Quantum Field Theory[1]

R. Clarkson Dr. D. G. C. McKeon[2]

January 13, 2003

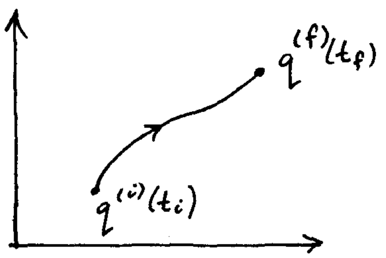
[1]Notes taken by R. Clarkson for Dr. McKeon's Field Theory (Parts I and II) Class.
[2]email: dgmckeo2@uwo.ca

Contents

Chapter 1

Constraint Formalism

1.1 Principle of Least action:



Configuration space:
$(q_1, q_2, \ldots, q_n)$

with $q^{(i)}(t_i)$ & $q^{(f)}(t_f)$ fixed, the classical path is the one that minimizes

$$\underbrace{S}_{action} = \int_{t_i}^{t_f} L(q_i(t), \dot{q}_i(t))dt \quad (L = \text{Lagrangian}) \tag{1.1.1}$$

i.e. if $q_i(t) = q_i^{classical}(t) + \varepsilon\ \delta q(t)$.
As $S = S(\varepsilon)$ has a minimum at $\varepsilon = 0$,

$$\begin{aligned} \frac{dS(0)}{d\varepsilon} &= 0 \\ \{\delta q(t_i) &= 0\} \\ \{\delta q(t_f) &= 0\} \end{aligned} \tag{1.1.2}$$

$$\begin{aligned}\frac{dS(0)}{d\varepsilon} &= 0 \\ &= \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) \\ &\quad \text{Int. by parts} \\ &= \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\end{aligned}$$

and, as δq_i is arbitrary,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \tag{1.1.3}$$

1.2 Hamilton's Equations

(Legendre transforms)

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.2.1}$$

$$H = p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{1.2.2}$$

Note that H does not explicitly depend on $\dot{q}_i$. i.e.

$$\begin{aligned}\frac{\partial H}{\partial \dot{q}_i} &= p_i - \frac{\partial L}{\partial \dot{q}_i} \\ &= 0\end{aligned}$$

Thus $H = H(p_i, q_i)$.
<u>Now</u>:

$$dH = \underbrace{\frac{\partial H}{\partial q_i}}_{A} dq_i + \underbrace{\frac{\partial H}{\partial p_i}}_{B} dp_i \tag{1.2.3}$$

$$= \underbrace{p_i d\dot{q}_i}_{cancels} + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i}_{cancels}$$

but if we insert (1.2.1) → (1.1.3), we get $\frac{\partial L}{\partial q_i} = \frac{d}{dt}(p_i) = \dot{p}_i$,

$$\therefore \; dH = \underbrace{\dot{q}_i}_{A} dp_i - \underbrace{\dot{p}_i}_{B} dq_i \tag{1.2.4}$$

From (1.2.3) and (1.2.4), we have:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.2.5}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.2.6}$$

1.3 Poisson Brackets

$$\begin{aligned} A &= A(q_i, p_i) && (1.3.1) \\ B &= B(q_i, p_i) && (1.3.2) \end{aligned}$$

$$\{A, B\}_{PB} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right) \quad (1.3.3)$$

So, we have

$$\begin{aligned} \dot{q}_i &= \{q_i, H\} && (1.3.4) \\ \dot{p}_i &= \{p_i, H\} && (1.3.5) \end{aligned}$$

We can generalize this:

$$\frac{d}{dt}A(q_i(t), p_i(t)) = \{A, H\} \quad (1.3.6)$$

$\longrightarrow$ Suppose in defining

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{(1.2.1)} \quad (1.3.7)$$

we cannot solve for $\dot{q}_i$ in terms of p_i. i.e. in

$$H = p_i\dot{q}_i - L(q_i, \dot{q}_i)$$

ex: With S.H.O.

$$\begin{aligned} L &= \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 \\ p &= \frac{\partial L}{\partial \dot{q}} = m\dot{q} \end{aligned}$$

$$\begin{aligned} \rightarrow \therefore H &= p\dot{q} - L \\ &= p\left(\frac{p}{m}\right) - \left[\frac{m}{2}\left(\frac{p}{m}\right)^2 - \frac{kq^2}{2}\right] \\ &= \frac{p^2}{2m} - \frac{kq^2}{2} \end{aligned}$$

i.e. <u>can</u> solve for $\dot{q}$ in terms of p here.

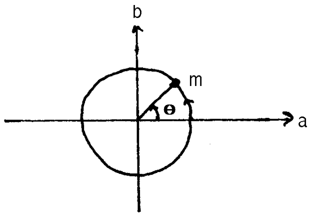
Suppose we used the cartesian coord's to define system

$$L = \frac{m}{2}\dot{a}^2 + \frac{m}{2}\dot{b}^2 + \lambda_1(\dot{a} - b) + \lambda_2(\dot{b} + a)$$

i.e.

$$\begin{aligned} (a(t), b(t)) &= (\cos(\theta(t)), \sin(\theta(t))) \\ \therefore (\dot{a}(t), \dot{b}(t)) &= (-\sin(\theta(t))\,\dot{\theta}(t), \cos(\theta(t))\,\dot{\theta}(t)) \end{aligned}$$

ex.



$a^2 + b^2 = 1$

scale $\dot{\theta} = const. = 1$

$$\therefore \dot{a} = -b \;\; / \;\; \dot{b} = a$$

Dynamical Variables: $a,\ b,\ \lambda_1,\ \lambda_2$

$$\frac{\partial L}{\partial \lambda_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}_i} = 0 = \left\{\begin{matrix} \dot{a} - b & i = 1 \\ \dot{b} + a & i = 2 \end{matrix}\right\} (\lambda_1,\ \lambda_2) \rightarrow \text{Lagrangian Multipliers}$$

The trouble comes when we try to pass to Hamiltonian;

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\left.\begin{matrix} p_{\lambda_1} = 0 \\ p_{\lambda_2} = 0 \end{matrix}\right\} \rightarrow \text{ Cannot solve for } \dot{\lambda}_i \text{ in terms of } p_{\lambda_i} \text{, because these are 2 \underline{constraints}}$$

i.e. if

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.3.8}$$

cannot be solved, then

$$\frac{\partial p_i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \text{ cannot be inverted.}$$

In terms of Lagrange's equations:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(q_i, \dot{q}_i) &= \frac{\partial L}{\partial q_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j \\ \text{So } \underbrace{\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}}_{*}\ddot{q}_j &= -\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial L}{\partial q_i} \end{aligned} \tag{1.3.9}$$

$* \rightarrow q_i, \dot{q}_i$ specified at $t = t_o$

$$\therefore q_i(t_0 + \delta t) = \underbrace{q_i(t_0)}_{given} + \underbrace{\dot{q}_i(t_0)}_{given}\delta t + \frac{1}{2!}\ddot{q}_i(t_0)\ (\delta t)^2 + \frac{1}{3!}\left(\frac{d^3}{dt^3}q_i(t_0)\right)(\delta t)^3 + \ldots$$

Can solve for $\ddot{q}_i(t_0)$ if we can invert $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$.
Thus, if a constraint occurs, $\ddot{q}_i(t_0)$ cannot be determined from the initial conditions using

Lagrange's equations. i.e. from our example,

$$\lambda_i(t_0+\delta t) = \lambda_i(t_0) + \ldots + \underbrace{\ddot{\lambda}_i}_{*}(t_0)$$

$* \to$ cannot be determined.

1.4 Dirac's Theory of Constraints

If $p_i = \frac{\partial L}{\partial \dot{q}_i}$ implies a constraint $\chi_i(q_i, p_i) = 0$ (i.e. from our example $p_{\lambda_1} = p_{\lambda_2} = 0$), then we can define

$$H_0 = p_i\dot{q}_i - L(q_i, \dot{q}_i) \quad \text{Constraints hold here} \tag{1.4.1}$$

(So, for our example, (scaling $m = 1$),

$$\begin{aligned}
L &= \frac{1}{2}(\dot{a}^2 + \dot{b}^2) + \lambda_1(\dot{a} - b) + \lambda_2(\dot{b} + a) \\
p_{\lambda_1} &= 0 = p_{\lambda_2} \\
p_a &= \dot{a} + \lambda_1 \\
p_b &= \dot{b} + \lambda_2 \\
\therefore H_0 &= p_a\dot{a} + p_b\dot{b} + \overbrace{p_{\lambda_i}\dot{\lambda}_i}^{=0} - L \\
&= p_a(p_a - \lambda_1) + p_b(p_b - \lambda_2) - \frac{1}{2}\left[(p_a - \lambda_1)^2 + (p_b - \lambda_2)^2\right] - \\
&\quad \lambda_1[p_a - \lambda_1 - b] - \lambda_2[p_b - \lambda_2 + a])
\end{aligned}$$

The constraints must hold for all t, thus

$$\begin{aligned}
\frac{d}{dt}\chi_i(q,p) &= 0 \qquad (1.4.2) \\
&= \{\chi_i, H\}_{PB} \approx 0 \;\; \text{zero if } \chi_i = 0 \;\; (\text{“weakly” equal to zero})
\end{aligned}$$

where $H = H_0 + c_i\chi_i$.
Sept. 15/99
So, we have:

$$\begin{aligned}
H_0 &= p_i\dot{q}_i - L(q, \dot{q}) && (1.4.3) \\
H &= H_0 + c_i\chi_i(q_i, p_i) && (1.4.4) \\
\frac{d}{dt}\chi_i &= [\chi_i, H_0 + c_i\chi_i] \approx 0 \to \begin{array}{c}\text{This consistency condition \underline{could} lead} \\ \text{to some additional constraints}\end{array} && (1.4.5)
\end{aligned}$$

The constraints coming from the definition $p_i = \frac{\partial L}{\partial \dot{q}_i}$ are called <u>primary</u> constraints.

Additional constraints are called secondary. We could in principle also have tertiary constraints, etc.. (In practice, tertiary constraints don't arise).

Suppose we have constraints χ_i. They can be divided into First class and Second class constraints.

$$\text{first class constraints} \rightarrow \text{label } \phi_i \tag{1.4.6}$$
$$\text{second class constraints} \rightarrow \text{label } \theta_i \tag{1.4.7}$$

For a first class constraint, ϕ_i, $[\phi_i, \chi_j] \approx 0 = \alpha_k^{ij} x_k$ (for all j).

θ_i is second class if it is not first class.

We know that

$$\begin{aligned}
H &= H_0 + c_i\chi_i \\
&= H_0 + a_i\phi_i + b_i\theta_i \\
\frac{d}{dt}\chi_i &= [\chi_i, H] \approx 0 \\
\text{Thus } \frac{d}{dt}\phi_i &= [\phi_i, H_0 + a_j\phi_j + b_j\theta_j] \\
&= [\phi_i, H_0] + a_j[\phi_i, \phi_j] + \phi_j[\phi_i, a_j] + b_j[\phi_i, \theta_j] + \theta_j[\phi_i, b_j] \\
&\approx 0 \\
&\approx [\phi_i, H_0] \\
&\quad \text{(true for any } a_i, b_j) \\
\frac{d}{dt}\theta_i &= [\theta_i, H_0] + \overbrace{a_j[\theta_i, \phi_j]}^{\approx 0} + \overbrace{\phi_j[\theta_i, a_j]}^{\approx 0} + b_j[\theta_i, \theta_j] + \underbrace{\theta_j}_{\approx 0}[\theta_i, b_j] \\
&\approx 0 \\
&\approx [\theta_i, H_0] + b_j[\theta_i, \theta_j]
\end{aligned}$$

this fixes b_j.

Note we have not fixed a_i.

Hence for each first class constraint there is an arbitrariness in H_0. To eliminate this arbitrariness we impose extra conditions on the system. (These extra conditions are called gauge conditions).

We call these gauge conditions γ_i (one for each first class constraint ϕ_i).

Full set of constraints: $\{\phi_i, \theta_i, \gamma_i\} = \{\Theta_i\}$

$$H = H_0 + a_i\phi_i + b_i\theta_i + c_i\gamma_i \tag{1.4.8}$$

Provided $\{\phi_i, \gamma_j\} \not\approx 0$, then the condition

$$\frac{d}{dt}\Theta_i \approx 0 \tag{1.4.9}$$

fixes $a_i,\ b_i,\ c_i$ (All arbitrariness is eliminated).

Dirac Brackets (designed to replace Poisson Brackets so as to eliminate all constraints from the theory).
Note:

$$\begin{aligned}[\theta_i,\theta_j] &\not\approx 0 \quad \text{(Could be weakly zero for particular } i,j \text{ but not in general (overall)).}\\ d_{ij} &= [\theta_i,\theta_j] = -[\theta_j,\theta_i] \text{(Antisymmetric matrix)}\\ \therefore \det(d_{ij}) &\neq 0\end{aligned}$$

Thus i, j must be even. Hence there are always an even number of 2^{nd} class constraints. Now we define the Dirac Bracket.

$$[A,B]^* = [A,B] - \sum_{i,j}[A,\theta_i]d_{ij}^{-1}[\theta_j,B] \tag{1.4.10}$$

Properties of the Dirac Bracket

1.

$$\begin{aligned}[\theta_i,B]^* &= [\theta_i,B] - \sum_{k,l} \underbrace{\underbrace{[\theta_i,\theta_k]}_{d_{ik}} d_{kl}^{-1}}_{\delta_{il}}[\theta_l,B]\\ &= [\theta_i,B] - [\theta_i,B]\\ &= 0\end{aligned} \tag{1.4.11}$$

2. We know that

$$0 = [[A,B],C] + [[B,C],A] + [[C,A],B] \tag{1.4.12}$$

We can show that

$$0 = [[A,B]^*,C]^* + [[B,C]^*,A]^* + [[C,A]^*,B]^* \tag{1.4.13}$$

3. If A is some 1^{st} class quantity, i.e. if $[A,\chi_i] \approx 0$ for any constraint χ_i, then

$$\begin{aligned}[A,B]^* &= [A,B] - \sum_{ij} \overbrace{[A,\theta_i]}^{\approx 0} d_{ij}^{-1}[\theta_j,B]\\ &= [A,B]\end{aligned} \tag{1.4.14}$$

Note that if

$$H = H_0 + a_i\phi_i + b_i\theta_i \tag{1.4.15}$$

then H itself is first class. Hence,

$$\frac{d\mathcal{C}}{dt} = [\mathcal{C}, H] \tag{1.4.16}$$

Thus by (3) above,

$$\frac{d\mathcal{C}}{dt} \approx [\mathcal{C}, H]^*. \tag{1.4.17}$$

But $[\theta_i, \mathcal{C}]^* = 0$ by (1). Thus, in H, we can set $\theta_i = 0$ before computing $[\mathcal{C}, H]^*$.
i.e.

If we want to find $\frac{d\mathcal{C}}{dt}$ we can use $[\mathcal{C}, H]^*$ and take H to be just $H = H_0 + a_i\phi_i + \overbrace{b_i\theta_i}^{=0}$ (Provided we exchange P.B. for Dirac B.).
Thus if we use the Dirac Bracket, we need not determine b_i. If we include the gauge condition, we can treat

$$\Theta_i = \{\phi_i, \theta_i, \gamma_i\} \tag{1.4.18}$$

as a large set of 2^{nd} class constraints, and if

$$D_{ij} = \{\Theta_i, \Theta_j\} \tag{1.4.19}$$

then

$$[A, B]^* = [A, B] - \sum_{ij}[A, \Theta_i]D^{-1}_{ij}[\Theta_j, B] \tag{1.4.20}$$

Sept. 17/99
So, so far:

$$\begin{aligned}
L &= \frac{1}{2}(\dot{a}^2 + \dot{b}^2 - a^2 - b^2) + \lambda_1(\dot{a} - b) + \lambda_2(\dot{b} + a) \\
p_a &= \frac{\partial L}{\partial \dot{a}} = \dot{a} + \lambda_1 \\
p_b &= \frac{\partial L}{\partial \dot{b}} = \dot{b} + \lambda_2 \\
p_{\lambda_1} &= p_{\lambda_2} = 0 \Rightarrow \text{Constraint} \\
H_0 &= p_i\dot{q}_i - L \\
&= p_a\dot{a} + p_b\dot{b} + \overbrace{p_{\lambda_1}\dot{\lambda}_1}^{0} + \overbrace{p_{\lambda_2}\dot{\lambda}_2}^{0} - \left[\frac{1}{2}(\dot{a}^2 + \dot{b}^2 - a^2 - b^2) + \lambda_1(\dot{a} - b) + \lambda_2(\dot{b} + a)\right] \\
&\quad \rightarrow \text{can't make any sense of this (can't express} \\
&\qquad \dot{\lambda}_i \text{ in terms of } p_{\lambda_i}\text{) unless we impose constraints.} \\
&= p_a(p_a - \lambda_1) + p_b(p_b - \lambda_2) - \left[\frac{1}{2}((p_a - \lambda_1)^2 + (p_b - \lambda_2)^2 - a^2 - b^2) + \right. \\
&\qquad \left. \lambda_1(p_a - \lambda_1 - b) + \lambda_2(p_b - \lambda_2 + a)\right] \\
&= \frac{(p_a - \lambda_1)^2}{2} + \frac{(p_b - \lambda_2)^2}{2} + \frac{1}{2}(a^2 + b^2) + \lambda_1 b - \lambda_2 a
\end{aligned}$$

$$\frac{dp_{\lambda_1}}{dt} = [p_{\lambda_1}, H] \approx 0$$

$$\left.\begin{array}{l} p_a - \lambda_1 - b = 0 \\ p_b - \lambda_2 + a = 0 \end{array}\right\} \text{Secondary constraints}$$

(tertiary constraints don't arise)

$$\begin{aligned}
\theta_1 &= p_{\lambda_1} \\
\theta_2 &= p_{\lambda_2} \\
\theta_3 &= p_a - \lambda_1 - b \\
\theta_4 &= p_b - \lambda_2 + a \\
&\rightarrow \text{These are all Second class - i.e. } [\theta_1, \theta_3] = 1 = [\theta_2, \theta_4] \\
&(\text{No first class constraints } \rightarrow \text{no gauge condition}). \\
H &= H_0 + c_i\theta_i \quad \rightarrow c_i \text{ fixed by the condition } \dot{\theta}_i = [\theta_i, H] \approx 0
\end{aligned}$$

- or could move to Dirac Brackets, and let $\theta_i = 0$.
<u>Need</u>:

$$\begin{aligned}
d_{ij} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & -2 & 0 \end{bmatrix} = [\theta_i, \theta_j] \\
\rightarrow [X,Y]^* &= [X,Y] - [X,\theta_i]d_{ij}^{-1}[\theta_j, Y]
\end{aligned}$$

Can eliminate constraints sequentially instead of all at once (easier).
Eliminate $\theta_3 \& \theta_4$ initially.

$$\begin{aligned}
\theta_1^{(1)} &= p_a - \lambda_1 - b \\
\theta_2^{(1)} &= p_b - \lambda_2 + a \\
d_{12}^{(1)} &= [\theta_1^{(1)}, \theta_2^{(1)}] \\
&= 2 \\
\therefore d_{ij}^{(1)} &= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \rightarrow \quad (d_{ij}^{(1)})^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \\
\therefore [X,Y]^* &= [X,Y] - [X,\theta_1^{(1)}] \overbrace{d_{12}^{-1}}^{-1/2} [\theta_2^{(1)}, Y] - [X, \theta_2^{(1)}] \overbrace{d_{21}^{-1}}^{1/2} [\theta_1^{(1)}, Y]
\end{aligned}$$

Now we need to eliminate

$$\begin{aligned}
\theta_1^{(2)} &= p_{\lambda_1} \\
\theta_2^{(2)} &= p_{\lambda_2}
\end{aligned}$$

$$\begin{aligned}
d_{ij}^{(2)} &= [\theta_i^{(2)}, \theta_j^{(2)}]^* \quad \text{(note *)} \\
\rightarrow d_{12}^{(2)} &= 0 - [p_{\lambda_1}, p_a - \lambda_1 - b] \left(-\frac{1}{2}\right) [p_b - \lambda_2 + a, p_{\lambda_2}] \\
&= [p_{\lambda_1}, \lambda_1] \left(-\frac{1}{2}\right) [p_{\lambda_2}, \lambda_2] \\
&= -\frac{1}{2} = -d_{21}^{(2)} \\
d_{ij}^{(2)} &= \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \longrightarrow (d_{ij}^{(2)})^{-1} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}
\end{aligned}$$

Finally,

$$[X, Y]^{**} = [X, Y]^* - [X, \theta_i^{(2)}]^* (d_{ij}^{(2)})^{-1} [\theta_j^{(2)}, Y]^*$$

We can finally see that

$$\begin{aligned}
[a, p_a]^{**} &= 1 \\
[b, p_b]^{**} &= 1
\end{aligned}$$

and all other fundamental Dirac Brackets are zero. i.e.

$$\begin{aligned}
[a, p_b] &= 0 \\
\text{as} & \\
p_a - \lambda_1 - b &= 0 \\
p_b - \lambda_2 + a &= 0 \\
\text{and} & \\
H_0 &= \frac{b^2}{2} + \frac{(-a)^2}{2} + \frac{a^2 + b^2}{2} + (p_a - b)b - (p_b + a)a \\
&= p_a b - p_b a
\end{aligned}$$

So,

$$\begin{aligned}
H &= p_a b - p_b a \\
\frac{da}{dt} &= [a, H]^{**} \\
&= [a, p_a b - p_b a]^{**} \\
&= b \\
\\
\frac{db}{dt} &= [b, H]^{**} \\
&= -a
\end{aligned}$$

If we have gauge conditions & first class constraints ϕ_i:

$$H = H_0 + a_i \phi_i + \overbrace{b_i \theta_i}^{0}$$

1^{st} stage: $\rightarrow$ get rid of 2^{nd} class constraints. Do this by

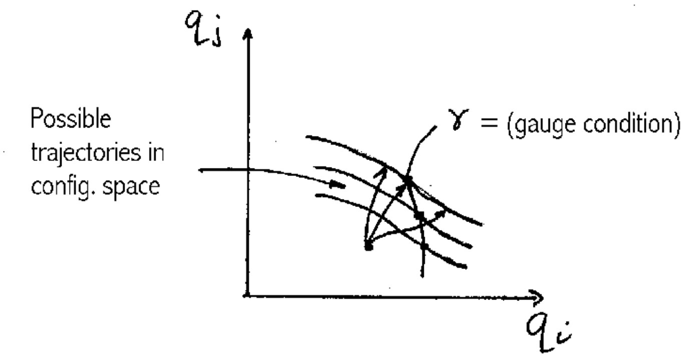
$$[\ ] \rightarrow [\quad]^*$$

At this stage,

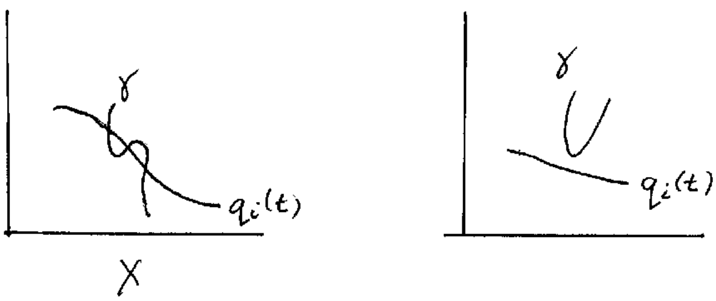
$$H = H_0 + a_i\phi_i$$

As the a_i's are not fixed,

$$\begin{aligned}\frac{dA}{dt} &= [A, H]^* \\ &\approx [A, H_0]^* + a_i [A, \phi_i]^* \\ &\quad a_i \rightarrow \text{Arbitrariness}\end{aligned}$$

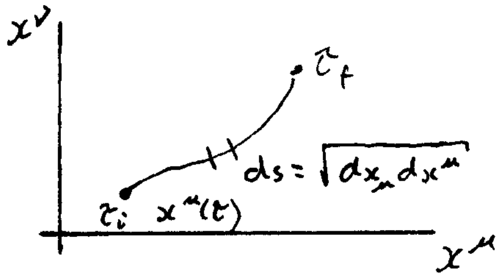


$\gamma = 0$ must intersect $q_i(t)$ at one & only one point.



$\rightarrow$ "Gribov Ambiguity" (to be avoided).

Sept. 21/99
Relativistic Free Particle



$$\begin{aligned} S &\propto \text{ arc length from } x^\mu(\tau_i) \text{ to } x^\mu(\tau_f) \\ &= -m\int_{\tau_i}^{\tau_f} \sqrt{\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx_\mu}{d\tau}\right)} d\tau \end{aligned}$$

The $m =$ const. of proportionality $\rightarrow g_{\mu\nu} = (+,-,-,-)$.

$$S = -m\int_{\tau_i}^{\tau_f} d\tau\sqrt{\dot{x}^2} \tag{1.4.21}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{m\dot{x}_\mu}{\sqrt{\dot{x}^2}} \tag{1.4.22}$$

Constraints

$$\begin{aligned} p_\mu p^\mu &= \frac{m^2\dot{x}_\mu\dot{x}^\mu}{\dot{x}^2} \\ 0 &= p^2 - m^2 \end{aligned} \tag{1.4.23}$$

$$\begin{aligned} H_0 &= p_\mu\dot{x}^\mu - L \\ &= -\frac{m\dot{x}_\mu\dot{x}^\mu}{\sqrt{\dot{x}^2}} - (-m\sqrt{\dot{x}^2}) \\ &= 0 \quad ! \end{aligned}$$

$$\begin{aligned} H &= H_0 + U_i\chi_i \\ &= \gamma(p^2 - m^2) \quad \text{(Pure Constraint!)} \end{aligned}$$

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} \\ \frac{dx^\mu}{d\tau} &= \kappa(2p^\mu) \end{aligned} \tag{1.4.24}$$

This arbitrariness in $\dot{x}^\mu$ is a reflection of the fat that in S, τ is a freely chosen parameter, i.e.

$$\begin{aligned} S &= -m\int d\tau\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} && (1.4.25)\\ \tau &\to \tau(\tau') \quad d\tau = \frac{d\tau}{d\tau'}d\tau' \\ \therefore \frac{dx^\mu}{d\tau} &= \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau} && (1.4.26)\\ S &= -m\int d\tau'\sqrt{\frac{dx^\mu}{d\tau'}\frac{dx_\mu}{d\tau'}} && (1.4.27)\end{aligned}$$

$\longrightarrow$ Now let $\kappa = \frac{1}{2}\frac{d\tau'}{d\tau}$. Thus, (insert κ into (1.4.24))

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{d\tau'}{d\tau}p^\mu \ \text{ and equate this with (1.4.26)} && (1.4.28)\\ \frac{dx^\mu}{d\tau'} &= p^\mu && (1.4.29)\end{aligned}$$

Gauge fixing in this case corresponds to a choice of the parameter τ.

-The formalism of Dirac actually breaks down for gauge choices γ which are dependent on "time" (which in this case means on τ).

(Note, we can think of this reparameterization invariance $\tau \to \tau(\tau')$ as being a form of diffeomorphism invariance in $0+1$ dimensions, i.e. $x^\mu(\tau)$ is a scalar field moving in $0+1$ dimensions, and has a so-called "tangent space" which is four dimensional.

Thus, this is a simpler version of G.R. where we have scalars $\phi'(x^\mu)$ moving in $3+1$ dimensions with the diffeomorphism invariance $x^\mu \to x^\mu(x'^\mu)$. Techniques in G.R. & in the single particle case often overlap.

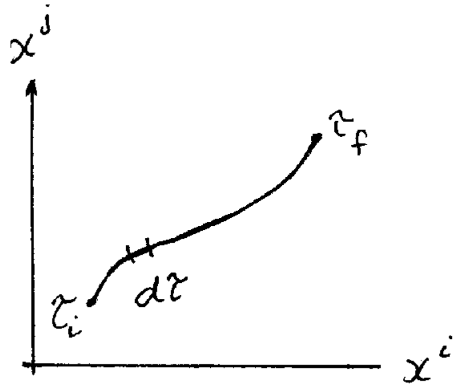
eq. of motion:

$$\begin{aligned} L &= -m\sqrt{\dot{x}^2}\\ 0 &= \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}\end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{-m\dot{x}_\mu}{\sqrt{\dot{x}^2}}\right) &= 0\\ \frac{m\ddot{x}^\mu}{\sqrt{\dot{x}^2}} &= 0\end{aligned}$$

Now identify τ with the arc length along the particle's trajectory:



$$ds^2 = dx_\mu dx^\mu$$

If $d\tau^2 = ds^2$, then

$$\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = 1 \tag{1.4.30}$$

$$\dot{x}^2 = 1 \tag{1.4.31}$$

(τ is called the "proper time" in this instance).
i.e.
if $d\underline{x} = 0$, $ds^2 = dt^2 = d\tau^2$.
In this case, the equation of motion becomes

$$m\ddot{x}^\mu = 0$$

The corresponding action is

$$S = \frac{m}{2}\int_{\tau_i}^{\tau_f} d\tau\ \dot{x}^2 \leftarrow \text{absence of } \sqrt{\ } \text{ means this is \underline{not} invariant under } \tau \to \tau(\tau').$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} &= m\ddot{x}^\mu = 0 \\ H &= p_\mu \dot{x}^\mu - L \end{aligned}$$

where $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m\dot{x}_\mu$

$$\begin{aligned} H &= p_\mu \left(\frac{p^\mu}{m}\right) - \frac{m}{2}\left(\frac{p_\mu}{m}\right)^2 \\ H &= \frac{p_\mu p^\mu}{2m} \end{aligned}$$

Other gauge choice:

$$\tau = x^4 = t(\text{breaks Lorentz invariance})$$

Work directly from the action:

$$S = -m \int d\tau \sqrt{\frac{dt}{d\tau}\frac{dt}{d\tau} - \frac{d\vec{r}}{d\tau}\frac{d\vec{r}}{d\tau}}$$

If we've chosen $\tau = t$, then

$$S = -m \int dt \sqrt{1 - \vec{v}^2} \quad ; \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

eq. of motion:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} &= 0 \\ \therefore \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) &= 0 \\ \vec{p} &= \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} \quad \text{(momentum)} \\ H &= \vec{p}\cdot\dot{\vec{r}} - L \\ &= \vec{p}\cdot\vec{v} - L \end{aligned}$$

but $\vec{p}^2 = \frac{m^2\vec{v}^2}{1-\vec{v}^2}$

$$\begin{aligned} \therefore \vec{p}^2(1-\vec{v}^2) &= m^2\vec{v}^2 \\ \vec{p}^2 - \vec{p}^2\vec{v}^2 &= m^2\vec{v}^2 \\ \vec{p}^2 &= \vec{v}^2(\vec{p}^2 + m^2) \\ \vec{v}^2 &= \frac{\vec{p}^2}{m^2+\vec{p}^2} \\ 1 - \vec{v}^2 &= 1 - \frac{\vec{p}^2}{m^2+\vec{p}^2} = \frac{m^2 + \cancel{\vec{p}^2} - \cancel{\vec{p}^2}}{m^2+\vec{p}^2} \\ \therefore \sqrt{1-\vec{v}^2} &= \sqrt{\frac{m^2}{\vec{p}^2+m^2}} \end{aligned}$$

Thus,

$$\begin{aligned} \vec{v} &= \frac{\vec{p}}{\sqrt{m^2+\vec{p}^2}} \\ H &= \vec{p}\cdot\frac{\vec{p}}{\sqrt{m^2+\vec{p}^2}} - (-m)\frac{m}{\sqrt{m^2+\vec{p}^2}} \\ &= \sqrt{\vec{p}^2+m^2} \Rightarrow E = \text{numerical value of H} = \frac{m}{\sqrt{1-v^2}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

Note:

$$\begin{aligned} E^2 &= \vec{p}^2 + m^2 \\ (E^2 - \vec{p}^2) - m^2 &= 0 \\ \therefore p_\mu &= (\vec{p}, E) \end{aligned}$$

$$\begin{aligned} p_\mu p^\mu - m^2 &= 0 \\ & \text{Limit} \quad m \to 0 \\ S &= -m \int d\tau \, \sqrt{\dot{x}^2} \\ &\to 0 \; ?? \end{aligned}$$

i.e.

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) \to 0 \; ???$$

$|\vec{v}| \to 1$ as $m \to 0$.

Circumvent by introducing a Lagrange multiplier e.

$$\begin{aligned} S &= -\frac{1}{2}\int \left(\frac{\dot{x}^2}{e} + m^2 e\right) \, d\tau \\ & e = e(\tau) \; , \quad x^\mu = x^\mu(\tau) \end{aligned}$$

$m^2 \to 0$ is well defined in S. As

$$\frac{d}{d\tau}\underbrace{\frac{\partial L}{\partial \dot{e}}}_{=0} - \frac{\partial L}{\partial e} = 0$$

So,

$$-\frac{\dot{x}^2}{e^2} + m^2 = 0 \to e = \sqrt{\frac{\dot{x}^2}{m^2}}$$

Thus,

$$\begin{aligned} S &= -\frac{1}{2}\int d\tau \, \left[\frac{m\dot{x}^2}{\sqrt{\dot{x}^2}} + m^2\sqrt{\frac{\dot{x}^2}{m^2}}\right] \\ &= -m\int d\tau \, \sqrt{\dot{x}^2} \end{aligned}$$

Can see that e field can be eliminated $\to$ really just a Lagrange multiplier that insures $S \neq 0$ when $m = 0$.

Note:

The action

$$S = -\frac{1}{2}\int d\tau \, \left(\frac{\dot{x}^2}{e} + m^2 e\right) \tag{1.4.32}$$

is invariant under

$$\tau \rightarrow \tau + f(\tau) \tag{1.4.33}$$
$$\delta x^\mu = \dot{x}^\mu f(\tau) \tag{1.4.34}$$
$$\delta e = \dot{f} e + \dot{e} f \tag{1.4.35}$$

Sept. 22/99
So, for the above system:

$$L = -\frac{1}{2}\left(\frac{\dot{x}^2}{e} + m^2 e\right)$$

eqn's of motion:

$$0 = -\frac{\dot{x}^2}{e^2} + m^2 \quad \left(\text{from } \frac{d}{d\tau}\frac{\partial L}{\partial \dot{e}} - \frac{\partial L}{\partial e} = 0\right)$$
$$0 = \frac{d}{d\tau}\left(\frac{\dot{x}^\mu}{e}\right) \quad \left(\text{from } \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0\right)$$
$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{\dot{x}_\mu}{e} \quad \text{(no constraint } \rightarrow \text{ can solve for } \dot{x}_\mu \text{ in terms of } p_\mu\text{).}$$
$$p_e = \frac{\partial L}{\partial \dot{e}} = 0 \qquad \text{(primary constraint)}$$
$$H_0 = p_\mu \dot{x}^\mu + \overbrace{p_e \dot{e}}^{=0} - L$$
$$= p_\mu(-p^\mu e) - \left[-\frac{1}{2}\left(\frac{\dot{x}^2}{e} + m^2 e\right)\right] \quad ; \quad \dot{x}^2 = x_\mu x^\mu = -p^2 e^2$$
$$= -\frac{1}{2}e(p^2 - m^2)$$
$$\dot{p}_e = 0 = [p_e, H_0] \quad \left\{= \left[p_e, \frac{1}{2}e(p^2 - m^2)\right]\right\}$$
$$0 = -\frac{1}{2}(p^2 - m^2) \quad \text{(Secondary Constraint)}$$

Both $p_e = 0$ (gauge condition $e = 1$) and $p^2 - m^2 = 0$ (already discussed) are first calss.
Note: Remember that

$$S = \int d^4x \, \sqrt{g} \, g_{\mu\nu} \left(\partial^\mu \phi^A(x)\right)\left(\partial^\nu \phi^A(x)\right) \tag{1.4.36}$$

- action for a scalar field $\phi^A(x)$ in 3+1 Dim.

Vierbein ("deals with 4-d")

$$g_{\mu\nu} = e_\mu^a e_{a\nu}$$
$$\sqrt{g} = \sqrt{\det(g_{\mu\nu})} = [\det(e_{\mu\nu})]^{-1} = e^{-1}$$

In $0+1$ dimensions

$$S = \int d\tau \, \frac{1}{e}\left(\frac{d}{d\tau}\phi^A\right)\left(\frac{d}{d\tau}\phi^A\right)$$
$$= \int d\tau \, \frac{(\dot{\phi}^A)^2}{e} \longrightarrow \int d\tau \, \frac{(\dot{x}^\mu)^2}{e}$$

$e \rightarrow$ "Einbein" (assoc. with 1 dim)

1.5 Quantizing a system with constraints

$$[A, B]_{PB} \rightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}]_{\text{commutator (c)}} \tag{1.5.1}$$

ex.

$$[q, p]_{PB} = 1 \tag{1.5.2}$$
$$[\hat{q}, \hat{p}]_c = i\hbar \tag{1.5.3}$$

If there are constraints $\xi_i(q, p)$ then,

$$\xi_i(\hat{q}, \hat{p}) \; |\psi >_{\text{phys.}} = 0 \tag{1.5.4}$$

ex. For $L = -\frac{1}{2}\left(\frac{\dot{x}^2}{e} + m^2 e\right)$

$$\chi_1 = p_e \quad \chi_2 = p^2 - m^2 \quad \text{(2 constraints)}$$

Quantization conditions will be

$$[x_\mu, p^\nu] = i\hbar\delta_\mu^\nu \tag{1.5.5}$$

$$\delta_\mu^\nu \rightarrow (+,+,+,+)$$
$$\left(\text{re: } p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} \rightarrow \text{ Contrav.} = \text{der. of covar.}\right)$$

$$(p^2 - m^2) \; |\psi >_{phys} = 0 \quad \text{(Klein-Gordon eq.)} \tag{1.5.6}$$

$$p_\mu = -i\hbar\frac{\partial}{\partial x^\mu} \tag{1.5.7}$$

If $\Phi(x) =< x|\psi >_{phys}$ then

$$\left[\left(-i\hbar\frac{\partial}{\partial x^\mu}\right)^2 - m^2\right]\Phi(x) = 0$$

<u>Classical Motivation for Spin</u>

Brint, deVecchia & How, Nuclear P. 118, pg. 76 (1977)

Chapter 2

Grassmann Variables

(Casa/booni, N.C. 33A)

$\theta_1,\ \theta_2 \rightarrow$ Grassmann Variables (call θ if only 1 present), where;

$$\theta_1\theta_2 = -\theta_2\theta_1 \tag{2.0.1}$$

$$\theta_1^2 = -\theta_1^2 = 0 \tag{2.0.2}$$

ex. (these can only be ...)

$$f(x_1,\theta) = a(x) + b(x)\theta \tag{2.0.3}$$

$$f(x,\theta_1,\theta_2) = a(x) + b_i(x)\theta_i + \frac{1}{2}c(x)\epsilon_{ij}\theta_i\theta_j \tag{2.0.4}$$

$$(\rightarrow \epsilon_{ij} = -\epsilon_{ji})$$

<u>Calculus</u>

$$\frac{d\theta}{d\theta} = 1 \quad \frac{d\kappa}{d\theta} = 0(\kappa = \text{const.}) \tag{2.0.5}$$

$$\frac{d}{d\theta_1}(\theta_1\theta_2) = \overbrace{\frac{d\theta_1}{d\theta_2}}^{=1}\theta_2 + \theta_1\overbrace{\left(-\frac{d\theta_2}{d\theta_1}\right)}^{=0}$$

(-ve sign in 2^{nd} term because we're moving $\frac{d}{d\theta_1}$ through to θ_2)

$$= \theta_2 \tag{2.0.6}$$

Similarly

$$\frac{d}{d\theta_2}(\theta_1\theta_2) = \frac{d\theta_1}{d\theta_2}\theta_2 + \theta_1\left(-\frac{d\theta_2}{d\theta_2}\right)$$

$$= -\theta_1 \tag{2.0.7}$$

2.1 Integration

$$\int d\theta \leftrightarrow \frac{d}{d\theta} \quad \text{i.e. Integration \& differentiation are \underline{identical}} \tag{2.1.1}$$

$$\text{i.e.} \int d\theta \ \ c = 0 \quad \text{(int. of a constant c)} \tag{2.1.2}$$

$$\int d\theta\ \theta = 1 \tag{2.1.3}$$

For example;

$$\begin{aligned}\int d\theta_1\ d\theta_2\ (\theta_1\theta_2) &= \frac{d}{d\theta_1}\left(\frac{d}{d\theta_2}\theta_1\theta_2\right)\\ &= \frac{d}{d\theta_1}(-\theta_1)\\ &= -1\\ &= -\int d\theta_2\ d\theta_1\ \theta_1\theta_2\\ &= -\int d\theta_1\ d\theta_2\theta_2\theta_1\end{aligned}$$

Another example:

$$\begin{aligned}\int d\theta_1 d\theta_2 f(x,\theta_1,\theta_2) &= \int d\theta_1 d\theta_2\left(\underbrace{a}_{(i)} + \underbrace{b_i\theta_i}_{(ii)} + \frac{1}{2}\epsilon_{ij}c\ \theta_i\theta_j\right)\\ &\quad (i)\to 0\ \ \text{(const.)} \qquad (ii)\to 0\ \ (\theta_i \to \text{int. over } \theta_j \text{ gives } 0)\\ &= c\int d\theta_1 d\theta_2\left(\frac{1}{2}\theta_1\theta_2 - \frac{1}{2}\theta_2\theta_1\right)\\ &= -c(x)\end{aligned}$$

Delta function:

$$\begin{aligned}\int d\theta\ \delta(\theta) f(x,\theta) &= f(x,0)\\ \to \int d\theta\ \delta(\theta)\,[a(x)+b(x)\theta] &= a(x)\\ \Longrightarrow \delta(\theta) &= \theta \end{aligned} \tag{2.1.4}$$

The following is an example to demonstrate the properties of different kinds of statistical models. Suppose there are three students:

(T) Tom

(D) Dick

(H) Harry

How many ways can two prizes be awarded to T, D, H?

1. Suppose there are two medals, (distinguishable awards) a Newton medal (N) and a Shakespear medal (S).

T	D	H
N	S	
S	N	
N		S

T	D	H
S		N
	N	S
	S	N

T	D	H
NS		
	NS	
		NS

   So, there are 9 different ways to award 2 distinguishable medals.

2. Two silver dollars (2 medals, indistinguishable)

T	D	H
$	$	
$		$
	$	$

T	D	H
$ $		
	$ $	
		$ $

   So, there are 6 ways to award 2 indistinguishable medals.

3. Two positions (P) on football team (2 medals, indistinguishable). But! → now makes no sense for one person to receive 2 "medals" (one player can't have 2 positions).

T	D	H
P	P	
P		P
	P	P

   There are 3 ways to award to indistinguishable yet distinct "medals".

Now call the $\left\{\begin{smallmatrix}\text{prizes} \leftrightarrow \text{particles} \\ \text{students} \leftrightarrow \text{states}\end{smallmatrix}\right.$

1. = Maxwell-Bolzmann statistics

2. = Bose-Einstein

3. = Fermi-Dirac

Sept. 24/99
Returning to our discussion of Grassmann variables, the Lagrangian is now

$$L = L\left(x^\mu(\tau), \theta_a(\tau), \dot{x}^\mu(\tau), \dot{\theta}_a(\tau)\right) \tag{2.1.5}$$

where

$$0 = \theta_a(\tau)\theta_b(\tau') + \theta_b(\tau')\theta_a(\tau) \tag{2.1.6}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} \text{ and } \Pi^a = \frac{\partial L}{\partial \dot{\theta}^a} \tag{2.1.7}$$

$$H = H(x^\mu, \theta_a, p_\mu, \Pi^a) = \dot{q}^\mu p_\mu + \underbrace{\dot{\theta}^a \Pi_a}_{*} - L \tag{2.1.8}$$

* - order important

$$\begin{aligned}
\delta \int_{\tau_0}^{\tau_f} d\tau\ L &= \int_{\tau_0}^{\tau_f} d\tau \left[\delta x^\mu \frac{\partial L}{\partial x^\mu} + \delta\theta \frac{\delta L}{\delta \theta^a} + \underbrace{\delta \dot{x} \frac{\partial L}{\partial \dot{x}^\mu}}_{\text{int. by parts}} + \underbrace{\delta \dot{\theta}^a \frac{\partial L}{\partial \dot{\theta}^a}}_{\text{int. by parts}} \right] \\
& \rightarrow \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}\ , \quad \rightarrow \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\theta}^a} = \frac{\partial L}{\partial \theta^a} \\
&= \delta \int d\tau \left(\dot{q}^\mu p_\mu + \dot{\theta}^a \Pi_a - H(q,p,\theta,\Pi) \right) \\
&= \int d\tau \left(\delta\dot{q}\ p + \dot{q}\ \delta p + \delta\dot{\theta}\ \Pi + \dot{\theta}\ \delta\Pi - \delta q\ \frac{\partial H}{\partial q} - \delta p \frac{\partial H}{\partial P} - \delta\theta\ \frac{\partial H}{\partial \theta} - \delta\Pi\ \frac{\partial H}{\partial \Pi} \right) \\
& \text{int. by parts:} \\
&= \int d\tau \left(-\dot{p}\ \delta q + \dot{q}\ \delta p - \dot{\Pi}\ \delta\theta + \dot{\theta}\ \delta\Pi - \delta q\ \frac{\partial H}{\partial q} - \delta p \frac{\partial H}{\partial p} - \delta\theta\ \frac{\partial H}{\partial \theta} - \delta\Pi\ \frac{\partial H}{\partial \Pi} \right)
\end{aligned}$$

And so, for this to be zero, we must have:

$$\dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}\ , \quad \dot{\Pi}^a = -\frac{\partial H}{\partial \theta_a} \tag{2.1.9}$$

$$\dot{q}_\mu = +\frac{\partial H}{\partial p^\mu}\ , \quad \dot{\theta}^a = -\frac{\partial H}{\partial \Pi^a} \tag{2.1.10}$$

2.2 Poisson Bracket

$A(q,p,\theta,\Pi)$, $B(q,p,\theta,\Pi)$
Note:

- Anything Grassmann is odd
- Anything else is even

For the order of two quantities:

- Even/Even (doesn't matter)
- Even/Odd (doesn't matter)
- Odd/Odd → Switch the two, you pick up a -ve sign.

$$[A,B]_{PB} = \begin{cases} \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} + \frac{\partial A}{\partial \theta}\frac{\partial B}{\partial \Pi} - \frac{\partial B}{\partial \theta}\frac{\partial A}{\partial \Pi} & \text{1. A,B "even"} \\ \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} + \frac{\partial A}{\partial \theta}\frac{\partial B}{\partial \Pi} + \frac{\partial A}{\partial \Pi}\frac{\partial B}{\partial \theta} & \\ \hline \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} + \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} - \frac{\partial A}{\partial \theta}\frac{\partial B}{\partial \Pi} - \frac{\partial B}{\partial \theta}\frac{\partial A}{\partial \Pi} & \text{2. A,B "Odd"} \\ \hline \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} + \frac{\partial A}{\partial \theta}\frac{\partial B}{\partial \Pi} + \frac{\partial B}{\partial \theta}\frac{\partial A}{\partial \Pi} & \text{3. A "Even", B "Odd"} \end{cases}$$

With this, we find that (Not trivial!):

$$0 = [[A,B],C] + [[B,C],A] + [[C,A],B] \tag{2.2.1}$$

Look at the "spinning particle" (Not superparticle).
Simplest action involving Grassmann Variables.

(Brink et. al. NP B118)

$$\begin{aligned} L &= L(\phi^\mu(\tau), \psi^\mu(\tau)) \qquad (\phi^\mu \equiv x^\mu(\tau), \psi \to \text{Grassmann}) \\ &= \frac{1}{2}\left[\frac{\dot\phi^\mu \dot\phi_\mu}{e} - \mathrm{i}\psi^\mu(\tau)\ \dot\psi_\mu(\tau)\right] \ ; \quad \left(\text{cf } \ L = -\frac{1}{2}\left(-\frac{\dot x^2}{2} + m^2 e\right)\right) \end{aligned} \tag{2.2.2}$$

Note:

1. $\dot\psi^\mu \dot\psi_\mu = 0$ (why we can't have two $\dot\psi$ in L).
2. i → needed so that $L = L^+$

$$\begin{aligned} \underline{\text{i.e.}}\quad L &= (-\mathrm{i}\psi\dot\psi) \qquad (\text{Re: } (AB)^+ = B^+A^+) \\ L^+ &= (+\mathrm{i})(\dot\psi^+\psi^+) = \mathrm{i}\dot\psi\psi = -\mathrm{i}\psi\dot\psi = L \end{aligned}$$

3. The following

$$\tau \to \tau + f(\tau) \tag{2.2.3}$$
$$\delta\phi^\mu = \dot\phi f \tag{2.2.4}$$
$$\delta e = \dot f e + \dot e f \tag{2.2.5}$$
$$\delta\psi^\mu = \dot\psi^\mu f \tag{2.2.6}$$

is an invariance of $S = \int d\tau\ L$ (reparametrization invariance)

Quantizing this leads to negative norm states in the Hilbert space associated with $\psi^0(\tau)$.

$$< \psi^0|\psi^0 > \quad < \quad 0 \quad \text{(Don't want.)}$$

$$\left(\underline{\text{ex}} \quad L \quad = \quad -\frac{1}{2}\partial_\mu A^\lambda \, \partial^\mu A_\lambda \;\rightarrow A^\lambda A_\lambda = (A^0)^2 - \vec{A}^2 \quad (A^0 = \text{-ve norm state})\right)$$

We eliminate the unwanted negative norm state by building in an extra symmetry.

$$L = \frac{1}{2}\left(\frac{\dot{\phi}^2}{e} - \mathrm{i}\psi_\mu\dot{\psi}^\mu - \frac{\mathrm{i}}{e}\chi\dot{\phi}^\mu\psi_\mu\right) \quad ; \quad \chi(\tau) \;\text{ is Grassmann} \tag{2.2.7}$$

1. Add in $\delta\chi = \dot{f}\chi + \dot{\chi}f$

2. We have another invariance ($\alpha = \alpha(\tau)$ is Grassmann)

$$\delta_\alpha\phi^\mu \;=\; \mathrm{i}\alpha\psi^\mu \tag{2.2.8}$$

$$\delta_\alpha\psi^\mu \;=\; \alpha\left(\frac{\dot{\phi}^\mu}{e} - \frac{\mathrm{i}}{2e}\chi\psi^\mu\right) \tag{2.2.9}$$

$$\delta_\alpha e \;=\; \mathrm{i}\alpha\chi \tag{2.2.10}$$

$$\delta_\alpha\chi \;=\; 2\dot{\alpha} \tag{2.2.11}$$

$$\delta_\alpha L \;=\; \frac{d}{d\tau}(...) \quad \text{He didn't recall} \tag{2.2.12}$$

- ϕ - Scalar in 0+1 dimensions.
- ψ^μ - Spinor in 0+1 dimensions.
- e - "einbein" in 0+1 dimensions.
- χ - "gravitino" in 0+1 dimensions.

L - analogous to SUGRA L in 3+1 dimensions.

2.3 Quantization of the spinning particle.

$$\Pi_\mu \;=\; \frac{\partial L}{\partial\dot{\psi}^\mu} = \frac{\partial}{\partial\dot{\psi}^\mu}\left(-\frac{\mathrm{i}}{2}\psi_\nu\dot{\psi}^\nu\right) \tag{2.3.1}$$

$$\;=\; \frac{\mathrm{i}}{2}\psi_\mu \rightarrow \quad \text{Second Class constraint} \tag{2.3.2}$$

$$\theta_\mu = \Pi_\mu - \frac{\mathrm{i}}{2}\psi_\mu = 0 \tag{2.3.3}$$

$$\therefore\!\rightarrow [\theta_\mu, \theta_\nu]_{PB} \neq 0 \tag{2.3.4}$$

Sept. 28/99

We could proceed with the Dirac formalism. We could also cheat and use the equations of motion. Recall:

$$L = \frac{1}{2}\left(\frac{\dot{\phi}_\mu\dot{\phi}^\mu}{e} - \mathrm{i}\psi_\mu\dot{\psi}^\mu - \frac{\mathrm{i}}{e}\chi\dot{\phi}_\mu\psi^\mu\right) \tag{2.3.5}$$

$$\delta_\alpha\phi^\mu = \mathrm{i}\alpha\psi^\mu \tag{2.3.6}$$

$$\delta_\alpha\psi^\mu = \alpha\left(\frac{\dot{\phi}^\mu}{e} - \frac{\mathrm{i}}{2e}\chi\psi^\mu\right) \tag{2.3.7}$$

$$\delta_\alpha e = \mathrm{i}\alpha\chi \tag{2.3.8}$$

$$\delta_\alpha\chi = 2\dot{\alpha} \tag{2.3.9}$$

Eq. of motion for e:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{e}}\right) = \frac{\partial L}{\partial e} \tag{2.3.10}$$

$$0 = \dot{\phi}^2 - \mathrm{i}\chi\dot{\phi}_\mu\psi^\mu \tag{2.3.11}$$

For χ :

$$\dot{\phi}^\mu\psi_\mu = 0 \quad \rightarrow \text{ eliminates -ve norm states If } \phi^0 = v,\ v\psi_\nu = 0 \tag{2.3.12}$$

We could have gotten these two equations from the Hamiltonian formalism because these are the constraint equations that follow from the primary constraints

$$p_e = \frac{\partial L}{\partial\dot{e}} = 0 \tag{2.3.13}$$

$$p_\chi = \frac{\partial L}{\partial\dot{\chi}} = 0 \tag{2.3.14}$$

These are both first class constraints $\rightarrow$ gauge conditions.

$$\left.\begin{array}{l} p_\chi = 0 \\ p_e = 0 \end{array}\right\} \leftarrow \left.\begin{array}{l} \chi = 0 \\ e = 1 \end{array}\right\} \text{ “proper time” guage conditions.} \tag{2.3.15}$$

The equations of motion for:

ϕ^μ:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\phi}^\mu}\right) = \frac{\partial L}{\partial\phi^\mu} \tag{2.3.16}$$

$$0 = \frac{d}{d\tau}\left(\frac{2\dot{\phi}^\mu}{e} - \frac{\mathrm{i}\chi\psi^\mu}{e}\right) \tag{2.3.17}$$

$\underline{\psi^\mu}$:

$$0 = 2\dot{\psi}^\mu - \frac{\chi\dot{\phi}^\mu}{e} \tag{2.3.18}$$

It is possible to show that the Dirac Brackets are

$$[\phi_\mu, p_\nu]^* = g_{\mu\nu} \tag{2.3.19}$$

$$[\psi_\mu, \psi_\nu]^* = -\mathrm{i}g_{\mu\nu} \left(\text{Remember that } \; p_{\psi_\mu} = \frac{\partial L}{\partial \dot{\psi}_\mu} = \frac{\mathrm{i}}{2}\psi_\mu\right) \tag{2.3.20}$$

Quantization:

1. $[\ ,\ ]^* \to \frac{1}{\mathrm{i}\hbar}[\ ,\ ]_{(anti)-commutator}$
   Thus: (letting $\hbar = 1$)

$$\left[\hat{\phi}_\mu, \hat{p}_\nu\right]_- = \mathrm{i}g_{\mu\nu} \to \hat{\phi}_\mu\hat{p}_\nu - \hat{p}_\nu\hat{\phi}_\mu = \mathrm{i}g_{\mu\nu} \tag{2.3.21}$$

$$\left[\hat{\psi}_\mu, \hat{\psi}_\nu\right]_+ = g_{\mu\nu} \to \hat{\psi}_\mu\hat{\psi}_\nu + \hat{\psi}_\nu\hat{\psi}_\mu = g_{\mu\nu} \tag{2.3.22}$$

2. $\chi_i|\psi> \;= 0$ for any constraint χ_i.

In the gauge $\chi = 0,\ e = 1$

$$p_\mu = \frac{\partial L}{\partial \dot{\phi}_\mu} = \dot{\phi}^\mu - \underbrace{\frac{\mathrm{i}}{2e}\chi\psi^\mu}_{=0} \tag{2.3.23}$$

Thus:

$$\dot{\phi}^2 = 0 \Rightarrow \hat{p}^2|\Psi> \;= 0 \tag{2.3.24}$$

$$\dot{\phi}\cdot\psi = 0 \Rightarrow \hat{p}\cdot\hat{\psi}|\Psi> \;= 0 \tag{2.3.25}$$

If we let

$$\hat{\psi}_\mu = \frac{\gamma_\mu}{\sqrt{2}} \tag{2.3.26}$$

and as $\dot{\psi}^\mu = 0$ in the Heisenberg picture, then γ^μ is a constant and satisfies the algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (\{\ ,\ \} \to \text{anticommutator}). \tag{2.3.27}$$

If the coordinate representation

$$\Psi(\phi) =< \phi|\Psi > \tag{2.3.28}$$

then

$$[\hat{\phi}_\mu, \hat{p}_\nu] = \mathrm{i}g_{\mu\nu} \tag{2.3.29}$$

$$\to p_\nu = -\mathrm{i}\frac{\partial}{\partial\phi^\nu} \text{ and } \; \hat{p}\cdot\hat{\psi}|\Psi> \;= 0 \text{ becomes}$$

$$-\mathrm{i}\gamma^\mu\frac{\partial}{\partial\phi^\mu}\Psi(\phi) = 0 \tag{2.3.30}$$

This is the massless Dirac equation.

For the massive Dirac equation,

$$S = \frac{1}{2}\int d\tau \left(\frac{\dot{\phi}^2}{e} - \mathrm{i}\psi \cdot \dot{\psi} - \frac{\mathrm{i}\chi\dot{\phi}\cdot\psi}{e} + em^2 + \mathrm{i}\psi_5\dot{\psi}_5 + \mathrm{i}m\chi\psi_5\right) \tag{2.3.31}$$

Following the same sort of argument as above, we find,

$$\rightarrow \left[\gamma_\mu\left(-\mathrm{i}\frac{\partial}{\partial\phi_\mu}\right) - m\right]\Psi(\phi) = 0$$

<u>Dirac's Approach</u>

Schrödinger Equation: (note first order time derivative)

$$\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} = H(x,p)\psi(x,t) \tag{2.3.32}$$

$$\rightarrow \vec{p} = -\mathrm{i}\hbar\nabla \tag{2.3.33}$$

Klein-Gordon equation (K.G.):

$$0 = \left(p_\mu p^\mu - m^2\right)\psi \tag{2.3.34}$$

$$p_\mu p^\mu = \nabla^2 - \underbrace{\frac{\partial^2}{\partial t^2}}_{2^{nd} order} \quad (c, \hbar = 1) \tag{2.3.35}$$

<u>From Dirac equation</u>

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + m\beta)\psi \tag{2.3.36}$$

For ψ to also satisfy the K.G. equation, then

$$\left(\mathrm{i}\frac{\partial}{\partial t}\right)^2\psi = (\vec{\alpha}\cdot\vec{p} + m\beta)^2\psi \tag{2.3.37}$$

$$-\frac{\partial^2}{\partial t^2}\psi = \left[\frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)p_ip_j + 2m(\alpha_i\beta + \beta\alpha_i)p_i + m^2\beta^2\right]\psi \tag{2.3.38}$$

Thus

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{2.3.39}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.3.40}$$

$$\beta^2 = 1 \tag{2.3.41}$$

$$\beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix} \quad , \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{2.3.42}$$

$$\text{where } \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; , \; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; , \; \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \; , \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.3.43}$$

Eddington showed that all representations of $\alpha_i,\ \beta$ in 4-d are unitarily equivalent.

Inclusion of the electromagnetic field:

$$\left.\begin{array}{r}\vec{p}\to\vec{p}-e\vec{A}\\ H\to H+e\Phi\end{array}\right\}(\vec{A},\ \Phi\to\ \text{electromagnetic potentials}) \tag{2.3.44}$$

Thus,

$$\mathrm{i}\frac{\partial\psi}{\partial t}=\left[\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta m+e\Phi\right]\psi \tag{2.3.45}$$

Aside:

1. This can be generalized to n dimensions.
   dimension of $(\vec{\alpha},\beta)$ is $\left\{\begin{smallmatrix}2^{n/2} & (n=\text{even})\\ 2^{(n-1)/2} & (n=\text{odd})\end{smallmatrix}\right\}$

2. In even dimensions all $(\vec{\alpha},\ \beta)$ are unitarily equivalent, but in odd # dimensions there are 2 sets $(\vec{\alpha},\beta),\ (+\vec{\alpha},-\beta)$ (not unitarily equivalent to each other $\to$ all sets are unitarily equivalent to one or the other, not both).

Sept. 29/99
Backtrack

$$\begin{aligned}[q,p]&=\mathrm{i}\hbar\\ \to[x_i,p_j]&=\mathrm{i}\hbar\delta_{ij}\\ p_j&=-\mathrm{i}\hbar\frac{\partial}{\partial x_j} &(2.3.46)\\ \mathrm{i}\hbar\frac{\partial\psi}{\partial t}&=H\psi\quad(H\to E) &(2.3.47)\\ \therefore E&=\mathrm{i}\hbar\frac{\partial}{\partial t} &(2.3.48)\end{aligned}$$

But also,

$$\begin{aligned}x^\mu&=(\vec{x},t)\ \ (c=1)\\ p^\mu&=(\vec{p},E)\end{aligned}$$

(consistent with (2.3.46),(2.3.48) provided $p_\mu=\mathrm{i}\frac{\partial}{\partial x^\mu}$).

i.e. (Metric $(-,-,-,+)$)

$$\begin{aligned}
p_\mu &= g_{\mu\nu}p^\nu \\
&= (-\vec{p},E) \leftarrow (\vec{p} = -\mathrm{i}\nabla \ , \quad E = \mathrm{i}\frac{\partial}{\partial t}) \\
&= \mathrm{i}\partial_\mu \\
&= \mathrm{i}\frac{\partial}{\partial x^\mu} \\
&= \mathrm{i}\left(\nabla, \ \frac{\partial}{\partial t}\right) \\
\mathrm{i}\frac{\partial\psi}{\partial t} &= \left[\vec{\alpha}\cdot(\vec{p}-e\vec{A}) + \beta m + e\Phi\right]\psi \\
\rightarrow \vec{\alpha} &= \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \ , \quad \beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore \psi &= 4 - \text{components} \\
&= \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}
\end{aligned} \tag{2.3.49}$$

We had

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \frac{\vec{p}^2}{2m}\psi$$

Pauli equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \left[\frac{(\vec{p}-e\vec{A})^2}{2m} + e\Phi + \frac{\vec{\sigma}\cdot\vec{B}}{2m}\right]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{2.3.50}$$

Let ψ be:

$$\psi = \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\chi}_1 \\ \tilde{\chi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{\phi} \\ \tilde{\chi} \end{bmatrix}}_{\substack{\text{Slowly} \\ \text{varying}}} = \begin{bmatrix} \phi \\ \chi \end{bmatrix} \underbrace{e^{-\mathrm{i}mt}}_{*} \tag{2.3.51}$$

* - Principle time dependence for “slowly moving” particles. ($\hbar = 1$).
i.e.

$$\mathrm{i}\frac{\partial\psi}{\partial t} = H\psi = E\psi \tag{2.3.52}$$

$$\rightarrow \quad \psi \sim e^{-\mathrm{i}Et} = e^{-\mathrm{i}mt} \text{ in rest frame.} \tag{2.3.53}$$

Thus,

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial t}\left\{\begin{bmatrix}\phi\\ \chi\end{bmatrix}e^{-\mathrm{i}mt}\right\} &= \left\{\begin{bmatrix}0 & \vec{\sigma}\cdot\vec{\Pi}\\ \vec{\sigma}\cdot\vec{\Pi} & 0\end{bmatrix}+\begin{bmatrix}m & 0\\ 0 & -m\end{bmatrix}+\begin{bmatrix}e\Phi & 0\\ 0 & e\Phi\end{bmatrix}\right\}\begin{bmatrix}\phi\\ \chi\end{bmatrix}e^{-\mathrm{i}mt}\\
\begin{bmatrix}\mathrm{i}\frac{\partial\phi}{\partial t}+m\phi\\ \mathrm{i}\frac{\partial\chi}{\partial t}+m\chi\end{bmatrix} &= \begin{bmatrix}\vec{\sigma}\cdot\vec{\Pi}\chi+m\phi+e\Phi\psi\\ \vec{\sigma}\cdot\vec{\Pi}\phi-m\chi+e\Phi\chi\end{bmatrix} \qquad (2.3.54)
\end{aligned}$$

(Assume $\frac{\partial\chi}{\partial t},\ e\Phi\chi\approx 0\rightarrow(\chi$ is “small”))
Thus from 2^{nd} equation,

$$\chi=\frac{\vec{\sigma}\cdot\vec{\Pi}\phi}{2m} \qquad (2.3.55)$$

Hence:

$$\mathrm{i}\frac{\partial\phi}{\partial t}=\left[\frac{(\vec{\sigma}\cdot\vec{\Pi})^2}{2m}+e\Phi\right]\phi \qquad (2.3.56)$$

Hence,

$$(\vec{\sigma}\cdot\vec{\Pi})^2=\left(\begin{bmatrix}(p-eA)_3 & \{(p-eA)_1-\mathrm{i}(p-eA)_2\}\\ \{(p-eA)_1+\mathrm{i}(p-eA)_2\} & (p-eA)_3\end{bmatrix}\right)^2 \qquad (2.3.57)$$

But:

$$\begin{aligned}
\sigma_i\sigma_j &= \delta_{ij}+\mathrm{i}\epsilon_{ijk}\sigma_k \qquad (2.3.58)\\
(\sigma^i)_{ab}(\sigma^i)_{cd} &= 2\delta_{ad}\delta_{bc}-\delta_{ab}\delta_{cd} \qquad (2.3.59)\\
\therefore(\vec{\sigma}\cdot\vec{\Pi})^2 &= (\sigma_i\sigma_j)\Pi_i\Pi_j\\
&= (\delta_{ij}+\mathrm{i}\epsilon_{ijk}\sigma_k)(p-eA)_i(p-eA)_j\\
&= (\vec{p}-e\vec{A})^2-\mathrm{i}(\vec{p}-e\vec{A})\times(\vec{p}-e\vec{A})\cdot\vec{\sigma}\\
&\quad \rightarrow[p_i,A_j]=-\mathrm{i}\frac{\partial A_j}{\partial x^i}\quad(\vec{p}=-\mathrm{i}\nabla)\\
&= (\vec{p}-e\vec{A})^2+e(\nabla\times\vec{A})\cdot\vec{\sigma}\\
&= (\vec{p}-e\vec{A})^2+e\vec{B}\cdot\vec{\sigma}
\end{aligned}$$

<u>“Conserved Current”</u>

$$\begin{aligned}
\mathrm{i}\frac{\partial\psi}{\partial t} &= (\vec{\alpha}\cdot\vec{p}+\beta m)\psi=(-\mathrm{i}\nabla\cdot\vec{\alpha}+\beta m)\psi \qquad (2.3.60)\\
&\text{and}\\
(-\mathrm{i})\frac{\partial\psi^+}{\partial t} &= \psi^+(\vec{\alpha}^+\cdot\vec{p}^++\beta^+m)\ ;\quad(\vec{\alpha}^+=\vec{\alpha},\ \beta^+=\beta)\\
&= \psi^+(\vec{\alpha}\cdot\vec{p}^++\beta m)\\
&= (+\mathrm{i}\nabla\psi^+\cdot\vec{\alpha}+\psi^+\beta m) \qquad (2.3.61)
\end{aligned}$$

$\psi^+\times$ (2.3.60)

$$\rightarrow \mathrm{i}\psi^+\frac{\partial\psi}{\partial t} = \psi^+(-\mathrm{i}\nabla\cdot\vec{\alpha}\psi) + \psi^+\beta m\psi \tag{2.3.62}$$

(2.3.61) $\times\psi$

$$\rightarrow -\mathrm{i}\left(\frac{\partial\psi^+}{\partial t}\right)\psi = \left(\mathrm{i}(\nabla\cdot\psi^+)\cdot\vec{\alpha}\right)\psi + \psi^+\beta m\psi \tag{2.3.63}$$

Subtract the two

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial t}\left(\psi^+\psi\right) &= -\mathrm{i}\nabla\cdot(\psi^+\vec{\alpha}\psi) \Rightarrow \frac{\partial}{\partial t}(\psi^+\psi) + \nabla\cdot(\psi^+\vec{\alpha}\psi) = 0 \\
\rightarrow \underline{\text{i.e.}}\quad 0 &= \frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} \\
\text{where}\quad j^\mu &= (\psi^+\vec{\alpha}\psi,\ \psi^+\psi) \\
&= (\vec{j},\rho) \\
\rightarrow \partial_\mu j^\mu &= 0
\end{aligned} \tag{2.3.64}$$

$$\psi^+\psi \quad\rightarrow\quad \text{probability density} \tag{2.3.65}$$

$$\psi^+\vec{\alpha}\psi \quad\rightarrow\quad \text{probability flux.} \tag{2.3.66}$$

Free particle solution in the frame of reference where $\vec{p}\psi = 0$ ("rest frame"):

$$\begin{aligned}
\underline{i.e.} - \mathrm{i}\nabla\psi &= 0 \\
\therefore \psi &= \psi(t) \\
\mathrm{i}\frac{\partial}{\partial t}\begin{bmatrix}\phi(t)\\ \chi(t)\end{bmatrix} &= \beta m\begin{bmatrix}\phi\\ \chi\end{bmatrix} \quad \rightarrow \beta = \begin{bmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{bmatrix} \\
&= \begin{bmatrix} m & 0\\ 0 & -m\end{bmatrix}\begin{bmatrix}\phi\\ \chi\end{bmatrix}
\end{aligned} \tag{2.3.67}$$

$$\therefore\quad \phi(t) = e^{-\mathrm{i}mt}\phi_0 \tag{2.3.68}$$

$$\chi(t) = e^{+\mathrm{i}mt}\chi_0 \tag{2.3.69}$$

$$\psi(t) = \begin{bmatrix} e^{-\mathrm{i}mt}\phi_0 \\ e^{+\mathrm{i}mt}\chi_0\end{bmatrix} \tag{2.3.70}$$

$$\rightarrow e^{+\mathrm{i}mt} = \text{"negative energy" (associated with } -\sqrt{\vec{p}^2+m^2})$$

K.G. equation

$$(-\vec{p}^2 + p_0^2 - m^2)\psi = 0 \tag{2.3.71}$$

$$p_0\psi = \pm\sqrt{\vec{p}^2+m^2}\psi \tag{2.3.72}$$

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi \tag{2.3.73}$$

Oct. 1/99

$$\psi = \begin{bmatrix} e^{-\mathrm{i}mt}\psi_1^0 \\ e^{-\mathrm{i}mt}\psi_2^0 \\ e^{+\mathrm{i}mt}\psi_3^0 \\ e^{+\mathrm{i}mt}\psi_4^0 \end{bmatrix} \rightarrow \text{ to solve } i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi, \quad \text{perform a "Boost".} \tag{2.3.74}$$

Rewrite the Dirac equation in a covariant form.

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (-\mathrm{i}\vec{\alpha}\cdot\nabla + \beta m)\psi \tag{2.3.75}$$

$$0 = \left[\mathrm{i}\frac{\partial}{\partial t}(\beta\psi) + \mathrm{i}\nabla\cdot(\beta\vec{\alpha}) - m\right]\psi \tag{2.3.76}$$

Remember that

$$\mathrm{i}\partial_\mu = \left(\mathrm{i}\nabla, \mathrm{i}\frac{\partial}{\partial t}\right) \tag{2.3.77}$$

Then we have

$$\left(\mathrm{i}\partial_\mu\gamma^\mu - m\right)\psi = 0 \tag{2.3.78}$$

where $\gamma^0 = \beta = +\gamma_0 \qquad \gamma^i = \beta\alpha^i = -\gamma_i$

$$\text{As} \quad \{\alpha^i, \alpha^j\} = 2\delta^{ij} \tag{2.3.79}$$

$$\{\beta, \alpha^i\} = 0 \tag{2.3.80}$$

$$\beta^2 = 1 \tag{2.3.81}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad ; \qquad g^{\mu\nu} = \begin{bmatrix} - & & & \\ & - & & \\ & & - & \\ & & & + \end{bmatrix} \tag{2.3.82}$$

Different Representations are:

$$\text{Standard:} \qquad \gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{2.3.83}$$

$$\text{Chiral Representation:} \qquad \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{2.3.84}$$

Often, $a_\mu\gamma^\mu$ is written $\not{a}$.

Lorentz transformation: $\rightarrow$ Either a Boost or a rotation.
Boost in x-direction:

$$x' = \frac{x - vt}{\sqrt{1-v^2}}, \qquad t' = \frac{-vx + t}{\sqrt{1-v^2}} \tag{2.3.85}$$

$$y' = y \tag{2.3.86}$$

$$z' = z \tag{2.3.87}$$

Rotation (example):

$$\begin{aligned} x' &= x\cos(\theta) - y\sin(\theta) \\ y' &= x\sin(\theta) + y\cos(\theta) \end{aligned}$$

We know

$$\begin{aligned} g^{\mu\nu}x'^{\mu}x'^{\nu} &= g_{\mu\nu}x^{\mu}x^{\nu} \\ &= t^2 - x^2 - y^2 - z^2 \end{aligned} \tag{2.3.88}$$

$$\underline{\text{Linear:}} \quad x'^{\mu} = a^{\mu}_{\nu}x^{\nu} \tag{2.3.89}$$

$$g_{\mu\nu}a^{\mu}_{\lambda}a^{\nu}_{\sigma} = g_{\lambda\sigma} \tag{2.3.90}$$

→ The Lorentz transformation is an O(3,1) transformation.

Under the Lorentz transformation we set

$$\psi'(x') = S(a)\psi(x) \quad (S(a) = 4\times 4 \text{ matrix.})$$

With

$$0 = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}} - m\right)\psi'(x') \tag{2.3.91}$$

$$0 = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x} - m\right)\psi(x) \tag{2.3.92}$$

(i.e. should have both the same masses and the same γ^{μ} matrices in each equation → <u>not</u> a γ'^{μ}, as in $\left(\mathrm{i}\gamma'^{\mu}\frac{\partial}{\partial x'^{\mu}} - m\right)\psi'(x') = 0$)
Note first:

$$\begin{aligned} \frac{\partial}{\partial x^{\mu}} &= \frac{\partial x'^{\lambda}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\lambda}} \\ &= a^{\lambda}_{\mu}\frac{\partial}{\partial x'^{\lambda}} \end{aligned}$$

Hence, (substitute into (2.3.92)),

$$0 = \left[\mathrm{i}\gamma^{\mu}\left(a^{\lambda}_{\mu}\frac{\partial}{\partial x'^{\lambda}}\right) - m\right]S^{-1}(a)\psi'(x')$$

→ multiply on left by $S(a)$.

$$0 = \left[\mathrm{i}\left(S(a)\gamma^{\mu}S^{-1}(a)\right)a^{\lambda}_{\mu}\frac{\partial}{\partial x'^{\lambda}} - m\right]\psi'(x') \tag{2.3.93}$$

But change of variables should leave us with an equation in the same form.

$$\therefore \gamma^\mu = S(a)\gamma^\rho S^{-1}(a) a^\mu_\rho \tag{2.3.94}$$

From this we determine $S(a)$

Suppose the Lorentz transformation is infinitesimal:

$$a^\mu_\nu = \delta^\mu_\nu + \Delta w^\mu_\nu \tag{2.3.95}$$

But

$$g_{\mu\nu} a^\mu_\lambda a^\nu_\sigma = g_{\lambda\sigma}$$

$$\therefore g_{\lambda\sigma} = g_{\mu\nu}(\delta^\mu_\lambda + \Delta w^\mu_\lambda)(\delta^\nu_\sigma + \Delta w^\nu_\sigma) \tag{2.3.96}$$

To order Δw, we see

$$0 = g_{\mu\nu}(\Delta w^\mu_\lambda \delta^\nu_\sigma + \delta^\mu_\lambda \Delta w^\nu_\sigma)$$

$$\text{or} \quad \Delta w_{\lambda\sigma} = -\Delta w_{\sigma\lambda} \tag{2.3.97}$$

Now take

$$S(a) = S(\mathbb{I} + \Delta w)$$

$$= \mathbb{I} - \frac{\mathrm{i}}{4}\sigma_{\mu\nu}\Delta w^{\mu\nu} \tag{2.3.98}$$

$\rightarrow$ Need $\sigma_{\mu\nu}$. We also have

$$S^{-1}(a) = \mathbb{I} + \frac{\mathrm{i}}{4}\sigma_{\mu\nu}\Delta w^{\mu\nu} \tag{2.3.99}$$

Hence

$$\gamma^\mu = \left(\mathbb{I} - \frac{\mathrm{i}}{4}\sigma_{\alpha\beta}\Delta w^{\alpha\beta}\right)\gamma^\rho\left(\mathbb{I} + \frac{\mathrm{i}}{4}\sigma_{\gamma\delta}\Delta w^{\gamma\delta}\right)\left(\delta^\mu_\rho + \Delta w^\mu_\rho\right) \tag{2.3.100}$$

$\rightarrow$ must hold to order Δw.
Consequently,

$$0 = \gamma^\rho \Delta w^\mu_\rho + \frac{\mathrm{i}}{4}\left(\gamma^\mu \sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^\mu\right)\Delta w^{\alpha\beta}$$

Hence

$$\frac{\mathrm{i}}{4}\left(\gamma^\mu \sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^\mu\right) = \frac{-1}{2}\left(\gamma_\beta \delta^\mu_\alpha - \gamma_\alpha \delta^\mu_\beta\right)$$

Solution:

$$\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{2.3.101}$$

For a finite Lorentz transformation, let

$$\Delta w^{\mu\nu} = \frac{1}{N} w^{\mu\nu} \qquad (2.3.102)$$

$$\therefore \quad S(a) = \lim_{N\to\infty} \left(1 - \frac{\mathrm{i}}{4}\sigma_{\mu\nu}\frac{w^{\mu\nu}}{N}\right)^N$$

$$= \exp\{-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}w^{\mu\nu}\} \qquad (2.3.103)$$

and if

$$x'^{\mu} = a^{\mu}_{\nu} x^{\nu} \qquad (2.3.104)$$

then

$$\psi'(x') = \exp\left\{-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}w^{\mu\nu}\right\}\psi(x) \qquad (2.3.105)$$

If

$$\Delta w_{\mu\nu} = \Delta w(\mathbb{I})_{\mu\nu}$$

$\rightarrow \Delta w =$ scalar,

$(\mathbb{I})_{\mu\nu} = 4 \times 4$ matrix characterizing the Lorentz transformation.

then for a rotation about the x-axis,

$$x' = x - \Delta w \; y \qquad (2.3.106)$$

$$y' = \Delta w \; x + y \qquad (2.3.107)$$

($\rightarrow$ infinitesimal version of "reg." rotation.)
In this case,

$$\Delta w_{\mu\nu} = \Delta w \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(where rows/columns $= [x, y, z, t]$)

Thus

$$S(a) = \exp\left\{-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}w^{\mu\nu}\right\}$$

$$= e^{-\mathrm{i}\sigma_{\mu\nu}w/2}$$

But,

$$\begin{aligned}
\sigma_{21} &= \frac{\mathrm{i}}{2}(\gamma_2\gamma_1 - \gamma_1\gamma_2) \\
&= \mathrm{i}\gamma_2\gamma_1 \\
&= \mathrm{i}\begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \\
&= (-1)\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \\
\text{Thus} & \\
S(a) &= \exp\left\{\frac{\mathrm{i}}{2}w\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}\right\} \qquad (2.3.108)
\end{aligned}$$

If $w = 2\pi$ (1 revolution)

$$S(w = 2\pi) = \exp\left\{\mathrm{i}\pi\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}\right\} \quad ; \quad \sigma_3^2 = 1$$

We know,

$$\begin{aligned}
e^{\vec{a}\cdot\vec{\sigma}} &= 1 + (\vec{a}\cdot\vec{\sigma}) + \frac{(\vec{a}\cdot\vec{\sigma})^2}{2!} + \ldots \\
& \quad \text{But } \rightarrow (\vec{a}\cdot\vec{\sigma})^2 = \vec{a}^2 \quad [\sigma_i\sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k] \\
&= \left(1 + \frac{\vec{a}^2}{2!} + \frac{(\vec{a}^2)^2}{4!} + \ldots\right) + \frac{\vec{a}\cdot\vec{\sigma}}{|\vec{a}|}\left(|\vec{a}| + \frac{|\vec{a}|^3}{3!} + \ldots\right) \\
&= \cosh|\vec{a}| + \sinh|\vec{a}|\frac{\vec{\sigma}\cdot\vec{a}}{|\vec{a}|} \\
\text{and so} \quad S(w = 2\pi) &= \exp\left\{\mathrm{i}\pi\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}\right\} \\
&= \underbrace{\cosh(\mathrm{i}\pi)}_{=-1} + \underbrace{\sinh(\mathrm{i}\pi)}_{=0}\sigma_3 \\
&= -1
\end{aligned}$$

Thus if $w = 2\pi$, $\psi'(x') = -\psi(x)$, and if $w = 4\pi$,

$$\psi'(x') = +\psi(x) \qquad (2.3.109)$$

$\therefore$ All physical quantities need an even # of ψ's.
$\rightarrow$ in 360^0 rotation,

$$\begin{aligned}
\psi\psi\psi &\Rightarrow -\psi\psi\psi \\
\psi\psi\psi\psi &\Rightarrow +\psi\psi\psi\psi
\end{aligned}$$

Oct. 5/99
So, we have:
Dirac Equation:

$$0 = (\mathrm{i}\gamma^\mu \partial_\mu - m)\,\psi \tag{2.3.110}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad ; \quad g^{\mu\nu} = (+ - - -) \tag{2.3.111}$$

$$\psi \rightarrow \exp\left\{-\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right\}\psi \tag{2.3.112}$$

$$\psi^+ \rightarrow \psi^+\left(\exp\left\{-\frac{\mathrm{i}}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right\}\right)^+ \tag{2.3.113}$$

$$\text{with } \sigma_{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{2.3.114}$$

Properties:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (\gamma^0)^+ \qquad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} = -(\gamma^i)^+$$

$$\begin{aligned} \sigma_{0i} &= \frac{\mathrm{i}}{2}\left[\gamma_0, \gamma_i\right] \\ &= \mathrm{i}\gamma_0\gamma_i \\ &= -\sigma_{i0} \end{aligned}$$

i.e.

$$\begin{aligned} (\sigma_{0i})^+ &= (\mathrm{i}\gamma_0\gamma_i)^+ \\ &= (-\mathrm{i})(\gamma_i^+\gamma_0^+) \\ &= (-\mathrm{i})(-\gamma_i)(\gamma_0) \\ &= -\sigma_{0i} \end{aligned}$$

$$\sigma_{ij} = \sigma_{ij}^+$$

Thus

$$\gamma_0\sigma_{\mu\nu}^+\gamma_0 = \sigma_{\mu\nu} \tag{2.3.115}$$

i.e. $\mu = 0,\ \nu = i$

$$\gamma_0\sigma_{0i}^+\gamma_0 = \gamma_0(-\sigma_{0i})\gamma_0 = (\gamma_0)^2\sigma_{0i} = \sigma_{0i}$$

$\mu = i, \nu = j$

$$\gamma_0\sigma_{ij}^+\gamma_0 = \gamma_0(+\sigma_{ij})\gamma_0 = \gamma_0^2\sigma_{ij} = \sigma_{ij}$$

Thus if

$$\bar{\psi} = \psi^+\gamma_0$$

then under a Lorentz transformation,

$$\begin{aligned}
\bar\psi &\rightarrow \left(\exp\left\{-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right\}\psi\right)^{+}\gamma_0 \\
&= \psi^{+}\left(\exp\left\{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}\right)^{+}\gamma_0 \\
&= \psi^{+}\left(\exp\left\{+\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu+}\right\}\right)\gamma_0 \quad ; \qquad \gamma_0^2 = 1 \\
&= \underbrace{\psi^{+}\gamma_0}_{\bar\psi}\underbrace{\gamma_0\exp\left\{\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu+}\right\}\gamma_0}_{\exp\left\{\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}} \\
\therefore \quad \bar\psi &\rightarrow \bar\psi\exp\left\{\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}
\end{aligned}$$

Hence

$$\left.\begin{aligned}\psi &\rightarrow U\psi \\ \bar\psi &\rightarrow \bar\psi U^{-1}\end{aligned}\right\} \rightarrow \quad U = \exp\left\{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}$$

Thus, (for ex.)

$$\bar\psi\psi \rightarrow \bar\psi U^{-1}U\psi = \bar\psi\psi$$

(i.e. Lorentz invariant).

