction of IRs of these algebras by the Poincaré and Galilei algebras. We also consider the connection of *P(1,n)*-invariant equations with Poincaré- and Galilei-invariant equations for a particle of variable mass and spin.

24. THE GROUP *P(1,4)*

24.1. Introduction

As was shown in Chapter 1 the maximal (in the Lie sense) invariance group of the Dirac and KGF equations is the ten parameter Poincaré group *P(1,3)*. Here we consider a natural generalization of this group to the case of a space with more dimensions. The generalized Poincaré group *P(1,n)* can be defined as a semidirect product of the groups *SO(1,n)* and T where T is the additive group of $(n+1)$-dimensional vectors $p_0, p_1, ..., p_n$ and *SO(1,n)* is a connected component of unity in the group of all linear transformations of T into T preserving the quadratic form p_0^2-p_1^2-...-p_n^2.

The groups *P(1,n)*, $n>3$ can be used to describe physical systems with variable masses and spins, e.g., the systems of two coupling relativistic particles. Thus in Chapter 7 we use *P(1,6)*-invariant wave equations to describe coupled states of two particles of arbitrary spin. Besides the generalized Poincaré groups have direct connection to the problem of the extension of S-matrix over the mass shell [82, 240] and to description of particles with internal structure [61, 136]. These groups are also closely connected with the modern multidimensional models appearing in the string theory.

The group *P(1,n)*, $n>3$ includes important (from the physical viewpoint) subgroups, such as the Poincaré group *P(1,3)* and Galilei group *G(1,3)*. Inasmuch as IRs of the group *P(1,n)* are reducible in respect to *P(1,3)* and *G(1,3)* it is natural to consider the problem of reduction of these representations by IRs of the Poincaré and Galilei group. Such a reduction makes it possible to answer the question what kind of relativistic (and nonrelativistic) particles is described by the equation invariant under the group *P(1,n)* and to find the energy spectrum and the spin states of such particles. These and related questions are considered in Sections 25-27. The present section is devoted to the description of IRs of the group *P(1,4)*.

24.2. The Algebra *AP(1,n)*

The Lie algebra of the group *P(1,n)* is defined by the following commutation relations

$$[P_m, P_n]=0, \quad [P_m, J_{nk}]=i(g_{mn}P_k-g_{mk}P_n),$$
$$\left[J_{mn}, J_{m'n'}\right]=i(g_{mn'}J_{nm'}+g_{nm'}J_{mn'}-g_{mm'}J_{nn'}-g_{nn'}J_{mm'}), \tag{24.1}$$

where $m, n, m', n'=0,1,...,n$, g_{kl} is a metric tensor for (1+n)-dimensional Minkowsky space: $g_{kl}=diag(1,-1,-1,...,-1)$.

The problem of construction of local representations of the group *P(1,n)* reduces to the description of nonequivalent representations of the Lie algebra (24.1) in terms of selfadjoint operators. We restrict ourselves to investigation of IRs of this algebra. To classify IRs it is necessary to find independent Casimir operators of the algebra (24.1) and to determine their spectra.

Finding the Casimir operators of the algebra *AP(1,n)* cannot be considered as a trivial generalization of the corresponding procedure for the algebra *AP(1,3)*. In particular for $n\neq3$ there is not an $(n+1)$-dimensional analog of the Lubanski-Pauli vector instead of which we have to consider tensors of an appropriate rank.

In this section we find all the independent Casimir operators of the algebra *AP(1,4)*.

We define the fundamental antisymmetric tensor of rank 3

$$V_{\lambda\rho\sigma}=P_\lambda J_{\rho\sigma}+P_\sigma J_{\lambda\rho}+P_\rho J_{\sigma\lambda}. \tag{24.2}$$

It is a tensor $V^{\lambda\rho\sigma}$ which is a natural generalization of the Lubanski-Pauli vector to the case of the generalized Poincaré groups. In the case when $J_{\mu\nu}$, P_μ belong to the algebra *AP(1,3)* we can assign to (24.2) the vector $W_\mu=\varepsilon_{\mu\nu\rho\sigma}V^{\nu\rho\sigma}/6$ (compare with (4.2)). For the algebra *AP(1,4)* the tensor (24.2) is equivalent to the tensor of rank 2:

$$W_{\mu\nu}=\frac{1}{6}\varepsilon_{\mu\nu\rho\sigma\lambda}V^{\rho\sigma\lambda}\equiv\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma\lambda}P^\rho J^{\sigma\lambda}. \tag{24.3}$$

Here $\varepsilon_{\mu\nu\rho\sigma\lambda}$ is the unit antisymmetrical tensor, $\varepsilon_{01234}=1$.

Besides $W_{\mu\nu}$ we consider the vector $\Gamma_\sigma=J_{\sigma\mu}P^\mu$. It follows from(24.1)-(24.3) that the operator $W_{\mu\nu}$ satisfies the relations

$$P_\mu W^{\mu\nu}=0, \quad [P_\mu, W_{\nu\lambda}]=0,$$
$$[W_{\mu\nu}, J_{\lambda\sigma}]=i(g_{\mu\sigma}J_{\nu\lambda}+g_{\nu\lambda}J_{\mu\sigma}-g_{\mu\lambda}J_{\nu\sigma}-g_{\nu\sigma}J_{\mu\lambda}), \tag{24.4}$$
$$[W_{\mu\alpha}, W_{\nu\beta}]=i(g_{\mu\nu}\varepsilon_{\alpha\beta\rho\sigma\lambda}+g_{\alpha\beta}\varepsilon_{\mu\nu\rho\sigma\lambda}-g_{\mu\beta}\varepsilon_{\alpha\nu\rho\sigma\lambda}-g_{\alpha\nu}\varepsilon_{\mu\beta\rho\sigma\lambda})W^{\rho\sigma}P^\lambda.$$

As to the vector Γ_σ it satisfies the relations (4.4) with an appropriate possible values of subindices.

It follows from the above that the scalar operators

$$C_1 = P_\mu P^\mu, \tag{24.5a}$$

$$C_2 = \frac{1}{2} W_{\mu\nu} W^{\mu\nu} = \frac{1}{2} P_\mu P^\mu J_{\nu\lambda} J^{\nu\lambda} - P^\mu P_\nu J_{\mu\sigma} J^{\nu\sigma}, \tag{24.5b}$$

$$C_3 = -\frac{1}{4} J_{\mu\nu} W^{\mu\nu} = -\frac{1}{8} \varepsilon_{\mu\nu\rho\sigma\lambda} J^{\mu\nu} J^{\rho\sigma} P^\lambda \tag{24.5c}$$

commute with any basis element of the algebra *AP(1,4)*. Moreover using (24.4) and relations analogous to (4.4) it is not difficult to show that formulae (24.5) give all the independent Casimir operators belonging to the enveloping algebra of the algebra *AP(1,4)*. The operators C_1 and C_2 are nothing but generalization of the corresponding Casimir operators (4.2) and (4.6) of the Poincaré algebra, but the operator C_3 has not any analog in the algebra *AP(1,3)*.

We note that besides the operators (24.5) there exist additional Casimir operators for different classes of IRs. These Casimir operators do not belong to the enveloping algebra of the algebra *AP(1,4)*, as in the case of the representations of the algebra *AP(1,3)*.

24.3. Nonequivalent Realizations of the Tensor $W_{\mu\nu}$

As was done by the description of IRs of the Poincaré algebra (see Section 4) we find all the possible (up to equivalence) realizations of the tensor $W_{\mu\nu}$ and then determine the explicit form of the corresponding operators P_μ and $J_{\mu\sigma}$.

We will seek for representations of the tensor $W_{\mu\nu}$ in the basis of eigenfunctions $|\tilde{p},\lambda\rangle$ of the commuting operators P_μ:

$$P_\mu |\tilde{p},\lambda\rangle = p_\mu |\tilde{p},\lambda\rangle, \quad \tilde{p} = (p_0, p_1, p_2, p_3, p_4), \tag{24.6}$$

where the symbol λ denotes eigenvalues of commuting operators forming a complete set together with P_μ. In the basis $|\tilde{p},\lambda\rangle$ the commutation relations for $W_{\mu\nu}$ take the form

$$[W_{\mu\alpha}, W_{\nu\beta}] = i(g_{\mu\nu}\varepsilon_{\beta\alpha\rho\lambda\sigma} + g_{\alpha\beta}\varepsilon_{\nu\mu\rho\lambda\sigma} - g_{\mu\beta}\varepsilon_{\nu\alpha\rho\lambda\sigma} - g_{\nu\alpha}\varepsilon_{\beta\mu\rho\lambda\sigma}) W^{\rho\sigma} p^\sigma, \tag{24.7}$$

$$[W_{\mu\nu}, p_\sigma] = 0, \quad W_{\mu\nu} p^\nu = 0, \tag{24.8}$$

where p_σ are arbitrary real numbers.

Relations (24.7) define a Lie algebra $A_{\tilde{p}}$ whose structure constants depend on $\tilde{p}$. Besides it will be demonstrated that for all p_μ satisfying one of the following conditions

$$p_\mu p^\mu > 0, \tag{24.9}$$

$$p_\mu p^\mu=0, \tag{24.10}$$

$$p_\mu p^\mu<0, \tag{24.11}$$

the algebras $A_{\tilde{p}}$ turn out to be isomorphic.

To describe constructively the representations of the algebra $A_{\tilde{p}}$ we transform the commutation relations (24.7) to such a form that the structure constants will depend on the single parameter $c_1=p_\mu p^\mu$. For this purpose we will use the linear transformation

$$p_\mu \to p'_\mu=R_{\mu\nu}p^\nu, \quad W_{\mu\nu}\to W'_{\mu\nu}=R_{\mu\lambda}R_{\nu\sigma}W^{\lambda\sigma} \tag{24.12}$$

where

$$R_{00}=1, \quad R_{0k}=R_{k0}=0, \quad k=1,2,3,4,$$

$$R_{kc}=-\delta_{kc}+\frac{\theta_{kc}}{2p}-\frac{\theta_{kn}\theta_{nc}}{2p(2p+p_1+p_2+p_3+p_4)}, \tag{24.13}$$

$$\theta_{kc}=p_k-p_c, \quad p=(p_1^2+p_2^2+p_3^2+p_4^2)^{1/2}.$$

This transformation is invertible, moreover

$$R^{-1}_{\mu\nu}=R_{\mu\nu}(-\theta_{kc}) \tag{24.14}$$

and it is nothing but the rotation of a reference frame, corresponding to the transition to the basis in which

$$p'_0=p_0, \quad p'_1=p'_2=p'_3=p'_4=p/2. \tag{24.15}$$

Using (24.7), (24.12) and (24.15) it is not difficult to find the commutation relations for $W'_{\mu\nu}$. These relations can be simplified essentially by passing to the new variables $W'_{\mu\nu}\to(\eta_a, \lambda_a)$. Using the fact that only six out of ten components of $W'_{\mu\nu}$ are linearly independent (see (24.8)), it is convenient to set

$$W'_{0a}=-p\left(\frac{1}{2}\eta+\eta_a\right), \quad W'_{04}=\frac{1}{2}p\eta,$$

$$W'_{ab}=\varepsilon_{abc}[\xi_c+\frac{1}{4}p_0(\eta_a-\eta_b)], \quad \xi_c=\frac{1}{2}(\lambda-\lambda_c), \tag{24.16}$$

$$W'_{4a}=\frac{1}{2}[\varepsilon_{abc}(\lambda_b-\lambda_c)-p_0(\eta-\eta_a)],$$

where

$$\eta=\eta_1+\eta_2+\eta_3, \quad \lambda=\lambda_1+\lambda_2+\lambda_3, \quad a,b=1,2,3. \tag{24.17}$$

Substituting (24.12), (24.15) and (24.16) into (24.7) we come to the following algebra of the operators η_a and λ_a:

$$[\eta_a,\eta_b]=i\varepsilon_{abc}\eta_c, \quad [\eta_a,\lambda_b]=i\varepsilon_{abc}\lambda_c, \quad [\lambda_a,\lambda_b]=ic_1\varepsilon_{abc}\eta_c. \tag{24.18}$$

The relations (24.16) define the isomorphism of the Lie algebra formed by the components of $W'_{\mu\nu}$ to the algebra of the operators λ_a, η_a which are characterized by the commutation relations (24.18). The structure constants of the algebra (24.18) depend on a single parameter c_1 (i.e., on the eigenvalue of the Casimir operator C_1). In the space of an IR of the algebra *AP(1,4)* this eigenvalue is fixed and relations (24.18) determine the Lie algebra.

If we choose a representation of the operators η_a and λ_a then formulae (24.16) and (24.12) determine the corresponding representation of the tensor $W'_{\mu\nu}$ in the frame of reference where components p_μ have the form (24.15). To obtain this tensor in an arbitrary frame of reference we use the transformation

$$W'_{\mu\nu}\rightarrow W_{\mu\nu}=UR^{-1}_{\mu\mu'}R^{-1}_{\nu\nu'}W'_{\mu'\nu'}U^{\dagger} \tag{24.19}$$

where U is the unitary operator

$$U=\exp\left(\frac{i\eta_k p_k}{\hat{p}}\arctan\frac{\hat{p}}{p_1+p_2+p_3+p_4}\right), \quad k=1,2,3,4, \quad \eta_4=\eta, \tag{24.20}$$
$$\hat{p}=[(p-p_2)^2+(p_3-p)^2+(p_2-p_3)^2+(p_4-p)^2+(p_4-p_2)^2+(p_4-p_3)^2]^{1/2},$$

and $R^{-1}_{\mu\nu}$ are given in (24.13) and (24.14). We obtain

$$W_{0k}=\Sigma_{kl}p_l, \quad W_{kl}=p_0\Sigma_{kl}+\frac{1}{p^2}[p_0(p_kW_{0l}-p_lW_{0k})-p_k\lambda_{ln}p_n+p_l\lambda_{kn}p_n], \tag{24.21}$$

where

$$\Sigma_{ab}=\varepsilon_{abc}\eta_c, \quad \Sigma_{4a}=\eta_a, \quad \lambda_{ab}=\varepsilon_{abc}\lambda_c, \quad \lambda_{4a}=\lambda_a. \tag{24.22}$$

Formulae (24.21) and (24.22) give a representation of the tensor $W_{\mu\nu}$ in terms of the operators λ_a and η_a satisfying the algebra (24.18). To find nonequivalent realizations of $W_{\mu\nu}$ it is sufficient to describe nonequivalent representations of this algebra which turn out to be equivalent to the well-known Lie algebras of the group of orthogonal 4×4 matrices *O(4),* Lorentz group *O(1,3)* and Euclidean group *E(3)*, depending on the value of the parameter c_1.

THEOREM 24.1. The Lie algebra (24.18) is isomorphic to the algebra *AO(4)* if $c_1>0$, *AE(3)* if $c_1=0$ and *AO(1,3)* if $c_1<0$.

PROOF. The formulated isomorphism can be established explicitly by the relations

$$\eta_a=\Sigma_a, \quad \lambda_a=m\Sigma'_a, \quad \textit{if} \quad c_1=m^2>0; \tag{24.23a}$$

$$\eta_a=\Sigma_a, \quad \lambda_a=T_a, \quad if \quad c_1=0; \qquad (24.23b)$$
$$\eta_a=\Sigma_a, \ \lambda_a=k\xi_a, \ if \ c_1=-k^2<0,$$

where $\{\Sigma_a, \Sigma_a'\}$, $\{\Sigma_a, T_a\}$ and $\{\Sigma_a, \xi_a\}$ are basis elements of the algebras *AO(4), AE(3)* and *AO(1,3)*, which satisfy the commutation relations

$$\left[\Sigma_a,\Sigma_b\right]=i\varepsilon_{abc}\Sigma_c,$$
$$\left[\Sigma_a,\Sigma_b'\right]=i\varepsilon_{abc}\Sigma_c', \quad \left[\Sigma_a',\Sigma_b'\right]=i\varepsilon_{abc}\Sigma_c, \qquad (24.24)$$
$$\left[\Sigma_a,T_b\right]=i\varepsilon_{abc}T_c, \quad [T_a,T_b]=0,$$
$$\left[\Sigma_a,\xi_b\right]=i\varepsilon_{abc}\xi_c, \quad \left[\xi_a,\xi_b\right]=-i\varepsilon_{abc}\Sigma_c.$$

It is not difficult to make sure that the relations (24.24) follow from (24.18), (24.23), and vice versa (24.18) is a consequence of (24.23) and (24.24). ■

We restrict ourselves by IRs of the algebras *AO(4), AE(3)* and *AO(1,3)* which have been well studied and described (see, e.g., [20], [197]). The necessary information about these IRs is given in Sections 4 and 12.

24.4. The Basis of an IR.

To describe constructively the nonequivalent tensors $W_{\mu\nu}$ and the corresponding basis elements of the algebra *AP(1,3)* it is necessary to define a basis in a space of the IR. It is convenient to choose such a basis in a form of eigenfunctions of a complete set of commuting operators.

We choose the following set of commuting operators

$$P_0, \quad P_1, \quad P_2, \quad P_3, \quad P_4, \quad \tilde{S}_3, \quad \tilde{S}_a\tilde{S}_a, \quad C, \qquad (24.25)$$

where

$$\tilde{S}_a=\varepsilon_{abc}P_bW_{0c}-P_4W_{0a}+P_aW_{04}, \qquad (24.26)$$

and C denotes all the Casimir operators existing in the representation considered, i.e., C_1, C_2, C_3 (24.5) and additional invariant operators for the different classes of IRs.

It follows from (24.21) that the operators (24.26) are expressed via Σ_a and p:

$$\tilde{S}_a=p^2\Sigma_a. \qquad (24.27)$$

We will denote the common eigenfunctions of the commuting operators (24.25) by the symbol $|c,\tilde{p},l,m\rangle$ where $c=(c_1,c_2,c_3,...)$ are eigenvalues of the Casimir operators, $\tilde{p}=(p_0,p_1,p_2,p_3,p_4)$ are eigenvalues of the operators P_μ, l and m characterize eigenvalues of the operators $\tilde{S}_3$ and $\tilde{S}_a\tilde{S}_a$ so that

$$\begin{aligned}&C_a|c,\tilde{p},l,m\rangle=c_a|c,\tilde{p},l,m\rangle,\quad a=1,2,...,\\&P_\mu|c,\tilde{p},l,m\rangle=p_\mu|c,\tilde{p},l,m\rangle,\\&\tilde{S}_a\tilde{S}_a|c,\tilde{p},l,m\rangle=p^4l(l+1)|c,\tilde{p},l,m\rangle,\\&S_3|c,\tilde{p},l,m\rangle=m|c,\tilde{p},l,m\rangle\end{aligned}\tag{24.28}$$

were l are positive integers or half integers, $m=-l,-l+1,...,l$.

In a space of an IR of the algebra $AP(1,4)$ c_a are fixed and possible values of p_μ are restricted by the relation $p_\mu p^\mu=c_1$.

We impose here the following normalization conditions for $c_1\neq 0$

$$\langle c,\tilde{p},l,m|c,\tilde{p}',l',m'\rangle=2\hat{p}_0\delta(p_4-p_4')\delta(\boldsymbol{p}-\boldsymbol{p}')\delta_{ll'}\delta_{mm'},\tag{24.29}$$

where $\hat{p}_0=\sqrt{\boldsymbol{p}^2+c_1}$. If $c_1<0$ then it is convenient to require

$$\langle c,\tilde{p},l,m|c,\tilde{p}',l',m'\rangle=2\hat{p}_4\delta(p_0-p_0')\delta(\boldsymbol{p}-\boldsymbol{p}')\delta_{ll'}\delta_{mm'},\tag{24.30}$$

where $\hat{p}_4=\sqrt{p_0^2-\boldsymbol{p}^2-c_1}$.

The orthonormalized basis defined by the relations (24.28) -(24.30) will be used below to determine an explicit form of the basis elements of the algebra $AP(1,4)$.

24.5. The Explicit Form of the Basis Elements of the Algebra *AP(1,4)*

Let us determine a general form of the operators P_μ, $J_{\mu\nu}$ corresponding to the tensors $W_{\mu\nu}$ described above. It is not difficult to make sure that without loss of generality we can set

$$\begin{aligned}&P_\mu=p_\mu,\quad J_{kl}=x_kp_l-x_lp_k+\Sigma_{kl},\\&J_{0k}=x_0p_k-x_kp_0+\frac{1}{p^2}(\lambda_{kl}p_l-p_0\Sigma_{kl}p_l),\end{aligned}\tag{24.31}$$

where p_μ are independent variables satisfying the relation $p_\mu p^\mu=c_1$, $x_\mu=-i\partial/\partial p^\mu$, λ_{kl} and Σ_{kl} are the matrices realizing an IR of the algebra (24.18), (24.22). The operators (24.31) satisfy the commutation relations (24.1) and therefore form a representation of the algebra $AP(1,4)$. On the other hand we obtain the representation (24.21) for the tensor $W_{\mu\nu}$ after the substitution (24.31) into (24.3). Finally substituting (24.21) and (24.31) into (24.5) we come to the following expressions for the Casimir operators

$$C_1=p_\mu p^\mu=c_1,\quad C_2=-(c\eta_a\eta_a+\lambda_a\lambda_a),\quad C_3=\eta_a\lambda_a.\tag{24.32}$$

According to (24.23) these operators reduce to one of the following forms:

$$C_1=m^2>0,\quad C_2=-m^2(\Sigma^2+\Sigma'^2),\quad C_3=m\Sigma\cdot\Sigma',\tag{24.33a}$$

$$C_1=0,\quad C_2=-\boldsymbol{T}^2,\quad C_3=\boldsymbol{T}\cdot\Sigma,\tag{24.33b}$$

$$C_1=-k^2<0, \quad C_2=k^2(\Sigma^2-\xi^2), \quad C_3=k\Sigma\cdot\xi. \tag{24.33c}$$

We can see that the basic Casimir operators of the algebra *AP(1,4)* in the realization (24.31) reduce to the invariant operators of the groups *O(4), E(3)* and *O(1,3)* which are the little groups of the group *P(1,4)* [135]. Thus any possible set of the eigenvalues of the operators C_1, C_2 and C_3 can be assigned to the representation of the algebra *AP(1,4)* given by the operators (24.31).

The basis elements of the algebra *AP(1,4)* for any class of IRs can be represented in the form (24.31). The matrices λ_{kl} and Σ_{kl} (included into (24.31)) satisfy the algebra (24.18), (24.22) which is isomorphic to the algebras *AO(4), AE(3)* or *AO(1,3)* for timelike, lightlike and spacelike vectors p_μ.

The realization (24.31) is distinguished by the simple form which is common for all the classes of the IRs. So this realization differs favorable from another known realizations of the algebra *AP(1,4)*.

24.6. Connection with Other Realizations

Let us consider IRs of the generalized Poincaré algebras in more detail and discuss connections of the realization (24.31) with other representations of this algebra.

a) $P_\mu P^\mu=c_1=m^2>0$. In this case the algebra of the matrices Σ_{kl} and λ_{kl} is isomorphic to the algebra *AO(4)*. The IRs of the last are finite dimensional and can be realized by the square matrices of dimension $(2j+1)(2\tau+1)\times(2j+1)(2\tau+1)$, where j and τ are positive integers or half of integers, which are connected with the eigenvalues of the Casimir operators C_2 and C_3 as follows:

$$\begin{aligned} c_2&=-m^2(l_0^2+l_1^2-1)=-2m^2[j(j+1))+\tau(\tau+1)], \\ c_3&=2ml_0l_1=2m[j(j+1))-\tau(\tau+1)]. \end{aligned} \tag{24.34}$$

The explicit form of the matrices Σ_{kl}, λ_{kl} in the basis $|c,\tilde{p},l,m\rangle$ is given by the relations

$$\begin{aligned} \Sigma_{ab}|c,\tilde{p},s,\mu\rangle&=\varepsilon_{abc}\Sigma_{4c}|c,\tilde{p},s,\mu\rangle=S_{ab}|c,\tilde{p},s,\mu\rangle, \\ \lambda_{ab}|c,\tilde{p},s,\mu\rangle&=\varepsilon_{abc}\lambda_{4c}|c,\tilde{p},s,\mu\rangle=-i\varepsilon_{abc}S_{0c}|c,\tilde{p},s,\mu\rangle, \end{aligned} \tag{24.35}$$

where $S_{\mu\nu}$ are the matrices realizing the representation $D(l_0,l_1)$ of the algebra *AO(1,3)* and acting on the vector $|c,\tilde{p},l,m\rangle$ according to (4.64).

The considered representations have the additional (energy sign) Casimir operator:

$$C_4=\frac{P_0}{|P_0|}=\frac{p_0}{\hat{p}_0}, \quad \hat{p}_0=E=\sqrt{p^2+m^2}\ . \tag{24.36}$$

In the space of an IR $C_4=\varepsilon=\pm1$ and $p_0=\varepsilon E$.

Using (24.23), (24.36) one can represent the operators (24.31) in the following form:

$$P_0=\varepsilon E,\quad P_k=p_k,\quad J_{kl}=x_k p_l-x_l p_k+\Sigma_{kl},$$
$$J_{0k}=x_0 p_k-x_k\varepsilon E+\frac{1}{p^2}(m\lambda_{kl}p_l-\varepsilon E\,\Sigma_{kl}p_l), \tag{24.37}$$

where λ_{kl}, Σ_{kl} are the matrices (24.35), (4.64).

The operators (24.37) are Hermitian with respect to the scalar product

$$(\Psi_1,\Psi_2)=\sum_{l,m}\int\frac{d^4p}{2E}\Psi_1^*(\boldsymbol{p},p_4,l,m)\Psi_2(\boldsymbol{p},p_4,l,m), \tag{24.38}$$

where $l=l_0,l_0+1,...,|l_1|-1$, $m=-l,-l+1,...,l$.

Using the unitary transformation $P_\mu\to P_\mu{}^K=UP_\mu U^\dagger$, $J_{\mu\nu}\to J_{\mu\nu}{}^K=UJ_{\mu\nu}U^\dagger$, where

$$U=\exp\left[\frac{i}{4}p\,\varepsilon_{abc}(\Sigma_{ab}-\varepsilon\lambda_{ab})p_c\arctan\frac{2|\boldsymbol{p}|p_4}{p_4^2-\boldsymbol{p}^2}\right], \tag{24.39}$$

the generators (24.37) reduce to the form found in [135,136]:

$$P_0^C=\varepsilon E,\quad P_l^C=p_l^C,\quad J_{kl}^C=x_k p_l-x_l p_k+S_{kl},$$
$$J_{0k}^C=x_0 p_k-\varepsilon E x_k-\varepsilon\frac{S_{kl}p_l}{E+m} \tag{24.40}$$

where $S_{ab}=\Sigma_{ab}$ and $S_{4a}=\lambda_{4a}$ are basis elements of the algebra $AO(4)$. The representation (24.40) is a natural generalization of the canonical Shirokov-Foldy representation (4.50) and we will call it therefore canonical for the sake of brevity.

b) $P_\mu P^\mu=c_1=0$. Such representations also have the additional Casimir operator (24.36) (with $m\equiv 0$). Thus an explicit realization of P_μ, $J_{\mu\nu}$ can be chosen in the form (see (24.23), (24.25), (24.28)):

$$P_0=\varepsilon E,\quad P_a=p_a,\quad J_{kl}=x_k p_l-x_l p_k+\Sigma_{kl},$$
$$J_{0k}=x_0 p_k-\varepsilon E x_k+(T_{kl}p_l-\varepsilon p\Sigma_{kl}p_l)/p^2, \tag{24.41}$$

where $\Sigma_{ab}=\varepsilon_{abc}\Sigma_c$, $\Sigma_{4a}=\Sigma_a$, $T_{ab}=\varepsilon_{abc}T_c$, $T_{4a}=T_a$, Σ_a and T_a are the matrices satisfying the algebra (24.24).

The algebra (24.24) for Σ_a, T_a is isomorphic to the Lie algebra of Euclidean group $E(3)$ and has two Casimir operators given in (24.33b). The eigenvalues of these operators that corresponding to unitary representations of the group $E(3)$ are equal to

$$-c_2=r^2\geq 0,\quad c_3=l_0 r,\quad l_0=0,\frac{1}{2},1,\frac{3}{2},.... \tag{24.42}$$

Representations of the algebra $AE(3)$ qualitatively differ for the cases $r^2=0$ and $r^2>0$. If $r^2=0$ then $T_a=0$, $T_{kl}=0$ and the algebra $AE(3)$ reduces to $AO(3)$. The last algebra generates the additional Casimir operator

$$C_5=\Sigma_a\Sigma_a \tag{24.43}$$

whose eigenvalues equal $s(s+1)$, s are positive integers or half integers. The corresponding representations are of finite dimensions and realized by the $(2s+1)\times(2s+1)$ matrices given in (4.31) (where $S_a=\Sigma_a$). Besides this the basis elements of the algebra *AP(1,4)* have the form (24.41) where $T_{kl}\equiv 0$ [135,136]. These operators are Hermitian with respect to the scalar product

$$(\Psi_1,\Psi_2)=\sum_\mu\int\frac{d^4p}{2p}\Psi_1^*(\boldsymbol{p},p_4,\mu)\Psi_2(\boldsymbol{p},p_4,\mu), \tag{24.44}$$

where $\mu=-s,-s+1,...,s$ are the eigenvalues of the matrix Σ_3, s is an integer or a half integer which determines an eigenvalue of the Casimir operator (24.43).

If $-c_2=r^2>0$ then unitary representations of the group *E(3)* are of infinite dimensions. The IRs are characterized by the numbers r, l_0 satisfying the conditions (24.42) and are denoted below by $D(l_0, r)$. The corresponding matrices Σ_a and T_a have the following form:

$$\Sigma_a=S_a,\quad T_a=\eta_a, \tag{24.45}$$

where S_a and η_a are given in (12.38).

Thus the basis elements of the IRs of the algebra *AP(1,4)* which corresponds to $c_1=0$, $c_2\neq 0$ can be chosen in the form (24.41) where T_a and Σ_a are the operators (24.45). In contrast with the case $c_2=0$ such representations are infinite dimensional in respect with the index l since the unitary IRs of the group *E(3)* are infinite dimensional.

c) $P_\mu P^\mu=c_1=-k^2<0$. In this case the matrices λ_{kl} and Σ_{kl} from (24.31) form a Lie algebra isomorphic to *AO(1,3)*. The Hermitian IRs of this algebra are infinite dimensional.

Using the isomorphism (24.23) we obtain from (24.31) the following explicit expressions of the basis elements of the algebra *AP(1,4)*:

$$\begin{gathered}P_\mu=p_\mu,\quad J_{kl}=x_kp_l-x_lp_k+\Sigma_{kl},\\ J_{0k}=x_0p_k-x_kp_0+(k\xi_{kl}p_l-p_0\Sigma_{kl}p_l)/p^2,\end{gathered} \tag{24.46}$$

where $\xi_{ab}=\varepsilon_{abc}S_{0c}$, $\xi_{4a}=S_{0a}$, $\Sigma_{ab}=S_{ab}$, $\Sigma_{4a}=\varepsilon_{abc}S_{bc}/2$, S_{ab} and S_{0a} are the basis elements of the algebra *AO(1,3)*, satisfying relations (2.18b). The explicit form of the matrices S_{ab} and S_{0a} which belonging to the IR $D(l_0,l_1)$ is given by formulae (4.64)-(4.66). Besides that for the Hermitian infinite-dimensional representations the parameter l_1 in (4.63) which defines the Casimir operators eigenvalues (4.61) is purely imaginary.

25. REPRESENTATIONS OF THE ALGEBRA *AP(1,4)* IN THE POINCARÉ-BASIS

25.1. Subgroup Structure of the Group *P(1,4)*

Investigation of a subgroup structure of the basic symmetry groups is one of the main problems of a group-theoretic analysis of equations of quantum mechanics. This problem is of great interest both for the group theory and for many applications, e.g., for finding the exact solutions of linear and nonlinear equations.

The subgroup structure of the basic groups *E(1,3), P(1,3), O(1,3)* has been described in papers [415,396]. The subgroups of the group *P(1,4)* were studied in [92,93]. It turns out that this group includes more than 400 connected subgroups, the main ones of which are:

a) the Euclidean group *E(4)* in the four-dimensional space;

b) the Poincaré group *P(1,3)*;

c) the Galilei group *G(1,3)*.

In other words it is the group *P(1,4)* which unites naturally the groups of motions of the relativistic and nonrelativistic quantum mechanics and the symmetry group of the Euclidean field theory.

Generally speaking IRs of the group *P(1,4)* are reducible with respect to its subgroups. So the problem of reduction of a representation of the generalized Poincaré group by its subgroups evokes a great interest.

We mean that the reduction of the group *P(1,4)* by its subgroups is a transformation of the group generators (basis elements of the algebra *AP(1,4)*) to such a basis where the Casimir operators of the corresponding subgroups (subalgebras) are diagonal. Transition to such a basis allows to answer the questions what sort of representations of a subalgebra are included into given IR of the algebra *AP(1,4)* and what is the multiplicity of these representations and also allows to find an explicit form of the basis elements of the algebra *AP(1,4)* in such a realization where the Casimir operators of the considered subalgebras are diagonal.

25.2. Poincaré-Basis

The operators P_μ, $J_{\mu\nu}$ ($\mu,\nu=0,1,2,3$) form a subalgebra of the algebra *AP(1,4)*. This subalgebra is isomorphic to the Lie algebra of the Poincaré group, i.e., to the algebra *AP(1,3)*. If P_μ and $J_{\mu\nu}$ belong to an IR of the algebra *AP(1,4)* then the representation of the subalgebra *AP(1,3)* is reducible because the Casimir operators (4.6)

$$C_1^{\prime}=P_0^2-\boldsymbol{P}^2\equiv c_1+P_4^2,\quad C_2^{\prime}=W_\mu W^\mu\equiv W_{4\mu}W^{4\mu} \tag{25.1}$$

are not multiples of the unit operator. Here W_μ is the Lubanski-Pauli vector (4.2); according to (24.3) $W_\mu=W_{4\mu}$.

Since the subalgebra *AP(1,3)* evokes the greatest interest in the algebra *AP(1,4)* it is desirable to find such realizations of this algebra that representations of the subalgebra *AP(1,3)* reduce to direct sums of the IRs. The basis of the IRs having the above-mentioned property will be called the Poincaré-basis (or *P(1,3)*-basis).

Let us give the more exact definitions.

DEFINITION 25.1. We say that an IR of the algebra *AP(1,4)* is defined in the Poincaré-basis if

1) the Casimir operators of the subalgebra *AP(1,4)* are diagonal;

2) the space of an IR of the algebra *AP(1,4)* is decomposed into a direct sum of Hilbert spaces invariant under IRs of the algebra *AP(1,3)*.

In the following subsections we define constructively the Poincaré-basis and find the explicit form of the corresponding operators P_μ, $J_{\mu\nu}$ for all the classes of IRs.

25.3. Reduction *P(1,4)* $\rightarrow$ *P(1,3)* of IRs of Class *I*

Consider IRs of the algebra *AP(1,4)* corresponding to positive values of the Casimir operator C_1: $P_\mu P^\mu=\kappa^2>0$. We start from the canonical realizations of such representations given by formulae (24.40).

The operators (24.40) are defined in the basis $|\kappa,l_0,l_1,\varepsilon;\boldsymbol{p},p_4,s,\lambda>$ formed by the eigenvectors of the Casimir operators (24.33a), (24.36) and the commuting operators P_1, P_2, P_3, P_4, Σ^2 and Σ_3. We refer to that basis as the canonical one.

Evidently there exist many other bases for IRs of the algebra *AP(1,4)*. Among them is the Poincaré basis which is very useful for physical applications. It can be defined as a set of eigenfunctions of the operators P_1, P_2, P_3, Casimir operators (25.1) and the operator S_3,

$$S_3=\frac{W_{43}}{M}-\frac{P_3W_{4a}P_a}{M(E+M)},\quad M=\sqrt{\kappa^2+P_4^2},\quad a=1,2,3. \tag{25.2}$$

To denote these eigenfunctions we use the symbol $|\kappa,l_0,l_1,\varepsilon;\boldsymbol{p},m,s,s_3>$ where four first numbers define eigenvalues of the Casimir operators (10.5), (10.36); the others characterize the eigenvalues of P_a, C'_1, C'_2 of (25.1) and S_3 of (25.2).

We normalize the basis vectors according to

$$\langle\kappa,l_0,l_1,\varepsilon;\boldsymbol{p},m,s,s_3|\kappa,l_0,l_1,\varepsilon;\boldsymbol{p}',m',s',s_3'\rangle=2E\delta(m-m')\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ss'}\delta_{s_3s_3'}.$$

This corresponds to the following scalar product

$$(\varphi_1,\varphi_2)=\sum_s\int\limits_{\kappa^2}^{\infty}\frac{dm^2}{2m}\int\frac{d^3p}{2E}\varphi_1^{\dagger}(\boldsymbol{p},m,s,s_3)\,\varphi_2(\boldsymbol{p},m,s,s_3),\tag{25.3}$$

where $\varphi_1(\boldsymbol{p},m,s,s_3)$, $\varphi_2(\boldsymbol{p},m,s,s_3)$ are vectors from a space of an IR of the algebra *AP(1,4)* defined in the Poincaré basis. It is not difficult to see that the bilinear form (25.3) is nothing but a sum over the discrete variable *s* and the integral with respect to m of the scalar products defined in orthogonal subspaces of the IRs of the algebra *AP(1,3)* corresponding to the eigenvalues m^2 and $-m^2s(s+1)$ of the Casimir operators (4.6).

Our task is to determine the possible eigenvalues of the operators (25.1), to find an explicit form of the operators P_n, J_{mn} in the Poincaré-basis and to find the unitary operator connecting the basises $|\kappa,l_0,l_1,\varepsilon;\boldsymbol{p},p_4,s,\lambda>$ and $|\kappa,l_0,l_1,\varepsilon;\boldsymbol{p},m,s,s_3>$.

We will see further on that the operators P_μ, $J_{\mu\nu}$ of (24.40) with $\mu,\nu\neq 4$ (these operators form the subalgebra *AP(1,3)*) can be transformed to the canonical form (4.50), where $m=(\kappa^2+p_4^2)^{1/2}$ and S_{ab} are matrices belonging to the IR $D(l_0,l_1)$ of the algebra *AO(4)*. By the reduction of the representation $D(l_0,l_1)$ by the algebra *AO(3)* we obtain a direct sum of the representations $D(s)$, $l_0\leq s\leq|l_1|-1$. It follows from the above that a Hilbert space *H* of the IR $D^{\varepsilon}(\kappa,l_0,l_1)$ of the algebra *AP(1,4)* is decomposed to the IRs $D^{\varepsilon}(m,s)$ of the algebra *AP(1,3)*, moreover

$$\kappa^2\leq m^2<\infty,\quad l_0\leq s\leq|l_1|-1.\tag{25.4}$$

We will search for an operator *V* connecting the canonical and Poincaré bases in the form

$$V=R\exp\left(i\frac{S_{4a}p_a}{|\boldsymbol{p}|}\theta\right),\tag{25.5}$$

where *R* and θ are unknown functions of $|\boldsymbol{p}|$ and p_4, S_{4a} are matrices belonging to a representation of the algebra *AO(4)* (they are included into the generators (24.40)).

The operators (25.5) can be used to define a class of IRs equivalent to (24.40):

$$P_n'=VP_nV^{-1},\quad J_{mn}'=VJ_{mn}V^{-1}.\tag{25.6}$$

By definition the operator (25.5) has to diagonalize the Casimir operators (25.1). We impose the stronger conditions on *V*, requiring that P'_μ, $J'_{\mu\nu}$ ($\mu,\nu\neq 4$) reduce to the canonical Shirokov-Foldy form (4.50). Substituting (24.40), (4.50) into (25.6) we come to the following equations for *V*

$$VP_0V^{-1}=\varepsilon\sqrt{p^2+m^2}\ ,\quad VP_aV^{-1}=p_a\tag{25.7a}$$

$$VJ_{ab}V^{-1}=J_{ab}\equiv x_ap_b-x_bp_a+S_{ab},$$

$$VJ_{0a}V^{-1}\equiv V\left(tp_a-\varepsilon Ex_a-\varepsilon\frac{S_{ab}p_b+S_{a4}p_4}{E+\kappa}\right)V^{-1}=tp_a-\varepsilon Ex_a-\varepsilon\frac{S_{ab}p_b}{E+m}, \tag{25.7b}$$

$m=\sqrt{\kappa^2+p_4^2}$, $\quad p=|\boldsymbol{p}|=\sqrt{p_1^2+p_2^2+p_3^2}$.

The conditions (25.7a) for the operator (25.5) are obviously satisfied. As to the equation (25.7b), we use the Campbell-Hausdorf formula (13.16) and obtain for the operators in the l.h.s. of it:

$$Vx_aV^{-1}=x_a+i\frac{\partial R}{\partial p_a}R^{-1}+\frac{p_aS_{4b}p_b}{p^2}\left(\frac{\partial\theta}{\partial p}-\frac{\sin\theta}{p}\right)+\frac{S_{ab}p_b}{p^2}(1-\cos\theta)+\frac{1}{p}S_{4a}\sin\theta,$$

$$VS_{4a}V^{-1}=S_{4a}\cos\theta+\frac{p_aS_{4b}p_b}{p^2}(1-\cos\theta)+\frac{S_{ab}p_b}{p}\sin\theta, \tag{25.8}$$

$$VS_{ab}p_bV^{-1}=S_{ab}p_b\cos\theta+\left(\frac{1}{p}p_aS_{4b}p_b-pS_{4a}\right)\sin\theta.$$

Substituting (25.8) into (25.7b) and equating coefficients of the linearly independent matrices we come to the following equations for θ and R:

$$\frac{\partial R}{\partial p_a}=0,\quad E(E+\kappa)\sin\theta+pp_4\cos\theta-p^2\sin\theta=0,$$

$$E(E+\kappa)(1-\cos\theta)-pp_4\sin\theta+p^2\cos\theta=\frac{p^2(E+\kappa)}{E+m},$$

$$E(E+\kappa)\left(\frac{\partial\theta}{\partial p}-\frac{\sin\theta}{p}\right)-p_4(1-\cos\theta)+p\sin\theta=0.$$

The general solutions of these equations are

$$R=R(p_4),\quad \theta=2\arctan\frac{pp_4}{(E+m)(m+\kappa)}. \tag{25.9}$$

Setting $R=(m/p_4)^{1/2}$ (which corresponds to the scalar product (25.3)) we obtain

$$V=\sqrt{\frac{m}{p_4}}\exp\left(2i\frac{S_{4a}p_a}{p}\arctan\frac{pp_4}{(E+m)(m+\kappa)}\right). \tag{25.10}$$

The operator (25.10) transforms P_μ, $J_{\mu\nu}$ ($\mu,\nu\neq 4$) to the canonical Shirokov-Foldy form (4.50). The explicit form of the remaining generators (i.e., J'_{04} and J'_{4a}) can be easily found using (25.6), (25.8), (25.9) and the relation

$$Vx_4V^{-1}=x_4-\frac{i\kappa^2}{2m^2p_4}+\frac{S_{4b}p_b(\kappa E-p_4^2)}{Em^2(E+\kappa)} \tag{25.11}$$

with the subsequent change $p_4\to\varepsilon'(m^2\text{-}\kappa^2)$, $\varepsilon'=p_4/|p_4|=\pm1$.

The results given above allow to formulate the following assertion [336]:

THEOREM 25.1. The space H of the IR $D^{\varepsilon}(\kappa,l_0,l_1)$ of the algebra $AP(1,4)$ with $P_kP^k>0$ reduces to the direct sum of the subspaces corresponding to the IRs $D^{\varepsilon}(m,s)$ of the algebra $AP(1,3)$ with the eigenvalues of the Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ given in (25.4). The operator connecting the canonical and Poincaré bases is given in (25.10), the basis elements of the algebra $AP(1,4)$ in the Poincaré-basis are

$$P_0'=\varepsilon\sqrt{p^2+m^2},\quad P_4'=\varepsilon'\sqrt{m^2-\kappa^2},\quad P_a'=p_a,$$

$$J_{ab}'=x_ap_b-x_bp_a+S_{ab},\quad J_{0a}'=x_0p_a-iP_0'\frac{\partial}{\partial p_a}-\varepsilon\frac{S_{ab}p_b}{E+m}, \tag{25.12a}$$

$$J_{04}'=x_0P_4'-iP_0'\left[\varepsilon'\sqrt{1-\frac{\kappa^2}{m^2}},\frac{\partial}{\partial m}\right]_+-\frac{\varepsilon\kappa}{m^2}S_{4a}p_a,$$

$$J_{4a}'=\frac{i}{2}p_a\left[\varepsilon'\sqrt{1-\frac{\kappa^2}{m^2}},\frac{\partial}{\partial m}\right]_+-iP_4'\frac{\partial}{\partial p_a}+\frac{\kappa p_aS_{4b}p_b}{m^2(E+m)}+P_4'\frac{S_{ab}p_b}{m(E+m)}+\frac{\kappa}{m}S_{4a}. \tag{25.12b}$$

The operators (25.12) are Hermitian with respect to the scalar product (25.3). They realize a representation of the algebra $AP(1,4)$ which corresponds to diagonal Casimir operators of the subalgebra $AP(1,3)$. ■

25.4. Reduction $P(1,4)\rightarrow P(1,2)$

In some physical problems (where the Poincaré-invariance is broken but the symmetry under the subgroup $P(1,2)$ is preserved) it is more convenient to use the $P(1,2)$-basis in which the Casimir operators of the algebra $AP(1,2)$ are diagonal. So it is interesting to continue the reduction of the algebra $AP(1,4)$ up to the subalgebra $AP(1,2)$. That reduction can be made by the transition to a basis where the operators P_0, J_{12}, P_a, J_{01}, J_{02} have the following (canonical) form:

$$P_0=\varepsilon E=\varepsilon\sqrt{p_1^2+p_2^2+m_1^2},\quad m_1^2=m^2+p_3^2,$$

$$P_\alpha=p_\alpha,\quad \alpha=1,2,\quad J_{12}=i\left(p_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial p_2}\right)+S_{12}, \tag{25.13a}$$

$$J_{01}=x_0p-iP_0\frac{\partial}{\partial p_1}-\frac{S_{12}p_2}{E+m_1},\quad J_{02}=x_0p_2-iP_0\frac{\partial}{\partial p_2}+\frac{S_{12}p_1}{E+m_1}. \tag{25.13b}$$

Here S_{12} is a matrix belonging to the IR $D(l_0,l_1)$ of the algebra $AO(4)$. To find an explicit form of the remaining basis elements of the algebra $AP(1,4)$ it is sufficient to find an operator $\tilde{V}$ satisfying the conditions

$$\tilde{V}P_0^{/}\tilde{V}^{-1}=\varepsilon E,\quad \tilde{V}P_\alpha^{/}\tilde{V}^{-1}=p_\alpha,\quad \tilde{V}J_{12}^{/}\tilde{V}^{-1}=J_{12}^{/},$$
$$\tilde{V}J_{0\alpha}^{/}\tilde{V}^{-1}=x_0 p_\alpha-i\varepsilon E\frac{\partial}{\partial p_\alpha}-\varepsilon\frac{S_{\alpha\beta}p_\beta}{E+m}, \tag{25.14}$$

where P'_0, P'_α, J'_{12}, J'_α are the generators (25.12). Representing $\tilde{V}$ in the form

$$\tilde{V}=\tilde{R}\exp\left(i\frac{S_{3\alpha}p_\alpha}{|p|_3}\tilde{\theta}\right),\quad |p|_3=\sqrt{p_1^2+p_2^2}, \tag{25.15}$$

where $\tilde{R}$ and $\tilde{\theta}$ are functions of p_3, $|p|_3$, m, and making the calculations analogous to ones in (25.8)-(25.10) we obtain

$$\tilde{\theta}=2\arctan\frac{p_3|p|_3}{(E+m)(m+m_1)},\quad \tilde{R}=\sqrt{m_1/p_3}\ . \tag{25.16}$$

This choice leads to the following scalar product

$$(\varphi_1,\varphi_2)=\int_{\kappa^2}^{\infty}\frac{dm^2}{2m}\int_{m^2}^{\infty}\frac{dm_1^2}{2m_1}\int\frac{d^2p}{2E}\varphi_1^\dagger\varphi_2. \tag{25.17}$$

Using the transformation operator (25.15), (25.16) and changing the variables $p_3\to\varepsilon''(m_1^2-m^2)^{1/2}$, we can find the explicit form of the generators J_{03} and J_{43} in the *P(1,2)*-basis:

$$J_{03}=x_0 m_1\lambda_{mm_1}-\frac{i}{2}\varepsilon E\left[\lambda_{mm_1},\frac{\partial}{\partial m_1}\right]-\frac{mS_{3a}p_a}{m_1^2},$$
$$J_{43}=-\frac{im}{2}\left[\lambda_{\kappa m}\lambda_{mm_1},\frac{\partial}{\partial m}\right]+\frac{\kappa m_1}{m^2}S_{43}, \tag{25.18}$$

where

$$\lambda_{\kappa m}=\varepsilon^{/}\sqrt{1-\kappa^2/m^2}\ ,\quad \lambda_{mm_1}=\varepsilon^{//}\sqrt{1-m^2/m_1^2}\ ,\quad \varepsilon^{/},\varepsilon^{//}=\pm1.$$

The remaining generators of the group *P(1,4)* can be obtained from (25.13), (25.18) using the commutation relations (25.1).

