QUANTUM FIELD THEORY

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Chapter 1

Lagrangian Field Theory

1.1 Units

We start with the most basic thing of all, namely units and concentrate on the units most widely used in particle physics and quantum field theory (natural units). We also mention the units used in General Relativity, because these days it is likely that students will study this subject as well.

Some useful quantities are [PPDB]:
$\hbar \equiv \frac{h}{2\pi} = 1.055 \times 10^{-34} J\ sec = 6.582 \times 10^{-22} MeV\ sec$
$c = 3 \times 10^8 \frac{m}{sec}$.
$1\ eV = 1.6 \times 10^{-19} J$
$\hbar c = 197 MeV\ fm$
$1 fm = 10^{-15} m$
$1 barn = 10^{-28} m^2$
$1 mb = .1 fm^2$

1.1.1 Natural Units

In particle physics and quantum field theory we are usually dealing with particles that are moving fast and are very small, i.e. the particles are both relativistic and quantum mechanical and therefore our formulas have lots of factors of c (speed of light) and $\hbar$ (Planck's constant). The formulas considerably simplify if we choose a set of units, called *natural units* where c and $\hbar$ are set equal to 1.

In *CGS* units (often also called *Gaussian* [Jackson appendix] units), the basic quantities of length, mass and time are centimeters (cm), gram (g), seconds (sec), or in *MKS* units these are meters (m), kilogram (kg), seconds. In natural units the units of length, mass and time are all expressed in GeV.

Example With $c \equiv 1$, show that $sec = 3 \times 10^{10} cm$.

Solution $c = 3 \times 10^{10} cm\ sec^{-1}$. If $c \equiv 1$

$$\Rightarrow \quad sec \quad = \quad 3 \times 10^{10} cm$$

We can now derive the other conversion factors for natural units, in which $\hbar$ is also set equal to unity. Once the units of length and time are established, one can deduce the units of mass from $E = mc^2$. These are

$$\begin{aligned} sec &= 1.52 \times 10^{24} GeV^{-1} \\ m &= 5.07 \times 10^{15} GeV^{-1} \\ kg &= 5.61 \times 10^{26} GeV \end{aligned}$$

(The exact values of c and $\hbar$ are listed in the [Particle Physics Booklet] as $c = 2.99792458 \times 10^8 m/sec$ and $\hbar = 1.05457266 \times 10^{-34} Jsec= 6.5821220 \times 10^{-25} GeV\ sec$.)

Example Deduce the value of Newton's gravitational constant G in natural units.

Solution It is interesting to note that the value of G is one of the least accurately known of the fundamental constants. Whereas, say the mass of the electron is known as [Particle Physics Booklet] $m_e = 0.51099906 MeV/c^2$ or the fine structure constant as $\alpha = 1/137.0359895$ and c and $\hbar$ are known to many decimal places as mentioned above, the best known value of G is [PPDB] $G = 6.67259 \times 10^{-11} m^3 kg^{-1} sec^{-2}$, which contains far fewer decimal places than the other fundamental constants.

Let's now get to the problem. One simply substitutes the conversion factors from before, namely

$$\begin{aligned} G &= 6.67 \times 10^{-11} m^3 kg^{-1} sec^{-2} \\ &= \frac{6.67 \times 10^{-11}(5.07 \times 10^{15} GeV^{-1})^3}{(5.61 \times 10^{26} GeV)(1.52 \times 10^{24} GeV^{-1})^2} \\ &= 6.7 \times 10^{-39} GeV^{-2} \\ &= \frac{1}{M_{Pl}^2} \end{aligned}$$

where the Planck mass is defined as $M_{Pl} \equiv 1.22 \times 10^{19} GeV$.

Natural units are also often used in *cosmology* and *quantum gravity* [Guidry 514] with G given above as $G = \frac{1}{M_{Pl}^2}$.

1.1.2 Geometrical Units

In classical General Relativity the constants c and G occur most often and *geometrical units* are used with c and G set equal to unity. Recall that in natural units everything was expressed in terms of GeV. In geometrical units everything is expressed in terms of cm.

Example Evaluate G when $c \equiv 1$.

Solution

$$\begin{aligned} G &= 6.67 \times 10^{-11} m^3 kg^{-1} sec^{-2} \\ &= 6.67 \times 10^{-8} cm^3 g^{-1} sec^{-2} \end{aligned}$$

and when $c \equiv 1$ we have $sec = 3 \times 10^{10} cm$ giving

$$\begin{aligned} G &= 6.67 \times 10^{-8} cm^3 g^{-1} (3 \times 10^{10} cm)^{-2} \\ &= 7.4 \times 10^{-29} cm\ g^{-1} \end{aligned}$$

Now imposing $G \equiv 1$ gives the geometrical units

$$\begin{aligned} sec &= 3 \times 10^{10} cm \\ g &= 7.4 \times 10^{-29} cm \end{aligned}$$

It is important to realize that geometrical and natural units are *not* compatible. In natural units $c = \hbar = 1$ and we *deduce* that $G = \frac{1}{M_{Pl}^2}$ as in a previous Example. In geometrical units $c = G = 1$ we *deduce* that $\hbar = 2.6 \times 10^{-66} cm^2$. (see Problems) Note that in these units $\hbar = L_{Pl}^2$ where $L_{Pl} \equiv 1.6 \times 10^{-33} cm$. In particle physics, gravity becomes important when energies (or masses) approach the Planck mass M_{Pl}. In gravitation (General Relativity), quantum effects become important at length scales approaching L_{Pl}.

1.2 Covariant and Contravariant vectors

The subject of covariant and contravariant vectors is discussed in [Jackson], which students should consult for a thorough introduction. In this section we summarize the basic results.

The metric tensor that is used in this book is

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Contravariant vectors are written in 4-dimensional form as

$$A^\mu = (A^o, A^i) = (A^o, \vec{A})$$

Covariant vectors are formed by "lowering" the indices with the metric tensor as in

$$A_\mu = g_{\mu\nu} A^\nu = (A_o, A_i) = (A^o, -A^i) = (A^o, -\vec{A})$$

noting that

$$A_o = A^o$$

Thus

$$\boxed{A^\mu = (A^o, \vec{A}) \qquad\qquad A_\mu = (A^o, -\vec{A})}$$

Now we discuss derivative operators, denoted by the covariant symbol ∂_μ and defined via

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = (\partial_o, \partial_i) = (\frac{\partial}{\partial x^o}, \frac{\partial}{\partial x^i}) = (\frac{\partial}{\partial t}, \vec{\bigtriangledown})$$

The contravariant operator ∂^μ is given by

$$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu = (\partial^o, \partial^i) = g^{\mu\nu}\partial_\nu = (\frac{\partial}{\partial x_o}, \frac{\partial}{\partial x_i}) = (\frac{\partial}{\partial x^o}, -\frac{\partial}{\partial x^i}) = (\frac{\partial}{\partial t}, -\vec{\bigtriangledown})$$

Thus

$$\boxed{\partial_\mu = (\tfrac{\partial}{\partial t}, \vec{\bigtriangledown}) \qquad\qquad \partial^\mu = (\tfrac{\partial}{\partial t}, -\vec{\bigtriangledown})}$$

The length squared of our 4-vectors is

$$A^2 \equiv A_\mu A^\mu = A^\mu A_\mu = A_o^2 - \vec{A}^2$$

and

$$\partial^2 \equiv \partial_\mu \partial^\mu \equiv \Box^2 = \frac{\partial^2}{\partial t^2} - \bigtriangledown^2 \tag{1.1}$$

Finally, note that with our 4-vector notation, the usual quantum mechanical replacements

$$p^i \rightarrow i\hbar\partial^i \equiv -i\hbar\vec{\bigtriangledown}$$

and

$$p^o \rightarrow i\hbar\partial^o = i\hbar\frac{\partial}{\partial t}$$

can be succintly written as

$$\boxed{p^\mu \rightarrow i\hbar\partial^\mu}$$

giving (with $\hbar = 1$)

$$p^2 \rightarrow -\Box^2$$

1.3 Classical point particle mechanics

1.3.1 Euler-Lagrange equation

Newton's second law of motion is

$$\vec{F} = \frac{d\vec{p}}{dt}$$

or in component form (for each component F_i)

$$F_i = \frac{dp_i}{dt}$$

where $p_i = m\dot{q}_i$ (with q_i being the generalized position coordinate) so that $\frac{dp_i}{dt} = \dot{m}\dot{q}_i + m\ddot{q}_i$. (Here and throughout this book we use the notation $\dot{x} \equiv \frac{dx}{dt}$.) If $\dot{m} = 0$ then $F_i = m\ddot{q}_i = ma_i$. For conservative forces $\vec{F} = -\vec{\bigtriangledown}U$ where U is the scalar potential. Rewriting Newton's law we have

$$-\frac{dU}{dq_i} = \frac{d}{dt}(m\dot{q}_i)$$

Let us define the Lagrangian $L(q_i, \dot{q}_i) \equiv T - U$ where T is the kinetic energy. In freshman physics $T = T(\dot{q}_i) = \frac{1}{2}m\dot{q}_i^2$ and $U = U(q_i)$ such as the harmonic oscillator $U(q_i) = \frac{1}{2}kq_i^2$. That is in freshman physics T is a function only of velocity $\dot{q}_i$ and U is a function only of position q_i. Thus $L(q_i, \dot{q}_i) = T(\dot{q}_i) - U(q_i)$. It follows that $\frac{\partial L}{\partial q_i} = -\frac{dU}{dq_i}$ and $\frac{\partial L}{\partial \dot{q}_i} = \frac{dT}{d\dot{q}_i} = m\dot{q}_i = p_i$. Thus Newton's law is

$$F_i = \frac{dp_i}{dt}$$
$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$$

with the canonical momentum defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

The next to previous equation is known as the Euler-Lagrange equation of motion and serves as an alternative formulation of mechanics [Goldstein]. It is usually written

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

or just

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

We have obtained the Euler-Lagrange equations using simple arguments. A more rigorous derivation is based on the calculus of variations [Ho-Kim47, Huang54,Goldstein37, Bergstrom284] as follows.

In classical point particle mechanics the action is

$$S \equiv \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt$$

where the Lagrangian L is a function of generalized coordinates q_i, generalized velocities $\dot{q}_i$ and time t.

According to *Hamilton's principle*, the action *has a stationary value for the correct path of the motion* [Goldstein36], i.e. $\delta S = 0$ for the correct path. To see the consequences of this, consider a variation of the path [Schwabl262, BjRQF6]

$$q_i(t) \to q_i'(t) \equiv q_i(t) + \delta q_i(t)$$

subject to the constraint $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The subsequent variation in the action is (assuming that L is not an explicit function of t)

$$\delta S = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i)dt = 0$$

with $\delta \dot{q}_i = \frac{d}{dt}\delta q_i$ and integrating the second term by parts yields

$$\begin{aligned} \int \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i dt &= \int \frac{\partial L}{\partial \dot{q}_i} d(\delta q_i) \\ &= \frac{\partial L}{\partial \dot{q}_i}\delta q_i|_{t_1}^{t_2} - \int \delta q_i d(\frac{\partial L}{\partial \dot{q}_i}) \\ &= 0 - \int \delta q_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})dt \end{aligned} \tag{1.2}$$

where the boundary term has vanished because $\delta q_i(t_1) = \delta q_i(t_2) = 0$. We are left with

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\right] \delta q_i dt = 0$$

which is true for an arbitrary variation δq_i indicating that the integral must be zero, which yields the Euler-Lagrange equations.

1.3.2 Hamilton's equations

We now introduce the Hamiltonian H defined as a function of p and q as

$$H(p_i, q_i) \equiv p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{1.3}$$

For the simple case $T = \frac{1}{2}m\dot{q}_i^2$ and $U \neq U(\dot{q}_i)$ we have $p_i \frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$ so that $T = \frac{p_i^2}{2m}$ and $p_i \dot{q}_i = \frac{p_i^2}{m}$ so that $H(p_i, q_i) = \frac{p_i^2}{2m} + U(q_i) = T + U$ which is the total energy. Hamilton's equations of motion immediately follow as

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$

Now $L \neq L(p_i)$ and $\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$ so that our original definition of the canonical momentum above gives

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i$$

1.4 Classical Field Theory

Scalar fields are important in cosmology as they are thought to drive inflation. Such a field is called an inflaton, an example of which may be the Higgs boson. Thus the field ϕ considered below can be thought of as an inflaton, a Higgs boson or any other scalar boson.

In both special and general relativity we always seek covariant equations in which space and time are given equal status. The Euler-Lagrange equations above are clearly not covariant because special emphasis is placed on time via the $\dot{q}_i$ and $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$ terms.

Let us replace the q_i by a field $\phi \equiv \phi(x)$ where $x \equiv (t, \mathbf{x})$. The generalized coordiante q has been replaced by the field variable ϕ and the discrete index i has been replaced by a continuously varying index x. In the next section we shall show how to derive the Euler-Lagrange equations from the action defined as

$$S \equiv \int L dt$$

which again is clearly not covariant. A covariant form of the action would involve a Lagrangian density $\mathcal{L}$ via

$$S \equiv \int \mathcal{L} d^4x = \int \mathcal{L} d^3x dt$$

where $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ and with $L \equiv \int \mathcal{L} d^3x$. The term $-\frac{\partial L}{\partial q_i}$ in the Euler-Lagrange equation gets replaced by the covariant term $-\frac{\partial \mathcal{L}}{\partial \phi(x)}$. Any time derivative $\frac{d}{dt}$ should be replaced with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ which contains space as well as time derivatives. Thus one can guess that the covariant generalization of the point particle Euler-Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

which is the covariant Euler-Lagrange equation for a field ϕ. If there is more than one scalar field ϕ_i then the Euler-Lagrange equations are

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0}$$

To *derive* the Euler-Lagrange equations for a scalar field [Ho-Kim48, Goldstein548], consider an arbitrary variation of the field [Schwabl 263; Ryder 83; Mandl & Shaw 30,35,39; BjRQF13]

$$\phi(x) \to \phi'(x) \equiv \phi(x) + \delta\phi(x)$$

again with $\delta\phi = 0$ at the end points. The variation of the action is (assuming that $\mathcal{L}$ is not an explicit function of x)

$$\delta S = \int_{X_1}^{X_2} \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^4x = 0$$

where X_1 and X_2 are the 4-surfaces over which the integration is performed. We need the result

$$\delta(\partial_\mu \phi) = \partial_\mu \delta\phi = \frac{\partial}{\partial x^\mu} \delta\phi$$

which comes about because $\delta\phi(x) = \phi'(x) - \phi(x)$ giving

$$\partial_\mu \delta\phi(x) = \partial_\mu \phi'(x) - \partial_\mu \phi(x) = \delta \partial_\mu \phi(x)$$

showing that δ commutes with differentiation ∂_μ. Integration by parts on the second term is a bit more complicated than before for the point particle case, but the final result is (see Problems)

$$\delta S = \int_{X_1}^{X_2} \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi \; d^4x = 0$$

which holds for arbitrary $\delta\phi$, implying that the integrand must be zero, yielding the Euler-Lagrange equations.

In analogy with the canonical momentum in point particle mechanics, we define the *covariant momentum density*

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

so that the Euler-Lagrange equations become

$$\partial_\mu \Pi^\mu = \frac{\partial \mathcal{L}}{\partial \phi}$$

The canonical momentum is defined as

$$\Pi \equiv \Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

The energy momentum tensor is (analagous to the definition of the point particle Hamiltonian)

$$T_{\mu\nu} \equiv \Pi_\mu \partial_\nu \phi - g_{\mu\nu} \mathcal{L}$$

with the Hamiltonian density

$$H \equiv \int \mathcal{H} d^3x$$
$$\mathcal{H} \equiv T_{00} = \Pi\dot{\phi} - \mathcal{L}$$

In order to illustrate the foregoing theory we shall use the example of the classical, massive Klein-Gordon field.

Example The massive Klein-Gordon Lagrangian density is

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$$
$$= \frac{1}{2}[\dot{\phi}^2 - (\bigtriangledown\phi)^2 - m^2\phi^2]$$

A) Derive expressions for the covariant momentum density and the canonical momentum.
B) Derive the equation of motion in position space and momentum space.
C) Derive expressions for the energy-momentum tensor and the Hamiltonian density.

Solution **A)** The covariant momentum density is more easily evaluated by re-writing $\mathcal{L}_{KG} = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$. Thus $\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\nu}(\delta^\alpha_\mu\partial_\nu\phi + \partial_\mu\phi\delta^\alpha_\nu) = \frac{1}{2}(\delta^\alpha_\mu\partial^\mu\phi + \partial^\nu\phi\delta^\alpha_\nu) = \frac{1}{2}(\partial^\alpha\phi + \partial^\alpha\phi) = \partial^\alpha\phi$. Thus for the Klein-Gordon field we have

$$\Pi^\alpha = \partial^\alpha\phi$$

giving the canonical momentum $\Pi = \Pi^0 = \partial^0\phi = \partial_0\phi = \dot{\phi}$,

$$\Pi = \dot{\phi}$$

B) Evaluating $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$, the Euler-Lagrange equations give the field equation as $\partial_\mu\partial^\mu\phi + m^2\phi$ or

$$(\Box^2 + m^2)\phi = 0$$
$$\ddot{\phi} - \bigtriangledown^2\phi + m^2\phi = 0$$

which is the Klein-Gordon equation for a free, massive scalar field. In momentum space $p^2 = -\Box^2$, thus

$$(p^2 - m^2)\phi = 0$$

(Note that some authors [Muirhead] define $\Box^2 \equiv \bigtriangledown^2 - \frac{\partial^2}{\partial t^2}$ different from (1.1), so that they write the Klein-Gordon equation as $(\Box^2 - m^2)\phi = 0$ or $(p^2 + m^2)\phi = 0$.)

C) The energy momentum tensor is

$$\begin{aligned} T_{\mu\nu} &\equiv \Pi_\mu \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \\ &= \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \\ &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2) \end{aligned}$$

Therefore the Hamiltonian density is $\mathcal{H} \equiv T_{00} = \dot{\phi}^2 - \frac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2)$ which becomes [Leon]

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}[\Pi^2 + (\bigtriangledown\phi)^2 + m^2\phi^2] \end{aligned}$$

1.5 Noether's Theorem

Noether's theorem provides a general and powerful method for discussing symmetries of the action and Lagrangian and directly relating these symmetries to conservation laws.

Many books [Kaku] discuss Noether's theorem in a piecemeal fashion, for example by treating internal and spacetime symmetries separately. It is better to develop the formalism for *all* types of symmetries and then to extract out the spacetime and internal symmetries as special cases. The best discussion of this approach is in [Goldstein, Section 12-7, pg. 588] and [Greiner FQ, section 2.4, pg. 39]. Another excellent discussion of this general approach is presented in [Schwabl, section 12.4.2, pg. 268]. However note that the discussion presented by [Schwabl] concerns itself only with the symmetries of the Lagrangian, although the general spacetime and internal symmetries are properly treated together. The discussions by [Goldstein] and [Greiner] treat the symmetries of both the Lagrangian and the action as well.

In what follows we rely on the methods presented by [Goldstein].

The theory below closely follows [Greiner FQ 40]. We prefer to use the notation of [Goldstein] for fields, namely $\eta_r(x)$ or $\eta_r(x_\mu)$ rather than using

$\phi_r(x)$ or $\psi_r(x)$ because the latter notations might make us think of scalar or spinor fields. The notation $\eta_r(x)$ is completely general and can refer to scalar, spinor or vector field components.

We wish to consider how the Lagrangian and action change under a coordinate transformation

$$x_\mu \to x'_\mu \equiv x_\mu + \delta x_\mu$$

Let the corresponding change in the field (*total variation*) be [Ryder83, Schwabl263]

$$\eta'_r(x') \equiv \eta_r(x) + \Delta\eta_r(x)$$

and the corresponding change in the Lagrangian

$$\mathcal{L}'(x') \equiv \mathcal{L}(x) + \Delta\mathcal{L}(x)$$

with

$$\mathcal{L}(x) \equiv \mathcal{L}(\eta(x), \partial^\mu \eta(x),\ x)$$

where $\quad \partial^\mu \eta(x) \equiv \frac{\partial\eta(x)}{\partial x_\mu} \quad$ and [1]

$$\mathcal{L}'(x') \equiv \mathcal{L}(\eta'(x'), \partial^{\mu'} \eta'(x'), x')$$

(Note: no prime on $\mathcal{L}$ on right hand side) where $\partial^{\mu'}\eta'(x') \equiv \frac{\partial\eta'(x')}{\partial x'_\mu}$

Notice that the variations defined above involve two transformations, namely the change in coordinates from x to x' and also the change in the *shape* of the function from η to η'.