Lorentz transformations involving parity (discrete transformations).

$$\begin{aligned}
(x,y,z,t) &\rightarrow (-x,-y,-z,t) \\
\psi(x) &\rightarrow \psi'(x') = S(a)\psi(x) \\
\underline{\text{Re.}} \qquad S(a)a^{\mu}_{\nu}\gamma^{\nu}S^{-1}(a) &= \gamma^{\mu} \qquad (2.3.116)
\end{aligned}$$

$$x'^{\mu} = a^{\mu}_{\nu}x^{\nu} \qquad (2.3.117)$$

$$a^{\mu}_{\nu} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix} \qquad (2.3.118)$$

$$S\gamma^{i}S^{-1} = -\gamma^{i} \qquad (2.3.119)$$

$$S\gamma^{0}S^{-1} = +\gamma^{0} \qquad (2.3.120)$$

Thus

$$\begin{aligned}
S &= e^{\mathrm{i}\phi}\gamma^0 \\
&= P \quad \text{(Parity operator)} \qquad (2.3.121)
\end{aligned}$$

<u>i.e.</u>

$$\psi'(x') = e^{\mathrm{i}\phi}\gamma^0\psi(x) \quad \text{(for parity operation)} \qquad (2.3.122)$$

Complete set of γ matrices (4×4 matrices).

$$\begin{array}{ccccc} \mathbb{I} & \gamma^\mu & \gamma_5 & \gamma^\mu\gamma_5 & \sigma^{\mu\nu} \\ (1) & (4) & (1) & (4) & (6) \end{array} = (16 \ \text{ total})\ (\text{total linearly independent}) \quad (2.3.123)$$

$$\rightarrow \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \quad (2.3.124)$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right] \quad (2.3.125)$$

$$\therefore \quad \left\{\gamma^\mu, \gamma^5\right\} = 0 \quad (\text{anticommute!}) \quad (2.3.126)$$

$$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad (2.3.127)$$

$$\begin{aligned} \gamma_5 &= \mathrm{i}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \\ &= \mathrm{i}\begin{bmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{bmatrix}\begin{bmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (2.3.128)$$

$$\begin{aligned} \gamma^0\gamma_5 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned} \quad (2.3.129)$$

$$\begin{aligned} \gamma^i\gamma_5 &= \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \end{aligned} \quad (2.3.130)$$

$$\sigma^{0i} = \mathrm{i}\gamma^0\gamma^i = \mathrm{i}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} = \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix} \quad (2.3.131)$$

$$\sigma^{ij} = \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix} = -\mathrm{i}\epsilon^{ijk}\begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix} \quad (2.3.132)$$

Note, for example,

$$\begin{aligned} \gamma^\alpha\gamma^\beta\gamma^\rho &= g^{\alpha\beta}\gamma^\rho - g^{\alpha\rho}\gamma^\beta + g^{\beta\rho}\gamma^\alpha - \mathrm{i}\epsilon^{\kappa\alpha\beta\rho}\gamma_\kappa\gamma_5 \\ & \quad (\epsilon_{0123} = +1) \end{aligned}$$

Transformation properties of bilinears in ψ, $\bar{\psi}$:

$$\bar{\psi}\psi \quad \rightarrow \quad \text{scalar}$$

$$i.e. \quad \left.\begin{array}{c} \psi \rightarrow S\psi \\ \bar{\psi} \rightarrow \bar{\psi}S^{-1} \end{array}\right\} \therefore \quad \bar{\psi}\psi \rightarrow \bar{\psi}\psi$$

$$\bar{\psi}\gamma_5\psi \quad \rightarrow \quad \bar{\psi}S^{-1}\gamma_5 S\psi$$

- If $S = \exp\left\{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}$; $[\gamma_5, \sigma^{\mu\nu}] = 0$

  $$\underline{\text{i.e.}} \quad \gamma_5 \frac{\mathrm{i}}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) - \frac{\mathrm{i}}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\gamma_5 = 0$$

  Thus, $[\gamma_5, S] = 0$

  $$\begin{aligned} \bar{\psi}\gamma_5\psi \quad \underbrace{\longrightarrow}_{S} & \quad \bar{\psi}\gamma_5 S^{-1} S\psi \\ = & \quad \bar{\psi}\gamma_5\psi \end{aligned}$$

- If $S = P = e^{\mathrm{i}\phi}\gamma^0$ (i.e. if S = parity operator)

  $$\begin{aligned} \bar{\psi}\gamma_5\psi \underbrace{\longrightarrow}_{P} \bar{\psi}\gamma^0 e^{-\mathrm{i}\phi}\gamma_5 e^{\mathrm{i}\phi}\gamma^0\psi &= -\bar{\psi}\gamma_5\gamma_0\gamma_0\psi \\ &= -\bar{\psi}\gamma_5\psi \end{aligned}$$

  (Picks up -ve → "Pseudo Scalar").

So also, one can show that

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi \quad & \underbrace{\longrightarrow}_{S} \quad a^\mu_\nu\bar{\psi}\gamma^\nu\psi \\ \bar{\psi}\gamma^0\psi \quad & \underbrace{\longrightarrow}_{P} \quad \bar{\psi}\gamma^0\psi \\ \bar{\psi}\gamma^i\psi \quad & \underbrace{\longrightarrow}_{P} \quad -\bar{\psi}\gamma^i\psi \end{aligned}$$

(The above are <u>vectors</u>)

and also

$$\begin{aligned} \bar{\psi}\gamma^\mu\gamma_5\psi \quad & \underbrace{\longrightarrow}_{S} \quad a^\mu_\nu\bar{\psi}\gamma^\nu\gamma_5\psi \\ \bar{\psi}\gamma^0\gamma_5\psi \quad & \underbrace{\longrightarrow}_{P} \quad -\bar{\psi}\gamma^0\gamma_5\psi \\ \bar{\psi}\gamma^i\gamma_5\psi \quad & \underbrace{\longrightarrow}_{P} \quad +\bar{\psi}\gamma^i\gamma_5\psi \end{aligned}$$

(The above are <u>axial vectors</u>)

and

$$\begin{aligned}
\bar{\psi}\sigma^{\mu\nu}\psi &\underbrace{\longrightarrow}_{S} a^{\mu}_{\lambda}a^{\nu}_{\kappa}\bar{\psi}\sigma^{\lambda\kappa}\psi \\
\bar{\psi}\sigma^{0i}\psi &\underbrace{\longrightarrow}_{P} -\bar{\psi}\sigma^{0i}\psi \\
\bar{\psi}\sigma^{ij}\psi &\underbrace{\longrightarrow}_{P} +\bar{\psi}\sigma^{ij}\psi
\end{aligned}$$

(The above are Tensors)

$$\begin{aligned}
\epsilon_{0123} &= +1 \text{ (Right Handed System)} \\
\epsilon^{\mu\nu\lambda\sigma} &\underbrace{\longrightarrow}_{S} a^{\mu}_{\alpha}a^{\nu}_{\beta}a^{\lambda}_{\gamma}a^{\sigma}_{\delta}\epsilon^{\alpha\beta\gamma\delta} \\
\epsilon^{\mu\nu\lambda\sigma} &\underbrace{\longrightarrow}_{P} -\epsilon^{\mu\nu\lambda\sigma} \quad (\epsilon_{0123} = -1 \rightarrow \text{ Left handed system})
\end{aligned}$$

Thus as

$$\sigma^{\mu\nu}\gamma_5 = \mathrm{i}\epsilon^{\mu\nu\lambda\sigma}\sigma_{\lambda\sigma}$$

we have

$$\bar{\psi}\sigma^{\mu\nu}\gamma_5\psi \text{ is a pseudotensor}$$

(in tensor ex., above, S eq. would hold, but signs on P's change).

Oct. 6/99

Useful Identities

$$\begin{aligned}
\gamma_\mu\gamma_\alpha\gamma^\mu &= [\underbrace{\gamma_\mu\gamma_\alpha + \gamma_\alpha\gamma_\mu}_{2g_{\mu\alpha}} - \gamma_\alpha\gamma_\mu]\gamma^\mu \\
&= 2g_{\mu\alpha}\gamma^\mu - \gamma_\alpha\gamma_\mu\gamma^\mu \ ; \quad \gamma_\mu\gamma^\mu = \frac{1}{2}(\gamma_\mu\gamma^\mu + \gamma^\mu\gamma_\mu) = \frac{1}{2}(2g^{\mu}_{\mu}) \\
&= 2\gamma_\alpha - \gamma_\alpha\frac{1}{2}(2(4)) \\
&= -2\gamma_\alpha \qquad (2.3.133) \\
\gamma_\alpha\gamma_\beta\gamma_\gamma &= g_{\alpha\beta}\gamma_\gamma - g_{\alpha\gamma}\gamma_\beta + g_{\beta\gamma}\gamma_\alpha - \mathrm{i}\epsilon_{\kappa\alpha\beta\gamma}\gamma^\kappa\gamma_5 \\
&= g_{\mu\alpha}\gamma^\mu - g^{\mu}_{\mu}\gamma_\alpha + g^{\mu}_{\alpha}\gamma_\mu - \mathrm{i}\underbrace{\epsilon^{\mu}_{\kappa\mu\alpha}}_{=0}\gamma^\kappa\gamma_5 \\
&= \gamma_\alpha - 4\gamma_\alpha + \gamma_\alpha \\
&= -2\gamma_\alpha \qquad (2.3.134)
\end{aligned}$$

So also,

$$\begin{aligned}
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu &= 4g_{\alpha\beta} \qquad (2.3.135) \\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\delta\gamma^\mu &= -2\gamma_\delta\gamma_\beta\gamma_\alpha \qquad (2.3.136) \\
\mathrm{Tr}[\gamma_\mu\gamma_\beta] &= 4g_{\mu\beta} \qquad (2.3.137) \\
\mathrm{Tr}[\gamma_\mu\gamma_\beta\gamma_\gamma\gamma_\delta] &= 4(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\delta}) \qquad (2.3.138)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}[\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}] &= \mathrm{Tr}[\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\gamma_5\gamma_5]\ ; \quad \gamma_5\gamma_5 = 1 \\
&= \mathrm{Tr}[\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\gamma_5] \ \text{(Using property of traces)} \\
&= (-1)^{2n+1}\mathrm{Tr}[\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\gamma_5] \ \text{(Moving } \gamma_5 \text{ through all (2n+1) matrices)} \\
\therefore \quad \mathrm{Tr}[\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}] &= 0 \qquad (2.3.139)
\end{aligned}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\beta\gamma_5] = 0 \qquad (2.3.140)$$

$$\mathrm{Tr}[\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5] = 4\mathrm{i}\epsilon_{\alpha\beta\gamma\delta} \qquad (2.3.141)$$

$$\begin{aligned}
\gamma_5 &= \gamma^5 \\
&= \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3
\end{aligned}$$

2.4 General Solution to the free Dirac Equation

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0 \qquad (2.4.1)$$

If

$$\gamma^0 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} = -\gamma_i \qquad (2.4.2)$$

then in the rest frame $\left(i.e. \frac{\partial}{\partial x^i}\psi = 0\right)$

$$\psi = \alpha_1 \underbrace{\begin{bmatrix} e^{-\mathrm{i}mt} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\psi_1^0} + \alpha_2 \underbrace{\begin{bmatrix} 0 \\ e^{-\mathrm{i}mt} \\ 0 \\ 0 \end{bmatrix}}_{\psi_2^0} + \alpha_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ e^{\mathrm{i}mt} \\ 0 \end{bmatrix}}_{\psi_3^0} + \alpha_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{\mathrm{i}mt} \end{bmatrix}}_{\psi_4^0} \qquad (2.4.3)$$

Using Boosts:

$$\begin{aligned}
S &= \exp\left\{-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right\} \quad \rightarrow \ \text{Can express in closed form.} \qquad (2.4.4) \\
\psi_i &= \exp\left\{-\frac{\mathrm{i}}{2}\omega\sigma_0\right\}\psi_i^0 \quad \rightarrow \ \cosh(\omega) = \frac{1}{\sqrt{1-v^2}}\ , \quad \sinh(\omega) = \frac{-v}{\sqrt{1-v^2}} \qquad (2.4.5)
\end{aligned}$$

Recall, if

$$\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1-v^2}} = \cosh(\tilde{\omega}x) - \sinh(\tilde{\omega}t) \\
t' &= \frac{-vx + t}{\sqrt{1-v^2}} = -\sinh(\tilde{\omega}x) + \cosh(\tilde{\omega}t) \\
\text{then} \qquad \cosh(\tilde{\omega}) &= \frac{1}{\sqrt{1-v^2}} \\
\sinh(\tilde{\omega}) &= \frac{v}{\sqrt{1-v^2}}
\end{aligned}$$

But $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$, So,

$$\begin{aligned}
\sigma_{01} &= \mathrm{i}\gamma_0\gamma_1 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \\
&= \mathrm{i} \begin{bmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \qquad (2.4.6) \\
\therefore \quad S &= \exp\left\{ -\frac{\mathrm{i}}{2}\omega\gamma_0\gamma_1 \right\} \\
&= \exp\left\{ -\frac{\omega}{2} \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \right\} \qquad (2.4.7)
\end{aligned}$$

Recall, $\sigma_1^2 = 1 \ \rightarrow \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}
\therefore \quad S &= 1 - \frac{\omega}{2} \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} + \frac{1}{2!}\left(\frac{\omega}{2}\right)^2 \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}^2 - \frac{1}{3!}\left(\frac{\omega}{2}\right)^3 \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}^3 + \cdots \\
&= \begin{bmatrix} \cosh\left(\frac{\omega}{2}\right) & -\sinh\left(\frac{\omega}{2}\right)\sigma_1 \\ -\sinh\left(\frac{\omega}{2}\right)\sigma_1 & \cosh\left(\frac{\omega}{2}\right) \end{bmatrix} \quad \text{(Closed form)} \qquad (2.4.8)
\end{aligned}$$

But $\tanh(\omega) = -v$, and recall the trig identities,

$$\begin{aligned}
\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \quad ; \qquad x = y = \theta \\
\therefore \quad \sinh(2\theta) &= 2\sinh(\theta)\cosh(\theta) \\
\cosh(2\theta) &= \cosh^2(\theta) + \sinh^2(\theta)
\end{aligned}$$

then

$$\begin{aligned}
\tanh\left(\frac{\omega}{2}\right) &= \frac{\tanh(\omega)}{1 + \sqrt{1 - \tanh^2(\omega)}} \\
&= \frac{-v}{1 + \sqrt{1 - v^2}}
\end{aligned}$$

But $p = \frac{mv}{\sqrt{1-v^2}} \qquad E = \frac{m}{\sqrt{1-v^2}}$,

$$\begin{aligned}\therefore \quad \tanh\left(\frac{\omega}{2}\right) &= \frac{-p}{E+m} \\ \cosh\left(\frac{\omega}{2}\right) &= \sqrt{\frac{E+m}{2m}}\end{aligned}$$

Hence,

$$\begin{aligned} S &= \exp\left\{-\frac{\mathrm{i}}{2}\omega\sigma_{01}\right\} \quad \text{(x-direction)} \\ &= \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 1 & 0 & 0 & \frac{p}{E+m} \\ 0 & 1 & \frac{p}{E+m} & 0 \\ 0 & \frac{p}{E+m} & 1 & 0 \\ \frac{p}{E+m} & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{2.4.9}$$

For a boost in an arbitrary direction:

$$(p_\pm = p_1 \pm \mathrm{i}p_2) \tag{2.4.10}$$

$$S = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 1 & 0 & \frac{p_3}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & \frac{p_3}{E+m} \\ \frac{p_3}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & \frac{p_3}{E+m} & 0 & 1 \end{bmatrix} \tag{2.4.11}$$

Call the columns of the above matrix $W_1(p),\ W_2(p),\ W_3(p),\ W_4(p)$. General solutions to the Dirac equation are:

$$\begin{aligned} \psi_1 &= W_1(p)e^{-\mathrm{i}p\cdot x} \qquad \psi_2 = W_2(p)e^{-\mathrm{i}p\cdot x} \\ \psi_3 &= W_3(p)e^{\mathrm{i}p\cdot x} \qquad \psi_4 = W_4(p)e^{\mathrm{i}p\cdot x} \quad (\text{Where } \mathrm{i}mt \to \mathrm{i}p\cdot x) \end{aligned}$$

<u>Properties of $W_i(p)$</u>

1.

$$\begin{aligned} (\not{p} - \epsilon_r m)W_r(p) &= 0\ ; \quad \epsilon_r = \begin{cases} +1, & r = 1,2 \\ -1, & r = 3,4 \end{cases} \\ \bar{W}(p)(\not{p} - \epsilon_r m) &= 0\ ; \qquad (\bar{W} = W^+\gamma_0) \end{aligned}$$

2. $\bar{W}^r(p)W^{r'}(p) = \delta_{rr'}\epsilon_r$

3. $(W^r(p))^+ \cdot W^{r'}(p) = \frac{E}{m}\delta^{rr'} \to \quad 4^{th}$ component of a vector.

$$\begin{aligned} & \bar{W}^r\gamma^\mu W^{r'} \\ =\ & (W^r)^+\gamma^0\gamma^\mu W^{r'} \\ & \to \mu = 0 \to \gamma^0\gamma^0 = 1 \\ =\ & (W^r)^+W^{r'} \\ \text{and} \quad & \frac{p^\mu}{m} = \frac{(\vec{p}, E)}{m} \end{aligned}$$

4. Completeness:

$$\sum_{r=1}^{4} = \epsilon_r W_\alpha^r(p)\bar{W}_\beta^r(p) = \delta_{\alpha\beta}$$

Oct. 8/99
So we have the following;

$$W_1^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-\mathrm{i}mt} \quad , \quad W_2^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-\mathrm{i}mt} \tag{2.4.12}$$

$$W_3^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{\mathrm{i}mt} \quad , \quad W_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{\mathrm{i}mt} \tag{2.4.13}$$

Now, boost to a frame where they have momentum p. For example,

$$W_1(p) = \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+\mathrm{i}p_y}{E+m} \end{bmatrix} \sqrt{\frac{E+m}{2m}} e^{-\mathrm{i}p\cdot x} \tag{2.4.14}$$

$$\Sigma_{03} W_{1,3}^0 = W_{1,3}^0 \tag{2.4.15}$$

$$\Sigma_{03} W_{2,4}^0 = -W_{2,4}^0 \tag{2.4.16}$$

$$\text{Where } \Sigma_{03} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = \begin{bmatrix} \sigma_z & \\ & \sigma_z \end{bmatrix} \tag{2.4.17}$$

To obtain an eigenstate of $\vec{\sigma} \cdot \hat{S}$ with eigenvalue ± 1, we consider

$$e^{-\frac{\mathrm{i}}{2}\phi \hat{S}\cdot\vec{\sigma}} W_{1,2}^0 = W_{1,2}^{0,s} \quad \text{(also for 3, 4)} \tag{2.4.18}$$

$$\text{where} \quad \cos\phi = \hat{S} \cdot \hat{n}_z \tag{2.4.19}$$

Upon boosting $W_i^{(0,s)}$, we obtain,

$$U(p, u_z) = W_1^{(u_z)}(p) \tag{2.4.20}$$

$$U(p, -u_z) = W_2^{(u_z)}(p) \tag{2.4.21}$$

$$V(p, -u_z) = W_3^{(u_z)}(p) \tag{2.4.22}$$

$$V(p, u_z) = W_4^{(u_z)}(p) \tag{2.4.23}$$

where now as

1. we have,

$$
\begin{aligned}
(\not{p} - \varepsilon_r m)W_r^{(s)}(p) &= 0 &\quad (2.4.24)\\
\text{so}\quad (\not{p} - m)U(p,u_z) &= 0 &\quad (2.4.25)\\
\text{and}\quad (\not{p} + m)V(p,u_z) &= 0 &\quad (2.4.26)
\end{aligned}
$$

2. and, as

$$
\begin{aligned}
\vec{\sigma}\cdot\hat{S}W_{1,3}^{(0,s)} &= W_{1,3}^{(0,s)} &\quad (2.4.27)\\
\vec{\sigma}\cdot\hat{S}W_{2,4}^{(0,s)} &= -W_{2,4}^{(0,3)} &\quad (2.4.28)
\end{aligned}
$$

$\Rightarrow$ in the rest frame, ($p_r = (0,0,0,m)$),

$$
\begin{aligned}
\vec{\sigma}\cdot\hat{S}u(0,0,0,m;s) &= u(p,s) &\quad (2.4.29)\\
\vec{\sigma}\cdot\hat{S}v(0,0,0,m;s) &= -v(p,s) &\quad (2.4.30)\\
\text{For example}\quad \vec{\sigma}\cdot\hat{S}u(0,0,0,m;-s) &= -u(p,-s)
\end{aligned}
$$

In general, any spinor which is a solution to the Dirac equation can be described as a linear superposition of these four eigenstates characterized by

1. $p_\mu (p^2 = m^2)$ "Mass shell condition"
2. Sign of p_0 (i.e. in the rest frame, $\frac{p_0}{m} = \pm 1$)
3. Eigenvalue of $\vec{\sigma}\cdot\hat{S}$ in the rest frame.

Thus,

$$
\begin{aligned}
\psi(\vec{x},t) &= \int d^4k\ \psi(k_m) &\quad (2.4.31)\\
&= \int d^4k\ \delta(k^2 - m^2)\sum_{s=\pm}\{b(k,s)u(k,s) + d^*(k,s)v(k,s)\} &\quad (2.4.32)
\end{aligned}
$$

Note if,

$$
\begin{aligned}
(\mathrm{i}\gamma\cdot\partial - m)\psi &= 0 &\quad (2.4.33)\\
\text{then}\quad (\mathrm{i}\gamma\cdot\partial + m)(\mathrm{i}\gamma\cdot\partial - m)\psi &= 0 &\quad (2.4.34)\\
(-\gamma\cdot\partial\gamma\cdot\partial - m^2)\psi &= 0 &\quad (2.4.35)\\
\Big(\text{Note:}\quad \gamma\cdot\partial\ \gamma\cdot\partial = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu &= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2\Big)\\
\therefore (\partial^2 + m^2)\psi &= 0\quad \text{(K.G. Eqn.)} &\quad (2.4.36)\\
\text{i.e., as}\quad p_\mu = \mathrm{i}\partial_\mu\ , &\qquad (p^2 - m^2)\psi(p) = 0 &\quad (2.4.37)
\end{aligned}
$$

Aside:

$$\begin{aligned}
\text{Recall} \quad & \int_{-\infty}^{\infty} dx\ \delta(cx) f(x) \quad ; \quad y = cx \\
= & \int_{-\infty}^{\infty} \frac{dy}{|c|}\ \delta y f\left(\frac{y}{c}\right) \\
= & \frac{f(0)}{|c|} \Rightarrow \delta(cx) = \frac{\delta(x)}{|c|}
\end{aligned}$$

So, note that;

$$\int_{-\infty}^{\infty} dx\ \delta(x^2 - a^2) f(x) = \int_{-\infty}^{\infty} dx\ \delta\left((x-a)(x+a)\right) f(x) \quad (\text{let} \quad a > 0) \tag{2.4.38}$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} dx\ \delta\left((x-a)(2a)\right) f(x) + \int_{-a-\varepsilon}^{-a+\varepsilon} dx\ \delta\left((x+a)(-2a)\right) f(x) \tag{2.4.39}$$

$$= \frac{f(a)}{|2a|} + \frac{f(-a)}{|-2a|} \tag{2.4.40}$$

$$= \frac{f(a) + f(-a)}{|2a|} \tag{2.4.41}$$

$$\text{thus} \quad \delta(x^2 - a^2) = \frac{\delta(x+a) + \delta(x-a)}{|2a|} \tag{2.4.42}$$

Hence for

$$\int d^4k\ \delta(k^2 - m^2)\psi(k) = \int d^3k \int_{-\infty}^{\infty} dk_0\ \delta(k_0^2 - \vec{k}^2 - m^2)\psi(\vec{k}, k_0) \tag{2.4.43}$$

$$= \int d^3k \int dk_0 \left(\frac{\delta(k_0^2 + \sqrt{\vec{k}^2 + m^2})}{2\sqrt{\vec{k}^2 + m^2}} + \frac{\delta(k_0^2 - \sqrt{\vec{k}^2 + m^2})}{2\sqrt{\vec{k}^2 + m^2}} \right) \psi(k) \tag{2.4.44}$$

Thus,

$$\psi(\vec{x}, t) = \int d^4k\ \delta(k^2 - m^2) \sum_{s=\pm} \left[b(k,s) u(k,s) \theta(k_0) e^{-\mathrm{i}k\cdot x} + d^*(k,s) v(k,s) \theta(-k_0) e^{+\mathrm{i}k\cdot x} \right] \tag{2.4.45}$$

Integrate over k_0 and insert normalizing factor $2\sqrt{\frac{m}{(2\pi)^3}}$, $(E = \sqrt{\vec{k}^2 + m^2})$

- +ve energy solution:

$$\psi^+(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \sum_{s=\pm} b(k,s) e^{-\mathrm{i}k\cdot x} u(k,s) \theta(k_0) \tag{2.4.46}$$

- -ve energy solution:

$$\psi^-(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \sum_{s=\pm} d^*(k,s) e^{\mathrm{i}k\cdot x} v(k,s)\theta(-k_0) \tag{2.4.47}$$

Oct 12/99

2.5 Charge Conjugation

Add in an external potential (recall $\partial\!\!\!/ = \gamma^\mu \partial_\mu$)

$$\begin{aligned} 0 &= (\mathrm{i}\,\partial\!\!\!/ - e\,A\!\!\!/ - m)\,\psi(\vec{x},t) \quad (p_\mu \to p_\mu - eA_\mu \qquad (2.5.1) \\ &\to \mathrm{i}\frac{\partial}{\partial t}\psi = \left[\vec{\alpha}\cdot(\vec{p} - e\vec{A}) + \beta m + e\Phi\right]\psi \end{aligned}$$

We can show that there is a wave function $\psi_c(\vec{x},t)$ satisfying,

$$0 = (\mathrm{i}\,\partial\!\!\!/ + e\,A\!\!\!/ - m)\psi_c \tag{2.5.2}$$

Taking the complex conjugate of (2.5.1),

$$\begin{aligned} 0 &= (\gamma_\mu^*(-\mathrm{i}\partial^\mu - eA^\mu) - m)\psi^* \\ &= C\gamma_0\left[\gamma_\mu^*(\mathrm{i}\partial^\mu + eA^\mu) + m\right](C\gamma_0)^{-1}(C\gamma_0)\psi^* \end{aligned}$$

If we now set

$$\begin{aligned} (c\gamma_0)(\gamma_\mu^*)(C\gamma_0)^{-1} &= -\gamma_\mu \qquad (2.5.3) \\ \text{and}\ \ \psi_c &= C\gamma_0\psi^* \qquad (2.5.4) \end{aligned}$$

then we recover (2.5.2). Now, as

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & \mathrm{i}\sigma^i \\ -\mathrm{i}\sigma^i & 0 \end{bmatrix} \tag{2.5.5}$$

we find that as

$$\begin{aligned} \gamma^0\gamma^{\mu *}\gamma^0 &= \gamma^{\mu T} \\ (C\gamma^0)^{-1} &= \gamma^0 C \quad \text{then (2.5.3) becomes} \\ C\gamma_\mu^T C^{-1} &= -\gamma_\mu \qquad (2.5.6) \end{aligned}$$

A solution to this equation is

$$\begin{aligned} C &= i\gamma^2\gamma^0 \qquad (2.5.7) \\ &= C^* = -C^{-1} = -C^+ = -C^T \qquad (2.5.8) \end{aligned}$$

With this,

$$(\gamma_\mu(\mathrm{i}\partial^\mu + eA^\mu) - m)\psi_c = 0 \tag{2.5.9}$$

Note that ψ and ψ_c transform in the same way under a Lorentz transformation. If,

$$\begin{aligned} \psi'(x') &= S(a)\psi(x) \ ; \quad S(a) = \exp(\frac{-\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}) && (2.5.10)\\ \text{then } \psi_c'(x') &= S(a)\psi_c(x) && (2.5.11)\\ & && (2.5.12) \end{aligned}$$

Note that as,

$$\begin{aligned} \bar{\psi} &= \psi^+\gamma^0 = (\psi^*)^T\gamma^0 && (2.5.13)\\ \text{then } \psi_c &= C\bar{\psi}^* && (2.5.14) \end{aligned}$$

If we now set

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}}\sum_{s=\pm}\left\{b(p,s)u(p,s)e^{-\mathrm{i}p\cdot x} + d^*(p,s)v(p,s)e^{+\mathrm{i}p\cdot x}\right\} \tag{2.5.15}$$

then

$$\begin{aligned} v_c(p,s) &= C\bar{u}^T(p,s) && (2.5.16)\\ u_c(p,s) &= C\bar{v}^T(p,s) && (2.5.17) \end{aligned}$$

Consequently, charge conjugation takes one from a positive energy solution associated with a charge e to a negative energy solution associated with a charge $-e$. (You can map one solution to another by (2.5.16), (2.5.17).

2.6 Majorana Spinors

A Majorana spinor satisfies an extra condition that

$$\psi = \psi_c \quad \text{(Lorentz invariant condition)} \tag{2.6.1}$$

As

$$\begin{aligned} \psi_c &= C\bar{\psi}^T && (2.6.2)\\ &= \int \frac{d^3k}{(2\pi)^{3/2}}\sum_{s=\pm}(b^*C\bar{u}^T e^{\mathrm{i}p\cdot x} + dC\bar{v}^T e^{-\mathrm{i}p\cdot x}) && (2.6.3) \end{aligned}$$

then we see that if ψ satisfies the Majorana condition, then

$$b(p,s) = d(p,s) \tag{2.6.4}$$

Note that in Euclidean space, $\psi_c = C\psi^{*T}$, and if $\psi = \psi_c$, then $\psi = 0$ (only way). i.e. it's not always possible to have Majorana condition.

2.7 Time Reversal

Recall that the Parity operation takes $(x, y, z, t) \rightarrow (-x, -y, -z, t)$

$$\psi'(x') = P\psi(x) \tag{2.7.1}$$
$$\rightarrow P = e^{i\phi}\gamma^0 \tag{2.7.2}$$

With time, the situation is slightly different. We must preserve,

$$i\frac{\partial\psi}{\partial t} = H\psi = \left[\vec{\alpha}\cdot(-i\nabla - e\vec{A}) + \beta m + e\Phi\right]\psi \tag{2.7.3}$$

and this must be equivalent to ($\vec{r}' = \vec{r},\ t' = -t$)

$$i\frac{\partial\psi'(t')}{\partial t'} = H'\psi'(t') \tag{2.7.4}$$
$$\text{with } H' = \left[\vec{\alpha}\cdot(-i\nabla - e\vec{A}') + \beta m + e\Phi'\right] \tag{2.7.5}$$
$$\text{recall: } A_\mu(x) = \int dx'\ D_R(x - x')\dot{\gamma}_\mu(x') \tag{2.7.6}$$
$$\Box A_\mu = -4\pi j_\mu \tag{2.7.7}$$

Here, $\Phi'(t') = \Phi(t)$, $\vec{A}'(t') = -\vec{A}(t)$. In going from $t \rightarrow -t$, we have the same physical process, but using a camera that's running backwards (sequence of events in reverse order).

A Wigner time transformation takes us from $\psi(t)$ to $\psi'(t')$.

$$\psi'(t') = \mathcal{T}\psi(t) \tag{2.7.8}$$
$$\text{where } \mathcal{T}O\mathcal{T}^{-1} = TO^*T^{-1} \tag{2.7.9}$$

with $0 =$ any operator, $\mathcal{T}$ is anti-unitary, and T is an ordinary matrix. i.e.

$$\mathcal{T}i\mathcal{T}^{-1} = T(-i)T^{-1}$$
$$= -i \tag{2.7.10}$$
$$\mathcal{T}\psi = T\psi^* \tag{2.7.11}$$

Starting with (2.7.3),

$$\mathcal{T}i\frac{\partial\psi}{\partial t} = \mathcal{T}\left[\vec{\alpha}\cdot(-i\nabla - e\vec{A}) + \beta m + e\Phi\right]\underbrace{\mathcal{T}^{-1}\mathcal{T}}\psi \tag{2.7.12}$$
$$-i\frac{\partial T\psi^*}{\partial t} = \left[(T\vec{\alpha}^*T^{-1})(i\nabla - e\vec{A}) + T\beta^*T^{-1}m + e\Phi\right](T\psi^*) \tag{2.7.13}$$
$$\text{but } t = -t'$$
$$+i\frac{\partial(T\psi^*)}{\partial t} = \left[(T\vec{\alpha}^*T^{-1})(i\nabla + e\vec{A}') + T\beta^*T^{-1}m + e\Phi'\right](T\psi*) \tag{2.7.14}$$

This is the same as (2.7.4) provided:

1. $\psi'(t') = T\psi^*(t)$
2. $T\vec{\alpha}^* T^{-1} = -\vec{\alpha}$
3. $T\beta^* T^{-1} = +\beta$

The solution is:

$$T = -\mathrm{i}\alpha_1\alpha_3 \tag{2.7.15}$$
$$= +\mathrm{i}\gamma^1\gamma^3 \tag{2.7.16}$$

The Dirac equation is invariant under an $SO(3,1)$ transformation $\psi'(x') = S(a)\psi(x)$ (as well as $P, \mathcal{C}, T \rightarrow$Parity, Charge Conj., time). Under the rotation subgroup $SO(3)$, these are spinors with $s = 1/2$.

Oct. 13/99

Chapter 3

Bargmann-Wigner Equations

The Bargman-Wigner fields are fields of higher spin ($s = 1, 3/2, 2, 5/2, \ldots$) that are built up out of the spin (1/2) fields.

$$\text{Spin } 1/2: \quad (\mathrm{i}\gamma_\mu \partial^\mu - m)_{\alpha\beta}\psi_\beta = 0 \tag{3.0.1}$$

These higher spin fields are associated with wave functions:

$$\begin{aligned} \psi_{\alpha_1\alpha_2\ldots\alpha_{2s}}(x) &\rightarrow \text{spin } s \text{ , totally symmetric in these } 2s \text{ indices} \\ \text{i.e. } \psi_\alpha &\rightarrow \text{spin } 1/2 \\ \psi_{\alpha\beta} = \psi_{\beta\alpha} &\rightarrow \text{spin } 1 \text{ etc.} \end{aligned} \tag{3.0.2}$$

The wave functions satisfy $2s$ equations:

$$0 = \left[\mathrm{i}(\gamma_\mu)_{\alpha_1\alpha_1'}\partial^\mu - m\delta_{\alpha_1\alpha_1'}\right]\psi_{\alpha_1'\alpha_2\ldots\alpha_{2s}} \tag{3.0.3}$$

$$0 = \left[\mathrm{i}(\gamma_\mu)_{\alpha_2\alpha_2'}\partial^\mu - m\delta_{\alpha_2\alpha_2'}\right]\psi_{\alpha_1\alpha_2'\ldots\alpha_{2s}} \tag{3.0.4}$$

$$\vdots$$

$$0 = \left[\mathrm{i}(\gamma_\mu)_{\alpha_{2s}\alpha_{2s}'}\partial^\mu - m\delta_{\alpha_{2s}\alpha_{2s}'}\right]\psi_{\alpha_1\alpha_2\ldots\alpha_{2s}'} \tag{3.0.5}$$

In the frame where $\frac{\partial}{\partial x^i}\psi = 0$ (rest frame $\vec{p} = 0$), we get

$$0 = \left[\mathrm{i}\gamma^0_{\alpha_i\alpha_i'}\frac{\partial}{\partial x^0} - m\delta_{\alpha_i\alpha_i'}\right]\psi_{\alpha_1\ldots\alpha_i'\ldots\alpha_{2s}} \tag{3.0.6}$$

with $\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the solutions generalize the spin 1/2 solution,

$$W_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-\mathrm{i}mt}, \quad W_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-\mathrm{i}mt}, \quad W_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{+\mathrm{i}mt}, \quad W_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{+\mathrm{i}mt} \tag{3.0.7}$$

<u>When there are $2s$ indices</u>

<u>+ve energy solutions</u>

(1) $\delta_{1\alpha_1}\delta_{1\alpha_2}\ldots\delta_{1\alpha_{2s}} = \psi^{(1)}_{\alpha_1\ldots\alpha_{2s}}$

(2)

$$\begin{aligned} & \delta_{2\alpha_1}\delta_{1\alpha_2}\delta_{1\alpha_3}\ldots\delta_{1\alpha_{2s}} \\ + & \ \delta_{1\alpha_1}\delta_{2\alpha_2}\delta_{1\alpha_3}\ldots\delta_{1\alpha_{2s}} \\ + & \ \ldots \\ + & \ \delta_{1\alpha_1}\delta_{1\alpha_2}\delta_{1\alpha_3}\ldots\delta_{2\alpha_{2s}} = \psi^{(2)}_{\alpha_1\ldots\alpha_{2s}} \end{aligned}$$

(3) $\delta_{2\alpha_1}\delta_{2\alpha_2}\delta_{1\alpha_3}\ldots\delta_{1\alpha_{2s}} + \binom{2s}{2}$ symmetrized terms.

(⋮)

(2s+1) $\delta_{2\alpha_1}\delta_{2\alpha_2}\delta_{2\alpha_3}\ldots\delta_{2\alpha_{2s}} = \psi^{(2s+1)}_{\alpha_1\ldots\alpha_{2s}}$

Similarly, there are $(2s+1)$ negative energy solutions.

The operator $\left(\frac{1}{2}\sigma_z\right)_{\alpha\beta}$ operating on ψ_β is generalized to

$$\begin{aligned} 2\left(\Sigma_z\right)_{\alpha_1\beta_1,\alpha_2\beta_2,\ldots,\alpha_{2s}\beta_{2s}} &= \left(\sigma_z\right)_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\ldots\delta_{\alpha_{2s}\beta_{2s}} \\ &\quad + \delta_{\alpha_1\beta_1}\left(\sigma_z\right)_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\ldots\delta_{\alpha_{2s}\beta_{2s}} \\ &\quad \vdots \\ &\quad + \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\ldots\left(\sigma_z\right)_{\alpha_{2s}\beta_{2s}} \end{aligned} \tag{3.0.8}$$

This acts on $\psi_{\beta_1\ldots\beta_{2s}}$

$$\Sigma_z\psi^{(1)} = s\psi^{(1)} \tag{3.0.9}$$

$$\Sigma_z\psi^{(2)} = \left(s-\frac{1}{2}\right)\psi^{(2)} \tag{3.0.10}$$

$$\Sigma_z\psi^{(3)} = (s-1)\,\psi^{(3)} \tag{3.0.11}$$

$$\vdots$$

$$\Sigma_z\psi^{(2s+2)} = (-s)\,\psi^{(2s+2)} \tag{3.0.12}$$

Thus $\psi^{(i)}$ are the $(2s+1)$ eigenvectors of (Σ_z). These are the spin eigenstates. Now, let's examine the spin-one equations $(s=1)$. Consider $\psi_{\alpha\beta}(x) = \psi_{\beta\alpha}(x)$. We have,

$$0 = \left(\mathrm{i}\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta}\right)\psi_{\beta\gamma} \tag{3.0.13}$$

$$0 = \left(\mathrm{i}\gamma^\mu_{\gamma\delta}\partial_\mu - m\delta_{\gamma\delta}\right)\psi_{\alpha\delta} \tag{3.0.14}$$

$$C^{-1}\gamma_\mu C = -\gamma^T_\mu \quad \left(C = \mathrm{i}\gamma_0\gamma_2 = C^* = -C^{-1} = -C^+ = -C^T\right) \tag{3.0.15}$$

With C, we see that

$$\begin{aligned}
(\gamma_\mu C)_{\alpha\beta} &= (\gamma_\mu C)_{\beta\alpha} \quad \text{- 4 matrices} \\
(\Sigma_{\mu\nu} C)_{\alpha\beta} &= (\Sigma_{\mu\nu} C)_{\beta\alpha} \quad \text{- 6 matrices} \quad \left(\Sigma_{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma_\mu, \gamma_\nu\right]\right)
\end{aligned}$$

For a total of 10 symmetric 4×4 matrices. We can then have,

$$\psi_{\alpha\beta}(x) = mA^\mu(x)\left(\gamma_\mu C\right)_{\alpha\beta} + \frac{1}{2}F^{\mu\nu}(x)\left(\Sigma_{\mu\nu} C\right)_{\alpha\beta} \tag{3.0.16}$$

Substitute this into (3.0.13), (3.0.14), and add;

$$\begin{aligned}
0 &= \left(\mathrm{i}\gamma_\mu \partial^\mu - m\right)_{\alpha\delta}\left[mA^\nu\left(\gamma_\nu C\right)_{\delta\beta} + \frac{1}{2}F^{\mu\nu}\left(\Sigma_{\mu\nu} C\right)_{\delta\beta}\right] \\
&\quad + \left[mA^\nu\left(\gamma_\nu C\right)_{\alpha\delta} + \frac{1}{2}F^{\mu\nu}\left(\Sigma_{\mu\nu} C\right)_{\alpha\delta}\right]\left(\mathrm{i}\left(\gamma_\mu^T\right)_{\delta\beta}\overleftarrow{\partial}^\mu - m\delta_{\delta\beta}\right)
\end{aligned} \tag{3.0.17}$$

Now, right multiplying by $C^{-1}_{\beta\epsilon}$ (remember that $C\gamma_\mu^T C^{-1} = -\gamma_\mu$). This gives,

$$\begin{aligned}
0 &= 2m\left[\gamma_\mu, \gamma_\nu\right]\left(\partial^\mu A^\nu\right) + \mathrm{i}\left[\gamma_\mu, \Sigma_{\lambda\sigma}\right]\left(\partial^\mu F^{\lambda\sigma}\right) \\
&\quad - 2m^2\gamma_\mu A^\mu - m\Sigma_{\lambda\sigma}F^{\lambda\sigma}
\end{aligned} \tag{3.0.18}$$

but recall that we know,

$$\left[\gamma_\mu, \gamma_\nu\right] = -2\mathrm{i}\Sigma_{\mu\nu} \tag{3.0.19}$$

$$\left[\gamma_\mu, \Sigma_{\lambda\sigma}\right] = 2\mathrm{i}\left(g_{\mu\lambda}\gamma_\sigma - g_{\mu\sigma}\gamma_\lambda\right) \tag{3.0.20}$$

So,

$$\begin{aligned}
0 &= 2m\Sigma_{\mu\nu}\partial^\mu A^\nu - 2\left(g_{\mu\lambda}\gamma_\sigma - g_{\mu\sigma}\gamma_\lambda\right)\partial^\mu F^{\lambda\sigma} - 2m^2\gamma_\mu A^\mu \\
&\quad - m\Sigma_{\mu\nu}F^{\mu\nu} \\
0 &= m\Sigma_{\mu\nu}\left[\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - F^{\mu\nu}\right] + \gamma^\mu\left[-4\partial_\lambda F^\lambda_\mu - 2m^2 A_\mu\right]
\end{aligned}$$

Thus, we get the Proca Equations (Massive spin 1 equations).

$$\boxed{\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
\partial_\mu F^{\mu\nu} &= -\frac{m^2}{2}A^\nu
\end{aligned}} \tag{3.0.21}$$

If we now set $m = 0$, we recover the free Maxwell equations:

$$F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.0.22}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{3.0.23}$$

Where, recall that (3.0.22) gives

$$\nabla \cdot \vec{B} = 0 \quad , \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

and (3.0.23) gives

$$\nabla \cdot \vec{E} = 0 \quad , \quad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$$

with, $F^{ij} = \epsilon_{ijk} B_k$, $F^{0i} = E^i$.

Oct. 15/99

Transformation of the field $A_\mu(x)$ under a Lorentz transformation.

$$\begin{aligned} \text{c.f.} \quad \psi'(x') &= \exp\left\{-\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right\}\psi(x) \\ &\rightarrow \quad \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right] \end{aligned}$$

Since,

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu} \quad \rightarrow (+,-,-,-) \\ \therefore \left[\sigma^{\mu\nu}, \sigma^{\lambda\rho}\right] &= A\left(g^{\mu\lambda}\sigma^{\nu\rho} - g^{\nu\lambda}\sigma^{\mu\rho} + g^{\nu\rho}\sigma^{\mu\lambda} - g^{\mu\rho}\sigma^{\nu\lambda}\right) \\ &= -2\mathrm{i}\left(g^{\mu\lambda}\sigma^{\nu\rho} - \ldots\right) \end{aligned}$$

(Aside - if $\mu = \lambda = 1$, $\nu = 2$, $\rho = 3$,

$$\begin{aligned} LS &= \left[\sigma^{12}, \sigma^{13}\right] \\ &= \left[\mathrm{i}\gamma^1\gamma^2, \mathrm{i}\gamma^1\gamma^3\right] \\ &= -\left(\gamma^1\gamma^2\gamma^1\gamma^3 - \gamma^1\gamma^3\gamma^1\gamma^2\right) \\ &= -\gamma^2\gamma^3 + \gamma^3\gamma^2 \\ &= -\left[\gamma^2, \gamma^3\right] \\ &= 2\mathrm{i}\sigma^{23} \\ RS &= Ag^{11}\sigma^{23} \\ &= -A\sigma^{23} \Rightarrow A = -2\mathrm{i}) \end{aligned}$$

For A_μ, since it is a vector, under an infinitesimal Lorentz transformation, we have,

$$x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu \quad (\omega^{\mu\nu} = -\omega^{\nu\mu}) \tag{3.0.24}$$

$$\begin{aligned} A'_\mu(x') &= A_\mu(x)\omega_{\mu\nu}A^\nu(x) \\ &= A_\mu(x) + \frac{\mathrm{i}}{u}\left(\omega^{\lambda\sigma}S_{\lambda\sigma}\right)_{\mu\nu}A^\nu(x) && (3.0.25) \\ \text{Where} \quad \left(S_{\lambda\sigma}\right)_{\mu\nu} &= 2\mathrm{i}\left(g_{\lambda\mu}g_{\sigma\nu} - g_{\lambda\nu}g_{\sigma\mu}\right) && (3.0.26) \end{aligned}$$

But now,

$$
\begin{aligned}
(S_{\lambda\sigma})(S_{\alpha\beta}) - (S_{\alpha\beta})(S_{\lambda\sigma}) &\equiv (S_{\lambda\sigma})_\mu^\kappa (S_{\alpha\beta})_{\kappa\nu} - (S_{\alpha\beta})_\mu^\kappa (S_{\lambda\sigma})_{\kappa\nu} \\
&= -2\mathrm{i}\left(g_{\lambda\alpha}S_{\sigma\beta} - g_{\sigma\alpha}S_{\lambda\beta} + g_{\sigma\beta}S_{\lambda\alpha} - g_{\lambda\beta}S_{\sigma\alpha}\right)_{\mu\nu} \qquad (3.0.27)
\end{aligned}
$$

i.e.

$$
A'_\mu(x') = \left(\exp\left\{\frac{\mathrm{i}}{4}\omega^{\lambda\sigma}S_{\lambda\sigma}\right\}\right)_{\mu\nu} A^\nu(x') \qquad (3.0.28)
$$

for a finite transformation.

ψ - spin 1/2 representation of $SO(3,1)$ (fundamental rep.)

A_μ - spin 1 representation of $SO(3,1)$ (Adjoint representation).

i.e. under a rotation,

$$
\psi \to \exp\left\{\mathrm{i}\frac{\vec{\sigma}}{2}\cdot\vec{\omega}\right\}\psi \quad \text{where} \quad
\begin{aligned}
\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] &= \mathrm{i}\epsilon_{ijk}\frac{\sigma_k}{2} \\
\left(\frac{\vec{\sigma}}{2}\right)^2 &= \frac{1}{2}\left(\frac{1}{2}+1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
A'_0(x') &= A_0(x) \\
A'_i(x') &= \left[e^{\mathrm{i}\vec{\omega}\cdot\vec{S}}\right]_{ij} A_j(x)
\end{aligned}
$$

where

$$
S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}, \quad
S_2 = \begin{bmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{bmatrix}, \quad
S_3 = \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.0.29)
$$

Note that,

$$
\begin{aligned}
(S_i)_{mn} &= \frac{1}{2}\epsilon_{ijk}(S_k)_{mn} \\
&= -\mathrm{i}\epsilon_{mij} \qquad (3.0.30)
\end{aligned}
$$

$$
[S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k \qquad (3.0.31)
$$

$$
\begin{aligned}
\vec{S}^2 &= 1(1+1) \\
&= s(s+1) \qquad (3.0.32)
\end{aligned}
$$

<u>Massless Particles</u>

Dirac equation:

$$0 = (\mathrm{i}\gamma^\mu \partial_\mu - m)\,\psi \tag{3.0.33}$$

$$\rightarrow \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \ , \ \ \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore 0 = \begin{bmatrix} -m & \left(\mathrm{i}\frac{\partial}{\partial t} - \vec{\sigma}\cdot\vec{p}\right) \\ \left(\mathrm{i}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\vec{p}\right) & -m \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} \quad , \quad (\vec{p} = -\mathrm{i}\nabla) \tag{3.0.34}$$

If $m = 0$, then we get the Decoupled (Weyl) equations;

$$\mathrm{i}\frac{\partial}{\partial t}\chi = \vec{\sigma}\cdot\vec{p}\chi \tag{3.0.35}$$

$$\mathrm{i}\frac{\partial}{\partial t}\phi = -\vec{\sigma}\cdot\vec{p}\phi \tag{3.0.36}$$

More generally, if we let $p_+ = \frac{1+\gamma_5}{2}$, $p_- = \frac{1-\gamma_5}{2}$ $\rightarrow$ where $\{\gamma_5, \gamma_\mu\} = 0$, $\gamma_5^2 = 1$,

$$\begin{aligned} p_+ + p_- &= 1 \\ (p_+)(p_-) &= 0 \\ (p_+)^2 &= p_+ \\ (p_-)^2 &= p_- \end{aligned}$$

So,

$$p_+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad p_- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.0.37}$$

$$\rightarrow p_+\psi = \begin{bmatrix} 0 \\ \chi \end{bmatrix} \qquad p_-\psi = \begin{bmatrix} \phi \\ 0 \end{bmatrix} \tag{3.0.38}$$

If

$$\psi_\pm = p_\pm \psi \tag{3.0.39}$$

then

$$0 = \mathrm{i}\gamma\cdot\partial\psi_+ - m\psi_- \tag{3.0.40}$$

$$0 = \mathrm{i}\gamma\cdot\partial\psi_- - m\psi_+ \tag{3.0.41}$$

For massless particles, we can't go to the rest frame, i.e.

$$\begin{aligned} p^2 = (p^0)^2 - \vec{p}^2 &= m^2 \\ &= 0 \end{aligned}$$

$\rightarrow$ there is no rest frame, so look at the frame where

$$p_\mu = (0, 0, p, p) \ \ ; \quad (p_1 = p_2 = 0 \ , \ \ p_3 = p_0 = p) \tag{3.0.42}$$

In this frame,

$$p_+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\chi_1\\\chi_2\end{bmatrix} = p_+\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\chi_1\\\chi_2\end{bmatrix} \tag{3.0.43}$$

(first of Weyl equations)

$$\chi = \begin{bmatrix}\chi_1\\0\end{bmatrix} \tag{3.0.44}$$

(only <u>one</u> state of negative energy)

$$p_-\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix} = -p_-\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix} \tag{3.0.45}$$

$$\phi = \begin{bmatrix}0\\\phi_2\end{bmatrix} \tag{3.0.46}$$

(only <u>one</u> state of positive energy)

So, Weyl equations are:

- Negative energy solutions:

$$\mathrm{i}\frac{\partial}{\partial t}\chi = \vec{\sigma}\cdot\vec{p}\chi \tag{3.0.47}$$

- Positive energy solutions:

$$\mathrm{i}\frac{\partial}{\partial t}\phi = -\vec{\sigma}\cdot\vec{p}\phi \tag{3.0.48}$$

Oct. 19/99

Can we define "spin" of a massless particle, seeing that there is no rest frame? - Employ the Pauli-Lubanski tensor.

$$S_\mu = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\lambda\sigma}\sigma^{\lambda\sigma}k^\nu \tag{3.0.49}$$

$$\rightarrow \sigma^{\lambda\sigma} = \frac{\mathrm{i}}{2}\left[\gamma^\lambda,\gamma^\sigma\right]$$

In the rest frame of a massive (m) particle,

$$k^\mu = (0,0,0,m) \tag{3.0.50}$$

$$S_\mu = \frac{\mathrm{i}}{2}\epsilon_{\mu 0\nu\lambda}\sigma^{\nu\lambda}m \tag{3.0.51}$$

(0 in ϵ index $\rightarrow \mu,\nu,\lambda$ all spatial.)

$$\epsilon^{0123} = +1 \tag{3.0.52}$$

$$S_i = \frac{\mathrm{i}}{2}\epsilon_{ijk}\sigma^{jk}m$$

$$= m\begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix} \tag{3.0.53}$$

Note:

$$\begin{aligned}\sigma_{ij} &= \frac{\mathrm{i}}{2}\epsilon_{ijk}\begin{bmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{bmatrix}\\ \sigma_i &= \begin{bmatrix}0 & \mathrm{i}\sigma_i\\ \mathrm{i}\sigma_i & 0\end{bmatrix}\end{aligned}$$

Thus, in the rest frame $\frac{1}{2}S_\mu$ reduces to $\frac{m\vec{\sigma}}{2}$.

$\frac{1}{2}S_\mu$ generalizes spin to an arbitrary frame. For a massless particle moving along the z-axis (with $k^2 = 0,\ k^\mu = (0, 0, k, k)$), then we have,

$$\begin{aligned}S_1 &= \frac{\mathrm{i}}{2}\epsilon_{1\lambda\sigma\nu}k^\nu\sigma^{\lambda\sigma}\\ &= \frac{\mathrm{i}}{2}\left(\epsilon_{1\lambda\sigma 3}k + \epsilon_{1\lambda\sigma 0}k\right)\sigma^{\lambda\sigma}\\ &\quad \text{etc. for } \ S_2,\ S_3,\ S_4\end{aligned} \tag{3.0.54}$$

Now consider the two massless solutions to the Bargmann Wigner equations

$$W^{(1)}_{\alpha_1\alpha_2\ldots\alpha_{2s}} = \delta_{\alpha_1 2}\delta_{\alpha_2 2}\ldots\delta_{\alpha_{2s}2} \tag{3.0.55}$$

$$W^{(2)}_{\alpha_1\alpha_2\ldots\alpha_{2s}} = \delta_{\alpha_1 3}\delta_{\alpha_2 3}\ldots\delta_{\alpha_{2s}3} \tag{3.0.56}$$

Generalize the Pauli Lubanski tensor,

$$\begin{aligned}(S_\mu)_{\alpha_1\beta_1,\ldots,\alpha_{2s}\beta_{2s}} &= (S_\mu)_{\alpha_1\beta_1}\left(\delta_{\alpha_2\beta_2}\ldots\delta_{\alpha_{2s}\beta_{2s}}\right)\\ &\quad + (S_\mu)_{\alpha_2\beta_2}\left(\delta_{\alpha_1\beta_1}\delta_{\alpha_3\beta_3}\ldots\delta_{\alpha_{2s}\beta_{2s}}\right)\\ &\quad \ldots\\ &\quad + (S_\mu)_{\alpha_{2s}\beta_{2s}}\left(\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_{2s-1}\beta_{2s-1}}\right)\end{aligned} \tag{3.0.57}$$

It can now be shown that

$$S_1W^{(1)} = 0 \qquad S_3W^{(1)} = -2skW^{(1)} \tag{3.0.58}$$

$$S_2W^{(1)} = 0 \qquad S_3W^{(2)} = +2skW^{(2)} \tag{3.0.59}$$

$$S_1W^{(2)} = 0 \qquad S_4W^{(1)} = +2skW^{(1)} \tag{3.0.60}$$

$$S_2W^{(2)} = 0 \qquad S_4W^{(2)} = -2skW^{(2)} \tag{3.0.61}$$

But $k_\mu = (0, 0, -k, k)$ and thus $\frac{1}{2}S_\mu = \pm sk_\mu$.

Hence for a massless particle, there are two solutions to the Bargmann-Wigner equations, and these two solutions correspond to spin states whose eigenvalues are $\pm S$, and whose direction is in the direction of k_μ.

Thus there are only two polarizations for any massless particle of spin S. We lose states through the introduction of gauge symmetry. ex.

- massless spin 1/2 - 2 states
  massive spin 1/2 - $2\left(\frac{1}{2}\right)+1=2$ states
- massless spin 1 - 2 states
  massive spin 1 - $(2s+1)=3$ states
- massless spin 3/2 - 2 states
  massive spin 3/2 - $(2s+1)=5$ states

Chapter 4

Gauge Symmetry and massless spin one particles

Remember that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.0.1}$$

$$\partial_\mu F^{\mu\nu} = -m^2 A^\nu \to 0 \text{ as } m^2 \to 0 \tag{4.0.2}$$

If $m^2 \to 0$, there is the gauge invariance,

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{4.0.3}$$

We now have,

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.0.4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.0.5}$$

which are derived from the action

$$S = -\frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\mu\nu} \to \left(\mathcal{L} = -\frac{1}{4}F^2\right) \tag{4.0.6}$$

(the -ve sign is present to keep the energy +ve, and the magnitude 1/4 is convention). i.e.

$$\begin{aligned}
\frac{\delta S}{\delta A_\mu(y)} &= -\frac{1}{4}\int d^4s \; \left(2F^{\alpha\beta}\frac{\delta F_{\alpha\beta}}{\delta A_\mu(y)}\right) \\
&= -\frac{1}{2}\int d^4x \; F^{\alpha\beta}\frac{\delta}{\delta A_\mu(y)}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\
&= -\frac{1}{2}\int d^4x \; F^{\alpha\beta}(x)\left(2\partial_\alpha \delta^4(x-y)\; g_{\mu\beta}\right) \\
&\quad \left\{\to \frac{\delta A_\beta(x)}{\partial A_\mu(y)} = g_{\mu\beta}\delta^4(x-y)\right\} \\
&= \partial_\alpha F^{\alpha\beta}(x) \\
&= 0
\end{aligned}$$

Just as with

$$\mathcal{L} = \mathcal{L}\left(q_i(t), \dot{q}_i(t)\right) \quad \text{and} \quad S = \int dt \; \mathcal{L}\left(q_i(t), \dot{q}_i(t)\right) \tag{4.0.7}$$

we have

$$\begin{aligned} S &= \int d^4x \; \mathcal{L}(A_\mu, \partial_\nu A_\nu) \\ &= \int dt \underbrace{\int d^3x \; \mathcal{L}\left(A_\mu(\vec{x},t), \nabla A_\mu(\vec{x},t), \frac{\partial}{\partial t}A_\mu(\vec{x},t)\right)}_{L} \end{aligned} \tag{4.0.8}$$

$$q_i(t) \;\leftrightarrow\; A_\mu(\vec{x},t), \nabla A_\mu(\vec{x},t) \tag{4.0.9}$$

$$\dot{q}_i(t) \;\leftrightarrow\; \frac{\partial}{\partial t}A_\mu(\vec{x},t) \tag{4.0.10}$$

and, where previously we had i discrete, we now have a continuum (μ) number of degrees of freedom. So, just as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \longrightarrow \pi_\mu(\vec{x},t) \;=\; \frac{\partial L}{\partial\left(\frac{\partial}{\partial t}A^\mu(\vec{x},t)\right)} \to \text{ momentum density} \tag{4.0.11}$$

$$\left(\text{c.f. } \; \partial_\mu = \frac{\partial}{\partial x^\mu}\right)$$

Oct. 20/99

Now, as

$$\frac{\partial A_\mu(\vec{x}',t)}{\partial A_\nu(\vec{x},t)} \;=\; g_{\mu\nu}\delta^3(\vec{x}-\vec{x}') \tag{4.0.12}$$

$$\begin{aligned} \therefore \pi_\mu(\vec{x},t) &= \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A^\mu}{\partial t}\right)} \\ &= \frac{\partial}{\partial\left(\frac{\partial A^\mu}{\partial t}\right)}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ &= -\frac{1}{2}F_{\lambda\nu}\frac{\partial}{\partial\left(\frac{\partial A^\mu}{\partial t}\right)}\left(F^{\lambda\nu}\right) \\ &\qquad \to \frac{\partial}{\partial t} = \partial_0 \\ &= -\frac{1}{2}F_{\lambda\nu}\frac{\partial}{\partial(\partial_0 A^\mu)}F^{\lambda\nu} \end{aligned}$$

But,

$$\begin{aligned} \to F_{\lambda\nu}F^{\lambda\nu} &= F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} + F_{00}F^{00} \\ &= 2F_{0i}F^{0i} + \underbrace{F_{ij}F^{ij}}_{\text{No t dep.}} \end{aligned}$$

$$\therefore \pi_\mu = -F_{0i}\frac{\partial}{\partial(\partial_0 A^\mu)}F^{0i} \quad (4.0.13)$$

$$\rightarrow F^{0i} = \partial^0 A^i - \partial^i A^0 \quad (4.0.14)$$

Thus,

$$\pi^0 = 0 \quad (4.0.15)$$

$$\begin{aligned} \pi^j &= -F^{0i}\frac{\partial}{\partial(\partial_0 A^\mu)}F_{0i} \\ &= -F^{0i}\frac{\partial}{\partial(\partial_0 A^j)}(\partial_0 A_i - \partial_i A_0) \\ &= -F^{0j} \end{aligned} \quad (4.0.16)$$

Remember,

$$A^\mu = \left(\vec{A},\phi\right) \quad (A^0=\phi, A^i=\vec{A}) \quad (4.0.17)$$

$$\begin{aligned} \vec{E} &= -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \\ &= -\partial_i A^0 - \partial_0 A^i \\ &= +\partial^i A^0 - \partial^0 A^i \\ &= -F^{0i} \end{aligned} \quad (4.0.18)$$

$$\boxed{\therefore \pi^j = E^j = -F^{0j} = +F_{0j}} \quad (4.0.19)$$

(4.0.19) is Canonical momentum (Not mechanical momentum). Canonically conjugate to A^3. i.e. A^i has canonically conjugate momentum $\pi^j = E^j$.

$$\boxed{\begin{aligned} A_i &= -A^i \\ A_0 &= +A^0 \end{aligned}} \quad (4.0.20)$$

$$\boxed{\pi^0 = 0} \quad (4.0.21)$$

i.e. A^0 has no conjugate momentum.

Define a Poison bracket,

$$[q_i,p_j] = \delta_{ij} \Rightarrow [A,B] = \sum_i \left\{\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right\} \quad (4.0.22)$$

Similarly

$$[F(A_\mu,\pi_\mu), G(A_\nu,\pi_\nu)] = \int d^3x\, g^{\mu\nu}\left[\frac{\delta F}{\delta A^\mu(\vec{x},t)}\frac{\delta G}{\delta\pi^\nu(\vec{x},t)} - \frac{\delta F}{\delta\pi^\mu(\vec{x},t)}\frac{\delta G}{\delta A^\nu(\vec{x},t)}\right] \quad (4.0.23)$$

4.1 Canonical Hamiltonian Density

$$
\begin{aligned}
\mathcal{H} &= \pi^\mu \frac{\partial}{\partial t} A_\mu - \mathcal{L} \quad (\text{re.} \quad H = p_i \dot{q}_i - L) && (4.1.1)\\
&= \underbrace{\pi^0}_{=0} \partial_0 A_0 + \pi^i \partial_0 A_i - \left(-\frac{1}{4} \left(2F_{0i}F^{0i} + F_{ij}F^{ij} \right) \right) && (4.1.2)\\
& \quad (\rightarrow \pi^0 = \text{primary constraint})\\
\mathcal{H} &= \pi^i \left(\underbrace{\partial_0 A_i - \partial_i A_0}_{F^{0i}} + \partial_i A_0 \right) + \frac{1}{2} F_{0i}F^{0i} + \frac{1}{4} F_{ij}F^{ij}
\end{aligned}
$$

Remember that,

$$
\begin{aligned}
(\vec{B})_i &= (\nabla \times \vec{A})_i = \epsilon_{ijk}(\partial_j A_k)\\
\epsilon^{imn} B_i &= \epsilon^{imn} \epsilon_{ijk} \partial_j A_k\\
\rightarrow \epsilon_{imn}\epsilon_{ijk} &= \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}
\end{aligned}
$$

So that

$$
\begin{aligned}
\epsilon^{imn} B_i &= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) \partial_j A_k\\
&= \partial^m A^n - \partial^n A^m\\
&= F^{mn} && (4.1.3)
\end{aligned}
$$

$$
\boxed{\therefore F_{mn} = \epsilon_{mnl} B_l} \qquad (4.1.4)
$$

Consequently:

$$
\begin{aligned}
\mathcal{H} &= \pi^i \underbrace{F^{0i}}_{\pi^i} + \pi^i \partial_i A_0 + \frac{1}{2} \underbrace{F_{0i}F^{0i}}_{\pi^i(-\pi^i)} + \frac{1}{4} \underbrace{(\epsilon_{ijk}B_k)(\epsilon_{ijl}B_l)}_{2B_iB_i}\\
&= \frac{1}{2}\pi^i\pi^i + \frac{1}{2} B_i B_i + \pi^i \partial_i A_0 \ ; \quad \pi^j = E^j\\
\mathcal{H} &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \pi^i \partial_i A_0 && (4.1.5)
\end{aligned}
$$

Primary constraint $(\pi^0(\vec{x}, t) = 0$. Thus,

$$
\frac{\partial}{\partial t} \pi^0 = 0 = \left[\pi^0(\vec{x}, t), \mathcal{H}(\vec{y}, t) \right] \qquad (4.1.6)
$$

<u>But</u> we have the fundamental poisson bracket

$$
[A_\alpha(\vec{x}, t), \pi_\beta(\vec{y}, t)] = g_{\alpha\beta} \delta^3(\vec{x} = \vec{y}) \quad (\text{c.f.} \quad [q_i, p_j] = \delta_{ij}) \qquad (4.1.7)
$$

$$\begin{aligned}
\therefore \frac{\partial}{\partial t}\pi^0 = 0 &= \left[\pi^0(\vec{x},t), \pi^i(\vec{y},t)\frac{\partial}{\partial y^i}A_0(\vec{y},t)\right] \\
0 &= \pi^i(\vec{y},t)\frac{\partial}{\partial y^i}\delta^3(\vec{x}-\vec{y}) \\
0 &= +\nabla\cdot\vec{E}(\vec{y},t)\delta^3(\vec{x}-\vec{y}) \qquad (4.1.8)
\end{aligned}$$

$\therefore$ Secondary constraint $\nabla\cdot\vec{E} = 0$ (Gauss's Law).

Oct. 22/99

So,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} && (4.1.9) \\
\pi^i &= -E_i = -F_{0i} && (4.1.10) \\
\pi^0 &= 0 \qquad \left.\begin{array}{c}\pi^0 = 0 \\ \partial_i E_i = 0\end{array}\right\} \text{ First class Constraints} && (4.1.11) \\
B_i &= \frac{1}{2}\epsilon_{ijk}F_{jk} && (4.1.12) \\
\mathcal{H}_0 &= \frac{\vec{E}^2+\vec{B}^2}{2} + A^0\partial_i E_i && (4.1.13) \\
H &= \int d^3x\,\mathcal{H} && (4.1.14)
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_T &= \mathcal{H}_0 + c_i\phi_i \\
&= \tfrac{1}{2}(\vec{E}^2+\vec{B}^2) + A_0\partial_i E_i + c_1\Pi_0 + c_2\partial_i E_i \quad ; \quad \text{can absorb } c_2 \to A_0 \qquad (4.1.15)
\end{aligned}$$

$\therefore A^0$ - Lagrange multiplier for the constraint $\partial_i E_i = 0$.
We need two gauge conditions:

- Coulomb gauge (usual choice)

$$\begin{aligned}
A_0 &= 0 && (4.1.16) \\
\partial_i A_i &= 0 && (4.1.17)
\end{aligned}$$

- Axial gauge also possible

$$\begin{aligned}
A_0 &= 0 && (4.1.18) \\
A_3 &= 0 \text{ (say)} && (4.1.19)
\end{aligned}$$

Now, replace Poisson Brackets by Dirac Brackets. The fundamental P.B. is:

$$\begin{aligned}
\{A_i(\vec{x},t), \Pi_j(\vec{y},t)\} &= \delta_{ij}\delta^3(\vec{x}-\vec{y}) && (4.1.20) \\
\Theta_i &= \{\phi_i \ldots \gamma_i\} && (4.1.21)
\end{aligned}$$

The Dirac Bracket is then:

$$\begin{aligned} \{A,B\}^* &= \{A,B\} - \sum_{i,j}\{A,\Theta_i\}d_{ij}\{\Theta_j,B\} \qquad (4.1.22)\\ \text{where } d_{ij}^{-1} &= \{\Theta_i,\Theta_j\} \end{aligned}$$

So,

$$\begin{aligned} \{\Pi_0(\vec{x},t), A^0(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y}) \qquad (4.1.23)\\ \left\{\frac{\partial}{\partial x_i}\Pi_i(\vec{x},t), \frac{\partial}{\partial y_i}A_j(\vec{y},t)\right\} &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^i}\left(-\delta_{ij}\delta^3(\vec{x}-\vec{y})\right)\\ &= \nabla_x^2\delta^3(\vec{x}-\vec{y}) \qquad (4.1.24) \end{aligned}$$

We now need $(\nabla^2)^{-1} = G(\vec{x}-\vec{y})$. i.e.

$$\begin{aligned} \nabla^2 G(\vec{x}-\vec{y}) &= \delta^3(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3}e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (4.1.25)\\ \text{Write } G(\vec{x}-\vec{y}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}g(\vec{k}) \quad \text{(Fourier transf.)} \qquad (4.1.26)\\ \therefore \nabla^2 G(\vec{x}-\vec{y}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\underbrace{\left(-\vec{k}^2 g(\vec{k})\right)}_{=1}e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (4.1.27) \end{aligned}$$

Aside:

$$\begin{aligned} \delta(x) &= 0 \quad (x\neq 0)\\ \int_{-\infty}^{\infty} dx\ \delta(x) &= 1 \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} dx\ \sqrt{\frac{a}{\pi}}e^{-ax^2} &= \sqrt{\frac{a}{\pi}}\left(\sqrt{\frac{\pi}{a}}\right)\\ &= 1\\ \lim_{a\to\infty}\sqrt{\frac{a}{\pi}}e^{-ax^2} &= \begin{cases} 0 & \text{if } x\neq 0\\ \infty & \text{if } x=0 \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned}
\delta(x) &= \lim_{a\to\infty}\sqrt{\frac{a}{\pi}}e^{-ax^2}\\
\text{So}\quad \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{\mathrm{i}kx} &= \lim_{a\to 0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{\mathrm{i}kx-ak^2} \quad ak^2 \ \text{ is regulator}\\
&= \lim_{a\to 0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\{-a\left(k-\tfrac{\mathrm{i}x}{2a}\right)\}\exp\{-\tfrac{x^2}{4a}\}\\
&= \lim_{a\to 0}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}e^{-a(k')^2}\exp\{-\frac{x^2}{4a}\}\\
&= \lim_{a\to 0}\sqrt{\frac{\pi}{a}}\frac{1}{2\pi}\exp\{-\frac{x^2}{4a}\}\\
&= \delta(x)
\end{aligned}$$

end of aside.
Now, recall (4.1.27)

$$\therefore g(\vec{k}) = -\frac{1}{\vec{k}^2} \tag{4.1.28}$$

Thus,

$$G(\vec{x}-\vec{y}) = \int\frac{d^3k}{(2\pi)^3}\left(\frac{-1}{\vec{k}^2}\right)e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \tag{4.1.29}$$

Say $(\vec{x}-\vec{y})$ is along k_3 axis.

$$\begin{aligned}
G(\vec{x}-\vec{y}) &= \int_0^{2\pi}d\phi\int_{-1}^{+1}d(\cos\theta)\int_0^{\infty}\frac{dk}{(2\pi)^3}k^2\left(\frac{-1}{k^2}\right)e^{\mathrm{i}kr\cos\theta} \quad ; \quad r=|\vec{x}-\vec{y}|\\
&= \frac{1}{(2\pi)^2}\int_{-1}^{1}dz\int_0^{\infty}dk\ e^{\mathrm{i}krz} \quad ;(z=\cos\theta)\\
&\quad \text{can extend k integral to } (-\infty)\to \ \text{int. over z makes even}\\
&= \frac{-1}{2(2\pi)^2}\int_{-1}^{1}dz\int_{-\infty}^{\infty}dk\ e^{\mathrm{i}k(rz)}\\
&= \frac{-1}{4\pi}\int_{-1}^{1}dz\ \delta(rz)\\
&= \frac{-1}{4\pi r}\int_{-\infty}^{\infty}dz'\ \delta(z') \quad ;z'=rz\\
&= \frac{-1}{4\pi r}
\end{aligned} \tag{4.1.30}$$

Thus,

$$G(\vec{x}-\vec{y}) = \frac{-1}{4\pi|\vec{x}-\vec{y}|} \tag{4.1.31}$$

One can verify that

$$\nabla^2\left(\frac{-1}{4\pi|\vec{x}-\vec{y}|}\right)=0 \quad , \quad \vec{x}\neq\vec{y} \tag{4.1.32}$$

and that

$$\begin{aligned}
\int d^3\vec{x}\ \nabla^2\left(\frac{-1}{4\pi|\vec{x}|}\right) &= \frac{-1}{4\pi}\int d^3\vec{x}\ \nabla\cdot\left(\nabla\frac{-1}{|\vec{x}|}\right) \\
&\quad \text{by Gauss's Law (Radius of Sphere } = R) \\
&= \frac{-1}{4\pi}\lim_{R=\infty}\int_{Sphere} d\Omega\ R^2\hat{n}\cdot\left(\nabla\frac{1}{|\vec{x}|}\right) \\
&= 1
\end{aligned} \tag{4.1.33}$$

Thus our Dirac Brackets are

$$\begin{aligned}
\{F(\vec{x},t),G(\vec{y},t)\}^* &= \{F(\vec{x},t),G(\vec{y},t)\} \\
&\quad -\int d^3\vec{x}'\ d^3\vec{y}'\ \left[\left\{F(\vec{x},t),\frac{\partial}{\partial x'^i}\Pi_i(\vec{x}',t)\right\}G(\vec{x}'-\vec{y}')\left\{\frac{\partial}{\partial y'_j}A_j(\vec{y}',t),G(\vec{y},t)\right\}\right. \\
&\quad \left. -\left\{G(\vec{x},t),\frac{\partial}{\partial x'^i}\Pi_i(\vec{x}',t)\right\}G(\vec{x}'-\vec{y}')\left\{\frac{\partial}{\partial y'_j}A_j(\vec{y}',t),F(\vec{y},t)\right\}\right]
\end{aligned} \tag{4.1.34}$$

Hence;

$$\begin{aligned}
\{A_i(\vec{x},t),\Pi_j(\vec{y},t)\}^* &= \left(\delta_{ij}\delta(\vec{x}-\vec{y})-\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}G(\vec{x}-\vec{y})\right) \\
&= \underbrace{\left(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\right)\delta(\vec{x}-\vec{y})}_{\delta^3_{ij\perp}(\vec{x}-\vec{y})} \quad \rightarrow \frac{1}{\nabla^2}=G(\vec{x}-\vec{y})
\end{aligned} \tag{4.1.35}$$

$$\tag{4.1.36}$$

Consistency:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\{A_i(\vec{x},t),\Pi_j(\vec{y},t)\}^* &= \frac{\partial}{\partial x_i}\left(\delta_{ij}-\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{\nabla^2}\right)\delta(\vec{x}-\vec{y}) \\
&= 0
\end{aligned} \tag{4.1.37}$$

$$\{A_i(\vec{x},t),\Pi_j(\vec{y},t)\}^* = \delta^3_{ij\perp}(\vec{x}-\vec{y}) \tag{4.1.38}$$

Oct. 26/99

$$\mathcal{H}_0 = \frac{1}{2}(\Pi^2+\mathbf{B}^2)+c_1\Pi_0+c_2\partial_i\Pi_i \tag{4.1.39}$$

$$\begin{aligned}
\rightarrow \dot{A}_0 &= [A_0,\mathcal{H}] \\
&= [A_0,c_1\Pi_0]=c_1(\text{Entirely arbitrary prior to fixing gauge})
\end{aligned} \tag{4.1.40}$$

$$\begin{aligned}
\rightarrow \dot{A}_i &= [A_i,\mathcal{H}] \\
&= [A_i,\tfrac{1}{2}\Pi^2+c_2\partial_j\Pi_j]=A_i-\partial_i c_2
\end{aligned} \tag{4.1.41}$$

where the usual gauge invariance in A is recovered;

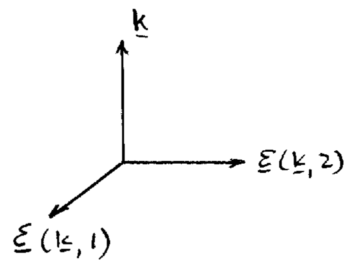
$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \Lambda \\ c_1 &= \partial_0 \Lambda \\ -\partial_i c_2 &= \partial_i \Lambda \end{aligned} \tag{4.1.42}$$

The Fourier expansion of $\mathbf{A}(\mathbf{x}, t)$ in the Coulomb gauge ($\partial_i A_i = 0$) is

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^4} \delta k^2 \sum_{\lambda=1}^{2} \{\varepsilon(\mathbf{k}, \lambda)[a(\mathbf{k}, \lambda)e^{-ik\cdot x} + a^*(\mathbf{k}, \lambda)e^{+ik\cdot x}]\} \tag{4.1.43}$$

1. $\delta(k^2)$ is the mass shell condition (i.e. ensures that $\Box\mathbf{A} = 0 \rightarrow \partial_\mu F^{\mu\nu} = 0 \Rightarrow \Box\mathbf{A} = 0$ if $\partial_i A_i = 0$)

2. $\varepsilon(\mathbf{k}, \lambda) \cdot \mathbf{k} = 0 \quad (\lambda = 1, 2)$ (Ensures $\partial_i A_i = 0$). and



$$\begin{aligned} \varepsilon(\mathbf{k}, \lambda) \cdot \varepsilon(\mathbf{k}, \lambda') &= \delta_{\lambda\lambda'} \\ \rightarrow \varepsilon(\mathbf{k}, 1) \times \varepsilon(\mathbf{k}, 2) &= \frac{\mathbf{k}}{|\mathbf{k}|} \end{aligned}$$

3. $\mathbf{A} = \mathbf{A}^* \Rightarrow$ is real. Thus,

$$ae^{-ik\cdot x} + a^*e^{ik\cdot x}$$

occurs.