25.5. Reduction of IRs for the Case c_1=0

In this subsection we transform the IRs of the Classes *II* and *III* (corresponding to $P_\mu P^\mu$=0) of the algebra *AP(1,4)* into the Poincaré basis, starting from the realization (24.41).

First let us consider IRs corresponding to zero eigenvalues of the Casimir operator C_2 of (24.33), i.e., the IRs of the Class *II*. The corresponding operators P_n, J_{mn}

have the following form according to (24.41) (with $T_a\equiv 0$):

$$P_0=\varepsilon E,\quad P_k=p_k,\quad E=p=\sqrt{p_1^2+p_2^2+p_3^2+p_4^2},$$
$$J_{kl}=x_k p_l-x_l p_k+\Sigma_{kl},\quad J_{0k}=x_0 p_k-\varepsilon p x_k-\varepsilon\frac{\Sigma_{kl}p_l}{p}, \tag{25.19}$$

where Σ_{kl} form the IR $D(s\ 0)$ of the algebra $AO(4)$.

It is not difficult to make sure that the Casimir operators of the subalgebra $AP(1,3)$ (this subalgebra is formed by the generators P_μ, $J_{\mu\nu}$ ($\mu,\nu\neq 4$)) are diagonal in the representation (25.19):

$$P_\mu P^\mu=p_4^2=m^2,\quad -W_\mu W^\mu=m^2\Sigma_a\Sigma_a=m^2 s(s+1),$$

where $\Sigma_a=\Sigma_{4a}=\varepsilon_{abc}\Sigma_{bc}/2$. Since that to obtain an IR in the Poincaré basis it is sufficient to make the substitution $p_4\to\varepsilon' m$, $\varepsilon'=\pm 1$ in (25.19). As a result we can obtain the following realization

$$P_0=\varepsilon E,\quad P_a=p_a,\quad P_4=\varepsilon' m,\quad E=\sqrt{\boldsymbol{p}^2+m^2},$$
$$J_{ab}=x_a p_b-x_b p_a+\Sigma_{ab}, \tag{25.20}$$
$$J_{0a}=x_0 p_a-\varepsilon E x_a-\varepsilon\frac{\Sigma_{ab}p_b+\varepsilon'\Sigma_{a4}m}{E},$$

$$J_{4a}=\varepsilon'\left(p_a\frac{\partial}{\partial m}-m\frac{\partial}{\partial p_a}\right)+\Sigma_{4a},\quad J_{04}=\varepsilon'\left(x_0 m-\varepsilon p\frac{\partial}{\partial m}\right)-\varepsilon\frac{\Sigma_{4a}p_a}{E}. \tag{25.21}$$

For any fixed value of m the operators (25.20) form a basis of the IR $D^\varepsilon(m,s)$ of the algebra $AP(1,3)$ in the realization (25.20) which had been considered in [136].

Using the transformation (25.5) where

$$V=V'=\exp\left(\frac{i\Sigma_a p_a}{|\boldsymbol{p}|}\arctan\frac{|\boldsymbol{p}|}{m}\right), \tag{25.22}$$

we can reduce (25.20), (25.21) to the form (25.12) where $\kappa\equiv 0$ but S_{kl} are matrices belonging to the IR $D(s\ 0)$ of the algebra $AO(4)$. We conclude from this fact that the result, formulated in the Theorem 24.1 is valid for the IRs of Class *II* also (besides, for the last $\kappa=0$, $l_0=s$, $l_1=s+1$).

Just now we consider IRs of the Class *III* corresponding to $P_nP^n=0$, $W_{mn}W^{mn}/2=-r^2<0$. Basis elements of such a representation can be chosen in the form (24.41). With the help of the unitary transformation (25.5), where

$$V=\exp\left(-i\frac{\boldsymbol{T}\cdot p}{|\boldsymbol{p}|p_4}\right)V',$$

V' is the operator (25.22) with the corresponding infinite-component matrices Σ_a, and the change $p_4 \rightarrow \varepsilon' m$ we come to the following realization

$$\begin{aligned}
&P_0'=\varepsilon E,\quad P_a'=p_a,\quad P_4'=\varepsilon' m,\quad E=\sqrt{\boldsymbol{p}^2+m^2},\\
&J_{ab}'=x_a p_b-x_b p_a+\Sigma_{ab},\\
&J_{0a}'=x_0 p_a-\varepsilon E x_a-\varepsilon\frac{\Sigma_{ab}p_b}{E+m},\\
&J_{4a}'=\varepsilon'\left(ip_a\frac{\partial}{\partial m}-im\frac{\partial}{\partial p_a}+\frac{\Sigma_{ab}p_b}{E+m}+\frac{E}{m^2}T_a\right)+\frac{(2E+m)p_a\boldsymbol{T}\cdot\boldsymbol{p}}{m^3(E+m)},\\
&J_{04}'=\varepsilon'\left(x_0 m-i\varepsilon E\frac{\partial}{\partial m}\right)-\frac{\boldsymbol{T}\cdot\boldsymbol{p}}{m^2}.
\end{aligned}\tag{25.23}$$

The operators (25.23) are defined in the Poincaré basis $|r,\lambda,\varepsilon;\,\boldsymbol{p},m,l,l_3>$ which is formed by the eigenvectors of the complete set of the commuting operators C_1, C_2, C_3 (24.5), C_4 (24.36) and $P_1, P_2, P_3, P_\mu P^\mu=P_4^2$, $W^\mu W^\mu=-m^2\Sigma_{ab}\Sigma^{ab}$, Σ_{12}. Moreover, the eigenvalues of the Casimir operators of the subalgebra *AP(1,3)* take the form (see (12.36)-(12.38))

$$\begin{aligned}
&P_\mu P^\mu|r,\lambda,\varepsilon;\,\boldsymbol{p},m,s,s_3\rangle=m^2|r,\lambda,\varepsilon;\,\boldsymbol{p},m,s,s_3\rangle,\\
&W_\mu W^\mu|r,\lambda,\varepsilon;\,\boldsymbol{p},m,s,s_3\rangle=-m^2 s(s+1)|r,\lambda,\varepsilon;\,\boldsymbol{p},m,s,s_3\rangle
\end{aligned}\tag{25.24}$$

where

$$0\leq m^2<\infty,\quad s=l_0,\ l_0+1,\ \ldots\ .\tag{25.25}$$

The operators (25.23) are Hermitian with respect to the scalar product (25.3), where $\kappa=0$ and s takes the values given in (25.25).

Let us summarize the results presented [336]:

THEOREM 25.2. The space of the IR $D^\varepsilon(r,\lambda)$ of Class *III* of the algebra *AP(1,4)* is decomposed into the direct integral of subspaces corresponding to the IRs $D^\varepsilon(m,s)$ of the algebra *AP(1,3)*; moreover, the possible eigenvalues of the Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ are given in (25.24). The generators P_n and J_{mn} are given in the Poincaré basis by formulae (25.23) where T_a and Σ_a are the infinite dimension matrices (24.45),(12.38). ■

25.6. Reduction of Representations of Class *IV*

Consider IRs of Class *IV* of the algebra *AP(1,4)*. These representations correspond to negative eigenvalues of the Casimir operator $C_1=P_nP^n$: $C_1=-k_2<0$.

A specific feature of transformation of such IRs to the Poincaré basis is the following: by reducing them by the subalgebra *AP(1,3)* we come to representations belonging to the different classes depending on the eigenvalues of the operator $M^2=P_4^2-k^2$.

We consider the case when the domain of eigenvalues p_4 is restricted by the condition $p_4^2>k^2$.

We start from the realization (24.46) of IRs of Class *IV*. It is not difficult to make sure that the corresponding Casimir operators of the subalgebra *AP(1,3)* are not diagonal. But using the transformation (25.5), where

$$V=\exp\left(-i\frac{S_{0a}P_a}{|\boldsymbol{p}|}\tanh^{-1}\frac{k|\boldsymbol{p}|}{p_0p_4}\right)\exp\left[i\frac{S_ap_a}{|\boldsymbol{p}|}\arctan\frac{|\boldsymbol{p}|}{p_4}+\frac{i\pi}{2}(1-\varepsilon)\right],$$

where

$$E=\sqrt{\boldsymbol{p}^2+m^2},\quad m^2=p_4^2-k^2,\quad S_{0a}=\xi_a,\quad S_a=\Sigma_a=\frac{1}{2}\varepsilon_{abc}S_{bc},$$

and changing $p_0\to\varepsilon E$, $p_4\to\varepsilon'(m^2+k^2)^{1/2}$, we reduce (24.46) to the following form

$$P_0=\varepsilon\sqrt{\boldsymbol{p}^2+m^2},\quad P_a=p_a,\quad P_4=\varepsilon'\sqrt{m^2+k^2},$$

$$J_{ab}=x_ap_b-x_bp_a+S_{ab},\quad J_{0a}=x_0p_a-\varepsilon Ex_a-\varepsilon\frac{S_{ab}p_b}{E+m},$$

$$J_{04}=-i\varepsilon E\left[\sqrt{1+\frac{k^2}{m^2}},\frac{\partial}{\partial m}\right]_+ +x_0P_4+\frac{kS_{0a}p_a}{m^2}, \tag{25.26}$$

$$J_{a4}=-\frac{i\varepsilon'}{2}p_a\left[\sqrt{1+\frac{k^2}{m^2}},\frac{\partial}{\partial m}\right]_+ +x_aP_4+\frac{k}{m}S_{0a}+\left(\varepsilon'\sqrt{1+\frac{k^2}{m^2}}+\varepsilon\frac{kE}{m^2}\right)\frac{S_{ab}p_b}{E+m}.$$

The operators P_μ, $J_{\mu\nu}$ of (25.26) have the canonical Wigner-Shirokov form, however, the matrices S_{ab} belong to the IR $D(l_0,l_1)$ of the algebra *AO(1,3)*. Hence we conclude IRs of the Class *IV* of the algebra *AP(1,4)* are decomposed into the direct sums of IRs $D^\varepsilon(s,m)$ by the reduction $P(1,4)\to P(1,3)$. Moreover,

$$m^2=p_4^2-k^2>0,\quad s=l_0,l_0+1,l_0+2,\ldots,|l_1|-1.$$

In analogous way it is possible to consider the cases $|p_4|=k$ and $|p_4|\le k$

[336]. We will not write out the corresponding cumbersome formulae here.

25.7. Reduction *P(1,n)* → *P(1,3)*

The results presented above can be generalized directly to the cases of the groups *P(1,n)* defined in $(n+1)$-dimensional Minkowsky spaces. Here we consider IRs of the algebra *AP(1,n)* corresponding to the positive values of the main Casimir operator $P^2=P_0^2-P_1^2-P_2^2-...-P_n^2$ and find a realization of such IRs in the Poincaré basis.

First we will show that the IRs of the algebra *AP(1,n)* can be defined in the *P(1,n-1)*-basis. The canonical realization of an IR corresponding to $P^2=\kappa^2>0$ is defined by the following relations [136]:

$$P_0=\varepsilon E=\varepsilon\sqrt{p_k p_k+\kappa^2},\quad P_k=p_k,\quad k=1,2,3...,n,$$

$$J_{ab}=x_a p_b-x_b p_a+S_{ab},\quad a,b=1,2,3,...,n-1, \tag{25.27}$$

$$J_{0a}=x_0 p_a-\varepsilon E x_a-\varepsilon\frac{S_{ab}p_b+S_{an}p_n}{E+\kappa},$$

$$J_{0n}=x_0 p_n-\varepsilon E x_n-\varepsilon\frac{S_{na}p_a}{E+\kappa},\quad J_{an}=x_a p_n-x_n p_a+S_{an}. \tag{25.28}$$

Here S_{kl} are matrices which form the IR $D(m_1,m_2,...m_{[n/2]})$ of the algebra *AO(n)*, m_1, m_2, ... are the Gelfand-Zetlin numbers. The operators (25.27), (25.28) are Hermitian with respect to the following scalar product

$$(\Psi_1,\Psi_2)=\int\frac{d^n p}{2E}\Psi_1^\dagger\Psi_2. \tag{25.29}$$

The subalgebra *P(1,n-1)* is a linear span of the basis elements (25.27). In the *P(1,n-1)*-basis these elements by definition have to have a form of a direct sum of the generators of the IRs of the group *P(1,n-1)*. If S_{kl} are defined in the Gelfand-Zetlin basis $O(n)\supset O(n-1)\supset O(n-2)$ ... then we can choose the transformed operators (25.27) in the form

$$P_0^{/}=\varepsilon E=\varepsilon\sqrt{p_a p_a+m_n^2},\quad m_n^2=\kappa^2+p_n^2,\quad P_a^{/}=p_a,$$
$$J_{ab}^{/}=J_{ab}=x_a p_b-x_b p_a+S_{ab},\quad J_{0a}^{/}=x_0 p_a-\varepsilon E x_a-\varepsilon\frac{S_{ab}p_b}{E+m_n}. \tag{25.30}$$

The problem of finding of a realization of the generators P_n, J_{mn} in the *P(1,n-1)*-basis reduces to the construction of an isometric operator which connects the realizations (25.27) and (25.30). In analogy with (25.5)-(25.11) it is possible to show that this operator has the form

$$V_n = \sqrt{\frac{m_n}{p_n}}\exp\left(i\frac{S_{na}p_a}{|p|_n}\theta_n\right), \tag{25.31}$$

where

$$\theta_n = 2\arctan\frac{p_n|p|_n}{(E+m_n)(m_n+\kappa)}, \quad |p|_n = \sqrt{p_1^2+p_2^2+\ldots+p_{n-1}^2}\ . \tag{25.32}$$

Using the following identities (which are easily verified with the Cambell-Hausdorf formula (13.16))

$$x_a' = V_n x_a V_n^{-1} = x_a - \frac{p_a p_n S_{nb} p_b}{Em_n(E+m_n)(E+\kappa)} + \frac{S_{ab}p_b(m_n-\kappa)}{m_n(E+m_n)(E+\kappa)} + \frac{p_n S_{na}p_a}{m_n(E+\kappa)}, \tag{25.33}$$

$$V_n p_k V_n^{-1} = p_k,$$

we obtain

$$V_n P_\mu V_n^{-1} = P_\mu', \quad V_n J_{\mu\nu} V_n^{-1} = J_{\mu\nu}'$$

where P'_μ, $J'_{\mu\nu}$ are the operators (25.30). Analogously using the substitution $p_n \to \varepsilon_n(m_n^2 - \kappa^2)^{1/2}$, $\varepsilon_n = \pm 1$ one can find the remaining basis elements of the algebra *AP(1,n)* in the following form:

$$P_n' = \varepsilon_n\sqrt{m_n^2-\kappa^2},$$

$$J_{0n}' = x_0 P_n' - \frac{i}{2}\varepsilon E\left[\frac{P_n'}{m_n}, \frac{\partial}{\partial m_n}\right]_+ - \frac{\kappa S_{na}p_a}{m_n^2}, \tag{25.34}$$

$$J_{na}' = \frac{ip_a}{2}\left[\frac{P_n'}{m_n}, \frac{\partial}{\partial m_n}\right]_+ - P_n' x_a + \frac{\kappa p_a S_{nb} p_b}{m_n^2(E+m_n)} + \frac{P_n' S_{ab} p_b}{m_n(E+m_n)} + \frac{\kappa}{m_n} S_{na}.$$

So we have found the explicit expressions for the generators of the group *P(1,n)* in the basis *P(1,n-1)*. These generators are Hermitian with respect to the following scalar product

$$(\phi_1,\phi_2) = \sum_\eta \int_\kappa^\infty dm_n \int \frac{d^{n-1}p}{2E}\phi_1^\dagger(\eta,m)\phi_2(\eta,m). \tag{25.35}$$

Here η is a set of numbers characterizing the IRs of the algebra *AO(n-1)* appearing by the reduction *AO(n)* $\to$ *AO(n-1)*.

Let us now define a representation of the algebra *AP(1,n)* in the *P(1,n-2)*-basis. Using the above results we conclude that the operator

$$V_{n-1}=\sqrt{\frac{m_{n-1}}{p_{n-1}}}\exp\left(i\frac{2S_{n-1a}p_a}{|p|_{n-1}}\arctan\frac{p_{n-1}|p|_{n-1}}{(E+m_n)(m_n+m_{n-1})}\right),$$

$$m_{n-1}=\sqrt{\kappa^2+p_n^2+p_{n-1}^2},\quad |p|_{n-1}=\sqrt{p_1^2+p_2^2+\ldots+p_{n-2}^2},\quad a=1,2,\ldots,n-2,$$

transforms (25.30) to the following form

$$\begin{aligned}
&P_0=E=\varepsilon\sqrt{|p|_{n-1}^2+m_{n-1}^2},\quad P_a=p_a,\quad a=1,2,\ldots,n-2,\\
&P_{n-1}=\varepsilon_{n-1}\sqrt{m_{n-1}^2-m_n^2},\quad P_n=\varepsilon_n\sqrt{m_n^2-\kappa^2},\\
&J_{ab}=x_ap_b-x_bp_a+S_{ab},\\
&J_{0a}=x_0p_a-\varepsilon Ex_a-\varepsilon\frac{S_{ab}p_b}{(E+m_{n-1})},\\
&J_{0n-1}=x_0P_{n-1}-\frac{i}{2}\varepsilon E\left[\frac{P_{n-1}}{m_{n-1}},\frac{\partial}{\partial m_{n-1}}\right]_+-\frac{m_nS_{n-1a}p_a}{m_{n-1}^2}.
\end{aligned}\tag{25.36}$$

To define the basis elements of the algebra *AP(1,n)* in the *P(1,n-2)*-basis it is sufficient to find the explicit form of just one more operator, i.e., J_{nn-1}, since the remaining generators can be obtained using the commutation relations (24.1). Using the identities

$$V_{n-1}x_nV_{n-1}^{-1}=x_n-\frac{p_np_{n-1}S_{n-1a}p_a}{m_nm_{n-1}^2E}-\frac{ip_n}{2m_{n-1}^2},$$

$$V_{n-1}S_{nn-1}V_{n-1}^{-1}=\frac{S_{nn-1}(m_{n-1}^2+Em_n)}{m_{n-1}(E+m_n)}+\frac{S_{na}p_aP_{n-1}}{m_{n-1}(E+m_n)}$$

and the last of the relations (25.33) with $n\to n$-1, $\kappa\to m_n$ we obtain

$$J_{nn-1}=\frac{i}{2}\left[\frac{P_nP_{n-1}}{m_n},\frac{\partial}{\partial m_{n-1}}\right]_++\frac{\kappa m_{n-1}}{m_n^2}S_{nn-1}.\tag{25.37}$$

The operators (25.36), (25.37) are Hermitian with respect to the scalar product

$$(\phi_1,\phi_2)=\int_\kappa^\infty dm_n\int_{m_n}^\infty dm_{n-1}\sum_\alpha\int\frac{d^{n-2}p}{E}\phi_1^\dagger(m_{n-1},\alpha)\phi_2(m_{n-1},\alpha),$$

where α denotes the sets of numbers that numerating IRs of the algebra *AO(n-2)* appearing by the reduction *AO(n)* $\to$ *AO(n-1)* $\to$ *AO(n-2)*.

A representation of the algebra *AP(1,n)* in the *P(1,n-k)*-bases can be determined in an analogous way. Starting from (25.36), (25.37) and making the consequent transformations $P_\mu\to V_{n-1}P_\mu(V_{n-1})^{-1}$, $J_{mk}\to V_{n-1}J_{mk}V_{n-1}^{-1}$, where

$$V_{n-l}=\sqrt{\frac{m_{n-l}}{p_{n-l}}}\exp\left(\frac{2iS_{n-la}p_a}{|p|_{n-l}}\arctan\frac{p_{n-l}|p|_{n-l}}{(E+m_{n-l})(m_{n-l}+m_{n-l+1})}\right),$$

$$|p|_{n-l}=\sqrt{p_1^2+p_2^2+\ldots+p_{n-l-1}^2}\,,\quad l=2,3,\ldots,k,$$

$$m_{n-l}=\sqrt{\kappa^2+p_n^2+p_{n-1}^2+\ldots+p_{n-l}^2}\,,\quad a=1,2,\ldots,n-l-1$$

and using the above results we obtain

$$P_0=\varepsilon E=\varepsilon\sqrt{|p|_{n-k+1}^2+m_{n-k+1}^2}\,,\quad P_a=p_a,$$

$$P_{n-\alpha}=\varepsilon_{n-\alpha}\sqrt{m_{n-\alpha}^2-m_{n-\alpha+1}^2}\,,\quad J_{ab}=x_a p_b-x_b p_a+S_{ab},$$

$$J_{0a}=x_0 p_a-\varepsilon E x_a-\varepsilon\frac{S_{ab}p_b}{E+m_{n-k+1}},\ \alpha\le k,\ a,b\le n-k,$$

$$J_{0\,n-\alpha}=x_0P_{n-\alpha}-\frac{i\varepsilon}{2}\left[\frac{P_{n-\alpha}}{m_{n-\alpha}},\frac{\partial}{\partial m_{n-\alpha}}\right]_+-\frac{m_{n-\alpha+1}S_{n-\alpha\,a}p_a}{m_{n-\alpha}^2},\tag{25.38}$$

$$J_{n-\alpha\,n-\alpha+1}=\frac{i}{2}\left[\frac{P_{n-\alpha}P_{n-\alpha+1}}{m_{n-\alpha}},\frac{\partial}{\partial m_n}\right]_+ +\frac{\kappa m_{n-\alpha+1}}{m_{n-\alpha}^2}S_{n-\alpha\,n-\alpha+1}.$$

The generators (25.38) are Hermitian with respect to the following scalar product

$$(\phi_1,\phi_2)=\int_\kappa^\infty dm_n\int_{m_n}^\infty dm_{n-1}\ldots\int_{m_{n-k+1}}^\infty dm_{n-k}\sum_\lambda\int\frac{d^{n-k}p}{2E}\phi_1^\dagger(m_{n-k},\lambda)\phi_2(m_{n-k},\lambda).$$

They form a representation of the algebra *AP(1,n)* in the basis *P(1,n)⊃P(1,n-1)⊃ ...⊃P(1,n-k)*. In the case *n-k*=3 formulae (25.38) defines an IR of the algebra *AP(1,n)* in the Poincaré-basis.

26. REPRESENTATIONS OF THE ALGEBRA *AP(1,4)* IN THE *G(1,3)*- AND E(4)-BASISES

26.1. The *G(1,3)*-Basis

As was noted in Subsection 24.1 the algebra *AP(1,4)* includes the subalgebras *AP(1,3)*, A*G(1,3)* and *AE(4)*, i.e., the Lie algebras of the main groups of quantum physics.

In the previous section we have found the generators of the generalized Poincaré group in the Poincaré basis. However it is very interesting for physical applications to describe representations of the algebra *AP(1,4)* in the Galilei basis characterized by a diagonal form of the Casimir operators of Galilean subalgebra.

Here* we obtain an explicit form of basis elements of the algebra *AP(1,4)* in the *G(1,3)*-basis for any class of IRs. The unitary operator is found also which makes the reduction of IRs of the Poincaré group by the Galilei group in (1+2)- dimensional space (i.e., the reduction $P(1,3) \to G(1,2)$). Such a reduction plays the central role in the null-plane formalism (see, e.g., [275]).

In Subsection 25.5 we find IRs of the algebra *AP(1,4)* in the *E(4)*-basis in which the Casimir operators of the four-dimensional Euclidean group are diagonal.

To select the Galilei subalgebra from the algebra *AP(1,4)* we should come to the basis

$$
\begin{aligned}
&\hat{P}_0=\frac{1}{2}(P_0-P_4), \quad M=P_0+P_4, \quad \hat{P}_a=P_a,\\
&J_a=\frac{1}{2}\varepsilon_{abc}J_{bc}, \quad G_a^{\pm}=J_{0a}\pm J_{4a}, \quad K=J_{04},
\end{aligned}
\tag{26.1}
$$

where as usually a=1,2,3.

It follows from (24.1) that the operators (26.1) satisfy the following commutation relations

$$
\begin{aligned}
&\left[\hat{P}_0,\hat{P}_a\right]=\left[\hat{P}_0,M\right]=\left[\hat{P}_a,M\right]=\left[\hat{P}_a,\hat{P}_b\right]=0,\\
&\left[\hat{P}_0,J_a\right]=\left[M,J_a\right]=\left[G_a^+,G_b^+\right]=\left[M,G_a^+\right]=0,\\
&\left[\hat{P}_a,J_b\right]=i\varepsilon_{abc}\hat{P}_c, \quad \left[G_a^+,\hat{P}_b\right]=i\delta_{ab}M,\\
&\left[J_a,J_b\right]=i\varepsilon_{abc}J_c, \quad \left[\hat{P}_0,G_b^+\right]=i\hat{P}_b,
\end{aligned}
\tag{26.2}
$$

$$
\begin{aligned}
&\left[\hat{P}_0,G_a^-\right]=\left[G_a^-,G_b^-\right]=0, \quad \left[G_a^-,M\right]=-2i\hat{P}_a,\\
&\left[G_a^-,J_b\right]=i\varepsilon_{abc}G_c^-, \quad \left[G_a^-,\hat{P}_b\right]=-2i\delta_{ab}\hat{P}_0,\\
&\left[G_a^+,G_b^-\right]=2i(\varepsilon_{abc}J_c+\delta_{ab}K), \quad \left[\hat{P}_a,K\right]=\left[J_a,K\right]=0,\\
&\left[\hat{P}_0,K\right]=-i\hat{P}_0, \quad \left[M,K\right]=iM, \quad \left[G_a^{\pm},K\right]=\pm iG_a^{\pm}.
\end{aligned}
\tag{26.3}
$$

The relations (26.2) characterize the Lie algebra of the Galilei group *G(1,3)* (compare with (11.6)). This algebra has three main Casimir operators presented in (11.14). Using (26.1) we represent these operators in the form

*For more details see [150]

$$\hat{C}_1=2M\hat{P}_0-\hat{P}_a\hat{P}_a\equiv C_1, \quad \hat{C}_3=M\equiv P_0+P_4,$$
$$\hat{C}_2=(M\boldsymbol{J}-\hat{P}\times G^{+})^2\equiv(W_{4a}+W_{0a})(W_{4a}+W_{0a}), \tag{26.4}$$

where C_1 is the Casimir operator of the algebra *AP(1,4)* of (24.5a), W_{4a}, W_{0a} are the components of the tensor $W_{\mu\nu}$ of (24.3).

Our task is to transform the realizations of the algebra *AP(1,4)* described in Section 24 to such a basis where the operators (26.4) are diagonal. This basis can be formed by a complete set of eigenfunctions of the commuting operators P_1, P_2, P_3, $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ and $\Sigma_3=W_{43}+W_{03}$ with eigenvalues p_1, p_2, p_3, $2mm_0$, $m^2s(s+1)$, m and ms_3. To denote such eigenfunctions we use the notation $|\boldsymbol{p},m_0,s,m,s_3;c>$, where c is a set of the eigenvalues of the Casimir operators of the algebra *AP(1,4)* characterizing the IR.

We normalize the basis vectors according to

$$\langle \boldsymbol{p},m_0,s,m,s_3;c\,|\,\boldsymbol{p}',m_0',s',m',s_3';c\rangle=2m\delta(m-m')\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,\delta_{ss'}\,\delta_{s_3s_3'}. \tag{26.5}$$

This normalization leads to the following definition of the scalar product in the Hilbert space spanned on the basis $|\boldsymbol{p},m_0,s,m,s_3;c>$:

$$(\Phi_1,\Phi_2)=\sum_s\int_{\lambda_1}^{\lambda_2}\frac{dm}{2m}\int d^3p\Phi_1^\dagger(m,\boldsymbol{p},s,s_3)\Phi_2(m,\boldsymbol{p},s,s_3). \tag{26.6}$$

The domains of m and s are different for the different classes of the IRs and will be defined further on.

26.2. Representations with $P_nP^n>0$

We start from the realizations (24.40) of these representations. Our aim is to find the explicit form of the generators P_n, J_{mn} in the Galilei basis and define the unitary operator connecting this basis with the canonical one.

Substituting (24.40) into (26.2)-(26.4) we obtain

$$\hat{P}_0=\frac{1}{2}(\varepsilon E-p_4), \quad M=\varepsilon E+p_4, \quad E=\sqrt{p^2+p_4^2+\kappa^2},$$
$$J_a=\varepsilon_{abc}\left(x_bp_c+\frac{1}{2}S_{bc}\right),$$
$$G_a^{+}=(x_4+x_0)p_a-Mx_a-\frac{\varepsilon S_{ab}p_b-S_{4a}(E+\kappa+\varepsilon p_4)}{E+\kappa}, \tag{26.7a}$$
$$\hat{C}_1=\kappa^2, \quad \hat{C}_3=M,$$

$$\hat{C}_2=\left\{\boldsymbol{S}^2[M(E+\kappa)-\varepsilon\boldsymbol{p}^2]^2+[p^2\boldsymbol{N}^2-(\boldsymbol{p}\cdot\boldsymbol{N})^2](E+\kappa+\varepsilon p_4)^2+\right.$$
$$\left.+(\boldsymbol{p}\cdot\boldsymbol{S})^2[2\varepsilon M(E+\kappa)-\boldsymbol{p}^2]\right\}(E+\kappa)^{-2},$$
$$K=x_0p_4-\varepsilon Ex_4-\varepsilon\frac{S_{4a}p_a}{E+\kappa}, \tag{26.7b}$$
$$G_a^-=(x_0-x_4)p_a-2\hat{P}_0x_a-\frac{\varepsilon S_{ab}p_b+S_{4a}(E+\kappa-\varepsilon p_4)}{E+\kappa},$$

where

$$S_a=\frac{1}{2}\varepsilon_{abc}S_{bc},\quad N_a=S_{4a},\quad x_k=-i\frac{\partial}{\partial p^k}.$$

The Casimir operators $\hat{C}_1$, $\hat{C}_3$ are diagonal in the canonical basis $|\boldsymbol{p},m_0,s,m,s_3;\kappa,l_0,l_1,\varepsilon>$, but $\hat{C}_2$ is a matrix depending on $\boldsymbol{p}$ and p_4. To diagonalize this matrix we use an operator

$$U=\exp\left(i\frac{S_{4a}p_a}{|\boldsymbol{p}|}\theta\right), \tag{26.8}$$

where θ is an unknown so far function of $\boldsymbol{p}$ and p_4.

Using the unitary operator (26.8) we can define a series of the representations equivalent to (26.7):

$$\hat{P}'_\mu=U_1\hat{P}_\mu U_1^\dagger=\hat{P}_\mu,\quad J'_a=U_1J_aU_1^\dagger=J_a,\quad M'=U_1MU_1^\dagger=M,$$
$$(G_a^+)'=U_1G_a^+U_1^\dagger=(x_0+x_4)p_a-x'_aM-\frac{\varepsilon S'_{ab}p_b-S'_{4a}(E+\kappa+\varepsilon p_4)}{E+\kappa}, \tag{26.9}$$
$$(G_a^-)'=U_1G_a^-U_1^\dagger,\quad K'=U_1KU_1^\dagger,$$

where x'_a, x'_4, S'_{ab}, S'_{4a} are given by formulae (25.8) (with R=1). Moreover $(G_a^+)'$ has an extremely simple form if we choose

$$\theta=2\arctan\frac{|\boldsymbol{p}|}{E+\kappa+\varepsilon p_4}+\frac{1}{2}(1-\varepsilon)\pi, \tag{26.10}$$

Indeed, we have in this case

$$(G_a^+)'=(x_0+x_4)p_a-Mx_a. \tag{26.11}$$

Substituting (26.9) and (26.11) into (26.4) we make sure that the corresponding Casimir operator $\hat{C}'_2$ is diagonal:

$$\hat{C}_2^{\prime}=M^2\boldsymbol{S}^2\equiv M^2(S_{12}^2+S_{31}^2+S_{23}^2)$$

where $\boldsymbol{S}^2$ is a diagonal matrix with eigenvalues $s(s+1)$ ($l_0\le s\le|l_1|-1$).

To find the explicit form of the generators of the group *P(1,4)* in the Galilei basis $|\boldsymbol{p},m_0,s,m,s_3;\kappa,l_0,l_1,\varepsilon>$ it is sufficient to substitute (25.8), (25.10),(26.8) into (26.11) and to make the change of variables $(p_1,p_2,p_3,p_4)\to(p_1,p_2,p_3,m)$ where $m=E+\varepsilon p_4$. Besides that

$$\frac{\partial}{\partial p_4}\to\varepsilon\frac{m}{E}\frac{\partial}{\partial m},\quad \frac{\partial}{\partial p_a}\to\frac{\partial}{\partial p_a}+\frac{p_a}{E}\frac{\partial}{\partial m},$$

and the operators (26.9) take the following form:

$$\begin{gathered}
\hat{P}_0^{\prime}=m_0+\varepsilon\frac{\boldsymbol{p}^2}{2m},\quad \hat{P}_a^{\prime}=p_a,\quad M^{\prime}=\varepsilon m,\\
J_a^{\prime}=\varepsilon_{abc}x_b p_c+S_a,\quad (G_a^{+})^{\prime}=x_0p_a-i\varepsilon m\frac{\partial}{\partial p_a},\\
K^{\prime}=-im\frac{\partial}{\partial m}+x_0\left(\frac{\varepsilon m}{2}-\hat{P}_0\right),\\
(G_a^{-})^{\prime}=x_0p_a-2i\left[\varepsilon p_a\frac{\partial}{\partial m}+\hat{P}_0^{\prime}\frac{\partial}{\partial p_a}\right]+\frac{2(S_{a4}\kappa-S_{ab}p_b)\varepsilon}{m},
\end{gathered}\tag{26.12}$$

$$\hat{C}_1^{\prime}=\kappa^2,\quad \hat{C}_2^{\prime}=m^2s(s+1),\quad \hat{C}_3^{\prime}=\varepsilon m,\tag{26.13}$$

where

$$\kappa\le m<\infty,\quad m_0=\varepsilon\frac{\kappa^2}{2m},\quad l_0\le s\le|l_1|-1.$$

So we come to the realization (26.12) of the IR $D^{\varepsilon}(\kappa,l_0,l_1)$ of the algebra *AP(1,4)*. The distinguishing feature of this realization is that the operators $\hat{P}_\mu^{\prime}$, $J_a^{\prime}$, $(G_a^{+})^{\prime}$ and $M^{\prime}$ form an IR of the Galilei algebra in the canonical realization (12.15) for any fixed values of m and s. The operators (26.12) are Hermitian with respect to the scalar product (26.6) where $\lambda_1=\kappa_1$, $\lambda_2\to\infty$, $l_0\le s\le|l_1|$-1.

Let us formulate the above results in the form of the following assertion [150]:

THEOREM 26.1. The Hilbert space of the IR $D^{\varepsilon}(\kappa,l_0,l_1)$ of the algebra *AP(1,4)* is decomposed into the direct integral of subspaces corresponding to the IRs of the algebra A*G(1,3)* labelled by the eigenvalues (26.13) of the Casimir operators. The explicit form of the generators of the group *P(1,4)* in the Galilei basis and of the transition operator connecting the canonical and *G(1,3)*-bases are given in formulae (26.12), (26.8), (26.10). ■

26.3. The Representations of Classes *II-IV*

Let us show that these representations can be transformed into the Galilei basis also and find the corresponding realizations of basis elements of the algebra *AP(1,4)* in the explicit form.

First we consider the IRs corresponding to $P_nP^n=0$ and belonging to Classes *II* or *III*. The corresponding basis elements can be chosen in the form (24.41). So we have in the basis (26.1)

$$\begin{aligned}
&\hat{P}_0=\frac{1}{2}(\varepsilon p-p_4),\quad M=\varepsilon p+p_4,\quad \hat{P}_a=p_a,\quad J_a=\varepsilon_{abc}x_bp_c+\Sigma_a,\\
&G_a^+=(x_4+x_0)p_a-Mx_a+\frac{1}{p^2}(\varepsilon_{abc}p_b(T_c-\varepsilon p\Sigma_c)+\Sigma_a(p^2+\varepsilon pp_4)),\\
&K=x_0p_4-\varepsilon px_4+\frac{1}{p^2}(T_ap_a-\varepsilon p\Sigma_ap_a),\\
&G_a^-=(x_0-x_4)p_a-2\hat{P}_0x_a+\frac{1}{p^2}(\varepsilon_{abc}p_b(T_c-\varepsilon p\Sigma_c)+\Sigma_a(\varepsilon pp_4-p^2)).
\end{aligned} \tag{26.14}$$

To transform (26.14) to the Galilei basis we use the operator

$$V=\exp\left[-i\frac{\boldsymbol{T}\cdot\boldsymbol{p}}{p(p+\varepsilon p_4)}+i\frac{\Sigma\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\left(\arctan\frac{|\boldsymbol{p}|}{p_4}+\frac{1}{2}(1-\varepsilon)\pi\right)\right]. \tag{26.15}$$

As a result of the transformation (26.9), (26.15) and change of variables $(p_1,p_2,p_3,p_4)\to(p_1,p_2,p_3,m)$, $m=p+\varepsilon p_4$ we obtain

$$\begin{aligned}
&\hat{P}_0=\varepsilon\frac{\boldsymbol{p}^2}{2m},\quad M=\varepsilon m,\quad \hat{P}_a=p_a,\\
&J_a=\varepsilon_{abc}(x_bp_c+\Sigma_a),\quad G_a^+=x_0p_a-Mx_a-i\varepsilon m\frac{\partial}{\partial p_a},\\
&K=x_0\left(\frac{\varepsilon m}{2}-\hat{P}_0\right)-im\frac{\partial}{\partial m}+\frac{(m^2-\boldsymbol{p}^2)\boldsymbol{T}\cdot\boldsymbol{p}}{\boldsymbol{p}^2(m^2+\boldsymbol{p}^2)}.
\end{aligned} \tag{26.16}$$

The realization (26.16) is defined in the Galilei basis $|\boldsymbol{p},m_0,s,m,s_3;r,\lambda\rangle$ in which the Casimir operators of the subalgebra *AG(1,3)* are diagonal. Indeed according to (26.4), (26.16) we have

$$\hat{C}_1\equiv0,\quad \hat{C}_2=m^2\Sigma_a\Sigma_a,\quad \hat{C}_3=\varepsilon m,$$

and thus the eigenvalues of these operators are

$$c_1=m_0=0, \quad c_2=m^2 s(s+1), \quad c_3=m,$$
$$0\leq m<\infty, \qquad s=\lambda,\lambda+1,\ldots, \quad r^2\neq 0. \tag{26.17}$$

If a representation of the algebra *AP(1,4)* is characterized by the zero eigenvalue of C_2 (i.e., belonging to Class *II*) then $r^2=0$, $T_a\equiv 0$, and the form of the generators (26.16) is simplified essentially.

The operators (26.16) are Hermitian with respect to the scalar product (26.6) where $\lambda_1=0$, $\lambda_2\to\infty$. Thus either the sum with respect to s reduces to the single term (for $c_2\equiv 0$) or s runs over all the values given in (26.17).

We see that the IRs of Classes *II* and *III* can be defined in the Galilei basis also. In comparison with the representation of Class *I* there are two specific features: the possibility of zero eigenvalues of the mass operator and the infinite number of spin states appearing by the reduction *P(1,4)* $\to$ *G(1,3)*.

In conclusion we present the explicit form of the basis elements of an IR of the algebra *AP(1,4)*, which belong to Class *IV*, in the Galilei basis. This form is given in (26.12) and (26.18):

$$K'=x_0\left(\frac{\varepsilon m}{2}-\hat{P}_0\right)-im\frac{\partial}{\partial m},$$
$$(G_a^-)'=2\left[i\left(\varepsilon p_a\frac{\partial}{\partial m}-\hat{P}_0\frac{\partial}{\partial p_a}\right)-\frac{S_{ab}p_b-kS_{0a}}{\varepsilon m}\right] \tag{26.18}$$

where $S_{\mu\nu}$ are generators of an IR of the Lorentz group, $m_0=-k^2/(2\varepsilon m)$, $-\eta^2<m<0$, $0<m<\infty$.

26.4. Covariant Representations

One of the most interesting problems appearing with the reduction *P(1,4)* $\to$ *G(1,3)* is the transformation to the Galilean basis of the covariant representations, which are characterized by the following form of the basis elements

$$P_n=p_n, \quad J_{mn}=x_m p_n-x_n p_m+S_{mn}, \tag{26.19}$$

where S_{mn} are matrices realizing a representation of the algebra *AO(1,4)*; x_m and p_n are the canonically conjugated variables satisfying the relations $[p_m, x_n]=iq_{mn}$. We will not concretize the realization of p_m and x_n so the further results are available both for the x- and for p-representations.

The operators (26.19) belong to the class M_1, and therefore generate the local finite transformations of the group *P(1,4)*. The covariant representations of the algebra *AP(1,4)* are used for the description of *P(1,4)*-invariant wave equations, see Section 27.

We restrict ourselves to the case when the spectrum of the Casimir operator

$C_1=P_nP^n$ is positive. Such a situation is typical for the cases for which the space of the representation (26.19) is defined as a completion of a set of the solutions of the *P(1,4)*-invariant wave equation. If we go over to the basis (26.1) then we obtain according to (26.19)

$$\hat{P}_0=\frac{1}{2}(p_0-p_4),\quad M=p_0+p_4,\quad \hat{P}_a=p_a, \tag{26.20a}$$

$$J_a=\varepsilon_{abc}x_bp_c+S_a,\quad G_a^+=\tilde{x}_0p_a-x_aM+\lambda_a^+,$$

$$K=\tilde{x}_4M-\tilde{x}_0\hat{P}_0+S_{04},\quad G_a^-=2\tilde{x}_4p_a-2x_a\hat{P}_0+\lambda_a^-, \tag{26.20b}$$

where

$$S_a=\frac{1}{2}\varepsilon_{abc}S_{bc},\quad \lambda^\pm=S_{0a}\pm S_{4a},\quad \tilde{x}_0=x_0-x_4,\quad \tilde{x}_4=\frac{1}{2}(x_0+x_4). \tag{25.20c}$$

It is not difficult to make sure that the Casimir operators of the Galilei subalgebra (formed by the operators (26.20a)) are not diagonal. To transform the representation (26.19) to the Galilean basis we use the operator

$$V=\exp\left(i\frac{\lambda_a^+p_a}{M}\right). \tag{26.21}$$

After the transformation (26.9) (where $U_1\rightarrow V$, V is the operator (26.21)) we come to the realization

$$\begin{gathered}\hat{P}'_0=\frac{1}{2}(p_0-p_4),\quad M'=p_0+p_4,\quad \hat{P}'_a=p_a,\\ J'_a=\varepsilon_{abc}x_bp_c+S_a,\quad (G_a^+)'=\tilde{x}_0p_a-x_aM,\\ K'=\tilde{x}_4M-\tilde{x}_0\hat{P}'_0+S_{04},\\ (G_a^-)'=2\tilde{x}_4p_a-2x_a\hat{P}'_0+2\lambda_a^--\frac{2}{M}(S_{ab}p_b+S_{40}p_a)-4\lambda_a^+\frac{\hat{P}_0}{M}+2\lambda_a^+\frac{\boldsymbol{p}^2}{M^2}.\end{gathered} \tag{26.22}$$

Besides that, the corresponding Casimir operators (26.4) take the form

$$\hat{C}_1=p_\mu p^\mu=2M\hat{P}_0-\boldsymbol{p}^2,\quad \hat{C}_2=M^2S_aS_a,\quad \hat{C}_3=M. \tag{26.23}$$

According to (26.23) the eigenvalues of $\hat{C}_1$ coincide with the eigenvalues of the operator $C_1=P_nP^n$ (which are positive by definition), the eigenvalues of $\hat{C}_2$ coincide with the ones of the matrix $\boldsymbol{S}^2$ multiplied by M^2 and the eigenvalues of $\hat{C}_3$ lie in the interval $(c_1)^{1/2}\leq c_3<\infty$.

We see that a covariant representation of the algebra *AP(1,4)* can be transformed to the form (26.22) for which the Casimir operators of the subalgebra *AG(1,3)* turn out to be diagonal. The operator (26.21) can be used for a diagonalization of the wave equations being invariant under the group *P(1,4)*, see Subsection 26.2.

26.5. The *E(4)*-Basis

Besides the subalgebras *AP(1,3)* and A*G(1,3)*, the Lie algebra of Euclidean group in the four-dimensional space is very interesting from the point of view of physics. This group (denoted by *E(4)*) plays an important role in the quantum field theory and quantum statistics.

In this section we consider the reduction of an IR of the algebra *AP(1,4)* by the algebra *AE(4)*. We will see such a reduction can be made in a relatively simple way without using any transformations of the kind (26.9).

The subalgebra *AE(4)* is formed by ten generators P_k and J_{kl} (k,l=1,2,3,4) which satisfy the following commutation relations according to (24.1):

$$\begin{gathered}[P_k, P_l]=0, \quad [P_k, J_{nl}]=i(\delta_{kl}P_n-\delta_{kn}P_l),\\ \left[J_{kl}, J_{k'l'}\right]=i(\delta_{kk'}J_{ll'}+\delta_{ll'}J_{kk'}-\delta_{kl'}J_{lk'}-\delta_{lk'}J_{kl'}).\end{gathered} \tag{26.24}$$

This algebra has the two Casimir operators

$$\hat{C}_1=P_1^2+P_2^2+P_3^2+P_4^2, \quad \hat{C}_2=W_1^2+W_2^2+W_3^2+W_4^2, \tag{26.25}$$

where $W_k=\varepsilon_{klnm}P_lJ_{mn}$. Substituting (24.31) into (26.25) we obtain

$$\hat{C}_1=p_1^2+p_2^2+p_3^2+p_4^2=p^2, \quad \hat{C}_2=p^2\Sigma_a\Sigma_a, \tag{26.26}$$

where Σ_a are matrices which realize a representation of the algebra *AO(3)* (24.24). We can see the realization (24.31) of the algebra *AP(1,4)* corresponds to the diagonal Casimir operators of the subalgebra *AE(4)* since the matrix $\Sigma_a\Sigma_a$ can be chosen diagonal without loss of generality.