However there are other transformations (such as internal symmetries or gauge symmetries) that change the shape of the wave function at a single point. Thus the *local variation* is defined as (same as before)

$$\eta'_r(x) \equiv \eta_r(x) + \delta\eta_r(x)$$

[1]This follows from the *assumption* of *form invariance* [Goldstein 589]. In general the Lagrangian gets changed to

$$\mathcal{L}(\eta_r(x), \partial_\nu \eta_r(x), x) \to \mathcal{L}'(\eta'_r(x'), \partial_{\nu'} \eta'_r(x'), x')$$

with $\partial_{\nu'} \equiv \frac{\partial}{\partial x'^\nu}$
The assumption of form invariance [Goldstein 589] says that the Lagrangian has the *same functional form* in terms of the transformed quantities as it does in the original quantities, namely

$$\mathcal{L}'(\eta'_r(x'), \partial_{\nu'} \eta'_r(x'), x') = \mathcal{L}(\eta'_r(x'), \partial_{\nu'} \eta'_r(x'), x')$$

The local and total variations are related via

$$\begin{aligned}\delta\eta_r(x) &= \eta'_r(x) - \eta_r(x)\\ &= \eta'_r(x) - \eta'_r(x') + \eta'_r(x') - \eta_r(x)\\ &= -[\eta'_r(x') - \eta'_r(x)] + \Delta\eta_r(x)\end{aligned}$$

Recall the Taylor series expansion

$$\begin{aligned}f(x) &= f(a) + (x-a)f'(a) + ...\\ &= f(a) + (x-a)\frac{\partial f(x)}{\partial x}\mid_{x=a} + ...\end{aligned}$$

or

$$f(x) - f(a) \approx (x-a)\frac{\partial f}{\partial a}$$

which gives

$$\begin{aligned}\eta(x') - \eta(x) &\approx (x'-x)\frac{\partial \eta(x')}{\partial x'}\mid_{x'=x}\\ &\equiv (x'-x)\frac{\partial \eta}{\partial x} = \delta x\frac{\partial \eta}{\partial x}\end{aligned}$$

Thus

$$\delta\eta_r(x) = \Delta\eta_r(x) - \frac{\partial \eta'_r}{\partial x_\mu}\delta x_\mu$$

To lowest order $\eta'_r \approx \eta_r$. We do this because the second term is second order involving both $\partial\eta'$ and δx_μ. Thus finally we have the relation between the total and local variations as (to first order)

$$\boxed{\delta\eta_r(x) = \Delta\eta_r(x) - \frac{\partial \eta_r}{\partial x_\mu}\delta x_\mu}$$

Now we ask whether the variations Δ and δ commute with differentiation. (It turns out δ does commute but Δ does not.) From the definition $\delta\eta(x) \equiv \eta'(x) - \eta(x)$ it is obvious that (see before)

$$\frac{\partial}{\partial x_\mu}\delta\eta(x) = \delta\frac{\partial \eta(x)}{\partial x_\mu}$$

showing that δ "commutes" with $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$. However Δ does not commute, but has an additional term, as in (see Problems) [Greiner FQ41]

$$\frac{\partial}{\partial x_\mu}\Delta\eta(x) = \Delta\frac{\partial\eta(x)}{\partial x_\mu} + \frac{\partial\eta(x)}{\partial x^\nu}\frac{\partial\delta x^\nu}{\partial x_\mu}$$

Let us now study invariance of the *action* [Goldstein 589, Greiner FQ 41]. The *assumption* of *scale invariance* [Goldtein 589] says that the *action* is invariant under the transformation [2] (i.e. transformation of an ignorable or cyclic coordinate)

$$\begin{aligned} S' &\equiv \int_{\Omega'} d^4x'\ \mathcal{L}'(\eta'_r(x'_\mu),\ \partial_{\nu'}\eta'_r(x'_\mu), x'_\mu) \\ &= \int_{\Omega} d^4x\ \mathcal{L}(\eta_r(x_\mu),\ \partial_\nu\eta_r(x_\mu),\ x_\mu) \equiv S \end{aligned}$$

Demanding that the action is invariant, we have (in shorthand notation)

$$\delta S \equiv \int_{\Omega'} d^4x'\,\mathcal{L}'(x') - \int_{\Omega} d^4x\,\mathcal{L}(x) \equiv 0$$

Note that *this* δS is defined *differently* to the δS that we used in the derivation of the Euler-Lagrange equations. Using $\mathcal{L}'(x') \equiv \mathcal{L}(x) + \Delta\mathcal{L}(x)$ gives

$$\delta S \equiv \int_{\Omega'} d^4x'\,\Delta\mathcal{L}(x) + \int_{\Omega'} d^4x'\,\mathcal{L}(x) - \int_{\Omega} d^4x\,\mathcal{L}(x) = 0$$

We transform the volume element with the Jacobian

$$d^4x' = \left|\frac{\partial x'^{\,\mu}}{\partial x^\nu}\right| d^4x$$

using $x'^{\,\mu} = x^\mu + \delta x^\mu$ which gives [Greiner FQ 41]

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[2] Combining *both* form invariance and scale invariance gives [Goldstein 589]

$$\delta S \equiv S' - S = \int_{\Omega'} d^4x'\,\mathcal{L}(\eta'_r(x'), \partial_{\nu'}\eta'_r(x'), x') - \int_{\Omega} d^4x\,\mathcal{L}(\eta_r(x), \partial_\nu\eta_r(x), x) = 0$$

In the first integral x' is just a dummy variable so that
$\int_{\Omega'} d^4x\ \mathcal{L}(\eta'_r(x), \partial_\nu\eta'_r(x), x) - \int_\Omega d^4x\ \mathcal{L}(\eta_r(x), \partial_\nu\eta_r(x),\ x) = 0$
which [Goldstein] uses to derive current conservation.

$$d^4x' = \left|\frac{\partial x'^{\,\mu}}{\partial x^\nu}\right| d^4x = \begin{vmatrix} 1+\frac{\partial \delta x^0}{\partial x^0} & \frac{\partial \delta x^0}{\partial x^1} & \cdots \\ \frac{\partial \delta x^1}{\partial x^0} & 1+\frac{\partial \delta x^1}{\partial x^1} & \vdots \\ \vdots & \cdots & 1+\frac{\partial \delta x^3}{\partial x^3} \end{vmatrix} d^4x$$

$$= (1+\frac{\partial \delta x^\mu}{\partial x^\mu})d^4x$$

to *first* order only. Thus the variation in the action becomes

$$\begin{aligned} \delta S &= \int_\Omega (1+\frac{\partial \delta x^\mu}{\partial x^\mu})d^4x \;\; \Delta\mathcal{L}(x) + \int_\Omega (1+\frac{\partial \delta x^\mu}{\partial x^\mu})d^4x \;\; \mathcal{L}(x) - \int_\Omega d^4x\ \mathcal{L}(x) \\ &= \int_\Omega d^4x\ \Delta\mathcal{L}(x) + \int_\Omega d^4x\ \mathcal{L}(x)\frac{\partial \delta x^\mu}{\partial x^\mu} \end{aligned}$$

to first order. The second order term $\frac{\partial \delta x^\mu}{\partial x^\mu}\Delta\mathcal{L}(x)$ has been discarded. Using the relation between local and total variations gives

$$\begin{aligned} \delta S &= \int_\Omega d^4x \;\; (\delta\mathcal{L}(x) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x_\mu) + \int_\Omega d^4x\ \mathcal{L}(x)\frac{\partial \delta x^\mu}{\partial x^\mu} \\ &= \int_\Omega d^4x\ \{\delta\mathcal{L}(x) + \frac{\partial}{\partial x^\mu}\left[\mathcal{L}(x)\delta x^\mu\right]\} \end{aligned}$$

Recall that $\mathcal{L}(x) \equiv \mathcal{L}(\eta_r(x), \partial^\mu \eta_r(x))$. Now express the local variation $\delta\mathcal{L}$ in terms of total variations of the field as

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \eta_r}\ \delta\eta_r + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\ \delta(\partial^\mu \eta_r) \\ &= \quad \text{''} \quad + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\ \partial^\mu\ \delta\eta_r \end{aligned}$$

because δ "commutes" with $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$. Now add zero,

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \eta_r}\ \delta\eta_r - \left[\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\right]\ \delta\eta_r + \left[\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\right]\ \delta\eta_r + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\partial^\mu\delta\eta_r \\ &= \left[\frac{\partial \mathcal{L}}{\partial \eta_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\right]\delta\eta_r + \partial^\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\delta\eta_r\right] \end{aligned}$$

Note: the summation convention is being used for *both* μ and r. This expression for $\delta\mathcal{L}$ is substituted back into $\delta S = 0$, but because the region of integration is abritrary, the integrand itself has to vanish. Thus the integrand is

$$\left[\frac{\partial \mathcal{L}}{\partial \eta_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\right] \delta\eta_r + \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)} \delta\eta_r + \mathcal{L}\delta x_\mu\right] = 0$$

The first term is just the Euler-Lagrage equation which vanishes. For η_r use the relation between local and total variations, so that the second term becomes

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)} \left(\Delta\eta_r - \frac{\partial \eta_r}{\partial x_\nu}\, \delta x_\nu\right) + \mathcal{L}\delta x_\mu\right] = 0$$

which is the continuity equation

$$\partial^\mu\, j_\mu = 0$$

with [Schwabl 270]

$$\boxed{\begin{aligned} j_\mu &\equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\, \Delta\eta_r - \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)} \partial_\nu \eta_r - g_{\mu\nu}\mathcal{L}\right] \delta x^\nu \\ &\equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\, \Delta\eta_r - T_{\mu\nu}\delta x^\nu \end{aligned}}$$

with the energy-momentum tensor defined as [Scwabl 270]

$$\boxed{T_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)} \partial_\nu \eta_r - g_{\mu\nu}\mathcal{L}}$$

The corresponding conserved charge is (See Problems)

$$\boxed{Q \equiv \int d^3x\, j_0(x)}$$

such that

$$\frac{dQ}{dt} = 0$$

Thus we have $j_0(x)$ is just the charge density

$$j_0(x) \equiv \rho(x)$$

This leads us to the statement,

> *Noether's Theorem:* Each continuous symmetry transformation that leaves the Lagrangian invariant is associated with a conserved current. The *spatial* integral over this current's zero component yields a conserved charge. [Mosel 16]

1.6 Spacetime Symmetries

The symmetries we will consider are *spacetime* symmetries and *internal* symmetries. *Super* symmetries relate both of these. The simplest spacetime symmetry is 4-dimensional translation invariance, involving space translation and time translation.

1.6.1 Invariance under Translation

[GreinerFQ 43] Consider translation by a constant factor ϵ_μ,

$$x'_\mu = x_\mu + \epsilon_\mu$$

and comparing with $x'_\mu = x_\mu + \delta x_\mu$ gives $\delta x_\mu = \epsilon_\mu$. The shape of the field does not change, so that $\Delta\eta_r = 0$ (which is properly justified in Schwabl 270) giving the current as

$$j_\mu = -\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\,\frac{\partial\eta_r}{\partial x^\nu} - g_{\mu\nu}\mathcal{L}\right)\epsilon^\nu$$

with $\partial^\mu j_\mu = 0$. *Dropping off the constant factor* ϵ^ν lets us write down a modified current (called the energy-momentum tensor)

$$T_{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\partial_\nu\eta_r - g_{\mu\nu}\mathcal{L}$$

with

$$\partial^\mu T_{\mu\nu} = 0$$

In general j_μ has a conserved charge $Q \equiv \int d^3x\ j_o(x)$. Thus $T_{\mu\nu}$ will have 4 conserved charges corresponding to $T_{00},\ T_{01},\ T_{02},\ T_{03}$ which are just the energy E and momentum $\vec{P}$ of the field. In 4-dimensional notation [GreinerFQ 43]

$$P^\nu = (E, \vec{P}) = \int d^3x\ T^{0\nu} = constant.$$

with

$$\frac{dP^\nu}{dt} = 0$$

The above expression for $T_{\mu\nu}$ is the same result we obtained before where we wrote

$$T_{\mu\nu} = \pi_\mu\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{1.4}$$

with

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} \tag{1.5}$$

1.6.2 Angular Momentum and Lorentz Transformations

NNN: below is old Kaku notes. Need to revise; Schwabl and Greiner are best (they do J=L+S)

Instead of a simple translation $\delta x^i = a^i$ now consider a rotation $\delta x^i = a^{ij}x^j$. Lorentz transformations are a generalisation of this rotation, namely $\delta x^\mu = \epsilon^\mu{}_\nu x^\nu$. Before for spacetime translations we had $\delta x^\mu = a^\mu$ and therefore $\delta\phi = \frac{\partial\phi}{\partial x_\mu}\delta x^\mu = \delta x^\mu \partial_\mu \phi = a^\mu \partial_\mu \phi$. Copying this, the *Lorentz transformation* is

$$
\begin{aligned}
\delta x^\mu &= \epsilon^\mu{}_\nu x^\nu \\
\delta\phi &= \epsilon^\mu{}_\nu x^\nu \partial_\mu \phi \\
\delta\partial_\rho\phi &= \epsilon^\mu{}_\nu x^\nu \partial_\mu \partial_\rho \phi
\end{aligned}
$$

Now repeat same step as before, and we get the conserved *current*

$$\mathcal{M}^{\rho\mu\nu} = T^{\rho\nu}x^\mu - T^{\rho\mu}x^\nu$$

with

$$\partial_\rho\, \mathcal{M}^{\rho\mu\nu} = 0$$

and the conserved charge

$$M^{\mu\nu} = \int d^3x \;\; \mathcal{M}^{0\mu\nu}$$

with

$$\frac{d}{dt}M^{\mu\nu} = 0$$

For rotations in 3-d space, the $\frac{d}{dt}M^{ij} = 0$ corresponds to conservation of angular momentum.

1.7 Internal Symmetries

[Guidry 91-92]
One of the important theorems in Lie groups is the following :

> *Theorem*: *Compact* Lie groups can always be represented by finite-dimensional unitary operators. [Tung p.173,190]

Thus using the notation $U(\alpha_1, \alpha_2, ...\alpha_N)$ for an element of an N-parameter Lie group (α_i are the group parameters), we can write any group element as

$$\begin{aligned} U(\alpha_1, \alpha_2, ...\alpha_N) &= e^{i\alpha_i X_i} \\ &\approx 1 + i\epsilon_i X_i + ... \end{aligned}$$

where the latter approximation is for infinitessimal group elements $\alpha_i = \epsilon_i$. The X_i are linearly independent Hermitian operators (there are N of them) which satisfy the Lie algebra

$$[X_i, X_j] = i\ f_{ijk} X_k$$

where f_{ijk} are the structure constants of the group.

These group elements act on wave functions, as in [Schwabl 272]

$$\begin{aligned} \eta(x) \to \eta'(x) &= e^{i\alpha_i X_i}\eta(x) \\ &\approx (1 + i\epsilon_i X_i)\eta(x) \end{aligned}$$

giving

$$\delta\eta(x) = \eta'(x) - \eta(x) = i\epsilon_i X_i \eta(x)$$

Consider the Dirac equation $(i\not\partial - m)\psi = 0$ with 4-current, $j^\mu = \bar{\psi}\gamma^\mu\psi$. This is derived from the Lagrangian $\mathcal{L} = \bar{\psi}(i\not\partial - m)\psi$ where $\bar{\psi} \equiv \psi^\dagger\gamma^0$. *Now, for α_i = constant, the Dirac Lagrangian $\mathcal{L}$ is invariant under the transformation $\psi \to \psi' = e^{i\alpha_i X_i}\psi$. This is the significance of group theory in quantum mechanics.* Noether's theorem now tells us tht we can find a corresponding conserved charge and conserved current.

The Noether current, with $\delta x^\nu = 0$ and therefore $\delta\eta(x) = \Delta\eta(x)$ [Schwabl 272], is

$$\begin{aligned} j_\mu &= \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\delta\eta_r \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)} i\epsilon_i\ X_i\ \eta_r \end{aligned}$$

and *again dropping off the constant factor* ϵ_i define a new curent (and throw in a minus sign so that we get a positive current in the example below)

$$j_i^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_r)} i \; X_i \; \eta_r$$

which obeys a continuity equation

$$\partial_\mu \, j_i^\mu = 0$$

Example Calculate j_i^μ for the isospin transformation $e^{i\alpha_i X_i}$ for the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \equiv \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

Solution

$$j_i^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} i X_i \psi$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} \; i \; \gamma^\mu$$

giving

$$j_i^\mu = -i\bar{\psi}\gamma^\mu \; iX_i\psi$$

or

$$\boxed{j_i^\mu = \bar{\psi}\gamma^\mu X_i \psi}$$

where X_i is the group generator. (Compare this to the ordinary Dirac probabilty current $j^\mu = (\rho, \vec{j}\,) = \bar{\psi}\gamma^\mu\psi$).

From the previous example we can readily display conservation of charge. Write the $U(1)$ group elements as

$$U(\theta) = e^{i\theta q}$$

where the charge q is the generator. Thus the conserved current is

$$j^\mu = q\bar{\psi}\gamma^\mu\psi$$

where $J^\mu = (\rho, \vec{j}\,) = \bar{\psi}\gamma^\mu\psi$ is just the probability current for the Dirac equation. The conserved charge is [Mosel 17,34]

$$\begin{aligned} Q &= \int d^3x\, j^0 = q\int d^3x\,\bar{\psi}\;\gamma^0\;\psi \\ &= q\int d^3x\,\rho \end{aligned}$$

(Note that $\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \Sigma_{i=1}^4|\psi_i|^2$ where ψ_i is each component of ψ. Thus ρ is positive definite.)

See also [Mosel 17,34; BjRQM 9]. Note that *if* ψ is normalized so that [Strange 123]

$$\int d^3x\,\rho = \int d^3x\,\psi^\dagger\psi = 1$$

then we have $Q = q$ as required. This is explained very clearly in [Gross 122-124]. However often different normalizations are used for the Dirac wave functions [Muirhead 72, Halzen & Martin 110]. For example [Halzen & Martin 110] have

$$\int d^3x\,\rho = \int d^3x\,\psi^\dagger\psi = u^\dagger u = 2E$$

See also [Griffiths 223].

1.8 Summary

1.8.1 Covariant and contravariant vectors

Contravariant and covariant vectors and operators are

$$A^\mu = (A^o, \vec{A}) \qquad\qquad A_\mu = (A^o, -\vec{A})$$

and

$$\partial_\mu = (\tfrac{\partial}{\partial t}, \vec{\bigtriangledown}) \qquad\qquad \partial^\mu = (\tfrac{\partial}{\partial t}, -\vec{\bigtriangledown})$$

1.8.2 Classical point particle mechanics

The point particle canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and the EL equations are

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

The point particle Hamiltonian is

$$H(p_i, q_i) \equiv p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{1.6}$$

giving Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad\qquad -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

1.8.3 Classical field theory

For classical fields ϕ_i, the EL equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$$

The covariant momentum density is

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

and *the* canonical momentum is

$$\Pi \equiv \Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

The energy momentum tensor is (analagous to point particle Hamiltonian)

$$T_{\mu\nu} \equiv \Pi_\mu \partial_\nu \phi - g_{\mu\nu}\mathcal{L}$$

with the Hamiltonian density

$$\mathcal{H} \equiv T_{00} = \Pi\dot{\phi} - \mathcal{L}$$

1.8.4 Noether's theorem

The conserved ($\partial^\mu j_\mu = 0$) Noether current is

$$j_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\,\Delta\eta_r - T_{\mu\nu}\delta x^\nu$$

with

$$T_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_r)}\partial_\nu \eta_r - g_{\mu\nu}\mathcal{L}$$

The conserved ($\frac{dQ}{dt} = 0$) charge is

$$Q \equiv \int d^3x\, j_0(x)$$

which is just the charge density,

$$j_0(x) \equiv \rho(x)$$

If we consider the *spacetime symmetry* involving *invariance under translation* then we can derive $T_{\mu\nu}$ from j_μ. The result for $T_{\mu\nu}$ agrees with that given above. For the Klein-Gordon Lagrangian this becomes

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{1.7}$$

The momentum is (with $E \equiv H$)

$$P^\nu = (H, \vec{P}) = \int d^3x\, T^{0\nu} = constant \tag{1.8}$$

where $\frac{dP^\nu}{dt} = 0$.