Note:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ \delta(x^2-a^2)f(x) &= \int_{a-\epsilon}^{a+\epsilon} dx\ \delta((x+a)(x-a))f(x) \\
&\quad + \int_{-a-\epsilon}^{-a+\epsilon} dx\ \delta((x+a)(x-a))f(x) \\
&= \int_{a-\epsilon}^{a+\epsilon} dx\ \delta(2a(x-a))f(x) + \int_{-a-\epsilon}^{-a+\epsilon} dx\ \delta(-2a(x+a))f(x) \\
&\quad \text{but recall that } \int_{-\infty}^{\infty} dx\ \delta(cx)f(x) = \frac{f(x)}{|c|} \\
&= \frac{f(a)}{|2a|} + \frac{f(-a)}{|2a|} \qquad (4.1.44)
\end{aligned}$$

Hence

$$\delta(x^2-a^2) = \frac{\delta(x-a)+\delta(x+a)}{|2a|} \qquad (4.1.45)$$

$$\therefore \mathbf{A}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_0}{(2\pi)} \delta(k_0^2-\mathbf{k}^2) \sum_{\lambda=1}^{2} \varepsilon(\mathbf{k},\lambda)[ae^{-\mathrm{i}k\cdot x} + a^* e^{\mathrm{i}k\cdot x}] \qquad (4.1.46)$$

Rescaling a, a^* and setting $\omega_k = \sqrt{\mathbf{k}^2}$, $\varepsilon_{\mathbf{k},\lambda} = \varepsilon(\mathbf{k},\lambda)$, etc., we get

$$\begin{aligned}
\therefore \mathbf{A}(\mathbf{x},t) &= \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon_{\mathbf{k},\lambda} \left[a_{\mathbf{k},\lambda} e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+\omega_k t)} + a_{\mathbf{k},\lambda} e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_k t)} \right. \\
&\qquad \left. + a^*_{\mathbf{k},\lambda} e^{+\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+\omega_k t)} + a^*_{\mathbf{k},\lambda} e^{+\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_k t)} \right] \qquad (4.1.47)
\end{aligned}$$

If $\mathbf{k} \to -\mathbf{k}$, and if

$$\varepsilon_{\mathbf{k},\lambda} = \begin{cases} \varepsilon_{-\mathbf{k},\lambda} & (\lambda=1) \\ -\varepsilon_{-\mathbf{k},\lambda} & (\lambda=2) \end{cases} \qquad (4.1.48)$$

$$\varepsilon_{-\mathbf{k},1} \times \varepsilon_{-\mathbf{k},2} = \frac{(-\mathbf{k})}{|\mathbf{k}|} \qquad (4.1.49)$$

then

$$\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+\omega_k t)} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon_{-\mathbf{k},\lambda} a_{-\mathbf{k},\lambda} e^{+\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_k t)} \qquad (4.1.50)$$

Now set $a_{-\mathbf{k},\lambda} = \left\{ \begin{smallmatrix} a^*_{k,\lambda} & \lambda=1 \\ -a^*_{k,\lambda} & \lambda=2 \end{smallmatrix} \right.$. Thus everything collapses to

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon_{k,\lambda} \left[a_{k,\lambda} e^{-\mathrm{i}k\cdot x} + a^*_{k,\lambda} e^{\mathrm{i}k\cdot x} \right] \qquad (4.1.51) \\
&\quad \text{where } \ k\cdot x = \mathbf{k}\cdot\mathbf{x} - \omega_k t
\end{aligned}$$

(we need not use lane wave basis; any complete set of functions will do).
Recall the Klein-Gordon equation:

$$0 = (\Box + m^2)\phi \tag{4.1.52}$$
$$0 = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi \tag{4.1.53}$$

Let $\theta = \frac{\partial \phi}{\partial t} \rightarrow \therefore \frac{\partial \theta}{\partial t} = (\nabla^2 - m^2)\phi$

$$\Psi_1 = \frac{1}{\sqrt{2}}\left(\phi + \frac{\mathrm{i}}{m}\theta\right) \qquad \Psi_2 = \frac{1}{\sqrt{2}}\left(\phi - \frac{\mathrm{i}}{m}\theta\right) \tag{4.1.54}$$

Let

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{4.1.55}$$
$$\therefore \mathrm{i}\frac{\partial \Psi}{\partial t} = \underbrace{\left[(\tau_3 + \mathrm{i}\tau_2)\frac{(-\mathrm{i}\nabla)^2}{2m} + m\tau_3\right]}_{H}\Psi \tag{4.1.56}$$
$$\rightarrow \tau_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.1.57}$$

K.G. equation follows from an action

$$S = \int d^4x \underbrace{\left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2\right]}_{\mathcal{L}} \tag{4.1.58}$$

Now,

$$\rightarrow \Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi \tag{4.1.59}$$
$$\begin{aligned} \mathcal{H} &= \Pi(\partial_0\phi) - \mathcal{L} \\ &= \Pi \cdot (\partial_0\phi) - \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - m^2\phi^2\right) \\ &= \frac{1}{2}\left[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \end{aligned} \tag{4.1.60}$$

Fundamental P.B.

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = \delta^3(\mathbf{x} - \mathbf{y}) \tag{4.1.61}$$

$$(\Box + m^2)\phi = j \quad \text{Describes} \quad \Pi, K, \ldots \tag{4.1.62}$$
$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = \mathbf{p}2m\psi \rightarrow \quad \text{we've treated as single particle} \tag{4.1.63}$$
$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = \left(\frac{\mathbf{p}^2}{2m} + V\right)\psi \rightarrow \quad \text{in a classical potential} \tag{4.1.64}$$

Dirac showed that the potential V itself, when it's due to an E.M. field, should be quantized. Make relativistic by considering Dirac eq.

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (\alpha\cdot\mathbf{p} + \beta m)\psi \tag{4.1.65}$$

- the above equation doesn't describe just 1 particle anymore, but in fact now describes 4
- this one equation describes ψ particles (spin $\frac{1}{2}$)
- if you put c's and $\hbar$'s back in, $\hbar$'s don't all cancel, $\rightarrow \hbar$ floating around

Maxwell's equations (Fiddle with Bargmann-Wigner (spin 1), and get these)

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.1.66}$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.1.67}$$

$\hbar$'s cancel, if $m = 0 \rightarrow$ classical equations. (2 particles/polarizations). i.e. in order to include relativity, you end up having to describe more than 1 particle. Wave function describing single particle $\rightarrow$ actually quantizing wave function itself (2^{nd} quantization).

Oct. 27/99

Chapter 5

(2^{nd}) Quantization, Spin and Statistics

5.1 Harmonic Oscillator

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) \quad ; \quad \omega = \sqrt{\frac{k}{m}} \tag{5.1.1}$$

Quantize:

$$[q, p] = \mathrm{i} \; ; \quad (\hbar = 1) \tag{5.1.2}$$

$$\dot{p} = [p, H] = -\omega^2 q \tag{5.1.3}$$

$$\dot{q} = [q, H] = p \tag{5.1.4}$$

Define:

$$a = \frac{\omega q + \mathrm{i}p}{\sqrt{2\omega}} \; ; \quad (\text{ annihilation}) \tag{5.1.5}$$

$$a^\dagger = \frac{\omega q - \mathrm{i}p}{\sqrt{2\omega}} \; ; \quad (\text{creation}) \tag{5.1.6}$$

Note that:

$$\begin{aligned} [a, a^\dagger] &= 1 \\ [a, a] &= [a^\dagger, a^\dagger] = 0 \\ H &= \omega\left(a^\dagger a + \frac{1}{2}\right) \end{aligned}$$

and

$$\begin{aligned} [H, a^\dagger] &= \omega a^\dagger \\ [H, a] &= -\omega a \end{aligned}$$

Suppose

$$\begin{aligned} H|n\rangle &= \omega_n|n\rangle & (5.1.7)\\ H(a^\dagger|n\rangle) &= ([H,a^\dagger]+a^\dagger H)|n\rangle \\ &= (\omega a^\dagger|n\rangle + a^\dagger\omega_n|n\rangle) \\ &= (\omega_n+\omega)(a^\dagger|n\rangle) & (5.1.8)\\ \text{and similarly}\quad H(a|n\rangle) &= (\omega_n-\omega)(a|n\rangle) & (5.1.9)\end{aligned}$$

More generally,

$$H(a^m|n\rangle) = (\omega_n - m\omega)(a^m|n\rangle) \tag{5.1.10}$$

For the energy to have a lower bound, there must be a state $|0\rangle$ such that $a|0\rangle = 0$.

In this case,

$$\begin{aligned} H|0\rangle &= \omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle \quad ; \quad a|0\rangle = 0 \\ &= \frac{1}{2}\omega|0\rangle \quad \text{(Consistent with uncertainty principle).} \\ &= \frac{1}{2}\left(p^2+\omega^2q^2\right)|0\rangle \to \Delta p\ \Delta q \geq \frac{\hbar^2}{4} & (5.1.11)\end{aligned}$$

i.e. Vacuum state $|0\rangle$ is <u>lowest</u> energy state, <u>not</u> nothing.

$$\begin{aligned} a^\dagger|0\rangle &\propto |1\rangle \\ (a^\dagger)^n|0\rangle &\propto |n\rangle \\ H|n\rangle &= \left(n+\frac{1}{2}\right)\omega|n\rangle \quad (n=0,1,\ldots) & (5.1.12)\end{aligned}$$

For $\langle n|m\rangle = \delta_{nm}$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{(n+1)}}|0\rangle \tag{5.1.13}$$

For our real K.G. field:

$$\begin{aligned} 0 &= \left(\Box + m^2\right)\psi \quad \text{has a solution} \\ \phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{-\mathrm{i}k\cdot x} + a_k^* e^{\mathrm{i}k\cdot x}\right) & (5.1.14)\end{aligned}$$

(note that this is only a scalar field here, $\therefore$ don't need polarization vectors $\varepsilon_{k,\lambda}$). For a complex field, we have the same treatment for real and complex parts.

$$\begin{aligned} \to k\cdot x &= \omega_k t - \underline{k}\cdot\underline{x} \\ \text{with}\ \ \omega_k &= \sqrt{\underline{k}^2+m^2} \\ f_k &= \frac{e^{-\mathrm{i}k\cdot x}}{\sqrt{(2\pi)^3 2\omega_k}}\end{aligned}$$

As

$$\int \frac{d^3k}{(2\pi)^3} e^{\mathrm{i}\underline{k}\cdot\underline{x}} = \delta^3(x) \tag{5.1.15}$$

then

$$\int d^3x \ f_k^* \phi(\underline{x}, t) = \frac{1}{2\omega_k} \left[a_k + a_{-k}^* e^{2\mathrm{i}\omega_k t}\right] \tag{5.1.16}$$

Remember that

$$\begin{aligned} \Pi(\underline{x}, t) &= \frac{\partial \phi(\underline{x}, t)}{\partial t} \\ \therefore \int d^3x \ f_k^* \ \Pi(\underline{x}, t) &= -\frac{\mathrm{i}}{2} \left[a_k - a_{-k}^* e^{2\mathrm{i}\omega_k t}\right] \end{aligned} \tag{5.1.17}$$

$$\tag{5.1.18}$$

Thus,

$$a_k = \mathrm{i} \int d^3x \underbrace{\left[f_k^*(\underline{x}, t) \left(\partial_t \phi(\underline{x}, t)\right) - \left(\frac{\partial}{\partial t} f_k^*(\underline{x}, t)\right) \phi(\underline{x}, t)\right]}_{\Rightarrow f_k^* \overleftrightarrow{\partial} \phi} \tag{5.1.19}$$

If we now quantize

$$[\hat{\phi}(\underline{x}, t), \hat{\Pi}(\underline{y}, t)] = \mathrm{i}\hbar \delta^3(\underline{x} - \underline{y}) \quad (\text{ let } \ \hbar = 1) \tag{5.1.20}$$

then we can show that

$$[\hat{a}(\underline{k}), \hat{a}(\underline{k}')] = 0 \tag{5.1.21}$$

$$[\hat{a}^*(\underline{k}), \hat{a}^*(\underline{k}')] = 0 \tag{5.1.22}$$

$$[\hat{a}(\underline{k}), \hat{a}^*(\underline{k}')] = \delta^3(\underline{k} - \underline{k}') \tag{5.1.23}$$

and

$$\begin{aligned} H &= \int d^3x \ \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right] \\ &= \int d^3x \ \left[\omega_k \left(a^*(\underline{k}) a(\underline{k}) + \frac{1}{2}\right)\right] \end{aligned} \tag{5.1.24}$$

→ Harmonic oscillator! (Note the $\frac{1}{2}$ term above will diverge → at each point, get this vacuum oscillation → $\sum$ over all points, get ∞.)

States in the Hilbert space of these operators:

$$|0_k, 0_{k'}, 0_{k''}, \ldots\rangle = \text{vacuum} \ \equiv |0\rangle \tag{5.1.25}$$

$$\frac{a_l^\dagger}{\sqrt{2!}}|0\rangle = |0_k, \ldots, 1_l, \ldots\rangle \tag{5.1.26}$$

i.e. the $a_l^\dagger$ "creates" a K.G. particle of momentum l. So,

$$\frac{(a_l^\dagger)^2}{\sqrt{3!}}\frac{(a_{l'}^\dagger)}{\sqrt{2!}}|0\rangle \tag{5.1.27}$$

is a state with one K.G. particle of momentum l' and two of momentum l. Notation: When we quantize, the $*$ goes to $\dagger$;

$$a_k^* \longrightarrow a_k^\dagger \tag{5.1.28}$$

As it is only the difference between energy levels that is observed, we take,

$$:H: \quad \rightarrow \quad \text{normal order of } H \tag{5.1.29}$$

$$:H: \quad = \quad \int d^3k\ \omega_k(a_k^\dagger a_k) \tag{5.1.30}$$

(excitations above lowest energy state)

As

$$H = \frac{1}{2}\int d^3k\ \omega_k\left(a_k^\dagger a_k + a_k a_k^\dagger\right) \tag{5.1.31}$$

$\rightarrow$ $:H:$ is obtained by rewriting H so that all annihilation operators are on the right. More generally, $:\phi(x)\phi(y):$ - ordered so that, again, all annihilation operators are on the right hand side. As,

$$\mathrm{i}\int d^3x f_k^* \overleftrightarrow{\partial} f_{k'} \quad = \quad \delta^3(\underline{k}-\underline{k}') \tag{5.1.32}$$

$$\phi^- \quad = \quad \int d^3\underline{x}\ a_k^\dagger f_k^* \tag{5.1.33}$$

$$\phi^+ \quad = \quad \int d^3\underline{x}\ a_k f_k \tag{5.1.34}$$

Oct. 29/99

$$\begin{aligned} :\phi(x)\phi(y): &= :\left(\phi^-(x)+\phi^+(x)\right)\left(\phi^-(y)+\phi^+(y)\right): \\ &= \phi^-(x)\phi^+(y) + \underbrace{\phi^-(y)\phi^+(x)}_{\text{inversion}} + \phi^-(x)\phi^-(y) + \phi^+(x)\phi^+(y) \end{aligned}$$

$$
\begin{aligned}
[\phi(x),\phi(y)] &= \left[(\phi^+(x)+\phi^-(x)),(\phi^+(y)+\phi^-(y))\right] \\
&= \left[\int d^3k\ \left(a_k^\dagger f_k^*(x)+a_k f_k(x)\right),\int d^3k'\left(a_{k'}^\dagger f_{k'}^*(y)+a_{k'}f_{k'}(y)\right)\right] \\
&= \int \frac{d^3k\ d^3k'}{(2\pi)^3\sqrt{2\omega_k\cdot\omega_{k'}}}\left(\underbrace{\left[a_k,a_{k'}^\dagger\right]}_{\delta_{kk'}}e^{-\mathrm{i}(k\cdot x-k'\cdot y)}+\underbrace{\left[a_k^\dagger,a_{k'}\right]}_{-\delta_{kk'}}e^{\mathrm{i}(k\cdot x-k'\cdot y)}\right) \\
&= \int\frac{d^3k}{(2\pi)^3(2\omega_k)}\left[e^{-\mathrm{i}k\cdot(x-y)}-e^{+\mathrm{i}k\cdot(x-y)}\right] \\
&= \frac{-\mathrm{i}}{(2\pi)^3}\int\frac{d^3k}{\omega_k}e^{\mathrm{i}\underline{k}\cdot(\underline{x}-\underline{y})}\sin\left[\omega_k(x_0-y_0)\right] \\
&\qquad \int\frac{d^3k}{2\omega_k}\to\int d^4k\ \delta(k^2-m^2) \\
&= \int\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\varepsilon(k_0)e^{-k\cdot(x-y)} \\
&\qquad \text{where}\ \ \varepsilon(k_0)=\begin{cases}+1 & k_0>0\\ -1 & k_0<0\end{cases} \\
&= \mathrm{i}\Delta(x-y) \qquad (5.1.35)
\end{aligned}
$$

Properties:

$$
\begin{aligned}
0 &= \left(\Box_x-m^2\right)\Delta(x-y)\to\ \text{expected, as}\ \ (\Box-m^2)\phi=0 \\
\Delta(x-y) &= -\Delta(y-x)\quad\to\ \text{Reasonable, as}\ \ [\phi(x),\phi(y)]=-\left[\phi(y),\phi(x)\right]
\end{aligned}
$$

<u>Note:</u>

$$\Delta(\underline{x}-\underline{y},0)=0$$

i.e.

$$\to\int\frac{d^4k}{(2\pi)^3}\delta(k_0^2-\underline{k}^2-m^2)\varepsilon(k_0)e^{-\mathrm{i}\underline{k}\cdot(\underline{x}-\underline{y})}=0$$

Thus,

$$\Delta(x-y)=0\ \ \text{if}\ \ (x-y)^2<0 \qquad (5.1.38)$$

Hence,

$$[\phi(x),\phi(y)]=0\ \ \text{if}\ \ (x-y)^2<0 \qquad (5.1.39)$$

as required by causality. If x, y are separated temporally, (i.e. within light cone), then the order of x, y (in time) is the same $\forall$ observers. If they are separated by spatial separation, (outside light cone), order (time order) not necessarily the same.

5.2 Feynman Propagator

$$\langle 0|T\phi(x)\phi(y)|0\rangle \tag{5.2.1}$$

Hence,

$$T\phi(x)\phi(y) = \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x) \tag{5.2.2}$$

$$\left[\theta(x) = \begin{Bmatrix} 1 & x > 0 \\ 0 & x < 0 \end{Bmatrix}\right] \tag{5.2.3}$$

This is in fact relativistically invariant.

$$\begin{aligned}
T\phi(x)\phi(y) &= \langle 0| \left[\theta(x_0 - y_0)(\phi_x^+ + \phi_x^-)(\phi_y^+ + \phi_y^-) + \theta(y_0 + x_0)(\phi_y^+ + \phi_y^-)(\phi_x^+ + \phi_x^-)\right] |0\rangle \\
&\quad \text{where } \ \phi_x^+ \equiv \phi^+(x), \quad \text{etc.} \\
&\quad \text{so, in first term above, } \phi_x^+,\ \phi_y^- \to 0 \quad \text{and in second term, } \phi_y^+,\ \phi_x^- \to 0 \\
&\to a_k|0\rangle = 0 \quad \langle 0|a_k^+ = 0 \\
&= \langle 0| \left[\theta(x_0 - y_0)\phi_x^-\phi_y^+ + \theta(y_0 - x_0)\phi_y^-\phi_x^+\right] |0\rangle \\
&\to \frac{1}{\sqrt{2}} a_k^\dagger |0\rangle = |1_k\rangle \\
&\quad \langle 0| \frac{1}{\sqrt{2}} a_k = \langle 1_k| \\
&\vdots \\
&= \int \frac{d^3k}{(2\pi)^3(2\omega_k)} \left[\theta(x_0 - y_0 e^{-ik\cdot(x-y)} + \theta(y_0 - x_0)e^{ik\cdot(x-y)}\right]
\end{aligned} \tag{5.2.4}$$

(Note that

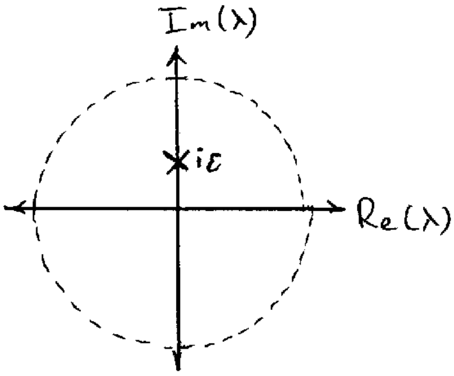
$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi i} \frac{e^{it\lambda}}{\lambda - i\varepsilon}\ ; \quad \varepsilon > 0 \tag{5.2.5}$$

$$\begin{aligned}
\to \lambda &\to k_0 \\
\theta(t) &= \int_{-\infty}^{\infty} \frac{dk_0}{2\pi i} \frac{e^{itk_0}}{k_0 - i\varepsilon}
\end{aligned} \tag{5.2.6}$$

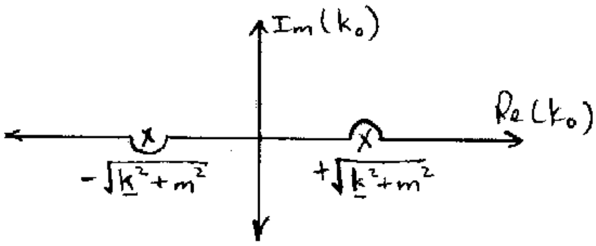
$$T(\phi(x)\phi(y)) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} \tag{5.2.7}$$

Note that $k_0^2 - \underline{k}^2 - m^2 + i\varepsilon \to$ poles at $k_0 = \pm\sqrt{\underline{k}^2 + m^2 - i\varepsilon}$. Hence for $\int dk_0$ we have the contour Hence

$$\begin{aligned}
\langle 0|T\phi(x)\phi(y)|0\rangle &= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} \\
&= i\Delta_F(x - y) \quad \text{(Feynman Propagator)}
\end{aligned} \tag{5.2.8}$$



$$
\begin{array}{lll}
t<0 & \rightarrow & \text{no poles}, \quad \therefore \theta(t)=0 \\
t>0 & \rightarrow & 1 \text{ pole}, \quad \therefore \text{ residue } \theta(t)=1
\end{array}
$$



Note:

$$(\Box_x + m^2)\Delta_F(x-y) = -\delta^4(x-y) \tag{5.2.9}$$

Remember

$$(\Box + m^2)\Delta(x-y) = 0 \tag{5.2.10}$$

We can repeat this procedure for the vector field. Classical P.B.

$$[A_i(\underline{x},t), \Pi_j(\underline{x},t)] = g_{ij}\delta^3(\underline{x}-\underline{y}) \quad (i,j \rightarrow \underline{\text{spatial}} \text{ indices}) \tag{5.2.11}$$

$$\downarrow$$

$$\left[\hat{A}_i(\underline{x},t), \hat{\Pi}_j(\underline{x},t)\right] = -\mathrm{i}\delta_{ij}\delta^3(\underline{x}-\underline{y}) \tag{5.2.12}$$

But, in order to eliminate the constraints, we need the classical Dirac Bracket

$$\left[A_i(\underline{x},t),\Pi_j(\underline{y},t)\right]^* = -\left(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\right)\delta^3(\underline{x}-\underline{y}) \quad \text{(Coulomb gauge)} \tag{5.2.13}$$

which goes to the commutator,

$$\left[\hat{A}_i(\underline{x},t),\hat{\Pi}_j(\underline{y},t)\right] = -\mathrm{i}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\right)\delta^3(\underline{x}-\underline{y}) \tag{5.2.14}$$

We'll work with this commutator.
Nov. 2/99

$$A_i(\underline{x},t) = \int \frac{d^3\underline{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1}^{2}(\underline{\varepsilon}(\underline{k},\lambda))\left(a(\underline{k},\lambda)e^{-\mathrm{i}k\cdot x}+a^\dagger(\underline{k},\lambda)e^{\mathrm{i}k\cdot x}\right) \tag{5.2.15}$$

$$\begin{aligned}\text{Re: }\ &\underline{\varepsilon}(\underline{k},1)=\underline{\varepsilon}(-\underline{k},1)\\ &\underline{\varepsilon}(\underline{k},2)=-\underline{\varepsilon}(-\underline{k},2)\\ &\underline{\varepsilon}(\underline{k},1)\times\underline{\varepsilon}(\underline{k},2)=\frac{\underline{k}}{|\underline{k}|}\end{aligned}$$

These $\underline{\varepsilon}$ are appropriate for the Coulomb gauge - the gauge condition is taken care of by the two conditions with $\underline{k}\to-\underline{k}$, above. Also, once again we have a superposition of harmonic oscillators making up $\underline{A}(\underline{x},t)$;

$$[a(\underline{k},\lambda),a(\underline{k}',\lambda')] = 0 \tag{5.2.16}$$

$$\left[a^\dagger(\underline{k},\lambda),a^\dagger(\underline{k}',\lambda')\right] = 0 \tag{5.2.17}$$

$$\left[a(\underline{k},\lambda),a^\dagger(\underline{k}',\lambda')\right] = \delta^3(\underline{k}-\underline{k}')\delta_{\lambda\lambda'} \tag{5.2.18}$$

Recall ($\hbar=1$)

$$H = \int d^3x\ :\frac{1}{2}(\underline{E}^2+\underline{B}^2):$$

$$= \int d^3\underline{k}\omega_k\sum_{\lambda=1}^{2}a^\dagger(\underline{k},\lambda)a(\underline{k},\lambda) \tag{5.2.19}$$

$$\text{(i.e. } E = \hbar\omega_k N_k \quad \text{Einstein photoelectric effect)}$$

$$\underline{p} = \int d^3x\ :\underline{E}\times\underline{B}:$$

$$= (\hbar)\int d^3\underline{k}\ \underline{k}\sum_{\lambda=1}^{2}a^\dagger(\underline{k},\lambda)a(\underline{k},\lambda) \tag{5.2.20}$$

$$(\underline{p} = \hbar\underline{k}N_k \quad \text{de Broglie}) \tag{5.2.21}$$

Remember that, for a Harmonic Oscillator, if

$$N=a^\dagger a \tag{5.2.22}$$

then

$$N|n\rangle = n|n\rangle \tag{5.2.23}$$

General expression for $A_\mu(\underline{x},t)$ and its time ordered product:

$$\text{Now} \quad A_\mu(\underline{x},t) = \int \frac{d^3\underline{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1}^{2}\varepsilon_\mu(\underline{k},\lambda)\left(a^\dagger(\underline{k},\lambda)e^{\mathrm{i}k\cdot x} + a(\underline{k},\lambda)e^{-\mathrm{i}k\cdot x}\right) \tag{5.2.24}$$

$$\begin{aligned} \text{So} \quad \langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{-\mathrm{i}k\cdot(x-y)}}{k^2+\mathrm{i}\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) \\ &= \mathrm{i}D_{F_{\mu\nu}}(x-y) \end{aligned} \tag{5.2.25}$$

In the Coulomb gauge, if

$$\varepsilon_\mu(\underline{k},\lambda) = (0,\underline{\varepsilon}(\underline{k},\lambda)) \quad (\underline{k}\cdot\underline{\varepsilon}=0) \tag{5.2.26}$$

Let $\eta_\mu = (1,0,0,0)$. In frame where ε_μ has the above form,

$$\tilde{k}_\mu = \frac{k_\mu - k\cdot\eta\;\eta_\mu}{\sqrt{(k\cdot\eta)^2-\eta^2}} \tag{5.2.27}$$

Note

$$\begin{aligned} \tilde{k}\cdot\eta &= 0 \qquad \eta\cdot\varepsilon = 0 \\ \tilde{k}\cdot\varepsilon &= 0 \end{aligned} \tag{5.2.28}$$

k_μ, $\varepsilon_\mu(1)$, $\varepsilon_\mu(2)$, η_μ are 4 orthonormal vectors, all defined with respect to frame where $\varepsilon_\mu(\underline{k},\lambda) = (0,\underline{\varepsilon}(\underline{k},\lambda))$.
Now,

$$\sum_{\lambda=1}^{2}\varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) = -g_{\mu\nu} + \eta_\mu\eta_\nu - \tilde{k}_\mu\tilde{k}_\nu \tag{5.2.29}$$

Reduces to

$$= \delta_{ij} - \frac{k_ik_j}{\underline{k}^2} \quad \text{in frame where} \quad \varepsilon_\mu = (0,\underline{\varepsilon}) \tag{5.2.30}$$

Hence in the Coulomb gauge,

$$\mathrm{i}D_{F_{\mu\nu}}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-\mathrm{i}k\cdot(x-y)}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \underbrace{\frac{k^2\eta_\mu\eta_\nu + k_\mu k_\nu - k\cdot\eta(k_\mu\eta_\nu + k_\nu\eta_\mu)}{-k^2+(\eta\cdot k)^2}}_{*}\right] \tag{5.2.31}$$

$*$ is an artifact of using the Coulomb gauge and cannot affect any physical process.

5.3 Quantizing the Dirac Field

We have

$$\phi(x) \to a_k,\ a^\dagger_{k'} \tag{5.3.1}$$

with states:

$$|n_{k_1}, n_{k_2}, \dots, n_{k_m}\rangle = \frac{(a^\dagger_{k_1})^{n_{k_1}}}{\sqrt{(n_{k_1}+1)!}} \cdots \frac{(a^\dagger_{k_m})^{n_{k_m}}}{\sqrt{(n_{k_m}+1)!}} |0, \dots, 0\rangle \tag{5.3.2}$$

where $n_{k_1}, \dots, n_{k_m} = 1, 2, \dots, \infty$. For electrons, Pauli suggested that no two electrons can be in the same state.

Fermionic Harmonic Oscillator

$$\begin{aligned} H &= \omega\left(b^\dagger b + \frac{1}{2}\right) && (5.3.3)\\ &\{b, b\} = \left\{b^\dagger, b^\dagger\right\} = 0 && (5.3.4)\\ &\left\{b, b^\dagger\right\} = 1 = bb^\dagger + b^\dagger b && (5.3.5)\end{aligned}$$

where the above are now <u>anti</u>-commutator relations. From this,

$$\begin{aligned} [b, H] &= \left[b, \omega\left(b^\dagger b + \frac{1}{2}\right)\right] \\ &= \omega(bb^\dagger b - b^\dagger bb) \quad ; \quad bb^\dagger = 1 - b^\dagger b \\ &= \omega(b - 2b^\dagger bb \quad ; \quad bb + bb = 0 = \{b, b\} \\ &= \omega b - \omega b^\dagger \{b, b\} \\ &= \omega b \end{aligned} \tag{5.3.6}$$

If

$$H|n\rangle = \omega_n |n\rangle \tag{5.3.7}$$

$$\begin{aligned} H(b|n\rangle) &= (bH - \omega b)|n\rangle \\ &\quad (\text{where} \quad [b, H] = bH - Hb = \omega b) \\ &= (\omega_n - \omega)(b|n\rangle) \end{aligned} \tag{5.3.8}$$

So also,

$$H(b^\dagger|n\rangle) = (\omega_n + \omega)(b^\dagger|n\rangle) \tag{5.3.9}$$

For the energy of the system to be bounded below,

$$b|0\rangle = 0 \tag{5.3.10}$$

For this state;

$$\begin{aligned} H|0\rangle &= \omega\left(b^\dagger b + \frac{1}{2}\right)|0\rangle \\ &= \frac{\omega}{2}|0\rangle \quad \left(\omega_0 = \frac{\omega}{2}\right) \end{aligned} \tag{5.3.11}$$

Now,

$$b^\dagger|0\rangle = |1\rangle \tag{5.3.12}$$

$$\begin{aligned} H|1\rangle &= \omega(b^\dagger b + 1/2)|1\rangle \\ &= \omega(b^\dagger b + 1/2)b^\dagger|0\rangle \quad (b^\dagger b b^\dagger = b^\dagger(bb^\dagger + b^\dagger b - b^\dagger b) = b^\dagger) \\ &= \omega(b^\dagger + (1/2)b^\dagger)|0\rangle \\ &= \frac{3\omega}{2}(b^\dagger|0\rangle) = \frac{3\omega}{2}|1\rangle \end{aligned} \tag{5.3.13}$$

But $(b^\dagger)^2|0\rangle = 0$ as $\{b^\dagger, b^\dagger\} = 0$.

Nov. 3/99

Consider,

$$H = \omega\left(b^\dagger b - \frac{1}{2}\right) \tag{5.3.14}$$

$$\begin{aligned} \{b, b^\dagger\} &= 1 \\ b|0\rangle &= 0\ , \quad b^\dagger|0\rangle = |1\rangle\ , \quad (b^\dagger)^2|1\rangle = 0 \end{aligned}$$

$$\begin{aligned} H|0\rangle &= -\frac{\omega}{2}|0\rangle \\ H|1\rangle &= \frac{\omega}{2}|1\rangle \end{aligned}$$

$$\begin{aligned} H' &= \omega\left(-b^\dagger b + \frac{1}{2}\right) = -H \\ &= \omega\left(\underbrace{-b^\dagger b - bb^\dagger}_{=-1} + bb^\dagger + \frac{1}{2}\right) \\ &= \omega\left[bb^\dagger - \frac{1}{2}\right] \end{aligned} \tag{5.3.15}$$

There is a symmetry between b and $b^\dagger$;

$$b^\dagger|0\rangle' = 0 \tag{5.3.16}$$

$$b|0\rangle = |1\rangle' \tag{5.3.17}$$

Consequently though,

$$\begin{aligned} H|0\rangle' &= \omega\left(b^\dagger b - \frac{1}{2}\right)|0\rangle' \\ &= \omega\left[\underbrace{b^\dagger b + bb^\dagger}_{1} - bb^\dagger - \frac{1}{2}\right]|0\rangle' \\ &= \frac{\omega}{2}|0\rangle' \end{aligned} \tag{5.3.18}$$

$$\begin{aligned} H|1\rangle' &= \omega\left(b^\dagger b - \frac{1}{2}\right)(b|0\rangle') \\ &= -\frac{\omega}{2}|1\rangle'^2 \end{aligned} \tag{5.3.19}$$

$$|0\rangle \leftrightarrow |1\rangle' \tag{5.3.20}$$

$$|1\rangle \leftrightarrow |0\rangle' \tag{5.3.21}$$

with the operator H, b destroys a state $|1\rangle$ while this operator b becomes a creation operator for a system with Hamiltonian H'.
Now for the electron field ψ.

$$\psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\left[b(p,s)e^{-\mathrm{i}p\cdot x}u(p,s) + d^*(p,s)e^{\mathrm{i}p\cdot x}v(p,s)\right] \tag{5.3.22}$$

$$(\not p - m)u = 0 = (\not p + m)v \tag{5.3.23}$$

$$\sum_{s=\pm} u_\alpha(p,s)\bar{u}_\beta(p,s) = \left(\frac{\not p + m}{2m}\right)_{\alpha\beta} \tag{5.3.24}$$

$$\sum_{s=\pm} v_\alpha(p,s)\bar{v}_\beta(p,s) = \left(\frac{\not p + m}{2m}\right)_{\alpha\beta} \tag{5.3.25}$$

$$\bar{u}(p,s)v(p,s') = 0 \tag{5.3.26}$$

$$\bar{u}(p,s)u(p,s') = -\bar{v}(p,s)v(p,s') = \delta_{ss'} \tag{5.3.27}$$

$$u^\dagger(p,s)u(p,s') = v^\dagger(p,s)v(p,s') = \frac{E_p}{m}\delta_{ss'} \tag{5.3.28}$$

Quantizing ψ

$$(\mathrm{i}\,\not\partial - m)\psi = 0 \tag{5.3.29}$$

This can be derived from the Lagrangian

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\,\not\partial - m)\psi \qquad (\bar{\psi} = \psi^\dagger\gamma_0) \tag{5.3.30}$$

provided ψ and $\bar{\psi}$ are treated as being independent.

$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = \bar{\psi}(\mathrm{i}\gamma_0)_\alpha = \mathrm{i}\psi^\dagger_\alpha \tag{5.3.31}$$

$$\bar{\Pi}_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\psi}_\alpha)} = 0 \tag{5.3.32}$$

Eventually the constraint formalism gives,

$$\left\{\psi_\alpha(\underline{x},t),\Pi_\beta(\underline{y},t)\right\}_{\text{quantum}} \leftrightarrow \left[\psi_\alpha(\underline{x},t),\Pi_\beta(\underline{y},t)\right]^*_{\text{Classical Dirac Bracket}} \tag{5.3.33}$$

The quantization of the E.M. field gives

$$\boxed{\{\psi_\alpha(\underline{x},t),\Pi_\beta(\underline{y},t)\} = \mathrm{i}\delta^3(\underline{x}-\underline{y})} \tag{5.3.34}$$

$$\boxed{\{\psi_\alpha(\underline{x},t),\psi^\dagger_\beta(\underline{y},t)\} = \delta^3(\underline{x}-\underline{y})} \tag{5.3.35}$$

As $b,\ d^\dagger$ are now operators, we obtain,

$$\left.\begin{array}{l}\{b(p,s),b^\dagger(p',s')\} = \delta^3(\underline{p}-\underline{p}')\delta_{ss'}\\ \{d(p,s),d^\dagger(p',s')\} = \delta^3(\underline{p}-\underline{p}')\delta_{ss'}\end{array}\right\}\ \text{Fermionic creation and annihilation operators} \tag{5.3.36}$$

$b^\dagger(p,s)$ $(b(p,s))$ creates (destroys) an electron with energy $E_p = +\sqrt{\underline{p}^2+m^2}$ and spin s, momentum $\underline{p}$.

$d(p,s)$ $(d^\dagger(p,s))$ creates (destroys) a positron with energy $-E_p = -\sqrt{\underline{p}^2+m^2}$, spin s and momentum $-\underline{p}$. or

$d(p,s)$ $(d^\dagger(p,s))$ annihilates (creates) a positron with energy $E_p = +\sqrt{\underline{p}^2+m^2}$, spin s and momentum $+\underline{p}$. i.e. Now,

$$: H := \int d^3x\ :\left(\Pi_\alpha\frac{\partial\psi_\alpha}{\partial t} - \mathcal{L}\right): \tag{5.3.37}$$

$$\left(\text{Here}\quad :\psi_\alpha\psi_\beta := \psi^+_\alpha\psi^+_\beta + \psi^-_\alpha\psi^-_\beta + \psi^+_\alpha\psi^-_\beta - \psi^+_\beta\psi^-_\alpha\right)$$

We find that

$$H = \sum_{s=\pm}\int d^3p\ E_p\left(b^\dagger(p,s)b(p,s) + d^\dagger(p,s)d(p,s)\right) \leftarrow \text{ Positive Definite Hamiltonian} \tag{5.3.39}$$

where this is a positive definite Hamiltonian because the $b(p,s)$ destroys an electron of energy E_p, and the $d(p,s)$ destroys a positron of momentum E_p.

If we had quantized using a commutator, we would have gotten;

$$H = \sum_{s=\pm}\int d^3p\ E_p\left(b^\dagger(p,s)b(p,s) - d^\dagger(p,s)d(p,s)\right) \tag{5.3.40}$$

where the negative sign would mean there is no lower bound on the spectrum of H. Consider the current

$$\begin{aligned} j_\mu &= \bar{\psi}\gamma_\mu\psi \quad \text{Vector - eventually this will be the e.m. current.} \quad (5.3.41) \\ :Q:=:j_0: &= :\bar{\psi}\gamma_0\psi: \\ &= :\psi^\dagger\psi: \\ &= \int d^3p \, \left(b^\dagger(p,s)b(p,s) - d^\dagger(p,s)d(p,s)\right) \quad (5.3.42) \end{aligned}$$

with $b^\dagger(p,s)b(p,s)$ the positive contribution from electrons, and $d^\dagger(p,s)d(p,s)$ term the negative contribution from positrons.
Nov. 6/99

$$\begin{aligned} \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} &= \mathrm{i}(\mathrm{i}\,\partial\!\!\!/ + m)\Delta(x-y) \quad (5.3.43) \\ [\phi(x), \phi(y)] &= \mathrm{i}\Delta(x-y) \\ &= 0 \ \text{ if } \ (x-y)^2 < 0 \quad (5.3.44) \end{aligned}$$

If we used commutators for $\psi_\alpha(\underline{x},t)$ and $\Pi_\beta(\underline{y},t)$, $[\psi_\alpha(x), \psi_\beta(y)] \neq 0$ for $(x-y)^2 < 0$. This would be inconsistent. So also,

$$\begin{aligned} \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta|0\rangle &= \mathrm{i}S_{F\alpha\beta}(x-y) \quad (5.3.45) \\ S_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\varepsilon}(p\!\!\!/ + m) \quad \text{(Scalar case)} \quad (5.3.46) \end{aligned}$$

Since $(p\!\!\!/ + m)(p\!\!\!/ - m) = p\!\!\!/^2 - m^2 = p^2 - m^2$, (last = due to anti-commutation relations)

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-\mathrm{i}p\cdot(x-y)}}{p\!\!\!/ - m + \mathrm{i}\varepsilon} \quad (5.3.47)$$

For Fermi Dirac particles

$$Ta(t)b(t') = \theta(t-t')a(t)b(t') - \theta(t'-t)b(t')a(t) \quad (5.3.48)$$

Chapter 6

Interacting Fields

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2}_{\text{Free field}} - \underbrace{\frac{\lambda \phi^4}{4!}}_{\text{Interaction}} \tag{6.0.1}$$

The equation of motion is

$$\partial^2 \phi + m^2 \phi + \frac{\lambda \phi^3}{6} = 0 \tag{6.0.2}$$

(Most important type of interaction $\rightarrow$ Gauge).

6.1 Gauge Interaction

Schrodinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi \quad \psi - \psi \;\text{ complex} \tag{6.1.1}$$

Probability density $\psi^*(\underline{x}, t)\psi(\underline{x}, t)$ is invariant under a phase change

$$\psi(\underline{x}, t) \rightarrow e^{\mathrm{i}\Lambda}\psi(\underline{x}, t) \tag{6.1.2}$$

where Λ is a constant. If $\psi(\underline{x}, t) \rightarrow e^{\mathrm{i}\Lambda(\underline{x},t)}\psi(\underline{x}, t)$ is an invariant then $\psi^*\psi$ is unaltered, but

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi \quad \text{is } \underline{\text{changed}} \tag{6.1.3}$$

Introduce $A_\mu(\underline{x}, t)$ and replace ∂_μ by $D_\mu = \partial - \mathrm{i}eA_\mu$, and let $A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu \Lambda$ if $\psi \rightarrow e^{\mathrm{i}\Lambda}\psi$. i.e.

$$D_\mu \psi \rightarrow e^{\mathrm{i}\Lambda} D_\mu \psi \tag{6.1.4}$$

A_μ is the electromagnetic potential; ψ becomes complex.

$$\mathcal{L} = (\partial_\mu \psi^*)(\partial^\mu \phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \tag{6.1.5}$$

$$\partial_\mu \phi \rightarrow D_\mu \phi$$

$$\mathcal{L} = \left[(\partial_\mu + \mathrm{i}eA_\mu)\,\phi^*\right]\left[(\partial^\mu - \mathrm{i}eA^\mu)\,\phi\right] - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \tag{6.1.6}$$

This is invariant under

$$\begin{aligned}\phi &\rightarrow e^{\mathrm{i}\Lambda}\phi \\ \phi^* &\rightarrow e^{-\mathrm{i}\Lambda}\phi \\ A_\mu &\rightarrow A_\mu + \frac{1}{e}\partial_\mu \Lambda\end{aligned}$$

Also, add in

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (6.1.7)\\ &\quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \frac{\mathrm{i}}{e}\left[D_\mu, D_\nu\right](f) \quad (f = \text{ test function}) \\ &= \frac{\mathrm{i}}{e}\left[(\partial_\mu - \mathrm{i}eA_\mu), (\partial_\nu - \mathrm{i}eA_\nu)\right](f) \\ &= A_{\nu,\mu} - A_{\mu,\nu} \qquad (6.1.8)\end{aligned}$$

We can also include spinors

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\left(\partial_\mu - \mathrm{i}eA_\mu\right)\gamma^\mu - m\right)\psi \qquad (6.1.9)$$

So also there is an interaction between spinors and real scalars:

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\ \partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\mu^2}{2}\phi - \frac{\lambda\phi^4}{4!} - \mathrm{i}g\bar{\psi}\psi\phi \qquad (6.1.10)$$

where the last term is the yukawa interaction (the i is required for $\mathcal{L} = \mathcal{L}^\dagger$. (you must have a complex scalar field to interact with the electromagnetic field).

6.2 Heisenberg Picture of Q.M.

$$\begin{aligned}A_H &= A_h(x,t) \quad \text{(Heisenberg operator)} \qquad (6.2.1)\\ \mathrm{i}\frac{\partial}{\partial t}A_H &= [A_H, H] \qquad (6.2.2)\\ \left(\text{c.f.} \quad \frac{\partial A}{\partial t}\right. &= \left.[A,H]_{PB} \rightarrow \frac{1}{\mathrm{i}}[\ \ ]_{\text{Commutator}}\right) \qquad (6.2.3)\end{aligned}$$

Solution

$$A_H(x,t) = e^{\mathrm{i}Ht}A_H(x,0)e^{-\mathrm{i}Ht} \qquad (6.2.4)$$

Heisenberg states $|\psi\rangle_H$ $\left(\frac{\partial}{\partial t}|\psi\rangle_H = 0\right)$. Matrix elements

$$\langle\phi|A_H|\psi\rangle_H = \langle\phi|e^{\mathrm{i}Ht}A_H(x,0)e^{-\mathrm{i}Ht}|\psi\rangle_H \qquad (6.2.5)$$

Let $A_H(x,0) = A_S(x) \leftarrow$ Schrodinger operator

$$|\psi_s\rangle = e^{-iHt}|\psi\rangle_H = |\psi(t)\rangle_S \tag{6.2.6}$$

$$\text{Now } \frac{\partial}{\partial t}A_s(x) = 0$$

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_S = H|\psi(t)\rangle_S \tag{6.2.7}$$

Dirac's <u>interaction representation</u>,

$$A_{ip}(x,t) = e^{iH_0t}A_S(x)e^{-iH_0t} \tag{6.2.8}$$

$$H = H_0 + H_I = e^{iH_0t}e^{-iHt}A_H(\underline{x},t)e^{iHt}e^{-iH_0t}$$

$$|a(t)\rangle = e^{iH_0t}|a(t)\rangle_S \tag{6.2.9}$$

$$\begin{aligned}\text{Thus } \langle\phi_H|A_H|\phi_H\rangle &= \langle\phi_S|A_S|\phi_S\rangle \\ &= \langle\phi_{ip}|A_{ip}|\psi_{ip}\rangle\end{aligned} \tag{6.2.10}$$

We note that

$$\frac{i\partial A_{ip}}{\partial t} = i\frac{\partial}{\partial t}\left[e^{iH_0t}A_Se^{-iH_0t}\right] = [A_{ip}, H_0] \tag{6.2.11}$$

($A_{ip}(\underline{x},t)$ evolves as if there were <u>no</u> interactions). <u>So, also</u>

$$\begin{aligned}i\frac{\partial}{\partial t}|a(t)\rangle_{ip} &= i\frac{\partial}{\partial t}\left[e^{iH_0t}|a(t)\rangle_S\right] \\ &= \left[-H_0e^{iH_0t}|a(t)\rangle_S + e^{iH_0t}\left(i\frac{\partial}{\partial t}|a(t)\rangle_S\right)\right] \\ &= -H_0e^{iH_0t}|a(t)\rangle_S + e^{iH_0t}\left(H_0+H_I\right)|a(t)\rangle_S \\ &= \left(e^{iH_0t}H_Ie^{-iH_0t}\right)e^{iH_0t}|a(t)\rangle_S \\ &= H_I^{ip}|a(t)\rangle_{ip}\end{aligned} \tag{6.2.12}$$

Nov. 9/99

Now, drop the "*ip*" (we're always working in the interaction picture from now on). So,

$$\begin{aligned}&i\tfrac{\partial}{\partial t}|a(t)\rangle = H_I|a(t)\rangle \\ &i\tfrac{\partial}{\partial t}A(t) = [A(t), H_0]\end{aligned} \tag{6.2.13}$$

is in the interaction picture. Suppose,

$$|a(t)\rangle = U(t,t_0)|a(t_0)\rangle \tag{6.2.14}$$

where $U(t,t_0)$ is an evolution operator. <u>Note:</u>

1. $U(t,t) = 1$
2. $U(t,t_0)^{-1} = U(t_0,t)$

3. By (6.2.13),

$$\begin{aligned}
\mathrm{i}\partial_t U(t,t_0)|a(t_0)\rangle &= H_I U(t,t_0)|a(t_0)\rangle \\
&\quad \text{Notice } |a(t_0)\rangle \text{ has no t dependence} \\
\therefore \mathrm{i}\frac{\partial}{\partial t}U(t,t_0) &= H_I(t)U(t,t_0) \qquad (6.2.15)
\end{aligned}$$

Take the Hermitian conjugate

$$-\mathrm{i}\frac{\partial}{\partial t}U^\dagger(t,t_0) = U^\dagger(t,t_0)H_I \quad (H_I^\dagger = H_I) \tag{6.2.16}$$

Thus, from $\big(U^\dagger(6.2.15) + (6.2.16)U\big)$

$$\frac{\partial}{\partial t}\left[U^\dagger(t,t_0)U(t,t_0)\right] = 0 \tag{6.2.17}$$

and

$$U^\dagger(t,t_0)U(t,t_0) = 1 \quad (\text{i.e. } = \text{constant, let constant } = 1) \tag{6.2.18}$$

So we have

$$U^{-1}(t,t_0) = U^\dagger(t,t_0) \;\; (\text{i.e. } U \text{ is unitary}) \tag{6.2.19}$$

We can integrate (6.2.15)

$$U(t,t_0) = K - \mathrm{i}\int_{t_0}^{t} dt'\; H_I(t')U(t',t_0) \tag{6.2.20}$$

As $U(t,t) = 1$, $\therefore K = 1$. We can iterate this equation,

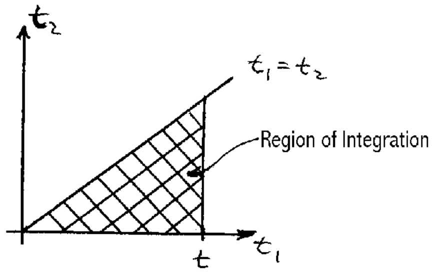
$$\begin{aligned}
U(t,t_0) &= 1 - \mathrm{i}\int_{t_0}^{t} dt_1\; H_I(t_1) + (-\mathrm{i})^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; H_I(t_1)H_I(t_2) \\
&\quad + (-\mathrm{i})^3\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_3} dt_3\; H_I(t_1)H_I(t_2)H_I(t_3) + \ldots \qquad (6.2.21)
\end{aligned}$$

This satisfies (6.2.15). Examine: We can also write as:

$$\frac{(-\mathrm{i})^2}{2}\left[\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\; H_I(t_2)H_I(t_1)\right] \tag{6.2.22}$$

where the first region of integration is as in diagram above, and the second region of integration is the same diagram with the region reflected in the $t_1 = t_2$ line. But this is also,

$$= \frac{(-1)^2}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left[T H_I(t_1)H_I(t_2)\right] \tag{6.2.23}$$



$$(-\mathrm{i})^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2)$$

So also

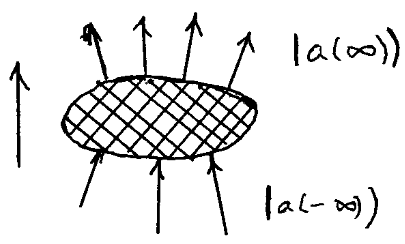
$$\begin{aligned} & (-\mathrm{i})^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \; H_I(t_1) H_I(t_2) H_I(t_3) \\ = & \frac{(-\mathrm{i})^3}{3!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3 \; T H_I(t_1) H_I(t_2) H_I(t_3) \end{aligned}$$

In general, then,

$$\begin{aligned} U(t, t_0) &= 1 + T\frac{(-\mathrm{i})}{1!} \int_{t_0}^{t} dt_1 \; H_I(t_1) + T\frac{(-\mathrm{i})^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; H_I(t_1) H_I(t_2) \\ & \quad + \ldots + T\frac{(-1)^n}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n \; H_I(t_1) \ldots H_I(t_n) \\ &= T \exp\left[-\mathrm{i} \int_{t_0}^{t} dt_1 \; H_I(t_1)\right] \end{aligned} \tag{6.2.24}$$

Consider now a scattering problem:

- In this we go from a state $|a(t = -\infty))$ to a state at $(a(t = +\infty)|$
- At $t = \pm\infty$, we have free particles (Adiabatic Approximation)



At $t = -\infty$, we have a free particle state, $|a(t = -\infty))$, and at $t = +\infty$, we have a free particle state $|b(t = +\infty))$, and what we want to compute is $(b(\infty)|a(\infty))$ - This gives the amplitude for $|a(-\infty))$ evolving into $|b(\infty))$. But,

$$|a(\infty)) = U(\infty, -\infty)\ |a(-\infty)) \tag{6.2.25}$$

This means we want,

$$(b(\infty)|\ U(\infty, -\infty)\ |a(-\infty)) = S_{ba} \ \rightarrow \ \text{(S-matrix)} \tag{6.2.26}$$

To evaluate this, we need,

$$(b(\infty)|\ \frac{(-\mathrm{i})^n}{n!} T \int_{t_0}^{t} dt_1 \ \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n \ H_I(t_1) \ldots H_I(t_n)\ |a(-\infty)) \tag{6.2.27}$$

6.3 Wick's Theorem

This converts Time-ordered products to normal-ordered products. (actual theorem works for any two orderings of operators).
Trivial Case: (n=1)

$$T H_I(t_1) =: H_I(t_1) : \tag{6.3.1}$$

n=2 (let $t_1 > t_2$).

$$\begin{aligned} T H_I(t_1) H_I(t_2) &= \ : \left(H_I^+(t_1) + H_I^-(t_1)\right) \left(H_I^+(t_2) + H_I^-(t_2)\right) : \\ & \quad \text{(let's let } \ H_I(t_1) = H_1, \ \text{ etc.)} \\ &= H_1^+ H_2^+ + H_1^+ H_2^- + \underbrace{H_1^- H_2^+}_{\text{Not N.O.'d}} + H_1^- H_2^- \end{aligned} \tag{6.3.2}$$

Now,

$$H_1^- H_2^+ = \begin{cases} \underbrace{H_1^- H_2^+ - H_2^+ H_1^-}_{\left[H_1^-, H_2^+\right]} - \underbrace{H_2^+ H_1^-}_{\text{N.O.'d}} & \text{(Bosonic)} \\ \underbrace{H_1^- H_2^+ + H_2^+ H_1^-}_{\{H_1^-, H_2^+\}} - \underbrace{H_2^+ H_1^-}_{\text{N.O.'d}} & \text{(Fermionic)} \end{cases} \tag{6.3.3}$$

Thus,

$$T H_1 H_2 =: H_1 H_2 : + \begin{cases} \left[H_1^-, H_2^+\right] & \leftarrow \text{(Bosonic)} \\ \left\{H_1^-, H_2^+\right\} & \leftarrow \text{(Fermionic)} \end{cases} \tag{6.3.4}$$

But this (anti)-commutator is a c-number. In general these are just $\langle 0 | T H_1 H_2 | 0 \rangle$. i.e. We have ($t_1 > t_2$),

$$\begin{aligned} \left[H_1^-, H_2^+\right] &= \langle 0 | H_1^- H_2^+ | 0 \rangle \\ &= \langle 0 | T H_1^- H_2^+ | 0 \rangle \\ &= \langle 0 | T H_1 H_2 | 0 \rangle \end{aligned} \tag{6.3.5}$$

For <u>both</u> $t_1 > t_2$ and $t_2 > t_1$, we have

$$T H_I(t_1)H_I(t_2) =: H_I(t_1)H_I(t_2) : +\langle 0|T H_I(t_1)H_I(t_2)|0\rangle \tag{6.3.6}$$

Nov. 10/99
Inductively, we can show that

$$\begin{aligned}
T\chi(x_1)\dots\chi(x_n) &= :\chi(x_1)\chi(x_2)\dots\chi(x_n): \\
&+\sum_{i<j}(0|\,T\chi(x_i)\chi(x_j)\,|0) :\chi(x_1)\dots\chi(x_{i-1})\chi(x_{i+1})\dots\chi(x_{j-1})\chi(x_{j+1})\dots\chi(x_n): \\
&+\sum_{\text{all possible 2 pairs}}(0|\,T\chi(x_i)\chi(x_j)\,|0)\,(0|\,T\chi(x_k)\chi(x_l)\,|0)\underbrace{:\chi(x_1)\dots\chi(x_n):}_{\chi(x_{ijkl})\text{ all excluded}} \\
&+\dots \\
&+\sum_{\text{all possible n/2 pairs}}(0|\,T\dots|0)\dots(0|\,T\dots|0) :\chi:
\end{aligned} \tag{6.3.7}$$

This is useful, as

$$(0|:\chi(x_1)\dots\chi x_n:|0) = 0 \tag{6.3.8}$$

Consider the case where

$$H_I = \int d^3x\ \mathcal{H}(x) \tag{6.3.9}$$

$$\mathcal{H}(x) = \mathrm{i}\kappa\bar{\psi}\gamma_5\psi\phi \quad (\text{ Pseudo-scalar Yakawa coupling }) \tag{6.3.10}$$

i.e.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\mu^2}{2}\phi^2 + \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \mathrm{i}\kappa\bar{\psi}\gamma_5\psi\phi \tag{6.3.11}$$

(Note that $\phi(-\underline{x},t) = -\phi(+\underline{x},t)$). Hence:

$$\begin{aligned}
T\exp\left\{-\mathrm{i}\int dt\ H_I(t)\right\} &= T\exp\left\{-\mathrm{i}\int d^4x\mathcal{H}_I(x)\right\} \\
&= T\Bigg[1-\mathrm{i}(\mathrm{i}\kappa)\int d^4x\ (\bar{\psi}(x)\gamma_5\psi(x)\phi(x)) \\
&\quad + \frac{(+\kappa)^2}{2!}\int d^4x_1\ d^4x_2\bar{\psi}(x_1)\gamma_5\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\gamma_5\psi(x_2)\phi(x_2) \\
&\quad + \dots\Bigg]
\end{aligned} \tag{6.3.12}$$

Chapter 7

Electron-Positron Scattering

Now suppose $|a(-\infty))$ consists of an electron of momentum p and polarization s, and a positron of momentum q and polarization t; and suppose $|a(\infty))$ consists of an electron of momentum p' and polarization s', and a positron of momentum q' and polarization t'.

Thus, we have

$$
\begin{aligned}
|a(-\infty)) &= b^\dagger_{ps} d^\dagger_{qt}\ |0) \longrightarrow |0) = b_{ps} d_{qt}\ |a(-\infty)) && (7.0.1)\\
|b(-\infty)) &= b^\dagger_{p's'} d^\dagger_{q't'}\ |0) \longrightarrow (0|\ b_{p's'} d_{q't'} = (b(-\infty)| && (7.0.2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& (b(\infty)|\ T e^{-\mathrm{i}\int d^4x \mathcal{H}(x)}\ |a(-\infty)) \\
= \ & (b(\infty)|\ T \left(1 + \kappa \int d^4x\ \bar{\psi}(x)\gamma_5\psi(x)\phi(x) + \frac{\kappa^2}{2!}\int d^4x_1\ d^4x_2 \ldots\right) |a(-\infty)) \quad (7.0.3)
\end{aligned}
$$

Upon applying Wick's theorem, the only surviving contributions to this matrix element will be those terms with destruction operators for e^- and e^+ acting on $|a(-\infty))$ and creation operators for e^- and e^+ acting on $(b(\infty)|$. So,

$$
\begin{aligned}
& (b(\infty)|\ T e^{-\mathrm{i}\int d^4x \mathcal{H}(x)}\ |a(-\infty)) \\
= \ & (b(\infty)|a(-\infty)) + \kappa \left[(b(\infty)| \int d^4x\ \ : \bar{\psi}(x)\gamma_5\psi(x)\phi(x) :\ |a(-\infty)) \right. \\
& \left. + (b(\infty)| \int d^4x\ \ (0|\ T\bar{\psi}_\alpha(x)\psi_\beta(x)\ |0)\,\gamma_{\alpha\beta} : \phi(x) :\ |a(-\infty)) \right] \\
= \ & 0 \ \text{ i.e. } \ a_p\ |a(-\infty)) = 0 \quad a_p \rightarrow \text{ destruction op. for } \ \phi(x)
\end{aligned}
$$

Term of order κ^2: The only surviving term.

$$
\frac{\kappa^2}{2}\int d^4x_1\ d^4x_2 \left\{ (b(\infty)| \left[\underbrace{(0|\ T\phi(x_1)\phi(x_2)\ |0)}_{\mathrm{i}\Delta_F(x_1 - x^2)} : \bar{\psi}(x_1)\gamma_5\psi(x_1)\bar{\psi}(x_2)\gamma_5\psi(x_2) : \right] |a(-\infty)) \right\} \quad (7.0.4)
$$

The following $(\psi^+,\ \bar{\psi}^+)$ act on $|a(-\infty))$ to give $|0)$ by (7.0.1);

$$\psi^+ = \int \frac{d^3p''}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{p''}}}\sum_{s''=\pm} b(p'',s'')u(p'',s'')e^{-\mathrm{i}p''\cdot x} \tag{7.0.5}$$

$$\bar{\psi}^+ = \int \frac{d^3p''}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{p''}}}\sum_{s''=\pm} d(p'',s'')\bar{v}(p'',s'')e^{-\mathrm{i}p''\cdot x} \tag{7.0.6}$$

where the ψ^+ represents e^- (b operator) and the $\bar{\psi}^+$ represents e^+ (d operator). So also, by (7.0.2),

$$\psi^+ = \int \frac{d^3p''}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{p''}}}\sum_{s''=\pm} b(p'',s'')u(p'',s'')e^{-\mathrm{i}p''\cdot x} \tag{7.0.7}$$

$$\bar{\psi}^+ = \int \frac{d^3p''}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{p''}}}\sum_{s''=\pm} d(p'',s'')\bar{v}(p'',s'')e^{-\mathrm{i}p''\cdot x} \tag{7.0.8}$$

which act on $(b(\infty)|$ to give $(0|$. ($\bar{\psi}^-$ has b^- in it $\rightarrow e^-$.)

Nov. 12/99

So, the only surviving term is:

$$\begin{aligned} = \frac{\kappa^2}{2}\int & dx_1 dx_2\,(0|\,T\phi(x_1)\phi(x_2)\,|0)\,(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}\,\cdot \\ & \cdot\,(b(\infty)|\,\big[\bar{\psi}^-_\alpha(x_1)\psi^-_\beta(x_1)\bar{\psi}^+_\gamma(x_2)\psi^+_\delta(x_2) \\ & \quad + \bar{\psi}^-_\gamma(x_2)\psi^-_\delta(x_2)\bar{\psi}^+_\alpha(x_1)\psi^+_\beta(x_1) \\ & \quad - \bar{\psi}^-_\alpha(x_1)\psi^-_\delta(x_2)\bar{\psi}^+_\gamma(x_2)\psi^+_\beta(x_1) \\ & \quad - \bar{\psi}^-_\gamma(x_2)\psi^-_\beta(x_1)\bar{\psi}^+_\alpha(x_1)\psi^+_\delta(x_2)\big]\,|a(-\infty)) \end{aligned} \tag{7.0.9}$$

where the first two terms give identical contributions, and the last two terms give identical contributions (the minus sign on the last two terms is due to Fermi-Dirac statistics). ex.

$$\psi^-(x) = \sum_{s=\pm}\int d^3p\sqrt{\frac{m}{E_p}}d^\dagger(p,s)v(p,s)e^{\mathrm{i}p\cdot x}$$

and so on ...

Recall

$$\begin{aligned} |a(-\infty)) &= d^\dagger_{q't'}b^\dagger_{p's'}\,|0) \\ \Rightarrow |0) &= d_{q't'}b_{p's'}\,|a(-\infty)) \end{aligned}$$

So also,

$$(0|\,T\phi(x_1)\phi(x_2)\,|0) = \frac{\mathrm{i}}{(2\pi)^4}\int d^4k\,\frac{e^{-\mathrm{i}k\cdot(x_1-x_2)}}{k^2-\mu^2+\mathrm{i}\varepsilon} \tag{7.0.11}$$

Thus,

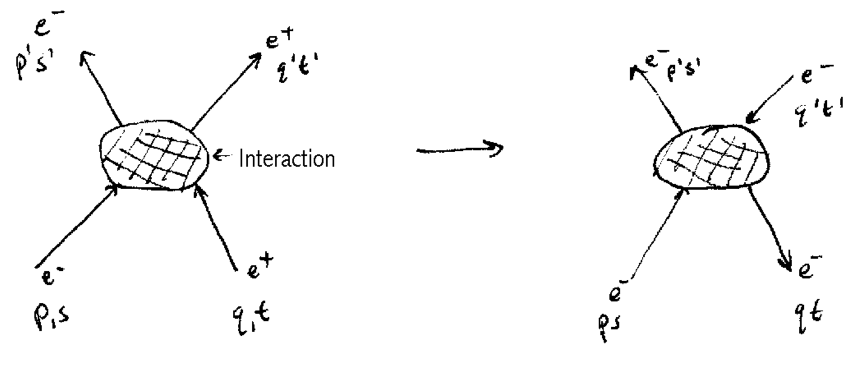
$$
\begin{aligned}
S_{ba} &= \kappa^2 \int d^4x_1\, d^4x_2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{i}e^{-\mathrm{i}k\cdot(x_1-x_2)}}{(k^2-\mu^2+\mathrm{i}\varepsilon)} \sqrt{\frac{m^4}{E_pE_qE_{p'}E_{q'}}}\cdot \\
&\quad \cdot \Big[(\bar{u}(p',s')\gamma_5 v(q',t'))\,(\bar{v}(q,t)\gamma_5 u(p,s))\, e^{\mathrm{i}(p'\cdot x_1+q'\cdot x_1-q\cdot x_2-p\cdot x_2)} \\
&\quad - (\bar{u}(p',s')\gamma_5 u(p,s))\,(\bar{v}(q,t)\gamma_5 v(q',t'))\, e^{\mathrm{i}(p'\cdot x_1-p\cdot x_1+q'\cdot x_2-q\cdot x_2)} \Big]
\end{aligned} \tag{7.0.12}
$$

Integrate over x_1 and x_2:

$$
\rightarrow \int \frac{d^4x}{(2\pi)^4} e^{\mathrm{i}k\cdot x} = \delta^4(k) \tag{7.0.13}
$$

$$
\begin{aligned}
S_{ba} &= \mathrm{i}\kappa^2 \sqrt{\frac{m^2}{E_pE_qE_{p'}E_{q'}}} \int d^4k\, \frac{1}{k^2-\mu^2-\mathrm{i}\varepsilon}\cdot \\
&\quad \cdot \big[(\bar{u}_{p's'}\gamma_5 v_{q't'})(\bar{v}_{qt}\gamma_5 u_{ps})\delta^4(-k+p'+q')\delta^4(k-q-p) \\
&\quad -(\bar{u}_{p's'}\gamma_5 u_{ps})(\bar{v}_{qt}\gamma_5 v_{q't'})\delta^4(-k+p'-p)\delta^4(k+q'-q) \big] \\
&= \mathrm{i}\kappa^2 \sqrt{\frac{m^2}{E_pE_qE_{p'}E_{q'}}} \left[\delta^4(p'+q'-p-q)\right] \left[\frac{(\bar{u}_{p's'}\gamma_5 v_{q't'})(\bar{v}_{qt}\gamma_5 u_{ps})}{(p'+q')^2-\mu^2+\mathrm{i}\varepsilon} \right. \\
&\quad \left. - \frac{(\bar{u}_{p's'}\gamma_5 u_{ps})(\bar{v}_{qt}\gamma_5 v_{q't'})}{(p'-p)^2-\mu^2+\mathrm{i}\varepsilon} \right]
\end{aligned} \tag{7.0.14}
$$

Pictorial Representation



The relative minus sign in figure 7.0.2 comes from interchanging two fermions. We don't actually observe the intermediate particles, and so the fact that they are off mass shell doesn't effect the calculation. After integrating over x_1 and x_2, and going to momentum space, we get

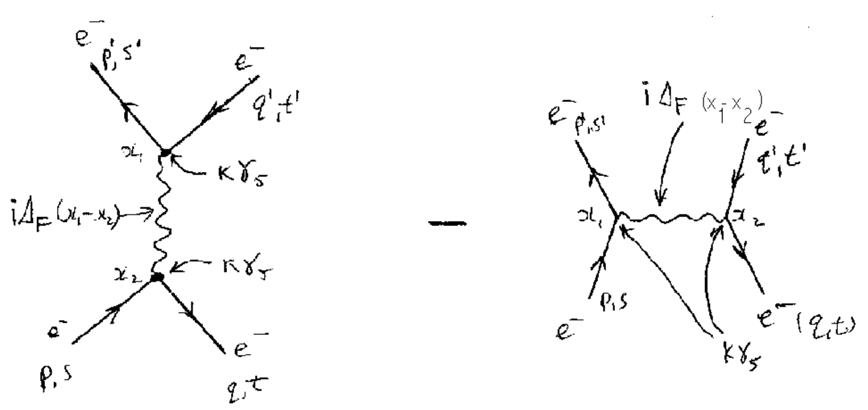


Figure 7.0.1: Electron Scattering

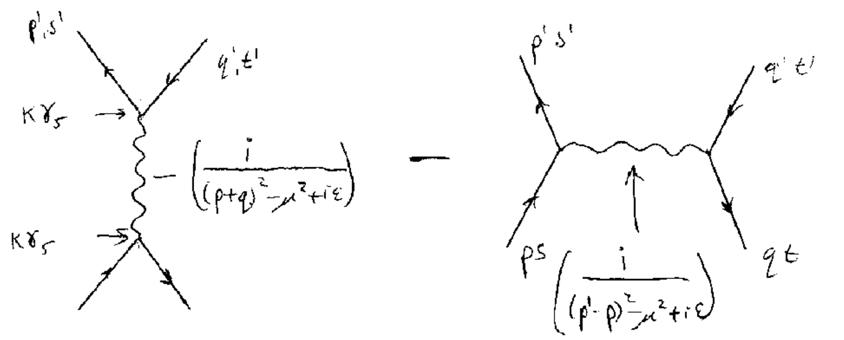


Figure 7.0.2: Electron Scattering (Momentum Space, after integration)

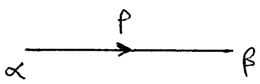
The integral over $x_1,\ x_2$ leads to momentum conservation at each vertex (total momentum incoming = total outgoing momentum).
Nov 16/99

Chapter 8

Loop Diagrams

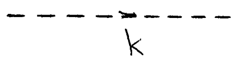
8.1 Feynman Rules in Momentum Space

Propagator for spinor



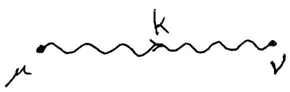
$$\frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon}$$

(Spin 0)



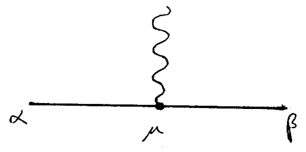
$$\frac{\mathrm{i}}{k^2 - \mu^2 + \mathrm{i}\varepsilon}$$

(Spin 0)



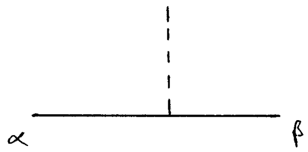
$$\frac{-\mathrm{i}g_{\mu\nu} + \text{ (gauge dependent)}}{k^2 + \mathrm{i}\varepsilon}$$

(Spin 1)



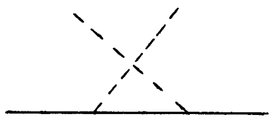
$-\mathrm{i}e(\gamma_\mu)_{\alpha\beta}$

(Vertex)



$-\kappa(\gamma_5)_{\alpha\beta}$

(Vertex)

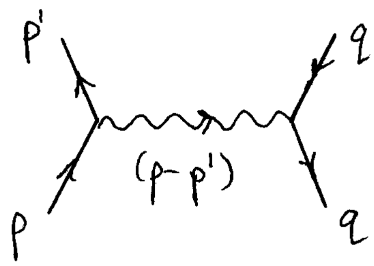


$-\mathrm{i}\lambda$

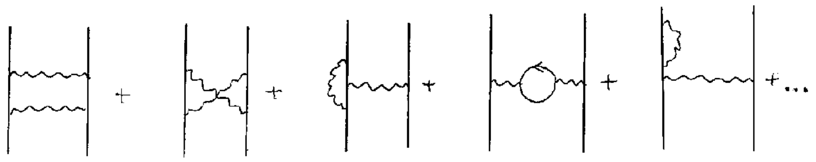
(Vertex)

- momentum is conserved at each vertex
- for each loop, there is an overall factor of $\int \frac{d^4k}{(2\pi)^4} \rightarrow$ (This leads to infinities).

Now, take an example:



One more photon added:
(i.e. many different possibilities)
example:



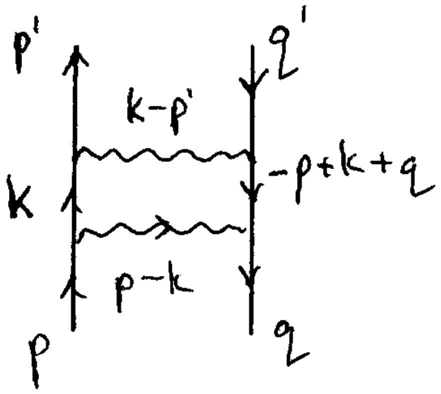


Figure 8.1.1: we only ever observe the external particles $\rightarrow \therefore$ don't know where particles in loop are $\therefore$ must sum over all possibilities $\rightarrow$ Uncertainty Principle.

$(p+q' = q+p')$. The External lines (spinors) give

$$\underbrace{\sqrt{\frac{m}{E_p V}}u(p,s)}_{\substack{\text{incoming } e^- \\ \text{momentum } p, \text{ spin } s}}\underbrace{\sqrt{\frac{m}{E_p V}}\bar{u}(p,s)}_{\text{outgoing } e^-}\underbrace{\sqrt{\frac{m}{E_p V}}v(p,s)}_{\text{Outgoing } e^+}\underbrace{\sqrt{\frac{m}{E_p V}}\bar{v}(p,s)}_{\text{incoming } e^+} \tag{8.1.1}$$

where $\frac{1}{V} \rightarrow$ unit probability in a box of volume V.

- Factor of (-1) for each fermion loop
- If we exchange fermions to go from one diagram to another, their relative phase is (-1).
- Overall factor of $(2\pi)^4\delta(p_i - p_f)$

For example, $e^-e^+ \rightarrow e^-e^+$ (with respective momenta $p, q \rightarrow p', q'$.)

Consider one particular pair of diagrams, which are of order e^4.

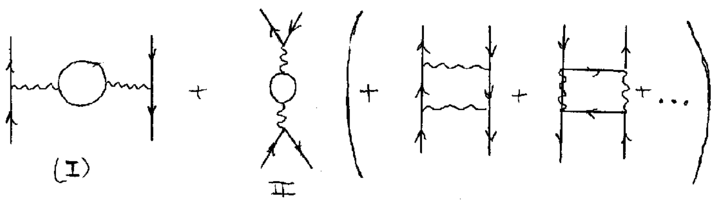
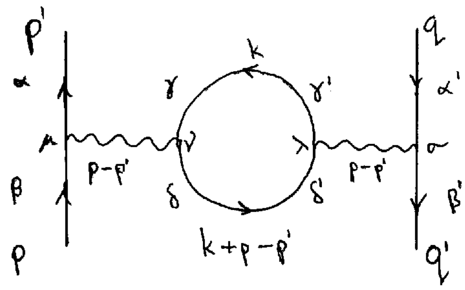


Figure 8.1.2: For Diagram I



$$
\begin{aligned}
& (-1)\int \frac{d^4k}{(2\pi)^4}\left[\sqrt{\frac{m}{E_{p'}V}}\bar{u}_\alpha(p',s')\right]\cdot(-\mathrm{i}e(\gamma_\mu)_{\alpha\beta})\cdot\left[\sqrt{\frac{m}{E_pV}}u_\beta(p,s)\right]\cdot\left(\frac{-\mathrm{i}g_{\mu\nu}}{(p-p')^2+\mathrm{i}\varepsilon}\right)\cdot \\
& \quad\cdot\left[(-\mathrm{i}e\gamma_\nu)_{\gamma\delta}\left(\frac{\mathrm{i}}{(\not k+\not p-\not p')-m+\mathrm{i}\varepsilon}\right)_{\delta\delta'}(-\mathrm{i}e\gamma_\lambda)_{\delta'\gamma'}\left(\frac{\mathrm{i}}{\not k-m+\mathrm{i}\varepsilon}\right)_{\gamma'\gamma}\right]\cdot \\
& \quad\cdot\left(\frac{-\mathrm{i}g_{\lambda\sigma}}{(p-p')^2+\mathrm{i}\varepsilon}\right)\sqrt{\frac{m}{E_qV}}\bar{v}_{\alpha'}(q,t)(-\mathrm{i}e\gamma_\sigma)_{\alpha'\beta'}\sqrt{\frac{m}{E_{q'}V}}v_{\beta'}(q',t') \\
= & (-1)\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta(p+q-p'-q')\frac{m^2}{\sqrt{E_pE_{p'}E_qE_{q'}}}\frac{1}{V^2}\left[\bar{u}(p',s')\gamma_\mu u(p,s)\right]\cdot \\
& \cdot\left(\frac{1}{(p-p')^2+\mathrm{i}\varepsilon}\right)^2\mathrm{Tr}\left(\gamma_\mu\frac{1}{(\not k+\not p-\not p')-m+\mathrm{i}\varepsilon}\gamma_\lambda\frac{1}{\not k-m+\mathrm{i}\varepsilon}\right)\bar{v}(q,t)\gamma_\lambda v(q',t') \quad (8.1.2)
\end{aligned}
$$

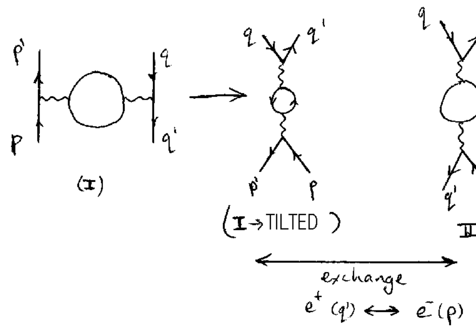
Note that (1) the (-1) is the fermion loop factor, and (2) the spinor propagator is $\frac{\not p+m}{p^2-m^2+\mathrm{i}\varepsilon}\Rightarrow\frac{1}{\not p-m+\mathrm{i}\varepsilon}$

Thus II becomes

$$(-1)(-1)\int\frac{d^4k}{(2\pi)^4}\ldots(u(p,s)\leftrightarrow v(q',t')) \qquad (8.1.3)$$

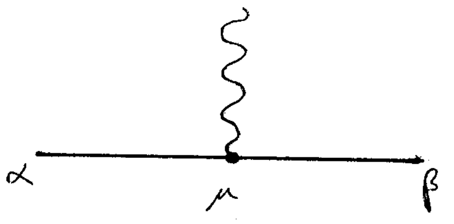
where the second (-1) will come from interchanging the spinors.