Let us denote eigenvectors of the complete set of the commuting operators P_1, P_2, P_3, P_4, $\hat{C}_1$, $\hat{C}_2$, Σ_3 and the Casimir operators of the algebra *AP(1,4)* by $|\tilde{p},s,s_3;c\rangle$. Thus

$$\begin{gathered}\hat{C}_1|\tilde{p},s,s_3;c\rangle=p^2|\tilde{p},s,s_3;c\rangle, \quad \hat{C}_2|\tilde{p},s,s_3;c\rangle=p^2s(s+1)|\tilde{p},s,s_3;c\rangle,\\ \Sigma_3|\tilde{p},s,s_3;c\rangle=s_3|\tilde{p},s,s_3;c\rangle, \quad P_k|\tilde{p},s,s_3;c\rangle=p_k|\tilde{p},s,s_3;c\rangle,\\ C_\alpha|\tilde{p},s,s_3;c\rangle=c_\alpha|\tilde{p},s,s_3;c\rangle,\end{gathered} \tag{26.27}$$

where $s_3=-s,-s+1,\dots,s$; s are positive integers or half-integers, c_α are eigenvalues of the Casimir operators of the algebra *AP(1,4)*, $\tilde{p}=(p_1,p_2,p_3,p_4)$, $c=(c_1,c_2\dots)$.

We impose on $|\tilde{p},s,s_3;c>$ the normalization condition

$$\langle\tilde{p},s,s_3;c|\tilde{p}',s',s_3';c\rangle=M_c\delta^4(\tilde{p}-\tilde{p}')\delta_{ss'}\delta_{s_3s_3'},$$

where $M_c=2P_0$, if $c_1\geq0$ and $M_c=2P_4$, if $c_1<0$. The vectors $|\tilde{p},s,s_3;c>$ form the orthonormalized basis called below *E(4)*-basis.

The explicit form of the generators of the group *P(1,4)* in the *E(4)*-basis is given by formulae (24.37), (24.41) and (24.46) for $c_1>0$, $c_1=0$ and $c_1<0$ respectively. Besides eigenvalues of the Casimir operators of the subalgebra *AE(4)* are infinitely degenerated and given by formulae (26.27) where $0\leq p^2<\infty$ and the values of s coincide with the numbers labelling IRs of the group *O(3)* which arise by the reduction $O(4)\rightarrow O(3)$, $E(3)\rightarrow O(3)$ and $O(1,3)\rightarrow O(3)$ for $c_1>0$, $c_1=0$ and $c_1<0$ correspondingly.

26.6 Representations of the Poincaré Algebra in the *G(1,2)*-Basis

Let us now consider the IRs of the Poincaré algebra and solve the problem of a transformation of these representations to such a basis where the Casimir operators of the subalgebra *AG(1,2)* (i.e., the Lie algebra of the Galilei group in the space of two spatial dimensions) are diagonal. Such a basis turns out to be very important for different applications. It is this basis which is used implicitly in the null-plane formalism [275].

A transformation of an IR of Class *I* ($P_\mu P^\mu=\kappa^2>0$) to the *G(1,2)*-basis can be made in a complete analogy with the transformation of one for the algebra *AP(1,4)* considered in Subsection 25.2. We will start from the canonical realization (4.50) of this IR. The basis elements of the subalgebra *AG(1,2)* can be expressed via P_μ, $J_{\mu\nu}$ with the help of the following relations

$$\hat{P}_0=\frac{1}{2}(P_0-P_3),\quad M=P_0+P_3,\quad \hat{P}_\alpha=p_\alpha,$$
$$\hat{J}_3=J_{12},\quad G_\alpha^+=J_{0\alpha}+J_{3\alpha},\quad \alpha=1,2. \tag{26.28}$$

We choose the remaining basis elements of the algebra *AP(1,3)* in the form

$$G_\alpha^-=J_{0\alpha}-J_{3\alpha},\quad K=J_{03}. \tag{26.29}$$

The operators (26.28), (26.29), (4.50) realize an IR of the Poincaré algebra in the canonical Shirokov-Foldy basis. To come to the *G(1,2)*-basis we use the transformation (26.9) where

$$U=\exp\left[\frac{2iS_{3\alpha}p_\alpha}{|p|_2}\left(\arctan\frac{|p|_2}{|p_0|+\varepsilon p_3+\kappa}+\frac{(1-\varepsilon)\pi}{4}\right)\right],$$

and make the change of the variables $(p_1,p_2,p_3)\rightarrow(p_1,p_2,m)$, $m=\varepsilon p_3+(p_1^2+p_2^2+\kappa^2)$. As a result we obtain the realization of the algebra *AP(1,4)* in a basis where the Casimir operators of the subalgebra *AG(1,2)* are diagonal, since

$$\hat{P}'_0=\frac{\varepsilon}{2m}(\kappa^2+|p|_2^2),\quad \hat{P}'_\alpha=p_\alpha,\quad |p|_2=(p_1^2+p_2^2)^{1/2},$$

$$J'_3=i\left(p_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial p_2}\right)+S_{12},\quad M'=\varepsilon m, \tag{26.30}$$

$$(G_\alpha^+)'=x_0 p_\alpha-i\varepsilon m\frac{\partial}{\partial p_\alpha},\quad \kappa\leq m<\infty,$$

$$K'=x_0\left(\frac{\varepsilon m}{2}-\hat{P}'_0\right)-im\frac{\partial}{\partial m},$$
$$(G_\alpha^-)'=-2\left[i\left(\varepsilon p_\alpha\frac{\partial}{\partial m}+\hat{P}'_0\frac{\partial}{\partial p_\alpha}\right)+\varepsilon\frac{S_{\alpha\beta}p_\beta+S_{3\alpha}\kappa}{m}\right], \tag{26.31}$$

$$\hat{C}'_1\equiv 2M(\hat{P}'_0-\hat{P}'^2_1-\hat{P}'^2_2)=\varepsilon\kappa^2,\quad \hat{C}'_3\equiv M'=\varepsilon m,$$
$$\hat{C}'_2\equiv(J'_{12}M'-\hat{P}'_1(G_2^+)'+\hat{P}'_2(G_1^+)')^2=m^2S_{12}^2. \tag{26.32}$$

We do not present exact calculations which are analogous to (26.7)-(26.13).

The operators (26.30) coincide with the generators of the kinematical group used in the null-plane formalism (see, e.g., [257]). So we have found the explicit connection of these generators with the Poincaré group generators in the Shirokov-Foldy representation.

Using the results given in Subsections 26.3, 26.4 it is not difficult to transform the IRs of Classes *II-IV* to the *G(1,2)*-basis. We do not present the correspondent calculations here but consider the class of covariant representations which is the most interesting from the point of view of physical applications.

Thus we start from the realization (2.22) of the algebra *AP(1,3)* where $S_{\mu\nu}$ are matrices realizing an arbitrary representation of the algebra *AO(1,3)*. The transformation operator for the case $P_\mu P^\mu>0$ has the form (compare with (26.21))

$$V=\exp\left[i\frac{(S_{0\alpha}+S_{3\alpha})p_\alpha}{M}\right],\quad M=p_0+p_3. \tag{26.33}$$

As a result of the transformation (26.9), (26.33) the operators (26.28), (26.29), (2.22) reduce to the form

$$\hat{P}_0=\frac{1}{2}(p_0-p_3),\quad M=p_0+p_3,\quad \hat{P}_\alpha=p_\alpha,$$

$$J_3=i\left(p_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial p_2}\right)+S_{12},\quad G_\alpha^+=\tilde{x}_0 p_\alpha-x_\alpha M+\lambda_\alpha^+, \tag{26.34}$$

$$K=\tilde{x}_3 M-\tilde{x}_0\hat{P}_0+S_{03},$$

$$G_\alpha^-=2\left(\tilde{x}_3 p_\alpha-x_\alpha \hat{P}_0^{/}+\frac{1}{2}\lambda_\alpha^- -\frac{S_{\alpha\beta}p_\beta+S_{30}p_\alpha}{M}-\lambda_\alpha^+\frac{2\hat{P}_0}{M}+\lambda_\alpha^+\frac{p_1^2+p_2^2}{M^2}\right), \tag{26.35}$$

where $\lambda_\alpha^\pm=S_{0\alpha}\pm S_{3\alpha}$,

It is not difficult to make sure the Casimir operators of the subalgebra *AG(1,2)* being a linear span of the operators (26.34) are diagonal.

The operator (26.33) can be used by solving of different problems in the null-plane formalism. Namely using this operator we can diagonalize the systems of the Poincaré-invariant motion equations for the particles of arbitrary spins interacting with the same types of external fields (e.g., the field of a plane wave, the homogeneous magnetic field and others). More precisely the corresponding transformation operator can be obtained from (26.33) by the change $p_\mu \rightarrow p_\mu - eA_\mu$ where A_μ is a vector-potential of an external field. Examples of the equations admitting such a diagonalization are present in Section 28.

27. WAVE EQUATIONS INVARIANT UNDER GENERALIZED POINCARÉ GROUPS

27.1. Preliminary Notes

In this section we discuss the wave equations having the symmetry under the generalized Poincaré group (mainly under the group *AP(1,4)*). Such equations are of great interest for physics inasmuch as they are invariant under both the Poincaré and the Galilei groups and can be interpreted as motion equations of a relativistic (or Galilean) particle with variable mass [136,333].

Until now the theory of *P(1,4)*-invariant equations is far from the complete. Strictly speaking, only certain classes of such equations were completely described (see, e.g., [136,115]). So we restrict ourselves by considering of the simplest (and the most important!) equations of the Dirac, Kemmer-Duffin type and some others.

Let us consider a system of first order partial differential equations:

$$L\Psi(x)\equiv(\Gamma_m P^{\,m}-\kappa)\Psi(x)=0 \tag{27.1}$$

where $x=(x_0, x_1, ..., x_n)$, $p^m=i\partial/\partial x_m$, Γ_m are numeric matrices. The equation (27.1) by definition is invariant under the algebra *AP(1,n)* if the operator L satisfies the relations

$$[L,P_m]=[L,J_{mk}]=0 \tag{27.2}$$

where P_m, J_{mk} are the generators of the group *P(1,4)* in the covariant realization (26.19).

Substituting (26.19), (27.1) into (27.2) we come to the following equations for Γ_m:

$$[\Gamma_m,S_{ln}]=i(g_{ml}\Gamma_n-g_{mn}\Gamma_l). \tag{27.3}$$

Here S_{mn} are matrices which realize a representation of the algebra *AO(1,n)*, $g_{mn}=diag(1,-1,-1, ...)$.

Thus the problem of the description of the *P(1,n)*-invariant equations of the kind (27.1) reduces to the solution of the equations (27.3). We will not search for a general solution of these relations restricting ourselves by considering of some particular (but important) examples.

27.2. The Generalized Dirac Equation

In the case n=4 the simplest (i.e., realized by matrices of minimal dimension) solution of (27.3) has the form

$$\Gamma_m=\gamma_m, \quad S_{ln}=\frac{i}{4}[\gamma_l, \gamma_n], \tag{27.4}$$

where γ_m are the Dirac matrices of dimension 4×4, satisfying the Clifford algebra (2.3). Substituting (27.4) into (27.1) we come to the generalized Dirac equation in the (1+4)-dimensional Minkowsky space:

$$(\gamma_m p^m-\kappa)\Psi(x)=0, \quad m=0,1,2,3,4. \tag{27.5}$$

This equation has a manifest symmetry under the algebra *AP(1,4)*. But in contrast to the corresponding equation (2.1) the generalized Dirac equation turns out not to be invariant under the transformation $x_a\rightarrow -x_a$, a=1,2,3 [135]. An equation being invariant under the complete (i,e. including all possible reflections of independent variables) group $\tilde{P}(1,4)$ can be obtained from (27.5) by doubling a number of the components of the wave function and by making the change [136]

$$\gamma_m\rightarrow\Gamma_m=\begin{pmatrix}\gamma_m & 0\\ 0 & \gamma_m\end{pmatrix}. \tag{27.6}$$

Let us discuss symmetries of the equation (27.5) and its possible interpretations.

In a complete analogy with Section 2 it is possible to show the algebra *AP(1,4)* is the maximal IA of the equation (27.5) in the class M_1. To answer the question what kind of representations of this algebra is realized on the set of solutions of (27.5) we use the fact that the corresponding generators (26.19) can be represented in the form

$$
\begin{aligned}
&P_0\Psi=H\Psi, \quad H=\gamma_0\gamma_k p_k+\gamma_0 m, \quad P_k\Psi=p_k\Psi,\\
&J_{kl}\Psi=(x_k p_l-x_l p_k+S_{kl})\Psi, \\
&J_{0k}\Psi=\left(x_0 p_k-\frac{1}{2}[x_k, H]_+\right)\Psi,
\end{aligned}
\tag{27.7}
$$

where Ψ is a solution of (27.5), k,l=1,2,3,4. The Casimir operators (24.5), (24.36) reduce to the following form in accordance with (27.7):

$$
\begin{aligned}
&C_1=P_\mu P^\mu=\kappa^2,\\
&C_2=\frac{1}{2}W_{mn}W^{mn}=-\frac{1}{2}\kappa^2 S_{kl}S_{kl},\\
&C_3=-\frac{1}{4}J_{mn}W^{mn}=\frac{1}{4}\kappa\varepsilon_{klk'l'}S_{kl}S_{k'l'},\\
&C_4=H/E, \quad E=\sqrt{p_1^2+p_2^2+p_3^2+p_4^2+\kappa^2}\,.
\end{aligned}
\tag{27.8}
$$

The eigenvalues of C_4 are equal to ±1. In order to determine the eigenvalues of C_2 and C_3 we choose the realization of γ-matrices given by the relations (2.4), (2.17). Then $S_{kl}S_{kl}=\gamma_0\varepsilon_{klmn}S_{kl}S_{mn}/4=3$, and $C_2\Psi=-2\kappa C_3\Psi=3\kappa^2\Psi$, from which it follows that the operators (27.7) realize the representation $D^+_\kappa(1/2\ 0)\oplus D^-_\kappa(0\ 1/2)$ of the algebra $AP(1,4)$.

Since the algebra $AP(1,4)$ includes the subalgebras $AP(1,3)$ and $AG(1,3)$ the generalized Dirac equation (27.5) turns out to be invariant under the Poincaré and Galilei groups. Let us demonstrate the connection of this equation with the Dirac equation for a particle with a variable mass. Representing $\Psi(x)$ in the form

$$
\Psi(x)=\int\exp(ip_4x_4)\Psi_{p_4}(x_0,\boldsymbol{x})dp_4,
\tag{27.9}
$$

we obtain from (27.5) the following equation

$$
[\gamma_0 p_0-\gamma\cdot\boldsymbol{p}-(\kappa+\gamma_4 p_4)]\Psi_{p_4}=0.
$$

Multiplying this equation by $(\kappa-\gamma_4 p_4)m^{-1}$ where $m=(\kappa^2+p_4^2)^{1/2}$ we obtain

$$
(\gamma'_\mu p^\mu-m)\Psi_{p_4}=0,
$$

where γ'_μ are matrices satisfying the Clifford algebra (2.3) moreover $\gamma'_\mu=(\kappa-\gamma_4 p_4)m^{-1}\gamma_\mu$.

We see that for any fixed value of p_4 the equation (27.5) is equivalent to the Dirac equation for a particle of mass $m=(\kappa^2+p_4^2)^{1/2}$. Therefore it is possible to interpret (27.5) as a motion equation for a relativistic particle with a variable mass.

The other possibility of an interpretation of the equation (27.5) is connected with its symmetry under the Galilei group. Using the new variables (26.20c) we can rewrite this equation in the form

$$\left(\tilde\beta_0 i\frac{\partial}{\partial\tilde x_0}-\tilde\beta_a p_a+2i\tilde\beta_5\frac{\partial}{\partial\tilde x_4}-\kappa\right)\Psi(\tilde x_0,\boldsymbol{x},\tilde x_4)=0 \tag{27.10}$$

where

$$\tilde\beta_0=\gamma_0+\gamma_4,\quad \tilde\beta_5=\gamma_0-\gamma_4,\quad \tilde\beta_a=\gamma_a. \tag{27.11}$$

The equation (27.10) is invariant under the Galilei transformations of the variables $(\tilde x_0,\boldsymbol{x})$ (see Subsection 11.4). If we impose on Ψ the following Galilei-invariant additional condition

$$i\frac{\partial}{\partial\tilde x_0}\Psi(\tilde x_0,\boldsymbol{x},\tilde x_4)=m\Psi(\tilde x_0,\boldsymbol{x},\tilde x_4), \tag{27.12}$$

then for any fixed value of m the equation (27.10) reduces to the form

$$L\Psi\equiv\left(\tilde\beta_0 i\frac{\partial}{\partial\tilde x_0}-\tilde\beta_a p_a+2\tilde\beta_5 m-\kappa\right)\Psi=0. \tag{27.13}$$

In the case $\kappa=0$ formulae (27.11), (27.13) define the Levi-Leblond [275] equation for a Galilean particle of spin 1/2.

We note that using the operator (26.21), we can transform (27.13) to the canonical diagonal form because

$$VLV^{-1}=\tilde\beta_0\left(i\frac{\partial}{\partial\tilde x_0}-\frac{\boldsymbol{p}^2}{2m}\right)+2\tilde\beta_5 m-\kappa. \tag{27.14}$$

According to (27.14) the function $\Phi_+=(1/2)(1+\gamma_0\gamma_4)V\Psi$ satisfies the Schrödinger equation

$$\left(i\frac{\partial}{\partial\tilde x_0}-\frac{\boldsymbol{p}^2}{2m}-\frac{\kappa^2}{2m}\right)\Phi_+=0$$

and the function $\Phi_-=(1/2)(1-\gamma_0\gamma_4)V\Psi$ is expressed via Φ_+: $\Phi_-=\gamma_0(\kappa/2m)\Phi_+$.

We see the generalized Dirac equation (27.5) can serve as a base of a description of a relativistic particle with a variable mass or of a Galilean particle. Such an interpretation is admissible also for the generalized Dirac equation which includes potentials of an external field. For example replacing p_k by p_k-eA_k in the equation (27.5), where $A_k=A_k(n_l x^l)$ ($n_0=n_4=1$, $n_1=n_2=n_3=0$) is a plane wave potential, we come to the equation preserving the symmetry under Galilei transformations. To solve such an equation it is convenient to use the variables (26.20c) and to decompose the wave function by the complete set of the functions satisfying (27.12). However if $A_k=A_k(x_4)$ then the corresponding equation can be used as a model of relativistic particles which possess the mass spectrum.

The Dirac equation in (1+4)-dimensional space is considered in more detail in [115,136].

In conclusion we have discussed briefly the Dirac-type equations being invariant under the groups *P(1,n)*. By this are meant the equations of the form (27.5) where the summation over m is extended to the case when $0 \le m \le n$, and γ_m are matrices realizing a representation of the Clifford algebra of dimension $n+1$.

The *P(1,n)*-invariant equations (27.5) exist for any $n>0$. Minimal dimension of the matrices γ_m is equal to $2^{[(n+1)/2]} \times 2^{[(n+1)/2]}$ for any given n where $[a]$ is the entire part of a. In the case of half integer n the corresponding equations are invariant under the complete group $\tilde{P}(1,n)$ including all the possible reflections of coordinates. However if n is integer then it is necessary to double the number of the components of the wave function in order to satisfy the requirement of *P*-invariance [115,136].

The Dirac equations being invariant under the groups *P(1,n)*, $n=5,6,\ldots$ can also serve as a mathematical models of the particles with a variable mass. Furthermore, these equations can be used to describe many-particle systems (see [136] and Subsection 32.5 of the present book).

26.3. The Generalized Kemmer-Duffin-Petiau Equation

The KDP equation also admits a direct generalization to the case of (1+4)-dimensional Minkowsky space since the equations (27.4) are obviously satisfied by the matrices

$$\Gamma_\mu = \beta_\mu, \quad \Gamma_4 = \beta_4, \quad S_{mn} = i[\beta_m, \beta_n]. \tag{27.15}$$

Here β_μ, β_4 are the 10×10 KDP matrices which can be chosen, e.g., in the form (6.22), (6.24).

The equation (27.3), (27.15) is invariant under the algebra *AP(1,4)* and thus under the Poincaré and Galilei algebras. Like the five-dimensional analog of the Dirac equation (27.5), this equation is non-invariant under the complete group $\tilde{P}(1,4)$ including space-time reflections.

The generalized KDP equation can be interpreted as an equation of motion of a relativistic particle with a variable mass and spin $s=1$. To make such an interpretation more clear we represent solutions of this equation in the form (27.9). In this way we come to the following equation

$$[\beta_0 p_0 - \beta_a p_a - \kappa - \beta_4 p_4]\Psi_{p_4} = 0. \tag{27.16}$$

For any fixed value of p_4 the matrix $\kappa+\beta_4 p_4$ is invertible; moreover

$$M = m(\kappa + \beta_4 p_4)^{-1} = \frac{m}{\kappa} + \frac{p_4^2}{\kappa m}\beta_4^2 - \frac{p_4}{m}\beta_4,$$

where $m=(\kappa^2+p^2)^{1/2}$. Multiplying (27.16) by M we come to the equivalent equation

$$(\beta'_\mu p^\mu - m)\Psi_{p_4} = 0, \tag{27.17}$$

where $\beta'_\mu = M\beta_\mu$ are new matrices satisfying the KDP algebra (6.20). The equation (27.17) is the KDP equation for a spin-one particle and the mass $m=(\kappa^2+p_4^2)^{1/2}$.

The KDP equation in (1+4)-dimensional Minkowsky space can be interpreted as a motion equation of a Galilean particle of the spin 1 and a variable mass. To make such an interpretation it is convenient to come to the new variables (26.20c) and to impose the Galilei-invariant additional condition (27.12) on Ψ. As a result we come to the equation (27.13) where

$$\tilde\beta_0 = \beta_0 + \beta_4, \quad \tilde\beta_5 = \beta_0 - \beta_4, \quad \tilde\beta_a = \beta_a. \tag{27.18}$$

For any fixed value of m the equation (27.13), (27.18) coincides with the Galilei-invariant equation for a particle of spin 1, see Subsection 13.3.

Thus the generalized KDP equation can be used for the description of relativistic and Galilean particles with variable masses. Besides, the corresponding Galilei-invariant model of a spin-one particle takes into account the spin-orbit coupling in the frames of the minimal interaction principle, see Subsection 13.4.

We note that the KDP equation admits a generalization to the case of the Minkowsky space of arbitrary dimension $1+n$. The corresponding representations of the β-matrices were described in [265].

The almost evident generalizations to the case of an $(1+n)$-dimensional Minkowsky space is admitted by the Bhabha equation also, i.e., by the equations of the form (6.2) where $\beta_5=1$, and $\beta_\mu=S_{5\mu}$ are matrices realizing a representation of the algebra *AO(1,5)* together with $S_{\mu\nu}=\mathrm{i}[\beta_\mu,\beta_\nu]$. The corresponding equation being invariant under the group *P(1,4)* has the form

$$(S_{5\mu} p^\mu - S_{54} p_4 + \kappa)\Psi = 0. \tag{27.19}$$

This formula defines generalized Bhabha equations in the (1+4)-dimensional space, in particular these equations include the Dirac and KDP ones. In the case when S_{mn} realize an arbitrary representation of the algebra *AO(1,5)* the equations (27.19) can be interpreted as mathematical models of relativistic or Galilean particles with variable masses like the generalized Dirac and KDP equations.

27.4. Covariant Systems of Equations

One of possible formulations of wave equations invariant under the group *P(1,4)* presupposes using the covariant systems of the form [11, 395]

$$p_n\Psi(x) = P_n\Psi(x), \quad P_n = L_{nm} p^m + \kappa L_n, \tag{27.20}$$

where L_n and $L_{mn} = -L_{nm}$ are numerical matrices.

We have already seen in Subsection 2.2 that the Dirac equation also can be reduced to the form (27.20), refer to (2.12). In contrast to the standard formulation (2.1) none of the equations (2.12) is invariant under Lorentz transformations but is transformed into a linear combination of these equations with the different values of n.

Covariant systems of equations in (1+4)-dimensional Minkowsky space are discussed in papers [11,395]. Here we consider a class of such equations corresponding to the choice $L_{mn} = S_{mn}/id$, $L_n = S_{5n}/id$ where S_{mn} are matrices belonging to the algebra $AO(1,5)$. In other words we consider the following systems of the covariant equations

$$L_n \Psi(x) = 0, \quad L_n = p_n - \frac{1}{id}(S_{nm} p^m + \kappa S_{n5}). \tag{27.21}$$

These equations have a manifestly covariant form since the operators L_n evidently satisfy the relations

$$[P_m, L_n] = 0, \quad [J_{mn}, L_l] = i(g_{nl} L_m - g_{ml} L_n),$$

where J_{mn}, P_m are the operators of (26.19).

Let us require that $\Psi(x)$ should satisfy the KGF equation componentwise. Multiplying (27.21) by p^m and summing up over m we obtain the following additional condition for $\Psi(x)$:

$$\left(\frac{1}{id} S_{n5} p^n - \kappa\right)\Psi(x) = 0. \tag{27.22}$$

It is possible to show [11] that this equation is a necessary condition of a consistency of the system (27.21) if the matrices S_{mn} are finite-dimensional.

Thus let S_{mn} be finite-dimensional matrices realizing an IR of the algebra $AO(1,4)$ and (27.21) be the system of the covariant equations corresponding to this IR. It turns out that this system is consistent only in the following exceptional cases [352]:

$$\begin{aligned} &a) \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \qquad S_{5\mu} = \gamma_\mu, \qquad \mu,\nu = 0,1,2,3,4, \\ &b),\ c) \quad S_{\mu\nu} = i[\beta_\mu, \beta_\nu], \qquad S_{5\mu} = \beta_\mu, \end{aligned} \tag{27.23}$$

where γ_n are 4×4 Dirac matrices, β_n are KDP matrices of dimension 10×10 (case b)) or 6×6 (case c)).

In case a) it is necessary to set d=1/2 (otherwise the system (27.21) is inconsistent). This leads to the system which is equivalent to the generalized Dirac equation (see Subsection 27.2). In cases b) and c) the system (27.21) possesses nontrivial solutions when d=1 only. In case b) the corresponding system turns out to be equivalent to the generalized KDP equation for particles of spin 1 (see Subsection 26.3). In the case c) we have the covariant system of equations describing particles of spin 0.

However if the matrices S_{mn} realize an IR of the algebra *AO(1,5)* which is not equivalent to the representations enumerated in (27.23) then the corresponding equations (27.22) are inconsistent. The proof of this statement is given in [352]. We see the class of covariant finite dimensional systems of the form (27.21) is exhausted by the representatives corresponding to the matrices (27.23). The analogous result is correct in respect to the covariant systems of equations in the frames of the Poincaré group [352].

Covariant systems of equations are with better prospects in the case when the corresponding matrices S_{mn} realize an infinite dimensional Hermitian representations of the algebra *AO(1,4)*. A well-known example of such an equation in the (1+3)-dimensional Minkowsky space is the Dirac equation with positive energies [82]*.

The other types of the wave equations invariant under the generalized Poincaré groups have been investigated in papers [115,136,181]. There are the equations of the Bargman-Wigner type, equations with a proper time, equations invariant under the representations of the Classes *II-IV* etc. An analysis of such equations is beyond the scope of the present book.

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*In papers [352] the covariant infinite-component Dirac equation is generalized to the case of particles of arbitrary spin.

6. EXACT SOLUTIONS OF LINEAR AND NONLINEAR EQUATIONS OF MOTION

We obtain exact solutions of Poincaré- and Galilei-invariant equations of motion of a particle of arbitrary spin s interacting with some particular classes of external fields. Besides we present solutions of a number of nonlinear equations of modern theoretical physics. The key to finding these solutions is using symmetries of the equation considered.

28. EXACT SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS FOR PARTICLES OF ARBITRARY SPIN

28.1. Introduction

The main purpose of this and the following sections is the construction of solutions of motion equations of arbitrary spin particles in the external electromagnetic field.

The number of known exact solutions of relativistic wave equations is very small even for the cases s=0 and s=1/2. The most interesting of them (from the physical point of view) are the following problems: a particle in the Coulomb field [73, 77], a particle in the plane-wave field [407], a particle in the constant magnetic field [365] and a particle in the Redmond field [370] (the last is a combination of the constant magnetic and plane-wave fields). Finally we mention the problem of interaction of a charged particle with the magnetic monopole field which also admits exact solutions for s=0, 1/2 [217, 402] and for arbitrary spin [164].

The construction of exact solutions for particles of spin s>1/2 is complicated in accordance with increasing of number of components of the corresponding wave function. Besides there appear additional difficulties, i.e., causality violation [405], the absence of stable solutions in the Coulomb problem [401] and many others. In spite of that a number of exact solutions for the vector particles have been obtained in the papers [46, 47, 266, 267, 270].

The problems enumerated in the above have physically reasonable solutions for arbitrary spin particles if we start from the motion equations (10.10). The most important of these solutions are considered in the following.

28.2. Free Motion of Particles

The knowledge of the explicit form of wave functions of non-interacting particles is necessary for many problems of theoretical physics [41,42]. That is why we present the explicit solutions of (10.10) for the case of absence of an external field (i.e., for $A_\mu=F_{\mu\sigma}=0$).

Instead of the equations (10.10) it is more convenient to consider the equivalent system (10.30) which takes the following form

$$(p_\mu p^\mu - m^2)\psi^{(+)}=0, \tag{28.1a}$$

$$P_s\psi^{(+)}=\psi^{(+)}, \quad P_s=\frac{1}{4s}\left[S_{ab}S_{ab}-2s(s-1)\right], \tag{28.1b}$$

$$\psi^{(-)}=\frac{1}{m}\Gamma_\mu p^\mu \psi^{+}, \quad \psi^{(\pm)}=\frac{1}{2}(1\pm i\Gamma_4)\psi. \tag{28.1c}$$

We will use the representation (9.14) for the matrices Γ_μ and choose $S_{\mu\nu}$ in the following form

$$S_{ab}=\begin{pmatrix}\tilde{S}_{ab} & 0\\ 0 & \tilde{S}_{ab}\end{pmatrix}, \quad S_{0a}=\begin{pmatrix}S'_{0a} & 0\\ 0 & S''_{0a}\end{pmatrix} \tag{28.2}$$

where $\{\tilde{S}_{ab},S'_{0a}\}$ and $\{\tilde{S}_{ab},S''_{0a}\}$ are matrices realizing the representations *D(s 0)*⊕ *D(s-1 0)* and *D(s-1/2 1/2)* of the algebra *AO(1,3)*. These matrices can be written in the form (compare with (4.64))

$$\tilde{S}_{ab}=\varepsilon_{abc}\hat{S}_c=\begin{pmatrix}S_c & 0\\ 0 & S_c\end{pmatrix}, \quad S'_{0a}=-i\hat{S}_a, \quad S''_{0a}=-i\begin{pmatrix}(s-1)S_a & K_a^\dagger\\ K_a & (s+1)S'_a\end{pmatrix} \tag{28.3}$$

where S_a and S'_a are generators of the IRs *D(s)* and *D(s-1)* of the group *O(3)*, $K_a=K_a^s$ are matrices of dimension $(2s-1)\times(2s+1)$ given in (4.66). Then we obtain the following realization for the matrices σ_a of (9.14)

$$\sigma_a=\frac{1}{2}\varepsilon_{abc}\tilde{S}_{bc}-iS''_{0a}=\frac{1}{s}\begin{pmatrix}S_a & -K_a^\dagger\\ K_a & -S'_a\end{pmatrix}. \tag{28.4}$$

Let us find the general solution of the system (28.1) in the representation (9.14), (28.2)-(28.4). It is convenient to represent this solution in the form of the Fourier integral

$$\psi(x_0,\boldsymbol{x})=\int\frac{\partial^3 p}{2E}\left(\tilde{\psi}_+\exp\left[i(\boldsymbol{p}\cdot\boldsymbol{x}-Ex_0)\right]+\tilde{\psi}_-\exp\left[i(\boldsymbol{p}\cdot\boldsymbol{x}+Ex_0)\right]\right), \tag{28.5}$$

where $E=(p^2+m^2)^{1/2}$ and the Fourier transforms $\tilde{\psi}_\pm$ satisfy according (28.1) the following conditions

$$(\varepsilon\Gamma_0 E-\Gamma\cdot\boldsymbol{p})\tilde{\psi}_\varepsilon^{(+)}=m\tilde{\psi}_\varepsilon^{(-)},\quad \tilde{\psi}_\varepsilon^{(\pm)}=\frac{1}{2}(1\pm i\Gamma_4)\tilde{\psi}_\varepsilon, \tag{28.6}$$

$$P_s\tilde{\psi}_\varepsilon^{(+)}=\tilde{\psi}_\varepsilon^{(+)},\quad \varepsilon=\pm 1.$$

In the representation (9.14), (28.2)-(28.4) $\tilde{\psi}_\varepsilon^{(\pm)}$ are columns of the following form

$$\tilde{\psi}_\varepsilon^{(+)}=\begin{pmatrix}\tilde{\Phi}_\varepsilon^{(+)}\\ 0\end{pmatrix},\quad \tilde{\psi}_\varepsilon^{(-)}=\begin{pmatrix}0\\ \tilde{\Phi}_\varepsilon^{(-)}\end{pmatrix} \tag{28.7}$$

where $\tilde{\Phi}_\pm^{(\varepsilon)}$ are $4s$-component spinors, 0 are zero columns having $4s$ rows. From (28.6), (9.14), (28.2)-(28.4) we conclude that

$$\tilde{\Phi}_\varepsilon^{(+)}=\begin{pmatrix}\varphi_s^\varepsilon\\ 0\end{pmatrix},\quad \tilde{\Phi}_\varepsilon^{(-)}=\frac{1}{m}(\varepsilon E+\sigma\cdot\boldsymbol{p})\tilde{\Phi}_\varepsilon^{(+)}=\frac{1}{m}\begin{pmatrix}\left(E+\frac{1}{s}\boldsymbol{S}\cdot\boldsymbol{p}\right)\varphi_\varepsilon^s\\ -\frac{1}{s}\boldsymbol{K}\cdot\boldsymbol{p}\,\varphi_\varepsilon^s\end{pmatrix}. \tag{28.8}$$

Here φ_s^ε is an arbitrary $(2s+1)$-component spinor, 0 is the zero column having $4s$ rows.

Thus the general solution of the system (10.10) in the case $A_\mu=F^{\lambda\sigma}=0$ can be represented in the form (28.5) where

$$\tilde{\psi}_\varepsilon=\tilde{\psi}_{(\varepsilon)}^{+}+\tilde{\psi}_{(\varepsilon)}^{-} \tag{28.9}$$

and $\tilde{\psi}_\varepsilon^{(\pm)}$ are the spinors given in (28.7), (28.8). This solution is defined up to arbitrary functions φ_s^ε each having $2s+1$ components. These functions can be represented in the form

$$\varphi_s^\varepsilon=\sum_{\nu=-s}^{s} b_\nu^\varepsilon(\boldsymbol{p})\frac{m}{\sqrt{2E\left(E+\frac{\nu}{s}p_3\right)}}\eta_\nu^s \tag{28.10}$$

where η_ν^s are normalized eigenvectors of the matrix S_3:

$$\eta_s^s=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix},\ \eta_{s-1}^s=\begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix},\ \ldots,\ \eta_{-s}^s=\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}, \tag{28.11}$$

$\dfrac{m}{\sqrt{2E(E+\frac{\nu}{s}p_3)}}$ is a normalizing multiplier, $b_\nu^\varepsilon(\boldsymbol{p})$ are arbitrary square integrable functions.

Substituting (28.10) into (28.5), (28.7), (28.8) and using the explicit form (4.65), (4.66) of the matrices **S** and **K**, we obtained the corresponding general solutions in the form

$$\psi=\sum_{\nu,\varepsilon}\int\frac{d^3p}{2E}b_\nu^\varepsilon(\boldsymbol{p})\tilde{\psi}_\varepsilon^\nu\exp[i(\boldsymbol{p}\cdot\boldsymbol{x}-\varepsilon x_0E)] \tag{28.12}$$

where

$$\tilde{\psi}_\varepsilon^\nu=\frac{m}{\sqrt{2E(E+\frac{\nu}{s}p_3)}}\begin{pmatrix}\eta_\nu^s\\ \hat{0}\\ \chi_\nu^{s\varepsilon}\\ \tilde{\chi}_\nu^s\end{pmatrix},$$

$$\chi_\nu^{s\varepsilon}=(\varepsilon E+\frac{\nu}{s}p_3)\eta_\nu^s+\frac{1}{2s}\sqrt{s(s+1)-\nu(\nu+1)}\,(p_1-ip_2)\eta_{\nu+1}^s+ \tag{28.13}$$

$$+\frac{1}{2s}\sqrt{s(s+1)-\nu(\nu-1)}\,(p_1+ip_2)\eta_{\nu-1}^s,$$

$$\tilde{\chi}_\nu^s=-\frac{1}{s}\sqrt{s^2-\nu^2}\,p_3\eta_\nu^{s-1}-\frac{1}{2s}\sqrt{s(s-1)-\nu(\nu-1)+2\nu s}\,(p_1-$$

$$-ip_2)\eta_{\nu+1}^{s-1}-\frac{1}{2s}\sqrt{s(s-1)+\nu(\nu+1)-2\nu s}\,(p_1-ip_2)\eta_{\nu-1}^{s-1},$$

$\tilde{0}$ is a column containing $2s$-1 zeros.

The spinors $\tilde{\psi}_\varepsilon^{(\pm)}$ satisfy the normalization condition

$$\tilde{\psi}_\nu^{(\varepsilon)\dagger}\psi_{\nu'}^{(\varepsilon')}=\delta_{\varepsilon\varepsilon'}\delta_{\nu\nu'}$$

and form a basis of solutions of the Dirac-type equations for a particle of arbitrary spin.

28.3. Relativistic Particle of Arbitrary Spin in the Homogeneous Magnetic Field

Consider the movement of a charged particle in the constant and homogeneous magnetic field. Without loss of generality, we choose the vector of the magnetic field strength be parallel to the third projection of the particle momentum, i.e.,

$$F^{0\alpha}=F^{23}=F^{31}=0,\quad F^{12}=H_3=H \tag{28.14}$$

where $F_{\mu\sigma}$ is the tensor of the electromagnetic field. The corresponding four-vector π can be chosen in the form

$$\pi_0=p_0,\quad \pi_1=p_1-eHx_2,\quad \pi_2=p_2\quad \pi_3=p_3. \tag{28.15}$$

As in the case of a free particle we will solve the equivalent system (10.30) instead of the equations (10.10). Using the representation (9.14), (28.2)-(28.4) we conclude that the general form of solutions corresponding to the energy ε is given by the following formula

$$\psi=\begin{pmatrix} \Phi_s \\ \hat{0} \\ \frac{1}{m}\left(\varepsilon+\frac{1}{s}\boldsymbol{S}\cdot\pi\right)\Phi_s \\ -\frac{1}{ms}\boldsymbol{K}\cdot\pi\Phi_s \end{pmatrix} \tag{28.16}$$

where $\tilde{0}$ is a column including $2s$-1 zeros, Φ_s is a $(2s+1)$-component spinor satisfying the equation

$$\left[p^2+e^2H^2x_2^2-eH\left(\frac{1}{s}S_3+2x_2p_1\right)\right]\Phi_s=(\varepsilon^2-m^2)\Phi_s. \tag{28.17}$$

It is convenient to expand Φ_s in a complete set of eigenvectors of the matrix S_3:

$$\Phi_s=\sum_\nu \varphi_\nu^s\eta_\nu^s \tag{28.18}$$

where η^s_ν are the spinors of (28.11), φ^s_ν are unknown functions which must satisfy the following equations according to (28.13):

$$\left[p^2+e^2H^2x_2^2-eH\left(\frac{\nu}{s}+2x_2p_1\right)\right]\varphi_\nu^s=(\varepsilon^2-m^2)\varphi_\nu^s. \tag{28.19}$$

We search for solutions of (28.19) in the form

$$\varphi_\nu^s=\exp[i(p_1x_1+p_3x_3)]f_\nu^s(x_2) \tag{28.20}$$

where p_1 and p_3 are constants. As a result we obtain the following equation for $f^{\prime s}_\nu(x_2)$:

$$\left[-\frac{d^2}{dx_2^2}+(eHx_2-p_1)^2-e\frac{\nu}{s}H\right]f_\nu^s(x_2)=(\varepsilon^2-m^2-p_3^2)f_\nu^s(x_2). \tag{28.21}$$

With the help of the change of variables

$$x_2=\frac{1}{eH}(p_1+\sqrt{eH}\,y),\quad \varepsilon^2-m^2-p_3^2+e\frac{\nu}{s}H=e\xi H$$

we reduce (28.21) to the equation for the harmonic oscillator

$$\left(-\frac{d^2}{dy^2}+y^2\right)f_\nu^s(y)=\xi f_\nu^s(y). \tag{28.22}$$

Requiring, that $f_\nu^s(y)$ tends to zero for $y \to \pm\infty$, we obtain the well-known expression for ξ [3]

$$\xi=2n+1, \quad n=0,1,..., \tag{27.23}$$

from which it follows that

$$\varepsilon^2=m^2+p_3^2+eH(2n+1-\nu/s). \tag{28.24}$$

The relation (28.24) generalizes the well-known formula [3,31] for energy levels of an electron in the constant and homogeneous magnetic field to the case of a particle of arbitrary spin.

To clarify the physical meaning of the expression (28.24) for particle energies, we consider the non-relativistic approximation

$$|\varepsilon|=\sqrt{m^2+p_3^2+eH(2n+1-\nu/s)}\cong m+\frac{p_3^2}{2m}+\Omega\left(n+\frac{s-\nu}{2s}\right), \tag{28.25}$$

where $\Omega=eH/m$ is the cyclotronic frequency.

We see $|\varepsilon|$ includes the kinetic energy of the particle movement along the magnetic field, and the quantized part of energy depending on two discrete parameters, i.e., n and ν. When n=1/2, ν can take two values: $\nu=\pm1/2$. The corresponding quantized part of energy determines the Landau levels and turns out to be proportional to an integer. Besides any level (excepting the ground one) is twice generated.

For the case s=0 the Landau levels are nondegenerated and proportional to half integers. For s=1 these levels are proportional to either integers or half integers besides all the integer levels are twice degenerated (the ground level and half integer ones are nondegenerated). For s>1 the energy spectrum is more complicated but also includes generated levels corresponding to $\nu=\pm s$. These levels are proportional to integers.

Let find the explicit form of the wave function satisfying the equation (10.10). The solutions of (28.22), (28.23) are given by the formula [3]

$$f_\nu^s=C_\nu^s U_n(y), \tag{28.26}$$

where C^s_ν is a normalizing multiplier, $U_n(y)$ is the Hermit function

$$U_n(y)=\frac{(eH)^4}{(2^n n!\pi^{1/2})^{1/2}}e^{-y^2/2}H_n(y), \tag{28.27}$$

$H_n(y)$ is the Hermit polynomial which can be written in the form

$$H_n(y)=(-1)^n e^{y^2}\left(\frac{\partial}{\partial y}\right)^n e^{-y^2}, \quad n=0,1,\ldots \ . \tag{28.28}$$

Using formulae (4.65), (4.66) defining the matrix elements of $\boldsymbol{S}$ and $\boldsymbol{K}$, bearing in mind the recurrence relations

$$\left(\frac{\partial}{\partial y}\pm y\right)U_n=\pm(2n+1\mp 1)^{1/2}U_{n\mp 1} \tag{28.29}$$

and the following identity

$$\boldsymbol{A}\cdot\boldsymbol{p}\Phi^s_\mu\equiv\left(A_3p_3+\frac{\sqrt{eH}}{2}\left[(A_1-iA_2)\left(y+\frac{\partial}{\partial y}\right)+(A_1+A_2)\left(y-\frac{\partial}{\partial y}\right)\right]\right)\Phi^s_\mu$$

($\boldsymbol{A}=\boldsymbol{S}$ or $\boldsymbol{A}=\boldsymbol{K}$), we obtain

$$\psi_{\nu n}=\frac{\exp[i(p_1x_1+p_3x_3)]}{\sqrt{2L_1L_2\varepsilon\left(\varepsilon+\frac{\nu}{s}p_3\right)}}\begin{pmatrix} mU_n(y)\eta^s_\nu \\ \hat{0} \\ \varphi_{\nu n} \\ \chi_{\nu n}\end{pmatrix} \tag{28.30}$$

where $\tilde{0}$ is a $(2s-1)$-component zero column, L_1 and L_3 are normalization constants,

$$\varphi_{\nu n}=\left(\varepsilon+\frac{\nu}{s}p_3\right)U_n(y)\eta^s_\nu+\frac{1}{2s}\sqrt{[s(s+1)-\nu(\nu+1)]2eH(n+1)}\,U_{n+1}(y)\eta^s_{\nu+1}+$$
$$+\frac{1}{2s}\sqrt{[s(s+1)-\nu(\nu-1)]2enH}\,U_{n-1}(y)\eta^s_{\nu-1},$$

$$\chi_{\nu n}=-\frac{1}{s}\sqrt{s^2-\nu^2}\,p_3U_n(y)\eta^{s-1}_\nu-$$
$$-\frac{1}{2s}\sqrt{[s(s-1)-\nu(\nu-1)+2\nu s]2eH(n+1)}\,U_{n+1}(y)\eta^{s-1}_{\nu+1}+$$
$$+\frac{1}{2s}\sqrt{[s(s-1)+\nu(\nu-1)-2\nu s]2eHn}\,U_{n-1}(y)\eta^{s-1}_{\nu-1}.$$

Here U_n is the Hermit function (28.27), $\eta^s{}_\nu$ are the spinors (28.11), $y=(eH)^{1/2}(x_2-p_1/eH)$.

The wave functions (28.30) are normalized in accordance with the following relation

$$\int_o^{L_1}dx_1\int_0^{L_3}dx_3\int_{-\infty}^{\infty}\psi^\dagger_{\nu n}\psi_{\nu n}dx_2=1$$

where L_1 and L_2 are arbitrary numbers included into the normalization constant in (28.30).

So using the equations of the Dirac type (10.10), we have obtained exact solution of the problem of interaction of charged spinning particle with the constant and homogeneous magnetic field for any value of spin.

The equations (10.10) admit exact solutions also for the cases of the constant homogeneous electric field, the combination of the electric and magnetic fields mentioned above [400] and the field of the magnetic monopole [164]. We do not represent here the corresponding cumbersome formulae but consider the problem of motion of any spin particle in the constant electric field in Section 30.

28.4. A Particle of Arbitrary Spin in the Field of the Plane Electromagnetic Wave

The equation (10.10) admits exact solutions in the important case of the external field reducing to the plane wave [322].

The plane wave field characterized by the wave vector k_μ ($k_\mu k^\mu=0$) is determined by the following vector-potential

$$A_\mu=A_\mu(\varphi),\quad \varphi=k_\mu x^\mu \tag{28.31}$$

satisfying the Lorentz gauge condition

$$p_\mu A^\mu=-ik_\mu A'^\mu=0, \tag{28.32}$$

where the prime denote the derivative with respect to φ.