For *internal symmetries* the group elements can be written

$$U(\alpha_1, \alpha_2, ...\alpha_N) = e^{i\alpha_i X_i} \approx 1 + i\epsilon_i X_i + ...$$

where

$$[X_i, X_j] = i\ f_{ijk} X_k$$

The group elements act on wave functions

$$\eta \to \eta' = e^{i\alpha_i X_i}\eta \approx (1 + i\epsilon_i X_i)\eta$$

giving

$$\delta\eta = \eta' - \eta = i\epsilon_i X_i \eta$$

The Noether current, for an internal symmetry ($\delta x^\nu = 0$ and therefore $\delta\eta = \Delta\eta$) becomes

$$j_i^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu \eta_r)} i\ X_i\ \eta_r$$

For the Dirac Lagrangian, invariant under $e^{i\alpha_i X_i}$, with $\alpha_i = constant$, this becomes

$$j_i^\mu = \bar{\psi}\gamma^\mu X_i \psi$$

1.9 References and Notes

General references for units (Section 1.1) are [**Aitchison and Hey, pg. 526-531; Halzen and Martin, pg. 12-13;** Guidry, pg.511-514; Mandl and Shaw, pg. 96-97; Griffiths, pg. 345; Jackson, pg. 811-821; Misner, Thorne and Wheeler, pg. 35-36]. References for Natural units (Section 1.1.1) are [Guidry, pg. 511-514; Mandl and Shaw, pg. 96-97; Griffiths, pg. 345; Aitchison and Hey, pg. 526-531; Halzen and Martin, pg. 12-13] and references for Geometrical units (Section 1.1.2) are [Guidry, pg. 514; MTW, pg. 35-36].

For a complete introduction to covariant and contravariant vectors see [Jackson]

The most commonly used metric is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This metric is used throughout the present book and is also used by the following authors: [Greiner, Aitchison & Hey, Kaku, Peskin & Schroeder, Ryder, Bjorken and Drell, Mandl & Shaw, Itzykson & Zuber, Sterman, Chang, Guidry, Griffiths, Halzen & Martin and Gross].

Another less commonly used metric is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This metric is *not* used in the present book, but it is used by [Weinberg and Muirhead].

The best references for the classical field ELE and Noether's theorem are [Schwabl, Ryder]

Chapter 2

Symmetries & Group theory

2.1 Elements of Group Theory

SUSY nontrivially combines both spacetime and internal symmetries.

2.2 SO(2)

In SO(2) the *invariant* is $x^2 + y^2$. We write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$x^{i'} = O^{ij} x^j$$

For small angles this is reduced to

$$\delta x = \theta y \ \text{ and } \ \delta \text{y} = -\theta \text{x}$$

or

$$\delta x^i = \theta \epsilon^{ij} x^j$$

where ϵ^{ij} is antisymmetric and $\epsilon^{12} = -\epsilon^{21} = 1$

Read Kaku p. 36-38

2.2.1 Transformation Properties of Fields

References: [Kaku 38; Greiner FQ 95, 96; Elbaz 192; Ho-Kim 28; Mosel 21]

Consider a transformation U which transforms a quantum state [Elbaz l92]

$$|\alpha' > \equiv U|\,\alpha >$$

$$< \alpha'\,| = < \alpha|\,U^\dagger$$

To conserve the norm, we impose

$$< \alpha'\,|\beta' > = < \alpha|\,\beta >$$

which means that U is unitary, i.e. $UU^\dagger = 1$ implying $U^\dagger = U^{-1}$.

Now consider transformation of an operator O, with expectation value

$$< O > \equiv < \alpha|O|\alpha > = < \alpha'|O'|\alpha' >$$

This gives

$$< O > = < \alpha|O|\alpha > = < \alpha|U^\dagger O' U|\alpha >$$

giving the transformation rule for the operator as

$$O = U^\dagger O' U$$

or

$$O' = UOU^\dagger$$

To summarize, if a quantum state transforms as [Elbaz 192]

$$\psi' = U\psi$$

or

$$|\alpha' > = U|\alpha >$$

then an operator transforms as

$$O' = UOU^\dagger$$

Fields can be grouped into different categories [Mosel 21] according to their behavior under *general* Lorentz transformations, which include spatial rotation, Lorentz boost transformations and also the discrete transformations of space reflection, time reversal and space-time reflection. The general Lorentz transformation is written [Mosel 21, Kaku 50]

$$x'_\mu = \Lambda_{\mu\nu} x^\nu \tag{2.1}$$

but let's write it more generally (in case we consider other transformations)

$$x'_\mu = a_{\mu\nu} x^\nu \tag{2.2}$$

$\Lambda_{\mu\nu}$ or $a_{\mu\nu}$ for rotation, boost, space inversion is very nicely discussed in [Ho-Kim 19-22]. A *scalar* field transforms under (2.1) or (2.2) as

$$\phi'(x') = \phi(x)$$

[Mosel 21; Ho-Kim 26; Greiner 95, 96]. Now if $\phi(x)$ *is an operator* then its transformation is also written as [Greiner 96, Kaku 38]

$$\phi'(x') = U\phi(x')U^\dagger$$

Thus a *scalar* field transforms under (2.2) as

$$\boxed{\phi'(x') = U\phi(x')U^\dagger = \phi(x)}$$

A *vector* field transforms as [Ho-Kim 30; Kaku 39]

$$\boxed{\phi'_\mu(x') = a^\nu_\mu \phi_\nu(x) \equiv U\phi_\mu(x')U^\dagger}$$

i.e. it just transforms in the same way as an ordinary 4-vector.

2.3 Representations of SO(2) and U(1)

Read Kaku 39-42, 741-748

SO(2) can be *defined* as the set of transformations that leave $x^2 + y^2$ invariant. There is a homomorphism between SO(2) and U(1). A U(1) transformation can be written $\psi' = e^{i\theta}\psi = U\psi$. This will leave the inner product $\psi^*\phi$ invariant. Thus the group U(1) can be *defined* as the set of transformations that leave $\psi^*\phi$ invariant [Kaku 742, Peskin 496]

2.4 Representations of SO(3) and SU(1)

SO(3) leaves $x^2 + x^2 + z^2$ invariant.

Read Kaku 42-45.

2.5 Representations of SO(N)

Students should read rest of chapter in Kaku.

NNN NOW do a.m.

Chapter 3

Free Klein-Gordon Field

NNN write general introduction

3.1 Klein-Gordon Equation

Relativistic Quantum Mechanics (RQM) is the subject of studying relativistic wave equations to replace the non-relativistic Schrodinger equation. The two prime relativistic wave equations are the Klein-Gordon equation (KGE) and the Dirac equation. (However these are only valid for 1-particle problems whereas the Schrodinger equation can be written for many particles.)

Our quantum wave equation (both relativistic and non-relativistic) is written

$$\hat{H}\psi = \hat{E}\psi$$

with

$$\hat{H} \equiv \hat{T} + \hat{U}$$

and

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Non-relativistically we have $\hat{T} = \frac{\hat{p}^2}{2m}$ with $\hat{p} = -i\hbar\vec{\nabla}$ giving

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

The time-independent Schrodinger equation simply has E_b instead of $\hat{E}$ where E_b is the *binding energy*, i.e.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi = E_b\psi$$

However in special relativity we have (with $c = 1$)

$$E = T + m$$

and

$$E^2 = p^2 + m^2$$

giving

$$T = \sqrt{p^2 + m^2} - m$$

With the replacement $\vec{p} \to -i\hbar\vec{\nabla}$, the relativistic version of the free particle ($U = 0$) Schrodinger equation would be ($\hat{T}\psi = E_b\psi$)

$$\left(\sqrt{-\hbar^2\nabla^2 + m^2} - m\right)\psi = E_b\psi$$

In the non-relativistic case $E = E_b$, but relativistically $E = E_b + m$, giving

$$\sqrt{-\hbar^2\nabla^2 + m^2}\,\psi = E\psi$$

which is called the *Spinless Salpeter equation.* There are two problems with this equation; firstly the operator $\sqrt{-\hbar^2\nabla^2 + m^2}$ is non-local [Landau 221] making it very difficult to work with in coordinate space (but actually it's easy in momentum space) and secondly the equation is not *manifestly covariant* [Gross, pg. 92]. Squaring the Spinless Salpeter operator gives the Klein-Gordon equation (KGE)

$$\begin{aligned}(-\hbar^2\nabla^2 + m^2)\phi &= E^2\phi \\ &= -\hbar^2\frac{\partial^2}{\partial t^2}\phi\end{aligned}$$

or (with $\hbar = c = 1$, see Halzen & Martin, pg. 12, 13; Aitchison & Hey, pg. 526-528)

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0$$

Recall the wave equation (with $y'' \equiv \partial^2 y/\partial x^2$, $\ddot{y} \equiv \partial^2 y/\partial t^2$)

$$y'' - \frac{1}{c^2}\ddot{y} = 0$$

where c is the wave velocity. Thus the KGE ($c \neq 1$) is (with $\nabla^2 = \partial^2/\partial x^2$)

$$\phi'' - \frac{1}{c^2}\ddot{\phi} = m^2\phi$$

which is like a massive (inhomogeneous) wave equation. The KGE is written in manifestly covariant form as

$$\boxed{(\Box^2 + m^2)\phi = 0}$$

which in momentum space is (using $p^2 \to -\Box^2$)

$$\boxed{(p^2 - m^2)\phi = 0}$$

A *quick route* to the KGE is with the relativistic formula $p^2 \equiv p_\mu p^\mu = m^2$ $(\neq \vec{p}^2)$ giving $p^2 - m^2 = 0$ and $(p^2 - m^2)\phi = 0$ and $p^2 \to -\Box^2$ giving $(\Box^2 + m^2)\phi = 0$. The KGE can be written in terms of 4-vectors, $(p^2 - m^2)\phi = 0$ and is therefore *manifestly covariant*. Finally, note that the KGE is a *1-particle* equation!

(Note that some authors [Muirhead] use $g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and so have $\Box^2 \equiv \nabla^2 - \frac{\partial^2}{\partial t^2}$ and $(\Box^2 - m^2)\phi = 0$ or $(p^2 + m^2)\phi = 0$ for the Klein-Gordon equation.)

3.2 Probability and Current

The Klein-Gordon equation was historically rejected because it predicted a negative probability density. In order to see this let's first review probability and current for the Schrodinger equation. Then the KG example will be easier to understand.

3.2.1 Schrodinger equation

The free particle Schrodinger equation (SE) is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}$$

The complex conjugate equation is (SE*)

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$$

Multiply SE by ψ^* and SE* by ψ

$$-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi = i\hbar\,\psi^*\frac{\partial\psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\psi\nabla^2\psi^* = -i\hbar\,\psi\frac{\partial\psi^*}{\partial t}$$

and subtract these equations to give

$$\begin{aligned} - \quad \frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) &= i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) \\ &= i\hbar\frac{\partial}{\partial t}(\psi^*\psi) \\ = \quad - \quad \frac{\hbar^2}{2m}\vec{\nabla}\cdot[\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi] \end{aligned}$$

or

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\hbar}{2mi}\vec{\nabla}\cdot[\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi] = 0$$

which is just the continuity equation $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ *if*

$$\boxed{\begin{aligned} &\rho \equiv \psi^*\psi \\ &\vec{j} \equiv \frac{\hbar}{2mi}[\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi] \end{aligned}}$$

which are the probability density and current for the Schrodinger equation.

3.2.2 Klein-Gordon Equation

The free particle KGE is $(\Box^2 + m^2)\phi = 0$ or (using $\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2$)

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0$$

and the complex conjugate equaiton is (KGE*)

$$\frac{\partial^2\phi^*}{\partial t^2} - \nabla^2\phi^* + m^2\phi^* = 0$$

Multiplying KGE by ϕ^* and KGE* by ϕ gives

$$\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi^*\nabla^2\phi + m^2\phi^*\phi = 0$$

$$\phi\frac{\partial^2\phi^*}{\partial t^2} - \phi\nabla^2\phi^* + m^2\phi\phi^* = 0$$

and subtract these equations to give

$$\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2} - \phi^*\nabla^2\phi + \phi\nabla^2\phi^* = 0$$

$$= \frac{\partial}{\partial t}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) - \vec{\nabla} \cdot \left[\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right]$$

which is the continuity equation $\partial\rho/\partial t + \vec{\nabla} \cdot \vec{j} = 0$ *if*

$$\boxed{\begin{aligned} \rho &\equiv \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \\ \vec{j} &\equiv \phi \vec{\nabla}\phi^* - \phi^* \vec{\nabla}\phi \end{aligned}} \quad \text{or} \quad \boxed{j^\mu = \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*}$$

but to get this to match the SE wave function we should define $\vec{j}$ in the same way, i.e.

$$\boxed{\vec{j} \equiv \frac{\hbar}{2mi}\left[\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right]} = \frac{-i\hbar}{2m}\left(\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right)$$

which is $\frac{-\hbar}{2mi}$ times $\vec{j}$ above. Thus for $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$ to hold we must have

$$\boxed{\rho \equiv \frac{i\hbar}{2m}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)}$$

The problem with ρ (in *both* expressions above) is that it is *not* positive definite and therefore cannot be interpreted as a probability density. This is one reason why the KGE was discarded. (Note: because ρ can be either positive or negative it can be interpreted as a *charge* density. See [Landau, pg. 227]

Notice how this *problematic* ρ comes about because the KGE is *2nd order in time*. We have $\frac{\partial \rho}{\partial t}$ and ρ itself constains $\frac{\partial}{\partial t}$; this does not happen with the SE or DE.

By the wave, note that we can form a 4-vector KG current

$$\boxed{j^\mu = \frac{i\hbar}{2m}\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right)}$$

3.3 Classical Field Theory

Reference: [Schwabl, Chapter 13; Kaku, Chapter 3]

Some of the key results for the free and real Klein-Gordon field were worked out in an Example in Chapter 1. Let's remind outselves of these results.

The massive Klein-Gordon Lagrangian was

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$$

which gave the equation of motion in position and momentum space as

$$(\Box^2 + m^2)\phi = 0$$
$$(p^2 - m^2)\phi = 0$$

The covariant momentum density was

$$\Pi^\mu = \partial^\mu\phi$$

giving *the* canonical momentum

$$\Pi \equiv \Pi^o = \dot{\phi}(x)$$

The Hamiltonian density was (with $H \equiv \int d^3x\,\mathcal{H}$)

$$\mathcal{H} = \frac{1}{2}[\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2]$$

Finally, the momentum operator of the Klein-Gordon field is (see Problems)

$$\vec{P} = -\int d^3x\,\dot{\phi}(x)\vec{\nabla}\phi(x)$$

3.4 Fourier Expansion & Momentum Space

[See Greiner FQ 76 , Jose and Saletin 589]

As with the non-relativistic case we expand plane wave states as

$$\phi(\vec{x},t) = \int d\tilde{k}\; a(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}$$

but now with

$$d\tilde{k} = N_k d^3k$$

where N_k is a normalization constant, to be determined later.

Substitute $\phi(\vec{r},t)$ into the KGE

$$\left(\Box^2 + m^2\right)\phi = 0$$

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi = 0$$

giving

$$\int d\tilde{k}\left(\ddot{a}+k^2a+m^2a\right)e^{i\vec{k}\cdot\vec{x}}=0$$

Defining

$$\boxed{\omega\equiv\omega(\vec{k})=\sqrt{k^2+m^2}}$$

and requiring the integrand to be zero gives

$$\ddot{a}+\omega^2a=0$$

which is a 2nd order differential equation, with Auxilliary equation

$$r^2+\omega^2=0$$

or

$$r=\pm-i\omega$$

The *two solutions* $\pm$ *are crucial*! They can be interpreted as *positive and negative energy, or as particle and antiparticle*. In the non-relativistic (NR) case we only got one solution. Thus [Teller, pg. 67]

$$a(\vec{k},t)=c(\vec{k})e^{i\omega t}+a(\vec{k})e^{-i\omega t}$$

giving our original expansion as

$$\phi(\vec{x},t)=\int d\tilde{k}\left[a(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}+c(\vec{k})e^{i(\vec{k}\cdot\vec{x}+\omega t)}\right]$$

(*Remember that* $\omega\equiv\omega(\vec{k})=\sqrt{k^2+m^2}$) Now the Schrodinger wave function $\psi(\vec{x},t)$ is complex but *the KG wave function* $\phi(\vec{x},t)$ *is real.* (The Schrodinger equation has an i in it, but the KGE does not.) Thus [Greiner, pg. 77]

$$\phi(\vec{x},t)=\phi^*(\vec{x},t)=\phi^\dagger(\vec{x},t)$$

giving

$$\int d\tilde{k}\left[a(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}+c(\vec{k})e^{i(\vec{k}\cdot\vec{x}+\omega t)}\right]$$
$$=\int d\tilde{k}\left[a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}+c^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}+\omega t)}\right]$$

Re-write this as

$$\int d\tilde{k}\left[a(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}+c(-\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right]$$
$$=\int d\tilde{k}\left[a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}+c^\dagger(-\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]$$

where we have made the substitution $\vec{k} \to -\vec{k}$ in two terms.

(This does **not** mean $k \to -k$; it cannot. k is the magnitude of $\vec{k}$. We are simply reversing the *direction* (magnitude) of $\vec{k}$. This does *not* affect the volume element d^3k.)

Obviously then

$$c(-\vec{k}) = a^\dagger(\vec{k})$$

$$c^\dagger(-\vec{k}) = a(\vec{k})$$

Re-writing our expansion as

$$\phi(\vec{x}, t) = \int d\tilde{k} \left[a(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)} + c(-\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \right]$$

gives [same conventions as Kaku 152, Mandl and Shaw 44, Schwabl 278]

$$\boxed{\begin{aligned} \phi(\vec{x}, t) &\equiv \phi^+(x) + \phi^-(x) \\ &= \int d\tilde{k} \left[a(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)} + a^\dagger(\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \right] \\ &= \int d\tilde{k} \left[a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right] \\ &= \sum_{\vec{k}} \frac{1}{\sqrt{2\omega V}} \left[a_{\vec{k}} e^{-ik\cdot x} + a^\dagger_{\vec{k}} e^{ik\cdot x} \right] \end{aligned}} \tag{3.1}$$

[see Greiner FQ 79 for box vs. continuum normalization] where $k \cdot x \equiv k^\mu x_\mu = k^o x_o - \vec{k} \cdot \vec{x} = \omega t - \vec{k} \cdot \vec{x}$.

The conjugate momentum is

$$\boxed{\begin{aligned} \Pi(\vec{x}, t) = \dot{\phi}(\vec{x}, t) = &\ -i \int d\tilde{k}\ \omega \left[a(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)} - a^\dagger(\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \right] \\ &= -i \int d\tilde{k}\ \omega \left[a(\vec{k}) e^{-ik\cdot x} - a^\dagger(\vec{k}) e^{ik\cdot x} \right] \\ &\equiv \Pi^+(x) + \Pi^-(x) \end{aligned}} \tag{3.2}$$

Note that we are still studying *classical field theory* when we Fourier expand these classical fields [Goldstein 568, Jose & Saletan 588].

3.5 Klein-Gordon QFT

Recall the commutation relations from non-relativistic quantum mechanics, namely

$$[x, p] = i\hbar$$

and

$$[a, a^\dagger] = 1$$

To develop the QFT we impose similar relations, but this time for fields. The *equal time* commutation relations

$$[\phi(\vec{x}, t), \Pi(\vec{x}', t)] = i\delta(\vec{x} - \vec{x}') \tag{3.3}$$

and

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\Pi(\vec{x}, t), \Pi(\vec{x}', t)] = 0 \tag{3.4}$$

where we had

$$\Pi = \dot{\phi}$$

We now need to find the commutation relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$. We *expect* the usual results

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') \tag{3.5}$$

and

$$[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0 \tag{3.6}$$

3.5.1 Indirect Derivation of $a, a^\dagger$ Commutators

[Greiner, pg. 77]

To check whether our expection above is correct evaluate

$$\begin{aligned}
&[\phi(\vec{x}, t), \Pi(\vec{x}', t)] \\
&= [\phi^+(\vec{x}, t) + \phi^-(\vec{x}, t), \Pi^+(\vec{x}', t) + \Pi^-(\vec{x}', t)] \\
&= [\phi^+(\vec{x}, t), \Pi^+(\vec{x}', t)] + [\phi^+(\vec{x}, t), \Pi^-(\vec{x}', t)] \\
&\quad + [\phi^-(\vec{x}, t), \Pi^+(\vec{x}', t)] + [\phi^-(\vec{x}, t), \Pi^-(\vec{x}', t)] \\
&= -i \int d\tilde{k} \int d\tilde{k}'\ \omega' \Big\{ [a(\vec{k}), a(\vec{k}')] e^{-i(k\cdot x + k'\cdot x')} - [a(\vec{k}), a^\dagger(\vec{k}')] e^{-i(k\cdot x - k'\cdot x')} \\
&\quad + [a^\dagger(\vec{k}), a(\vec{k}')] e^{i(k\cdot x - k'\cdot x')} - [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] e^{i(k\cdot x + k'\cdot x')}] \Big\}
\end{aligned}$$

where $k \cdot x = \omega t - \vec{k} \cdot \vec{x}$ and $t' \equiv t$ and $\omega' \equiv \omega(\vec{k}') = \sqrt{\vec{k}'^{\,2} + m^2}$.