Figure 8.1.3: For Diagram II



8.2 Combinatoric Factors

(Always (+1) in QED, as all particles entering a vertex are distinguishable).



$$-e\bar{\psi}\gamma_\mu \psi A^\mu$$

(3 distinct fields)

But, not so lucky in other theories - for example

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2}_{\text{Scalar Propagator}} - \frac{\lambda}{4!}\phi^4 \tag{8.2.1}$$

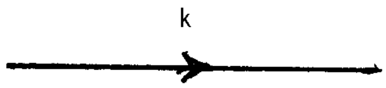
with the 4! in the ϕ^4 term to take into account the indistinguishability of the 4 ϕ's. The Feynman diagrams are:

Consider the one-loop correction to $\phi(p)\phi(q) \to \phi(p')\phi(q')$, above. There are three possible diagrams that have the same topology; note that there is no factor of (-1) here, though, as these are scalars (bosons) here.

Nov. 17/99

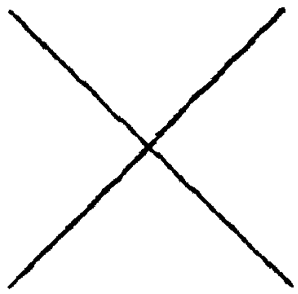
External Photons; recall, for electrons there was the factor

$$e^- \quad - \quad \sqrt{\frac{m}{E_p V}}u(p,s) \tag{8.2.2}$$



$$\frac{\mathrm{i}}{k^2 - \mu^2 + \mathrm{i}\varepsilon}$$

Scalar Propagator



$-\mathrm{i}\lambda$

Vertex

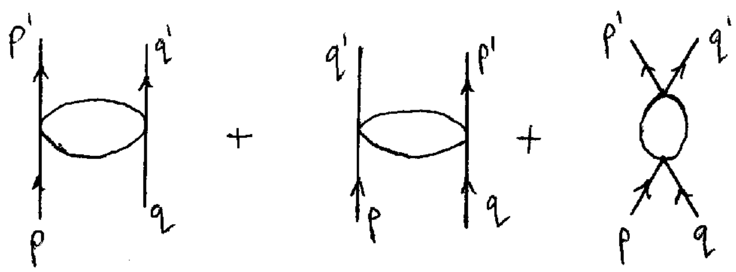
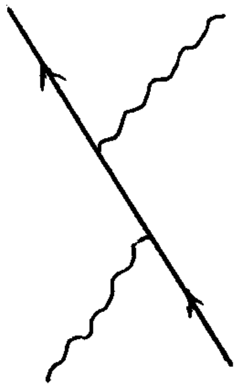


Figure 8.2.1: 1-loop correction to $\phi\phi \to \phi\phi$

Symmetry Factors

recall the diagram for $\phi\phi \to \phi\phi$, figure 8.2.1 above.

As shown in figure 8.2.2, the p leg can attach to either vertex, on any one of the eight different legs, then the p' leg can attach to the same vertex to any of the 3 remaining legs. After that the q leg can attach to any of the 4 legs of the remaining vertex, followed by the q' leg attaching to any of the 3 legs of that vertex; finally, one of the remaining legs on the first



$$\frac{\varepsilon_\mu(k,\lambda)}{\sqrt{2\omega_k V}}$$

$$e^-\gamma \to e^-\gamma$$

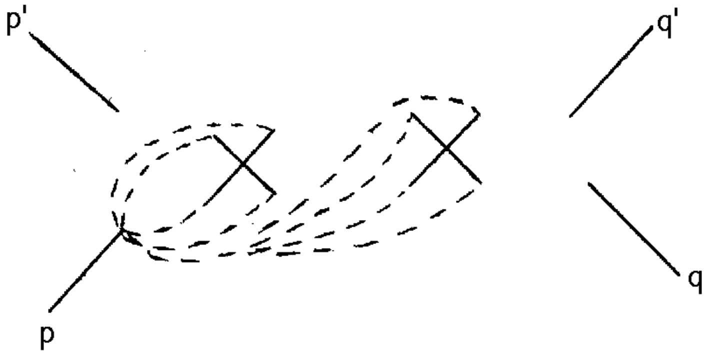


Figure 8.2.2: Symmetry Factor diagram (example)

vertex can attach to either remaining leg on the other vertex, giving in the numerator of the symmetry factor $(8\times 3)\times(4\times 3)\times(2)$. In the denominator, we must take into account that there are 4 possible legs on each vertex $(4!\times 4!)$, and that we can switch vertices $(2!)$; thus, the symmetry factor is

$$\frac{(8\times 3)(4\times 3)(2)}{4!\;4!\;2!} = \frac{1}{2} = \text{ Symmetry factor} \tag{8.2.3}$$

8.3 Cross Sections From Matrix elements

Consider $e^-p \to e^-p$, $\quad (p = \text{proton})$. The proton is treated as a "heavy" electron with opposite charge. To lowest order,

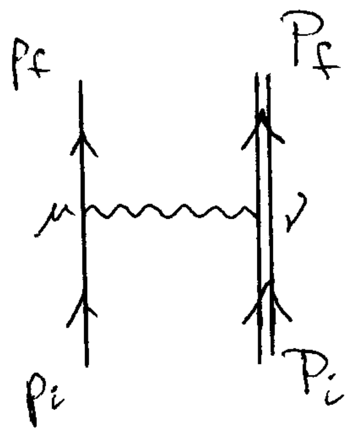


Figure 8.3.1: Electron-Proton scattering

$$
\begin{aligned}
S_{fi} = (a(-\infty)|b(\infty)) \quad = \quad & \left(\sqrt{\frac{m}{\mathcal{E}_f V}}\bar{u}(p_f, s_f)\right)(-\mathrm{i}e\gamma_\mu)\left(\sqrt{\frac{m}{\mathcal{E}_i V}}u(p_i, s_i)\right)\left(\frac{-\mathrm{i}g_{\mu\nu}}{(p_f - P_i)^2 + \mathrm{i}\varepsilon}\right) \\
& \cdot \left(\sqrt{\frac{M}{E_f V}}\bar{u}(P_f, s_f)\right)(+\mathrm{i}e\gamma_\nu)\left(\sqrt{\frac{M}{E_i V}}u(P_i, s_i)\right) \\
& (2\pi)^4\delta^4(p_i + P_i - p_f - P_f) \qquad (8.3.1)
\end{aligned}
$$

(For $e^-e^- \to e^-e^-$, we would have to account for the fact that we can exchange e^-'s (identical particles). - also note the "$+''$ in front of the second e above - this is because the proton is positive.) The probability of this transition occurring per unit volume of space per unit time is

$$\frac{|S_{fi}|^2}{VT} = W_{fi} \qquad (8.3.2)$$

Note:

$$
\begin{aligned}
\left[\delta^4(p_i + P_i - p_f - P_f)\right]^2 \quad &= \quad \delta^4(0)\delta^4(p_i + P_i - p_f - P_f) \\
\text{As} \quad \delta(p) = \int_{-\infty}^{\infty}\frac{dx}{(2\pi)}e^{\mathrm{i}p\cdot x} \quad &\Rightarrow \quad \delta(0) = \int\frac{dx}{(2\pi)} \sim \frac{V}{2\pi} \\
\therefore \left[\delta^4(p_i + P_i - p_f - P_f)\right]^2 \quad &= \quad \frac{VT}{(2\pi)^4}\delta^4(p_i + P_i - p_f - P_f) \qquad (8.3.3)
\end{aligned}
$$

Thus,

$$\begin{aligned} W_{fi} &= \overbrace{(2\pi)^4\delta^4(p_i + P_i - p_f - P_f)\frac{m^2M^2}{V^4E_iE_f\mathcal{E}_i\mathcal{E}_f}}^{\text{Kinematics}} \cdot \\ &\quad \cdot \underbrace{\left|(-\mathrm{i}e)(\mathrm{i}e)(\bar{u}_{p_fs_f}\gamma_\mu u_{p_i,s_i})\left(\frac{-\mathrm{i}}{(p_f - p_i)^2 + \mathrm{i}\varepsilon}\right)(\bar{u}_{P_fs_f}\gamma_\nu u_{P_is_i})\right|^2}_{|M_{fi}|^2} \end{aligned} \tag{8.3.4}$$

The number of states in a volume d^3p_f, d^3P_f is going to be

$$(Vd^3p_f)(Vd^3P_f)W_{fi} \tag{8.3.5}$$

Divide through by $J_{\text{inc}}\frac{1}{V}$, where J_{inc} = flux of incident particles = # of particles per unit area that run by each other per unit time for collinear beams.

$$J_{\text{inc}} = \frac{|v_i - V_i|}{V} \tag{8.3.6}$$

$\frac{1}{V}$ is the number of target particles per unit volume. (as $\psi_i\sqrt{\frac{m}{EV}}u(p,s)$). The cross section is thus

$$\begin{aligned} d\sigma_{fi} &= V^2d^3p_f\ d^3P_f\ (W_{fi})\frac{V}{J_{\text{inc}}} \\ &= d^3p_f\ d^3P_f(2\pi)^4\delta^4(p_i + P_i - p_f - P_f)\frac{m^2M^2}{E_iE_f\mathcal{E}_i\mathcal{E}_f}\frac{1}{|v_i - V_i|}|M_{fi}|^2 \end{aligned} \tag{8.3.7}$$

Note that

$$\frac{mM}{E_i\mathcal{E}_i|v_i - V_i|} = \frac{mM}{\sqrt{(p_i - P_i)^2 - m^2M^2}} \quad \text{(Lorentz invariant)} \tag{8.3.8}$$

So also,

$$\frac{d^3p}{(2E_p)} = \int dp_0\ \delta(p^2 - m^2)\theta(p_0)d^3p \quad \rightarrow \text{ also Lorentz invariant} \tag{8.3.9}$$

Thus $d\sigma$ is Lorentz invariant.
For unpolarized beams,

- Average over the polarizations of the incoming particles
- Sum over outgoing polarizations

i.e.

$$\frac{1}{2}\sum_{s_i}\frac{1}{2}\sum_{S_i}\cdot\sum_{s_f}\sum_{S_f}d\sigma_{fi} \tag{8.3.10}$$

Remember that,

$$\sum_{s=\pm} u_\alpha(p,s)\bar{u}_\beta(p,s) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \tag{8.3.11}$$

$$\left(\sum_{s=\pm} v_\alpha(p,s)\bar{v}_\beta(p,s) = \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}\right) \tag{8.3.12}$$

Thus,

$$\frac{1}{4}\sum_{\substack{s_i,S_i\\ s_f,S_f}} |M_{fi}|^2 = \frac{1}{4}\mathrm{Tr}\left[\frac{\not{p}_f+m}{2m}\gamma_\mu\frac{\not{p}_i+m}{2m}\gamma_\nu\right]\mathrm{Tr}\left[\frac{\not{P}_f+M}{2M}\gamma_\mu\frac{\not{P}_i+M}{2M}\gamma_\nu\right]\frac{e^4}{(p_f-p_i)^4} \tag{8.3.13}$$

Recall the relations:

$$\mathrm{Tr}[\not{a}\ \not{b}] = 4a\cdot b \tag{8.3.14}$$

$$\mathrm{Tr}[\not{a}\ \not{b}\ \not{c}] = 0 \tag{8.3.15}$$

$$\mathrm{Tr}[\not{a}\ \not{b}\ \not{c}\ \not{d}] = 4\left(a\cdot bc\cdot d - a\cdot cb\cdot d + a\cdot db\cdot c\right) \tag{8.3.16}$$

And so

$$\frac{1}{4}\sum_{\substack{s_i,S_i\\ s_f,S_f}} = \frac{e^4}{2m^2M^2(p_f-p_i)^2}\left(P_f\cdot p_f P_i\cdot p_i + P_f\cdot p_i P_i\cdot p_f - m^2 P_f\cdot P_i - M^2 p_f\cdot p_i + 2m^2M^2\right) \tag{8.3.17}$$

Nov. 19/99

$$d\sigma = \frac{mM}{\sqrt{(p_i\cdot P_i)^2 - m^2M^2}}(2\pi)^4\delta^4(p_i+P_i-p_f-P_f)\frac{md^3p_f}{(2\pi)^3\mathcal{E}_f}\frac{Md^3P_f}{(2\pi)^3E_f}|M_{fi}|^2 \tag{8.3.18}$$

Consider,

$$\begin{aligned} d^3p_f &= d\Omega\ |\underline{p}_f|^2 d|\underline{p}_f| \\ &= d\Omega\ |\underline{p}_f|^2\mathcal{E}_f d\mathcal{E}_f \end{aligned} \tag{8.3.19}$$

In the frame of reference where

$$p_f^\mu = (\underline{p}_f, \mathcal{E}_f) \tag{8.3.20}$$

$$p_i^\mu = (\underline{p}_i, \mathcal{E}_i) \tag{8.3.21}$$

$$P_f^\mu \approx P_i^\mu = (\underline{0}, M) \quad \rightarrow \text{ Proton at rest during this scattering} \tag{8.3.22}$$

Note that if $\underline{p}_i\cdot\underline{p}_f = |\underline{p}_i||\underline{p}_f|\cos\theta$, then

$$\begin{aligned} q^2 &= (p_f-p_i)^2 \\ &= -4\mathcal{E}_i\mathcal{E}_f\sin^2\left(\frac{\theta}{2}\right) \end{aligned} \tag{8.3.23}$$

Hence,

$$d\sigma = \frac{\pi^2\alpha^2}{m^2\mathcal{E}_i\mathcal{E}_f\sin^4\left(\frac{\theta}{2}\right)}\left[1+\sin^2\left(\frac{\theta}{2}\right)\left(-\frac{1}{2}\frac{q^2}{M^2}-1\right)\right] \tag{8.3.24}$$

where $\alpha = \frac{e^2}{4\pi}$ if $q^2 \ll M^2$. The first term (the "1") above is the classical Rutherford term.

8.4 Higher order corrections

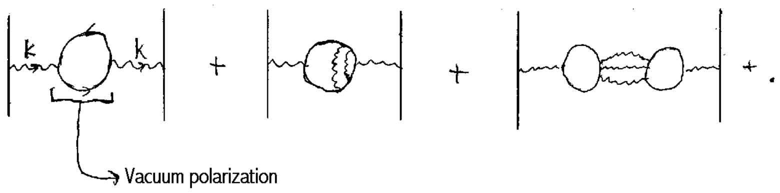


Figure 8.4.1: Higher order corrections

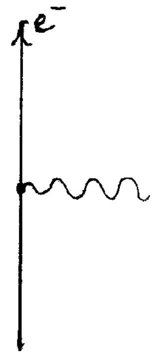


Figure 8.4.2: Generates coulomb field → classically

The loop (solid) is generated as the e^- travels - it "Polarizes" - the e^-e^+ created in vacuum.
The Loop contribution is

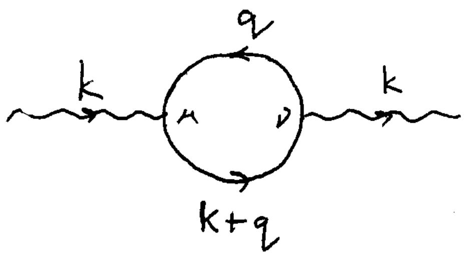


Figure 8.4.3: Loop contribution

$$
\begin{aligned}
I &= (-1)\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left[(-\mathrm{i}e\gamma_\mu)\frac{\mathrm{i}}{(\not k+\not q)-m+\mathrm{i}\varepsilon}(-\mathrm{i}e\gamma_\nu)\left(\frac{\mathrm{i}}{\not q-m+\mathrm{i}\varepsilon}\right)\right]\\
&= -e^2\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left[\frac{\gamma_\mu((\not k+\not q)+m)\gamma_\nu(\not q+m)}{((k+q)^2-m^2)(q^2-m^2)}\right]\\
&= -e^2\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left[\frac{\gamma_\mu(\not k+\not q)\gamma_\nu\not q+m^2\gamma_\mu\gamma_\nu}{((k+q)^2-m^2)(q^2-m^2)}\right]\\
&= -4e^2\int \frac{d^4q}{(2\pi)^4}\left\{\frac{(k+q)_\mu q_\nu-(k+q)\cdot q g_{\mu\nu}+(k+q)_\nu q_\mu+m^2g_{\mu\nu}}{((k+q)^2-m^2)(q^2-m^2)}\right\} \qquad (8.4.1)
\end{aligned}
$$

Now, use the relation:

$$
\frac{1}{ab}=\int_0^1 dx\ \frac{1}{[ax+b(1-x)]^2} \qquad (8.4.2)
$$

$\Big($ example:

$$
\begin{aligned}
\int\frac{dt}{t(t+1)} &= \int dt\ \int_0^1 dx\ \frac{1}{[tx+(t+1)(1-x)]^2}\\
&= \int dt\ \int_0^1 dx\ \frac{1}{[t+(1-x)]^2}\\
&= \int_0^1 dx\ \left[\frac{-1}{(t+(1-x))}+K\right]\\
&= \ln\left[t+(1-x)\right]\Big|_0^1+K\\
&= \ln(t)-\ln(t+1)+K\Big) \qquad (8.4.3)
\end{aligned}
$$

$$\begin{aligned} I &= -4e^2 \int \frac{d^4q}{(2\pi)^4} \overbrace{\left[(k+q)_\mu q_\nu + (k+q)_\nu q_\mu + (-(k+q)\cdot q + m^2) g_{\mu\nu}\right]}^{\text{Quadratically divergent}} \\ &\quad \cdot \int_0^1 dx \ \frac{1}{[x((k+q)^2 - m^2) + (1-x)(q^2-m^2)]^2} \\ &= -4e^2 \int \frac{d^4q}{(2\pi)^4} N_{\mu\nu} \int_0^1 dx \ \frac{1}{[\underbrace{q^2 + 2k\cdot qx}_{(q+kx)^2 - k^2x^2} + k^2x - m^2]^2} \end{aligned}$$

Shift variable of integration (not straightforward) to $q' = q + kx$.

$$\begin{aligned} I &= -4e^2 \int \frac{d^4q'}{(2\pi)^4} \int_0^1 dx \ \frac{1}{[q'^2 + k^2x(1-x) - m^2]^2} \left[(k+q'-xk)_\mu (q'-xk)_\nu \right. \\ &\quad \left. + (k+q'-xk)_\nu (q'-xk)_\mu + (-(k+q'-xk)\cdot(q'-xk) + m^2) g_{\mu\nu}\right] \end{aligned}$$

We can drop terms odd in q (because it's integrating over odd terms over an area symmetric about origin).

$$I = -4e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \ \frac{2q_\mu q_\nu - 2x(1-x)k_\mu k_\nu + (-q^2 + x(1-x)k^2 + m^2) g_{\mu\nu}}{[q^2 + k^2x(1-x) - m^2]^2}$$

where

$$\begin{aligned} q_\mu q_\nu &= \frac{1}{4} g_{\mu\nu} q^2 \\ \to \int d^4q \ f(q^2) q_\mu q_\nu &= A g_{\mu\nu} \\ &\to \text{ multiply by } g_{\mu\nu} \\ \int d^4q \ f(q^2) q^2 &= 4A \\ \therefore \int d^4q \ f(q^2) q_\mu q_\nu &= \frac{g_{\mu\nu}}{4} \int d^4q \ f(q^2) q^2 \end{aligned}$$

$$\frac{1}{ab} = \int_0^1 dx \ \frac{1}{[ax + (1-x)b]^2} \tag{8.4.5}$$

$$\begin{aligned} \frac{1}{abc} &= \int_0^1 dx \frac{1}{[xa + (1-x)b]^2} \frac{1}{c} \\ &\quad \text{take } \frac{d}{da} \text{ of (8.4.5)} \\ \frac{1}{a^2b} &= \int_0^1 dy \frac{2y}{[ay + (1-y)b]^3} \\ \therefore \frac{1}{abc} &= \int_0^1 dx \int_0^1 dy \frac{2y}{\{y[ax + (1-x)b] + (1-y)c\}^3} \end{aligned} \tag{8.4.6}$$

Nov. 23/99
In general (where $\Gamma(x) = \int_0^\infty dt\ t^{x-1}e^{-t}$),

$$\begin{aligned}\frac{1}{D_1^{a_1} D_2^{a_2} \dots D_n^{a_n}} &= \frac{r(a_1 + a_2 + \dots + a_n)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_n)} \cdot \\ &\cdot \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 dx_n \; \frac{\delta(1 - x_1 - \dots - x_n)x_1^{a_1-1}\dots x_n^{a_n-1}}{[x_1 D_1 + x_2 D_2 + \dots + x_n D_n]^2}\end{aligned} \tag{8.4.7}$$

Recall the loop contribution from last time: figure 8.4.3, with equation

$$I = -4e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx\ \frac{[\frac{1}{2}g_{\mu\nu}q^2 - 2k_\mu k_\nu x(1-x) + (-q^2 + x(1-x)k^2 + m^2)g_{\mu\nu}}{[q^2 + x(1-x)k^2 - m^2]^2} \tag{8.4.8}$$

How do we make sense out of a divergent integral? We first insert some parameter into the theory to render this finite in a way consistent with the symmetries of the theory (Regularization). Then absorb this parameter into the quantities that characterize the theory. (mass, couplings, external wave functions, etc.) (Renormalization).

8.5 Renormalization

Recall the mass on a string problem (see figure 10.4.1):

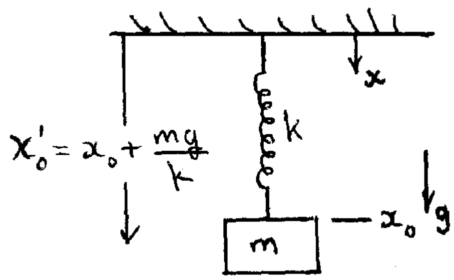


Figure 8.5.1: Loop contribution

The Lagrangian for this system is characterized by $m,\ k,\ x_0$;

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - x_0)^2 \tag{8.5.1}$$

Now, if we turn on a gravitational field,

$$\begin{aligned}
\mathcal{L} &= \frac{m\dot{x}^2}{2} - \frac{k(x-x_0)^2}{2} + mgx \\
&= \frac{m}{2}\dot{x}^2 - \frac{k}{2}\left[x - x_0 - \frac{mg}{k}\right]^2 + \underbrace{\frac{k}{2}\left[\left(\frac{mg}{2}\right)^2 - \frac{2x_0 mg}{k}\right]}_{\text{Constant}} \\
&= \frac{m\dot{x}^2}{2} - \frac{k}{2}\left(x - x_0'\right)^2 + \text{constant} \qquad (8.5.2) \\
& \quad x_0' = x + \frac{mg}{k}
\end{aligned}$$

x_0 has been renormalized by an amount $\frac{mg}{k}$!

8.6 Regularization

1. Cut-off (Λ)

$$\int d^4q \to \int^{|q|<\Lambda} d^4q \qquad (8.6.1)$$

Inserting the cutoff destroys gauge invariance.

$$S = \int d^4x \left(-\frac{1}{4}F^2 + \bar{\psi}(\mathrm{i}\gamma - m)\psi\right) \qquad (8.6.2)$$

$$|x| > \frac{1}{\Lambda} \quad \text{is not } \underline{\text{gauge invariant}}$$

2. Pauli Villars

$$S \to S_{PV} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\gamma - m)\psi + \bar{\Psi}(\mathrm{i}\gamma M)\Psi\right] \qquad (8.6.3)$$

   (a) Bosonic spinor
   (b) $M \to \infty$
   (c) Still have gauge invariance

Now we must consider

(figure 8.6.1)

$$= -4e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{\frac{1}{2}g_{\mu\nu}q^2 - 2k_\mu k_\nu x(1-x) + (-q^2 + x(1-x)k^2 + m^2)g_{\mu\nu}}{[q^2 + x(1-x)k^2 - m^2]^2}$$
$$\frac{\frac{1}{2}g_{\mu\nu}q^2 - 2k_\mu k_\nu x(1-x) + (-q^2 + x(1-x)k^2 + M^2)g_{\mu\nu}}{[q^2 + x(1-x)k^2 - M^2]^2} \qquad (8.6.4)$$

Figure 8.6.1: Loop contribution

This is Finite! (Divergence reappears as $M^2 \to \infty$. The dependence on M^2 disappears when we renormalize. Combine terms using

$$\frac{1}{a^n} - \frac{1}{b^n} = -n(a-b)\int_0^1 dx\, \frac{1}{[xa+(1-x)b]^{n+1}} \tag{8.6.5}$$

$$\text{let } \; y = x(a-b)+b$$

$$= -n(a-b)\int_a^b \frac{dy/(a-b)}{y^{n+1}} = -n\left[\frac{y^{-n}}{-n}\right]_b^a = \frac{1}{a^n} - \frac{1}{b^n} = LHS \tag{8.6.6}$$

Actually computing the integrals involves the following general integral. In n dimensions ($n = 1, 2, 3, \ldots$) (in Minkowski space);

$$\int \frac{d^n q}{(2\pi)^n}\frac{(q^2)^a}{(q^2-m^2)^b} = \frac{\mathrm{i}}{(4\pi)^{n/2}}(-1)^{a-b}(m^2)^{n/2+a-b}\frac{\Gamma\left(\frac{n}{2}+a\right)\Gamma\left(b-a-\frac{n}{2}\right)}{\Gamma(a)\Gamma(b)} \tag{8.6.7}$$

Note:

$$\int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_{n-1}\, \frac{(q^2)^a}{(q^2-m^2)^b} \tag{8.6.8}$$

$$q^2 = q_0^2 - q_1^2 - q_2^2 - \ldots - q_{n-1}^2$$

Let $q_0 = \mathrm{i}q_n$ (Wick Rotation - i.e. Cartesian co-ordinates in Minkowski space, giving Euclidean space)

$$q_E = q_n^2 + q_1^2 + \ldots + q_{n-1}^2$$

So (8.6.8) becomes

$$= \mathrm{i}\int_{-\infty}^{\infty} dq_n \ldots \int_{-\infty}^{\infty} dq_{n-1} = \mathrm{i}(-1)^{a-b}\int d^n q_e\, \frac{(q_E^2)^a}{(q_E^2+m^2)^b} \tag{8.6.10}$$

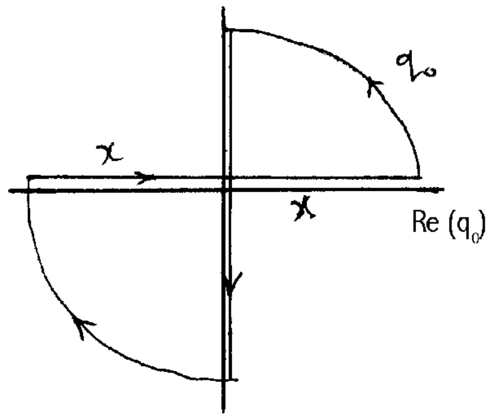


Figure 8.6.2: Integral around contour is zero - $\int_{-\infty}^{\infty} dq_0 \leftrightarrow \int_{-i\infty}^{i\infty} dq_0$

$$
\begin{aligned}
q_n &= q\cos\theta_1 \\
q_1 &= q\sin\theta_1\cos\theta_2 \\
q_2 &= q\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
q_{n-1} &= q\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1} \\
q^2 &= q_1^2+q_2^2+\ldots q_{n-1}^2+q_n^2
\end{aligned}
$$

i.e.

$$\int d^n q_E = \int_0 dq\ q^{n-1}\int_0^{2\pi} d\theta_1 \int_0^{\phi} d\theta_2 \sin\theta_2 \int_0^{\pi} d\theta_3\ \sin^2\theta_3 \ldots \int_0^{\pi} d\theta_{n-1}\ \sin^{n-2}\theta_{n-1} \quad (8.6.11)$$

Nov. 24/99

$$
\begin{aligned}
I &= \int \frac{d^n q}{(2\pi)^n}\frac{(q^2)^a}{(q^2-m^2)^b} \\
&= i(-1)^{a-b}\int \frac{d^N q_E}{(2\pi)^n}\frac{(q_E^2)^a}{(q_E^2+m^2)^b} \\
&= i(-q)^{a-b}\int_0^{\infty} dq\ q^{n-1}\int_0^{2\pi} d\theta_1\ \int_0^{\pi} d\theta_2\ \sin(\theta_2)\ldots \\
&\qquad \int_0^{\pi} d\theta_{n-1}\sin^{n-2}(\theta_{n-1})\frac{(q^2)^a}{(q^2+m^2)^b} \qquad (8.6.12)
\end{aligned}
$$

But,

$$\int_0^\pi d\theta\ \sin^m\theta = \frac{\sqrt{\pi}\ \ \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$\int_0^\infty dt\ \frac{t^{m-1}}{(t^2+a^2)^n} = \frac{\Gamma(m)\Gamma(n-m)}{(a^2)^{n-m}\Gamma(n)}$$

$$I = \frac{\mathrm{i}}{(4\pi)^{n/2}}(m^2)^{n/2+a-b}\frac{\Gamma\left(\frac{n}{2}+a\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(b-a-\frac{n}{2}\right)}{\Gamma(b)} \tag{8.6.13}$$

This is held to be true for n non-integer valued also. Substitute this into the integral for the Pauli Villars loop (figure 8.6.1).

$$\frac{1}{a^n} - \frac{1}{b^n} = -n(a-b)\int_0^1 dx\ \frac{1}{[xa+(1-x)b]^{n+1}} \tag{8.6.14}$$

i.e. Formally, $(n=4,\ a=1,\ b=2)$

$$\int \frac{d^4q}{(2\pi)^4}\frac{q^2}{[q^2+x(1-x)k^2-m^2]^2} = \frac{\mathrm{i}}{(2\pi)^2}\left(m^2-x(1-x)k^2\right)^{2+1-2}\frac{\Gamma(2+1)}{\Gamma(2)}\frac{\Gamma(2-1-2)}{\Gamma(2)} \tag{8.6.15}$$

But $\Gamma(-1)$ is at a pole of $\Gamma(x)$. ($\Gamma(x)$ has poles at $x = 0,\ -1,\ -2,\ -3,\ \ldots$) We can just let the number of dimensions be "n", not 4. We have,

$$\begin{aligned} I_{PV} &= \int \frac{d^nk}{(2\pi)^n}\frac{q^2}{[q^2+x(1-x)k^2-m^2]^2} \\ &= \frac{\mathrm{i}}{(4\pi)^{n/2}}(m^2-x(1-x)k^2)^{n/2+1-2}\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(2-1-\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(2)} \\ &= \frac{\mathrm{i}}{(4\pi)^{n/2}}\left(m^2-x(1-x)k^2\right)^{n/2-1}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(2)} \end{aligned}$$

Now let $\varepsilon = 2-\frac{n}{2} \to 0$ as $n \to 4$ $\left(\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right)$

$$I_{PV} = \frac{\mathrm{i}}{(4\pi)^{2-\varepsilon}}(m^2-x(1-x)k^2)^{-\varepsilon+1}(2-\varepsilon)\Gamma(\varepsilon-1) \tag{8.6.16}$$

But,

$$\Gamma(1+\varepsilon) = \exp\left[-\gamma\varepsilon + \sum_{n=2}^{\infty}\frac{(-1)^n\varepsilon^n\zeta(n)}{n}\right] \tag{8.6.17}$$

$$\to \zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$$

$$\gamma = \pm\lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \int_1^N\frac{dt}{t}\right) = 0.577\ldots$$

From this,

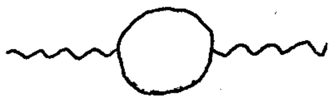
$$\Gamma(\varepsilon) = \frac{\Gamma(1+\varepsilon)}{2} = \frac{1}{\varepsilon} - \gamma\mathcal{O}(\varepsilon) \tag{8.6.18}$$

$$\begin{aligned} a^{\varepsilon} &= e^{\varepsilon \ln a} \\ &= 1 + (\varepsilon \ln a) + \frac{\varepsilon^2 \ln^2 a}{2!} + \dots \end{aligned} \tag{8.6.19}$$

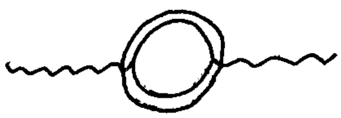
Thus our integral becomes

$$\begin{aligned} I_{PV} &= \frac{\mathrm{i}}{(4\pi)^2}(m^2 - x(1-x)k^2)\Bigg[(1+\varepsilon\ln(4\pi))\left(1-\varepsilon(m^2-x(1-x)k^2)\right)\cdot \\ &\quad \cdot\left(1-\frac{\varepsilon}{2}\right)\Bigg]\left(\frac{1}{\varepsilon-1}\right)\left(\frac{1}{\varepsilon}-\gamma+\dots\right) \\ &= \frac{-2\mathrm{i}}{(4\pi)^2}(m^2-x(1-x)k^2)\left(\frac{1}{\varepsilon}-\gamma+\ln(4\pi)-\ln(m^2-x(1-x)k^2)+\frac{1}{2}\right)+\mathcal{O}(\varepsilon) \end{aligned}$$

Note that the first term $\frac{1}{\varepsilon}$ will be a pole as $\varepsilon \to 0$ (other terms finite). This is only a portion of



Combine this with the integral for

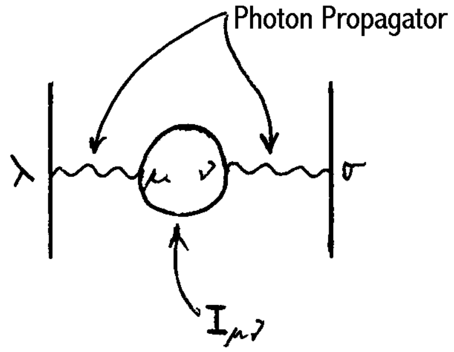


and will see that all terms of order $\frac{1}{\varepsilon}$ will cancel.

We note that as $M^2 \ggg m^2$, the combination of these two diagrams (recall figure 8.6.1) is,

$$I_{\mu\nu}(k) = \frac{2i\alpha}{\pi}(k_\mu k_\nu - g_{\mu\nu}k^2)\int_0^1 dx\ x(1-x)\ln\left(\frac{M^2}{m^2 - x(1-x)k^2}\right) \tag{8.6.20}$$

where $\alpha = \frac{e^2}{4\pi\hbar c}$ (note that the ln still diverges as $M^2 \to \infty$.) Note that the result is proportional to $k_\mu k_\nu - g_{\mu\nu}k^2$.



Now, photon propagator is

$$\frac{i(g_{\lambda\mu} + \text{ gauge dependant parts})}{k^2 + i\varepsilon} \tag{8.6.21}$$

$\to$ gauge dependant parts $\sim k_\lambda k_\mu$. But,

$$\begin{aligned} k^\mu I_{\mu\nu} &= k^\mu(k_\mu k_\nu - g_{\mu\nu}k^2)f(k^2) \\ &= (k^2 k_\nu - k_\nu k^2)f(k^2) \\ &= 0 \end{aligned}$$

Thus gauge dependant parts of the photon propagator don't contribute to the physical process.

This form of $I_{\mu\nu}$ does not arise if we were to use cutoffs (i.e. cutoffs are not gauge invariant.) Note also that the dependance on M^2 is logarithmic. i.e.

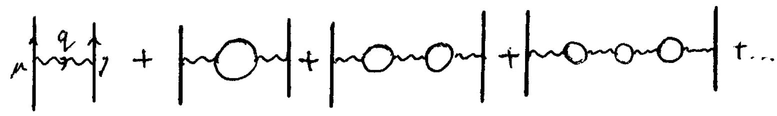
$$\sim \int \frac{d^4q[Ag_{\mu\nu}q^2 + (Bx(1-x)k^2 + Cm^2)g_{\mu\nu}]}{[q^2 + x(1-x)k^2 - m^2]^2}$$

Would expect quadratic divergence $\sim M^2$. (M^2 terms all cancelled.) Quadratic divergences proportional to $g_{\mu\nu}M^2$ <u>all cancel</u>. → still logarithmically divergent.

Nove 26/99

Elimination of $\ln M^2$. Remember the photon propagator itself. (μ^2 → some mass scale introduced →arbitrary (important)) Write

$$\begin{aligned} I_{\mu\nu} &= \frac{\mathrm{i}\alpha}{3\pi}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\left[\ln\left(\frac{M^2}{\mu^2}\right) - 6\int_0^1 dz\ z(1-z)\ln\left(\frac{m^2 - q^2 z(1-x)}{\mu^2}\right)\right] \\ &= -\frac{\mathrm{i}\alpha}{3\pi}q^2 \underbrace{\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)}_{g^T_{\mu\nu}}\left[\ln\left(\frac{M^2}{\mu^2}\right) + \Pi(q^2, m^2\mu^2)\right] \qquad (8.6.22) \end{aligned}$$



$$\begin{aligned} &= (-\mathrm{i}e\gamma_\mu)\Bigg\{\frac{-\mathrm{i}g_{\mu\nu}}{q^2} + \frac{-\mathrm{i}g_{\mu\lambda}}{q^2}\left[\frac{-\mathrm{i}\alpha q^2}{3\pi}g^T_{\lambda\sigma}\left(\ln\left(\frac{M^2}{\mu^2}\right) + \Pi\right)\right]\left(\frac{-\mathrm{i}g_{\lambda\nu}}{q^2}\right) \\ &\quad + \left(\frac{-\mathrm{i}g_{\mu\lambda}}{q^2}\right)\left[\frac{-\mathrm{i}\alpha q^2}{3\pi}g^T_{\lambda\sigma}\left(\ln\left(\frac{M^2}{\mu^2}\right) + \Pi\right)\right]\cdot \\ &\quad \cdot\left(-\frac{\mathrm{i}g_{\sigma\kappa}}{q^2}\right)\left[\frac{-\mathrm{i}\alpha q^2 g^T_{\kappa\rho}}{3\pi}\left(\ln\left(\frac{M^2}{\mu^2}\right) + \Pi\right)\right]\left(\frac{-\mathrm{i}g_{\rho\nu}}{q^2}\right) + \ldots\Bigg\}(-\mathrm{i}e\gamma_\nu) \qquad (8.6.23) \end{aligned}$$

This is a geometric series, $a + ar + ar^2 + \ldots = \frac{a}{1-r}$, $|r| < 1$. So,

$$\begin{aligned} &= (-\mathrm{i}e\gamma_\mu)\Bigg[\frac{-\mathrm{i}g_{\mu\nu}}{q^2} + \frac{-\mathrm{i}g_{\mu\lambda}}{q^2}\left(-\frac{\alpha}{3\pi}g^T_{\lambda\nu}\left(\ln\left(\frac{M^2}{\mu^2}\right) + \Pi\right)\right) \\ &\quad + \frac{-\mathrm{i}g_{\mu\lambda}}{q^2}\left(-\frac{\alpha}{3\pi}g^T_{\lambda\nu}\left(\ln\left(\frac{M^2}{\mu^2} + \Pi\right)\right)\right)^2 + \ldots\Bigg](-\mathrm{i}e\gamma_\nu) \\ &= (-\mathrm{i}e\gamma_\mu)\left[\frac{-\mathrm{i}g^T_{\mu\nu}}{q^2}\frac{1}{1 + \frac{\alpha}{3\pi}\left(\ln\left(\frac{M^2}{\mu^2}\right) + \Pi(q^2, m^2, \mu^2)\right)}\right](-\mathrm{i}e\gamma_\nu) \\ &= \frac{-e^2}{\left(1 + \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{\mu^2}\right)\right)}\gamma_\mu\left[\frac{-\mathrm{i}g^T_{\mu\nu}}{q^2\left[q + \frac{\frac{m}{3\pi}}{1 + \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{\mu^2}\right)}\left(\Pi(q^2, m^2, \mu^2)\right)\right]}\right]\gamma_\nu \qquad (8.6.24) \end{aligned}$$

(neglecting terms of order $\alpha^2 = \left(\frac{e^2}{4\pi}\right)^2$). Now, Define:

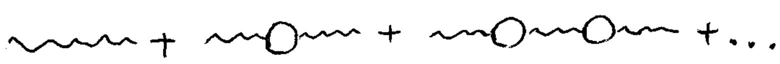
$$e_R^2 = \frac{e^2}{1 + \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{\mu^2}\right)} \approx e^2\left(1 - \frac{\alpha}{3\pi}\ln\left(\frac{M^2}{\mu^2}\right)\right) \tag{8.6.25}$$

(Renormalized Charge). Thus, continuing (8.6.24)

$$= -e_R^2\gamma_\mu\left\{\frac{-\mathrm{i}g^T_{\mu\nu}}{q^2\left(1 + \frac{\alpha_R}{3\pi}\Pi(q^2, m^2, \mu^2)\right)}\right\}\gamma_\nu \tag{8.6.26}$$

We now let $M^2 \to \infty$.

- Note that there is still a dependence on μ^2. (Changes in μ^2 give rise to corresponding changes in α_R.
- Also, in
$$g^T_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$
$\to (-\mathrm{i}e_R\gamma_\mu)\left(\frac{q_\mu q_\nu}{q^2}\right)(-\mathrm{i}e\gamma_\nu)$ gives zero provided the ends of the electron legs are on shell.
- Also
$$\Pi(q^2, \mu^2, m^2) = -6\int_0^1 dz\ z(1-z)\ln\left(\frac{m^2 - q^2z(1-z)}{\mu^2}\right)$$
is such that $q^2\Pi(q^2, m^2, \mu^2) \to 0$ as $q^2 \to 0$. Hence,



still has a pole at $q^2 = 0$. Thus the photon is massless.

In 2-D, if $m^2 = 0$, $\Pi(q^2, \mu^2)$ is finite, and $\Pi(q^2) = \frac{e^2}{\pi q^2}$, then the above diagram sum becomes

$$= -\frac{e^2\gamma_\mu g^T_{\mu\nu}\gamma_\nu}{q^2\left(1 - \frac{e^2}{\pi q^2}\right)}$$

$$= -\frac{e^2\gamma_\mu g^T_{\mu\nu}\gamma_\nu}{q^2 - \frac{e^2}{\pi}}$$

Thus the photon develops a mass in 2-D due to the radiative corrections. (e has dimensions of mass in 2-D). i.e.

$$S = \int d^2x\mathcal{L}$$

$$\underbrace{S}_{[\mu]=0} = \int \underbrace{d^2x}_{-2}\left(-\frac{1}{4}\left(\underbrace{\partial_\mu}_{1}\underbrace{A_\nu}_{0}-\partial_\nu A_\mu\right)^2+\underbrace{\bar{\psi}}_{\frac{1}{2}}(\underbrace{\not\partial}_{+1}-\mathrm{i}\underbrace{e}_{1}\not A)\psi\right) \quad (8.6.29)$$

8.7 Noether's Theorem

For every infinitesimal change which leaves the action invariant there exists a conserved current which leads to a conserved charge.
Change of type I

$$\mathcal{L} = \mathcal{L}(\phi_A(x), \partial_\mu\phi_A(x)) \quad (8.7.1)$$

$$x^\mu \to x'^\mu = x^\mu \quad (8.7.2)$$

$$\phi_A(x) \to \phi'_A(x) = \phi_A + \varepsilon_{AB}\phi_B(x) \quad (8.7.3)$$

with $\varepsilon_{AB} \to$ infinitesimal. (ex. we had a gauge transformation

$$\left.\psi(x) \to e^{\mathrm{i}\theta(x)}\psi(x) = \lim_{N\to\infty}\left(1+\frac{\mathrm{i}\theta}{N}\right)^N\psi(x) \approx (1+\mathrm{i}\underbrace{\theta(x)}_{\varepsilon_{AB}})\psi(x)\right)$$

Now,

$$\begin{aligned} S &= \int d^4x\ \mathcal{L}(\phi_A, \partial_\mu\phi_A) \\ &= \int d^4x\ \mathcal{L}(\phi'_A, \partial_\mu\phi'_A) \end{aligned} \quad (8.7.4)$$

where both $\mathcal{L}(\phi_A, \partial_\mu\phi_A)$, $\mathcal{L}(\phi'_A, \partial_\mu\phi'_A)$ have the same functional dependence. (For example, $f(x,y) = x^2+y^2$; let $x' = x\cos\theta - y\sin\theta$, $y' = x\sin\theta + y\cos\theta$, and we still get $f(x',y') = x'^2+y'^2$ (invariant) - whereas $g(x,y) = xy \neq g(x',y')$).
Nov 30/99
Suppose $\mathcal{L}$ is invariant under an infinitesimal translation $x^\mu \to x^\mu + \varepsilon^\mu$.

$$S = \int d^4x\ \mathcal{L}(\phi(x), \partial_\mu\phi(x)) \quad (8.7.5)$$

$$= \int d^4x'\ \mathcal{L}(\phi(x'), \partial'_\mu\phi(x')) \quad (8.7.6)$$

in the same way as

$$\begin{aligned} S &= \int d^4x\ \mathcal{L}(x) = \int d^4x'\ \mathcal{L}(x') \\ &\to d^4x' = \frac{\partial(x'^1 \ldots x'^0)}{\partial(x^1 \ldots x^0)} d^4x = d^4x \end{aligned} \tag{8.7.7}$$

Now,

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(x') - \mathcal{L}(x) \\ &= \mathcal{L}(x^\mu + \varepsilon^\mu) - \mathcal{L}(x^\mu) \\ &= \varepsilon^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} \qquad (8.7.8) \\ &\underline{\text{or}} \\ &= \mathcal{L}(\phi(x'), \partial'_\mu \phi(x')) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \qquad (8.7.9) \end{aligned}$$

If now,

$$\delta\phi(x) = \phi(x') - \phi(x) = \phi(x^\mu + \varepsilon^\mu) - \phi(x) = \varepsilon^\mu \frac{\partial \phi}{\partial x^\mu} \tag{8.7.10}$$

then

$$\begin{aligned} \delta\mathcal{L} &= \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \delta(\partial_\mu \phi(x)) \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \\ &= \left(\varepsilon^\mu \frac{\partial \phi}{\partial x^\mu}\right) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \frac{\partial}{\partial x^\mu} \left(\varepsilon^\nu \frac{\partial \phi}{\partial x^\nu}\right) \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi(x))} \end{aligned} \tag{8.7.11}$$

However, we also have the equations of motion.

$$\delta \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = 0 \tag{8.7.12}$$

$$\to \phi = \phi_{cl} + \delta\phi \tag{8.7.13}$$

i.e.

$$\begin{aligned} &\int d^4x \left[\mathcal{L}(\phi_{cl} + \delta\phi, \partial\phi_{cl} + \partial_\mu \delta\phi)\right] \\ = &\int d^4x \left[\mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl}) + \delta\phi \frac{\delta \mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl})}{\delta\phi} + (\partial_\mu \delta\phi(x)) \frac{\delta \mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl})}{\partial(\partial_\mu \phi)} + \ldots\right] \\ &\text{If } \ \delta\phi \to 0 \text{ as } x^\mu \to \infty \\ = &\int d^4x \left[\mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl}) + \underbrace{\delta\phi(x) \left(\frac{\delta \mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl})}{\delta\phi} - \frac{\partial}{\partial x^\mu} \frac{\delta \mathcal{L}(\phi_{cl}, \partial_\mu \phi_{cl})}{\delta(\partial_\mu \phi(x))}\right)}_{=0 \text{ if } \phi_{cl} \text{ is extremum of } S} + \ldots\right] \end{aligned} \tag{8.7.14}$$

Hence as $\delta\phi$ is arbitrary,

$$0 = \frac{\delta\mathcal{L}}{\delta\phi} - \frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta(\partial\phi)} \tag{8.7.15}$$

on the classical path. Recall (8.7.11),

$$\begin{aligned}\delta\mathcal{L} &= \left(\varepsilon^\nu\frac{\partial\phi}{\partial x^\nu}\right)\left(\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right) + \frac{\partial}{\partial x^\mu}\left(\varepsilon^\nu\frac{\partial\phi}{\partial x^\nu}\right)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \\ &= \frac{\partial}{\partial x^\mu}\left[\varepsilon^\nu\frac{\partial\phi}{\partial x^\nu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right]\end{aligned} \tag{8.7.16}$$

(8.7.8) and (8.7.16) must be identical. As a result,

$$\begin{aligned}\varepsilon^\mu\frac{\partial\mathcal{L}}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu}\left(g_{\mu\nu}\varepsilon^\nu\mathcal{L}\right) \\ &= \frac{\partial}{\partial x^\mu}\left[\varepsilon^\nu\frac{\partial\phi}{\partial x^\nu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right]\end{aligned}$$

Thus

$$0 = \frac{\partial}{\partial^\mu}\underbrace{\left[-g_{\mu\nu}\mathcal{L} + \frac{\partial\phi}{\partial x^\nu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right]}_{T_{\mu\nu}\quad\text{Energy-Momentum Tensor}} \tag{8.7.17}$$

Or,

$$\frac{\partial}{\partial x^\mu}T_{\mu\nu} = 0 \tag{8.7.18}$$

Thus, there are 4 conserved currents, ($\mu = 1,\ 2,\ 3,\ 0$), one for each <u>invariance</u>. This only holds if ϕ that it depends on is a solution to the classical equation. (ex. $k_\mu(g_{\mu\nu}k^2 - k_\mu k_\nu) = 0$ always, but $k_\mu(g_{\mu\nu}m^2 - k_\mu k_\nu) = 0$ only if $k^2 = m^2$.)

There are 4 conserved currents; one fore each direction in which we have an infinitesimal invariance.
For

$$\begin{aligned}T_{00} &= -g_{00}\mathcal{L} + \frac{\partial\phi}{\partial x^0}\frac{\delta\mathcal{L}}{\delta(\partial_0\phi)} \\ &= -\mathcal{L} + \frac{\partial\phi}{\partial t}\Pi = \mathcal{H} \\ (H &= -L + p\dot{q})\end{aligned}$$

Now, if

$$\partial_\mu j^\mu = 0 \tag{8.7.19}$$

Then,

$$\begin{aligned}\int d^3x\ \partial_\mu j^\mu &= 0 \\ 0 &= \int d^3x\ \left[\nabla\cdot \underline{j} + \frac{\partial}{\partial t} j^0\right] \\ &= \int d\underline{S}\cdot\underline{j} + \frac{\partial}{\partial t}\int d^3x\ j^0(\underline{x},t) \quad (\text{ by Gauss})\end{aligned}$$

The first term is 0 if $j \to 0$ as $|\underline{x}| \to \infty$. Let $Q(t) = \int d^3x\ j^0(\underline{x},t)$.

$$\therefore \frac{\partial}{\partial t} Q(t) = 0 \quad (\text{“Charge” conservation}) \tag{8.7.20}$$

For $\nu = 0$,

$$\begin{aligned}Q(t) &= \int d^3x\ T_{00}(\underline{x},t) \\ &= \int d^3x\ \mathcal{H}\end{aligned} \tag{8.7.21}$$

(The “charge” here is the energy of the field.) For $\nu = i$

$$\int d^3x T_{0i}(\underline{x},t) = P_i(t) \to \quad \text{Mechanical momentum of field} \tag{8.7.22}$$

Infinitesimal internal transformations

$$\begin{aligned}x^\mu &\to x'^\mu = x^\mu \\ \mathcal{L} &= \mathcal{L}(\phi_A(x), \partial_\mu\phi_A(x))\end{aligned} \tag{8.7.23}$$

Invariance under

$$\phi_A \to \phi'_A = \phi_A + \varepsilon_{AB}\phi_B \tag{8.7.24}$$

(ex. $A_\mu \to A_\mu + \partial_\mu\Lambda$, $\psi \to (1 + \mathrm{i}\Lambda)\psi \Rightarrow$ Gauge transformation.)

$$\begin{aligned}S &= \int d^4x\ \mathcal{L}(\phi_A, \partial_\mu\phi_A) = \int d^4x\ \mathcal{L}(\phi'_A, \partial_\mu\phi'_A) \\ &= \int d^4x\ \left[\mathcal{L}(\phi_A,\partial_\mu\phi_A) + \varepsilon_{AB}\phi_B\frac{\delta\mathcal{L}}{\delta\phi_A} + (\partial\varepsilon_{AB}\phi_B)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_A)} + \dots\right]\end{aligned} \tag{8.7.25}$$

Also if ϕ_A satisfies $\frac{\delta\mathcal{L}}{\delta\phi_A} = \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_A)}$.
Hence,

$$\begin{aligned}0 &= \int d^4x\ \left[(\varepsilon_{AB}\phi_B)\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_A)} + (\partial_\mu\varepsilon_{AB}\phi_B)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_A)}\right] \\ 0 &= \int d^4x\ \frac{\partial}{\partial x^\mu}\left[\varepsilon_{AB}\phi_B\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_A)}\right]\end{aligned} \tag{8.7.26}$$

Thus we have a conserved current,

$$J_\mu^{AB} = \left[\phi_B \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_A)}\right] \tag{8.7.27}$$

ex. if $\mathcal{L} = \bar{\psi}(\mathrm{i}\not{\partial} - \not{A})\psi - \frac{1}{4}F^2$, then $j_\mu = \bar{\psi}\gamma_\mu\psi$.

Dec. 1/99

$$\begin{aligned} x'^\mu &= x^\mu + \varepsilon^\mu \\ \phi'_A(x) &= \phi_A(x) + \varepsilon_{AB}\phi_B(x) \end{aligned}$$

Consider a general transformation in which,

$$x'^\mu = x^\mu + \delta x^\mu \tag{8.7.28}$$
$$\psi(x') = \psi(x) + \delta\psi(x) \tag{8.7.29}$$

ex. consider a rotation of a vector field $A_i(x_i)$ in 2-D.

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \approx x - \theta y \quad \text{for small } \theta \\ y' &= x\sin\theta + y\cos\theta \approx \theta x + y \\ \rightarrow \delta x &= -\theta y \ (\text{ small } \theta) \\ \rightarrow \delta y &= \theta x \ (\text{ small } \theta) \end{aligned}$$

At the same time,

$$\begin{aligned} A'_x(x', y') &= A_x(x, y)\cos\theta - A_y(x, y)\sin\theta \\ A'_y(x', y') &= A_x(x, y)\sin\theta + A_y(x, y)\cos\theta \end{aligned}$$

$$\begin{aligned} \delta A_x &= A'_x(x', y') - A_x(x, y) \\ &= \underbrace{A'_x(x', y') - A_x(x', y')} + \underbrace{A_x(x', y') - A_x(x, y)} \\ &= \bar{\delta} A_x(x', y') + [A_x(x - \theta y, y + \theta x) - A_x(x, y)] \\ &= \bar{\delta} A_x(x', y') + \left[-\theta y \frac{\partial}{\partial x} + \theta x \frac{\partial}{\partial y}\right] A_x(x, y) \end{aligned} \tag{8.7.30}$$

Invariance of the action S

$$\begin{aligned} S &= \int d^4x \ \mathcal{L}(\psi(x), \partial_\mu \psi(x)) \\ &= \int d^4x' \mathcal{L}(\psi'(x'), \partial'_\mu \psi'(x')) \end{aligned}$$

In general, $d^4x' = |J|d^4x$,

$$\begin{aligned}
\rightarrow |J| &= \left|\det\left[\frac{\partial(x'^1\ldots x'^0)}{\partial(x^1\ldots x^0)}\right]\right| \\
&= \left|\det\begin{bmatrix} \frac{\partial x'^1}{\partial x^1} & \ldots & \frac{\partial x'^1}{\partial x^0} \\ \vdots & & \vdots \\ \frac{\partial x'^0}{\partial x^1} & \ldots & \frac{\partial x'^0}{\partial x^0}\end{bmatrix}\right| \\
&= \left|\det\begin{bmatrix} \left(1+\frac{\partial\delta x'^1}{\partial x^1}\right) & \frac{\partial\delta x'^1}{\partial x^2} & \ldots & \frac{\partial\delta x'^0}{\partial x^1} \\ \vdots & \left(1+\frac{\partial\delta x'^2}{\partial x^2}\right) & & \vdots \\ & & \ddots & \\ & & & \left(1+\frac{\partial\delta x'^0}{\partial x^0}\right)\end{bmatrix}\right| \\
&= \left|1+\frac{\partial\delta x^\mu}{\partial x^\mu}+\mathcal{O}\left((\delta x^\mu)^2\right)\right| \qquad (8.7.31)
\end{aligned}$$

So also, if we call

$$\mathcal{L}'(x') = \mathcal{L}(\psi'(x'), \partial'_\mu\psi'(x')) \qquad (8.7.32)$$

$$\begin{aligned}
\mathcal{L}'(x') - \mathcal{L}(x) &= \delta\mathcal{L} \\
&= \underbrace{\mathcal{L}'(x') - \mathcal{L}(x')}_{\bar{\delta}\mathcal{L}(x)} + \underbrace{\mathcal{L}(x') - \mathcal{L}(x)}_{\delta x^\mu\frac{\partial\mathcal{L}(x)}{\partial x^\mu}} \qquad (8.7.33)
\end{aligned}$$

Now,

$$\begin{aligned}
\bar{\delta}\mathcal{L}(x) &= \mathcal{L}'(x') - \mathcal{L}(x') \\
&= \mathcal{L}(\psi'(x'), \partial'_\mu\psi'(x')) - \mathcal{L}(\psi(x'), \partial'_\mu\psi(x')) \\
&\quad (\text{Re:}\quad \psi'(x') - \psi(x') = \bar{\delta}\psi(x')) \\
&= \mathcal{L}(\psi(x') + \bar{\delta}\psi(x'), \partial'_\mu\psi(x') + \partial'_\mu\bar{\delta}\psi(x')) - \mathcal{L}(\psi(x'), \partial'_\mu\psi(x'))
\end{aligned}$$

Hence,

$$\bar{\delta}\mathcal{L} = \bar{\delta}\psi(x')\frac{\delta\mathcal{L}}{\delta\psi(x')} + \partial'_\mu\bar{\delta}\psi(x')\frac{\delta\mathcal{L}}{\delta(\partial'_\mu\psi(x'))} \qquad (8.7.34)$$

By the equations of motion,

$$\begin{aligned}
\bar{\delta}\mathcal{L} &= \bar{\delta}\psi(x')\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta(\partial'_\mu\psi(x'))} + (\partial'_\mu\bar{\delta}\psi(x'))\frac{\partial\mathcal{L}}{\delta(\partial'_\mu\psi(x'))} \\
&= \frac{\partial}{\partial x'^\mu}\left(\bar{\delta}\psi(x')\frac{\delta\mathcal{L}}{\delta(\partial'_\mu\psi(x'))}\right) \qquad (8.7.35)
\end{aligned}$$

So, putting together,

$$\begin{aligned} 0 &= \int d^4x' \ \mathcal{L}(\psi'(x'), \partial'_\mu \psi'(x')) - \int d^4x \ \mathcal{L}(\psi(x), \partial\psi(x)) \ \ (\text{ subs. in (8.7.31), (8.7.33), (8.7.35)}) \\ &= \int d^4x \ \left[1 + \frac{\partial \delta x^\mu}{\partial x^\mu}\right] \left[\mathcal{L}(\psi(x), \partial_\mu\psi(x)) + \bar{\delta}\mathcal{L} + \delta x^\mu \frac{\partial \mathcal{L}}{\partial x^\mu}\right] - \int d^4x \ \mathcal{L}(\psi(x), \partial_\mu\psi(x)) \end{aligned}$$

To leading order,

$$\begin{aligned} 0 &= \int d^4x \ \left[\frac{\partial \delta x^\mu}{\partial x^\mu}\mathcal{L} + \delta x^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial}{\partial x^\mu}\left(\bar{\delta}\psi(x)\frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi(x))}\right)\right] \\ &= \int d^4x \ \left[\frac{\partial}{\partial x^\mu}\left(\delta x^\mu \mathcal{L} + \bar{\delta}\psi \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi(x))}\right)\right] \end{aligned}$$

As δx^μ and $\bar{\delta}\psi$ contain an arbitrary parameter, then

$$\partial_\mu j^\mu = 0 \tag{8.7.36}$$

where

$$j^\mu = \delta x^\mu \mathcal{L} + \bar{\delta}\psi \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \tag{8.7.37}$$

(Noether Current).
Suppose

$$\begin{aligned} \delta x^\mu &= x'^\mu - x^\mu \\ &= A_{\mu j}(x)\varepsilon^j \ \ (\text{translation: } \varepsilon^\mu) \end{aligned}$$

$$\begin{aligned} \delta\psi_A(x) &= \psi'_A(x') - \psi_A(x) \\ &= B_{Aj}(x)\varepsilon^j \end{aligned}$$

But

$$\begin{aligned} \bar{\delta}\psi_A(x) &= \psi'_A(x) - \psi_A(x) \\ &= \psi'_A(x') - \psi_A(x) + \psi'_A(x) - \psi'_A(x') \\ &= \delta\psi_A + \left(-\delta x^\mu \frac{\partial \psi'_A(x)}{\partial x^\mu}\right) \\ &\approx B_{Aj}\varepsilon^j - A_{\mu j}\varepsilon^j \frac{\partial \psi'_A}{\partial x^\mu} \\ &\approx B_{Aj}\varepsilon^j - A_{\mu j}\varepsilon^j \frac{\partial \psi}{\partial x^\mu} \end{aligned}$$

(can replace $\psi' \to \psi$ because ψ', ψ differ by a term which would lead to an over-all term $\varepsilon^2 \to$ no good). Thus,

$$j^\mu = (A_{\mu j}\varepsilon^j)\mathcal{L} + \left(B_{Aj}\varepsilon^j - A_{\mu j}\varepsilon^j \frac{\partial \psi_A}{\partial x^\mu}\right)\frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi_A)} \tag{8.7.38}$$

For each ε_j there is a different current;

$$j_k^\mu = A_{\mu k}\mathcal{L} - \left(B_{Ak} - A_{\mu k}\frac{\partial \psi_A}{\partial x^\mu}\right)\frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi_A)} \tag{8.7.39}$$

(Translations: $A_{\mu k} = g_{\mu k}$, $B_{Ak} = 0$. Angular momentum $\rightarrow$ rotations of space-time points and fields.)

Dec. 3/99

Recall the spin $\frac{1}{2}$ particle:

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\ \partial\!\!\!/ - m)\psi \tag{8.7.40}$$

$\psi =$ proton, neutron.

Yukawa postulated that the strong force was mediated by spin-zero pseudo-particles (mass μ, $V(r) \sim \frac{e^{-\mu r}}{r}$).

Heisenberg: postulated that the strong force treated the p and n exactly the same way and these two "nucleons" were just different states of the same "particle", and they corresponded to spin up and spin down states of the electron. (Not really spin $\rightarrow$ but same mathematical structure, "Isosspin").

$$\begin{aligned} \begin{bmatrix} \psi_n \\ \psi_p \end{bmatrix} &\rightarrow \quad \text{"Isospin doublet"} \\ \psi_n,\ \psi_p &\rightarrow \quad \text{four component Dirac sponors} \\ \psi &= \psi_{i\alpha} \quad ; i = 1,2 \text{ (isospin)} \quad , \quad \alpha = 1,\ldots,4 \quad \text{(Dirac)}. \end{aligned} \tag{8.7.41}$$

$$\mathcal{L} = \bar{\psi}_i(\mathrm{i}\ \partial\!\!\!/ - m)\psi_i - \mathrm{i}g\bar{\psi}_i\gamma_5(\underline{\phi}\cdot\underline{\tau}_{ij})\psi_j - \frac{1}{2}(\partial_\mu\underline{\phi})(\partial^\mu\underline{\phi}) - \frac{\mu^2}{2}\underline{\phi}^2 - \frac{\lambda}{4}(\underline{\phi}^2)^2 \tag{8.7.42}$$

with $\underline{\tau} =$ Pauli matrices, and $\phi_1,\ \phi_2,\ \phi_3$- 3 spin zero pseudo-scalars, "pions". This is invariant under

$$\begin{aligned} \psi &\rightarrow e^{\mathrm{i}\underline{\tau}\cdot\underline{\theta}/2}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}e^{-\mathrm{i}\underline{\tau}\cdot\underline{\theta}/2} \\ \underline{\tau}\cdot\underline{\phi} &\rightarrow e^{\mathrm{i}\underline{\tau}\cdot\underline{\theta}/2}\underline{\tau}\cdot\underline{\phi}\ \ e^{-\mathrm{i}\underline{\tau}\cdot\underline{\theta}/2} \end{aligned}$$

where $\underline{\theta} = (\theta_1, \theta_2, \theta_3) \rightarrow$ constant 3 component isovector. $\rho_\mu^1,\ \rho_\mu^2,\ \rho_\mu^3 \rightarrow$ 3 vector particles, also interact with the nucleons.

Postulate

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu\rho_\nu^i - \partial_\nu\rho_\mu^i)^2 - \frac{\mu^2}{2}\rho_\mu^i\rho^{i\mu} + \bar{\psi}\gamma^\mu\rho_\mu^i\tau^i\psi + (\text{ Rest of terms from last } \mathcal{L}) \tag{8.7.43}$$

If

$$\rho^i_\mu \tau^i \quad \rightarrow \quad e^{\mathrm{i}\tau^i\theta^i/2}\rho^i_\mu \tau^i e^{-\mathrm{i}\tau^i\theta^i/2} \tag{8.7.44}$$

For $\theta^i \approx 0$

$$\begin{aligned} &= \left(1+\frac{\mathrm{i}\tau^i\theta^i}{2}\right)\rho^j_\mu\tau^j\left(1-\frac{\mathrm{i}\tau^i\theta^i}{2}\right) \\ &= \left[\rho^j_\mu\tau^j + \frac{\mathrm{i}}{2}\theta^i\rho^j_\mu \underbrace{\left(\tau^i\tau^j - \tau^j\tau^i\right)}_{2\mathrm{i}\varepsilon^{ijk}\tau^k} + \mathcal{O}(\theta^2)\right] \end{aligned} \tag{8.7.45}$$

Thus,

$$\rho^i_\mu \rightarrow \rho^i_\mu + \varepsilon^{ijk}\rho^j\theta^k \tag{8.7.46}$$

Conserved current arises in

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu\rho^i_\nu - \partial_\nu\rho^i_\mu\right)^2 - \frac{\mu^2}{2}\rho^i_\mu\rho^{i\mu} \tag{8.7.47}$$

Here;

$$\begin{aligned} 0 = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\rho^i_\mu}\delta\rho^i_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho^i_\mu)}\delta\partial_\nu\rho^i_\mu \\ &= \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho^i_\mu)}\delta\rho^i_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho^i_\mu)}\delta\partial_\nu\rho^i_\mu \\ &= \partial_\nu \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho^i_\mu)}\delta\rho^i_\mu\right]}_{j_\nu} \end{aligned} \tag{8.7.48}$$

Thus as

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho^i_\mu)} = -\frac{1}{4}(2)(2)\left(\partial_\nu\rho^i_\mu - \partial_\mu\rho^i_\nu\right) \tag{8.7.49}$$

Thus,

$$\begin{aligned} j_\nu &= -\overbrace{\left(\partial_\nu\rho^i_\mu - \partial_\mu\rho^i_\nu\right)}^{f^i_{\nu\mu}}(\varepsilon^{ijk}\rho^j\theta^k) \\ j^k_\nu &= -f^i_{\nu\mu}\varepsilon^{ijk}\rho^j_\mu \end{aligned} \tag{8.7.50}$$

Couple j^k_μ to $A^{k\mu}$,

$$\begin{aligned} \mathcal{L}' &= \overbrace{-\frac{1}{4}f^i_{\mu\nu}f^{i\mu\nu} - \frac{\mu^2}{2}\rho^i_\mu\rho^{i\mu}}^{\mathcal{L}} + g j^k_\mu\rho^{k\mu} \\ &\quad \rho^i_\mu \rightarrow \rho^i_\mu + \varepsilon^{ijk}\rho^j_\mu\theta^k \ ; \quad \text{A new piece is added to } j^k_\mu \\ &= -\frac{1}{4}\left(\partial_\mu\rho^i_\nu - \partial_\nu\rho^i_\mu\right)^2 - \frac{\mu^2}{2}\rho^i_\mu\rho^{i\mu} - g(\partial_\nu\rho^i_\mu - \partial_\mu\rho^i_\nu)\varepsilon^{ijk}\rho^{j\mu}\rho^{k\nu} \end{aligned} \tag{8.7.51}$$

i.e.

$$\mathcal{L}^{\text{new}} = -g\varepsilon^{ijk}(\partial_\nu\rho^i_\mu - \partial_\mu\rho^i_\nu)\rho^{j\mu}\rho^{k\nu} \tag{8.7.52}$$

$$\begin{aligned} j^{\text{new}}_\nu &= \frac{\partial\mathcal{L}^{\text{new}}}{\partial(\partial_\nu\rho^i_\mu)}\delta\rho^i_\mu \\ &= -2g\varepsilon^{ijk}(\rho^{j\mu}\rho^{k\nu})(\varepsilon^{imn}\rho^m\theta^n) \end{aligned} \tag{8.7.53}$$

Again, as θ is arbitrary,

$$j^{\text{new}i}_\nu = -2g\varepsilon^{lmn}\rho^m_\mu\rho^n_\nu\varepsilon^{lji}\rho^{j\mu} \tag{8.7.54}$$

$$\begin{aligned} \mathcal{L}\text{"} &= \mathcal{L}' + gj^{\text{new}k}_\nu\rho^{k\nu} \\ &= -\frac{1}{4}(\partial_\mu\rho^i_\nu - \partial_\nu\rho^i_\mu)^2 - \frac{\mu^2}{2}\rho^k_\nu\rho^{k\nu} - g(\partial_\nu\rho^i_\mu - \partial_\mu\rho^i_\nu)\varepsilon^{ijk}\rho^j_\mu\rho^k_\nu + g(-2g\varepsilon^{lmn}\rho^m_\mu\rho^n_\nu\varepsilon^{lji}\rho^j_\mu)(\rho^{i\nu}) \\ &\quad \text{fourth term - only depends on } \rho, \text{ not derivative - need no more terms} \\ &= -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - \frac{\mu^2}{2}\rho^i_\mu\rho^{i\mu} \\ &\quad \text{where } F^i_{\mu\nu} = \partial\rho^i_\nu - \partial_\nu\rho^i_\mu - 2g\varepsilon^{ijk}\rho^j_\mu\rho^k_\nu \end{aligned} \tag{8.7.55}$$

<u>Rescale g by ?</u>

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu\rho^i_\nu - \partial_\nu\rho^i_\mu + g\varepsilon^{ijk}\rho^j_\mu\rho^k_\nu)^2 - \frac{\mu^2}{2}\rho^i_\mu\rho^{i\mu} \tag{8.7.56}$$

is invariant under $\rho^i_\mu \to \rho^i_\mu + \varepsilon^{ijk}\rho^j_\mu\theta^k$.
This can be generalized to an $SU(N)$ gauge series.

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + gf^{ijk}A^j_\mu A^k_\nu)^2 - \frac{\mu^2}{2}A^i_\mu A^{i\mu} \tag{8.7.57}$$

with $f^{ijk} \to$ structure functions for $SU(N)$ group.

$$\begin{aligned} \mathcal{L}_{\text{spin-2}} &= \mathcal{L}(h_{\mu\nu}) + T_{\mu\nu}h^{\mu\nu} + T'^{\mu\nu}h_{\mu\nu} + T''^{\mu\nu}h_{\mu\nu} + \ldots \\ h_{\mu\nu}(x) &= h_{\nu\mu} \end{aligned} \tag{8.7.58}$$

Define $T_{\mu\nu}$ (stress tensor) from $\mathcal{L}$. Couple

$$\begin{aligned} h_{\mu\nu}T^{\mu\nu} &\Rightarrow \mathcal{L} + \mathcal{L}' \\ T_{\mu\nu} &\to T_{\mu\nu} + T'_{\mu\nu} \end{aligned}$$

Sum $\mathcal{L}_{\text{spin-2}}$ (∞), get

$$\mathcal{L}_{\text{spin-2}} = R\sqrt{g} \quad \text{(Einstein- Lagrangian)} \tag{8.7.59}$$

If $\mu^2 = 0$, the gauge invariance becomes local (allows for renormalizability).

$$\rho^i_\mu(x) \to \rho^i_\mu(x) + \partial_\mu\frac{\theta^i(x)}{g} + \varepsilon^{ijk}\rho^j_\mu(x)\theta^k(x) \tag{8.7.60}$$

Chapter 9

Path Integral Quantization

recall:

9.1 Heisenberg-Dirac

$$\begin{aligned} \frac{dA(t)}{dt} &= [A(t), H]_{\text{PB}} \qquad (9.1.1) \\ &= \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad ; \quad \left(\frac{\partial H}{\partial t} = 0 \right) \qquad (9.1.2) \end{aligned}$$

With (9.1.1), we can go from a classical to a quantum mechanical (QM) operator,

$$A(t) \rightarrow \hat{A}(t) \ \ \text{(QM operator)} \qquad (9.1.3)$$

$$\begin{aligned} \frac{d\hat{A}(t)}{dt} &= \frac{1}{\mathrm{i}\hbar} \left[\hat{A}(t), \hat{H} \right] \\ &= \frac{1}{\mathrm{i}\hbar} \left[\hat{A}\hat{H} - \hat{H}\hat{A} \right] \\ [\ ,\ ]_{\text{PB}} &\rightarrow \frac{1}{\mathrm{i}\hbar} [\ ,\ ] \qquad (9.1.4) \\ \mathrm{i}\hbar \frac{\partial \hat{A}}{\partial t} &= \left[\hat{A}, \hat{H} \right] \qquad (9.1.5) \end{aligned}$$

(Note: will drop "ˆ" now, as we will deal with operators only from now on.) The solution to this is,

$$\begin{aligned} A(t) &= e^{\mathrm{i}Ht/\hbar} A(0) e^{-\mathrm{i}Ht/\hbar} \qquad (9.1.6) \\ &= A_H(t) \quad ; \text{(Heisenberg Variable - Time dependent)} \qquad (9.1.7) \end{aligned}$$

State vectors $|\psi_H\rangle$ are time independent

$$\langle\psi_H|A_H(t)|\psi_H\rangle = \underbrace{\langle\psi_H|e^{\mathrm{i}Ht/\hbar}}_{\langle\phi_s|}A(0)\underbrace{e^{-\mathrm{i}Ht/\hbar}|\phi_H\rangle}_{|\phi_s\rangle} \tag{9.1.8}$$

We go to Schrodinger state vectors by;

$$\begin{aligned}
|\phi_s(t)\rangle &= e^{-\mathrm{i}Ht/\hbar}|\phi_H\rangle && (9.1.9)\\
&\rightarrow \mathrm{i}\hbar\frac{d}{dt}|\phi_s(t)\rangle = H|\phi_s(t)\rangle \;\; \text{Schrodinger eq.} && (9.1.10)\\
\langle\phi_s(t)| &= \langle\phi_H|e^{\mathrm{i}Ht/\hbar} && (9.1.11)\\
A_s &= A(0) \rightarrow \;\text{time independent.} && (9.1.12)
\end{aligned}$$

Consider eigenstates of the Heisenberg operators.

$$\begin{aligned}
A_H(t)|a(t)\rangle &= a|a(t)\rangle && (9.1.13)\\
e^{\mathrm{i}Ht/\hbar}A(0)e^{-\mathrm{i}Ht/\hbar}|a(t)\rangle &= a|a(t)\rangle\\
A(0)\left[e^{-\mathrm{i}Ht/\hbar}|a(t)\rangle\right] &= a\left[e^{-\mathrm{i}Ht/\hbar}\right]|a(t)\rangle && (9.1.14)
\end{aligned}$$

Let $t = 0$,

$$A(0)|a(0)\rangle = a|a(0)\rangle \tag{9.1.15}$$

Hence

$$\begin{aligned}
|a(t)\rangle &= e^{\mathrm{i}Ht/\hbar}|a(0)\rangle && (9.1.16a)\\
\langle a(t)| &= \langle a(0)|e^{-\mathrm{i}Ht/\hbar} && (9.1.16b)
\end{aligned}$$

9.2 Wave Functions

Schrodinger wave function

$$\psi(q,t) = \langle q|\psi(t)\rangle_s \tag{9.2.1}$$

where $\mathrm{i}\hbar\frac{d}{dt}|\psi_s(t)\rangle = H|\psi_s(t)\rangle,\; |\psi(t)\rangle = e^{-\mathrm{i}Ht/\hbar}|\psi(0)\rangle$. Thus

$$\begin{aligned}
\psi(q,t) &= \underbrace{\langle q|e^{-\mathrm{i}Ht/\hbar}}_{(9.1.16b)}|\psi(0)\rangle\\
&= \langle q_H(t)|\psi(0)\rangle && (9.2.2)
\end{aligned}$$

(here $Q(t)|q_H(t)\rangle = q|q_H(t)\rangle \rightarrow$ because we have a complete set of eigenfunctions).
Note:

$$\int dq(t)\;|q_H(t)\rangle\langle q_H(t)| = 1 \quad \text{for any t} \tag{9.2.3}$$

This is the Completeness Relation. Let $0 < t_1 < t_2 < \ldots < t_n < t**$.

$$\psi(q,t) = \int dq'(0)\ \underbrace{\langle q_H(t)|q'_H(0)\rangle}_{\substack{K(t,0)\\ \to \text{ propagator}}}\underbrace{\langle q'_H(0)|\psi(0)\rangle}_{\psi(q,0)=\langle q|\psi(0)\rangle} \tag{9.2.4}$$

For

$$\begin{aligned} K_1(t,0) &= \langle q_H(t)|q'_H(0)\rangle \\ &= \int \langle q_H(t)|q_{Hn}(t_n)\rangle\langle q_{Hn}(t_n)|q_{H(n-1)}(t_{n-1})\rangle\langle q_{H(n-1)}(t_{n-1})|\ldots|q_{H1}(t_1)\rangle\cdot \\ &\quad \cdot\langle q_{H1}(t_1)|q'(0)\rangle dq_n(t_n)\ dq_{n-1}(t_{n-1})\ldots dq_1(t_1) \end{aligned} \tag{9.2.5}$$

Consider one part of above:

$$\begin{aligned} \langle q_{H(i+1)}(t_{i+1})|q_{Hi}(t_i)\rangle &= \langle q_{s(i+1)}(0)|e^{-\mathrm{i}Ht_{i+1}/\hbar}e^{+\mathrm{i}Ht_i/\hbar}|q_{si}(0)\rangle \\ &= \langle q_{s(i+1)}(0)|e^{-\mathrm{i}H(t_{i+1}-t_i)/\hbar}|q_{si}(0)\rangle \end{aligned} \tag{9.2.6}$$

Let $\varepsilon = t_{i+1} - t_i$. If $\varepsilon \approx 0$, then,

$$\langle q_{H(i+1)}(t_{i+1})|q_{Hi}(t_i)\rangle \sim \langle q_{s(i+1)}(0)|\left(1-\frac{\mathrm{i}H\varepsilon}{\hbar}\right)|q_{si}(0)\rangle \tag{9.2.7}$$

Remember

$$[q,p] = \mathrm{i}\hbar \tag{9.2.8}$$

Thus,

$$< q,p >= \frac{e^{\mathrm{i}pq/\hbar}}{\sqrt{2\pi\hbar}} \tag{9.2.9}$$

i.e.

$$\langle q|\hat{p}|\psi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|\psi\rangle \tag{9.2.10}$$

which is satisfied if $|\psi\rangle = |p\rangle$. So,

$$\langle q_{s(i+1)}(0)|\left(1-\frac{\mathrm{i}H\varepsilon}{\hbar}\right)|q_i(0)\rangle = \int dp_{i+1}\ dp_i\ \underbrace{\langle q_{i+1}(0)|p_{i+1}\rangle}_{\frac{e^{\mathrm{i}p_{i+1}q_{i+1}/\hbar}}{\sqrt{2\pi\hbar}}}\langle p_{i+1}|\left(1-\frac{\mathrm{i}H\varepsilon}{\hbar}\right)|p_i\rangle\underbrace{\langle p_i|q_i(0)\rangle}_{\frac{e^{-\mathrm{i}p_iq_i/\hbar}}{\sqrt{2\pi\hbar}}} \tag{9.2.11}$$

i.e. used

$$\langle q|\hat{p}|\psi(t)\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|\psi(t)\rangle - -\mathrm{i}\hbar\frac{\partial}{\partial q}\psi(q,t) \tag{9.2.12}$$

$$\begin{aligned}
\rightarrow \langle q|\,[\hat{q},\hat{p}]\,|\psi(t)\rangle &= \mathrm{i}\hbar\langle q|\psi(t)\rangle \\
&= q\langle q|\hat{p}|\psi(t)\rangle - \langle q|\hat{p}\hat{q}|\psi(t)\rangle \\
&= q\left(-\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|\psi(t)\rangle\right) - \left(-\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|\hat{q}|\psi(t)\rangle\right) \\
&= \left[q\left(-\mathrm{i}\hbar\frac{\partial}{\partial q}\right) - \left(-\mathrm{i}\hbar\frac{\partial}{\partial q}\right)q\right]\langle q|\psi(t)\rangle \\
&= \mathrm{i}\hbar\langle q|\psi(t)\rangle \qquad (9.2.13)
\end{aligned}$$

$$\begin{aligned}
[\hat{q},\hat{p}] &= \mathrm{i}\hbar \qquad (9.2.14)\\
\langle q|\hat{p}|\psi(t)\rangle &= \left[-\mathrm{i}\hbar\frac{\partial}{\partial q} + f(q)\right]\langle q|\psi(t)\rangle \qquad (9.2.15)
\end{aligned}$$

→ See Dirac - $f(q)$ can be absorbed into the phase of $\psi(q,t)$.