As in previous subsection we can represent solutions of the corresponding equations (10.10) in the form (28.16) where Φ_s is a $(2s+1)$-component spinor satisfying the second order equation (10.30a). In our case this equation takes the form

$$\left(p_\mu p^\mu+2eA_\mu p^\mu+e^2A_\mu A^\mu-m^2-\frac{e}{s}\boldsymbol{S}\cdot\boldsymbol{F}\right)\Phi_s=0 \tag{28.33}$$

where

$$\boldsymbol{F}=\boldsymbol{k}\times\boldsymbol{A}'-i(k_0\boldsymbol{A}'-\boldsymbol{k}A_0').$$

Like in [407], we look for the solution of (28.33) in the form

$$\Phi_s=\exp(-i\tilde{p}_\mu x^\mu)\psi(\varphi), \tag{28.34}$$

where $\tilde{p}_\mu$ is a constant four-vector besides without loss of generality $\tilde{p}^\mu\tilde{p}_\mu=m^2$. Then using the relations

$$p_\mu\psi(\varphi)=-ik_\mu\psi'(\varphi),\quad p_\mu p^\mu\psi(\varphi)=k_\mu k^\mu\psi''(\varphi)$$

we obtain from (28.33) the following equation for $\psi(\varepsilon)$:

$$2ik_\mu\tilde{p}_\mu\psi'+\left[-2e\tilde{p}_\mu A^\mu+e^2A_\mu A^\mu-\frac{e}{s}\boldsymbol{S}\cdot\boldsymbol{F}\right]\psi=0.$$

This equation is easily integrated

$$\psi=\exp\left(-i\int_0^{k_\mu k^\mu}\left[\frac{e}{k_\mu\tilde{p}^\mu}\tilde{p}_\mu A^\mu-\frac{e^2}{2k_\mu\tilde{p}^\mu}A_\sigma A^\sigma\right]d\varphi-\frac{ie\boldsymbol{S}\cdot\boldsymbol{F}}{2sk_\mu\tilde{p}^\mu}\right)U_p \tag{28.35}$$

where U_p is an arbitrary constant spinor.

The matrices $\boldsymbol{S}\cdot\boldsymbol{F}$ of (28.35) satisfy the relations

$$\begin{gathered}\prod_\lambda\left[(\boldsymbol{S}\cdot\boldsymbol{F})^2-\lambda^2\boldsymbol{F}\right]=0,\quad \lambda=1/2,3/2,\ldots,s\\ \boldsymbol{S}\cdot\boldsymbol{F}\prod_\lambda\left[(\boldsymbol{S}\cdot\boldsymbol{F})^2-\nu^2\boldsymbol{F}\right]=0,\quad \nu=1,2,\ldots,s.\end{gathered} \tag{28.36}$$

besides the first formula is available for half integer s and the second formula is valid for integer s. Inasmuch as $\boldsymbol{F}^2=k_\mu k^\mu A'_\nu A'^\nu$ and $k_\mu k^\mu=0$ the conditions (28.36) reduce to the form $(\boldsymbol{S}\cdot\boldsymbol{F})^{2s+1}=0$, and we have from (28.34),(28.35)

$$\Phi^s=\exp(iS)U_p\sum_{n=0}^{2s}\frac{1}{n!}\left(ie\frac{\boldsymbol{S}\cdot\boldsymbol{F}}{2sk\mu\tilde{p}_\mu}\right)^n, \tag{28.37}$$

where S is the classical action of the particle moving in the plane wave field [271]

$$S=-\tilde{p}_\nu x^\nu-\int_0^{k_\mu\tilde{p}^\mu}\left(\frac{e}{k_\mu\tilde{p}^\mu}\tilde{p}_\nu A^\nu-\frac{e^2}{2k_\mu\tilde{p}^\mu}A_\nu A^\nu\right)d\varphi.$$

We can choose an arbitrary spinor U_p of (28.36) in accordance with the requirement that the corresponding wave function (28.16) reduces to the plane wave solution for a free particle if $A_\mu\to 0$. As a result, we obtain the following solutions

$$\tilde{\psi}^{\nu\varepsilon}=\sum_{n=0}^{2s}\frac{1}{n!}\left(ie\frac{\boldsymbol{S}\cdot\boldsymbol{F}}{2sk_\mu\tilde{p}_\mu}\right)^n\exp(iS)\tilde{\psi}^\nu_\varepsilon$$

where $\tilde{\psi}^\nu_\varepsilon$ are the spinors (28.13).

We see that in contrast to the Volkov solution for an electron in the field of the plane electromagnetic wave [407] the solutions of relativistic equations for particles of arbitrary spin depend on the fields strength non-linearly (as a polynomial of order $2s$).

We note that the equations (10.10) can be solved exactly also for the case of the Redmond field which is a combination of the constant magnetic field and the field of the plane wave [400].

29. RELATIVISTIC PARTICLE OF ARBITRARY SPIN IN THE COULOMB FIELD

29.1. Separation of Variables in a Central Field

The problem of description of a spin particle in a central field is one of the basic problems of quantum mechanics. In the case of an electron in the Coulomb field it is the problem of the hydrogen atom.

The problem of description of the hydrogen-type system in which the spin of orbital particle is more than 1/2 is not of theoretical interest only, since such relatively stable particles as W^--boson, Ω^--hyperon in principle can play a role of orbital particles in exotic atoms.

Following [322] we present exact solutions of the equations (10.10) for a charged particle of arbitrary spin in the Coulomb field. It will be shown these equations do not lead to the difficulties connected with a possibility of particle falling to the center (such a situation is non-admissible from quantum mechanics point of view; nevertheless it is predicted e.q. by the KDP equation [401]).

Consider equations (10.30) for the case of the Coulomb field where the vector-potential reduces to the form (22.8). Such equations admit solutions in separated variables besides the general scheme of obtaining these solutions is valid for any central potential $A_0=A_0(x)$, $x=|\boldsymbol{x}|$.

We use the representation (10.31) for the equation (10.30a). Solutions corresponding to a state with energy ε are represented in the form

$$\Phi^s=\exp(-i\varepsilon t)\Phi^s(\boldsymbol{x}), \tag{29.1}$$

where $\Phi^s(\boldsymbol{x})$ is a $(2s+1)$-component spinor satisfying the equation

$$\left[\left(\varepsilon-\frac{ze^2}{x}\right)^2-p^2-m^2-\frac{ize^2}{s}\frac{\boldsymbol{S}\cdot\boldsymbol{x}}{x^3}\right]\Phi^s(\boldsymbol{x})=0. \tag{29.2}$$

This equation admits solutions in separated variables. Indeed, bearing in mind the symmetry of (29.2) in respect with the group *O(3)* we represent $\Phi^s(\boldsymbol{x})$ as a linear combination of spherical spinors

$$\Phi^s(\boldsymbol{x})=\sum_\lambda \varphi^\lambda(x)\Omega^s_{jj-\lambda m}, \tag{29.3}$$

where $\left\{\Omega^s_{jj-\lambda m}=\Omega^s_{jj-\lambda m}(\boldsymbol{x}/x)\right\}$ is a complete set of eigenfunctions of the commuting operators $\boldsymbol{J}^2$, $\boldsymbol{L}^2$, J_3 and $\boldsymbol{S}^2$ ($\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S}$, $\boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p}$), so that

$$
\begin{aligned}
&\boldsymbol{J}^2\Omega^s_{jj-\lambda m}=j(j+1)\Omega^s_{jj-\lambda m},\\
&\boldsymbol{L}^2\Omega^s_{jj-\lambda m}=(j-\lambda)(j-\lambda+1)\Omega^s_{jj-\lambda m},\\
&\boldsymbol{S}^2\Omega^s_{jj-\lambda m}=s(s+1)\Omega^s_{jj-\lambda m},\\
&J_3\Omega^s_{jj-\lambda m}=m\Omega^s_{jj-\lambda m}.
\end{aligned}
\tag{29.4}
$$

Here j are arbitrary positive integers or half integers,

$$
m=-j,-j+1,...,j,\quad \lambda=-s,-s+1,...,-s+2m_{sj},\quad m_{sj}=\min(s,j)=\begin{cases} s, & s\leq j\\ j, & s>j\end{cases}. \tag{29.5}
$$

We find it is more convenient to use the quantum number $\lambda=j$-l instead of the usual orbital quantum number l. The explicit expressions for the $\Omega^s_{jj-\lambda m}$ are given in Appendix 3.

The representation (29.3) makes it possible to reduce (29.2) to the system of ordinary differential equations for the function φ^λ. To find this system, it is necessary to know the action of the operator $\boldsymbol{S}\cdot\boldsymbol{x}$ on spherical spinors. It is not difficult to make sure that

$$
\boldsymbol{S}\cdot\hat{\boldsymbol{x}}\Omega^s_{jj-\lambda m}=d^{sj}_{\lambda'\lambda}\Omega^s_{jj-\lambda' m} \tag{29.6}
$$

where $d^{sj}_{\lambda'\lambda}$ are numeric coefficients, $\hat{\boldsymbol{x}}=\boldsymbol{x}/$x, besides the summation is imposed over the repeated indices λ'. Indeed, $\boldsymbol{S}\cdot\boldsymbol{x}/x$ commutes with $\boldsymbol{J}$ and hence with $\boldsymbol{J}^2$ and J_3. This is why in the l.h.s. and r.h.s. of (29.6) we have an eigenfunction of $\boldsymbol{J}^2$ and J_3 corresponding to the same eigenvalues. Expressing such an eigenfunction via the complete set of the spherical spinor we come to the relation (29.6).

The exact values of the coefficients $d^{sj}_{\lambda'\lambda}$ are calculated in Appendix 3 and can be represented in the form

$$
d^{sj}_{\lambda'\lambda}=-\frac{1}{2}\left(\delta_{\lambda'+1\,\lambda}a^{sj}_{s-\lambda}+\delta_{\lambda'-1\,\lambda}a^{sj}_{s-\lambda+1}\right), \tag{29.7}
$$

where

$$
a^{sj}_{\mu}=\left(\frac{\mu(2j+1-\mu)(2s+1-\mu)(2j+2s+2-\mu)}{(2s+2j+1-2\mu)(2s+2j+3-2\mu)}\right)^{1/2} \tag{29.8}
$$

besides the possible values of λ, $\lambda'+1$, $\lambda'-1$ are given in (29.5).

Substituting (29.3) into (29.2) and using (29.6) and the following identity

$$
p^2=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\right)-\frac{\boldsymbol{L}^2}{x^2} \tag{29.9}
$$

we come to the following equations for radial functions

$$D\varphi^{\lambda}=x^{-2}b^{sj}_{\lambda'\lambda}\varphi^{\lambda'}, \tag{29.10}$$

where

$$D=\left(\varepsilon+\frac{\alpha}{x}\right)^2-m^2+\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2},$$
$$b^{sj}_{\lambda\lambda'}=[\lambda^2-\lambda(2j+1)]\delta_{\lambda\lambda'}+i\frac{\alpha}{s}d^{sj}_{\lambda\lambda'},\quad \alpha=ze^2. \tag{29.11}$$

Thus a problem of description of a charged particle of arbitrary spin in the Coulomb field reduces to solving the system of ordinary differential equation (29.10).

29.2. Solution of Equations for Radial Functions

The matrix $\left\| b^{sj}_{\lambda\lambda'}\right\|$ of (29.10) commutes with the operator D and is *normal*, i.e.,

$$\left(b^{sj}_{\lambda\lambda'}\right)^* b^{sj}_{\lambda\lambda''}=b^{sj}_{\lambda'\lambda}\left(b^{sj}_{\lambda''\lambda}\right)^*.$$

It means this matrix can be diagonalized and so the system (29.10) can be reduced to the following chain of noncoupled equations

$$D\tilde{\varphi}=x^{-2}b^{sj}_{\lambda}\tilde{\varphi} \tag{29.12}$$

where D is the operator (29.11), b^{sj}_{λ} are eigenvalues of the matrix $\left\| b^{sj}_{\lambda\lambda'}\right\|$. Any of the equations (29.12) in its turn reduces to the well-known equation [105]

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\left(\beta-\frac{z}{4}-\frac{k^2_{\lambda}}{4z}\right)y=0 \tag{29.13}$$

where

$$y=\left(\frac{z}{m^2-\varepsilon^2}\right)^{1/2}\tilde{\varphi},\quad z=2(m^2-\varepsilon^2)^{1/2}x,$$
$$\beta=\frac{\varepsilon\alpha}{(m^2-\alpha^2)^{1/2}},\quad k^2_{\lambda}=(2j+1)^2+4(b^{sj}_{\lambda})^2-4\alpha^2. \tag{29.14}$$

The equations (29.13) arise in the problem of the hydrogen atom (besides in this case $s=1/2$, $b^{sj}_{\lambda}=1/4\pm(2j+1-4\alpha^2)^{1/2}$). In the case of an arbitrary spin particle we have the only new feature that the parameter k_{λ} can take another values depending on s.

We represent the solutions of (29.13) corresponding to coupled states ($m^2>\varepsilon^2$) in the form

$$y=z^{k_{\lambda}/2}\exp(-z/2)F \tag{29.15}$$

and obtain the following equation for F

$$z\frac{d^2F}{dz^2}+(k_\lambda+1-z)\frac{dF}{dz}+\left(\beta-\frac{k_\lambda+1}{2}\right)F=0. \tag{29.16}$$

Formula (29.16) defines the equation for a degenerated hypergeometric function, thus the corresponding solutions have the form

$$F=C\mathscr{F}\left(\frac{k_\lambda+1}{2}-\beta,\,k_\lambda+1,\,z\right) \tag{29.17}$$

where C is an arbitrary constant, $\mathscr{F}$ is a degenerated hypergeometric function. The analytical expression for $\mathscr{F}$ $(a.b,z)$ is given by the following formula

$$\mathscr{F}(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)z^n}{\Gamma(b+n)n!},$$

where $\Gamma(a)$ is the Γ-function of Euler.

It follows from the boundary condition for the solutions (29.15) at infinity [31] that the argument $a=(k_\lambda+1)/2-\beta$ has to be a negative integer or zero, i.e.,

$$\beta=(k_\lambda+1)/2+n',\quad n'=0,1,\ \dots\ . \tag{29.18}$$

So we have obtained the solutions of the equations (29.12) in the form

$$\tilde{\varphi}_\lambda=C(m^2-\varepsilon^2)^{\frac{k_\lambda+1}{4}}x^{\frac{k_\lambda-1}{2}}e^{-(m^2-\varepsilon^2)^{1/2}x}\mathscr{F}\left(-n',\,k_\lambda+1,\,2(m^2-\varepsilon^2)^{1/2}x\right) \tag{29.19}$$

where the index λ labels the solutions corresponding to possible values of k_λ. A solution of (29.2) can be written in the form

$$\Phi^s=U_{\lambda\lambda'}\tilde{\varphi}_{\lambda'}\Omega^s_{jj-\lambda m} \tag{29.20}$$

where $U_{\lambda\lambda'}$ are the matrix elements of the unitary matrix diagonalizing $\left\|b^{sj}_{\lambda\lambda'}\right\|$ of (29.11) so that

$$\sum_{\lambda''}U_{\lambda\lambda''}U^*_{\lambda'\lambda''}b^{sj}_{\lambda''}=b^{sj}_{\lambda\lambda'}.$$

The corresponding solutions of (10.10) in the representation (9.14), (28.2)-(28.4) have the form

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \psi_1=\begin{pmatrix}\Phi^s\\ \hat{0}\end{pmatrix},\quad \psi_2=\frac{1}{m}(\sigma_0\varepsilon+\sigma\cdot\boldsymbol{p})\psi_1,$$

where $\hat{0}$ is the zero column including $2s$-1 rows, σ are the $4s\times4s$ matrices of (28.4), σ_0 is the unite matrix of dimension $4s\times4s$.

Substituting the matrix $(\sigma\cdot\boldsymbol{x})(\sigma\cdot\boldsymbol{x})/x^2\equiv\sigma_0$ between $\sigma\cdot\boldsymbol{p}$ and ψ_1 we obtain

$$\sigma\cdot\boldsymbol{p}\psi_1\equiv\left[-i\sigma\cdot\hat{\boldsymbol{x}}\left(\frac{\partial}{\partial x}+\frac{2}{x}\right)+\frac{i}{2x}(\sigma\cdot\boldsymbol{x}\times\boldsymbol{p})\sigma\cdot\hat{\boldsymbol{x}}\right]\psi_1.$$

The action of $\sigma\cdot\boldsymbol{x}/x$ and $\sigma\cdot\boldsymbol{x}\times\boldsymbol{p}$ on spherical spinors is described in Appendix 3 (see (A.3.10)) thus we obtain finally

$$\psi_2 = U_{\lambda\lambda'}\Big\{\varepsilon\tilde{\varphi}_{\lambda'}\Omega^{s}_{jj-\lambda m} + \\ + \left[-\frac{i}{s}d^{sj}_{\lambda'\lambda''}\left(\frac{d}{dx}+\frac{2}{x}\right)+\frac{i}{2s^2x}\left(d^{sj}_{\lambda'\lambda''}f^{sj}(\lambda)+ib^{s-1j}_{\lambda'\lambda''}g^{sj}(\lambda)\right)\right]\tilde{\varphi}_{\lambda''}\Omega^{s}_{jj-\lambda m} + \\ + \left[\frac{1}{s}d^{s-1j}_{\lambda'\lambda''}\left(\frac{\partial}{\partial x}+\frac{2}{x}\right)+\frac{1}{2s^2x}\left(d^{sj}_{\lambda'\lambda''}g^{sj}(\lambda)+ib^{s-1j}_{\lambda'\lambda''}f^{s-1j}(\lambda)\right)\right]\tilde{\varphi}_{\lambda''}\Omega^{s-1}_{jj-\lambda m}\Big\}. \tag{29.22}$$

where $d^{sj}_{\lambda'\lambda}$, $b^{sj\lambda}_{\lambda'}$, $f^{sj}(\lambda)$ and $g^{sj}(\lambda)$ are the coefficients given in (29.7), (A 3.4).

Formulae (29.19)-(29.22) define solutions of the Dirac-type equations for a particle of arbitrary spin in the Coulomb field. For any value of spin s we can determine the exact value of an arbitrary constant C starting from a normalization condition for the corresponding solution.

29.3. Energy Levels of a Relativistic Particle of Arbitrary Spin in the Coulomb Field

Starting from the condition (29.18) it is not difficult to find possible values of the parameters ε defining the energy of the particle described. Indeed, according to (29.18) we have

$$\varepsilon = m\left[1+\frac{\alpha^2}{\left(n'+1/2+\left[(j+1/2)^2-\alpha^2-b^{sj}_{\lambda}\right]^{1/2}\right)^2}\right]^{1/2}. \tag{29.23}$$

Formula (29.23) defines the energy levels of arbitrary spin particle in the Coulomb field and so can be considered as a generalization of the Zommerfeld formula for an electron. The parameters b^{sj}_{λ} of (29.23) take the values coinciding with the roots of the characteristic equation for the matrix $\left\| b^{sj}_{\lambda\lambda'}\right\|$ of (29.11)

$$\det\left\|\left[\lambda^2-\lambda(2j+1)-b^{sj}_{\lambda}\right]\delta_{\lambda\lambda'}+\frac{i\alpha}{s}d^{sj}_{\lambda\lambda'}\right\| = 0 \tag{29.24}$$

where $d^{sj}_{\lambda'\lambda}$ are the coefficients (29.7).

Thus we have found energy spectrum of a particle of spin s in the field of a point charge. However the practical using of (29.23) is restricted by the fact that to find all the possible values of $d^{sj}_{\lambda'\lambda}$ it is necessary to solve the algebraic equation (29.4) of order $2s+1$ (for $j\geq s$) or of order $2j+1$ (for $s\geq j$). This equation can be solved in radicals in the cases $s\leq 3/2$ or $j\leq 3/2$ only.

To analyze the spectrum (29.23) for arbitrary s and j it is sufficient to consider approximate solutions of (29.24) defined up to the terms of order α^2. Such an approximation is reasonable for the cases when the charge z of a particle generating the Coulomb field satisfies the condition $z \ll 137$.

Representing b_λ^{sj} in the form

$$b_\lambda^{sj} = \lambda^2 - (2j+1)\lambda + \tilde{b}_\lambda^{sj}\alpha^2 + o(\alpha^4) \tag{29.25}$$

where $\tilde{b}_\lambda^{sj}$ are unknown coefficients we obtain from (29.24)

$$\tilde{b}_\lambda^{sj} = \frac{1}{8s^2}\left[\frac{\left(a_{\lambda+s}^{sj}\right)^2}{j-\lambda+1} - \frac{\left(a_{\lambda+s+1}^{sj}\right)^2}{j-\lambda}\right]. \tag{29.26}$$

Here a_σ^{sj} ($\sigma=\lambda+s$ or $\sigma=\lambda+s+1$) are the coefficients (29.8).

Using (29.25) and representing the r.h.s. of (29.23) as series in respect with α^2 we obtain with accuracy to α^4:

$$\varepsilon = m\left(1 - \frac{\alpha^2}{2n^2} + \frac{2\alpha^4\left(\tilde{b}_\lambda^{sj} - 1\right)}{n^3(2l+1)} + \frac{3}{8}\frac{\alpha^4}{n^4}\right), \tag{29.27}$$

where

$$n = n' + j - \lambda + 1 = 1,2,...; \quad l = j - \lambda = 0,1,...,n-1. \tag{29.28}$$

The relations (29.27), (29.28) define the fine structure of energy spectrum of arbitrary spin particle in the Coulomb field. The corresponding values of $\tilde{b}_\lambda^{sj}$ are easily calculated using (29.26), (29.8).

In addition to the nonessential constant term m formula (29.27) includes Balmer's term $-m\alpha^2/2n^2$, and the additional terms which are proportional to α^4 and represent the contribution of the spin-orbit coupling of a particle with an external field. The energy levels are labelled by three quantum numbers n, l and λ (the number j is expressed via l and λ: $j=\lambda+l$). The possible values of n and l are given in (29.28), and admissible values of λ for fixed l are

$$\lambda = -s, -s+1, ..., s-2m_{sl} \quad m_{sl} = \min(s,l). \tag{29.29}$$

We see that any energy level corresponding to the quantum number n is split to sublevels of the fine structure corresponding to possible values of l and λ. It is not difficult to calculate the number N_n of such sublevels which equals to

$$N_n = (2s+1)n - [(s+1/2)^2], \quad n \geq s+1; \quad N_n = n^2, \quad n \leq s, \tag{29.30}$$

where $[(s+1/2)^2]$ is the entire part of $(s+1/2)^2$.

We note that the energy levels corresponding to different sets of l and λ can to coincide in general. For instance for $s=1/2$ we have $\tilde{b}_\lambda^{sj} = 2\lambda/(2j+1)$, and the energy corresponding to the same l but different λ are equal one to another. The corresponding number of the fine structure sublevels is equal to n^2 besides any sublevel with $j\neq n-1/2$ is twice degenerated.

Let consider more precisely the cases s=0, 1/2, 3/2. The corresponding values of $\tilde{b}_\lambda^{sj}$ (calculated in accordance with (29.26), (29.8)) are

$$\tilde{b}_\lambda^{0j}=0,\quad \lambda=0;\quad \tilde{b}_\lambda^{1/2\,j}=\frac{2\lambda}{d_j},\quad \lambda=\pm\frac{1}{2},\quad d_j=2l+1;$$

$$\tilde{b}_\lambda^{1j}=\lambda\frac{d_j+\lambda}{2d_j(d_j-\lambda)}+\frac{2(1-\lambda^2)}{1-d_j^2},\quad \lambda=\begin{cases}1,0,-1, & j\neq 0,\\ -1 & j=0;\end{cases} \tag{29.31}$$

$$\tilde{b}_{\pm 3/2}^{3/2\,j}=\pm\frac{(d_j\pm 1)(5\pm 1)}{18(d_j\mp 1)(d_j\mp 2)}=-\tilde{b}_{\pm 1/2}^{3/2\,j}\mp\frac{2(d_j^2-4)}{9d_j(d_j^2-1)},\quad j\neq 1/2;$$

$$\tilde{b}_{-3/2}^{3/2\;1/2}=-\tilde{b}_{-1/2}^{3/2\;12}=-\frac{1}{54}.$$

We conclude from (29.31) that for the case s=0 formula (29.27) gives the known energy spectrum of a scalar particle (described by the KGF equation) in the Coulomb field, but when s=1/2, the relation (29.27) reduces to the Zommerfeld formula for the spectrum of the hydrogen atom [31].

Formulae (29.27), (29.31) define the fine structure of the energy spectrum for particles of spins s=1 and *3/2* also. In contrast to the case s=1/2 the spectrum is nondegenerated and the number of sublevels of the fine structure is defined in (29.30).

The energy levels corresponding to the ground state n=1 are nondegenerated. But if n=2 then we have two sublevels for s=0 (and s=1/2) and four sublevels for s=1 (and s=3/2). It is not difficult to calculate that the energy splitting ΔE (i.e., divergence between the highest and lowest sublevels) for n=2 is equal to the data given in the table

s	0	1/2	1	3/2
ΔE	$m\alpha^4/12$	$m\alpha^4/32$	$m\alpha^4/24$	$5m\alpha^4/72$

Let us return to exact formula (29.23). For the cases $s\leq 3/2$ or $j\leq 3/2$ it is possible to solve (29.24) in radicals and find the exact values of b_λ^{sj} for (29.23). We present the corresponding solutions for $s\leq 1$ and $j\leq 1$:

$$b_\lambda^{0j}=0,\quad b_\lambda^{1/2j}=1/4+\lambda\sqrt{d_j^2+4\alpha^2},\quad \lambda=\pm 1/2; \tag{29.32}$$

$$b_\lambda^{1j}=\frac{c}{3}+2\sqrt{-c}\cos\left[\frac{1}{3}\left(\gamma+\lambda\frac{\pi}{2}\right)\right],\quad \lambda=0,\pm 1,\ j\neq 0;\quad b_\lambda^{10}=0; \tag{29.33}$$

$$b_\lambda^{s0}=0, \quad b_\lambda^{s\,1/2}=\frac{1}{4}(d_s^2-3)\pm\frac{1}{2}\sqrt{d_s^2+\left(\frac{\alpha}{s}\right)^2}, \quad s\neq 0, \quad d_s=2s+1,$$
$$b_\lambda^{s1}=s(s+1)-2+\frac{d}{3}+2\sqrt{-d}\cos\left[\frac{1}{3}\left(\xi+\lambda\frac{\pi}{2}\right)\right], \tag{29.34}$$

where

$$\cos\gamma=\frac{b}{\sqrt{-c^3}}, \quad b=\frac{2}{3}\alpha^2+\frac{1}{3}d_j^2-\frac{1}{27}, \quad c=-\alpha^2+b+\frac{4}{27},$$
$$\cos\xi=\frac{f}{\sqrt{-d^3}}, \quad f=\frac{2}{3}\left(\frac{\alpha}{s}\right)^2+\frac{1}{3}d^2-\frac{1}{27}, \quad d=-\left(\frac{\alpha}{s}\right)^2+f+\frac{4}{27}.$$

Substituting (29.32) into (29.23) we come to the exact Zommerfeld formula for the hydrogen atom. The relations (29.23), (29.33) generalize this formula to the cases of a particle of spin 1 and for particles of any spin (but for $j\leq 1$ only). Exact formulae (29.23), (29.33) can be useful for the case of large z, i.e., for $ze^2\sim 1$.

Equation (29.24) can be solved in radicals for the cases $s=3/2$ (j is arbitrary) and $j=3/2$ (s is arbitrary). We do not present the corresponding cumbersome formulae here, see [322].

Summarizing we note that the Dirac-type equations (10.10) enable to solve the problem of motion of a spinning particle in the Coulomb field immediately for any value of a spin.

The movement of a spin-one particle in the Coulomb field was studied by Tamm [401] for the first time. Interesting contributions in solving the Coulomb problem for $s>1/2$ were made in papers [47,195,274]. In the above, we follow papers [322,164].

30. EXACT SOLUTIONS OF GALILEI-INVARIANT WAVE EQUATIONS

30.1. Preliminary notes

In this section we use Galilei-invariant wave equations to solve the problems of motion of charged particles of any spin in external fields.

In accordance with the result of Section 3 it is possible to describe a charged particle of arbitrary spin using different Galilei-invariant equations which are equivalent in the physically reasonable approximation $1/m^2$. This is why we will use the most convenient way in solving of any concrete problem.

We consider all the types of external fields studied in two previous sections. It makes it possible to compare the results obtained in the Poincaré- and Galilei-invariant approaches and to estimate the adequateness of Galilei-invariant models of spinning particles. Besides we solve the problem of motion of a particle of arbitrary spin in the combined field including the constant electric and magnetic components.

30.2. Nonrelativistic Particle in the Constant and Homogeneous Magnetic Field

To solve this problem we use equations in the Hamiltonian form (15.26), (15.28). We choose the vector of the magnetic field strength be parallel to p_3 and so π_μ and $F_{\mu\nu}$ are of the form given in (28.14),(28.15). As a result choosing for convenience $k_1=-1$ we come to the equation (15.28) where

$$\hat{H}(\pi,A_0)=\sigma_1 am+\frac{\pi^2}{2m}+\frac{e\boldsymbol{S}\cdot\boldsymbol{H}}{m}+2iak\boldsymbol{S}\cdot\pi+\frac{1}{m}(\sigma_1+i\sigma_2)[2ak^2(\boldsymbol{S}\cdot\pi)^2-ek_0\boldsymbol{S}\cdot\boldsymbol{H}]. \quad (30.1)$$

Here $k_0=ak^2$-k_2 is an arbitrary parameter and $A_0\equiv 0$.

We restrict ourselves to the case when the field strength is sufficiently small and satisfy the condition

$$\boldsymbol{H}<\frac{am^2}{2ek_0 s}. \quad (30.2)$$

Such a restriction is physically reasonable inasmuch as for a very large $\boldsymbol{H}$ Galilei invariant approach is not available a priori.

Let us transform $\hat{H}(\pi,A_0)$ to the representation in which it includes commuting terms only. It enables us to find eigenvalues of this operator without solving the motion equations. We will make the transformation in two steps.

Using the operator

$$V_1=\exp\left(\frac{i}{m}\eta\cdot\pi\right)\equiv 1+\frac{i}{m}\eta\cdot\pi,\quad V_1^{-1}=1-\frac{i}{m}\eta\cdot\pi \tag{30.3}$$

where η are matrices of (14.3), and taking into account the identity

$$\left[\pi^2+2e\boldsymbol{S}\cdot\boldsymbol{H},\,\eta\cdot\pi\right]=0$$

we obtain

$$H'=V_1\hat{H}V_1^{-1}=\sigma_1 am+\frac{\pi^2}{2m}+\frac{e\boldsymbol{S}\cdot\boldsymbol{H}}{m}+\frac{ek_0}{mk}\eta\cdot\boldsymbol{H}. \tag{30.4}$$

The following transformation will be made using the operator

$$V_2=\frac{1}{2}\left(1+\sigma_3\frac{h}{\sqrt{h^2}}\right),\quad V_2^{-1}=\frac{1}{2}\left(1+\frac{h}{\sqrt{h^2}}\sigma_3\right),$$

$$h=\sigma_1 am+\frac{ek_0}{mk}\eta\cdot\boldsymbol{H},\quad \left(\sqrt{h^2}\right)^{-1}=\sum_{\nu=-s}^{s}(a^2m^2-2eak_0\nu H)^{-1}\Lambda_\nu,\quad \Lambda_\nu=\prod_{\nu'\neq\nu}\frac{S_3-\nu'}{\nu-\nu'}, \tag{30.5}$$

which leads to the resulting Hamiltonian

$$H''=V_2H'V_2^{-1}=\frac{\pi^2}{2m}-\frac{e}{m}S_3H+\sigma_3(a^2m^2-2eak_0S_3H)^{1/2}. \tag{30.6}$$

All the terms of the last Hamiltonian commute and thus have a common system of eigenfunctions. The corresponding eigenvalues are (see Subsection 28.3)

$$\frac{\pi^2}{2m}\Phi=\frac{1}{2m}\left[(2n+1)eH+p_3^2\right]\Phi,\quad n=0,1,2,...,$$

$$S_3\Phi=\nu\Phi,\quad \nu=-s,-s+1,...,s;\quad \sigma_3\Phi=\varepsilon\Phi,\quad \varepsilon=\pm1. \tag{30.7}$$

Using (30.7) it is not difficult to find the eigenvalues of the Hamiltonian (30.6)

$$E_{n\nu\varepsilon p_3}=(2n+1-2\nu)\frac{eH}{2m}+\frac{p_3^2}{2m}+\varepsilon(a^2m^2-2eak_0\nu H)^{1/2}. \tag{30.8}$$

Formula (30.8) gives the energy spectrum of an arbitrary spin particle moving in the constant magnetic field directed along the third coordinate axis. This spectrum lies on the real axis and includes as the continuous ingredient $(p_3)^2/2m$ as the discrete terms labelled by the numbers n, ν and ε.

It is interesting to compare (30.8) with the spectrum of energies of a relativistic particle found in Subsection 28.3. Representing the last term of (30.8) as a series in powers of $1/m$ and restricting ourselves to the terms of order $1/m^2$ we obtain

$$E_{\varepsilon n\nu p_3}=\varepsilon m+\frac{p_3^2}{2m}+\omega\left(n+\frac{s-\nu}{2s}+\nu f(k_0)\right), \tag{30.9}$$

where $\omega=eH/m$ is the cyclotronic frequency, $f(k_0)=(2s+1+\varepsilon k_0)/2s$ is the parameter defining the deviation of the dipole moment of a particle (i.e., of the coefficient by the term $e\boldsymbol{S}\cdot\boldsymbol{H}/2m$ in the approximate Hamiltonian (10.26), (15.33)) from the value $q=1/s$.

Comparing (30.9) with (28.25) we come to the conclusion that in the case $f(k_0)=0$ the energy spectrum of a particle of arbitrary spin interacting with the constant and homogeneous magnetic field, predicted by the Galilei-invariant wave equations, coincides with the corresponding relativistic spectrum in the approximation $1/m^2$. If, however $f(k_0)\neq 0$ the relation (30.9) can be interpreted as energy spectrum of a particle with anomalous (distinct from $1/s$) magnetic moment.

Let us present the explicit form of the eigenfunction of the Hamiltonian (30.6). In analogy with (28.17)-(28.27) we set

$$\Phi_{n\nu 1p_3}=\exp[i(p_1x_1+p_3x_3)]\begin{pmatrix} C_\nu^+U_n(y)\eta_\nu^s \\ 0\end{pmatrix},$$

$$\Phi_{n\nu -1p_3}=\exp[i(p_1x_1+p_3x_3)]\begin{pmatrix} 0 \\ C_\nu^-U_n(y)\eta_\nu^s\end{pmatrix}.$$

Here $U_n(y)$ is the Hermit function (28.27), $y=(eH)^{1/2}(x_2-p_1/eH)$, η^s_ν are the spinors of (28.11), 0 is a $(2s+1)$-component zero column, $C_\nu^{\pm}$ are normalization constants. The set of eigenfunctions of the starting Hamiltonian (30.1) can be obtained by the transformation

$$\Phi_{n\nu\varepsilon p_3}\rightarrow\psi_{n\nu\varepsilon p_3}=V_2^{-1}V_1^{-1}\Phi_{n\nu\varepsilon p_3}$$

where V_1 and V_2 are the operators (30.3),(30.5).

30.3 Nonrelativistic Particle of Arbitrary Spin in Crossed Electric and Magnetic Fields

Let us consider a more complicated problem when the constant and homogeneous magnetic field is supplemented by the constant and homogeneous electric field being perpendicular to the magnetic one. We again start from the equations (15.26), (15.28) where $k_1=-1$. The corresponding vector-potential A_μ can be chosen in the form

$$A_0=-x_2E,\quad A_1=-x_2H,\quad A_2=A_3=0, \tag{30.10}$$

then

$$\boldsymbol{E}=(0,E,0),\quad \boldsymbol{H}=(0,0,H). \tag{30.10′}$$

The equations (15.26), (15.28) with the chosen potentials can be solved exactly. With the help of the transformation (30.4), (30.6) the corresponding

Hamiltonian (15.26) is reduced to the following form

$$H'' = \frac{\pi^2}{2m} + eA_0 - \frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{H} + \sigma_3(a^2m^2 - 2aek_0\boldsymbol{S}\cdot\boldsymbol{H})^{1/2} \tag{30.11}$$

or

$$H'' = \frac{\pi^2}{2m} - eEx_2 + \sum_{\nu=-s}^{s}\left[-\frac{e}{m}\nu H + \sigma_3(a^2m^2 - 2a\nu ek_0 H)^{1/2}\right]\Lambda_\nu \tag{30.12}$$

where Λ_ν are the projectors of (30.5).

The Hamiltonian H'' commutes with the matrices S_3, σ_3 and the operators P_1, P_3. This is why it is convenient to look for the corresponding eigenfunctions in the form

$$\psi_{\lambda\nu} = \exp[i(p_1x_1 + p_3x_3)]\chi_{\lambda\nu} \tag{30.13}$$

where $\chi_{\lambda\nu}=\chi_{\lambda\nu}(x_2)$ is an eigenvector of the commuting matrices σ_3 and S_3 (corresponding to the eigenvalues $\lambda=\pm1$ and $\nu=-s,-s+1,...,s$), p_1 and p_3 are eigenvalues of the operators P_1 and P_3.

Substituting (30.12), (30.13) into the Schrödinger equation for stationary states we come to the following system

$$H''\psi_{\lambda\nu} = \tilde{E}\psi_{\lambda\nu}$$

which reduces to the following sequence of the noncoupled equations for $\chi_{\lambda\nu}$:

$$\begin{aligned}\Big[&-\partial^2/\partial x_2^2 + (p_1 + eHx_2)^2 + p_3^2 - 2e\nu H + \\ &+2m\lambda(a^2m^2 - 2eak_0\nu H)^{1/2} - 2m(\tilde{E} - eEx_2)\Big]\chi_{\lambda\nu} = 0.\end{aligned} \tag{30.14}$$

Using the change of variables

$$y = \sqrt{|e|H}\left[x_2 + (mE - p_1H)/(eH^2)\right] \tag{30.15}$$

(30.14) reduces to the equations for the harmonic oscillator (compare with (28.22))

$$\left(-d^2/dy^2 + y^2\right)\chi_{\lambda\nu} = K_{\lambda\nu}\chi_{\lambda\nu}, \tag{30.16}$$

besides $K_{\lambda\nu}$ are connected with the eigenvalues $\tilde{E}$ of the Hamiltonian H'' as follows

$$eHK_{\lambda\nu} = (Hp_1 - mE)^2H^{-2} + 2m\tilde{E} - p_3^2 + 2e\nu H - 2m\lambda(a^2m^2 - 2e\nu ak_0H)^{1/2}. \tag{30.17}$$

Solutions of (30.16) are proportional to the Hermit functions (28.27) and the corresponding eigenvalues $K_{\lambda\nu}$ are equal to $2n+1$, $n=0,1,...$. Hence it follows the possible values of energy are

$$\tilde{E} = (2m)^{-1}\left[(2n+1-2\nu)eH + p_3^2\right] + (p_1E)/(2H) - m/2\left(E/H\right)^2 + \lambda(a^2m^2 - 2e\nu ak_0H)^{1/2}. \tag{30.18}$$

In the case E=0 formula (30.18) reduces to (30.8).

In the approximation $1/m^2$ $\tilde{E}$ reduces to the form

$$\tilde{E}=\lambda am+p_3^2+\omega\left(n+\frac{s-\nu}{2s}-\nu f(k_0)\right)+\frac{p_1 e}{2H}-\frac{m}{2}\left(\frac{E}{H}\right)^2 \tag{30.19}$$

where $f(k_0)=(2s+1+\lambda k_0)/2s$. In the case $k_0=0$, $s=1/2$ the energy levels (30.19) coincide with the levels predicted by the Dirac equation for the case of fields of the considered configuration.

We note that the equations (15.26), (15.28) can be solved exactly also for the case of parallel electric and magnetic fields not depending on x_μ. Then we can obtain solutions for arbitrary directed fields using Lorentz transformation.

30.4. Nonrelativistic Particle of Arbitrary Spin in the Coulomb Field

This problem also can be treated successively using Galilei-invariant wave equations.

We start with the first-order equations (15.1) with β-matrices corresponding to column R_3 of the Table 13.1 (pp.160-161). Choosing the vector-potential in the form $\boldsymbol{A}=0$, $A_0=ze/x$ and going with the help of the transformation (15.6), (15.7) to the representation (15.19), (15.20) we obtain the following system

$$\left(p_0+\frac{\alpha}{x}-\frac{p^2}{2m}-\varepsilon_0-\frac{\alpha}{s(s+1)\kappa_2 m}\frac{\boldsymbol{S}\cdot\boldsymbol{x}}{x^3}\right)\Phi^s=0, \tag{30.20}$$

$$\Phi_2^s=-\Phi^s,\quad \Phi_3^s=\left(\frac{\kappa^2}{2}+\frac{\alpha\boldsymbol{S}\cdot\boldsymbol{x}}{x^3 2ms(s+1)\kappa_2}\right)\Phi^s, \tag{30.21}$$
$$\chi_1^{s-1}=\chi_2^{s-1}=0,\quad \Phi^s=\Phi_1^s,\quad \alpha=ze^2.$$

We see that our problem reduces to solving the Schrödinger-type equation (25.20) for (2s+1)-component wave function Φ^s. The remaining components of the vector-function (15.18) are expressed via Φ^s according to (30.21). Further on, we set $\varepsilon_0=m$ without loss of generality.

Solutions of (30.20) can be found in accordance with the relativistic Coulomb problem. Representing Φ^s in the form of (29.1) where $\varepsilon\to\tilde{E}$ we come to the stationary Schrödinger equation

$$\left(\hat{E}+\frac{\alpha}{x}-m-\frac{p^2}{2m}-\frac{\alpha g}{2sm}\frac{\boldsymbol{S}\cdot\boldsymbol{x}}{x^3}\right)\Phi^s(\boldsymbol{x})=0, \tag{30.22}$$

where

$$g=\frac{2}{(s+1)\kappa_2}. \tag{30.23}$$

The equation (30.22) is invariant under the group *O(3)* and so admits solutions in separated variables. Expanding Φ^s in the spherical spinors according to (29.3) and using the representation (29.9) for p^2, we come to the following system of ordinary differential equations for radial functions

$$D'\varphi^{\lambda}=x^{-2}b_{\lambda\lambda'}\varphi^{\lambda'}, \tag{30.24}$$

where

$$D'\equiv 2m(\tilde{\varepsilon}+\alpha/x)+\frac{d^2}{dx^2}+\frac{2}{x}\frac{d}{dx}-\frac{j(j+1)}{x^2}-2m^2,$$

$$b_{\lambda\lambda'}=[\lambda^2-\lambda(2j+1)]\delta_{\lambda\lambda'}+\frac{g\alpha}{s}d^{sj}_{\lambda\lambda'}, \tag{30.25}$$

$$\lambda,\lambda'=-s,-s+1,\dots,-s+2n_{sj},\quad n_{sj}=\min(s,j)$$

and $d^{sj}_{\lambda'\lambda}$ are the matrix elements of the operator $\boldsymbol{S}\cdot\boldsymbol{x}/x$ in the basis of spherical spinors, see (29.7).

The matrix $\|b_{\lambda\lambda'}\|$ is diagonalizable so the equations (30.24) reduces to the chain of the following noncoupled equations

$$D'\varphi=x^{-2}b^{sj}\varphi$$

where D' is the operator (30.25), b^{sj} are eigenvalues of the matrix $\|b_{\lambda\lambda'}\|$.

Until now we have repeated reasoning from Section 29, where the analogous but relativistic problem was considered. Further on, bearing in mind differences between the operators D' (30.25) and D (29.11), we use a new (in comparison with (29.14)) change of variables

$$y=\sqrt{z}\,\varphi,\qquad z=2\sqrt{-2m(\tilde{E}-m)}\;x,\quad \beta=\sqrt{\frac{-m}{2(\tilde{E}-m)}}\,\alpha,\quad k^2=(2j+1)^2+4b^{sj}, \tag{30.26}$$

and again come to the equations (29.13) for y.

We see that the Galilei-invariant equations for a particle of an arbitrary spin in the Coulomb field generate the equation for eigenvalues β which is the same as in the case of relativistic Coulomb problem. But the energy spectrum of a Galilei particle differs from the relativistic one in accordance with another dependence of $\tilde{E}$ on β, compare (30.26) and (29.14). We obtain from (29.18), (30.26) the following energy eigenvalues:

$$\tilde{E}=m-\frac{m\alpha^2}{\left(\sqrt{\left(j+\frac{1}{2}\right)^2+2b^{sj}}+n'+\frac{1}{2}\right)^2},\quad n'=0,1,\dots\,. \tag{30.27}$$

Formula (30.27) defines energy levels of a nonrelativistic particle of arbitrary spin s in the Coulomb field. Here j takes integer (for integer s) or half integer (for half integer s) values defining the total angular momentum of a particle, besides the possible values of b^{sj} coincide with the roots of the characteristic equation (29.24) for the matrix $\|b_{\lambda\lambda'}\|$ of (30.25).

To analyze the spectrum (30.27) and to compare it with the spectrum predicted by relativistic motion equations we consider approximate solutions of the characteristic equation (29.24), (30.25). Representing the r.h.s. of (30.27) as series in powers of α^2 we obtain

$$\tilde{E}=m-\frac{m\alpha^2}{2n^2}-\frac{mg^2\alpha^4 b_\lambda^{sj}}{2n^3(l+1/2)}+o(\alpha^6), \tag{30.28}$$
$$n=n'+l+1=1,2,\ldots;\quad l=j-\lambda=0,1,\ldots,n-1.$$

Formula (30.28) defines a fine structure of the energy spectrum of an arbitrary spin particle in the Coulomb field. The values of the corresponding parameters b_λ^{sj} are easily calculated by formulae (29.8), (29.26).

Besides the constant term m the energy of a particle is defined by the Balmer term - $m\alpha^2/2n^2$ and by the correcting term of order α^4 connected with the spin. Further on it will be shown that this correcting term is caused by the spin-orbit, Darwin and quadruple couplings.

According to (30.28) any energy level corresponding to the given main quantum number n is splited into N_n sublevels of the fine structure corresponding to the admissible values of l and λ given in (30.28). The number N_n is given in (29.30). In the case s=1/2 formula (30.28) reduces to the form

$$\tilde{E}=m-\frac{m\alpha^2}{2n^2}+\frac{\lambda mg^2\alpha^4}{2n^3(l+1/2)(l+\lambda+1/2)},\quad \lambda=\pm 1/2. \tag{30.29}$$

In contrast with the energy spectrum predicted by the Dirac equation (see (29.27), (29.31) for s=1/2) the energy levels (30.29) are nondegenerated.

Comparing formula (30.28) with (29.27) obtained by the solution of the relativistic Coulomb problem we come to the conclusion that for g^2=-1

$$\tilde{E}=\varepsilon+\Delta\varepsilon,\quad \Delta\varepsilon=-\left(\frac{3}{4}-\frac{n}{l+1/2}\right)\frac{m\alpha^4}{2n^4} \tag{30.30}$$

where ε is energy of a relativistic particle given in (29.27).

We see the energy levels of a particle of arbitrary spin in the Coulomb field, obtained from the Galilei invariant wave equations, are shifted by the value $\Delta\varepsilon$ in comparison with the levels predicted by relativistic equations. This result is quite natural inasmuch as $\Delta\varepsilon$ coincides with the mean value of the relativistic correction to kinetic energy [18]: $\Delta\varepsilon=\langle p^4/8m^3\rangle$ where averaging is made by Schrödinger's wave functions.