Inserting (3.5) and (3.6) gives

$$[\phi(\vec{x},t),\Pi(\vec{x}',t)] = -i\int d\tilde{k}\int d\tilde{k}'\ \omega'\left[-\delta(\vec{k}-\vec{k}')e^{-i(k\cdot x-k'\cdot x')}\right.$$
$$\left.-\delta(\vec{k}'-\vec{k})e^{i(k\cdot x-k'\cdot x')}\right]$$
$$= i\int d^3k\ N_k^2\ \omega\left[e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}+e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right]$$

Now *if* [Greiner, pg. 77]

$$N_k = \frac{1}{\sqrt{2\omega(2\pi)^3}}$$

then we obtain

$$\begin{aligned}[\phi(\vec{x},t),\Pi(\vec{x}',t)] &= i\int d^3k\frac{1}{2\omega(2\pi)^3}\omega\left[e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}+e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right]\\ &= i\delta(\vec{x}-\vec{x}')\end{aligned}$$

as required. Here we have used the result,

$$\frac{1}{(2\pi)^3}\int d^3k\ e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta(\vec{x}-\vec{x}')$$

With the above normalization we have

$$\boxed{d\tilde{k} \equiv \frac{d^3k}{\sqrt{2\omega(2\pi)^3}}}$$

(which is *different* to IZ114, but same as Kaku). Using the result [IZ114, Kaku 64-65]

$$\frac{d^3k}{2\omega} = d^4k\ \delta(k^2-m^2)\ \theta(k^0)$$

gives

$$\boxed{d\tilde{k} = \sqrt{\frac{2\omega}{(2\pi)^3}}\ d^4k\delta(k^2-m^2)\ \theta(k^o)}$$

(which is different from IZ 114, but same as Kaku)

Note: Some authors [IZ114] use the normalization $N_k = \frac{1}{(2\pi)^3 2\omega}$ in which case $[a(\vec{k}),a^\dagger(\vec{k}')] = (2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{k}')$ [Greiner, pg. 77, footnote].

3.5.2 Direct Derivation of $a, a^\dagger$ Commutators

The best way to obtain the commutators directly is to invert the Fourier expansions to obtain a and $a^\dagger$ in terms of ϕ and Π. The commutators are then derived directly.

Inverting (3.1) and (3.2) gives [see Problems]

$$\boxed{\begin{aligned} a_{\vec{k}} &= \frac{1}{\sqrt{2\omega(2\pi)^3}} \int d^3x \, e^{ik\cdot x}[\omega\phi(x) + i\Pi(x)] \\ \Rightarrow \ a^\dagger_{\vec{k}} &= \frac{1}{\sqrt{2\omega(2\pi)^3}} \int d^3x \, e^{-ik\cdot x}[\omega\phi^\dagger(x) - i\Pi^\dagger(x)] \end{aligned}}$$

where $a_{\vec{k}} \equiv a(\vec{k})$ and $\phi(x) \equiv \phi(\vec{x}, t)$ and $k \cdot x \equiv \omega t - \vec{k} \cdot \vec{x}$ and by direct evaluation of the commutators we arrive at (3.5) and (3.6) [do Problem 5.7]

But remember for KGE ϕ is real and therefore $\phi = \phi^\dagger$ and $\Pi = \Pi^\dagger$ giving

$$\boxed{a^\dagger_{\vec{k}} = \frac{1}{\sqrt{2\omega(2\pi)^3}} \int d^3x \, e^{-ik\cdot x}[\omega\phi(x) - i\Pi(x)]}$$

3.5.3 Klein-Gordon QFT Hamiltonian

The 2nd quantized field Hamiltonian is (see Problems)

$$H = \int d^3k \frac{\omega}{2}(a^\dagger_{\vec{k}} \, a_{\vec{k}} + a_{\vec{k}} \, a^\dagger_{\vec{k}})$$

and using the commutator

$$[\, a_{\vec{k}} \, , a^\dagger_{\vec{k}'} \,] = \delta(\vec{k} - \vec{k}')$$

gives

$$\boxed{\begin{aligned} H &= \int d^3k \, \left(N_{\vec{k}} + \frac{1}{2}\right)\omega \\ \text{or} & \quad \sum_{\vec{k}} \left(N_{\vec{k}} + \frac{1}{2}\right)\omega \end{aligned}}$$

with

$$\boxed{N_{\vec{k}} \equiv a^\dagger_{\vec{k}} a_{\vec{k}}}$$

(see Problems)

We can also calculate the momentum $\vec{P}$ as [Kaku 66, 67; Schwabl 279]

$$\boxed{\begin{array}{c}\vec{P} = \int d^3k \left(N_{\vec{k}} + \frac{1}{2}\right)\vec{k} \\ \text{or } \sum_{\vec{k}} \left(N_{\vec{k}} + \frac{1}{2}\right)\vec{k}\end{array}}$$

3.5.4 Normal order

References [Kaku68, Mosel 27, Schwabl 280]
The previously derived Hamiltonian is actually infinite! This is beacuse the term

$$\int d^3k \frac{1}{2}\omega = \frac{1}{2}\int d^3k \sqrt{\vec{k}^{\,2} + m^2} = \infty$$

This is one of the first (of many) places where QFT gives infinite answers. Now because only energy *differences* are observable, we are free to simply throw away the infinite piece, and re-write the Hamiltonian as

$$H = \int d^3k \; N_{\vec{k}} \; \omega.$$

A *formal* way to always get rid of these infinite terms (there is also one in the previous expression for the momentum $\vec{P}$), is to introduce the idea of normal order.

> In a normal ordered product all annihilation operators are placed to the right hand side of all creation operators.

Two colons :: are used to denote a normal ordered product. For example [Schwabl 280]

$$: a_{\vec{k}_1} a_{\vec{k}_2} a^\dagger_{\vec{k}_3} : \; = a^\dagger_{\vec{k}_3} a_{\vec{k}_1} a_{\vec{k}_2}$$

$$: a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}} : \; = 2 a^\dagger_{\vec{k}} a_{\vec{k}}$$

In calculating the Hamiltonian (see Problems) we arrived at

$$H = \int d^3k (a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}) \frac{\omega}{2}$$

and using the commutator

$$[\, a_{\vec{k}} \, , a^\dagger_{\vec{k}'} \,] = \delta(\vec{k} - \vec{k}')$$

gave

$$
\begin{aligned}
H &= \int d^3k(a^\dagger_{\vec{k}} a_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + 1)\frac{\omega}{2} \\
&= \int d^3k(a^\dagger_{\vec{k}} a_{\vec{k}} + 1/2)\omega
\end{aligned}
$$

which is infinite. If we define H to be normal ordered then

$$
\begin{aligned}
:H: &= \int d^3k : (a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}) : \frac{\omega}{2} \\
&= \int d^3k(a^\dagger_{\vec{k}} a_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}})\frac{\omega}{2} \\
&= \int d^3k a^\dagger_{\vec{k}} a_{\vec{k}} \omega
\end{aligned}
$$

which is finite. *Note that normal ordering is equivalent to treating the boson operators as if they had vanishing commutator.* [Schwabl 280]. (nnn Are we back to a *classical* theory?)

3.5.5 Wave Function

[Kaku 68-69]

Now that we have expanded the KG field and Hamiltonian in terms of creation and destruction operators, we need something for them to operate on. These are just the many-body states introduced earlier [Bergstrom & Goobar 289]

$$|\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle = \prod_i |n_{\vec{k}_i}\rangle \tag{3.7}$$

with

$$|n_{\vec{k}_i}\rangle = \frac{1}{\sqrt{n_{\vec{k}}!}} \left(a^\dagger_{\vec{k}}\right)^{n_{\vec{k}}} |0\rangle$$

where the vacuum state is defined via

$$a_{\vec{k}}|0\rangle = 0$$

The physical interpretation [Bergstrom & Goobar 290] is provided by

$$H|\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle = \sum_{\vec{k}} n_k \, \epsilon(\vec{k})|\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle$$

where $\epsilon(\vec{k}) = \hbar\omega(\vec{k})$. *Equation (3.7) is interpreted as a many particle state where* $n_{\vec{k}_1}$ *have momentum* $\vec{k}_1$, $n_{\vec{k}_2}$ *have momentum* $\vec{k}_2$ *etc.*

Thus a 1-particle state is written

$$|1_{\vec{k}}\rangle \equiv |\vec{k}\rangle = a^\dagger_{\vec{k}}|0\rangle$$

or

$$\langle 1_{\vec{k}}| \equiv \langle\vec{k}| = \langle 0|a_{\vec{k}}$$

The states are normalized as

$$\langle\vec{k}|\vec{k}'\rangle = \delta(\vec{k} - \vec{k}')$$

giving

$$\langle 0|a_{\vec{k}} a^\dagger_{\vec{k}'}|0\rangle = \delta(\vec{k} - \vec{k}').$$

3.6 Propagator Theory

[Halzen and Martin 145-150, Kaku, Bj RQM Chapter 6]

The Klein-Gordon equation for a free particle is

$$(\Box^2 + m^2)\phi = 0$$

or with the replacement $\Box^2 \to -p^2$ in momentum space

$$(p^2 - m^2)\phi = 0$$

Let's write the non-free KGE as

$$(\Box^2 + m^2)\phi = J(x)$$

where $J(x)$ is referred to as a *source* term. This is solved with the Green function method by defining a propagator $\Delta_F(x-y)$ as

$$\boxed{(\Box^2 + m^2)\Delta_F(x-y) \equiv -\delta^4(x-y)}$$

so that the solution is

$$\phi(x) = \phi_0(x) - \int d^4y\, \Delta_F(x-y)J(y)$$

where $\phi_0(x)$ is the solution with $J = 0$. (see Problems) Define the Fourier transform

$$\Delta_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}\Delta_F(k)$$

Our usual method of solution for propagators or Green functions is to solve for $\Delta_F(k)$ and then do a contour integral to get $\Delta_F(x-y)$ rather than solving for $\Delta_F(x-y)$ directly. The Problems show that

$$\Delta_F(k) = \frac{1}{k^2 - m^2}$$

Example Derive the momentum space Green function for the Schrodinger equation.

Solution The free particle Schrodinger equation is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\psi = 0$$

and with the inclusion of a source this is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\psi = J(x) \equiv -U(\vec{x})\psi(\vec{x},t)$$

or

$$\left(+\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{\partial}{\partial t}\right)\psi = -J(x) = U(\vec{x})\psi(x)$$

Define the Green function

$$\left(\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{\partial}{\partial t}\right)G_0(x-x') = \delta^4(x-x')$$

and define the Fourier transform

$$G_0(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}G_0(p)$$

Now operate on this to give, and using $p^\mu \equiv (\omega, \vec{p})$, with $\hbar = 1$

$$\begin{aligned}\left(\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{\partial}{\partial t}\right)G_0(x-x') &= \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{2m}\,(-i\vec{p})^2 + i(-i\omega)\right]e^{-ip\cdot(x-x')}G_0(p) \\ = \delta^4(x-x') &= \int \frac{d^4p}{(2\pi)^4}\left[-\frac{\vec{p}^2}{2m} + \omega\right]e^{-ip\cdot(x-x')}G_0(p)\end{aligned}$$

but recall that

$$\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}$$

which implies

$$\boxed{G_0(p) = \frac{1}{\omega - \vec{p}^2/2m}}$$

Example Derive the position space Green function leaving the pole *on* the Real axis.

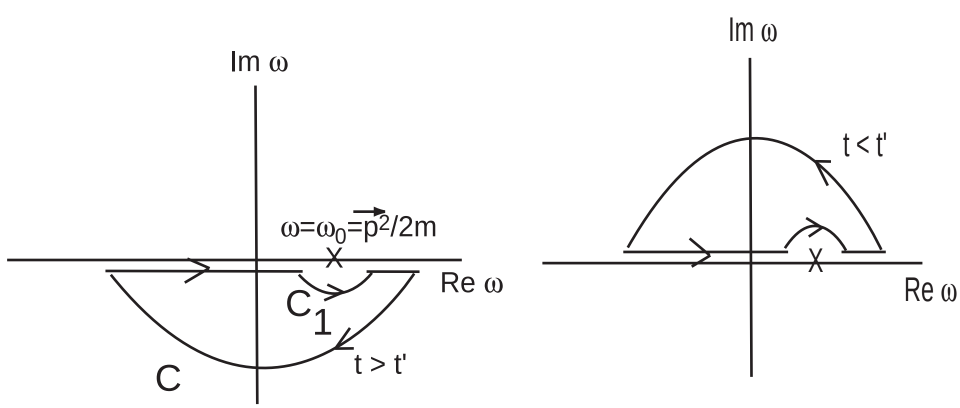
Solution Substituting for $G_0(p)$ into $G_0(x-x')$ we need to evaluate ($\hbar = 1$)

$$\begin{aligned} G_0(x-x') &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{\omega - \vec{p}^2/2m} e^{-ip\cdot(x-x')} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega - \vec{p}^2/2m} \end{aligned}$$

and we see that the integrand is singular at $\omega = \vec{p}^2/2m$. This simple pole is shown in the figure. We need to decide whether to integrate in the upper half plane (UHP) or the lower half plane (LHP). This is dictated by the *boundary conditions* as follows. Write $\omega \equiv \text{Re}\ \omega + i\ \text{Im}\ \omega$, so that

$$e^{-i\omega(t-t')} = e^{i\,\text{Re}\,\omega(t-t')} e^{+\,\text{Im}\,\omega(t-t')}$$

For $t - t' > 0$, the term $e^{\text{Im}\,\omega(t-t')}$ will blow up for Im $\omega > 0$ but will go to zero for Im $\omega < 0$. Thus the boundary condition $t-t' > 0$ dictates we integrate in the LHP. Similarly for $t-t' < 0$ we use the UHP. This is shown in the figure. [See also Halzen and Martin 148]



Now apply the Cauchy Residue Theorem, and remember *counterclockwise* integration is a *positive* sign. In the left figure the coutour does not enclose any poles so that

$$0 = \int_{-\infty}^{\omega_0-\delta} + \int_{C_1} + \int_{\omega_0+\delta}^{\infty} + \int_C$$

and with $\int\limits_C = 0$ (Jordan's lemma) and $\lim\limits_{\delta\to 0}$ we have

$$\int_{-\infty}^{\infty} = -\int_{C_1}$$

Thus for $t > t'$

$$\begin{aligned} G_0(x-x')_{t>t'} &= -+\pi i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{1}{2\pi} e^{-i\omega_0(t-t')} \\ &= -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\omega_0(t-t')} \end{aligned}$$

with $\omega_0 \equiv \vec{p}^2/2m$. Similarly for $t < t'$ we obtain

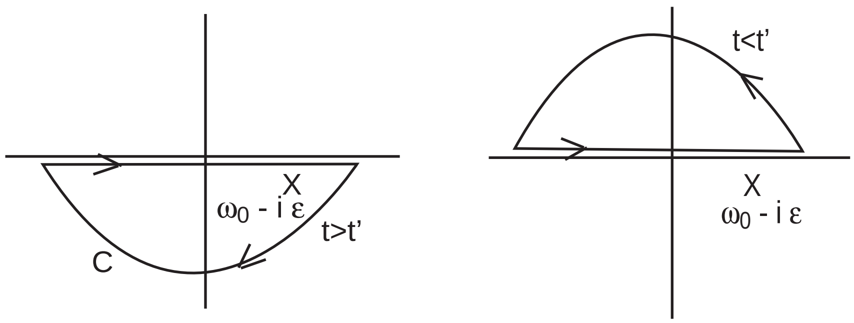
$$\begin{aligned} G_0(x-x')_{t<t'} &= --\pi i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{1}{2\pi} e^{-i\omega_0(t-t')} \\ &= +\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\omega_0(t-t')} \end{aligned}$$

But this *violates causality* ! If the wave pulse is sent out *at* t' we do expect a signal at a later time $t > t'$ but certainly not at an earlier time $t < t'$.

Because of the singularity, the integral is not well defined until we specify the limiting process. Let's now try a different method for evaluating the integral that *is* consistent with the boundary condition, i.e. we *want* $G_0(x-x')_{t<t'} = 0$.

Example Evaluate the position space Green function by shifting the pole off the real axis by a small amount ϵ, and take $\lim_{\epsilon \to 0}$.

Solution We can shift the pole either above or below the real axis. We choose to shift it below because then the upper contour will enclose *no* poles and will give a zero integral consistent with our boundary conditions. This is shown in the figure.



For $t > t'$ we have $\int_{-\infty}^{\infty} + \int_C = 2\pi i \Sigma$ Residues and with $\int_C = 0$ we get

$$\begin{aligned} G_0(x-x')_{t>t'} &= -2\pi i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega - \vec{p}^2/2m + i\epsilon} \\ &= -i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\omega_0(t-t')} \end{aligned}$$

with $\lim_{\epsilon \to 0} e^{\epsilon(t-t')} = 0$, which is exactly *double* our previous answer. For $t < t'$ we get

$$G_0(x-x')_{t<t'} = 0$$

because no poles are enclosed and this now obeys the boundary condition.

An excellent discussion of all of these issues can be found in [Arfken 4th ed. 417, 427; Landau 84-88; Bj RQM 85; Cushing 315; Greiner QED 27; Halzen and Martin 145-150]

The key integral that we have been considering is of the form

$$I = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega - \omega_0}$$

and with the pole shifted below the real axis this is

$$I(\epsilon) = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega - \omega_0 + i\epsilon}$$

and let's summarize our results as

$$\begin{aligned} I_{t>0} &= -\pi i \, e^{-i\omega_0 t} \\ I_{t<0} &= +\pi i \, e^{-i\omega_0 t} \end{aligned}$$

and

$$\begin{aligned} I(\epsilon)_{t>0} &= -2\pi \, i \, e^{-i\omega_0 t} \\ I(\epsilon)_{t<0} &= 0 \end{aligned}$$

When we leave the pole on the real axis the integral $\int\limits_{-\infty}^{\infty}$ is actually the Cauchy Principal Value defined as [Arfken 4th ed., p. 417; Landau 88]

$$\boxed{P \int_{-\infty}^{\infty} f(x) dx \equiv \lim_{\delta \to 0} \left[\int_{-\infty}^{x_0 - \delta} f(x) dx + \int_{x_0 + \delta}^{\infty} f(x) dx \right]}$$

with the pole at $x = x_0$. As discussed by [Arfken 417, 418] the Cauchy Principal value is actually a cancelling process. See also [Landau 85-88]. This is because in the vicinity of the simple pole we have

$$f(x) \approx \frac{a}{x - x_0}$$

which is odd relative to x_0 and so the large singular regions cancel out [Arfken 4th ed., 417, 418].