Jan. 13/2000

i.e.

$$\left[-\mathrm{i}\hbar\frac{\partial}{\partial q} + f(q)\right]\exp\left\{\frac{\mathrm{i}}{\hbar}\int^{q} f(q')dq'\right\}\psi(q,t) = \left[-\mathrm{i}\hbar\frac{\partial}{\partial q}\psi(q,t)\right]\exp\left\{\frac{\mathrm{i}}{\hbar}\int^{q} f(q')dq'\right\} \qquad (9.2.16)$$

Re: $K(q',t';q,t)$,

$$\begin{aligned}
\psi(q',t') &= \int dq\; K(q',t';q,t)\psi(q,t) \qquad (9.2.17)\\
\rightarrow K(q',t';q,t) &= \langle (q',t')_H|(q,t)_H\rangle \\
&= \int dq_1\ldots dq_n\; \langle (q',t')_H|(q_n,t_n)_H\rangle\langle (q_nt_n)_H|(q_{n-1}t_{n-1})_H\rangle\ldots\langle (q_1t_1)_H|(q,t)_H\rangle \qquad (9.2.18)
\end{aligned}$$

$$\begin{aligned}
\rightarrow \langle (q^{i+1}t^{i+1})_H|(q^it^i)_H\rangle &= \langle q_s^{i+1}|e^{\mathrm{i}H(t^{i^1}-t^i)/\hbar}|q_s^i\rangle \\
&\approx \langle q_s^{i+1}|\left(1-\frac{\mathrm{i}}{\hbar}H(\hat{q},\hat{p})\varepsilon\right)|q_s^i\rangle \\
&= \int dp^i\; \langle q^{i+1}|\left(1-\frac{\mathrm{i}\varepsilon}{\hbar}H(\hat{q},\hat{p})\right)|p^i\rangle\langle p^i|q^i\rangle \qquad (9.2.19)
\end{aligned}$$

$$\begin{aligned}
\langle q|\hat{p}|p\rangle &= -\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|p\rangle \qquad (9.2.20)\\
\rightarrow \langle q|p\rangle &= Ke^{\mathrm{i}pq/\hbar}\\
\therefore \int dp\; \langle q'|p\rangle\langle p|q\rangle &= \int dq\; e^{\mathrm{i}p\cdot(q'-q)/\hbar}\\
&= \delta(q'-q) \qquad (9.2.21)\\
\Rightarrow K &= \frac{1}{\sqrt{2\pi\hbar}} \qquad (9.2.22)
\end{aligned}$$

$$\begin{aligned}
\therefore \langle (q^{i+1}t^{i+1})_H | (q^i t^i)_H \rangle &= \int \frac{dp^i}{\sqrt{2\pi\hbar}} \langle q^{i+1} | \left(1 - \frac{\mathrm{i}\varepsilon}{\hbar} H(\hat{q},\hat{p})\right) |p^i\rangle e^{-\mathrm{i}p^i q^i/\hbar} \\
&\downarrow \\
\rightarrow \big[\langle q^{i+1}| H(\hat{q},\hat{p}) |p^i\rangle &\approx H(q_{i+1},p_i)\langle q^{i+1}|p^i\rangle\big] \\
&\downarrow \quad (\text{ Have to worry about ordering of } \hat{p}'s,\ \hat{q}'s.) \\
\therefore \langle (q^{i+1}t^{i+1})_H | (q^i t^i)_H \rangle &= \int \frac{dp^i}{\sqrt{2\pi\hbar}} e^{\mathrm{i}p^i(q^{i+1}-q^i)/\hbar} \left(1 - \frac{\mathrm{i}\varepsilon}{\hbar} H(q_{i+1},p_i)\right) \qquad (9.2.23)
\end{aligned}$$

(The H term is important in curved space). Thus, $K(q',t';q,t)$ is;

$$\begin{aligned}
K(q',t';q,t) &= \int dq_1 \ldots dq_n \int \frac{dp_0}{2\pi\hbar} \cdots \frac{dp_n}{2\pi\hbar} e^{\mathrm{i}p_0(q_1-q)/\hbar} e^{\mathrm{i}p_1(q_2-q_1)/\hbar} \ldots e^{\mathrm{i}p_n(q'-q_n)/\hbar} \cdot \\
&\quad \cdot \left(1 - \frac{\mathrm{i}\varepsilon}{\hbar}H(q_1,p_0)\right)\left(1 - \frac{\mathrm{i}\varepsilon}{\hbar}H(q_2,p_1)\right)\ldots\left(1 - \frac{\mathrm{i}\varepsilon}{\hbar}H(q',p_n)\right) \\
&\simeq \int dq_1 \ldots dq_n \int \frac{dp_0}{2\pi\hbar} \cdots \frac{dp_n}{2\pi\hbar} \exp\left\{ \frac{\mathrm{i}}{\hbar}\Big[p_0(q_1-q_0) - \varepsilon H(q_1,p_0) \right. \\
&\qquad \left. + p_1(q_2-q_1) - \varepsilon H(q_2,p_1) + \ldots + p_n(q'-q_n) - \varepsilon H(q',p_n)\Big] \right\}
\end{aligned}$$

Let $\frac{q_{i+1}-q_i}{\varepsilon} \approx \dot{q}(t_i)$. Thus,

$$K(q',t';q,t) = \int dq_1 \ldots dq_n \int \frac{dp_0}{2\pi\hbar} \cdots \frac{dp_n}{2\pi\hbar} \exp\left\{ \frac{\mathrm{i}}{\hbar} \sum_{i=0}^{n} \overbrace{\varepsilon\left[p_i(\dot{q}_i(t_i)) - H(q_{i+1},p_i)\right]}^{\text{Riemann Sum}} \right\} \tag{9.2.24}$$

See figure 9.2.1. As $\varepsilon \rightarrow 0$,

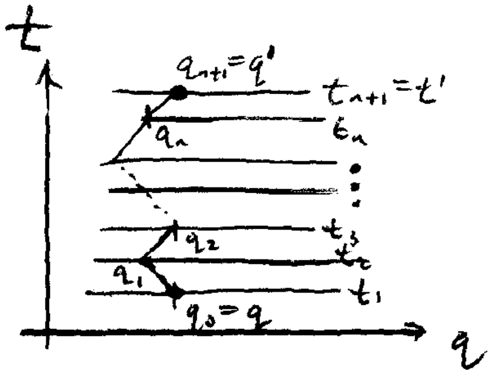


Figure 9.2.1: Particle path, where the $q_{n+1} = q'$, $q_0 = q$ are fixed, and the other q's (and p's) are all integrated over.

$$K \rightarrow \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t) \int \mathcal{D}p(t)\ \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_t^{t'} dt\ \ [p(t)\dot{q}(t) - H(q(t),p(t))] \right\} \tag{9.2.25}$$

Which is the Feynman Path Integral (Dirac/Feynman Papers → in a collection edited by Schwinger).

Suppose,

$$H(q,p) = \frac{p^2}{2m} + V(q) \tag{9.2.26}$$

i.e.

$$\int \frac{dp^i}{2\pi\hbar} \;\; e^{\frac{\mathrm{i}}{\hbar}\left[p_i(q_{i+1}-q_i)-\varepsilon\left(\frac{p_i^2}{2m}\right)\right]} \tag{9.2.27}$$

- this is a Gaussian $\Big($ Recall:

$$\begin{aligned}\int_{-\infty}^{\infty} dx\; e^{-ax-bx^2} &= \int_{-\infty}^{\infty} e^{-b\left(x+\frac{a}{2b}\right)^2+\frac{a^2}{4b}} \\ &= \sqrt{\frac{\pi}{b}} e^{a^2/4b}\Big)\end{aligned} \tag{9.2.28}$$

"Ignore" i in exponent of integral by putting in a convergence factor. i.e.,

$$\begin{aligned}&\int \frac{dp^i}{2\pi\hbar}\;\exp\left\{\tfrac{\mathrm{i}}{\hbar}\left[p_i(q_{i+1}-q_i) - \varepsilon\left(\tfrac{p_i^2}{2m}\right) + \mathrm{i}\delta p^2\right]\right\} \quad (\text{let } \delta \to 0) \\ &= \frac{1}{2\pi\hbar}\sqrt{\frac{\pi}{\frac{\mathrm{i}\varepsilon}{2m\hbar}}}\;\exp\left[\frac{\left(\frac{-\mathrm{i}(q_{i+1}-q_i)}{\hbar}\right)^2}{4\left(\frac{\mathrm{i}\varepsilon}{2m\hbar}\right)}\right] \\ &= \sqrt{\frac{m}{2\mathrm{i}\pi\hbar\varepsilon}}\;\exp\left\{\frac{\mathrm{i}m\left(q_{i+1}-q_i\right)^2}{2\varepsilon\hbar}\right\}\end{aligned} \tag{9.2.29}$$

Plug this in $\forall\ p_i$'s ,

$$\begin{aligned}K(q',t';q,t) &= \int dq_1\ldots dq_n\;\underbrace{\left(\sqrt{\frac{m}{2\mathrm{i}\pi\hbar\varepsilon}}\right)^{n+1}}_{N}\;\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{i=0}^{n}\left[\frac{m(q_{i+1}-q_i)^2}{2\varepsilon} - \varepsilon V(q_{i+1})\right]\right\} \\ &= N\int dq_1\ldots dq_n\;\exp\left\{\frac{\mathrm{i}\varepsilon}{\hbar}\sum_{i=0}^{n}\left[\frac{m}{2}\underbrace{\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^2}_{(\dot{q}_i(t_i))^2} V(q_{i+1})\right]\right\}\end{aligned}$$

$$L(q,\dot{q}) = \frac{\dot{q}^2}{2m} - V(q) \to \text{ Dim. of Energy } = \text{ mass } \times \frac{l^2}{t^2}$$

$$\therefore K(q',t';q,t) = N\int_{q(t)=q}^{q(t')=q'} dq_1\ldots dq_n\;\exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} dt\; L(q(t),\dot{q}(t))\right\} \tag{9.2.30}$$

Jan. 17/2000
As $\hbar \to 0$, the dominant contribution when integrating over all paths comes from the path that minimizes

$$S = \int_t^{t'} d\tau\ L(q, \dot{q}) \tag{9.2.31}$$

i.e.

$$\delta \int_t^{t'} d\tau\ L(q, \dot{q}) = 0 \tag{9.2.32}$$

(Hamilton's Principle of Least Action).

9.3 Free Particle

For a free particle, we mean $V = 0$.

$$\begin{aligned}
\langle q', t'|q, t\rangle &= N \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(\tau)\ \exp\left\{ \mathrm{i} \int_t^{t'} d\tau\ L(\dot{q}(\tau), q(\tau)) \right\} \\
&\quad (\hbar = m = 1) \\
&= N \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(\tau)\ \exp\left\{ \mathrm{i} \int_t^{t'} d\tau\ \frac{\dot{q}^2(\tau)}{2} \right\}
\end{aligned} \tag{9.3.1}$$

<u>But</u> recall;

$$\begin{aligned}
\langle (q't')_H|(qt)_H\rangle &= \left(\langle q'|e^{-\mathrm{i}\hat{H}t'}\right)\left(e^{+\mathrm{i}\hat{H}t}|q\rangle\right) \quad ; \quad \text{free particle } \to H = \frac{p^2}{2} \\
&= \int dp\ \langle q'|e^{-\mathrm{i}\hat{p}^2 t'/2}|p\rangle\langle p|e^{\mathrm{i}\hat{p}^2 t/2}|q\rangle \\
&= \int dp\ e^{-\mathrm{i}p^2(t-t')/2}\ \underbrace{\langle q'|p\rangle}_{\frac{e^{\mathrm{i}q'p}}{\sqrt{2\pi}}}\underbrace{\langle p|q\rangle}_{\frac{e^{-\mathrm{i}qp}}{\sqrt{2\pi}}} \\
&= \int \frac{dp}{2\pi} e^{-\mathrm{i}p^2(t-t')/2} e^{\mathrm{i}p(q'-q)-\delta p^2}
\end{aligned} \tag{9.3.2}$$

where $\delta \to 0^+$ (inserted so integral converges). Now, recall the Gaussian:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ e^{-ax^2-bx} &= \int_{-\infty}^{\infty} dx\ e^{-a\left(x+\frac{b}{2a}\right)^2} e^{b^2/4a} \\
&= \sqrt{\frac{\pi}{a}} e^{b^2/4a}
\end{aligned} \tag{9.3.3}$$

Here we have $a = \delta + \frac{\mathrm{i}(t'-t)}{2}$, $b = -\mathrm{i}(q'-q)$. So,

$$\begin{aligned}
\therefore \langle (q',t')_H|(q,t)_H\rangle &= \frac{1}{2\pi}\sqrt{\frac{\pi}{\delta + \frac{\mathrm{i}(t'-t)}{2}}}\exp\left\{-\frac{(q'-q)^2}{4\left(\delta+\frac{\mathrm{i}(t'-t)}{2}\right)}\right\} \\
&= \frac{1}{\sqrt{2\pi\mathrm{i}(t'-t)}}\exp\left\{\frac{\mathrm{i}(q'-q)^2}{2(t'-t)}\right\} \qquad (9.3.4)
\end{aligned}$$

Remember that,

$$\psi(q',t') = \int dq\; K(q',t';q,t)\psi(q,t) \qquad (9.3.5)$$

where

$$K(q',t';q,t) = \langle (q',t')_H|(q,t)_H\rangle \qquad (9.3.6)$$

but also

$$\mathrm{i}\frac{\partial}{\partial t}\psi(q,t) = -\frac{1}{2}\left(\frac{\partial^2}{\partial q^2}\right)\psi(q,t) \qquad (9.3.7)$$

Hence,

$$\mathrm{i}\frac{\partial}{\partial t'}K(q',t';q,t) = -\frac{1}{2}\frac{\partial^2}{\partial q'^2}K(q',t';q,t) \qquad (9.3.8)$$

And

$$\lim_{t'\to t} K(q',t';q,t) = \delta(q'-q)\delta(t'-t) \qquad (9.3.9)$$

If we start putting in potentials, it's awfully difficult to work out, especially when you put in boundary conditions. It's easier to consider vacuum to vacuum transitions (in order to eliminate having to consider $q(t) = q$ and $q(t') = q'$ → i.e. the B.C.'s). Examine,

$$\langle (q't')_H|(qt)_H\rangle^J = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t)\; e^{\mathrm{i}\int_t^{t'} d\tau\; [L(\dot{q},q)+J(\tau)q(\tau)]} \qquad (9.3.10)$$

(where $J =$ "source") Look at, (now dropping subscript "H" → always in Heisenberg eigenstates);

$$\langle (Q',T')_H|(Q,T)_H\rangle^J = \int dq'\; dq\; \langle Q'T'|q't'\rangle^J\langle q't'|qt\rangle^J\langle qt|QT\rangle^J \qquad (9.3.11)$$

<u>Now</u>

$$\begin{aligned}
\langle Q'T'|q't'\rangle &= \langle Q'|e^{-\mathrm{i}\hat{H}T'}e^{\mathrm{i}\hat{H}t'}|q'\rangle \\
&\quad \text{If } \hat{H}|n\rangle = E_n|n\rangle \\
&= \sum_n \langle Q'|e^{-\mathrm{i}E_nT'}|n\rangle\langle n|e^{\mathrm{i}E_nt'}|q'\rangle \\
&\quad \text{If } \langle q|n\rangle = \phi_n(q) \\
&= \sum_n e^{-\mathrm{i}E_n(T'-t')}\phi_n(Q')\phi_n^*(q') \qquad (9.3.12)
\end{aligned}$$

and similarly for $\langle q,t|Q,T\rangle^J$.

$$\therefore \langle Q'T'|QT\rangle = \int dq\ dq' \sum_{n,n'} e^{-\mathrm{i}E_n(T'-t')}\ e^{-\mathrm{i}E_{n'}(t-T)}\phi_n(Q')\phi_n^*(q')\phi_{n'}^*(Q)\phi_{n'}(q)\langle q't'|qt\rangle^J \tag{9.3.13}$$

Let $T' \to -\mathrm{i}\infty,\ T \to +\mathrm{i}\infty \to$ The vacuum becomes the dominant contribution. (i.e. only the vacuum state $|0\rangle$ survives - projects only the vacuum state) - holds for any t, t'.

$$\langle Q'T'|QT\rangle = \int dq\ dq'\ \underbrace{e^{-\mathrm{i}E_0(-t'+t)}}_{*}\phi_0(Q')\phi_0^*(Q)e^{-\mathrm{i}E_0(T'-T)}\ \underbrace{\phi_0^*(q')\langle q't'|qt\rangle\phi_0(q)}_{*} \tag{9.3.14}$$

("$*$" $\to$ involves integral over q, q'). If $\phi_0(q)e^{-\mathrm{i}E_0t} = \phi_0(q,t)$,

$$\begin{aligned} \to \int dq\ dq'\ \phi_0^*(q',t')\langle q't'|qt\rangle^J\phi_0(q,t) &= \lim_{\substack{T'\to-\mathrm{i}\infty \\ T\to+\mathrm{i}\infty}} \frac{\langle Q'T'|QT\rangle^J}{\phi_0^*(Q,T)\phi_0(Q',T')} && (9.3.15)\\ &= \langle 0,t'|0,t\rangle && (9.3.16)\end{aligned}$$

(with $\langle Q'T'|QT\rangle^J$ = Path integral, and $\phi_0^*(Q,T)\phi_0(Q',T')$ = Normal factor N). So,

$$\langle 0,\infty|0,-\infty\rangle^J = \lim_{\substack{T'\to-\mathrm{i}\infty \\ T\to+\mathrm{i}\infty}} N\int_{Q(T)=Q}^{Q(T')=Q'} \mathcal{D}Q(\tau)\ \exp\left\{\mathrm{i}\int_T^{T'} d\tau\ \Big[L(\dot{Q}(\tau),Q(\tau)) + J(\tau)Q(\tau)\Big]\right\} \tag{9.3.17}$$

Jan. 19/2000

Consider now,

$$\langle q',t'|\hat{q}(t_n)\hat{q}(t_{n-1})\ldots\hat{q}(t_1)|q,t\rangle = \langle q',t'|T\hat{q}(t_1)\hat{q}(t_2)\ldots\hat{q}(t_n)|q,t\rangle \tag{9.3.18}$$

where T is the time ordering operator (largest times to the left), and where $t' > t_n > \ldots > t_1 > t$. Proceed as we did with the QM Path Integral. i.e.

$$\begin{aligned}\langle q't'|\hat{q}(t_n)\ldots\hat{q}(t_1)\langle qt| &= \int dq_n\ldots dq_1\ \langle q',t'|\overbrace{\hat{q}(t_n)|q_nt_n\rangle}^{q_n|q_nt_n\rangle}\langle q_nt_n|\overbrace{\hat{q}(t_{n-1})|q_{n-1}t_{n-1}\rangle}^{q_{n-1}|q_{n-1}t_{n-1}\rangle}\cdot\\ &\qquad \langle q_{n-1}t_{n-q}|\ldots\overbrace{\hat{q}(t_1)|q_1t_1\rangle}^{q_1|q_1t_1\rangle}\langle q_1t_1|qt\rangle && (9.3.19)\end{aligned}$$

Thus, just as

$$\langle q't'|qt\rangle = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t)\ \exp\left\{\mathrm{i}\int_t^{t'} d\tau\ L(\dot{q}(\tau),q(\tau))\right\} \tag{9.3.20}$$

in this case we obtain

$$\langle q't'|T\hat{q}(t_1)\ldots\hat{q}(t_n)|qt\rangle = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t)\ q(t_1)q(t_2)\ldots q(t_n)\exp\left\{\mathrm{i}\int_t^{t'} d\tau\ L(\dot{q}(\tau),q(\tau))\right\} \tag{9.3.21}$$

(where $t' \geqslant t_i \geqslant t \;\; \forall \; i = 1, \ldots, n$). Add $J(\tau)q(\tau)$ to L:

$$\begin{aligned}
\langle q't'|T\hat{q}(t_1)\ldots\hat{q}(t_n)|qt\rangle &= \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t)\; q(t_1)\ldots q(t_n)\exp\left\{\mathrm{i}\int_t^{t'} d\tau\;\; [L(\dot{q}(\tau), d(\tau)) + J(\tau)q(\tau)]\right\}\Bigg|_{J=0} \\
&= \frac{\delta}{\mathrm{i}\delta J(t_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(t_n)}\int_{q(t)=q}^{q(t')=q'} \mathcal{D}q(t)\;\;\exp\left\{\mathrm{i}\int_t^{t'} d\tau\,[L(\dot{q}(\tau), q(\tau))\right. \\
&\qquad \left. + J(\tau)q(\tau)]\right\}\Bigg|_{J=0} \qquad (9.3.22)
\end{aligned}$$

Thus,

$$\langle 0, \infty|Tq(t_1)\ldots(t_n)|0, -\infty\rangle = \frac{\delta}{\mathrm{i}\delta J(t_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(t_n)}\lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} N\langle Q'T'|QT\rangle \qquad (9.3.23)$$

With $Q = Q' = 0$,

$$\begin{aligned}
&\langle 0, \infty|Tq(t_1)\ldots q(t_n)|0, -\infty\rangle \\
&= \frac{\delta}{\mathrm{i}\delta J(t_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(t_n)} N \lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} \int_{Q(T)=0}^{Q(T')=0} \mathcal{D}Q(T)\;\;\underbrace{\exp}_{*}\left\{\mathrm{i}\int_T^{T'} d\tau\;\;\Big[L(\dot{Q}(\tau), Q(\tau))\right. \\
&\qquad \left. + J(\tau)Q(\tau)\Big]\right\} \qquad (9.3.24)
\end{aligned}$$

" $*$ " Oscillates?

$$\begin{aligned}
\to L = \frac{\dot{Q}^2}{2} - V(Q) &= \frac{1}{2}\left(\frac{d}{d\tau}Q\right)^2 - V(Q) \\
* \;\; &\to \int_0^{ic\infty} dx\; e^{\mathrm{i}x} \quad \mathrm{i}x = -y \\
&= \mathrm{i}\int_0^{+\infty} dy\; d^{-y} \quad \text{(well defined)} \qquad (9.3.25)
\end{aligned}$$

Now let $\tau = -\mathrm{i}\tau_E$, (E = Euclidean). So, we get

$$\begin{aligned}
&\langle 0, \infty|Tq(t_1)\ldots q(t_n)|0, -\infty\rangle \\
&= \frac{1}{\mathrm{i}^n}\left(\frac{\delta}{\delta J(t_1)}\cdots\frac{\delta}{\delta J(t_n)}\right) N \cdot \\
&\quad \cdot \lim_{\substack{T'_E\to\infty \\ T_E\to-\infty}} \int_{q(-\infty)=0}^{q(\infty)=0} \mathcal{D}q(t)\;\;\exp\left\{+\int_{T_E}^{T'_E} d\tau_E\;\;\left[-\frac{\dot{Q}^2}{2} - V(Q) + J(\tau)Q(\tau)\right]\right\}\Bigg|_{J(\tau)=0} \qquad (9.3.26)
\end{aligned}$$

The same procedure can be applied in QFT. Here,

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{1}{\mathrm{i}^n}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} N\int \mathcal{D}\phi(x)\ e^{\int d^4x_E\ [\mathcal{L}_E(\partial_\mu\phi,\phi)+J(x)\phi(x)]}\Bigg|_{J=0} \tag{9.3.27}$$

(where $\phi(x)\to 0$ as $x\to\pm\infty$.) If

$$\mathcal{L}_M(\partial_\mu\phi,\phi) = \frac{1}{2}\Bigg[\underbrace{(\partial_0\phi)^2-(\nabla\phi)^2}_{A}-m^2\phi^2\Bigg]-\frac{\lambda}{4!}\phi^4 \tag{9.3.28}$$

where $A=(\partial_\mu\phi)(\partial^\mu\phi)\to(+,-,-,-)$ (Bj. and Drell), or $=-(\partial_\mu\phi)(\partial^\mu\phi)\to(-,+,+,+)$. (and $\mathcal{L}_M$ is the Minkowski lagrangian). And,

$$\mathcal{L}_E = \frac{1}{2}\left(-(\partial_4\phi)^2-(\nabla\phi)^2-m^2\phi^2\right)-\frac{\lambda}{4!}\phi^4 \tag{9.3.29}$$

with $\underbrace{x^0}_{(M)} = \underbrace{-\mathrm{i}x^4}_{(E)} \to x_E^2=(x^4)^2+(\underline{x})^2$.

Let

$$Z_E(J) = N\int \mathcal{D}\phi \exp\left\{d^4x_E\ \left[-\frac{1}{2}(\partial_\mu^E\phi)^2-\frac{\mu^2\phi^2}{2}-\frac{\lambda\phi^4}{4!}+J(x)\phi(x)\right]\right\} \tag{9.3.30}$$

(Generating Functional) - (call the mass "μ" now). So,

$$\langle 0|T\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)|0\rangle = \frac{1}{\mathrm{i}^n}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z_E(J)\Bigg|_{J=0} \tag{9.3.31}$$

Jan. 21/2000

Consider,

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ e^{-a^2x^2+jx-\lambda x^4}\bigg|_{j=0} &= \int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty} \frac{(-\lambda x^4)^n}{n!} e^{-a^2x^2+jx}\bigg|_{j=0} \\
&= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\left(\frac{d}{dj}\right)^{4n} \int_{-\infty}^{\infty} dx\ e^{-a^2x^2+jx}\bigg|_{j=0} \\
&= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\frac{d^{4n}}{dj^{4n}} \int_{-\infty}^{\infty} dx\ \exp\left\{-a^2\left(x-\frac{j}{2a^2}\right)^2 + \left(\frac{j^2}{4a^2}\right)\right\}\bigg|_{j=0} \\
&\rightarrow \text{ Let } x' = x - \frac{j}{2a^2} \\
&= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\frac{d^{4n}}{dj^{4n}} \underbrace{\int_{-\infty}^{\infty} dx'\ e^{-a^2x'^2}}_{\frac{\sqrt{\pi}}{a}} e^{j^2/(4a^2)}\bigg|_{j=0} \\
&= \frac{\sqrt{\pi}}{a}\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\frac{d^{4n}}{dj^{4n}} e^{j^2/(4a^2)}\bigg|_{j=0} \\
&= \frac{\sqrt{\pi}}{a}\left[1 + \frac{-\lambda}{1!}\frac{d^4}{dj^4}\bigg|_{j=0} + \frac{(-\lambda)^2}{2!}\frac{d^8}{dj^8}\bigg|_{j=0} + \ldots\right] e^{j^2/(4a^2)} \\
&= \frac{\sqrt{\pi}}{a}\left[1 + \left(\frac{-\lambda}{1!}\right)\frac{3}{4a^2} + \ldots\right] \qquad (9.3.32)
\end{aligned}$$

Note:

$$\rightarrow \frac{d}{dj}e^{j^2/(4a^2)} = \frac{2j}{4a^2}e^{j^2/(4a^2)} \quad \bigg\| \quad \frac{d^2}{dj^2}e^{j^2/(4a^2)} = \left[\frac{2}{4a^2} + \left(\frac{2j}{4a^2}\right)^2\right] e^{j^2/(4a^2)}$$

$$\ldots\text{and so} \qquad \frac{d^4}{dj^4}e^{j^2/(4a^2)}\bigg|_{j=0} = \frac{8}{16a^4} + \frac{4}{16a^4} = \frac{3}{4a^4}$$

Thus,

$$\int_{-\infty}^{\infty} dx\ e^{-a^2x^2+jx-\lambda x^4} = \frac{\sqrt{\pi}}{a}\left[1 + \left(\frac{-\lambda}{1!}\right)\frac{d^4}{dj^4}\bigg|_{j=0} + \frac{(-\lambda)^2}{2!}\frac{d^8}{dj^8}\bigg|_{j=0} + \ldots\right] e^{j^2/(4a^2)} \qquad (9.3.33)$$

with the $(-\lambda)$ representing a dot (vertex), and $\left(\frac{1}{a^2}\right)$ being the loops. (see figure 9.3.1) For

$$\begin{aligned}
Z(J) &= \int \mathcal{D}\phi\ \exp\left\{\int d^4x\ \left[-\frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right) - \frac{\mu^2}{2}\phi^2 - \frac{\lambda\phi^4}{4!} + J\phi\right]\right\} \\
&= \sum_{k=0}^{\infty}\left(-\frac{\lambda}{4!}\right)^k \left(\int d^4x\ \frac{\delta^4}{\delta J^4(x)}\right)^k \int \mathcal{D}\phi\ \exp\left\{d^4x\ \left[\frac{1}{2}\phi\left(\partial^2-\mu^2\right)\phi + J\phi\right]\right\} \qquad (9.3.34)
\end{aligned}$$

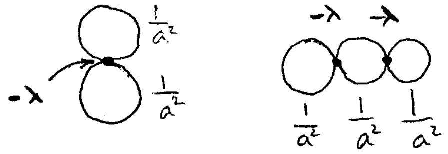


Figure 9.3.1: Simple Feynman Diagram example

But now,

$$\begin{aligned} \to & \int d^4x \ \left[\frac{1}{2}\phi\left(\partial^2-\mu^2\right)\phi + J\phi\right] \quad \text{(Now, complete square)} \\ = & \int d^4x \ \left[\frac{1}{2}\left(\phi + J\left(\frac{1}{\partial^2-\mu^2}\right)\right)\left(\partial^2-\mu^2\right)\left(\phi+\left(\frac{1}{\partial^2-\mu^2}\right)J\right) - \frac{1}{2}J\left(\frac{1}{\partial^2-\mu^2}\right)J\right] \end{aligned} \tag{9.3.35}$$

$$\text{Let} \quad \phi' = \phi + \left(\frac{1}{\partial^2-\mu^2}\right)J$$

So,

$$Z(J) = \sum_{k=0}^{\infty}\left(\frac{-\lambda}{4!}\right)\left(\int d^4x \ \frac{\delta^4}{\delta J^4(x)}\right)^k \int \mathcal{D}\phi'(x) \ e^{\int d^4x \ \left[\frac{1}{2}\phi'\left(\partial^2-\mu^2\right)\phi' - \frac{1}{2}J\left(\frac{1}{\partial^2-\mu^2}\right)J\right]} \tag{9.3.36}$$

Here, $\frac{1}{\partial^2-\mu^2} \Rightarrow \left(\partial^2-\mu^2\right)\frac{1}{(\partial^2-\mu^2)} = 1$. So,

$$\frac{1}{\partial^2-\mu^2} = G(x,y) \to (\partial_x^2-\mu^2)G(x,y) = \delta^4(x-y) \tag{9.3.37}$$

Thus,

$$J\frac{1}{\partial^2-\mu^2}J = \int d^4x \ d^4y \quad J(x)G(x,y)J(y) \tag{9.3.38}$$

Also, if,

$$G(x,y) = \int \frac{d^4k}{(2\pi)^4}g(k)e^{ik\cdot(x-y)} \tag{9.3.39}$$

Then using (9.3.37) ,

$$\begin{aligned} \left(\partial^2-\mu^2\right)\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}g(k) &= \underbrace{\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}}_{\delta(x-y)} \\ (-k^2-\mu^2)g(k) &= 1 \\ \to G(x,y) &= \int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{(-k^2-\mu^2)} \quad \text{(Non-singular integral)} \end{aligned} \tag{9.3.40}$$

Note that this provides meaning of the last term in the exponent of $Z(J)$. (also, $k^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2$).
For the other part of the exponent,

$$\int \mathcal{D}\phi \ e^{\int d^4x \ \left[-\frac{1}{2}\phi'(\partial^2-\mu^2)\phi'\right]} \tag{9.3.41}$$

Consider,

$$\int d^nx \ e^{-\underline{x}\underline{\mathbf{M}}\underline{x}} \quad ; \quad \underline{\mathbf{M}}^T = \underline{\mathbf{M}} \ \ (\underline{\mathbf{M}}^* = \underline{\mathbf{M}})$$

$$\rightarrow \ \underline{\mathcal{O}}^{-1}\underline{\mathbf{M}}\underline{\mathcal{O}} = \underline{\mathcal{D}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \ (\lambda_i = \text{ eigenvalues of } \underline{\mathcal{M}})$$

$$\underline{\mathcal{O}}^{-1} = \underline{\mathcal{O}}^T$$

Let

$$\underline{y} \ = \ \underline{\mathcal{O}}^{-1}\underline{x} \tag{9.3.42}$$

$$d^n\underline{y} \ = \ \det\left(\underline{\mathcal{O}}^{-1}\right) d^n\underline{x}$$

$$= \ (1)d^n\underline{x} \tag{9.3.43}$$

($\det(\underline{\mathcal{O}})$ = Product of eigenvalues of $\underline{\mathcal{O}}$). So,

$$\int d^n\underline{x} \ e^{-\underline{x}^T\underline{\mathbf{M}}\underline{x}} \ \Rightarrow \ \int d^n\underline{y} \ e^{-\underline{y}\underline{\mathbf{D}}\underline{y}}$$

$$= \int_{-\infty}^{\infty} d\underline{y}_1 \ e^{-\lambda_1\underline{y}_1^2} \int_{-\infty}^{\infty} d\underline{y}_2 \ e^{-\lambda_2\underline{y}_2^2} \cdots \int_{-\infty}^{\infty} d\underline{y}_n \ e^{-\lambda_n\underline{y}_n^2}$$

$$= \sqrt{\frac{\pi}{\lambda_1}}\sqrt{\frac{\pi}{\lambda_2}}\cdots\sqrt{\frac{\pi}{\lambda_n}}$$

$$= \frac{\pi^{n/2}}{\det^{1/2}(\underline{\mathcal{M}})} \tag{9.3.44}$$

So,

$$\int \mathcal{D}\phi' \ e^{d^4x \ \left[-\frac{1}{2}\phi'\left(\partial^2-\mu^2\right)\phi'\right]} = N\det{}^{1/2}(\partial^2-\mu^2) \tag{9.3.45}$$

and so,

$$Z_0(J) = N\det{}^{1/2}(\partial^2-\mu^2)e^{-\frac{1}{2}\int d^4x \ d^4y \ J(x)G(x,y)J(y)} \tag{9.3.46}$$

Thus,

$$Z(J) = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\left(\int d^4w \ \frac{\delta^4}{\delta J^4(w)}\right)^n N\det{}^{-1/2}(\partial^2-\mu^2)e^{+\frac{1}{2}\int d^4x \ d^4y \ J(x)\tilde{G}(x,y)J(y)} \tag{9.3.47}$$

where $\tilde{G}(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+\mu^2} = -G(x,y)$.

Jan. 24/2000

Recall;

$$Z[J] = \int \mathcal{D}\phi \exp\left\{\int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda\phi^4}{4!} + J\phi\right]\right\} \tag{9.3.48}$$

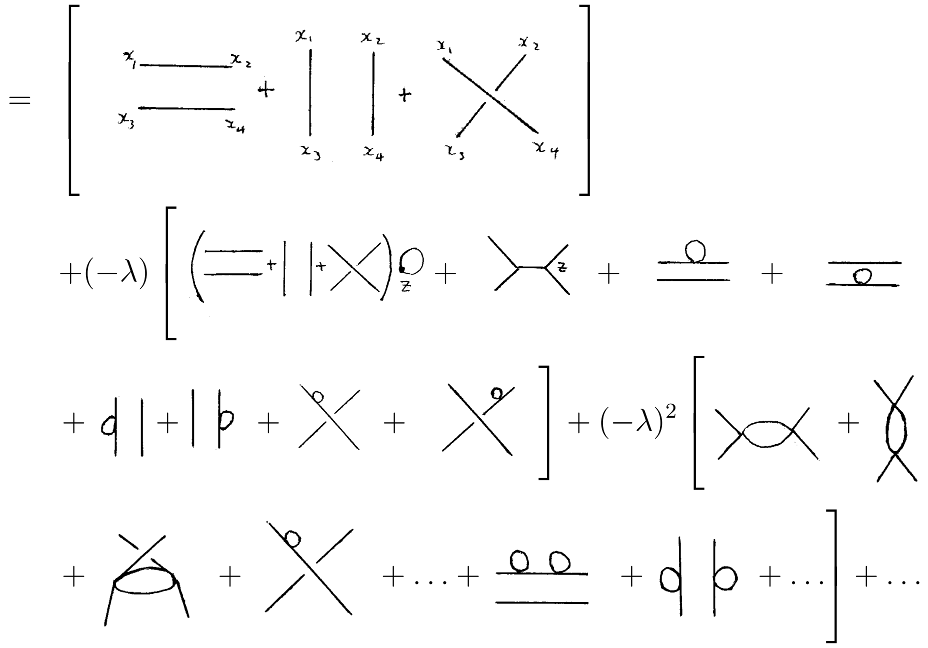
$$Z_0[J] = Ne^{(1/2)\int d^4xd^4yJ(x)G(x,y)J(y)} \quad (\lambda = 0) \tag{9.3.49}$$

$$\begin{aligned}
&(\mathrm{i})^n\langle 0|T\hat{\phi}(x_1)\dots\hat{\phi}(x_n)|0\rangle \\
&= \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z[J]\bigg|_{J=0} \\
&= \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} \sum_{j=0}^{\infty} \frac{1}{(j!)}\left(-\frac{\lambda}{4!}\right)^j \left(\int d^4z\, \frac{\delta^4}{\delta J^4(z)}\right)^j Ne^{(1/2)\int d^4xd^4y\ J(x)J(y)G(x,y)}\bigg|_{J=0}
\end{aligned} \tag{9.3.50}$$

For example, say $n = 4$ (and let $\frac{\delta}{\delta J_1} = \frac{\delta}{\delta J(x_1)}$, etc.),

$$\begin{aligned}
&(\mathrm{i})^n\langle 0|T\hat{\phi}(x_1)\dots\hat{\phi}(x_4)|0\rangle \\
&= \frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_4}\left\{1 + \left(-\frac{\lambda}{4!}\right)\int d^4z\frac{\delta^4}{\delta J_z^4} + \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\int d^4z_1\frac{\delta^4}{\delta J_{z_1}^4}\int dz_2\frac{\delta^4}{\delta J_{z_2}^4} + \dots\right\}\cdot \\
&\qquad \cdot\exp\left\{\frac{1}{2}\int d^4xd^4yJ(x)J(y)G(x,y)\right\}\bigg|_{J=0} \\
&= \underbrace{[G(x_1-x_2)G(x_3-x_4) + G(x_1-x_3)G(x_2-x_4) + G(x_1-x_4)G(x_2-x_3)]}_{*}(-\lambda)^0 \\
&\qquad + (-\lambda)\left[G(0)\,[\ *\ ] + \int d^4z\ G(x_1-z)G(x_2-z)G(x_3-z)G(x_4-z) + \dots\right] + \dots
\end{aligned}$$

or, diagrammatically, this is

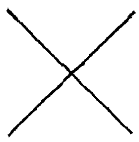


(9.3.51)

where,

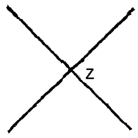
$\rightsquigarrow G(x,y)$

(9.3.52)



$\rightsquigarrow (-\lambda)$

9.4 Feynman Rules



$$= \quad -\lambda \tag{9.4.1}$$

$$= \quad G(x-y) \tag{9.4.2}$$

- Integrate over all internal points z
- Take into account symmetry factors and number of diagrams of a given topology

$$Z_0[J] \quad = \quad N e^{(1/2)\int d^4x \ d^4y \ J(x)G(x-y)J(y)} \tag{9.4.3}$$

$$Z[J] \quad = \quad \sum_j \frac{1}{j!}\left(\frac{-\lambda}{4!}\int \frac{\delta^4}{\delta J_z^4} d^4z\right)^j Z_0[J] \tag{9.4.4}$$

It can be shown (c.f. Cheng and Li) ... Just as $\left[\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\right]$ gives rise to all N point Feynman diagrams, so also for the connected diagrams,

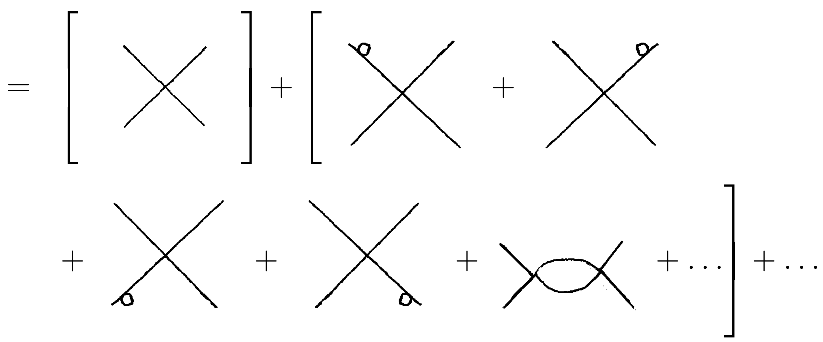
$$(\mathrm{i})^n\langle 0|T\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)|0\rangle_{\text{connected}} = \frac{\delta}{\delta J_1}\ldots\frac{\delta}{\delta J_n}W[J] \tag{9.4.5}$$

where

$$W[J] = \ln\left(Z[J]\right) \tag{9.4.6}$$

For example, in $n=4$,

$$(\mathrm{i})^4\langle 0|T\hat{\phi}(x_1)\ldots\hat{\phi}(x_4)|0\rangle_{\text{connected}}$$



(9.4.7)

(i.e. we don't get diagrams with separate "disconnected" parts) $\Rightarrow$ (*) and disconnected diagrams can be shown to contribute only to the phase of the Green's Function (G.F.).

We thus only need the connected G.F. to calculate physical cross sections. Futhermore, if

$$\Phi(x) = \frac{\delta W[J]}{\delta J(x)} \tag{9.4.8}$$

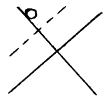
and

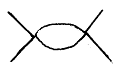
$$\Gamma[\Phi] = W[J] - \int d^4x \; J(x)\Phi(x) \tag{9.4.9}$$

then

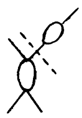
$$\frac{\delta}{\delta\Phi(x_1)} \cdots \frac{\delta}{\delta\Phi(x_n)} \Gamma[\phi] \tag{9.4.10}$$

gives rise to the one-particle irreducible (1PI) connected Green's Functions. $\rightarrow$ i.e. can't cut diagrams into 2 parts by cutting a single internal line. For example,

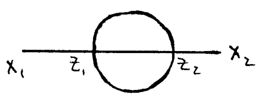

$\rightarrow$ Not 1PI


$\rightarrow$ is 1PI


$\rightarrow$ 1PI


$\rightarrow$ Not 1PI

This is useful in calculations because we can compute each 1PI part and connect the two to get non-1PI parts.
For example,



$$= (-\lambda)^2 \int d^4z_1 d^4z_2 \; G(x_1 - z_1)G(x_2 - z_2)G^3(z_1 - z_2) \tag{9.4.11}$$

$\rightarrow$ # = 1 (one such diagram). Symmetry factor (# of ways we can connect these lines) $S = \frac{1}{6}$. (Recall, for the symmetry factor: for the numerator, x_1 can go to 8 points (four points per vertex), then x_2 can go to 4 points (four points on the second vertex), and then one of the remaining three lines on one of the vertices can go to 3 possible lines on the other vertex, then one of the remaining two lines can go to 2 possible lines on the other, and the

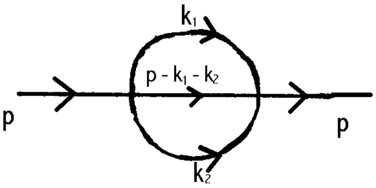


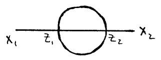
Figure 9.4.1: Momentum Space diagram - note momentum conserved at each vertex

final line can go to the 1 line on the other vertex - for the denominator, there are two 4! factors (four points per vertex, two vertices), and we can switch the vertices, so there is also a 2!; thus,

$$
\begin{aligned}
S &= \frac{8 \times 4 \times 3 \times 2 \times 1}{4!\; 4!\; 2!} \\
&= \frac{1}{6}
\end{aligned}
$$

Note that the 4! in $\frac{-\lambda}{4!}\phi^4$ per vertex is accounted for in the symmetry factor, as is the $\frac{1}{2!}$ that would arise from the sum (see (9.4.4))).
Convert to p-space,

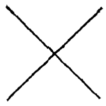
$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+\mu^2} \tag{9.4.12}$$

In momentum space (for ) the number of integrals $\equiv$ number of loops.

$$
\begin{aligned}
\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle &= \frac{1}{6}(-\lambda)^2 \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x_1-x_2)} \left[\frac{1}{(p^2+m^2)^2} \int d^4k_1\, d^4k_2 \left(\frac{1}{k_1^2+m^2} \right) \cdot \right. \\
&\quad \left. \cdot \left(\frac{1}{k_2^2+m^2} \right) \frac{1}{(k_1+k_2+p)^2+m^2} \right]
\end{aligned} \tag{9.4.13}
$$

(see figure 9.4.1)

In Minkowski space, we have the same diagrams, with,



$$= \quad -\mathrm{i}\lambda \tag{9.4.14}$$

$$= \quad \mathrm{i}\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{\mathrm{i}k\cdot(x-y)}}{k^2-\mu^2+\mathrm{i}\varepsilon} \tag{9.4.15}$$

Note: $k^2 = k_0^2 - \underline{k}^2$, $\quad\therefore$ pole at $k_0 = \pm\sqrt{\underline{k}^2+\mu^2}$.

$$\langle 0|T\hat{\phi}(x_1)\dots\hat{\phi}(x_n)|0\rangle_{\text{Minkowski}} = \frac{\delta}{\mathrm{i}\delta J(x_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(x_n)}Z_m[J] \tag{9.4.16}$$

9.5 Path Integrals for Fermion Fields

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\ \not{\partial} - m)\psi \tag{9.5.1}$$

Canonical Quantization $\rightarrow \{\psi,\bar{\psi}\} = \mathrm{i}\hbar\delta(\quad)$. In the path integral, ψ and $\bar{\psi}$ are Grassmann Variables. i.e.

$$Z(\eta,\bar{\eta}) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left\{\mathrm{i}\int d^4x\ \left[\mathcal{L}+\bar{\eta}\psi+\bar{\psi}\eta\right]\right\} \tag{9.5.2}$$

It's important that ψ, $\bar{\psi}$ are treated independently. (Note, though, if $\underline{\psi=\psi_c} = \bar{C}\bar{\psi}^T$ (majorana), then ψ, $\bar{\psi}$ are no longer independent).

9.6 Integration over Grassmann Variables

Recall,

$$\theta_1\theta_2 = -\theta_2\theta_1 \quad (\theta_i^2 = 0) \tag{9.6.1}$$

$$\begin{aligned}\frac{d}{d\theta_a}(\theta_m\theta_n) &= \frac{d\theta_m}{d\theta_a}\theta_n + \theta_m\left(-\frac{d\theta_n}{d\theta_a}\right)\\ &= \delta_{ma}\theta_n - \delta_{na}\theta_m\end{aligned} \tag{9.6.2}$$

If we have $\theta_1\dots\theta_n$, then

$$p(x,\theta_1,\theta_2,\dots,\theta_n) = p_0(x) + p_i(x)\theta_i + p_{ij}\theta_i\theta_j + \dots + p_{i_1,\dots,i_n}(x)\theta_i,\dots,\theta_{i_n} \tag{9.6.3}$$

(note no $(n+1)$ term due to $\theta_i^2 = 0$) where $p_{ij} = -p_{ji},\ p_{ijk} = -p_{jik}$, etc.. We have,

$$\begin{aligned} p(x,\theta) &= p_0 + p_1\theta \\ \rightarrow \int d\theta\ p(x,\theta) &= \int d\theta\ [p_0 + p_1\theta] \\ &= \int d\theta\ [p(x,\theta+\alpha)] \quad (\alpha \rightarrow \text{ a displacement}) \end{aligned} \tag{9.6.4}$$

For this to hold,

$$\int d\theta\ \theta = 1 \tag{9.6.5}$$

$$\int (1) d\theta = 0 \tag{9.6.6}$$

Thus, (9.6.4) is

$$= \int d\theta\, [p_0 + p_1 \cdot (\theta + \alpha)] \tag{9.6.7}$$

Thus, $\int d\theta \Leftrightarrow \frac{d}{d\theta}$ (i.e. are the same operator). Also, note that the only term that will survive in the integral will be the one with n θ's (all other terms won't have enough θ's). So,

$$\begin{aligned} \int d\theta_1 \ldots d\theta_n\ [p(x,\theta_1,\ldots,\theta_n)] &= \int d\theta_n \ldots d\theta_1\ [p_0 + p_i\theta_i + p_{ij}\theta_i\theta_j + \ldots + p_{i_1,\ldots i_n}\theta_{i_1}\ldots\theta_{i_n}] \\ &= \int d\theta_n \ldots d\theta_1\ p_{i_1\ldots i_n}\theta_{i_1}\ldots\theta_{i_n} \\ &= \underbrace{\epsilon_{i_1\ldots i_n} p_{i_1\ldots i_n}}_{n!\ \text{terms}} \\ &= n! p_{123\ldots n} \end{aligned} \tag{9.6.8}$$

(the $\epsilon_{i_1\ldots i_n}$ is present because order of θ's may not be $i_1, i_2, \ldots$). If we now change variables of integration,

$$\tilde{\theta}_i = a_{ij}\theta_j \tag{9.6.9}$$

then,

$$\begin{aligned} \int d\theta_i p(x,\theta_i) &= \int d\tilde{\theta}_i\ p(x,\tilde{\theta}_i) \quad \text{(expect)} \\ &= \int d\tilde{\theta}_i\ p_{i_1\ldots i_n}\tilde{\theta}_{i_1}\ldots\tilde{\theta}_{i_n} \\ &= \int d\tilde{\theta}_i\ (p_{i_1\ldots i_n}) a_{i_1 j_1}\ldots a_{i_n j_n}\theta_{j_1}\ldots\theta_{j_n} \qquad (9.6.10) \\ &= \int d\theta_1 \ldots d\theta_n\ (p_{i_1\ldots i_n})\theta_{i_1}\ldots\theta_{i_n} \qquad (9.6.11) \end{aligned}$$

and (9.6.10) must be equal to (9.6.11). Thus,

$$\int d\theta_1 \dots d\theta_n \; p_{1\dots n}\theta_1 \dots \theta_n = \int d\tilde{\theta}_1 \dots d\tilde{\theta}_n \; \epsilon_{i_1\dots i_n} a_{i_1 j_1} \dots a_{i_n j_n} p_{1\dots n}\theta_1 \dots \theta_n \tag{9.6.12}$$

Hence

$$\begin{aligned} d\tilde{\theta}_1 \dots d\tilde{\theta}_n &= d\theta_1 \dots d\theta_n \underbrace{\epsilon_{i_1\dots i_n} a_{i_1 j_1} \dots a_{i_n j_n}}_{=\det(\underline{a})} \\ &= \det(\underline{a}) d\theta_1 \dots d\theta_n \\ \text{i.e.} \quad d^n\tilde{\theta} &= \underbrace{\det\left[\frac{\partial(\tilde{\theta}_1 \dots \tilde{\theta}_n)}{\partial(\theta_1 \dots \theta_n)}\right]}_{*} d^n\theta \end{aligned} \tag{9.6.13}$$

where "*" is the reciprocal of the normal jacobian; recall

$$\int d^n x \; f(x) = \int d^n\tilde{x} \; \det\left[\frac{\partial(x^1 \dots x^n)}{\partial(\tilde{x}^1 \dots \tilde{x}^n)}\right] f(\tilde{x}) \tag{9.6.14}$$

$$\text{i.e.} \quad d^n x = \det\left[\frac{\partial(x^1 \dots x^n)}{\partial(\tilde{x}^1 \dots \tilde{x}^n)}\right] d^n\tilde{x} \tag{9.6.15}$$

Also note,

$$\begin{aligned} & \int d\theta_1 \dots d\theta_n d\bar{\theta}_1 \dots d\bar{\theta}_n e^{\bar{\theta}_i C_{ij}\theta_j} \quad \text{(expand exponential)} \\ = & \int d\theta_1 \dots d\theta_n d\bar{\theta}_1 \dots d\bar{\theta}_n \; \left[\frac{1}{n!}(\bar{\theta}_i C_{ij}\theta_j)^n\right] \\ = & \frac{1}{n!}\epsilon_{i_1\dots i_n}\epsilon_{j_1\dots j_n} C_{i_1 j_1} \dots C_{i_n j_n} \\ = & \frac{1}{n!}(n! \; \det(\underline{C})) \\ = & \det(\underline{C}) \end{aligned} \tag{9.6.16}$$

(Recall,

$$\begin{aligned} \int d\theta_1 \dots d\theta_n e^{\theta_i C_{ij}\theta_j} \quad &= \quad \det{}^{1/2}(\underline{C}) \\ \text{but} \quad & \int_{-\infty}^{\infty} dx_1 \dots dx_n d\bar{x}_1 \dots d\bar{x}_n e^{-\bar{x}_i C_{ij} x_j} \\ & \text{If} \;\; \lambda_i > 0 \;\; \text{for} \;\; C_{ij} \;\; \text{then} \\ & = \frac{\pi^n}{\det\underline{C}} \quad \to \bar{x}/x's \;\; \text{distinct}, \;\; \therefore \;\; \text{not} \; \frac{\sqrt{\pi^n}}{\sqrt{\det(\underline{C})}} \end{aligned}$$)

For $n = 2$,

$$\begin{aligned}
\int d\theta_1 d\theta_2 \ \exp\left\{[\theta_1, \theta_2]\begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}\right\} &= \int d\theta_1 d\theta_2 \ \left[1 + [\theta_1, \theta_2]\begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}\right] \\
&= \int d\theta_1 d\theta_2 \ [\theta_1 \gamma \theta_2 - \theta_2 \gamma \theta_1] \\
&= \gamma \\
&= \det{}^{(1/2)}\left\{\begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}\right\} \qquad (9.6.17)
\end{aligned}$$

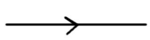
Hence, for,

$$\begin{aligned}
Z(\eta, \bar{\eta}) &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{\mathrm{i}\int d^4x \ \left[\bar{\psi}(\not{p} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\} \\
&= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{\mathrm{i}\int d^4x \ \left[\left(\bar{\psi} + \bar{\eta}\left(\frac{1}{\not{p} - m}\right)\right)(\not{p} - m)\left(\left(\frac{1}{\not{p} - m}\right)\eta + \psi\right)\right.\right. \\
&\qquad \left.\left. -\bar{\eta}\left(\frac{1}{\not{p} - m}\right)\eta\right]\right\} \quad (\text{ recall } \ p \equiv \mathrm{i}\partial) \\
&= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{\mathrm{i}\int d^4x \ \left[\bar{\psi}(\not{p} - m)\psi + \bar{\eta}\left(\frac{1}{\not{p} - m}\right)\eta\right]\right\} \quad (\psi = \psi(x^\mu)) \\
&= \underbrace{\det(\not{p} - m)}_{\text{normalization}} \exp\left\{\mathrm{i}\int d^4x \ \left[\bar{\eta}\left(\frac{1}{\not{p} - m}\right)\eta\right]\right\} \qquad (9.6.18)
\end{aligned}$$

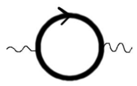
$\Big($ Recall:

$$Z[J] = \frac{1}{\det^{(1/2)}(p^2 - m^2)} \exp\left\{\frac{\mathrm{i}}{2}\int dx \ J\left(\frac{1}{p^2 - m^2}\right)J\right\}\Big)$$

Only fallout from having to take into account that $\bar{\eta}$, η are Grassmann.



$$\frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon} \quad \text{Propagator} \qquad (9.6.20)$$



$\rightarrow$ Closed loop of fermions $\rightarrow$ factor of (-1) (9.6.21)

(9.6.22)

Jan. 27/2000

9.7 Gauge Invariance

(1st Principle) $U(1)$

$$\begin{aligned} S &= \int d^4x \left(-\frac{1}{2}\phi(\partial^2+\mu^2)\phi\right) \\ &= \int d^4x\, \frac{1}{2}\left((\partial_\mu\phi)^2-\mu^2\phi^2\right) \quad (\text{ 1 degree of freedom } \phi) \end{aligned} \tag{9.7.1}$$

If $\phi \to$ complex,

$$S=\int d^4x\, \left[(\partial_\mu\phi^*)(\partial_\mu\phi)-\mu^2\phi^*\phi\right] \quad (\text{ 2 degrees of freedom } \phi^*,\ \phi) \tag{9.7.2}$$

We demand

$$\phi \to e^{\mathrm{i}\Lambda(x)\phi} \tag{9.7.3}$$

be an invariance.

$$\mathcal{L}=(\partial_\mu\phi^*)(\partial^\mu\phi)-m^2\phi^*\phi \tag{9.7.4}$$

Invariant under a global transformation,

$$\phi \to e^{\mathrm{i}\Lambda}\phi \quad (\Lambda \to \text{ const.}) \tag{9.7.5}$$

If $\Lambda=\Lambda(x)$,

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)^2+\left[\left(\partial_\mu+\mathrm{i}eA_\mu\right)\phi^*\right]\left[\left(\partial^\mu-\mathrm{i}eA^\mu\right)\phi\right]-m^2\phi^*\phi \tag{9.7.6}$$

where

$$A_\mu \to A_\mu+\frac{1}{e}\partial_\mu\Lambda \tag{9.7.7}$$

$$D_\mu = \partial-\mathrm{i}eA_\mu \quad \to \text{ covariant derivative} \tag{9.7.8}$$

$$D_\mu\phi \to UD_\mu\phi \tag{9.7.9}$$

$$\begin{aligned} (D_\mu D_\nu-D_\nu D_\mu)\phi &\Rightarrow U(D_\mu D_\nu-D_\nu D_\mu)\phi \\ &= -U(\mathrm{ie})F_{\mu\nu}\phi \quad ; \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \end{aligned} \tag{9.7.10}$$

Advantage of looking at E.M. this way $\to$ can generalize,

$$\begin{bmatrix} p \\ n \end{bmatrix} = N \quad \to \quad \text{Heisenberg postulate} \tag{9.7.11}$$

$$N = N^i_\alpha \tag{9.7.12}$$

where α is the Dirac index, and $i=1,\ 2\ \ (1\equiv\ \uparrow,\ 2\equiv\ \downarrow$. The invariance is of the form (forgetting Dirac index);

$$N^i \to U^{ij}(\Lambda)N^j \tag{9.7.13}$$

where

$$U^{ij}(\Lambda) = \left(e^{\mathrm{i}\underline{\tau}\cdot\underline{\Lambda}/2}\right)^{ij} \tag{9.7.14}$$

and

$$\underline{\tau} = \frac{1}{2}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} \tag{9.7.15}$$

$\bar{N}^i \left(\mathrm{i}\, \partial\!\!\!/ \delta^{ij} - m\delta^{ij}\right) N^j$ is globally $SU(2)$ invariant. Suppose $\underline{\Lambda} = \underline{\Lambda}(x^\mu)$. Add in $A^{ij}_\mu = A^a_\mu(\tau^a)^{ij}$. So,

$$\begin{aligned} &\rightarrow \quad \bar{N}\left[\mathrm{i}\left(\partial\!\!\!/ - \frac{\mathrm{i}}{2} g\, A\!\!\!/\right) - m\right] N \quad ; \qquad A\!\!\!/ = A^a_\mu(\tau^a)_{ij}\gamma^\mu_{\alpha\beta} \\ &= \quad \bar{N}^i_\alpha\left[\mathrm{i}\left(\partial^\mu\gamma^\mu_{\alpha\beta}\delta^{ij} - \frac{\mathrm{i}}{2} g A^a_\mu(\tau^a)^{ij}\gamma^\mu_{\alpha\beta}\right) - m\delta^{ij}\delta_{\alpha\beta}\right] N^j_\beta \end{aligned} \tag{9.7.16}$$

If $N \rightarrow UN$, $\bar{N} \rightarrow \bar{N}U^{-1}$, then,

$$\begin{aligned} \bar{N}\mathrm{i}\left(\partial\!\!\!/ - \frac{\mathrm{i}g}{2}\, A\!\!\!/\right) N \quad &\rightarrow \quad \bar{N}\mathrm{i}U^{-1}\left[\partial\!\!\!/ - \frac{\mathrm{i}g}{2}\, A\!\!\!/\right] UN \\ &= \quad \bar{N}\gamma^\mu \mathrm{i}\left[U^{-1}\partial_\mu - \frac{\mathrm{i}g}{2} U^{-1}A_\mu\right] UN \\ &= \quad \mathrm{i}\bar{N}U^{-1}\gamma^\mu\left[\overbrace{(\partial_\mu U) + (U\partial)}^{\partial(UN)} - \frac{\mathrm{i}g}{2} A_\mu U\right] N \\ &= \quad \mathrm{i}\bar{N}\left[\gamma^\mu\right]\left[\partial_\mu - \frac{\mathrm{i}g}{2} A'_\mu\right] N \end{aligned} \tag{9.7.17}$$

(Notation: note that $(\partial_\mu A) = A_{,\mu}$ i.e. "," $\equiv$ derivative wrt. μ.) where,

$$\begin{aligned} U^{-1}(\partial_\mu U) - \frac{\mathrm{i}g}{2} U^{-1}A_\mu U \quad &= \quad -\frac{\mathrm{i}gA'_\mu}{2} \\ U^{-1}U_{,\mu} - \frac{\mathrm{i}g}{2} U^{-1}A_\mu U \quad &= \quad -\frac{\mathrm{i}g}{2} A'_\mu \end{aligned} \tag{9.7.18}$$

For $\Lambda^a \approx 0$,

$$U^{ij} \approx \delta^{ij} + \mathrm{i}\Lambda^a(\tau^a)^{ij} \tag{9.7.19}$$

$$U^{-1} = 1 - \mathrm{i}\underline{\Lambda}\cdot\underline{\tau} \tag{9.7.20}$$

$$\begin{aligned} -\frac{\mathrm{i}g}{2}\underline{A}'_\mu\cdot\underline{\tau} \quad &= \quad \left(1 - \mathrm{i}\underline{\tau}\cdot\underline{\Lambda}\right)\left(\mathrm{i}\underline{\tau}\cdot\underline{\Lambda}_{,\mu}\right) - \frac{\mathrm{i}g}{2}\left(1 - \mathrm{i}\underline{\tau}\cdot\underline{\Lambda}\right)\left(\underline{A}_\mu\cdot\underline{\tau}\right)\left(1 + \mathrm{i}\underline{\tau}\cdot\underline{\Lambda}\right) \\ &= \quad \mathrm{i}\underline{\tau}\cdot\underline{\Lambda}_{,\mu} - \frac{\mathrm{i}g}{2}\left[-\mathrm{i}\underline{\tau}\cdot\underline{\Lambda}\underline{\tau}\cdot\underline{A}_\mu + \mathrm{i}\underline{\tau}\cdot\underline{A}_\mu\underline{\tau}\cdot\underline{\Lambda}\right] \end{aligned} \tag{9.7.21}$$

But,

$$
\begin{aligned}
\underline{\tau}\cdot\underline{\Lambda}\ \underline{\tau}\cdot\underline{A}-\underline{\tau}\cdot\underline{A}\ \underline{\tau}\cdot\underline{\Lambda} &= \Lambda^a A^b_\mu\left(\tau^a\tau^b-\tau^b\tau^a\right)\\
&= \mathrm{i}\epsilon^{abc}\Lambda^a A^b_\mu\tau^c \quad (\underline{\tau}=\frac{1}{2}\underline{a})\\
&= \mathrm{i}\underline{\Lambda}\times\underline{A}\cdot\underline{\tau} \qquad (9.7.22)
\end{aligned}
$$

So,

$$
-\frac{\mathrm{i}g}{2}\underline{A}'_\mu\cdot\underline{\tau}=\mathrm{i}\underline{\tau}\cdot\underline{\Lambda}_{,\mu}-\frac{\mathrm{i}g}{2}\left[-\mathrm{i}(2\mathrm{i})(\underline{\Lambda}\times\underline{A}_\mu)\cdot\underline{\tau}\right] \qquad (9.7.23)
$$

Hence,

$$
\begin{aligned}
-\frac{\mathrm{i}g}{2}\underline{A}'_\mu &= \mathrm{i}\underline{\Lambda}^a_{,\mu}+\mathrm{i}\left(\frac{g}{2}\right)\left(\underline{A}_\mu\times\underline{\Lambda}\right)^a\\
\therefore A'^a_\mu &= -\frac{2}{g}\Lambda^a_{,\mu}-\epsilon^{abc}A^b_\mu\Lambda^c \qquad (9.7.24)
\end{aligned}
$$

In other words, we have 3 photon fields, 1 for each spin matrix. ($A_\mu\to A_\mu+\frac{1}{e}\partial_\mu\Lambda$).
Consider three scalars ϕ^a $(a=1,\ 2,\ 3)$.

$$
\mathcal{L}=\frac{1}{2}\left[(\partial_\mu\phi^a)(\partial^\mu\phi^a)-m^2\phi^a\phi^a\right] \qquad (9.7.25)
$$

For an invariance under $\phi^a\to\phi^a+g\epsilon^{abc}\phi^b\Lambda^c(x)$, for $\Lambda^a\approx 0$, then,

$$
\mathcal{L}=\left[\left(\partial_\mu\delta^{ab}+g\epsilon^{acb}A^c_\mu\right)\phi^b\right]\left[\left(\partial_\mu\delta^{ap}+g\epsilon^{aqp}A^q_\mu\right)\phi^p\right]-m^2\phi^a\phi^a \qquad (9.7.26)
$$

with

$$
A'^a_\mu=A^a_\mu+\partial_\mu\Lambda^a+g\epsilon^{abc}A^b_\mu\Lambda^c \qquad (9.7.27)
$$

(again, $SU(2)$, but $SU(2)$ rep. that is isomorphic to $SU(3)$). Define:

$$
D^{ab}_\mu=\partial_\mu\delta^{ab}+g\epsilon^{amb}A^m_\mu \qquad (9.7.28)
$$

(Analogous to $D\mu=\partial_\mu-\mathrm{i}eA_\mu$ $\to$ do the same thing $D_\mu D_\nu-D_\nu D_\mu=\mathrm{i}eF_{\mu\nu}$),

$$
\begin{aligned}
\mathcal{L} &= (D^{am}\phi^m)(D^{an}\phi^n)-m^2\phi^a\phi^a \qquad (9.7.29)\\
A'^a_\mu &= A^a_\mu+D^{ab}_\mu\Lambda^b \qquad (9.7.30)\\
\phi'^a &= \phi^a+g\epsilon^{abc}\phi^b\Lambda^c \qquad (9.7.31)
\end{aligned}
$$

Now,

$$
D^{ab}_\mu(A)\phi\to(\delta^{ab}+g\epsilon^{abc}\Lambda^c)(D^{bm}\phi^m) \qquad (9.7.32)
$$

under the gauge transformations.
Consider,

$$
(D^{ab}_\mu D^{bc}_\nu-D^{ab}_\nu D^{bc}_\mu)\phi\ \overset{\rightarrow}{(9.7.32)}\ (\delta^{ab}+g\epsilon^{abc}\Lambda^c)\left[(D^{bm}_\mu D^{mn}_\nu-D^{bm}_\nu D^{mn}_\mu)\phi^n\right] \qquad (9.7.33)
$$

But now,

$$\begin{aligned} D_\mu^{ab} D_\nu^{bc} - D_\nu^{ab} D_\mu^{bc} &= (\partial_\mu \delta^{ab} + g\epsilon^{apb} A_\mu^p)(\partial_\nu \delta^{bc} + g\epsilon^{bqc} A_\nu^q) \\ &\quad - (\partial_\nu \delta^{ab} + g\epsilon^{apb} A_\nu^p)(\partial_\mu \delta^{bc} + g\epsilon^{bqc} A_\mu^q) \\ &= g\epsilon^{abc}\left((\partial_\mu A_\nu^b) - (\partial_\nu A_\mu^b) + g\epsilon^{bmn} A_\mu^m A_\nu^n\right) \\ &\quad (\text{c.f.}\quad \epsilon^{abc}\epsilon^{bmn} = -\delta^{am}\delta^{cn} + \delta^{an}\delta^{cm}) \\ &= g\epsilon^{abc} F_{\mu\nu}^b \end{aligned} \tag{9.7.34}$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \quad (U(1) \text{ case}) \tag{9.7.35}$$

$$F_{\mu\nu}^a \rightarrow (\delta^{ab} + \epsilon^{abc}\Lambda^c) F_{\mu\nu}^b \tag{9.7.36}$$

Lagrangian for A_μ^a

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \tag{9.7.37}$$

(Gauge invariant).

Jan. 31/2000

Chapter 10

Quantizing Gauge Theories

We could look at canonical quantization → $A^a_\mu(x)$; Define:

$$\pi^{a\mu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\mu)} \tag{10.0.1}$$

- Identify constraints
- insert gauge conditions for each of the 1^{st} class constraints
- Form Dirac Brackets
- From these, determine commutators

<u>Problems</u> with this in practice are;

1. Manifest Lorentz invariance is lost (Important)
2. For the Coulomb Gauge (for example), where $\partial_i A^a_i(\underline{x}, t) = 0$, in forming the Dirac Brackets → will involve

$$\left[\partial_i (D^{ab}_\mu)\right]^{-1} \tag{10.0.2}$$

where $D^{ab}_\mu = \partial_i \delta^{ab} + \underbrace{g\epsilon^{apb} A^p_i}_{*}$ (" $*$ " part drops out in abelian case).

10.1 Quantum Mechanical Path Integral

Use the Q.M Path Integral (QMPI).

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}(A) F^{a\mu\nu}(A) + J^a_\mu A^{a\mu} \tag{10.1.1}$$

where for now, we will eliminate the second term $J^a_\mu A^{a\mu}$ → purely classical $\mathcal{L}$. Note that we have full gauge invariance $A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a + g\epsilon^{abc} A^b_\mu \Lambda^c$. Consider,

$$Z[J] = \int \mathcal{D}A^a_\mu \exp\left\{ \mathrm{i} \int d^4x \ \left(\mathcal{L} + J^a_\mu A^{a\mu}\right)\right\} \tag{10.1.2}$$

$$\begin{aligned} Z[J] &= \int \mathcal{D}A^a_\mu \exp\left\{\mathrm{i}\int d^4x \left[-\frac{1}{4}\Big(\overbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2}^{\mathcal{L}_0} + 2g\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\epsilon^{abc}A^b_\mu A^c_\nu \right.\right. \\ &\quad \left.\left. + g^2\epsilon^{amn}A^p_\mu A^q_\nu A^m_\mu A^n_\nu\Big) + J^a_\mu A^{a\mu}\right]\right\} \end{aligned} \qquad (10.1.3)$$

but, $\mathcal{L}_0 = \frac{1}{2}A^a_\mu\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A^a_\nu$, and,

$$\int \mathcal{D}A^a_\mu \exp\left\{\mathrm{i}\int d^4x\left[\frac{1}{2}A^a_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A^a_\nu + J^a_\mu A^{a\mu}\right]\right\} \sim \det{}^{(-1/2)}\left[\delta^{ab}(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\right] \qquad (10.1.4)$$

and,

$$\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)\left(\partial_\nu\Lambda\right) = 0 \qquad (10.1.5)$$

$\rightarrow \partial^\nu\Lambda$ is an eigenvector with vanishing eigenvalue! Hence,

$$\det\left[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right] = 0 \qquad (10.1.6)$$

(Dirac put in $-\frac{1}{2}\left[\partial_\mu A^{a\mu}\right]^2$ into (10.1.2), (in exponential integral) $\rightarrow$ eliminates $\partial^\mu\partial^\nu$ $\rightarrow$ but he found inconsistencies). To see the Faddeev-Popov procedure, consider,

$$\int d^n\underline{x}e^{-\underline{x}^T\underline{N}\underline{x}} = \frac{\pi^{n/2}}{\det^{1/2}}(\underline{N}) \qquad (10.1.7)$$

$$\left.\begin{array}{r} \rightarrow \det(\underline{N}) = \lambda_1\ldots\lambda_n \\ \rightarrow \int dx_1\ldots dx_n e^{-x_1^2\lambda_1}\ldots e^{-x_n^2\lambda_n} \end{array}\right\} \text{Only if } \lambda_i > 0\forall i \qquad (10.1.8)$$

Suppose,

$$\underline{N}\ \underline{m}(\underline{x})\underline{\lambda}_0 = 0 \qquad (10.1.9)$$

$\underline{m}(\underline{x})\underline{\lambda}_0 \rightarrow$ eigenvector corresponding to eigenvalue zero. Note that,

$$\int d^n\underline{y}\ \delta(\underline{A}\underline{y} + \underline{b})f(\underline{y}) \qquad (10.1.10)$$

Let $\underline{z} = \underline{A}\underline{y} + \underline{b}$, so $d^n\underline{y} = \frac{1}{\det\underline{A}}d^n\underline{z}$. So,

$$\begin{aligned} \int d^n\underline{y}\ \delta(\underline{A}\underline{y} + \underline{b})f(\underline{y}) &= \frac{1}{\det\underline{A}}\int d^n\underline{z}\ \delta^n(\underline{z})f(\underline{A}^{-1}(\underline{z} - \underline{b})) \\ &= \frac{f(-\underline{b})}{\det(\underline{A})} \end{aligned} \qquad (10.1.11)$$

Thus,

$$\delta(\underline{A}\underline{y} + \underline{b}) = \frac{\delta(\underline{y} + \underline{A}^{-1}\underline{b})}{\det(\underline{A})} \qquad (10.1.12)$$

(c.f. 1-D → $\delta(ax) = \frac{1}{|a|}\delta(x)$). Consider

$$1 = \int d\underline{\lambda}_0 \; \delta\left[\underline{L}(\underline{x} + \underline{m}(\underline{x})\underline{\lambda}_0)\right] \; \det(\underline{L} \; \underline{m}(\underline{x})) \tag{10.1.13}$$

Insert this into

$$\int d^n\underline{x} \; e^{-\underline{x}^T\underline{N} \; \underline{x}} = \int d^n\underline{x} \int d\underline{\lambda}_0 \delta\left[\underline{L}(\underline{x} - \underline{m}(\underline{x})\underline{\lambda}_0)\right] \; \det\left[\underline{L} \; \underline{m}(\underline{x})\right] e^{-\underline{x}^T\underline{N} \; \underline{x}} \tag{10.1.14}$$

Let $\underline{x} \to \underline{x} - \underline{m}(\underline{x})\underline{\lambda}_0$. Then,

$$\begin{aligned}
\int d^n\underline{x} \; e^{-\underline{x}^T\underline{N} \; \underline{x}} &= \int d^n\underline{x} \int d\underline{\lambda}_0 \; \delta(\underline{L} \; \underline{x}) \; \det\left[\underline{L} \; \underline{m}(\underline{x})\right] e^{-\underline{x}^T\underline{N} \; \underline{x}} \\
&\quad \text{(note no change in } \; e^{-\underline{x}^T\underline{N} \; \underline{x}} \; \text{ because } \; \underline{N}(\underline{m} \; \underline{\lambda}_0) = 0) \\
&= \int d\underline{\lambda}_0 \int d^n\underline{x} \; \delta(\underline{L} \; \underline{x}) \; \det\left(\underline{L} \; \underline{m}(\underline{x})\right) e^{-\underline{x}^T\underline{N} \; \underline{x}}
\end{aligned} \tag{10.1.15}$$

(The $\int d\underline{\lambda}_0$ contains the ∞ occurring in the integral over the eigenvector with vanishing eigenvalue). Note:

1. $\det\left[\underline{L} \; \underline{m}(\underline{x})\right] = \int d\underline{c} d\underline{\bar{c}} \; e^{\underline{\bar{c}}^T(\underline{L} \; \underline{m}(\underline{x}))\underline{c}}$(where $\underline{c}$, $\underline{\bar{c}}$ → Grassmann vectors).
2. $\delta(\underline{L} \; \underline{x}) = \lim_{\alpha\to 0} \sqrt{\frac{\pi}{\alpha}} e^{-\alpha(\underline{L} \; \underline{x})^2}$

So,

$$\begin{aligned}
\int d^n\underline{x} e^{-\underline{x}^T\underline{N} \; \underline{x}} &= \int d\underline{\lambda}_0 \int d^n\underline{x} \; \lim_{\alpha\to 0} \sqrt{\frac{\pi}{\alpha}} e^{-\underline{x}^T\underline{N} \; \underline{x}} \int d\underline{c} \; d\underline{\bar{c}} \; e^{\underline{\bar{c}}^T\underline{L} \; \underline{m}(\underline{x})\underline{c} - \alpha(\underline{L} \; \underline{x})^2} \\
&= \underbrace{\int d\underline{\lambda}_0 \lim_{\alpha\to 0} \sqrt{\frac{\pi}{\alpha}}}_{\text{absorb into Normalizing factor}} \int d^n\underline{x} \; d\underline{c} \; d\underline{\bar{c}} \; e^{\left[-\underline{x}^T\left(\underline{N} + \alpha\underline{c}^T \; \underline{L}\right)\underline{x} + \underline{\bar{c}}^T\underline{L} \; \underline{m}(\underline{x})\underline{c}\right]}
\end{aligned} \tag{10.1.16}$$

- Choose $\underline{L}$ so that $(\underline{N} + \alpha\underline{L}^T\underline{L})$ is invertible (i.e. $(\underline{N} + \alpha\underline{L}^T\underline{L})$ has no vanishing eigenvalue).
- $\underline{\bar{c}}^T\underline{L} \; \underline{m}(\underline{x})\underline{c}$ cancels the contribution of eigenvectors with vanishing eigenvalues ("ghost fields").

Feb. 2/2000

The condition is $\underline{N}(\underline{m}(\underline{x})\underline{\lambda}_0) = 0$

$$\begin{aligned}
1 &= \int d\underline{\lambda}_0 \delta(\underline{L}(\underline{x} - \underline{m}(\underline{x})\underline{\lambda}_0)) \det(\underline{L} \; \underline{m}(\underline{x})) \\
&\quad \underline{x} \to \underline{x} - \underline{m}(\underline{x})\underline{\lambda}_0 \\
&= \int d\underline{\lambda}_0 \int d^n\underline{x} \; \underbrace{\delta(\underline{L} \; \underline{x})}_{A} \underbrace{\det(\underline{L} \; \underline{m}(\underline{x}))}_{B} e^{-\underline{x}^T\underline{N} \; \underline{x}}
\end{aligned}$$

$$\begin{aligned} A &\rightarrow \lim_{\alpha\rightarrow\infty}\sqrt{\frac{\pi}{\alpha}}e^{-(\underline{L}\ \underline{x})^2/\alpha} \\ B &\rightarrow \int d\underline{c}d\underline{\bar{c}}\ e^{-\underline{\bar{c}}^T\underline{L}\ \underline{m}(\underline{x})\underline{c}} \end{aligned}$$

10.2 Gauge Theory Quantization

Recall the path integral

$$\int \mathcal{D}A^a_\mu(x)e^{\int d^4x\ \left(-\frac{(F^a_{\mu\nu}(x))^2}{4}+J^a_\mu A^{a\mu}\right)} \tag{10.2.1}$$

If $J=0$,

$$\begin{aligned} A^a_\mu &\rightarrow A^a_\mu + D^{ab}_\mu(A)\Omega^b(x) \qquad (10.2.2) \\ \text{(Analogy } \underline{x} &\leftrightarrow A^a_\mu(x)\ \text{ from above)} \\ \text{With } D^{ab}_\mu(A) &= \partial_\mu\delta^{ab} + g\epsilon^{apb}A^b_\mu \end{aligned}$$

Insert a factor of "1".

$$\begin{aligned} 1 &= \int d\Omega^a(x)\ \delta(\underbrace{\partial_\mu}_{\underline{L}}(\underbrace{A^a_\mu(x)}_{\underline{x}} + \underbrace{D^{ab}_\mu(A)\Omega^b}_{\underline{m}(\underline{x})\underline{\lambda}_0}))\det(\underbrace{\partial_\mu D^{ab}_\mu(A)}_{\underline{L}\underline{m}(\underline{x})}) \qquad (10.2.3) \\ &= \int d\Omega^a(x)\ \delta(\partial^\mu A^{\Omega_a}_\mu(x))\det(\partial^\mu D^{ab}_\mu(A)) \end{aligned}$$

Where,

$$\begin{aligned} \rightarrow A^{\Omega_a}_\mu(x) &= U^{-1}(\Omega)(A^a_\mu+\partial_\mu)U(\Omega)\ \ ; \quad U(\Omega)\approx\delta^{ab}+\epsilon^{apb}\Omega^p \\ &\approx A^a_\mu + D^{ab}_\mu(A)\Omega^b \end{aligned}$$

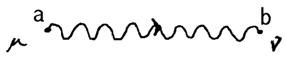
Let $A^{\Omega_a}_\mu \rightarrow A^a_\mu$ (reverse gauge transf. $\rightarrow -\Omega$)

$$\begin{aligned} 1 &= \int d\Omega^a \int dA^a_\mu\ \delta(\partial^\mu A^a_\mu)\det(\partial^\mu D^{ab}_\mu(A)e^{\mathrm{i}\int d^4x\ \left(-\frac{1}{4}(F^a_{\mu\nu})^2\right)} \\ &= \underbrace{\lim_{\alpha\rightarrow 0}\sqrt{\frac{\pi}{2\alpha}}\int d\Omega^a}_{\substack{\text{Absorb into}\\ \text{Normalization}}}\int dA^a_\mu\int dc^a d\bar{c}^a\ \exp\left\{\mathrm{i}\int d^4x\ \left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}-\frac{1}{2\alpha}(\partial_\mu A^{a\mu})^2-\bar{c}^a\partial^\mu D^{ab}_\mu(A)c^b\right]\right\} \end{aligned} \tag{10.2.4}$$

Recall that the c's are grassmann. Also, note that, in the last term, $D^{ab}_\mu(A)=\partial_\mu\delta^{ab}+g\epsilon^{apb}A^p_\mu$, and so ghost fields are important (ghosts are free fields) → Mathematical constructs, Longitudinal mode of vector particle. Note that in the $U(1)$ case (gauge group), the Ghost Lagrangian is $\bar{c}\partial_\mu\partial^\mu c = \bar{c}\partial^\nu c$.