So Galilei invariant wave equations take into account all the effects predicted by the Poincaré-invariant equations (i.e., the spin-orbit, Darwin and quadruple couplings) except relativistic correction to the kinetic energy. This conclusion is in a good accordance with the result of the analysis of the corresponding approximate Hamiltonians of (10.26).

We repeat that the spin-orbit, Darwin and quadruple couplings can be described in frames of a Galilei-invariant approach, and therefore it is not necessary to interpret these couplings as purely relativistic effects.

In conclusion, we note that the Galilei invariant wave equations can be solved exactly also for some other types of external fields, e.q., for the field of the magnetic monopole. The last problem is formulated and solved in analogy with [164].

31. NONLINEAR EQUATIONS INVARIANT UNDER THE POINCARÉ AND GALILEI GROUPS

31.1. Introduction

In this section we present a brief survey of the main results of papers [128-131, 166-180, 183-192, 6*, 10*, 11*, 14*-25*] where nonlinear equations are investigated which have the same symmetries as the linear equations of D'Alembert, Dirac, Schrödinger and Maxwell.

The superposition principle is not valid for nonlinear differential equations so it is important to select equations being invariant under wide groups from the class, e.g., first and second order partial differential equations. The symmetry properties of such equations enable us to construct wide classes of solutions starting from a known particular solution.

Later, we will consider symmetry properties and exact solutions of a number of nonlinear equations of modern theoretical and mathematical physics. We hope it will be a useful demonstration of power of symmetry methods in application to practically interesting problems.

31.2. Symmetry Analysis and Exact Solution of Scalar Nonlinear Wave Equations

Consider the nonlinear d'Alembert equation

$$p_\mu p^\mu U + F(U) = 0, \tag{31.1}$$

where $F(U)$ is a smooth function, $U=U(x_0,x_1,...x_n)$ is a real scalar function.

Let us denote by $\tilde{P}(1,n)$ the extended generalized Poincaré group, i.e., the group *P(1,n)* completed by the one-parametric group of the scale transformations (1.48).

THEOREM 31.1 [175]. The equation (31.1) is invariant under the group $\tilde{P}(1,n)$ in the two following cases only

$$F(U)=F_1(U)=\lambda_1 U^r, \quad r\neq 1, \tag{31.2}$$

$$F(U)=F_2(U)=\lambda_2 \exp(U), \tag{31.3}$$

where λ_1, λ_2, r are arbitrary numbers. The corresponding generators of the group *P(1,n)* have the form (1.6) (where $\mu, \sigma = 0, 1, \ldots, n$), and the dilatation generator is given by the following formulae

$$D=D_1=x_\mu p^\mu-\frac{2iU}{1-r}\frac{\partial}{\partial U}, \tag{31.4}$$

$$D=D_2=x_\mu p^\mu-2i\frac{\partial}{\partial U}. \tag{31.5}$$

Corollary 1. The equation (31.1) with the nonlinearity (31.3) in the two-dimensional space *R(1,1)* is invariant under the infinite-dimensional Lie algebra including the subalgebra $A\tilde{P}(1,n)$. Such symmetry makes it possible to construct the Liouville solution [175]

$$U(x_0,x_1)=\ln\left(\frac{-8\dot{f}_1(x_0+x_1)\dot{f}_2(x_0-x_1)}{\lambda_2[f_1(x_0+x_1)+f_2(x_0-x_1)]^2}\right) \tag{31.6}$$

where f_1 and f_2 are arbitrary smooth functions, and the dots denote the derivations in respect with the corresponding arguments.

Corollary 2. The solutions (31.6) have singularity in the point $\lambda_2=0$, so it is impossible to apply the standard method of a small parameter to the equation (31.1), (31.3). Singular solutions of the two-dimensional Liouville equations are investigated in paper [363].

Corollary 3. If $r=(n+3)/(n-1)$ then the equation (31.1) with the nonlinearity (31.2) is invariant under the conformal algebra *AC(1,n)* whose basis element have the form (1.6), (31.4), and K_μ of (1.16).

To construct exact solutions of the equation (31.1) we use the ansatz

$$U(x)=f(x)\varphi(\omega)+g(x) \tag{31.7}$$

proposed in [124] and realized effectively for a number of nonlinear equations of mathematical physics [125, 166-168, 173-176]. Here $\varphi(\omega)$ is a function has to be determined which depends on invariant variables $\omega=(\omega_1,\omega_2,\ldots\omega_n)$. The explicit form of $f(x)$ and $g(x)$ is determined from the condition of "separation of variables", i.e., from the requirement the variables $x=(x_0,x_1,...x_n)$ have not to be present explicitly in the equation for $\varphi(\omega)$ obtained from (31.1), (31.7). The invariant variables $\omega(x)$ are the first integrals of the corresponding Euler-Lagrange equation [124,171,186].

Let us represent one of possible ansätze (31.7) for the equation (31.1), which reduces it to the equation with three independent variables:

$$U(x)=[(c\cdot x)^2+(d\cdot x)^2]^{2/(l-r)}\varphi(\omega_1,\omega_2,\omega_3),\quad r\neq 1,$$
$$\omega_1=\frac{a\cdot x}{b\cdot x},\ \ \omega_2=[(c\cdot x)^2+(d\cdot x)^2](a\cdot xb\cdot x)^{-1},\ \ \omega_3=\ln[(c\cdot x)^2+(d\cdot x)^2]+2\theta\arctan\frac{c\cdot x}{d\cdot x}, \tag{31.8}$$

where $a=(a_0,a_1,a_2,a_3)$, $b=(b_0,b_1,b_2,b_3)$, $c=(c_0,c_1,c_2,c_3)$, $d=(d_0,d_1,d_2,d_3)$ are parameters satisfying the conditions

$$a^2=-b^2=-c^2=-d^2=1,\quad a\cdot x\neq 0,\quad b\cdot x\neq 0,\quad c\cdot x\neq 0,$$
$$a\cdot b=a\cdot c=a\cdot d=b\cdot d=b\cdot c=c\cdot d=0,\quad a^2=a\cdot a,\quad a\cdot b=a_0b_0-a_1b_1-a_2b_2-a_3b_3,$$

θ is an arbitrary parameter.

The ansatz (31.8) reduces the equations (31.1) (31.2) to the linear equation with variable coefficients:

$$(1+\omega_1^2)\frac{\partial^2\varphi}{\partial\omega_1^2}-4(1+\theta^2)\frac{\partial^2\varphi}{\partial\omega_3^2}-\omega_3^3\frac{\partial^2\varphi}{\partial\omega_2\partial\omega_3}+\omega_2^2[\omega_2(\omega_1+\omega_1^{-1})-4]\frac{\partial^2\varphi}{\partial\omega_2^2}-$$
$$-2\omega_2^2(1+\omega_1^2)\frac{\partial^2\varphi}{\partial\omega_1\partial\omega_2}-2\omega_1\omega_2\frac{\partial\varphi}{\partial\omega_2}+4k\frac{\partial\varphi}{\partial\omega_3}-k^2\varphi+\lambda_1\varphi^r=0,\quad k=\frac{2}{r-1}. \tag{31.9}$$

To find exact solutions of (31.9) is not an easy thing so it is convenient to reduce this equation to the equation with two independent variables, the last can be reduced to an ordinary differential equation (ODE). In some cases the obtained ODE can be solved analytically.

We represent two multiparametric families of solutions obtained in accordance with the above scheme

$$U=[(a\cdot x)^2+b\cdot xc\cdot x]^{1/(1-r)},$$
$$a\cdot b=a\cdot c=b^2=c^2=0,\quad 2a^2=b\cdot c=\frac{\lambda(r-1)^2}{r-3},\quad r\neq 3; \tag{31.10}$$
$$U=[\Phi(a\cdot x)+b\cdot x]^2/(1-r),$$

where Φ is an arbitrary smooth function,

$$a\cdot a=a\cdot b=0,\quad b\cdot b=-\frac{1}{2}\lambda_1(1-r^2)(1+r)^{-1},\quad r\neq -1.$$

Formula (31.10) gives solutions of (31.1) expressed via an arbitrary function. Such a solution is available for the corresponding initial or boundary values problems.

More general classes of exact solutions of (31.1) are constructed in [171,175]. Nonlinear equations for a complex scalar field $U(x)$ are considered in [131,35*].

Let us consider also the following generalization of the d'Alembert equation

$$\partial_\mu\partial^\mu U + e(\partial_\mu A^\mu + A^\mu\partial_\mu)U + 2eA^\mu\partial_\mu U + e^2 A_\mu A^\mu U = 0. \qquad (31.11)$$

In the case e=0 we come to the usual linear d'Alembert equation, but for nonzero e we have nothing but the equation with minimal interaction

$$\pi^\mu\pi_\mu U = 0.$$

Treating (31.11) as a nonlinear equation for U and A_μ it is possible to prove the following assertions*

THEOREM 31.2. The maximal IA of the equation (31.11) is generated by the following operators

$$P_\mu = p_\mu = i\partial^\mu, \quad J_{\mu\sigma} = x_\mu p_\sigma - x_\sigma p_\mu + S_{\mu\sigma},$$

$$D = x_\mu p^\mu - iA_\mu\partial_{A_\mu} + ikU\partial_U, \quad K_\mu = 2x_\mu D - x_\sigma x^\sigma p_\mu + 2S_{\mu\sigma}x^\sigma + \lambda p_{A_\mu},$$

$$Q = a(x)U\partial_U - (\partial_\mu a(x))\partial_{A_\mu}, \quad I = U\partial_U,$$

where $a(x)$ is an arbitrary differentiable function, λ and k are arbitrary parameters satisfying the condition $2k+\lambda=2-n$, n is the number of independent variables x_μ.

THEOREM 31.3. The equation (31.11) admits the most extended IA in the class of equations

$$\partial_\mu\partial^\mu U + B(\partial_\sigma A^\sigma)U + CA^\mu(\partial^\mu U) + DA_\mu A^\mu U = 0$$

where B, C, D are arbitrary constants.

We see that the principle of minimal coupling leads to the most symmetric equation in the class considered. The corresponding IA is very extended because its basis elements depend on an arbitrary function $a(x)$.

31.3. Symmetries and Exact Solutions of the Nonlinear Dirac Equation

The following equation is a natural generalization of the linear Dirac equation:

$$[\gamma^\mu p_\mu + F(\bar\psi,\psi)]\psi = 0, \qquad (31.12)$$

where $F(\bar\psi,\psi)$ is an arbitrary 4×4 matrix whose elements are smooth functions of the field variables.

A description of all the matrices F for which the equation (31.12) be invariant

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*These results where obtained by Dr. Zhdanov under the groups *P(1,3)*, $\tilde{P}(1,n)$ or *C(1,3)* is given in the following assertion.

THEOREM 31.4 [172]. The equation (31.12) is invariant under one of the following groups: a) the Poincaré group, b) extended Poincaré group, c) conformal group iff

$$a)\ F(\bar{\psi},\psi)=F_1+F_2\gamma_4+F_3\gamma^\mu\,\bar{\psi}\gamma_4\gamma_\mu\psi+F_4S^{\nu\mu}\,\bar{\psi}\gamma_4S_{\nu\mu}\psi, \tag{31.13}$$

where F_1, F_2, F_3, F_4 are arbitrary scalar functions of $\bar{\psi}\psi$, $\bar{\psi}\gamma_4\psi$;

b) F has form (31.13) where

$$F_i=(\bar{\psi}\psi)^{1/2k}G_i,\quad i=1,2;\quad F_j=(\bar{\psi}\psi)^{(1-2k)/2k}G_j,\quad j=3,4, \tag{31.14}$$

G_i, G_j are arbitrary functions of $\bar{\psi}\psi/\bar{\psi}\gamma_4\psi$, k is an arbitrary constant;

c) F has the form (31.13) where

$$F_i=(\bar{\psi}\psi)^{-1/3}G_i,\quad i=1,2;\quad F_j=(\bar{\psi}\psi)^{-2/3}G_j,\quad j=3,4. \tag{31.15}$$

The simplest conform-invariant equations of the class (31.12), (31.13), (31.15) have the form

$$[\gamma_\mu p^\mu+\lambda(\bar{\psi}\psi)^{1/3}]\psi=0, \tag{31.16}$$

$$[\gamma_\mu p^\mu+\lambda\gamma^\mu\,\bar{\psi}\gamma_\mu\psi(\bar{\psi}\gamma_\nu\psi\bar{\psi}\gamma^\nu\psi)^{-1/3}]\psi=0, \tag{31.17}$$

The equation (31.16) was obtained by Gürsey [212] for the first time.

Corollary. There exist first order equations for the spinor field, which have a more extended symmetry than the Dirac equation (31.12). An example of such an equation is the following system [125,126]

$$\bar{\psi}\gamma^\mu\,\psi p_\mu\psi=0$$

which is invariant under the infinite-dimensional algebra.

Consider the $\tilde{P}(1,n)$-invariant nonlinear equation

$$\left[\gamma_\mu p^\mu+\lambda(\bar{\psi}\psi)^{1/(2k)}\right]\psi=0, \tag{31.18}$$

where λ and $k\neq0$ are arbitrary constants. We search for solutions of (31.18) in the form [124,125]

$$\psi=A(x)\varphi(\omega) \tag{31.19}$$

where $A(x)$ is a 4×4 matrix, $\varphi(\omega)$ is a four-component function depending on three new variables $\omega=\{\omega_1(x),\omega_2(x),\omega_3(x)\}$.

The ansatz (31.19) leads to the equation for $\varphi(\omega)$, which depends on ω only if $A(x)$ and ω_i satisfy the equations [186-192]

$$\left(\xi^{\mu}(x)\frac{\partial}{\partial x^{\mu}}+\eta(x)\right)A(x)=0,\quad \xi^{\mu}(x)\frac{\partial\omega_i}{\partial x^{\mu}}=0,\quad i=1,2,3,$$

where $\xi^{\mu}(x)$, $\eta(x)$ are coefficients of infinitesimal operators of the group $\tilde{P}(1,n)$.

Without going into details we present two solutions [186]:

$$A(x)=(x_0-x_2)^{-k}\exp\left\{\frac{1}{2a}\gamma_1(\gamma_2-\gamma_0)\ln(x_0-x_2)\right\},$$
$$\omega_1=(x_0^2-x_1^2-x_2^2)x_3^{-2},\quad \omega_2=(x_0-x_2)x_3^{-1},\quad \omega_3=ax_1(x_0-x_2)^{-1}-\ln(x_0-x_2),\quad a\neq 0. \tag{31.20}$$

If $a=0$ then

$$A(x)=(2x_0+2x_1+\beta)^{-k/2}\exp\left[\frac{1}{4}\gamma_0\gamma_1\ln(2x_0+2x_1+\beta)-\frac{1}{2}\gamma_2\gamma_3\arctan\frac{x_2}{x_3}\right],\quad \beta\neq 0,$$
$$\omega_1=(2x_0+2x_1+\beta)\exp[2(x_1-x_0)\beta^{-1}],\quad \omega_2=(2x_0+2x_1+\beta)(x_2^2+x_3^2)^{-1}, \tag{31.21}$$
$$\omega_3=b\ln(x_2^2+x_3^2)+2\arctan\frac{x_2}{x_3}.$$

Here β and b are arbitrary parameters.

Let us represent the explicit form of solutions of (31.18) for three values of the parameter k [186].

The case $k=1/2$:

$$\psi=[(a\cdot z)^2+(b\cdot z)^2]^{-1/4}\exp\left(-\frac{1}{2}\gamma\cdot a\gamma\cdot b\arctan\frac{a\cdot z}{b\cdot z}\right)\exp\left(i\lambda\frac{\tilde{\chi}\chi}{2(1+\theta^2)}(\gamma\cdot b+\right.$$
$$\left.+\theta\gamma\cdot a)\left[\ln((a\cdot z)^2+(b\cdot z)^2)+2\theta\arctan\frac{a\cdot z}{b\cdot z}\right]\right)\chi \tag{31.22}$$

where $z_{\mu}=x_{\mu}+\theta_{\mu}$, $\theta=(\theta_{\mu}\theta^{\mu})^{1/2}$, a_{μ}, b_{μ}, θ_{μ} are arbitrary parameters satisfying the conditions $a\cdot a\equiv a_{\mu}a^{\mu}=-1$, $b\cdot b=-1$, $a\cdot b=0$, χ is a constant spinor.

The case $k\neq 1/2$:

$$\psi=[(a\cdot z)^2+(b\cdot z)^2]^{-1/4}\exp\left(-\frac{1}{2}\gamma\cdot a\gamma\cdot b\arctan\frac{a\cdot z}{b\cdot z}\right)\times$$
$$\times\exp\left(\frac{2ik\lambda}{2k-1}\gamma\cdot b(\tilde{\chi}\chi)^{1/(2k)}[(a\cdot z)^2+(b\cdot z)^2)]^{(1-2k)/(4k)}\right)\chi. \tag{31.23}$$

The case $k=1/3$:

$$\psi=\gamma\cdot x(x\cdot x)^{-2}\exp[i\lambda k\gamma\cdot\beta\beta\cdot x(x\cdot x)^{-1}]\chi,\quad \beta\cdot\beta>0,\quad x\cdot x\neq 0. \tag{31.24}$$

The relations (31.20)-(31.24) define the multiparameter families of exact solutions of the equation (31.18).

If k=1/3 then the equation (31.18) is conform-invariant. It means that if we know a particular solution of (31.18) then it is possible to generate other solution using conformal transformations.

Using the given ansätze, families of exact solutions of the equations of classical electrodynamics

$$(\gamma^\mu p_\mu + e\gamma_\mu A^\mu + m)\psi = 0,$$
$$p_\nu p^\nu A_\mu - p_\mu p_\nu A^\nu = e\bar{\psi}\gamma_\mu\psi$$

and many other nonlinear equations have been obtained in [168,188].

A new conformal invariant equation for the spinor field

$$\left\{\gamma_\mu p^\mu - \lambda_1 W_1^{1/4} - \lambda_2(\bar{\psi}\psi)^{1/3}\right\}\psi = 0$$

(where C_2 is the invariant of the electromagnetic field (31.34)) was proposed in [30*].

We do not consider two-dimension nonlinear integrable equations which were investigated by great many of authors (see [350] and the references cited there in).

NOTE. In this book we consider only linear representations of algebras *AP(1,3)* and *AC(1,3)*. But there exist nonlinear equations invariant under nonlinear realizations
of the conformal algebra. Thus on the solutions of the eikonal equation

$$\frac{\partial U}{\partial x_\mu}\frac{\partial U}{\partial x^\mu} = 1$$

the following nonlinear representation of the algebra *AC(1,4)* is realized [36*]

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\sigma} = x_\mu p_\sigma - x_\sigma p_\mu, \quad P_U \equiv p_U = -i\frac{\partial}{\partial U},$$

$$J_{04} = x_0 p_U - U p_0, \quad J_{4\mu} = U p_\mu - x_\mu p_U,$$

$$D = x_\mu p^\mu - U p_U, \quad K_\mu = 2x_\mu D - \left(x_\mu x^\mu - U^2\right) p_\mu.$$

Nonlinear representations of the algebras *AP(1,1)*, *AP(1,2)*, *AP(2,2)*, *AC(2,2)* are found in [37*-39*]. Moreover, the nonlinear representation of the algebras *AP(2,2)* and *AC(2,2)* is realized by the operators P_μ, $J_{\mu\sigma}$, K_μ, D, which have the form (2.22), (2.42) where the Greek indices run over the values 1, 2, 3, 4,

$$S_{ab} = \varepsilon_{abc}\Sigma_c, \quad S_{4a} = \varepsilon\Sigma_a, \quad \varepsilon = \pm 1; \quad K = 0,$$

$$\Sigma_1 = -i\sin U\partial_U, \quad \Sigma_2 = i\cos U\partial_U, \quad \Sigma_3 = -i\partial_U,$$

and the covariant summation over the repeating indices is imposed using the metric tensor $g_{\mu\sigma}=1$, $\mu=\sigma=1,2$; $g_{\mu\sigma}=-1$, $\mu=\sigma=3,4$; $g_{\mu\sigma}=0$, $\mu\neq\sigma$.

31.4. Equations of Schrödinger Type Invariant Under the Galilei Group

Here we consider the following nonlinear generalization of the Schrödinger equation

$$\left(p_0 - \frac{p^2}{2m}\right)U + F(x,U,U^*) = 0 \qquad (31.25)$$

where F is an arbitrary differentiable function.

The maximal IA of (31.25) for F=0 is the algebra $AG_2(1,3)$ whose basis elements are given in (11.5). We denote by $AG_1(1,3)$ the subalgebra including the basis elements P_0, P_a, J_a, G_a, D of (11.5).

Of course symmetry of the equation (31.25) depends on the structure of F.

THEOREM 31.5 [124,176]. The equation (31.25) is invariant under the following algebras:

$AG(1,3)$, iff $F=\Phi(|U|)U$, Φ is an arbitrary smooth function;

$AG_1(1,3)$, iff $F=\lambda|U|^kU$, λ and $k\neq 0$ are arbitrary parameters besides the corresponding generator of scale transformations has the form $D=2x_0p_0-\boldsymbol{x}\cdot\boldsymbol{p}+2i/k$;

$AG_2(1,3)$, iff $F=\lambda|U|^{3/4}U$.

More general (than (31.25)) equations of Schrödinger type are investigated in [129]. One such equation has the form

$$\left(p_0 - \frac{p^2}{2m}\right)U + \lambda U \frac{\partial(UU^*)}{\partial x_a}\frac{\partial(UU^*)}{\partial x_a}(UU^*)^{-2} = 0. \qquad (31.26)$$

This equation is invariant under the algebra $AG_1(1,3)$. A wide classes of exact solutions of (31.26) have been constructed in [14] using continuous subgroups of the Galilei group [127].

In conclusion, we represent three ansätze and families of solutions of the equation [176]

$$\left(p_0 - \frac{p^2}{2m}\right)U + \lambda|U|^{4/3}U = 0. \qquad (31.27)$$

We search for solutions of (31.27) in the form (31.7) where φ is a function of invariant variables $\omega=(\omega_1,\omega_2,\omega_3)$. The explicit form of $f(x)$ and ω which enables us to reduce (31.27) to the equation with three independent variables is given by the following formulae

$$f(x)=(1-x_0^2)^{-3/4}\exp\left[\frac{imx_0x^2}{2(1-x_0^2)}\right],$$

$$\omega_1=a\cdot x(1-x_0^2)^{-1/2},\quad \omega_2=x^2(1-x_0^2)^{-1},\quad \omega_3=\arctan x_0+\arctan\frac{b\cdot x}{c\cdot x}, \tag{31.28}$$

$$x^2=x\cdot x,\quad b\cdot x=b_1x_1+b_2x_2+b_3x_3,$$

$$f(x)=x_0^{-3/2}\exp\left(\frac{-imx^2}{2x_0}\right),$$

$$\omega_1=a\cdot x x_0^{-1},\quad \omega_2=x^2x_0^{-2},\quad \omega_3=x_0^{-1}+\arctan\frac{b\cdot x}{c\cdot x}, \tag{31.29}$$

$$f(x)=x_0^{-3/4},\quad \omega_1=a\cdot x x_0^{-1/2},\quad \omega_2=b\cdot x x_0^{-1/2},\quad \omega_3=c\cdot x x_0^{-1/2}$$

where a, b, c are constant vectors satisfying the conditions $a\cdot a=b\cdot b=c\cdot c=1$, $a\cdot b=a\cdot c=b\cdot c=0$.

Exact solutions of (31.27) are constructed using the ansätze (31.28)-(31.30). The simplest solutions are

$$U=(1-x_0^2)^{3/4}\exp\left[\frac{imx^2}{2(1-x_0)}\right],\quad \lambda=\frac{3}{2}i,$$

$$U=x_0^{-3/2}\exp\left[-\frac{im}{2}(x^2-r\cdot x)x_0^{-1}\right],\quad r^2=-\frac{8\lambda}{m},$$

$$U=x_0^{-3/2}\varphi(\omega_1)\exp\left(\frac{imx^2}{2x_0}\right),\quad \omega_1=\frac{a\cdot x}{x_0}.$$

The function $\varphi(\omega_1)$ is determined by the elliptic integral

$$\int_0^{\varphi}\frac{\partial\tau}{(k_1+\tau^{10/3})^{1/2}}=\left(\frac{6}{5}\lambda m\right)^{1/2}(\omega_1+k_2),$$

k_1 and k_2 are arbitrary constants.

Formulae (31.30) present multiparametrical families of exact solutions of the nonlinear equation (31.26).

Let $U=U(x_0,\boldsymbol{x})$ is a solution of (31.26). To obtain other solutions we can use the following formulae

$$U_1=U(x_0,\boldsymbol{x}+\boldsymbol{v}x_0)\exp\left[im\left(\frac{\boldsymbol{v}^2x_0}{2}+\boldsymbol{v}\cdot\boldsymbol{x}\right)\right],$$

$$U_2=U\left(\frac{x_0}{1-ax_0},\ \frac{\boldsymbol{x}}{1-ax_0}\right)(1-ax_0)^{-3/2}\exp\left[\frac{iam\boldsymbol{x}^2}{2(1-ax_0)}\right]$$

where a, $\boldsymbol{v}$ are arbitrary constants. These formulae reflect the fact that the equation (31.26) is invariant under the Galilei group. For more details see [171, 176].

Symmetry properties and component reduction of the equation (31.25) with the logarithmic nonlinearity

$$F(x,U,U^*)=(\lambda_1+i\lambda_2)U\ln(U^*U),\quad \lambda_2\neq 0$$

was investigated in papers [32*, 33*]. Besides the generators of the Galilei group this equation admits the additional SO

$$Q=\exp(2i\lambda_2 x_0)\left[U\frac{\partial}{\partial U}+U^*\frac{\partial}{\partial U^*}-\frac{i\lambda_1}{\lambda_2}\left(U\frac{\partial}{\partial U}-U^*\frac{\partial}{\partial U^*}\right)\right].$$

This SO generates similarity transformations of coordinates $x_0\to x_0'=x_0$, $x_a\to x_a'$, and nontrivial one-parametrical transformation for U:

$$U\to U'=\exp\left[\theta\left(1-i\frac{\lambda_1}{\lambda_2}\right)\exp\left(2\lambda_2 x_0\right)\right]U.$$

Lie symmetry and reduction of multidimensional systems of the Schrödinger type was investigated in papers [129, 14*] where, in particular, one-dimension Galilei-invariant equations (31.25) with nonlinearity

$$F=U|U|^4h\left(|U|_x|U|^3\right)$$

(where h is an arbitrary smooth function) was considered.

31.5. Symmetries of Nonlinear Equations of Electrodynamics

The electromagnetic field in a medium is described by Maxwell's equations

$$i\frac{\partial\boldsymbol{D}}{\partial x_0}=-\boldsymbol{p}\times\boldsymbol{H},\quad i\frac{\partial\boldsymbol{B}}{\partial x_0}=\boldsymbol{p}\times\boldsymbol{E},\quad \boldsymbol{p}\cdot\boldsymbol{D}=0,\quad \boldsymbol{p}\cdot\boldsymbol{B}=0. \tag{31.31}$$

where $\boldsymbol{D}$ and $\boldsymbol{B}$ are vectors of induction, $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors of strengths. The underdetermined system (31.31) has to be completed by constitutive equations (equations connecting $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{H}$) which reflect the properties of the medium.

Following [182] we present here some results connecting symmetries of equations (31.31) supplemented by constitutive equations.

First we note that the system (31.31) without additional conditions is invariant under an infinite-dimensional Lie algebra including the subalgebra *AIGL(4,R)* [157].

Let us represent constitutive equations in the form of the following functional relations

$$\boldsymbol{E}=\Phi(\boldsymbol{D},\boldsymbol{H}), \quad \boldsymbol{B}=\boldsymbol{F}(\boldsymbol{D},\boldsymbol{H}), \tag{31.32}$$

where Φ and $\boldsymbol{F}$ are arbitrary smooth functions of $\boldsymbol{D}$ and $\boldsymbol{H}$.

THEOREM 31.6. The system of equations (31.31), (31.32) is invariant under the group *P(1,3)* iff

$$\boldsymbol{D}=M\boldsymbol{E}+N\boldsymbol{B}, \quad \boldsymbol{H}=M\boldsymbol{B}-N\boldsymbol{E} \tag{31.33}$$

where $M=M(C_1,C_2)$, $N=N(C_1,C_2)$ are arbitrary functions of the invariants of the electromagnetic field,

$$C_1=\boldsymbol{E}^2-\boldsymbol{B}^2, \quad C_2=\boldsymbol{B}\cdot\boldsymbol{E}. \tag{31.34}$$

Proof is given in [182]. ■

In the case $M=L^{-1}$, $N=\boldsymbol{B}\cdot\boldsymbol{E}L^{-1}$, $L=(1-\boldsymbol{B}^2-\boldsymbol{E}^2-(\boldsymbol{B}\cdot\boldsymbol{E}))^{1/2}$ the system (31.31), (31.33) coincides with the Born-Infeld equations [48].

If we set in (31.33) $M=\varepsilon$, $N=-\mu\boldsymbol{B}\cdot\boldsymbol{E}$, ε, μ are constants, then the constitutive equations reduce to the form

$$\boldsymbol{D}=\varepsilon\left[1+\frac{\mu^2(\boldsymbol{E}\cdot\boldsymbol{H})^2}{\varepsilon(\varepsilon+\mu\boldsymbol{E}^2)^2}\right]\boldsymbol{E}-\frac{\mu\boldsymbol{E}\cdot\boldsymbol{H}}{\varepsilon(\varepsilon+\mu\boldsymbol{E}^2)}\boldsymbol{H}, \quad \boldsymbol{B}=\frac{1}{\varepsilon}\boldsymbol{H}-\frac{\mu}{\varepsilon}\frac{\boldsymbol{E}\cdot\boldsymbol{H}}{(\varepsilon+\mu\boldsymbol{E}^2)}\boldsymbol{E}.$$

A popular form of constitutive equations is

$$\boldsymbol{B}=\mu(\boldsymbol{E},\boldsymbol{H})\boldsymbol{H}, \quad \boldsymbol{D}=\varepsilon(\boldsymbol{E},\boldsymbol{H})\boldsymbol{E}.$$

It follows from Theorem 31.6 that the condition of Poincaré-invariance generates the following restrictions for μ and ε:

$$\varepsilon\mu=1.$$

If $\boldsymbol{B}=\Phi(\boldsymbol{H})$, $\boldsymbol{D}=\boldsymbol{F}(\boldsymbol{E},\boldsymbol{H})$ then in accordance with Theorem 31.6

$$\boldsymbol{D}=\mu\boldsymbol{E}, \quad \boldsymbol{B}=\mu^{-1}\boldsymbol{H} \quad (\mu=const).$$

The restrictions imposed by the requirement of the conformal invariance are formulated in the following assertion.

THEOREM 31.7. The system of equations (31.31), (31.33) is invariant under the conformal group if

$M=M(C_1/C_2),\ \ N=N(C_1/C_2).$

For the proof see [182].

An example of the conformal-invariant constitutive equations is given by the following formulae

$$\boldsymbol{D}=\sqrt{\frac{\mu \boldsymbol{H}^2}{\mu \boldsymbol{E}^2-\boldsymbol{E}\cdot\boldsymbol{H}}}\,\boldsymbol{E},\quad \boldsymbol{B}=\sqrt{\frac{\mu \boldsymbol{E}^2-\boldsymbol{E}\cdot\boldsymbol{H}}{\mu \boldsymbol{H}^2}}\,\boldsymbol{H}.$$

Corollary 1. The nonlinear Born-Infeld equations are not invariant under the group *C(1,3)*.

Corollary 2. A class of nonlinear equations for the electromagnetic field was proposed in [124], which we write in the form

$$D^{\mu}F_{\mu\nu}=j_{\nu},\quad D^{(\alpha}F^{\mu\nu)}=0,$$

$$j_{\nu}=A_{\nu\alpha\beta}F^{\alpha\beta},\quad D^{\mu}=A_1\frac{\partial}{\partial x_{\mu}}+A_2F^{\mu\nu}\frac{\partial}{\partial x^{\nu}}+A_3\frac{\partial F^{\mu\nu}}{\partial x^{\nu}},$$

where A_a, $A_{\sigma\alpha\beta}$ are arbitrary functions of invariants of the electromagnetic field and of $(\partial C_{\alpha}/\partial x_{\mu})(\partial C_{\alpha}/\partial x^{\mu})$, $\alpha=1,2$.

Let us show one more nonlinear generalization of Maxwell's equations [29*]:

$$i\frac{\partial \boldsymbol{E}}{\partial t}=-v\boldsymbol{p}\times\boldsymbol{H},\quad i\frac{\partial \boldsymbol{H}}{\partial t}=v\boldsymbol{p}\times\boldsymbol{E}, \tag{31.35}$$

$$\boldsymbol{p}\cdot\boldsymbol{E}=0,\quad \boldsymbol{p}\cdot\boldsymbol{H}=0,$$

where $v=v(\boldsymbol{H}^2,\boldsymbol{E}^2,\boldsymbol{E}\cdot\boldsymbol{H})$ is modula of the velocity of propagation of the electromagnetic field. The density ρ and velocity $\boldsymbol{v}$ can be defined as follows [29*]

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_l)}{\partial x_l}=0,\quad \rho=\rho(\boldsymbol{E}\cdot\boldsymbol{H},\boldsymbol{E}^2,\boldsymbol{H}^2), \tag{31.36}$$

$$\rho v_l=a(\boldsymbol{E}^2-\boldsymbol{H}^2,\boldsymbol{E}\cdot\boldsymbol{H})\varepsilon_{lkn}E_kH_n,\quad k,l,n=1,2,3,$$

where a is a smooth function of the invariants of the electromagnetic field. The last relation coinsides with the Pointing formula, if $\rho=(\boldsymbol{E}^2+\boldsymbol{H}^2)/2$. In this case (31.36) coinsides with the continuity equation.

The essential difference of the system (31.35)-(31.37) from the classical Maxwell equations is that this system is nonlinear.

By studying symmetries of the nonlinear equations for the vector-potential

$$p_{\sigma}p^{\sigma}A_{\mu}-p_{\mu}(p^{\sigma}A_{\sigma})=0,$$

$$(p_{\mu}-ieA_{\mu})A^{\mu}=0$$

a new vector representation of the algebra *AC(1,3)* was found [10*]. The corresponding basis elements are P_μ, $J_{\mu\sigma}$, D of (2.22), (2.42) where

$$S_{\mu\sigma}=-i\left(A_\mu\frac{\partial}{\partial A^\sigma}-A_\sigma\frac{\partial}{\partial A^\mu}\right),\quad K=iA_\mu\frac{\partial}{\partial A_\mu},$$

but the remaining generators K_μ are singular in respect with the coupling constant e:

$$K_\mu=2x_\mu D-x^\sigma x_\sigma P_\mu+2x^\sigma S_{\mu\sigma}-2\frac{i}{e}\frac{\partial}{\partial A_\mu}.$$

In [29*] nonlinear generalizations of different wave equations (d'Alembert, KGF, Dirac) was proposed. In particular, the equation

$$\frac{\partial^2\boldsymbol{E}}{\partial t^2}+v^2p^2\boldsymbol{E}=0,\quad \frac{\partial^2\boldsymbol{H}}{\partial t^2}+v^2p^2\boldsymbol{H}=0$$

was considered. Moreover, the velocity of propagation of the electromagnetic field is defined by the equation

$$\lambda_1 v_\alpha\frac{\partial v_\mu}{\partial x^\alpha}+\lambda_2\partial_\nu\partial^\nu v_\mu=0$$

where λ_1, λ_2 are some parameters.

It follows from the above that the nonlinear description of the dynamics of the classical electromagnetic field can be developed in different ways. Until now we have not canonical equations for nonlinear electrodynamical processes [30*].

31.6. Galilei Relativity Principle and Nonlinear Heat Equations

It is generally accepted to think that nonlinear heat processes are described by the equation

$$\frac{\partial}{\partial x_0}+\frac{\partial}{\partial x_a}\left[C(U)\frac{\partial U}{\partial x_a}\right]=0 \tag{31.37}$$

where $U=U(x_0,x_1,x_2,x_3)$ and $C(U)$ are real functions.

Group properties of the corresponding one-dimensional equation (a=1, $C(U)$=const) where investigated by Sofus Lie. The group analysis of one-dimensional equation with arbitrary C was carried out by Ovsiannikov [355], group properties of the three-dimensional equation (31.37) are described in [85].

It was pointed out in [6*] that any of the equations (31.37) is not invariant under Galilei transformations. Thus if the Galilei relativity principle is valid for heat processes, then it is necessary either to search for other equations or to look for such subsets of solutions of (31.37) which are invariant under Galilei transformations. The first ability was discussed in [6*]. Here we consider the second possibility and demonstrate that it is possible to add such additional conditions to (31.37) that the corresponding system of equations be Galilei invariant.

Let us consider a nonlinear equation of the second order

$$L(x,U,U_1,U_2,...,U_n)=0, \quad x\in R(1,n),$$
$$U_1=\left(\frac{\partial U}{\partial x_0},\frac{\partial U}{\partial x_1},\dots,\frac{\partial U}{\partial x_n}\right), \quad U_2=\left(\frac{\partial^2 U}{\partial x_0^2},\frac{\partial^2 U}{\partial x_0\partial x_1},\dots,\frac{\partial^2}{\partial x_n^2}\right), \dots \tag{31.38}$$

DEFINITION 31.1 (S.Lie). The equation (31.38) is invariant under the operators

$$X=\xi^\mu(x,U)\frac{\partial}{\partial x^\mu}+\eta(x,U)\frac{\partial}{\partial U}, \quad \mu=0,1,...,n,$$

if

$$\tilde{X}L\big|_{L=0}=0, \quad or \quad \tilde{X}L=\lambda(x,U,U_1,...,U_n)L$$

where $\tilde{X}$ is the corresponding prolongation of the operator X, λ is an arbitrary smooth function (for definition of prolongation of operators see [355]).

Let an operator Q does not belong to the IA of the equation (31.38) and its prolongation is given by the formulae

$$\tilde{Q}L=\lambda_0 L+\lambda_1 L_1,$$
$$\tilde{Q}L_1=\lambda_2 L+\lambda_3 L_1, \tag{31.39}$$

besides the following equation is satisfied

$$L_1\equiv L_1(x,U,U_1,U_2)=0. \tag{31.40}$$

DEFINITION 31.2 [167, 6*]. We say the equation (31.38) is *conditionally invariant* if it is invariant under the operator Q together with the additional condition (31.40), i.e., if the relations (31.39) hold.

The additional condition (31.40) selects such subsets of solutions of the equation (31.38), which have a more extended symmetry than the complete set of solutions. Of course we suppose that the system (31.38), (31.40) is compatible.

The main point of the approach connected with conditional symmetry is to find a way to select such additional conditions which extend the symmetry of the starting equation.

DEFINITION 31.3 [183]. We say the equation (31.36) is *Q-invariant* if

$$\tilde{Q}L=\lambda_0 L+\lambda_1(QU). \tag{31.41}$$

Let us formulate the assertion about the conditional symmetry of the equation (31.37).

THEOREM 31.8 [180]. The equation (31.37) is conditionally invariant under the generators of the Galilei group

$$G_a = x_0 \frac{\partial}{\partial x_a} + M(U) x_a \frac{\partial}{\partial U} \tag{31.42}$$

if (31.40) has the form

$$L_1 \equiv \frac{\partial U}{\partial x_0} + \frac{1}{2} M^{-1}(U) \frac{\partial U}{\partial x^a} \frac{\partial U}{\partial x^a} = 0, \quad M(U) = \frac{1}{2} U C^{-1}(U). \tag{31.43}$$

THEOREM 31.9 [180]. The equation (31.37) is Q-invariant under the operators (31.42) if

$$C(U) = \frac{1}{2m} U^r, \quad M(U) = 2mr^{-n-2} U^{1-r}$$

where n is the number of space variables, $m \neq 0$, $r \neq -2n^{-1}$ are arbitrary constants.

Corollary. The Galilei relativity principle is valid for the overdetermined system including (31.37) and the following equation:

$$\frac{\partial U}{\partial x_0} + \frac{1}{2} M^{-1}(U) \frac{\partial U}{\partial x^a} \frac{\partial U}{\partial x^a} = 0 \tag{31.44}$$

where M is a function defined in (31.43).

We note that the system (31.37), (31.44) reduces to the system of the Laplace and Hamilton-Jacobi equations

$$\Delta W = 0, \quad \frac{\partial W}{\partial x_0} + \frac{1}{2m} (\nabla W)^2 = 0, \quad W = 2m \int C(U) U^{-1} dU. \tag{31.45}$$

THEOREM 31.10. The maximal (in Lie sense) IA of the equations (31.45) is $AG_1(1,3)$. The two bases of this IA are (with i=1 or i=2):

$$P_0^{(i)} = \frac{\partial}{\partial x_0}, \quad P_a^{(i)} = \frac{\partial}{\partial x_a}, \quad J_{ab}^{(i)} = x_a p_b - x_b p_a,$$

$$D^{(1)} = 2x_0 P_0 + x_a P_a, \quad D^{(2)} = 2WP_{n+1} + x_a P_a,$$

$$G_a^{(1)} = x_0 P_a + m x_a P_{n+1}, \quad G_a^{(2)} = WP_a + m x_a P_0, \quad a,b = 1,2,\dots,n.$$

The proof reduces to using of the standard Lie algorithm. ■

We note that the operators $G_a^{(1)}$ generate usual Galilei transformations. As to $G_a^{(2)}$, they generate the following finite transformations [180]

$$x_0' = \frac{m}{2} \tau^2 U + m x_a \tau_a + x_0, \quad x_a' = \tau_a U + x_a, \quad U'(x') = U(x),$$

where $\tau^2=\tau_a\tau_a$, τ_a are group parameters. We see that independent variables are transformed in nonusual manner.

Conditionally invariant systems of equation of the Schrödinger type were investigated in [171].

Now it is established that all the main equations of mathematical and theoretical physics (Maxwell, d'Alembert, Dirac, Schrödinger, KdV, Born-Infeld, Navier-Stokes, etc.) have non-trivial conditional symmetries [34*].

31.7. Conditional Symmetry and Exact Solutions of the Boussinesq Equation

It is known that the maximal IA of the Boussinesq equation

$$U_{00}+\frac{1}{2}\Delta U^2+\Delta^2 U=0, \quad U=U(x), \quad x\in R_{1+n} \tag{31.47}$$

(where $U_{00}=\partial^2 U/\partial x_0^2$) is the Lie algebra of the extended Euclid group with the following basis elements

$$P_0=\frac{\partial}{\partial x_0}, \quad P_a=\frac{\partial}{\partial x_a},$$
$$J_{ab}=x_aP_b-x_bP_a, \quad D=2x_0P_0+x_aP_a-2U\partial_U. \tag{31.48}$$

All the nonequivalent ansätze reducing the two-dimensional (n=1) equation (31.47) to an ODE, which are generated by this IA, have the form

$$U=\varphi(\omega), \quad \omega=a_0x_0+a_1x_1, \quad a_0,a_1=const;$$
$$U=x_0^{-1}\varphi(\omega), \quad \omega=x_1x_0^{-1/2}. \tag{31.49}$$

Using the ansatz

$$U=\varphi(\omega)-4\mu^2x_0^2, \quad \omega=x_1+\mu x_0^2, \quad \mu=const \tag{31.50}$$

Olver and Rosenau [351] reduced the two-dimensional equation (31.47) to the ODE

$$\varphi'''+\varphi\varphi'+2\mu\varphi=8\mu^2\omega+C. \tag{31.51}$$

The operator corresponding to this ansatz has the form

$$Q=P_0-2\lambda x_0P_1-8\lambda^2x_0\partial_U, \quad \lambda=-2\mu$$

and does not belong to the IA (31.48).

It is natural to call the ansatz (31.50) non-Lie since it does not follow from the group properties of the equation (31.47).

Following [127] and using the concept of conditional invariance we will describe non-Lie ansätze reducing (31.48) to ODE.

Using the direct substitution method Clarkson and Kruskal [68] had described ansätze of the form (31.50) reducing the two-dimensional Boussinesq equation to ODE. The distinguishing feature of our approach from the approach used in [68] is that the conditional invariance idea makes to be clear the reason of appearing of such ansätze and gives the regular procedure to find non-Lie ansätze. Moreover, the conditional invariance makes it possible to construct such ansätze which cannot be obtained in the way proposed in [68].

First we consider the two-dimensional equation (31.47)

$$U_{00}+UU_{11}+U_1^2+U_{1111}=0. \tag{31.52}$$

THEOREM 31.11. The equation (31.51) is Q-invariant under the operator

$$Q=A(x)\partial_0+B(x)\partial_1+[\alpha(x)U+\beta(x)]\partial_U \tag{31.53}$$

if the functions $A(x)$, $B(x)$, $\alpha(x)$, $\beta(x)$ satisfy the following equations.

Case 1. A is non-zero (without loss of generaliry we set A=1).

$$\alpha=-2B_1, \quad \alpha_1=B_{11} \quad \beta=-2B(B_0+2BB_1),$$

$$\beta_1=\frac{1}{2}B_{00}+(\alpha B)_0+B_1(B_0-BB_1+4\alpha B), \tag{31.54}$$

$$\beta_{11}=-(\partial_0+4B_1)(\alpha_0+\alpha^2), \quad \beta_0-2B_0\beta_1+4B_1(\beta_0-B\beta_1+\alpha\beta)+2\alpha_0\beta=0;$$

Case 2. A=0, B=1

$$\alpha=0, \quad \alpha_{11}+5\alpha\alpha_1+2\alpha^3=0,$$

$$\beta_{11}+3\alpha\beta_1+4\alpha^2\beta+5\alpha_1\beta+5\alpha_{11}(\alpha^2-\alpha_1)+5\alpha\alpha_1(\alpha_1+2\alpha^2)=0, \tag{31.55}$$

$$\beta_{1111}+4\alpha_{111}\beta+6\alpha_{11}(\beta_1+\alpha\beta)+4\alpha_1[(\alpha^2+\alpha_1)\beta+(\beta_1+\alpha\beta)_1+\beta_{00}+3\beta\beta_1+2\alpha\beta^2]=0.$$

The proof is carried out using formulae (5.7.8) from [171].

In Case 1 there exist the general solution of (31.54) which leads to the following operator (31.53)

$$Q=\partial_0+(ax_1+b)\partial_1-2[aU+a(a'+2a^2)x_1^2+(a'b+ab'+4a^2b)x_1+b(b'+2ab)]\partial_U$$

where $a=a(x_0)$ and $b=b(x_0)$ are solutions of the differential equations

$$a''+2aa'-4a^3=0, \quad b''+2ab'-4a^2b=0.$$

In accordance with the explicit form of a and b we have several operators

$$Q_1=\partial_0+x_0\partial_1-2x_0\partial_U \quad (a=0,\ b=x_0);$$

$$Q_2=x_0\partial_0-(x_1+6x_0^5)\partial_1+2[U+3(x_1^2x_0^{-2}-24x_0^8+2x_1x_0^3]\partial_U \quad \left(a=-\frac{1}{x_0},\ b=6x_0^5\right); \tag{31.56a}$$

$$Q_3=2x_0\partial_0+(x_1-3x_0^2)\partial_1-2(U-3x_1+9x_{)}^2)\partial_U \quad \left(a=(2x_0)^{-1},\ b=-\frac{3}{2}x_0\right);$$

$$Q_4=2W\partial_0+W'[x_1\partial_1-(2U+Wx_1^2)\partial_U] \quad \left(a=\frac{W'}{2W'},\ b=0\right); \tag{31.56b}$$

$$Q_5=2W\partial_0+W'(x_1+\Omega)\partial_1-[2W'U+WW'(x_1+\Omega)^2+x_1+\Omega]\partial_U,$$

where $a=W'/2W$, $b=a\Omega$, $\Omega=\int W(W')^{-2}dx_0$, $W=W(x_0)$ is the Weierstrasse function satisfying the equation $W''=W^2$.