The Principal Value Prescription (i.e. our first method with the pole left *on* the axis) and the $i\epsilon$ Prescription are related. Consider [Merzbacher, old edition]

$$\frac{1}{\omega \pm i\epsilon} = \frac{\omega \mp i\epsilon}{\omega^2 + \epsilon^2} = \frac{\omega}{\omega^2 + \epsilon^2} \mp \frac{i\epsilon}{\omega^2 + \epsilon^2}$$

and using [Merzbacher 3rd ed., p. 631]

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \frac{\epsilon}{x^2 + \epsilon^2}$$

we have

$$\frac{1}{\omega \pm i\epsilon} = \frac{\omega}{\omega^2 + \epsilon^2} \mp i\pi\, \delta(\omega)$$

The first term on the right hand side becomes $\frac{1}{\omega}$ as $\epsilon \to 0$ except if $\omega = 0$. If $f(\omega)$ is a well behaved function we have [Merzbacher, old ed., p. 85]

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(\omega) \frac{\omega}{\omega^2 + \epsilon^2} d\omega &= \lim_{\epsilon \to 0} \Bigg[\int_{-\infty}^{-\epsilon} f(\omega) \frac{d\omega}{\omega} + \int_{\epsilon}^{\infty} f(\omega) \frac{d\omega}{\omega} \\
&\qquad + \int_{-\epsilon}^{\epsilon} f(\omega) \frac{\omega d\omega}{\omega^2 + \epsilon^2} \Bigg] \\
&= P \int_{-\infty}^{\infty} f(\omega) \frac{d\omega}{\omega} + f(0) \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{\omega d\omega}{\omega^2 + \epsilon^2} \\
&= P \int_{-\infty}^{\infty} f(\omega) \frac{d\omega}{\omega} + 0
\end{aligned}$$

where the last integral vanishes because the integrand is an odd function of ω. Thus we can write [Merzbacher, old ed., p. 85]

$$\boxed{\lim_{\epsilon \to 0} \frac{1}{\omega \pm i\epsilon} = P\frac{1}{\omega} \mp i\pi\, \delta(\omega)}$$

and we can also write [Cushing 315] (with the pole at $\omega = \omega_0$ instead of $\omega = 0$)

$$\boxed{P \int \frac{f(\omega) d\omega}{\omega - \omega_0} = \int \frac{f(\omega) d\omega}{\omega - \omega_0 \pm i\epsilon} \pm i\pi\, f(\omega_0)}$$

Example Verify the above formula for our previous integral
$I = \int d\omega \frac{e^{-i\omega t}}{\omega - \omega_0}$.

Solution For $t > 0$ we had

$$P \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0} = -\pi i\, e^{-i\omega_0 t}$$

and

$$\int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0 + i\epsilon} = -2\pi i\, e^{-i\omega_0 t}$$

Now $f(\omega_0) = e^{-i\omega_0 t}$, so that

$$\begin{aligned} \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0 + i\epsilon} + i\pi\, f(\omega_0) &= -2\pi i\, e^{-i\omega_0 t} + \pi i\, e^{-i\omega_0 t} \\ &= -\pi i\, e^{-i\omega_0 t} \\ &= P \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0} \end{aligned}$$

in agreement with the formula. For $t < 0$ we had

$$P \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0} = +\pi i\, e^{-i\omega_0 t}$$

and

$$\int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0 + i\epsilon} = 0$$

so that

$$\begin{aligned} \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0 + i\epsilon} + i\pi\, f(\omega_0) &= 0 + i\pi\, e^{-i\omega_0 t} \\ &= P \int \frac{e^{-i\omega t} d\omega}{\omega - \omega_0} \end{aligned}$$

also in agreement with the formula.

Example Show that the θ function defined as

$$\theta(t-t') = \begin{cases} 1 & \text{if} \quad t > t' \\ 0 & \text{if} \quad t < t' \end{cases}$$

can be written as

$$\boxed{\theta(t-t') = \lim_{\epsilon \to 0} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t')}}{\omega + i\epsilon}}$$

Solution We showed previously that with $I(\epsilon) \equiv \int\limits_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega - \omega_0 + i\epsilon}$ we get

$$\begin{aligned} I(\epsilon)_{t>0} &= -2\pi i \, e^{-i\omega_0 t} \\ I(\epsilon)_{t<0} &= 0 \end{aligned}$$

Replacing t with $t - t'$ and $\omega_0 = 0$ gives $I = \int d\omega \frac{e^{-i\omega(t-t')}}{\omega + i\epsilon}$ and

$$\begin{aligned} I_{t>t'} &= -2\pi i \\ I_{t<t'} &= 0 \end{aligned}$$

which verifies the above integral representation of θ.

Now let's finish the job of evaluating the position space Green function [Bj RQM 84, Greiner QED 27, Kaku 73-74]

Example Evaluate the position space Green function, consistent with the causality boundary condition, in terms of $\theta(t-t')$. Also write the answer in terms of the plane wave states

$$\phi_p(x) \equiv \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}$$

Solution To satisfy causality we use the $i\epsilon$ prescription to evaluate the integral which gave

$$G_0(x-x')_{t>t'} = -i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\omega_0(t-t')}$$

and

$$G_0(x-x')_{t<t'} = 0$$

with $\omega_0 \equiv \vec{p}^2/2m$. This can be written

$$G_0(x-x') = -i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')-i\omega_0(t-t')}\theta(t-t')$$

Using the plane wave states we have

$$\begin{aligned} \phi_p(x)\phi_p^*(x') &= \frac{1}{(2\pi)^3}e^{-ip\cdot x}e^{ip\cdot x'} = \frac{1}{(2\pi)^3}e^{-ip\cdot(x-x')} \\ &= \frac{1}{(2\pi)^3}e^{-i[\omega(t-t')-\vec{p}\cdot(\vec{x}-\vec{x}')]} \end{aligned}$$

giving

$$\boxed{G_0(x-x') = -i\theta(t-t')\int d^3p\,\phi_p(x)\phi_p^*(x')}$$

[Bj RQM 86, Kaku 74]

Finally the d^3p integrating gives (see Problems) [Bj RQM 86]

$$\boxed{G_0(x-x') = -i\left(\frac{m}{2\pi i(t-t')}\right)^{3/2} e^{\frac{im|\vec{x}-\vec{x}'|^2}{2(t-t')}}\theta(t-t')}$$

See footnote in [Bj RQM 86] discussing the Schrodinger equation and the Diffusion equation.

Klein-Gordon Propagator

We originally wrote the Klein-Gordon propagator $\Delta_F(k) = \frac{1}{k^2-m^2}$ but then digressed on a lengthy discussion of the Schrodinger propagator in order to illustrate the integration techniques. We now return to the Klein-Gordon propagator. Consider the following $i\epsilon$ prescription

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon}$$

Write this as

$$\frac{1}{k^2 - m^2 + i\epsilon} = \frac{1}{k_0^2 - \vec{k}^2 - m^2 + i\epsilon} = \frac{1}{k_0^2 - E^2 + i\epsilon} \quad \text{with} \quad E \equiv +\sqrt{\vec{k}^2 + m^2}$$

We can show (see Problems)

$$\frac{1}{k^2 - m^2 + i\epsilon} = \frac{1}{k_0^2 - E^2 + i\epsilon} = \frac{1}{2k_0}\left[\frac{1}{k_0 - E + i\epsilon} + \frac{1}{k_0 + E - i\epsilon}\right]$$

Recall that the momentum space propagator $\Delta_F(k)$ was defined in terms of the Fourier transform

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \Delta_F(k)$$

Performing the integration as before we get, with $d\tilde{\tilde{k}} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k}$ (see Problems)

$$\boxed{\begin{aligned} \Delta_F(x-x') &= -i\theta(t-t')\int d\tilde{\tilde{k}}\, e^{-ik\cdot(x-x')} - i\theta(t'-t)\int d\tilde{\tilde{k}}\, e^{ik\cdot(x-x')} \\ &= -i\theta(t-t')\int d\tilde{\tilde{k}}\, \phi_k(x)\phi_k^*(x') - i\theta(t'-t)\int d\tilde{\tilde{k}}\, \phi_k^*(x)\phi_k(x') \end{aligned}}$$

Example Show that $\boxed{i\Delta_F(x-x') = \langle 0|T\phi(x)\phi(x')|0\rangle}$
(Note this is also equal to

$$= \langle 0|T\phi(x)\phi^\dagger(x')|0\rangle$$

for Hermitian fields [BjRQF42].)

Solution The time ordered product is

$$\begin{aligned} TA(t_1)B(t_2) &\equiv \begin{cases} A(t_1)B(t_2) & \text{if} \quad t_1 > t_2 \\ B(t_2)A(t_1) & \text{if} \quad t_2 > t_1 \end{cases} \\ &\equiv \theta(t_1-t_2)A(t_1)B(t_2) + \theta(t_2-t_1)B(t_2)A(t_1) \end{aligned}$$

Thus for the fields

$$T\phi(x)\phi(x') = \theta(t-t')\phi(x)\phi(x') + \theta(t'-t)\phi(x')\phi(x)$$

or

$$\langle 0|T\phi(x)\phi(x')|0\rangle = \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

Thus we want to evaluate $\langle 0|\phi(x)\phi(x')|0\rangle$ and $\langle 0|\phi(x')\phi(x)|0\rangle$. Now

$$\phi(x) = \int d\tilde{k}\,(a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x})$$

with $d\tilde{k} \equiv \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}$ and $\omega \equiv \sqrt{\vec{k}^2 + m^2}$ and $\omega' \equiv \sqrt{\vec{k}'^2 + m^2}$. Evaluate

$$\begin{aligned} &\langle 0|\phi(x)\phi(x')|0\rangle \\ &= \frac{\langle 0|}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega}}\frac{d^3k'}{\sqrt{2\omega'}}(a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x})(a_{\vec{k}'}e^{-ik'\cdot x'} + a^\dagger_{\vec{k}'}e^{ik'\cdot x'})|0\rangle \\ &= \int d\tilde{k}\,d\tilde{k}'\,\langle 0|a_{\vec{k}}a_{\vec{k}'}e^{-i(k\cdot x+k'\cdot x')} + a_{\vec{k}}a^\dagger_{\vec{k}'}e^{-i(k\cdot x-k'\cdot x')} \\ &\qquad +a^\dagger_{\vec{k}}a_{\vec{k}'}e^{i(k\cdot x-k'\cdot x')} + a^\dagger_{\vec{k}}a^\dagger_{\vec{k}'}e^{i(k\cdot x+k'\cdot x')}|0\rangle \end{aligned}$$

However $a_{\vec{k}}|0\rangle = 0$ and $\langle 0|a^\dagger_{\vec{k}} = 0$. Thus the 1st, 3rd and 4th terms are zero. We are only left with

$$\langle 0|a_{\vec{k}}a^\dagger_{\vec{k}'}|0\rangle = \delta(\vec{k}-\vec{k}')$$

Continuing

$$\begin{aligned}\langle 0|\phi(x)\phi(x')|0\rangle &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega}}\frac{d^3k'}{\sqrt{2\omega'}}\delta(\vec{k}-\vec{k}')e^{-i(\omega t-\vec{k}\cdot\vec{x}-\omega' t'+\vec{k}'\cdot\vec{x}')}\\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{i\vec{k}.(\vec{x}-\vec{x}')-i\omega(t-t')}\\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{-ik\cdot(x-x')}\end{aligned}$$

Obviously we also have

$$\begin{aligned}\langle 0|\phi(x')\phi(x)|0\rangle &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{-ik\cdot(x'-x)}\\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{ik\cdot(x-x')}\end{aligned}$$

Thus

$$\begin{aligned}&\langle 0|T\phi(x)\phi(x')|0\rangle = \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle\\ &\quad= \theta(t-t')\int \frac{d^3k}{(2\pi)^3 2\omega}e^{-ik\cdot(x-x')} + \theta(t'-t)\int \frac{d^3k}{(2\pi)^3 2\omega}e^{ik\cdot(x-x')}\\ &\quad= i\Delta_F(x-x')\end{aligned}$$

Other Propagators

Recall that

$$\begin{aligned}\phi(x) &= \int d\tilde{k}[a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x}] \\ &\equiv \phi^+(x) + \phi^-(x) \quad \text{[Kaku 152; Mandl \& Shaw 44]}\end{aligned}$$

From this we can define other propagators (see Problems)

$$\boxed{i\Delta^\pm(x-y) \equiv [\phi^\pm(x), \phi^\mp(y)] = \pm\int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} e^{\mp ik\cdot(x-y)}}$$

where $\omega_{\vec{k}} \equiv +\sqrt{\vec{k}^2 + m^2}$ giving (see Problems)

$$\boxed{i\Delta(x-y) \equiv [\phi(x), \phi(y)] = i\Delta^+(x-y) + i\Delta^-(x-y)}$$

or just $\Delta(x) = \Delta^+(x) + \Delta^-(x)$. Thus one has [Mandl & Shaw 51]

$$\begin{aligned}\Delta(x) &= \Delta^+(x) + \Delta^-(x) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{\omega_{\vec{k}}} \sin k\cdot x \\ &= \frac{-i}{(2\pi)^3}\int d^4k\, \delta(k^2 - m^2)\epsilon(k_0)e^{-ik\cdot x}\end{aligned}$$

with $\epsilon(k_0) = \frac{k_0}{|k_0|} = \begin{cases} +1 & \text{for} \quad k_0 > 0 \\ -1 & \text{for} \quad k_0 < 0 \end{cases}$

Note also that $\Delta^-(x) = -\Delta^+(-x)$

These propagators are related to the Feynman propagator by [Mandl and Shaw 54]

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x)$$

or [Schwabl 283]

$$\Delta_F(x-x') = \theta(t-t')\Delta^+(x-x') - \theta(t'-t)\Delta^-(x-x')$$

which can be written [Schwabl 283, Mandl and Shaw 54]

$$\Delta_F(x) = \pm\Delta^\pm(x) \quad \text{for} \quad t \gtrless 0$$

Example Show that

$$\Delta_F(x-x') = \theta(t-t')\Delta^+(x-x') - \theta(t'-t)\Delta^-(x-x')$$

where

$$i\Delta_F(x-x') = \langle 0|T\phi(x)\phi(x')|0\rangle$$

and

$$i\Delta^\pm(x-y) \equiv [\phi^\pm(x), \phi^\mp(y)].$$

Solution

$$\begin{aligned} i\Delta_F(x-x') &= \langle 0|T\phi(x)\phi(x')|0\rangle \\ &= \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle \end{aligned}$$

Thus to do this problem we really need to show that

$$\langle 0|\phi(x)\phi(x')|0\rangle = [\phi^+(x), \phi^-(x')]$$

and

$$\langle 0|\phi(x')\phi(x)|0\rangle = -[\phi^-(x), \phi^+(x')]$$

Let's only do the first of these. The question is, how do these relations come about ? The answer is easy. Commutators are just c-numbers, i.e. classical functions or numbers. Thus $\langle 0|c|0\rangle = c\langle 0|0\rangle = c$ for any c-number. Therefore

$$\begin{aligned} [\phi^+(x), \phi^-(x')] &= \langle 0|[\phi^+(x), \phi^-(x')]|0\rangle \\ &= \langle 0|\phi^+(x)\phi^-(x')|0\rangle - \langle 0|\phi^-(x')\phi^+(x)|0\rangle \\ &= \langle 0|\phi^+(x)\phi^-(x')|0\rangle \end{aligned}$$

because $\phi^+|0\rangle = 0$.

Now $\phi = \phi^+ + \phi^-$. Thus

$$\phi|0\rangle = \phi^+|0\rangle + \phi^-|0\rangle = 0 + \phi^-|0\rangle = \phi^-|0\rangle$$

and similarly

$$\langle 0|\phi = \langle 0|\phi^+$$

giving

$$[\phi^+(x), \phi^-(x')] = \langle 0|\phi(x)\phi(x')|0\rangle$$

as required. (Proof of the other relation follows similarly).

3.7 Complex Klein-Gordon Field

[Schwabl 285; Kaku 69; Huang 24-25]

[W. Pauli and V. F. Weisskopf, Helv. Phys. Acta, **7**, 709 (1934)]

Charged particles cannot be described with a real scalar field because it only has one component, whereas a minimum of two components is needed to describe charge. Define

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \text{ and } \phi^\dagger \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

where ϕ_1 and ϕ_2 are real fields. The normalization factor $\frac{1}{\sqrt{2}}$ is chosen so that ϕ_i has the same renormalization as the real scalar field discussed before. A suitable classical Lagrangian is (with $|\phi|^2 = \phi\phi^\dagger$) [Huang 24]

$$\boxed{\mathcal{L} = \partial_\mu\phi^\dagger\, \partial^\mu\phi - m^2|\phi|^2}$$

$$= \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu\phi_i\, \partial^\mu\phi_i - m^2\phi_i^2)$$

which is just the sum of the Lagrangian for the real fields.

The Euler-Lagrange equations give the equations of motion [Schwabl 285]

$$(\Box^2 + m^2)\phi = 0 \text{ and } (\Box^2 + m^2)\phi^\dagger = 0$$

with the conjugate momenta being [Kaku 70; Schwabl 285]

$$\Pi = \dot{\phi}^\dagger \text{ and } \Pi^\dagger = \dot{\phi}$$

with the equal time commutation relations [Kaku 70; Schwabl 285]

$$[\phi(\vec{x},t), \Pi(\vec{x}',t)] = [\phi(\vec{x},t), \dot{\phi}^\dagger(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$$
$$[\phi^\dagger(\vec{x},t), \Pi^\dagger(\vec{x}',t)] = [\phi^\dagger(\vec{x},t), \dot{\phi}(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$$

and [Schwabl 285]

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\Pi(\vec{x},t), \Pi(\vec{x}',t)] = 0$$
$$[\phi^\dagger(\vec{x},t), \phi^\dagger(\vec{x}',t)] = [\Pi^\dagger(\vec{x},t), \Pi^\dagger(\vec{x}',t)] = 0$$

We Fourier expand *each* component $\phi_i(x)$ exactly as before [Kaku 70; Huang 24]

$$\phi_i(x) = \int d\tilde{k}\, (a_{i\vec{k}} e^{-ik\cdot x} + a^\dagger_{i\vec{k}} e^{ik\cdot x})$$

where the commutation relations are [Kaku 70; Huang 25]

$$[a_{i\vec{k}}, a^\dagger_{j\vec{k}'}] = \delta(\vec{k} - \vec{k}')\delta_{ij}$$

However rather than treating the i fields separately, we can combine them as we did with the wave functions $\phi(x)$. Define [Kaku 70; Huang 25]

$$a_{\vec{k}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{k}} + ia_{2\vec{k}}) \text{ and } b_{\vec{k}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{k}} - ia_{2\vec{k}})$$

so that the Fourier expansion now becomes [Huang 25; Schwabl 285]

$$\boxed{\phi(x) = \int d\tilde{k}\,(a_{\vec{k}}e^{-ik\cdot x} + b^\dagger_{\vec{k}}e^{ik\cdot x})}$$

(Exercise: Prove this result) and for the other field, obviously it is

$$\phi^\dagger(x) = \int d\tilde{k}(a^\dagger_{\vec{k}}e^{ik\cdot x} + b_{\vec{k}}e^{-ik\cdot x})$$

The new commutation relations are [Kaku 70; Schwabl 286; Huang 25]

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = [b_{\vec{k}}, b^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}$$

and

$$[a_{\vec{k}}, a_{\vec{k}'}] = [b_{\vec{k}}, b_{\vec{k}'}] = [a_{\vec{k}}, b_{\vec{k}'}] = [a_{\vec{k}}, b^\dagger_{\vec{k}'}] = 0$$

There are now two occupation-number operators, for particles a and for particles b [Schwabl 286]

$$N_{a\vec{k}} \equiv a^\dagger_{\vec{k}}a_{\vec{k}} \text{ and } N_{b\vec{k}} \equiv b^\dagger_{\vec{k}}b_{\vec{k}}$$

where $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ create and annihilate a particles and $b^\dagger_{\vec{k}}$ and $b_{\vec{k}}$ create and annihilate b particles. The vacuum state is defined by [Schwabl 286]

$$a_{\vec{k}}|0\rangle = b_{\vec{k}}|0\rangle = 0$$

The four momentum is [Schwabl 286; Mandl and Shaw 49; Greiner FQ 93]

$$P^\mu = (H, \vec{P}) = \sum_{\vec{k}} k^\mu(N_{a\vec{k}} + N_{b\vec{k}})$$

giving

$$H = \sum_{\vec{k}} \omega_{\vec{k}}(N_{a\vec{k}} + N_{b\vec{k}})$$

and

$$\vec{P} = \sum_{\vec{k}} \vec{k}(N_{a\vec{k}} + N_{b\vec{k}})$$

3.7.1 Charge and Complex Scalar Field

Recall the real scalar field Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ and the complex scalar field Lagrangian

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2|\phi|^2 = \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu\phi_i\partial^\mu\phi_i - m^2\phi_i^2)$$

with $|\phi|^2 \equiv \phi\phi^\dagger$. The complex scalar field Lagrangian (and action) is invariant under the transformation

$$\phi(x) \to e^{iq\theta}\phi(x) \qquad \phi^\dagger(x) \to e^{-iq\theta}\phi^\dagger(x)$$

which generates a $U(1)$ symmetry [Kaku 70]. Recall that before we had (for small $\alpha_i = \epsilon_i$)

$$\eta_r(x) \to \eta_r'(x) = e^{i\alpha_i X_i}\eta_r(x) \approx (1 + i\epsilon_i X_i)\eta_r(x)$$

giving

$$\delta\eta_r(x) = \eta_r'(x) - \eta_r(x) = i\epsilon_i X_i\eta_r(x)$$

Our two fields are $\eta_1 \equiv \phi$ and $\eta_2 \equiv \phi^\dagger$ giving (with small $\theta = \epsilon$)

$$\delta\phi(x) = iq\epsilon\phi(x) \quad \text{and} \quad \delta\phi^\dagger(x) = -iq\epsilon\phi^\dagger(x)$$

Recall the Noether current

$$j_\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\,\Delta\eta_r - T_{\mu\nu}\delta x^\nu$$

with

$$T_{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\partial_\nu\eta_r - g_{\mu\nu}\mathcal{L}$$

Also recall the relation between local and total variations

$$\delta\eta_r(x) = \Delta\eta_r(x) - \frac{\partial\eta_r}{\partial x_\mu}\delta x_\mu$$

For $\delta x^\nu = 0$, we have $\delta\eta(x) = \Delta\eta(x)$ [Schwabl 272] and therefore

$$\begin{aligned} j_\mu &= \frac{\partial\mathcal{L}}{\partial(\partial^\mu\eta_r)}\delta\eta_r \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^\dagger)}\delta\phi^\dagger \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}iq\epsilon\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^\dagger)}(-i)q\epsilon\phi^\dagger \end{aligned}$$

and with $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi\phi^\dagger = g_{\mu\nu}\partial^\nu \phi^\dagger \partial^\mu \phi - m^2 \phi\phi^\dagger$ gives

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = g_{\mu\nu}\partial^\nu \phi^\dagger = \partial_\mu \phi^\dagger$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^\dagger)} = \partial_\mu \phi$$

Thus the Noether current is

$$j_\mu = (\partial_\mu \phi^\dagger) iq\epsilon\phi - (\partial_\mu \phi) iq\epsilon\phi^\dagger$$

Once again we drop the constant factor ϵ and insert a minus sign to define a new current [Kaku 71, Mosel 19, Schwabl 286, Huang 25]

$$j_\mu = iq[(\partial_\mu \phi)\phi^\dagger - (\partial_\mu \phi^\dagger)\phi]$$

which agrees exactly with the 4-current derived previously!