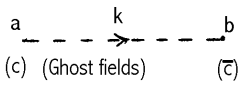
10.3 Feynman Rules

Here are the Feynman Rules.

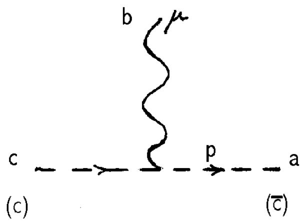


$$-i\delta^{ab}\left[\frac{(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2})}{k^2}+\frac{\alpha k_\mu k_\nu}{k^4}\right]$$

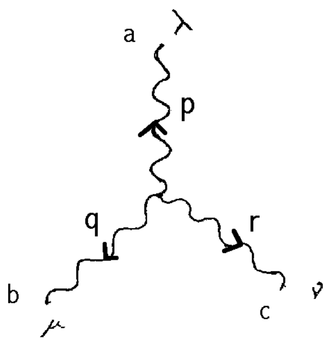
Where the last term is the α dependance - (longitudinal part of propagator).



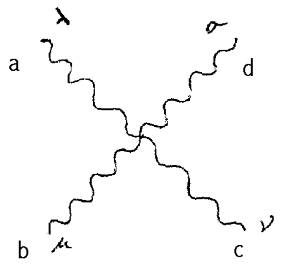
$$-\frac{i\delta^{ab}}{k^2}$$



$$g\epsilon^{abc}p^{\mu}$$



$$-g\epsilon^{abc}\left\{(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}\right\}$$



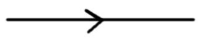
$$
\begin{aligned}
-\mathrm{i}g^2 \quad &\left\{f^{abc}f^{cde}\left(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu}\right)\right. \\
&\quad f^{abe}f^{bde}\left(g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu}\right) \\
&\quad \left. f^{ade}f^{cbe}\left(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\mu}g_{\sigma\nu}\right)\right\}
\end{aligned}
$$

Feb. 3/2000

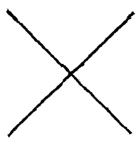
10.4 Radiative Corrections

$$
\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda\phi^4}{4!} \tag{10.4.1}
$$

and recall the Feynman rules are,



$\frac{\mathrm{i}}{p^2-m^2}$



$-\mathrm{i}\lambda$

Recall that, in general,

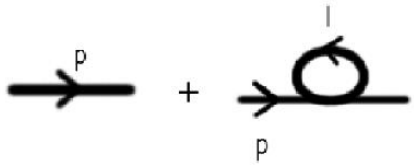
$$
\int \frac{d^n\ell}{(2\pi)^n}\frac{(\ell^2)^a}{(\ell^2-m^2)^b} = \frac{\mathrm{i}(-1)^{a-b}}{(4\pi)^{n/2}}\frac{\Gamma\left(a+\frac{n}{2}\right)\Gamma\left(a-b-\frac{n}{2}\right)(m^2)^{b-a+n/2}}{\Gamma(b)\Gamma\left(\frac{n}{2}\right)} \tag{10.4.2}
$$

$$
\to \int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}}{(\ell^2-m^2)} = \frac{\mathrm{i}(-1)}{(4\pi)^{n/2}}\underbrace{\Gamma\left(1-\frac{n}{2}\right)}_{\text{pole at } n=4}(m^2)^{n/2-1} \tag{10.4.3}
$$

Also, in general,

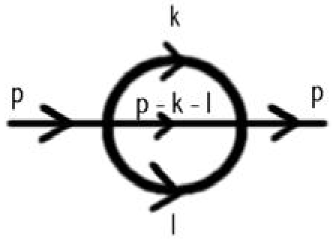
$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xa+(1-x)b]^{\alpha+\beta}} \tag{10.4.4}$$

The 1PI contribution to the 2-pt. function is



$$= \frac{\mathrm{i}}{p^2-m^2} + \left(\frac{\mathrm{i}}{p^2-m^2}\right)\frac{1}{2}(-\mathrm{i}\lambda)\int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}}{(\ell^2-m^2)}\left(\frac{\mathrm{i}}{p^2-m^2}\right) \tag{10.4.5}$$

Which can be easily integrated. Another loop integral is (dropping external leg contribution for now);



$$\begin{aligned}
&= \frac{(-\mathrm{i}\lambda)^2}{6}\int \frac{d^nk\ d^n\ell}{(2\pi)^{2n}} \frac{1}{(k^2-m^2)(\ell^2-m^2)[(p-k-\ell)^2-m^2]} \\
&= \frac{(-\mathrm{i}\lambda)^2}{6}\int \frac{d^nk\ d^n\ell}{(2\pi)^{2n}} \frac{1}{(\ell^2-m^2)}\int_0^1 dx \frac{1}{[(1-x)(k^2-m^2)+x((p-k-\ell)^2-m^2)]^2} \\
&= \frac{(-\mathrm{i}\lambda)^2}{6}\int \frac{d^nk\ d^n\ell}{(2\pi)^{2n}} \frac{1}{(\ell^2-m^2)}\int_0^1 dx \frac{1}{[k^2-2xk(p-\ell)+x(p-\ell)^2-m^2]^2} \\
&= \frac{(-\mathrm{i}\lambda)^2}{6}\int \frac{d^nk\ d^n\ell}{(2\pi)^{2n}} \frac{1}{(\ell^2-m^2)}\int_0^1 dx \frac{1}{[(\underbrace{k-x(p-\ell)}_{k'})^2+x(1-x)(p-\ell)^2-m^2]} \\
&= \frac{(-\mathrm{i}\lambda)^2}{6}\int \frac{d^nk'\ d^n\ell}{(2\pi)^{2n}} \frac{1}{(\ell^2-m^2)}\int_0^1 dx \frac{1}{[k'^2+x(1-x)(p-\ell)^2-m^2]^2}
\end{aligned}$$

The integral over k' can be evaluated; recall (10.4.2) (*Note: General integral correct in*

(10.4.2), but check signs on a's and b's).

$$= \frac{(-\mathrm{i}\lambda)^2}{6}\int_0^1 dx \int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}}{(4\pi)^{n/2}}\frac{\Gamma\left(2-\frac{n}{2}\right)}{(\ell^2-m^2)(m^2-x(1-x)(p-\ell)^2)^{2-n/2}}$$

$$= \frac{(-\mathrm{i}\lambda)^2}{6}\int_0^1 dx \int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}}\frac{1}{(\ell^2-m^2)}(-x(1-x))^{-2+n/2}\frac{1}{\left((\ell-p)^2-\frac{m^2}{x(1-x)}\right)^{2-n/2}}$$

Now, recall (10.4.4).

$$= \frac{(-\mathrm{i}\lambda)^2}{6}\int_0^1 dx \int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}}(-x(1-x))^{-2+n/2}\frac{\Gamma\left(1+2-\frac{n}{2}\right)}{\Gamma(1)\Gamma\left(2-\frac{n}{2}\right)}\cdot$$
$$\int dy \frac{y^{1-1}(1-y)^{2-n/2-1}}{\left[y(\ell^2-m^2)+(1-y)\left((\ell-p)^2-\frac{m^2}{x(1-x)}\right)\right]^{3-n/2}}$$

$$= \int_0^1 dx\int_0^1 dy \int \frac{d^n\ell}{(2\pi)^n}\frac{\mathrm{i}\Gamma\left(3-\frac{n}{2}\right)(-x(1-x))^{-2+n/2}(1-y)^{1-n/2}}{(4\pi)^{n/2}\left[\ell^2+y(1-y)p^2-m^2y-\frac{m^2(1-y)}{x(1-x)}\right]^{3-n/2}}$$

$$= \int_0^1 dx\int_0^1 dy \frac{\mathrm{i}}{(4\pi)^{n/2}}\Gamma\left(3-\frac{n}{2}\right)(-x(1-x))^{-2+n/2}(1-y)^{1-n/2}\frac{\mathrm{i}}{(4\pi)^{n/2}}(-1)^{-3+n/2}\Gamma\left(3-\frac{n}{2}-\frac{n}{2}\right)\cdot$$
$$\left[-y(1-y)p^2+m^2y+\frac{m^2(1-y)}{x(1-x)}\right]^{n/2+n/2-3}$$

$$= \int_0^1 dx\int_0^1 dy\left(\frac{\mathrm{i}}{(4\pi)^{n/2}}\right)^2\Gamma\left(3-\frac{n}{2}\right)\Gamma(3-n)(-1)^{-5+n}(x(1-x))^{-2+n/2}(1-y)^{1-n/2}\cdot$$
$$\left[-y(1-y)p^2+m^2y+\frac{m^2(1-y)}{x(1-x)}\right]^{n-3}$$

Now, the integrals over x, y are problems. But, if $m^2\to 0$,

$$= \left(\frac{\mathrm{i}}{(4\pi)^{n/2}}\right)^2\int_0^1 dx\int_0^1 dy\Gamma\left(3-\frac{n}{2}\right)\Gamma(3-n)(-(-1)^n)(x(1-x))^{-2+n/2}(1-y)^{1-n/2}\cdot$$
$$(-1)^{n-3}\left(y(1-y)p^2\right)^{n-3}$$

$$= \left(\frac{\mathrm{i}}{(4\pi)^{n/2}}\right)^2\int_0^1 dx\int_0^1 dy\Gamma\left(3-\frac{n}{2}\right)\Gamma(3-n)(x(1-x))^{-2+n/2}y^{n-3}(1-y)^{n/2-2}(p^2)^{n-3}$$

Now, using (10.4.4), we have $a=b=1$, so

$$= \left(\frac{\mathrm{i}}{(4\pi)^{n/2}}\right)^2\left[\Gamma\left(3-\frac{n}{2}\right)\underbrace{\Gamma(3-n)}_{\text{Pole at } n=4}\right]\left[\frac{\Gamma^2\left(\frac{n}{2}-1\right)}{\Gamma(n-2)}\right]\left[\frac{\Gamma(n-2)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{3n}{2}-3\right)}\right](p^2)^{n-3}$$

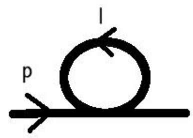
The $(p^2)^{n-3}$ can be expanded as $(1 + (n-3)\ln(p^2) + \ldots)$, and recall,

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + \ldots \tag{10.4.6}$$

Thus, the 2-loop integral is

$$= \frac{A}{2} + \left(B\ln(p^2) + C\right) + \mathcal{O}(\varepsilon) \tag{10.4.7}$$

($\ln(p^2)$ is singular if $p^2 = m^2 = 0$). At one loop,



$$\begin{aligned} &\propto \int \frac{d^n\ell}{(2\pi)^n}\frac{1}{(\ell^2 - m^2)} \\ &= \frac{A}{2} + B \end{aligned} \tag{10.4.8}$$

B is independent of p^2 (this is a fluke).

Feb. 7/2000

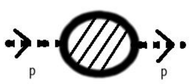
$$\left\{\sigma^{\mu\nu}, \sigma^{\lambda\rho}\right\} = -2\left(g^{\mu\rho}g^{\lambda\nu} - g^{\mu\lambda}g^{\rho\nu}\right) + 2\mathrm{i}\epsilon^{\mu\nu\lambda\rho}\gamma_5 \tag{10.4.9}$$

where $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right]$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and

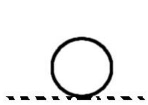
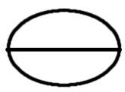
$$\gamma_5 = \frac{1}{4!}\epsilon^{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho \tag{10.4.10}$$

Let

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{\mu_0^2\phi_0^2}{2} - \frac{\lambda_0\phi^4}{4!} \tag{10.4.11}$$



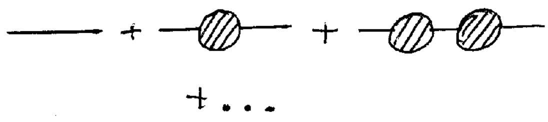
$= -\mathrm{i}\Sigma(p)$

$=$  $+$  $+$  $+ \ldots$

$=$ Sum of all 2pt 1PI diagrams.

Let

$$\langle 0|T\phi_0(x)\phi_0(y)|0\rangle_{\text{connected}} =$$



$$\begin{aligned}
&= \frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon} + \frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon} \\
&\quad + \frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon} + \dots \\
&= a + ar + ar^2 + \dots \\
&= \frac{a}{1-r^2} \\
&= \frac{\frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon}}{1-(-\mathrm{i}\varepsilon)\frac{\mathrm{i}}{p^2-\mu_0^2+\mathrm{i}\varepsilon}} \\
&= \frac{\mathrm{i}}{p^2-\mu_0^2-\Sigma(p^2)+\mathrm{i}\varepsilon} \\
&= \mathrm{i}\Delta(p^2)
\end{aligned} \tag{10.4.12}$$

Eliminate the divergence (at least to one loop order).
Follow treatment by Lurié
"on-shell" Renormalization scheme $\to$ μ^2 is the mass of the field $\phi(x)$ ($p^2 = \mu^2 \to$ "on shell")

$$\Sigma(p^2) = \underbrace{\Sigma(\mu^2)}_{\substack{\text{When } p^2=\mu^2 \\ \text{diverges}}} + (p^2-\mu^2)\underbrace{\Sigma'(\mu^2)}_{\substack{\text{log} \\ \text{divergence}}} + \underbrace{\text{Converging terms}}_{\Sigma_c(p^2)} \tag{10.4.13}$$

where $\Sigma'(p^2) = \frac{d}{dp^2}\Sigma(p^2)$. Thus,

$$\mathrm{i}\Delta(p^2) = \frac{\mathrm{i}}{\underbrace{p^2-\mu_0^2-\Sigma(\mu^2)}_{*} - (p^2-\mu^2)\Sigma'(\mu^2) - \Sigma_c(p^2)} \tag{10.4.14}$$

* - let $\mu^2 = \mu_0^2 + \Sigma(\mu^2)$ ($\mu^2 \to$ finite, μ_0^2, $\Sigma(\mu^2)$ diverge if $n = 4$). This is mass renormalization, eliminates quadratic divergences. (c.f. recall figure 10.4.1, from previous notes from QFTI). Thus,

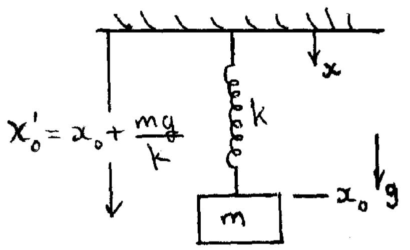


Figure 10.4.1: Recall mass on string example. x_0 gets renormalized by m_g. $x_0' = x_0 + \frac{mg}{k}$

$$\begin{aligned} i\Delta(p^2) &= \frac{i}{(p^2-\mu^2)(1-\Sigma'(\mu^2)) - \Sigma_c(p^2)} \\ &\quad \to \text{ Pole occurs at } p^2 = \mu^2 \\ &= \frac{i\left(\frac{1}{1-\Sigma'(\mu^2)}\right)}{p^2-\mu^2-\frac{\Sigma_c(p^2)}{1-\Sigma'(\mu^2)}} \\ &\simeq \frac{i\left(\frac{1}{1-\Sigma'(\mu^2)}\right)}{p^2-\mu^2-\Sigma_c(p^2)} \quad \text{to lowest order in } \lambda_0 \end{aligned} \tag{10.4.15}$$

Let

$$Z_\phi = \frac{1}{1-\Sigma'(\mu^2)} \approx 1 + \Sigma'(\mu^2) \ \ \text{(log divergent)} \tag{10.4.16}$$

$$\begin{aligned} \to i\Delta(p^2) &= \frac{iZ_\phi}{p^2-\mu^2-\Sigma_c(p^2)} \\ &= \text{F.T.}\{\langle 0|T\phi_0(x)\phi_0(y)|0\rangle_{\text{connected}}\} \end{aligned} \tag{10.4.17}$$

(F.T. = Fourier Transf.). Now let

$$\phi(x) = Z_\phi^{-1/2}\phi_0(x) \tag{10.4.18}$$

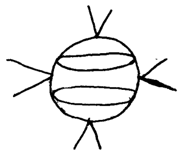
(re-scaling of ϕ_0 field in $\mathcal{L}$). Hence:

$$\text{F.T.}\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{i}{p^2-\mu^2-\Sigma_c(p^2)} \ \ \text{(Finite)} \tag{10.4.19}$$

What diagrams do diverge?
The superficial degree of divergence $= D$. For any diagram,

$$\begin{aligned} &\sim (g)^{\#} d^4\ell_1 \dots d^4\ell_L \, \frac{1}{\ell_1^2-\mu^2} \cdots \frac{1}{\ell_L^2-\mu^2} \quad ; \quad (g^{\#}) \to \text{ coupling const.} \\ &= 4L - 2I \text{ (only if } g \text{ is dimensionless)} \end{aligned} \tag{10.4.20}$$

- $L = \#$ of loops
- $I = \#$ of internal lines
- $V = \#$ of vertices



$$= \langle 0|T\phi_1 \ldots \phi_4|0\rangle \tag{10.4.21}$$

$$4V = 2I + E \quad (E = \ \# \text{ of external legs}) \tag{10.4.22}$$

The 4 is in front of the V because each vertex has 4 lines going out. Check: (figure 10.4.2) In figure 10.4.2, $L = 2,\ I = 3,\ V = 2$. Thus,

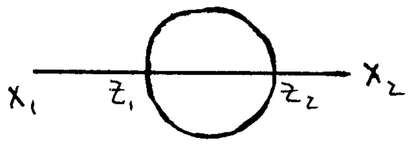


Figure 10.4.2: E = 2, I = 3, V = 2

$$\begin{aligned} 4(2) &= 2(3) + 2 \\ 8 &= 6 + 2 \ \ \text{Check} \end{aligned} \tag{10.4.23}$$

Also,

$$\begin{aligned} L &= I - (V - 1) \\ &= I - V + 1 \end{aligned} \tag{10.4.24}$$

Hence,

$$\begin{aligned} D &= 4L - 2I \\ 4V &= 2I + E \rightarrow 2I = 4V - E \rightarrow \therefore 4I - 8V = -2E \\ \therefore D &= 4L - (4V - E) \\ &= 4(I - V + 1) - 4V + E \\ &= 4I - 8V + 4 + E \\ &= -2E + 4 + E \\ D &= 4 - E \rightarrow \text{ Depends only on E!!} \end{aligned} \tag{10.4.25}$$

Thus, $D = 4 - E$ as λ is dimensionless.

- $D = 4$ if $(E = 0) \rightarrow$ not possible
- $D = 3$ if $(E = 1) \rightarrow$ not possible $\phi \rightarrow -\phi$, $\therefore L$ even in ϕ. There must be a symmetry, $\therefore E \neq 1$
- $D = 2$ if $(E = 2) \rightarrow$ Quadratic divergence + Logarithmic
- $D = 0$ if $(E = 4) \rightarrow$ Log divergence
- $D < 0$ if $(E > 4) \rightarrow$ Converges (Higher pt. diagrams are ok).

Now, consider for example a ϕ^p theory: Use the following scalar field theory in field theory in four dimensions

$$L = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^p \tag{10.4.26}$$

(where $p \geq 3$ is a positive integer) to explain the difference between theories that are: renormalizable, nonrenormalizable and super-renormalizable. (Simply consider the (superficial) degree of divergence of the Feynman diagrams).
The superficial "degree of divergence" is given by

$$D = 4L - 2I \tag{10.4.27}$$

where

- $L = \#$ of loops
- $I = \#$ of internal lines
- $E = \#$ of external lines
- $V = \#$ of vertices

Each loop momentum k has a volume element d^4k associated with it, and each (scalar) internal line is associated with a propagator, which for large $|\not{k}|$ behaves like $|\not{k}|^{-2}$. Now, each vertex has p lines emerging from it, and each internal line removes two of these, so

$$E = pV - 2I \tag{10.4.28}$$

and the number of loops is given by

$$L = I - V + 1 \tag{10.4.29}$$

(only $V - 1$ conservation constraints, due to overall momentum conservation). So, subs. (10.4.29) into (10.4.27),

$$\begin{aligned} D &= 4(I - V + 1) - 2I \\ &= 4I - 4V + 4 - 2I \\ &= 2I - 4V + 4 \;; \qquad \text{Subs. in (10.4.28)} \\ &= pV - E - 4V + 4 \\ &= 4 - E + (p - 4)V \end{aligned} \tag{10.4.30}$$

The conditions are:

- If $D < 0 \rightarrow$ super-renormalizable
- if $D = 0 \rightarrow$ renormalizable
- if $D > 0 \rightarrow$ non-renormalizable

So, as can be seen from (10.4.30), if $p < 4$, then D will become negative very quickly as V increases, so $p = 3$ theories would be super-renormalizable; for $p = 4$, we get the usual ϕ^4 theory, which is renormalizable; and for $p > 4$, the final term will fast outweigh the first two, D will be positive as more and more vertices are added, and the theory will be (superficially) non-renormalizable.

Feb. 9/2000

Recall:

$$\text{F.T.}\langle 0|T\phi_0(x)\phi_0(y)|0\rangle = \frac{\mathrm{i}Z_\phi}{p^2 - \mu^2 - \Sigma_c(p^2)} \tag{10.4.31}$$

where

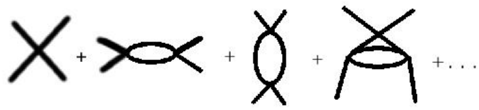
$$\Sigma_c(\mu^2) = 0 \tag{10.4.32}$$

$$\phi_0 = Z_\phi^{1/2}\phi \tag{10.4.33}$$

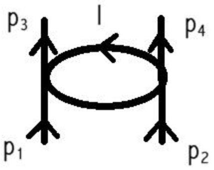
$$\text{F.T.}\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{\mathrm{i}}{p^2 - \mu^2 - \Sigma_c(p^2)} \tag{10.4.34}$$

The four-point function:

$$\text{F.T.}\langle 0|T\phi_0(x_1)\ldots\phi_0(x_4)|0\rangle =$$



$$= -\mathrm{i}\lambda_0 + \Gamma(t) + \Gamma(s) + \Gamma(u) + \ldots \tag{10.4.35}$$



$$= \frac{1}{2}(-\mathrm{i}\lambda_0)^2 \int \frac{d^n\ell}{(2\pi)^n}\left(\frac{\mathrm{i}}{\ell^2 - \mu_0^2}\right)\left(\frac{\mathrm{i}}{(\ell + p_1 - p_3)^2 - \mu_0^2}\right)$$

$$= \Gamma\left((p_1 - p_3)^2\right)$$

$$= \Gamma(t) \tag{10.4.36}$$

(recall the Mandelstem Variables:)

- $s = (p_1 + p_2)^2$
- $t = (p_1 - p_3)^2$

- $u = (p_1 - p_4)^2$

These variables satisfy:

$$\begin{aligned}
s+t+u &= p_1^2 - 2p_1p_3 + p_3^2 + p_1^2 + 2p_1p_2 + p_2^2 + p_1^2 - 2p_1p_4 + p_4^2 \\
&\quad \text{But } \; p_1 + p_1 = p_3 + p_4 \; , \quad p_i^2 = \mu^2 \\
&= 6\mu^2 + 2(-p_1p_3 + p_1p_2 - p_1p_4) \\
&= 6\mu^2 + 2[-p_1p_3 + p_1p_2 - p_1(p_1 + p_2 - p_3)] \\
&= 6\mu^2 - 2p_1^2 \\
&= 4\mu^2
\end{aligned} \tag{10.4.37}$$

The "truncated" Fourier transform of $\langle 0|\phi_0(x_1)\ldots\phi_0(x_4)|0\rangle$ means the external legs are removed. Hence,

$$\text{Truncated F.T.}\langle 0|\phi_0(x_1)\ldots\phi_0(x_4)|0\rangle = -\mathrm{i}\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u) \tag{10.4.38}$$

where

$$\Gamma(s) \sim \int \frac{d^4\ell}{\ell^4} \sim \int \frac{dx}{x} \to \text{ log divergent} \tag{10.4.39}$$

(c.f. $D = 4 - E \ldots E = 4 \to D = 0$). Somehow must get rid of divergences in $\Gamma(s)$, $\Gamma(t)$, $\Gamma(u)$. Expand $\Gamma(s)$, $\Gamma(t)$, $\Gamma(u)$ about $s = \frac{4\mu^2}{3}$, $t = \frac{4\mu^2}{3}$, $u = \frac{4\mu^2}{3}$.

$$\begin{aligned}
\text{Truncated F.T.}\langle 0|\phi_0(x_1)\ldots\phi_0(x_4)|0\rangle &= -\mathrm{i}\lambda_0 + 3\Gamma\left(\frac{4\mu^2}{3}\right) + \underbrace{\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)}_{\text{Finite}} \\
&\quad \to \text{ Now let } \; -\mathrm{i}Z_\lambda^{-1}\lambda_0 = -\mathrm{i}\lambda_0 + 3\Gamma\left(\frac{4\mu^2}{3}\right) \\
&= -\mathrm{i}Z_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)
\end{aligned} \tag{10.4.40}$$

What we want to compute is

$$\langle 0|T\phi(x_1)\ldots\phi(x_4)|0\rangle \quad (\phi \text{ are Renormalized fields}) \tag{10.4.41}$$

But

$$\phi(x) = Z_\phi^{-1/2}\phi_0(x) \tag{10.4.42}$$

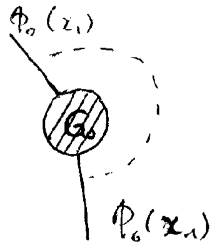
Recall

$$\text{F.T.}\langle 0|T\phi_0(x)\phi_0(y)|0\rangle = \frac{\mathrm{i}Z_\phi}{p^2 - \mu^2 - \Sigma_c(\mu^2)} \tag{10.4.43}$$

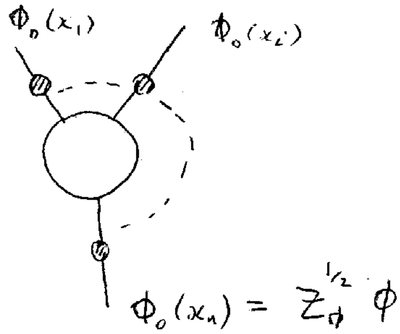
Now:

$$G_0(x_1,\dots,x_4) = \text{F.T.}\langle 0|\phi_0(x_1)\dots\phi_0(x_4)|0\rangle$$

$$=$$



$$=$$



(where the circle in the line in the second diagram $= \frac{-\mathrm{i}Z_\phi}{p_i^2-\mu^2}$. $\rightarrow \Gamma_0(x_1,\dots,x_n) =$ Truncated n-pt. unrenormalized G.F.

$$\begin{aligned} G(x_1,\dots,x_n) &= \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \\ &= Z_\phi^{-n/2}\langle 0|T\phi_0(x_1)\dots\phi_0(x_n)|0\rangle \\ &= Z_\phi^{-n/2}G_0(x_1\dots x_n) \end{aligned} \tag{10.4.44}$$

But then

$$\Gamma(x_1,\dots,x_n) = Z_\phi^{n/2}\Gamma_0(x_1,\dots,x_n) \tag{10.4.45}$$

We have

$$\text{F.T. } \Gamma_0(x_1,\dots,x_n) = -\mathrm{i}Z_\lambda^{-1}\lambda_0 + \tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u) \tag{10.4.46}$$

Thus,

$$\text{F.T. } \Gamma(x_1,\dots,x_n) = Z_\phi^2\Big[-\mathrm{i}Z_\lambda^{-1}\lambda_0 + \overbrace{\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)}^{\mathcal{O}(\lambda_0^2)}\Big] \tag{10.4.47}$$

(recall that $Z_\phi^2 \sim \big(1+\frac{\lambda_0}{2}\big)^2$). To order λ_0^2,

$$\text{F.T. } \Gamma(x_1,\dots,x_4) \approx -\mathrm{i}Z_\phi^2 Z_\lambda^{-1}\lambda_0 + \tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u) \tag{10.4.48}$$

Let $\lambda = Z_\phi^2 Z_\lambda^{-1} \lambda_0$ (Renormalized Coupling - where Z_ϕ^2 arises from 2 pt. function, and Z_λ^{-1} arises from 4 pt. function - $Z_\phi^2 \to \infty$, $Z_\lambda^{-1} \to \infty$ (cancel)). So,

$$\text{F.T. } \Gamma(x_1, \ldots, x_4) = -\mathrm{i}\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \quad \text{(Finite)} \tag{10.4.49}$$

aside

$$\begin{aligned} \text{F.T.}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \frac{\mathrm{i}}{p^2 - \mu^2 - \Sigma_c(p^2)} \\ &= \text{F.T. } G(x_1, x_2) \qquad (10.4.50) \\ \text{F.T. } \Gamma(x_1, x_2) &= \frac{p^2 - \mu^2 - \Sigma_c(p^2)}{\mathrm{i}} \qquad (10.4.51) \end{aligned}$$

end aside

10.5 Divergences at Higher orders

10.5.1 Weinberg's Theorem

A Feynman integral is convergent if its degree of divergence is negative and the degree of divergence of any integral associated with a subdiagram is also negative. Recall: ϕ^4: $D = 4 - E$

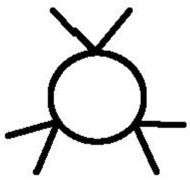


Figure 10.5.1: Convergence

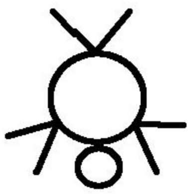


Figure 10.5.2: Subdiagram is divergent: $\therefore$ whole diagram is divergent

In figure 10.5.3,

1. All subdiagrams have $D < 0$

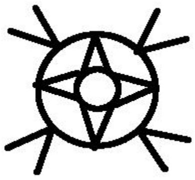


Figure 10.5.3:

2. $D < 0$ for entire diagram

W.Th. $\Rightarrow$ Overall diagram convergent! (i.e. no unanticipated divergences will appear).

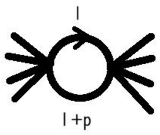
Thus, all divergences can be eliminated by an iterative procedure (i.e. first eliminate divergences in subdiagrams, and then in the diagrams as a whole (if there are any)).

Feb. 10/2000

Consider

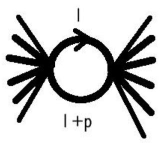
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda\phi^6}{6!} \qquad (4-D) \tag{10.5.1}$$

What goes wrong? Consider:



$$\sim \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{\ell^2 - m^2}\right)\left(\frac{1}{(\ell+p)^2 - m^2}\right) \sim \text{ divergent} \tag{10.5.2}$$

We can add in $-\frac{\lambda_8\phi^8}{8!}$ to $\mathcal{L}$ to absorb this divergence. Once we have $\lambda\phi^8$, we can examine the 12-point function:



$$\sim \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{\ell^2 - m^2}\right)\left(\frac{1}{(\ell+p)^2 - m^2}\right) \tag{10.5.3}$$

We now need $-\frac{\lambda_{12}\phi^{12}}{12!}$ in order to absorb this new divergence. This goes on and on - in total, there will be an infinite number of vertices to absorb all of these divergences.

Consider a diagram with

$n_i \rightarrow$ # of vertices of type i

$b_i \rightarrow$ # of Bosons in the i^{th} vertex

$f_i \rightarrow$ # of Fermions in the i^{th} vertex

$d_i \rightarrow$ # of derivatives in the i^{th} vertex

$B \rightarrow$ # of external Bosons

$I_B \rightarrow$ # of internal Bosons

$F \rightarrow$ # of external Fermions

$I_F \rightarrow$ # of internal Fermions

$L \rightarrow$ # of loops

$D \rightarrow$ degree of divergence

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi\partial_\mu\phi \rightarrow b &= 1 \\ f &= 2 \\ d &= 1 \end{aligned}$$

$$\begin{aligned} \bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} \rightarrow b &= 1 \\ f &= 2 \\ d &= 1 \end{aligned}$$

(where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$) So,

$$\begin{aligned} 2I_B + B &= \sum_i n_i b_i \\ 2I_F + F &= \sum_i n_i f_i \\ \text{\# of loops } L &= I_B + I_F - \sum_i n_i + 1 \end{aligned}$$

(where the "1" is present in L because one of the δ-functions is superfluous).

$$D = 4L - 2I_B - I_F + \sum_i n_i d_i$$

Now eliminate L, I_B, I_F.

$$D = 4 - B - \frac{3}{2}F + \sum_i n_i \delta_i \tag{10.5.5}$$

where $\delta_i = b_i + \frac{3}{2}f_i + d_i - 4$.

The i^{th} vertex:

$$\mathcal{L}_I = g \int \underbrace{d^4x}_{p^{-4}} \underbrace{\psi_1 \ldots \psi_{f_i}}_{p^{3f_i/2}} \underbrace{\partial_{\mu_1} \ldots \partial_{\mu_{d_i}}}_{p^{d_i}} \underbrace{\phi_1 \ldots \phi_{b_i}}_{p^{b_i}} \tag{10.5.6}$$

This means that the dimension of g must be $p^{-\delta_i}$ in order to keep overall dimensionless. (recall:

$$\int \underbrace{d^4x}_{p^{-4}} \underbrace{\bar{\psi}}_{p^{3/2}} \underbrace{\partial\!\!\!/}_{p^{+1}} \underbrace{\psi}_{p^{3/2}}$$

→ dimensionless). With a dimensionless coupling constant, $\delta_i = 0$ and

$$D = 4 - B - \frac{3}{2}F \tag{10.5.8}$$

and then there are only a certain number of divergent diagrams.

For $\delta_i > 0$, the degree of divergence increases with more vertices of type n_i (BAD). (ex. $\lambda\phi^6$ in 4 dimensions). → NON-RENORMALIZABLE THEORY!

$\delta_i < 0$; ex. $\lambda\phi^3$ in 4 dimensions - $\delta_i = 3 - 4 = -1 < 0$ → Only a finite # of divergent diagrams. In fact:

are the only fundamental divergent diagrams. → SUPERRENORMALIZABLE.

This (above) treatment is a bit "primitive" - we are only counting powers of momenta in diagrams to determine divergence.

In Gauge theories, there is a cancellation of divergences between 2 different diagrams. (See for example Figure 10.5.4)

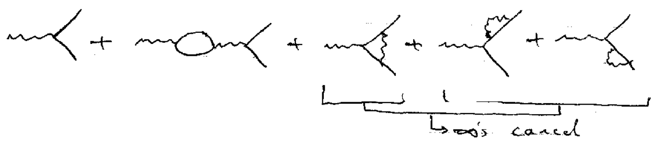
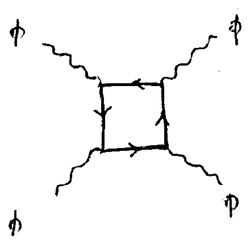


Figure 10.5.4: Example of gauge theories where (here) the divergences in last 3 diagrams cancel

For example

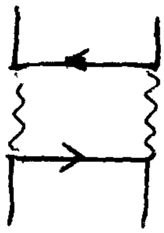
$$\mathcal{L} = \bar{\psi}(\not{p} - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 - g\bar{\psi}\psi\phi - \frac{\lambda}{4!}\phi^4 \tag{10.5.9}$$

(with g dimensionless). There is still a diagram that is divergent:



$$\int \frac{d^4\ell}{(\not{\ell})^4} \sim \text{ log divergent} \tag{10.5.10}$$

We require $-\frac{\lambda}{4!}\phi^4$ term to absorb this. The following diagram



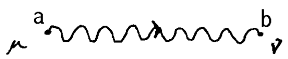
$$- \kappa(\bar{\psi}\psi)^2 \tag{10.5.11}$$

is convergent. $\left(\int \frac{d^4\ell}{\ell^6}\right)$

For massive vectors, the lagrangian contribution is

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 - \frac{m^2}{2} V_\mu V^\mu \tag{10.5.12}$$

Note that the last term isn't gauge invariant, and so we're stuck with a longitudinal polarization ($\left(\frac{k_\mu k_\nu}{m^2}\right)$) in the propagator; The propagator is:

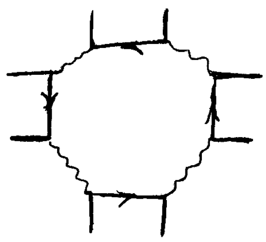


$$-\frac{\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right)}{k^2 - m^2 + \mathrm{i}\varepsilon} \tag{10.5.13}$$

The integral goes like $\sim (k^2)^0 \sim$ constant. This causes divergences.

$$\mathcal{L}_I = \bar{\psi}\gamma_\mu\psi V^\mu + g(\bar{\psi}\psi)^4 \tag{10.5.14}$$

Then we could have



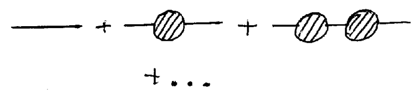
$$\int \frac{d^4\ell}{(\not{\ell})^4} \sim \text{ log divergent} \tag{10.5.15}$$

(Higgs Mechanism)

10.6 Renormalization Group

There is an arbitrariness inherent in renormalization. ex. Self-energy in ϕ^4:

$$\mathrm{i}\Delta(p) = \frac{\mathrm{i}}{p^2 - \mu_0^2 - \Sigma(p^2)} = \tag{10.6.1}$$



If we're just interested in eliminating the infinities, we could expand about some point κ^2. Let

$$\begin{aligned}
\Sigma(p^2) &= \underbrace{\Sigma(\kappa^2)}_{\substack{\text{Quadratic}\\ \text{divergence}}} + (p^2-\kappa^2)\underbrace{\Sigma'(\kappa^2)}_{\substack{\text{log}\\ \text{divergence}}} - \underbrace{\frac{1}{2!}(p^2-\kappa^2)\Sigma''(\kappa^2)}_{\substack{=\Sigma_c(p^2,\kappa^2)\\ \text{(converges)}}} + \ldots \\
&= \frac{1}{p^2 - \mu_0^2 + \Sigma(\kappa^2) - (p^2-\kappa^2)\Sigma'(\kappa^2) - \Sigma_c(p^2,\kappa^2)}
\end{aligned}$$

Wave function renormalization is done first:

$$= \frac{\frac{\mathrm{i}}{(1-\Sigma'(\kappa^2))}}{\left[p^2 - \left(\frac{\mu_0^2+\Sigma(\kappa^2)-\kappa^2}{1-\Sigma'(\kappa^2)}\right) - \underbrace{\frac{\Sigma_c(p^2,\kappa^2)}{(1-\Sigma'(\kappa^2))}}_{\approx\Sigma_c(p^2,\kappa^2)}\right]}$$

$$\text{Let}\quad \mu^2 = \frac{\mu_0^2 + \Sigma(\kappa^2) - \kappa^2}{1-\Sigma'(\kappa^2)}$$

$$\mathrm{i}\Delta(p) = \frac{\mathrm{i}Z_\phi}{p^2-\mu^2-\Sigma_c(p^2,\kappa^2)} \quad ; \quad Z_\phi = \frac{1}{1-\Sigma'(\kappa^2)} \sim 1 + \Sigma'(\kappa^2) \tag{10.6.2}$$

where $\kappa \rightarrow$ arbitrary. Note

1. $\mu^2 \rightarrow$ no longer the physical mass of ϕ ($\Sigma_c(\mu^2,\kappa^2) \neq 0, \quad \rightarrow \Sigma_c(\kappa^2,\kappa^2) = 0$). μ_p^2 is the physical mass, given by
$$\mu_p^2 - \mu^2 - \Sigma_c(\mu_p^2,\kappa^2) = 0 \tag{10.6.3}$$

2. How does something "physical" depend on κ?

See "New methods for the renormalization group", J.C. Collins and A.J. Macfarlane, Phys. Rev. D, Vol 10, Number 4, 15 August 1974.
Feb. 14/2000
Momentum subtraction $\rightarrow$ is difficult to work out in practice.

"Minimal Subtraction" in conjunction with dimensional regularization is the easiest way to renormalize. c.f. Collins and MacFarlane.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_0\right)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \tag{10.6.4}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{m^2}{2}\phi_0^2 \quad (m \rightarrow \text{ renormalized}) \tag{10.6.5}$$

$$\mathcal{L}_I = \underbrace{-\frac{(m_0^2 - m^2)}{2}\phi_0^2}\, - \frac{\lambda_0}{4!}\phi_0^4 \tag{10.6.6}$$

Again, we have,

$$\text{F.T.}\langle 0|T\phi_0\phi_0|0\rangle_{\text{Truncated}} =$$

$$-\mathrm{i}\Sigma(p^2) = \frac{1}{2}(-\mathrm{i}\lambda_0)\underbrace{\mu^{4-n}}_{*}\int \frac{d^n \ell}{(2\pi)^n}\frac{\mathrm{i}}{(\ell^2 - m^2)} + \left(\frac{-\mathrm{i}}{2}\right)(m^2 - m_0^2) \tag{10.6.7}$$

$* \rightarrow$ thus $\lambda_0 = \mu^{4-2n+n} = \mu^{4-n}$. Keep in mind, we're working in n dimensions. The Action is:

$$S = \int \underbrace{d^n x}_{a} \left(\frac{1}{2}(\underbrace{\partial_\mu}_{b}\, \phi_0)^2 - \frac{\lambda_0}{4!}\phi_0^4 \right) \tag{10.6.8}$$

Where the dimensions are: $[a] = \mu^{-n}$, $[b] = \mu^{+2}$. Since the dimension of the action must be zero, this means that the dimension of $[\phi_0] \sim \mu^{-(2+n)/2}$.
Note: as expected, if $n = 4$, λ_0 is dimensionless.

$$\begin{aligned}
-\mathrm{i}\Sigma(p^2) &= \frac{1}{2}(-\mathrm{i}\lambda_0)\mu^{4-n}\int \frac{d^n \ell}{(2\pi)^n}\frac{\mathrm{i}}{(\ell^2 - m^2)} + \frac{-\mathrm{i}}{2}(m^2 - m_0^2) \\
\Sigma(p) &= \frac{\mu^{4-n}\lambda_0}{2(4\pi)^{n/2}}\Gamma\left(1 - \frac{n}{2}\right)(m^2)^{-1+n/2} + \frac{1}{2}(m^2 - m_0^2) \\
&= \frac{\lambda_0 m^2}{(4\pi)^{n/2}}\Gamma\left(1 - \frac{n}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{-2+n/2} + \frac{1}{2}(m^2 - m_0^2)
\end{aligned}$$

Let $n \rightarrow 4$

$$= \frac{\lambda_0 m^2}{(4\pi)^{n/2-2+2}}\frac{\Gamma\left(2 - \frac{n}{2}\right)}{\left(1 - \frac{n}{2}\right)}\left(\frac{m^2}{\mu^2}\right)^{-2+n/2} + \frac{1}{2}(m^2 - m_0^2)$$

Let $\varepsilon = 2 - \frac{n}{2}$ ($\varepsilon \to 0$ as $n \to 4$).

$$\begin{aligned}
&= \frac{\lambda_0 m^2}{(4\pi)^2} \underbrace{(1+\varepsilon\ln(4\pi)+\ldots)}_{(4\pi)^{-n/2+2}} \underbrace{\left(\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right)}_{\Gamma\left(2-\frac{n}{2}\right)}(-1)\underbrace{\left(1+\varepsilon+\mathcal{O}(\varepsilon^2)\right)}_{\left(1-\frac{n}{2}\right)^{-1}} \cdot \\
&\quad \underbrace{\left(1-\varepsilon\ln\left(\frac{m^2}{\mu^2}\right)+\mathcal{O}(\varepsilon)\right)}_{\left(\frac{m^2}{\mu^2}\right)^{-2+n/2}} + \frac{1}{2}(m^2 - m_0^2) \\
&= \frac{\lambda_0 m^2}{(4\pi)^2}(-1)\left(\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m^2}{\mu^2}\right)\right) + \frac{1}{2}(m^2 - m_0^2) + \mathcal{O}(\varepsilon)
\end{aligned}$$

Let $m_0^2 = m^2\left(1 - \frac{2\lambda_0}{(4\pi)^2\varepsilon}\right)$.

$$\begin{aligned}
&= -\frac{\lambda_0}{(4\pi)^2}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m^2}{\mu^2}\right)\right] \\
&= \text{Finite}
\end{aligned} \tag{10.6.10}$$

(where the $\ln\left(\frac{m^2}{\mu^2}\right)$ is a residual dependence on renormalization).

We could also have absorbed the $\ln(4\pi)$, γ_E in to m_0^2, or μ^2. Let $\mu'^2 = \mu^2 e^{\gamma_E - \ln(4\pi)}$.

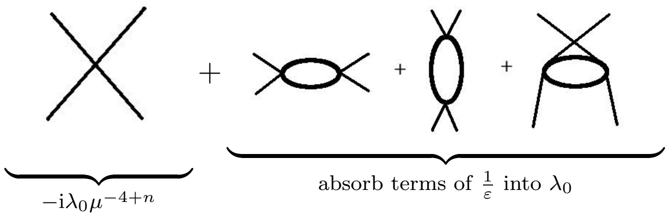
$$\to \ln\left(\frac{m^2}{\mu^2}\right) = \ln\left(\frac{m^2}{\mu'^2}\right) + \ln\left(e^{(\gamma_E - \ln(4\pi))}\right) \tag{10.6.11}$$

So we'd get

$$\Sigma(p) = -\frac{\lambda_0}{(4\pi)^2}\left(-\ln\left(\frac{m^2}{\mu'^2}\right)\right) \tag{10.6.12}$$

($\to$ modified minimal subtraction: gets rid of $\ln(4\pi)$ and γ_E.)

So also, for the four-point function:



(10.6.13)

Net Result:

$$\lambda_0 \mu^{-4+n} = \lambda + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda, m, \mu)}{(n-4)^\nu} \tag{10.6.14}$$

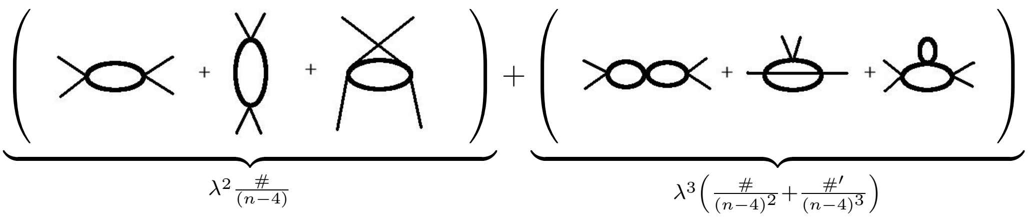
$$m_0 = m\left(1 + \sum_{\nu=1}^{\infty} \frac{b_\nu(\lambda, m, \mu)}{(n-4)^\nu}\right) = m Z_\nu \tag{10.6.15}$$

$$Z_\phi = 1 + \sum_{\nu=1}^{\infty} \frac{c_\nu(\lambda, m, \mu)}{(n-4)^\nu} \tag{10.6.16}$$

Note: a_ν, b_ν, c_ν in minimal subtraction are independent of m^2 and μ^2 in practice.
Note also:

$$a_i(\lambda) = \sum_{j=i+1}^{\infty} a_{ij} \lambda^j \qquad \begin{pmatrix} i = \# \text{ of powers of } \frac{1}{n-4} \\ j = \# \text{ of loops} \end{pmatrix} \tag{10.6.17}$$

i.e.



(10.6.18)

(and the sum goes higher, etc.). So also,

$$b_i = \sum_{j=i+1}^{\infty} b_{ij} \lambda^j \tag{10.6.19}$$

$$c_i = \sum_{j=i+1}^{\infty} c_{ij} \lambda^j \tag{10.6.20}$$

Note that $\frac{-\lambda_0 \phi_0^4}{4!} \to \lambda_0$ occurs in Feynman rules, but isn't finite. ex: see figure 10.6.1.

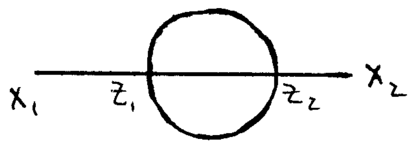


Figure 10.6.1: Overall divergence comes only partially from the diagram → λ_0 also has poles due to calculated graph at 1-loop order.

Net result:

$$\Gamma_R(p, m, \lambda, \mu) = \lim_{n \to 4} \tilde{\Gamma}_R(p, m(n), \lambda(n), \mu, n) \tag{10.6.21}$$

But now,

$$\begin{aligned}\rightarrow \tilde{\Gamma}_R(p, m(n), \lambda(n), \mu, n) &= \tilde{\Gamma}_R\left[p, \lambda\left(\lambda_0\mu^{n-4}, n\right) m_0 Z_m^{-1}\left(\lambda(\lambda_0)\right), \mu, n\right] \\ &= Z_\phi^{-N/2}\Gamma_0(p_i, m_0, \lambda_0, n) \end{aligned} \quad (10.6.22)$$

(N is the # of external legs - Note μ dependence vanishes. $\therefore$ can't be μ dependent on L.H.S.)

Feb. 16/2000

or

$$\tilde{\Gamma}_R = Z_\Gamma(\lambda_0\mu^{n-4}, n)\Gamma_0(p, \lambda_0(n), m_0(n), n) \quad (10.6.23)$$

with $Z_\Gamma = Z_\phi^{N/2}$.

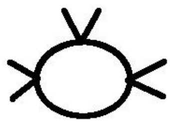
$$\begin{aligned}\rightarrow \mu\frac{d}{d\mu}\Gamma_0 &= 0 \\ &= \mu\frac{d}{d\mu}\left[\lim_{n\to 4} Z_\Gamma^{-1}\tilde{\Gamma}_R(p, \lambda(\partial_0\mu^{n-4}, n), m(\lambda_0\mu^{n-4}, n), \mu, n)\right] \\ 0 &= \left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + Z_m m\mu\frac{\partial Z_m^{-1}}{\partial\mu}\frac{\partial}{\partial m} - \mu\frac{\partial Z_\Gamma}{\partial\mu}Z_\Gamma^{-1}\right]\tilde{\Gamma}_R\end{aligned}$$

Now set $\left(\beta(\lambda) \equiv \mu\frac{\partial\lambda}{\partial\mu}\right)$, $\left(-\gamma_m m \equiv Z_m m\mu\frac{\partial Z_m^{-1}}{\partial\mu}\right)$, $\left(-\gamma_\Gamma \equiv -\mu\frac{\partial Z_\Gamma}{\partial\mu}Z_\Gamma^{-1}\right)$, and let $n \rightarrow 4$.

$$0 = \left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma_m m\frac{\partial}{\partial m} - \gamma_\Gamma\right)\Gamma \quad (10.6.24)$$

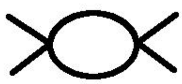
Note: as Γ is finite, β, γ_m and γ_Γ are finite.

Suppose we just look at the "engineering dimension" (depends only on the # of external legs), D_Γ, of $\Gamma = \Gamma(p, \lambda, m, \mu)$.



$D_\Gamma = -2$

$D_\Gamma = +2$



$D_\Gamma = 0$

Consider $p^\mu = \kappa p_0^\mu$ ($p_0^\mu \rightarrow$ reference momentum).

10.6.1 Euler's Theorem

$$f(\lambda x, \lambda y) \quad = \quad \lambda^D f(x,y) \quad \to \frac{d}{d\lambda} \text{ at } \lambda = 1 \tag{10.6.25}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f \quad = \quad D\lambda^{D-1}F \tag{10.6.26}$$

From this, if $\Gamma = \Gamma(\kappa p_0, \lambda, m, \mu)$, then

$$\left(\kappa\frac{\partial}{\partial \kappa} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu}\right)\Gamma = D_\Gamma \Gamma \tag{10.6.27}$$

Now, eliminate $\mu\frac{\partial}{\partial\mu}$ from (10.6.24), (10.6.27):

$$0 = \left[\kappa\frac{\partial}{\partial\kappa} - \beta(\lambda)\frac{\partial}{\partial\lambda} + (1+\gamma_m)m\frac{\partial}{\partial m} + (\gamma_\Gamma - D_\Gamma)\right]\Gamma(\kappa p_0, \lambda, m, \mu) \tag{10.6.28}$$

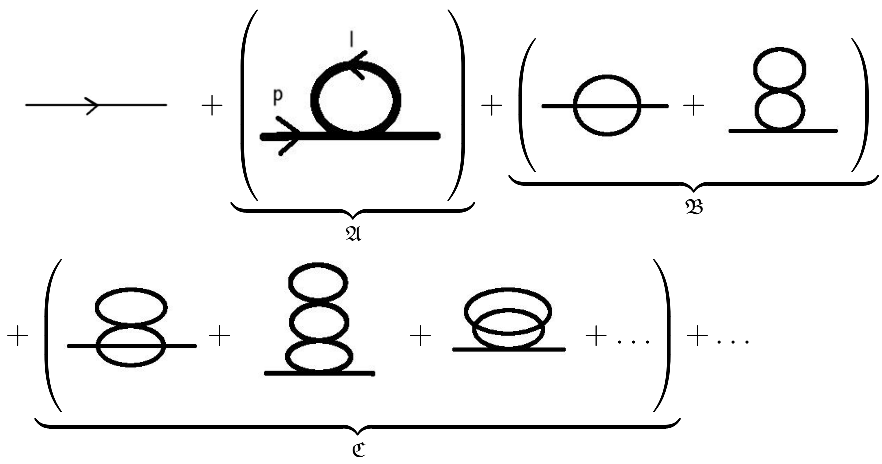
- Can trade in dependence on mass scale parameter μ for dependence on scale in front of momentum p_0.
- By rescaling momentum (changing κ) $\to$ will effectively change value of coupling constant/mass parameter (β, m).
- $\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}$, $-\gamma_m = \mu Z_m \frac{\partial Z_m^{-1}}{\partial\mu}$.

Formal Solution

$$\Gamma(\kappa p_0, \lambda, m, \mu) = \kappa^{D_\Gamma}\exp\left(-\int_1^\kappa \gamma_\lambda(\bar\lambda(\kappa'))\frac{d\kappa'}{\kappa'}\right)\Gamma(p_0, \bar\lambda(\kappa), \bar m(\kappa), \mu) \tag{10.6.29}$$

with $\left(\kappa\frac{\partial\bar\lambda(\kappa)}{\partial\kappa} = \beta(\bar\lambda(\kappa))\right)$, $\left(\kappa\frac{\partial \bar m(\kappa)}{\partial\kappa} = -\left[1+\gamma_m(\bar\lambda(\kappa))\right]\bar m(\kappa)\right)$ and with boundary conditions $\bar\lambda(1) = \lambda$, $\bar m(1) = m$.

For example, consider the 2-pt. function:



The net result of renormalization will be:

$$\mathfrak{A} = \lambda\left(A_1 \ln\left(\frac{p^2}{\mu^2}\right) + B_1\right) \tag{10.6.30}$$

$$\mathfrak{B} = A_2 \ln^2\left(\frac{p^2}{\mu^2}\right) + B_2 \ln\left(\frac{p^2}{\mu^2}\right) + C_2 \tag{10.6.31}$$

$$\mathfrak{C} = A_3 \ln^3\left(\frac{p^2}{\mu^2}\right) + B_3 \ln^2\left(\frac{p^2}{\mu^2}\right) + C_3 \ln\left(\frac{p^2}{\mu^2}\right) + D_3 \tag{10.6.32}$$

$\rightarrow$ we're just taking the leading order terms, $(A's)$, $\rightarrow$ all terms subsumed into $\bar{\lambda}(\kappa)$. Thus,

$$\begin{aligned} \kappa\frac{\partial\bar{\lambda}(\kappa)}{\partial\kappa} &= \beta(\bar{\lambda}(\kappa)) \\ &= \underbrace{\beta_1\lambda^2}_{\substack{\text{lowest}\\ \text{order}}} + \beta_2\lambda^3 + \beta_3\lambda^4 + \ldots \end{aligned} \tag{10.6.33}$$

For perturbation theory to make sense, $\bar{\lambda}(\kappa)$ must be small.

- If,

$$\lim_{\kappa\to\infty} \bar{\lambda}(\kappa) = 0 \tag{10.6.34}$$

  we have asymptotic freedom $\rightarrow$ (Yang-Mills Theory)

- If

$$\lim_{\kappa\to 0} \bar{\lambda}(\kappa) = 0 \tag{10.6.35}$$

  we have infrared freedom $\rightarrow$ (almost everything else).

Feb. 17/2000
Recall

$$\kappa\frac{\partial\bar{\lambda}(\kappa)}{\partial\kappa} = \beta(\bar{\lambda}(\kappa)) \tag{10.6.36}$$

$$\kappa\frac{\partial\bar{m}(\kappa)}{\partial\kappa} = -\left[1 + \gamma_m(\bar{\lambda}(\kappa))\right]\bar{m}(\kappa) \tag{10.6.37}$$

<u>In Dimensional Regularization</u>

$$\lambda_0\mu^{n-4} = \lambda + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda)}{(n-4)^\nu} \tag{10.6.38}$$

The μ^{n-4} was introduced so that λ is dimensionless.

$$\begin{aligned} \lambda_0 &\rightarrow \mu^{4-n} \\ \beta(\lambda) &= \mu\frac{\partial\lambda}{\partial\mu} \end{aligned}$$

Now, how do we get $\beta(\lambda)$?

$$\beta(\lambda) = x_0 + x_1(n-4) + x_2(n-4)^2 + \ldots \tag{10.6.39}$$

$$\mu\frac{\partial}{\partial\mu}(\lambda_0\mu^{n-4}) = \underbrace{\mu\frac{\partial\lambda}{\partial\mu}}_{\beta(\lambda)}\frac{\partial}{\partial\lambda}\left[\lambda + \sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(n-4)^\nu}\right] \tag{10.6.40}$$

Subs. in $\beta(\lambda)$

$$(n-4)\underbrace{\lambda_0\mu^{n-4}}_{\text{subs. (10.6.38)}} = \left[x_0 + x_1(n-4) + x_2(n-4)^2 + \ldots\right]\left[1 + \sum_{\nu=1}^{\infty}\frac{a'_\nu(\lambda)}{(n-4)^\nu}\right]$$

$$(n-4)\left[\lambda + \sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(n-4)^\nu}\right] = \left[x_0 + x_1(n-4) + \ldots\right]\left[1 + \sum_{\nu=1}^{\infty}\frac{a'_\nu(\lambda)}{(n-4)^\nu}\right]$$

Thus,

$$\begin{aligned}(n-4)\lambda + a_1 + \frac{a_2}{(n-4)} + \frac{a_3}{(n-4)^2} + \ldots &= \left[x_0 + x_1(n-4) + x_2(n-4)^2 + \ldots\right] + \\ &+ \left[\frac{x_0a'_1}{(n-4)} + x_1a'_1 + x_2a'_1(n-4) + \ldots\right] \\ &+ \left[\frac{x_0a'_2}{(n-4)^2} + \frac{x_1a'_2}{(n-4)} + x_2a'_2 + \ldots\right] + \ldots\end{aligned}$$

We set $x_2 = x_3 = x_4 = \ldots = 0$.

$$\begin{aligned}(n-4)\lambda + a_1 + \frac{a_2}{(n-4)} + \ldots &= x_1(n-4) + (x_0 + x_1a'_1) \\ &= +(x_0a'_1 + x_1a'_2)\frac{1}{(n-4)} \\ &\quad +(x_0a'_2 + x_1a'_3)\left(\frac{1}{(n-4)}\right)^2\end{aligned} \tag{10.6.41}$$

Matching powers of $(n-4)$:

$$\left\{\begin{aligned} x_1 &= \lambda \\ x_0 + x_1a'_1 &= a_1\end{aligned}\right\} \quad \begin{aligned} x_1 &= \lambda \\ x_0 &= a_1 - \lambda a'_1\end{aligned} \quad \to \beta(\lambda) = [a_1 - \lambda a'_1] + \lambda(n-4)$$

$$\left\{\begin{aligned} x_0a'_1 + x_1a'_2 &= a_2 \\ &\vdots \\ x_0a'_\nu + x + 1a'_{\nu+1} & \end{aligned}\right\} \quad \begin{array}{c}\text{constraint eq's that fix } a_2,\ a_3,\ \ldots, a_\nu \\ \text{in terms of } a_1\end{array}$$

(Note as $n \to 4$, $\beta(\lambda) = (a_1 - \lambda a'_1)$). Thus,

$$\lambda_0\mu^{n-4} = \lambda\sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(n-4)^\nu} \tag{10.6.42}$$

- only a_1 is needed for $\beta(\lambda)$
- a_1 fixes a_2, a_3, ...

Remember that $\lambda\phi^4$ in 4-D has the form:

$$a_1(\lambda) = \underbrace{a_{12}}_{\substack{\text{From 1-loop}\\\text{diagrams}}}\lambda^2 + \underbrace{a_{13}}_{\text{2-loop}}\lambda^3 + \underbrace{a_{14}}_{\text{3-loop}}\lambda^4 + \ldots \tag{10.6.43}$$

Recall that:

$$a_\nu = \sum_{j=\nu+1}^{\infty} a_{\nu j}\lambda^j \tag{10.6.44}$$

ex: for

$$\begin{aligned} a_2 &= x + 0a_1' + x_1 a_2' \\ a_{23}\lambda^3 + a_{24}\lambda^4 + \ldots &= \Big[\big(a_{12}\lambda^2 + a_{13}\lambda^3 + \ldots \\ &\quad -\lambda\frac{d}{d\lambda}\left(a_{12}\lambda^2 + a_{13}\lambda^3 + \ldots\right)\Big)\frac{d}{d\lambda}\left(a_{12}\lambda^2 + a_{13}\lambda^3 + \ldots\right)\Big] \\ &\quad + \lambda\left[\frac{d}{d\lambda}\left(a_{23}\lambda^3 + a_{24}\lambda^4 + \ldots\right)\right] \end{aligned}$$

Here, a_{2j} can be solved for in terms of a_{1j}. Similarly, we can show that, if

$$Z_m = 1 + \sum_{\nu=1}^{\infty}\frac{b_\nu}{(n-4)^\nu} \quad , \quad Z_\phi = 1 + \sum_{\nu=1}^{\infty}\frac{c_\nu}{(n-4)^\nu} \tag{10.6.45}$$

Then (***)

$$\gamma_m(\lambda) = \frac{\lambda b_1'}{m} \tag{10.6.46}$$

$$\gamma_\Gamma = \lambda c_1' \tag{10.6.47}$$

(c.f. Collins & MacFarlane)

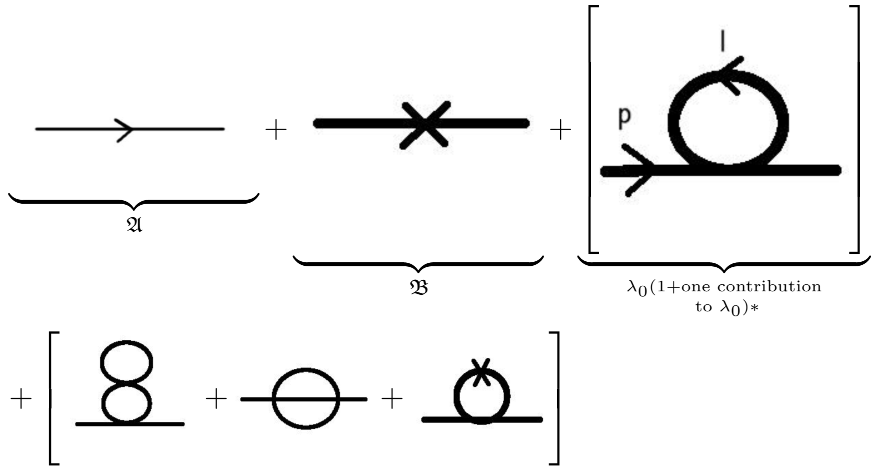
10.6.2 Explicit Calculations

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \tag{10.6.48}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{10.6.49}$$

$$\mathcal{L}_I = -\frac{1}{2}(m_0^2 - m^2)\phi^2 - \frac{\lambda_0\phi^4}{4!} \tag{10.6.50}$$

Two point function



(* - because it is itself renormalized). We find that, to 1-loop order,

$$S_F^{-1} = \overbrace{(p^2 - m^2)}^{\mathfrak{A}} - \overbrace{m^2\left(\frac{b_{11}\lambda}{(n-4)}\right)}^{\mathfrak{B}} - \overbrace{\frac{m^2\lambda^2}{16\pi^2(n-4)} + \text{ finite }}^{\text{1-loop contrib.}} + (\text{two-loop}) \tag{10.6.51}$$

i.e.

$$m_0^2 = m^2\left(1 + \sum_{\nu=1}^{\infty}\frac{b_\nu(\lambda)}{(n-4)^\nu}\right) \tag{10.6.52}$$

(we just have to take lowest order term. $(b_2(\lambda) = b_{11}\lambda + b_{12}\lambda^2 + \ldots)$. $\rightarrow$ S_F^{-1} will be finite if we choose

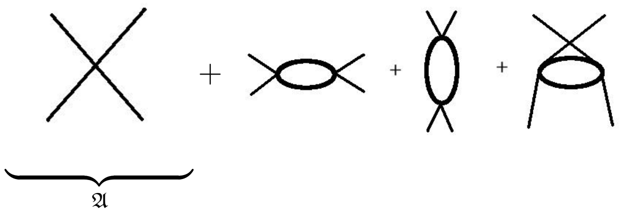
$$\left[b_{11} = \frac{-\lambda}{16\pi^2}\right] \tag{10.6.53}$$

Note, no divergence $\propto$ p^2 (could be, in principle, but it just doesn't occur in ϕ^4 theory).

$\therefore$ $Z_\phi = 1$ to one-loop order

<u>For the 4-pt. function</u>

Γ =



+ (two-loop)

$$= \lambda_0 + \frac{\lambda_0^2}{(n-4)}\left(\frac{3}{16\pi^2}\right) + (\text{finite}) \tag{10.6.54}$$

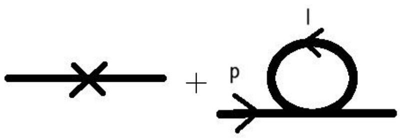
but,

$$\begin{aligned}\lambda_0 &= \lambda + \frac{a_{12}\lambda^2}{(n-4)} + \dots \\ \Gamma &= \lambda + \frac{a_{12}\lambda^2}{(n-4)} + \frac{\lambda^2}{(n-4)}\frac{3}{16\pi^2} + \dots\end{aligned}$$

→ arrange for pole from $\mathfrak{A}$ and from 1-loops to cancel

$$\Rightarrow \left[a_{12} = -\frac{3}{16\pi^2}\right] \tag{10.6.55}$$

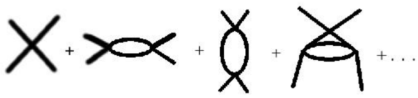
Feb. 28/2000
Recall:



$$= \quad -m^2\left(\frac{b_{11}\lambda}{(n-4)} + \frac{b_{12}\lambda^2}{(n-4)} + \dots\right) + \left(\dots\right)$$

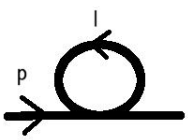
$$S_F^{-1} = p^2 - m^2\left(1 + \frac{b_{11}\lambda}{(n-4)}\right) - \frac{m^2\lambda}{16\pi^2(n-4)} + \text{ finite } + \mathcal{O}(\lambda^2) \tag{10.6.56}$$

where $b_{11} = -\frac{1}{16\pi^2}$, $Z = 1$.

$\Gamma \quad =$  $+ \mathcal{O}(\lambda^3)$

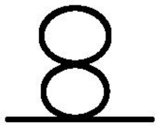
$$= \lambda + \frac{\lambda^2 a_{12}}{(n-4)} + \frac{3\lambda^2}{(n-4)16\pi^2} + \text{ finite } + \mathcal{O}(\lambda^3) \tag{10.6.57}$$

with $\Rightarrow a_{12} = -\frac{3}{16\pi^2}$. Higher order contributions to the 2-pt. function are:



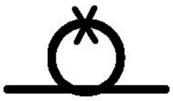
$$\begin{aligned} &= \frac{1}{2}\frac{1}{(4\pi)^{n/2}}\Gamma\left(1-\frac{n}{2}\right)(m^2)^{(n/2-1)}\mu^{-4+n}\left(\lambda + \frac{a_{12}\lambda^2}{(n-4)}\right) \\ &= \mathcal{O}(\lambda) + \frac{m^2\lambda}{32\pi^2}\left[\frac{2a_{12}\lambda}{(n-4)^2} + \frac{2+(\gamma-1)a_{12}\lambda}{(n-4)} + \text{ finite }\right] \end{aligned} \tag{10.6.58}$$

(where the vertex of the loop with the propagator is $\lambda_0 = \lambda + \frac{a_{12}\lambda^2}{(n-4)}$).



$$= -\frac{\mu^{8-4n}}{4(2\pi)^{2n}} \int d^n k \; d^n \ell \frac{1}{(k^2 - m^2)} \frac{1}{(\ell^2 - m^2)^2}$$

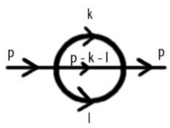
$$= \frac{m^2\lambda^2}{(16\pi^2)^2} \frac{1}{(n-4)^2} + \frac{(4\gamma - 2)m^2\lambda^2}{(32\pi^2)^2(n-4)} + \text{ finite} \qquad (10.6.59)$$



$$= \mathcal{O}(\lambda) + \frac{\lambda^2 m^2 a_{11}}{16\pi^2}\left(\frac{1}{(n-4)^2}\right) + \frac{\gamma b_{11}\lambda^2 m^2}{32\pi^2(n-4)} + \text{ finite} \quad (10.6.60)$$

$$\left(\text{Recall } \mathcal{L} \;=\; -\frac{1}{2}(m_0^2 - m^2)\phi^2 + \ldots\right)$$

(where the cross at the top of the loop is due to the mass insertion).



$$= \text{(Collins)}$$

$$= \frac{m^2\lambda^2}{(16\pi^2)^2}\frac{1}{(n-4)^2} + \frac{\lambda^2}{(16\pi^2)^2(n-4)}\left[\underbrace{\frac{p^2}{12}}_{*} - \frac{m^2}{2} + (\gamma - 1)m^2\right] + \text{finite} \quad (10.6.61)$$

($* \Rightarrow$ <u>wave function renormalization</u>)

All together (to 2-loop order - only showing two loop order here):

$$S_p^{-1} = p^2\left[1 - \frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{(n-4)}\right] - m^2\left[1 + \frac{\lambda^2}{(n-4)}\underbrace{\left(b_{12} - \frac{1}{2(16\pi^2)^2}\right)}_{\substack{\text{would think} \\ b_{12} = \frac{1}{2(16\pi^2)^2}}}\right.$$

$$\left. + \frac{\lambda^2}{(n-4)^2}\underbrace{\left(b_{22} - \frac{2}{(16\pi^2)^2}\right)}_{\substack{\text{and here} \\ b_{22} = \frac{2}{(16\pi^2)^2}}}\right] \qquad (10.6.62)$$

First!! Remember that

$$\text{FT } \langle 0|T\phi\phi|0\rangle \to \frac{Z_\phi}{(p^2 - m^2 - \text{ finite })} \qquad (10.6.63)$$

Thus,

$$S^{-1} = Z_\phi^{-1}\left[p^2 - m^2 - \text{ finite }\right] \tag{10.6.64}$$

Thus,

$$Z_\phi^{-1} = 1 - \frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{(n-4)} \tag{10.6.65}$$

So,

$$\begin{aligned} S^{-1} &= Z_\phi^{-1}\left[p^2 - m^2 - \text{ finite }\right] \\ &= \left(1 - \frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{(n-4)}\right)\left[p^2 - m^2\left(1 + \frac{\lambda^2}{(n-4)}\left(b_{12} - \frac{1}{2(16\pi^2)^2} + \frac{1}{12(16\pi^2)^2}\right)\right.\right. \\ &\quad \left.\left. + \frac{\lambda^2}{(n-4)^2}\left(b_{22} - \frac{2}{(16\pi^2)^2}\right)\right)\right] \end{aligned} \tag{10.6.66}$$

Thus,

$$Z_\phi = 1 + \frac{\lambda^2}{12(16\pi^2)}\frac{1}{(n-4)} \tag{10.6.67}$$

$$\left.\begin{array}{rcl} b_{22} &=& \frac{2}{(16\pi^2)^2} \\ b_{12} &=& \frac{5}{12(16\pi^2)^2} \end{array}\right\} m_0^2 = m^2\left[1 + \frac{1}{(n-4)}\left(\frac{-\lambda}{16\pi^2} + \frac{5\lambda^2}{12(16\pi^2)^2}\right.\right. \left.\left. + \frac{2\lambda^2}{(16\pi^2)^2}\frac{1}{(n-4)^2}\right) + \mathcal{O}(\lambda^3)\right] \tag{10.6.68}$$

$$\lambda_0 = \mu^{4-n}\left[\lambda - \frac{3\lambda^2}{16\pi^2}\frac{1}{(n-4)} + \mathcal{O}(\lambda^3)\right] \tag{10.6.69}$$

Thus,

$$a_1(\lambda) = -\frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \tag{10.6.70}$$

$$\begin{aligned} \beta(\lambda) &= \left(1 - \lambda\frac{\partial}{\partial\lambda}\right)a_1(\lambda) \\ &= \frac{3\lambda^2}{16\pi^2} \end{aligned} \tag{10.6.71}$$

Thus the running coupling is the solution to:

$$\begin{aligned} \kappa\frac{\partial\bar{\lambda}(\kappa)}{\partial\kappa} &= \beta(\bar{\lambda}(\kappa)) \\ \int_{\kappa_0}^{\kappa}\frac{d\kappa'}{\kappa'} &= \int_{\lambda}^{\bar{\lambda}(\kappa)}\frac{d\bar{\lambda}}{\beta(\bar{\lambda})} \end{aligned} \tag{10.6.72}$$

$\Bigg($ originally

$$\begin{aligned}
\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}\right)\Gamma(p,\mu,\lambda) &= 0(?) \\
\kappa &\rightarrow \Gamma(\kappa p_0,\mu,\lambda)\rightarrow\Gamma(p_0,\mu,\bar{\lambda}(\kappa)) \\
\left(\kappa\frac{\partial}{\partial\kappa}+\beta(\lambda)\frac{\partial}{\partial\lambda}\right)\Gamma &\overbrace{=}^{?} 0\Bigg)
\end{aligned} \tag{10.6.73}$$

The boundary conditions are: $\bar{\lambda}(\kappa_0)=\lambda$. So,

$$\begin{aligned}
\ln(\kappa')\Big|_{\kappa_0}^{\kappa} &= \int_{\lambda}^{\bar{\lambda}(\kappa)}\frac{d\bar{\lambda}}{\left(\frac{3\bar{\lambda}^2}{16\pi^2}\right)} \\
\ln\left(\frac{\kappa}{\kappa_0}\right) &= \frac{16\pi^2}{3}\left(-\frac{1}{\bar{\lambda}}\right)\Bigg|_{\lambda}^{\bar{\lambda}(\kappa)} \\
\frac{3}{16\pi^2}\ln\left(\frac{\kappa}{\kappa_0}\right) &= -\frac{1}{\bar{\lambda}(\kappa)}+\frac{1}{\lambda} \\
\bar{\lambda}(\kappa) &= \frac{1}{\frac{1}{\lambda}-\frac{3}{16\pi^2}\ln\left(\frac{\kappa}{\kappa_0}\right)} \\
\bar{\lambda}(\kappa) &= \frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\ln\left(\frac{\kappa}{\kappa_0}\right)}\approx\left\{\begin{array}{c}0^- \text{ as } \kappa\rightarrow\infty \text{ (unstable)}\\ 0^+ \text{ as } \kappa\rightarrow 0^+ \text{ (stable *)}\end{array}\right\}
\end{aligned} \tag{10.6.74}$$

$*\rightarrow$ i.e. perturbation theory acceptable for <u>small</u> momenta.
Thus, as $\kappa\rightarrow 0^+$, perturbation theory <u>reliable</u>.

$$\kappa\frac{\partial\bar{\lambda}(\kappa)}{\partial\kappa}=\beta(\bar{\lambda}(\kappa)) \tag{10.6.75}$$

The plot of this can be seen in Figure 10.6.2. If we include higher orders, we get Figure 10.6.2.

Mar. 1/2000

$$\kappa\frac{\partial\bar{\lambda}(\kappa)}{\partial\kappa}=\frac{3}{16\pi^2}\bar{\lambda}^2(\kappa)\rightarrow\int_{\lambda}^{\bar{\lambda}(\kappa)}\frac{d\bar{\lambda}'(\kappa)}{\lambda'^2(\kappa)}=\frac{3}{16\pi^2}\int_{\kappa_0}^{\kappa}\frac{d\kappa'}{\kappa'} \tag{10.6.76}$$

$$\bar{\lambda}(\kappa)=\frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\ln\left(\frac{\kappa}{\kappa_0}\right)} \tag{10.6.77}$$

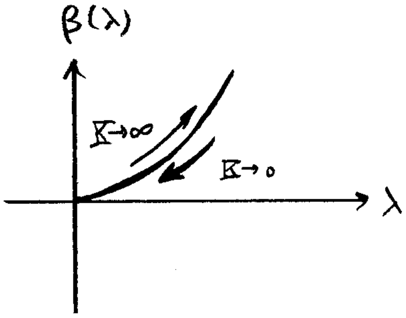


Figure 10.6.2: Beta function plot

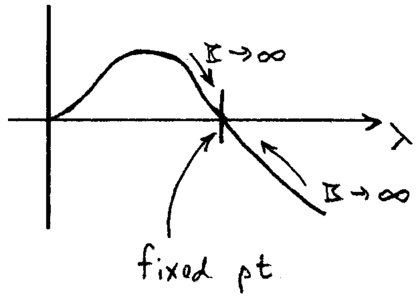


Figure 10.6.3: Beta function plot 2 - note $\beta = 0$!! - depends on where you start. ($\bar{\lambda}(\kappa) \to \lambda_{\text{fixed}}$ as $\kappa \to \infty$)

B.C. $\bar{\lambda}(\kappa_0) = \lambda$.
We often let $\lambda = \infty$ at $\kappa_0 = \Lambda$. i.e. $\bar{\lambda}(\Lambda) = \infty$. Thus,

$$\begin{aligned}
\int_{\infty}^{\bar{\lambda}(\kappa)} \frac{d\lambda'}{\lambda'^2} &= \frac{3}{16\pi^2} \ln(\kappa') \Big|_{\Lambda}^{\kappa} \\
-\frac{1}{\lambda'} \Big|_{\infty}^{\bar{\lambda}(\kappa)} &= \frac{3}{16\pi^2} \ln\left(\frac{\kappa}{\Lambda}\right) \\
&= -\frac{3}{16\pi^2} \ln\left(\frac{\Lambda}{\kappa}\right) \\
\bar{\lambda}(\kappa) &= \frac{16\pi^2}{3\ln\left(\frac{\Lambda}{\kappa}\right)} \qquad (10.6.78)
\end{aligned}$$

Note that:

- $\kappa \to 0$, $\bar{\lambda}(\kappa) \to 0^+$
- $\kappa \to \infty$, $\bar{\lambda}(\kappa) \to 0^-$ (unacceptable)
- $\kappa \approx \Lambda$, $\bar{\lambda}(\kappa) \to \infty \to$ perturbation breaks down.