In Case 2 we obtain only a few particular solutions of (31.54) corresponding to the operators

$$Q_6=x_0^2\partial_1+(x_0^5-2x_1)\partial_U \qquad (\alpha=0,\quad \beta=x_0^3-2x_1x_0^{-2});$$

$$Q_7=\partial_1+(\Lambda-\frac{1}{3}Wx_1)\partial_U \quad \left(\alpha=0,\quad \beta=\Lambda-\frac{1}{3}Wx_1\right); \tag{31.57}$$

$$Q_8=x_1\partial_1+2U\partial_U \qquad \left(\alpha=\frac{2}{x_1},\quad \beta=0\right);$$

$$Q_9=x_1^3\partial_1+2(x_1^2U+24)\partial_U \quad (\alpha=2x_1^{-1},\quad \beta=48x_1^{-3})$$

where $\Lambda=\Lambda(x_0)$ is the Lame functions satisfying the equation $\Lambda''=W\Lambda$.

Using the operators (31.55), (31.56) we find the ansätze:

$$
\begin{aligned}
&1.\ \ U=\varphi(\omega)-4x_0^2, && \omega=x_1+x_0^2;\\
&2.\ \ U=x_0^2\varphi(\omega)-\left(\frac{x_0}{x_1}+6x_0^4\right)^2, && \omega=x_0(x_1+x_0^5);\\
&3.\ \ U=x_0^{-1}\varphi(\omega)+2(x_1+x_0^2), && \omega=x_0^{-1/2}(x_1+x_0^2);\\
&4.\ \ U=W^{-1}\varphi(\omega)-\frac{1}{6}Wx_1^2, && \omega=W^{-1/2}x_1;\\
&5.\ \ U=W^{-1}\varphi(\omega)-\frac{1}{4}W^{-2}(W')^2(x_1+\Omega)^2, && \omega=W^{-1/2}x_1-\frac{1}{2}\int W'\Omega\,dx_0;\\
&6.\ \ U=\varphi(\omega)-x_0^{-2}x_1^2+x_0^3x_1, && \omega=x_0;\\
&7.\ \ U=\varphi(\omega)-\frac{1}{6}x_1^2W+\Lambda x_1, && \omega=x_0;\\
&8.\ \ U=\varphi(\omega)x_1^2, && \omega=x_0;\\
&9.\ \ U=\varphi(\omega)x_1^2-12x_1^{-2}, && \omega=x_0.
\end{aligned}
\tag{31.58}
$$

Substituting them into (31.51) we come the following reduced equations:

1. Equation (31.51) with $\mu=1$,
2. Equation (31.51) with $\mu=15$,
3. The following equation with $\lambda=6$,
4. $\varphi^{IV}+\varphi\varphi''+(\varphi')^2+\frac{\lambda}{6}(\omega^2\varphi''+7\omega\varphi'+8\varphi)=0,$ (31.59)
5. $\varphi^{IV}+\varphi\varphi''+(\varphi')^2+\frac{\lambda}{2}(\omega\varphi'+2\varphi-\lambda\omega^2)=0,$
6. $\varphi''-2\omega^{-2}\varphi+\omega^6=0,$
7. $\varphi''-\frac{1}{3}W\varphi+\Lambda^2=0,$

8,9. $\varphi''+6\varphi^2=0.$

Solving (31.59) we obtain from (31.58) solutions of the equation (31.52). We present some of them:

$$U=-\frac{1}{6}x_1^2W,\quad U=-12x_1^{-2},\quad U=-\frac{1}{6}x_1^2W-12x_1^{-2},$$

$$U=2(x_1-x_0^2),\quad U=2(x_1-x_0^2)-12(x_1+x_0^2)^{-2}.$$

Let us consider also the multidimensional Boussinesq equation.

THEOREM 31.12. The equation (31.52) with n=6 is invariant under the conformal algebra *AC(1,6)* whose basis elements have the form

$$P_a=\frac{\partial}{\partial x_a},\quad J_{ab}=x_aP_b-x_bP_a,\quad D=x_aP_a-4U\partial_U,$$

$$K_a=2x_aD-x^2P_a\qquad (a,b=1,2,\dots,6)$$

if U satisfies the additional condition $\Delta U+U^2/2=0$.

One of ansätze obtained using the operator K_a has the form

$$U=(x^2)^{-2}\varphi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=\frac{b\cdot x-b^2x^2}{x^4},$$

where b_a are constant numbers. The corresponding reduced equations is

$$\varphi_{11}=0,\quad 2\omega_2\varphi_{22}+5\varphi_2=\frac{1}{4b^2}\varphi^2. \tag{31.60}$$

A particular solution of the equations (31.60) is $\varphi=-4b^2(\omega_2)^{-1}$. The corresponding solution of (31.53) has the form

$$U=4\left[x^2-(\alpha\cdot x)^2\right]^{-1}\qquad (\alpha=const;\ \alpha^2=1).$$

For more details about conditional symmetry of Boussinesq equation and other nonlinear equations of mathematical physics, see [171].

31.8. Exact Solutions of the Linear and Nonlinear Schrödinger Equation

In this subsection we apply higher order SOs admitted by the Schrödinger equation to construct exact solutions of it. In this way it is possible to find solutions as of the linear Schrödinger equation with a wide class of potentials as of the nonlinear Schrödinger equation. In particular we find soliton solutions and propose a new method of their generation using higher order SO.

The higher order SOs of the Schrödinger equation were described in Section 21. We restrict ourselves to the potentials satisfying the equation (21.22a). The corresponding SO (21.36) commute with the Hamiltonian so it is convenient to search for solutions of the Schrödinger equation in the form

$$\Psi(t,x)=\exp(-iEt)\Psi(x) \tag{31.61}$$

where $\Psi(x)$ are eigenfunctions of the commuting operators H and Q:

$$H\Psi(x)=E\Psi(x), \tag{31.62a}$$

$$Q\Psi(x)=\lambda\Psi(x). \tag{31.62b}$$

If relation (31.62a) is satisfied than (31.62b) reduces to the first order equation

$$\left(E+\frac{U}{2}+\frac{\omega_0}{2}\right)\Psi'=\left(\frac{1}{4}U'+i\lambda\right)\Psi \tag{31.63}$$

which is easily integrated:

$$\Psi=A\sqrt{U+2E+\omega_0}\,\exp\left(2i\lambda\int\frac{dx}{U+2E+\omega_0}\right) \tag{31.64}$$

where A is an arbitrary constant. On the other hand, substituting the expression for Ψ' from (31.63) into (31.62a) we obtain the compatibility condition for the system (31.62) in the form of the following *algebraic* relation for E and λ

$$\lambda^2=E^3+cE^2+\frac{1}{4}(c^2+b)E+\frac{1}{4}(bc-C), \tag{31.65}$$

C is an arbitrary constant (the first integral of the Weierstrasse equation, see (21.25)).

Thus using the third order SO it is possible to find solutions of the Schrödinger equation solving the first-order ordinary differential equation (31.63) and the algebraic equation (31.65). This remarkable simple procedure is admissible thanks to hidden symmetry of the equation considered.

Moreover, it happens that the third-order SO of the linear Schrödinger equation enables us to find exact solutions of the corresponding nonlinear equation

$$i\partial_t\Psi=\frac{1}{2}p^2\Psi+\frac{1}{2A^2}(\Psi^*\Psi)\Psi. \tag{31.66}$$

In accordance with (31.61), (31.64)

$$\Psi^*\Psi=A^2(U+2E+\omega_0) \quad or \quad U=\frac{1}{A^2}\Psi^*\Psi-2E-\omega_0, \tag{31.67}$$

from which it follows that the functions

$$\Psi=\exp\left(\frac{i}{2}ct\right)\Psi(x) \tag{31.68}$$

(where $\Psi(x)$ are functions defined in (31.64)) are exact solutions of the nonlinear Schrödinger equation (31.66).

The equation (31.66) is invariant under the Galilei transformations which enables us to generate a more extended family of solutions starting from (31.68):

$$\Psi=A\sqrt{U(x-vt)+2E+\omega_5}\exp\left(i\left[(\omega_0-v^2)\frac{t}{2}+vx+2\lambda\int_0^{x-vt}\frac{dy}{U(y)+2E+\omega_5}\right]\right). \tag{31.69}$$

Here U is a Weierstrasse function i.e. an arbitrary solution of the equation (21.25), v, ω_0, ω_5, λ and E>0 are arbitrary parameters satisfying the condition (31.65).

Thus using third order SOs of the linear Schrödinger equation we obtain a wide class of exact solutions of the nonlinear Schrödinger equation (31.66). In particuliar we find soliton solutions corresponding to the potential given in (21.26). Indeed, this potential satisfies (21.25) with the following values of arbitrary parameters: $\omega_0=-\nu^2$, $\omega_5=-\nu^4$, $C=\nu^6$, thus relation (31.65) reduces to the form

$$\lambda^2=E^2(E+\nu^2)\equiv E^2\varepsilon. \tag{31.70}$$

The corresponding integral included in (31.69) is easily calculated, which enables us to represent solutions in the following form

$$\Psi=\frac{A\nu}{\cosh[\nu(x-vt)]}\exp\left\{i\left[\left(\nu^2-\frac{v^2}{2}\right)t+vx+\varphi_0\right]\right\}, \quad E=0; \tag{31.71}$$

$$\Psi=A\left\{\nu\tanh[\nu(x-vt)]\pm i\sqrt{\varepsilon}\right\}\exp\left[i\left(\nu^2-\frac{v^2}{2}\right)t+(v\mp\sqrt{\varepsilon})x+\varphi_0\right], \quad E\neq 0,\ \varepsilon\geq 0. \tag{31.72}$$

Formula (31.71) presents a quickly decreasing one-soliton solution [31*]. In the case $v=\pm\sqrt{\varepsilon}$ the solution (31.72) describes solitons with a finite density [31*].

In general the linear Schrödinger equation with a potential satisfying (21.22a) does not possesses any non-trivial (different from time displacements) Lie symmetry.

Nevertheless its solutions admit group generating in frames of the concept of conditional symmetry. Indeed, these solutions satisfy (31.67), and the equation (21.1) with the additional condition (31.67) *is* invariant under the Galilei transformations. This conditional symmetry enables us to generate new solutions:

$$\Psi = A\sqrt{U(x-vt)+2E+\omega_5}\,\exp\left\{i\left[(-E-v^2)\frac{t}{2}+vx+2\lambda\int_0^{x-vt}\frac{dy}{U(y)+2E+\omega_5}\right]\right\}. \tag{31.73}$$

The functions (31.73) satisfy the Schrödinger equation with a potential $U(x-vt)$ where $U(x)$ is a solution of the equation (21.22a).

The next generation of new solutions can be made using the third order SO. Indeed, if $U(x)$ satisfies (21.22a) then $U(x-vt)$ satisfies the Boussinesq equation (21.33), so the corresponding SO does exist. Moreover, this SO can be represented in the form

$$Q=p^2+\frac{1}{4}[3U+\omega_0+6v^2,p]_+ +\frac{3}{2}vU\equiv 2pH+\frac{1}{2}(U+\omega_0+6v^2)p+\frac{3}{2}vU+\frac{i}{4}U'. \tag{31.74}$$

Acting by the operator (31.74) on Ψ of (31.73) we obtain a family of solutions

$$\Psi' = Q\Psi = a\Psi + iv^2\Psi_1 \tag{31.75}$$

where $a=\lambda+2Ev+\omega_0 v/2-4v^3$ is a constant multiplier, Ψ is given in (31.73),

$$\Psi_1=\frac{U'+4i\lambda}{2(2E+U+c)}\Psi. \tag{31.76}$$

The first term in the r.h.s. of (31.75) is an evident solution of the Schrödinger equation (inasmuch as Ψ is a solution), but (31.76) is the essentially new solution obtained using the third order SO.

It is interesting to note that if Ψ is the soliton solution

$$\Psi=\frac{\nu A}{\cosh[\nu[x-vt)]}\exp\left\{i\left[-\frac{v^2}{2}t+vx+\varphi_0\right]\right\}, \tag{31.77}$$

then the generated solution (31.76) has the form

$$\Psi_1=\frac{\nu^2 A}{\cosh^2[\nu[x-vt)]}\sqrt{\cosh^2[\nu(x-vt)]-1}\,\exp\left\{i\left[-\frac{v^2}{2}t+vx+\varphi_0\right]\right\}, \tag{31.78}$$

i.e., is a soliton solution also.

Consequently using the SO (31.74) we can generate new and new solutions.

We see higher order SOs are effective instruments for solving equations of motion and generating new solutions starting from known ones.

7. TWO-PARTICLE EQUATIONS

Here we consider equations describing systems of two interacting particles and being invariant under the Galilei or Poincaré group. We investigate non-Lie symmetries of such equations and find exact solutions of some of them.

32. TWO-PARTICLE EQUQTIONS INVARIANT UNDER THE GALILEI GROUP

32.1. Preliminary notes

Equations describing a particle motion in the given external fields correspond to the idealized physical situation when we can neglect the particle influence to the field. But in many cases such an influence is very essential, e.g., it is the case for the Kepler problem if the mass of a particle generating the central field is comparable with the mass of an orbital or scattered particle. The last case is a typical example of a two-particle problem in which it is necessary to take into account motions of two interacting objects simultaneously.

A formulation of a two-particle problem in frames of relativistic quantum theory clashes with great difficulties of mathematical and logical nature. We can say with some provisos there is no satisfactory relativistic theory of two-particle systems until now. Meanwhile the need in developing of such a theory is very large since a wide class of really existing particles (mesons) is usually interpreted as a coupled states of two "more elementary" objects, i.e., quarks.

To describe a two-particle system it is usually to use the covariant Bethe-Solpeter equation [33] or quasi-relativistic equations of the type of Breit [54] or Kraichick-Foldy [260]. The alternative possibilities are to use the quasipotential approach of Logunov, Tavchelidze, Todorov and Kadyshevsky [238, 283] or the theory of the direct interaction which was proposed in the classical Dirac work [80] (about the current situation in this theory see, e.g., [69, 394]). We have had not a possibility to analyze special features of any of the mentioned approaches. Here we note only the solutions of the Bethe-Solpeter equation depend on the additional parameter ("proper time"of the system) whose physical meaning is unclear, and the theory of the direct interaction a priory is not available in the cases when it is necessary to take into account the finiteness of velocity of signal propagation. As to the Breit equation, it is invariant neither under Lorentz nor Galilei transformations. So this equation does not satisfy any relativity principle accepted in modern physics. Nevertheless this equation is a good mathematical model of quasirelativistic two-particle systems and is used widely by describing as hydrogen type systems with electromagnetic interaction as strongly coupled systems of the quark-antiquark type [220, 65] (in the last case the Breit potential has to be modified of course).

In this section we consider two-particle equations for particles of arbitrary spins which satisfy the Galilei relativity principle [155]. It will be shown such equations can be used successively by describing two-particle coupled systems, and take into account various fine effects caused by the spin. In particular we present a Galilei-invariant analog of the Breit equation, which leads to a correct hyperfine structure of spectrum of hydrogen-type systems.

32.2. Equations for Spinless Particles

Consider the simplest version of a two-particle system when the particle spins are equal to zero and an interaction between particles is absent. In quantum mechanics such a system is described by the following equation

$$\frac{\partial}{\partial x_0}\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)})=\left(\frac{\boldsymbol{p}_{(1)}^2}{2m_1}+\frac{\boldsymbol{p}_{(2)}^2}{2m_2}\right)\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}) \tag{32.1}$$

where $\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)})$ is the wave function of a system, depending on the coordinates of the first ($\boldsymbol{x}_{(1)}$) and second ($\boldsymbol{x}_{(2)}$) particles, m_1 and m_2 are masses of the first and second particles, $\boldsymbol{p}_{(\alpha)}=-i\partial/\partial\boldsymbol{x}_{(\alpha)}$. In the following we use the indices (1) and (2) to denote all the quantities connected with the particles with numbers 1 and 2.

The equation (32.1) is invariant under the Galilei algebra* whose basis elements are

$$\begin{aligned}
&P_0=H=\frac{\boldsymbol{p}_{(1)}^2}{2m_1}+\frac{\boldsymbol{p}_{(2)}^2}{2m_2},\quad \boldsymbol{P}=\boldsymbol{p}_{(1)}+\boldsymbol{p}_{(2)},\\
&\boldsymbol{J}=\boldsymbol{J}_1+\boldsymbol{J}_2=\boldsymbol{x}_{(1)}\times\boldsymbol{p}_{(1)}+\boldsymbol{x}_{(2)}\times\boldsymbol{p}_{(2)},\\
&\boldsymbol{G}=\boldsymbol{G}_1+\boldsymbol{G}_2=x_0(\boldsymbol{p}_{(1)}+\boldsymbol{p}_{(2)})-m_1\boldsymbol{x}_{(1)}-m_2\boldsymbol{x}_{(2)}.
\end{aligned} \tag{32.2}$$

The representation of the algebra *AG(1,3)*, spanned on the basis (32.2), is reducible. To reduce this representation it is convenient to come to the new (center of mass or c.m.) variables

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*In reality the symmetry of (30.1) is more extensive. The maximal IA of this equation in the class M_1 for $m_1\neq m_2$ is defined by the relation $A=[AG'(1,3)\oplus AG'(1,3)]\Subset\{P_0,D\}$ where $AG'(1,3)$ is the nine-dimensional Lie algebra including all the basis elements of the Galilei algebra except P_0, $\{P_0,D\}$ is the two-dimensional subalgera including $P_0=i\partial/\partial x_0$ and $D=2x_0p_0-\boldsymbol{x}_{(1)}\boldsymbol{p}_{(1)}-\boldsymbol{x}_{(2)}\boldsymbol{p}_{(2)}$. If however $m_1=m_2$ then A coincides with the Lie algebra of the Schrödinger group in the space of dimension (1+6).

$$\boldsymbol{x}=\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}, \quad \boldsymbol{X}=\frac{m_1\boldsymbol{x}_{(1)}+m_2\boldsymbol{x}_{(2)}}{m_1+m_2}$$

or

$$\boldsymbol{x}_{(1)}=\boldsymbol{X}+\frac{m_2\boldsymbol{x}}{m_1+m_2}, \quad \boldsymbol{x}_{(2)}=\boldsymbol{X}-\frac{m_1\boldsymbol{x}}{m_1+m_2}. \tag{32.3}$$

For the momentum operators we obtain

$$\boldsymbol{p}_{(1)}=\boldsymbol{p}+\frac{m_1\boldsymbol{P}}{m_1+m_2}, \quad \boldsymbol{p}_{(2)}=-\boldsymbol{p}+\frac{m_2\boldsymbol{P}}{m_1+m_2}, \tag{32.4}$$

where

$$\boldsymbol{p}=-i\frac{\partial}{\partial\boldsymbol{x}}, \quad \boldsymbol{P}=-i\frac{\partial}{\partial\boldsymbol{X}}.$$

Substituting (32.4) into (32.2) we come to the following realization

$$\hat{P}_0=H+\frac{\boldsymbol{P}^2}{2M}+E, \quad \hat{\boldsymbol{P}}=P, \quad \hat{\boldsymbol{J}}=\boldsymbol{X}\times\boldsymbol{P}+\boldsymbol{P}, \quad \hat{\boldsymbol{G}}=x_0\boldsymbol{P}-M\boldsymbol{X}, \tag{32.5}$$

where

$$M=m_1+m_2, \quad E=\frac{\boldsymbol{p}^2}{2\mu}, \quad \mu=\frac{m_1m_2}{m_1+m_2}, \quad \boldsymbol{S}=\boldsymbol{x}\times\boldsymbol{p}. \tag{32.6}$$

The operators (32.5) have the same form as one-particle generators, compare with (12.15). Besides the role of coordinate and momentum is played by c.m. coordinate and total momentum, instead of spin and internal energy we have internal angular momentum $\boldsymbol{x}\times\boldsymbol{p}$ and $\boldsymbol{p}^2/2\mu$.

It is not difficult to make sure the Casimir operators (11.14) have the following form for the representation (32.5)

$$C_3=\frac{\boldsymbol{p}^2}{2\mu}, \quad C_1=M, \quad C_2=M^2(\boldsymbol{x}\times\boldsymbol{p})^2,$$

and are characterized by the following eigenvalues

$$0\leq c_3<\infty, \quad c_1=M, \quad c_3=M^2l(l+1), \quad l=0,1,\dots\ .$$

So in non-relativistic quantum mechanics, the system of two noninteracting spinless particles corresponds to the IR of the algebra *A[G(1,3)⊗G(1,3)]* which reduces to the direct integral in respect with c_3 and direct sum in respect with c_2 of IRs of the algebra *AG(1,3)*. In other words the system under consideration can be interpreted as a quasiparticle whose mass is equal to the sum of masses of constituent particles, spin can take arbitrary integer values and internal energy can take arbitrary positive value.

An equation for interacting particles can be obtained from (32.1) by adding an interaction potential to the Hamiltonian. As a result we have

$$i\frac{\partial}{\partial x_0}\psi(x_0,\boldsymbol{x},\boldsymbol{X})=\left(\frac{\boldsymbol{P}^2}{2M}+\frac{\boldsymbol{p}^2}{2\mu}+V\right)\psi(x_0,\boldsymbol{x},\boldsymbol{X}). \tag{32.7}$$

Requiring the symmetry of (32.7) under the Galilei algebra we obtain the following conditions for V:

$$[V,\boldsymbol{P}]=[V,\boldsymbol{X}]=[V,\boldsymbol{S}]=0. \tag{32.8}$$

where $\boldsymbol{P}$, $\boldsymbol{X}$, $\boldsymbol{S}$ are given in (32.3)-(32.6).

It follows from (32.8) a potential is a scalar function of the internal variables $\boldsymbol{x}$ and $\boldsymbol{p}$:

$$V=V(\boldsymbol{x}^2,\boldsymbol{p}^2,\boldsymbol{x}\cdot\boldsymbol{p}). \tag{32.9}$$

Formulae (32.7), (32.9) define the general form of two-particle Schrödinger equation satisfying the Galilei relativity principle. The corresponding Galilei group generators are given by formulae (32.5) where $H=\boldsymbol{P}^2/2M+\boldsymbol{p}^2/2\mu+V$.

The equations (32.7), (32.9) have high symmetry, being invariant under the 13-dimensional Lie algebra isomorphic to $A[G(1,3)\otimes O(3)]$. Besides the Galilei generators (32.5), this IA includes the generators $\boldsymbol{S}$ of (32.6).

The symmetry under the Galilei algebra makes it possible to separate the motion of the center of mass of the system. Representing the wave function in the form $\psi=\chi(\boldsymbol{X})\varphi(\boldsymbol{x})$ we come to the one-particle Schrödinger equation with reduced mass:

$$\left(\frac{\boldsymbol{p}^2}{2\mu}+V\right)\varphi(\boldsymbol{x})=E\varphi(\boldsymbol{x}); \tag{32.10}$$

moreover, $\chi(\boldsymbol{X})$ satisfies the free Schrödinger equation in c.m. variables:

$$\frac{i\partial}{\partial x_0}\chi=\left(\frac{\boldsymbol{P}^2}{2M}+E\right)\chi,$$

E is a constant of separation of variables.

The equation (32.10) is invariant under the algebra $AO(3)$ whose basis elements coincide with $\boldsymbol{S}$ of (32.6). This invariance enables to separate angular variables and so to reduce (32.10) to the system of ordinary differential equations for radial wave functions.

Thus a Galilei-invariant Schrödinger equation for a pair of interacting spinless particles can be reduced always to ordinary differential equation for radial wave functions in the c.m. frame. This conclusion does not depend on an explicit form of the interaction potential but follows from the analysis of symmetries of the two-particle Schrödinger equation (32.7).

We note that the equation (32.1), like (32.7), can be represented as a system of first-order partial differential equations:

$$(\beta_\lambda p^\lambda - \beta_7 M)\psi = 0, \quad \lambda = 0,1,...,6, \tag{32.11}$$

where we denote $p_0 = i\partial/\partial x_0$, $\boldsymbol{p}_{(1)} = (p_1, p_2, p_3)$, $\boldsymbol{p}_{(2)} = (p_4, p_5, p_6)$, ψ is a seven-component wave function, β_λ, β_7 are the 7×7 matrices of the following form

$$\beta_0 = \begin{pmatrix} 1 & \tilde{0} & \tilde{0} \\ \tilde{0}^\dagger & \hat{0} & \hat{0} \\ \tilde{0}^\dagger & \hat{0} & \hat{0} \end{pmatrix}, \quad \beta_a = \begin{pmatrix} 0 & \lambda_a & \tilde{0} \\ \lambda_a^\dagger & \hat{0} & \hat{0} \\ \tilde{0}^\dagger & \hat{0} & \hat{0} \end{pmatrix}, \quad \beta_{3+a} = \begin{pmatrix} 0 & \tilde{0} & \lambda_a \\ \tilde{0}^\dagger & \hat{0} & \hat{0} \\ \lambda_a^\dagger & \hat{0} & \hat{0} \end{pmatrix}, \quad \beta_7 = \begin{pmatrix} 0 & \tilde{0} & \tilde{0} \\ \tilde{0}^\dagger & \frac{m_1}{M}\hat{I} & \hat{0} \\ \tilde{0}^\dagger & \hat{0} & \frac{m_2}{M}\hat{I} \end{pmatrix}, \tag{32.12}$$

$\hat{0}$ and $\hat{I}$ are the zero and unit matrices of dimension 3×3, $\tilde{0}$ is the zero row matrix of dimension 1×3, λ_a are the matrices (6.18). Multiplying (32.11) by β_0 and $(1-\beta_0)$ and expressing the functions $\psi_2 = (1-\beta_0)\psi$ via $\psi_1 = \beta_0\psi$ we come to the Schrödinger equation (32.1) for ψ_1. The remaining components of ψ (i.e., ψ_2) are expressed via ψ_1: $\psi_2 = -\beta_7(\beta_a p_a/m_1 + \beta_{3+a} p_a/m_2)\psi_1$.

The equations (32.11) are Galilei-invariant besides the corresponding basis elements of the algebra *AG(1,3)* have the form

$$P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_a}, \tag{32.13}$$
$$\boldsymbol{J} = \boldsymbol{X}\times\boldsymbol{P} + \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{S}', \quad \boldsymbol{G} = x_0\boldsymbol{P} - M\boldsymbol{X} + \eta$$

where

$$S_a' = i\varepsilon_{abc}(1-\beta_0)(\beta_b\beta_c + \beta_{3+b}\beta_{3+c}),$$
$$\eta_a = (1-\beta_0)(\beta_a + \beta_{3+a}), \quad a = 1,2,3.$$

It is possible to show formulae (32.11), (32.12) define the simplest (i.e., including the minimal number of equations) Galilei-invariant system of first-order equations which reduces to the two-particle Schrödinger equation for scalar particles.

32.3. Equations for Systems of Particles of Arbitrary Spin

In analogy with the two-particle equations for scalar particles we will construct motion equations for particles of arbitrary spins. In this subsection we consider equations in the Schrödinger form and then the systems of equations of the kind (32.11).

We start with the one-particle Schrödinger equations (11.1), (14.9a) for a particle of arbitrary spin. In analogy with (32.1) we write the equation for two noncoupled particles in the form

$$i\frac{\partial}{\partial x_0}\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)})=(H_{s_1}+H_{s_2})\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}) \tag{32.14}$$

where $\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)})$ is a $4(2s_1+1)(2s_2+1)$-component wave function, H_{s_1} and H_{s_2} are Hamiltonians of the first and second particles, i.e., the operators (14.9a) where we set $k=a=1$ for simplicity.

The equation (32.14) is evidently Galilei-invariant. The corresponding Galilei group generators have the form of a sum of the one-particle generators (12.18), (14.3), i.e.,

$$\begin{aligned}
&P_0=H_{s_1}+H_{s_2},\quad \boldsymbol{P}=\boldsymbol{p}_{(1)}+\boldsymbol{p}_{(2)},\\
&\boldsymbol{J}=\boldsymbol{x}_{(1)}\times\boldsymbol{p}_{(1)}+\boldsymbol{x}_{(2)}\times\boldsymbol{p}_{(2)}+\boldsymbol{S}_{(1)}+\boldsymbol{S}_{(2)},\\
&\boldsymbol{G}=x_0\boldsymbol{P}-m_1\boldsymbol{x}_{(1)}-m_2\boldsymbol{x}_{(2)}+\eta_{(1)}+\eta_{(2)}.
\end{aligned} \tag{32.15}$$

The operators (32.15) realize a reducible representation of the algebra *AG(1,3)*. It is not difficult to make sure the transition to the c.m. variables does not reduce these operators to a direct sum of generators of IRs in contrast to the case of zero mass particles. The corresponding Casimir operator C_3 (11.14) is not diagonalized in this way, i.e. we have to transform (32.15) to the representation where the internal energy operator

$$C_3=E=H_{s_1}+H_{s_2}-\frac{\boldsymbol{P}^2}{2(m_1+m_2)}$$

does not depend on the total momentum [155]. Using the following transformation operator (compare with (14.13))

$$U=\left(1+\frac{i}{M}\eta_{(1)}\cdot\boldsymbol{P}\right)\left(1+\frac{i}{M}\eta_{(2)}\cdot\boldsymbol{P}\right)=\exp\left(\frac{i}{M}\eta\cdot\boldsymbol{P}\right),\quad \eta=\eta_{(1)}+\eta_{(2)},\quad M=m_1+m_2,$$

we obtain from (32.14) the following equivalent equation

$$i\frac{\partial}{\partial x_0}\Phi=\hat{H}\Phi,\quad \Phi=U\psi,\quad \hat{H}=UHU^{-1}=\frac{\boldsymbol{P}^2}{2M}+\hat{E}, \tag{32.16}$$

where $\hat{E}$ is the transformed operator of internal energy

$$\begin{aligned}
\hat{E}=&\sigma_1^{(1)}m_1+\sigma_2^{(2)}m_2+2\sigma_3^{(1)}\boldsymbol{S}_{(1)}\cdot\boldsymbol{p}-2\sigma_3^{(2)}\boldsymbol{S}_{(2)}\cdot\boldsymbol{p}+\\
&+\frac{\boldsymbol{p}^2}{2}\left[\frac{1}{\mu}-\left(\sigma_1^{(1)}-i\sigma_2^{(1)}\right)\frac{1}{m_1}-\left(\sigma_1^{(2)}-i\sigma_2^{(2)}\right)\frac{1}{m_2}\right],
\end{aligned} \tag{32.17}$$

$\boldsymbol{p}$ and μ are the relative momentum and reduced mass. Simultaneously the generators (32.15) are transformed to the form (32.5) where

$$\boldsymbol{S}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S}_{(1)}+\boldsymbol{S}_{(2)}, \tag{32.18}$$

and $\hat{E}$ is defined in (32.17).

Thus we have obtained the Galilei-invariant equations for a system of two particles of arbitrary spins in the form (32.16). It follows from (32.18) that such a system can be considered as a quasiparticle with variable spin defined as a result of adding of three angular momenta: $\boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p}$, $\boldsymbol{S}_{(1)}$ and $\boldsymbol{S}_{(2)}$.

The equation (32.16) and the above interpretation can be used as a base of constructing of equations for a pair of coupling particles of arbitrary spins. We search for such equations in the following form

$$i\frac{\partial}{\partial x_0}\Phi=(\hat{H}+V)\Phi, \tag{32.19}$$

where $\hat{H}$ is the Hamiltonian of noninteracting particles (32.16), V is an interaction potential. The condition of Galilei-invariance of (32.19) reduces to the requirement of commutativity of V with the generators (32.5), (32.18). This requirement can be written in the form (32.8), (32.18).

It follows from the above that potential V has to be a scalar function of variables $\boldsymbol{x}$ and $\boldsymbol{p}$; moreover, there is no restriction on the exact form of this function imposed by the condition of Galilei invariance.

We see the condition of Galilei invariance admits a very wide class of interaction potentials for arbitrary spin particles. The examples of physically interesting potentials are considered in Subsection 32.5.

32.4 Two-Particle Equations of First Order

It is known that partial differential equations of order $N>1$ can be reduced to equivalent equations including first order derivatives only. So it is interesting to consider Galilei-invariant equations describing pairs of arbitrary spin particles and having the form (32.11).

The problem of describing of equations of the form (32.11) being invariant under the Galilei group can be formulated in analogy with the corresponding one-particle problem considered in Section 13. The general form of the Galilei group generators defined on the set of solutions of these equations is given by formulae (32.13) where

$$\boldsymbol{S}'=\boldsymbol{S}_{(1)}+\boldsymbol{S}_{(2)}, \tag{32.20}$$

$\boldsymbol{S}_{(1)}$ and $\boldsymbol{S}_{(2)}$ are the spin matrices of the first and second particle, besides,

$$\left[S^a_{(1)},S^b_{(2)}\right]=0,$$

η are numerical matrices satisfying relations (12.20) together with $\boldsymbol{S}'$.

Requiring the equations (32.11) admit the Lagrangian formulation and be invariant under the algebra (32.13), (32.20), we come to the following equations for matrices β_λ, β_7 (compare with (13.9))

$$\left[S'_a,\beta_0\right]=0,\quad \left[S'_a,\beta_7\right]=0,$$
$$\eta^\dagger_a\beta_7-\beta_7\eta_a=-i\left(\frac{m_1}{M}\beta_a+\frac{m_2}{M}\beta_{3+a}\right), \tag{32.21a}$$

$$\eta^\dagger_a\beta_b-\beta_b\eta_a=-i\delta_{ab}\beta_0,\quad \left[S'_a,\beta_b\right]=i\varepsilon_{abc}\beta_c,$$
$$\eta^\dagger_a\beta_{3+b}-\beta_{3+b}\eta_a=-i\delta_{ab}\beta_0,\quad \left[S'_a,\beta_{3+b}\right]=i\varepsilon_{abc}\beta_{3+c}. \tag{32.21b}$$

We do not present detailed calculations which are analogous to given in Subsections 13.2, 13.3.

Thus the problem of describing of Galilei-invariant and two-particle equations of the form (32.11) reduces to the purely algebraic problem of finding of all the non-equivalent matrices S'_a, η_a, β_7, β_λ satisfying (12.20), (32.21). In comparison with the corresponding one-particle problem we have the additional matrices β_{3+a} which have to satisfy (32.32b). Besides we require the wave function ψ of (32.11) satisfies the two-particle Schrödinger equation (32.1) componentwise (the last has to be a consequence of (32.11)).

Even the simplest (i.e., realized by matrices of the minimal dimension) solutions of (32.21) for arbitrary values of spins s_1 and s_2 are very cumbersome. This is why we restrict ourselves to considering the cases $s_1=s_2=1/2$ and $s_1=0$, $s_2=1/2$.

For $s_1=s_2=1/2$ we obtain the solutions of (32.21) in the form

$$\beta_0=\frac{1}{2}(1-\Gamma_0),\quad \beta_7=(1+\Gamma_0),\quad \beta_a=\sqrt{\frac{M}{m_1}}\Gamma_0\Gamma_a,\quad \beta_{3+a}=\sqrt{\frac{M}{m_2}}\Gamma_0\Gamma_{3+a},$$
$$\left(S_{(1)}\right)_a=\frac{i}{4}\varepsilon_{abc}\Gamma_b\Gamma_c,\ \left(S_{(2)}\right)_a=\frac{i}{4}\varepsilon_{abc}\Gamma_{3+b}\Gamma_{3+c},\ \eta_a=\frac{i}{2M}(1+\Gamma_0)\left(\sqrt{m_1}\,\Gamma_a+\sqrt{m_2}\,\Gamma_{3+a}\right) \tag{32.22}$$

where Γ_0, Γ_a, Γ_{3+a} (a=1,2,3) are matrices of dimension 8×8 satisfying the Clifford algebra (8.2). The explicit realization of these matrices can be chosen say in the form (17.25), (17.27).

The equation (32.11) with the matrices (32.22) is Galilei-invariant and describes a pair of noninteracting particles of masses m_1, m_2 and spins $s_1=s_2=1/2$. Indeed, expressing $\psi_2=(1/2)(1-\beta_0)\psi$ via $\psi_1=\beta_0\psi$ we come to the two-particle Schrödinger equation in the c.m. variables. Besides the corresponding operator of the system spin is given by formulae (32.18), (32.22).

For s_1=0, s_2=1/2 we obtain the simplest solution of (32.21) in the form of the following 6×6 matrices

$$\beta_0=\begin{pmatrix}\sigma_0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \beta_a=\sqrt{\frac{M}{m_1}}\begin{pmatrix}0 & \sigma_a & 0\\ \sigma_a & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \beta_{3+a}=\sqrt{\frac{M}{m_2}}\begin{pmatrix}0 & 0 & \sigma_a\\ 0 & 0 & 0\\ \sigma_a & 0 & 0\end{pmatrix},$$

$$\beta_7=2(1-\beta_0),\quad \eta_a=\frac{i}{2\sqrt{M}}\begin{pmatrix}0 & 0 & 0\\ \sqrt{m_1}\,\sigma_a & 0 & 0\\ \sqrt{m_2}\,\sigma_a & 0 & 0\end{pmatrix},\quad S_a=\frac{1}{2}\begin{pmatrix}\sigma_a & 0 & 0\\ 0 & \sigma_a & 0\\ 0 & 0 & \sigma_a\end{pmatrix} \tag{32.23}$$

where σ_μ are the Pauli matrices, 0 are the zero matrices of dimension 2×2. The corresponding equation (32.11) is Galilei-invariant and can be interpreted as a motion equation of a system of two noninteracting particles of spins 0,1 and masses m_1, m_2.

The equations present above admit an obvious generalization to the case of interacting particles by the change $p_0 \to p_0+V$ where V is an interaction potential being a scalar function of internal variables. These equations make it possible to take into account interaction of a system with an external field using the standard change

$$p_0\to p_0-e_1A_0(\boldsymbol{x}_{(1)},x_0)-e_2A_0(\boldsymbol{x}_{(2)},x_0),\quad \boldsymbol{p}_{(\alpha)}\to\boldsymbol{p}_{(\alpha)}-e_\alpha A(\boldsymbol{x}_\alpha, x_0), \tag{32.24}$$

where $(A_0,\boldsymbol{A})$ is the vector-potential of the electromagnetic field. It is necessary to emphasize the change (32.24) preserves the Galilei-invariance of the corresponding equation (32.11) if we postulate Galilean transformation law (15.3) for the electromagnetic field.

Examples of two particle interaction potentials and external fields being interesting from the physical point of view are considered in the following subsection.

32.5. Equations for Interacting Particles of Arbitrary Spin

Let us analyze the possibilities arising by using the equations considered above for description of pairs of interacting particles. We assume the interaction potential depends on the interparticle distance only, i.e., $V=V(x)$.

The first-order equations considered in 31.4 reduces to the two-particle Schrödinger equation if we make the change $p_0 \to p_0$-V in (32.11) and then delete redundant components of the wave function. Such equations do not take into account any spin effects and so are not of a great interest.

Equations in the form (32.19) are more interesting. In the c.m. frame these equations take the form

$$H\psi=(\hat{E}+V)\psi=p_0\psi, \tag{32.25}$$

where $\hat{E}$ is the operator (32.17).

It is convenient to analyze (32.25) in a representation where the Hamiltonian H has a quasidiagonal form (i.e., commutes with the matrices $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ standing near the mass terms). Using the standard Barker-Glover-Chraplivy procedure [18, 67] (see Subsection 32.3) we obtain according to (33.7) (in our case $\gamma_0^{(1)}=\sigma_1^{(1)}$, $\gamma_0^{(2)}=\sigma_1^{(2)}$ (E,E) are the terms commuting with $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ etc.)

$$H'=\sigma_1^{(1)}m_1+\sigma_2^{(2)}m_2+\frac{\boldsymbol{p}^2}{2\mu}+V+i\left(\sigma_1^{(1)}\frac{1}{m_1}\frac{\boldsymbol{S}_{(1)}\cdot\boldsymbol{x}}{x}-\sigma_1^{(2)}\frac{1}{m_2}\frac{\boldsymbol{S}_{(2)}\cdot\boldsymbol{x}}{x}\right)\frac{\partial V}{\partial x}. \tag{32.26}$$

To snake off the non-Hermitian terms in the second row of (32.26) we make an additional transformation

$$H'\to H''=VH'V^{-1},\quad V=\exp\left(\sigma_1^{(1)}\frac{\boldsymbol{S}_{(1)}\cdot\boldsymbol{p}}{m_1}-\sigma_1^{(2)}\frac{\boldsymbol{S}_{(2)}\cdot\boldsymbol{p}}{m_2}\right) \tag{32.27}$$

and obtain the following Hamiltonian, neglecting terms of order $1/m_im_jm_j$:

$$\begin{aligned}H''=p^2/2\mu+V+\frac{\sigma_1^{(1)}\sigma_1^{(2)}}{3m_1m_2}\left[\left(\boldsymbol{S}_{(1)}\cdot\boldsymbol{S}_{(2)}-3\boldsymbol{S}_{(1)}\cdot\hat{\boldsymbol{x}}\boldsymbol{S}_{(2)}\cdot\hat{\boldsymbol{x}}\right)\left(\frac{\partial^2V}{\partial x^2}-\frac{1}{x}\frac{\partial V}{\partial x}\right)-\boldsymbol{S}_{(1)}\cdot\boldsymbol{S}_{(2)}\Delta V\right]+\\+\sum_{i=1,2}\left[\sigma_1^{(i)}m_i+\frac{\partial V}{\partial x}\frac{\boldsymbol{S}_{(i)}\cdot\boldsymbol{x}\times\boldsymbol{p}}{2m_ix}+\frac{1}{2}\left(\frac{\boldsymbol{S}_{(i)}\cdot\hat{\boldsymbol{x}}}{2m_ix}\right)^2\left(\frac{\partial^2V}{\partial x^2}-\frac{1}{x}\frac{\partial V}{\partial x}\right)+\frac{s_i(s_i+1)}{2m_i^2x}\frac{\partial V}{\partial x}\right].\end{aligned} \tag{32.28}$$

Thus starting from the Hamiltonian of (32.19) with a central potential $V=V(x)$ we come to the approximate quasidiagonal Hamiltonian H'' including the terms representing the spin-orbit and quadruple couplings (compare (32.28) with (10.26)).

If however the interaction potential depends on spins, then the Galilei-invariant equations (32.19) can describe more fine effects also, e.g., effects connected with the retardation of the electromagnetic waves. As an example, consider the equation (32.19) for particles of spins $s_1=s_2=1/2$ where $V=V_B$ is the Breit potential (33.3) (we use the temporary notation $i\gamma_4^{(i)}=\sigma_3^{(i)}$, i=1,2).

The equation (32.19) with the Breit potential is manifestly invariant under the Galilei group inasmuch as V_B of (33.3) satisfies the relations (32.8). Let us demonstrate this equation is a good model of a pair of interacting particles of spins $s_1=s_2=1/2$. Considering this equation in the rest frame we come to the relation (32.25) where $\hat{E}$ is the operator (32.17), V_B is the Breit potential (33.3). Transforming H to the form (33.7) and then using the additional transformation (32.27) we obtain

$$H''=\sigma_1^{(1)}m_1+\sigma_2^{(2)}m_2+\frac{\boldsymbol{p}^2}{2\mu}+\frac{e^2}{x}-\frac{e^2\sigma_1^{(1)}\sigma_1^{(2)}}{2\mu M}\left(\boldsymbol{p}\cdot\frac{1}{x}\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{x}\frac{1}{x^3}\boldsymbol{x}\cdot\boldsymbol{p}\right)+$$
$$+\frac{e^2}{x^3}\left[\frac{\sigma_1^{(1)}\sigma_1^{(2)}}{2m_1m_2}(\boldsymbol{S}_{(1)}+\boldsymbol{S}_{(2)})+\frac{1}{2m_1^2}\boldsymbol{S}_{(1)}+\frac{1}{2m_2^2}\boldsymbol{S}_{(2)}\right]\cdot\boldsymbol{x}\times\boldsymbol{p}-\frac{e^2\sigma_1^{(1)}\sigma_1^{(2)}}{m_1m_2}\left[\frac{1}{x^3}\boldsymbol{S}_{(1)}\cdot\boldsymbol{S}_{(2)}-\right.\tag{32.29}$$
$$\left.-\frac{3(\boldsymbol{S}_{(1)}\cdot\boldsymbol{x})(\boldsymbol{S}_{(2)}\cdot\boldsymbol{x})}{x^5}-\left(\frac{8\pi}{3}\boldsymbol{S}_{(1)}\cdot\boldsymbol{S}_{(2)}+\frac{\pi}{2}\frac{m_2^2+m_1^2}{m_1m_2}\right)\delta(\boldsymbol{x})\right]+o(e^4).$$

All the terms of the Hamiltonian (32.29) have a clear physical interpretation and describe the well-known physical effects appearing in systems of two interacting particles. We postpone the discussion of this Hamiltonian to Subsection 33.4 but note that on the set of functions corresponding to the positive eigenvalues of $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ (i.e., on the subspace corresponding the positive rest energy of any of particles) this Hamiltonian can be represented in the following form

$$H''=H_B'+\left(\frac{1}{8m_1^2}+\frac{1}{8m_2^2}\right)p^4,\tag{32.30}$$

where H'_B is the approximate Breit Hamiltonian (33.8).

We see the approximate Hamiltonian (32.29), obtained by diagonalization of the Galilei-invariant two-particle equation with the Breit potential, includes all the terms of the approximate Breit Hamiltonian excepting the relativistic correction to the kinetic energy. This means that this equation takes into account all the physical effects predicted by the Breit equation, i.e., spin-orbit coupling, retardation of the electromagnetic field etc. The correction to the kinetic energy only ($\sim p^4$), which has essentially relativistic nature, is absent. This correction can be taken into account by adding an additional term (depending on $\boldsymbol{x}$) to the Breit potential. An example of such a generalized Breit potential is given in [155].

Galilei-invariant equations of first order also can be used successively for description of pairs of interacting particles but the corresponding potential has to be more complicated. Thus, for example, starting from the equations (32.11), (32.22) and changing $p_0 \rightarrow p_0-V$, where [324]

$$V=-e_1e_2\left\{\frac{1}{x}+\frac{1}{2m_1m_2}\left(a\boldsymbol{p}\cdot\frac{1}{x}\boldsymbol{p}+b\boldsymbol{p}\cdot\boldsymbol{x}\frac{1}{x^3}\boldsymbol{x}\cdot\boldsymbol{p}\right)+\right.$$
$$\left.+\frac{i}{2}\left(\frac{\boldsymbol{S}_{(1)}}{m_1}-\frac{\boldsymbol{S}_{(2)}}{m_2}\right)\cdot\frac{\boldsymbol{x}}{x^3}-\frac{1}{m_1m_2}\left[\frac{\boldsymbol{S}\cdot\boldsymbol{L}}{x_3}+\left(\frac{1}{2}\boldsymbol{S}^2-\frac{3}{4}\right)\delta(\boldsymbol{x})\right]\right\}, \tag{32.31}$$

where

$$\boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p},\quad a=1-\frac{1}{4}\delta,\quad b=1+\frac{3}{2}\delta,\quad \delta=\frac{m_1^3+m_2^3}{m_1m_2(m_1+m_2)},$$

we come to the Galilei-invariant equations which reduce to the two-particle Schrödinger equation whose Hamiltonian is equivalent to the Breit Hamiltonian in the approximation $1/m_im_j$.