The conserved charge ($\frac{dQ}{dt} = 0$) is therefore [Kaku 71, Mosel 19]

$$\begin{aligned} Q &= \int d^3x\, j_o = iq \int d^3x (\dot{\phi}\phi^\dagger - \dot{\phi}^\dagger \phi) \\ &= q \int d^3k (a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) \\ &= q \int d^3k (N_{a\vec{k}} - N_{b\vec{k}}) \end{aligned}$$

(Exercise: Prove this result). *Here j_o matches exactly our expression for ρ derived previously*! [Kaku 71] gives an excellent discussion of the physical interpretation of this charge. Also note that "charge" is used in a generic sense [Huang 26] since electromagnetic coupling has not yet been turned on.

Compare this to our result from Chapter 1 which was (with *probability* current density $j^\mu = (\rho, \vec{j}\,) = \bar{\psi}\gamma^\mu \psi$)

$$\begin{aligned} Q &= \int d^3x\, j^0 = q \int d^3x\, \bar{\psi}\ \gamma^0\ \psi \\ &= q \int d^3x\, \rho \end{aligned}$$

where the *probability* density was $\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$. This was an expression in terms of the classical Dirac field ψ. Above we now have Q in terms of the quantized field operators $a, a^\dagger, b, b^\dagger$ for the complex scalar field.

3.8 Summary

Two useful integrals are:

$$\frac{1}{(2\pi)^3}\int d^3x e^{\pm i(\vec{k}-\vec{k}').\vec{x}} = \delta(\vec{k}-\vec{k}')$$

$$\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

3.8.1 KG classical field

The massive Klein-Gordon Lagrangian is

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$$

giving the KGE (using $p^2 \to -\Box^2$)

$$(\Box^2 + m^2)\phi = 0$$
$$(p^2 - m^2)\phi = 0$$

The covariant momentum density is

$$\Pi^\mu = \partial^\mu\phi$$

giving *the* canonical momentum

$$\Pi \equiv \Pi^o = \dot{\phi}(x)$$

and Hamiltonian density

$$\mathcal{H} = \frac{1}{2}[\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2]$$

and momentum

$$\vec{P} = -\int d^3x\, \dot{\phi}(x)\vec{\nabla}\phi(x)$$

The Fourier expansion of the KG field is

$$\phi(\vec{x},t) \equiv \phi^+(x) + \phi^-(x) = \int d\tilde{k}\left[a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x}\right]$$

The conjugate momentum is

$$\begin{aligned}\Pi(\vec{x},t) = \dot{\phi}(\vec{x},t) &\equiv \Pi^+(x) + \Pi^-(x) \\ &= -i\int d\tilde{k}\,\omega\left[a_{\vec{k}}e^{-ik\cdot x} - a^\dagger_{\vec{k}}e^{ik\cdot x}\right]\end{aligned}$$

In the above equations (with $k^0 \equiv \omega = +\sqrt{\vec{k}^2 + m^2}$)

$$d\tilde{k} \equiv \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} = \sqrt{\frac{2\omega}{(2\pi)^3}}\, d^4k\delta(k^2 - m^2)\, \theta(k^o)$$

The integrals are cast into discrete form with the replacement

$$\int d^3k \to \Sigma_{\vec{k}}.$$

3.8.2 Klein-Gordon Quantum field

To develop the QFT we impose the *equal time* commutation relations

$$[\phi(\vec{x}, t), \Pi(\vec{x}', t)] = i\delta(\vec{x} - \vec{x}')$$

and

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\Pi(\vec{x}, t), \Pi(\vec{x}', t)] = 0$$

which imply

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}')$$

and

$$[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

Defining

$$N_{\vec{k}} \equiv a^\dagger_{\vec{k}} a_{\vec{k}}$$

the Hamiltonian and momentum can be re-written as

$$H = \int d^3k \left(N_{\vec{k}} + \frac{1}{2} \right) \omega$$

$$\vec{P} = \int d^3k \left(N_{\vec{k}} + \frac{1}{2} \right) \vec{k}$$

The vacuum state is defined via

$$a_{\vec{k}}|0\rangle = 0$$

and the many body states are

$$|\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle = \prod_i |n_{\vec{k}_i}\rangle$$

with

$$|n_{\vec{k}_i}\rangle = \frac{1}{\sqrt{n_{\vec{k}}!}} \left(a^\dagger_{\vec{k}} \right)^{n_{\vec{k}}} |0\rangle$$

$$H|\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle = \sum_{\vec{k}} n_k\, \epsilon(\vec{k}) |\cdots n_{\vec{k}_i} \cdots n_{\vec{k}_j} \cdots\rangle$$

where $\epsilon(\vec{k}) = \hbar\omega(\vec{k})$.

3.8.3 Propagator Theory

The non-free KGE is

$$(\Box^2 + m^2)\phi = J(x)$$

The Feynman propagator in position space is defined as

$$(\Box^2 + m^2)\Delta_F(x - y) \equiv -\delta^4(x - y)$$

so that the solution to the non-free KGE is

$$\phi(x) = \phi_0(x) - \int d^4y\, \Delta_F(x - y)J(y)$$

where $\phi_0(x)$ is the solution with $J = 0$. The momentum space Feynman propagator is defined through the Fourier transform

$$\Delta_F(x - y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \Delta_F(k)$$

The usual method of solution for propagators or Green functions is to solve for $\Delta_F(k)$ and then do a contour integral to get $\Delta_F(x - y)$ rather than solving for $\Delta_F(x - y)$ directly. We get

$$\Delta_F(k) = \frac{1}{k^2 - m^2}$$

which is the *inverse of the momentum operator in the free KGE*. This finally leads to

$$\Delta_F(x - x') = -i\theta(t - t') \int d\tilde{k}\, e^{-ik\cdot(x-x')} - i\theta(t' - t) \int d\tilde{k}\, e^{ik\cdot(x-x')}$$

One can also show that

$$\begin{aligned} i\Delta_F(x - x') &= \langle 0|T\phi(x)\phi(x')|0\rangle \\ &= \langle 0|T\phi(x)\phi^\dagger(x')|0\rangle \end{aligned}$$

where the second line is true for Hermitian fields [BjRQF42].

Other useful propagators are

$$i\Delta^\pm(x - y) \equiv [\phi^\pm(x), \phi^\mp(y)] = \pm \int d\tilde{\tilde{k}} e^{\mp ik\cdot(x-y)}$$

giving

$$i\Delta(x - y) \equiv [\phi(x), \phi(y)] = -i\Delta^+(x - y) + i\Delta^-(x - y)$$

with

$$\Delta(x) \quad = \quad \Delta^+(x) + \Delta^-(x) = -\int \frac{d^3k}{(2\pi)^3\omega_{\vec{k}}} \sin k \cdot x$$

These other propagators are related to the Feynman propagator by

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x)$$

or

$$\Delta_F(x) = \pm\Delta^\pm(x) \quad \text{for} \qquad t \gtrless 0$$

3.8.4 Complex KG field

Not summarized; see text.

3.9 References and Notes

Good references for this chapter are [**Teller**, Bergstrom and Goobar, Roman, Halzen and Martin, Muirhead, Leon, Goldstein].

Chapter 4

Dirac Field

We have seen that the KGE gives rise to *negative energies* and *non-positive definite probabilities* and for these reasons was discarded as a fundamental quantum equation. These problems arise because the KGE is *non-linear* in $\frac{\partial}{\partial t}$, unlike the SE. Dirac thus sought a relativistic quantum equation linear in $\frac{\partial}{\partial t}$, like the SE. We shall see that therefore Dirac was forced to invent a matrix equation.

The 'derivation' of the DE presented here follows [Griffiths, pg. 215]. Dirac was searching for an equation linear in E or $\frac{\partial}{\partial t}$. Instead of starting from

$$p^2 - m^2 = 0$$

his strategy was to factor this relation, which is easy if we only have p^0 (i.e. $\vec{p} = 0$), namely (with $p^0 = p_0$)

$$(p^0 - m)(p^0 + m) = 0$$

and obtain two first order equations

$$p^0 - m = 0 \quad \text{or} \quad p^0 + m = 0$$

However it's more difficult if $\vec{p}$ is included. Then we are looking for something of the form $p^2 - m^2 = p^\mu p_\mu - m^2 = (\beta^\mu p_\mu + m)(\gamma^\nu p_\nu - m)$ where β^μ and γ^ν are 8 coefficients to be determined. The RHS is

$$\beta^\mu \gamma^\nu p_\mu p_\nu + m(\gamma^\nu - \beta^\nu) p_\nu - m^2$$

For equality with LHS we don't want terms linear in p_ν, thus $\gamma^\nu = \beta^\nu$, leaving

$$p^2 = p^\mu p_\mu = \gamma^\mu \gamma^\nu p_\mu p_\nu$$

i.e. (with $(p^1)^2 = (p_1)^2$)

$$
\begin{aligned}
(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = \; & (\gamma^0)^2(p^0)^2 + (\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 \\
& +(\gamma^3)^2(p^3)^2 + (\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0p_1 \\
& +(\gamma^0\gamma^2 + \gamma^2\gamma^0)p_0p_2 + (\gamma^0\gamma^3 + \gamma^3\gamma^0)p_0p_3 \\
& +(\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 \\
& +(\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3
\end{aligned}
$$

Here's the problem; we could pick $\gamma^0 = 1$ and $\gamma^1 = \gamma^2 = \gamma^3 = i$ but we can't get rid of the 'cross terms'. At this point Dirac had a brilliant inspiration; what if the γ's are *matrices* instead of *numbers*? Since matrices don't commute we might be able to find a set such that

$$
(\gamma^0)^2 = 1 \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1
$$

$$
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \;\text{ for }\; \mu \neq \nu
$$

Or, more succinctly, $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$ which is called a *Clifford Algebra* where the curly brackets denote the anti-commutator $\{A, B\} \equiv AB + BA$. It turns out that this *can* be done, but the smallest set of matrices are 4×4. These are

$$
\boxed{\gamma^0 \equiv \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \vec{\gamma} \equiv \beta\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^\mu \equiv (\beta, \beta\vec{\alpha})}
$$

Let's introduce everything else for completeness

$$
\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

$$
\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{array}{c} {\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0 \\ \text{i.e. } {\gamma^0}^\dagger = \gamma^0,\; {\gamma^i}^\dagger = -\gamma^i \end{array}
$$

$$
\beta^{-1} = \beta \quad \text{and} \quad \gamma^5 = \gamma_5
$$

These conventions are used by the following authors: [Griffiths, Kaku, Halzen & Martin].

Thus the Dirac equation is $\boxed{(p - m)\psi = 0}$ with $\boxed{A \equiv \gamma^\mu A_\mu}$. In coordinate space (using $p^\mu \to i\partial^\mu$) it is $\boxed{(i\partial - m)\psi = 0}$. In non-covariant notation it is

$$
\boxed{\begin{array}{c} H\psi = i\hbar\dfrac{\partial\psi}{\partial t} \\ \text{with} \quad H \equiv \vec{\alpha}\cdot\vec{p} + \beta m \end{array}}
$$

The DE can be written in terms of 4-vectors $(\not{p} - m)\psi = 0$ and is therefore *manifestly covariant*.

4.1 Probability & Current

For the SE and KGE we used SE* and KGE* to derive the continuity equation. For *matrices* the generalization of complex conjugate ($*$) is Hermitian conjugate ($\dagger$) which is the transpose of the complex conjugate. The DE is

$$(i\not{\partial} - m)\psi = 0 = (i\gamma^\mu\partial_\mu - m)\psi$$

and DE† is (using $(AB)^\dagger = B^\dagger A^\dagger$)

$$\begin{aligned} &\psi^\dagger(i\not{\partial} - m)^\dagger = 0 \\ =\ &\psi^\dagger(-i\gamma^{\mu\dagger}\partial_\mu - m) = 0 \quad \text{where } \partial_\mu^\dagger = \partial_\mu \\ =\ &\psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\partial_\mu - m) = 0 \quad \text{using } \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \end{aligned}$$

We want to introduce the *Dirac adjoint* (ψ is a column matrix)

$$\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0} \quad (\bar{\psi} \text{ is a } \textit{row} \text{ matrix!})$$

Using $\gamma^0\gamma^0 = 1$ we get

$$\psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\partial_\mu - m\gamma^0\gamma^0) = 0$$

$$\Rightarrow \bar{\psi}(i\gamma^\mu\gamma^0\partial_\mu + m\gamma^0) = 0$$

and cancelling out γ^0 gives, the *Dirac adjoint equation*,

$$\boxed{\bar{\psi}(i\not{\partial} + m) = 0} \quad \Leftrightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

Some sloppy authors write this as $(i\not{\partial} + m)\bar{\psi} = 0$ but this *cannot* be because $\bar{\psi}$ is a row matrix! The notation $\overleftarrow{\not{\partial}}$ however means that $\not{\partial}$ operates on $\bar{\psi}$ to the left, i.e. $\boxed{\bar{\psi}\overleftarrow{\not{\partial}} \equiv (\partial_\mu\bar{\psi})\gamma^\mu}$.

It's very important not to get confused with this. The DE and DE† are explicitly

$$\boxed{\begin{array}{lcl} (i\not{\partial} - m)\psi = 0 & \Leftrightarrow & i\gamma^\mu\partial_\mu\psi - m\psi = 0 \\ \bar{\psi}(i\not{\partial} + m) = 0 & \Leftrightarrow & i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0 \end{array}}$$

Now let's derive the continuity equation. Multiply DE from the left by $\bar{\psi}$ and DE† from the right by ψ. Now one could get *very* confused writing

$$\bar{\psi}(i\not{\partial} - m)\psi = 0$$
$$\bar{\psi}(i\not{\partial} + m)\psi = 0$$

whereas what is *really* meant is [Halzen & Martin, pg. 103]

$$\bar{\psi}(i\gamma^\mu \partial_\mu \psi - m\psi) = 0$$
$$(i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi})\psi = 0$$

Adding these gives

$$\bar{\psi}\gamma^\mu \partial_\mu \psi + (\partial_\mu \bar{\psi})\gamma^\mu \psi = 0 = \partial_\mu(\bar{\psi}\gamma^\mu \psi)$$

giving

$$\boxed{j^\mu = (\bar{\psi}\gamma^\mu \psi)} \equiv (\rho, \vec{j})$$

or

$$\rho = \bar{\psi}\gamma^0 \psi = \psi^\dagger \psi = \sum_{i=1}^{4} |\psi_i|^2$$

which is now *positive definite*! The 3-current is

$$\vec{j} = \bar{\psi}\vec{\gamma}\psi$$

[see also Mosel 17, 34]

4.2 Bilinear Covariants

ψ can be written $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ and one can try to construct a scalar, such as $\psi^\dagger \psi = (\psi_1^* \ \psi_2^* \ \psi_3^* \ \psi_4^*) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ but this is *not* a Lorentz scalar (it's got *all* + signs).

Rather, define the Dirac adjoint

$$\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*)}$$

and $\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$ *is* a Lorentz scalar (see Bjorken and Drell]. One can prove that the following quantities transform as indicated:

$\bar{\psi}\psi$	scalar	(1 component)
$\bar{\psi}\gamma^5\psi$	pseudoscalar	(1 component)
$\bar{\psi}\gamma^\mu\psi$	vector	(4 component)
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector	(4 component)
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric 2nd rank tensor	(6 component)

where $\sigma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

4.3 Negative Energy and Antiparticles

4.3.1 Schrodinger Equation

The free particle SE is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}$$

which has solution

$$\begin{aligned}\psi(x,t) &= (C\cos kx + D\sin kx)\, e^{\frac{i}{\hbar}Et} \equiv \psi_E \\ &= \left(Ae^{ikx} + Be^{-ikx}\right) e^{\frac{i}{\hbar}Et} \text{ in 1-dimension.}\end{aligned}$$

Substituting gives

$$\frac{\hbar k^2}{2m}\psi(x,t) = -E\psi(x,t) \text{ or } \left(E + \frac{\hbar^2 k^2}{2m}\right)\psi = 0$$

yielding

$$E = -\frac{\hbar^2 k^2}{2m}$$

However it also has solution $\psi(x,t) = (Ae^{ikx} + Be^{-ikx})e^{-\frac{i}{\hbar}Et} \equiv \psi_{-E}$

Substituting gives

$$\frac{\hbar^2 k^2}{2m}\psi(x,t) = +E\psi(x,t) \text{ or } \left(E - \frac{\hbar^2 k^2}{2m}\right)\psi = 0$$

yielding

$$E = +\frac{\hbar^2 k^2}{2m}$$

ψ_E and ψ_{-E} are *different* solutions. The first one ψ_E corresponds to positive energy and the second one ψ_{-E} to negative energy. We are *free* to toss away one *solution* as unphysical and only keep ψ_E.

However for KGE and DE the *same* solution ψ gives *both* positive and negative energy. Of course we are *not* free to toss away one energy because it's in the solution. Whereas in the SE we got *two different* solutions for positive and negative energy. If we want to get rid of negative energy in the SE we toss away one *solution*. However in KGE and DE we always get both positive and negative energy for *all* solutions. The only way to toss away negative energy is to toss away all solutions; i.e. toss out the whole equation!

$\star$Also reason why Einstein did not reject SR; $E = \pm\sqrt{\vec{p}^2 + m^2}$ were *separate* solutions; not part of *same* solution.