<u>In QCD</u>

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}\left(\mathrm{i}\,\partial\!\!\!/ - g\tau^a \,A\!\!\!/^a\right)\psi + \phi_i^*\left(\partial_\mu - \mathrm{i}egT^aA^a_\mu\right)^2_{ij}\phi_j \qquad (10.6.79)$$

$$\beta(g) = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}C_2(v) + \frac{4}{3}C_2(F) + \frac{1}{3}C_2(S)\right] \qquad (10.6.80)$$

$$\left[T^a, T^b\right] = c^{abc}T^c \qquad (10.6.81)$$

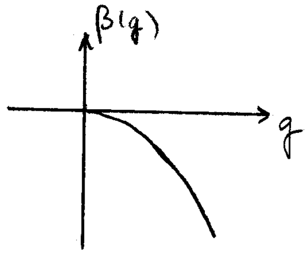


Figure 10.6.4: Beta function plot for pure Yang-Mills.

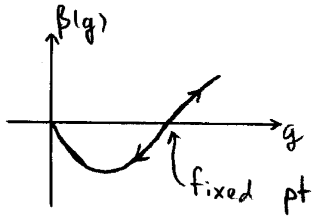


Figure 10.6.5: Beta function plot for pure Yang-Mills - two-loop order.

Then,

$$C_2(v)\delta_{ab} = c_{amn}c_{bmn} = n \text{ for } SU(n) \tag{10.6.82}$$

$$\begin{aligned}\delta_{ab}C_2(F) &= \mathrm{Tr}(\tau^a\tau^b)\\ &= \frac{1}{2} \text{ for quarks in QCD}\end{aligned} \tag{10.6.83}$$

$$\begin{aligned}C_2(S)\delta^{ab} &= \mathrm{Tr}(T^aT^b)\\ &= \frac{1}{2} \text{ for complex scalars in the fundamental rep. for } SU(3)\end{aligned} \tag{10.6.84}$$

For pure $SU(n)$ Yang-Mills theory,

$$\beta(g) = -\frac{11}{3}n\frac{g^3}{16\pi^2} \tag{10.6.85}$$

whose graph looks like Figure 10.6.4 Thus,

$$\kappa\frac{d\bar{g}(\kappa)}{d\kappa} = -\frac{11}{3}\frac{n\bar{g}^3(\kappa)}{16\pi^2} \tag{10.6.86}$$

$$\frac{d\bar{g}}{d\kappa} < 0 \text{ as } \kappa \to \infty \to \text{ asymptotic freedom} \tag{10.6.87}$$

$$\int_\infty^{\bar{g}(\kappa)} \frac{dg'}{g'^3} = -\frac{11n}{3(4\pi)^2}\int_\Lambda^\kappa \frac{d\kappa'}{\kappa'} \tag{10.6.88}$$

(B.C. $\bar{g}(\Lambda) = \infty$)

$$-\frac{1}{2g'^2}\Bigg|_\infty^{\bar{g}(\kappa)} = -\frac{11n}{3(4\pi)^2}\ln\left(\frac{\kappa}{\Lambda}\right)$$

$$\bar{g}^2 = \frac{3(4\pi)^2}{22n\ln\left(\frac{\kappa}{\Lambda}\right)}\left.\begin{matrix}\kappa\to\infty, & \bar{g}^2\to 0\\ \kappa<\Lambda, & \bar{g}^2(\kappa)<0 \ \text{(Not acceptable)}\end{matrix}\right\} \tag{10.6.89}$$

For N_f flavours of quarks, (see Figure 10.6.5)

$$\beta(g) = \frac{g^3}{16\pi^2}\left[\underbrace{-\frac{11}{3}\overbrace{(3)}^{SU(3)} + \frac{4}{3}\left(\frac{1}{2}\right)N_f}_{-11+\frac{2N_f}{3}<0 \text{ if } N_f<16\frac{1}{2} \text{ flavours}} \right] \tag{10.6.90}$$

Chapter 11

Spontaneous Symmetry Breaking

$\phi_4^4 \rightarrow$ (the subscript is the number of dimensions).

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!} \tag{11.0.1}$$

Symmetry: $\phi \rightarrow -\phi$ (must be respected. ex: if we have a 5 pt. function $\rightarrow$ Green's functions are negative, $\therefore$ must be 0).

$$\mathcal{H} = \Pi\partial_t\phi - \mathcal{L} \tag{11.0.2}$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)} - \frac{\partial \phi}{\partial t}$$

$$= \underbrace{\frac{1}{2}\left[\Pi^2 + (\nabla\phi)^2\right]}_{\text{Kinetic part}} + \underbrace{\frac{m^2}{2}\phi^2 + \frac{\lambda\phi^4}{4!}}_{\text{Potential } V(\phi)} \tag{11.0.3}$$

Lowest Energy state:

$$\text{KE}\begin{cases} \geq 0 \\ = 0 \text{ if } \phi = \text{ const.} \end{cases} \tag{11.0.4}$$

$$\begin{aligned} \text{Energy} &= V(\phi) \\ &= \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!}\ ; \quad \lambda > 0 \text{ for } E \text{ to be bounded below} \end{aligned} \tag{11.0.5}$$

There are two cases: we can have a positive mass squared ($m^2 > 0$ - Figure 11.0.1) or a negative mass squared ($m^2 < 0$ - Figure 11.0.2). For the first case,

$$\frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!} \tag{11.0.6}$$

For the second, we get

$$\begin{aligned} \frac{dV}{d\phi} &= m62\phi + \frac{\lambda\phi^3}{6} \\ &= 0 \text{ if } \left.\begin{matrix} \phi = 0 \\ \phi = \pm\sqrt{\frac{-6m^2}{\lambda}} \end{matrix}\right\} \text{(2 possible vacua)} \end{aligned} \tag{11.0.7}$$

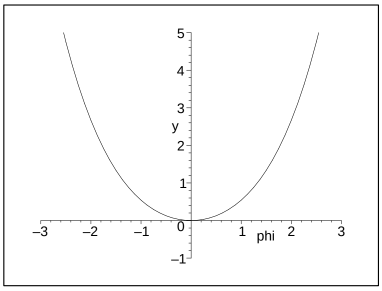


Figure 11.0.1: Positive mass ($m^2 > 0$) plot.

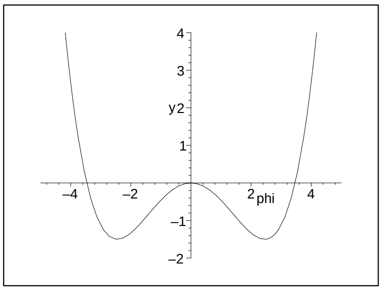


Figure 11.0.2: Negative mass ($m^2 < 0$) plot

Let's consider excitations above $\phi_0 = +\sqrt{\frac{-6m^2}{\lambda}}$.

$$\phi = \phi' + \sqrt{-\frac{6m^2}{\lambda}} \tag{11.0.8}$$

The Lagrangian for ϕ' is:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[\partial_\mu(\phi' + \phi_0)\right]^2 - \frac{m^2}{2}\left(\phi' + \phi_0\right)^2 - \frac{\lambda}{4!}\left(\phi' + \phi_0\right)^4 \\
&= \frac{1}{2}(\partial_\mu\phi')^2 - \frac{m^2}{2}\left(\phi'^2 + \underbrace{2\phi'\phi_0}_{*} + \phi_0^2\right) - \frac{\lambda}{4!}\left(\phi'^4 + 4\phi'^3\phi_0 + 6\phi'^2\phi_0^2 + \underbrace{4\phi'\phi_0^3}_{*} + \phi_0^4\right) \\
&\quad *'s \text{ cancel using } \phi_0 \text{ definition} \\
&= \frac{1}{2}(\partial_\mu\phi')^2 - \frac{m^2}{2}\phi'^2 - \frac{\lambda}{4!}\left(\phi'^4 + 4\phi'^3\sqrt{-\frac{6m^2}{\lambda}} + 6\phi'^2\left(-\frac{6m^2}{\lambda}\right)\right) + \text{ const} \\
&= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 + \left(-\frac{m^2}{2} + \frac{3m^2}{2}\right)\phi'^2 - \frac{\lambda}{4!}\phi'^4 - \frac{\lambda}{3!}\phi'^3\sqrt{-\frac{6m^2}{\lambda}} \\
&= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - \frac{1}{2}(-2m^2)\phi'^2 - \frac{\lambda}{3!}\sqrt{-\frac{6m^2}{\lambda}}\underbrace{\phi'^3}_{\dagger} - \frac{\lambda}{4!}\phi'^4
\end{aligned} \tag{11.0.9}$$

$\rightarrow$ mass of ϕ' is $(-2m^2) > 0$.

† breaks the $\phi \leftrightarrow -\phi$ symmetry. This can be seen in Figure 11.0.2 - the bottom of the two wells will have ground states $|0_-\rangle$, $|0_+\rangle$ - this ground state doesn't respect symmetry. For example,

$$\begin{aligned}
\hat{F}\phi\hat{F}^{-1} &= -\phi \text{ (suppose)} \\
\hat{F}|0\rangle &= |0\rangle \;\text{ if } (m^2 > 0) \\
\hat{F}|0_+\rangle &= |0_-\rangle \;\text{ if } (m^2 < 0)
\end{aligned}$$

$\left(\frac{|0_-\rangle+|0_+\rangle}{\sqrt{2}} \rightarrow \text{symmetry restored}\right)$.

Mar. 2/2000

Heisenberg (1-D) Ferromagnet:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \rightarrow & \rightarrow & \leftarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \rightarrow \end{array} \tag{11.0.10}$$

(where $\sigma = +1$ for positions $(1+2)$, $\sigma = -1$ for positions $(3+4)$,...).

$$\begin{aligned} H &= -\sum_i \kappa \sigma_i \sigma_{i+1} \\ &= -\kappa \left[(+1)^2 + (+1)(-1) + (-1)^2 + \ldots\right] \end{aligned} \tag{11.0.11}$$

There is an "up-down" symmetry in this Hamiltonian (no preference for up/down).

The lowest energy states are "all up" or "all down".

→ Generalize to an $\mathcal{O}(N)$ model.

$$\underline{\phi} = (\phi^1, \phi^2, \ldots, \phi^N) \tag{11.0.12}$$

$\mathcal{L}$ is invariant under

$$\underline{\phi} \rightarrow \underline{R}\underline{\phi} \quad (\underline{R}^T = \underline{R}^{-1}) \tag{11.0.13}$$

$\underline{R}$ is a global orthogonal matrix.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \underline{\phi})(\partial^\mu \underline{\phi}) - \frac{m^2}{2}\underline{\phi}\cdot\underline{\phi} - \frac{\lambda}{4!}(\underline{\phi}\cdot\underline{\phi})^2 \tag{11.0.14}$$

The potential for this Lagrangian is

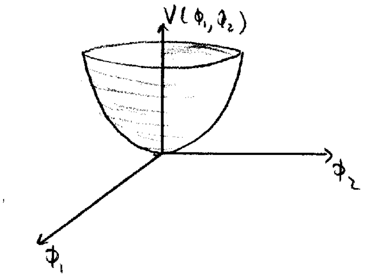


Figure 11.0.3: $\mathcal{O}(N)$ model for $m^2 > 0$.

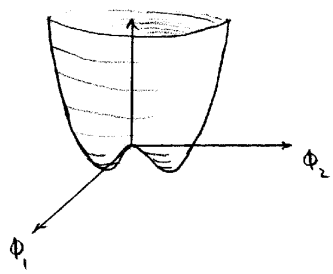


Figure 11.0.4: $\mathcal{O}(N)$ model ($m^2 < 0$) plot "Mexican Hat" potential

$$V(\underline{\phi}) = \frac{m^2}{2}\underline{\phi}^2 + \frac{\lambda}{4!}((\underline{\phi})^2)^2 \tag{11.0.15}$$

(where $\lambda > 0$ for stability). This can be plotted - see figures 11.0.3 (for $m^2 > 0$) and 11.0.4 (for $m^2 < 0$).

$$\begin{aligned}\frac{\partial V}{\partial \phi^i} &= m^2\phi_i + \frac{\lambda}{4!}(2\underline{\phi}^2)(2\phi_i)\\ &= \left(m^2 + \frac{\lambda}{6}\underline{\phi}^2\right)\phi_i\\ &= 0 \quad \text{at} \quad \phi_i = 0 \ \underline{\text{or}} \ \text{at} \quad |\underline{\phi}| = \sqrt{\frac{-6m^2}{\lambda}}\end{aligned} \tag{11.0.16}$$

Suppose,

$$\begin{aligned}\underline{\phi}_0 &= V\delta_{i1} \quad (\text{ in the "one" direction})\\ \underline{\phi} &= (V+\phi'_1, \phi'_2, \ldots, \phi'_N)\\ \mathcal{L} &= \frac{1}{2}(\partial_\mu\underline{\phi}')(\partial^\mu\underline{\phi}') - \frac{m^2}{2}\underbrace{\left[(V+\phi'_1)^2 + \phi'_2 + \ldots + \phi'_N\right]}_{\substack{=V^2+2V\phi'_1+\phi'^2_1+\ldots\\ =V^2+2V\phi'_1+\underline{\phi}'^2}} - \frac{\lambda}{4!}\left[(V+\phi'_1)^2 + \phi'^2_2 + \ldots + \phi'^2_N\right]^2\\ &= \frac{1}{2}(\partial_\mu\underline{\phi}')^2 - \frac{m^2}{2}\left[V^2 + 2V\phi'_1 + \underline{\phi}'^2\right] - \frac{\lambda}{4!}\left[V^2 + 2V\phi'_1 + \underline{\phi}'^2\right]^2\\ &= \frac{1}{2}(\partial_\mu\underline{\phi}')^2 - \frac{\lambda}{4!}\left[2V^2\underline{\phi}'^2 + 4V^2\phi'^2_1\right] + \text{ interaction terms}\\ &= \frac{1}{2}(\partial_\mu\underline{\phi}')^2 - \frac{m^2}{2}\underline{\phi}'^2 - \frac{\lambda}{4!}\left(\frac{-6m^2}{\lambda}\right)\left(2\underline{\phi}'^2 + 4\phi'^2_1\right)\\ &= \frac{1}{2}(\partial_\mu\underline{\phi}')^2 - \frac{m^2}{2}\underline{\phi}'^2 + m^2\left(\frac{1}{2}\underline{\phi}'^2 + \phi'^2_1\right)\\ &= \frac{1}{2}(\partial_\mu\underline{\phi}')^2 - \frac{1}{2}(-2m^2)\phi'^2_1\end{aligned} \tag{11.0.17}$$

Thus

- $\phi'_1 \to$ mass $(-2m^2)$
- $\phi'_2, \ldots, \phi'_N \to$ massless ($N-1$ massless Goldstone Bosons).

In general, if a system of N scalars has an invariance under a group G, and if the ground state of the theory has an invariance under a group $G' \subset G$, then there is a massless excitation above this ground state for each generator of G'.

$\to$ Not realized in Nature.

Here we have

$$G \to O(N) \quad \text{(Orthogonal group, N-dimensional)} \tag{11.0.18}$$

(see figures 11.0.5,11.0.6). Figure 11.0.7 $\to$ No massless scalars (spin 0) particles observed

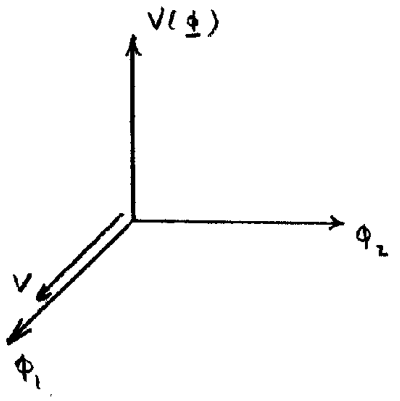


Figure 11.0.5: So $G' \to O(N-1)$

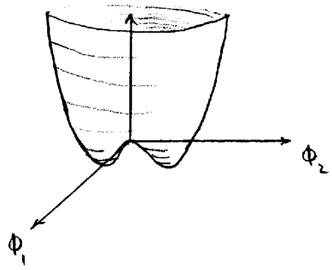


Figure 11.0.6: Excitation where you just roll around bottom of well (Correspond to Goldstone Bosons)

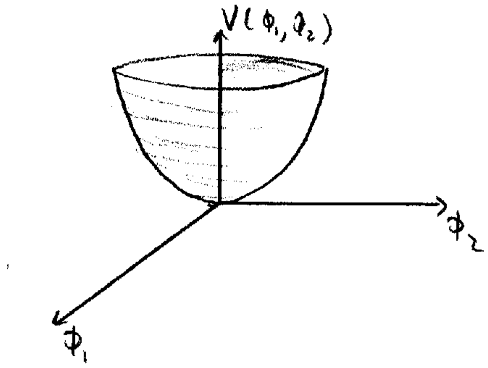


Figure 11.0.7: $m^2 > 0$ - (No massless excitations).

in nature.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2 \quad \text{(Massive spin-1 particle)} \tag{11.0.19}$$

$$\mathrm{i}D_{\mu\nu} = -\frac{\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right)}{k^2 - m^2} \quad \text{(whole term is transverse)} \tag{11.0.20}$$

where the $\frac{k_\mu k_\nu}{m^2}$ term comes from the longitudinal polarization (Destroys Renormalizability) - the gauge invariance is ruined by m^2.

In the Higgs Mechanism, the massless Goldstone Bosons are absorbed by the longitudinal polarization of the massless vectors to which they are coupled to give massive vector particles.

However, gauge invariance is not sacrificed, so renormalizability is retained. The massive scalars left over are all the "Higgs" particles.

Mar. 6/2000

11.1 $\mathcal{O}(2)$ Goldstone model:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi^i\right)\left(\partial^\mu \phi^i\right) - \frac{m^2}{2}\phi^i\phi^i - \frac{\lambda}{4!}(\phi^i\phi^i)^2 \\ &\rightarrow \phi^i = (\phi^1, \phi^2) \quad \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \end{aligned} \tag{11.1.1}$$

Let $\phi = \frac{\phi^1 + \mathrm{i}\phi^2}{\sqrt{2}}$, $\phi^* = \frac{\phi^1 - \mathrm{i}\phi^2}{\sqrt{2}}$.

$$\begin{aligned} \mathcal{L} &= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2\phi^*\phi - \frac{\lambda}{6}(\phi^*\phi)^2 \\ & \phi \rightarrow e^{\mathrm{i}\theta}\phi \end{aligned} \tag{11.1.2}$$

Local gauge invariance

$$\phi \rightarrow e^{\mathrm{i}\theta(x)}\phi \tag{11.1.3}$$

Introduce a gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[(\partial + \mathrm{i}eA_\mu)\phi^*\right]\left[(\partial^\mu - \mathrm{i}eA^\mu)\phi\right] - m^2\phi^*\phi - g(\phi^*\phi)^2 \tag{11.1.4}$$

If $m^2 < 0$ - (see figure 11.1.1).

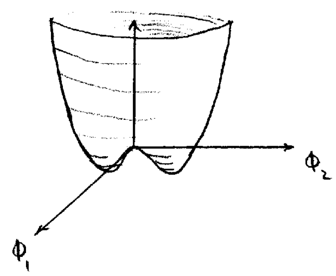


Figure 11.1.1: $m^2 < 0, \rightarrow V(\phi) = m^2\phi^*\phi + g(\phi^*\phi)^2$

$$\frac{\partial V}{\partial \phi} = 0 \quad \Rightarrow |\phi| = \sqrt{\frac{-m^2}{2g}} \tag{11.1.5}$$

Choose

$$\phi_{min} = \sqrt{\frac{-m^2}{2g}} \quad \text{(Real)} \tag{11.1.6}$$

$$\begin{aligned} \phi &= \sqrt{\frac{-m^2}{2g}} + \phi' \\ &= \frac{V}{\sqrt{2}} + \phi' \end{aligned} \tag{11.1.7}$$

Now,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + \left[(\partial_\mu + \mathrm{i}eA_\mu)\left(\frac{V}{\sqrt{2}} + \phi'^*\right)\right]\left[(\partial^\mu - \mathrm{i}eA^\mu)\left(\frac{V}{\sqrt{2}} + \phi'\right)\right] \\ &\quad - m^2\left(\frac{V}{\sqrt{2}} + \phi'^*\right)\left(\frac{V}{\sqrt{2}} + \phi'\right) - g\left[\left(\frac{V}{\sqrt{2}} + \phi'^*\right)\left(\frac{V}{\sqrt{2}} + \phi'\right)\right]^2 \\ &= -\frac{1}{4}F_{\mu\nu}^2 + [(\partial_\mu + \mathrm{i}eA_\mu)\phi'^*][(\partial_\mu - \mathrm{i}eA_\mu)\phi'] + \frac{V}{\sqrt{2}}[(\mathrm{i}eA_\mu(\partial^\mu\phi') - (\partial_\mu\phi'^*)(\mathrm{i}eA^\mu)] \\ &\quad + \underbrace{\frac{V^2}{2}e^2A_\mu A^\mu}_{*} - m^2\left[\phi'^*\phi' + \frac{V}{\sqrt{2}}(\phi' + \phi'^*) + \frac{V^2}{2}\right] - g\left[\frac{V^2}{2} + \frac{V}{\sqrt{2}}(\phi' + \phi'^*) + \phi'^*\phi'\right]^2 \end{aligned}$$

(Note that it would appear, from the * term, that $A_\mu A^\mu$ has a mass V^2e^2). Now, let $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi')^*(D^\mu\phi') + \frac{\mathrm{i}eV}{\sqrt{2}}[A_\mu\partial^\mu\phi' - (\partial_\mu\phi'^*)A^\mu\phi'] + \frac{e^2V^2}{2}A_\mu A^\mu - m^2\phi'^*\phi' \\ &\quad - g\Bigg[\underbrace{\frac{V^2}{2}(\phi' + \phi'^*)^2 + 2\left(\frac{V^2}{2}\right)\phi^*\phi'}_{\text{appears that } \mathrm{Im}(\phi') \text{ is massless}} + \frac{2V}{\sqrt{2}}(\phi' + \phi'^*)\phi'^*\phi' + (\phi'^*\phi')^2\Bigg] + \text{const.} \end{aligned} \tag{11.1.8}$$

To determine the actual degrees of freedom, set

$$\phi(x) = \left(\frac{\eta(x) + V}{\sqrt{2}}\right)e^{\mathrm{i}\xi(x)/V} \tag{11.1.9}$$

where $\eta(x)$, $\xi(x)$ are <u>real</u>. Then, let

$$\phi_v(x) = e^{-\mathrm{i}\xi/V}\phi(x) \tag{11.1.10}$$

$$B_\mu = A_\mu - \frac{1}{e}\partial_\mu\left(\frac{\xi(x)}{V}\right) \tag{11.1.11}$$

where (11.1.10,11.1.11) are a gauge transformation. Then,

$$\phi_v(x) = \frac{\eta(x) + V}{\sqrt{2}} \tag{11.1.12}$$

Now we find that

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \eta)^2 - \frac{1}{2}(-2m^2)\eta^2 - \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 + \frac{1}{2}(eV)^2 B_\mu B^\mu \\ &\quad + \frac{1}{2}e^2 B_\mu B^\mu(\eta^2 + 2V\eta) - gV^2\eta^2 - \frac{g}{4}\eta^4 \end{aligned} \tag{11.1.13}$$

- $\xi(x) \to$ disappeared (Goldstone boson has vanished)
- $\eta(x) \to$ mass $(-2m^2) \to$ Higgs
- $B_\mu \to$ massive vector (3 polarizations) - mass $(eV)^2$

There are still only 3 parameters that characterize the theory, $\eta,\ e,\ g$. (recall $V = \sqrt{\frac{-m^2}{2g}}$).

This is the form of the Lagrangian in the "U" (unitary) gauge - physical degrees of freedom are apparent.

Renormalizability is only manifest in "other" gauges (f-gauges) (Note, though, that since the "original theory" was renormalizable, this one is also).

In general, for a non-Abelian Field,

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi_i^* + \mathrm{i}gT^a_{ij}A^a_\mu \phi_j^*)(\partial^\mu \phi_i - \mathrm{i}gT^b_{il}A^{b\mu}\phi_l)\right] - V(\phi_i) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{11.1.14}$$

If we say

- $n \to \#$ of scalar fields ϕ_i ($2n$ if $\phi_i \to$ complex).

and if

- $\frac{\partial V}{\partial \phi_i} = 0$ if $\phi_i = \frac{V_i}{\sqrt{2}} \neq 0$ $(i = 1, \ldots, n')$ $(n' < n)$

then there are then n' massive Higgs and $(n - n')$ massive vector particles.

Electroweak $SU(2) \times U(1)$ model

$$\phi_i \Rightarrow i = 1,\ 2 \ \ \text{(Complex doublet of } SU(2)\text{)} \tag{11.1.15}$$

$(n = 4)$,

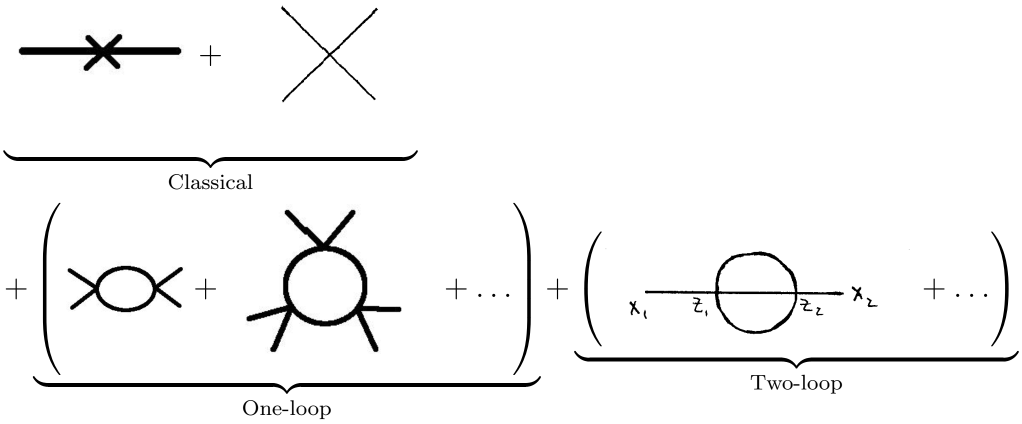
- $n' = 1 \to$ One Higgs particle
- $\to$ 4 vectors, (4-1) = 3 are massive, and 1 is massless (photon).

Mar. 8/2000

11.2 Coleman Weinberg Mechanism

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad (11.2.1)$$
$$\rightarrow V = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

V_{tot} =



= Sum of 1PI (irreducible) diagrams with an arbitrary number of external legs where momenta $= 0$

Remember (Generator function of all diagrams)

$$Z[J] = \int d\phi \; e^{(\mathrm{i}/\hbar)\int dx \; (\mathcal{L}+J\phi)} \qquad (11.2.2)$$
$$= e^{\mathrm{i}W(J)/\hbar} \qquad (11.2.3)$$

where $W(J)$ is the generating functional of all connected diagrams.

$$\Phi_c(x) = \frac{\delta W(J)}{\delta J(x)} \quad \rightarrow \text{ Legendre Transformation} \qquad (11.2.4)$$
$$\text{Renormalized} \rightarrow \Gamma(\Phi_c) = W(J) - \int dx \; J(x)\Phi_c(x) \qquad (11.2.5)$$

(where Φ_c is the 1PI generating functional). <u>Define</u> the effective potential to be

$$V(\phi) = -\sum_{n=0}^{\infty} \frac{1}{n!}\Gamma(0,\ldots,0)\phi^n \qquad (11.2.6)$$

- We first of all renormalize $\Gamma(\phi)$

- We next define

$$m_R^2 = \left.\frac{d^2V}{d\phi^2}\right|_{\phi=\kappa} \tag{11.2.7}$$

$$\lambda_R = \left.\frac{d^4V}{d\phi^4}\right|_{\phi=\kappa'} \tag{11.2.8}$$

(m_R = Renormalized mass).

Thus,

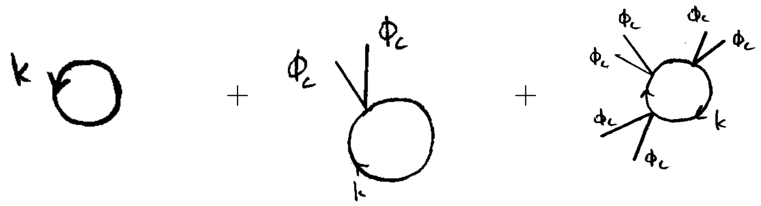
$$V = V(\phi, \lambda_R, m_R^2) \tag{11.2.9}$$

We can now determine if there is Spontaneous Symmetry breaking.

Note that even if V_{cl} has $\phi_{\min} = 0$ (i.e. no spontaneous symmetry breakdown), it is conceivable that V_{tot} does generate the breakdown. If there is spontaneous symmetry and the Higgs mechanism has generated massive vectors, then there will be no divergences in this theory with vectors.

- Heuristic argument for the renormalizability of models with the Higgs mechanism for generating massive vectors.

11.3 One loop Effective Potential in $\lambda\phi^4$ model



$\phi_c -$ constant field (external momenta $= 0$)

First loop: $\displaystyle\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{i}}{(k^2 - m^2 + \mathrm{i}\varepsilon)}$ 

n$^{\text{th}}$ Loop: $\displaystyle S_n \int \frac{d^4k}{(2\pi)^4}(-\mathrm{i}\lambda)^n \left(\frac{\mathrm{i}}{(k^2 - m^2 + \mathrm{i}\varepsilon)}\right)^n$ ($S_n \to$ Symmetry factor)

$$2n \text{ legs } \to \frac{(2n)!}{(2n)2^n}$$

(where, in the denominator, the $2n$ is because we can start anywhere in the circle, and the 2^n is because the legs are interchangeable). Thus,

$$\Gamma(\underbrace{0,\ldots,0}_{\substack{2n \text{ zero}\\ \text{momenta}}}) = \frac{(2n)!}{2^n(2n)}\int\frac{d^4k}{(2\pi)^4}\left(\frac{\lambda}{k^2-m^2+\mathrm{i}\varepsilon}\right)^n \tag{11.3.2}$$

$$n = 0 \text{ (constant term)} \tag{11.3.3}$$

Therefore,

$$\begin{aligned}
V(\phi) &= -\sum_{n=1}^{\infty}\frac{1}{(2n)!}\Gamma(0,\ldots,0)\phi_c^{2n}\\
&= -\sum_{n=1}^{\infty}\frac{1}{(2n)!}\frac{(2n)!}{2^n(2n)}\int d^4k\left(\frac{\lambda\phi_c^2}{k^2-m^2+\mathrm{i}\varepsilon}\right)^n\\
&= -\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\lambda\frac{\phi_c^2}{2}}{k^2-m^2+\mathrm{i}\varepsilon}\right)^n\\
&\quad (n = 1, 2 diverges)\\
&= -\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}\ln\left(1-\frac{\frac{\lambda\phi_c^2}{2}}{k^2-m^2+\mathrm{i}\varepsilon}\right)\\
&\quad (\text{as } \ln(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots)\\
&= -\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}\left[\ln\left(k^2-m^2-\frac{\lambda\phi_c^2}{2}\right)-\underbrace{\ln\left(k^2-m^2+\mathrm{i}\varepsilon\right)}_{\text{constant}}\right]
\end{aligned} \tag{11.3.4}$$

The minus sign means we must still renormalize. Note that the second (constant) term is a problem in G.R. - $\int d^4x\ \sqrt{g_\infty}k \leftarrow$ Cosmological constant $= 0$ actually.

<u>Alternative Derivation</u>

$$Z(J) = \int d\phi\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int d^4x\ \ (L(\phi)+J(\phi))\right\} \tag{11.3.5}$$

Let $\phi(x) = \phi_c + h(x) =$ (constant field) + (quantum fluctuations about ϕ_c).

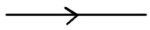
$$Z(J) = \int dh\exp\left\{\frac{\mathrm{i}}{\hbar}\int\left[\mathcal{L}(\phi_c+h)+Jh\right]d^4x\right\} \tag{11.3.6}$$

Consider only diagrams with external ϕ_c's and are 1PI to get $V(\phi_c)$. But,

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(\phi_c+h)\\
&= \frac{1}{2}(\partial_\mu h)^2-\frac{m^2}{2}(\phi_c+h)^2-\frac{\lambda}{4!}(\phi_c^2+2\phi_c h+h^2)^2\\
&= \frac{1}{2}(\partial_\mu h)^2-\frac{m^2}{2}(\phi_c^2+2\phi_c h+h^2)-\frac{\lambda}{4!}\left(\phi_c^4+4\phi_c^3h+6\phi_c^2h^2+\mathcal{O}(h^3)\right)
\end{aligned}$$

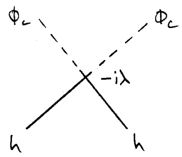
The only terms that will contribute to $V(\phi_c)$ at one-loop order are $\mathcal{O}(h^2)$. i.e. One-Loop

$$\frac{1}{2}(\partial_\mu h)^2 - \frac{m^2}{2}h^2 \quad \Rightarrow$$



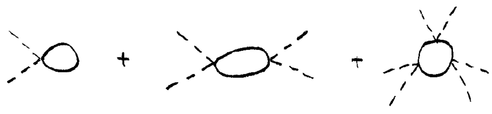
$(< hh >)$ (11.3.7)

$$-\frac{\lambda}{4}\phi_c^2 h^2 \quad \Rightarrow$$



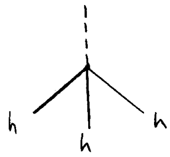
(11.3.8)

One loop

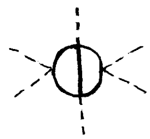


Two-loop

$$\left(h^3\phi_c \right.$$



$\left. \right)$,



(11.3.9)

Then,

$$\begin{aligned}
\hat{Z} &= \int dh \exp\left\{\frac{\mathrm{i}}{\hbar}\int d^4x \left[\frac{1}{2}(\partial_\mu h)^2 - \frac{m^2}{2}h^2 - \frac{\lambda}{4}\phi_c^2 h^2\right]\right\} \\
&= \int dh \ \exp\left\{\frac{\mathrm{i}}{\hbar}\int d^4x \left[\frac{1}{2}\ln\left(-p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right) h\right]\right\} \\
&= \det{}^{-1/2}\left(p^2 + m^2 + \frac{\lambda\phi_c^2}{2}\right)\kappa \qquad (11.3.10)
\end{aligned}$$

Mar. 9/2000

$$\begin{aligned}
V(\phi) &= \mathrm{i}\int \frac{d^4k}{(2\pi)^4}\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\frac{1}{2}\lambda\phi^2}{k^2 - m^2 + \mathrm{i}\varepsilon}\right) \\
&= \frac{\mathrm{i}}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(k^2 - m^2 - \frac{1}{2}\lambda\phi^2 + \mathrm{i}\varepsilon\right) \qquad (11.3.11)
\end{aligned}$$

Recall we can get the same thing from:

$$\begin{aligned} Z(J) &= \int dh \exp\left\{\frac{\mathrm{i}}{\hbar}\int d^4x \; \frac{1}{2}h(x)\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)h(x)\right\} \\ &= \det{}^{-1/2}\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right) \\ &\quad \text{(1-loop 1PI contribution to } Z \text{ if } \phi_c \text{ is a constant)} \\ &= e^{\mathrm{i}W} \end{aligned} \tag{11.3.12}$$

where we define

$$\begin{aligned} \mathrm{i}W &= \ln\left(\det{}^{-1/2}\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)\right) \\ W &= \frac{\mathrm{i}}{2}\ln\left(\det\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)\right) \\ &= \Gamma(\phi_c) - \underbrace{\int dx \; \phi_c J}_{=0} \end{aligned} \tag{11.3.13}$$

Thus,

$$\Gamma(\phi_c) = \frac{\mathrm{i}}{2}\ln\left(\det\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)\right) \tag{11.3.14}$$

Note:

$$\begin{aligned} \ln(\det(\mathcal{O})) &= \ln(\det(U^\dagger \mathcal{O} U)) \quad ; \; \mathcal{O} = \mathcal{O}^\dagger \\ &= \ln\left(\det\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)\right) \\ &= \ln\left(\lambda_1 \ldots \lambda_n\right) \\ &= \ln(\lambda_1) + \ldots + \ln(\lambda_n) \end{aligned} \tag{11.3.15}$$

and

$$\begin{aligned} \mathrm{Tr}(\ln(\mathcal{O})) &= \mathrm{Tr}(\ln(U^\dagger \mathcal{O} U)) \\ &= \mathrm{Tr}\left(\ln\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) \\ &= \ln(\lambda_1) + \ldots + \ln(\lambda_n) \end{aligned} \tag{11.3.16}$$

Then,

$$\mathrm{Tr}(\ln(\mathcal{O})) = \ln(\det(\mathcal{O})) \tag{11.3.17}$$

So

$$\begin{aligned}
\Gamma(\phi_c) &= \frac{\mathrm{i}}{2}\ln\left(\det\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)\right) \\
&= \frac{\mathrm{i}}{2}\mathrm{Tr}\left(\ln\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)\right) \\
&= \frac{\mathrm{i}}{2}\int dp\ \langle p|\ln\left(p^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right)|p\rangle \\
&= \frac{\mathrm{i}}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(k^2 - m^2 - \frac{\lambda\phi_c^2}{2}\right) \qquad (11.3.18)
\end{aligned}$$

Now, to compute $V(\phi_c)$ in closed form.

11.4 Dimensional Regularization

$$V(\phi_c) = \frac{\mathrm{i}}{2}\int \frac{d^nk}{(2\pi)^2n}\ln\left(k^2 - m^2 - \frac{\lambda\phi_c^2}{2!} + \mathrm{i}\varepsilon\right) \qquad (11.4.1)$$

$$\to \ln(a + \mathrm{i}\varepsilon) - \ln(b + \mathrm{i}\varepsilon) = \int_0^\infty \frac{d(\mathrm{i}t)}{\mathrm{i}t}\left(e^{\mathrm{i}(b+\mathrm{i}a)t} - e^{\mathrm{i}(a+\mathrm{i}\varepsilon)}\right) \qquad (11.4.2)$$

i.e.:

$$\begin{aligned}
\frac{d}{da}\ln(a + \mathrm{i}\varepsilon) &= \frac{d}{da}\int_0^\infty \frac{d(\mathrm{i}t)}{\mathrm{i}t}e^{\mathrm{i}(a+\mathrm{i}\varepsilon)t}(-1) \\
\frac{1}{a + \mathrm{i}\varepsilon} &= -\int_0^\infty d(\mathrm{i}t)\ e^{\mathrm{i}(a+\mathrm{i}\varepsilon)t} \\
&= -\left.\frac{e^{\mathrm{i}(a+\mathrm{i}\varepsilon)t}}{a + \mathrm{i}\varepsilon}\right|_0^\infty \\
&= \frac{1}{a + \mathrm{i}\varepsilon} \qquad (11.4.3)
\end{aligned}$$

$$\begin{aligned}
f_L(a,b) &= f_R(a,b) + k \qquad (11.4.4) \\
\frac{\partial f_L}{\partial a} &= \frac{\partial f_R}{\partial a} \qquad (11.4.5) \\
\frac{\partial f_L}{\partial b} &= \frac{\partial f_R}{\partial b} \qquad (11.4.6) \\
\frac{\partial k}{\partial a} &= \frac{\partial k}{\partial b} = 0 \qquad (11.4.7)
\end{aligned}$$

$$\begin{aligned}
V(\phi_c) &= \frac{\mathrm{i}}{2}\int \frac{d^nk}{(2\pi)^n}\left[\ln\left(k^2-m^2-\frac{\lambda\phi_c^2}{2}+\mathrm{i}\varepsilon\right)-\ln\left(k^2-m^2+\mathrm{i}\varepsilon\right)\right] \\
&= \frac{\mathrm{i}}{2}\int \frac{d^nk}{(2\pi)^n}\int \frac{d(\mathrm{i}t)}{\mathrm{i}t}\left(e^{\mathrm{i}(k^2-m^2+\mathrm{i}\varepsilon)t}-e^{\mathrm{i}\left(k^2-m^2-\frac{\lambda\phi_c^2}{2}+\mathrm{i}\varepsilon\right)t}\right) \\
&= \frac{\mathrm{i}}{2}\int \frac{d^nk}{(2\pi)^n}\int_0^\infty \frac{d(\mathrm{i}t)}{\mathrm{i}t}e^{\mathrm{i}\left(k^2-m^2-\frac{\lambda\phi_c^2}{2}+\mathrm{i}\varepsilon\right)t} \qquad (11.4.8)
\end{aligned}$$

$$\left(\begin{matrix}\text{Will often} \\ \text{see written}\end{matrix} \quad \frac{1}{H+\mathrm{i}\varepsilon} = -\int_0^\infty \frac{d(\mathrm{i}t)}{\mathrm{i}t}e^{\mathrm{i}(H+\mathrm{i}\varepsilon)t} \to \begin{matrix}\text{Not true strictly speaking} \\ \text{(11.4.2) is true}\end{matrix}\right)$$

However,

$$a^{-s}\Gamma(s) = \int_0^\infty dt\ t^{s-1}e^{-at} \qquad (11.4.9)$$

Hence,

$$\int \frac{d^nk}{(2\pi)^n}\underbrace{e^{\mathrm{i}(k^2+\mathrm{i}\varepsilon)t}}_{\text{No angular comp.}} = \frac{\mathrm{i}}{(4\pi\mathrm{i}t)^{n/2}} \qquad (11.4.10)$$

Thus, one loop

$$\begin{aligned}
-V(\phi_c) &= -\frac{\mathrm{i}}{2}\int d^4x\ \Gamma(0\ldots0)\frac{\phi^n}{n!} \\
&= -\frac{\mathrm{i}}{2}\int_0^\infty \frac{d(\mathrm{i}t)}{\mathrm{i}t}\frac{\mathrm{i}}{(4\pi\mathrm{i}t)^{n/2}}\ e^{\mathrm{i}\left(-m^2-\frac{\lambda\phi_c^2}{2}+\mathrm{i}\varepsilon\right)t} \\
&\quad \to \text{Let } \mathrm{i}t = u \\
&= \frac{1}{2}\int_0^\infty \frac{du}{u^{1+n/2}}\frac{1}{(4\pi)^{n/2}}\ e^{-\left(m^2+\frac{\lambda\phi_c^2}{2}\right)u} \\
&= \frac{1}{2(4\phi)^{n/2}}\Gamma\left(-\frac{n}{2}\right)\left(m^2+\frac{\lambda\phi_c^2}{2}\right)^{n/2} \qquad (11.4.11)
\end{aligned}$$

This diverges as $n \to 4$ ($\Gamma\left(-\frac{n}{2}\right) \to \Gamma(-2) \to \Gamma$ diverges at negative integers). So, let $\varepsilon = 2-\frac{n}{2}$.

$$\begin{aligned}
\Gamma\left(-\frac{n}{2}\right) &= \Gamma(-2+\varepsilon) \\
&= \frac{\Gamma(-1+\varepsilon)}{-2+\varepsilon} \\
&= \frac{1}{(-2+\varepsilon)}\frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} \\
&\quad (\Gamma(x+1) = x\Gamma(x)) \\
\to \Gamma(\varepsilon) &= \frac{1}{\varepsilon} - \gamma_E + \ldots \qquad (11.4.12)
\end{aligned}$$

$$\begin{aligned}
V(\phi_c) &= \frac{1}{2}\frac{1}{(4\pi)^{2-\varepsilon}}\left(\frac{1}{2-\varepsilon}\right)\left(\frac{1}{1-\varepsilon}\right)\left(\frac{1}{\varepsilon}-\gamma_E+\ldots\right)\left(m^2+\frac{\lambda\phi_c^2}{2}\right)^{2-\varepsilon} \\
&= \frac{1}{2}\left[\frac{1}{(4\pi)^2}\left(1+\varepsilon\ln(4\pi)+\mathcal{O}(\varepsilon^2)\right)\right]\left[\frac{1}{2-3\varepsilon+\ldots}\right]\left[\frac{1}{\varepsilon}-\gamma_E+\ldots\right] \\
&\qquad \left[m^2+\frac{\lambda\phi_c^2}{2}\right]^2\left(1-\varepsilon\ln\left(m^2+\frac{\lambda\phi_c^2}{2}\right)+\ldots\right) \\
&= \frac{1}{4}\frac{1}{(4\pi)^2}\left(m^2+\frac{\lambda\phi_c^2}{2}\right)^2\left[\frac{1}{\varepsilon}+\frac{3}{2}-\gamma_E+\ln(4\pi)-\ln\left(m^2+\frac{\lambda\phi_c^2}{2}\right)+\mathcal{O}(\varepsilon)\right] \qquad (11.4.13)
\end{aligned}$$

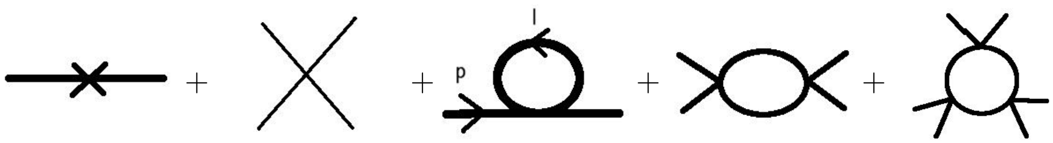
So,

$$\begin{aligned}
V_{\mathrm{TOT}} &= \underbrace{\frac{m^2\phi_c^2}{2}}_{*}+\underbrace{\frac{\lambda\phi_c^2}{4}}_{**}+\frac{(-1)}{64\pi^2}\left(m^4+\underbrace{m^2\lambda\phi_c^2}_{*}+\underbrace{\frac{\lambda^2\phi_c^4}{4}}_{**}\right)\cdot \\
&\qquad \left[\frac{1}{\varepsilon}-\ln\left(m^2+\frac{\lambda\phi_c^2}{2}\right)+\ln(4\pi)-\gamma_E+\frac{3}{2}+\mathcal{O}(\varepsilon)\right] \qquad (11.4.14)
\end{aligned}$$

So, to one loop order,

$$\frac{m_R^2}{2} = \frac{m^2}{2}+\left(\frac{-1}{64\pi^2}\right)\left(\frac{m^2\lambda}{\varepsilon}\right) \qquad (11.4.15)$$

$$\frac{\lambda_R}{4!} = \frac{\lambda}{4!}+\frac{\lambda^2}{4}\left(\frac{-1}{64\pi^2}\right)\left(\frac{1}{\varepsilon}\right) \qquad (11.4.16)$$



$+\, ldots$

$$\therefore V_{\mathrm{Tot}}^{\mathrm{finite}} = \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\left(\frac{\lambda\phi_c^2}{\mu^2}\right)+k\right) \qquad (11.4.18)$$

(lim $m^2 \to 0$). Thus, as $m^2 \to 0$,

$$V^{1\ \mathrm{loop}}(\phi_c) = \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\left(\frac{\lambda\phi_c^2}{\mu^2}\right)+k\right) \qquad (11.4.19)$$

$$V^{\mathrm{Tot}} = \frac{\lambda\phi_c^4}{4!}+\frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\left(\frac{\lambda\phi_c^2}{\mu^2}\right)+k\right) \qquad (11.4.20)$$

(See Figure 11.4.1)

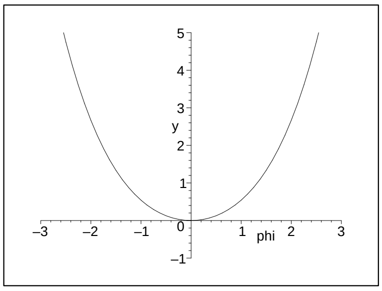


Figure 11.4.1: $V^{c'}(\phi_c) = \frac{\lambda\phi_c^4}{4!}$

$$
\begin{aligned}
\frac{dV^{\text{Tot}}(\phi_c)}{d\phi_c} &= \frac{\lambda\phi_c^3}{6} + \frac{\lambda^2}{256\pi^2}\left[4\phi_c^3\left(\ln\left(\frac{\lambda\phi_c^2}{\mu^2}\right)+k\right)+\phi_c^4\frac{1}{\phi_c}\right] \\
&= \phi_c^3\left[\frac{\lambda}{6}+\frac{\lambda^2}{256\pi^2}\left(4\ln\left(\frac{\lambda\phi_c^2}{\mu^2}\right)+k\right)+1\right] \\
&= \phi_c^3\left[\frac{\lambda}{6}+\frac{\lambda^2}{64\pi^2}\ln\left(\frac{\phi_c^2}{\mu'^2}\right)\right] \\
&= 0\left\{\begin{array}{rcl} \text{if } \phi_c &=& 0 \\ \underline{\text{or}}\ \ \phi_c^2 &=& \mu'^2 e^{-(64\pi^2/6\lambda)} \\ &=& \mu'^2 e^{-(32\pi^2/3\lambda)} \end{array}\right\}
\end{aligned}
\tag{11.4.21}
$$

→ Changes in μ are compensated for by changing λ.
Mar. 13/2000
Recall:

$$
\begin{aligned}
V_{\text{tot}} &= \frac{\lambda\phi^4}{4!} + \frac{\lambda^2}{256\pi^2}f^4\left(\ln\left(\frac{f^2}{\mu^2}\right)+k\right) \\
&= \frac{\lambda f^4}{4!} + \frac{\lambda^2 f^4}{256\pi^2}\ln\left(\frac{f^2}{\mu'^2}\right) \qquad (f = \phi_c \text{ from last time})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_{\text{tot}}}{\partial f} &= 4f^3\left(\frac{\lambda}{4!}+\frac{\lambda^2}{256\pi^2}\ln\left(\frac{f^2}{\mu^2}\right)\right)+\frac{\lambda^2 f''}{256\pi^2}\left(\frac{1}{f}\right) \\
0 &= f^3\left(\frac{\lambda}{6}+\frac{4\lambda^2}{256\pi^2}\ln\left(\frac{f^2}{\mu^2}\right)+\frac{\lambda^2}{256\pi^2}\right)
\end{aligned}
$$

If $f = 0$, or

$$
\begin{aligned}
\ln\left(\frac{f^2}{\mu^2}\right) &= \left(-\frac{\lambda}{6} - \frac{\lambda^2}{256\pi^2}\right)\frac{256\pi^2}{4\lambda^2} \\
&= \frac{-42\frac{2}{3}\pi^2}{4\lambda} - \frac{1}{4} \\
&= \frac{-21\frac{1}{3}\pi^2}{2\lambda} - \frac{1}{4} \\
&= -\frac{64\pi^2}{6\lambda} - \frac{1}{4} \\
&= -\frac{32\pi^2}{3\lambda} - \frac{1}{4} \\
\rightarrow \ln\left(\frac{f^2}{\mu'^2}\right) &= -\frac{32\pi^2}{3\lambda} \\
f^2_{\text{min}} &= \mu'^2 \exp\left\{-\frac{32\pi^2}{3\lambda}\right\} \qquad (11.4.22)
\end{aligned}
$$

At this value of f_{min}, we have,

$$V_{\text{TREE}} \ll V_{1-\text{loop}} \qquad (11.4.23)$$

Thus we are beyond the region where perturbation theory can be trusted.

Note: For scalar electrodynamics (See Coleman and Weinberg).

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \qquad (11.4.24)$$

$$
\begin{aligned}
V_{\text{Classical}} &= m^2\phi^*\phi + \lambda(\phi^*\phi)^2 \quad , \quad V_{\text{Tot}} = V_{\text{cl}} + V_{1-\text{loop}} = V_{\text{tot}}(m^2, \lambda, e, f) \\
D_\mu &= \partial_\mu + \mathrm{i}eA_\mu
\end{aligned}
$$

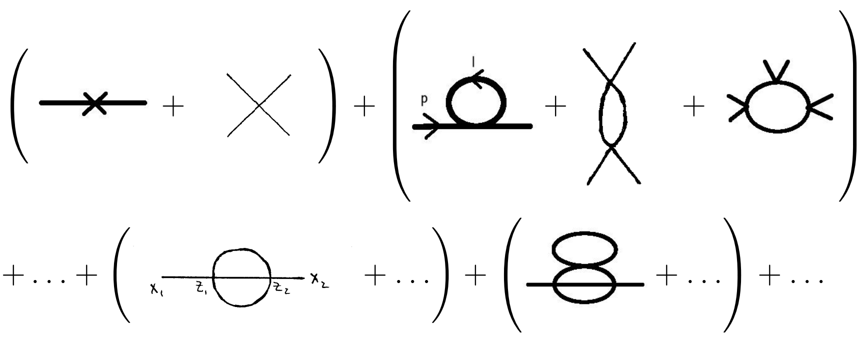
Here,

$$\frac{\partial V_{\text{Tot}}}{\partial f} = 0 \quad \text{at} \quad f = f_{\text{min}} \qquad (11.4.25)$$

This fixes λ in terms of e, but introduces a new parameter f_{min} into the resting effective potential.

"Dimensional Transmutation" (Coleman and Weinberg).

V_{tot} to higher loop order

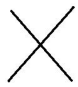


(11.4.26)

The loop expansion is an expansion in powers of $\hbar$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda\phi^4}{4!} \tag{11.4.27}$$

$$Z = \int D\phi e^{(\mathrm{i}/\hbar)\int d^4x \ \mathcal{L}} \tag{11.4.28}$$

Vertices $\sim \frac{1}{\hbar} \quad \rightarrow$  $\rightarrow -\frac{\mathrm{i}\lambda}{\hbar}$

Propagator $\sim \hbar$ 

$\left(\frac{\partial^2 + m^2}{\hbar}\right)^{-1} = \underbrace{\frac{\hbar}{k^2 - m^2}}_{\text{Mom. space}}$

For a given diagram

$$\sim (\hbar)^{I-V} \qquad \begin{array}{l} \text{I} \rightarrow \# \text{ of internal lines} \\ \text{V} \rightarrow \# \text{ of vertices} \end{array} \tag{11.4.29}$$

But in a given diagram,

$$\begin{aligned} L &= \# \text{ of loops} \\ &= I - (V-1) \\ L &= I - V + 1 \end{aligned} \tag{11.4.30}$$

(Where the V is present because the δ-function at each vertex imposes a restriction, but the (-1) in the second line is because one of the δ-functions is superfluous due to "overall" δ-function).
Thus, a given diagram

$$\sim (\hbar)^{L-1} \tag{11.4.31}$$

So, for example, the expression (11.4.26) will have the following powers of $\hbar$:

$$\left(\frac{1}{\hbar}\right) + \left(\hbar^0\right) + \ldots + (\hbar) + \left(\hbar^2\right) + \ldots \tag{11.4.32}$$

11.5 Spontaneous Symmetry Breaking in Gauge Theories

Choice of gauge

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \qquad (11.5.1) \\ \phi(x) &= V + \phi'(x) \quad (V = V^*) \end{aligned}$$

The kinetic energy for scalar:

$$\begin{aligned}\text{K.E.} &= [\partial_\mu\phi'^* - \mathrm{i}eA_\mu(V+\phi'^*)]\,[\partial_\mu\phi' + \mathrm{i}eA_\mu(V+\phi')] \\ &= \partial\phi'^*\partial\phi' + e^2V^2A_\mu + \underbrace{\mathrm{i}e(\partial_\mu\phi'^* - \partial_\mu\phi')VA^\mu}_{\text{leads to mixed propagator}} + \mathcal{O}(\phi^3)\end{aligned} \tag{11.5.2}$$

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}(\partial A)^2 \quad (gf = \text{gauge fixing}) \tag{11.5.3}$$

(Feynman gauge). We can eliminate the mixed propagator by working in the R_ξ gauge.

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial\cdot A - \mathrm{i}\xi eV(\phi'^* - \phi'))^2 \tag{11.5.4}$$

$$\text{If } \phi' = \phi_1 + \mathrm{i}\phi_2$$

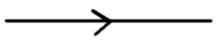
$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial\cdot A - 2\xi eV\phi_2)^2 \tag{11.5.5}$$

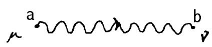
(recall previously, $\phi = \left(\frac{V+\eta}{\sqrt{2}}\right)e^{\mathrm{i}\rho/\sqrt{2}V}$, where the imaginary (exponential) part is eliminated by gauge transf.).

$$\begin{aligned}\mathcal{L}^{(2)} + \mathcal{L}_{gf} &= -\frac{1}{4}(F_{\mu\nu})^2 + (\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + e^2V^2A_\mu A^\mu \quad \Big(\text{From (11.5.2)}\Big) \\ &\quad + 2eVA^\mu(\partial_\mu\phi_2) - \frac{1}{2\xi}(\partial\cdot A)^2 + 2m(\partial_\mu A^\mu)\phi_2 \\ &\quad - 2\xi(eV)^2\phi_2^2 + 2m^2\phi_1^2\end{aligned} \tag{11.5.6}$$

The $A_\mu - \phi_2$ propagator has disappeared.

ϕ_1 Propagator  $$\frac{\mathrm{i}}{k^2+2m^2} \tag{11.5.7}$$

ϕ_2 Propagator  $$\frac{\mathrm{i}}{k^2-\xi m^2} \quad \left[M^2 = \frac{(eV)^2}{2}\right] \tag{11.5.8}$$

A_μ Propagator 

$$\frac{-\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu(1-\xi)}{k^2-\xi M^2}\right)}{k^2-M^2} \tag{11.5.9}$$

Mar. 15/2000
Recall:

$$\phi(x) = V + \phi_1 + \mathrm{i}\phi_2 \tag{11.5.10}$$

$$\begin{aligned} i\Delta_1 &= \frac{i}{k^2 - 2\mu^2} && (11.5.11) \\ i\Delta_2 &= \frac{i}{k^2 - \xi M^2} && (11.5.12) \\ i\Delta_{\mu\nu} &= \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu (1-\xi)}{k^2-\xi M^2}\right)}{k^2 - M^2} && (11.5.13) \end{aligned}$$

R_ξ gauge

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial \cdot A + \xi M \phi_2)^2 \quad (11.5.14)$$

- In the limit $\xi \to 0$

$$\begin{aligned} i\Delta_1 &= \frac{i}{k^2 - 2\mu^2} \quad \text{(Higgs Field)} && (11.5.15) \\ i\Delta_2 &= \frac{i}{k^2} \quad \text{(Serves to act as the longitudinal mode of the vector)} && (11.5.16) \\ i\Delta_{\mu\nu} &= \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{k^2 - M^2} \quad \text{(Purely Transverse)} && (11.5.17) \end{aligned}$$

  This is the "renormalized" (R) gauge.

- If $\xi \to \infty$,

$$\begin{aligned} i\Delta_1 &= \frac{i}{k^2 - 2\mu^2} && (11.5.18) \\ i\Delta_2 &= 0 && (11.5.19) \\ i\Delta_{\mu\nu} &= \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)}{k^2 - M^2} \quad \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right) && (11.5.20) \end{aligned}$$

  → Unitary ("U") gauge.

- $\xi = 1$ (Feynman-'tHooft gauge).

$$\begin{aligned} i\Delta_1 &= \frac{i}{k^2 - 2\mu^2} && (11.5.21) \\ i\Delta_2 &= \frac{i}{k^2 - M^2} && (11.5.22) \\ i\Delta_{\mu\nu} &= \frac{-ig_{\mu\nu}}{k^2 - M^2} && (11.5.23) \end{aligned}$$

  → Easy to calculate with.

Note that Physical effects are independent of ξ.

Chapter 12

Ward-Takhashi-Slavnov-Taylor Identities

The WTST identities are relations between different Green's Functions that follow from gauge invariance. (We'll look at Slavnov's approach).

$$I(a,\underline{b}) = \int d^3\underline{x}\ e^{-ax^2+\underline{b}\cdot\underline{x}} \tag{12.0.1}$$

If

$$\begin{aligned} \underline{x} &\rightarrow \underline{x} + \underline{\varepsilon}\times\underline{x} \quad (\underline{\varepsilon}\approx 0) \\ \underline{x}^2 &\rightarrow \underline{x}^2 \end{aligned}$$

Let

$$\underline{x} = \underline{x}' + \underline{\varepsilon}\times\underline{x}' \tag{12.0.2a}$$

$$d^3\underline{x} = d^3\underline{x}' \quad , \qquad \underline{x}'^2 = \underline{x}^2 \tag{12.0.2b}$$

So,

$$\begin{aligned} I(a,\underline{b}) &= \int d^3\underline{x}'\ e^{-a\underline{x}'^2+\underline{b}\cdot(\underline{x}'+\underline{\varepsilon}\times\underline{x}')} \quad (\text{Re: } \varepsilon\sim 0) \\ &\approx \int d^3\underline{x}'\ e^{-a\underline{x}'^2+\underline{b}\cdot\underline{x}'}\,(1+\underline{b}\cdot\underline{\varepsilon}\times\underline{x}') \end{aligned} \tag{12.0.3}$$

(relabel $\underline{x}'\rightarrow\underline{x}$, and note that the exponent times the "1" in the last line $= I(a,\underline{b})$.)

$$\begin{aligned} 0 &= \int d^3\underline{x}\ e^{-a\underline{x}^2+\underline{b}\cdot\underline{x}}(\underline{b}\cdot\underline{\varepsilon}\times\underline{x}) \\ \rightarrow 0 &= \underline{b}\times\int d^3\underline{x}\ \underline{x}e^{-a\underline{x}^2+\underline{b}\cdot\underline{x}} \\ \Rightarrow 0 &= \frac{\partial}{\partial b_a}\left(\epsilon_{ijk}b_j\int d^3x\ x_k e^{-a\underline{x}^2+b_m x_m}\right) \\ 0 &= \int d^3x\,(\epsilon_{iak}x_k+\epsilon_{ijk}b_j x_k x_a)\,e^{-a\underline{x}^2+\underline{b}\cdot\underline{x}} \end{aligned} \tag{12.0.4}$$

Perform analogous operation in the functional integral.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\,\not{\partial} - e\,\not{A} - m)\psi + J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta - \frac{1}{2}(\partial\cdot A)^2 \tag{12.0.5}$$

$$Z(J_\mu,\bar{\eta},\eta) = \int d\psi\; d\bar{\psi}\; dA_\mu\; e^{\mathrm{i}\int dx\left(\mathcal{L}_{cl}+J\cdot A+\bar{\eta}\psi+\bar{\psi}\eta-\frac{1}{2\alpha}(\partial\cdot A)^2\right)} \tag{12.0.6}$$

Use transformation analogous to (12.0.2a,12.0.2b).

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\Omega \tag{12.0.7a}$$

$$\psi \rightarrow e^{-\mathrm{i}\Omega}\psi \tag{12.0.7b}$$

$$\bar{\psi} \rightarrow \bar{\psi}e^{\mathrm{i}\Omega} \tag{12.0.7c}$$

$$\mathcal{L}_{cl} \rightarrow \mathcal{L}_{cl} \quad \text{(analogous to } \underline{x}'^2 \rightarrow \underline{x}^2) \tag{12.0.7d}$$

$$dA_\mu\; d\psi\; d\bar{\psi} \rightarrow dA_\mu\; d\psi\; d\bar{\psi} \quad \text{(Note demonstrating this \underline{NOT} trivial)} \tag{12.0.7e}$$

Thus,

$$\begin{aligned} Z(J,\bar{\eta},\eta) &= \int dA_\mu\; d\psi\; d\bar{\psi}\; \exp\left\{\mathrm{i}\int dx\; \left[\mathcal{L}_{cl} + J\cdot(A+d\Omega) + \bar{\eta}e^{-\mathrm{i}\Omega}\psi + \bar{\psi}e^{\mathrm{i}\Omega}\eta \right.\right. \\ &\qquad \left.\left. - \frac{1}{2\alpha}(\partial A + \partial^2\Omega)^2\right]\right\} \end{aligned} \tag{12.0.8}$$

For $\Omega \approx 0$ (i.e. expanding exponent)

$$\begin{aligned} &= \int dA_\mu\; d\psi\; d\bar{\psi}\; \exp\left\{\mathrm{i}\int dx\; \left[\mathcal{L}_{cl} + J\cdot A + \bar{\eta}\psi + \bar{\psi}\eta - \frac{1}{2\alpha}(\partial\cdot A)^2\right]\right\}\cdot \\ &\quad \cdot\left\{1 + \int dx\; \left(\mathrm{i}\bar{\psi}(x)\Omega(x)\eta(x) - \mathrm{i}\bar{\eta}(x)\Omega(x)\psi(x) \right.\right. \\ &\quad \left.\left. \underbrace{- \frac{\mathrm{i}}{\alpha}(\partial\cdot A(x))(\partial^2\Omega(x) + \mathrm{i}J_\mu(x)\partial^\mu\Omega(x))}_{=0}\right)\right\} \end{aligned}$$

Let

$$\Omega(x) = \int d^4y D_F(x-y)\kappa(y) \tag{12.0.9}$$

$$\partial^2\Omega(x) = \kappa(x) \tag{12.0.10}$$

Thus,

$$
\begin{aligned}
0 &= \int dA_\mu\, d\psi\, d\bar{\psi}\; e^{\mathrm{i}\int dx\, \left[\mathcal{L}_{cl}+\bar{\psi}\eta+\bar{\eta}\psi+J\cdot A-\frac{1}{2\alpha}(\partial\cdot A)^2\right]} \int d^4x \left[\mathrm{i}\bar{\psi}(x)\eta(x)D_F(x-y)\right. \\
&\quad \left. -\mathrm{i}\bar{\eta}(x)\psi(x)D_F(x-y)+\mathrm{i}J_\mu(x)\frac{\partial}{\partial x^\mu}D_F(x-y)-\frac{\mathrm{i}}{\alpha}(\partial\cdot A(x))\delta^4(x-y)\right] \\
0 &= \int d^4x \left[\mathrm{i}\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\right)\eta(x)D_F(x-y)-\mathrm{i}\bar{\eta}(x)\left(\frac{\delta}{\mathrm{i}\delta\bar{\eta}(x)}\right)D_F(x-y)\right. \\
&\quad \left. -\mathrm{i}\frac{\partial}{\partial x^\mu}\left(\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\right)\delta^4(x-y)+J_\mu(x)\frac{\partial}{\partial x^\mu}D_F(x-y)\right]\cdot Z(J^\mu,\bar{\eta},\eta)
\end{aligned}
$$

$\left(\psi \to \frac{\delta}{\mathrm{i}\delta\bar{\eta}(x)}\right)$. Acto on this with $\left(\frac{\delta}{\mathrm{i}\delta J_\nu(x)}\right)$ and then let $J_\mu = \eta = \bar{\eta} = 0$.

$$
\begin{aligned}
0 &= \int d^4x \left[(-\mathrm{i})\frac{\partial}{\partial x^\mu}\left(\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\mathrm{i}\delta J_\nu(x)}\right)\delta(x-y)Z[J,0,0]+\right. \\
&\quad \left. \mathrm{i}\frac{\partial}{\partial x^\nu}D_F(x-y)Z[0,0,0]\right] \qquad (12.0.11)
\end{aligned}
$$

Mar. 16/2000
From last time:

$$
Z = \int dA_\mu\, d\psi\, d\bar{\psi}\; \exp\left\{\mathrm{i}\int dx\left(\mathcal{L}_{cl}+J\cdot A+\bar{\psi}\eta+\bar{\eta}\psi-\frac{1}{2\alpha}(\partial\cdot A)^2\right)\right\} \qquad (12.0.12)
$$

$$
\begin{aligned}
A_\mu &\to A_\mu+\frac{1}{e}\partial_\mu\Omega \qquad \mathcal{L}_{\bar{\psi}\psi}=\bar{\psi}\left(\mathrm{i}\,\partial\!\!\!/ - e\,A\!\!\!/ - m\right)\psi \\
\psi &\to e^{-\mathrm{i}\Omega}\psi \\
\bar{\psi} &\to \bar{\psi}e^{\mathrm{i}\Omega} \qquad (12.0.13)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int dA_\mu d\psi d\bar{\psi} e^{\mathrm{i}\int dx\left(\mathcal{L}_{cl}+J\cdot A+\bar{\psi}\eta+\bar{\eta}\psi-\frac{1}{2\alpha}(\partial\cdot A)^2\right)}\left[\frac{1}{e}J\cdot\partial\Omega+\mathrm{i}\Omega\bar{\psi}\eta-\mathrm{i}\Omega\bar{\eta}\psi\right.\\
&\quad\left.-\frac{1}{\alpha}(\partial\cdot A)\left(\frac{1}{e}\partial^2\Omega\right)\right]\\
&\to \Omega(x)=\int d^4y D_F(x-y)\kappa(y)\\
0 &= \int dA_\mu d\psi d\bar{\psi}\int dx\ \left[\frac{1}{e}J_\mu(x)\frac{\partial}{\partial x^\mu}D_F(x-y)+\mathrm{i}\bar{\psi}_\alpha(x)\eta_\alpha(x)D_F(x-y)\right.\\
&\quad\left.-\mathrm{i}\bar{\eta}_\alpha(x)\psi_\alpha(x)D_F(x-y)-\frac{1}{e\alpha}\frac{\partial}{\partial x^\mu}A_\mu(x)\delta(x-y)\right]e^{\mathrm{i}\int dx\ (\mathcal{L}_{cl}+\ldots)}\\
0 &= \left[\frac{1}{e}J_\mu(x)\frac{\partial}{\partial x^\mu}D_F(x-y)+\mathrm{i}\frac{\delta}{\mathrm{i}\delta\eta_\alpha(x)}\eta_\alpha(x)D_F(x-y)-\mathrm{i}\bar{\eta}_\alpha(x)\frac{\delta}{\mathrm{i}\delta\bar{\eta}_\alpha(x)}D_F(x-y)\right.\\
&\quad\left.-\frac{1}{e\alpha}\frac{\partial}{\partial x^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\delta(x-y)\right]Z\\
&\to \times\text{ by }\frac{\delta}{\delta J_\nu(z)}\quad\text{and let } J=\eta=\bar{\eta}=0\\
0 &= \int dx\ \left[\frac{1}{e}g_{\mu\nu}\delta(x-z)\frac{\partial}{\partial x^\mu}D_F(x-y)-\frac{1}{e\alpha}\frac{\partial}{\partial x^\mu}\delta^4(x-y)\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\delta J_\nu(z)}\right]Z[J,0,0]\Bigg|_{J=0}
\end{aligned}
$$

But,

$$
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle=\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}Z[J,0,0]\Bigg|_{J=0}
$$

$$
\begin{aligned}
0 &= -\frac{1}{\alpha}\int d^4x\ \left[\frac{\partial}{\partial x^\mu}\mathrm{i}\langle 0|TA_\mu(x)A_\nu(y)|0\rangle\delta^4(x-y)\left(-\frac{1}{\alpha}\right)+\delta(x-z)\frac{\partial}{\partial x^\nu}D_F(x-y)\right]\\
0 &= -\frac{\mathrm{i}}{\alpha}\frac{\partial}{\partial y^\mu}\langle 0|TA_\mu(y)A_\nu(z)|0\rangle+\frac{\partial}{\partial z^\nu}D_F(z-y)\\
0 &= \frac{\mathrm{i}}{\alpha}\frac{\partial}{\partial y^\mu}\langle 0|TA_\mu(y)A_\nu(z)|0\rangle+\frac{\partial}{\partial y^\nu}D_F(y-z) \qquad (12.0.15)
\end{aligned}
$$

Thus if

$$
\begin{aligned}
\langle 0|TA_\mu(y)A_\nu(z)|0\rangle &= \int\frac{d^4k}{(2\pi)^4}e^{\mathrm{i}k\cdot(y-z)}\mathrm{i}\Pi_{\mu\nu}(k) \qquad (12.0.16)\\
D_F(y-z) &= \int\frac{d^4k}{(2\pi)^4}\frac{e^{\mathrm{i}k\cdot(y-z)}}{k^2+\mathrm{i}\varepsilon} \qquad (12.0.17)
\end{aligned}
$$

Hence

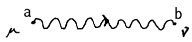
$$
0=-\frac{k_\mu}{\alpha}\Pi_{\mu\nu}(k)+\frac{(\mathrm{i}k_\nu)}{k^2} \qquad (12.0.18)
$$

$\langle 0|TA_\mu(y)A_\nu(z)|0\rangle_{FT}$ =  +  + 

+  + ...

$$= \mathrm{i}\Pi_{\mu\nu}(k) \tag{12.0.19}$$

Here,


$$= \frac{-\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu(1-\alpha)}{k^2}\right)}{k^2 + \mathrm{i}\varepsilon} \tag{12.0.20}$$

Hence,

$$0 = -\frac{k_\mu}{\alpha}\left[\frac{-\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu(1-\alpha)}{k^2}\right)}{k^2 + \mathrm{i}\varepsilon}\right] + \frac{\mathrm{i}k_\nu}{k^2}$$

but $k_\mu\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) = 0$. Thus,

$0 = k_\mu$ [ +  +  + ...]

And so,

 +  + ...

must be proportional to

$$g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{12.0.23}$$

i.e.

 +  + ... $= (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$

$$= k^2 g_{\mu\nu}\Pi_1(k^2) - k_\mu k_\nu \Pi_2(k^2) + m^2\Pi_3(k^2)g_{\mu\nu}$$

Must have $\rightarrow$ $\Pi_1(k^2) = -\Pi_2(k^2)$

$\Pi_3(k^2) = 0$

Thus, all divergences are proportional to $(g_{\mu\nu}k^2 - k_\mu k_\nu)$. $\rightarrow$ No divergences proportional to $g_{\mu\nu}$ alone.
If,

$$\begin{aligned} \mathcal{L}_{cl} = -\frac{1}{4}F_{\mu\nu}^2 &= \frac{1}{4}\left(k_\mu A_\nu(k) - k_\nu A_\mu(k)\right)^2 \\ &= \frac{1}{2}A_\nu(k^2 g_{\mu\nu} - k_\mu k_\nu)A_\nu \end{aligned}$$

$\rightarrow$ divergences and $\mathcal{L}_{cl} \sim k^2 g_{\mu\nu} - k_\mu k_\nu$.

Thus all divergences in the vacuum polarization can be removed by a wave function renormalization.

$$A_\mu^{\text{bare}} = Z_3^{1/2} A_\mu^{\text{Renormalized}} \tag{12.0.24}$$

i.e. if $\Pi_1 \neq -\Pi_2$, $\Pi_3 \neq 0$,

$$= \underbrace{\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right) \overbrace{\Pi_1}^{\text{log div.}}}_{\substack{\text{divergence can be absorbed into} \\ A_\mu(g_{\mu\nu}k^2 - k_\mu k_\nu)A_\nu}} + \underbrace{(\Pi_2 + \Pi_1) \overbrace{k_\mu k_\nu}^{\text{log div.}}}_{\text{need } A_\mu \partial_\mu \partial_\nu A_\nu} + \underbrace{\overbrace{m^2 \Pi_3 g_{\mu\nu}}^{\substack{\text{quadratic} \\ \text{div.}}}}_{m_R^2 A_\mu A_\nu} \tag{12.0.25}$$

- Π_1, $\Pi_2 \rightarrow$ Log. div.
- $\Pi_3 \rightarrow$ quadratic

Recall:

$$\begin{aligned} &= (-1)(-\mathrm{i}e)^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left[\frac{\mathrm{i}(\not\ell + m)}{\ell^2 - m^2 + \mathrm{i}\varepsilon}\gamma_\mu \frac{\mathrm{i}(\not\ell + \not k + m)}{(\ell + k)^2 - m^2 + \mathrm{i}\varepsilon}\gamma_\nu\right] && (12.0.26) \\ &= \text{appears to be quadratic divergent} && (12.0.27) \end{aligned}$$

Gauge invariance overrides simple power counting arguments.

- Quadratic divergence disappears
- Automatically satisfied in dimensional Regularization.

Mar. 20/2000

12.1 Dimensional Regularization with Spinors

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^a}{(k^2 - m^2)^b} = \frac{\mathrm{i}}{(4\pi)^{n/2}} (-1)^{ab} (m^2)^{(n/2)+a-b} \frac{\Gamma\left(\frac{n}{2} + a\right)\Gamma\left(b - a - \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(b)} \tag{12.1.1}$$

with spinors,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad ; \quad g^\mu{}_\mu = n \tag{12.1.2}$$

$$\begin{aligned}
\gamma_\mu \gamma^\alpha \gamma^\eta &= \left[\frac{\gamma_\mu \gamma^\mu + \gamma^\mu \gamma_\mu}{2g_\mu{}^\alpha} - \gamma^\alpha \gamma_\mu \right] \gamma^\mu \\
&= 2\gamma^\alpha - \gamma^\alpha \gamma_\mu \gamma^\mu \\
&= 2\gamma^\alpha - \gamma^\alpha \left(\frac{1}{2} \left(\gamma_\mu \gamma^\mu + \gamma^\mu \gamma_\mu \right) \right) \\
&= 2\gamma^\alpha - \gamma^\alpha \frac{1}{2} (2g^\mu{}_\mu) \\
&= (2-n)\gamma^\alpha && (12.1.3a) \\
&\quad \to n = 4 \\
&= -2\gamma^\alpha && (12.1.3b)
\end{aligned}$$

12.1.1 Spinor Self-Energy

$$\begin{aligned}
0 &= \Bigg[-\frac{1}{\alpha} \frac{\partial}{\partial y^\mu} \left(\frac{\delta}{\mathrm{i}\delta J^\mu(y)} \right) + \mathrm{i}e \int dx\ \bar{\xi}(x) \frac{\delta}{\mathrm{i}\delta \bar{\xi}(x)} D_F(x-y) + \mathrm{i}e \int dx\ \xi(x) \frac{\delta}{\mathrm{i}\delta \xi(x)} D_F(x-y) \\
&\quad + \int dx\ J_\mu(x) \frac{\partial}{\partial x^\mu} D_F(x-y) \Bigg] Z[J_\mu, \xi(x), \bar{\xi}(x)] && (12.1.4)
\end{aligned}$$

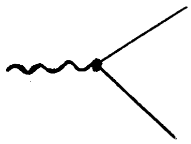
Take $\frac{\delta}{\delta\xi(w)} \frac{\delta}{\delta\bar{\xi}(z)}$ of (12.1.4) $\xi = \bar{\xi} = J_\mu = 0$. We get,

$$-\frac{1}{\alpha} \frac{\partial}{\partial x_\mu} \langle 0|TA_\mu(x)\bar{\psi}(y)\psi(z)|0\rangle = e \left[D_F^{(0)}(x-y) - D_F^{(0)}(x-z) \right] \underbrace{\langle 0|T\bar{\psi}(y)\psi(z)|0\rangle}_{S_F(y-z)} \tag{12.1.5}$$

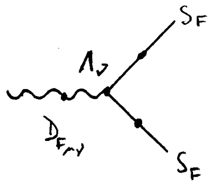
(Note that $\Box D_F^{(0)}(x-y) = \delta(x-y)$). In F.T. space, if

$$\begin{aligned}
\langle 0|TA_\mu(x)\bar{\psi}(y)\psi(z)|0\rangle &= e \int dx'\ dy'\ dz'\ D_F(x-x') S_F(y-y') \Lambda_\mu(x', y', z') S_F(z'-z) && (12.1.6) \\
&\quad \to \partial^x_\mu D^F_{\mu\nu}(x-x') = \alpha \frac{\partial}{\partial x_\nu} D_F^{(0)}(x-x')
\end{aligned}$$

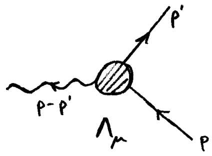
→ amputating external legs:



$=$



$$\rightarrow (p-p')\Lambda_\mu(p,p') \;=\; \left(S_F^{-1}(p')\right) - \left(S_F^{-1}(p)\right)$$



In the limit $p \rightarrow p'$,

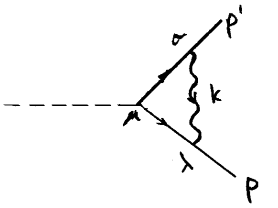
$$\Lambda_\mu(p,p) = \frac{\partial}{\partial p^\mu} S_F^{-1}(p) \qquad \underline{\text{Ward Identity}} \tag{12.1.7}$$

At tree level, let $\Lambda_\mu = \gamma_\mu$

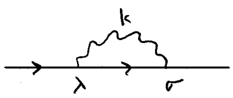
$$\begin{aligned} S_F^{-1} &= \not p - m \\ \rightarrow \frac{\partial}{\partial p^\mu} S_F^{-1} &= \gamma_\mu = \Lambda_\mu \quad \text{at tree level} \end{aligned}$$

(trivial at tree level to get Ward identity). One loop:

$$\Lambda_\mu = \int \frac{d^4k}{(2\pi)^4}(-\mathrm{i}e\gamma_\lambda)\left(\frac{\mathrm{i}}{\not p + \not k - m}\right)(-\mathrm{i}e\gamma_\mu)\left(\frac{\mathrm{i}}{\not p' - \not k - m}\right)(-\mathrm{i}e\gamma_\sigma)\left(\frac{-\mathrm{i}g^{\lambda\sigma}}{k^2 - \mathrm{i}\varepsilon}\right) \tag{12.1.8}$$



$S_F =$



$$= \int \frac{d^4k}{(2\pi)^4}(-\mathrm{i}e\gamma_\lambda)\left(\frac{\mathrm{i}}{\not p + \not k - m}\right)(-\mathrm{i}e\gamma_\sigma)\left(\frac{-\mathrm{i}g^{\lambda\sigma}}{k^2 - \mathrm{i}\varepsilon}\right) \tag{12.1.9}$$

→ Bj. and Drell

(One loop order, evident (12.1.7) holds also).
In general, as $\not p \to m$,

$$S_F(p) \quad \to \quad \frac{Z_2}{\not p - m - \Sigma_c(p)} \qquad (12.1.10)$$
$$\left(\Sigma_c(p) \to 0 \text{ as } p^2 \to m^2\right)$$

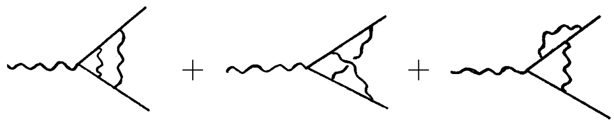
And vertex function,

$$\bar{u}(p)\Lambda_\mu(p,p)u(p) = Z_1^{-1}\bar{u}(p)\gamma_\mu u(p) \qquad (12.1.11)$$

Thus our Ward identity gives $Z_1 = Z_2$.
Tells us, if we compute a self-energy diagram,

$$\psi_{\text{Bare}} = Z_2^{1/2}\psi_{\text{Renormalized}} \quad , \quad A_\mu^{\text{Bare}} = Z_3^{1/2}A_\mu^{\text{Ren.}} \qquad (12.1.13)$$

And look at all vertex functions,

Z_1 :  (12.1.14)

(Z_1 parameterizes ∞'s in diagrams).

$$e_{\text{Bare}} = \frac{Z_1}{Z_3^{1/2}Z_2}e_{\text{Renormalized}} \qquad (12.1.15)$$

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B\right)^2 + \bar{\psi}^B\left(\not p - m^B\right)\psi - e^B A_\mu^B \bar{\psi}_B\gamma^\mu\psi_B \\
&= -\frac{1}{4}Z_3\left(\partial_\mu A_\nu^R - \partial_\nu A_\mu^R\right)^2 + Z_2\bar{\psi}^R\left(\not p - m^R + \delta m\right)\psi^R \\
&\quad - \frac{Z_1}{Z_3^{1/2}Z_2}e^R(Z_3^{1/2}A_\mu^R)Z_2\bar{\psi}^R\gamma^\mu\psi^R \qquad (12.1.16)
\end{aligned}$$

(What we would expect from renormalization)

We also have

$$\begin{aligned}
Z_1 &= Z_2 \qquad (12.1.17) \\
e^B &= \frac{Z_1}{Z_3^{1/2}Z_2}e^R \\
&= Z_3^{-1/2}e^R \qquad (12.1.18)
\end{aligned}$$

Suppose we have 2 fermions (e^-, proton):

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A^B_\nu - \partial_\nu A^B_\mu\right)^2 + \bar{\psi}^B\left(\not{p} - m^B\right)\psi - e^B A^B_\mu \bar{\psi}_B \gamma^\mu \psi_B \\ &\quad + \bar{\Psi}^B\left(\not{p} - M^B\right)\Psi^B - \underbrace{e^B A^B_\mu \bar{\Psi}_B \gamma^\mu \Psi_B}_{e^B \text{ for proton}} \end{aligned} \quad (12.1.19)$$

Renormalization of proton will be different. e^B for proton $\rightarrow$ Ward identity insures $Z_1 = Z_2$. (Lurie).
i.e. e^B same for e^-, proton, $\rightarrow$ Ward identity ensures e^R will be same $\{$ Z_1 for $e^- \neq Z_1$ for p, but Z_2 for $e^- \neq Z_2$ for p either$\}$.