Let us summarize. Two-particle equations being invariant under the Galilei group can serve as good mathematical models of quantum mechanical systems consisting in two interacting particles of arbitrary spins. These equations give a correct description (not only qualitative but also quantitative) of these systems and take into account such fine effects as a spin-orbit coupling and even the retardation of the electromagnetic potentials.

The advantages of the Galilei-invariant approach in comparison with quasirelativistic ones are not only of methodological nature (because the corresponding theory satisfies (Galilean) relativity principle) but are connected with the relative simplicity of the corresponding mathematical means which make it possible to obtain solutions of many problems for any values of spins immediately. One such problem is considered in Section 34.

33. QUASI-RELATIVISTIC AND POINCARÉ-INVARIANT TWO-PARTICLE EQUATIONS

33.1 Preliminary notes

We demonstrated in the above the system of two interacting particles can be described with a good accuracy by Galilei-invariant wave equations. Since such equations do not have a universal reputation yet, we present alternative approaches to two-particle problems here. Besides familirizing readers with these approaches, we will try to attain clear formulation of bounds of application of Galilei-invariant two-particle equations by solving concrete physical problems.

We will consider only such two-particle equations which do not include an additional parameter (i.e., the proper time). For the equations with two time variables (equations of the Bethe-Salpeter type), see, e.g., [33].

In this section we present the main information about the Dirac-Breit equation [54] and its generalizations [266-269] and propose our formulations of quasi-relativistic models of two-particle interaction. We deduce equations for the radial wave function of parapositronium and give a formulation of two-particle problem in frames of our constructive model of direct interaction.

32.2. The Breit Equation

A two-particle quasirelativistic equation for spin 1/2 particles was proposed by Breit [54] in 1929 for the first time. This equation has the form

$$i\frac{\partial}{\partial x_0}\psi=(H^{(1)}+H^{(2)}+V_B)\psi, \tag{33.1}$$

where $\psi=\psi(x_0,\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)})$ is a 16-component wave function,

$$\begin{gathered} H^{(i)}=\gamma_0^{(i)}\gamma_a^{(i)}\pi_a^{(i)}+\gamma_0^{(i)}m_i+e^{(i)}A_0(\boldsymbol{x}_{(i)},x_0),\\ \pi_a^{(i)}=-i\frac{\partial}{\partial x_a^{(i)}}-e^{(i)}A_a(\boldsymbol{x}_{(i)},x_0),\quad i=1,2, \end{gathered} \tag{33.2}$$

$\{\gamma_\mu^{(1)}\}$ and $\{\gamma_\mu^{(2)}\}$ are two commuting sets of the Dirac matrices, A_μ is the vector-potential of an external field. The symbol V_B denotes in (33.1) the interaction (Breit) potential

$$\begin{gathered} V_B=\frac{e^{(1)}e^{(2)}}{x}\left[1-2\gamma_4^{(1)}\gamma_4^{(2)}(\boldsymbol{S}_{(1)}\cdot\boldsymbol{S}_{(2)}+\boldsymbol{S}_{(1)}\cdot\hat{\boldsymbol{x}}S_{(2)}\cdot\hat{\boldsymbol{x}})\right],\\ \boldsymbol{x}=\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)},\quad \hat{\boldsymbol{x}}=\frac{\boldsymbol{x}}{|\boldsymbol{x}|},\quad S_a^{(i)}=\frac{i}{4}\varepsilon_{abc}\gamma_b^{(i)}\gamma_c^{(i)}. \end{gathered} \tag{33.3}$$

The physical reasons used by Breit to deduce the equation (33.1) was the following. As it was shown by Darwin in 1920 (see [54]) the classical approximate Hamiltonian of a system of two charged particles in the external electromagnetic field has the form

$$H=H_{(1)}+H_{(2)}+\frac{e^{(1)}e^{(2)}}{x}-\frac{e^{(1)}e^{(2)}}{2m_1m_2}\left[\frac{\boldsymbol{p}_{(1)}\cdot\boldsymbol{p}_{(2)}}{x}+\frac{\boldsymbol{p}_{(1)}\cdot\boldsymbol{x}\,\boldsymbol{p}_{(2)}\cdot\boldsymbol{x}}{x^3}\right], \tag{33.4}$$

where $H_{(1)}$ and $H_{(2)}$ are the Hamiltonians of the first and second particles, $\boldsymbol{p}_{(i)}$ are classical momenta. The last two terms of (33.4) arise by taking into account the retardation of the potentials of the electromagnetic field. Changing $H_{(1)}$ and $H_{(2)}$ by the Dirac Hamiltonians and the velocities $\boldsymbol{v}_{(i)}=\boldsymbol{p}_{(i)}/m_i$ by the operators $\boldsymbol{V}_{(i)}=[H_{(i)},\boldsymbol{x}_{(i)}]$ we come to the equation (33.4)*.

There were proposed other ways of deduction of the Breit equation using quantum electrodynamics approach. We will not discuss these ways (see, e.g., [33]) but note that the main argument of validity of the equation (33.1) is the correspondence of the physical effects predicted by this equation to experimental data (of course, up to accuracy which can be expected from an approximate equation). But the principal defect of the Breit equation is the absence of symmetry under either the Poincaré or Galilei group.

In the following we discuss the Breit equation and its quasirelativistic limit and deduce the system of equations for radial functions by separating the angular variables.

33.3 Transformation to the Quasidiagonal Form

To simplify applying of the perturbation theory and to clarify the physical sense of the terms included in the Hamiltonian it is convenient to transform the Breit equation to the representation where the Hamiltonian commutes with the matrices $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, i.e., has a quasidiagonal form. As in the case of one-particle equations (see Section 10) such a diagonalization can be carried out approximately only, using series of consequent transformations.

In papers [18, 67] there was proposed a general method of diagonalization of two-particle Hamiltonians of the kind

$$H=\gamma_0^{(1)}m_1+\gamma_0^{(2)}m_2+(EE)+(OE)+(EO)+(OO), \tag{33.5}$$

where (EE) are terms commuting with $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, (OO) are terms anticommuting with $\gamma_0^{(1)}$ and $\gamma_0^{(2),}$ (EO) are the terms commuting with $\gamma_0^{(1)}$ and anticommuting with $\gamma_0^{(2),}$ and (OE) are terms anticommuting with $\gamma_0^{(1)}$ and commuting with $\gamma_0^{(2)}$, besides it is supposed that all these terms are "small" in comparison with m_1 and m_2. In the case of the Breit Hamiltonian

$$(EE)=e^{(1)}A_0(\boldsymbol{x}_{(1)},x_0)+e^{(2)}A_0(\boldsymbol{x}_{(2)},x_0)+\frac{e^{(1)}e^{(2)}}{x},\quad (OE)=\gamma_0^{(1)}\gamma_a^{(1)}(p_a^{(1)}-e^{(1)}A_a(\boldsymbol{x}_{(1)},x_0)), \tag{33.6}$$

$$(EO)=\gamma_0^{(2)}\gamma_a^{(2)}(p_a^{(2)}-e^{(2)}A_a(\boldsymbol{x}_{(2)},x_0)),\quad (AA)=V_B.$$

It is shown in [18] the Hamiltonian (33.5) can be transformed to the following form being quasidiagonal up to the terms of order $1/m_im_j$:

*It was Dirac who informed Breit about the correspondence $\boldsymbol{p}_{(i)}\rightarrow m[H_{(i)},\boldsymbol{x}]=\mathrm{i}m\gamma_0^{(i)}\gamma^{(i)}$ [54]

$$H' = \gamma_0^{(1)} m_1 + \gamma_0^{(2)} m_2 + (EE) + \frac{1}{2m_1}\gamma_0^{(1)}(OE)^2 + \frac{1}{2m_2}\gamma_0^{(2)}(EO)^2 - \frac{1}{8m_1^3}\gamma_0^{(1)}(EO)^4 -$$

$$-\frac{1}{8m_2^3}\gamma_0^{(2)}(EO)^4 + \frac{1}{8m_1^2}[[(OE),(EE)],OE] + \frac{1}{8m_2^2}[[(EO),(EE)],EO] + \tag{33.7}$$

$$+\frac{1}{4m_1 m_2}\gamma_0^{(1)}\gamma_0^{(2)}[[(OE),(OO)]_+,(EO)]_+ + \frac{1}{2(m_1^2-m_2^2)}(\gamma_0^{(1)} m_1 + \gamma_0^{(2)} m_2)(OO)^2 + \ldots\ .$$

It is not difficult to make sure that the first three lines of (33.7) give a sum of an approximate Hamiltonian of Foldy and Wouthuysen, compare with (10.26) for s=1/2. The remaining terms represent the contribution of the Breit potential.

Substituting (33.6) into (33.7) and choosing

$$A_\mu = 0, \quad \boldsymbol{P} = \boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)} = 0, \quad \boldsymbol{p}^{(1)} = -\boldsymbol{p}^{(2)} = \boldsymbol{p}, \quad \gamma_0^{(1)} \to 1, \quad \gamma_0^{(2)} \to 1$$

(i.e., considering the Hamiltonian in the rest frame and on the subset of eigenvectors of the commuting matrices $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, besides the corresponding eigenvalues are equal to +1) we obtain

$$H' = m_1 + \frac{p^2}{2m_1} - \frac{p^4}{8m_1^3} + m_2 + \frac{p^2}{2m_2} - \frac{p^4}{8m_2^3} + \frac{e^{(1)}e^{(2)}}{x} -$$

$$-\frac{\pi e^{(1)}e^{(2)}\delta(\boldsymbol{x})}{2}\left(\frac{1}{m_{(1)}^2} + \frac{1}{m_{(2)}^2}\right) - \frac{e^{(1)}e^{(2)}}{2x^3}\left(\frac{\boldsymbol{S}^{(1)}}{m_{(1)}^2} + \frac{\boldsymbol{S}^{(2)}}{m_{(2)}^2}\right)\cdot\boldsymbol{x}\times\boldsymbol{p} +$$

$$+\frac{e^{(1)}e^{(2)}}{m_{(1)}m_{(2)}}\left\{\frac{1}{2}\left[\boldsymbol{p}\frac{1}{x}\boldsymbol{p} + (\boldsymbol{p}\cdot\boldsymbol{x})\frac{1}{x^3}\boldsymbol{x}\cdot\boldsymbol{p}\right] - \frac{1}{x^3}\left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right)\cdot\boldsymbol{x}\times\boldsymbol{p} + \right. \tag{33.8}$$

$$\left. + \frac{\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}}{x^3} - \frac{3(\boldsymbol{S}^{(1)}\cdot\boldsymbol{x})(\boldsymbol{S}^{(2)}\cdot\boldsymbol{x})}{x^5} - \frac{8\pi}{3}\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}\delta(\boldsymbol{x})\right\} + O(e_1^2 e_2^2).$$

Any term in (33.8) has exact physical sense. The terms in the first line define the kinetic energy of a two-particle system, the terms from the second line correspond to interaction of any of particles with the Coulomb field generated by another particle (i.e., interactions of point charged particles plus spin orbit and Darwin interactions). The remaining terms of (33.8) represent essentially two-particle interactions and correspond to the classical relativistic correction to the Coulomb interaction, caused by the retardation of the electromagnetic field and spin-orbit coupling of the total spin of the system with the field generated by this system. The last three terms of (33.8) correspond to an interaction between the spin dipole moments of the particles.

In conclusion we note that to describe real two-particle systems, the modified Breit equation is usually used which takes into account anomalous magnetic moments of particles. For two-quark systems, the Breit potentials has to be changed by another one guaranteeing the confinement of constituent parts of a system [65, 220]. We consider some examples of such generalized equations in Subsections 33.4 and 33.5.

33.4. The Breit Equation for Particles of Equal Masses

Consider the equation (33.1) for the case when an external field is absent and the particle masses are equal, $m_1=m_2=m$. The total momentum operator $\boldsymbol{P}=\boldsymbol{p}_{(1)}+\boldsymbol{p}_{(2)}$ is a constant of motion for such an equation, so without loss of generality we can restrict ourselves to considering the rest frame of references where $\boldsymbol{p}_{(1)}+\boldsymbol{p}_{(2)}=0$, $\boldsymbol{p}_{(1)}=-\boldsymbol{p}_{(2)}=\boldsymbol{p}$.

Let us analyze the generalized Breit equation [266]

$$i\frac{\partial}{\partial x_0}\psi=\Big\{\gamma_0^{(1)}\gamma_a^{(1)}p_a+\gamma_0^{(1)}m-\gamma_0^{(2)}\gamma_a^{(2)}p_a+\gamma_0^{(2)}m+ \\ +V+2\gamma_4^{(1)}\gamma_4^{(2)}\left[\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}+(\boldsymbol{S}^{(1)}\cdot\hat{\boldsymbol{x}})(\boldsymbol{S}^{(2)}\cdot\hat{\boldsymbol{x}})\right]V'\Big\}\,\psi, \tag{33.9}$$

where V and V' are arbitrary functions of x, $\hat{\boldsymbol{x}}=\boldsymbol{x}/x$.

When $V=V'=e^{(1)}e^{(2)}/x$, formula (33.9) defines the Breit equation in the above representation. Since the explicit form of V and V' is not essential for further reasoning, we do not concretize it for the time being.

Let us demonstrate that the equation (33.9) decomposes into two noncoupled subsystems corresponding to the values $s=0$ and $s=1$ of the total spin of a system described. This means the states corresponding to $s=1$ (i.e., orthostates) and $s=0$ (i.e., parastates) are independent and transitions between these states are forbidden.

Multiplying the operator in the brackets from the left and the right by $\gamma_0^{(1)}$ we come to the following equivalent equation for $\psi'=\gamma^{(1)}\psi$

$$i\frac{\partial}{\partial x_0}\psi'=\Big\{\left(\gamma_0^{(1)}\gamma_a^{(1)}+\gamma_0^{(2)}\gamma_a^{(2)}\right)p_a+\left(\gamma_0^{(1)}+\gamma_0^{(2)}\right)m+V+ \\ +2\gamma_4^{(1)}\gamma_4^{(2)}\left[\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}+(\boldsymbol{S}^{(1)}\cdot\hat{\boldsymbol{x}})(\boldsymbol{S}^{(2)}\cdot\hat{\boldsymbol{x}})\right]V'\Big\}\psi'=0. \tag{33.10}$$

Any of matrices in (33.10) can be expressed via the 16×16 KDP matrices which are connected with γ-matrices by the following relations:
The matrices β_μ satisfy the KDP algebra (6.20) and realize a reducible representation

$$\beta_\mu=\frac{1}{2}\left(\gamma_\mu^{(1)}+\gamma_\mu^{(2)}\right),\quad \mu=0,1,2,3,4. \tag{33.11}$$

of this algebra, reducing into three IRs realized by 10×10, 5×5 and 1×1 matrices (the last is the trivial zero representation).

Let us denote

$$S_{0a}=i[\beta_0,\beta_a], \tag{33.12}$$

then

$$2\gamma_4^{(1)}\gamma_4^{(2)}\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}\equiv\frac{1}{2}\gamma_0^{(1)}\gamma_a^{(1)}\gamma_0^{(2)}\gamma_a^{(2)}=-\frac{3}{2}-S_{0a}S_{0a},$$

$$2\gamma_4^{(1)}\gamma_4^{(2)}(\boldsymbol{S}^{(1)}\cdot\hat{\boldsymbol{x}})(\boldsymbol{S}^{(2)}\cdot\hat{\boldsymbol{x}})\equiv\frac{1}{2}\gamma_0^{(1)}\gamma_a^{(1)}\hat{x}_a\gamma_0^{(2)}\gamma_b^{(2)}\hat{x}_b=-\frac{1}{2}-\left(S_{0a}\hat{x}_a\right)^2$$

and (33.10) takes the form

$$i\frac{\partial}{\partial x_0}\psi'=(-2iS_{0a}p_a+2\beta_0 m+W)\psi'=0, \tag{33.13}$$

$$W=V-[2+S_{0a}S_{0a}+(S_{0a}\hat{x}_a)^2]V'. \tag{33.14}$$

Choosing β_μ in the form of a direct sum of irreducible matrices of dimensions 10×10, 5×5 and 1×1 we obtain from (33.13) two noncoupled systems of ten and five equations and one-component equation corresponding to the zero matrices S_{0a} and β_μ. So (33.13) is decomposed into three independent subsystems which can be solved separately.

We note that the equation (33.13) reduces to the form

$$i\frac{\partial}{\partial x_0'}\psi'=\left(H^k+\frac{1}{2}W\right)\psi',$$

where $x'_0=2x_0$, H^K is the KDP Hamiltonian (6.35a). In other words, the Breit equation for equal mass particles in the rest frame is equivalent to the KDP equation in the Schrödinger form with the special potential $W/2$, where W is given in (33.14).

Consider the equation (33.11) for the case of 5×5 matrices β_μ. Choosing these matrices in the form (6.17) and representing the wave function in the form $\psi=column(\varphi_1,\varphi_2,\chi)$ we come to the following system of equations for stationary states

$$
\begin{aligned}
&(E-V+2V')\varphi_1-2m\varphi_2=0,\\
&(E-V-2V')\varphi_2-2m\varphi_1-2\boldsymbol{p}\cdot\chi=0,\\
&-2\boldsymbol{p}\varphi_2+(E+V'-V)\chi-V'\hat{\boldsymbol{x}}(\hat{\boldsymbol{x}}\cdot\chi)=0.
\end{aligned}
\tag{33.15}
$$

The system (33.15) reduces to the following equation for φ_2

$$
\left[p^2-\frac{\frac{\partial V}{\partial x}}{E-V}\frac{\partial}{\partial x}-\frac{(E-V)(E-V-2V')}{4}+\frac{m^2(E-V)}{E-V+2V'}-\frac{V'}{E-V+V'}\frac{\boldsymbol{L}^2}{x^2}\right]\varphi_2=0,
\tag{33.16}
$$

where $\boldsymbol{L}^2=(\boldsymbol{x}\times\boldsymbol{p})^2$. As to φ_1 and χ, they are expressed via φ_2 according to the following relations

$$
\chi=\left(\frac{2V'}{(E-V)(E-V+V')}[\hat{\boldsymbol{x}}(\hat{\boldsymbol{x}}\cdot\boldsymbol{p})-\boldsymbol{p}]+\frac{2}{E-V}\boldsymbol{p}\right)\varphi_2,\quad \varphi_1=\frac{E-V-2V'}{2m}\varphi_2-\frac{\boldsymbol{p}\cdot\chi}{m}.
$$

We see the Breit equation for parastates reduces to the equation (33.16) for the scalar function φ_2. This equation admits solutions in separated variables. Indeed, expanding φ_2 by the spherical functions

$$
\varphi_2=\varphi^{lm}(x)Y_{lm}(\hat{\boldsymbol{x}})
$$

and using the representation (29.9) for p^2 it is possible to reduce (33.16) to the ordinary differential equations for radial functions

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}+\frac{\frac{\partial V}{\partial x}}{E-V}\frac{\partial}{\partial x}+\frac{(E-V)(E-V-2V')}{4}-\frac{m^2(E-V)}{E-V+2V'}+\right.\\
&\qquad\left.+\frac{E-V'}{E-V+V'}\frac{l(l+1)}{x^2}\right)\varphi^{lm}=0.
\end{aligned}
\tag{33.17}
$$

So the problem reduces to solving equation (33.17) for any possible integer l. When V and V' coincide with the Coulomb potential, the equation (33.17) can be solved exactly [266].

The equation (33.17) was obtained in [266] but we present a more simple way of its deduction.

Starting from (33.13) and using the representation (6.22) for the β_μ-matrices it is not difficult to find the radial equations for orthostates. All the necessary formulae are given in Appendix 3. Using formulae (A.3.2) we can obtain radial equations corresponding to nonequal masses $m_1\neq m_2$ and to generalized Breit potential which is an arbitrary *O(3)*-invariant function of x. Such equations were considered in [65, 220, 267-269].

33.5. Two-Particle Equations Invariant Under the Group *P(1,6)*

The number of the space-time variables needed to describe a two-particle system is equal to 7 (this variables are particle coordinates and time). Thus a natural candidate for the role of the symmetry group of a two-particle system is the generalized Poincaré group *P(1,6)*, i.e., the group of motions of the (1+6)-dimensional Minkowski space (see Chapter 5).

An analog of the KGF equation invariant under the group *P(1,6)* has the form

$$(p_0^2-p_1^2-p_2^2-p_3^2-p_4^2-p_5^2-p_6^2-\kappa^2)\psi=0. \tag{33.18}$$

If we denote $\boldsymbol{P}=(p_1,p_2,p_3)$, $\boldsymbol{p}=(p_4,p_5,p_6)$, $\kappa=2m$ then the expression in the brackets is nothing but the correct relation between energy p_0, total momentum $\boldsymbol{P}$ and internal momentum $\boldsymbol{p}$ of a two-particle system with equal masses $m_1=m_2=m$ of particles [290]. If $m_1\neq m_2$ then it is convenient to choose $\boldsymbol{k}=(p_4,p_5,p_6)$ where $\boldsymbol{k}$ is a vector parallel to $\boldsymbol{p}$, defined by the following relation [238]

$$\boldsymbol{k}^2=-m_1m_2+\frac{m_1m_2}{(m_1+m_2)^2}\left(\sqrt{m_1^2+\boldsymbol{p}^2}+\sqrt{m_2^2+\boldsymbol{p}^2}\right)^2. \tag{33.19}$$

The equation (33.18) is manifestly invariant under the algebra *AP(1,6)* and its subalgebra *AP(1,3)*. But in order to interpret (33.18) as a two-particle equation it is necessary to make sure that the following product of the Poincaré group representations

$$D=D(m_1\ s_1)\otimes D(m_2\ s_2) \tag{32.20}$$

is realized on the set of solutions of this equation. It is the case when ψ of (33.18) has $(2s_1+1)(2s_2+1)$ components, moreover the corresponding generators of the Poincaré group can be chosen in the form

$$\begin{gathered}\hat{P}_0=\varepsilon E\equiv\varepsilon\sqrt{\boldsymbol{P}^2+M^2},\quad \hat{P}_a=p_a,\\ \boldsymbol{J}=\boldsymbol{X}\times\boldsymbol{P}+\hat{\boldsymbol{j}},\quad \boldsymbol{N}=x_0\boldsymbol{P}-\frac{1}{2}[\boldsymbol{X},P_0]_+-\varepsilon\frac{\boldsymbol{P}\times\hat{\boldsymbol{j}}}{E+M},\end{gathered} \tag{33.21}$$

where

$$\hat{\boldsymbol{j}}=\boldsymbol{x}\times\boldsymbol{p}+\hat{\boldsymbol{S}},\quad \hat{\boldsymbol{S}}=\boldsymbol{S}^{(1)}+\boldsymbol{S}^{(2)},\quad M^2=\boldsymbol{p}^2+\kappa^2,\quad \kappa=m_1+m_2. \tag{33.22}$$

Here $\boldsymbol{S}^{(1)}$ and $\boldsymbol{S}^{(2)}$ are the commuting matrices of spin of the first and second particles, $\boldsymbol{X}$ and $\boldsymbol{x}$ are coordinates canonically conjugated with $\boldsymbol{P}$ and $\boldsymbol{p}$ so that

$$[P_a,X_b]=[p_a,x_b]=-i\delta_{ab},\quad [P_a,x_b]=[p_a,X_b]=[P_a,p_b]=[X_a,x_b]=0.$$

The operators (33.21) commute with the operator in the brackets of (33.18) and so form an IA of this equation. To make this assertion evident we rewrite (33.18) in the following form

$$(P_0^2-\boldsymbol{P}^2-M^2)\psi=0. \tag{33.23}$$

In relativistic quantum theory we assign the space of the representation (33.23) to the space of states of a system of noninteracting particles with spins s_1 and s_2. Thus it is possible to interpret (33.18) (where ψ is a $(2s_1+1)(2s_2+1)$ -component wave function) as a motion equation for such a system.

The operator M^2 commutes with P_0^2 and $\boldsymbol{P}^2$ thus it is possible to consider the equations for eigenvectors of M^2:

$$(p_0^2-\boldsymbol{P}^2-m^2)\psi_m=0, \tag{33.24}$$

where ψ_m are solutions of the following equation

$$M^2\psi_m\equiv(\boldsymbol{p}^2+\kappa^2)\psi_m=m^2\psi_m. \tag{33.25}$$

Formula (33.24) gives the KFG equation in the variables x_0, $\boldsymbol{X}$, but relation (33.25) defines the eigenvalue problem besides $\kappa^2\leq m^2<\infty$. The equations (33.24), (33.25) are invariant under the algebra *AP(1,3)* whose basis elements have the form (33.21), and describe a system of two noninteracting particles with spins s_1 and s_2.

Thus starting with the simplest partial differential equation of the second order, which is invariant under the generalized Poincaré group *P(1,6)*, we come to the system (33.24), (33.25) which can be interpreted as motion equations for a pair of noninteracting relativistic particles. Of course the adequacy of such equations is comparative only inasmuch as the main interest is attracted by mathematical models of interacting particles. In the following section, we will consider a generalization of (33.24), (33.25) to the case of interacting particles.

In [119] the *P(1,6)*-invariant equation of the Dirac type was proposed, which has the form

$$(\Gamma_0 p_0-\Gamma_k p_k-\kappa)\psi=0, \quad k=1,2...,6,$$

where Γ_0, Γ_k are matrices of dimension 8×8, satisfying the Clifford algebra. This equation can be used for describing of two relativistic particles of spin 1/2. An analysis of this equation (and its possible generalizations in the case of arbitrary spin) lies out of frames of the present book.

33.6. Additional Constants of Motion for Two- and Three-Particle Equations

We show in Subsection 22.3 that there exist a SO of the Dirac type for any relativistic wave equation describing a particle with spin $s>0$ in a central field. Here it is demonstrated that such a SO exists for a number of relativistic and quasirelativistic two- and three-particle equations.

Consider the Breit equation (33.1). The obvious SO of this equation are

$$P_0=p_0,\quad P_a=p_a^{(1)}+p_a^{(2)},\quad J_a=\varepsilon_{abc}(X_aP_c+x_bp_c)+S_a \tag{33.26}$$

where

$$S_a=S_a^{(1)}+S_a^{(2)},\quad S_a^{(\alpha)}=\frac{i}{4}\gamma_b^{(\alpha)}\gamma_c^{(\alpha)}\varepsilon_{abc}. \tag{33.27}$$

It is shown in [341] that the operators (33.26) form a basis of the maximal IA of the Breit equation in the class M_1 (i.e., are the generators of the maximal Lie group admitted by this equation) if $m_1\neq m_2$.

To find an additional SO we consider the Breit equation in the c.m. frame, setting in (33.1) $\boldsymbol{p}^{(1)}=-\boldsymbol{p}^{(2)}=\boldsymbol{p}$. It is not difficult to verify that the operator [341, 161]

$$\hat{Q}=\gamma_0^{(1)}\gamma_0^{(2)}\left[2(\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{j}})^2-2\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{j}}-\hat{\boldsymbol{j}}^2\right] \tag{33.28}$$

where $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{j}}$ are defined in (33.22), is a SO of this equation.

The operator (33.28) satisfies the condition

$$\hat{Q}^2\equiv(\hat{\boldsymbol{j}})^2$$

and so its spectrum is given in (22.18). We emphasize that this operator does not belong to the enveloping algebra generated by the generators (33.26) and so is essentially non-Lie.

Like the Dirac SO for the Dirac equation (see (22.10)) the operator (33.28) can be used by solving the Breit equation in separable variables. We note that the SO (33.28) is admitted also by the generalized Breit equation with arbitrary *O(3)*- and *P*-invariant potential.

Apparently it is possible to continue the list of the equations for which the operator (33.28) is a motion constant. For instance this is the case for the relativistic Barut-Komi equation [19].

Following [161], we present new constants of motion for the equation describing two interacting particles with spins 1/2 and 1 [269] and for the three-particle equation of Krolikowsky [267]. These constants have the form

$$Q=\Gamma d_{3/2} \tag{33.29}$$

where $d_{3/2}$ is the operator given in (22.29). Moreover for the two-particle equation proposed in [269] $\Gamma=\gamma_0(1-2\beta_0)$, $S_a=i\varepsilon_{abc}(\gamma_b\gamma_c+\beta_b\beta_c)$, γ_μ and β_μ are commuting sets of the Dirac and KDP matrices.

For the three-particle equation [267]

$$\Gamma=\gamma_0^{(1)}\gamma_0^{(2)}\gamma_0^{(3)},\quad S_a=\frac{i}{4}\varepsilon_{abc}\left[\gamma_b^{(1)}\gamma_c^{(1)}+\gamma_b^{(2)}\gamma_c^{(2)}+\gamma_b^{(3)}\gamma_c^{(3)}\right], \tag{33.30}$$

where $\gamma_\mu^{(1)}$, $\gamma_\mu^{(2)}$ and $\gamma_\mu^{(3)}$ are commuting sets of the Dirac matrices.

The spectrum of the operators (33.29), (33.30) coincides with the spectrum of the operator $d_{3/2}$ of (21.30).

34. EXACTLY SOLVABLE MODELS OF TWO-PARTICLE SYSTEMS

34.1. Nonrelativistic Model

Consider the two-particle Schrödinger equation of the kind

$$i\frac{\partial}{\partial t}\psi = H\psi \equiv \left(\frac{\boldsymbol{P}^2}{2M} + \frac{\boldsymbol{p}^2}{2\mu} + \hat{V}\right)\psi, \tag{34.1}$$

where ψ is a $(2s_1+1)(2s_2+1)$-component wave function, $\hat{V}$ is the interaction potential of the following form

$$\hat{V} = \frac{\alpha}{x}\left(-1 + i\frac{k_{(1)}\boldsymbol{S}_{(1)}\cdot\boldsymbol{x}}{m_1 x^2} + i\frac{k_{(2)}\boldsymbol{S}_{(2)}\cdot\boldsymbol{x}}{m_2 x^2}\right), \tag{34.2}$$

α, $k_{(1)}$ and $k_{(2)}$ are dimensionless constants.

The equation (34.1) is manifestly invariant under Galilean transformations inasmuch as the potential (34.2) satisfies the conditions (32.8), (32.18). The corresponding generators of the group *G(1,3)* are given by formulae (32.5), (32.18).

The considered equation is a particular case of two-particle equations in the Schrödinger form discussed in Subsection 31.1. The Hamiltonian H of (34.1) in the c.m. frame can be obtained from (32.26) by a special choosing of V, e_1 and e_2 and restricting ourselves to the subset of eigenfunctions of the matrices $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$, corresponding to the eigenvalue +1.

In accordance with the results of Subsection 32.5 the considered model is realistic enough and takes into account such fine effects as spin-orbit, Darwin and quadruple couplings. Other important merit of this model is that it is exactly solvable for any values of spins of constitutive particles.

In the following we find the corresponding exact solutions which can be obtained also for the more complicated (relativistic) model considered in the next subsection.

34.2. Relativistic Two-Particle Model

To describe a relativistic two-particle system we use the Bakamjian-Tomas model [10] whose essence is expand in the following.

Let $(P_\mu, \boldsymbol{J}, \boldsymbol{N})$ be Poincaré group generators describing the kinematics of a system of two non-interacting particles. Then a system of interacting particles corresponds to the generators $(P'_\mu, \boldsymbol{J}', \boldsymbol{N}')$ obtained from $(P_\mu, \boldsymbol{J}, \boldsymbol{N})$ by the change

$$M \rightarrow M+V \tag{34.3}$$

where M is the mass operator determined as a Casimir operator:

$$M^2 = P_0^2 - \boldsymbol{P}^2,$$

and V is a potential of "instantaneous" interaction satisfying the conditions

$$[V, P_\mu] = [V, \boldsymbol{J}] = [V, \boldsymbol{N}] = 0. \tag{34.34}$$

Using the representation (33.21) it is not difficult to show that V has to be a scalar function of internal variables only:

$$V = V(\boldsymbol{x}, \boldsymbol{p}), \quad [V, \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}] = 0. \tag{34.5}$$

Here we consider a constructive model of a potential of instantaneous interaction. On one hand this model is based on equations invariant under the group *P(1,6)* (see Subsection 33.6); on the other hand, it includes manifestly covariant equations in c.m. variables.

Let us define the interaction potential of (34.3) by the following relation

$$\left(\hat{M} + \frac{\alpha}{x}\right)^2 = \boldsymbol{k}^2 + \mu^2 - i\frac{\alpha}{x^3}(k_{(1)}\boldsymbol{S}_{(1)}\cdot\boldsymbol{x} - k_{(2)}\boldsymbol{S}_{(2)}\cdot\boldsymbol{x}), \tag{34.6}$$

where $\boldsymbol{x}$ are variables canonically conjugated with $\boldsymbol{k}$, $\boldsymbol{k}$ is a vector related to the internal momenta by the condition (33.19), so that

$$\boldsymbol{k}^2 + \mu^2 = \tilde{M}^2, \quad \tilde{M}^2 = \frac{\mu}{\kappa}M, \quad \kappa = m_{(1)} + m_{(2)}. \tag{34.7}$$

Formulae (34.5)-(34.7) define a potential V as an implicit function of $\boldsymbol{x}$, $\boldsymbol{k}$, $\boldsymbol{S}_{(1)}$ and $\boldsymbol{S}_{(2)}$. It is not difficult to make sure that such defined potential satisfies the conditions (34.4).

The considered potential can be introduced into the *P(1,6)*-invariant equations (33.24), (33.25), moreover, the last takes the form

$$\left[\left(\tilde{M} + \frac{\alpha}{x}\right)^2 - \boldsymbol{k}^2 - \mu^2 + i\frac{\alpha}{x^3}(k_{(1)}\boldsymbol{S}_{(1)}\cdot\boldsymbol{x} - k_{(2)}\boldsymbol{S}_{(2)}\cdot\boldsymbol{x})\right]\psi_{\tilde{m}} = 0, \tag{34.8}$$

and can be considered as eigenvalue equation for the operator $\hat{M}$.

So starting with the *P(1,6)*-invariant equations (33.24), (33.25) describing a system of two noninteracting particles we go over to the equations (33.25), (34.8) which can be interpreted as a mathematical model of interacting particles of arbitrary spins. The choice of an interaction potential is caused by an additional requirement that the motion equations have to admit a manifestly covariant formulation and describe the electromagnetic interaction of particles. Such a formulation is discussed further on.

Let us demonstrate that in spite of lack of manifestly covariance the equation (34.8) can be rewritten in the covariant notations. For this purpose we consider the following system of equations for a $(2s_1+1)(2s_2+1)$-component wave function:

$$(\pi_\mu\pi^\mu-\mu^2-e\Sigma_{\mu\nu}F^{\mu\nu})\psi=0, \tag{34.9}$$

where $\pi_\mu=k_\mu-A_\mu(x_0,\boldsymbol{x})$, A_μ and $F^{\mu\nu}$ are the vector-potential and tensor of the electromagnetic field. The symbol $\Sigma_{\mu\nu}$ denotes a matrix tensor of valence 2 defined in the space of the IR $D(s_1\, s_2)$ of the Lorentz group.

Formula (34.9) defines the general form of the equation obtained from the KGF equation by the "minimal" change $k_\mu \to \pi_\mu$ and by taking into account the "anomalous" interaction linear in respect with strengths of an external field. When $s_1=0$ or $s_2=0$, (34.9) reduces to the form (10.31) (besides the corresponding $\Sigma_{\mu\nu}$ are proportional to the Lorentz group generators $S_{\mu\nu}$. For arbitrary s_1 and s_2 we can set without loss of generality

$$\Sigma_{\mu\nu}=k_{(1)}S^{(1)}_{\mu\nu}+k_{(2)}S^{(2)}_{\mu\nu}, \tag{34.10}$$

where $k_{(1)}$ and $k_{(2)}$ are arbitrary coefficients, $S_{\mu\nu}^{(1)}$ and $S_{\mu\nu}^{(2)}$ are commuting sets of matrices being generators of the IR $D(s_1\; 0)$ and $D(0\; s_2)$ of the Lorentz group.

The equations (34.9), (34.10) can be interpreted as motion equations of a charged quasiparticle with variable spin *s*, $s_1+s_2\leq s\leq|s_1-s_2|$. When the vector-potential A_μ reduces to the Coulomb potential, these equations reduce to the form (34.8) if we write them for stationary states. So these equations can be interpreted as a two-particle equation for particles of spins s_1 and s_2.

Thus we come to a constructive model of the direct interaction for two-particles systems. This model is based on the manifestly covariant equation (34.9), (34.10) which is the simplest relativistic-invariant equation for a $(2s_1+1)(2s_2+1)$ -component wave function ψ, taking into account the minimal electromagnetic interaction and the anomalous interaction of the Pauli type. One of the merits of this model is that it is exactly solvable for the cases of the Coulomb and magnetic monopole potentials.

34.3. Solutions of Two-Particle Equations

The operator in square brackets of (34.8) commutes with the total angular momentum operator $\boldsymbol{j}$ of (33.22) so this equation admits solutions in separable variables. Such solutions can be found by the scheme used in Subsection 29.1 for one-particle equations.

To separate angular variables we represent $\psi_{\tilde{m}}$ as a linear combination of the spherical spinors

$$\psi_{\tilde{m}}=\sum_{s,\lambda_s} \varphi_s^{\lambda_s}\Omega^s_{ij-\lambda_s m}, \tag{34.11}$$

where $\Omega^s_{...}$ are eigenfunctions of the operators $\boldsymbol{j}^2$, $\boldsymbol{S}^2$, $(\boldsymbol{x}\times\boldsymbol{p})^2$ and j_3 (33.22), see (29.4). In contrast with the corresponding formula (29.3) where s was fixed we suppose summing over s by the values $|s_1\text{-}s_2|\leq s\leq s_1+s_2$, the domain of λ_μ for any fixed s coincides with the possible values of λ of (29.5).

Substituting (34.11) into (34.8) and separating angular variables we come to the following system of ordinary differential equations for the radial functions

$$D\varphi_s^{\lambda_s}=x^{-2}b^{ss'}_{\lambda_s\lambda'_{s'}}\varphi_{s'}^{\lambda'_{s'}}, \tag{34.12}$$

where D is the operator (29.11), $b^{ss'}_{\lambda_s\lambda'_{s'}}$ are the following coefficients

$$b^{ss'}_{\lambda_s\lambda'_{s'}}=[\lambda_s^2-\lambda_s(2j+1)]\delta_{ss'}\delta_{\lambda_s\lambda'_{s'}}+i\alpha k_{(1)}B^{js\lambda_s}_{(1)s'\lambda'_{s'}}-i\alpha k_{(2)}B^{js\lambda_s}_{(2)s'\lambda'_{s'}} \tag{34.13}$$

and $B^{sj\lambda_s}_{(i)s'\lambda'_{s'}}$ are the coefficients given in (A.3.7).

The system (34.12) is easily integrated. $\|b^{ss'}_{\lambda_s\lambda'_{s'}}\|$ is a diagonalizable (normal) matrix so (34.12) reduces to the series of noncoupled equations of the kind

$$D\varphi_{\tilde{m}}=x^{-2}b^{js_1s_2}\varphi_{\tilde{m}}, \tag{34.14}$$

where D is the operator (29.11), $b^{\cdots}$ are eigenvalues of the matrix $\|b^{ss'}_{\lambda_s\lambda'_{s'}}\|$.

Equations of the form (34.14) had already been considered in Subsection 29.2. Repeating reasoning present after formula (29.12) (but changing $b^{sj}_\lambda\rightarrow b^{s_1s_2j}$, $m\rightarrow\mu$) we come to the conclusion that the following values of $\varepsilon=\tilde{m}$ correspond to coupled states (compare with (29.23)):

$$\varepsilon=\mu\left[1+\frac{\alpha^2}{\left(n'+1/2+\left[\left(j+\frac{1}{2}\right)^2-\alpha^2+b^{js_1s_2}\right]^{1/2}\right)^2}\right]^{1/2},\quad n'=0,1,2,..., \tag{34.15}$$

and the explicit form of the corresponding eigenfunctions is given by the following formula

$$\varphi_\varepsilon = C(\mu^2-\varepsilon^2)^{(k+1)/4} x^{(k-1)/2} \exp\left[-(\mu^2-\varepsilon^2)^{1/2}x\right] \mathscr{F}(-n',k+1,2(\mu^2-\varepsilon^2)^{1/2}x).$$

Here $\mathscr{F}$ is a degenerated hypergeometric function, C is an arbitrary constant, $k^2=(2j+1)^2+4(b^{js_1s_2})^2-4\alpha^2$.

So we have described solutions of (34.8) and find the corresponding eigenvalues of the operator (34.6). The analysis of the spectrum (34.15) is present in Subsection 34.4.

Solutions of the Galilei-invariant equations (34.1), (34.2) can be found in analogy with the corresponding one-particle problem considered in Subsection 30.4. Considering these equations for stationary states in the rest frame, we come to the following system

$$\left[\tilde{E}+\frac{\alpha}{x}-\frac{\boldsymbol{p}^2}{2\mu}-\frac{i\alpha}{x^3}\left(\frac{\tilde{k}_{(1)}}{m_1}\boldsymbol{S}_{(1)}\cdot\boldsymbol{x}+\frac{\tilde{k}_{(2)}}{m_2}\boldsymbol{S}_{(2)}\cdot\boldsymbol{x}\right)\right]\psi_{\tilde{E}}(\boldsymbol{x})=0. \tag{34.16}$$

Representing solutions in the form (34.11) we come to the equations (34.12) for radial functions where

$$D=2\mu\left(\tilde{E}+\frac{\alpha}{x}\right)+\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}, \tag{34.17}$$

$$b^{ss'}_{\lambda_s\lambda'_{s'}}=[\lambda_s^2-\lambda_s(2j+1)]\delta_{ss'}\delta_{\lambda_s\lambda'_{s'}}-\frac{i\tilde{k}_{(1)}\mu\alpha}{m_1}B^{js\lambda_s}_{(1)s'\lambda'_{s'}}-\frac{i\tilde{k}_{(2)}\mu\alpha}{m_2}B^{js\lambda_s}_{(2)s'\lambda'_{s'}} \tag{34.18}$$

$B^{js\lambda_s}_{(i)s'\lambda'_{s'}}$ are the coefficients (A.3.7). Diagonalizing the matrix $\left\| b^{ss'}_{\lambda_s\lambda'_{s'}} \right\|$ we reduce these equations to the form (34.14) where D is the operator (34.17), $b^{js_1s_2}\rightarrow\tilde{b}^{js_1s_2}$, the last are the eigenvalues of this matrix.

Repeating the arguments present in Subsection 30.4 after formula (30.25) we obtain the possible values of energy of a pair of interacting particles of arbitrary spins in the form

$$\tilde{E}=\frac{\mu\alpha^2}{\left(\sqrt{(j+1/2)^2+\tilde{b}^{js_1s_2}}+n'+1/2\right)^2}, \quad n'=0,1,...; \quad j=1/2,3/2... \ \textit{or}\ j=0,1,... \ . \tag{34.19}$$

The discussion of (34.19) is given in the following subsection. Here we present the explicit form of solutions of (34.14), (34.17), (34.18):

$$\varphi^s_{\lambda_s}=\left(\sqrt{-2\mu\tilde{E}}\ x\right)^{(k-1)/2}\exp\left(-\sqrt{-2\mu\tilde{E}}\ x\right)\mathscr{F}(-n',k+1,2\sqrt{-2\mu\tilde{E}}\ x),$$

where $\mathscr{F}$ is a degenerated hypergeometric function, C is an arbitrary number, $k^2=(2j+1)^2+4b^{js_1s_2}$.

33.4. Discussing the Spectra of the Two-Particle Models

We have obtained formulae (34.15), (34.19) giving the possible energy values of the considered two-particle models. The analysis of these formulae is complicated by the circumstance that the parameters $b^{\cdots}$ are defined as roots of algebraic equations of order $(2s_1+1)(2s_2+1)$ if $j\geq s_1+s_2$. Thus it is convenient to consider approximate solutions which can be represented in the form

$$b_{\lambda_s}^{js_1s_2}=\lambda_s^2-(2j+1)\lambda_s+b_{\lambda_s}^{sj}\alpha^2+o(\alpha^4),\quad \tilde{b}_{\lambda_s}^{js_1s_2j}=\lambda_s^2-(2j+1)\lambda_s+\tilde{b}_{\lambda_s}^{sj}\alpha^2+o(\alpha^4), \tag{34.20}$$

where $b_{\lambda_s}^{sj}$, $\tilde{b}_{\lambda_s}^{sj}$ are numeric parameters whose values for given s_1, s_2 and j can be found by equating coefficients near α^2 in the characteristic equations.

Substituting (34.20) into (34.15), (34.19) and representing the obtained expressions as series in powers α^2 we come to the following formulae for energy levels, valid up to α^4:

$$\varepsilon=\mu\left(1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4\left(b_{\lambda_s}^{s\lambda_s+l}-1\right)}{n^3(2l+1)}+\frac{3}{8}\frac{\alpha^4}{n^4}\right), \tag{34.21}$$

$$\tilde{E}=-\mu\left(\frac{\alpha^2}{2n^2}+\frac{\alpha^4\tilde{b}_{\lambda_s}^{s\lambda_s+l}}{n^3(2l+1)}\right), \tag{34.22}$$

where

$$\begin{gathered} n=1,\ 2,\ldots,\ s=s_{(1)}+s_{(2)},\ s_{(1)}+s_{(2)}-1,\ldots,|s_{(1)}-s_{(2)}|,\\ \lambda_s=-s,-\ s+1,\ldots-s+2min(s,j),\ l=j-\lambda_s=0,\ 1,\ldots,\ n-1. \end{gathered} \tag{34.23}$$

Formulae (34.21), (34.22) differ from (29.27), (30.28) by the change $m\to\mu$ besides the parameters $b_{\lambda_s}^{sj}$, $\tilde{b}_{\lambda_s}^{sj}$ are defined as coefficients of approximate solutions (34.20) of the characteristic equations for the matrices whose elements are given in (34.13), (34.18).

According to (34.21), (34.22) the spectra of energies of the considered two-particle models are defined by Balmer's term $-\mu\alpha^2/2n^2$ and by the additional terms of order α^4 besides the last describe the fine splitting of the energy levels. The analysis of possible values of the quantum numbers *n, l, s* and λ_s given in (34.23) makes it possible to calculate the number of sublevels of the fine structure:

$$N_n^{s_1s_2}=\sum_{s=|s_1-s_2|}^{s+s_2}N_n^s,$$

where N_n^s are the numbers given in (29.30). We note that the energy levels corresponding to different *l, s* and λ_s can coincide in general and be degenerated.

Let us consider in more detail the cases when the particle spins are: A.

$s_1=0$, s_2 is arbitrary; B. $s_1=s_2=1/2$.