4.3.2 Klein-Gordon Equation

The free particle KGE is

$$(\Box^2 + m^2)\phi = 0$$

with

$$\Box^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

A solution is

$$\phi = Ne^{ip\cdot x} = Ne^{i(Et - \vec{p}\cdot\vec{x})}$$

Substituting gives

$$(E^2 - \vec{p}^{\,2} + m^2)\phi = 0$$

which implies

$$\Rightarrow E^2 = \vec{p}^{\,2} + m^2$$

or

$$E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

Thus the *single* solution $\phi = Ne^{ip\cdot x}$ has *both* positive and negative energy solutions. Another solution is

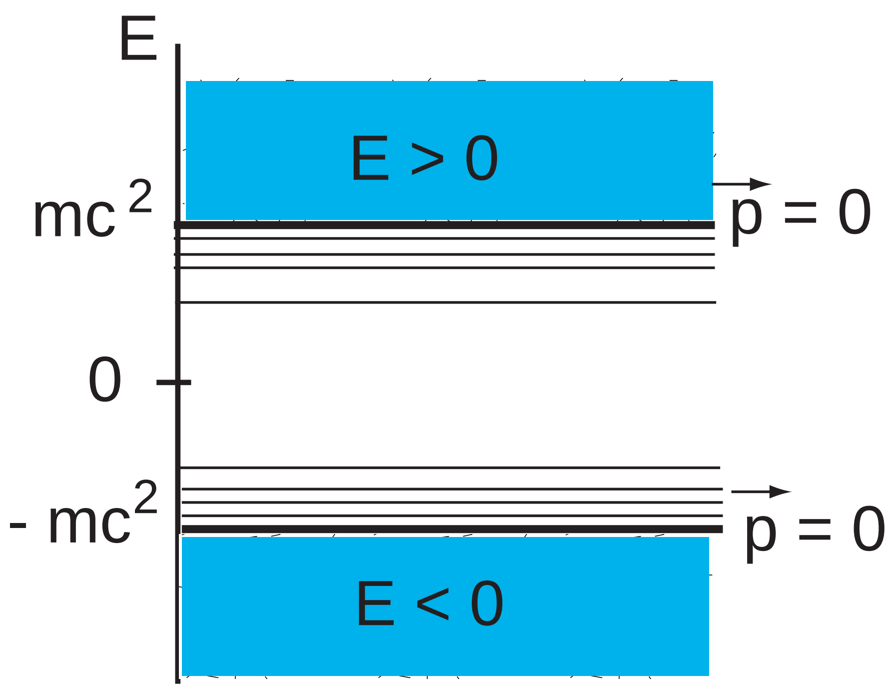
$$\phi = Ne^{-ip\cdot x} = Ne^{-i(Et - \vec{p}\cdot\vec{x})}$$

Substituting *also* gives the same as above, namely $(E^2 - \vec{p}^{\,2} + m^2)\phi = 0$ or

$$E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

and again the *single* solution $\phi = Ne^{-ip\cdot x}$ has *both* positive and negative energy solutions.

The *interpretation* of these states is as follows. For $\vec{p} = 0$ (particle at rest) then $E = \pm m$. For $\vec{p} \neq 0$, there will be a continuum of states above and below $E = \pm m$ [Landau, pg. 225], with bound states appearing in between. In QM there would be transitions to the negative energy continuum to infinite negative energy. This also happens with the Dirac equation. However, the DE describes, automatically, particles with spin. Dirac's *way out* of the negative energy catastrophe was to postulate that the negative energy sea was filled with fermions and so the ???



However for the KGE the negative energies are a catastrophe.

Also we can now clearly see the problem with ρ as calculated with the KGE. Recall $\rho = \frac{i\hbar}{2m}\left(\phi^* \frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$. For $\phi = e^{\pm ip\cdot x}$ we get $\boxed{\rho = \mp\frac{\hbar}{m}\phi^*\phi E}$ which gives *negative* ρ for positive or negative E. (Halzen & Martin, pg. 74]

4.3.3 Dirac Equation

Let's look for plane wave solutions of the form $\boxed{\psi(x) = w(\vec{p})e^{-ip\cdot x}}$. Substituting into $(i\partial\!\!\!/ - m)\psi = 0$ gives the momentum space DE $\boxed{(p\!\!\!/ - m)w = 0}$. Actually, let's first look at rest frame ($\vec{p} = 0$) solutions (RF). Writing the DE as

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{with } H = \vec{\alpha}\cdot\vec{p} + \beta m$$

independent DE is $H\psi = E\psi$, which in the RF is

$$Hw = \beta m w = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} w = Ew,$$

The eigenvalues are $E = m, m, -m, -m$ with eigenvectors [Halzen & Martin, pg. 104]

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus the DE has positive ($E = +m$) and negative ($E = -m$) energy solutions! (We shall look at $\vec{p} \neq 0$ solutions in a moment)

Let's summarize so far (Aitchison & Hey, pg. 71)

	Probability (ρ)	Energies for *same* solution ψ
SE	+	+
KGE	−	+, −
DE	+	+, −

Thus *both* the KGE and DE have negative energies.

The DE describes *fermions* (see next section). Dirac's idea was that the negative energy sea was filled with fermions and via PEP prevented the negative energy cascade of positive energy particles. Dirac realized if given particles in sea energy of $2mc^2$ create holes. Dirac Sea not taken seriously until positron discovered!! (1932 Anderson)

4.4 Free Particle Solutions of Dirac Equation

Before proceeding recall the following results:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} P_z & p_- \\ p_+ & -p_z \end{pmatrix} \equiv \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Let's now look at $\vec{p} \neq 0$ solutions (i.e. not in rest frame). As before, we look for solutions of the form $\psi(x) = w(\vec{p})e^{-ip\cdot x}$ Substituting into $(i\not\partial - m)\psi = 0$ gives the momentum space DE $(\not p - m)w = 0$. Using $\not p = \begin{pmatrix} E & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E \end{pmatrix}$ gives, with $w \equiv \begin{pmatrix} w_A \\ w_B \end{pmatrix}$

$$(\not p - m)w = \begin{pmatrix} E - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E - m \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix} = 0 = \begin{bmatrix} (E - m)\, w_A - \vec{\sigma} \cdot \vec{p}\, w_B \\ \vec{\sigma} \cdot \vec{p}\, w_A - (E + m)\, w_B \end{bmatrix}$$

giving

$$w_A = \frac{\vec{\sigma} \cdot \vec{p}}{E - m}\, w_B \quad \text{and} \quad w_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m}\, w_A$$

Combining yields

$$w_A = \frac{(\vec{\sigma} \cdot \vec{p}\,)^2}{E^2 - m^2} w_A = \frac{\vec{p}^{\,2}}{E^2 - m^2} w_A$$

because $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^{\,2}$. Thus

$$E^2 = \vec{p}^{\,2} + m^2 \quad \text{or} \quad E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

Thus again we see the *negative energy solutions* ! (this time for the $\vec{p} \neq 0$ DE)

Our final solutions are

$$w = \begin{pmatrix} w_A \\ w_B \end{pmatrix} = \begin{pmatrix} w_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} w_A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m} w_B \\ w_B \end{pmatrix}$$

with w_A and w_B left unspecified, which means we are free to choose them [Aitchison & Hey 69] as

$$w_A, w_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have the following possibilities

$$\text{Pick } w_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow w_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{E+m}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \quad (1) \quad p_\pm \equiv p_x \pm i p_y$$

$$\text{Pick } w_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow w_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{E+m}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \quad (2)$$

$$\text{Pick } w_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow w_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{E-m}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \quad (3)$$

$$\text{Pick } w_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow w_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{E-m}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \quad (4)$$

But the question is, do we use $E = +\sqrt{\vec{p}^{\,2}+m^2}$ or $E = -\sqrt{\vec{p}^{\,2}+m^2}$? Well, for (1) and (2) we *must* use $E = +\sqrt{\ }$ otherwise $\frac{1}{E+m}$ blows up for $\vec{p} = 0$. For (3) and (4) we *must* use $E = -\sqrt{\ }$ otherwise $\frac{1}{E-m}$ blows up for $\vec{p} = 0$. The term $E = +\sqrt{\ }$ is called the *particle* solution. The term $E = -\sqrt{\ }$ is called the *antiparticle* solution.

Rewrite as, with $\chi^{(1)} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^{(2)} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$w^{(1)}(\vec{p}) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi^{(1)}$$

$$w^{(2)}(\vec{p}) = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi^{(2)}$$

$$\underline{\text{with}} \quad E \equiv +\sqrt{\vec{p}^{\,2}+m^2}$$

and also

$$w^{(3)}(\vec{p}) = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_+}{E-m} \\ 1 \\ 0 \end{pmatrix} = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m} \\ 1 \end{pmatrix} \chi^{(1)}$$

$$w^{(4)}(\vec{p}) = N \begin{pmatrix} \frac{p_-}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m} \\ 1 \end{pmatrix} \chi^{(2)}$$

$$\underline{\text{with}} \quad E \equiv -\sqrt{\vec{p}^{\,2}+m^2}$$

or

$$\left.\begin{array}{l} w^{(s)}(\vec{p}) = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi^{(s)} \;\text{ with } E \equiv +\sqrt{\vec{p}^{\,2}+m^2} \\ \\ w^{(s+2)}(\vec{p}) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m} \\ 1 \end{pmatrix} \chi^{(s)} \;\text{ with } E \equiv -\sqrt{\vec{p}^{\,2}+m^2} \end{array}\right\} s = 1,2$$

[Halzen & Martin, pg. 105]

But all free particles carry *positive* energy! Thus re-interpret $w^{(3)}$ and $w^{(4)}$ as *positive energy antiparticle* states

$$w^{(s+2)}(\vec{p}) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{-\sqrt{\vec{p}^{\,2}+m^2}-m} \\ 1 \end{pmatrix} \chi^{(s)}$$

$$w^{(s+2)}(-\vec{p}) = N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{-\sqrt{\vec{p}^{\,2}+m^2}-m} \\ 1 \end{pmatrix} \chi^{(s)} = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{\vec{p}^{\,2}+m^2}+m} \\ 1 \end{pmatrix} \chi^{(s)}$$

$$\therefore \text{ Define } \begin{array}{ll} u(p,s) = u^{(s)}(p) \equiv w^{(s)}(\vec{p}) & u^{(1,2)}(p) = w^{(1,2)}(\vec{p}) \\ v(p,s) = v^{(s)}(p) \equiv w^{(s+2)}(-\vec{p}) & v^{(1,2)}(p) = w^{(3,4)}(-\vec{p}) \end{array}$$

(Me & Kaku)

everyone else: $v^{(1,2)}(p) = w^{(4,3)}(-\vec{p})$

Note: Bj writes $u(p,s)$ and $w(\vec{p})$. Gross writes $u(\vec{p},s)$

$$\left.\begin{array}{ll} u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} & u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \\ \\ v^{(2)} = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} & v^{(1)} = N \begin{pmatrix} \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \end{array}\right\} \text{all with } E \equiv +\sqrt{\vec{p}^{\,2}+m^2}$$

$= v^{(1)}$ (Kaku) $\quad = v^{(2)}$ (Kaku) $\quad$ Kaku different from everyone else

$$\boxed{u = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi \qquad v = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi}$$

$$\boxed{\begin{array}{l} u^{(1)}, v^{(2)} \to \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u^{(2)}, v^{(1)} \to \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}}$$

$$\text{Kaku } N = \sqrt{\frac{E+m}{2m}} \qquad \vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix}$$

Halzen & Martin, pg. 107

$$\begin{array}{ll} \psi(x) = w(\vec{p})e^{-ip\cdot x} & \psi(x) = w(-\vec{p})e^{-i(-p\cdot x)} \\ \Rightarrow \psi(x) = u(\vec{p})e^{-ip\cdot x} & \psi(x) = v(p)e^{+ip\cdot x} \\ (\not{p} - m)w(\vec{p}) = 0 & (-\not{p} - m)w(-\vec{p}) = 0 \\ \Rightarrow \boxed{(\not{p} - m)u = 0} & \Rightarrow \boxed{(\not{p} + m)v = 0} \end{array}$$

The adjoints satisfy

$$\bar{u}(\not{p} - m) = 0 \quad \text{and} \quad \bar{v}(\not{p} + m) = 0$$

Normalization [Muirhead, pg. 71ff; Griffiths, pg. 220 footnotes]

The most common conventions are, with $\boxed{E \equiv +\sqrt{\vec{p}^2 + m^2}}$ everywhere below

Bjorken & Drell, Kaku, Greiner, Sterman

$$u^\dagger u = \frac{E}{m} \Rightarrow N = \sqrt{\frac{E+m}{2m}} \text{ but spurious difficulties when } m \to 0$$

Griffiths, Halzen & Martin

$$u^\dagger u = 2E \Rightarrow N = \sqrt{E+m}$$

Bogoliubov & Shirkov

$$u^\dagger u = 1$$

These odd-looking normalizations come by specifying $\bar{u}u$. Using the result $\boxed{\bar{u}\gamma^\mu u = \frac{p^u}{m}\bar{u}u}$ (Problem 4.7) we have $u^\dagger \gamma^0 \gamma^\mu u = \frac{p^\mu}{m}\bar{u}u$ and with $\mu = 0$ and $\gamma^0\gamma^0 = 1$ we get

$$\boxed{u^\dagger u = \frac{E}{m}\bar{u}u}$$

Thus $\boxed{\bar{u}u = 1} \Rightarrow u^\dagger u = \frac{E}{m}$. Actualy the more general normalization, in this case, is $\boxed{\bar{u}^{(r)}u^{(s)} = \delta_{rs}}$ (see Problem 4.8).

Alternatively we can specify $\boxed{\bar{v}v = -1}$, i.e. $\boxed{\bar{v}^{(r)}v^{(s)} = -\delta_{rs}}$ [Kaku, pg. 754]

4.5 Classical Dirac Field

The Dirac Lagrangian is [Mosel 34]

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

from which one can obtain the Dirac equation and its adjoint (see Problems).

NNN - now handwriting

4.5.1 Noether spacetime current

4.5.2 Noether internal symmetry and charge

4.5.3 Fourier expansion and momentum space

References: [Mosel 35, Greiner FQ 123, Peskin 52, Huang 123, Schwabl 290, Kaku 86]

We now wish to expand the Dirac field in terms of creation and annihilation operators. Most books just write down the answer (as we shall) but [Greiner FQ 123] *derives* the result very clearly using the same method that we used for the Klein-Gordon field where one *proves* that the expansion must contain two terms.

The best discussion as to why the Dirac creation and annihilation operators must obey *anticommutation relations* is given in [Peskin 52-56] and [Greiner FQ 129].

Because of the normalization of our Dirac spinors (see previous chapter) we will have a different normalization constant in our Fourier expansion of the Dirac field, as compared to the KG case. Because of this the measure for *fermions* will be [IZ 114, 147, 703]

NNN - now handwriting

4.6 Dirac QFT

4.6.1 Derivation of $b, b^\dagger, d, d^\dagger$ Anticommutators

4.7 Pauli Exclusion Principle

4.8 Hamiltonian, Momentum and Charge in terms of creation and annihilation operators

4.8.1 Hamiltonian

4.8.2 Momentum

4.8.3 Angular Momentum

4.8.4 Charge

4.9 Propagator theory

4.10 Summary

4.10.1 Dirac equation summary

4.10.2 Classical Dirac field

4.10.3 Dirac QFT

4.10.4 Propagator theory

4.11 References and Notes

Mandl & Shaw, Teller, Sakurai QM, Leon, Merzbacher

S matrix and G function Bj RQM 83,97,100

S matrix without SE, Bj RQF 177

Chapter 5

Electromagnetic Field

5.1 Review of Classical Electrodynamics

5.1.1 Maxwell equations in tensor notation

5.1.2 Gauge theory

5.1.3 Coulomb Gauge

5.1.4 Lagrangian for EM field

5.1.5 Polarization vectors

References: [Weidner and Sells, 1st ed.,p.975; GriffithsEM, 1st ed., p. 350; Jackson, 2nd ed. p. 274; Schwabl 313; Mandl and Shaw 129; Guidry 86,87; GreinerFQ 161, 177]. Note that Guidry and Schwabl are excellent.

In general transverse waves can be linearly or circularly polarized. A nice elementary discussion can be found in [Weidner and Sells, 1st ed.,p.975; GriffithsEM, 1st ed., p. 350]. A circularly polarized wave can be made out of two linearly polarized waves if the two linear waves are out of phase. (If they are in phase then they combine to form a linearly polarized wave at a different angle.) To produce a linearyly polarized wave on a rope just jiggle the rope up and down for a linearly polarized wave in the vertical direction. The circularly polarized wave can be produced by moving one's hand, which is holding the rope, in a circular fashion. One can rotate clockwise or counterclockwise to produce circularly polarized states of opposie helicity.

An elliptically polarized wave is formed from two linearly polarized waves out of phase but with each linearly polarized wave having a differnt amplitude. The amplitudes are the same for a circular wave.

Just as the elliptically or circularly polarized wave can be created from two linear polarized waves (linear combination of them), similarly a linearly polarized wave can be built from a linear combination of two circularly polarized wave. Thus any wave in general can be represented in terms of linearly polarized basis states or circularly polarized basis states. This is discussed from a mathematical point of view in [Jackson, 2nd ed. p. 274]. Even though it should be obvious, note that the treatment of polarization vectors from the point of view of classical electrodynamics [Jackson] is the *same* as the quantum field treatments. That is, the polarization vectors used are the *same* in the classical and quantum case.

In general the linear polarization basis states are described by *real* polarization vectors $\vec{\epsilon}_1$ and $v\epsilon_2$. (As we shall see below there are only two, not four, possible states for photons.) The circularly polarized states are described by *complex* basis vectors, say written as $\vec{\epsilon}_+$ and $\vec{\epsilon}_-$.

These states are written in terms of each other

$$\vec{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\vec{\epsilon}_1 \pm i\epsilon_2)$$

The circularly polarized states are complex, which is an easy way of allowing for a phase difference between the two linearly polarized states out of which the circular states are constructed. [Jackson, 2nd ed. p. 274]

Finally, in describing our quantum field states for the photon it does not matter whether we use the linearly polarized basis states or the circularly polarized ones. Different authors choose different basis states. We will know what states a particlar author is using by simply recognizing wether the states are real (linear polarization) or complex (circular polarization).

5.1.6 Linear polarization vectors in Coulomb gauge

5.1.7 Circular polarization vectors

5.1.8 Fourier expansion

5.2 Quantized Maxwell field

5.2.1 Creation & annihilation operators

5.3 Photon propagator

5.4 Gupta-Bleuler quantization

5.5 Proca field

Chapter 6

S-matrix, cross section & Wick's theorem

Our Feynman diagram series is going to come from an expansion of the so-called Scattering Matrix (or S-matrix). Each term in this S-matrix expansion will represent a particular Feynman diagram. The S-matrix expansion looks very similar to something which occurs in non-relativistic quantum mechanics, namely the expansion of the time evolution operator. Thus we will study that first.

6.1 Schrodinger Time Evolution Operator

Recall the time-dependent Schrodinger equation (SE)

$$H|\alpha(t)\rangle = i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle$$

Define the *time evolution operator* [Mandl & Shaw, pg. 101, Leon, pg. 63, Sakurai]

$$|\alpha(t)\rangle \equiv \mathcal{U}(t,t_0)|\alpha(t_0)\rangle$$

and upon substitution into the SE gives

$$H\mathcal{U}(t,t_0)|\alpha(t_0)\rangle = i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0)|\alpha(t_0)\rangle$$

but $\frac{\partial}{\partial t}$ does not operate on $|\alpha(t_0)\rangle$ because t_0 is fixed. Thus we can 'cancel' $|\alpha(t_0)\rangle$ to arrive at the operator equation

$$H\mathcal{U}(t,t_0) = i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0)$$

A *schematic solution* is

$$\boxed{\mathcal{U}(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}}$$

However let's solve the equation more rigorously. We solve for $\mathcal{U}(t,t_0)$ by integrating both sides to give

$$\mathcal{U}(t,t_0) - \mathcal{U}(t_0,t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} H(t_1)\mathcal{U}(t_1,t_0)dt_1$$

and with the boundary condition

$$\mathcal{U}(t_0,t_0) = 1$$

we have

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\; H(t_1)\mathcal{U}(t_1,t_0)$$

This is solved by iteration as follows

$$\begin{aligned}
\mathcal{U}(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\; H(t_1)\mathcal{U}(t_1,t_0)\\
&= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\; H(t_1)\left[1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2\; H(t_2)\mathcal{U}(t_2,t_0)\right]\\
&= 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt_1\; H(t_1) + \left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\; H(t_1)H(t_2)\mathcal{U}(t_2,t_0)\\
&= 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt_1\; H(t_1) + \left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\; H(t_1)H(t_2)\times\\
&\qquad\qquad\times\left[1 - \frac{i}{\hbar}\int_{t_0}^{t_2} dt_3\; H(t_3)\mathcal{U}(t_3,t_0)\right]\\
&= 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt_1\; H(t_1) + \left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\; H(t_1)H(t_2)\\
&\qquad + \left(\frac{-i}{\hbar}\right)^3\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\; H(t_1)H(t_2)H(t_3) + \cdots
\end{aligned}$$

or

$$\boxed{\mathcal{U}(t,t_0) = 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\; H(t_1)\cdots H(t_n)}$$

which is the first version of our expansion of the time evolution operator. Two other ways of writing this are

$$\boxed{\begin{aligned}\mathcal{U}(t,t_0) &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n\; T[H(t_1)\cdots H(t_n)] \\ &\equiv T\; e^{-i/\hbar\int_{t_0}^{t}H(t')dt'}\end{aligned}}$$

[Leon, pg. 64; Merzbacher, pg. 475] where the last expression is just a *formal* way of writing the formula above [Leon, pg. 64] and recall

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = 1+\sum_{n=1}^{\infty}\frac{1}{n!}x^n$$

. Hatfield, pg. 52-53 gives a good example as to *why* we introduce time ordered product. See also Guidry, pg. 99.