This has the effect that if 2 different charged fields have the same bare charge, then they will have same renormalized charge.

12.2 Yang-Mills Theory

Mar. 22/2000

$$Z(J_\mu) = \int \mathcal{D}A_\mu \Delta_F(A_\mu) \exp\left\{\mathrm{i}\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\alpha}(\partial\cdot A)^2 + J\cdot A\right)\right\} \quad (12.2.1)$$

$$\Delta_F(A) = \det(\partial \cdot D(A)) \quad , \quad D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{apb} A^p_\mu \quad (12.2.2)$$

$$\Delta^{-1}_F(A) = \int d\Omega\ \delta(\partial A^\Omega) \quad , \quad A^{a\Omega}_\mu = A^a_\mu + D^{ab}_\mu(A)\Omega^b \quad (12.2.3)$$

$$\Delta^{-1}_F(A^{\Omega'}) = \int d\Omega\ \delta(\partial\cdot A^{\Omega'\Omega}) \quad (12.2.4)$$

But, $A^{\Omega'\Omega}_\mu = A^{\Omega''}_\mu$ and $\int d\Omega \rightarrow$ integrate over <u>all</u> gauge transformations.

$$\begin{aligned}\Delta^{-1}_F(A^{\Omega'}) &= \int d\Omega''\delta(\partial\cdot A^{\Omega''}) \\ &= \Delta^{-1}_F(A_\mu) \end{aligned} \quad (12.2.5)$$

Thus,

$$\Delta_F(A_\mu) = \Delta_f(A^\Omega_\mu) \quad (12.2.6)$$

In the functional integral for Z, let,

$$A_\mu \rightarrow A_\mu + D_\mu\Omega \quad (12.2.7)$$

$$DA_\mu \rightarrow DA^\Omega_\mu = DA_\mu \quad (12.2.8)$$

Thus,

$$Z[J] = \int \mathcal{D}A_\mu \Delta_F(A_\mu) \exp\left\{\mathrm{i}\int dx \left(\mathcal{L}_{YM} - \frac{1}{2\alpha}(\partial\cdot A^\Omega)^2 + J\cdot A^\Omega\right)\right\} \quad (12.2.9)$$

For small Ω,

$$\begin{aligned} Z[J] &= \int \mathcal{D}A_\mu \Delta_F(A) \exp\left\{ \mathrm{i} \int dx \ \left(\mathcal{L}_{YM} - \frac{1}{2\alpha}(\partial \cdot A^\Omega)^2 + J \cdot A \right) \right\} \cdot \\ &\quad \cdot \left[1 + 2\left(\frac{-1}{2\alpha}\right)(\partial \cdot A)(\partial \cdot D\Omega) + J \cdot (D\Omega) \right] \\ 0 &= \int \mathcal{D}A_\mu \Delta_F(A) e^{\mathrm{i}\int dx \ \left[\mathcal{L}_{YM} - \frac{1}{2\alpha}(\partial \cdot A)^2 + J\cdot A\right]} \int d^4x \left[-\frac{1}{\alpha}(\partial_\mu A^{\mu a}(x)) \overbrace{(\partial_\nu D^{ab}_\nu(A)\Omega^b(x))}^{\kappa^b} \right. \\ &\quad \left. + J^a_\mu(x) D^{ab\mu}(A) \underbrace{\Omega^b(x)}_{M^{bc}\kappa^c} \right] \end{aligned} \tag{12.2.10}$$

If

$$\partial_\mu D^{ab\mu}(A) M^{bc}(x,y) = \delta^\mu(x-y)\delta^{ac} \tag{12.2.11}$$

and

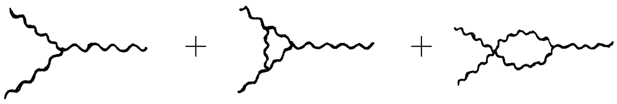
$$\begin{aligned} \kappa^a(x) &= \int d^4y \ M^{ab}(x,y)\Omega^b(y) \\ &\Rightarrow \ \partial_\mu D^{ab\mu}\kappa^b(x) = \Omega^a(x) \end{aligned} \tag{12.2.12}$$

Then, as $A^a_\mu(x) \to \frac{1}{\mathrm{i}}\frac{\delta}{\delta J^a_\mu(x)}$, we have,

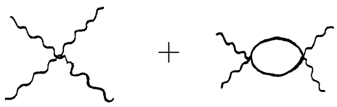
$$0 = \left\{ \int d^4x \ \left[-\frac{1}{\alpha}\left(\frac{\partial}{\partial x^\mu}\frac{\delta}{\mathrm{i}\delta J^a_\mu(x)}\right)\kappa^a(x) \right] + \int d^4x \ J^a_\mu(x) D^{ab\mu}\left(\frac{\delta}{\mathrm{i}\delta J}\right) \underbrace{(M^{bc})^{-1}(x,y)}_{\text{Messy}} \kappa^b(y) \right\} Z \tag{12.2.13}$$

From the resulting Ward identities, (WTST identities), it was proven that the coupling constant g could be renormalized by considering <u>any</u> vertex into which it enters, and the same result would arise.

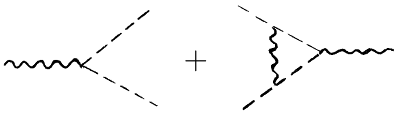
i.e.



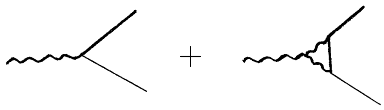
$+\dots$ $(A-A-A$ vertex)



$+\dots$ $(A-A-A-A$ vertex)



$+\dots$ $(c-\bar{c}-A$ vertex)



$+\dots$ $(\psi-\bar{\psi}-A$ vertex)

If $g_B = Z_g g_R$, Z_g is the same for all 4 vertices $\Rightarrow$ Gauge invariance of Lagrangian has not been altered. $\rightarrow$ Renormalization doesn't break gauge invariance.

12.3 BRST Identities

WSTS $\rightarrow$ Were derived by using a local, linear transformation on A^a_μ that left $\mathcal{L}_{YM}$ and $\Delta_F(A)$ invariant.

BRST identities involve global non-linear transformations which leave $\mathcal{L}_{YM} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{gf}$ invariant (gf = gauge fixing).

$$Z[J^a_\mu, \eta^a, \bar{\eta}^a] = \int \mathcal{D}A^a_\mu \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\left\{ \mathrm{i} \int d^4x \left(-\frac{1}{4}(F^a_{\mu\nu}(A))^2 - \bar{c}^a D^{ab}_\mu(A)\partial^\mu c^b - \frac{1}{2\alpha}(\partial\cdot A^a)^2 + J^a_\mu A^{a\mu} + \bar{c}^a\eta^a + \bar{\eta}^a c^a \right)\right\} \tag{12.3.1}$$

Remarkably,

$$\mathcal{L}_{eff} = \mathcal{L}_{YM} + \mathcal{L}_{gh} + \mathcal{L}_{gf} \tag{12.3.2}$$

is invariant under

$$\begin{aligned} A_\mu^a &\to A_\mu^a + D_\mu^{ab}(A)c^b\varepsilon \quad (\varepsilon,\ c \to \text{ Grassmann}) &(12.3.3)\\ c^a &\to c^a - \frac{1}{2}f^{abc}c^bc^c\varepsilon &(12.3.4)\\ \bar{c}^a &\to \bar{c}^a + \frac{1}{\alpha}(\partial\cdot A^a)\varepsilon &(12.3.5)\end{aligned}$$

(The Proof involves the Jacobi identity).

To derive the BRST identities, let $A \to A + (Dc)\varepsilon$,etc., in the functional integral, and to leading order in ε,

$$\begin{aligned} Z[J] &= \int dA\ dc\ d\bar{c}\ \exp\left\{\mathrm{i}\int dx\ \ (\mathcal{L}_{eff} + J\cdot A + \bar{\eta}c + \bar{c}\eta)\right\}\left[1 + \int d^4x\ \left(J_\mu^a D^{ab\mu}(A)c^b\varepsilon\right.\right.\\ &\quad \left.\left. + \frac{1}{\alpha}(\partial\cdot A^a)c^a + \frac{1}{2}\bar{\eta}^a f^{abc}c^bc^c\varepsilon\right)\right]\\ 0 &= \int d^4x\left[J_\mu^a(x)D^{ab}\left(\frac{\delta}{\mathrm{i}\delta J}\right)\left(\frac{\delta}{\mathrm{i}\delta\bar{\eta}^b(x)}\right) + \frac{1}{\alpha}\frac{\partial}{\partial x^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu^a}\left(\frac{\delta}{\mathrm{i}\delta\bar{\eta}^a(x)}\right)\right.\\ &\quad \left. - \frac{1}{2}f^{abc}\bar{\eta}^a(x)\left(\frac{\delta}{\mathrm{i}\delta\bar{\eta}^b(x)}\right)\left(\frac{\delta}{\mathrm{i}\delta\bar{\eta}^c(x)}\right)\right]Z[j,\eta,\bar{\eta}] \qquad (12.3.6)\end{aligned}$$

This can be streamlined:

$$\delta_{BRST}\overbrace{\langle 0|T\mathcal{O}_1\ldots\mathcal{O}_N|0\rangle}^{\text{Any Green's Fn.}} = 0 \qquad (12.3.7)$$

Consider

$$\begin{aligned} 0 &= \delta_{BRST}\langle 0|TA_\mu^a(x)\bar{c}^b(y)|0\rangle\\ 0 &= \langle 0|T\left(D_\mu^{ac}(A)c^c(x)\varepsilon\bar{c}^b(y)\right)|0\rangle\\ &\quad +\langle 0|\left(TA_\mu^a(x)\frac{1}{\alpha}\frac{\partial}{\partial y^\nu}A_\nu^b(y)\varepsilon\right)|0\rangle\\ \frac{1}{\alpha}\frac{\partial}{\partial y^\nu}\langle 0|TA_\mu^a(x)A_\nu^b(y)|0\rangle &= \langle 0|TD_\mu^{ab}(A)c^c(x)\bar{c}^b(y)|0\rangle \qquad (12.3.8)\end{aligned}$$

Typical of BRST identity → Relates two Green's Functions.
Note: $\langle 0|TA_\mu^a(x)A_\nu^b(y)|0\rangle$ is transverse only if $\alpha = 0$ (Landau gauge).
Mar. 23/2000

12.4 Background Field Quantization

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{12.4.1}$$

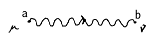
$$Z[j] = \int d\phi \, \exp\left\{ \mathrm{i} \int d^4x \, (\mathcal{L}(\phi) + J\phi) \right\} \tag{12.4.2}$$

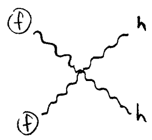
write $\phi(x) = f(x) + h(x)$, ($f(x)$ = Classical Background, and $h(x)$ = Quantum Correction).

$$\tilde{Z}(f, J) = \int \mathcal{D}h \, \exp\left\{ \mathrm{i} \int d^4x \, (\mathcal{L}(f+h) + Jh) \right\} \tag{12.4.3}$$

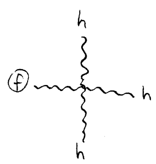
$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu (f+h)\right)^2 - \frac{m^2}{2}(f+h)^2 - \frac{\lambda}{4!}(f+h)^4 \\ &= \underbrace{\frac{1}{2}(\partial_\mu f)^2 - \frac{m^2}{2}f^2 - \frac{\lambda}{4!}f^4}_{\text{Classical Lagrangian}} + \underbrace{f(\ldots)}_{\substack{\text{Can be} \\ \text{neglected}}} \\ &\quad + \underbrace{\frac{1}{2}(\partial_\mu h)^2 - \frac{m^2 h^2}{2}}_{\mathfrak{A}} - \underbrace{\frac{\lambda}{4!}(6f^2h^2)}_{\mathfrak{B}} + \frac{\lambda}{4!}(\underbrace{4fh^3}_{\mathfrak{C}} + \underbrace{h^4}_{\mathfrak{D}}) \end{aligned} \tag{12.4.4}$$

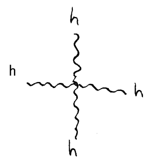
With

$\mathfrak{A}$ =  $= \dfrac{\mathrm{i}}{p^2 - m^2}$

$\mathfrak{B}$ = 

$\to$ 4 - pt.

$\mathfrak{C}$ = 

$\mathfrak{D}$ = 

(12.4.5)

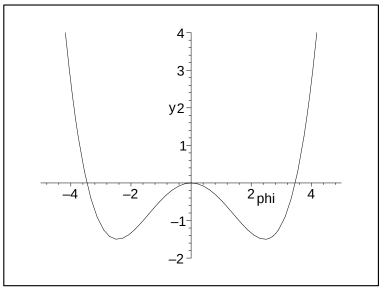


Figure 12.4.1: $\phi = V + \phi' \to V-$ background - (Just some const. previously). Where $V(\phi) = \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!}$

Aside: Recall the Figure 12.4.1:

If we consider only diagrams with external fields, we will reproduce $Z(J)$. Let $h \to h - f$ in $\tilde{Z}$.

$$\begin{aligned} \tilde{Z}(f, J) &= \int \mathcal{D}h \ \exp\left\{ \mathrm{i} \int dx \ \ (\mathcal{L}(h) + J(h - f)) \right\} \\ &= e^{-\mathrm{i}\int dx \ Jf} Z(J) \end{aligned} \tag{12.4.6}$$

$$\left. \begin{aligned} W(J) &= -\mathrm{i}\ln(Z(J)) \\ \tilde{W}(f, J) &= -\mathrm{i}\ln(\tilde{Z}(f, J)) \end{aligned} \right\} \text{Generating functional for connected diagrams} \tag{12.4.7}$$

Thus,

$$\begin{aligned} \tilde{W}(f, J) &= -\mathrm{i}\ln\left(e^{-\mathrm{i}\int d^4x \ Jf} Z(J)\right) \\ &= -\mathrm{i}\ln\left(Z(J)\right) - \mathrm{i}\ln\left(e^{-\mathrm{i}\int d^4x \ Jf}\right) \\ &= W(J) - \int d^4x \ Jf \end{aligned} \tag{12.4.8}$$

$$\Phi \equiv \frac{\delta W(J)}{\delta J} \tag{12.4.9}$$

$$\Gamma(\Phi) = W(J) - \int d^4x \ J\Phi \tag{12.4.10}$$

$$= \text{Generating functional for 1PI diagrams}$$

So also,

$$\tilde{\Phi} \equiv \frac{\delta \tilde{W(J)}}{\delta J} \tag{12.4.11}$$

$$\tilde{\Gamma}(\Phi) = \tilde{W}(J) - \int d^4x \ J\tilde{\Phi} \tag{12.4.12}$$

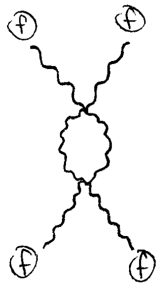
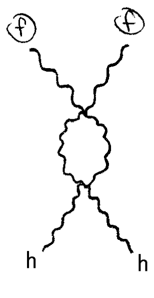
$$\begin{aligned} \tilde{\Phi} &= \frac{\delta}{\delta J}\left(W - \int dx \ Jf\right) \\ &= \Phi - f \\ \therefore \Phi &= \tilde{\Phi} + f \end{aligned} \tag{12.4.13}$$

So also

$$\begin{aligned}
\tilde{\Gamma}(f,\tilde{\Phi}) &= \tilde{W}(f,J) - \int dx\ J\tilde{\phi} \\
&= W(J) - \int d^4x\ Jf - \int d^4x\ J\tilde{\Phi} \\
&= W(J) - \int d^4x\ \Phi \\
&= \Gamma(\Phi)
\end{aligned} \tag{12.4.14}$$

Hence,

$$\begin{aligned}
\Gamma(\Phi) &= \tilde{\Gamma}(\tilde{f},\tilde{\Phi}) \\
&\text{or} \\
\Gamma(f+\tilde{\Phi}) &= \tilde{\Gamma}(f,\tilde{\Phi}) \\
\rightarrow \text{ if } \tilde{\Phi} = 0 & \\
\Gamma(f) &= \tilde{\Gamma}(f,0)
\end{aligned} \tag{12.4.15}$$

$\Gamma(\Phi)$	$\tilde{\Gamma}(\tilde{\Phi})$	
$\mathcal{L} = \mathcal{L}(\phi)$		
		
		

If we make the identification $\phi \rightarrow f$, we can ignore the diagrams on the left in $\tilde{\Gamma}$ column,

and get the diagram on the right.
Applying this to YM theory,

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}(V))^2 \quad (12.4.16)$$

$$Z[J] = \int dV\ \Delta_F \exp\left\{\mathrm{i}\int dx\ (\mathcal{L} + \mathcal{L}_{gf} + JA)\right\} \quad (12.4.17)$$

$$V^a_\mu(x) = A^a_\mu(x) + Q^a_\mu(x)$$

$$\tilde{Z}[A^a_\mu, J^a_\mu] = \int \mathcal{D}Q\ \exp\left\{\mathrm{i}\int dx\ \left(\mathcal{L}(A+Q) + \mathcal{L}_{gf}(A+Q) + J\cdot Q\right)\right\} \quad (12.4.18)$$

$$\begin{aligned} \delta V^a_\mu &= D^{ab}_\mu(V)\Omega^b \\ &= \partial_\mu \Omega^a + g f^{abc} V^b_\mu \Omega^c \end{aligned}$$

This can become:

(i) $\left\{\begin{array}{l} \delta A^a_\mu = 0 \\ \delta Q^a_\mu = \partial_\mu\Omega^a + g f^{abc}(A^b_\mu + Q^b_\mu)\Omega^c \end{array}\right\}$ Choose $\mathcal{L}_{gf}$ so that this gauge invariance is broken.

(ii) $\left\{\begin{array}{l} \delta A^a_\mu = \partial_\mu\Omega^a + g f^{abc} A^a_\mu \Omega^c \\ \delta Q^a_\mu = g f^{abc} Q^b_\mu \Omega^c \end{array}\right\}$ Leave this intact.

Homerkemp gauge,

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}(D^{ab}_\mu(A) Q^b_\mu)^2 \quad (12.4.19)$$

Leaves type **(ii)** gauge invariance unbroken.

$$\tilde{\Gamma}(A^a_\mu, \tilde{Q}^a_\mu) = \Gamma(A^a_\mu + \tilde{A}^a_\mu) \quad (12.4.20)$$

in same way as in scalar case $\rightarrow \tilde{\Gamma}(f, \tilde{\Phi}) = \Gamma(f + \Phi)$.
$\rightarrow$ Let $\tilde{Q} = 0$. Thus,

$$\Gamma(A^a_\mu) = \underbrace{\tilde{\Gamma}(A^a_\mu, 0)}_{*} \quad (12.4.21)$$

* - Invariant under $A^a_\mu \rightarrow A^a_\mu + D^{ab}_\mu(A)\Omega^b$. Thus,

$$\tilde{\Gamma}(A^a_\mu) = \tilde{\Gamma}(\mathrm{Tr}\left[(F^a_{\mu\nu}(A))^2\right] + \mathrm{Tr}\left[(F^a_{\mu\nu}(A))^3\right]) \quad (12.4.22)$$

Mar. 27/2000
Recall: if we have

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}(D^{ab}_\mu(A) Q^{b\mu})^2 \quad (12.4.23)$$

then $Z[A^a_\mu] \Rightarrow$ invariant under $\delta A^a_\mu = D^{ab}_\mu(A)\Omega^b$. This gauge fixing ($\mathcal{L}_{gf}$) breaks the invariance.

$$\begin{aligned} \delta A^a_\mu &= 0 && (12.4.24) \\ \delta Q^a_\mu &= D^{ab}_\mu(A+Q)\Omega^b && (12.4.25) \end{aligned}$$

If we follow through with the standard way of deriving the ghost Lagrangian, we find that,

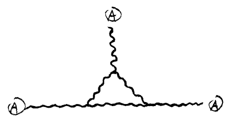
$$\Rightarrow \mathcal{L}_{ghost} = -\bar{c}^a D^{ab}_\mu(A) D^{bc}_\mu(A+Q)c^c \tag{12.4.26}$$

(where $\bar{c}, c$ are Grassmann ghost fields). Z depends only on $F^a_{\mu\nu}(A)F^a_{\mu\nu}(A)$, etc. (i.e. another would be $\epsilon^{abc}F^a_{\mu\lambda}F^b_{\lambda\nu}F^c_{\nu\mu} \rightarrow$ must construct Z out of gauge invariant quantities).

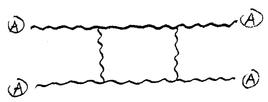
But divergences, by simple power counting, can arise only in 2, 3 and 4 point functions.

$\sim$ quadratic divergent $\displaystyle\int d^4k \frac{k_\mu k_\nu}{k^2k^2}$ (12.4.27)

($k_\mu,\ k_\nu$ at each vertex)

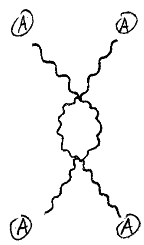


$\sim$ Linear $\displaystyle\int d^4k \frac{k_\mu k_\mu k_\mu}{k^2k^2k^2}$ (12.4.28)



$\sim$ log (12.4.29)

$+$



$\sim$ log (12.4.30)

The divergences in all three diagrams can only serve to renormalize

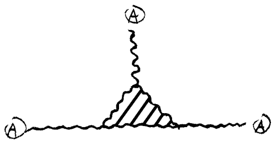
$$F^a_{\mu\nu}F^{a\mu\nu} \rightarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{12.4.31}$$

We can get divergences in (12.4.28,12.4.29) diagrams automatically if we know the divergences in (12.4.27). Hence, the divergences in $Z(A)$ are of the form,

$$\sum_{j=1}^{\infty} \frac{a_j}{\varepsilon^j} F^a_{\mu\nu}(A)F^a_{\mu\nu}(A) \tag{12.4.32}$$

ex:

$$\epsilon^{abc} F^a_{\mu\lambda} F^b_{\lambda\nu} F^c_{\nu\mu} \quad \rightarrow \quad \epsilon^{abc} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \left(\partial_\lambda A^a_\nu - \partial_\nu A^a_\lambda\right) \left(\partial_\nu A^a_\mu - \partial_\mu A^a_\nu\right)$$



$$ppp \int d^4k \frac{1}{k^2k^2k^2} \quad \left(\begin{array}{c}\text{Not divergent -} \\ \text{no renormalization req.}\end{array}\right) \tag{12.4.33}$$

$F^a_{\mu\nu} F^{a\mu\nu} \rightarrow$ one power of external momentum associated with it.

$$\rightarrow \underbrace{\int d^4k \frac{k_\mu k_\nu}{k^2k^2k^2} p_\lambda}_{\sim \text{ log div.}} \rightarrow \text{ Could have } \underbrace{\int d^4k \frac{k_\mu k_\nu k_\lambda}{k^2k^2k^2}}_{\sim \text{ linearly div.}} \tag{12.4.34}$$

From the two point function,

$$A^{a \text{ Bare}}_\mu = Z_A^{1/2} A^{a \text{ Ren.}}_\mu \tag{12.4.35}$$

From the three point function,

$$g^{\text{Bare}} = Z_g g^{\text{Ren.}} \tag{12.4.36}$$

Thus $Z[A]$ has a contribution

$$\begin{aligned} & \left(\partial_\mu A^{a \text{ Bare}}_\nu - \partial_\nu A^{a \text{ Bare}}_\mu + g^{\text{Bare}} \epsilon^{abc} A^{b \text{ Bare}}_\mu A^{c \text{ Bare}}_\nu\right)^2 \\ = & \left(Z_A^{1/2}\left[\partial_\mu A^{a \text{ Ren.}}_\nu - \partial_\nu A^{a \text{ Ren.}}_\mu + Z_A^{1/2} Z_g g^{\text{Ren.}} \epsilon^{abc} A^{b \text{ Ren.}}_\mu A^{c \text{ Ren.}}_\nu\right]\right)^2 \end{aligned} \tag{12.4.37}$$

In order for this to remain gauge invariant,

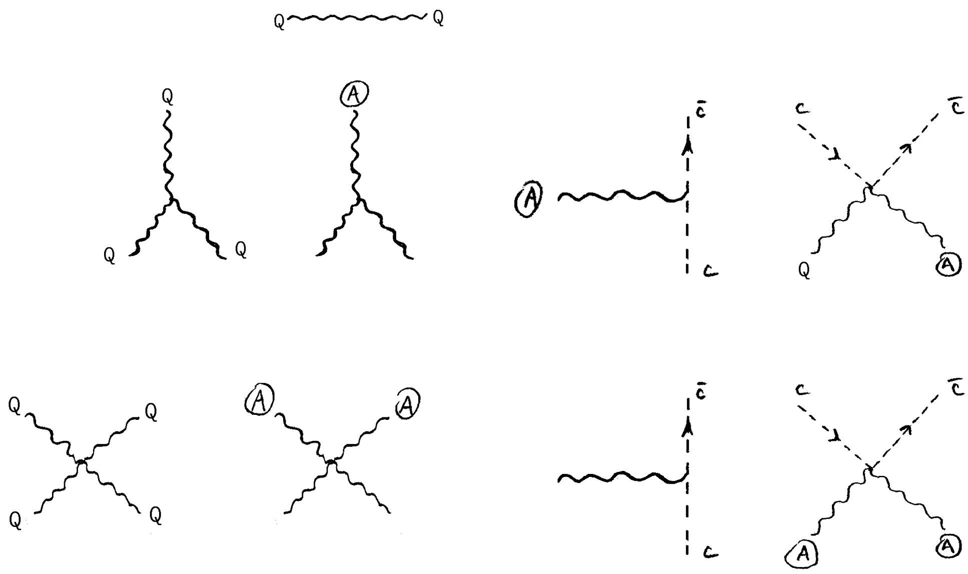
$$Z_A^{1/2} Z_g = 1 \tag{12.4.38}$$

So, $Z_A^{1/2}$ and Z_g are <u>not</u> independent in background field quantization!

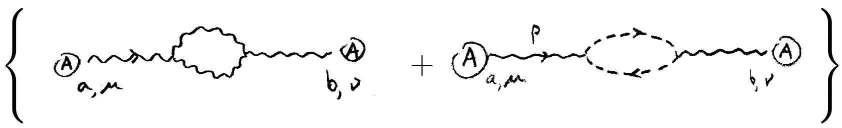
But Z_A can be determined from the 2 point function alone.

Note also - the relation between 2, 3 and 4 point functions implied by the gauge invariance of $Z(A)$ is the Ward identities in the context of background field quantization.

The Feynman rules are:



For Z_A to one loop order



$\rightarrow$ Have the divergent parts

$$\frac{\mathrm{i}g^2\delta^{ab}c_2}{(4\pi)^2}\frac{10}{3\varepsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu) + \frac{\mathrm{i}g^2}{(4\pi)^2}\delta^{ab}c_2\frac{1}{3\varepsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu) \tag{12.4.39}$$

$$\rightarrow \beta(g) = -\frac{11}{3}\frac{c_2}{(4\pi)^2}g^3 \tag{12.4.40}$$

(Details: See Abbott - Nuclear Physics 1981).

Chapter 13

Anomalies

A theory has an anomaly if a symmetry of the classical Lagrangian is broken by quantum effects.
"Most important one": $\rightarrow$ Chiral anomaly for massless fermions.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) + \bar{\psi}\left(\mathrm{i}\,\partial\!\!\!/ - A\!\!\!/ - B\!\!\!/\gamma_5\right)\psi \tag{13.0.1}$$

There are <u>two</u> invariances in this Lagrangian,

1. $\psi(x) \rightarrow e^{\mathrm{i}\Omega(x)}\psi(x)$ provided $A_\mu \rightarrow A_\mu - \partial_\mu\Omega(x)$
2. $\psi(x) \rightarrow e^{\mathrm{i}\Omega_5(x)\gamma_5}\psi(x)$ provided $B_\mu \rightarrow B_\mu - \partial_\mu\Omega_5(x)$

i.e.

$$F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu \rightarrow \partial_\mu(B_\nu - \underbrace{\partial_\nu\Omega_5}_{*}) - \partial_\nu(B_\mu - \underbrace{\partial_\mu\Omega_5}_{*}) \tag{13.0.2}$$

* - cancel, provided derivatives commute with each other.

$$\psi(x) \rightarrow e^{\mathrm{i}\Omega_5(x)\gamma_5}\psi(x) \tag{13.0.3}$$
$$\psi^\dagger(x) \rightarrow \psi^\dagger e^{-\mathrm{i}\Omega_5(x)\gamma_5^\dagger} = \psi^\dagger e^{-\mathrm{i}\Omega_5(x)\gamma_5} \tag{13.0.4}$$
$$\bar{\psi} = \psi^\dagger\gamma_0 \rightarrow \psi^\dagger e^{-\mathrm{i}\Omega_5(x)\gamma_5}\gamma_0 = \bar{\psi}e^{\mathrm{i}\Omega_5(x)\gamma_5} \tag{13.0.5}$$

Thus,

$$\begin{aligned}
\bar{\psi}\left(\mathrm{i}\,\partial\!\!\!/ - B\!\!\!/\gamma_5\right)\psi &\rightarrow \bar{\psi}e^{\mathrm{i}\Omega_5\gamma_5}\left[\mathrm{i}\,\partial\!\!\!/ - (B\!\!\!/ - \partial\!\!\!/\Omega_5)\gamma_5\right]e^{\mathrm{i}\Omega_5\gamma_5}\psi \\
&= \left(\bar{\psi}e^{\mathrm{i}\Omega_5\gamma_5}\right)e^{-\mathrm{i}\Omega_5\gamma_5}\left[\mathrm{i}\left(\partial\!\!\!/ + \mathrm{i}\,\partial\!\!\!/\Omega_5\gamma_5\right) - \left(B\!\!\!/ - \partial\!\!\!/\Omega_5\right)\gamma_5\right]\psi \\
&= \bar{\psi}\left(\mathrm{i}\,\partial\!\!\!/ - B\!\!\!/\gamma_5\right)\psi
\end{aligned} \tag{13.0.6}$$

Note:

$$\bar{\psi}\psi \rightarrow \left(\bar{\psi}e^{\mathrm{i}\Omega_5\gamma_5}\right)\left(e^{\mathrm{i}\Omega_5\gamma_5}\psi\right) \neq \bar{\psi}\psi \tag{13.0.7}$$

$\therefore$ Chiral gauge invariance is <u>not</u> respected by the mass term in a Lagrangian.

In Euclidean space,

$$\gamma_\mu = \gamma_\mu^\dagger \quad \rightarrow \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{13.0.8}$$

So $\bar{\psi} = \psi^\dagger$ (not $\bar{\psi} = \psi^\dagger\gamma^0$ as in Minkowski), and so $\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - B\!\!\!/\gamma_5)\psi$ is <u>not</u> chirally gauge invariant, but $\bar{\psi}\psi$ <u>is</u>.

Mar. 29/2000

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^2(A) - \frac{1}{4}F^2(B) + \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - e\,A\!\!\!/ - g\,B\!\!\!/)\psi &\quad (13.0.9)\\
\psi &\rightarrow e^{\mathrm{i}e\Omega(x)}\psi \quad A_\mu \rightarrow A_\mu - \partial_\mu\Omega &\quad (13.0.10)\\
\psi &\rightarrow e^{\mathrm{i}g\Omega_5(x)}\psi \quad B_\mu \rightarrow B_\mu - \partial_\mu\Omega_5 &\quad (13.0.11)
\end{aligned}$$

$$\begin{aligned}
\partial_\mu j^\mu(x) = 0 \qquad j^\mu = \bar{\psi}\gamma^\mu\psi &\quad (13.0.12)\\
\partial_\mu j_5^\mu(x) = 0 \qquad j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi &\quad (13.0.13)
\end{aligned}$$

The Green's Functions are external in the fields A_μ, B_μ. i.e.

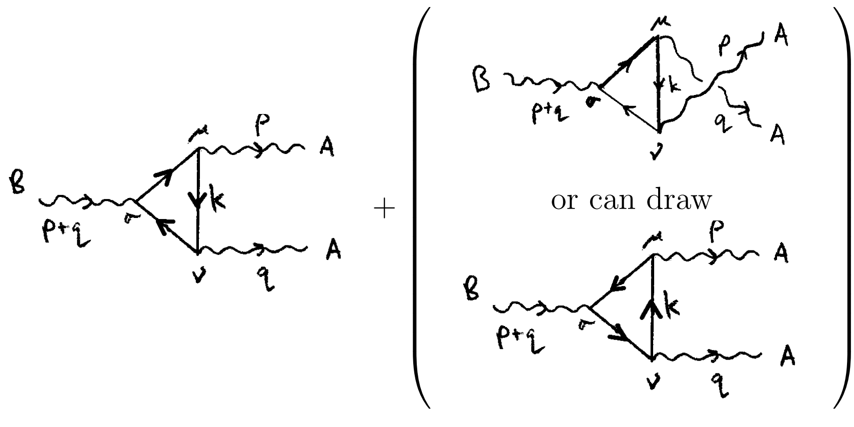
$$\begin{aligned}
G_{\mu_1,\ldots\mu_n,\nu_1,\ldots,\nu_m}(x_1\ldots x_n, y_1\ldots y_m) &= \langle 0|TA_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)B_{\nu_1}(y_1)\ldots B_{\nu_m}(y_m)|0\rangle &\quad (13.0.14)\\
\frac{\partial G}{\partial x_{\mu_i}} &= 0 = \frac{\partial G}{\partial y_{\nu_j}} &\quad (13.0.15)
\end{aligned}$$

$\rightarrow$ Remembera $CA_\mu C^{-1} = -A_\mu \rightarrow n$ is even , $CB_\mu C^{-1} = +B_\mu \rightarrow m$ is even or odd.

Examine

$$\langle 0|TA_\mu(x_1)A_\nu(x_2)B_\sigma(y)|0\rangle \tag{13.0.16}$$

In momentum space at one-loop order,



(13.0.17)

$$\begin{aligned}
R_{\mu\nu\sigma}(p,q) &= (-1)(-\mathrm{i}e)^2(-\mathrm{i}g)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left\{(-\mathrm{i}\gamma_\sigma\gamma_5)\frac{\mathrm{i}(k\!\!\!/ + p\!\!\!/)}{(k+q)^2}(-\mathrm{i}\gamma_\nu)\frac{\mathrm{i}\,k\!\!\!/}{k^2}(\mathrm{i}\gamma_\mu)\frac{\mathrm{i}(k\!\!\!/ - p\!\!\!/)}{(k-p)^2}\right.\\
&\left.+(-\mathrm{i}\gamma_\sigma\gamma_5)\frac{\mathrm{i}(k\!\!\!/ + p\!\!\!/)}{(k+p)^2}(-\mathrm{i}\gamma_\mu)\left(\frac{\mathrm{i}\,k\!\!\!/}{k^2}\right)(-\mathrm{i}\gamma_\nu)\frac{\mathrm{i}(k\!\!\!/ - q\!\!\!/)}{(k-q)^2}\right\} \qquad (13.0.18)
\end{aligned}$$

(the first term in the trace refers to the second diagram, and the second term to the first diagram). By using

$$\not{a}\,\not{b}\,\not{c} \;=\; a\cdot b\,\not{c} - a\cdot c\,\not{b} + b\cdot c\,\not{a} + \mathrm{i}\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma \gamma^\delta\gamma_5$$
$$\left(\epsilon^{0123} = +1\;,\quad \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\right)$$

→ can show that the two traces are identical. Examine: $q^\nu R_{\mu\nu\sigma}(p,q)$. Note:

$$\begin{aligned}
q_\nu\left(\frac{(\not{k}+\not{q})}{(k+q)^2}\gamma^\nu\frac{\not{k}}{k^2}\right) &= q_\nu\left(\frac{1}{(\not{k}+\not{q})}\gamma^\nu\frac{1}{\not{k}}\right)\\
&= \frac{1}{(\not{k}+\not{q})}\left((\not{k}+\not{q})-\not{k}\right)\frac{1}{\not{k}}\\
&= \left(1-\frac{\not{k}}{(\not{k}+\not{q})}\right)\frac{1}{\not{k}}\\
&= \left(\frac{1}{\not{k}}-\frac{1}{(\not{k}+\not{q})}\right) \qquad (13.0.19)
\end{aligned}$$

We finally obtain

$$\begin{aligned}
q^\nu R_{\mu\nu\sigma}(p,q) &= -e^2 g\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left\{\gamma_\sigma\gamma_5\frac{1}{\not{k}}\gamma_\mu\frac{1}{(\not{k}-\not{p})} - \gamma_\sigma\gamma_5\frac{1}{(\not{k}+\not{q})}\gamma_\mu\frac{1}{(\not{k}-\not{p})}\right.\\
&\left. -\gamma_\sigma\gamma_5\frac{1}{(\not{k}+\not{p})}\gamma_\mu\frac{1}{\not{k}} + \gamma_\sigma\gamma_5\frac{1}{(\not{k}+\not{p})}\gamma_\mu\frac{1}{(\not{k}-\not{q})}\right\} \qquad (13.0.20)
\end{aligned}$$

Now, $k \to k+p$ in the first term in the trace, and $k \to k+p-q$ in the second term - $q^\nu R_{\mu\nu\sigma}(p,q) \stackrel{?}{=} 0$ upon this shifting of variables.

But, it is quadratically divergent (actually just linearly divergent because of $\epsilon_{\alpha\beta\gamma\delta}$ appearing upon taking the trace), so care must be taken when shifting variables. ex.

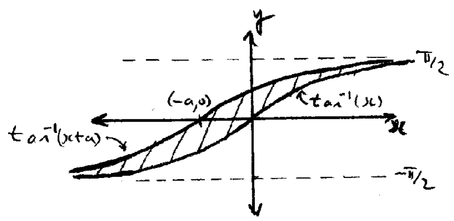


Figure 13.0.1: Plot of $\tan^{-1}(x+a)$ and $\tan^{-1}(x)$

$$\begin{aligned}
I = \int_{-\infty}^{\infty} dx\;\left(\tan^{-1}(x+a)-\tan^{-1}(x)\right) &\stackrel{?}{=} 0(\text{shifting } x\to x-a)\\
&\quad\text{but (see figure 13.0.1)}\\
&= \lim_{\substack{\Lambda_+\to\infty\\ \Lambda_-\to-\infty}}\int_{\Lambda_-}^{\Lambda_+} dx\;\left(\tan^{-1}(x+a)-\tan^{-1}(x)\right)
\end{aligned}$$

Or

$$\tan^{-1}(x+a) = \tan^{-1}(x) + a\frac{d}{dx}\tan^{-1}(x) + \frac{a^2}{2!}\frac{d^2}{dx^2}\tan^{-1}(x) + \dots \tag{13.0.21}$$

Substitute this back into I

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dx \left[\tan^{-1}(x) + a\frac{d}{dx}\tan^{-1}(x) + \frac{a^2}{2!}\frac{d^2}{dx^2}\tan^{-1}(x) + \dots - \tan^{-1}(x)\right] \\ &= \left[a\tan^{-1}(x) + \frac{a^2}{2!}\frac{d}{dx}\tan^{-1}(x) + \frac{a^3}{3!}\frac{d^2}{dx^2}\tan^{-1}(x) + \dots\right]_{-\infty}^{\infty} \\ &= a\left(\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + 0 + \dots\right) \\ &= \pi a \end{aligned} \tag{13.0.22}$$

In general, if $\int d^4k\ f(k^2)$ is linearly divergent, then,

$$\begin{aligned} \int d^4k\ \left[f((k+a)^2) - f(k^2)\right] &= a^\lambda \int d^4k\ \frac{\partial}{\partial k^\lambda} f(k^2) \\ &= \mathrm{i}\int d\Omega_k\ k^3 a\cdot k^n f(k^2)\Big|_{k^2\to\infty} \quad \text{by Gauss} \\ &= 2\pi^2 \mathrm{i} a\cdot\hat{k} f(k^2)\Big|_{k^2\to\infty} \end{aligned} \tag{13.0.23}$$

($\int d\Omega_k = 2\pi^2$ in 4-d, recall in 3-d, $\int d\Omega_k = 4\pi$). Go back to

$$\begin{aligned} R_{\mu\nu\sigma}(a) &= -e^2 g \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left\{\gamma_\sigma\gamma_5 \frac{1}{(\not{k}+\not{a}_1+\not{q})}\gamma_\nu \frac{1}{(\not{k}+\not{a}_1)}\gamma_\mu \frac{1}{(\not{k}+\not{a}_1-\not{p})}\right. \\ &\quad \left.\gamma_\sigma\gamma_5 \frac{1}{(\not{k}+\not{a}_2+\not{p})}\gamma_\mu \frac{1}{(\not{k}+\not{a}_2)}\gamma_\nu \frac{1}{(\not{k}+\not{a}_2-\not{q})}\right\} \\ R_{\mu\nu\sigma}(a) - R_{\mu\nu\sigma}(0) &= -e^2 g \frac{(2\pi^2 \mathrm{i})}{(2\pi)^4} \lim_{k^2\to\infty} \mathrm{Tr}\left[a_1^\lambda k^2 k^\lambda \gamma_\sigma\gamma_5 \frac{1}{(\not{k}+\not{q})}\gamma_\nu \frac{1}{\not{k}}\gamma_\mu \frac{1}{(\not{k}-\not{p})}\right. \\ &\quad \left. + a_2^\lambda k^2 k^\lambda \gamma_\sigma\gamma_5 \frac{1}{(\not{k}+\not{p})}\gamma_\mu \frac{1}{\not{k}}\gamma_\nu \frac{1}{(\not{k}-\not{q})}\right] \\ &\ \vdots \\ &= \frac{e^2 g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(a_1 - a_2)^\kappa \end{aligned} \tag{13.0.24}$$

$$a_1^\kappa - a_2^\kappa = Ap^\kappa + Bq^\kappa \tag{13.0.25}$$

$$\Rightarrow R_{\mu\nu\sigma}(a) = R_{\mu\nu\sigma}(0) + \frac{e^2 g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}\left(Ap^\kappa + Bq^\kappa\right) \tag{13.0.26}$$

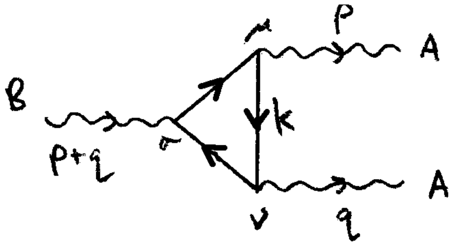
So, (by same analogy), by surface terms,

$$q^\nu R_{\mu\nu\sigma}(p,q) = \frac{e^2 g}{\pi^2}\left(\epsilon_{\nu\alpha\mu\beta} q^\alpha p^\beta\right) \tag{13.0.27}$$

(does not appear to be $= 0$).

Mar. 30/2000

Recall:



$$R_{\mu\nu\sigma}(a) - R_{\mu\nu\sigma}(0) \;=\; \frac{e^2g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(Ap^\kappa + Bq^\kappa)$$

$$\begin{aligned} q^\nu R_{\mu\nu\sigma}(0) &= -e^2g\frac{(2\pi^2\mathrm{i})}{(2\pi)^4}\lim_{k^2\to\infty}\Big[(-p^\lambda)k^2k_\lambda \mathrm{Tr}\left(\frac{\gamma_\sigma\gamma_5\gamma_\alpha\gamma_\mu\gamma_\beta}{k^4}p^\alpha k^\beta\right) \\ &\quad - (p-q)^\lambda k^2 k_\lambda \mathrm{Tr}\left(\frac{\gamma_\sigma\gamma_5\gamma_\alpha\gamma_\mu\gamma_\beta}{k^4}(q^\alpha k^\beta - k^\alpha p^\beta)\right)\Big] \\ &= -\frac{e^2g}{\pi^2}\epsilon_{\sigma\alpha\mu\beta}q^\alpha p^\beta \quad \text{(unsure of sign - see (13.0.34))} \end{aligned} \tag{13.0.28}$$

In the same way,

$$p_\mu R_{\mu\nu\sigma}(0) = -\frac{e^2g}{\pi^2}\epsilon_{\sigma\alpha\nu\beta}p^\alpha q^\beta \tag{13.0.29}$$

And lastly the axial vertex,

$$(p+q)^\sigma R_{\mu\nu\sigma}(0) = 0 \tag{13.0.30}$$

Maybe can choose A, B so we get conservation of momentum at all 3 vertices.

$$\begin{aligned} q_\nu R_{\mu\nu\sigma}(a) &= q_\nu\left[R_{\mu\nu\sigma}(0) + \frac{e^2q}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(Ap^\kappa + Bq^\kappa)\right] \\ &= \frac{e^2g}{\pi^2}\left[\epsilon_{\sigma\alpha\mu\beta}q^\alpha p^\beta + \epsilon_{\beta\mu\alpha\sigma}q^\alpha A p^\beta\right] \\ &= \frac{e^2g}{\pi^2}\left[\epsilon_{\sigma\alpha\mu\beta}(q^\alpha p^\beta)(1+A)\right] \end{aligned} \tag{13.0.31}$$

So also,

$$\begin{aligned} p_\mu R_{\mu\nu\sigma}(a) &= p_\mu\left[R_{\mu\nu\sigma}(0) + \frac{e^2g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(Ap^\kappa + Bq^\kappa)\right] \\ &= -\frac{e^2g}{\pi^2}\epsilon_{\sigma\alpha\nu\beta}p^\alpha q^\beta + \frac{e^2g}{\pi^2}\epsilon_{\beta\alpha\nu\sigma}p^\alpha q^\beta B \\ &= \frac{e^2g}{\pi^2}\epsilon_{\sigma\alpha\nu\beta}p^\alpha q^\beta(-1-B) \quad \text{(unsure of signs - see (13.0.34))} \end{aligned} \tag{13.0.32}$$

And,

$$\begin{aligned}(p+q)^\sigma R_{\mu\nu\sigma}(a) &= (p+q)^\sigma\left[R_{\mu\nu\sigma}(a)+\frac{e^2 g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(Ap^\kappa+Bq^\kappa)\right]\\ &= 0+\frac{e^2 g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(Ap^\kappa q^\sigma+Bq^\kappa p^\sigma)\\ &= \frac{e^2 g}{\pi^2}\epsilon_{\kappa\mu\nu\sigma}(A-B)p^\kappa q^\sigma \end{aligned} \tag{13.0.33}$$

It should be: If

$$\begin{aligned} q^\nu R_{\mu\nu\sigma}(a) &= 0\\ p^\mu R_{\mu\nu\sigma}(a) &= 0 \end{aligned} \tag{13.0.34}$$

then

$$(p+q)^\sigma R_{\mu\nu\sigma}(a) = -\frac{e^2 g}{2\pi^2}\epsilon_{\mu\nu\lambda\sigma}p^\lambda q^\sigma \tag{13.0.35}$$

So, if we do have conservation of vector current, then axial current not conserved. i.e. If,

$$\partial_\mu j^\mu = 0 \tag{13.0.36}$$

then

$$\partial_\mu j^\mu_s = e^2\frac{\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}(A)F^{\lambda\sigma}(A)}{(4\pi)^2} \tag{13.0.37}$$

(Note: anomaly independent of mass). This is the modification of $\partial\cdot j_5=0$ that is required to account for $(p+q)^\sigma R_{\mu\nu\sigma}\neq 0$.

$$\partial^\mu\bar\psi\gamma_\mu\gamma_5\psi = 0 \tag{13.0.38}$$

Note: The presence of the anomaly accounts for the decay of the π^0 ($\pi^0\to\gamma\gamma$). i.e. If ψ were massive,

$$\begin{aligned}\partial\cdot\bar\psi\gamma_\mu\gamma_5\psi &= 2m\bar\psi\gamma_5\psi\\ &= \bar\psi\left[\partial^\mu\gamma_\mu\gamma_5\psi\right]+\left[\partial_\mu\bar\psi\gamma^\mu\right]\gamma_5\psi\end{aligned}$$

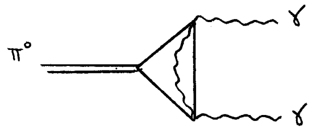
But,

$$\begin{aligned}(\mathrm{i}\,\partial\!\!\!/ - A\!\!\!/ - B\!\!\!/\gamma_5)\psi - m\psi &= 0\\ \partial\!\!\!/\psi &= \frac{1}{\mathrm{i}}\left[(A\!\!\!/ + B\!\!\!/\gamma_5)\psi + m\psi\right]\end{aligned} \tag{13.0.39}$$

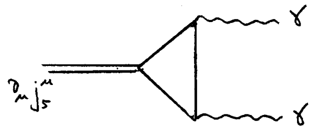
Similarly,

$$\begin{aligned}\psi^\dagger\overleftarrow{\partial\!\!\!/} &= -\frac{1}{\mathrm{i}}\left[\psi^\dagger(A\!\!\!/^\dagger + B\!\!\!/^\dagger\gamma_5)+m\psi^\dagger\right]\\ \bar\psi\overleftarrow{\partial\!\!\!/} &= -\frac{1}{\mathrm{i}}\left[\bar\psi(A\!\!\!/ - B\!\!\!/\gamma_5)+m\psi^\dagger\right]\end{aligned} \tag{13.0.40}$$

$$\begin{aligned}\pi^0 &\sim \bar{\psi}\gamma_5\psi \quad \text{(pseudoscalar)} \\ &= \frac{1}{2m}\partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi \\ &= \frac{1}{2m}\partial_\mu j^{\mu 5}\end{aligned} \tag{13.0.41}$$



$\rightarrow$



$\rightarrow 0$ if $\partial \cdot j_5 = 0$ (13.0.42)

(i.e. says π^0 cannot go to $\gamma^0\gamma^0$).
But the anomaly says this $\nrightarrow 0$ if

$$\partial \cdot j_5 = \frac{e^2\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}}{(4\pi)^2} \tag{13.0.43}$$

If j_5^μ couples to a vector B_μ, and j_5 is not conserved because of the anomaly, then renormalizability is lost.

$$\bar{\psi}_1(\mathrm{i}\,\partial\!\!\!/)\psi_1 + \bar{\psi}_2(\mathrm{i}\,\partial\!\!\!/)\psi_2 - g\bar{\psi}_1\gamma_\mu\gamma_5\psi_1 B^\mu + g\bar{\psi}_2\gamma_\mu\gamma_5\psi_2 B^\mu \quad ; \quad B_\mu \rightarrow B_\mu - \partial_\mu\Omega_5 \tag{13.0.44}$$

Anomalous contributions to $\partial_\mu j_5^\mu$ cancel in this case, and renormalizability is no longer a problem.

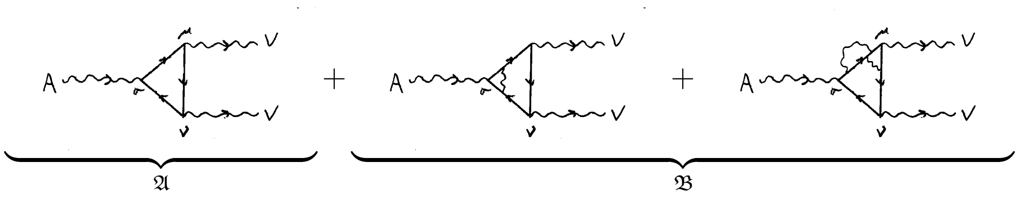
In the standard model, having equal numbers of quarks and leptons ensures the cancellation of anomalies.

Apr. 3.2000

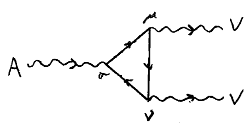
13.1 Anomalies

1. "Adler Bardeen" theorem $\rightarrow$ Anomalies are exclusively one-loop effects. i.e.

$$\langle 0|T \underbrace{V_\mu(x)V_\nu(y)}_{\text{Transverse}} \underbrace{A_\sigma(z)}_{\text{Anomaly}} |0\rangle \tag{13.1.1}$$



(13.1.2)



$$\rightarrow \partial_\mu j_5^\mu = \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} \tag{13.1.3}$$

All of the anomaly can be shown to be from the $\mathfrak{A}$ graph only. No anomalous contribution comes from $\mathfrak{B}$, though they are needed in order to ensure conservation of momentum at all three vertices. So, no $\frac{e^4}{(4\pi)^4}$ correction for $\mathfrak{B}$.

2.

$$Z(J,K) = \int \mathcal{D}A_\mu \mathcal{D}B_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{\frac{\mathrm{i}}{\hbar}\int d^4x \ [\mathcal{L}_{cl} + \mathcal{L}_{gf} + A\cdot J + B \cdot J_5]\right\} \quad (13.1.4)$$

$$\mathcal{L}_{cl} = \bar{\psi}(\not{p} - m - e \not{A} - g \not{B}\gamma_5)\psi - \frac{1}{4}F^2(A) - \frac{1}{4}F^2(B)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Omega \quad , \quad \psi \rightarrow e^{\mathrm{i}e\Omega(x)}\psi \quad (13.1.5)$$

$$B_\mu \rightarrow B_\mu + \partial_\mu \Omega_5 \quad , \quad \underbrace{\psi \rightarrow e^{\mathrm{i}g\Omega_5\gamma_5}\psi}_{*} \quad (13.1.6)$$

* - this gauge transformation does affect measure → there is a non-trivial Jacobian associated with this gauge transformation.

(Fujikawa showed that this Jacobian gives the "anomalous" term in the WTST identity for the axial current.)

→ anomaly arises in trying to make this gauge transformation in this infinite measure → $(\mathcal{D}\psi\mathcal{D}\bar{\psi})$ → integrating over all possibilities.

Chapter 14

Instantons

Instantons are generated in a 2 stage process.

1. Make Wick rotation to Euclidean space
2. Solve resulting equations of motion ("Instanton")

14.1 Quantum Mechanical Example

ex: (see figure 14.1.1)

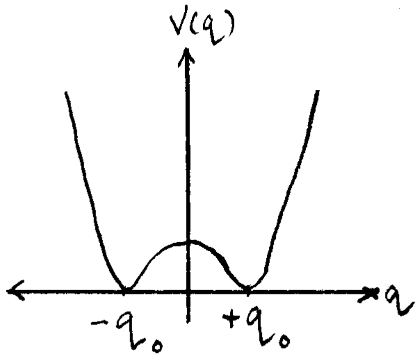


Figure 14.1.1: Classically, if in the lowest energy state at the bottom (one of the $V(q) = 0 \to \pm q_0$ positions), particle would just sit there.

$$V(q) = (q^2 - q_0^2)^2 \tag{14.1.1}$$

"Vacuum tunnelling"

$$\begin{aligned} |q(t \to -\infty)\rangle &= |-q_0\rangle & (14.1.2) \\ |q(t \to \infty)\rangle &= |+q_0\rangle & (14.1.3) \end{aligned}$$

Compute transition (Q.M.) $\to$ Path integral.

$$\langle q_f | e^{-\mathrm{i}Ht} | q_i \rangle \tag{14.1.4}$$

is the amplitude for being in state $|q_i\rangle$ at $t=0$ and being in state $|q_f\rangle$ at $t=t$.

Path integral:

$$\begin{aligned} \langle q_f|e^{-\mathrm{i}Ht}|q_i\rangle &= \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t d\tau\ L(q(\tau))\right\} \qquad (14.1.5) \\ \to L(q(\tau)) &= \frac{1}{2}\dot{q}^2(\tau) - V(q(\tau)) \\ &= \frac{1}{2}\dot{q}^2(\tau) - (q^2-q_0^2)^2 \end{aligned}$$

Let $\tau \to -\mathrm{i}\tau$.

$$\begin{aligned} \langle q_f|e^{-\mathrm{i}Ht}|q_i\rangle &= \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q(t)\ \exp\left\{\frac{1}{\hbar}\int_0^t d\tau\ \left(-\frac{\dot{q}^2}{2} - (q^2-q_0^2)^2\right)\right\} \\ &= \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q(t)\ \exp\left\{-\frac{1}{\hbar}S_E\right\} \qquad (14.1.6) \\ S_E &= \int_0^t d\tau\ \left(\frac{\dot{q}^2}{2} + (q^2-q_0^2)^2\right) = S_{\text{Euclidean}} \end{aligned}$$

Dominant path: path that satisfies $\frac{\delta S_E}{\delta q(\tau)} = 0$ with the appropriate boundary conditions.

<u>Now then</u>, if the

$$\begin{aligned} t_i &\to -\infty \qquad q(t) \to -q_0 \\ t_f &\to +\infty \qquad q(t) \to +q_0 \end{aligned}$$

The solution to the equations of motion that satisfies the above is

$$q_{cl}(t) = q_0 \tanh(\sqrt{2}q_0 t) \qquad (14.1.7)$$

The more general solution is

$$q_{cl}(t) = q_o \tanh(\sqrt{2}q_0(t - \underbrace{t_0}_{*})) \qquad (14.1.8)$$

* - localized at an "instant" $t_0 \to$ "instanton".

We will now consider

$$\langle q(+\infty) = +q_0|q(-\infty) = -q_0\rangle = \int \mathcal{D}(\delta q)\ e^{-[S_E(q_{cl}+\delta q)]} \qquad (14.1.9)$$

$\delta q = 0$ at $t \to \pm\infty$ $\to$ expand about q_{cl}.

$$\begin{aligned} \langle q(+\infty) \to +q_0|q(-\infty) \to -q_0\rangle &= \int \mathcal{D}(\delta q)\exp\left\{-S_E(q_{cl}) - \int d\tau\left[(\delta q(\tau))^2\frac{D^2S_E}{(D\delta q(\tau))^2} + \dots\right]\right\} \\ &= e^{-S_E(q_{cl})}\left[1 + \mathcal{O}(\hbar)\right] \qquad (14.1.10) \\ &\quad \text{here}\ \ S_E(q_{cl}) = 4\sqrt{2}\frac{q_0^3}{3\hbar} \\ \Bigg\{ &= e^{-S_E(q_{cl})}\left[1 + \det\left(\frac{D^2S(q_{cl})}{(Dq(\tau))^2}\right) + \dots\right]\Bigg\} \qquad (14.1.11) \end{aligned}$$

Suppose we have 2 spinors X, Y. Consider two functions f_0 and f_1.

$$f_i : \ X \to Y \quad (i = 0, 1) \tag{14.1.12}$$

$f_0(x),\ f_1(x)$ are homotopic if there exists a function $F(x,t)$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ and $F(x,t)$ is continuous in t in $(0 \leq t \leq 1)$.

Suppose we took

$$\begin{aligned} X : &\quad 0 < \theta \leq 2\pi & (14.1.13)\\ Y : &\quad Z; Z \in \mathcal{C} \quad , \quad |Z| = 1 & (14.1.14) \end{aligned}$$

Consider

$$\begin{aligned} f_\alpha(\theta) &= e^{\mathrm{i}(\theta+\alpha)} & (14.1.15)\\ f_\beta(\theta) &= e^{\mathrm{i}(\theta+\beta)} & (14.1.16) \end{aligned}$$

Homotopic $\to F(x,t) = e^{\mathrm{i}(\theta + t\beta + (1-t)\alpha)} \to$ Homotopy connecting $f_\alpha,\ f_\beta$. But can also write two functions not homotopic.

$$\begin{aligned} f_\alpha(\theta) &= e^{\mathrm{i}(\theta+\alpha)} & (14.1.17)\\ f_\beta(\theta) &= e^{\mathrm{i}(2\theta+\beta)} & (14.1.18) \end{aligned}$$

Not homotopic (still functions that map $X \to Y$, but can't continuously deform one into the other).

We have in general different classes of functions characterized by integers "n". These are of the form,

$$f_\alpha^n(\theta) = e^{\mathrm{i}(n\theta+\alpha)} \quad ; \quad n = \pm 1, \pm 2, \ldots \quad \text{(sign indicates direction of winding.)} \tag{14.1.19}$$

In general, if two functions have different n's, Not homotopic.

$$\begin{aligned} n &\to \text{Pontryagin index ("Winding number")}\\ n &= -\mathrm{i} \int_0^{2\pi} d\theta \, \frac{f'(\theta)}{f(\theta)} \qquad (14.1.20) \end{aligned}$$

Apr. 5/2000

Recall:

$$\begin{aligned} X : &\to \theta \in [0, 2\pi)\\ Y : &\to Z,\ |Z| = 1\\ F^{(n)}(\theta) &= e^{\mathrm{i}n\theta} \quad n = \pm 1, \pm 2, \ldots\\ n &= -\mathrm{i} \int_0^{2\pi} d\theta \, \frac{f'(\theta)}{f(\theta)} \quad \text{(counts \# of times each pt. in X mapped onto Y)} \end{aligned}$$

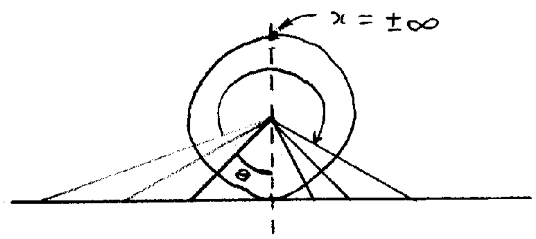


Figure 14.1.2: $X :\to (-\infty, \infty)$

We can let X be $(-\infty, \infty)$ provided $+\infty$ and $-\infty$ are mapped on to the same point. (See figure (14.1.2)). Consider,

$$f(x) = \exp\left\{\frac{\mathrm{i}\pi x}{\sqrt{x^2+\lambda^2}}\right\} \tag{14.1.21}$$

This maps $-\infty < x < \infty$ onto Z where $|Z| = 1$.

$$n = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \frac{f'(x)}{f(x)} \quad \text{(Winding number for this mapping)} \tag{14.1.22}$$

Two functions with different winding numbers have different homotopies.

Consider the mapping from $X = S_3$ to $Y = SU(2)$. i.e. $S_3 \to$ Set of all points with $x_1^2 + x_2^2 + x_3^2 + x_0^2 = 1 = x_0^2 + \underline{x}^2$ (4-d sphere).

$SU(2) \to$ set of 2×2 matrices U with $UU^\dagger = 1$.

$$f(x_0, \underline{x}) = x_0 + \mathrm{i}\underline{x} \cdot \underline{\sigma} \tag{14.1.23}$$

$$\begin{aligned} \to (x_0 + \mathrm{i}\underline{x}\cdot\underline{\sigma})^\dagger (x_0 + \mathrm{i}\underline{x}\cdot\underline{\sigma}) &= (x_0 - \mathrm{i}\underline{x}\cdot\underline{\sigma})(x_0 + \mathrm{i}\underline{x}\cdot\underline{\sigma}) \\ &= 1 \quad (\text{if } x_0^2 + \underline{x}^2 = 1) \end{aligned}$$

Winding number for this mapping is,

$$n = -\frac{1}{24\pi^2}\int d\theta_1\, d\theta_2\, d\theta_3 \,\mathrm{Tr}\,\{\epsilon_{ijk} A_i A_j A_k\} \tag{14.1.24}$$

- Where θ_i are the polar angles characterizing S_3
- and $A_i = f^{-1}(\theta)\frac{\partial}{\partial\theta_i} f(\theta)$

(Coleman's lectures on Instantons).

We now extend X to be the entire three dimensional Euclidean space with all points at ∞ identified with a single point.

Here,

$$f_1(x) = \exp\left\{\frac{\mathrm{i}\pi\underline{x}\cdot\underline{\pi}}{\sqrt{x_0^2+\underline{x}^2+\lambda^2}}\right\} \tag{14.1.25}$$

is such a mapping. The winding number is

$$n = -\frac{1}{24\pi^2}\int d^3x\ \epsilon_{ijk}\mathrm{Tr}\left\{A_iA_jA_k\right\} \tag{14.1.26}$$

where

$$A_i(x) = f^{-1}\frac{\partial}{\partial x^i}f \tag{14.1.27}$$

Recall: Gauge theories,

$$A_\mu(x) \to U^{-1}(A_\mu(x) + \partial_\mu)U \tag{14.1.28}$$

(if A_μ originally 0 $\to$ would get pure gauge function $\to$ very similar to above).

14.2 Classical Solutions to $SU(2)$ YM field equations in Euclidean Space

$$\begin{aligned} S_{YM} &= \int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}(A)F^{a\mu\nu}(A)\right) \\ & F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk}A^j_\mu A^k_\nu \end{aligned} \tag{14.2.1}$$

Equations of motion:

$$D^{ij}_\mu(A)F^j_{\mu\nu}(A) = 0 \qquad \to D^{ij}_\mu = \partial_\mu\delta^{ij} + \epsilon^{imj}A^m_\mu \tag{14.2.2}$$

Bit of a short-cut to solve this (not in every case). “Bianchi Identity”.

$$\begin{aligned} *F^i_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^i_{\lambda\sigma} \quad \text{(Dual)} \\ \to D^{ij}_\mu *F^i_{\mu\nu} &= 0 \end{aligned} \tag{14.2.3}$$

Thus, if

$$F^i_{\mu\nu} = \pm *F^i_{\mu\nu} \tag{14.2.4}$$

(ex: in E.M. $\to$ would be analogous to requiring $\underline{E} = \underline{B}$) then,

$$\begin{aligned} D^{ij}_\mu F^j_{\mu\nu} &= \pm D^{ij}_\mu *F^j_{\mu\nu} \\ &= 0 \end{aligned} \tag{14.2.5}$$

Let

$$A^i_m = (\epsilon_{imk}\partial_k \mp \delta_{mi}\partial_0)\ln[f(x_0, x_m)] \tag{14.2.6}$$

$$A^i_0 = \pm\partial_i\ln[f(x_0, x_m)] \tag{14.2.7}$$

($m \rightarrow$ spacial index, $i \rightarrow$ group index). If $F = \pm *F$, then

$$\begin{aligned} f^{-1}\partial^2 f &= 0 \\ f &= \sum_{i=1}^{n} \frac{\lambda_i^2}{(x-x_i)^2} \quad (\text{“n Instanton solution”}) \\ &\rightarrow \quad \text{Size } \rightarrow \lambda_i \\ &\quad\ \text{position } \rightarrow x_i \end{aligned} \tag{14.2.8}$$

Perform a gauge transformation on the “one instanton” solution.

$$A'_\mu = U^{-1}(A_\mu + \partial_\mu)U \tag{14.2.9}$$

with

$$U = \exp\left\{ \frac{\mathrm{i}\underline{x}\cdot\underline{\sigma}}{\sqrt{x^2+\lambda^2}}\theta \right\} \tag{14.2.10}$$

$$\rightarrow \tan(\theta) = \left(\frac{x_0}{\sqrt{x^2+\lambda^2}} - \frac{n\pi}{2} \right) \tag{14.2.11}$$

(n is integer, not the n from above). Note, as

$$\begin{aligned} |\underline{x}| &\rightarrow \infty\ , \quad A_0'^i \rightarrow 0 \quad (\text{not easy to show}) \\ x_0 &\rightarrow +\infty\ , \quad A_m'^i \rightarrow \Omega_n^{-1}\frac{\partial}{\partial x^m}\Omega_n \\ x_0 &\rightarrow -\infty\ , \quad A_m'^i \rightarrow \Omega_{n-1}^{-1}\frac{\partial}{\partial x^m}\Omega_{n-1} \end{aligned} \tag{14.2.12}$$

where $\Omega_n = \left[\exp\left\{\frac{-\mathrm{i}\pi\underline{x}\cdot\underline{\sigma}}{\sqrt{x^2+\lambda^2}}\right\}\right]^n$.

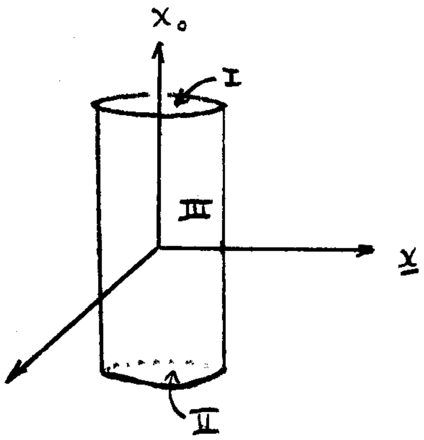


Figure 14.2.1: I, II, III are all 3-d spaces. In all 3 spaces, $A_m^i \rightarrow U^{-1}\partial_m U$ (i.e. A_m^i is pure gauge function).

To show this, consider $F_{\mu\nu}(A)$; $F_{\mu\nu} \to 0$ as $|x_\mu| \to \infty$. Consider U as a mapping from 3-d Euclidean space onto $SU(2)$.

$$\begin{aligned} n &= \frac{1}{24\pi^2} \int d^3x \; \epsilon_{ijk} \mathrm{Tr} \{A_i A_j A_k\} \\ &\to A_i = U^{-1} \frac{\partial}{\partial x^i} U \end{aligned} \tag{14.2.13}$$

In region III (figure 14.2.1);

$$\begin{aligned} n_{III} &= \frac{1}{24\pi^2} \int_{III} d^3x \; \epsilon_{ijk} \mathrm{Tr} \{A_i A_j A_k\} \\ &\quad \text{where } i,j,k \text{ has one index “0” because we're on one side of sphere} \\ &= 0 \quad \text{(on III)} \end{aligned} \tag{14.2.14}$$

But, from (14.2.12),

$$\begin{aligned} n_I &= \frac{1}{24\pi^2} \int_I d^3x \; \epsilon_{ijk} \mathrm{Tr} \{A_i A_j A_k\} \\ &= n \quad ((i,j,k) \text{ are a mixture of 1,2,3 (no 0) in integral}) \end{aligned} \tag{14.2.15}$$

and

$$\begin{aligned} n_{II} &= \frac{1}{24\pi^2} \int_{II} d^3x \; \epsilon_{ijk} \mathrm{Tr} \{A_i A_j A_k\} \\ &= n - 1 \end{aligned} \tag{14.2.16}$$

See figure 14.2.2.

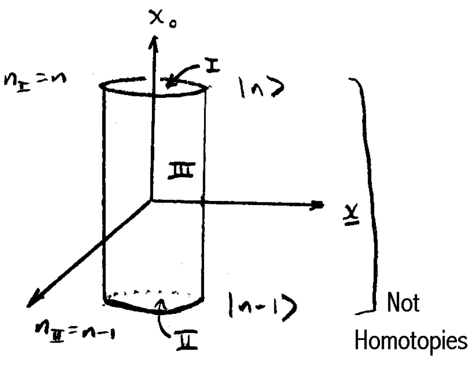


Figure 14.2.2:

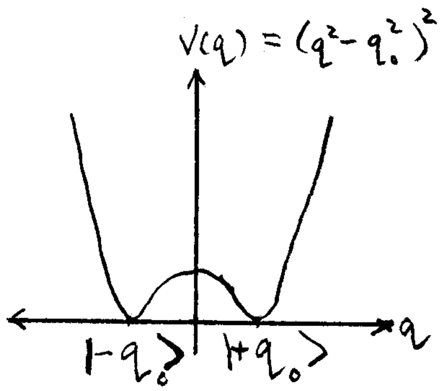


Figure 14.2.3:

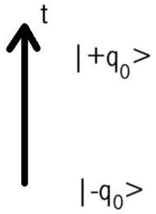


Figure 14.2.4: $q(t) = q_0 \tan(\sqrt{2} q_0 t) \to$ instanton $\to$ interpolated from $-\infty \to +\infty$.

Apr. 6/2000
For completeness, One-instanton:

$$\begin{aligned} A_\mu(x) &= \frac{x^2}{x^2+\lambda^2}\Omega^{-1}\partial_\mu\Omega \quad ; \quad \Omega = \frac{x_4 \pm i\underline{x}\cdot\underline{\sigma}}{\sqrt{x^2}} \\ &\to \Omega^{-1}\partial_\mu\Omega \quad \text{(pure gauge) as } |x| \to \infty \end{aligned} \tag{14.2.17}$$

$$m = \frac{1}{24\pi^2}\int d^3x \; \epsilon_{ijk}\mathrm{Tr}(A_i A_j A_k) \quad ; \quad A_i = f^{-1}\partial_i f \tag{14.2.18}$$

Note:

$$K_\mu = 4\epsilon_{\mu\nu\lambda\sigma}\mathrm{Tr}\left(A_\nu\partial_\lambda A_\sigma + \frac{2}{3}A_\nu A_\lambda A_\sigma\right) \quad \text{(not gauge invariant)} \tag{14.2.19}$$

$\ldots$ satisfies

$$\partial_\mu K_\mu = 2\mathrm{Tr}\left\{F_{\mu\nu} *F_{\mu\nu}\right\} \quad ; \quad *F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma} \tag{14.2.20}$$

Thus,

$$\begin{aligned} \mathrm{Tr}\left\{\int d^4x \; F_{\mu\nu} *F_{\mu\nu}\right\} &= \int d^4 \; \frac{1}{2}\partial_\mu K_\mu \\ &= \frac{1}{2}\int_{S_\infty} d^3\sigma_\mu \; K_\mu \quad \text{(By Gauss' Theorem)} \end{aligned} \tag{14.2.21}$$

But,

$$A_\mu \to \Omega^{-1}\partial_\mu\Omega \quad \text{as } |x| \to \infty \tag{14.2.22}$$

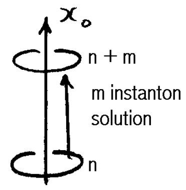
If $A_\mu = \Omega^{-1}\partial_\mu\Omega$, then,

$$K_\mu = \frac{4}{3}\epsilon_{\mu\nu\lambda\sigma}\mathrm{Tr}\left\{(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\lambda\Omega)(\Omega^{-1}\partial_\sigma\Omega)\right\} \tag{14.2.23}$$

Thus,

$$\mathrm{Tr}\left\{\int d^4x \; F_{\mu\nu} *F_{\mu\nu}\right\} = \frac{1}{2}\int_{S_\infty} d^3\sigma_\mu \; \frac{4}{3}\epsilon_{\mu\nu\lambda\sigma}\underbrace{\mathrm{Tr}\left\{(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\lambda\Omega)(\Omega^{-1}\partial_\sigma\Omega)\right\}}_{m\text{ (above)}} \tag{14.2.24}$$

$$\begin{aligned}\mathrm{Tr}\left\{\int d^4x\ F_{\mu\nu} * F_{\mu\nu}\right\} &= \frac{1}{2}\left(\frac{4}{3}\right)\left(24\pi^2\right) m \\ &= 16\pi^2 m \end{aligned} \qquad (14.2.25)$$



m is instanton #.

Sectors of the vacuum are labelled by the <u>winding number</u> n.

$$|n\rangle \qquad (14.2.26)$$

Perform a gauge transformation

$$T_1|n\rangle = |n+1\rangle \qquad (14.2.27)$$

But,

$$[T_1, H] = 0 \qquad (14.2.28)$$

as H is gauge invariant. Thus the vacuum must be an eigenstate of T_1.

Consider the "θ vacuum".

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{\mathrm{i}n\theta}|n\rangle \qquad (14.2.29)$$

i.e.

$$\begin{aligned} T_1|\theta\rangle &= \sum_{n=-\infty}^{\infty} e^{\mathrm{i}n\theta}|n+1\rangle \\ &= e^{-\mathrm{i}\theta}\sum_{n=-\infty}^{\infty} e^{\mathrm{i}(n+1)\theta}|n+1\rangle \\ &= e^{\mathrm{i}\theta}|\theta\rangle \end{aligned} \qquad (14.2.30)$$

Thus we have a vacuum labelled by this parameter θ.
Vacuum-vacuum transition.

$$\begin{aligned} \langle\theta'|e^{-\mathrm{i}Ht}|\theta\rangle &= \delta(\theta'-\theta)\langle\theta'|e^{-\mathrm{i}Ht}|\theta\rangle && (14.2.31) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\langle m|e^{-\mathrm{i}m\theta'}e^{-\mathrm{i}Ht}e^{\mathrm{i}n\theta}|n\rangle \\ &= \sum_{m,n} e^{\mathrm{i}(n\theta-m\theta')}\langle m|e^{-\mathrm{i}Ht}|n\rangle \\ &\quad \theta=\theta' \ \text{ from (14.2.31)} \\ &= \sum_{\nu=-\infty}^{\infty} e^{\mathrm{i}\nu\theta}\int_{\mathcal{A}}^{\mathcal{B}}\mathcal{D}A_\mu e^{\mathrm{i}\int d^4x\ \mathcal{L}_{YM}} \quad (m = n+\nu) && (14.2.32) \end{aligned}$$

($[\mathcal{A} = A_\mu \to \Omega_n^{-1}\partial_\mu\Omega_n$ at $t = -\infty] \Rightarrow n$, $[\mathcal{B} = A_\mu \to \Omega_m^{-1}\partial_\mu\Omega_m$ at $t = +\infty] \Rightarrow m$)

The dominant instanton for this transition has instanton number ν.

$$\nu = \frac{1}{24\pi^2}\mathrm{Tr}\left\{\int d^4x\ F_{\mu\nu} *F_{\mu\nu}\right\} \tag{14.2.33}$$

$$\begin{aligned}\langle\theta'|e^{-\mathrm{i}Ht}|\theta\rangle &= \sum_{\nu=-\infty}^{\infty}\int_n^m \mathcal{D}A_\mu \exp\left\{\mathrm{i}\int d^4x\left(\mathcal{L}_{YM} + \underbrace{\frac{\theta}{24\pi^2}\mathrm{Tr}\left\{F_{\mu\nu} *F_{\mu\nu}\right\}}_{\text{(Violates Parity, CP)}}\right)\right\} \\ &\to \mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}\end{aligned} \tag{14.2.34}$$

$\theta \to$ leads to effective term in action, which leads to the Parity, CP violation. $\theta < 10^{-9}$ (electric dipole moment of neutron).