A. If $s_1=0$ then the matrices $\left\| b^{ss'}_{\lambda_s\lambda'_{s'}} \right\|$, $\left\| \tilde{b}^{ss'}_{\lambda_s\lambda'_{s'}} \right\|$ reduce to the form obtained from $\left\| b^{sj}_{\lambda\lambda'} \right\|$ of (29.11) by the changes $1/s\to k_{(2)}$ and $1/s\to \mathrm{k}_{(2)}\mu/m_2$. The corresponding parameters included into (34.21), (34.22) take the form

$$b^{sj}_{\lambda_s}=s^2k^2_{(2)}\tilde{b}^{sj}_{\lambda},\quad \tilde{b}^{sj}_{\lambda_s}=\frac{\mu^2s^2\tilde{k}^2_{(2)}}{m_2^2}\tilde{b}^{sj}_{\lambda},\quad s=s_2, \tag{34.24}$$

where $\tilde{b}^{sj}_{\lambda}$ are the parameters given in (29.26). Substituting (34.24) into (34.21), (34.22) we obtain

$$\varepsilon=\varepsilon_\mu+\frac{\mu\alpha^4\tilde{b}^{s_2\lambda+l}_{\lambda}}{n^3(2l+1)}\left(k^2_{(2)}s_2^2-1\right), \tag{34.25}$$

$$\tilde{E}=\tilde{E}_\mu-\mu+\frac{\mu\alpha^4\tilde{b}^{s\lambda+l}_{\lambda}}{n^3(2l+1)}\left(\frac{s^2k^2_{(2)}\mu^2}{m_2^2}-1\right), \tag{34.26}$$

where ε_μ and $\tilde{E}_\mu$ are the energy values obtained from (29.27) and (30.28) by the change $m\to\mu$.

We see in the case $s_1=0$ the formulae for energy levels of the considered models include the terms ε_μ and $\tilde{E}_\mu$ obtained by multiplying of the corresponding one-particle levels (29.27) and (30.28) by $m_1/(m_1+m_2)$. Besides these formulae include additional terms of order α^4 which are nullified if $k_{(2)}=\tilde{k}_{(2)}\mu/m_2=\pm 1/s$. In other words for the last values of arbitrary parameters formulae (34.25), (33.26) reduce to the formulae for energy levels of a particle of arbitrary spin in the Coulomb field, multiplied by the coefficient $m_1/(m_1+m_2)$ in order to take into account finiteness of the mass of a particle generating the field. Besides that formula (34.26) includes nonessential constant term $-\mu$. The relation between relativistic and nonrelativistic formulae is the same as in the corresponding one-particle problems (see Subsection 29.6), i.e., for $k_{(2)}=\tilde{k}_{(2)}\mu/m_2$ the levels (34.25) and (33.26) differ by the value $\Delta\varepsilon$ (30.31) giving the relativistic correction to the kinetic energy.

For other values of the parameters $k_{(2)}$ and $\tilde{k}_{(2)}$ formulae (34.25), (34.26) take into account the anomalous (Pauli) interaction of the orbital particle with the electromagnetic field.

B. For $s_1=s_2=1/2$ the characteristic equation for the matrix (34.13) takes the form

$$\left(b^j_{\lambda_s}\right)^4-2\left(b^j_{\lambda_s}\right)^3-[4j(j+1)-\alpha^2(c^2+d^2)]\left(b^j_{\lambda_s}\right)^2-2\alpha^2d^2\left(b^j_{\lambda_s}\right)+\alpha^2c^2d^2=0, \tag{34.27}$$

where

$$b_{\lambda_s}^{j}=b_{\lambda_s}^{\frac{1}{2}\frac{1}{2}j},\quad c=\frac{1}{2}(k_{(1)}+k_{(2)}),\quad d=\frac{1}{2}(k_{(1)}-k_{(2)}),\quad j\neq 0, \tag{34.28}$$

Representing solutions of (34.27) in the form (34.20) and equating the coefficients for the lowest orders of α^2 we obtain the following values of the parameters $b_{\lambda_s}^{sj}$:

$$b_0^{0l}=-\frac{d^2}{4l(l+1)}\left(1-\sqrt{1+4l(l+1)\frac{c^2}{d^2}}\right),\quad b_0^{1l}=-\frac{d^2}{4l(l+1)}\left(1+\sqrt{1+4l(l+1)\frac{c^2}{d^2}}\right), \tag{34.29}$$
$$b_1^{1l+1}=\frac{(l+1)c^2+(l+2)d^2}{2(l+1)(2l+3)},\quad b_{-1}^{1l-1}=-\frac{c^2l+d^2(l-1)}{2l(2l-1)}.$$

In the exceptional case j=0 the index λ_s can take two values λ_1=-1 and λ_0=0 (see (34.23)) and the corresponding characteristic equation has the form

$$\left(b_{\lambda_s}^{s_1s_20}\right)^2-2b_{\lambda_s}^{s_1s_20}+\alpha^2c^2=0,$$

from which we obtain the coefficients of (34.20)

$$b_0^{00}=-b_{-1}^{10}=c^2/2. \tag{34.30}$$

Formulae (34.25), (34.29), (34.30) define possible values of energies of the considered model of two interacting particles of spins 1/2. Due to existing of two arbitrary parameters c and d there are wide possibilities of modelling of spectra of hydrogen type systems. We consider in detail two cases: the system of a particle and antiparticle and the system including one heavy and one light particle.

In the case of the system of "particle +antiparticle" it is natural to set $k_{(1)}$=-$k_{(2)}$=d. Then c=0 and formulae (34.29) take the form

$$b_0^{0l}=0,\quad b_0^{1l}=-\frac{d^2}{2l(l+1)},\quad b^{1l+1}=\frac{d^2(l+2)}{2l(l+1)(2l+3)},\quad b_{-1}^{1l-1}=-d^2\frac{(l-1)}{2l(2l-1)}. \tag{34.31}$$

Substituting (34.31) into (34.25) we obtain the corresponding energy levels:

$$\varepsilon=\mu\left(1-\frac{\alpha^2}{2n^2}\right)+W(s,j,l),\quad j=l+\lambda_s, \tag{34.32}$$

where $W(s,j,l)$ are the corrections of order α^4, defining the fine structure of the spectrum:

$$W(0,l,l)=\mu\alpha^4\left(\frac{3}{8n^4}-\frac{1}{n^3(2l+1)}\right),$$
$$W(1,l,l)=W(0,l,l)-\frac{\mu\alpha^4d^2}{2n^3l(l+1)(2l+1)},\quad l\neq 0, \tag{34.33}$$

In the case d^2=1 formulae (34.33), (34.34) are in excellent accord with the

$$W(1,l,l+1)=W(0,l,l)+\frac{\mu\alpha^4(l+2)d^2}{2n^3(l+1)(2l+1)(2l+3)},$$
$$W(1,l,l-1)=W(0,l,l)-\frac{\mu\alpha^4(l-1)d^2}{2n^3 l(2l+1)(2l-1)}. \qquad (34.34)$$

corresponding corrections for the positronium energy levels calculated in frames of quantum electrodynamics [33]. Namely (34.33) differ from the corresponding corrections for positronium by the value ΔW not depending on l and s ($l\neq 0$)

$$\Delta W=\frac{\mu\alpha^4}{32n^4}\approx 0{,}03\frac{\mu\alpha^4}{n^4}. \qquad (34.35)$$

Formulae (34.34) are less similar with the known results. But if we choose arbitrary parameters c and d in (34.26) in the form $c^2=2d^2=2$ then relations (34.21), (34.29) give the correct dependence of W on l,s and j up to the common displacement (34.35) (compare with the corresponding results for energy levels of positronium given in [3, 33]).

Thus the mathematical model of a pair of interacting particles, based on the equation (34.9), is consistent enough and makes it possible to obtain a correct dependence of a fine structure of the positronium spectrum on the quantum numbers l, s and j. The displacement (34.35) is not large and is of the same value for any sublevel corresponding to a fixed value of the main quantum number (if $l\neq 0$). For $l=0$ the divergence of (34.32)-(34.34) with the known results reaches 17% of the value of fine splitting. Such a divergence is clear inasmuch as the considered model does not take into account the interchange interaction.

We note that in the case $c=d=1$ formulae (34.29) take the form

$$b_0^{0l}=b_1^{1l+1}=\frac{1}{2(l+1)}, \quad b_0^{1l}=b_{-1}^{1l-1}=-\frac{d^2}{l}$$

and the relation (34.25) reduces to the form obtained from (29.27) by changing m to the reduced mass μ. In other words in the case $k_{(2)}\to 0$ we obtain the known formula of the fine structure of the spectrum of a hydrogen type system besides, it takes into account the finiteness of the mass of a central particle. So the relation (34.25) includes the case corresponding to a system of a heavy and light particle.

It follows from the above the considered model of two-particle interaction is realistic enough to be used successively for a description of two-particle systems with arbitrary spins.

The analysis of the spectrum (34.22) can be made in a complete analogy with the above. For $|\tilde{k}_{(i)}\mu|=|k_{(i)}m_i|$ the energy levels of (34.22) differ from the levels (34.21) by the value $\Delta\varepsilon$-μ where $\Delta\varepsilon$ is given in (30.30) and corresponds to the relativistic correction for the kinetic energy.

APPENDIX 1

LIE ALGEBRAS, SUPERALGEBRAS AND PARASUPERALGEBRAS

Here we recall the basic definitions related to Lie algebras, super- and parasuperalgebras (for more details, see [20, 26, 411]).

We denote by A a finite-dimensional vector space over the field K of real (complex) numbers. The vector space A is called a Lie algebra over K if it is closed with respect to the binary operation $(x,y)\to[x,y]$ which satisfies the following axioms

$$\begin{aligned} &[\alpha x+\beta y,z]=\alpha[x,z]+\beta[y,z],\\ &[x,y]=-[y,x], \qquad\qquad (A.1.1)\\ &[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \end{aligned}$$

for all $x,y,z\in A$.

The operation [,] is called the Lie multiplication (Lie brackets), and the last of the relations (A.1.1) is called the Jacobi identity. A Lie algebra is called commutative or Abelian if $[x,y]=0$ for any $x,y\in A$. A subspace N of the algebra A is called a subalgebra if $[N,N]\subset N$ and an ideal if $[A,N]\subset N$.

Let A and B be two arbitrary Lie algebras and Φ be a mapping of A into B. This mapping is called a homomorphism if

$$\phi(\alpha x+\beta y)=\alpha\phi(x)+\beta\phi(y), \quad x,y\in A, \quad \alpha,\beta\in K$$

$$\phi([x,y])=[\phi(x),\phi(y)], \quad x,y\in A.$$

A one-to-one homomorphism of one algebra onto another is called isomorphism, and it is said, that the corresponding algebras A and B are isomorphic.

A representation of a Lie algebra is a homomorphism $x\to T(x)$ of this algebra into a set of linear operators T defined on some linear space H:

$$\alpha x+\beta y\to\alpha T(x)+\beta T(y),$$

$$[x,y]\to[T(x),T(y)]=T(x)T(y)-T(y)T(x).$$

The last condition guarantees validity of the Jacobi identity for the algebras of operators T. If the representation space H is infinite-dimensional then it is assumed additionally that for all $x\in A$ the operators $T(x)$ have a common invariant domain D dense in H.

A representation is called irreducible if H does not have a subspace invariant under the operator $T(x)$ for all $x\in A$. If H has invariant subspaces the corresponding representation is called reducible (and if all such subspaces are mutually orthogonal it is called completely reducible).

A *superalgebra SA* is a vector space over the field K of complex (or real) numbers. Besides, this space is graded and closed with respect to the following binary operation

$$(x_\sigma,y_{\sigma'})\to[x_\sigma,y_{\sigma'}]'=[x_\sigma,y_{\sigma'}]'_{f_{\sigma\sigma'}} \tag{A.1.2}$$

whose explicit form depends on values of grading indices σ and σ′. This binary operation has to satisfy the axioms

$$[\alpha x_\sigma+\beta y_{\sigma'},z_{\sigma''}]'=\alpha[x_\sigma,z_{\sigma''}]'+\beta[y_{\sigma'},z_{\sigma''}]',$$

$$[x_\sigma,y_{\sigma'}]'=(-1)^{f_{\sigma\sigma'}+1}[y_{\sigma'},x_\sigma]',$$

$$[x_\sigma,[y_{\sigma'},z_{\sigma''}]']'+[y_{\sigma'},[z_{\sigma''},x_\sigma]']'+[z_{\sigma''},[x_\sigma,y_{\sigma'}]']'=0$$

where $\alpha,\beta\in K$, $x_\sigma,y_{\sigma'},z_{\sigma''}\in SA$.

We consider the simplest version of gradation, i.e., the so-called Z_2-gradation when any element of a *SA* is labelled by one of two possible values 0, 1 of the index σ besides $f_{\sigma\sigma'}=\sigma\sigma'$. Then the binary operation (A.1.2) reduces to commutator for $\sigma\sigma'=0$ and anticommutator for $\sigma\sigma'=1$.

In other words a superalgebra is a vector space consisting in elements of two kinds: even (*E*) corresponding to σ=0 and odd (*O*) corresponding to σ=1. This space is closed under the binary operation (A.1.2) which is nothing but a commutator for even-even and odd-odd elements and anticommutator for odd-odd elements. Furthermore, commutators of even-even and anticommutators of odd-odd elements are even and commutators of even-odd elements are odd. Such "multiplication rules" are formulated in (18.1).

We will not enter into details and discuss the main properties of so defined algebraic objects but note that the principal results of the theory of Lie algebras are extended to the case of superalgebras.

A *parasuperalgebra PSA* is a graduated vector space in which bi- and trilinear generalized Lie brackets are defined [26, 88, 372]. Thus in Rubakov-Spiridonov [372] formulation of the parasupersymmetric quantum mechanics the following relations for even and odd elements are postulated

$$[E,E]\sim E,\quad [E,O]\sim O,\quad \{O,O,O\}\sim EO \tag{A.1.3}$$

where

$$\{A,B,C\}=A[B,C]_+ +B[A,C]_+ +C[A,B]_+.$$

An alternative definition of parasuperalgebra (which is not equivalent to (A.1.3)) was proposed by Beckers and Debergh [26]. It is characterized by the double commutation relations instead of the double anticommutators.

Generalized Lie structures called *PSA* arises in parasupersymmetric quantum mechanics [26, 372]. We notice such structures are weaker then *SA* in the sense that a *SA* can be always considered as a *PSA* (e.g., relations (A.1.3) follow from (18.1)) but the converse is not true.

A simple example of *PSA* is the set of the KDP matrices satisfying (6.20), and matrices $S_{\mu\sigma}=i[ß_\mu, ß_\sigma]$, besides $ß_\mu$ are odd and $S_{\mu\sigma}$ are even elements.

APPENDIX 2

GENERALIZED KILLING TENSORS

1. Let us consider the following system of overdetermined partial differential equations:

$$\partial^{(a_{j+1}}\partial^{a_{j+2}}\ldots\partial^{a_{j+s}}K^{a_1a_2\ldots a_j)}=0 \qquad \text{(A.2.1)}$$

where $K^{\cdots}$ is a symmetric tensor of valence j, depending on m variables $x_1,x_2,\ldots,x_m$, $\partial^a=\partial/\partial x_a$, and symmetrization is imposed over the indices in paratheses.

The equation (A.2.1) arises naturally in problems of description of higher-order SO of partial differential equations, see, e.g., (16.33).

In the cases s=1, j is arbitrary and s=1, j=1 relation (A.2.1) reduces to the equation for Killing tensor [408] and Killing vector [249] correspondingly. We shall call a symmetric tensor $K^{\cdots}$ satisfying (A.2.1) with arbitrary s *a generalized Killing tensor of valence j and order s* [328, 342].

Here we present the explicit form of solutions of (A.2.1) for $m\leq 4$.

Consider the case s=1, when (A.2.1) reduces to the form (1.11). This system is overdetermined and includes C^{j+1}_{j+m} equations for C^{j}_{j+m-1} unknowns. The corresponding solutions have the form [328]

$$K^{a_1a_2\ldots a_j}=\sum_{l=0}^{j}\hat{\lambda}^{a_1a_2\ldots a_l[a_{l+1}b_1]\ldots[a_jb_{j-l}]}x_{b_1}x_{b_2}\ldots x_{b_{j-l}} \qquad \text{(A.2.2)}$$

where $\hat{\lambda}^{\cdots}$ are arbitrary tensors symmetric under the permutations $a_j\longleftrightarrow a_k$ $(i,k=1,2,\ldots,j)$, antisymmetric under the permutations a_{l+i} with b_i $(1\leq i\leq j\text{-l})$. Besides, any cyclic permutation of three indices nullifies this tensor. The number of linearly independent solutions N_j^m is equal to

$$N_j^{\ m}=\frac{1}{m}C^{m-1}_{j+m-1}C^{m-1}_{j+m}. \qquad \text{(A.2.3)}$$

Expanding $\hat{\lambda}^{\cdots}$ in basic tensors (see definitions in Section 16) we come to the representation (16.13). Expanding $\hat{\lambda}^{\cdots}$ in irreducible tensors having zero traces over any pair of indices it is possible to represent $K^{a_1a_2\ldots a_j}$ in a form including irreducible parameters only [342].

Killing tensors of arbitrary valence j and order s are polynomials on x_μ of order $s+j$-1, depending on N_{sj}^{m} arbitrary parameters besides

$$N_{sj}^{m}=\frac{s}{m}C_{j+m-1}^{m-1}C_{j+s+m-1}^{m-1}. \tag{A.2.4}$$

The explicit expressions of these tensors can be obtained from the recurrence relations

$$K_{s}^{a_1a_2...a_j}=K_{s-1}^{a_1a_2...a_j}+K_{s-1}^{a_1a_2...a_{j+1}}x_{a_{j+1}} \tag{A.2.5}$$

where $K_{s-1}^{a_1a_2...a_j}$ and $K_{s-1}^{a_1a_2...a_{j+1}}$ are the Killing tensors of order s-1 and valences j and j+1 correspondingly.

2. Consider one more class of overdetermined equations

$$[\partial^{(a_{j+1}}\partial^{a_{j+2}}...\partial^{a_{j+s}}\tilde{K}^{a_1a_2...a_j)}]^{TL}=0 \tag{A.2.6}$$

where $\tilde{K}^{a_1a_2...a_j}$ is a symmetric traceless tensor of valence j, depending on m variables, and the symbol $[...]^{TL}$ denotes the traceless part of the tensor in the square brackets (in our case this tensor is symmetric and has the valence $R=j+s$):

$$\begin{aligned}&[T^{a_1a_2...a_R}]^{TL}=T^{a_1a_2...a_R}+\\&+\sum_{\alpha=1}^{R/2}(-1)^{\alpha}K_{\alpha}\prod_{i=1}^{\alpha}g^{(a_{2i-1}a_{2i}}T^{a_{2\alpha+1}a_{2\alpha+2}...a_R)b_1b_2...b_{2\alpha-1}b_{2\alpha}}g_{b_1b_2}g_{b_3b_4}\cdots g_{b_{2\alpha-1}b_{2\alpha}}\end{aligned} \tag{A.2.7}$$

where

$$K_{\alpha}=\frac{1}{2^{\alpha}(R-2\alpha)!}\prod_{i=1}^{\alpha}\frac{1}{i![2(R-i)+m-2]}. \tag{A.2.8}$$

In the case s=1 (A.2.6) reduces to the equation for the conformal Killing tensor [408]. We call a symmetric traceless tensor $\tilde{K}^{a_1a_2...a_j}$ satisfying (A.2.6) *a generalized conformal Killing tensor of valence j and order s*.

The equations (A.2.6) were analyzed and solved in [328]. The numbers $\tilde{N}_{sj}^{m}$ of linearly independent solutions are given in (A.2.9):

$$\begin{aligned}&m=3,\quad \tilde{N}_{sj}^{3}=s(2j+1)(2j+2s+1)(2j+s+1)/6,\\&m=4,\quad \tilde{N}_{sj}^{4}=s(j+1)^2(j+s+1)^2(2j+2+s)/12.\end{aligned} \tag{A.2.9}$$

If m=2, then there exist an infinite number of solutions of the system (A.2.6).

First we represent solutions of (A.2.6) for the case s=1, m>2:

$$\begin{aligned}\tilde{K}^{a_1a_2...a_j}=\Bigg[&\sum_{l,k=0}^{j}\sum_{j=o}^{l-l-k}(-1)^{i}C_{j-l-k}^{i}\lambda^{b_1b_2...b_{j-l-k-i}(a_1a_2...a_{l+i}[a_{l+i+1}d_1]...[a_{l+i+k}d_k]}\times\\&\times x^{a_{l+k+i+1}}x^{a_{l+k+i+2}}...x^{a_j)}x_{d_1}x_{d_2}...x_{d_k}x_{b_1}x_{b_2}...x_{b_{j-l-k-i}}(x^2)^{i}\Bigg]^{TL}\end{aligned} \tag{A.2.10}$$

where $l+1\leq j$, $\lambda^{\cdot\cdot}$ are arbitrary basic tensors. Solutions of (A.2.6) corresponding to arbitrary s can be expressed via (A.2.10):

$$\tilde{K}_{(s)}^{a_1a_2\ldots a_j}=\sum_{i=0}^{s}\left[\tilde{K}_i^{a_1a_2\ldots a_j}(x^2)^{i-1}+\sum_{\alpha=0}^{s-i}f_{i-1\alpha}^{a_1a_2\ldots a_j}(x^2)^{\alpha}\right]. \tag{A.2.11}$$

Here $\tilde{K}_i^{a_1a_2\ldots a_j}$ are conformal Killing tensors of valence j and order s=1 (see (A.2.10)), the index i labels independent solutions with different powers of x^2, $f_{i-1}^{a_1a_2\ldots a_j}$ are the tensors described in the following.

For m=3 independent tensors $f_{i-1\alpha}^{a_1a_2\ldots a_j}$ are labeled by integers c, $0\leq c\leq 2j$, besides

$$f_{i\alpha}^{a_1a_2\ldots a_j}=\sum_c\left[\varepsilon_c\hat{f}_{i\alpha c}^{\,a_1a_2\ldots a_j}+(1-\varepsilon_c)\hat{f}_{i\alpha c}^{\,b(a_1a_2\ldots a_{j-1}}\varepsilon^{a_j)bk}x_k\right]^{TL}, \tag{A.2.12}$$

where

$$\hat{f}_{i\alpha c}^{\,a_1a_2\ldots a_j}=\sum_{n=0}^{\{c/2\}}(-2)^nC_n^{\{c/2\}-n}\hat{\lambda}^{b_1b_2\ldots b_{\alpha+n}(a_1a_2\ldots a_{j-n}}\times$$
$$\times x^{\alpha_{j-n+1}}x^{\alpha_{j-n+2}}\ldots x^{a_j}x_{b_1}x_{b_2}\ldots x_{b_{\alpha n}}x^{2(\{c/2\}-n)},\quad \varepsilon_c=\frac{1}{2}[1-(-1)^c], \tag{A.2.13}$$

$\hat{\lambda}^{b_1\ldots}$ are arbitrary symmetric and traceless tensors.

In the case m=4 the tensors $f_{i\alpha}^{a_1a_2\ldots a_j}$ are labelled by pairs of integers $c=(c_1,c_2)$, satisfying the conditions $-j\leq c_1\leq j$, $0\leq c_2\leq[(j-|c_1|)/2]$. The corresponding explicit expressions of $f_{i\alpha}^{a_1a_2\ldots a_j}$ are

$$f_{i\alpha}^{a_1a_2\ldots a_j}=\left[\sum_{\alpha=0}^{n+2c_2}(-1)^2C_{n+2c_2}^{\alpha}(x^2)^{\alpha}\lambda^{b_1b_2\ldots b_{n-2c_2-i+\alpha}(a_1a_2\ldots a_{|c_1|-n+i}[a_{|c_1|-n+i+1}d_1]\ldots[a_{j+i-n}d_{j-|c_2|}]}\times\right.$$
$$\left.\times x^{a_{j-n+i+1}}x^{a_{j-n+i+2}}\ldots x^{a_j)}x_{b_1}x_{b_2}\ldots x_{b_{n+2c_2+\alpha-i}}x_{d_1}x_{d_2}\ldots x_{d_{j-|c_1|}}\right]^{TL}. \tag{A.2.14}$$

where $n=-c_1$, $c_1<0$ and $n=0$ for $c_1\geq 0$.

Let $K_{b_1b_2\ldots b_{R_2}}^{a_1a_2\ldots a_{R_2}a_{R_2+1}\ldots a_{R_2+R_1}}$ is an irreducible tensor of valence R_1+R_2, symmetric under the permutations $a_{R_2+i}\longleftrightarrow a_{R_2+i'}$, $i,i'\leq R_1$, $(a_\mu,b_\mu)\longleftrightarrow(a_{\mu'},b_{\mu'})$ and antisymmetric under the permutations $a_\mu\longleftrightarrow b_{\mu'}$, $\mu,\mu'\leq R_2$. We say $K_{b_1b_2\ldots b_{R_2}}^{a_1a_2\ldots a_{R_1+R_2}}$ is a generalized conformal Killing tensor of valence R_1+2R_2 and order s if it satisfies the following overdetermined system of PDE

$$\left[\partial^{(a_{j+1}}\partial^{a_{j+2}}\ldots\partial^{a_{j+s}}K_{b_1b_2\ldots b_{R_2}}^{a_1a_2\ldots a_{R_2}a_{R_2+1}\ldots a_j)}\right]^{TL}=0 \tag{A.2.15}$$

where $j=R_1+R_2$. In the case s=1, R_2=1 and R_2=0, s is any integer, (A.2.15) reduces to the equation for the conformal Killing tensor [418] and to the equation (A.2.6).

In this book, we use only generalized Killing tensors of order s=1. The number of linearly independent solutions of the equation (A.2.15) for s=1, m=4 is

Let us present the explicit form of some of the above defined tensors.

$$N_{R_1R_2}=\frac{1}{6}(R_1+1)(R_1+1+2R_2)(R_1+2)(R_1+2+2R_2)(2R_1+2R_2+3). \tag{A.2.16}$$

Denoting arbitrary irreducible tensors by Greek letters we obtain from (A.2.14) the conformal Killing tensor for four-dimension Minkowski space (m=4, s=1, j=2):

$$\tilde{F}^{ab}=\sum_{\alpha=0}^{4}(G_\alpha{}^{ab}+G_\alpha{}^{ba}-\frac{1}{2}g^{ab}G_\alpha{}^{dc}g_{dc}) \tag{A.2.17}$$

where

$$\begin{aligned}
&G_0{}^{ab}=\lambda_0^{ab},\quad G_1{}^{ab}=\lambda^{a[bc]}x_c+\lambda^a x^b,\\
&G_2{}^{ab}=\lambda_2^{ab}x^2-2x^a\lambda_2^{bc}x_c+\lambda x^a x^b+\lambda^{[ab][cd]}x_cx_d+x^a\lambda^{[bd]}x_d,\\
&G_3{}^{ab}=\lambda_3^{a[bc]}x_cx^2-2x^a\lambda_3^{d[ac]}x_dx_c+x^a\nu^b x^2-2x^a x^b\nu^c x_c,\\
&G_4{}^{ab}=\nu^{ab}x^4+4x^a x^b\nu^{cd}x_cx_d-4\nu^{ac}x^b x_c x^2.
\end{aligned} \tag{A.2.18}$$

We present also the explicit form of the generalized conformal Killing tensors for m=4, s=1, R_1=0,1, R_2=1,2.

R_1=0, R_2=1:

$$K_\nu^\mu=\lambda^{[\mu\nu]}+x^\mu\lambda^\nu-x^\nu\lambda^\mu+\varepsilon^{\mu\nu\rho\sigma}x_\rho\xi_\sigma+(x^\mu\eta^{[\nu\sigma]}-x^\nu\eta^{[\mu\sigma]})x_\sigma-\frac{1}{2}\eta^{[\mu\nu]}x^2. \tag{A.2.19}$$

R_1=0, R_2=0:

$$K_{\nu\sigma}^{\mu\rho}=\hat{K}_{\nu\sigma}^{\mu\rho 0}, \tag{A.2.20}$$

$$\begin{aligned}
\hat{K}_{\nu\rho}^{\mu\rho\lambda}=&2G_{\nu\sigma}^{\mu\rho\lambda}-G_{\rho\sigma}^{\nu\mu\lambda}-G_{\mu\sigma}^{\rho\nu\lambda}+\frac{1}{2}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})G_{mm}^{nn\lambda}-\\
&-\frac{3}{2}(g^{\mu\rho}G_{\nu\sigma}^{nn\lambda}+g^{\nu\sigma}G_{\mu\rho}^{nn\lambda}-g^{\mu\sigma}G_{\nu\rho}^{nn\lambda}-g^{\nu\rho}G_{\mu\sigma}^{nn\lambda}),
\end{aligned} \tag{A.2.21}$$

where

$$G_{\nu\sigma}^{\mu\rho 0}=G_{\nu\sigma}^{\mu\rho}+G_{\sigma\nu}^{\rho\mu}, \tag{A.2.22}$$

$$\begin{aligned}
G_{\nu\sigma}^{\mu\rho}=&\lambda^{[\mu\nu][\rho\sigma]}=x^\mu\lambda^{\nu[\rho\sigma]}-x^\nu\lambda^{\mu[\rho\sigma]}+(x^\mu\lambda^{\nu n}-x^\nu\lambda^{\mu n})\varepsilon^{\rho\sigma mn}x_m+\\
&+x^\mu x^\rho\eta^{\nu\sigma}-x^\nu x^\rho\eta^{\mu\sigma}+\varepsilon^{\rho\sigma mn}[(x^\mu\eta^{[\nu k]m}-x^\nu\eta^{[\mu k]m})x_kx_m-\\
&-\frac{1}{2}\eta^{[\mu\nu]m}x_nx^2]+(x^\mu x^\rho\lambda^{[\nu n][\sigma m]}-x^\nu x^\rho\lambda^{[\mu n][\sigma m]})x_mx_n-\\
&-2(x^\mu\lambda^{[\nu m][\rho\sigma]}-x^\nu\lambda^{[\mu m][\rho\sigma]})x_mx^2+\frac{1}{4}\lambda^{[\mu\nu][\rho\sigma]}x^4.
\end{aligned} \tag{A.2.23}$$

R_1=1, R_2=2:

$$K^{\mu\rho\lambda}_{\nu\sigma}=2\hat{K}^{\mu\rho\lambda}_{\nu\sigma}-\hat{K}^{\nu\rho\mu}_{\lambda\sigma}-\hat{K}^{\lambda\rho\nu}_{\mu\sigma}-\hat{K}^{\sigma\mu\rho}_{\lambda\nu}-\hat{K}^{\lambda\mu\sigma}_{\rho\nu}-$$
$$-g^{\mu\lambda}\hat{K}^{\rho nn}_{\sigma\nu}+g^{\nu\lambda}\hat{K}^{\rho nn}_{\sigma\mu}-g^{\lambda\rho}\hat{K}^{\mu nn}_{\nu\sigma}+g^{\lambda\sigma}\hat{K}^{\mu nn}_{\nu\rho}, \tag{A.2.24}$$

$\hat{K}^{\mu\rho\sigma}_{\nu\sigma}$ is defined in (A.2.21) where

$$G^{\mu\rho\lambda}_{\nu\sigma}=g^{\mu\nu\rho\sigma}_{\lambda}-g^{\nu\mu\rho\sigma}_{\lambda}-g^{\mu\nu\sigma\rho}_{\lambda}+g^{\nu\mu\sigma\rho}_{\lambda}+g^{\rho\sigma\mu\nu}_{\lambda}-g^{\rho\sigma\nu\mu}_{\lambda}-g^{\sigma\rho\mu\nu}_{\lambda}+g^{\sigma\rho\nu\mu}_{\lambda},$$

$$g^{\mu\nu\rho\sigma}_{\lambda}=\lambda^{\lambda[\mu\nu][\rho\sigma]}+x^{\mu}\lambda^{\lambda\nu[\rho\sigma]}+x^{\mu}\lambda^{[\nu\rho][\sigma\lambda]}+\lambda^{[\mu\nu][\rho\sigma][\lambda n]}x_n+$$
$$+x^{\mu}\lambda^{\nu\lambda m}x^{n}\varepsilon^{\rho\sigma}{}_{mn}+x^{\mu}x^{\rho}\eta^{\nu\sigma\lambda}+x^{\mu}\eta^{\nu[\rho\sigma][\lambda m]}+x^{\lambda}x^{\mu}\lambda^{\nu[\rho\sigma]}+$$
$$+x^{\lambda}x^{\mu}\lambda^{\nu m}\varepsilon^{\rho\sigma}{}_{mn}x^{n}+x^{\lambda}x^{\mu}x^{\rho}\eta^{\sigma\nu}+x^{\mu}x^{\rho}\eta^{\nu\sigma[\lambda n]}x_n+$$
$$+\varepsilon^{\rho\sigma}{}_{mn}(x^{\mu}\xi^{\lambda n[\nu k]}x_k x^{m}-\xi^{\lambda n[\mu\nu]}x^{m}x^{2})+(x^{\mu}\xi^{\nu n\lambda}x^{2}-$$
$$-2x^{\lambda}x^{\mu}\xi^{\nu nk}x_k)\varepsilon^{\rho\sigma}{}_{mn}x^{+}x^{\mu}x^{\rho}\xi^{\nu\sigma\lambda}x^{2}-2x^{\lambda}x^{\rho}x^{\mu}\xi^{\nu\sigma n}x_n+$$
$$+\varepsilon^{\rho\sigma}{}_{mn}(x^{\mu}\xi^{m[\nu k][\lambda f]}x_k x_f x^{n}-\xi^{m[\mu\nu][\lambda f]}x^{n}x_f x^{2})+$$
$$+x^{\lambda}\varepsilon^{\rho\sigma}{}_{mn}(x^{\mu}\xi^{[\nu k]m}x^{n}x_k-\eta^{[\mu\nu]m}x^{n}x^{2})+ \tag{A.2.25}$$
$$+\varepsilon^{\rho\sigma}{}_{mn}[x^{\mu}\zeta^{\lambda m[\nu k]}x_k x^{n}x^{2}-\zeta^{\lambda m[\mu\nu]}x^{n}x^{4}-2x^{\lambda}(x^{\mu}\zeta^{km[\nu f]}x_f-$$
$$-\eta^{km[\mu\nu]}x^{2}x_k x^{n}]+x^{\mu}(x^{\rho}\eta^{[\nu n][\sigma m][\lambda k]}x_m-\eta^{[\nu n][\rho\sigma][\lambda k]}x^{2})x_n x_k+$$
$$+\frac{1}{4}\eta^{[\mu\nu][\rho\sigma][\lambda k]}x_k x^{4}+x^{\lambda}[x^{\mu}x^{\rho}\eta^{[\nu n][\sigma m]}x_m x_n-x^{\mu}\lambda^{[\nu m][\rho\sigma]}x^{2}_{mx}+$$
$$+\frac{1}{4}\eta^{[\mu\nu][\rho\sigma]}x^{4}+\frac{1}{4}\zeta^{\lambda[\mu\nu][\rho\sigma]}x^{6}+x^{\mu}(x^{\rho}\zeta^{\lambda[\nu n][\sigma m]}x_n x_m x^{2}-$$
$$-x^{\mu}\zeta^{\lambda[\nu m][\rho\sigma]}x_m x^{4})-2x^{\lambda}[x^{\mu}(x^{\rho}\zeta^{k[\nu n][\sigma m]}x_m-$$
$$-\zeta^{k[\nu m][\rho\sigma]}x^{20}x_n x_k+\frac{1}{4}\zeta^{k[\mu\nu][\rho\sigma]}x_k x^{4}.$$

APPENDIX 3

MATRIX ELEMENTS OF SCALAR OPERATORS IN THE BASIS OF SPHERICAL SPINORS

Spherical spinors satisfying the relations (29.4) can be represented as columns whose m-th component is given by the relation

$$(\Omega^{s}_{jj-\lambda m})^{\mu}=C_{j-\lambda m-\mu s\mu}Y_{j-\lambda m-\mu}, \tag{A.3.1}$$

where Y_{lm} are the spherical functions

$$Y_{lm}(\hat{x})=\frac{1}{2\pi}\exp(im\varphi)(-1)^{\frac{m+|m|}{2}}\sqrt{\frac{(l+1/2)(l-|m|)!}{(l+|m|)}}P_l^{|m|}(\cos\theta),$$

$$P_l^{|m|}(\cos\theta)=\frac{1}{2^l l!}\sin^{|m|}\theta\frac{d^{|m|+1}}{(d\cos\theta)^{|m|}}(\cos^2\theta-1)^l,$$

φ and θ are the polar and azimuthal angles of the vectors $\hat{\boldsymbol{x}}=\boldsymbol{x}/x$, $C^{jm}_{jm_1s\mu}$ are the Wigner coefficients

$$C^{jm}_{lm_1s\mu}=\delta_{m_1+\mu\, m}\left[\frac{(2j+1)!(l+s-j)!(l-m_1)!(s-\mu)!(j+m)!(j-m)!}{(l+s+j+1)!(l-s+j)!(j+s-l)!(l+m_1)!(s+\mu)!}\right]^{1/2}$$
$$\times\sum_{n=0}^{j-m}(-1)^{l-m_1+n}\frac{(l+m_1+n)!(s+j-m_1-n)!}{n!(l-m_1-n)!(j-m-n)!(s-j+m_1+n)!}.$$

Such defined spherical spinors satisfy the normalization condition

$$\int\Omega^{s+}_{jj-\lambda m}\Omega^{s}_{j'j'-\lambda' m'}d\omega=\delta_{jj'}\delta_{\lambda\lambda'}\delta_{mm'}.$$

Let us present the action of some scalar operators on the spherical spinors. Following [323] we write

$$\boldsymbol{S}\cdot\hat{\boldsymbol{x}}\Omega^s_{jj-\lambda m}=\sum_{\lambda'}d^{sj}_{\lambda\lambda'}\Omega^s_{jj-\lambda' m},$$

$$\boldsymbol{S}\cdot\boldsymbol{p}\Omega^s_{jj-\lambda m}=\frac{-2i}{x}\sum_{\lambda'}\left[d^{sj}_{\lambda\lambda'}\left(1+\frac{1}{s}h^{sj}_{\lambda'}\right)+\frac{1}{s}b^{sj}_{\lambda\lambda'}q^{sj}_{\lambda'}\right]\Omega^s_{jj-\lambda' m},$$

$$\boldsymbol{S}\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^s_{jj-\lambda m}=h^{sj}_{\lambda}\Omega_{jj-\lambda m},$$

$$\boldsymbol{K}\cdot\hat{\boldsymbol{x}}\Omega^s_{jj-\lambda m}=\sum_{\lambda'}b^{sj}_{\lambda\lambda'}\Omega^{s-1}_{jj-\lambda' m},$$

$$\boldsymbol{K}^+\cdot\hat{\boldsymbol{x}}\Omega^{s-1}_{jj-\lambda m}=\sum_{\lambda'}b^{sj}_{\lambda'\lambda}\Omega^s_{jj-\lambda' m}, \tag{A.3.3}$$

$$\boldsymbol{K}\cdot\boldsymbol{p}\Omega^s_{jj-\lambda m}=\frac{-2i}{x}\sum_{\lambda'}\left[\frac{1}{s}d^{sj}_{\lambda\lambda'}q_{\lambda'}+b^{sj}_{\lambda\lambda'}(1-\frac{1}{s}h^{s-1j}_{\lambda'})\right]\Omega^s_{jj-\lambda' m},$$

$$\boldsymbol{K}^+\cdot\boldsymbol{p}\Omega^{s-1}_{jj-\lambda m}=\frac{2i}{x}\sum_{\lambda'}\left[\frac{1}{s}d^{sj}_{\lambda'\lambda}q^{sj}_{\lambda'}+b^{sj}_{\lambda'\lambda}\left(1-\frac{1}{s}h^{s-1j}_{\lambda'}\right)\right]\Omega^s_{jj-\lambda' m},$$

$$\boldsymbol{K}\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^s_{jj-\lambda m}=q^{sj}_{\lambda}\Omega^{s-1}_{jj-\lambda m},$$

$$\boldsymbol{K}^+\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^{s-1}_{jj-\lambda m}=q^{sj}_{\lambda}\Omega^s_{jj-\lambda m}$$

where S_a are generators of the IR *D(s)* of the group *O(3)*, K_a are the matrices defined by relations (12.26) and given in the explicit form in (4.65),

$$h_\lambda^{sj}=\frac{1}{2}(\lambda d_j-\lambda^2-s(s+1)),\quad d_j=2j+1,$$

$$q_\lambda^{sj}=\frac{1}{2}[(s^2-\lambda^2)(d_j+s-\lambda)(d_j-s-\lambda)]^{1/2}$$

$$b_{\lambda\lambda'}^{sj}=-\frac{1}{2}(\delta_{\lambda\lambda'-1}g_\lambda^{sj}-\delta_{\lambda\lambda'+1}f_\lambda^{sj}),\tag{A.3.4}$$

$$g_\lambda^{sj}=\sqrt{\frac{(2j+s-\lambda+1)(s-\lambda-1)(s-\lambda)(2j+s-\lambda)}{(2j-2\lambda-1)(2j-2\lambda+1)}},$$

$$f_\lambda^{sj}=\sqrt{\frac{(2j-s-\lambda+1)(2j-s-\lambda+2)(s+\lambda-1)(s+\lambda)}{(2j-2\lambda+1)(2j-2\lambda+3)}}$$

and the coefficients $d_{\lambda'\lambda}^{sj}$ are given in (29.7), (29.8).

Relations (A.3.2) are present, e.g., in [409] where another notations are used. Relations (A.3.3) complete (A.3.2) and make it possible to find the result of action of an arbitrary scalar operator $\eta\prod \boldsymbol{F}$ (where η is a vector matrix satisfying (12.20b), $\boldsymbol{F}$ is one of the operators $\boldsymbol{p}$, $\boldsymbol{x}$ or $\boldsymbol{x}\times\boldsymbol{p}$) on the spherical spinor. Using the representation (12.23) we can write

$$\eta\cdot\boldsymbol{F}\Omega_{jj-\lambda m}^{si}=(a_{ii'}^{s}\,\boldsymbol{S}^{(s)}\cdot\boldsymbol{F}+b_{ii'}^{s+1}\boldsymbol{K}^{(s+1)}\cdot\boldsymbol{F}+c_{ii'}^{s-1}\boldsymbol{K}^{(s)\dagger}\cdot\boldsymbol{F})\Omega_{jj-\lambda m}^{si}\tag{A.3.5}$$

where $a^{s}_{ii'}$, $b^{s+1}_{ii'}$ and $c^{s-1}_{ii'}$ are parameters defining the matrix η, the index i labels eigenvectors of $\boldsymbol{S}2$ corresponding to degenerated eigenvalues $s(s+1)$. Using (A.3.2)-(A.3.4) we obtain the explicit expression of the r.h.s. of (A.3.5).

Consider two commuting spin matrices $S_a^{(1)}$ and $S_a^{(2)}$ satisfying the relations

$$[S_a^{(\alpha)},S_b^{(\beta)}]=i\delta_{\alpha\beta}\varepsilon_{abc}S_c^{(\alpha)},\quad \sum_\alpha S_a^{(\alpha)}S_a^{(\alpha)}=s_a(s_a+1),\quad \alpha=1,2.\tag{A.3.6}$$

Without loss of generality we can choose

$$S_a^{(1)}=\frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc}-iS_{0a}\right),\quad S_a^{(2)}=\frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc}+iS_{0a}\right)$$

where $S_{\mu\sigma}$ are generators of the finite-dimensional IR $D(l_0,l_1)=D(|s_1-s_2|(s_1+s_2+1)sign(s_1-s_2))$ of the group $O(1,3)$, which are given in (4.65). Then according to (4.63), (A.3.2)-(A.3.5) we obtain

$$\boldsymbol{S}^{(\alpha)}\cdot\hat{\boldsymbol{x}}\Omega_{jj-\lambda m}^{s}=\sum_{s',\lambda'_{s'}}B_{(\alpha)s'\lambda'_{s'}}^{js\lambda_s}\Omega_{jj+\lambda' m}^{s'}\equiv\sum_{\lambda'}\Big(B_{ss}^{(\alpha)}d_{\lambda\lambda'}^{sj}\Omega_{jj-\lambda' m}^{s}+$$
$$+iB_{ss+1}^{(\alpha)}b_{\lambda'\lambda}\Omega_{jj-\lambda m}^{s+1}-iB_{s-1s}^{(\alpha)}b_{\lambda\lambda'}^{s-1j}\Omega_{jj-\lambda' m}^{s-1}\Big),\tag{A.3.7}$$

where $d_{\lambda'\lambda}^{sj}$, $b_{\lambda'\lambda}^{sj}$ are the coefficients (28.7), (A.3.4),

For the operator $\boldsymbol{S}^{(\alpha)}\cdot\boldsymbol{x}\times\boldsymbol{p}$ we obtain

$$B_{ss}^{(\alpha)}=\frac{1}{2s}+(-1)^{\alpha}\frac{[(s_1+s_2+1)(s_1-s_2)]^{1/2}}{2s(s+1)^{1/2}},\quad \alpha=1,2,$$

$$B_{s-1s}^{(\alpha)}=(-1)^{\alpha+1}\frac{[(s_1+s_2+1)^2-s^2]^{1/2}[s^2-(s_1-s_2)^2]^{1/2}}{2s(4s^2-1)^{1/2}}, \tag{A.3.8}$$

$$B_{ss+1}^{(\alpha)}=B_{s'-1s'}^{(\alpha)},\quad s=s_1+s_2,s_1+s_2-1,...,|s_1-s_2|,\quad s'=s+1.$$

$$\boldsymbol{S}^{(\alpha)}\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^{s}_{jj-\lambda m}=B_{ss}^{(\alpha)}h_{\lambda}^{sj}\Omega^{s}_{jj-\lambda m}+iB_{ss+1}^{(\alpha)}q_{\lambda}^{sj}\Omega^{s-1}_{jj-\lambda m}-iB_{s-1s}^{(\alpha)}q_{\lambda}^{s+1j}\Omega^{s+1}_{jj-\lambda m}, \tag{A.3.9}$$

all the coefficients included in (A.3.9) are given in (A.3.4), (A.3.8).

Let $s=1/2$, $S_a^{(2)}=\sigma_a/2$ where σ_a are the $4s\times 4s$ Pauli matrices (27.4). Then according to (A.3.7), (A.3.8)

$$\sigma\cdot\hat{\boldsymbol{x}}\,\Omega^{s}_{jj-\lambda m}=\frac{1}{s}\sum_{\lambda'}(d_{\lambda\lambda'}^{sj}\Omega^{s}_{jj-\lambda' m}-ib_{\lambda\lambda'}^{s-1j}\Omega^{s-1}_{jj-\lambda' m}),$$

$$\sigma\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^{s}_{jj-\lambda m}=\frac{1}{s}(h_{\lambda}^{sj}\Omega^{s}_{jj-\lambda m}-iq_{\lambda}^{sj}\Omega^{s-1}_{jj-\lambda m}), \tag{A.3.10}$$

$$\sigma\cdot\boldsymbol{x}\times\boldsymbol{p}\Omega^{s-1}_{jj-\lambda m}=\frac{1}{s}(h_{\lambda}^{s-1j}\Omega^{s-1}_{jj-\lambda m}+iq_{\lambda}^{sj}\Omega^{s}_{jj-\lambda m}).$$

Relations (A.3.2)-(A.3.5) can be used for separation of variables for a wide class of *O(3)*-invariant systems of partial differential equations. We use them (as well as (A.3.7)-(A.3.10)) in Subsections 29.2, 30.4, 34.3.

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LIST OF ABBREVIATIONS

FW - Foldy-Wouthuysen
IA - invariance algebra,
IR - irreducible representation,
KDP - Kemmer-Duffin-Petiau (algebra, equation, matrices),
KGF - Klein-Gordon-Fock (equation),
LHH - Levi-Leblond-Hagen-Hurley (equations),
ODE - ordinary differential equations,
PSA - parasuperalgebra,
PUA - projective unitary and antiunitary (representation),
SA - superalgebra,
SO - symmetry operator,
TST - Tamm-Sakata-Taketani (equation).

SUBJECT INDEX