6.1.1 Time Ordered Product

In the above formulae we introduced the time ordered product which we now define for two terms

$$\begin{aligned}\begin{matrix}T[A(t_1)B(t_2)]\\ (2!\ \text{combinations})\end{matrix} &\equiv \begin{cases}A(t_1)B(t_2) & \text{if}\quad t_1>t_2\\ B(t_2)A(t_1) & \text{if}\quad t_2>t_1\end{cases}\\ &\equiv \theta(t_1-t_2)A(t_1)B(t_2)+\theta(t_2-t_1)B(t_2)A(t_1)\end{aligned}$$

where

$$\theta(x-y)\equiv\begin{cases}1 & \text{if}\quad x>y\\ 0 & \text{if}\quad x<y\end{cases}$$

The time ordered product of three terms is defined as

$$\begin{matrix}T[A(t_1)B(t_2)C(t_3)]\\ (3!\ \text{combinations})\end{matrix} \equiv \begin{cases}A(t_1)B(t_2)C(t_3) & \text{if}\quad t_1>t_2>t_3\\ A(t_1)C(t_3)B(t_2) & \text{if}\quad t_1>t_3>t_2\\ B(t_2)A(t_1)C(t_3) & \text{if}\quad t_2>t_1>t_3\\ B(t_2)C(t_3)A(t_1) & \text{if}\quad t_2>t_3>t_1\\ C(t_3)A(t_1)B(t_2) & \text{if}\quad t_3>t_1>t_2\\ C(t_3)B(t_2)A(t_1) & \text{if}\quad t_3>t_2>t_1\end{cases}$$

$$\begin{aligned}&\equiv \theta(t_1-t_2)\theta(t_2-t_3)A(t_1)B(t_2)C(t_2)+\theta(t_1-t_3)\theta(t_3-t_2)A(t_1)C(t_3)B(t_2)\\ &\quad+\theta(t_2-t_1)\theta(t_1-t_3)B(t_2)A(t_1)C(t_3)+\theta(t_2-t_3)\theta(t_3-t_1)B(t_2)C(t_3)A(t_1)\\ &\quad+\theta(t_3-t_1)\theta(t_1-t_2)C(t_3)A(t_1)B(t_2)+\theta(t_3-t_2)\theta(t_2-t_1)C(t_3)B(t_2)A(t_1)\end{aligned}$$

Notice that there are $n!$ combinations for n operators. This explains the $\frac{1}{n!}$ term appearing in the expansion for $\mathcal{U}$.

[do Problem 8.1]

6.2 Schrodinger, Heisenberg and Dirac (Interaction) Pictures

Reference: [Mandl and Shaw, pg. 22]

One can do classical mechanics with either the Newtonian, Lagrange or Hamilton formulation of mechanics. In each formulation the *equations are different*. Although not exactly analogous, there are three popular ways to work in quantum mechanics known as the Schrodinger, Heisenberg or Dirac pictures. The Dirac picture is often also called the Interaction picture.

In the usual formulation of quantum mechanics via the Schrodinger equation, i.e. in the Schrodinger picture, the *operators are frozen in time and the states evolve in time*. The opposite is true in the Heisenberg picture.

In the Interaction picture, which we shall use extensively, the Hamiltonian is split into a fre particle piece and an interaction piece

$$H \equiv H_0 + H_I$$

then H_I evolves and H_0 is frozen.

Picture	Operators	State Vectors
Schrodinger	Frozen	Evolve
Heisenberg	Evolve	Frozen
Dirac (Interaction)	Evolve	Evolve

In time dependent perturbation theory one considers potentials like $\mathcal{U} = E_0 \cos \omega t$ (e.g. oscillating electromagnetic field) in the Schrodinger picture (SP). $\mathcal{U}$ is part of H and so what does it mean to say that operators are frozen in the SP? What we mean is that the state vectors $|\alpha\rangle$ obey an equation of motion $H|\alpha\rangle = i\hbar\frac{\partial}{\partial t}|\alpha\rangle$ and the operators do not. Vice-versa for the Heisenberg picture.

We shall label Schrodinger, Heisenberg, Interaction picture states as $|\alpha\rangle_S$, $|\alpha\rangle_H$, $|\alpha\rangle_I$ and operators as O^S, O^H, O^I.

Recall the Schrodinger picture of NRQM

$$H|\alpha(t)\rangle_S = i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle_S$$

and the Schrodinger state vectors evolve in time according to

$$|\alpha(t)\rangle_S \equiv \mathcal{U}(t, t_0)|\alpha(t_0)\rangle_S$$

where $\mathcal{U}(t, t_0)$ is our time evolution operator.

Define Heisenberg picture states as

$$|\alpha\rangle_H \equiv \mathcal{U}^\dagger|\alpha(t)\rangle_S = |\alpha(t_0)\rangle_S$$

which is clearly frozen in time. (We have used $\mathcal{U}\mathcal{U}^\dagger = 1$). *Define* Heisenberg operators

$$O^H(t) \equiv \mathcal{U}^\dagger O^S \mathcal{U}$$

which clearly evolves in time because O^S is frozen but $\mathcal{U}^\dagger$ and $\mathcal{U}$ carry time dependence.

Now the important thing about the Heisenberg and Schrodinger pictures is that *expectation values remain the same* in both pictures. We *must* have this for the physics to be the same. Expectation values are the same, i.e.

$${}_S\langle\beta(t)|O^S|\alpha(t)\rangle_S = {}_H\langle\beta|O^H(t)|\alpha\rangle_H$$

[do Problem 8.2]

6.2.1 Heisenberg Equation

In the Schrodinger representation the state vectors evolve in time and the Schrodinger equation describes their time evolution. In the Heisenberg picture the operators evolve in time, so what is the equation governing the operator time evolution? It is called the Heisenberg equation of motion (for operators). It is obtained by differentiating $O^H(t) \equiv \mathcal{U}^\dagger O^S \mathcal{U}$ to give

$$\boxed{[O^H(t), H] = i\hbar \frac{d}{dt} O^H(t)}$$

[do Problems 8.3 and 8.4]

6.2.2 Interaction Picture

We use the Dirac or Interaction representation if the Hamiltonian can be split into two parts; a free particle piece H_0 and an interaction piece H_I. (Unfortunately we have the notation H_I^I for the interaction Hamiltonian in the Interaction picture.) Thus

$$H = H_0 + H_I$$

where, at this stage, H_I means interaction Hamiltonian, not Interaction picture.

Define a time evolution operator for H_0 alone as

$$\mathcal{U}_0 \equiv \mathcal{U}_0(t, t_0) \equiv e^{-\frac{i}{\hbar} H_0 (t - t_0)}$$

and

$$|\alpha(t)\rangle_I \equiv \mathcal{U}_0^\dagger |\alpha(t)\rangle_S$$

and

$$O^I(t) \equiv \mathcal{U}_0^\dagger O^S \mathcal{U}_0$$

Now

$$H_0^I = H_0^S \equiv H_0$$

(exercise: show this)

Manipulating the previous two equations we get

$$[O^I(t), H_0] = i\hbar \frac{d}{dt} O^I(t)$$

and

$$H_I^I(t)|\alpha(t)\rangle_I = i\hbar \frac{d}{dt}|\alpha(t)\rangle_I$$

where

$$H_I^I(t) = \mathcal{U}_0^\dagger H_I^S \mathcal{U}_0 = e^{\frac{i}{\hbar} H_0 (t - t_0)} H_I^S e^{-\frac{i}{\hbar} H_0 (t - t_0)}$$

[do Problems 8.5, 8.6, 8.7]

Now define a time evolution operator for H_I alone as

$$|\alpha(t)\rangle_I \equiv \mathcal{U}_I |\alpha(t_0\rangle_I$$

which leads to

$$\boxed{H_I^I \mathcal{U}_I = i\hbar \frac{d}{dt} \mathcal{U}_I} \tag{6.1}$$

which has the rigorous solution

$$\mathcal{U}_I(t, t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, T[H_I(t_1) \cdots H_I(t_n)]$$

Now define the S-matrix

$$\boxed{S \equiv \mathcal{U}(\infty, -\infty)}$$

in other words

$$|\alpha(\infty)\rangle \equiv S|\alpha(-\infty\rangle = S|i\rangle$$

where $|i\rangle$ is the initial state. Thus

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \, T[H_I(t_1) \cdots H_I(t_n)$$

or in *covariant* form

$$\boxed{S = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int d^4x_1 d^4x_2 \cdots d^4x_n \, T[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2) \cdots \mathcal{H}_I(x_n)]}$$

which is the *Dyson expansion of the S-matrix*. This is an infinite series, each term of which gets represented as a Feynman diagram.

Note that this expansion of the S-matrix does *not* rely on the Schrodinger equation but comes from the general interaction picture and the operator equation of motion (6.1), which, although reminiscent of, is *not* the Schrodinger equation but rather the operator equation of motion in the Interaction picture [Bj RQF 177].

6.3 Cross section and S-matrix

Consider the reaction

$$1 + 2 \to 1' + 2' + 3' + \ldots n'$$

We shall be calculating an S-matrix element, but it always has common factors. Pulling these out we are left with a quantity $\mathcal{M}$ called the invariant amplitude. A set of Feynman rules actually gives $-i\mathcal{M}$ [Griffiths]. Cross sections are writtem directly in terms of $\mathcal{M}$. The relation between the S-matrix element and the invariant amplitude is given by [Greiner FQ 267, Greiner QED 221]

$$< f|S|i >= i(2\pi)^4 \, \delta^4(p_1 + p_2 - \sum_{i=1}^{n} p_i' \,) \, \mathcal{M} \, \Pi_{i=1}^2 \sqrt{\frac{N_i}{2E_i(2\pi)^3}} \, \Pi_{i=1}^n \sqrt{\frac{N_i'}{2E_i'(2\pi)^3}}$$

where the normalization factors are $N_i = 1$ for scalar bosons and photons and $N_i = 2m$ for fermions. [Greiner FQ 267]

A set of Feynman rules actually gives $-i\mathcal{M}$ [Griffiths, 1987]. The differential cross section is given directly in terms of the invariant amplitude as [RPP, pg. 176; PPB, pg. 219; Griffiths, pg. 198]

$$d\sigma = \frac{(2\pi)^4|\mathcal{M}|^2}{4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}} d\Phi_n(p_1 + p_2; p_3 \ldots p_{n+2})$$

with [RPP, pg. 175; PPB, pg. 215; Griffiths, pg. 198]

$$d\Phi_n(P; p_1 \ldots p_n) \equiv \delta^4\left(P - \sum_{i=1}^{n} p_i\right) \sum_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2E_i}$$

Thus for the reaction

$$1 + 2 \rightarrow 3 + 4 \cdots n$$

we have [Griffiths, pg. 198]

$$\begin{aligned} d\Phi_n(p_1 + p_2; p_3 \cdots p_n) = \;& \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots p_n) \times \\ & \times \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \cdots \frac{d^3p_n}{(2\pi)^3 2E_n} \end{aligned}$$

giving [Griffiths, pg. 198]

$$\begin{aligned} d\sigma = \;& \frac{(2\pi)^4|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \\ & \cdots \frac{d^3p_n}{(2\pi)^3 2E_n} \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots p_n) \end{aligned}$$

For the reaction

$$1 + 2 \rightarrow 3 + 4$$

$$d\sigma = \frac{(2\pi)^4|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \delta^4(p_1 + p_2 - p_3 - p_4)$$

with [Griffiths, pg. 200]

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

giving

$$\frac{d\sigma}{d^3p_4} = \frac{(2\pi)^4|\mathcal{M}|^2}{4\sqrt{}} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{1}{(2\pi)^3 2E_4} \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - p_4)$$

so that

$$\int \frac{d\sigma}{d^3p_4} d^3p_4 = d\sigma = \frac{2\pi|\mathcal{M}|^2}{8E_4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \frac{d^3p_3}{(2\pi)^3 2E_3} \delta(E_1 + E_2 - E_3 - E_4)$$

giving the Lorentz invariant differential cross section for production of particle 3 as

$$\boxed{\frac{d\sigma}{d^3p_3/E_3} = \frac{|\mathcal{M}|^2}{64\pi^2 E_4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \delta(E_1 + E_2 - E_3 - E_4)}$$

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6.4 Wick's theorem

6.4.1 Contraction

6.4.2 Statement of Wick's theorem

References and Notes
Mandl & Shaw, Teller, Sakurai QM, Leon, Merzbacher

S matrix and G function Bj RQM 83,97,100

S matrix without SE, Bj RQF 177
For a derivation of the S-matrix based on Green function techniques, and in- and out-states, see Bjorken and Dell, RQF 177; RQM 83, 97, 100.

Chapter 7

QED

7.1 QED Lagrangian

7.2 QED S-matrix

7.2.1 First order S-matrix

7.2.2 Second order S-matrix

From the Dyson expansion, the second order term in the S-matrix is (with $\hbar = 1$) [GreinerFQ 238]

$$
\begin{aligned}
S^{(2)} &= \frac{1}{2!}(-i)^2 \int d^4x_1 d^4x_2 T[\mathcal{H}(x_1)\mathcal{H}(x_2)] \\
&= \frac{1}{2!}(-iq)^2 \int d^4x_1 d^4x_2 T[\,:\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1):\;:\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2):\,] \\
&= \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\bar{\psi}(x_1)\gamma_\mu\psi(x_1)\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\mu(x_1)A^\nu(x_2): \qquad (a) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\bar{\psi}(x_1)\gamma_\mu\underline{\psi}(x_1)\underline{\bar{\psi}}(x_2)\gamma_\nu\psi(x_2)A^\mu(x_1)A^\nu(x_2): \qquad (b) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\underline{\bar{\psi}}(x_1)\gamma_\mu\psi(x_1)\bar{\psi}(x_2)\gamma_\nu\underline{\psi}(x_2)A^\mu(x_1)A^\nu(x_2): \qquad (c) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\bar{\psi}(x_1)\gamma_\mu\psi(x_1)\bar{\psi}(x_2)\gamma_\nu\psi(x_2)\underline{A}^\mu(x_1)\underline{A}^\nu(x_2): \qquad (d) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\bar{\psi}(x_1)\gamma_\mu\underline{\psi}(x_1)\underline{\bar{\psi}}(x_2)\gamma_\nu\psi(x_2)\underline{\underline{A}}^\mu(x_1)\underline{\underline{A}}^\nu(x_2): \qquad (e) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\underline{\bar{\psi}}(x_1)\gamma_\mu\psi(x_1)\bar{\psi}(x_2)\gamma_\nu\underline{\psi}(x_2)\underline{\underline{A}}^\mu(x_1)\underline{\underline{A}}^\nu(x_2): \qquad (f) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\underline{\bar{\psi}}(x_1)\gamma_\mu\underline{\underline{\psi}}(x_1)\underline{\underline{\bar{\psi}}}(x_2)\gamma_\nu\underline{\psi}(x_2)A^\mu(x_1)A^\nu(x_2): \qquad (g) \\
&+ \frac{(-iq)^2}{2!}\int d^4x_1 d^4x_2 \;:\underline{\bar{\psi}}(x_1)\gamma_\mu\underline{\underline{\psi}}(x_1)\underline{\underline{\bar{\psi}}}(x_2)\gamma_\nu\underline{\psi}(x_2)\underline{\underline{\underline{A}}}^\mu(x_1)\underline{\underline{\underline{A}}}^\nu(x_2): \qquad (h)
\end{aligned}
$$

In this expansion we have used the modification to Wick's theorem that says that "no equal time contractions are allowe" (no e.t.c. - see before). As previously mentioned Greiner states this somewhat differently [Greiner 238] but the result is the same. Greiner has a nice discussion [Greiner 238] showing that this prescription eliminates the so-called "tadpole" diagrams.

Also in the above expansion we don't include contributions of the form

$$\underline{\psi}(x_1)\underline{\psi}(x_2) = \underline{\bar{\psi}}(x_1)\underline{\bar{\psi}}(x_2) = 0$$

because they give zero contribution. [Schwabl 337, GreinerFQ 238]

We now need to introduce some additional Feynman diagrams corresponding to the above contractions. These are illustrated in the figure below. Note that the photon diagrams do not have an arrow associated with them since each photon is its own antiparticle. [Greiner FQ 236]

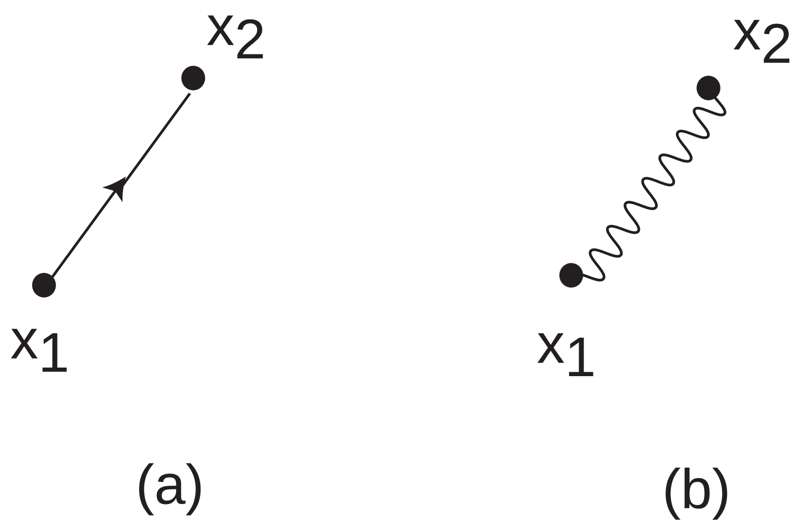

Fig. x.x
(a) Fermion contraction $\underline{\psi}(x_2)\underline{\bar{\psi}}(x_1) = iS_F(x_2 - x_1)$

(b) Photon contraction $\underline{A}^{\mu}(x_2)\underline{A}^{\nu}(x_1) = iS_F(x_2 - x_1)$

Using the above diagrams and also the ones shown previously we are in a position to draw the Feynman diagrams corresponding to all the terms in the second order S-matrix. These are shown in the Figure below.

Fig. Feynman diagrams for 2nd order S-matrix
NNN identical to GreinerFQ, fig 8.5, p. 239

7.2.3 First order S-matrix elements

We previously considered the first order S-matrix, and we wish now to evluate S-matrix *elements*. For definiteness let's consider the diagram of Fig. 7.4 (b) which corresponds to an electron radiating a photon. In that case the initial state would be creation of an electron from the vacuum, i.e.

$$|i>\equiv b^{\dagger}_{\vec{k}_1,s_1}|0>$$

and the final state consists of the scattered electron together with the produced photon, i.e.

$$|f>\equiv b^{\dagger}_{\vec{k}'_1,s'_1}a^{\dagger}_{\vec{k},\lambda}|0>$$

or

$$<f|\equiv<0|a_{\vec{k},\lambda}b_{\vec{k}'_1,s'_1}$$

We wish to evaluate the matrix element $<f|S|i>$ and the first order S-matrix $S^{(1)}$ contains 8 terms. However all but one (the second) of these terms will be zero, *if* the $|i>$ and $<f|$ states above are used. To illustrate this let's evaluate the first of the eight $<f|S^{(1)}|i>$ terms, denoted $<f|S^{(1)}_1|i>$.

7.2.4 Second order S-matrix elements

Electron-electron (Moeller) scattering

7.2.5 Invariant amplitude and lepton tensor

Electron-muon scattering

Invariant amplitude

7.3 Casimir's trick & Trace theorems

7.3.1 Average over initial states / Sum over final states

Polarized final states / Unpolarized initial staes

Unpolarized initial and final states

7.3.2 Casimir's trick